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# Home assignment 1: Riemann-Roch formula in dimension 1

(Partial progress)

**Exercise 1** Prove that the ring $\mathcal{O}_1$ of germs of holomorphic functions on $\mathbb{C}$is a principal ideal ring.

*Solution.* I assume that $\mathcal{O}_1$ is the ring of germs of holomorphic functions about $0 \in \mathbb{C}$. I will use Griffiths and Harris, Chapter 0, section *Weierstrass Theorems and Corollaries*. A ***Weierstrass polynomial*** in $w$ is a function

$$w^d + a_1 w^{d-1} + \ldots + a_d(z), \qquad a_i(0) = 0.$$

**Theorem** (Weierstrass Division Theorem) Let $g(z, w) \in \mathcal{O}_{n-1}[w]$ be a Weierstrass polynomial of degree $k$ in $w$. Then for any $f \in \mathcal{O}_n$ we can write

$$f = g \cdot h + r$$

with $r(z, w)$ a polynomial of degree $< k$ in $w$.

In words, we can express the germ of a holomorphic function around 0 as the the product of a Weierstrass polynomial of degree $k$ times some holomorphic function plus a polynomial of degree $< k$.

Then the proof is just mimicking the proof that the usual polynomial ring is a principal ideal domian.

Let $J$ be an ideal of $\mathcal{O}_1$ and $g \in J$ be a Weierstrass polynomial of lowest degree. Then by Weierstrass Division Theorem we get $h, r \in \mathcal{O}_1$ such that $f = g \cdot h + r$ but by the choice of $g$ we see $r$ must be zero. This means $J = \langle g \rangle$.

**Remark** (Based in Bruno's approach) The former proof using Weierstrass polynomials is, though correct, a bit of an overkill, since, as explained by Griffiths & Harris, Weierstrass polynomials are used to generalize the 1-dimensional situation, where by power series expansion we know that a holomorphic function has a unique local representation (why?)

$$f(z) = (z - z_0)^n u(z), \qquad u(z_0) \neq 0$$

So, any function $f \in \mathcal{O}_1$ can be expressed as

$$f(z) = z^n g(z), \qquad g(0) \neq 0$$

Then we notice that if $f$ is in some ideal $J \subset \mathcal{O}_1$, then the ideal $(z^n)$ must be contaied in $J$ since any $\phi(z)z^n \in (z^n)$ must also belong to $J$ since

$$z^n \phi(z) = \frac{f(z)\phi(z)}{g(z)} \in J$$

(why is it ok to divide by $g$? It is only non-zero at 0...) Then, like in our proof, let $n_0$ the least order of vanishing of non-zero elements of $J$, which cannot be zero for in that case we get $J = \mathcal{O}_1$. Then we get that $J = (z^{n_0})$ since any $f \in J$ can be written as

$$f(z) = z^n g(z) = z^{n_0} z^{n-n_0} g(z).$$

$\square$

**Exercise 1.2** (Invariant factors theorem) Let $R$ be a principal ideal ring. Prove that any finitely-generated $R$-module is a direct sum of cyclic $R$-modules. Use this result to deduce the Jordan normal form theorem, and to classify the finitely-generated abelian groups.

*Solution.* I follow Dummit and Foote to prove that

**Invariant Factors Theorem** Let $R$ be a principal ideal *domain* and let $M$ be a finitely generated $R$-module. Then

$$M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \ldots \oplus R/(a_m)$$

for some integer $r \geqslant 0$ and nonzero elements $a_1, a_2, \ldots, a_m$ of $R$ which are not units in $R$ and which satisfy the divisibility relation

$$a_1 | a_2 | \ldots | a_m$$

which strongly relies on

**Theorem 4** Let $R$ be a PID, $M$ a a free $R$-module of finite rank $n$ and $N$ a submodule of $M$. Then $N$ is a free module of rank $m \leqslant n$ and there exists a basis $y_1, y_2, \ldots, y_n$ of $M$ such that $a_1 y_1, a_2 y_2, \ldots, a_m y_m$ is a basis of $N$, where $a_1, a_2, \ldots, a_m$ are nonzero elements of $R$ with the divisibility relations

$$a_1 | a_2 | \ldots | a_m.$$

whose proof is rather involved.

