INFTY-CATEGORIES

github.com/danimalabares/infty-categories

Contents

1. Discrete fibrations

Definition 1.1. A functor $F : C \to B$ is a *discrete fibration* if for every object c in C and every morphism of the form $g : b \to F(c)$ in B there is a unique morphism $h : d \to c$ in C such that $F(h) = g$.

Exercise 1.2. Prove that discrete fibrations over a category C correspond to presheaves over C.

Proof. First suppose that we are given a presheaf X on C. Define a discrete fibration $F : C/X \to C$ by $(a, s) \mapsto a$ on objects and mapping a morphism $f : (a, s) \to (b, t)$ in C/X to the corresponding morphism $a \to b$ in C. To show F is a discrete fibration let $g : b \to a$ be a morphism in C. Consider $g^* = X(g) : X_a \to X_b$, and the section g^*s of X_b. Then the morphism $h : (b, g^*s) \to (a, s)$ is the only one mapping to g under h.

For the converse let $F : B \to C$ be a discrete fibration over C. To define a presheaf $X : C^{\mathrm{op}} \to \mathrm{Sets}$ let $c \in \mathrm{Ob}\, C$. We assign the set (for now I won't justify why this is a set) of objects in B mapped to c under F. To define the correspondence on morphisms, consider a map $f : c \to d$ in C^{op}. In other words, we have a map in C of the form $f^{\mathrm{op}} : d \to c$. Then to any object in b such that $F(b) = c$, by definition of discrete fibration, we have a unique morphism of B of the form $h : r \to b$ such that $F(h) = f^{\mathrm{op}}$. In particular this means that $F(r) = d$. This gives a function from $X(c)$ to $X(d)$. This situation is described in the following diagram:

$$\begin{tikzcd}
X(d) \ni & r \arrow[r, mapsto] \arrow[d, dotted, "\exists ! h"'] & d = F(r) \arrow[d, "f^{\mathrm{op}} = F(h)"] \\
X(c) \ni & b \arrow[r, mapsto, "F"'] & c = F(b)
\end{tikzcd}$$

To check functoriality of X defined in the previous paragraph suppose that $f : c \to d$ and $g : d \to e$ are two morphisms in C^{op}. Like before, we have maps $f^{\mathrm{op}} : d \to c$ and $g^{\mathrm{op}} : e \to d$.

$$\begin{tikzcd}
X(e) \ni & q \arrow[r, mapsto] \arrow[d, dotted, "\exists ! j"'] & e = F(q) \arrow[d, "g^{\mathrm{op}} = F(j)"] \\
X(d) \ni & r \arrow[r, mapsto] \arrow[d, dotted, "\exists ! h"'] & d = F(r) \arrow[d, "f^{\mathrm{op}} = F(h)"] \\
X(c) \ni & b \arrow[r, mapsto, "F"'] & c = F(b)
\end{tikzcd}$$

on the other hand, $gf : c \to e$ gives by the same construction a unique map $k : \hat{q} \to b$ such that $F(k) = f^{\mathrm{op}} g^{\mathrm{op}}$. To check that $\hat{q} = q$, observe that by functoriality of F, we have $F(hj) = F(h)F(j) = f^{\mathrm{op}} g^{\mathrm{op}} = F(k)$. By uniqueness of k, we conclude that $k = hj$ and thus $q = \hat{q}$. □

2. Simplicial sets

Definition 2.1. The *simplex category* is the category of ordinals, i.e. non-empty finite ordered sets

$$[n] = \{0 < 1 < \ldots < n\}, \quad n = 0, 1, \ldots$$

with order preserving maps of sets.

Objects of the simplicial category Δ are not called simplices. Instead, simplices are type of simplicial set:

Definition 2.2. A *simplicial set* is a presheaf on Δ, i.e. an element of $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets})$.

Definition 2.3. The *n-simplex* is the representable simplicial set $\mathrm{Hom}(-, [n]) := \Delta^n$.

So far the upshot for me is that simplicial sets are like a generalization of a triangulated topological space. The following construction shows how to associate to $S \in \mathsf{sSet}$ a topological space $|S| \in \mathsf{Top}$. In fact, the weak equivalences in sSet can be defined using weak equivalences in Top just like in Hatcher.

Here's some copy-paste from [Hau25, Chapter 1] transcribed by ChatGPT:

The category Δ is generated by

- the *face maps* $d_i : [n-1] \hookrightarrow [n]$ that skip $i \in [n]$,
- the *degeneracy maps* $s_i : [n+1] \xrightarrow{\text{surj.}} [n]$ that repeat $i \in [n]$,

subject to certain relations.

Definition 2.4. The *topological n-simplex* $|\Delta^n|$ is the topological space

$$|\Delta^n| := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1,\ 0 \leq x_i \leq 1\}$$

(with the subspace topology from $\mathbb{R}^{n+1}$). For $\varphi : [n] \to [m]$ we can define a continuous map $\varphi_* : |\Delta^n| \to |\Delta^m|$ by

$$\varphi_*(x_0, \ldots, x_n)_i = \sum_{j:\varphi(j)=i} x_j.$$

This gives a functor $|\Delta^\bullet| : \Delta \to \mathsf{Top}$.

We can then define the *singular simplicial set functor*

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{Set}_\Delta$$

as

$$\mathrm{Sing}(X) = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^\bullet|, X).$$

This has a left adjoint $|-| : \mathbf{Set}_\Delta \to \mathbf{Top}$, called the *geometric realization functor*, which is the unique colimit-preserving functor that extends $|\Delta^\bullet|$ via the Yoneda embedding. More concretely, we can define $|S|$ for a simplicial set S as the quotient of $\coprod_n S_n \times |\Delta^n|$ where we identify $(\sigma, \varphi_* p)$ with $(\varphi^* \sigma, p)$ for $\varphi : [n] \to [m]$, $\sigma \in S_n$ and $p \in |\Delta^m|$. Informally, we build the topological space $|S|$ out of simplices according to the "blueprint" S.

If we say that a morphism $S \to T$ in $\mathbf{Set}_\Delta$ is a weak equivalence if $|S| \to |T|$ is a weak homotopy equivalence, then the relative category consisting of $\mathbf{Set}_\Delta$ with these weak equivalences describes the same homotopy theory as that of topological spaces; for example, the counit map $|\mathrm{Sing}\, X| \to X$ for a topological space X is always a weak homotopy equivalence. We can also describe the weak equivalences of simplicial sets as homotopy equivalences (or describe them via homotopy groups) if we restrict to a class of nice objects, which we will introduce next.

Exercise 2.5. It should be possible to show that the geometric realization of Δ^n is in fact the topological n-simplex in $\mathbb{R}^n$, right?

I wonder what is the relationship between sSet and Top. How much information do simplicial sets give us about topological spaces? Most likely we will be able to see any tringulated space, say, a CW complex as a simplicial set. And also most likely, there are pathological topological spaces that cannot be retrieved by the geometric realization functor. But probably we just don't care, and forget Top (again).

Now I put some exercises.

Exercise 2.6 (Observation 1.4.7)**.** Show that the simplicial sets category Set_Δ has internal Hom S^T for simplicial sets S and T, given by

$$(S^T)_n := \mathrm{Hom}_{\mathsf{Set}_\Delta}(T \times \Delta^n, S)$$

Proof. We need to show, that S^T is the internal Hom in the category Set_Δ. Different notations for the internal Hom are $\mathrm{Map}(-,-)$, $\underline{\mathrm{Hom}}(-,-)$. It must be right adjoint to the functor $U \times -$. That is,

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(U \times S, T) \cong \mathrm{Hom}_{\mathsf{Set}_\Delta}(U, S^T)$$

(I think) I understand the statement correctly but I don't understand how to apply [Cis23, Theorem 1.1.10 (Kan)] nor [Cis23, Remark 1.1.11] to prove it.

□

Exercise 2.7 (1.1). If S is a Kan complex, then the relation defining $\pi_0 S$ is an equivalence relation.

Proof. (1) (Reflexivity.) Let $a \in S_0$. Consider the composition

$$\begin{array}{ccccc} [0] & \xrightarrow{d_0} & [1] & \xrightarrow{f_0} & [0] \\ 0 & \longmapsto & 1 & \longmapsto & 0 \end{array}$$

since this gives the identity we must have

$$\begin{array}{ccccc} S_0 & \xrightarrow{S(f_0)} & S_1 & \xrightarrow{S(d_0)} & [0] \\ a & \longmapsto & S(f_0)(a) & \longmapsto & a \end{array}$$

but we can replace d_0 by d_1 and we'd still get the identity, so that $S(d_1)$ also maps $S(f_0)(a)$ to a. In other words, for any $a \in S_0$ the 1-simplex $S(f_0)(a)$ is the desired one.

(2) (Symmetry.) Let $a, b \in S_0 \ldots$

□

Rather informally, I understand a category C to be an *enriched category over D* if for any objects c, d in C, $\mathrm{Hom}(c, d)$ is an object of D. Compositions of morphisms exist and are associative, and there is an identity morphism for every object c in C. (See `https://ncatlab.org/nlab/show/enriched+category` for a formal definition.)

Exercise 2.8 (1.2). Show that Cat_Δ can be described as the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ containing the functors whose simplicial sets of objects are constant.

Remark 2.9. The phrase "simplicial sets of objects are constants" means the following. Consider the functor $\mathsf{Cat} \to \mathsf{Set}$ that maps a category to its set of objects (I suppose we may take Set to be a universe), which induces for every functor in $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ a functor in $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$. We mean to say the latter map is constant.

Proof. We need to construct a fully faithful functor

$$F : \mathsf{Cat}_\Delta \to \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$$

whose image is the subcategory of functors whose simplicial sets of objects are constant.

To a Set_Δ-enriched category C associate the functor $F(C)$ which maps $[n]$ to the category C_n, which is defined as follows. The objects of C_n are the objects of C for all n. (Notice that once we define the functor completely, this property will make it indeed a functor whose simplicial sets of objects are constant.) For a, b in C, the morphisms of C_n are $\mathrm{Hom}(a, b)_n$. To a map $f : [n] \to [m]$ in Δ^{op}, define $F(C)$ to give the functor of C_m to C_n that fixes all objects and maps a map in $\mathrm{Hom}(a, b)_m$ to the induced map $\mathrm{Hom}(a, b)_n$ by the presheaf $\mathrm{Hom}(a, b)$.

Now let's define how F acts on morphisms. (This definition is just what it should be, but let's go over it.) Choose two Set_Δ-enriched categories C, D and consider their corresponding functors $F(C), F(D)$. Fix a morphism $\varphi \in \mathrm{Hom}_{\mathsf{Cat}_\Delta}(C, D)$. Define a morphism (of presheaves of categories) $F(\varphi) : F(C) \to F(D)$ defined as a collection of maps $F(C)_n \to F(D)_n$ given on objects by φ and on morphisms also given by φ, using that φ is a morphism of Cat_Δ to ensure naturality.

Functoriality of F follows from functoriality of each φ as in the previous paragraph.

Now let's confirm that F is faithful, that is, it induces injections on the Hom sets. Suppose $\varphi, \psi \in \mathrm{Hom}_{\mathsf{Cat}_\Delta}(C, D)$ are such that $F(\varphi) = F(\psi)$. By definition of $F(\varphi)$ and $F(\psi)$, it is immediate that φ and ψ coincide on objects. In fact, it is also immediate that they coincide on morphism and as simplicial sets by definition.

To prove F is fully faithful we only need to check surjectivity of the induced maps in Hom sets. Pick a morphism of presheaves of categories, denote it $F(\varphi)$, between two presheaves of categories $F(C)$ and $F(D)$, both of whose simplicial sets of objects are constant, namely two sets C and D. Then we can define two Set_Δ-enriched categories, which we also denote by C and D, by defining their objects to be the sets C and D, and their morphisms to be the collections of all the induced morphisms by $F(C)$ and $F(D)$ coming from morphisms of Δ^{op}. Then it is immediate that the set $\mathrm{Hom}(C, D)$ is indeed a simplicial set. Thus $C, D \in \mathsf{Cat}_{\mathsf{Set}_\Delta}$. Further, we can define a morphism $\varphi \in \mathrm{Hom}_{\mathsf{Cat}_\Delta}(C, D)$ which maps on objects as any of the induced maps by the morphism of presheaves of categories we started with (since both of the simplicial sets of objects of the corresponding categories are constant!) and on morphisms as well (any morphism of C was defined as the induced map by $F(C)$ coming from a map of Δ^{op}). It is clear that this morphism is mapped to $F(\varphi)$ under F. ◻

Exercise 2.10 (1.3)**.** Show that $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is fully faithful.

