k3

1 Class 1

The most important invariant of a k3 surface is intersection form.

There are three classes of manifolds

1. Smooth manifolds

$$\text{smooth manifolds} \xrightarrow{\text{forgetful functor}} \text{PL manifold} \longrightarrow \text{Topological manifolds}$$

Donaldson: contiually many non-equivalent smooth structures on $\mathbb{R}^4$. K3 surfaces has countably many smooth structures and only one of them is compatible with complex structure.

Definition. Intersection form. Given a quadratic form on a lattice $V_{\mathbb{Z}} = \mathbb{Z}^n$, so

$$q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$$

is ***unimodular*** if

$$V_{\mathbb{Z}} \xrightarrow{q} \mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z})$$

is an isomorphism.

Theorem (Universal coefficients formula)**.**

$$H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}^{b_{n-1}(M)} \oplus T_{n-1}(M)$$

$$h^n(M, \mathbb{Z}) = \mathbb{Z}^{b_n(M)} \oplus T_{n-1}(M)$$

Corollary. $H^2(X, \mathbb{Z})$ is torsion free if $\pi_1(X) = 0$ because

Definition. ***Signature*** is $m - n$ if q has signature (m, n).

Theorem (Rokhlm-Wu?)**.** Signature is divisible by 16 for simply-connected (something else).

Remark. The methods used in surgery break down in smooth case because strange topological objects like infinite sums of spheres arise.

Theorem (Freedman, 1982)**.** There are as many 4-manifolds as there are intersection forms. M simply connected 4 manifold homotopy class is uniquely determined by intersection dorm. Moreover, for every unimodular form there exists a unique M with this intersection form.

Theorem (Donaldson, 1986)**.** M smooth compact manifold with positive definite odd intersection form q. Then

$$\begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition. Bilinear symmetric form is ***indefinite*** if it is not positive definite nor negative definite.

Theorem (Classification of unimodular symmetric bilinear forms)**.** Odd are diagonalizable, while even are related to special Lie group E_8.

Definition. A ***K3 surface*** is a Kähler complex surface M with $b_1 = 0$ (simply connected) and $c_1(M, \mathbb{Z}) = 0$.

Kodaira did what André Weil couldn'g classify.

Theorem. K3 surfaces have trivial canonical bundle $K_M = \Lambda^2(\Omega^1 M)$.

2 Class 2

G topological group. ***Principal*** G ***bundle*** is a space with free G-action such that the quotient E/G is Housdorff. There are several conditions that make this work. And then you have Homotopy(X, BG) =equivalence classes of G-bundles. Vector bundles of a manifold are the same as maps from X to $B\,U(n)$.

Vector bundles up to stable equivalence are classified basically by Chern classes, so by the cohomology in $H^\bullet(B\,U) = Q[c_1, c_2, \dots, c_n$.

Now look at the loop space of X. Then $H^\bullet(\Omega X)$ is a free graded commutative algebra. Loop space has the interesting property that $\Omega\,U = B\,U$ and $\Omega B\,U = U$.

2.1 Bialgebras

Let A be a superalgebra (graded with antisymmetric product). Then we ask the axiom of coassociativity and that .

Example. G group, and $C(G)$ the ring of k-valued functions $C(G \times G) = C(G) \times C(G)$ so

$$\begin{aligned} G \times G &\longrightarrow G \\ C(G) &\longmapsto C(G) \otimes C(G) \end{aligned}$$

2.2 H-spaces

Definition. H-space is a space M with a map $\mu : M \times M$ to$M$that is homotopy associative,

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"] & M \end{tikzcd}$$

which is homotopy commutative. And with homotopy unit.

So it's like a homotopy algebra?

Example. The loop space.

2.3 Bialgebras of finite type

Definition. A bialgebra A is of ***finite type*** if it is the direct sum of $A = \bigoplus_{i \geqslant 0} A^i$ super-commutative and each A^1 is finite dimensional.

Remark. Free commutative algebra is polynomial algebra

Definition. $A = \mathbb{C}[x_1, \ldots, x_n, \ldots] \otimes \Lambda^\bullet(a_1, \ldots, a_n, \ldots)$ is a graded commutative free algebra. In the slides: it is $\mathrm{Sym}_{gr} V^*$ where V^* is a graded vector space.

Theorem (Hopf). A graded commutative bialgebra of finite type over k of 0 characteristic is free graded commutative as a k algebra.

