CATEGORIES

github.com/danimalabares/stack

Contents

1. Definitions

We recall the definitions, partly to fix notation.

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Mor}_{\mathcal{C}}(x, y)$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map $\mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $\mathrm{id}_x \in \mathrm{Mor}_{\mathcal{C}}(x, x)$ such that $\mathrm{id}_x \circ \phi = \phi$ and $\psi \circ \mathrm{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is associative, i.e., $(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi)$ whenever these compositions make sense.

Definition 1.2. A *functor* $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
(2) For every $x, y \in \mathrm{Ob}(\mathcal{A})$ a map $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in the following manner: $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for a composable pair (ϕ, ψ) of morphisms of $\mathcal{A}$ and $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

2. Monomorphisms

Definition 2.1. Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathcal{C}$.

(1) We say that f is a *monomorphism* if for every object W and every pair of morphisms $a, b : W \to X$ such that $f \circ a = f \circ b$ we have $a = b$.
(2) We say that f is an *epimorphism* if for every object W and every pair of morphisms $a, b : Y \to W$ such that $a \circ f = b \circ f$ we have $a = b$.

Definition 2.2. Let $\mathcal{C}$ be a category, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

Lemma 2.3. *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of PSh($\mathcal{C}$). A map is an isomorphism if and only if it is both injective and surjective.*

3. Presheaves

Definition 3.1. A *presheaf of sets* on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to *Sets*. *Morphisms of presheaves* are natural transformations of functors. The category of presheaves of sets is denoted *PSh*($\mathcal{C}$) or $\hat{\mathcal{C}}$.

4. Yoneda lemma

The Yoneda lemma says that the sections over a of a presheaf X can be described completely as natural transformations between the presheaf $\mathrm{Hom}(-, a)$ and X.

Definition 4.1. Let A be a category. The *Yoneda embedding* is the functor

$$h : A \to \hat{A}$$

whose value at an object a of A is the presheaf

$$h_a = \mathrm{Hom}_A(-, a).$$

In other words, the evaluation of the presheaf h_a at an object c of A is the set of maps from c to a.

Theorem 4.2 (Yoneda lemma)**.** *For any presheaf X over A, there is a natural bijection (in Sets I think!!)*

$$\begin{aligned} \mathrm{Hom}_{\widehat{A}}(h_a, X) &\xrightarrow{\simeq} X_a \\ (h_a \xrightarrow{u} X) &\longmapsto u_a(1_a) \end{aligned}$$

5. Internal Hom

Upshot. Internal Hom is when the Hom set of two objects in some category is in also an object of the category. Down-to-earth, that for two sheaves $\mathcal{F}, \mathcal{G}$, $U \mapsto \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U)$ is also a sheaf, called $\mathcal{H}om$.

I start with Stacks Project approach.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section **??** that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is rarely an isomorphism.

Cartesian closed category In the category of sets there is a bijection $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ that depends naturally on X, Y and Z. The notions related to this bijection are "Cartesian closed category", "currying" and "internal Hom".

Definition 5.1. A category $\mathcal{C}$ is *Cartesian closed* if:

(1) $\mathcal{C}$ has all finite products (Caveat: some require that $\mathcal{C}$ has all finite limits)
(2) For any object Y the functor $- \times Y$ has a right adjoint, which we will denote by $\mathrm{Map}(Y, -)$ or by $-^Y$.

Remark 5.2. By section 3 here, the second property above implies that we get a functor $\mathrm{Map}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, and moreover we get natural isomorphisms $\mathrm{Hom}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Hom}(X \times Y, Z)$ and $\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Map}(X \times Y, Z)$.

6. Product

Definition 6.1. A *product* in a category $\mathcal{C}$ is

$$\begin{tikzcd} P \arrow[r, "a"] \arrow[d, "b"'] & A \\ B & \end{tikzcd}$$

such that for every other

$$\begin{tikzcd} P' \arrow[r, "a'"] \arrow[d, "b'"'] & A \\ B & \end{tikzcd}$$

there exists a unique map $p : P' \to P$ such that $ap = a'$ and $bp = b'$.

