Daniel González Casanova Azuela
github.com/danimalabares/k3

Home assignment 7: Moser isotopy lemma

Exercise 7.1 Let M be a compact manifold, and V_0, V_1 two smooth volume forms which satifsy $\int_M V_0 = \int_M V_1$. Prove that there exists a diffeomorphism which satisfies $\Phi^* V_0 = V_1$.

Solution. First notice that the condition $\int_M V_0 = \int_M V_1$ implies that V_0 and V_1 are cohomologous. This follows from the fact that $H^n(M)$ is a real one-dimensional vector space, so $[V_1] = a[V_0]$ for some real number a. Then

$$\alpha \int_M [V_0] = \int_M [V_1] = \int_M [V_0]$$

which means that $a = 1$. This means that there exists a $(n-1)$-form η such that

$$V_1 - V_0 = d\eta.$$

Now define

$$V_t = V_0 + t\,(V_1 - V_0)\,, \qquad t \in [0,1].$$

Notice that V_t is a nowhere-vanishing top-form. This follows since V_0 and V_1 are nowhere-vanishing and their integrals over M coincide, so they are both always positive or always negative. Now if $V_t = 0$ we would have $t = 1/2$ but $V_{1/2} = 1/2(V_1 + V_0)$.

To solve this exercise we will immitate the proof of Moser lemma given in lecture 16, which coincides with the proof in Lee and in this StackExchange answer.

First notice that the contraction map

$$\begin{aligned} i_\bullet V_t : \mathfrak{X}(M) &\longrightarrow \Omega^{n-1}(M) \\ X &\longmapsto i_X V_t \end{aligned} \tag{1}$$

is a bundle isomorphism for every t. The proof of this fact was found in yet another StackExchange post. First we notice that this map is an isomorphism at every point. Indeed, if $i_{X_1} V_t = i_{X_2} V_t$, then

$$V_t(X_1 - X_2, \cdot, \ldots, \cdot) = 0.$$

This is not possible since if $X_1 - X_2 \neq 0$, such vector can be completed to a basis where V_t cannot be zero since it is a volume form. Further, $i_\bullet V_t$ is an isomorphism at every point since both T_pM and $V^{n-1}(T_pM)$ have dimension n.

We must also make sure that this operation is smooth. This can be checked in a local chart U where $V_t = g dx_1 \ldots dx_n$. Choose any differential form $\alpha = \sum f_i dx_1 \ldots \widehat{dx_i} \ldots dx_n \in$

$\Omega^{n-1}(U)$. Smoothness follows from the fact that the coordinate functions of its corresponding vector field are also smooth. Indeed, we claim that $i_X V_t = \alpha$ for

$$X = \sum(-1)^{i+1}\frac{f_i}{g}\partial_i,$$

whose coordinate functions are smooth.

This is most easily seen in the case $n = 2$. By Leibniz rule of interior multiplication,

$$\begin{aligned} i_X(g dx_1 \wedge dx_2) &= (i_X g dx_1) \wedge dx_2 - g dx_1 \wedge i_X dx_2 \\ &= f_1 dx_2 - f_2 dx_1 = \alpha \end{aligned}$$

A similar computation confirms the general case.

Having established the isomorphism in eq. (2), we see that $-\eta \in \Omega^{n-1}(M)$ can be seen as $i_{X_t} V_t$ for some vector field X_t for each t. Now we will show that the pullback $\varphi_t^* \omega_t$ is constant as t varies. The proof relies on the following result:

Proposition 22.15 (Lee) Let $\{X_t\}$ be an autonomous (time-dependent) vector field and choose for every t a k-form $\omega_t \in \Omega^k(M)$. Then

$$\frac{d}{dt}(\varphi_t^* \omega_t) = \varphi^* \left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right),$$

where φ_t is the isotopy corresponding to $\{X_t\}$, i.e. the family of flows associated to $\{X_t\}$.

So we have

$$\begin{aligned} \frac{d}{dt}\varphi_t^* V_t &= \varphi^* \left(\mathcal{L}_{X_t} V_t + \frac{d}{dt} V_t\right) \\ &= \varphi^* \left(d i_{X_t} V_t + \cancelto{0}{i_{X_t} d V_t} + \frac{d}{dt}\left(V_0 + t(V_1 - V_0)\right)\right) \\ &= \varphi^* \left(d i_{X_t} V_t + V_1 - V_0\right) \\ &= \varphi^* \left(d(-\eta) + d\eta\right) = 0 \end{aligned}$$

Evaluating at $t = 1$ we see that φ_1 is the desired isomorphism since

$$\varphi_1^* V_1 = \varphi_0^* V_0 = \mathrm{id}^* V_0 = V_0.$$

□

Problem 7.2 Let (M, I) be an almost complex manifold, and ω_0, ω_1 co-homologous symplectic forms which satisfy $\omega_i(x, Ix) > 0$ for any non-zero tangent vetor x (such forms are called ***taming***).