*Proof of Invariant factors theorem.* Choose a basis $\{x_i\}_{i=1}^n$ of $M$. Consider the free module $R^n$ along with a basis $\{b_i\}_{i=1}^n$. The homomorphism $\pi : R^n \to M$, $b_i \mapsto x_i$ is surjective, so we get $R^n / \ker \pi \cong M$.

Apply Theorem 4 for the module $R^n$ and its submodule $\ker \pi$. We get a basis $\{y_i\}_{i=1}^n$ of $R^n$ such that $\{a_i y_i\}_{i=1}^m$ is a basis of $\ker \pi$ for $a_i \in R$ for $i = 1, \ldots, m$ such that $a_1 | \ldots | a_m$. Then

$$M \cong R^n / \ker \pi = (Ry_1 \oplus \ldots \oplus Ry_n \Big/ (Ra_1 y_1 \oplus \ldots \oplus Ra_m y_m).$$

It remains to show that the quotient on the right-hand side of last equation is in fact the desired decomposition of $M$.

Consider the map

$$Ry_1 \oplus \ldots \oplus Ry_n \quad \longrightarrow \quad R/(a_1) \oplus \ldots \oplus R/(a_m) \oplus R^{n-m}$$

given by

$$(\alpha_1 y_1, \ldots, \alpha_n y_n) \quad \longmapsto \quad (\alpha_1 + (a_1), \ldots, \alpha_m + (a_m), \alpha_{m+1}, \ldots, a_n)$$

The kernel of this map is the set of elements which $\alpha_i \in (a_i)$ for all $i = 1, \ldots, m$. Thus we can write the kernel as

$$Ra_1 y_1 \oplus \ldots \oplus Ra_m y_m.$$

□

**Remark** Looks like I didn't use the divisibility condition $a_1 | \ldots | a_m$.

Now let's deduce the Jordan normal form theorem using again Dummit and Foote. For this we fix a vector space $V$ over a field $F$ and a linear transformation $T$. This makes $V$ into an $F[x]$-module by substitution of the variable $x$ by $T$.

The ***invariant factors*** are the elements $a_i$ from the last theorem, which in our present case can be shown to be monic polynomials $a_i(x)$ satisfying $a_1(x) | \ldots | a_n(x)$. Further, these elements are associated to the so-called ***elementary divisors***, which concern another similar formulation of the Invariant factors theorem. The elementary divisors are powers of the irreducible components of the $a_i(x)$.

If we assume that the $a_i(x)$ factor into linear polynomials, the elementary divisors can be written as $(x - \lambda)^k$, where $\lambda$ is one of the eigenvalues of $T$ (we consider only one of the eigenvalues at this point since we are constructing *one* of Jordan blocks of $T$ ).

Using the Invariant factors theorem for elementary divisors, we see that $V$ is the direct sum of finitely many cyclic modules of the form $F[x]/(x - \lambda)^k$. Such quotients have basis $\bar{x}^{k-1}, \bar{x}^{k-2}, \ldots, \bar{x}, 1$. Now we observe that the polynomials

$$(\bar{x} - \lambda)^{k-1}, (\bar{x} - \lambda)^{k-2}, \ldots, \bar{x} - \lambda, 1$$

are also a basis. This follows since expanding the latter in terms of the $\bar{x}_i$ gives a triangular matrix, which makes it invertible, so it's a valid change of coordinates.

