K3 surfaces, home assignment 3: the splitting principle

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Exercise 3.1. Let $\mathbb{C}P^\infty = \mathcal{G}r(1,\infty)$ be the union $\bigcup_i \mathbb{C}P^i$ where all maps $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3 \hookrightarrow ...$ are hyperplane embeddings. Prove that there exists a principal $U(1)$-bundle over $\mathbb{C}P^\infty$ with contractible total space. Prove that the cohomology of $\mathbb{C}P^\infty$ is a polynomial algebra with one generator in $H^2(\mathbb{C}P^\infty)$.

Definition 3.1. The **fundamental bundle** on $\mathbb{C}P^\infty = BU(1)$ is B_{fun}, isomorphic to $\mathcal{O}(1)$ on each $\mathbb{C}P^n \subset \mathbb{C}P^\infty$.

Exercise 3.2. Let X be a compact CW-space. Prove that any line bundle on X is isomorphic to $\phi^*(B_{\mathsf{fun}})$ for some continuous map $\phi: X \longrightarrow BU(1)$.

Exercise 3.3. Let B_{fun} be **the fundamental vector bundle** on $\mathcal{G}r(n)$, which has fiber W at any point of $\mathcal{G}r(n)$ corresponding to a subspace $W \subset \mathbb{C}^\infty$. Let X be a CW-space. Prove that any vector bundle on X is isomorphic to $\phi^*(B_{\mathsf{fun}})$ for some continuous map $\phi_B : X \longrightarrow \mathcal{G}r(n)$.

Exercise 3.4. Let $\Phi : (BU(1))^n \longrightarrow \mathcal{G}r(n)$ be a morphism such that the pullback of the fundamental bundle is the direct sum of n line bundles, obtained by lifting $\mathcal{O}(1)$ from each factor $BU(1)$. A complex vector bundle is called **split** if it obtained as a direct sum of complex line bundles. Prove that a vector bundle B on X is split if and only if $\phi_B : X \longrightarrow \mathcal{G}r(n)$ is factorized through Φ.

Exercise 3.5.

a. Let $\mathfrak{F}(V) \cong \mathbb{P}^{n-1} \times \mathbb{P}^{n-2} \times ... \times \mathbb{P}^1$ be the space of all orthogonal bases in $V = \mathbb{C}^{n+1}$ up to independent rescaling of each of the vectors (“the flag space”; we will denote it $\mathfrak{F}$). Denote by $\mathfrak{S}$ the smooth, locally trivial bundle over $\mathcal{G}r(n)$, with the fiber the flag space $\mathfrak{F}(V)$ in each subspace $V \in \mathcal{G}r(n)$. Prove that the pullback of the fundamental bundle B_{fun} to $\mathfrak{S}$ is split.

b. Prove that the induced map $H^*(\mathcal{G}r(n), \mathbb{Q}) \longrightarrow H^*(\mathfrak{S}, \mathbb{Q})$ is injective.

c. Deduce that $H^*(\mathfrak{S}, \mathbb{Q})$ as $H^*(\mathcal{G}r(n), \mathbb{Q})$-module is isomorphic to $H^*(\mathcal{G}r(n), \mathbb{Q}) \otimes H^*(\mathfrak{F})$.

Hint. Construct a cell decomposition of $\mathfrak{F}$ with all cells even-dimensional and use the Leray-Serre spectral sequence, or use the next exercise instead.

Exercise 3.6. Let $E \longrightarrow B$ be a locally trivial fibration with fiber F. Assume that $H^*(E)$ is equipped with a structure of a free $H^*(F)$-module in such a way that the restriction of $H^*(F) \cdot 1 \subset H^*(E)$ to a fiber $F \subset E$ is an isomorphism. Prove that $H^*(E) = H^*(F) \otimes H^*(B)$ as a $H^*(F)$-module.

Hint. Use the Leray-Serre spectral sequence.

Exercise 3.7. Consider the fibration $\mathfrak{S} \longrightarrow \mathcal{G}r(n)$ defined above. Prove that $H^*(\mathfrak{S})$ is equipped with a free $H^*(\mathfrak{F})$-action, in such a way that the restriction of $H^*(\mathfrak{F}) \cdot 1 \subset H^*(E)$ to a fiber $\mathfrak{F} \subset \mathfrak{S}$ is an isomorphism.

Hint. First, prove it for $\mathbb{P}^n$-bundles on $\mathcal{G}r(n, \infty)$.

Exercise 3.8. Let B be a vector bundle over X of rank n, and $\phi: \ X \longrightarrow \mathcal{G}r(n)$ the corresponding map. Consider the fibered product $X \times_{\mathcal{G}r(n)} \mathfrak{S}$ Prove that $X \times_{\mathcal{G}r(n)} \mathfrak{S}$ is isomorphic to $H^*(X) \otimes H^*(\mathfrak{F})$ as an $H^*(\mathfrak{F}))$-module.

Hint. Use the previous exercise.

Exercise 3.9. Let B be a vector bundle over X. Prove that there exists a space Y fibered over X such that the pullback map $H^*(X) \longrightarrow H^*(Y)$ is injective, and B is split on $H^*(Y)$.

Hint. Use the previous exercise.

Exercise 3.10. Prove that the natural map $\mathfrak{S} \longrightarrow \mathcal{G}r(n)$ can be factorized through $\Phi: \ BU(1)^n \longrightarrow \mathcal{G}r(n)$. Deduce that the pullback

$$\Phi^*: \ H^*(\mathcal{G}r(n)) \longrightarrow H^*(BU(1)^n)$$

is injective.

Hint. To prove that it can be factorized, show that the pullback of B_{fun} to $\mathfrak{S}$ is split, and use Exercise 3.4. Injectivity of the map $\Phi^*: \ H^*(\mathcal{G}r(n)) \longrightarrow H^*(BU(1)^n)$, should follow from the injectivity of the pullbach map $H^*(\mathcal{G}r(n), \mathbb{Q}) \longrightarrow H^*(\mathfrak{S}, \mathbb{Q})$ (prove it).

Exercise 3.11. Let Σ_n denote the symmetric group, and $\frac{BU(1)^\infty}{\Sigma_\infty} := \lim_n \frac{BU(1)^n}{\Sigma_n}$

a. Prove that the natural map $BU(1)^n \times BU(1)^m \longrightarrow BU(1)^{n+m}$ induces the structure of H-space on $\frac{BU(1)^\infty}{\Sigma_\infty}$.

b. Let B be a split bundle of rank n on X. Construct the fundamental bundle B_{fun} on $\frac{BU(1)^n}{\Sigma_n}$ and prove that there exists a map $X \longrightarrow \frac{BU(1)^n}{\Sigma_n}$ such that B is a pullback of B_{fun}.

c. Construct the natural map $\frac{BU(1)^\infty}{\Sigma_\infty} \longrightarrow BU$, and show that it is compatible with the H-structure and injective on cohomology.

d. Prove that the primitive generators of $H^{2k}(\frac{BU(1)^\infty}{\Sigma_\infty})$ are Newton polynomials $\sum z_i^k$, where z_i is the generator of $H^2(B(1))$ for a component number i in $BU(1)^n$.

e. Deduce that the pullback of the primitive generator $C_k \in H^{2k}(BU)$ is proportional to $\sum z_i^k \in H^*(BU(1)^n)$.