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Lecture notes on K3 surfaces

Contents

1 Class 1

The most important invariant of a k3 surface is intersection form.

There are three classes of manifolds

1. Smooth manifolds

$$\text{smooth manifolds} \xrightarrow{\text{forgetful functor}} \text{PL manifold} \longrightarrow \text{Topological manifolds}$$

Donaldson: contiually many non-equivalent smooth structures on $\mathbb{R}^4$. K3 surfaces has countably many smooth structures and only one of them is compatible with complex structure.

Definition Intersection form. