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Home assignment 7: Moser isotopy lemma

Exercise 7.1 Let M be a compact manifold, and V_0, V_1 two smooth volume forms which satifsy $\int_M V_0 = \int_M V_1$. Prove that there exists a diffeomorphism which satisfies $\Phi^* V_0 = V_1$.

Solution. We must show there is a family V_t of cohomologous volume forms joining V_0 and V_1. Define

$$V_t = tV_0 + (1-t)V_1.$$

Now since $H^n(M)$ is a real one-dimensional vector space, $[V_t] = \alpha[V_0]$ for some real number α.

To show that α must be 1 we need to integrate $[V_t]$, which we may do since V_t is a closed nowhere-vanishing top-form. Closedness is immediate. To see it is nowhere-vanishing just notice that both V_0 and V_1 are, and since they integrate to the same number they are both always positive or always negative.

Then

$$\int_M [V_t] = t\int_M [V_0] + (1-t)\int_M [V_1] = \int_M [V_0] = \int_M [V_1]$$

which means that $\alpha = 1$. Then by Moser's lemma there exists an isotopy φ_t with $\varphi_t^* V_t = V_0$. Taking $t = 1$ we obtain the result. $\square$

Problem 7.2 Let (M, I) be an almost complex manifold, and ω_0, ω_1 co-homologous symplectic forms which satisfy $\omega_i(x, Ix) > 0$ for any non-zero tangent vetor x (such forms are called ***taming***).

a. Prove that there exists a diffeomorphism Φ which satisfies $\Phi^*\omega_0 = \omega_1$.

b. Prove that $|x|_i^2 := \omega_i(X, Ix)$ is a Hermitian metric on M. Prove that a diffeomorphism that satisfies $\Phi^*\omega_0 = \omega_1$ defines an isometry

$$(M, |x|_1^2) \longrightarrow (M, |x|_0^2)$$

is Φ is compatible with I, that is, satisfies $d\Phi(Ix) = I(d\Phi(x))$

c. Find an example of $\omega_0, \omega_1 \in \Lambda^2(M)$ such that a diffeomorphism compatible with I and satisfying $\Phi^*\omega_0 = \omega_1$ does not exist.

Solution.

a. This is immediate from Moser's lemma defining $\omega_t := t\omega_0 + (1+t)\omega_1$.

b. According to Riemann Surfaces course, a hermitian form may be understood as a Riemannian metric h such that $h(\cdot,\cdot) = h(I\cdot, I\cdot)$. This happens for $h(\cdot,\cdot) = \omega(\cdot, I\cdot)$ if we require $\omega(\cdot,\cdot) = \omega(I\cdot, I\cdot)$. Indeed, symmetry holds since

$$\omega(x, Iy) = \omega(Ix, -y) = \omega(y, Ix)$$

and positive-definiteness is given as a hypothesis.

The condition that $\omega(\cdot, I\cdot)$ is a riemannian metric is called ***compatibility*** of ω and I in Silva. In this case we can also produce a hermitian metric in the sense of WolframMathWorld, namely, a positive-definite symmetric sesquilinear form. Indeed, define $g(\cdot,\cdot) := \omega(\cdot, I\cdot)$, then the form

$$h := g + \sqrt{-1}\omega$$

satisfies the required properties as follows.

(a) Additivity, $h(u_1 + u_2, v) = h(u_1, v) + h(u_1, v)$, is immediate.

(b) Homegenity on the first argument, $h(\lambda u, v) = \lambda h(u, v)$, is also immediate.