2.4 The cohomology algebra of U(n)

Claim. The cohomology algebra $H^*(U(n), \mathbb{Q})$ is a free graded commutative algebra with generators in degrees $1, 3, 5, \ldots, 2n-1$.

Demostração. Induction. $U(1)$ is clear because it is a circle. Then do Serre spectral sequence. Differentials vanish on the second page because there's only nonzero groups on even degrees! And we get that $E_2^{p,q} = H^p(S^{2n-1}) \otimes H^q(U(n-1))$. And then the sequence converges to that of the total space which is $U(n)$. $\square$

2.5 Grassman manifolds

Definition. The ***fundamental bundle*** B_{fun} is a rank n vector bundle over $\mathrm{Gr}(n, m)$.

Claim. B, B' vector bundles of rank n, $m-n$, $B \oplus B'$

$$\varphi : X \to \mathrm{Gr}(m, n)$$

$$\varphi(x) = B_x \subset B_x \oplus B'_x = \mathbb{K}^m$$

then $B = \varphi^* B_{\text{fun}}$.

Theorem. If you have B as a bundle on a manifold X then $B \oplus B'$ is trivial for some bundle B'.

Demostração. Embed the total space in a large enough euclidean space. □

Definition. $Gr(n, \infty) = Gr(n)$ is $\bigcup_{m=n_1}^{\infty} Gr(n, m) = Gr(n)$

Corollary. For every bundle B of rank n there is a function $\varphi : X \to Gr(n)$ such that $B = \varphi^* B_{fun}$.

Take a bundle $E \to X$ and G acts freely on E so E principal G bundle. Classifying space BG

Theorem (Atiyah-Bott)**.** Classifying space is unique up to homotopy equivalence.

2.5.1 The fundamental bundle

In class 4 I finally understood that

Definition. The ***fundamental bundle*** on the Grassmanian $Gr(n)$ (the Grassmanian is this space where points are linear spaces) is the vector bundle such that the fiber of one point (which is a vector space) is the vector space that is the point. It's very tautological.

Theorem (Did we prove this?)**.** Let B be a vector bundle of rank n on a cellular space X. Then there exists a continuous map $\varphi : X \to Gr(n)$ such that B is isomorphic to the pullback $\varphi^* B_{fun}$ of the fundamental bundle.

Remark. In fact $Gr(n)$ is the classifying space of vector bundles of rank n, in the sense that isomorphism classes of vector bundles of maps $\varphi : X \to Gr(n)$.

2.6 Stiefel spaces

Definition. $\mathbb{K}^\infty$ is the direct limit of $\mathbb{K}^n$ so its just the direct sum $\bigoplus_{i=n}^{\infty} \mathbb{K}$. Stiefel space is the space of orthonormal n-frames.

If we prove that Stiefel is contractible we obtain our classifying space so let's prove that. We have a fibration

$$U(n) \hookrightarrow St(n, \infty) \to Gr(n, \infty)$$

Theorem. $St(n)$ is contractible.

*Demostração***Step 1** Locally trivial fibration with contractible fiber and base $Y \to X$ then Y is contractible, this is so trivial.

Step 2 Fibration $St(n) \to St(n-1)$ with fiber S^∞

Step 3 Show that S^∞ is contractible.

Step 4 And then some map $\mathbb{R}$ that is not surjective, and construct homotopy of identity to a constant map.

$\square$

Exercise. If $X_\infty = \bigcup X_i$ is the inductive limit of contractible cellular spaces then it is contractible. Use Whitehead theorem.

Theorem (Important)**.** $\mathrm{Gr}(\infty) = \mathrm{B\,U}$.

2.7 Stable equivalence

Definition. Vector bundles V, W are stable equivalent if $V \oplus A \cong W \cong B$ for trivial vector bundles A and B.

Homotopy classes of equivalent vector bundles are in coorespondance with...

Theorem. $\mathrm{B\,U}$ is H-space.

Corollary. $H^*(\mathrm{B\,U}, \mathbb{Q})$ is a free supercommutative algebra.

Claim. $H^*(\mathrm{B\,U}$ is a free polynomial algebra generated by classes $c_1, c_2, \ldots$ in all even degrees.

3 Class 3

3.1 Reminder

Definition. ***Bialgebra*** is an algebra that is equipped with comultiplication, counit...

Remark. It is when the dual space also has an algebra structure, but we prefer to use the tensor notation.

