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Another proof of corollary 1

(Communicated by Daniela Paiva)

In Lecture 10: surfaces with Picard rank 1 we proved the following proposition:

Proposition A K3 surface M is isomorphic to a quartic if and only if $\mathrm{Pic}(M)$ contains a very ample bundle L with $(L, L) = 4$

And that leads to

Corollary Let M be a K3 surface such that $\mathrm{Pic}(M) = \mathbb{Z}$ and L the line bundle generating $\mathrm{Pic}(M)$. Assuma that $(L, L) = 4$. Then M is isomorphic to a quartic.

In this document I will prove the corollary.

Plan

1. L ample implies that the map associated to the linear system $|L|$ is an embedding $\varphi_{|L|} : M \hookrightarrow S \subset \mathbb{P}^n$
2. The fact that $L^2 = 4$ implies that $n = 3$ and that the degree of S is 4.

First recall that there is an isomorphism

$$\mathrm{Pic}(M) \cong \mathrm{Div}(M)/\mathrm{PDiv}(M)$$

where $\mathrm{Pic}(M)$ is the group of isomorphism classes of line bundles on M and

$$\mathrm{Div}(M) := \left\{ D = \sum a_i D_i : a_i \in \mathbb{Z}, D_i \text{ irreducible subvariety of codimension } 1 \right\}$$

and

$$\mathrm{PDiv}(M) = \{\mathrm{div}(f) : f \in \mathcal{M}_M\}$$

where $\mathcal{M}_M$ is the space of meromorphic functions on M. Recall that the (principal) divisor associated to a meromorphic function is a formal combination of the subvarieties where its zeroes and poles lie counted with multiplicity.

Note that the quotient by $\mathrm{PDiv}(M)$ ammounts to ***linear equivalence*** which is given by

$$D \sim D' \iff \exists f \in \mathcal{M}_M \text{ s.t. } D - D' = \mathrm{div}(f).$$

Now recall that a divisor E is called ***effective*** (denoted $E \geqslant 0$) if all its coefficients are greater or equal than zero, and that the ***linear system*** associated to a divisor D is

$$|D| = \{E \in \mathrm{Div}(M) : E \geqslant 0, E \sim D\}$$

which is a finitely generated vector space over the field of meromorphic functions with the product defined by

$$f \cdot E = \operatorname{div}(f) + E.$$

Indeed, $f \cdot E \in |D|$ because it is linearly equivalent to D:

$$f \cdot E - D = \operatorname{div}(f) + E - D = \operatorname{div}(f) - \operatorname{div}(g) \in \mathcal{M}_M$$

and also it is effective:

?

First observation after my talk is here.

Definition (Linear system=Linear series) Bruno:

$$|D| = H^0(M, \mathcal{O}_M(D))$$

Misha:

zero divisors of holomorphic sections of D

Stacks Project: k field, X proper scheme over k, $d, r \geqslant 0$. A ***linear series of degree*** d ***and dimension*** r is

a pair $(\mathcal{L}, V)$, $\mathcal{L}$ invertible $\mathcal{O}_M$-module and $V \subset H^0(M, \mathcal{O}_M(D))$ k-subvector space of dim $r+1$

The linear system defines a rational map

$$\begin{aligned} \varphi_{|D|} : M &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [f_0(x) : \ldots : f_n(x)] \end{aligned}$$

where f_i are generators of $|D|$. I'd like to have another look at how this map is constructed. See wiki to make sure it's not obvious.

Here's another look at this map:

$$\begin{aligned} \varphi_{|D|} : M &\longrightarrow \mathbb{P}\big((H^0(M, \mathcal{O}_M(D))^*\big) \\ x &\longmapsto [\mathrm{ev}_x] \end{aligned}$$

where

$$\begin{aligned} \mathrm{ev}_x : H^0(M, \mathcal{O}_M(D)) &\longrightarrow \mathcal{L}_x \\ s &\longmapsto s(x) \in \mathcal{L}_x \end{aligned}$$

meaning that

$$\mathrm{ev}_x \in \operatorname{Hom}_{\mathbb{C}}(H^0(M, \mathcal{O}_M(D)), \mathcal{L}_x)$$

Notice that this map could be undefined at points where all the f_i vanish. These conform the ***base locus*** of $|D|$ and each such point is called a ***base point***. If there are no base points we say $|D|$ is ***base-point free***.

Now we turn to our exercise, where L is the generator of $\operatorname{Pic}(M)$. Our objectives are

- If $|L|$ is base-point free, the above map $\varphi_{|L|}$ is a morphism (not only a rational map).
- If L is very ample then $\varphi_{|L|}$ is an embedding.
- The dimension of the projective space is $h^0(M, \mathcal{O}_M(L))$.

