

k3

1 Class 1

The most important invariant of a k3 surface is [intersection form](#).

There are three classes of manifolds

1. Smooth manifolds

$$\text{smooth manifolds} \xrightarrow{\text{forgetful functor}} \text{PL manifold} \longrightarrow \text{Topological manifolds}$$

Donaldson: countably many non-equivalent smooth structures on \mathbb{R}^4 . K3 surfaces has countably many smooth structures and only one of them is compatible with complex structure.

Definition. Intersection form. Given a quadratic form on a lattice $V_{\mathbb{Z}} = \mathbb{Z}^n$, so

$$q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \rightarrow \mathbb{Z}$$

is *unimodular* if

$$V_{\mathbb{Z}} \xrightarrow{q} \text{Hom}(V_{\mathbb{Z}}, \mathbb{Z})$$

is an isomorphism.

Theorem (Universal coefficients formula).

$$H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}^{b_{n-1}(M)} \oplus T_{n-1}(M)$$

$$h^n(M, \mathbb{Z}) = \mathbb{Z}^{b_n(M)} \oplus T_{n-1}(M)$$

Corollary. $H^2(X, \mathbb{Z})$ is torsion free if $\pi_1(X) = 0$ because

Definition. *Signature* is $m - n$ if q has signature (m, n) .

Theorem (Rokhlin-Wu?). Signature is divisible by 16 for simply-connected (something else).

Remark. The methods used in surgery break down in smooth case because strange topological objects like infinite sums of spheres arise.

Theorem (Freedman, 1982). There are as many 4-manifolds as there are intersection forms. A simply connected 4 manifold homotopy class is uniquely determined by intersection form. Moreover, for every unimodular form there exists a unique M with this intersection form.

Theorem (Donaldson, 1986). M smooth compact manifold with positive definite odd intersection form q . Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition. Bilinear symmetric form is *indefinite* if it is not positive definite nor negative definite.

Theorem (Classification of unimodular symmetric bilinear forms). Odd are diagonalizable, while even are related to special Lie group E_8 .

Definition. A **K3 surface** is a Kähler complex surface M with $b_1 = 0$ (simply connected) and $c_1(M, \mathbb{Z}) = 0$.

Kodaira did what André Weil couldn't classify.

Theorem. K3 surfaces have trivial canonical bundle $K_M = \Lambda^2(\Omega^1 M)$.

2 Class 2

G topological group. **Principal G bundle** is a space with free G -action such that the quotient E/G is Hausdorff. There are several conditions that make this work. And then you have $\text{Homotopy}(X, BG) = \text{equivalence classes of } G\text{-bundles}$. Vector bundles of a manifold are the same as maps from X to $BU(n)$.

Vector bundles up to stable equivalence are classified basically by Chern classes, so by the cohomology in $H^*(BU) = \mathbb{Q}[c_1, c_2, \dots, c_n]$.

Now look at the loop space of X . Then $H^*(\Omega X)$ is a free graded commutative algebra. Loop space has the interesting property that $\Omega U = BU$ and $\Omega BU = U$.

2.1 Bialgebras

Let A be a superalgebra (graded with antisymmetric product). Then we ask the axiom of coassociativity and that .

Example. G group, and $C(G)$ the ring of k -valued functions $C(G \times G) = C(G) \times C(G)$ so

$$\begin{aligned} G \times G &\longrightarrow G \\ C(G) &\longmapsto C(G) \otimes C(G) \end{aligned}$$

2.2 H-spaces

Definition. H-space is a space M with a map $\mu : M \times M \rightarrow M$ that is homotopy associative,

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mu \times \text{id}} & M \times M \\ \downarrow \text{id} \times \mu & & \downarrow \mu \\ M \times M & \xrightarrow{\mu} & M \end{array}$$

which is homotopy commutative. And with homotopy unit.

So it's like a homotopy algebra?

Example. The loop space.

2.3 Bialgebras of finite type

Definition. A bialgebra A is of *finite type* if it is the direct sum of $A = \bigoplus_{i \geq 0} A^i$ supercommutative and each A^1 is finite dimensional.

Remark. Free commutative algebra is polynomial algebra

Definition. $A = \mathbb{C}[x_1, \dots, x_n, \dots] \otimes \Lambda^\bullet(a_1, \dots, a_n, \dots)$ is a graded commutative free algebra. In the slides: it is $\text{Sym}_{\text{gr}} V^*$ where V^* is a graded vector space.