Let $\sum_{i\geqslant 0} A^i$ with $\dim A^i < \infty$. ***Free commutative algebra*** is a polynomial algebra. ***Free graded commutative algebra*** is

$$\widetilde{\mathrm{Sym}}^\bullet(W^\bullet \oplus V^\bullet) := \mathrm{Sym}^\bullet(W^\bullet) \otimes \Lambda^\bullet(V^\bullet)$$

where

$$W = \bigoplus_i W^{\mathrm{even}} \qquad V = \bigoplus_i V^{\mathrm{odd}}.$$

3.2 Hopf algebra

Definition. A bialgebra is a ***Hopf algebra*** when it is also equipped with an antipode map (S) such that the following diagram commutes

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\eta"] \arrow[dr, "\Delta"'] & & \mathbb{C} \arrow[rr, "u"] & & H \\
 & H \otimes H \arrow[rr, "id \otimes S"] & & H \otimes H \arrow[ur] &
\end{tikzcd}$$

[diagram from quantum group minicourse notes]

Example. The cohomology of the loop space, $H^\bullet(\Omega X)$.

3.3 Primitive elements in a bialgebra

Definition. An element of a bialgebra $x \in A$ is ***primitive*** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$$\begin{aligned}
\Delta(xy) &= \Delta(x)\Delta(y) \\
&= (1 \otimes x + x \cdot 1)(y \otimes 1 + y \otimes y) \\
&= 1 \otimes xy + xy \otimes 1 + x \otimes y + y \otimes x.
\end{aligned}$$

Remark. We trying to show that Hopf algebras? bialgebras? are generated by primitive elements?

Definition. $A^\bullet$ bialgebra, $\mathscr{P}^\bullet \subset A^\bullet$ space of primitive, and the natural embedding

$$\mathrm{Sym}_{gr}(\mathscr{P}^\bullet) \to A$$

We say that A is ***free up to defree*** k if

$$\bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P) \xrightarrow{\psi} A$$

is an embedding.

Lemma. Let $A^\bullet$ be a bialgebra which is free up to degree k. Then $A^\bullet$ is free up to degree $k+1$.

Proof.

Step 1 Choose a basis of P, $\{x_i\}$. Chose a polynomial condition $Q(x_1, \ldots, x_n) = 0$ of degree $k+1$. Write this as

$$Q = Q_m x_1^m + Q_{m-1} x_1^{m-1} + \ldots + Q_0.$$

that is

$$Q = \sum_{i=0}^{m} Q_i x_1^i$$

with Q_i invariant somehow. Then we apply comutiplication to obtain

$$\Delta(Q) = Q \otimes 1 + 1 \otimes Q + R$$

where R is some sort of reminder with bounded degree:

$$R \in \mathfrak{U} := \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P) \otimes \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P)$$

which follows from a similar computation of that which we did after defining primitive elements.

Step 2 Project to drop the terms that have $Q \otimes 1 + 1 \otimes Q$:

$$\Pi : \mathfrak{U} \to x_1 \otimes \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P)$$

since the x_i are primitive, i.e. $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, one has

$$\Delta(x_1^m) = (x_1 \otimes 1 + 1 \otimes x_1)^m$$

we get that

$$\Pi(\Delta(x_1^m)) = m x_1 \otimes x_1^{m-1}$$

while on the board it is written that

$$\Pi(\Delta(x_1^m)) = \Pi((x_1 \otimes 1 + 1 \otimes x_1)^m)$$

Step 3 Let $\Pi(R) := x_1 \otimes R_0$. Since $Q = 0$ in A, its component R_0 is also equal to 0. So $\Pi(\Delta(Q)) = 0$. Then

$$\begin{aligned} 0 &= \Pi\left(\Delta\left(\sum_m x_1^m \cdot Q_m\right)\right) \\ &= \sum_m x_1 \otimes x_1^{m-1} Q_m + \Pi(m x_1 \otimes x_1^{m-1} \cdot \Delta(Q_m)) \\ &= \sum_m x_1 \otimes x_1^{m-1} Q_m \end{aligned}$$

so that

$$\begin{aligned} x_1 \otimes x_1^{m-1} Q_m &= 0 \\ \implies x_1^{m-1} Q_m &= 0 \end{aligned}$$

So we conclude that

$$Q_m = 0$$

□

Remark. We just proved that for any subalgebra generated by finite elements, we didn't use that it is free.

