INFTY-CATEGORIES

github.com/danimalabares/infty-categories

Contents

1. Other exercises

Definition 1.1. A functor $F : C \to B$ is a *discrete fibration* if for every object c in C and every morphism of the form $g : b \to F(c)$ in B there is a unique morphism $h : d \to c$ in C such that $F(h) = g$.

Exercise 1.2. Prove that discrete fibrations over a category C correspond to presheaves over C.

Proof. First suppose that we are given a presheaf X on C. Define a discrete fibration $F : C/X \to C$ by $(a, s) \mapsto a$ on objects and mapping a morphism $f : (a, s) \to (b, t)$ in C/X to the corresponding morphism $a \to b$ in C. To show F is a discrete fibration let $g : b \to a$ be a morphism in C. Consider $g^* = X(g) : X_a \to X_b$, and the section g^*s of X_b. Then the morphism $h : (b, g^*s) \to (a, s)$ is the only one mapping to g under h.

For the converse let $F : B \to C$ be a discrete fibration over C. To define a presheaf $X : C^{\mathrm{op}} \to \mathrm{Sets}$ let $c \in \mathrm{Ob}\, C$. We assign the set (for now I won't justify why this is a set) of objects in B mapped to c under F. To define the correspondence on morphisms, consider a map $f : c \to d$ in C^{op}. In other words, we have a map in C of the form $f^{\mathrm{op}} : d \to c$. Then to any object in b such that $F(b) = c$, by definition of discrete fibration, we have a unique morphism of B of the form $h : r \to b$ such that $F(h) = f^{\mathrm{op}}$. In particular this means that $F(r) = d$. This gives a function from $X(c)$ to $X(d)$. This situation is described in the following diagram:

$$\begin{tikzcd}
X(d) \ni & r \arrow[r, mapsto] \arrow[d, dotted, "\exists! h"'] & d = F(r) \arrow[d, "f^{\mathrm{op}} = F(h)"] \\
X(c) \ni & b \arrow[r, mapsto, "F"'] & c = F(b)
\end{tikzcd}$$

To check functoriality of X defined in the previous paragraph suppose that $f : c \to d$ and $g : d \to e$ are two morphisms in C^{op}. Like before, we have maps

$f^{\mathrm{op}} : d \to c$ and $g^{\mathrm{op}} : e \to d$.

$$\begin{tikzcd} X(e) \ni & q \arrow[r, mapsto] \arrow[d, dotted, "\exists ! j"'] & e = F(q) \arrow[d, "g^{\mathrm{op}} = F(j)"] \\ X(d) \ni & r \arrow[r, mapsto] \arrow[d, dotted, "\exists ! h"'] & d = F(r) \arrow[d, "f^{\mathrm{op}} = F(h)"] \\ X(c) \ni & b \arrow[r, mapsto, "F"'] & c = F(b) \end{tikzcd}$$

on the other hand, $gf : c \to e$ gives by the same construction a unique map $k : \hat{q} \to b$ such that $F(k) = f^{\mathrm{op}} g^{\mathrm{op}}$. To check that $\hat{q} = q$, observe that by functoriality of F, we have $F(hj) = F(h)F(j) = f^{\mathrm{op}} g^{\mathrm{op}} = F(k)$. By uniqueness of k, we conclude that $k = hj$ and thus $q = \hat{q}$. □

2. Exercises of Rune, Chapter 1

Here's my progress so far on the exercises in [Hau25], Chapter 1.

Exercise 2.1 (Observation 1.4.7)**.** Show that the simplicial sets category Set_Δ has internal Hom S^T for simplicial sets S and T, given by

$$(S^T)_n := \mathrm{Hom}_{\mathsf{Set}_\Delta}(T \times \Delta^n, S)$$

Proof. We need to show, that S^T is the internal Hom in the category Set_Δ. Different notations for the internal Hom are $\mathrm{Map}(-,-)$, $\underline{\mathrm{Hom}}(-,-)$. It must be right adjoint to the functor $U \times -$. That is,

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(U \times S, T) \cong \mathrm{Hom}_{\mathsf{Set}_\Delta}(U, S^T)$$

(I think) I understand the statement correctly but I don't understand how to apply [Cis23, Theorem 1.1.10 (Kan)] nor [Cis23, Remark 1.1.11] to prove it.

□

Exercise 2.2 (1.1)**.** If S is a Kan complex, then the relation defining $\pi_0 S$ is an equivalence relation.

