

INFTY-CATEGORIES

github.com/danimalabares/infty-categories

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1. OTHER EXERCISES

Definition 1.1. A functor $F : C \rightarrow B$ is a *discrete fibration* if for every object c in C and every morphism of the form $g : b \rightarrow F(c)$ in B there is a unique morphism $h : d \rightarrow c$ in C such that $F(h) = g$.

Exercise 1.2. Prove that discrete fibrations over a category C correspond to presheaves over C .

Proof. First suppose that we are given a presheaf X on C . Define a discrete fibration $F : C/X \rightarrow C$ by $(a, s) \mapsto a$ on objects and mapping a morphism $f : (a, s) \rightarrow (b, t)$ in C/X to the corresponding morphism $a \rightarrow b$ in C . To show F is a discrete fibration let $g : b \rightarrow a$ be a morphism in C . Consider $g^* = X(g) : X_a \rightarrow X_b$, and the section g^*s of X_b . Then the morphism $h : (b, g^*s) \rightarrow (a, s)$ is the only one mapping to g under F .

For the converse let $F : B \rightarrow C$ be a discrete fibration over C . To define a presheaf $X : C^{\text{op}} \rightarrow \text{Sets}$ let $c \in \text{Ob}$. We assign the set (for now I won't justify why this is a set) of objects in B mapped to c under F . To define the correspondence on morphisms, consider a map $f : c \rightarrow d$ in C^{op} . In other words, we have a map in C of the form $f^{\text{op}}d \rightarrow c$. Then to any object in b such that $F(b) = c$, by definition of discrete fibration, we have a unique morphism of B of the form $h : r \rightarrow b$ such that $F(h) = f^{\text{op}}$. In particular this means that $F(r) = d$. This gives a function between $X(c)$ and $X(d)$. \square

2. EXERCISES OF RUNE, CHAPTER 1

Here's my progress so far on the exercises in [Hau25], Chapter 1.

Exercise 2.1 (Observation 1.4.7). Show that the simplicial sets category Set_Δ has internal Hom S^T for simplicial sets S and T , given by

$$(S^T)_n := \text{Hom}_{\text{Set}_\Delta}(T \times \Delta^n, S)$$

Proof. We need to show, as in [Cis23, Notation 1.1.13], that S^T is right adjoint to the functor $- \times S$. That is,

$$\text{Hom}_{\text{Set}_\Delta}(U, S^T) \cong \text{Hom}_{\text{Set}_\Delta}(U \times S, T)$$

(I think) I understand the statement correctly but I don't understand how to apply [Cis23, Theorem 1.1.10 (Kan)] nor [Cis23, Remark 1.1.11] to prove it. \square

Exercise 2.2 (1.1). If S is a Kan complex, then the relation defining $\pi_0 S$ is an equivalence relation.

Proof. I didn't understand well enough what is $\pi_0 S$. . . \square

Rather informally, I understand a category C to be an *enriched category over D* if for any objects c, d in C , $\text{Hom}(c, d)$ is an object of D . Compositions of morphisms exist and are associative, and there is an identity morphism for every object c in C . (See <https://ncatlab.org/nlab/show/enriched+category> for a formal definition.)

Exercise 2.3 (1.2). Show that Cat_Δ can be described as the full subcategory of $\text{Fun}(\Delta^{\text{op}}, \text{Cat})$ containing the functors whose simplicial sets of objects are constant.

Proof. To a Set_Δ -enriched category C associate the functor which maps $[n]$ to the category C_n defined as follows. The objects of C_n are the objects of C for all n . For a, b in C , the morphisms of C_n are $\text{Hom}(a, b)_n$. To complete the definition of our functor in $\text{Fun}(\Delta^{\text{op}}, \text{Cat})$ we must specify where to send a map $f : [n] \rightarrow [m]$ in Δ^{op} . It must go to a functor of C to C that fixes all objects and maps a map in $\text{Hom}(a, b)_n$ to the induced map $\text{Hom}(a, b)_m$ by the presheaf $\text{Hom}(a, b)$.

Now we need to show that every functor whose simplicial sets of objects is constant (i.e. constant functor $\Delta^{\text{op}} \rightarrow \text{Cat}$) can be expressed this way. Suppose F is such a functor with value C . Associate the Set_Δ -enriched category C with $\text{Hom}(a, b)_n$ given by $\text{Hom}_{F([n])}(a, b)$.

(Hopefully my understanding of the term “full subcategory” is not wrong?) \square

Exercise 2.4 (1.3). Show that $N : \text{Cat} \rightarrow \text{Set}_\Delta$ is fully faithful.

Proof. We need to show that for any categories A, B , $\text{Hom}(A, B) = \text{Fun}(A, B)$ is in “bijection” with $\text{Hom}_{\text{Set}_\Delta}(NA, NB)$. Recall that NA is the presheaf that maps $[n]$ to the set of composable sequence of n morphisms in A . Then to a functor $F : A \rightarrow B$ we associate the map that sends a sequence of n morphisms in A to the respective sequence of n morphisms in B after applying F to each object and map.

Conversely, given a morphism in Set_Δ from NA to NB we can reconstruct a functor from A to B by interpreting objects of A as NA_0 and maps as NA_1 . \square

REFERENCES

- [Cis23] Denis-Charles Cisinsky, *Higher categories and homotopical algebra*, 2023.
- [Hau25] Rune Haugseng, *Yet another introduction to infty-categories*, 2025.