Proof. We need to show that for any categories A, B, $\mathrm{Hom}(A, B) = \mathrm{Fun}(A, B)$ is in "bijection" with $\mathrm{Hom}_{\mathsf{Set}_\Delta}(NA, NB)$. Recall that NA is the presheaf that maps $[n]$ to the set of composible sequence of n morphisms in A. Then to a functor $F : A \to B$ we associate the map that sends a sequence of n morphisms in A to the respective sequence of n morphisms in B after applying F to each object and map.

Conversely, given a morphism in Set_Δ from NA to NB we can reconstruct a functor from A to B by interpreting objects of A as NA_0 and maps as NA_1. ◻

3. ∞-groupoids

Apparently the philosophy is that we will not formally construct ∞-categories (nor ∞-groupoids) but barely start using them. So we admit "facts" such as "there are objects called ∞-*groupoids*". We shall admit that although there are points (and paths), we cannot distinguish between points if there is a path joining two points. Thus we don't really have points but a set $\pi_0 X$ of path components.

There are also homotopies between paths, and homotopies between homotopies, and so on.

Also, there are *maps* or *morphisms* between groupoids, homotopies between morphisms, homotopies between homotopies, and so on. In fact, all those things form an ∞-groupoid we denote by $\mathsf{Map}(X, Y)$.

We can *compose* maps, and there is an *identity morphism* for every ∞-groupoid X. Composition is unital and associative in the only way that makes sense (?), which is up to homotopy.

Definition 3.1. An *equivalence* of groupoids is a pair of maps that may be composed not be the identities, but homotopical to the identities.

Dani: it looks like the main idea is to care about anything only up to homotopy.

Sets are groupoids and for any set

$$\mathrm{Hom}_{\mathsf{Set}}(\pi_0 X, S) \xrightarrow{\simeq} \mathsf{Map}(X, S)$$

is an equivalence.

Then the 1-point set ends up being the terminal ∞-groupoid. The empty set is also an ∞-groupoid, and it is the initial one.

Given morphisms $X \to Z$ and $Y \to Z$, there exists a pullback square

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

where "square" means not that it is commutative, but that there exists an homotopy between the compositions (commutative up to homotopy). Looks like it basically a fibre product (see Definition **??**) up to homotopy.

The *fibre* $f^{-1}(b)$ at b of a map $f : E \to B$ is defined as the pullback

$$\begin{tikzcd} f^{-1}(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ \{b\} \arrow[r] & B. \end{tikzcd}$$

The *product* $X \times Y$ of two ∞-groupoids X and Y is defined as the pullback

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y \arrow[r] & *. \end{tikzcd}$$

Composition of squares is another square, and the composition of two pullback squares is another pullback square.

Definition 3.2. For points $x, y \in X$, the *path space* $X(x, y)$ is the pullback

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \{x\} \arrow[d] \\ \{y\} \arrow[r] & X. \end{tikzcd}$$

Exercise 3.3 (2.1.1)**.** Assuming that pushouts exist and have the expected universal property, show that $\Omega^n_x X \simeq \mathsf{Map}_*(S^n, X)$, where the n-th sphere is the pushout

$$S^n := * \amalg_{S^{n-1}} *,$$

and the space of pointed maps is the pullback

$$\begin{tikzcd} \mathsf{Map}_*(S^n, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Map}(S^n, X) \arrow[d] \\ \{x\} \arrow[r] & \mathsf{Map}(*, X). \end{tikzcd}$$

Proof. Since both $\Omega^1_x X$ and $\mathsf{Map}_*(S^1, X)$ are pullbacks (and pullbacks are unique up to homotopy), it's enough to show that they are the pullback of the same diagram (up to homotopy).

First notice that $\mathsf{Map}(*, X) \simeq X$ in an obvious way: we identify a map $* \to X$ with the image of $*$. To identify $\mathsf{Map}(S^n, X)$ with $\{x\} \simeq \mathsf{Map}(*, X)$ pick a map $* \to X$. Now consider the universal property of pushouts:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \arrow[ddr] & \\ * \arrow[r] \arrow[drr] & S^n = * \amalg_{S^{n-1}} * \arrow[dr, dashed, "\exists!"] & \\ & & X \end{tikzcd}$$

□

Definition 3.4. The *n-th homotopy group* of a groupoid is $\pi_0 \Omega^n_x$.

$\pi_1(X, x)$ is a group, and $\pi_n(X, x)$ is an abelian group for $n > 1$.

Homotopy group detect equivalences: a map between groupoids is an equivalence if and only if all homotopy groups are isomorphic (as groups for $n > 1$ or as sets for $n = 0$).

Lemma 3.5. *Equivalences of ∞-groupoids satisfy the 3-for-2 property: if f and g are composable and two out of $f, g, f \circ g$ are equivalences, so is the third.*

Lemma 3.6. *A morphism $f : X \to Y$ is an equivalence if and only if $\pi_0 X \to \pi_0 Y$ is surjective and $X(x, x') \to Y(fx, fx')$ is an equivalence for all $x, x' \in X$.*

Lemma 3.7. *For an ∞-groupoid X, the map $X \to \pi_0 X$ is an equivalence if and only if $X(x, x')$ is either empty of contractible for all $x, x' \in X$.*

Lemma 3.8. *A groupoid X is contracible iff $X(x, x')$ is either empty or contractible for all $x, x' \in X$.*

We also state as a fact that for a map $f : E \to B$, a point $b \in B$ and a point $e \in f^{-1}(b)$, there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(f^{-1}(b), e) \longrightarrow \pi_n(E, e) \longrightarrow \pi_n(B, b) \longrightarrow \pi_{n-1}(f^{-1}(b), e) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow \pi_0(B),$$

with appropriate interpretation near the end since π_0 is only a set while π_1 is a group.

Proposition 3.9. *A map $f : E \to B$ is an equivalence if and only if all the fibers $f^{-1}(b)$ for $b \in B$ are contractible.*

Proof. This is immediate from homotopy long exact sequence and the fact that homotopy groups detect equivalences. □

Exercise 3.10 (2.2)**.** Use the 5-lemma to show that given a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

the morphism f is an equivalence if and only if the induced maps on the fibres $p^{-1}(b) \to q^{-1}(b)$ are equivalences for all $b \in B$.

Proof. For the converse implication,

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(B,b) \arrow[r] \arrow[d, "\simeq"] & \pi_n(p^{-1}(b),x) \arrow[r] \arrow[d, "\simeq"] & \pi_n(X,x) \arrow[r] \arrow[d, dotted, "\simeq"] & \pi_n(B,b) \arrow[r] \arrow[d, "\simeq"] & \pi_{n-1}(p^{-1}(b),x) \arrow[r] \arrow[d, "\simeq"] & \cdots \\ \cdots \arrow[r] & \pi_{n+1}(B,b) \arrow[r] & \pi_n(q^{-1}(b),f(x)) \arrow[r] & \pi_n(Y,f(x)) \arrow[r] & \pi_n(B,b) \arrow[r] & \pi_{n-1}(q^{-1}(b),f(x)) \arrow[r] & \cdots \end{tikzcd}$$

and for the forward implication just do the same with the map $\pi_n(p^{-1}(b),x) \to \pi_n(q^{-1},f(x))$ in the center. $\square$

Lemma 3.11. *A commutative square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback if and only if for every $x \in X$, the induced map on fibres is an equivalence.

Proof. We need to check that $X' \simeq Y' \times_Y X$. I think we use the universal property of pullbacks to get a map $X' \to Y' \times_Y X$. To check this is an equivalence we use Exercise 3.10 as in

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "f'"'] & & Y' \times_Y X \arrow[dl, "p"] \\ & X. & \end{tikzcd}$$

We see that the map on fibers is non other than $f'^{-1}(x) \to f^{-1}(g(x))\ldots$ but why? This ammounts to showing that $p^{-1}(x) \simeq f^{-1}(g(x))$.

$$\begin{tikzcd} f^{-1}(x) \arrow[r, dashed] \arrow[d] & p^{-1}(x) \arrow[r, "\simeq?"] \arrow[dd] & f^{-1}(g(x)) \arrow[dd] \\ X' \arrow[dr] \arrow[drr] \arrow[ddr, "f'"'] & & \\ & X \times_Y Y' \arrow[r] \arrow[d, "p"'] & Y' \arrow[d, "f"] \\ & X \arrow[r, "g"'] & Y \end{tikzcd}$$

$\square$

Exercise 3.12 (2.3)**.** Consider a commutative square...

Exercise 3.13 (2.4)**.** Suppose we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X'' \arrow[d] \\ Y \arrow[r] & Y' \arrow[r] & Y''. \end{tikzcd}$$

(1) If the right and composite squares are both pullbacks, then so is the left-hand square.
(2) If $\pi_0 Y \to \pi_0 Y'$ is surjective and the left and composite squares are both pullbacks, then so is the right-hand square.

Proof. (1) To use Corollary 2.1.23 we look at the fibers and apply Corollary 2.1.17: the 3-for-2 property.
(2) I'm not sure why is that condition on $\pi_0 Y \to \pi_0 Y'$... looks like the same argument should work.

□

4. Monomorphisms of ∞-groupoids

Monomorphisms can be easily recalled as the maps whose fibers are either empty or contractible, and, equivalently, as the maps that are injective on π_0 and the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \pi_0 X \arrow[r] & \pi_0 Y \end{tikzcd}$$

is a pullback.

In class I picked up that a map is monomorphism if and only if it's injective on π_0 and isomorphism in higher homotopy groups (after fixing a base point).

Monomorphisms of ∞-groupoids are...

Exercise 4.1 (2.6)**.** Show that monomorphisms are closed under base change.

This means that in a pullback square, if the vertical arrow is a monomorphism, then so is the other vertical arrow.

Remark 4.2. Using Exercise 3.10, it follows that f is a monomorphism if and only if for all $x, x' \in X$, the induced morphism on path spaces $X(x, x') \to Y(fx, fx')$ is an equivalence.

The following lemma is very similar to Proposition 3.9. The difference is that in that proposition we require all the fibers to be contractible, and in the following we allow some of them to be empty.

Lemma 4.3. *A morphism of ∞-groupoids $f : X \to Y$ is a monomorphism if and only if the fibres are all either empty of contractible.*

Proof. Using Lemma 3.11, which says a map is a pullback iff induced maps on fibers are equivalences. □

Remark 4.4. There is a caveat in Lemma 2.2.3 on the claim that the vertical map on the left in the following diagram is “diagonal map”

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d] \\
\bullet \arrow[r] & X \times X Y \times Y \arrow[r] & {}
\end{tikzcd}$$

This should be proved.

Proposition 4.5. *If $X \to Y$ is a monomorphism, then $\pi_0 X \to \pi_0 Y$ is a monomorphism of sets, and the commutative square*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
\pi_0 \arrow[r] & \pi_0 Y
\end{tikzcd}$$

is a pullback.

Exercise 4.6. Let X be an ∞-groupoid and consider a subset $S \subseteq \pi_0 X$. Show that if we form the pullback

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
S \arrow[r, hook] & \pi_0 X
\end{tikzcd}$$

then the induced map $\pi_0 Y \to S$ is an isomorphism.

Proof. Might be kind of nonsense but I thought this: think of π_0 as a functor from ∞-groupoids to Sets. Note that this functor preserves pullbacks almost tautologically: in fact, morphisms of groupoids are defined to be morphisms of the π_0's along with some homotopies and so on. Then $\pi_0 Y$ is the pullback of the square above after applying the functor π_0. But so is S by trivial reasons. Then by universal property of the pullback in Sets there is a unique morphism $S \to \pi_0 Y$ which inverts $\pi_0 Y \to S$, i.e.

$$\begin{tikzcd}
S \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& \pi_0 Y \arrow[r] \arrow[d] & \pi_0 X \arrow[d] \\
& S \arrow[r] & \pi_0 X
\end{tikzcd}$$

□

See remark 2.2.8. Looks like there is a factorization structure involved.

Exercise 4.7 (2.8)**.** Given a commutative triangle

$$\begin{tikzcd}
X \arrow[rr, "f"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& B &
\end{tikzcd}$$

of ∞-groupoids, the morphism f is a monomorphism if and only if for all $b \in B$, the induced map on fibres $X_b \to Y_b$ is a monomorphism.

Proof. By Lemma 2.2.3 it's enough to show that all the fibres of f are either empty or contractible. Let $y \in Y$. The map $X_{q(y)} \to Y_{q(y)}$ induced from f is a monomorphism by hypothesis, so that its fibers are empty or contractible. Since $f(q(y)) = p^{-1}(q)$, the fibers of the fiber map coincide with the fiber of the original map f. $\square$

Exercise 4.8 (2.9)**.** Show that if X is an ∞-groupoid, then so is $\mathsf{Fun}(\mathcal{C}, X)$ for any ∞-category $\mathcal{C}$.

Proof. We should use that ∞-categories are Cartesian closed. $\square$

5. ∞-categories

There are objects called ∞-*categories*. ...