a. Prove that there exists a diffeomorphism Φ which satisfies $\Phi^* \omega_0 = \omega_1$.

b. Prove that $|x|_i^2 := \omega_i(X, Ix)$ is a Hermitian metric on M. Prove that a diffeomorphism that satisfies $\Phi^*\omega_0 = \omega_1$ defines an isometry

$$(M, |x|_1^2) \longrightarrow (M, |x|_0^2)$$

is Φ is compatible with I, that is, satisfies $d\Phi(Ix) = I(d\Phi(x))$

c. Find an example of $\omega_0, \omega_1 \in \Lambda^2(M)$ such that a diffeomorphism compatible with I and satisfying $\Phi^*\omega_0 = \omega_1$ does not exist.

Solution.

a. This is immediate from Moser's lemma defining $\omega_t := t\omega_0 + (1+t)\omega_1$.

b. According to Riemann Surfaces course, a hermitian form may be understood as a Riemannian metric h such that $h(\cdot,\cdot) = h(I\cdot, I\cdot)$. This happens for $h(\cdot,\cdot) = \omega(\cdot, I\cdot)$ if we require $\omega(\cdot,\cdot) = \omega(I\cdot, I\cdot)$. Indeed, symmetry holds since

$$\omega(x, Iy) = \omega(Ix, -y) = \omega(y, Ix)$$

and positive-definiteness is given as a hypothesis.

The condition that $\omega(\cdot, I\cdot)$ is a riemannian metric is called ***compatibility*** of ω and I in Silva. In this case we can also produce a hermitian metric in the sense of WolframMathWorld, namely, a positive-definite symmetric sesquilinear form. Indeed, define $g(\cdot,\cdot) := \omega(\cdot, I\cdot)$, then the form

$$h := g + \sqrt{-1}\omega$$

satisfies the required properties as follows.

(a) Additivity, $h(u_1 + u_2, v) = h(u_1, v) + h(u_1, v)$, is immediate.

(b) Homegenity on the first argument, $h(\lambda u, v) = \lambda h(u, v)$, is also immediate.

(c) $h(u, v) = \overline{h(v, u)}$ is clear by anti-symmetry of Ω:

$$h(u, v) = g(u, v) + i\omega(u, v) = g(v, u) - i\omega(v, u) = \overline{h(v, u)}$$

(d) The property $h(u, \lambda v) = \bar{\lambda} h(u, v)$ follows easily from (b) and (c) since

$$h(u, \lambda v) = \overline{h(\lambda v, u)} = \bar{\lambda}\overline{h(v, u)} = \bar{\lambda} h(u, v)$$

(e) Positive-definiteness follows from positive-definiteness of g and antisymmetry of ω.

c. The conditions $\Phi^*\omega_0 = \omega_1$ and $\Phi_*(I\cdot) = I\Phi_*(\cdot)$ imply that

$$(\Phi^*\omega_0)(\cdot, I\cdot) = \omega_0(\Phi_*\cdot, \Phi_* I\cdot) = \omega_0\Phi_*\cdot, I\Phi_*\cdot) = \omega_1(\cdot, I\cdot) \tag{2}$$

Whether the metric is taken to be $\omega_i(\cdot, I\cdot)$ or $h_i = g_i + \sqrt{-1}\omega_i$, the result follows. In the latter case only note that

$$\begin{aligned}\Phi^* h_0 &= \Phi^*(g_0 + \sqrt{-1}\omega_0) = \Phi^* g_0 + \sqrt{-1}\Phi^*\omega_0 \\ &= \Phi^*\Big(\omega(\cdot, I\cdot)\Big) = g_1 + \sqrt{-1}\omega_1 = h_1\end{aligned}$$

by eq. (2).

d. I am intrigued to know the answer here.

□

Exercise 7.4 Let (M, V) be a connected manifold equipped with a volume form. Prove that the group of volume-preserving diffeomorphisms acts on M transitively.

Solution. I will sketch the proof of Boothby, found in MathOverflow.

We wish to define a vector field along γ whose flow Φ_t gives the desired volume preserving diffeomorphism at $t = 1$. Consider a diffeomorphism from $I \times B^{n-1}$ to a neighbourhood U of γ (this diffeomorphism is constructed from geodesic segments introducing a metric on M). The volume form on U may be expressed as

$$V = f dt \wedge dx_1 \wedge \ldots \wedge dx_{n-1}$$

for a positive function f. Now consider the vector field

$$X' = \frac{1}{f}\frac{\partial}{\partial t}.$$

X' may be extended to a vector field X in all of M using a partition of unity. It is immediate that the flow of this vector field takes p to q.