Nex we observe that action of multiplication by $x$, which is the same as applying $T$, maps these basis elements in the quotient as follows:

$$\begin{aligned}
1 &\mapsto 1\bar{x} = \lambda \cdot 1 + \bar{x} - \lambda \\
(\bar{x} - \lambda) &\mapsto \bar{x}^2 - \lambda\bar{x} = x\lambda \cdot (\bar{x} - \lambda) + (\bar{x} - \lambda)^2 \\
&\vdots \\
(\bar{x} - \lambda)^{k-2} &\mapsto \lambda \cdot (\bar{x} - \lambda)^{k-2} + (\bar{x} - \lambda)^{k-1} \\
(\bar{x} - \lambda)^{k-1} &\mapsto \lambda \cdot (\bar{x} - \lambda)^{k-1} + (\bar{x} - \lambda)^{k}
\end{aligned}$$

(where I'm not sure how to compute the last two). The last expression simplifies further since $(\bar{x}-\lambda)^k = 0$ in the quotient, so that multiplication of $x$ by the basis element $(\bar{x}-\lambda)^{k-1}$ is barely multiplication by $\lambda$. In sum, we have constructed the ***Jordan block***

$$J_\lambda = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & & & \\ & & \ddots & 1 & \\ & & & \lambda & \\ & & & & 1 \\ & & & & \lambda \end{pmatrix}$$

applying this procedure to the rest of the cyclic components of $V$ we obtain the matrix representation

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_t \end{pmatrix}$$

of $T$.

Finally, the classification of finitely generated abelian groups follows by direct application of the Invariant factors theorem choosing $R = \mathbb{Z}$, since abelian groups are in correspondence with $\mathbb{Z}$-modules. □

**Definition 1.2** A ***coherent sheaf*** on a complex manifold $M$ is a sheaf of modules over the sheaf $\mathcal{O}_M$ of holomorphic functions on $M$, which is locally finitely generated and locally finitely presented (that is, the sheaf of relations between its local generators is also locally finitely generated (I don't understand this)).

**Exercise 1.3** Let $C$ be a complex curve, and $x \in C$ a smooth point. Prove that any coherent sheaf on $C$ supported in $x$ is isomorphic to $\bigoplus_{i=1}^k \mathcal{O}_C/\mathfrak{m}^{d_i}$.

*Solution.* First notice that the local ring of regular functions $\mathcal{O}_x$ at a smooth point $x \in C$ is a principal ideal domain. This follows from exercise 1, since any neighbourhood of $x$ may be identified with a neighbourhood of $0 \in \mathbb{C}$.

**Remark** In more general algebraic geometry the statement also holds, namely, the local ring of regular functions $\mathcal{O}_x$ at a smooth point $x$ in a curve over a closed field $k$ is a principal ideal domain. This is because (see Hartshorne thm I.5.1) since $x$ is smooth, $\mathcal{O}_x$ is regular, meaning $\mathfrak{m}/\mathfrak{m}^2$ has the same dimension as $\mathcal{O}_x$ as a vector space over the field $\mathcal{O}_x/\mathfrak{m}$. This means that any element in $\mathfrak{m}$ will be generated by any element in $\mathfrak{m}/\mathfrak{m}^2$.

Now any coherent sheaf $\mathcal{F}$ supported in $x$ is, in particular, finitely generated over $\mathcal{O}_x$, so we may apply Invariant Factors Theorem at any neighbourhood $U \ni x$ to obtain

$$\mathcal{F}(U) = \mathcal{O}_x^r \oplus \mathcal{O}_x/(a_1) \oplus \ldots \mathcal{O}_X/(a_m)$$

It only remains to show that the quotients $\mathcal{O}_x/(\mathfrak{a}_i)$ are actually $\mathcal{O}_x/\mathfrak{m}^{d_i}$ for some positive integers $d_i$. This ammounts to showing that every ideal in $\mathcal{O}_x$ is a power of $\mathfrak{m}$. A straighfoward proof of this fact (provided by Bruno) is based on exercise 1.1:

*Proof of exercise 1.1.* The idea is that $\mathcal{O}_1$ is not only a principal ideal ring, but its ideals are in fact of the form $(z^{n_0+k}) = (z^{n_0})^k$.