Given an ∞-category $\mathcal{C}$, there exists a *localization* $\|\mathcal{C}\|$ to an ∞-groupoid, with a canonical map $\mathcal{C} \to \|\mathcal{C}\|$, (which is not equivalent to $\mathcal{C}$, but) such that for any ∞-groupoid X the induced map

$$\mathsf{Map}(\|\mathcal{C}\|, X) \to \mathsf{Map}(\mathcal{C}, X)$$

is an equivalence. Moreover $\|[1]\| \simeq *$.

It turns out that this concept of localization generalizes the notion of geometric realization for simplices.

We say that [0] and [1] "detect equivalences" in the following sense: a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if the maps

$$\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}, \qquad \mathsf{Map}([1], \mathcal{C}) \to \mathsf{Map}([1], \mathcal{D})$$

are equivalences of ∞-groupoids. (Because $\mathcal{C}^{\simeq} \simeq \mathsf{Map}([0], \mathcal{C})$.) To understand this you can think that $\mathsf{Map}([1], \mathcal{C})$ is like the space of all Homs, which is a groupoid, whose arrows are natural equivalences.

Lemma 5.1. *The following are equivalent for an ∞-category $\mathcal{C}$:*

(1) *$\mathcal{C}$ is an ∞-groupoid.*
(2) *$\mathcal{C}^{\simeq} \to \mathcal{C}$ is an equivalence.*
(3) *The map*
$$\mathcal{C}^{\simeq} \simeq \mathsf{Map}([0], \mathcal{C}) \to \mathsf{Map}([1], \mathcal{C})$$
induced by $[1] \to [0]$ is an equivalence of ∞-groupoids.
(4) *The functor*
$$\mathcal{C} \to \mathsf{Fun}([1], \mathcal{C})$$
induced by $[1] \to [0]$ is an equivalence of ∞-categories.

Proof. The first two conditions are equivalent by definition of $\mathcal{C}^{\simeq}$. The third condition is implied by the second one using the object $\|\cdot\|$ postulated above (which applies when $\mathcal{C}$ is a groupoid). **Caveat:** I think the fiber of this map is always a point!

($3 \implies 2$) actually makes sense: we can construct a commutative diagram with left and top arrows equivalences (why the left one is an equivalence?). By hypothesis, the one on the right is also an equivalence. Then the one on the bottom is too, and since [0] and [1] detect equivalences, we are done.

($3 \implies 4$) We need to apply $(-)^{\simeq}$ and $\mathsf{Map}([1], -) = \mathsf{Fun}([1], -)^{\simeq} \ldots$ $\square$

6. Segal spaces

Remark 6.1. Think of a Segal space as satisfying

$$\mathsf{Map}(\mathrm{Spine}^n, X) \simeq \mathsf{Map}(\Delta^n, X)$$

where Spine^n is a simplicial construction. In the case of a tetrahedon, Spine is defined as the chain of edges joining the four vertices 0,1,2,3.

The *Kan condition* is

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "i"] & X \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

for all i, and is called strict/weak whether the existence is unique or not. The *inner Kan condition* is if we require for $i \neq 0$ and $i \neq n$.

Why is this so important?

Definition 6.2. The *nerve* functor $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is defined by

$$\mathcal{C} \mapsto \mathsf{Hom}_{\mathsf{Cat}}([\bullet], \mathcal{C}),$$

so that $N\mathcal{C}_n$ is the set of all composable sequences of n morphisms.

Notice that $N\mathcal{C}_0$ is the set of all objects of $\mathcal{C}$ and $N\mathcal{C}_1$ is the set of all morphisms. This is crucial for the following exercise:

Exercise 6.3. $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is fully faithful.

Proof. To a functor $F \in \mathsf{Hom}(\mathcal{C}, \mathcal{D})$ we assign a simplicial set $N(F)_n : N(\mathcal{C})_n \to N(\mathcal{D})_n$ defined in the obvious way: we map a sequence

$$\bullet \xrightarrow{f_1} \bullet \to \cdots \bullet \xrightarrow{f_n} \bullet$$

to

$$\bullet \xrightarrow{F(f_1)} \bullet \to \cdots \bullet \xrightarrow{F(f_n)} \bullet$$

This functor is well defined as a functor of simplicial sets by functoriality of F. To check fully faithfullness we use the fact that we can reconstruct a category from $N(F)_0$ and $N(F)_1$. $\square$

Thus, the Kan condition allows to see categories as simplicial sets. Here are possible generalizations of this:

	strict	weak
all i	groupoids	∞-groupoids
inner	categories	∞-categories.

7. Segal condition

The *Segal condition* is

$$\mathsf{Map}([n], X) \xrightarrow{\simeq} \mathsf{Map}([1], X) \times_{\mathsf{Map}([0],X)} \cdots \times_{\mathsf{Map}([0],X)} \mathsf{Map}([1], X)$$

8. Lifting properties

The key to proving anything related to left orthogonality is not to use the usual lifting diagram but the equivalent definition involving Map.

Definition 8.1. For morphisms $\ell : A \to B$ and $r : X \to Y$ we say that r is *right orthogonal* to ℓ (and dually that ℓ is *left orthogonal* to r if for any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\ell"'] & X \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

the space of diagonal lifts $B \to X$ is contractible. (I think of this as: there exists a unique lift.)

The key observation is that a choice of horizontal arrows making the former diagram commute is the same as a choice of element in $\mathsf{Map}(B, Y) \times_{\mathsf{Map}(A,Y)} \mathsf{Map}(A, X)$. Indeed, given a map $A \to X$, we may poscompose with r to obtain a map $A \to Y$, and similarly precomposing with ℓ any map $B \to Y$. Intuitively, the pullback of these two maps is the set of pairs of maps $A \to X$ and $B \to Y$ that give the same map $A \to Y$, i.e. that the square commutes.

But a lift $B \to X$ also determines a map $A \to X$ and a map $B \to Y$. All this to say that a choice of horizontal arrows, i.e. an element of the pullback $\mathsf{Map}(B, Y) \times_{\mathsf{Map}(A,Y)} \mathsf{Map}(A, X)$, that comes from a choice of map $B \to X$ is the same as a fibre of the map

$$\mathsf{Map}(B, X) \to \mathsf{Map}(B, Y) \times_{\mathsf{Map}(A,Y)} \mathsf{Map}(A, X).$$

Then the definition is equivalent to this fibre being contractible. In turn, this is equivalent to the map being an equivalence. By putting $\mathsf{Map}(B, X)$ instead of the pullback, this is also equivalent to the commutative square

$$\begin{tikzcd} \mathsf{Map}(B, X) \arrow[r, "r_*"] \arrow[d, "\ell^*"'] & \mathsf{Map}(B, Y) \arrow[d, "\ell^*"] \\ \mathsf{Map}(A, X) \arrow[r, "r_*"'] & \mathsf{Map}(A, Y) \end{tikzcd}$$

being a pullback. So that's the right way to look at things, apparently.

Example 8.2. It's the same for a map $f : X \to Y$ to be a monomorphism and to be right orthogonal to $* \amalg * \to *$. To see it, draw the diagram of right orthogonality and then put it in the Map pullback form. Using that $\mathsf{Map}(* \amalg *, X) \simeq X \times X$, which follows from writing the definition of $* \amalg *$ as the pushout of the empty space, we obtain the definition of monomorphism.

Lemma 8.3. *Epimorphisms in Gpd_∞ are left orthogonal to monomorphisms.*

Exercise 8.4. Show that a map is left orthogonal to itself if and only if it is an equivalence.

Proof. We use Proposition 3.9. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted] & Y. \end{tikzcd}$$

If f is an equivalence then by the existence of the lift every fiber consists of only one point and thus it is contractible. Conversely, if every fiber is contractible we define the unique lift as the fiber of every point. ☐

Lemma 8.5. *Suppose $f : A \to B$ is left orthogonal to a map $r : X \to Y$. Then a map $g : B \to C$ is left orthogonal to r if and only if $g \circ f$ is.*

Proof. By 3-for-2 for pullbacks, i.e. Exercise 3.13. ☐

It is important to remember that being a monomorphism is the same as square with π_0 is pullback. That is the key to Lemma 2.4.3. For one implication we need to put surjectivity on π_0 to apply 3-for-2.

Look at the case for sets, it's also true that epimorphisms is orthogonal to monomorphisms.

Lemma 8.6. *Suppose we have a commutative diagram*

(8.6.1)
$$\begin{tikzcd} X \arrow[d, "f"] & Y \arrow[l] \arrow[r] \arrow[d, "g"'] & Z \arrow[d, "h"'] \\ X' & Y' \arrow[l] \arrow[r] & Z, \end{tikzcd}$$

such that each of the morphisms f, g and h is left orthogonal to a morphism $r : U \to V$. Then the induced morphism on pushouts $X \amalg_Y Z \to X' \amalg_{Y'} Z'$ is also left orthogonal to r.

Proof. The induced map on pushouts is

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \arrow[drr] & & \\ X \arrow[r] \arrow[dr] & X \amalg_Y Z \arrow[drr, dashed, "F"] & & Z' \arrow[d] \\ & X' \arrow[rr] & & X' \amalg_{Y'} Z' \end{tikzcd}$$

To construct a lift of the form

$$\begin{tikzcd} X \amalg_Y Z \arrow[r] \arrow[d] & U \arrow[d, "r"] \\ X' \amalg_{Y'} Z' \arrow[r] \arrow[ur, dotted] & V \end{tikzcd}$$

we use the universal product of pushout for $X' \amalg_{Y'} Z'$, that is, we shall be done once we construct a digram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X' \arrow[r] & U. \end{tikzcd}$$

Notice that this is indeed the case: the universal property of the pushout in homotopy sense means that the space of maps is contractible.

Such a map is obviously constructed by composing $Y' \to X'$ as in Diagram 8.6.1 followed by the lift of f. Likewise we construct a map $Y' \to V$. And in fact **it is not needed that g is orthogonal to r.** ☐

Proof.

$$\begin{tikzcd} X \arrow[d] & Y \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[d] & Y \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[d] & Y \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[d] \\ X' \amalg_Y Y' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[d] \\ X' & Y' \arrow[l] \arrow[r] & Z' \end{tikzcd}$$

In each row of vertical arrows there is one that is not trivial. We will be done once we show… using the next lemma.

Every triple of horizontal arrows gives a pushout. Every triple of vertical arrows gives a morphism of pushouts. In fact, we have that for every triple of horizontal arrows, the corresponding pushout morphism is a pushout of the corresponding nontrivial vertical arrow, e.g. on the top part we get

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \amalg_Y Z \arrow[d] \\ X' \arrow[r] & X' \amalg_Y Z \end{tikzcd}$$

☐

Lemma 8.7. *Suppose we have a pushout square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "f'"] \\ C \arrow[r] & B \amalg_A C \end{tikzcd}$$

where f is left orthogonal to a morphism r. Then f' is also left orthogonal to r.

Proof. Suppose there is a square

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f'"'] & U \arrow[d, "r"] \\ C \arrow[r] & B \amalg_A C. \end{tikzcd}$$

To find a unique up to homotopy lift of f', all we need to do is construct a square

$$\begin{tikzcd} A \arrow[r, "?"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "?"'] & U. \end{tikzcd}$$

Which we can construct easily by

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] & U \arrow[d, "r"] \\ C \arrow[r] \arrow[urr, dotted] & B \amalg_A C \arrow[r] & V. \end{tikzcd}$$

□

A more formal approach is:

Proof. First apply $\mathsf{Map}-, U)$. □

Definition 8.8. Recall that an object X is a *retract* of Y if there are maps $X \to Y \to X$ and a homotopy between the composite and the identity of X. Similarly, we say that a morphism of f' is a *retract* of f if there is a commutative diagram . . .

Lemma 8.9. *Suppose f' is a retract of f'. If f is left orthogonal to a morphism r, then so is f'.*

Proof. Suppose we have a lifting problem

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & U \arrow[d, "r"] \\ Y' \arrow[r] & V. \end{tikzcd}$$

Then we construct

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & U \arrow[d, "r"] \\ Y \arrow[r] \arrow[urr, dotted] & Y' \arrow[r] & V \end{tikzcd}$$

thus obtaining a lift by precomposing the bottom arrow with $Y' \to Y$ given in the definition of retract. □

Exercise 8.10. Show that any retract of an equivalence is an equivalence.

Proof. Recall that the definition of equivalence is that it has inverses such that the compositions are homotopic to the identity.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "f"] & \bullet \arrow[d, shift left, "f'"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r] & \bullet \arrow[u, shift left] \arrow[r] & \bullet
\end{tikzcd}$$

In a more formal way we obtain a cube diagram upon application of Map. □

9. Conservative functors and mapping spaces

The fibers of conservative functors are groupoids. Fibrations are conservative (Exercise 15.2) and so are fully faithful functors (Theorem 10.5)

Definition 9.1. A functor $\mathcal{C} \to \mathcal{D}$ is *conservative* if it is right orthogonal to $s_0 : [1] \to [0]$.