3.4 Algebras with filtration

Definition. A ***filtration on algebra*** is

$$A^\bullet \supset F_1 A^\bullet \supset F_2 A^\bullet \supset \dots$$

such that

$$F_i A^\bullet F_j \subset F_{i+j} A^\bullet$$

Definition. ***Associated graded*** to a filtered algebra is

$$A^\bullet_{gr} = \bigoplus_{i=0}^{\infty} \frac{F^1 A^\bullet}{F^{i+1} A^\bullet}$$

$$F^0 A^\bullet = A^\bullet$$

Definition. $I \subset A$ ideal then I-***adic filtration*** is the filtration by the degrees of the ideal

$$A \supset I \supset I^2 \supset I^3 \dots$$

Lemma. Choose an I-adic filtration. Then A_{gr} is generated by its first and second graded components $A/I \oplus I/I^2$.

Demostração. Indeed, $I^k/^{k+1}$ is generated by products of k elements in (I/I^2). □

Definition. A ***augmentation ideal*** in a bialgebra is the kernel of the counit homomorphism $\varepsilon : A \to k$. We denote it by $Z = \ker A$

Remark.

$$\Delta(x) = 1 \otimes x + x \otimes 1 \bmod Z \otimes Z$$

Why? Because

$$x = \varepsilon \otimes \mathrm{id}(\Delta(x)) \qquad \text{up to } Z \otimes A$$
$$\Delta(x) = 1 \otimes x \qquad \text{up to } A \otimes X$$
$$\Delta(x) = x \otimes 1$$

Ok, now we can prove Hopf theorem.

Theorem (Hopf theorem). A finite type bialgebra is generated by primitive elements.

In slides: Let A be a graded bialgebra of finite type over a field k of characteristic 0. Then A is a free graded commutative k-algebra.

Proof.

Step 1 I think this is the computation above.

Step 2 A_{gr} is a bialgebra.

Step 3 A_{gr} is multiplicative generated by Z^1/Z^2. All elements Z^1/Z_2 are primitive, so this algrebra A_{gr} is generated by primitive elements.

Step 4 Let $\{x_i\}$ be a basis of primitive elements of A_{gr}. Then lifts of A have no relations because A_{gr} is already generated by primitive elements. Then there are no relations also for elements in $A^\bullet$ (I think).

$\square$

3.5 Grassmanians (Reminder)

B vector bundle of rank n on X then there exists a map (essentialy unique) $\varphi : X \to \mathrm{Gr}(n)$ such that

$$\varphi^*(B_{\text{fun}} = B$$

which makes the Grassmanian a classifying space, and $\mathrm{Gr}(1) = B\,U(n)$.

The infinite Grassmanian is important.

3.6 BU as an H-space (Reminder)

Bott periodicity identifies the space of loops on U and $B\,U$.

Proposition. Embed $\mathbb{C}^\infty \times \mathbb{C}^\infty$ into $\mathbb{C}^\infty$ taking the basis vectors of the first copy to the even basis vectors and the basis of the second copy to the odd. Then for $L_1 \subset \mathbb{C}^\infty$, $L_2 \subset \mathbb{C}^\infty$, the map

$$L, L' \mapsto S(L, L')$$

defines a structure of H-space on the infinite Grassmanian $B\,U$.

Proof. Just show that H-associatity up to homotopy. $\square$

Corollary. $H^\bullet(B\,U, \mathbb{Q})$ is a free supercommutative algebra.

Proof. Follows from Hopf theorem. $\square$

3.7 Cohomology of BU

Claim. $H^\bullet(B\,U, \mathbb{Q})$ is a free polynomial algebra generated by classes $c_1, c_2, \ldots$ in all even degrees.

Demostração. Leray-Serre spectral sequence. $\square$

3.8 Chern classes: axiomatic definition

Definition. ***Chern classes*** are classes $c_i(B) \in H^{2i}(X)$ for $i = 0, 1, 2, \dots$

Chern classes are $c_i(B) \in H^{2i}(X)$ for a complex vector bundle B over X with axioms

a. $c_0(B) = 1$

b. Functoriality (commutes with bullbacks): for $\varphi : X \to Y$ with B bundle on Y,

$$\varphi^*(c_i(B)) = c_1(\varphi^*(B))$$

c. Define ***total Chern class*** $c_* := \sum_i c_i(B)$ then

$$c_i(B) \cdot c_i(B') = c_*(B \oplus B') \qquad \text{(Whitney)}$$

d. $\mathcal{O}(1)$ on $\mathbb{C}P^n$,

$$c_i(\mathcal{O}(1) = 1 + [H]$$

where $[H]$ is the fundamental class of a hyperplane section.