Theorem (Hopf). A graded commutative bialgebra of finite type over k of 0 characteristic is free graded commutative as a k algebra.

2.4 The cohomology algebra of $U(n)$

Claim. The cohomology algebra $H^*(U(n), \mathbb{Q})$ is a free graded commutative algebra with generators in degrees $1, 3, 5, \dots, 2n-1$.

Demonstração. Induction. $U(1)$ is clear because it is a circle. Then do Serre spectral sequence. Differentials vanish on the second page because there's only nonzero groups on even degrees! And we get that $E_2^{p,1} = H^p(S^{2n-1}) \otimes H^q(U(n-1))$. And then the sequence converges to that of the total space which is $U(n)$. \square

2.5 Grassman manifolds

Definition. The *fundamental bundle* B_{fun} is a rank n vector bundle over $\text{Gr}(n, m)$.

Claim. B, B' vector bundles of rank $n, m-n, B \oplus B'$

$$\varphi : X \rightarrow \text{Gr}(m, n)$$

$$\varphi(x) = B_x \subset B_x \oplus B'_x = \mathbb{K}^m$$

then $B = \varphi^* B_{\text{fun}}$.

Theorem. If you have B as a bundle on a manifold X then $B \oplus B'$ is trivial for some bundle B' .

Demonstração. Embed the total space in a large enough euclidean space. \square

Definition. $\text{Gr}(n, \infty) = \text{Gr}(n)$ is $\bigcup_{m=n_1}^{\infty} \text{Gr}(n, m) = \text{Gr}(n)$

Corollary. For every bundle B of rank n there is a function $\varphi : X \rightarrow \text{Gr}(n)$ such that $B = \varphi^* B_{\text{fun}}$.

Take a bundle $E \rightarrow X$ and G acts freely on E so E principal G bundle. Classifying space BG

Theorem (Atiyah-Bott). Classifying space is unique up to homotopy equivalence.

2.6 Stiefel spaces

Definition. \mathbb{K}^{∞} is the direct limit of \mathbb{K}^n so its just the direct sum $\bigoplus_{i=n}^{\infty} \mathbb{K}$. Stiefel space is the space of orthonormal n -frames.

If we prove that Stiefel is contractible we obtain our classifying space so let's prove that. We have a fibration

$$U(n) \hookrightarrow \text{St}(n, \infty) \rightarrow \text{Gr}(n, \infty)$$

Theorem. $\text{St}(n)$ is contractible.

Demonstração **Step 1** Locally trivial fibration with contractible fiber and base $Y \rightarrow X$ then Y is contractible, this is so trivial.

Step 2 Fibration $\text{St}(n) \rightarrow \text{St}(n-1)$ with fiber S^{∞}

Step 3 Show that S^{∞} is contractible.

Step 4 And then some map \mathbb{R} that is not surjective, and construct homotopy of identity to a constant map. \square

Exercise. If $X_{\infty} = \bigcup X_i$ is the inductive limit of contractible cellular spaces then it is contractible. Use Whitehead theorem.

Theorem (Important). $\text{Gr}(\infty) = BU$.

2.7 Stable equivalence

Definition. Vector bundles V, W are stable equivalent if $V \oplus A \cong W \cong B$ for trivial vector bundles A and B .

Homotopy classes of equivalent vector bundles are in corespondance with...

Theorem. $B\mathbb{U}$ is H-space.

Corollary. $H^*(B\mathbb{U}, \mathbb{Q})$ is a free supercommutative algebra.

Claim. $H^*(B\mathbb{U})$ is a free polynomial algebra generated by classes c_1, c_2, \dots in all even degrees.

3 Class 3

3.1 Reminder

Definition. *Bialgebra* is an algebra that is equipped with comultiplication, counit...

Remark. It is when the dual space also has an algebra structure, but we prefer to use the tensor notation.

Let $\sum_{i \geq 0} A^i$ with $\dim A^i < \infty$. *Free commutative algebra* is a polynomial algebra. *Free graded commutative algebra* is

$$\widetilde{\text{Sym}}^\bullet(W^\bullet \oplus V^\bullet) := \text{Sym}^\bullet(W^\bullet) \otimes \Lambda^\bullet(V^\bullet)$$

where

$$W = \bigoplus_i W^{\text{even}} \quad V = \bigoplus_i V^{\text{odd}}.$$

3.2 Hopf algebra

Definition. A bialgebra is a *Hopf algebra* when it is also equipped with an antipode map (S) such that the following diagram commutes

$$\begin{array}{ccccc}
 & H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H & \\
 \Delta \nearrow & & & & \searrow m \\
 H & \xrightarrow{\eta} & \mathbb{C} & \xrightarrow{u} & H \\
 \Delta \searrow & & & & \nearrow \\
 & H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H &
 \end{array}$$

[diagram from quantum group minicourse notes]

Example. The cohomology of the loop space, $H^*(\Omega X)$.