Exercise 9.2. The following are equivalent for a functor $F : \mathcal{C} \to \mathcal{D}$:

(1) F is conservative.
(2) The fibres of F are all ∞-groupoids.
(3) The commutative square

$$\begin{tikzcd}
\mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathcal{D}^{\simeq} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

is a pullback.

Proof. Maybe the easiest is to check that $3 \implies 2$. This is just because the square being a pullback means that the fibres are equivalent, and the fibres of the left vertical arrow are groupoids, while the fibres of the right vertical arrow are the fibres of F.

In a similar way we can prove $1 \iff 2$. Use that a map is an equivalence if and only if it is an equivalence upon groupoidification and $\mathsf{Map}([1], -)$ on the fibres of diagram in 3. Applying groupoidification gives trivial equivalences in the fibres. Applying $\mathsf{Map}([1], -)$ and getting equivalences on the fibres is equivalent to the square

$$\begin{tikzcd}
\mathcal{C} \simeq \mathsf{Map}([0], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}([1], \mathcal{C}) \arrow[d] \\
\mathcal{D} \simeq \mathsf{Map}([0], \mathcal{D}) \arrow[r] & \mathsf{Map}([1], \mathcal{D})
\end{tikzcd}$$

being a pullback, which is the definition of conservative.

if and only if the maps on fibres are equivalences. The fiber of the left vertical arrow is the groupoidification of the fiber of groupoidification of F, and the fiber of the right vertical arrow is the $\mathsf{Map}([1], -)$.

Also, by applying $\mathsf{Map}([1], -)$ to the square on 3 we immediately obtain the Map condition for being conservative, i.e. condition 1.

Finally to check that 2 $\implies$ 3 we apply $\mathsf{Map}([0], -)$ to obtain

$$\begin{tikzcd}
\mathsf{Map}([0], \mathcal{C}^{\simeq}) \arrow[r] \arrow[d] & \mathsf{Map}([0], \mathcal{C}) \arrow[d] \\
\mathsf{Map}([0], \mathcal{D}^{\simeq}) \arrow[r] & \mathsf{Map}([0], \mathcal{D})
\end{tikzcd}$$

To check this is a pullback we only need to check that the maps induced in fibers are equivalences by Lemma 3.11. But since the fibres of F are ∞-groupoids, they must equal the fiber of the map on the associated groupoids (it's a fact! Fibre and groupoidification commute). We do the same for [1] and use that [0] and [1] detect equivalences. $\square$

Condition 3 can be interpreted as: if a map gives an equivalence in $\mathcal{D}$, then it already was an equivalence in $\mathcal{C}$. For that we use

$$\begin{tikzcd}
\pi_0\mathcal{C}^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \pi_0\mathcal{D}^{\simeq} \arrow[d] \\
\pi_0\mathsf{Map}([1], \mathcal{C}) \arrow[r] & \pi_0\mathsf{Map}([1], \mathcal{D}).
\end{tikzcd}$$

Key tip: by Lemma (?) it is equivalent that a functor $F : \mathcal{C} \to \mathcal{D}$ is conservative and that the square [square with π_0] is a pullback.

Definition 9.3. The *arrow category* of $\mathcal{C}$ is $\mathsf{Ar}(\mathcal{C}) = \mathsf{Fun}([1], \mathcal{C})$.

Lemma 9.4.

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner"] & B \amalg_Y Z \arrow[d, "f \amalg_{id} id"] \\
D \arrow[r] & D \amalg_Y Z
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Z \arrow[d] \\
B \arrow[r] & B \amalg_Y Z \arrow[d] \\
D \arrow[r] & D \amalg_Y Z
\end{tikzcd}$$

$\square$

Now observe that the folowing is a particular case:

$$\begin{tikzcd}
{[1]} \arrow[r] \arrow[d, "s_0"] \arrow[dr, phantom, "\ulcorner"] & {[1] \amalg_{[0]} [1] = [2]} \arrow[d, "\mathrm{id} \amalg_{\mathrm{id}} s_0"'] \\
{[0]} \arrow[r] & {[1] \amalg_{[0]} [0] = [1]}
\end{tikzcd}$$

where the latter equality, namely $[1] \amalg_{[0]} [0] = [1]$, holds because it must be a fact that when one of the horizontal arrows of a pushout is an equivalence then the other one must be too.

Lemma 9.5. *In*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & V \arrow[r] \arrow[d] & B \amalg_Y Z \arrow[d, "f\amalg_{id} id"'] \\ C \arrow[r] & D \arrow[r] & D \amalg_Y Z \end{tikzcd}$$

We can put identity left or right and it will still be a pushout

We conclude that

$$\begin{tikzcd} {[1] \amalg [1]} \arrow[r] \arrow[d, "s_0 \amalg_{\emptyset} \mathrm{id}"] \arrow[dr, phantom, "\ulcorner"] & {[2] \amalg_{[1]} [2]} \arrow[d] \\ {[0] \amalg [1]} \arrow[r] \arrow[d, "\mathrm{id} \amalg s_0"] & {[1] \amalg_{[1]} [2]} \arrow[d, "\mathrm{id} \amalg_{\mathrm{id}} s_0"'] \\ {[0] \amalg [0]} \arrow[r] & {[1] \amalg_{[1]} [1] = [1]} \end{tikzcd}$$

Why do all this? A map that is right orthogonal to $[1] \to [0]$ is also right orthogonal to $s_0 \times \mathrm{id} : [1] \times [1] \to [1]$.

Proposition 9.6. *It is equivalent that $F : \mathcal{C} \to \mathcal{D}$ is conservative and that the square*

$$\begin{tikzcd} \mathsf{Fun}([0], \mathcal{C}) = \mathcal{C} \arrow[r] \arrow[d] & \mathsf{Fun}([0], \mathcal{D}) = \mathcal{D} \arrow[d] \\ \mathsf{Fun}([1], \mathcal{C}) = \mathsf{Ar}(\mathcal{C}) \arrow[r] & \mathsf{Fun}([1], \mathcal{D}) = \mathsf{Ar}(\mathcal{D}) \end{tikzcd}$$

is a pullback.

Proof. One implication is easy: just take $(-)^{\simeq}$ to obtain the Map condition for conservative functors.

For the other direction we can get to the exercise diagram by looking at

$$\begin{tikzcd} {[1]} & \emptyset \arrow[l] \arrow[r] \arrow[d] & {[1]} \arrow[d] \\ {[0]} & \emptyset \arrow[l] \arrow[r] & \emptyset \end{tikzcd}$$

Then we apply Map (I think) and get to

$$\begin{tikzcd} \mathsf{Map}([1], \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Map}([1], \mathcal{D}) \arrow[d] \\ \mathsf{Map}([1], \mathsf{Fun}([1], \mathcal{C}) \arrow[r] \arrow[d, "="] & \mathsf{Map}([1], \mathcal{D}) \arrow[d, "="] \\ \mathsf{Map}([1] \times [1], \mathcal{C}) \arrow[r] & \mathsf{Map}([1] \times [1], \mathcal{D}) \end{tikzcd}$$

Where have used currying. The latter diagram gives the pullback after applying $\mathsf{Map}([1], -)$. The pullback after applying $(-)^{\simeq}$ gives just the Map condition for conservativity. $\square$

Proposition 9.7. *$\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is conservative.*

Proof. We use Proposition 9.6 putting $\mathcal{C} = \mathsf{Ar}(\mathcal{C})$ and $\mathcal{D} = \mathcal{C} \times \mathcal{C}$. That is, we know that $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is conservative if and only if

$$\begin{tikzcd} \mathsf{Fun}([1], \mathcal{C}) = \mathsf{Ar}(\mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}([0] \amalg [0], \mathcal{C}) = \mathcal{C} \times \mathcal{C} \arrow[d] \\ \mathsf{Fun}([1], \mathsf{Fun}([1], \mathcal{C})) = \mathsf{Fun}([1] \times [1], \mathcal{C} \arrow[r] & \mathsf{Fun}[1], \mathcal{C} \times \mathcal{C}) = \mathsf{Fun}([1] \amalg [1], \mathcal{C}) \end{tikzcd}$$

where we have used

$$\begin{aligned} \mathcal{C} \times \mathcal{C} &= \mathsf{Fun}([0] \amalg [0], \mathcal{C}) \\ \mathsf{Fun}([1], \mathcal{C} \times \mathcal{C}) &= \mathsf{Fun}([1] \times ([0] \amalg [0]), \mathcal{C}) \\ &= \mathsf{Fun}([1] \amalg [1], \mathcal{C}) \end{aligned}$$

where the first equality is given just by applying $\mathsf{Fun}(-, \mathcal{C})$ to the basic pushout diagram, the second one by currying, and third one by using that these are also 1-categories.

The point is that once we look at this the right way we realise that it is nothing but the exercise diagram. So it's a pullback. $\square$

Definition 9.8. The *mapping space* is the fiber

$$\begin{tikzcd} \mathcal{C}(x, y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Ar}(\mathcal{C}) \arrow[d] \\ \{(x, y)\} \arrow[r] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

Exercise 9.9. Use the pushout decomposition of [2] to define composition maps

$$\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z).$$

(For extra credit, use the decomposition of [3] to show this is associative up to a specified homotopy.)

Proof.

$$\begin{tikzcd} \text{fiber} \arrow[rr, dotted, "\simeq"] \arrow[d] & & \mathcal{C}(x, y) \times \mathcal{C}(y, z) \arrow[d] \\ \mathsf{Fun}([2], \mathcal{C}) \arrow[r, "\simeq"] \arrow[d] & \mathsf{Fun}([1] \amalg_{[0]} [1], \mathcal{C}) \arrow[r, "\simeq"] \arrow[d] & \mathsf{Fun}([1], \mathcal{C}) \times_{\mathsf{Fun}([0], \mathcal{C})} \mathsf{Fun}([1], \mathcal{C}) \arrow[dl] \\ \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "\simeq"] & (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} (\mathcal{C} \times \mathcal{C}) & \end{tikzcd}$$

We shall define the fiber to be the composition $\mathcal{C}(x, y, z)$. This makes sense since the fiber on the rightmost part is indeed the product of the fibers, as is displayed in the diagram. This is because in general

$$fib(F) \times fib(G) \longrightarrow \mathcal{A} \times_Z \mathcal{B} \xrightarrow{F \times_{\mathrm{id}} G} \mathcal{C} \times_Z \mathcal{D}$$

$\square$

10. Fully faithful functors

Definition 10.1. A functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* if it is right orthogonal to $\partial[1] \to [1]$, that is, if the commutative square

$$\begin{tikzcd} \mathsf{Map}([1], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}([1], \mathcal{D}) \arrow[d] \\ (\mathcal{C}^{\simeq})^{\times 2} \arrow[r] & (\mathcal{D}^{\simeq})^{\times 2} \end{tikzcd}$$

is a pullback.

By Exercise 9.2, since $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is conservative, we get that the fibres are groupoids, so we know that we get equivalences on the fibres when we get a pullback, so really we have equivalences

$$\mathcal{C}(x, y) \xrightarrow{\simeq} \mathcal{D}(Fx, Fy)$$

Proposition 10.2. *$F : \mathcal{C} \to \mathcal{D}$ is fully faithful if and only if the commutative square*

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{D}) \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

Proof. At least one implication is easy: if it's a pullback, maps on fibres are equivalences. By the remark on the human-readble definition of fully faithfulness, we are done. □

Lemma 10.3. *If $F : \mathcal{C} \to \mathcal{D}$ is fully faithful, so is $F_* : \mathsf{Fun}(\mathcal{A}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{D})$ for any ∞-category $\mathcal{A}$.*

Exercise 10.4. Show that the following are equivalent for a commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C}' \arrow[d] \\ \mathcal{D} \arrow[r, "G"'] & \mathcal{D}' \end{tikzcd}$$

where F and G are fully faithful:

(1) The square is a pullback of ∞-categories.
(2) The square gives a pullback of ∞-groupoids on cores.
(3) The square gives a pullback of sets on $\pi_0(-)^{\simeq}$.

Proof. $1 \implies 2$ is immediate since pullbacks are detected on maps from [0] (and [1]).

For $2 \implies 1$ first apply $\mathsf{Map}([1], -)$, then we get

For $2 \implies 3$ we first need to make the diagram in the exercise after applying $(-)^{\simeq}$, then the arrows of F and G become monomorphisms. If we can prove that, then we apply some past exercise to see that the diagram is pullback iff the diagram on π_0. In fact, to prove those maps are monomorphisms we will use the theorem proved next, Theorem 10.5. □

Theorem 10.5. *F is fully faithfull if and only if F is conservative.*

Proof. We want to prove that F is right orthogonal to $\partial[1] \to [1]$ then it is right orthogonal to $[1] \to [0]$. First we apply some sort of 3-for-2 property (some past exercise) we look at

$$\partial[1] = [0] \amalg [0] \xrightarrow{\substack{\text{left ort.}\\ \text{to } F}} [1] \xrightarrow{\substack{\text{want to}\\ \text{prove r.o.}}} [0]$$

we realise that it's enough to prove right-orthogonality of the composition of the two maps above.

$$\begin{tikzcd}
{[1] \amalg [1]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {[3]} \arrow[d] \\
{[0] \amalg [0]} \arrow[r] & E \simeq *
\end{tikzcd}$$

where $E \simeq *$ by a fact.