(c) $h(u, v) = \overline{h(v, u)}$ is clear by anti-symmetry of Ω:

$$h(u, v) = g(u, v) + i\omega(u, v) = g(v, u) - i\omega(v, u) = \overline{h(v, u)}$$

(d) The property $h(u, \lambda v) = \bar{\lambda} h(u, v)$ follows easily from (b) and (c) since

$$h(u, \lambda v) = \overline{h(\lambda v, u)} = \bar{\lambda}\overline{h(v, u)} = \bar{\lambda} h(u, v)$$

(e) Positive-definiteness follows from positive-definiteness of g and antisymmetry of ω.

c. The conditions $\Phi^*\omega_0 = \omega_1$ and $\Phi_*(I\cdot) = I\Phi_*(\cdot)$ imply that

$$(\Phi^*\omega_0)(\cdot, I\cdot) = \omega_0(\Phi_*\cdot, \Phi_* I\cdot) = \omega_0\Phi_*\cdot, I\Phi_*\cdot) = \omega_1(\cdot, I\cdot) \tag{1}$$

Whether the metric is taken to be $\omega_i(\cdot, I\cdot)$ or $h_i = g_i + \sqrt{-1}\omega_i$, the result follows. In the latter case only note that

$$\begin{aligned}\Phi^* h_0 &= \Phi^*(g_0 + \sqrt{-1}\omega_0) = \Phi^* g_0 + \sqrt{-1}\Phi^*\omega_0 \\ &= \Phi^*\Big(\omega(\cdot, I\cdot)\Big) = g_1 + \sqrt{-1}\omega_1 = h_1\end{aligned}$$

by eq. (1).

d. I am intrigued to know the answer here.

□

Exercise 7.4 Let (M, V) be a connected manifold equipped with a volume form. Prove that the group of volume-preserving diffeomorphisms acts on M transitively.

Solution. I will sketch the proof of Boothby, found in MathOverflow.

We wish to define a vector field along γ whose flow Φ_t gives the desired volume preserving diffeomorphism at $t = 1$. Consider a diffeomorphism from $I \times B^{n-1}$ to a neighbourhood U of γ (this diffeomorphism is constructed from geodesic segments introducing a metric on M). The volume form on U may be expressed as

$$V = f dt \wedge dx_1 \wedge \ldots \wedge dx_{n-1}$$

for a positive function f. Now consider the vector field

$$X' = \frac{1}{f}\frac{\partial}{\partial t}.$$

X' may be extended to a vector field X in all of M using a partition of unity. It is immediate that the flow of this vector field takes p to q.

To show this flow is volume-preserving recall Cartan's formula that $\mathcal{L}_X V = di_X V + \cancelto{0}{i_X dV} = di_X V$ since V is a top-form. Thus, showing volume invariance ammounts to showing $d_X V = 0$, which holds since

$$i_{X'} f dt \wedge dx_1 \wedge \ldots \wedge dx_{n-1} = dx_1 \wedge \ldots \wedge dx_{n-1}$$

is closed. □

Exercise 7.5 Prove that the group of symplectomorphisms of (M, ω) acts transitively on M, for any connected symplectic manifold (M, ω).

Solution. This case is similar but requires a little more work: we intend to use Darboux coordinate charts, which can only be defined locally. We will split a path joining any two points into small pieces, find the symplectomorphisms locally and the compose them.

1. ***Strong local transitivity*** on M means that for every point $p \in M$ and neighbourhood U of p there are neighbourhoods V and W of p with $\overline{V} \subset W$ and $\overline{W} \subset U$, $\overline{W}$ compact, and for any $q \in V$ there is a symplectomorphism isotopy Φ_t of the identity map Φ_0 to Φ_1 such that $\Phi_1(p) = q$ and for all t, Φ_t leaves fixed every point outside $\overline{W}$.

2. If M is strongly locally transitive then it is transitive by symplectomorphisms. *Sketch of proof:* let γ be a path from p to q. For every $x \in \gamma$ choose $0 < \delta'_x < \delta''_x < \varepsilon$ such that the δ' and δ'' neighbourhoods of x satisfy the conditions of strong local transitivity. Since γ_1 is compact, there is a finite collection of points $p_1 = a_1, a_2, \ldots, a_r = q_1$ with volume-preserving diffeomorphisms mapping each point to the next. The composition of all such diffeomorphisms yields the desired volume-preserving map.

