Daniel González Casanova Azuela
github.com/danimalabares/k3

Home assignment 2: spectral sequences

The monodromy of Gauss-Manin local system

Definition 2.1. Let $\pi : E \to B$ be a locally trivial fibration with fiber F. The family of cohomology of fibers of π is locally trivial, (what does this mean precisely?) but it might have ***the monodromy***. In other words, the group $\pi_1(B)$ naturally acts on the algebra $H^*(F)$ by autmorphisms. To obtain this action, take a loop in B and trivialize the family π along small intervals of this loop; this gives an identification of $H^*(F)$ with itself, which might be non-trivial.

Remark (Understanding the monodromy action of cohomology). (From StackExchange) Let $f : X \to U$ be a proper surjective submersion and fix $u_0 \in U$.

For any path $\gamma \subset U_j$, there is a canonical diffeomorphism $\varphi_\gamma : f^{-1}(\gamma(0)) \to f^{-1}(\gamma(1))$, using ψ_j (by a theorem of Ehresmann, all the fibers of f are diffeomorphic).

Now, for any loop γ, split γ into paths $\gamma_i \subset U_i$ and you can compose these diffeomorphisms to get a diffeomorphism

$$\phi_{\gamma_n} \circ \ldots \circ \varphi_{\gamma_1} : f^{-1}(u_0) \to f^{-1}(\gamma(u_0))$$

It induces a map on homology: you can check that it is well defined up to homotopy.

Exercise 2.1. Let $\phi^* : \mathbb{Z} \to \operatorname{Aut}(H^*(F))$ be an automorphism induced by a homeomorphism $\phi : F \to F$. Construct a locally trivial family over a circle with monodromy in cohomology induced by ϕ^*.

Interpretation Given an action $\phi^* : \mathbb{Z} = \pi_1(S^1) \to \operatorname{Aut}(H^*(F))$, construct a fibre bundle such that ϕ^* is the monodromy action on cohomology.

Proof. Consider the standard torus fibration $T^2 \to S^1$. Any path in the circle can be thought of as an number $n \in \mathbb{Z}$. Perhaps the induced automorphism on cohomology is precisely the map $\mathbb{Z} \ni a \mapsto na \in \mathbb{Z}$. But I'm not looking for an automorphism of $\mathbb{Z}$… I need an automorphism of $H^\bullet(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$… □