To show this flow is volume-preserving recall Cartan's formula that $\mathcal{L}_X V = d i_X V + \cancelto{0}{i_X dV} = d i_X V$ since V is a top-form. Thus, showing volume invariance ammounts to showing $d_X V = 0$, which holds since

$$i_{X'} f dt \wedge dx_1 \wedge \ldots \wedge dx_{n-1} = dx_1 \wedge \ldots \wedge dx_{n-1}$$

is closed. □

Exercise 7.5 Prove that the group of symplectomorphisms of (M, ω) acts transitively on M, for any connected symplectic manifold (M, ω).

Solution. This case is similar but requires a little more work: we intend to use Darboux coordinate charts, which can only be defined locally. We will split a path joining any two points into small pieces, find the symplectomorphisms locally and the compose them.

1. ***Strong local transitivity*** on M means that for every point $p \in M$ and neighbourhood U of p there are neighbourhoods V and W of p with $\overline{V} \subset W$ and $\overline{W} \subset U$, $\overline{W}$ compact, and for any $q \in V$ there is a symplectomorphism isotopy Φ_t of the identity map Φ_0 to Φ_1 such that $\Phi_1(p) = q$ and for all t, Φ_t leaves fixed every point outside $\overline{W}$.

2. If M is strongly locally transitive then it is transitive by symplectomorphisms. *Sketch of proof:* let γ be a path from p to q. For every $x \in \gamma$ choose $0 < \delta'_x < \delta''_x < \varepsilon$ such that the δ' and δ'' neighbourhoods of x satisfy the conditions of strong local transitivity. Since γ_1 is compact, there is a finite collection of points $p_1 = a_1, a_2, \ldots, a_r = q_1$ with volume-preserving diffeomorphisms mapping each point to the next. The composition of all such diffeomorphisms yields the desired volume-preserving map.

3. It remains to show that any connected symplectic manifold is strongly locally transitive. Let U be a Darboux chart of p and take two neighbourhoods V' and W' as in the definition of strong local transitivity. For any $y \in V'$ we can define the vector field X in $\mathbb{R}^{2n}$ of vectors parallel to yx_0, where x_0 are the coordinates of p. The flow of this vector field takes takes x_0 to y at $t = 1$.

   To show that this vector field is ω-invariant we use again Cartan's formula to see it's enough to show that $di_X\omega = 0$. Now if $\dim M = 2m$ we have

   $$i_X\omega = \sum_{i=1}^{m} dx^i \wedge dx^{i+m}(X, \cdot) = 0$$

   (I'm not sure why does it vanish…)

□

Extra exercise: infinite product

Exercise 8 (Zorich and Cooke, 3.2.5, p. 148) An inifinite product $\prod_{k=1}^{\infty} e_k$ is said to converge if the sequence $\Pi_n = \prod_{k=1}^{n} e_k$ has a finite **nonzero** limit Π. We then set $\Pi = \prod_{k=1}^{\infty} e_k$. Show that

(b) if $\forall n \in \mathbb{N}(e_n > 0)$, then the infinite product converges $\prod_{n=1}^{\infty}$ converges if and only if the series $\sum_{n=1}^{\infty} \log e_n$ converges.

Remark The proof is straightforward using elementary properties of the logarithm ($\log(ab) = \log a + \log b$) and the exponential ($\exp(a + b) = \exp a + \exp b$), as well as their continuity. The caveat is that if the product converges to zero, taking logarithms of the partial products gives a series diverging to $-\infty$ (we say that the product ***diverges to zero***). The following exercise is an example of this.

Claim

$$\prod_{n\geqslant 0}\left|-1+\frac{1}{n+\frac{1}{2}}\right| = 0.$$

Proof. By taking exponent of the partial sums of the following series, we see it's enough to show that

$$\sum_{n\geqslant 0}\log\left|-1+\frac{1}{n+\frac{1}{2}}\right| = -\infty. \tag{3}$$

To show this first notice that $\left|-1+\frac{1}{n+1/2}\right|$ is just $1-\frac{1}{n+1/2}$. We can also quickly notice that $\log\left(1-\frac{1}{n+1/2}\right)$ is a sequence of negative numbers convering to 0.

(It was not immediate to me why the this series should diverge or converge. The following argument was provided by ChatGPT.)

We can prove the series diverges if we find a divergent series that bounds it from above. Such series is $\sum\frac{-1}{n+1/2}$. Indeed, it turns out that for every number $0\leqslant x<1$,

$$\log(1-x)\leqslant -x.$$

To see why, define the function $f(x)=\log(1-x)+x$ and differentiate to find

$$\frac{d}{dx}\log(1-x)+x = -\frac{1}{1-x}+1 = \frac{1}{1-x}+\frac{1-x}{1-x} = \frac{-x}{1-x}\leqslant 0,$$

and since also $f(0)=0$, we have that

$$\log(1-x)+x\leqslant 0, \qquad 0\leqslant x<1.$$

Finally just recall that $\sum_{n\geqslant 0}\frac{1}{n}$ diverges (and behaves like $\sum\frac{1}{n+1/2}$ for lage n). This confirms eq. (3).

□

References

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