

# Home assignment 2: spectral sequences

## The monodromy of Gauss-Manin local system

**Definition 2.1.** Let  $\pi : E \rightarrow B$  be a locally trivial fibration with fiber  $F$ . The family of cohomology of fibers of  $\pi$  is locally trivial, (what does this mean precisely?) but it might have *the monodromy*. In other words, the group  $\pi_1(B)$  naturally acts on the algebra  $H^*(F)$  by automorphisms. To obtain this action, take a loop in  $B$  and trivialize the family  $\pi$  along small intervals of this loop; this gives an identification of  $H^*(F)$  with itself, which might be non-trivial.

**Remark (Understanding the monodromy action of cohomology).** (From [StackExchange](#))  
Let  $f : X \rightarrow U$  be a proper surjective submersion and fix  $u_0 \in U$ .

For any path  $\gamma \subset U$ , there is a canonical diffeomorphism  $\phi_\gamma : f^{-1}(\gamma(0)) \rightarrow f^{-1}(\gamma(1))$ , using  $\psi_j$  (by a theorem of Ehresmann, all the fibers of  $f$  are diffeomorphic).

Now, for any loop  $\gamma$ , split  $\gamma$  into paths  $\gamma_i \subset U_i$  and you can compose these diffeomorphisms to get a diffeomorphism

$$\phi_{\gamma_n} \circ \dots \circ \phi_{\gamma_1} : f^{-1}(u_0) \rightarrow f^{-1}(\gamma(u_0))$$

It induces a map on homology: you can check that it is well defined up to homotopy.

**Exercise 2.1.** Let  $\phi^* : \mathbb{Z} \rightarrow \text{Aut}(H^*(F))$  be an automorphism induced by a homeomorphism  $\phi : F \rightarrow F$ . Construct a locally trivial family over a circle with monodromy in cohomology induced by  $\phi^*$ .

**Interpretation** Given an action  $\phi^* : \mathbb{Z} = \pi_1(S^1) \rightarrow \text{Aut}(H^*(F))$ , construct a fibre bundle such that  $\phi^*$  is the monodromy action on cohomology.

*Proof.* Consider the standard torus fibration  $T^2 \rightarrow S^1$ . Any path in the circle can be thought of as an number  $n \in \mathbb{Z}$ . Perhaps the induced automorphism on cohomology is precisely the map  $\mathbb{Z} \ni a \mapsto na \in \mathbb{Z}$ . But I'm not looking for an automorphism of  $\mathbb{Z}$ ... I need an automorphism of  $H^*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ ...

□

## Leray-Serre spectral sequence

**Exercise 2.4.** Let  $\pi : E \rightarrow B$  be a fibration with the fiber a torus. Assume the  $d_2 = 0$ . Prove that all differentials vanish.

*Proof.* Since  $d_2 = 0$ , we have that  $E_3^{p,q} = E_2^{p,q}$ . Then

$$d_3 : H^p(B) \otimes H^q(T) \longrightarrow H^{p+3}(B) \otimes H^{q-2}(T),$$

so the only way it could be non-zero is for  $q = 2$ , which implies that

$$H^p(B) \otimes H^2(T) \cong H^{p+3}(B) \otimes H^0(T) \iff H^p(B) \cong H^{p+3}(B)$$

But I don't see why this couldn't happen...

**Exercise 2.5.** Let  $\pi : E \rightarrow B$  be a fibration with the fiber a torus. Assume that the pullback map  $\pi^* : H^2(B) \rightarrow H^2(E)$  is injective. Prove that all differentials  $d_i$  vanish.

*Solution.* I'm not sure how to use the hypothesis since I usually deal with the total space after computing the  $E_\infty$  page via the filtration. . .  $\square$

**Exercise 2.8.** Let  $F = S^k$ , that is,  $\pi: E \rightarrow B$  is a sphere bundle. Prove that all differentials  $d_{k+1}$  vanish. Construct the *Gysin exact sequence*

$$\cdots \rightarrow H^p(B) \rightarrow H^{p+k+1}(B) \xrightarrow{\pi^*} H^{p+k+1}(E) \rightarrow H^{p+1}(B) \rightarrow \cdots$$

*Solution.* (This argument is adapted from the construction of Wang exact sequence found in [Wikipedia](#)). We have that  $E_2^{p,q} = H^p(B) \otimes H^q(S^k)$  can only be non-zero for  $q = 0, k$ . This means that the only non-zero differentials are of the form

$$d_{k+1} : E_2^{p,k} \cong H^p(B) \longrightarrow E_2^{p+k+1,0} \cong H^{p+k+1}(B)$$

$$\begin{array}{ccccccc}
H^0(B) \otimes H^k(S^k) & H^1(B) \otimes H^k(S^k) & & \cdots & & H^k(B) \otimes H^k(S^k) & \\
\vdots & & \searrow d_{k+1} & & \searrow d_{k+1} & & \vdots \\
H^0(B) \otimes H^0(S^k) & \cdots & & H^{k+1}(B) \otimes H^0(S^k) & & H^{k+2}(B) \otimes H^0(S^k) & 
\end{array}$$

which means that  $E^{k+1} = E^\infty$ . Since  $E^{k+1} = \ker d_{k+1} / \text{img } d_{k+1}$ , we can write