For completeness here's Stacks Project formulation:

Definition 6.2. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *product* of x and y is an object $x \times y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times y, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times y, y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms

$\alpha \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(w, y)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times y)$ making the diagram

$$\begin{tikzcd}
w \arrow[dr, dashed, "\gamma"] \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\
& x \times y \arrow[r, "q"'] \arrow[d, "p"'] & y \\
& x &
\end{tikzcd}$$

commute.

And some nice piece of information and a definition also from Stacks Project.

If a product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times y$ to be an object of $\mathcal{C}$ such that

$$h_{x \times y}(w) = h_x(w) \times h_y(w)$$

functorially in w. In other words the product $x \times y$ is an object representing the functor $w \mapsto h_x(w) \times h_y(w)$.

Definition 6.3. We say the category $\mathcal{C}$ *has products of pairs of objects* if a product $x \times y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

7. Coproducts of pairs

Definition 7.1. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *coproduct*, or *amalgamated sum* of x and y is an object $x \amalg y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $i \in \mathrm{Mor}_{\mathcal{C}}(x, x \amalg y)$ and $j \in \mathrm{Mor}_{\mathcal{C}}(y, x \amalg y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(x, w)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(y, w)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(x \amalg y, w)$ making the diagram

$$\begin{tikzcd}
& y \arrow[d, "j"] \arrow[ddr, "\beta"] & \\
x \arrow[r, "i"] \arrow[drr, "\alpha"'] & x \amalg y \arrow[dr, dashed, "\gamma"] & \\
& & w
\end{tikzcd}$$

commute.

If a coproduct exists it is unique up to unique isomorphism. This follows from the Yoneda lemma (applied to the opposite category) as the definition requires $x \amalg y$ to be an object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathcal{C}}(x \amalg y, w) = \mathrm{Mor}_{\mathcal{C}}(x, w) \times \mathrm{Mor}_{\mathcal{C}}(y, w)$$

functorially in w.

Definition 7.2. We say the category $\mathcal{C}$ *has coproducts of pairs of objects* if a coproduct $x \amalg y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

8. Fibre products

Definition 8.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_\mathcal{C}(x, y)$ and $g \in \mathrm{Mor}_\mathcal{C}(z, y)$. A *fibre product* of f and g is an object $x \times_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times_y z, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times_y z, z)$ making the diagram

$$\begin{tikzcd} x \times_y z \arrow[r, "q"'] \arrow[d, "p"'] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

commute, and such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(w, z)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times_y z)$ making the diagram

$$\begin{tikzcd} w \arrow[dr, dashed, "\gamma"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & x \times_y z \arrow[r, "q"'] \arrow[d, "p"] & z \arrow[d, "g"] \\ & x \arrow[r, "f"] & y \end{tikzcd}$$

commute.

If a fibre product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times_y z$ to be an object of $\mathcal{C}$ such that

$$h_{x \times_y z}(w) = h_x(w) \times_{h_y(w)} h_z(w)$$

functorially in w. In other words the fibre product $x \times_y z$ is an object representing the functor $w \mapsto h_x(w) \times_{h_y(w)} h_z(w)$.

Definition 8.2. We say a commutative diagram

$$\begin{tikzcd} w \arrow[r] \arrow[d] & z \arrow[d] \\ x \arrow[r] & y \end{tikzcd}$$

in a category is *cartesian* if w and the morphisms $w \to x$ and $w \to z$ form a fibre product of the morphisms $x \to y$ and $z \to y$.

Definition 8.3. We say the category $\mathcal{C}$ *has fibre products* if the fibre product exists for any $f \in \mathrm{Mor}_\mathcal{C}(x, y)$ and $g \in \mathrm{Mor}_\mathcal{C}(z, y)$.

Definition 8.4. A morphism $f : x \to y$ of a category $\mathcal{C}$ is said to be *representable* if for every morphism $z \to y$ in $\mathcal{C}$ the fibre product $x \times_y z$ exists.

