INFTY-CATEGORIES

github.com/danimalabares/infty-categories

Contents

1. Other exercises

Definition 1.1. A functor $F : C \to B$ is a *discrete fibration* if for every object c in C and every morphism of the form $g : b \to F(c)$ in B there is a unique morphism $h : d \to c$ in C such that $F(h) = g$.

Exercise 1.2. Prove that discrete fibrations over a category C correspond to presheaves over C.

Proof. First suppose that we are given a presheaf X on C. Define a discrete fibration $F : C/X \to C$ by $(a, s) \mapsto a$ on objects and mapping a morphism $f : (a, s) \to (b, t)$ in C/X to the corresponding morphism $a \to b$ in C. To show F is a discrete fibration let $g : b \to a$ be a morphism in C. Consider $g^* = X(g) : X_a \to X_b$, and the section g^*s of X_b. Then the morphism $h : (b, g^*s) \to (a, s)$ is the only one mapping to g under h.

For the converse let $F : B \to C$ be a discrete fibration over C. To define a presheaf $X : C^{\mathrm{op}} \to \mathrm{Sets}$ let $c \in \mathrm{Ob}\, C$. We assign the set (for now I won't justify why this is a set) of objects in B mapped to c under F. To define the correspondence on morphisms, consider a map $f : c \to d$ in C^{op}. In other words, we have a map in C of the form $f^{\mathrm{op}} : d \to c$. Then to any object in b such that $F(b) = c$, by definition of discrete fibration, we have a unique morphism of B of the form $h : r \to b$ such that $F(h) = f^{\mathrm{op}}$. In particular this means that $F(r) = d$. This gives a function from $X(c)$ to $X(d)$. This situation is described in the following diagram:

$$\begin{tikzcd}
X(d) \ni & r \arrow[r, mapsto] \arrow[d, dotted, "\exists! h"'] & d = F(r) \arrow[d, "f^{\mathrm{op}} = F(h)"] \\
X(c) \ni & b \arrow[r, mapsto, "F"'] & c = F(b)
\end{tikzcd}$$

To check functoriality of X defined in the previous paragraph suppose that $f : c \to d$ and $g : d \to e$ are two morphisms in C^{op}. Like before, we have maps

$f^{\mathrm{op}} : d \to c$ and $g^{\mathrm{op}} : e \to d$.

$$\begin{tikzcd}
X(e) \ni & q \arrow[r, mapsto] \arrow[d, dotted, "\exists! j"'] & e = F(q) \arrow[d, "g^{\mathrm{op}} = F(j)"] \\
X(d) \ni & r \arrow[r, mapsto] \arrow[d, dotted, "\exists! h"'] & d = F(r) \arrow[d, "f^{\mathrm{op}} = F(h)"] \\
X(c) \ni & b \arrow[r, mapsto, "F"'] & c = F(b)
\end{tikzcd}$$

on the other hand, $gf : c \to e$ gives by the same construction a unique map $k : \hat{q} \to b$ such that $F(k) = f^{\mathrm{op}} g^{\mathrm{op}}$. To check that $\hat{q} = q$, observe that by functoriality of F, we have $F(hj) = F(h)F(j) = f^{\mathrm{op}} g^{\mathrm{op}} = F(k)$. By uniqueness of k, we conclude that $k = hj$ and thus $q = \hat{q}$. $\square$

2. Exercises of Rune, Chapter 1

Here's my progress so far on the exercises in [Hau25], Chapter 1.

Exercise 2.1 (Observation 1.4.7)**.** Show that the simplicial sets category Set_Δ has internal Hom S^T for simplicial sets S and T, given by

$$(S^T)_n := \mathrm{Hom}_{\mathsf{Set}_\Delta}(T \times \Delta^n, S)$$

Proof. We need to show, that S^T is the internal Hom in the category Set_Δ. Different notations for the internal Hom are $\mathrm{Map}(-,-)$, $\underline{\mathrm{Hom}}(-,-)$. It must be right adjoint to the functor $U \times -$. That is,

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(U \times S, T) \cong \mathrm{Hom}_{\mathsf{Set}_\Delta}(U, S^T)$$

(I think) I understand the statement correctly but I don't understand how to apply [Cis23, Theorem 1.1.10 (Kan)] nor [Cis23, Remark 1.1.11] to prove it. $\square$

Exercise 2.2 (1.1)**.** If S is a Kan complex, then the relation defining $\pi_0 S$ is an equivalence relation.

Proof.

(1) (Reflexivity.) Let $a \in S_0$. Consider the composition

$$\begin{array}{c} [0] \xrightarrow{d_0} [1] \xrightarrow{f_0} [0] \\ 0 \longmapsto 1 \longmapsto 0 \end{array}$$

since this gives the identity we must have

$$\begin{array}{c} S_0 \xrightarrow{S(f_0)} S_1 \xrightarrow{S(d_0)} [0] \\ a \longmapsto S(f_0)(a) \longmapsto a \end{array}$$

but we can replace d_0 by d_1 and we'd still get the identity, so that $S(d_0)$ also maps $S(f_0)(a)$ to a. In other words, for any $a \in S_0$ the 1-simplex $S(f_0)(a)$ is the desired one.

(2) (Symmetry.) Let $a, b \in S_0 \ldots$

$\square$

Rather informally, I understand a category C to be an *enriched category over D* if for any objects c, d in C, $\mathrm{Hom}(c, d)$ is an object of D. Compositions of morphisms exist and are associative, and there is an identity morphism for every object c in C. (See `https://ncatlab.org/nlab/show/enriched+category` for a formal definition.)

Exercise 2.3 (1.2)**.** Show that Cat_Δ can be described as the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ containing the functors whose simplicial sets of objects are constant.

Proof. To a Set_Δ-enriched category C associate the functor which maps $[n]$ to the category C_n defined as follows. The objects of C_n are the objects of C for all n. For a, b in C, the morphisms of C_n are $\mathrm{Hom}(a, b)_n$. To complete the definition of our functor in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ we must specify where to send a map $f : [n] \to [m]$ in Δ^{op}. It must go to a functor of C to C that fixes all objects and maps a map in $\mathrm{Hom}(a, b)_n$ to the induced map $\mathrm{Hom}(a, b)_m$ by the presheaf $\mathrm{Hom}(a, b)$.

Now we need to show that every functor whose simplicial sets of objects is constant (i.e. constant functor $\Delta^{\mathrm{op}} \to \mathsf{Cat}$) can be expressed this way. Suppose F is such a functor with value C. Associate the Set_Δ-enriched category C with $\mathrm{Hom}(a, b)_n$ given by $\mathrm{Hom}_{F([n])}(a, b)$.

(Hopefully my understanding of the term "full subcategory" is not wrong?) $\square$

Exercise 2.4 (1.3)**.** Show that $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is fully faithful.

Proof. We need to show that for any categories A, B, $\mathrm{Hom}(A, B) = \mathrm{Fun}(A, B)$ is in "bijection" with $\mathrm{Hom}_{\mathsf{Set}_\Delta}(NA, NB)$. Recall that NA is the presheaf that maps $[n]$ to the set of composible sequence of n morphisms in A. Then to a functor $F : A \to B$ we associate the map that sends a sequence of n morphisms in A to the respective sequence of n morphisms in B after applying F to each object and map.

Conversely, given a morphism in Set_Δ from NA to NB we can reconstruct a functor from A to B by interpreting objects of A as NA_0 and maps as NA_1. $\square$

References

[Cis23] Denis-Charles Cisinsky, *Higher categories and homotopical algebra*, 2023.
[Hau25] Rune Haugseng, *Yet another introduction to infty-categories*, 2025.