Daniel González Casanova Azuela
github.com/danimalabares/k3

Home assignment 7: Moser isotopy lemma

Exercise 7.1 Let M be a compact manifold, and V_0, V_1 two smooth volume forms which satifsy $\int_M V_0 = \int_M V_1$. Prove that there exists a diffeomorphism which satisfies $\Phi^* V_0 = V_1$.

Solution. We must show there is a family V_t of cohomologous volume forms joining V_0 and V_1. Define

$$V_t = tV_0 + (1-t)V_1.$$

Now since $H^n(M)$ is a real one-dimensional vector space, $[V_t] = \alpha[V_0]$ for some real number α.

To show that α must be 1 we need to integrate $[V_t]$, which we may do since V_t is a closed nowhere-vanishing top-form. Closedness is immediate. To see it is nowhere-vanishing just notice that both V_0 and V_1 are, and since they integrate to the same number they are both always positive or always negative.

Then

$$\int_M [V_t] = t\int_M [V_0] + (1-t)\int_M [V_1] = \int_M [V_0] = \int_M [V_1]$$

which means that $\alpha = 1$. Then by Moser's lemma there exists an isotopy φ_t with $\varphi_t^* V_t = V_0$. Taking $t = 1$ we obtain the result. □

Problem 7.2 Let (M, I) be an almost complex manifold, and ω_0, ω_1 co-homologous symplectic forms which satisfy $\omega_i(x, Ix) > 0$ for any non-zero tangent vetor x (such forms are called ***taming***).

a. Prove that there exists a diffeomorphism Φ which satisfies $\Phi^*\omega_0 = \omega_1$.

b. Prove that $|x|_i^2 := \omega_i(X, Ix)$ is a Hermitian metric on M. Prove that a diffeomorphism that satisfies $\Phi^*\omega_0 = \omega_1$ defines an isometry

$$(M, |x|_1^2) \longrightarrow (M, |x|_0^2)$$

is Φ is compatible with I, that is, satisfies $d\Phi(Ix) = I(d\Phi(x))$

c. Find an example of $\omega_0, \omega_1 \in \Lambda^2(M)$ such that a diffeomorphism compatible with I and satisfying $\Phi^*\omega_0 = \omega_1$ does not exist.

Solution.

a. This is immediate from Moser's lemma defining $\omega_t := t\omega_0 + (1+t)\omega_1$.

b. According to Riemann Surfaces course, a hermitian form may be understood as a Riemannian metric h such that $h(\cdot,\cdot) = h(I\cdot, I\cdot)$. This happens for $h(\cdot,\cdot) = \omega(\cdot, I\cdot)$ if we require $\omega(\cdot,\cdot) = \omega(I\cdot, I\cdot)$. Indeed, symmetry holds since

$$\omega(x, Iy) = \omega(Ix, -y) = \omega(y, Ix)$$

and positive-definiteness is given as a hypothesis.

The condition that $\omega(\cdot, I\cdot)$ is a riemannian metric is called ***compatibility*** of ω and I in Silva. In this case we can also produce a hermitian metric in the sense of WolframMathWorld, namely, a positive-definite symmetric sesquilinear form. Indeed, define $g(\cdot,\cdot) := \omega(\cdot, I\cdot)$, then the form

$$h := g + \sqrt{-1}\omega$$

satisfies the required properties as follows.

(a) Additivity, $h(u_1 + u_2, v) = h(u_1, v) + h(u_1, v)$, is immediate.

(b) Homegenity on the first argument, $h(\lambda u, v) = \lambda h(u, v)$, is also immediate.

(c) $h(u, v) = \overline{h(v, u)}$ is clear by anti-symmetry of Ω:

$$h(u, v) = g(u, v) + i\omega(u, v) = g(v, u) - i\omega(v, u) = \overline{h(v, u)}$$

(d) The property $h(u, \lambda v) = \bar{\lambda} h(u, v)$ follows easily from (b) and (c) since

$$h(u, \lambda v) = \overline{h(\lambda v, u)} = \bar{\lambda}\overline{h(v, u)} = \bar{\lambda} h(u, v)$$

(e) Positive-definiteness follows from positive-definiteness of g and antisymmetry of ω.

c. The conditions $\Phi^*\omega_0 = \omega_1$ and $\Phi_*(I\cdot) = I\Phi_*(\cdot)$ imply that

$$(\Phi^*\omega_0)(\cdot, I\cdot) = \omega_0(\Phi_*\cdot, \Phi_* I\cdot) = \omega_0\Phi_*\cdot, I\Phi_*\cdot) = \omega_1(\cdot, I\cdot) \tag{1}$$

Whether the metric is taken to be $\omega_i(\cdot, I\cdot)$ or $h_i = g_i + \sqrt{-1}\omega_i$, the result follows. In the latter case only note that

$$\begin{aligned} \Phi^* h_0 &= \Phi^*(g_0 + \sqrt{-1}\omega_0) = \Phi^* g_0 + \sqrt{-1}\Phi^*\omega_0 \\ &= \Phi^*\Big(\omega(\cdot, I\cdot)\Big) = g_1 + \sqrt{-1}\omega_1 = h_1 \end{aligned}$$

by eq. (1).

d. I am intrigued to know the answer here.

□

References

Silva, A.C. da. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics no. 1764. Springer, 2001. ISBN: 9783540421955.