Lemma 8.5. *Let $\mathcal{C}$ be a category. Let $f : x \to y$, and $g : y \to z$ be representable. Then $g \circ f : x \to z$ is representable.*

Proof. Let $t \in \mathrm{Ob}(\mathcal{C})$ and $\varphi \in \mathrm{Mor}_{\mathcal{C}}(t, z)$. As g and f are representable, we obtain commutative diagrams

$$\begin{tikzcd}
y \times_z t \arrow[r, "q"'] \arrow[d, "p"'] & t \arrow[d, "\varphi"] \\
y \arrow[r, "g"] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
x \times_y (y \times_z t) \arrow[r, "q'"'] \arrow[d, "p'"'] & y \times_z t \arrow[d, "p"] \\
x \arrow[r, "f"] & y
\end{tikzcd}$$

with the universal property of Definition 8.1. We claim that $x \times_z t = x \times_y (y \times_z t)$ with morphisms $q \circ q' : x \times_z t \to t$ and $p' : x \times_z t \to x$ is a fibre product. First, it follows from the commutativity of the diagrams above that $\varphi \circ q \circ q' = g \circ f \circ p'$. To verify the universal property, let $w \in \mathrm{Ob}(\mathcal{C})$ and suppose $\alpha : w \to x$ and $\beta : w \to y$ are morphisms with $\varphi \circ \beta = g \circ f \circ \alpha$. By definition of the fibre product, there are unique morphisms δ and γ such that

$$\begin{tikzcd}
w \arrow[drr, bend left, "\beta"] \arrow[dr, dashed, "\delta"] \arrow[ddr, bend right, "f \circ \alpha"'] & & \\
& y \times_z t \arrow[r, "q"'] \arrow[d, "p"'] & t \arrow[d, "\varphi"] \\
& y \arrow[r, "g"] & z
\end{tikzcd}$$

and

$$\begin{tikzcd}
w \arrow[drr, bend left, "\delta"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, bend right, "\alpha"'] & & \\
& x \times_y (y \times_z t) \arrow[r, "q'"'] \arrow[d, "p'"'] & y \times_z t \arrow[d, "p"] \\
& x \arrow[r, "f"] & y
\end{tikzcd}$$

commute. Then, γ makes the diagram

$$\begin{tikzcd}
w \arrow[drr, bend left, "\beta"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, bend right, "\alpha"'] & & \\
& x \times_z t \arrow[r, "q \circ q'"'] \arrow[d, "p'"'] & t \arrow[d, "\varphi"] \\
& x \arrow[r, "g \circ f"] & z
\end{tikzcd}$$

commute. To show its uniqueness, let γ' verify $q \circ q' \circ \gamma' = \beta$ and $p' \circ \gamma' = \alpha$. Because γ is unique, we just need to prove that $q' \circ \gamma' = \delta$ and $p' \circ \gamma' = \alpha$ to conclude. We supposed the second equality. For the first one, we also need to use the uniqueness of delta. Notice that δ is the only morphism verifying $q \circ \delta = \beta$ and $p \circ \delta = f \circ \alpha$. We already supposed that $q \circ (q' \circ \gamma') = \beta$. Furthermore, by definition of the fibre product, we know that $f \circ p' = p \circ q'$. Therefore:

$$p \circ (q' \circ \gamma') = (p \circ q') \circ \gamma' = (f \circ p') \circ \gamma' = f \circ (p' \circ \gamma') = f \circ \alpha.$$

Then $q' \circ \gamma' = \delta$, which concludes the proof. $\square$

Lemma 8.6. *Let $\mathcal{C}$ be a category. Let $f : x \to y$ be representable. Let $y' \to y$ be a morphism of $\mathcal{C}$. Then the morphism $x' := x \times_y y' \to y'$ is representable also.*

Proof. Let $z \to y'$ be a morphism. The fibre product $x' \times_{y'} z$ is supposed to represent the functor

$$\begin{aligned} w & \mapsto & h_{x'}(w) \times_{h_{y'}(w)} h_z(w) \\ & = & (h_x(w) \times_{h_y(w)} h_{y'}(w)) \times_{h_{y'}(w)} h_z(w) \\ & = & h_x(w) \times_{h_y(w)} h_z(w) \end{aligned}$$

which is representable by assumption. □

9. Examples of fibre products

In this section we list examples of fibre products and we describe them.