Consider

$$K = \{0 < 1 < 2\} \amalg_{\{1<2\}} \{1 < 2 < 3\}$$

which is nothing more than $[2] \amalg_{[1]} [2]$. Putting $[2] \simeq [1] \amalg_{[0]} [1]$ we see that

$$K \simeq [1] \amalg_{[0]} [1] \amalg_{[1]} [1] \amalg_{[0]} [1] \simeq [1] \amalg_{[0]} [1] \amalg_{[0]} [1] \simeq [3]$$

Now we do

$$\begin{tikzcd}
\{0 < 2\} \amalg \{1 < 3\} \widetilde{K} \arrow[r] \arrow[d] & {[3]} \arrow[d] \\
* \amalg * \arrow[r] & E' \arrow[r] & E
\end{tikzcd}$$

where we already saw the outer square is a pushouts by definition of E, and E' is defined as the pushout of whatever it should be. Then, since $K \simeq [3]$, we apply the property of pushouts that if the map on the "total spaces" is equivalence then so is the map on "base spaces" (this is opposite to the property in pullback, which is just what we call "pullback of equivalences is equivalence", and then apply Fun to get to the property for pushouts).

Now back to our objective, by the 3-for-2-ish for left-orthogonality, now we aim to prove that

$$\{1 < 2\} \longrightarrow K \longrightarrow E' \longrightarrow E.$$

Now let H be the pushout

$$\begin{tikzcd}
\{0 < 2\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\
{[2]} \arrow[r] & H\ldots
\end{tikzcd}$$

□

Recall from Remark **??** that a map of ∞-groupoids $f : X \to Y$ is a monomorphism if and only if the induced map on mapping spaces $X(x, x') \to Y(fx, fx')$ is an equivalence. Which is strange: that's what we wanted for *fully faithful* functors, not monomorphisms. (But then again: that was before we defined ∞-categories.)

Lemma 10.6. *If $F : \mathcal{C} \to \mathcal{D}$ is fully faithful, then its underlying morphism of ∞-groupoids is a monomorphism of spaces.*

Proof. Fully faithful says $\partial[1] \to [1]$ is left orthogonal to F. But by Theorem **??** we also know it is conservative, so it's right orthogonal to $[1] \to [0]$. By 3-for-2, $\partial[1] \to [1] \to [0]$ is left orthogonal to F. But we know that this is equivalent to being an equivalence on underlying groupoids □

Definition 10.7. $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if $\pi_0 F : \pi_0\mathcal{C}^{\simeq} \to \pi_0\mathcal{D}^{\simeq}$ is surjective.

Remark 10.8. Suppose $F : \mathcal{C} \to \mathcal{D}$ gives an equivalence on cores. Let's show that F is fully faithful if and only if $\mathsf{Map}([1], \mathcal{C}) \to \mathsf{Map}([1], \mathcal{D})$ is an equivalence. Indeed: by definition of full faithfullness we get equivalences on all mapping spaces $\mathcal{C}(x, x') \simeq \mathcal{D}(fx, fx')$. Applying $(-)^{\simeq}$ we obtain a similar diagram on groupoids, namely

$$\begin{tikzcd} \mathsf{Map}([1], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}([1], \mathcal{D}) \arrow[d] \\ \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \arrow[r] & \mathcal{D}^{\simeq} \times \mathcal{D}^{\simeq}. \end{tikzcd}$$

The lower arrow is an equivalence by hypothesis and by Exercise 3.12 so is the upper arrow.

In conclusion, provided F is an equivalence on the cores, it is an equivalence of ∞-categories if and only if it is fully faithful.

Lemma 10.9. *A functor of ∞-categories is an equivalence if and only if it is fully faithful and essentially surjective.*

Proof. Apply Lemma 10.6 and then Remark 10.8. □

11. Full subcategories

This is the construction of a full subcategory from a subgroupoid of the underlying groupoid of the given category.

Given an ∞-category $\mathcal{C}$ and a monomorphism $i : X \hookrightarrow \mathcal{C}^{\simeq}$ of ∞-groupoids, then there exist an ∞-category $i^*\mathcal{C}$ and a functor of ∞-categories $\bar{i} : i^*\mathcal{C} \to \mathcal{C}$ such that $(\bar{i})^{\simeq} \simeq i$ (and $(i^*\mathcal{C}) \simeq X$ should also hold) and such that the following diagram

$$\begin{tikzcd} \mathsf{Map}(\mathcal{D}, i^*\mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{D}, \mathcal{C}) \arrow[d] \\ \mathsf{Map}(\mathcal{D}^{\simeq}, X) \arrow[r] & \mathsf{Map}(\mathcal{D}^{\simeq}, \mathcal{C}^{\simeq}). \end{tikzcd}$$

is a pullback.

Note that putting $\mathcal{D} = [1]$ we get that $\mathcal{D}^{\simeq} = \partial[1]$, meaning that $\bar{i}$ is fully faithful.

In the other direction, it's also true that any fully faithful functor is of that form:

Lemma 11.1. *If $j : \mathcal{D} \to \mathcal{C}$ is fully faithful then $j \simeq \overline{j^{\simeq}}$.*

Proof. By Lemma 10.6 we know that fully faithful gives monomorphism of underlying groupoids, which means that we can apply the construction above. Now, using the diagram above we can factor

$$\begin{tikzcd} \mathcal{D} \arrow[r, "f"] \arrow[rr, bend right, "j"'] & (j^{\simeq})^*\mathcal{C} \arrow[r, "\bar{i}"] & \mathcal{C} \end{tikzcd}$$

(Essentially by definition of pullback.) Further, by 3-for-2 in for fully faithful morphisms we may conclude that f is an equivalence. □

Lemma 11.2. *Essentially surjective functors are left orthogonal to fully faithful ones.*

We are almost done constructing a factorization system! Consider the functor $F : \mathcal{C} \to \mathcal{D}$ such that $F : \mathcal{C}^{\simeq} X \hookrightarrow \mathcal{D}^{\simeq}$. This gives

12. Equivalences in an infinity-category

For any ∞-category we always have $\mathcal{C} \to \mathsf{Ar}(\mathcal{C})$ induced from $[1] \to [0]$ by $\mathsf{Fun}(-, \mathcal{C})$. This is a fully faithful functor (picture).

Then we apply that

$$\mathcal{C} \xrightarrow[\text{}]{\substack{\text{essentially}\\\text{surjective}}} i^*\mathcal{D} \xrightarrow{\substack{\text{fully}\\\text{faithful}}} ?$$

for the case

$$C \xrightarrow{\simeq} \mathsf{Ar}_{\mathrm{eq}}(\mathcal{C}) \hookrightarrow \mathsf{Ar}(\mathcal{C}).$$

Recall

$$\begin{tikzcd}
\text{Spaces with a } G\text{-bundle} \arrow[rr] \arrow[dr] & & \text{Spaces} \\
& \text{Spaces admitting a } G\text{-bundle} \arrow[ur] &
\end{tikzcd}$$

So th Then we get

triangle diagram.

Lemma 12.1. *Suppose $f : x \to y$ is a morphism in an ∞-category. Then the following are equivalent:*

(1) *f is an equivalence.*
(2) *For all $c \in \mathcal{C}$, the morphism $f_*\mathcal{C}(c, x) \to \mathcal{C}(c, y)$ is an equivalence of ∞-groupoids.*
(3) *For all $c \in \mathcal{C}$, the morphism $f^*\mathcal{C}(y, c) \to \mathcal{C}(x, c)$ is an equivalence of ∞-groupoids.*

Proof. $2 \implies 1$. Put $c = x$ and then $c = y$. We obtain that $\mathcal{C}(x, x) \simeq \mathcal{C}(x, y)$ and $\mathcal{C}(y, x) \simeq \mathcal{C}(y, y)$.

Exercise. Check that $-*$ is "functorial". Consider

$$\begin{tikzcd}
C(x,y) \arrow[r] \arrow[d] & \mathcal{C}(c,y) \times \mathcal{C}(x,y) \arrow[d] & \mathcal{C}(c,x) \arrow[d] \\
pt \arrow[r, "f"'] & \mathcal{C}(x,y) \arrow[r] & pt
\end{tikzcd}$$

where the three squares are pullbacks. □

Lemma 12.2. *An ∞-category $\mathcal{C}$ is an ∞-groupoid if and only if every morphism in $\mathcal{C}$ is an aquicalence.*

Lemma 12.3. *A morphism $f \in \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is an equivalence if and only if its*

Exercise 12.4. The statement is equivalent to the fact that essentially surjective is left orthogonal to conservative! [Picture]

13. Monomorphisms of infinity categories

Exercise 13.1. Show that the following are equivalent for $F : \mathcal{C} \to \mathcal{D}$:

(1) F is a monomorphism.
(2) $F([i]) : \mathcal{C}([i]) \to \mathcal{C}([i])$ is a monomorphism of ∞-groupoids for $i = 0, 1$.
(3) F is right orthogonal to $[i] \amalg [i] \to [i]$ for $i = 0, 1$.

Proof. For (1) $\implies$ (2) just apply definition that pullbacks are detected on maps from [0] and [1].

Now (2) $\implies$ (3) is definition of orthogonality on Maps. ☐

Remark 13.2. By Observation 2.10.3, it really only matters what happens on [1].

Exercise 13.3. Use exercise dsadas

Proof. Recall we already proved that

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd}$$

then $E \to W'$ is mono iff $F_b \to F'_b$ is mono for all $b \in B$.

Then we have a square with bases $\mathcal{C}[0] \times \mathcal{C}[0]$ and $\mathcal{D}[0] \times \mathcal{D}[0]$. But those are different bases! Then we can construct the pullback

$$\begin{tikzcd} pt \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}[0] \times \mathcal{C}[0] \arrow[d] \\ pt \arrow[r] & \mathcal{D}[0] \times \mathcal{D}[0], \end{tikzcd}$$

indeed, it's a pullback *when it exists*, since monomorphisms have fibers that are either empty of points, but in this case, it cannot be a point (why?!). ☐

Now some intuition on what is a monomorphism in 1-categories: if $f \in \mathcal{C}$ is an isomorphism, then $F(f)$ is an isomorphism too.

Lemma 13.4. *Monomorphisms are conservative.*

Proof. consider

$$[1] \amalg [1] \to [1] \perp F \implies [1] \to [0] \perp F.$$

We want to argue that $[1] \to [0]$ is a retract of $[0] \amalg [1] \to [0]$.

$$\begin{tikzcd} {[1]} \arrow[r, "f"] \arrow[d] & {[0]} \arrow[d] \\ {[1] \amalg [1]} \arrow[r, "f \amalg \mathrm{id}"] \arrow[d, "\mathrm{id} \amalg \mathrm{id}"] & {[0] \amalg [1]} \arrow[d, "\text{left. o. to } F"] \\ {[1]} \arrow[r] & {[0]} \end{tikzcd}$$

Then the upper square is a pushout. The outer square it's all identities, so it's also a pushout. This gives that the two vertical lower arrows are orthogonal to each other. That's equivalent to the diagram in Observation 2.10.3. ☐

Lemma 13.5. *Fully faithful functors are conservative.*

Proof. Use that monomorphism iff monomorphism on groupoids and on mapping spaces. ☐

14. Subcategories

A similar construction to that of full subcategories yields the notion of subcategory.

15. Left and right fibrations

Definition 15.1. A functor of ∞-categories $p : \mathcal{E} \to \mathcal{B}$ is a *left fibration* if it is right orthogonal to the inclusion $\{0\} \to [1]$, and a *right fibration* if it is orthogonal to $\{1\} \to [1]$. Equivalently, p is a left or right fibration if the corresponding commutative square

$$\begin{tikzcd} \mathsf{Map}([1], \mathcal{E}) \arrow[r] \arrow[d] & \mathsf{Map}([1], \mathcal{B} \arrow[d] \\ \mathcal{E}^{\simeq} \arrow[r] & \mathcal{B}^{\simeq} \end{tikzcd}$$

is a pullback.

Exercise 15.2. Every left/right fibration is conservative.