Suppose we have a class $a \in H^\bullet(BU)$. Then for all B on X

$$\varphi : X \to BU$$

so

$$B \cong \varphi^*(B_{\text{fun}})$$

and so

$$\varphi_B^*(c_*) = c_*(B).$$

4 Class 4

4.1 Reminder

For each rank n bundle B on X there exists $\varphi_B : X \to \mathrm{Gr}(n, \infty) = BU(n)$ such that $\varphi_B^*(B_{\text{fun}} = B$.

The infinite grassmanian is classifying space for (?) stable bundles.

Some more review about H-space structure, primitive elements, a comment on last exercise of homework 2.

Chern classes of $\mathcal{O}(1)$ are hyperplane sections: $c_i(\mathcal{O}(1)) = 1 + [H]$.

4.2 The splitting principle

Exercise. Prove that $BU(1) = \mathbb{C}P^\infty$.

Solution. Hopf fibration on S^∞? It's easier, take $n = 1$, it's just by definition. □

Definition. The ***fundamental bundle*** on $BU(1)^n$ has fiber

$$\ell_1 \oplus \ell_2 \oplus \ldots \ell_n$$

where $\ell_i \in BU(1)$ are product $\ell_1 \times \ell_2 \times \ldots \times \ell_n$.

Remark. Chern classes of B_{fun} are uniquely determined by axioms, because every factor has Chern classes, and fibers are just sums, and pullbacks preserve sums...

$$B_{\text{fun}} = \bigoplus_i \pi_i \mathcal{O}(1)$$

where

$$pi_i : BU(1)^n \to BU(1)$$

is a projection.

Remark. $H^\bullet(BU(1))^n = \mathbb{Z}[z_1, \ldots, zn$ Here at least I remember that the cohomology of $\mathbb{C}P^\infty$ is just polynomials so it looks reasonable that the n-th power is polynomials in more cariables.

Theorem (Splitting principle). Let $\varphi_{\text{fun}} : BU(1)^n \to BU$, the ***fundamental map***, it induces embedding on cohomology up to degree $2n$. For all primer generator $\sigma_i \in H^2(BU)$, $\varphi_{\text{fun}}(\sigma_1) = \lambda \sum_i z_i^k$ with $\lambda \neq 0$.

So

$$\begin{tikzcd}
BU(1)^n \arrow[r] & BU \\
X \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Remark. Wiki Thus, the set of isomorphism classes of circle bundles over a manifold M are in one-to-one correspondence with the homotopy classes of maps from M to $\mathbb{C}P^\infty$

Theorem. Chern classes are unique (uniquely determined by axioms).

Proof.

Step 1 Every bundle is obtained as pullback of the fundamental bundle. So for $A \in H^\bullet(BU)$ and B bundle on X, $A(B) = \varphi_B^*(A) \subset H^\bullet(X)$ so $c_i(B)$ are obtained as pullbacks of c in the fundamental bundle.

Step 2

$$BU(1)^\infty \xrightarrow{\varphi_{\text{fun}}} BU$$

pullback of fundamental bundle is fundamental. (This map is defined from the former by induction).

$$\varphi_{\text{fun}}^*(c_i(B_{\text{fun}}) = c_i(B_{\text{fun}} \text{ on } BU)$$

The Chern classes of the fundamental bundle are already known. Since φ_{fun}^* is injective by the splitting principle we are done.

□

4.3 Primitive generators of $H^*(BU)$

Recall the H-space multiplication:

$$\begin{aligned} BU \times BU &\longrightarrow BU \\ L_1 \times L_2 &\longmapsto L_1 \oplus L_2 \end{aligned}$$

and the comultiplication

$$\Delta : H^\bullet(BU) \to H^\bullet(BU)$$

Generators of $H^\bullet(BU)$ are $c_{h_1}, c_{h_2}, \ldots$ with $c_{h_i} \in H^{2i}(BU)$ and we have the comultiplication $\Delta(c_{h_i}) = c_{h_i} \otimes 1 + 1 \otimes c_{h_i}$.

