

Home assignment 2: spectral sequences

The monodromy of Gauss-Manin local system

Definition 2.1. Let $\pi : E \rightarrow B$ be a locally trivial fibration with fiber F . The family of cohomology of fibers of π is locally trivial, (what does this mean precisely?) but it might have *the monodromy*. In other words, the group $\pi_1(B)$ naturally acts on the algebra $H^*(F)$ by automorphisms. To obtain this action, take a loop in B and trivialize the family π along small intervals of this loop; this gives an identification of $H^*(F)$ with itself, which might be non-trivial.

Remark (Understanding the monodromy action of cohomology). (From [StackExchange](#))
Let $f : X \rightarrow U$ be a proper surjective submersion and fix $u_0 \in U$.

For any path $\gamma \subset U$, there is a canonical diffeomorphism $\phi_\gamma : f^{-1}(\gamma(0)) \rightarrow f^{-1}(\gamma(1))$, using ψ_j (by a theorem of Ehresmann, all the fibers of f are diffeomorphic).

Now, for any loop γ , split γ into paths $\gamma_i \subset U_i$ and you can compose these diffeomorphisms to get a diffeomorphism

$$\phi_{\gamma_n} \circ \dots \circ \phi_{\gamma_1} : f^{-1}(u_0) \rightarrow f^{-1}(\gamma(u_0))$$

It induces a map on homology: you can check that it is well defined up to homotopy.

Exercise 2.1. Let $\phi^* : \mathbb{Z} \rightarrow \text{Aut}(H^*(F))$ be an automorphism induced by a homeomorphism $\phi : F \rightarrow F$. Construct a locally trivial family over a circle with monodromy in cohomology induced by ϕ^* .

Interpretation Given an action $\phi^* : \mathbb{Z} = \pi_1(S^1) \rightarrow \text{Aut}(H^*(F))$, construct a fibre bundle such that ϕ^* is the monodromy action on cohomology.

Proof. Consider the standard torus fibration $T^2 \rightarrow S^1$. Any path in the circle can be thought of as an number $n \in \mathbb{Z}$. Perhaps the induced automorphism on cohomology is precisely the map $\mathbb{Z} \ni a \mapsto na \in \mathbb{Z}$. But I'm not looking for an automorphism of \mathbb{Z} ... I need an automorphism of $H^*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$...

□

Leray-Serre spectral sequence

Exercise 2.4. Let $\pi : E \rightarrow B$ be a fibration with the fiber a torus. Assume the $d_2 = 0$. Prove that all differentials vanish.

Proof. Since $d_2 = 0$, we have that $E_3^{p,q} = E_2^{p,q}$. Then

$$d_3 : H^p(B) \otimes H^q(T) \longrightarrow H^{p+3}(B) \otimes H^{q-2}(T),$$

so the only way it could be non-zero is for $q = 2$, which implies that

$$H^p(B) \otimes H^2(T) \cong H^{p+3}(B) \otimes H^0(T) \iff H^p(B) \cong H^{p+3}(B)$$

But I don't see why this couldn't happen. . . □

Exercise 2.5. Let $\pi : E \rightarrow B$ be a fibration with the fiber a torus. Assume that the pullback map $\pi^* : H^2(B) \rightarrow H^2(E)$ is injective. Prove that all differentials d_i vanish.

Solution. I'm not sure how to use the hypothesis since I usually deal with the total space after computing the E_∞ page via the filtration. . . □

Exercise 2.6. Let $\pi : E \rightarrow B$ be a fibration with the fiber a complex projective space. Assume that $d_2 = 0$ and $d_3 = 0$. Prove that all differentials d_i vanish.

Solution. Since a complex projective space has cohomology equal to the coefficients in even dimensions and 0 in odd dimensions, we have the following second page of the spectral sequence:

\vdots							
6	$H^0(B)$	$H^1(B)$	$H^2(B)$	$H^3(B)$	$H^4(B)$	$H^5(B)$	
5							
4	$H^0(B)$	$H^1(B)$	$H^2(B)$	$H^3(B)$	$H^4(B)$	$H^5(B)$	
3							
2	$H^0(B)$	$H^1(B)$	$H^2(B)$	$H^3(B)$	$H^4(B)$	$H^5(B)$	
1							
0	$H^0(B)$	$H^1(B)$	$H^2(B)$	$H^3(B)$	$H^4(B)$	$H^5(B)$	\dots
	0	1	2	3	4	5	\dots

It is immediate that d_4 is also zero, meaning that $E_2 = E_3 = E_4 = E_5$. However the case of d_5 is not so obvious since we get a map

$$d_5 : H^0(B) \rightarrow H^5(B)$$

that could be non-zero. The same will happen for all odd-index differentials. □

Exercise 2.8. Let $F = S^k$, that is, $\pi : E \rightarrow B$ is a sphere bundle. Prove that all differentials d_{k+1} vanish. Construct the *Gysin exact sequence*

$$\dots \rightarrow H^p(B) \rightarrow H^{p+k+1}(B) \xrightarrow{\pi^*} H^{p+k+1}(E) \rightarrow H^{p+1}(B) \rightarrow \dots$$

Solution. (This argument is adapted from the construction of Wang exact sequence found in [Wikipedia](#)). We have that $E_2^{p,q} = H^p(B) \otimes H^q(S^k)$ can only be non-zero for $q = 0, k$. This means that the only non-zero differentials are of the form

$$d_{k+1} : E_2^{p,k} \cong H^p(B) \longrightarrow E_2^{p+k+1,0} \cong H^{p+k+1}(B)$$

$$\begin{array}{ccccccc} H^0(B) \otimes H^k(S^k) & H^1(B) \otimes H^k(S^k) & \dots & & H^k(B) \otimes H^k(S^k) & & \\ & \searrow d_{k+1} & & & \searrow d_{k+1} & & \\ \vdots & & & & & & \vdots \\ H^0(B) \otimes H^0(S^k) & \dots & & H^{k+1}(B) \otimes H^0(S^k) & & H^{k+2}(B) \otimes H^0(S^k) & \end{array}$$

which means that $E^{k+1} = E^\infty$. Since $E^{k+1} = \ker d_{k+1} / \text{img } d_{k+1}$, we can write