Proof. Put orthogonal diagram in Map form, but it doesn't match exactly. The same happens when we look at Exercise 9.2. By the way: shouldn't we use Fun instead of Map when we work with ∞-categories instead of ∞-groupoids? [This was solved in class, there must be a picture.] ☐

Lemma 15.3. *If $\mathcal{B}$ is an ∞-groupoid, then a functor $p : \mathcal{E} \to \mathcal{B}$ is a left/right fibration if and only if $\mathcal{E}$ is an ∞-groupoid. In particular, any morphism of ∞-groupoids is both a left and right fibration.*

Proof. The proof looks good: I think the vertical maps are induced from maps $[0] \to [1]$ and $[1] \to [0]$. But how is the former defined? We can just take any of the two maps $[0] \to [1]$: each of these concerns the left/right fibration case. We see that either choice implies that $\mathcal{E}$ is an ∞-groupoid, and that $\mathcal{E}$ being an ∞-groupoid implies both. ☐

Next we have a characterization:

Proposition 15.4. *p is a left/right fibration if and only if the commutative square*

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{E}) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{B}) \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{B} \end{tikzcd}$$

is a pullback.

Proof. This is complicated. ☐

But it gives a nice corollary:

Lemma 15.5. *If $p : \mathcal{E} \to \mathcal{B}$ is a left/right fibration, then so is $p_* : \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{B})$ for any category $\mathcal{C}$.*

Proof. It follows easily from the characterization. □

Now we characterize fibrations that are also equivalences:

Proposition 15.6. *The following are equivalent for a left (or right) fibration $p : \mathcal{E} \to \mathcal{B}$:*

(1) *p is an equivalence.*
(2) *The map on cores $p^{\simeq} : \mathcal{E}^{\simeq} \to \mathcal{B}^{\simeq}$ is an equivalence of ∞-groupoids.*
(3) *The fibre $\mathcal{E}_b$ is contractible for every $b \in \mathcal{B}$.*

Proof. Easy, uses previous exercises. □

Remark 15.7 (3-for-2). Given a commutative triangle

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"'] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

where p and q are both left (or both right) fibrations, then so if F. This follows easily by 3-for-2 for pullback squares.

The following proposition looks very similar to Proposition 15.6. The only difference is on the third condition: this time we look at the fibers of the base and total space over $\mathcal{B}$ (last time we asked that the fibers of the fibration itself).

Lemma 15.8. *Suppose given a commutative triangle*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"'] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

where p and q are both left (or both right) fibrations. Then the following are equivalent.

(1) *F is an equivalence.*
(2) *The underlying map of ∞-groupoids $\mathcal{E}^{\simeq} \to \mathcal{F}^{\simeq}$ is an equivalence.*
(3) *The functor on fibres $\mathcal{E}_b \to \mathcal{F}_b$ is an equivalence for all $b \in \mathcal{B}$.*

Proof. The first two conditions are given by Proposition 15.6 and Remark 15.7. For the last one we apply Exercise 3.10. □

Exercise 15.9. Show that a commutative square

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"'] & \mathcal{F} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

Proof. I think this would be proved essentially as Lemma 3.11, using the recently proved Lemma 15.8. (Notice that the fact that the fibres are groupoids follows from Exercise 15.2 and Exercise 9.2. It's true, with a single nuance: that we need to prove that the pullback p is a right/left fibration when q is. To prove this we need to use a cube diagram, see picture. □

16. Slice categories

Let $\mathcal{C}$ be a 1-category and $x \in \operatorname{Ob}\mathcal{C}$. Then the *slice category* $\mathcal{C}/x$ is the 1-category whose objects are morphisms $y \to x$ in $\mathcal{C}$, and morphisms are

$$\begin{tikzcd} y \arrow[dd] \arrow[dr] & \\ & x \\ y' \arrow[ur] & \end{tikzcd}$$

It turns out that the slice category is the pullback

$$\begin{tikzcd} \mathcal{C}/x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Ar(\mathcal{C}) \arrow[d, "ev_1"] \\ \{x\} \arrow[r] & \mathcal{C} \end{tikzcd}$$

We claim that the map ev_1 is a fibration. (Shown in class.)

Definition 16.1. For an ∞-category $\mathcal{K}$, the *left cone* $\mathcal{K}^{\triangleleft}$ and *right cone* $\mathcal{K}^{\triangleright}$ on $\mathcal{K}$ are defined by the pushouts

$$\mathcal{K}^{\triangleleft} = \{-\infty\} \amalg_{\mathcal{K}\times\{0\}} \mathcal{K} \times [1]$$
$$\mathcal{K}^{\triangleright} = \mathcal{K} \times [1] \amalg_{\mathcal{K}\times\{1\}} \{\infty\}.$$

The *cone point* is the object $\pm\infty$.

It can be shown (or maybe we take as a fact) that cores and mapping spaces in the left cone $\mathcal{K}^{\triangleleft}$ are given by

$$(\mathcal{K}^{\triangleleft})^{\simeq} \simeq \{-\infty\} \amalg \mathcal{K}^{\simeq},$$
$$\mathcal{K}^{\triangleleft}(x,y) = \begin{cases} \emptyset & x \in \mathcal{K}, y \simeq -\infty \\ *, & x \simeq -\infty, \\ \mathcal{K}(x,y), & x, y \in \mathcal{K}. \end{cases}$$

It looks like: there are no maps from any object to $-\infty$, there is a single map coming from $-\infty$ to any object (i.e., the space of maps from $-\infty$ to any object is contractible).

Lemma 16.2. *For ∞-categories $\mathcal{K}$ and $\mathcal{C}$ and an object x in $\mathcal{C}$, there are natural equivalences*

$$\mathsf{Fun}(\mathcal{K}, \mathcal{C}_{/x}) \simeq \mathsf{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \times_{\mathsf{Fun}(\{\infty\},\mathcal{C})} \{x\},$$
$$\mathsf{Fun}(\mathcal{K}, \mathcal{C}_{x/} \simeq \mathsf{Fun}(\mathcal{K}^{\triangleleft}, \mathcal{C}) \times_{\mathsf{Fun}(\{-\infty\},\mathcal{C})} \{x\}.$$

I think of this as describing the fiber of x from the projection $\mathcal{K}^{\triangleright} \to \{\infty\}$:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \arrow[d] \\ \{x\} \arrow[r] & \mathsf{Fun}(\{\infty\}, \mathcal{C}), \end{tikzcd}$$

i.e., the space of maps from ∞ to x.

Looking back at the definition of cone with $\mathcal{K} = [1]$, if we can show that

$$\begin{tikzcd} \mathcal{K} \times \{1\} \simeq [1] \times \{1\} \arrow[r] \arrow[d] & \mathcal{K} \times [1] = [1] \times [1] \arrow[d] \\ \{\infty\} \simeq [0] \arrow[r] & [2] \end{tikzcd}$$

is a pushout, we can conclude that $[1]^{\triangleright} \simeq [2]$. Consider the following diagram, where we want to check that the double lower square is a pushout:

$$\begin{tikzcd} & {[1]} \arrow[r] \arrow[d, "\substack{0\mapsto 0 \\ 1\mapsto 2}"] & {[2]} \arrow[d] \\ {[1]} \arrow[r, "\substack{0\mapsto 1 \\ 1\mapsto 2}"] \arrow[d] & {[2]} \arrow[r] \arrow[d, "\substack{0\mapsto 0 \\ 1\mapsto 1 \\ 2\mapsto 1}"] & {[2] \amalg_{[1]} [2] \simeq [1] \times [1]} \arrow[d] \\ {[0]} \arrow[r, "0\mapsto 1"'] & {[1]} \arrow[r, "\substack{0\mapsto 0 \\ 1\mapsto 2}"'] & {[2] \simeq [1] \amalg_{[0]} [1].} \end{tikzcd}$$

First observe that the lower left square is a pushout. This is by 2-for-3 since the upper right square is a pushout by pushout decomposition of $[1] \times [1]$ and because the two composite vertical arrows are the identity. To conclude it's enough to show that the lower left square is also a pushout.

Here's a summary of what else is in the chapter. Proposition 3.3.7 on two pullback squares. The exercise that the left and right cones of $[n]$ are both $[n+1]$. Corollaruy: $C_{/c} \to \mathcal{C}$ is right fibration if and only if $C_{c/} \to \mathcal{C}$ is left fibration. Corollary: for a morphism $f : x \to y$ in an ∞-category $\mathcal{C}$, we have $(C_{/y})_{/f} \xrightarrow{\sim} \mathcal{C}_{/x}$ and $(\mathcal{C}_{x/})_{f/} \xrightarrow{\sim} \mathcal{C}_{y/}$.

Lemma 16.3. *For a morphism $f : x \to y$ in an ∞-category we have equivalences $(C_{/y})_{f/}/ \simeq \mathcal{C}/x$.*

This one I like:

Lemma 16.4. *A functor $p : \mathcal{E} \to \mathcal{B}$ is a right fibration of and only if $\mathcal{E}_{/k} \to \mathcal{B}_{/p(x)}$ is an equivalence for all x, and a left fibration if and only if for every $x \in \mathcal{E}$, the induced functor $\mathcal{E}_{x/} \to \mathcal{B}_{p(x)/}$ is an equivalence.*

Proof. Did in class. Maybe this is already somewhere else in this document? □

17. Straightening for left and right fibrations

Suppose $p : \mathcal{E} \to \mathcal{B}$ is a left fibration. "Then p is supposed to determine a functor from $\mathcal{B}$ to ∞-groupoids."

First we note that a morphism $f : b \to b'$ in $\mathcal{B}$ we get maps on fibers $f_! : \mathcal{E}_b \to \mathcal{E}_{b'}$.

$$\begin{tikzcd} \mathcal{E}[0] \arrow[d] & \mathcal{E}[1] \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & \mathcal{E}[0] \arrow[d] \\ \mathcal{B}[0] & \mathcal{B}[1] \arrow[l, "s"'] \arrow[r, "t"'] & \mathcal{B}[0] \end{tikzcd}$$

where the left square is a pullback by definition of p being a left fibration, which in turn gives equivalences on maps on fibers, so that we may invert the left morphism in

$$\mathcal{E}_b \xleftarrow[\simeq]{} \mathcal{E}[1]_f \longrightarrow \mathcal{E}_{b'}.$$

Further, we see that $(\mathrm{id}_b)_!$ is homotopic to the identity of $\mathcal{E}_b$ and that $(gf)_! \simeq g_! f_!$. Dually, a right fibration gives the basic data for a contravariant functor from $\mathcal{B}^{\mathrm{op}}$ to ∞-groupoids.

We introduce the fact that there is an ∞-category Cat_∞ of ∞-categories. We denote Gpd_∞ the full subcategory spanned by the ∞-groupoids. For an ∞-category $\mathcal{B}$, we denote $\mathsf{LFib}(\mathcal{B})$ and $\mathsf{RFib}(\mathcal{B})$ the subcategories of $\mathsf{Cat}_{\infty/\mathcal{B}}$ of left and right fibrations.

Theorem 17.1 (Lurie)**.** *For an ∞-category $\mathcal{B}$, there is an equivalence*

$$Str^L_{\mathcal{B}} : \mathsf{LFib}(\mathcal{B}) \xrightarrow{\sim} \mathsf{Fun}(\mathcal{B}, \mathsf{Gpd}_\infty)$$

called the straightning equivalence for left fibrations. This is moreover natural in $\mathcal{B}$ with respect to precomposition of functors and pullback of fibrations.

18. Cartesian and cocartesian fibrations

The idea here is to generalize the notion of right fibration to some object whose fibers are not necessarily ∞-groupoids (recall that a fibration is conservative, which implies its fibers are ∞-groupoids).