Proof.
1. (Reflexivity.) Let $a \in S_0$. Consider the composition

$$\begin{tikzcd} {[0]} \arrow[r, "d_0"] & {[1]} \arrow[r, "f_0"] & {[0]} \\ 0 \arrow[r, mapsto] & 1 \arrow[r, mapsto] & 0 \end{tikzcd}$$

   since this gives the identity we must have

$$\begin{tikzcd} S_0 \arrow[r, "S(f_0)"] & S_1 \arrow[r, "S(d_0)"] & {[0]} \\ a \arrow[r, mapsto] & S(f_0)(a) \arrow[r, mapsto] & a \end{tikzcd}$$

   but we can replace d_0 by d_1 and we'd still get the identity, so that $S(d_1)$ also maps $S(f_0)(a)$ to a. In other words, for any $a \in S_0$ the 1-simplex $S(f_0)(a)$ is the desired one.
2. (Symmetry.) Let $a, b \in S_0 \ldots$

□

Rather informally, I understand a category C to be an *enriched category over D* if for any objects c, d in C, $\mathrm{Hom}(c, d)$ is an object of D. Compositions of morphisms exist and are associative, and there is an identity morphism for every object c in C. (See `https://ncatlab.org/nlab/show/enriched+category` for a formal definition.)

Exercise 2.3 (1.2)**.** Show that Cat_Δ can be described as the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ containing the functors whose simplicial sets of objects are constant.

Remark 2.4. The phrase "simplicial sets of objects are constants" means the following. Consider the functor $\mathsf{Cat} \to \mathsf{Set}$ that maps a category to its set of objects (I suppose we may take Set to be a universe), which induces for every functor in $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ a functor in $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$. We mean to say the latter map is constant.

Proof. We need to construct a fully faithful functor

$$F : \mathsf{Cat}_\Delta \to \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$$

whose image is the subcategory of functors whose simplicial sets of objects are constant.

To a Set_Δ-enriched category C associate the functor $F(C)$ which maps $[n]$ to the category C_n, which is defined as follows. The objects of C_n are the objects of C for all n. (Notice that once we define the functor completely, this property will make it indeed a functor whose simplicial sets of objects are constant.) For a, b in C, the morphisms of C_n are $\mathrm{Hom}(a, b)_n$. To a map $f : [n] \to [m]$ in Δ^{op}, define $F(C)$ to give the functor of C_m to C_n that fixes all objects and maps a map in $\mathrm{Hom}(a, b)_m$ to the induced map $\mathrm{Hom}(a, b)_n$ by the presheaf $\mathrm{Hom}(a, b)$.

Now let's define how F acts on morphisms. (This definition is just what it should be, but let's go over it.) Choose two Set_Δ-enriched categories C, D and consider their corresponding functors $F(C), F(D)$. Fix a morphism $\varphi \in \mathrm{Hom}_{\mathsf{Cat}_\Delta}(C, D)$. Define a morphism (of presheaves of categories) $F(\varphi) : F(C) \to F(D)$ defined as a collection of maps $F(C)_n \to F(D)_n$ given on objects by φ and on morphisms also given by φ, using that φ is a morphism of Cat_Δ to ensure naturality.

Functoriality of F follows from functoriality of each φ as in the previous paragraph.

Now let's confirm that F is faithful, that is, it induces injections on the Hom sets. Suppose $\varphi, \psi \in \mathrm{Hom}_{\mathsf{Cat}_\Delta}(C, D)$ are such that $F(\varphi) = F(\psi)$. By definition of $F(\varphi)$ and $F(\psi)$, it is immediate that φ and ψ coincide on objects. In fact, it is also immediate that they coincide on morphism and as simplicial sets by definition.

To prove F is fully faithful we only need to check surjectivity of the induced maps in Hom sets. Pick a morphism of presheaves of categories, denote it $F(\varphi)$, between two presheaves of categories $F(C)$ and $F(D)$, both of whose simplicial sets of objects are constant, namely two sets C and D. Then we can define two Set_Δ-enriched categories, which we also denote by C and D, by defining their objects to be the sets C and D, and their morphisms to be the collections of all the induced morphisms by $F(C)$ and $F(D)$ coming from morphisms of Δ^{op}. Then it is immediate that the set $\mathrm{Hom}(C, D)$ is indeed a simplicial set. Thus $C, D \in \mathsf{Cat}_{\mathsf{Set}_\Delta}$. Further, we can define a morphism $\varphi \in \mathrm{Hom}_{\mathsf{Cat}_\Delta}(C, D)$ which maps on objects as any of the induced maps by the morphism of presheaves of categories we started with (since both of the simplicial sets of objects of the corresponding categories are constant!) and on morphisms as well (any morphism of C was defined as the induced map by $F(C)$ coming from a map of Δ^{op}). It is clear that this morphism is mapped to $F(\varphi)$ under F. $\square$

Exercise 2.5 (1.3)**.** Show that $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is fully faithful.

Proof. We need to show that for any categories A, B, $\mathrm{Hom}(A,B) = \mathrm{Fun}(A,B)$ is in "bijection" with $\mathrm{Hom}_{\mathsf{Set}_\Delta}(NA, NB)$. Recall that NA is the presheaf that maps $[n]$ to the set of composible sequence of n morphisms in A. Then to a functor $F : A \to B$ we associate the map that sends a sequence of n morphisms in A to the respective sequence of n morphisms in B after applying F to each object and map.