Remark.

$$\varphi = (\varphi_1, \varphi_2) : X \to BU \times BU$$

and we can compose so we have

$$\varphi \circ \mu : X \to BU$$

what does this map do?

$$\begin{aligned} \varphi \circ \mu : X &\longrightarrow BU \\ \varphi^*(B_{\text{fun}} &\longmapsto B_1 \\ (\varphi \circ \mu)^*(B_{\text{fun}}) &= B_1 \oplus B_2 \end{aligned}$$

So then we have

$$\begin{aligned} \varphi^* : H^\bullet(BU) \otimes H^\bullet(BU) &\to H^\bullet(X) \\ \Delta : H^\bullet(BU) &\to H^\bullet(BU) \otimes H^\bullet(BU) \\ \Delta \circ \varphi^* : H^\bullet(BU) &\to H^\bullet(X) \end{aligned}$$

Corollary. For every $x \in H^\bullet(BU)$

$$X(B_1 \oplus B_2) = \Delta(x)(B_1, B_2)$$

Corollary. If $x \in H^*(BU)$ is primitive, then $x(B_1 \oplus B_2) = x(B_1) \oplus X(B_2)$.

Proof. $\Delta(x) = x \otimes 1 + 1 \otimes x$ so $\Delta(x)$ evaluated on (B_1, B_2) □

Remark. We will construct the full Chern class $c_*(B)$ as a pullback of a class $C \in H^*(BU)$.

Remark. Then take exponential. Let $\chi_i \in H^{2i}(BU)$ be a primitive generator. Use Hopf theorem to see that it is unique by a constant. Snce $\chi_i(B_1 \oplus B_2) = \chi_i(B_1) + \chi_i(B_2)$, the class $C = e^{\sum_i a_i \chi_i} = 1 + \ldots + \frac{\chi_n}{n!} + \ldots$ satisfies the Whitney formula.

To construct Chern classes satisfying the axioms it remains to arrange the coefficients a_i in such a way that $C(\mathcal{O}(1)) = 1 + [H]$ I think this means hyperplane section.

Lemma. An embedding

$$BU(1) \overset{\varphi}{\hookrightarrow} BU$$

with $\chi_i \in H^{2i}(BU)$ primitive generator. Then $\varphi^*(\chi_i) \neq 0$

Proof. $H^\bullet(BU) =$ symmetric polynomials in $H^i(BU(1))^n$, $\varphi_{\text{fun}}(x_N) = x \sum_{i=1}^n z_i^k$ so $\varphi(x_k) = \lambda x_1^k$. ☐

Remark. $\varphi^*(c_i(B_{\text{fun}}) = c_i(\mathcal{O}(1) = 1 + [H]$

Theorem. Choose generators $\chi_i \in H^2(BU)$ primitive. Then $\varphi^*(\sum_i \chi_i = \log(1 + [H])$ where the logarith is a formal power series, namely $\sum_{i=1}^\infty \frac{H^n}{n!}(-1)^n$.

That means $c(B_{\text{fun}}) = \exp\left(\sum_{\chi_i}\right)$.

5 Class 5

We want to study the space of line bundles on a surface.

5.1 K-group

Definition. Let V be the set of equivalence classes of vector bundles on X. Consider the free module generated by V (it's just V copies of Z):

$$\mathbb{Z}\langle V\rangle = \bigoplus_V \mathbb{Z}$$

And now consider

$$\frac{\mathbb{Z}\langle V\rangle}{[F_1] - [F_1] - [F_3]}$$

for all exact sequences

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

Equivalently, the relation is $[F_1] + [F_3] = [F_2]$.

Remark. We may give an H-structure to the set of homotopy classes of maps $X \to BU$ as follows $\varphi_1, \varphi_2 : X \to B\,U$

$$B_1 = \varphi^*(B_{\text{fun}})$$

define the H-product

$$\varphi := \varphi_1 \circ \varphi_2$$

such that

$$\varphi^*(B_{\text{fun}} = B_1 \oplus B_2$$

And then we have an isomorphism (that we are not going to use):

$$K(X) \xrightarrow{\text{hom}} \text{group of homotopy classes of maps from } X \text{ to } BU$$

This is because every bundle on X is the pullback of the fundamental bundle by some map. We need to check that the image of trivial bundle is trivial map (homotopic to constant?) and that it preserves the product.

Remark. The important thing of today is that that sum corresponds to addition

Remark. I guess I should first understand how is it that every bundle is the pullback of the fundamental bundle.

So for example for injectivity we need to show that if a map φ pulls back the fundamental bundle to the trivial bundle then φ is homotopic to identity. This is not obvious though.

The point is that that map is a bijection.

Claim. Chern classes are defined on $K(X)$ and satisfy Whitney formula (meaning Chern classes they pass to the quotient, right?)