3.3 Primitive elements in a bialgebra

Definition. An element of a bialgebra $x \in A$ is *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$$\begin{aligned}\Delta(xy) &= \Delta(x)\Delta(y) \\ &= (1 \otimes x + x \otimes 1)(y \otimes 1 + y \otimes y) \\ &= 1 \otimes xy + xy \otimes 1 + x \otimes y + y \otimes x.\end{aligned}$$

Remark. We trying to show that Hopf algebras? bialgebras? are generated by primitive elements?

Definition. A^\bullet bialgebra, $\mathcal{P}^\bullet \subset A^\bullet$ space of primitive, and the natural embedding

$$\text{Sym}_{\text{gr}}(\mathcal{P}^\bullet) \rightarrow A$$

We say that A is *free up to degree* k if

$$\bigoplus_{i \leq k} \text{Sym}_{\text{gr}}^i(P) \xrightarrow{\psi} A$$

is an embedding.

Lemma. Let A^\bullet be a bialgebra which is free up to degree k . Then A^\bullet is free up to degree $k + 1$.

Proof.

Step 1 Choose a basis of P , $\{x_i\}$. Chose a polynomial condition $Q(x_1, \dots, x_n) = 0$ of degree $k + 1$. Write this as

$$Q = Q_m x_1^m + Q_{m-1} x_1^{m-1} + \dots + Q_0.$$

that is

$$Q = \sum_{i=0}^m Q_i x_1^i$$

with Q_i invariant somehow. Then we apply comultiplication to obtain

$$\Delta(Q) = Q \otimes 1 + 1 \otimes Q + R$$

where R is some sort of reminder with bounded degree:

$$R \in \mathcal{U} := \bigoplus_{i \leq k} \text{Sym}_{\text{gr}}^i(P) \otimes \bigoplus_{i \leq k} \text{Sym}_{\text{gr}}^i(P)$$

which follows from a similar computation of that which we did after defining primitive elements.

Step 2 Project to drop the terms that have $Q \otimes 1 + 1 \otimes Q$:

$$\Pi : \mathcal{U} \rightarrow x_1 \otimes \bigoplus_{i \leq k} \text{Sym}_{\text{gr}}^i(P)$$

since the x_i are primitive, i.e. $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, one has

$$\Delta(x_1^m) = (x_1 \otimes 1 + 1 \otimes x_1)^m$$

we get that

$$\Pi(\Delta(x_1^m)) = mx_1 \otimes x_1^{m-1}$$

while on the board it is written that

$$\Pi(\Delta(x_1^m)) = \Pi((x_1 \otimes 1 + 1 \otimes x_1)^m)$$

Step 3 Let $\Pi(R) := x_1 \otimes R_0$. Since $Q = 0$ in A , its component R_0 is also equal to 0. So $\Pi(\Delta(Q)) = 0$. Then

$$\begin{aligned} 0 &= \Pi \left(\Delta \left(\sum_m x_1^m \cdot Q_m \right) \right) \\ &= \sum_m x_1 \otimes x_1^{m-1} Q_m + \Pi(mx_1 \otimes x_1^{m-1} \cdot \Delta(Q_m)) \\ &= \sum_m x_1 \otimes x_1^{m-1} Q_m \end{aligned}$$

so that

$$\begin{aligned} x_1 \otimes x_1^{m-1} Q_m &= 0 \\ \implies x_1^{m-1} Q_m &= 0 \end{aligned}$$

So we conclude that

$$Q_m = 0$$

□

Remark. We just proved that for any subalgebra generated by finite elements, we didn't use that it is free.

3.4 Algebras with filtration

Definition. A *filtration on algebra* is

$$A^\bullet \supset F_1 A^\bullet \supset F_2 A^\bullet \supset \dots$$

such that

$$F_i A^\bullet F_j \subset F_{i+j} A^\bullet$$

Definition. *Associated graded* to a filtered algebra is

$$A_{\text{gr}}^{\bullet} = \bigoplus_{i=0}^{\infty} \frac{F^i A^{\bullet}}{F^{i+1} A^{\bullet}}$$

$$F^0 A^{\bullet} = A^{\bullet}$$

Definition. $I \subset A$ ideal then *I-adic filtration* is the filtration by the degrees of the ideal

$$A \supset I \supset I^2 \supset I^3 \dots$$

Lemma. Choose an I-adic filtration. Then A_{gr} is generated by its first and second graded components $A/I \oplus I/I^2$.

