INFTY-CATEGORIES

github.com/danimalabares/infty-categories

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1. Other exercises

Definition 1.1. A functor $F: C \to B$ is a discrete fibration if for every object c in C and every morphism of the form $g: b \to F(c)$ in B there is a unique morphism $h: d \to c$ in C such that F(h) = g.

Exercise 1.2. Prove that discrete fibrations over a category C correspond to presheaves over C.

Proof. First suppose that we are given a presheaf X on C. Define a discrete fibration $F: C/X \to C$ by $(a,s) \mapsto a$ on objects and mapping a morphism $f: (a,s) \to (b,t)$ in C/X to the corresponding morphism $a \to b$ in C. To show F is a discrete fibration let $g: b \to a$ be a morphism in C. Consider $g^* = X(g): X_a \to X_b$, and the section g^*s of X_b . Then the morphism $h:(b,g^*s)\to(a,s)$ is the only one mapping to g under h.

For the converse let $F: B \to C$ be a discrete fibration over C. To define a presheaf $X: C^{\mathrm{op}} \to \mathrm{Sets}$ let $c \in \mathrm{Ob}$. We assign the set (for now I won't justify why this is a set) of objects in B mapped to c under F. To define the correspondence on morphisms, consider a map $f: c \to d$ in C^{op} . In other words, we have a map in C of the form $f^{op}d \to c$. Then to any object in b such that F(b) = c, by definition of discrete fibration, we have a unique morphism of B of the form $h: r \to b$ such that $F(h) = f^{op}$. In particular this means that F(r) = d. This gives a function between X(c) and X(d).

2. Exercises of Rune, Chapter 1

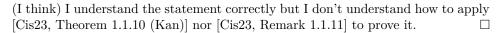
Here's my progress so far on the exercises in [Hau25], Chapter 1.

Exercise 2.1 (Observation 1.4.7). Show that the simplicial sets category $\operatorname{Set}_{\Lambda}$ has internal Hom S^T for simplicial sets S and T, given by

$$(S^T)_n := \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(T \times \Delta^n, S)$$

Proof. We need to show, as in [Cis23, Notation 1.1.13], that S^T is right adjoint to the functor $-\times S$. That is,

$$\operatorname{Hom}_{\operatorname{Set}_{\Delta}}(U, S^T) \cong \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(U \times S, T)$$



Exercise 2.2 (1.1). If S is a Kan complex, then the relation defining $\pi_0 S$ is an equivalence relation.

Proof. I didn't understand well enough what is $\pi_0 S...$

Rather informally, I understand a category C to be an enriched category over D if for any objects c,d in C, $\operatorname{Hom}(c,d)$ is an object of D. Compositions of morphisms exist and are associative, and there is an identity morphism for every object c in C. (See https://ncatlab.org/nlab/show/enriched+category for a formal definition.)

Exercise 2.3 (1.2). Show that Cat_{Δ} can be described as the full subcategory of $\mathsf{Fun}(\Delta^{\mathrm{op}},\mathsf{Cat})$ containing the functors whose simplicial sets of objects are constant.

Proof. To a Set_{Δ}-enriched category C associate the functor which maps [n] to the category C_n defined as follows. The objects of C_n are the objects of C for all n. For a, b in C, the morphisms of C_n are $\operatorname{Hom}(a, b)_n$. To complete the definition of our functor in $\operatorname{Fun}(\Delta^{\operatorname{op}}, \operatorname{Cat})$ we must specify where to send a map $f:[n] \to [m]$ in $\Delta^{\operatorname{op}}$. It must go to a functor of C to C that fixes all objects and maps a map in $\operatorname{Hom}(a, b)_n$ to the induced map $\operatorname{Hom}(a, b)_m$ by the presheaf $\operatorname{Hom}(a, b)$.

Now we need to show that every functor whose simplicial sets of objects is constant (i.e. constant functor $\Delta^{\mathrm{op}} \to \mathsf{Cat}$) can be expressed this way. Suppose F is such a functor with value C. Associate the Set_{Δ} -enriched category C with $\mathrm{Hom}(a,b)_n$ given by $\mathrm{Hom}_{F([n])}(a,b)$.

(Hopefully my understanding of the term "full subcategory" is not wrong?)

Exercise 2.4 (1.3). Show that $N : \mathsf{Cat} \to \mathsf{Set}_{\Delta}$ is fully faithful.

Proof. We need to show that for any categories A, B, $\operatorname{Hom}(A, B) = \operatorname{Fun}(A, B)$ is in "bijection" with $\operatorname{Hom}_{\operatorname{Set}_\Delta}(NA, NB)$. Recall that NA is the presheaf that maps [n] to the set of composible sequence of n morphisms in A. Then to a functor $F: A \to B$ we associate the map that sends a sequence of n morphisms in A to the respective sequence of n morphisms in B after applying F to each object and map.

Conversely, given a morphism in Set_Δ from NA to NB we can reconstruct a functor from A to B by interpreting objects of A as NA_0 and maps as NA_1 . \square

REFERENCES

[Cis23] Denis-Charles Cisinsky, Higher categories and homotopical algebra, 2023.

[Hau25] Rune Haugseng, Yet another introduction to infty-categories, 2025.