Exercise 18.1. Given a functor $p : \mathcal{E} \to \mathcal{B}$ and a morphism $f : x \to y$ in $\mathcal{E}$, show that for an object $z \in \mathcal{E}$ we get commutative squares

$$\begin{tikzcd} \mathcal{E}(y,z) \arrow[r, "f^*"] \arrow[d] & \mathcal{E}(x,z) \arrow[d] \\ \mathcal{B}(py,pz) \arrow[r, "p(f)^*"'] & \mathcal{B}(px,pz) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}(z,x) \arrow[r, "f_*"] \arrow[d] & \mathcal{E}(z,y) \arrow[d] \\ \mathcal{B}(pz,px) \arrow[r, "p(f)_*"'] & \mathcal{B}(pz,py) \end{tikzcd}$$

by taking appropriate fibers in a cube with top face

$$\begin{tikzcd} \mathcal{E}[2] \arrow[r, "d_1"] \arrow[d] & \mathcal{E}[1] \arrow[d] \\ \mathcal{B}[2] \arrow[r, "d_1"'] & \mathcal{B}[1]. \end{tikzcd}$$

Similarly we have the diagrams

$$\begin{tikzcd} \mathcal{E}_{y/} \arrow[r, "f^*"] \arrow[d] & \mathcal{E}_{x/} \arrow[d] \\ \mathcal{B}_{py/} \arrow[r, "p(f)^*"'] & \mathcal{B}_{px/} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_{/x} \arrow[r, "f_*"] \arrow[d] & \mathcal{E}_{/y} \arrow[d] \\ \mathcal{B}_{/px} \arrow[r, "p(f)_*"'] & \mathcal{B}_{/py}. \end{tikzcd}$$

Proof. We use the proof of Lemma 16.3 to construct the following complicated diagram:

$$\begin{tikzcd}
 & C_{/y} \arrow[r] & \mathsf{Ar}(\mathcal{C}) \arrow[r, "s"] & \mathcal{C} \\
(\mathcal{C}_{/y})_{/f} \arrow[ur, dashed] \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{C}_{/y}) \arrow[u, "s"] \arrow[r] \arrow[d, "t"] & \mathsf{Fun}([2],\mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}([1],\mathcal{C}) = \mathsf{Ar}(\mathcal{C}) \arrow[u, "s"'] \arrow[d, "t"] \\
pt \arrow[r] & \mathcal{C}_{/y} \arrow[r] \arrow[d] & \mathsf{Fun}([1],\mathcal{C}) \arrow[r, "s"] \arrow[d] & \mathcal{C} \\
 & pt \arrow[r] & \mathcal{C} &
\end{tikzcd}$$

This then leads to

$$\begin{tikzcd}
\mathcal{E}_{/x} \arrow[rr, "\text{right fib.}"] \arrow[dr] \arrow[ddd] & & \mathcal{E}_{/y} \arrow[dl] \arrow[ddd] \\
 & \mathcal{E} \arrow[d] & \\
 & \mathcal{B} & \\
\mathcal{B}_{px} \arrow[rr, "\text{right fib.}"'] \arrow[ur] & & \mathcal{B}_{py} \arrow[ul]
\end{tikzcd}$$

and in fact the commutativity of the first diagram in the exercise is given by taking the fibres of this last diagram by the diagonal arrows. □

From this we can define p-cartesian as in 1-categories:

Definition 18.2. Given a functor $p : \mathcal{E} \to \mathcal{B}$, we say a morphism $f : x \to y$ in $\mathcal{E}$ is *p-cocartesian* if for every object $z \in \mathcal{E}$, the commutative square

$$\begin{tikzcd}
\mathcal{E}(y,z) \arrow[r, "f^*"] \arrow[d] & \mathcal{E}(x,y) \arrow[d] \\
B(py,pz) \arrow[r, "p(f)^*"'] & \mathcal{B}(px,pz)
\end{tikzcd}$$

is a pullback. Dually, we say that f is *p-cartesian* if all squares

$$\begin{tikzcd}
\mathcal{E}(z,x) \arrow[r, "f_*"] \arrow[d] & \mathcal{E}(z,y) \arrow[d] \\
\mathcal{B}(pz,px) \arrow[r, "p(f)_*"'] & \mathcal{B}(pz,py)
\end{tikzcd}$$

are pullbacks.

This is basically the definition we had for cartesian (and cocartesian, by inverting the arrows) for 1-categories. Namely, for cocartesian, that given the solid diagram

$$\begin{tikzcd} & z \arrow[d, dashed, "\exists!"] \\ x \arrow[ur, "g"] \arrow[r, "f"'] & y \end{tikzcd} \qquad \begin{tikzcd} & p(z) \arrow[d, "\phi"] \\ p(x) \arrow[ur, "p(g)"] \arrow[r, "p(f)"'] & p(y) \end{tikzcd}$$

there is a unique dashed arrow.

Exercise 18.3. A morphism is p-cocartesian if and only if

$$\begin{tikzcd} \mathcal{E}_{y/} \arrow[r, "f^*"] \arrow[d] & \mathcal{E}_{x/} \arrow[d] \\ \mathcal{B}_{py/} \arrow[r, "p(f)^*"'] & \mathcal{B}_{px/} \end{tikzcd}$$

is a pullback.

Proof. We have constructed a diagram of two cubes on top of each other. In fact, this can be thought of as shifts in derived category. □

Lemma 18.4. *This looks like a 2-for-3 for cocartesian maps.*

Definition 18.5. Given a morphism $f : a \to b$ in $\mathcal{B}$ and an object $x \in \mathcal{E}$

Lemma 18.6. *Lifts of p-cartesian if and only if ... equivalence.*

Lemma 18.7. *Complicated monomorphism statement.*

Proof. We start by presenting a general situation. We start with a 2×2 square with homotopy fibers. We claim that it is well-defined, and proved it.

Now we imagine another situation: a cube with the bottom and top faces that are pullbacks. Then we take homotopy fibre and produce a cube on top of the original cube. We claim that the top face is also a pullback. We take homotopy fibres producing a cube to one of the sides, then take homotopy fibre of the homotopy fibres, which gives an equivalence since the maps induced on homotopy fibres are equivalences.

As for another equivalence for fibration, we have that the diagram

$$\begin{tikzcd} \mathsf{Ar}_{\mathrm{cart}}(\mathcal{E}) \arrow[r] \arrow[d, "\mathrm{ev}_1"'] & \mathsf{Ar}(\mathcal{B}) \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{E} \arrow[r] & \mathcal{B} \end{tikzcd}$$

is a pullback. That is, that

$$\mathsf{Ar}_{\mathrm{cart}}(\mathcal{E}) \simeq \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}).$$

This can also be represented as

$$
\begin{tikzcd}
\mathsf{Ar}_{\mathrm{cart}}(\mathcal{E}) \arrow[r, hook] & \mathsf{Ar}(\mathcal{E}) \arrow[dr] \arrow[drr] \arrow[ddr, "\mathrm{ev}_1"'] & & \\
& & \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{B}) \arrow[d] \\
& & \mathcal{E} \arrow[r] & \mathcal{B}
\end{tikzcd}
$$

Applying $\mathsf{Map}([1], -)$ we obtain

$$
\begin{tikzcd}
\mathcal{E}[1]_{\mathrm{cart}} \arrow[r, hook] \arrow[drr, bend right] & \mathcal{E}[1] \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& & \mathcal{E}^{\simeq} \times_{\mathcal{B}^{\simeq}} \mathcal{B}[1] \arrow[r] \arrow[d] & \mathcal{B}[1] \arrow[d] \\
& & \mathcal{E}^{\simeq} \arrow[r] & \mathcal{B}^{\simeq}
\end{tikzcd}
$$

where the arrow from $\mathcal{E}[1]$ to the pullback comes from the universal property of pullback.

□

Proposition 18.8. *Left fibrations are p-cocartesian and right fibrations are p-cartesian.*

The following lemma shows that for cocartesian fibrations right fibrations are equivalent fibrations in ∞-groupoids.

Lemma 18.9. *Suppose $p : \mathcal{E} \to \mathcal{B}$ a cocartesian fibration. Then the following are equivalent:*

(1) *p is a left fibration (which is the same as saying it's a fibration in ∞-groupoids).*
(2) *Every morphism in $\mathcal{E}$ is p-cocartesian.*
(3) *p is conservative.*
(4) *The fibers of p are ∞-groupoids.*

Proof. We already proved (1) implies (2) (?). For (2)implies (1) we consider

$$
\begin{tikzcd}
\mathcal{E}[1]_{\mathrm{cartesian}} \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & \mathcal{B}[1] \times_{\mathcal{B}} \mathcal{E}^{\simeq} \\
& \mathcal{E}[1] \arrow[ur, "\simeq"'] &
\end{tikzcd}
$$

We already know that (3) is equivalent to (4).

To prove that (**??**) implies (**??**) we use the property that conservative functors lift equivalences to equivalences (?).

□

Proposition 18.10. *If*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"'] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

is a morphism of cocartesian fibrations over $\mathcal{B}$ then F is an equivalence if and only if the induced map on fibres is an equivalence for all $b \in \mathcal{B}$.

Proposition 18.11. *Pullback of fibration is fibration.*

Exercise 18.12. Suppose we have a morphism of (co-)cartesian fibrations F

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "p"'] & \mathcal{E}' \arrow[d, "p'"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{B}'. \end{tikzcd}$$

Proof. Factor the above square as

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}' \times_{\mathcal{B}'} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d] \\ \mathcal{B} \arrow[r, "\mathrm{id}"'] & \mathcal{B} \arrow[r] & \mathcal{B}' \end{tikzcd}$$

and apply the last two propositions. □

Exercise 18.13. Pullback of three cocartesian fibrations...

19. Arrow infty-categories and fibration

Proposition 19.1. *The mapping space of two morphisms $f : x \to y$ and $g : x' \to y'$ in an ∞-category is the pullback*

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C})(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}(x,x') \arrow[d] \\ \mathcal{C}(y,y') \arrow[r, "f^*"'] & \mathcal{C}(x,y'). \end{tikzcd}$$

Upshot: morphisms of morphisms are a map of sources and map of targets such that composing with f^* or g_* accordingly give the same map.

More precisely, the idea is as in Section 8, where we defined lifting as a pullback. In the current situation: a choice of a map $x \to x'$ and a map $y \to y'$ such that $x \to x' \xrightarrow{g_*} y'$ coincides with $x \xrightarrow{f} y \to y'$, is just an element of $\mathcal{C}(x,x') \times_{\mathcal{C}(x,y')} \mathcal{C}(y,y')$. The proposition then says that morphisms from f to g are the same as elements in such pullback.

Proof. Rune starts by saying that we want to find the fibre at (f,g) of the map

$$\mathsf{Fun}([1], \mathsf{Fun}([1], \mathcal{C})) = \mathsf{Map}([1] \times [1], \mathcal{C}) \to \mathcal{C}[1]^{\times 2}.$$

Why do we want that? Because that's just the definition of the mapping space $\mathsf{Ar}(x,y)$ (see Definition 9.8).

To do it we notice that by the fact that in a general situation

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, dashed] & \mathcal{E}_b \arrow[r] \arrow[d] & \mathcal{F}_b \arrow[d] \\
\bullet \arrow[r] & \mathcal{E} \arrow[r] \arrow[dr] & \mathcal{F} \arrow[d] \\
& & \mathcal{B}
\end{tikzcd}$$

the dashed arrow is an equivalence (because the square is a pullback by 2-for-3:

$$\begin{tikzcd}
\mathcal{E}_\ell \arrow[r] \arrow[d] & \mathcal{F}_\ell \arrow[r] \arrow[d] & pt \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{B}),
\end{tikzcd}$$

so it's enough to find the fibre of the fibre map.

Since we want to compute $\mathsf{Ar}(x, y)$, which is by definition the fibre of $\mathsf{Ar}(\mathcal{C}) \times \mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$, a conservative map, which means that its fibres are groupoids, then the following diagram should be enough:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, dashed] & (\mathcal{C}(x,y) \times \mathcal{C}(y,y')) \times_{\mathbb{C}(x,y')} (\mathcal{C}(x,x'), \mathcal{C}(z,y')) \arrow[r] \arrow[d] & \mathcal{C}(x,y) \times \mathcal{C}(x',y') \arrow[d] \\
\bullet \arrow[r] & \mathsf{Map}([1] \times [1], \mathcal{C}) = \mathsf{Ar}(\mathsf{Ar}(\mathcal{C}))^{\simeq} \arrow[r] \arrow[dr] & \mathcal{C}[1]^{\times 2} \arrow[d] \\
& & \mathcal{B}
\end{tikzcd}$$

We find that the fibre we are seeking is

$$\{f\} \times \mathcal{C}(y, y') \times_{\mathcal{C}(x,y')} (\mathcal{C}(x, x') \times \{g\}). \tag{19.1.1}$$

The reason is given by the three diagrams

$$\mathcal{C}(x, y') \longrightarrow \mathcal{C}(x, y') \longrightarrow pt$$

$$\{f\} \times \mathcal{C}(y, y') \longrightarrow \mathcal{C}(x, y) \times \mathcal{C}(y, y') \longrightarrow \mathcal{C}(x, y) \ni f$$

$$\mathcal{C}(x, x') \times \{y\} \longrightarrow \mathcal{C}(x, x') \times \mathcal{C}(x', y') \longrightarrow \mathcal{C}(x', y') \ni g.$$

□

Next we observe that morphisms in $\mathsf{Ar}(\mathcal{C})$ are given by commutative squares

$$\begin{tikzcd}
y \arrow[r, "e"] \arrow[d, "f"'] & y' \arrow[d, "g"] \\
x \arrow[r, "h"'] & x'
\end{tikzcd} \tag{19.1.2}$$

which in fact are elements of the fibre we found in the last proposition, namely Equation 19.1.1. This just says that "the mapping spaces in $\mathsf{Ar}(\mathcal{C})$ are what we expected them to be"! (namely those commutative squares).