Recall that for any ideal $J \subset \mathcal{O}_1$ and $f \in \mathcal{O}_1$ we can express $f$ locally as $f(z) = z^k g(z)$ with $g(0) \neq 0$. Let $k_0 \neq 0$ be the smallest such number (if $k_0 = 0$, all $f \in J$ do not vanish at 0 and we obtain that $J = \mathcal{O}_1$) and $f_0 = z^{n_0} g_0$ the corresponding element in J. Then $J \subset (z^{k_0})$ since every $f_1(z) = z^{k_1} g_1(z)$ can be expressed as

$$f_1(z) = z^{k_0} z^{k_1-k_0}(z) g_1(z) = z^{k_0} h(z) \in (z^{k_0})$$

This shows every ideal is of the form $(z^{k_0})$. Finally, every ideal must be contained in the maximal ideal $(z^{n_0})$ (this is a straighforward consequence of Zorn's lemma). Then every ideal is of the form $(z^{k_0}) \subset (z^{n_0})$, so $z^{k_0} = z^{n_0+k}$ and we get the desired assertion that $(z^{k_0}) = (z^{n_0+k}) = (z^{n_0})^k$. □

**Remark** (by Altan) As in the previous remark, the fact that all ideals are powers of the maximal ideal in a local ring at a smooth point in an abstract algebraic variety also holds. This follows from Nakayama's lemma. (See also StackExchange: all principal ideals are powers of the maximal ideal when $\bigcap_{n \in \mathbb{N}} \mathfrak{m}_x^n = \{0\}$.)

□

**Exercise 1.4** (After date) Let C be a complex curve, and V an abelian group, freely generated by isomorphism classes of coherent sheaves on C. The ***Gothendieck K-group*** $K_0(C)$ is the quotient of V by its subgroup generated by relations $[F_1] + [F_3] = [F_2]$ for all exact sequences of coherent sheaves $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$

a. Let L be a line bundle and $0 \longrightarrow \mathcal{O}_C \longrightarrow L \longrightarrow R \longrightarrow 0$ be an exact sequence associated with a section $\ell \in H^0(C, L)$. Prove that $[L] - [\mathcal{O}_C] = \sum_i a_i [x_i]$, where $a_i \in \mathbb{Z}^{>0}$, $[x_i]$ are classes of skyscraper sheaves $\mathcal{O}_C/\mathfrak{m}_{x_i}$, and $\mathfrak{m}_{x_i}$ is the maximal ideal of a point $x_i$.

b. Prove that $K_0(C)$ is generated by $\mathcal{O}_C$ and the classes of skyscraper sheaves $\mathcal{O}_C/\mathfrak{m}_x$.

*Proof.*

a. Here I suppose that the associated exact sequence to the section $\ell$ is given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_C & \xrightarrow{m} & L & \longrightarrow & R = \operatorname{coker} m = L/\operatorname{img} m & \longrightarrow & 0 \\ & & f & \longmapsto & f \cdot \ell & \longmapsto & f \cdot \ell + \operatorname{img} m & & \end{array}$$

(See Stacks Project.) Since L is a line bundle, it is a locally free sheaf. This makes R a finitely generated module (see wiki: module is finitely generated iff it is a quotient of a free module of finite rank). Apply Invariant factors theorem to obtain $R = \sum_i \mathcal{O}_C/\mathfrak{m}^{d_i}$. In the Grothendieck group we get

$$[L] - [\mathcal{O}_C] = \left[\sum_i \mathcal{O}_C/\mathfrak{m}^{d_i}\right] = \sum [\mathcal{O}_X/\mathfrak{m}^{d_i}]$$

□

**Exercise 1.5** (After date) Let $C$ be a compact complex curve, and $F$ a coherent sheaf on $C$. We define the ***Euler characteristic*** of $F$ as

$$\chi(F) := \dim H^0(C, F) - H^1(C, F)$$

Prove that $\chi$ defines a group homomorphism $K_0(C) \to \mathbb{Z}$.