As a really trivial first example we observe that the category of sets has fibre products and hence every morphism is representable. Namely, if $f : X \to Y$ and $g : Z \to Y$ are maps of sets then we define $X \times_Y Z$ as the subset of $X \times Z$ consisting of pairs (x, z) such that $f(x) = g(z)$. The morphisms $p : X \times_Y Z \to X$ and $q : X \times_Y Z \to Z$ are the projection maps $(x, z) \mapsto x$, and $(x, z) \mapsto z$. Finally, if $\alpha : W \to X$ and $\beta : W \to Z$ are morphisms such that $f \circ \alpha = g \circ \beta$ then the map $W \to X \times Z$, $w \mapsto (\alpha(w), \beta(w))$ obviously ends up in $X \times_Y Z$ as desired.

In many categories whose objects are sets endowed with certain types of algebraic structures the fibre product of the underlying sets also provides the fibre product in the category. For example, suppose that X, Y and Z above are groups and that f, g are homomorphisms of groups. Then the set-theoretic fibre product $X \times_Y Z$ inherits the structure of a group, simply by defining the product of two pairs by the formula $(x, z) \cdot (x', z') = (xx', zz')$. Here we list those categories for which a similar reasoning works.

1. The category *Groups* of groups.
2. The category *G-Sets* of sets endowed with a left G-action for some fixed group G.
3. The category of rings.
4. The category of R-modules given a ring R.

10. Limits and colimits

The definition of product is actually a particular case of a limit.

Let $\mathcal{C}$ be a category. A *diagram* in $\mathcal{C}$ is simply a functor $M : \mathcal{I} \to \mathcal{C}$. We say that $\mathcal{I}$ is the *index category* or that M is an $\mathcal{I}$-diagram. We will use the notation M_i to denote the image of the object i of $\mathcal{I}$. Hence for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $M(\phi) : M_i \to M_{i'}$.

Definition 10.1. A *limit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\lim_{\mathcal{I}} M$ in $\mathcal{C}$ together with morphisms $p_i : \lim_{\mathcal{I}} M \to M_i$ such that

1. for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $p_{i'} = M(\phi) \circ p_i$, and
2. for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to M_i$ (indexed by $i \in \mathrm{Ob}(\mathcal{I})$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $q_{i'} = M(\phi) \circ q_i$ there exists a unique morphism $q : W \to \lim_{\mathcal{I}} M$ such that $q_i = p_i \circ q$ for every object i of $\mathcal{I}$.

Limits $(\lim_{\mathcal{I}} M, (p_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Products of pairs, fibre products, and equalizers are examples of limits. The limit over the empty diagram is a final object of $\mathcal{C}$. In the category of sets all limits exist. The dual notion is that of colimits.

Definition 10.2. A *colimit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\mathrm{colim}_{\mathcal{I}} M$ in $\mathcal{C}$ together with morphisms $s_i : M_i \to \mathrm{colim}_{\mathcal{I}} M$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $s_i = s_{i'} \circ M(\phi)$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $t_i : M_i \to W$ (indexed by $i \in \mathrm{Ob}(\mathcal{I})$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $t_i = t_{i'} \circ M(\phi)$ there exists a unique morphism $t : \mathrm{colim}_{\mathcal{I}} M \to W$ such that $t_i = t \circ s_i$ for every object i of $\mathcal{I}$.

Definition 10.3. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *product* $\prod_{i \in I} M_i$ is by definition $\lim_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is a final object of the category. The morphisms $p_i : \prod M_i \to M_i$ are called the *projection morphisms*.

Definition 10.4. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *coproduct* $\coprod_{i \in I} M_i$ is by definition $\mathrm{colim}_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the coproduct is an initial object of the category. Note that the coproduct comes equipped with morphisms $M_i \to \coprod M_i$. These are sometimes called the *coprojections*.

11. Simplicial sets

Definition 11.1. The *simplex category* is the category of ordinals, i.e. non-empty finite ordered sets

$$[n] = \{0 < 1 < \ldots < n\}, \quad n = 0, 1, \ldots$$

with order preserving maps of sets.

Objects of the simplicial category Δ are not called simplices. Instead, simplices are type of simplicial set:

Definition 11.2. A *simplicial set* is a presheaf on Δ, i.e. an element of $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets})$.

Definition 11.3. The *n-simplex* is the simplicial set $\mathrm{Hom}(-, [n]) := \Delta^n$.

No time to delve into the details, but there's a way to identify simplicial sets to topological spaces. And then we can actually define two simplicial sets to be weakly equivalent if the corresponding topological spaces are weakly equivalent. Wow!

References