# CATEGORIES

github.com/danimalabares/stack

Contents



## 1. Definitions

We recall the definitions, partly to fix notation.

**Definition 1.1.** A *category* $\mathcal{C}$ consists of the following data:

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Mor}_{\mathcal{C}}(x, y)$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map $\mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $\mathrm{id}_x \in \mathrm{Mor}_{\mathcal{C}}(x, x)$ such that $\mathrm{id}_x \circ \phi = \phi$ and $\psi \circ \mathrm{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is associative, i.e., $(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi)$ whenever these compositions make sense.

**Definition 1.2.** A *functor* $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
(2) For every $x, y \in \mathrm{Ob}(\mathcal{A})$ a map $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in the following manner: $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for a composable pair $(\phi, \psi)$ of morphisms of $\mathcal{A}$ and $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

## 2. Monomorphisms

**Definition 2.1.** Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathcal{C}$.

(1) We say that $f$ is a *monomorphism* if for every object $W$ and every pair of morphisms $a, b : W \to X$ such that $f \circ a = f \circ b$ we have $a = b$.
(2) We say that $f$ is an *epimorphism* if for every object $W$ and every pair of morphisms $a, b : Y \to W$ such that $a \circ f = b \circ f$ we have $a = b$.

**Definition 2.2.** Let $\mathcal{C}$ be a category, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets.

(1) We say that $\varphi$ is *injective* if for every object $U$ of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that $\varphi$ is *surjective* if for every object $U$ of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

**Lemma 2.3.** *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of $PSh(\mathcal{C})$. A map is an isomorphism if and only if it is both injective and surjective.*

## 3. Presheaves

**Definition 3.1.** A *presheaf of sets* on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to *Sets*. *Morphisms of presheaves* are natural transformations of functors. The category of presheaves of sets is denoted $PSh(\mathcal{C})$ or $\hat{\mathcal{C}}$.

## 4. Yoneda lemma

The Yoneda lemma says that the sections over $a$ of a presheaf $X$ can be described completely as natural transformations between the presheaf $\mathrm{Hom}(-, a)$ and $X$.

**Definition 4.1.** Let $A$ be a category. The *Yoneda embedding* is the functor

$$h : A \to \hat{A}$$

whose value at an object $a$ of $A$ is the presheaf

$$h_a = \mathrm{Hom}_A(-, a).$$

In other words, the evaluation of the presheaf $h_a$ at an object $c$ of $A$ is the set of maps from $c$ to $a$.

**Theorem 4.2** (Yoneda lemma)**.** *For any presheaf $X$ over $A$, there is a natural bijection (in Sets I think!!)*

$$\begin{aligned} \mathrm{Hom}_{\widehat{A}}(h_a, X) &\xrightarrow{\simeq} X_a \\ (h_a \xrightarrow{u} X) &\longmapsto u_a(1_a) \end{aligned}$$

## 5. Internal Hom

**Upshot.** Internal Hom is when the Hom set of two objects in some category is in also an object of the category. Down-to-earth, that for two sheaves $\mathcal{F}, \mathcal{G}$, $U \mapsto \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U)$ is also a sheaf, called $\mathcal{H}om$.

I start with Stacks Project approach.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section **??** that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by $f$ on $\mathcal{F}|_U$ or postcomposing with multiplication by $f$ on $\mathcal{G}|_U$

(it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x)$$

which is rarely an isomorphism.

**Cartesian closed category** In the category of sets there is a bijection $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ that depends naturally on $X$, $Y$ and $Z$. The notions related to this bijection are "Cartesian closed category", "currying" and "internal Hom".

**Definition 5.1.** A category $\mathcal{C}$ is *Cartesian closed* if:

(1) $\mathcal{C}$ has all finite products (Caveat: some require that $\mathcal{C}$ has all finite limits)
(2) For any object $Y$ the functor $- \times Y$ has a right adjoint, which we will denote by $\mathrm{Map}(Y, -)$ or by $-^Y$ .

*Remark* 5.2. By section 3 here, the second property above implies that we get a functor $\mathrm{Map}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, and moreover we get natural isomorphisms $\mathrm{Hom}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Hom}(X \times Y, Z)$ and $\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Map}(X \times Y, Z)$.

## References