**Exercise 1.7** Let $X$ be a complex manifold and $\mathrm{Pic}(X)$ the set of equivalence classes of vector bundles, equipped with multiplicative structure induced by the tensor product. The group $\mathrm{Pic}(X)$ is called the ***Picard group***of $X$.

a. Prove that the cohomology group $H^1(X, \mathcal{O}_X^*)$ is naturally identified with $\mathrm{Pic}(X)$.

*Solution.*

a. Recall that a line bundle $L$ may be reconstructed from the gluing functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, which can be thought as changes of coordinates on each fiber. These functions satisfy the consistency condition that $g_{\gamma\beta} g_{\beta\gamma} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

   On the other hand, a cochain in Čech cohomology is an assigment of an element $g_{\alpha\beta} \in \mathcal{O}_X^*$ for every intersection $U_\alpha \cap U_\beta$. Elements of $H^1(X, \mathcal{O}_X^*)$ are cocycles, meaning that coboundary operator vanishes. For 1-cocycles this is typically written as $(g_{\beta\gamma} - g_{\alpha\gamma} + \gamma_{\alpha\beta})|_{U_\alpha \cap U_\beta \cap U_\gamma} = 0$. In multiplicative notation this just says that $g_{\beta\gamma} g_{\alpha\gamma}^{-1} \gamma_{\alpha\beta} = 1$ which equivalent to the consistency condition for gluing functions.

b.

c. Hopf surfaces.

$$\begin{array}{c} \mathbb{C}^2 \backslash \mathbb{Z} = \langle \alpha \rangle \\ \big\downarrow {\scriptstyle \mathbb{C}^*/\langle \alpha \rangle} \\ \mathbb{C}P^1 \end{array}$$

   now $H^2(\text{Hopf surface}) = 0$. Topologically it is $S^1 \times S^3$.

Here's why it won't work on K3 surfaces:

$$a \in H^0(L)$$
$$c_1(L) = [\mathcal{D}] \in H^2(M, \mathbb{Z})$$
$$\mathcal{D} \text{ zero divisor of de ?}$$
$$\int_{\mathcal{D}} \omega^{n-1} = \langle [\mathcal{D}], \omega^{n-1} \rangle = 0$$

So this statement tells you that Euler number is the same as its first Chern class.

That

$$e(L_1 \otimes L_2) = e(L_1) + e(L_2)$$

follows from taking zero sections on $L_2$, on $L_2$ and then tensor product goes to addition.

□

**Exercise 1.8** Let $\mathbb{C}$ be a complex curve and $F$ a coherent sheaf on $\mathbb{C}$.

a. Prove that the restriction of $F$ to a certain open set $U \subset C$ is isomorphic to a vector bundle. Prove that the rank of this vector bundle is independant on the choice of $U$ when $\mathbb{C}$ is irreducible. This number is called the ***rank*** of $F$.

*Solution.*

a. (From StackExchange) I show how to construct a vector bundle from a *locally free* sheaf. Given an open cover $U_i$ such that $\mathcal{F}(U_i)$ is free for all $i$, we just need to find gluing functions $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$.

   From the definition of locally free, we have isomorphisms $f_i : \mathcal{F}(U_i) \to \mathcal{O}^n_{U_i}$. Restricting to the intersection $U_i \cap U_j$ we may define the functions

   $$f_{ij} = f_j|_{U_i \cap U_j} \circ f_i|^{-1}_{U_i \cap U_j} : \mathcal{O}^n_{U_i}|_{U_i \cap U_j} \to \mathcal{O}^n_{U_j}|_{U_i \cap U_j}.$$

   The claim in StackExchange is that every such map is induced by a gluing function, but I still cannot see why.

   The question indeed is to show why a coherent sheaf can be locally expressed as a locally free sheaf. We cover the manifold with sheafs that look like exercise 1.3. And that basically says that the sheaf is locally a direct sum of free and a finite number of torsion quotients $\mathcal{O}_X / \mathfrak{m}^{d_i}$.

   So

   $$0 \longrightarrow \text{Torsion} \longrightarrow \mathcal{F} \longrightarrow (\mathcal{F}^{**} \longrightarrow 0$$

   where $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, 0)$.

□

## References


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