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker d_{k+1} & \longrightarrow & H^p(B) & \xrightarrow{d_{k+1}} & H^{p+k+1}(B) \longrightarrow \text{coker } d_{k+1} \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \frac{\ker d_{k+1}}{\text{img } d_{k+1}} & & & & \frac{E^{p+k+1,0}}{\text{img } d_{k+1}} \\ & & \parallel & & & & \parallel \\ & & E_{k+1}^{p,k} & & & & E_{k+1}^{p+k+1,0} \\ & & \parallel & & & & \parallel \\ & & E_\infty^{p,k} & & & & E_\infty^{p+k+1,0} \end{array}$$

This is the "first half" of the Gysin sequence. For the other half we must compute the E_∞ terms. We use the filtration

$$H^n(E) = F^0 H^n \supset F^1 H^n \supset \dots \supset F^n H^n$$

that we know to satisfy

$$E_\infty^{p,q} \cong \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}.$$

We may write (I'm not completely sure why this works)

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_\infty^{p+k+1,0} & \longrightarrow & H^{p+k+1}(E) & \longrightarrow & E_\infty^{p+1,k} \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \frac{F^{p+k+1} H^{p+k+1}}{F^{p+k+2} H^{p+k+1}} & & & & \frac{F^{p+1} H^{p+k+1}}{F^{p+2} H^{p+k+1}} \end{array}$$

Putting this together with the first sequence we computed, we get that

$$\longrightarrow E_\infty^{p,k} \longrightarrow H^p(B) \xrightarrow{d_{k+1}} H^{p+k+1}(B) \longrightarrow E_\infty^{p+k+1,0} \longrightarrow H^{p+k+1}(E) \longrightarrow E_\infty^{p+1,k} \longrightarrow$$

and we simply remove the E_∞ terms to get the Gysin sequence

$$\longrightarrow H^p(B) \xrightarrow{d_{k+1}} H^{p+k+1}(B) \longrightarrow H^{p+k+1}(E) \longrightarrow H^{p+1}(B) \longrightarrow$$

Remark. I still cannot see why the map $H^{p+k+1}(B) \rightarrow H^{p+k+1}(E)$ is the map induced by the projection.

□

Exercise 2.11. Let $\pi : E \rightarrow B$ be a fibration with $B = S^k$. Prove that all differentials except d_k vanish. Construct an exact sequence

$$\dots \rightarrow H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F) \xrightarrow{\mu} H^{p+k}(E) \rightarrow H^{p+k+1}(F) \rightarrow \dots$$

where μ is multiplication by $\pi^* \text{Vol}_{S^k}$ and \tilde{d}_k is equal to d_k after the identification $H^p(F) = H^k(S^k) \otimes H^p(F) = E_2^{k,p}$

Solution. Like in Exercise 2.8 we see that the only non-zero differentials are

$$d_k : H^0(S^k) \otimes H^{k+p} \longrightarrow H^k(S^k) \otimes H^{p+1}(F)$$

since $E_2 = E_k$ is of the form

$$\begin{array}{ccccc} H^0(S^k) \otimes H^{k+1}(F) & \dots & & H^k(S^k) \otimes H^{k+1}(F) & \\ & \searrow & & & \\ H^0(S^k) \otimes H^k(F) & \dots & d_k & H^k(S^k) \otimes H^k(F) & \\ & \searrow & & & \\ \vdots & & d_k & \vdots & \\ H^0(S^k) \otimes H^2(F) & \dots & & H^k(S^k) \otimes H^2(F) & \\ & \searrow & & & \\ H^0(S^k) \otimes H^1(F) & \dots & & H^k(S^k) \otimes H^1(F) & \\ & \searrow & & & \\ H^0(S^k) \otimes H^0(F) & \dots & & H^k(S^k) \otimes H^0(F) & \end{array}$$

Again like in Exercise 2.8 we obtain a sequence

$$0 \longrightarrow E_{\infty}^{0,q} \longrightarrow H^q(F) \xrightarrow{d_k} H^{q-k+1}(F) \longrightarrow E_{\infty}^{k,q-k+1} \longrightarrow 0$$

Remark. The exercise has the map $H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F)$, but my computations suggest it should be $H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^{p+1}(F)$.

Then we compute the E_{∞} terms using a filtration

$$H^n(E) = F^0 H^n \supset F^1 H^n \supset \dots \supset F^n H^n, \quad E_{\infty}^{p,q} = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}$$

which yields

$$0 \longrightarrow E_{\infty}^{k-1,q-k+1} \longrightarrow H^q(E) \longrightarrow E_{\infty}^{0,q} \longrightarrow 0$$

and then we get

$$\dots \rightarrow H^q(E) \rightarrow H^q(F) \rightarrow H^{q-k+1}(F) \rightarrow H^{q+1}(E) \rightarrow H^{q+1}(F) \rightarrow \dots$$

Remark. As in Exercise 2.8, I don't know why the map $H^{q-k+1}(F) \rightarrow H^{q+1}(E)$ should be multiplication by the volume form of S^k .

□

Exercise last. Generators here (horizontal), generators there (vertical, 1,3,5), "Extend generators by Leibniz rule, and then they just kill everything"