**INFTY-CATEGORIES**

github.com/danimalabares/infty-categories

Contents



## 1. Other exercises

**Definition 1.1.** A functor $F : C \to B$ is a *discrete fibration* if for every object $c$ in $C$ and every morphism of the form $g : b \to F(c)$ in $B$ there is a unique morphism $h : d \to c$ in $C$ such that $F(h) = g$.

**Exercise 1.2.** Prove that discrete fibrations over a category $C$ correspond to presheaves over $C$.

*Proof.* First suppose that we are given a presheaf $X$ on $C$. Define a discrete fibration $F : C/X \to C$ by $(a, s) \mapsto a$ on objects and mapping a morphism $f : (a, s) \to (b, t)$ in $C/X$ to the corresponding morphism $a \to b$ in $C$. To show $F$ is a discrete fibration let $g : b \to a$ be a morphism in $C$. Consider $g^* = X(g) : X_a \to X_b$, and the section $g^*s$ of $X_b$. Then the morphism $h : (b, g^*s) \to (a, s)$ is the only one mapping to $g$ under $h$.

For the converse let $F : B \to C$ be a discrete fibration over $C$. To define a presheaf $X : C^{\text{op}} \to \text{Sets}$ let $c \in \operatorname{Ob} C$. We assign the set (for now I won't justify why this is a set) of objects in $B$ mapped to $c$ under $F$. To define the correspondence on morphisms, consider a map $f : c \to d$ in $C^{\text{op}}$. In other words, we have a map in $C$ of the form $f^{\text{op}} : d \to c$. Then to any object in $b$ such that $F(b) = c$, by definition of discrete fibration, we have a unique morphism of $B$ of the form $h : r \to b$ such that $F(h) = f^{\text{op}}$. In particular this means that $F(r) = d$. This gives a function from $X(c)$ to $X(d)$. This situation is described in the following diagram:

$$\begin{tikzcd}
X(d) \ni & r \arrow[r, mapsto] \arrow[d, dotted, "\exists! h"'] & d = F(r) \arrow[d, "f^{\text{op}} = F(h)"] \\
X(c) \ni & b \arrow[r, mapsto, "F"'] & c = F(b)
\end{tikzcd}$$

To check functoriality of $X$ defined in the previous paragraph suppose that $f : c \to d$ and $g : d \to e$ are two morphisms in $C^{\text{op}}$. Like before, we have maps

$f^{\mathrm{op}} : d \to c$ and $g^{\mathrm{op}} : e \to d$.

$$\begin{tikzcd}
X(e) \ni & q \arrow[r, mapsto] \arrow[d, dotted, "\exists! j"'] & e = F(q) \arrow[d, "g^{\mathrm{op}} = F(j)"] \\
X(d) \ni & r \arrow[r, mapsto] \arrow[d, dotted, "\exists! h"'] & d = F(r) \arrow[d, "f^{\mathrm{op}} = F(h)"] \\
X(c) \ni & b \arrow[r, mapsto, "F"'] & c = F(b)
\end{tikzcd}$$

on the other hand, $gf : c \to e$ gives by the same construction a unique map $k : q \to b$ such that $F(k) = f^{\mathrm{op}} g^{\mathrm{op}}$. While it's not clear whether $k$ equals $hj$ or not, the correspondence we are worried about is barely $X(c) \to X(e)$, $b \mapsto q$ which is satisfied in both constructions.

□

## 2. Exercises of Rune, Chapter 1

Here's my progress so far on the exercises in [Hau25], Chapter 1.

**Exercise 2.1** (Observation 1.4.7)**.** Show that the simplicial sets category $\mathrm{Set}_\Delta$ has internal Hom $S^T$ for simplicial sets $S$ and $T$, given by

$$(S^T)_n := \mathrm{Hom}_{\mathrm{Set}_\Delta}(T \times \Delta^n, S)$$

*Proof.* We need to show, as in [Cis23, Notation 1.1.13], that $S^T$ is right adjoint to the functor $- \times S$. That is,

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(U, S^T) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(U \times S, T)$$

(I think) I understand the statement correctly but I don't understand how to apply [Cis23, Theorem 1.1.10 (Kan)] nor [Cis23, Remark 1.1.11] to prove it. □

**Exercise 2.2** (1.1)**.** If $S$ is a Kan complex, then the relation defining $\pi_0 S$ is an equivalence relation.

*Proof.* I didn't understand well enough what is $\pi_0 S$... □

Rather informally, I understand a category $C$ to be an *enriched category over $D$* if for any objects $c, d$ in $C$, $\mathrm{Hom}(c, d)$ is an object of $D$. Compositions of morphisms exist and are associative, and there is an identity morphism for every object $c$ in $C$. (See `https://ncatlab.org/nlab/show/enriched+category` for a formal definition.)

**Exercise 2.3** (1.2)**.** Show that $\mathsf{Cat}_\Delta$ can be described as the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ containing the functors whose simplicial sets of objects are constant.

*Proof.* To a $\mathrm{Set}_\Delta$-enriched category $C$ associate the functor which maps $[n]$ to the category $C_n$ defined as follows. The objects of $C_n$ are the objects of $C$ for all $n$. For $a, b$ in $C$, the morphisms of $C_n$ are $\mathrm{Hom}(a, b)_n$. To complete the definition of our functor in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ we must specify where to send a map $f : [n] \to [m]$ in $\Delta^{\mathrm{op}}$. It must go to a functor of $C$ to $C$ that fixes all objects and maps a map in $\mathrm{Hom}(a, b)_n$ to the induced map $\mathrm{Hom}(a, b)_m$ by the presheaf $\mathrm{Hom}(a, b)$.

Now we need to show that every functor whose simplicial sets of objects is constant (i.e. constant functor $\Delta^{\mathrm{op}} \to \mathsf{Cat}$) can be expressed this way. Suppose

$F$ is such a functor with value $C$. Associate the $\mathrm{Set}_\Delta$-enriched category $C$ with $\mathrm{Hom}(a,b)_n$ given by $\mathrm{Hom}_{F([n])}(a,b)$.

(Hopefully my understanding of the term "full subcategory" is not wrong?) $\square$

**Exercise 2.4** (1.3)**.** Show that $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is fully faithful.

*Proof.* We need to show that for any categories $A, B$, $\mathrm{Hom}(A,B) = \mathrm{Fun}(A,B)$ is in "bijection" with $\mathrm{Hom}_{\mathsf{Set}_\Delta}(NA, NB)$. Recall that $NA$ is the presheaf that maps $[n]$ to the set of composible sequence of $n$ morphisms in $A$. Then to a functor $F : A \to B$ we associate the map that sends a sequence of $n$ morphisms in $A$ to the respective sequence of $n$ morphisms in $B$ after applying $F$ to each object and map.

Conversely, given a morphism in $\mathsf{Set}_\Delta$ from $NA$ to $NB$ we can reconstruct a functor from $A$ to $B$ by interpreting objects of $A$ as $NA_0$ and maps as $NA_1$. $\square$

## References

[Cis23] Denis-Charles Cisinsky, *Higher categories and homotopical algebra*, 2023.
[Hau25] Rune Haugseng, *Yet another introduction to infty-categories*, 2025.