Knowing these three things allows to see M as a projective variety, and then we only need to compute its degree, which should be 4. But first let's address the three points above. We shall use

Theorem 5 (Taken from Mori) Let X be a K3 surface defined over an algebraically closed field of characteristic $\neq 2$. Let H be a numerically effective divisor on X. Then one has

1. H is not base point free if and only if there exit irreducible curves E, Γ, and an integer $k \geqslant 2$ such that $H \sim kE + \Gamma$, $E^2 = 0$, $\Gamma^2 = -2$ and $E \cdot \Gamma = 1$. In this case [...]
2. Let $H^2 \geqslant 4$. Then H is very ample if and only if
   (a) there is no irreducible curve E such that $E^2 = 0$ and $E \cdot H = 1, 2$,
   (b) there is no irreducible curve E such that $E^2 = 2$, $H \sim 2E$, and
   (c) there is no irreducible curve E such that $E^2 = -2$, $E \cdot H = 0$.

I think that the base-point free condition does not follow from this statement since, supposing that such E and Γ exisy we can barely show that $k = 3$. So something might be missing.

However to show that L is ample we barely notice that the second and third case are impossible supposing that $E = kL$ since we get, in the second case that $k^2 \cdot 4 = 2$ and in the third that $k^2 \cdot 4 = -2$. The first case also can't happen since $E^2 = 0$ implies $k = 0$ which implies $E \cdot L = 0$.

Remark

In Misha's course we actually showed this in the following proposition and corollary from Lecture 10 (just after definition of ***line system***):

Theorem Let M be a K3 surface such that $\mathrm{Pic}(M) = \mathbb{Z}$, and L the line bundle generating $\mathrm{Pic}(M)$. Assume that $(L, L) > 2$. Then L or L^* is ample, base point free and the map $\psi : M \to \mathbb{P}H^0(M, L)^*$ is an embedding or a 2-sheeted ramified covering.

The second cas is when $|L|$ contains at least one hyperelliptic curve (and then all curves in $|L|$ are hyperelliptic). In fact, it is shown later that the hyperelliptic case corresponds to $(L, L) = 2$. This gives us the embedding we needed.

Also there's

Corollary 2 Let M be a K3 surface such that $\mathrm{Pic}(M) = \mathbb{Z}$ and L the line bundle generating $\mathrm{Pic}(M)$. Assume that $(L, L) > 0$. Then L is very ample.

Then we wish to calculate the dimension of the image of the projective embedding. It turns out that the projective space in the image of $\varphi_{|L|}$ is $\mathbb{P}(H^0(M, \mathcal{O}_M(L)) \cong |L|$, which means that we can compute its dimension via

$$\dim |L| = \dim_{\mathbb{C}} H^0(M, \mathcal{O}_M(L)) - 1.$$

And then we cite Saint-Donat, where he shows that by Riemann-Roch and Serre duality we have for any invertible sheaf L on a K3 that

$$h^0(\mathcal{O}_M(L)) = \dim |L| + 1 = 2 + \frac{L^2}{2} + h^1(\mathcal{O}_M(L))$$

and since L is ample, $h^1(M, \mathcal{O}_M(L)) = 0$ and we get

$$1 + \dim |L| = 2 + \frac{L^2}{2} = 4 \implies \dim |L| = 3$$

So that the codomain of $\varphi_{|L|}$ is $\mathbb{P}^3$.

Finally we compute the degree of M. This part is a little sketchy but here goes.

We have constructed an embeddinf $\varphi_{|L|} : M \hookrightarrow S \subseteq \mathbb{P}^3$. Its image S is a codimension-1 hypersurface in $\mathbb{P}^3$, which means that it is an element of $\mathrm{Pic}(\mathbb{P}^3) = \mathbb{Z}$. So $S \sim kH$, and this k is the degree. So I'm not sure how this is used.

Now to compute the degree we must intersect with two hyperplanes because we are in a 3-dimensional thing (so intersection of $n-1$ hyperplanes in an n-dimensional thing gives a line). For some reason it's enough to intersect S with H. Here H is the generator of $\mathrm{Pic}(\mathbb{P}^3)$ which is in fact pulled back to L in $\mathrm{Pic}(M)$ (by construction of the map $\varphi_{|L|}$.

$$\deg(S) = k = S \cdot H \cdot H = H^2 = L^2 = 4$$

Since we may pull back divisors by an isomorphism and the intersection number is preserved (see Beauville prop. 1.8)

Exercise (Sergey) $\deg X \geqslant \operatorname{codim} X + 1$ if X is not degenerate (not contained in hyperplane)