Demonstração. Indeed, I^k/I^{k+1} is generated by products of k elements in (I/I^2) . □

Definition. A *augmentation ideal* in a bialgebra is the kernel of the counit homomorphism $\varepsilon : A \rightarrow k$. We denote it by $Z = \ker \varepsilon$

Remark.

$$\Delta(x) = 1 \otimes x + x \otimes 1 \text{ mod } Z \otimes Z$$

Why? Because

$$\begin{aligned} x &= \varepsilon \otimes \text{id}(\Delta(x)) && \text{up to } Z \otimes A \\ \Delta(x) &= 1 \otimes x && \text{up to } A \otimes Z \\ \Delta(x) &= x \otimes 1 \end{aligned}$$

Ok, now we can prove Hopf theorem.

Theorem (Hopf theorem). A finite type bialgebra is generated by primitive elements.

In slides: Let A be a graded bialgebra of finite type over a field k of characteristic 0. Then A is a free graded commutative k -algebra.

Proof.

Step 1 I think this is the computation above.

Step 2 A_{gr} is a bialgebra.

Step 3 A_{gr} is multiplicatively generated by Z^1/Z^2 . All elements Z^1/Z^2 are primitive, so this algebra A_{gr} is generated by primitive elements.

Step 4 Let $\{x_i\}$ be a basis of primitive elements of A_{gr} . Then lifts of A have no relations because A_{gr} is already generated by primitive elements. Then there are no relations also for elements in A^{\bullet} (I think).

□

3.5 Grassmanians (Reminder)

A vector bundle of rank n on X then there exists a map (essentially unique) $\varphi : X \rightarrow \text{Gr}(n)$ such that

$$\varphi^*(B_{\text{fun}}) = B$$

which makes the Grassmanian a classifying space, and $\text{Gr}(1) = B U(n)$.

The infinite Grassmanian is important.

3.6 BU as an H-space (Reminder)

Bott periodicity identifies the space of loops on U and $B U$.

Proposition. Embed $\mathbb{C}^\infty \times \mathbb{C}^\infty$ into \mathbb{C}^∞ taking the basis vectors of the first copy to the even basis vectors and the basis of the second copy to the odd. Then for $L_1 \subset \mathbb{C}^\infty$, $L_2 \subset \mathbb{C}^\infty$, the map

$$L, L' \mapsto S(L, L')$$

defines a structure of H-space on the infinite Grassmanian $B U$.

Proof. Just show that H-associativity up to homotopy. □

Corollary. $H^*(B U, \mathbb{Q})$ is a free supercommutative algebra.

Proof. Follows from Hopf theorem. □

3.7 Cohomology of BU

Claim. $H^*(B U, \mathbb{Q})$ is a free polynomial algebra generated by classes c_1, c_2, \dots in all even degrees.

Demonstração. Leray-Serre spectral sequence. □

3.8 Chern classes: axiomatic definition

Definition. *Chern classes* are classes $c_i(B) \in H^{2i}(X)$ for $i = 0, 1, 2, \dots$

Chern classes are $c_i(B) \in H^{2i}(X)$ for a complex vector bundle B over X with axioms

- a. $c_0(B) = 1$
- b. Functoriality (commutes with pullbacks): for $\varphi : X \rightarrow Y$ with B bundle on Y ,

$$\varphi^*(c_i(B)) = c_i(\varphi^*(B))$$

- c. Define *total Chern class* $c_* := \sum_i c_i(B)$ then

$$c_i(B) \cdot c_i(B') = c_*(B \oplus B') \quad (\text{Whitney})$$

d. $\mathcal{O}(1)$ on \mathbb{CP}^n ,

$$c_1(\mathcal{O}(1)) = 1 + [H]$$

where $[H]$ is the fundamental class of a hyperplane section.

Suppose we have a class $a \in H^*(B U)$. Then for all B on X

$$\varphi : X \rightarrow B U$$

so

$$B \cong \varphi^*(B_{\text{fun}})$$

and so

$$\varphi_B^*(c_*) = c_*(B).$$

4 Class 4