K3 surfaces, home assignment 5: Positive forms and Riemann-Hodge pairing

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Definition 5.1. Throughout this handout, $V = \mathbb{R}^{2n}$ is a real vector space, $I \in \operatorname{End}(V)$ an operator which satisfies $I^2 = -\mathsf{Id}$ **("the complex structure operator"),** and $\Lambda^*(V^* \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus \Lambda^{p,q}(V^*)$ the Hodge decomposition of its Grassmann algebra. A (real) $(1,1)$-form $\omega \in \Lambda^{1,1}(V^*)$ is **Hermitian**, or **strictly positive** if $\omega(x, Ix) > 0$ for any non-zero $x \in V$. It is called **semi-Hermitian**, or **positive** if $\omega(x, Ix) \geqslant 0$ for any $x \in V$. A bivector $\eta \in \Lambda^{1,1}(V)$ is **positive** if $\eta(v, Iv) \geqslant 0$ for any non-zero $v \in V^*$.

5.1 Positive (p,p)-forms

Exercise 5.1. Let $\operatorname{Pos} \subset \Lambda^{1,1}(V^*)$ be the set of all positive (1,1)-forms, and Pos^n be the set of all non-zero volume forms obtained as n-th power of elements of Pos. Prove that Pos^n is connected.

Remark 5.1. The corresponding orientation on V is called **the orientation compatible with the complex structure operator**.

Exercise 5.2. Let $\omega \in \Lambda^{1,1}(V^*)$ be a 2-form on V, satisfying $\omega(x, Ix) \geqslant 0$, and $W \subset V$ the set of all vectors $v \in V$ such that $\omega(v, Iv) = 0$.

a. Prove that $W \subset V$ is I-invariant.

b. Prove that there exists a projection $\Pi : (V, I) \longrightarrow (V_1, I_1)$ commuting with the complex structure operator, and a Hermitian form ω_1 on V_1 such that $\omega(x, y) = \omega_1(\Pi(x), \Pi(y))$

Exercise 5.3. Let $g \in \mathsf{Sym}^2 V^*$ be an I-invariant, non-degenerate, symmetric 2-form on V. Such g is called **a pseudo-Hermitian metric**.

a. Prove that the form $\omega(x, y) := g(Ix, y)$ belongs to $\Lambda^{1,1}(V^*)$. This form is called **a pseudo-Hermitian (1,1)-form**.

b. Prove that the signature of g is $(2p, 2q)$, where $p + q = n$. In this case we say that **the signature of the pseudo-Hermitian form** ω is (p, q).

Exercise 5.4. Let $P^{p,q} \subset \Lambda^{1,1}(V^*)$, $p + q = n$, be the set of all (1,1)-forms associated with pseudo-Hermitian metrics of signature (p, q). Prove that $P^{p,q}$ is connected, or find a counterexample.

Exercise 5.5. Consider the set P of (1,1)-forms $\eta \in \Lambda^{1,1}(V^*)$ such that η^n is a positive volume form. Count the number of connected components of the set $P \subset \Lambda^{1,1}(V^*)$.

Definition 5.2. Consider the cone in $\Lambda^{p,p}(V)$ generated by $\sum_i \alpha_i \omega_i^p$ where ω_i are semi-Hermitian forms, and α_i are positive. This cone is called **the cone of strongly positive forms**. A form which belongs to the interior of this set is called **a strictly strongly positive (p,p)-form.**

Exercise 5.6. Let $\omega_1, ..., \omega_k$ be Hermitian forms. Prove that the form $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_k$ is strongly positive.

Exercise 5.7. Fix a positive volume form $\operatorname{Vol} \in \Lambda^{2n}(V^*)$. Consider an isomorphism $\Lambda^{1,1}(V) \to \Lambda^{n-1,n-1}(V^*)$ obtained by contracting Vol and η.

a. Prove that this isomorphism produces an isomorphism of the cone of positive bivectors and the cone of strongly positive forms.

b. Let $Q: \ \Lambda^{1,1}(V^*) \longrightarrow \Lambda^{n-1,n-1}(V^*)$ be a map taking ω to ω^{n-1}. Prove that Q defines a bijection between the set of Hermitian $(1,1)$-forms and the set of strictly strongly positive $(n-1,n-1)$-forms.

Exercise 5.8. Let ω be a Hermitian form. Prove that the map

$$R_\omega : \ \Lambda^{1,1}(V^*) \longrightarrow \Lambda^{n-1,n-1}(V^*)$$

taking η to $\eta \wedge \omega^{n-2}$ maps positive form to positive forms. Prove that it is bijective. Prove that it maps the set of positive (1,1)-forms to a proper subset of the set of positive $(n-1,n-1)$-forms, for any $n \geqslant 2$.

5.2 Riemann-Hodge pairing

Definition 5.3. For the duration of this subsection, fix a Hermitian form ω on (V, I), and let $\operatorname{Vol} := \omega^n \in \Lambda^{n,n}(V^*)$. **The Riemann-Hodge pairing** on $\Lambda^k(V^*)$, $k \leqslant n$ is the pairing $q(\eta, \eta') := \frac{\eta \wedge \eta' \wedge \omega^{n-k}}{\operatorname{Vol}}$.

Exercise 5.9. Prove that the Riemann-Hodge pairing is non-degenerate.

Exercise 5.10. Let $x \in V^*$ be a non-zero vector. Prove that $q(x, Ix) > 0$.

Exercise 5.11. Let V be an irreducible real representation of a compact Lie group, and g a non-degenerate bilinear symmetric form on V. Prove that g is positive definite or negative definite.

Exercise 5.12. Let $U(n) \subset GL(V)$ denote the group of matrices preserving I and ω, and $\mathfrak{u}(V) \subset \operatorname{End}(V)$ its Lie algebra.

a. Consider map $\Lambda^{1,1}(V) \longrightarrow \operatorname{End}(V^*)$ taking $\eta \in \Lambda^{1,1}(V)$ to the map $x \mapsto \omega(i_x\eta, -)$, where $i_x\eta \in V$ is the contraction of η with x. Prove that this map identifies $\Lambda^{1,1}(V)$ and $\mathfrak{u}(V^*)$.

b. Prove that $\mathfrak{u}(V) = \mathfrak{su}(V) \oplus \mathbb{R}$, where $\mathfrak{su}(V)$ is all elements of $\mathfrak{u}(V)$ with vanishing trace. Prove that the Lie algebra $\mathfrak{su}(V)$ is simple (has no proper ideals).

c. Let $\Lambda^{1,1}_0(V) \subset \Lambda^{1,1}(V)$ be the subspace corresponding to $\mathfrak{su}(V)$ under the isomorphism defined above. Prove that $\Lambda^{1,1}_0(V)$ is the orthogonal complement to ω under the standard Euclidean pairing on the Grassmann algebra.

d. Consider a $\Lambda^2(V)$ as the representation of $U(V)$, and let $W \subset \Lambda^*(V)$ be any irreducible component. Prove that the Riemann-Hodge pairing is sign-definite on W.

e. Prove that $\Lambda^{1,1}_0(V)$ is an irreducible representation of $U(V)$, and show that q is negative definite on $\Lambda^{1,1}_0(V)$.

f. Prove that q has signature $1, n^2 - 1$ on $\Lambda^{1,1}(V)$.