Exercise 19.2. Extend this description to show that for a morphism $\alpha : f \to g$ in $\mathsf{Ar}(\mathcal{C})$, given by a commutative square

$$\begin{tikzcd} x \arrow[r, "e"] \arrow[d, "f"'] & z \arrow[d, "g"] \\ y \arrow[r, "h"'] & w, \end{tikzcd}$$

precomposition with α fits in a commutative cube (where the left and right faces are pullbacks)

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C})(g,q) \arrow[rr, "\alpha^*"] \arrow[dr] \arrow[dd] & & \mathsf{Ar}(\mathcal{C})(f,q) \arrow[dr] \arrow[dd] & \\ & \mathcal{C}(z,u) \arrow[rr, "e^*"] \arrow[dd] & & \mathcal{C}(x,u) \arrow[dd] \\ \mathcal{C}(w,v) \arrow[rr, "h^*"] \arrow[dr] & & \mathcal{C}(y,v) \arrow[dr] & \\ & \mathcal{C}(z,v) \arrow[rr, "e^*"] & & \mathcal{C}(x,v) \end{tikzcd}$$

for $q : u \to v$.

Proof. By Proposition 19.1 we know that the left and right faces are pullbacks (and in particular comute). The front and back squares are commutative by associativity of composition (which in fact is not immediate to prove, but holds). The hardest part are the top and bottom faces. They commute by a careful application of Exercise 18.1. ☐

The next proposition looks like a technical statement justifying Exercise 19.2, and to be used for proving the next lemma.

Proposition 19.3. *For any ∞-category $\mathcal{C}$, the functor*

$$ev_1 : \mathsf{Ar}(\mathcal{C}) \to \mathcal{C}$$

is a cocartesian fibration. A morphism in $\mathsf{Ar}(\mathcal{C})$ given by a square **??** *is ev_1 cocartesian if and only if e is an equivalence.*

Lemma 19.4. *Given a functor $p : \mathcal{E} \to \mathcal{B}$, the following are equivalent for a morphism $\overline{f} : x \to y$ in $\mathcal{E}$ over $f : a \to b$ in $\mathcal{B}$:*

(1) *The morphism $\overline{f}$ is p-cocartesian.*
(2) *For every morphism ...*

The next proposition describes a situation in which we have a subcategory S of arrows and we construct a functor $p : \mathcal{E} \to \mathcal{B}$ such that S is the p-cocartesian maps.

Proposition 19.5. *For a functor $p : \mathcal{E} \to \mathcal{B}$ and a monomorphism $S \hookrightarrow \mathcal{E}[1]$, let $\mathsf{Ar}_S(\mathcal{E}) \subseteq \mathsf{Ar}(\mathcal{E})$ denote the full subcategory spanned by morphisms in S. Then the following are equivalent:*

(1) *p is a cocartesian fibration and S is the ∞-groupoid of p-cocartesian morphisms.*

(2) *The commutative square*

$$\begin{tikzcd} \mathsf{Ar}_S(\mathcal{E}) \arrow[r, "ev_0"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ \mathsf{Ar}(\mathcal{B}) \arrow[r, "ev_0"'] & \mathcal{B} \end{tikzcd}$$

is a pullback.

We obtain a corollary that seems to establish that $\mathsf{Fun}(\mathcal{K}, -)$ preserves cocartesian functors.

Lemma 19.6. *Suppose $p : \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. Then for any ∞-category $\mathcal{K}$, the functor*

$$p_* : \mathsf{Fun}(\mathcal{K}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{K}, \mathcal{B})$$

given by composition with p is also a cocartesian fibration. A natural transformation $\phi : \mathcal{K} \times [1] \to \mathcal{E}$ is p_*-cocartesian if and only if its components $\phi_a : [1] \to \mathcal{E}$ are p-cocartesian for all $a \in \mathcal{K}$.

Then there's the part on the cocartesian transport functor. I think it goes like this: we use Proposition 19.5 to identify $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \simeq \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E})$. We have learnt to read elements in the pullback as pairs in the product that project nicely; namely, elements of $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B})$ are nothing but pairs $(x, f : p(x) \to b)$ according to the diagram

$$\begin{tikzcd} \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Ar}(\mathcal{B}) \arrow[d, "s"] \\ \mathcal{E} \arrow[r] & \mathcal{B}. \end{tikzcd}$$

Take any such pair, and consider a cocartesian lift. This gives a map on $\mathcal{E}$ with source x. Denote the target by $f_! x$. This map,

$$\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \xrightarrow{\sim} \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \xrightarrow{\mathrm{ev}_1} \mathcal{E},$$

is called the *cocartesian transport functor*.

20. Representability and the Yoneda embedding

An object $x \in \mathcal{C}$ is *initial* if $\mathcal{C}(x, c)$ is contractible for all $c \in \mathcal{C}$ and *terminal* if $\mathcal{C}(c, x)$ is contractible for all c. A right fibration $p : \mathcal{E} \to \mathcal{B}$ is *representable* if $\mathcal{E}$ has a terminal object and a left fibration $p : \mathcal{E} \to \mathcal{B}$ is *corepresentable* if $\mathcal{E}$ has an initial object. We say that p is *represented* by $b \in \mathcal{B}$ is $\mathcal{E}$ has a terminal object that lies over b, and that $x \in \mathcal{E}$ *exhibits* p as represented by b if x is terminal and $p(x) \simeq b$.

Lemma 20.1. *The object x is initial if and only if $\mathcal{C}_{x/} \to \mathcal{C}$ is an equivalence, and terminal if and only if $\mathcal{C}_{/x} \to \mathcal{C}$ is an equivalence.*

Proof. Since $C_{x/} \to \mathcal{C}$ is a left fibration (why?) it's enough to show that the induced maps on fibers of the diagram

$$\begin{tikzcd} \mathcal{C}(x, y) \arrow[r] \arrow[d] & C_{x/} \arrow[d] \\ \{y\} \arrow[r] & \mathcal{C} \end{tikzcd}$$

is a puback. [See picture.] $\square$

It would be natural to define a left fibration $p : \mathcal{E} \to \mathcal{B}$ to be co-representable if its corresponding straightening $\mathcal{B} \to \mathrm{Grp}_\infty$

Recall from Exercise 1.2 that in 1-categories we have an equivalence between presheaves $F : \mathcal{B}^{\mathrm{op}} \to \mathsf{Sets}$ and discrete fibrations $p : \mathcal{E} \to \mathcal{B}$. We define F to be *representable* if $\mathcal{E}$ has terminal object.

Exercise 20.2. Show that this definition is equivalent to having $\mathrm{Hom}(-, b_0) \cong F$.

Proof. Recall from Exercise 1.2 that $\mathcal{E}$ is defined so that the fiber of $b \in \mathcal{B}$ is the set $F(b)$. More exactly, the objects of $\mathcal{E}$ are pairs (e, b) with $b \in \mathcal{B}$ and $e \in F(b)$. The morphisms are pairs $(f, F(f))$ for $f : b' \to b$ in $\mathcal{B}$.

Suppose that $\underline{e} = (e, p(e))$ is a terminal object of $\mathcal{E}$. We claim that $\mathrm{Hom}(-, p(e)) \cong F$. Let $b \in \mathcal{B}$. Let's check that $\mathrm{Hom}(b, p(e)) \simeq F(b)$ as sets. By definition of discrete fibration for every map $f : b \to p(e)$ there is a unique map $g : e' \to e$ such that $p(g) = f$. Then the element e' is in $F(b)$. To see that this correspondence is natural in b pick a map $h : b_1 \to b$. We have an induced map $F(h) : F(b) \to F(b_1)$ that for any map $f \in \mathrm{Hom}(b, p(e))$ puts the element we found e' to an element e_1' in the set b_1, which is the p-fiber of b_1 by definition of p.

$$\begin{tikzcd} f \arrow[d, mapsto] & \mathrm{Hom}(b, p(e)) \arrow[r] \arrow[d, "h^*"'] & F(b) \arrow[d, "F(h)"] & e' \arrow[d, mapsto] \\ fh & \mathrm{Hom}(b_1, p(e)) \arrow[r] & F(b_1) & e_1'. \end{tikzcd}$$

To confirm that the diagram commutes we need to check that this element e_1' is the source of the unique lift of the map fh. Since $\underline{e} = (e, p(e))$ is terminal, there is only one map in $\mathcal{E}$ with target $\underline{e}$ and thus the source of the lift of fh is uniquely determined.

Conversely, suppose that $\mathrm{Hom}(-, b_0) \cong F$ for some $b_0 \in \mathcal{B}$. To find a terminal object in $\mathcal{E}$ we look for a distinguished element in the fiber of b_0 (which is the set $F(b_0)$ by construction). We claim that the element corresponding to the identity in $\mathrm{Hom}(b_0, b_0) \cong F(b_0) = p^{-1}(b_0)$ is terminal. Let us denote this element by $\underline{e} = (e, p(e)) \in \mathcal{E}$. Let $f : \underline{e}' \to \underline{e}$ be any morphism in $\mathcal{E}$. This gives a map on $\mathcal{B}$ by $p(f) : p(e') \to p(e) = b_0$. Since p is a discrete fibration, there exists a unique morphism in $\mathcal{E}$ lifting $p(f)$, which must be f. $\square$

The following explanation is quite illuminating. Let's informally define a Lie groupoid to be a groupoid object in the category of differentiable manifolds (see nLab Lie groupoid and groupoid object). Actually, this just means that it is a representable presheaf of groupoids (perhaps with some other conditions). That is, a Lie groupoid is a representable functor $\mathrm{Diff}^{\mathrm{op}} \to \mathrm{Grpd}$.

Definition 20.3. We say a presheaf $\phi : \mathcal{B}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ is *representable* if the corresponding right fibration is representable, and that a copresheaf $\mathcal{B} \to \mathsf{Gpd}_\infty$ is *correpresentable* if the corresponding left fibration is corepresentable. We say that ϕ is *represented* by $b \in \mathcal{B}$ and that a point $x \in \phi(b)$ *exhibits* ϕ as represented by b.

Let's do a quick digression for Yoneda lemma. First recall the usual Yoneda lemma. The easiest way to remember this is that the yoneda map $a \to h_a := \mathrm{Hom}(-, a)$ is an embedding $\mathcal{C} \to \mathrm{Psh}$. But that's not very precise. What is precise is to say that

$$\mathrm{Nat}(\mathrm{Hom}(-,b),X) \cong X(b).$$

This is Yoneda lemma (see Categories, Lemma **??**). There are distinguished objects:

$$\begin{aligned} \mathrm{Nat}(\mathrm{Hom}(-,b),X) &\cong X(b) \\ \mathrm{id}_b &\mapsto \eta(\mathrm{id}_b)\eta(\mathrm{id}_b) \\ (f : c \to b) &\mapsto f^*(\eta(\mathrm{id}_b)) \end{aligned}$$

But we are interested in another version, let me put it as a lemma:

Lemma 20.4. *Yoneda lemma is equivalent to* $(\mathrm{Hom}(-,b), id_b)$ *is the initial object in* $\mathcal{C}$*, where* $\mathcal{C}$ *is defined as the category whose objects are presheaves* X *over a given category* $\mathcal{B}$ *together with a choice of object* $x \in X(b)$*, and morphisms are morphisms of presheaves that preserve the chosen object.*

Via straightening, there is more:

Lemma 20.5. *Yoneda lemma is equivalent to the category of discrete fibrations* $\mathcal{E} \to \mathcal{B}$ *with a choice of object* $e \in \mathcal{E}$ *over* b *having an initial object. (and such an initial object is a discrete fibration with a terminal object in* $\mathcal{E}$*).*

Left as an exercise.

Now recall Lemma 16.4, it says that the top arrow in the following diagram is an equivalence

$$\begin{tikzcd} \mathcal{E}_x \arrow[r, "\sim"] \arrow[d] & \mathcal{B}_{/b} \arrow[d] \\ \mathcal{E} \arrow[r, "p"] & \mathcal{B}. \end{tikzcd}$$

Invert the equivalence to obtain a map over $\mathcal{B}$:

$$\begin{tikzcd} \mathcal{B}_{/b} \arrow[rr] \arrow[dr] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{B}. & \end{tikzcd}$$

which takes id_b to x. This leads to a section $s_x : \mathcal{B}_{/b} \to \mathcal{E}$.

A proposition I did not understand properly, but seems not very hard to prove follows. And then,

Proposition 20.6. *id_x is initial in $\mathcal{C}_{x/}$ and terminal in $\mathcal{C}_{/x}$. In particular, $\mathcal{C}_{/x} \to \mathcal{C}$ is a representable right fibration and $C_{x/} \to \mathcal{C}$ is a corepresentable left fibration.*

Proof. Missing. □

This leads to a corollary:

Lemma 20.7. *A right fibration over $\mathcal{B}$ is representable if and only if it is equivalent to $\mathcal{B}_{/b} \to \mathcal{B}$ for some $b \in \mathcal{B}$, while a left fibration is corepresentable if and only if it is equivalent to $\mathcal{B}_{b/} \to \mathcal{B}$ for some $b \in \mathcal{B}$.*

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