Proof. Let B be a bundle on X so that $B = \varphi^*(B_{\text{fun}}$. We showed last time that there is a $c_\cdot \in H^0(BU)$ such that $c_\cdot(B) = \varphi^*(c_\cdot$. In fact we proved that $c_\cdot = \exp(\text{additive})$, but its actually Chern character, $c_\cdot = \exp(\text{Ch}_\cdot)$, in fact $\text{Ch}_\cdot(B_1 + B_2 \, \text{Ch}(B_1) + \text{Ch}_\cdot(B_2)$. □

5.2 Coherent sheaves

Definition. Let M be a complex manifold and $\mathcal{O}_M$ its structure sheaf (of holomorphic functions). A ***coherent sheaf*** is a sheaf of $\mathcal{O}_M$-modules, locally isomorphic to a quotient of a free sheaf $\mathcal{O}_M^n$ by a finitely generated $\mathcal{O}_M$-invariant subsheaf.

A ***coherent sheaf*** on a projective manifold. A ***projective manifold*** is $\text{Proj}(A^\bullet)$ where $A^\bullet$ is a graded ring. ***Coherent sheafes*** are sheaves of graded $A^\bullet$-modules.

Exercise. Let M be a projective manifold. Prove that any coherent sheaf F has a (projective) resolution

$$0 \longrightarrow B_n \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow F \longrightarrow 0$$

where B_i are vector bundles. This is called the ***syzygy resolution***

Solution. Every module has a projective resolution called ***Koszul resolution***. So what is Koszul resolution. First you have a resolution of a maximal ideal. For a maximal ideal it is clear since ... (Herieta? and) Eisenbud or even Bourbaki Homological algebra. □

5.3 Coherent sheaves and their Chern classes

So there's actually two K-groups. One is generated by bundles and the other by sheaves. For bundles, it is an algebra. For sheaves, it is a module over the other one. For Groethendick one was $K^\bullet$ and the other $K_\bullet$ but we don't know which is which.

Remark. After this is done, it's possible to prove that the K-group of coherent sheaves on a projective manifold is equal to the K-group generated by holomorphic vector bundles.

Definition. The ***Chern class*** of a coherent sheaf is the Chern class of the corresponding element in the K-group.

Remark (about singularities, see slides). Suppose we do resolution of a manifold and pullback a bundle

$$\begin{tikzcd} \tilde{M} \arrow[d, "\pi"] & \pi^* F \arrow[d] \\ M & F \end{tikzcd}$$

5.4 Euler characteristic of a coherent sheaf

Definition. Let F be a coherent sheaf. Its ***Euler characteristic*** is

$$\chi(F) = \sum_i (-1)^i \dim H^i(F)$$

But what is that cohomology? What is the space?

Claim. For any exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

we have

$$\chi(F_2) = \chi(F_1) + \chi(F_3)$$

Proof. Should be possible... □

Then

$$\chi : K(M) \to \mathbb{Z}$$

is a homomorphism.

5.5 Chern character

OK so last class we defined an homomorphism called χ that was additive. Now let's call it

$$c. = \exp(Ch.)$$

and it was additive

$$Ch.(B_1 \oplus B_2) = Ch.(B_1) + Ch.(B_2)$$

So the textbook definition is that ***Chern character*** on line bundles is

$$\exp(c_\bullet(L))$$

So c_1 is additive and if you pass to the exponent it will be multiplicative:

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$
$$\mathrm{Ch.}(L_2 \otimes L_2) = \mathrm{Ch.}(L_1) \cdot \mathrm{Ch.}(L_2)$$

5.6 Riemann-Roch-Hirzebruch theorem

Theorem (RRH). Let F be a coherent sheaf on a complex compact manifold M. Then $\chi(F)$ can be expressed through Chern classes of F and M as follows:

$$\chi(F) = \int_X \mathrm{Ch.}(F) \wedge \mathrm{Td.}(TM),$$

where $\mathrm{Td.}(TM)$ mdenotes the ***total Todd class of the tangent bundle*** TM, which is a sum of Chern classes.

$$\mathrm{Td.} = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_1}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4}{720} + \dots$$

5.7 K-group for complex curves

Lemma. K-group for complex curves is generated by line bundles.

Proof.

Step 1 For each F coherent sheaf, $L^n \otimes F$ has a section. So there is a monomorphism $L^{-N} \hookrightarrow F$.