Conversely, given a morphism in Set_Δ from NA to NB we can reconstruct a functor from A to B by interpreting objects of A as NA_0 and maps as NA_1. $\square$

3. Exercises of Rune, Chapter 2

Definition 3.1. For points $x, y \in X$, the *path space* $X(x,y)$ is the pullback

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \{x\} \arrow[d] \\ \{y\} \arrow[r] & X. \end{tikzcd}$$

Exercise 3.2 (2.1)**.** Assuming that pushouts exist and have the expected universal property, show that $\Omega^n_x X \simeq \mathsf{Map}_*(S^n, X)$, where the n-th sphere is the pushout

$$S^n := * \amalg_{S^{n-1}} *,$$

and the space of pointed maps is the pullback

$$\begin{tikzcd} \mathsf{Map}_*(S^n, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Map}(S^n, X) \arrow[d] \\ \{x\} \arrow[r] & \mathsf{Map}(*, X). \end{tikzcd}$$

Proof. Since both $\Omega^1_x X$ and $\mathsf{Map}_*(S^1, X)$ are pullbacks (and pullbacks are unique up to homotopy), it's enough to show that they are the pullback of the same diagram (up to homotopy).

First notice that $\mathsf{Map}(*, X) \simeq X$ in an obvious way: we identify a map $* \to X$ with the image of $*$. To identify $\mathsf{Map}(S^n, X)$ with $\{x\} \simeq \mathsf{Map}(*, X)$ pick a map $* \to X$. Now consider the universal property of pushouts:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \arrow[ddr] & \\ * \arrow[r] \arrow[drr] & S^n = * \amalg_{S^{n-1}} * \arrow[dr, dashed, "\exists!"] & \\ & & X \end{tikzcd}$$

$\square$

Exercise 3.3 (2.2)**.** Use the 5-lemma to show that given a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

the morphism f is an equivalence if and only if the induced maps on the fibres $p^{-1}(b) \to q^{-1}(b)$ are equivalences for all $b \in B$.

Proof. For the converse implication,

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(B,b) \arrow[r] \arrow[d, "\simeq"] & \pi_n(p^{-1}(b),x) \arrow[r] \arrow[d, "\simeq"] & \pi_n(X,x) \arrow[r] \arrow[d, dotted, "\simeq"] & \pi_n(B,b) \arrow[r] \arrow[d, "\simeq"] & \pi_{n-1}(p^{-1}(b),x) \arrow[r] \arrow[d, "\simeq"] & \cdots \\ \cdots \arrow[r] & \pi_{n+1}(B,b) \arrow[r] & \pi_n(q^{-1}(b),f(x)) \arrow[r] & \pi_n(Y,f(x)) \arrow[r] & \pi_n(B,b) \arrow[r] & \pi_{n-1}(q^{-1}(b),f(x)) \arrow[r] & \cdots \end{tikzcd}$$

and for the forward implication just do the same with the map $\pi_n(p^{-1}(b),x) \to \pi_n(q^{-1},f(x))$ in the center. $\square$

Exercise 3.4 (2.3)**. Note:** I think this problem is formulated wrong in the book (there's no g in the diagram!), so I think this is the correct version:

A commutative square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback if and only if for every $x \in X$, the induced map on fibres is an equivalence.

Proof. The proof is using Corollary 2.1.23 that such a square is a pullback if and only if for every $x \in X$ the induced map on fibres is an equivalence. This reduces everything to a situation completely analogous to Exercise 3.3 and we conclude by 5-lemma. $\square$

Exercise 3.5 (2.4)**.** Suppose we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X'' \arrow[d] \\ Y \arrow[r] & Y' \arrow[r] & Y''. \end{tikzcd}$$

(1) If the right and composite squares are both pullbacks, then so is the left-hand square.
(2) If $\pi_0 Y \to \pi_0 Y'$ is surjective and the left and composite squares are both pullbacks, then so is the right-hand square.

Proof. (1) To use the last exercise we only need to show that $Y \to Y''$ is an equivalence. But this is what Corollary 2.1.17: the 3-for-2 property.
(2) I'm not sure why is that condition on $\pi_0 Y \to \pi_0 Y' \ldots$ looks like the same argument should work.

$\square$

Exercise 3.6 (2.9)**.** Show that if X is an ∞-groupoid, then so is $\mathsf{Fun}(\mathcal{C}, X)$ for any ∞-category $\mathcal{C}$.

References

[Cis23] Denis-Charles Cisinsky, *Higher categories and homotopical algebra*, 2023.
[Hau25] Rune Haugseng, *Yet another introduction to infty-categories*, 2025.