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker d_{k+1} & \longrightarrow & H^p(B) & \xrightarrow{d_{k+1}} & H^{p+k+1}(B) \longrightarrow \operatorname{coker} d_{k+1} \longrightarrow 0 \\
& & \parallel & & & & \parallel \\
& & \frac{\ker d_{k+1}}{\operatorname{img} d_{k+1}} & & & & \frac{E^{p+k+1,0}}{\operatorname{img} d_{k+1}} \\
& & \parallel & & & & \parallel \\
& & E_{k+1}^{p,k} & & & & E_{k+1}^{p+k+1,0} \\
& & \parallel & & & & \parallel \\
& & E_{\infty}^{p,k} & & & & E_{\infty}^{p+k+1,0}
\end{array}$$

This is the “first half” of the Gysin sequence. For the other half we must compute the  $E_\infty$  terms. We use the filtration

$$H^n(E) = F^0 H^n \supset F^1 H^n \supset \dots \supset F^n H^n$$

that we know to satisfy

$$E_\infty^{p,q} \cong \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}.$$

We may write (I’m not completely sure why this works)

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_\infty^{p+k+1,0} & \longrightarrow & H^{p+k+1}(E) & \longrightarrow & E_\infty^{p+1,k} \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \frac{F^{p+k+1} H^{p+k+1}}{F^{p+k+2} H^{p+k+1}} & & & & \frac{F^{p+1} H^{p+k+1}}{F^{p+2} H^{p+k+1}} \end{array}$$

Putting this together with the first sequence we computed, we get that

$$\longrightarrow E_\infty^{p,k} \longrightarrow H^p(B) \xrightarrow{d_{k+1}} H^{p+k+1}(B) \longrightarrow E_\infty^{p+k+1,0} \longrightarrow H^{p+k+1}(E) \longrightarrow E_\infty^{p+1,k} \longrightarrow$$

and we simply remove the  $E_\infty$  terms to get the Gysin sequence

$$\longrightarrow H^p(B) \xrightarrow{d_{k+1}} H^{p+k+1}(B) \longrightarrow H^{p+k+1}(E) \longrightarrow H^{p+1}(B) \longrightarrow$$

**Remark.** I still cannot see why the map  $H^{p+k+1}(B) \rightarrow H^{p+k+1}(E)$  is the map induced by the projection.

□

**Exercise 2.11.** Let  $\pi : E \rightarrow B$  be a fibration with  $B = S^k$ . Prove that all differentials except  $d_k$  vanish. Construct an exact sequence

$$\dots \longrightarrow H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F) \xrightarrow{\mu} H^{p+k}(E) \longrightarrow H^{p+k+1}(F) \longrightarrow \dots$$

where  $\mu$  is multiplication by  $\pi^* \text{Vol}_{S^k}$  and  $\tilde{d}_k$  is equal to  $d_k$  after the identification  $H^p(F) = H^k(S^k) \otimes H^p(F) = E_2^{k,p}$

*Solution.* Like in Exercise 2.8 we see that the only non-zero differentials are

$$d_k : H^0(S^k) \otimes H^{k+p} \longrightarrow H^k(S^k) \otimes H^{p+1}(F)$$

since  $E_2 = E_k$  is of the form

$$\begin{array}{ccccc}
 H^0(S^k) \otimes H^{k+1}(F) & \cdots & & H^k(S^k) \otimes H^{k+1}(F) & \\
 & \searrow & & & \\
 H^0(S^k) \otimes H^k(F) & \cdots & d_k & H^k(S^k) \otimes H^k(F) & \\
 & \searrow & & & \\
 \vdots & & d_k & & \vdots \\
 H^0(S^k) \otimes H^2(F) & \cdots & & H^k(S^k) \otimes H^2(F) & \\
 H^0(S^k) \otimes H^1(F) & \cdots & & H^k(S^k) \otimes H^1(F) & \\
 H^0(S^k) \otimes H^0(F) & \cdots & & H^k(S^k) \otimes H^0(F) & 
 \end{array}$$

Again like in Exercise 2.8 we obtain a sequence

$$0 \longrightarrow E_{\infty}^{0,q} \longrightarrow H^q(F) \xrightarrow{d_k} H^{q-k+1}(F) \longrightarrow E_{\infty}^{k,q-k+1} \longrightarrow 0$$

**Remark.** The exercise has the map  $H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F)$ , but my computations suggest it should be  $H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^{p+1}(F)$ .

Then we compute the  $E_{\infty}$  terms using a filtration

$$H^n(E) = F^0 H^n \supset F^1 H^n \supset \cdots \supset F^n H^n, \quad E_{\infty}^{p,q} = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}$$

which yields

$$0 \longrightarrow E_{\infty}^{k-1,q-k+1} \longrightarrow H^q(E) \longrightarrow E_{\infty}^{0,q} \longrightarrow 0$$

and then we get

$$\cdots \longrightarrow H^q(E) \longrightarrow H^q(F) \longrightarrow H^{q-k+1}(F) \longrightarrow H^{q+1}(E) \longrightarrow H^{q+1}(F) \longrightarrow \cdots$$

**Remark.** As in Exercise 2.8, I don't know why the map  $H^{q-k+1}(F) \rightarrow H^{q+1}(E)$  should be multiplication by the volume form of  $S^k$ .

□

**Exercise last.** Generators here (horizontal), generators there (vertical, 1,3,5), "Extend generators by Leibniz rule, and then they just kill everything"