Step 2 The consider the localization to produce a short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

since $F = \bigoplus_i F_i$ for $F_i = \mathcal{O}_M / \mathfrak{m}_X^{a_i}$ so

$$0 \longrightarrow (\mathfrak{m}_X^{a_1} \longrightarrow \mathcal{O}_X \longrightarrow F_1 \longrightarrow 0$$

□

5.8 Riemann-Roch for complex curves

Theorem (Riemann-Roch for complex curves). Let F be a coherent sheaf on a compact complex curve of genus g. Then

$$\chi(F) = c_1(F) + \mathrm{rk}(F)(1 - g)$$

Proof. We want to see

$$c_1(L) = \deg(L)$$

Step 1 It suffices to prove for line bundles by the lemma.

Step 2 For degree 0 its easy beacuse $c_1(k_x) = 1$. For structure sheaf $\mathcal{O}_X$ we have rank is 1.

Step 3 Now let L be a line bundle. We have

$$0 \longrightarrow \mathcal{O}_M \longrightarrow L \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow L \longrightarrow \mathcal{O}_M \longrightarrow F \otimes L = F \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_M \longrightarrow L_1^N \otimes L \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow L_1^{-N} \longrightarrow L \longrightarrow F \longrightarrow 0$$

and the point is that many things "have sections". What does it mean to have sections.

□

5.9 Riemann-Roch-Hirzebruch for line bundles on complex surfaces

Definition. A ***complex surface*** is a compact complex manifold of dimension 2.

Notation

$$(L_1, L_2) = c_1(L_1) \wedge c_1(L_2)$$

and if D is a divisor we write

$$(D, L) = \deg_D L = \int_M [D] \wedge c_1(L)$$

Theorem (RRH for surfaces). L line bundle on surface and $K_X = \Omega^2(X)$ its canonical bundle. Then

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{(L - K_X, L)}{2}$$

where (A, B) denotes the intersection form applied to cohomology classes on X.

Proof.

Step 1 Let D a smooth curve of genus g and L_1, L_2 line bundles that fit in an exact sequence

$$0 \longrightarrow L_2 \longrightarrow L_2 \longrightarrow L_2|_D \longrightarrow 0$$

Then we use Rieman-Roch for curves gives

$$\chi(L_1) = \chi(L_2) + (L_2, D) + (1 - g)$$

Step 2 Let ND denote the normal bundle on D. The adjunction formula gives $K_D = K_X|_D \otimes KD$. Since $g - 1 = \deg K_D/2$, we obtain $1 - g = -(K_X + D, D)/2$.

Step 3 The next step goes as before, with Rieman-Roch in one dimension. Let $\chi'(L)$ be the RHS of **??** , namely $\chi'(L) = \chi(\mathcal{O}_X) + \frac{L-K_X, L)}{2}$. In step 1 we have $c_1(L_2) = c_1(L_1) + D$. Then

$$\begin{aligned}\chi'(L_2) - \chi'(L_1) &= \frac{1}{2}\left[(L_2 - K_X, L_2) - (L_2 - K_X - D, L_2 - D)\right] \\ &= (L_2, D) - (K_X + D, D)/2\end{aligned}$$

Step 4 Comparing Step 2 and Step 3, we get

$$\chi'(L_2) - \chi'(L_1) = \chi(L_2) - \chi(L_1)$$

Therefore, **??** is equivalent for L_2 and for L_1. We just need to manipulate bundles to reduce a bundle to... by building exact sequences.

Step 5 So suppose you have a smooth section of a bundle. Take an ample bundle A and do

$$0 \longrightarrow \mathcal{O}_X \longrightarrow A^N \longrightarrow A^N|_D \longrightarrow 0$$

$$0 \longrightarrow L \longrightarrow A^N \otimes L \longrightarrow A^N \otimes L|_D \longrightarrow 0$$

and then by step 4 we just need to deal with $A^N \otimes L$.

Step 6 It's very ample, it has many sections, including some that are smooth. Now we just assume L is $A^N \otimes L$. So

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L|_D \longrightarrow 0$$

so for bundles that have smooth sections the statement is free.

□

5.10 Applying the general formula to the curve case

We have

$$\mathrm{Ch.}(L) = 1 + c_1(L) + \frac{c_1^2(L)}{2}$$

$$\mathrm{Td.}(L) = 1 + \frac{c_1(TM)}{2} + \frac{c_1^2(M) + c_2}{12}$$

Now

$$\chi(L) - \chi(\mathcal{O}) = -\frac{(K_1(L), K)}{2} + \frac{c_1(L)^2}{2} = \frac{(L, K - L)}{2}$$

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