3. It remains to show that any connected symplectic manifold is strongly locally transitive. Let U be a Darboux chart of p and take two neighbourhoods V' and W' as in the definition of strong local transitivity. For any $y \in V'$ we can define the vector

field X in $\mathbb{R}^{2n}$ of vectors parallel to yx_0, where x_0 are the coordinates of p. The flow of this vector field takes takes x_0 to y at $t = 1$.

To show that this vector field is ω-invariant we use again Cartan's formula to see it's enough to show that $di_X\omega = 0$. Now if $\dim M = 2m$ we have

$$i_X\omega = \sum_{i=1}^{m} dx^i \wedge dx^{i+m}(X, \cdot) = 0$$

(I'm not sure why does it vanish...)

$\square$

Extra exercise: infinite product

Exercise 8 (Zorich and Cooke, 3.2.5, p. 148) An inifinite product $\prod_{k=1}^{\infty} e_k$ is said to converge if the sequence $\Pi_n = \prod_{k=1}^{n} e_k$ has a finite **nonzero** limit Π. We then set $\Pi = \prod_{k=1}^{\infty} e_k$. Show that

(b) if $\forall n \in \mathbb{N}(e_n > 0)$, then the infinite product converges $\prod_{n=1}^{\infty}$ converges if and only if the series $\sum_{n=1}^{\infty} \log e_n$ converges.

Remark The proof is straightforward using elementary properties of the logarithm ($\log(ab) = \log a + \log b$) and the exponential ($\exp(a + b) = \exp a + \exp b$), as well as their continuity. The caveat is that if the product converges to zero, taking logarithms of the partial products gives a series diverging to $-\infty$ (we say that the product ***diverges to zero***). The following exercise is an example of this.

Claim

$$\prod_{n\geqslant 0} \left| -1 + \frac{1}{n + \frac{1}{2}} \right| = 0.$$

Proof. By taking exponent of the partial sums of the following series, we see it's enough to show that

$$\sum_{n\geqslant 0} \log \left| -1 + \frac{1}{n + \frac{1}{2}} \right| = -\infty. \tag{2}$$

To show this first notice that $\left| -1 + \frac{1}{n+1/2} \right|$ is just $1 - \frac{1}{n+1/2}$. We can also quickly notice that $\log\left(1 - \frac{1}{n+1/2}\right)$ is a sequence of negative numbers convering to 0.

(It was not immediate to me why the this series should diverge or converge. The following argument was provided by ChatGPT.)

We can prove the series diverges if we find a divergent series that bounds it from above. Such series is $\sum \frac{-1}{n+1/2}$. Indeed, it turns out that for every number $0 \leqslant x < 1$,

$$\log(1-x) \leqslant -x.$$

To see why, define the function $f(x) = \log(1-x) + x$ and differentiate to find

$$\frac{d}{dx}\log(1-x) + x = -\frac{1}{1-x} + 1 = \frac{1}{1-x} + \frac{1-x}{1-x} = \frac{-x}{1-x} \leqslant 0,$$

and since also $f(0) = 0$, we have that

$$\log(1-x) + x \leqslant 0, \qquad 0 \leqslant x < 1.$$

Finally just recall that $\sum_{n \geqslant 0} \frac{1}{n}$ diverges (and behaves like $\sum \frac{1}{n+1/2}$ for lage n). This confirms eq. (2).

□

References

Boothby, William M. "Transitivity of the Automorphisms of Certain Geometric Structures". In: *Transactions of the American Mathematical Society* 137 (1969), pp. 93–100. ISSN: 00029947. URL: http://www.jstor.org/stable/1994789 (visited on 11/11/2024).

Silva, A.C. da. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics no. 1764. Springer, 2001. ISBN: 9783540421955.

Zorich, V.A. and R. Cooke. *Mathematical Analysis I*. Mathematical Analysis. Springer, 2004. ISBN: 9783540403869.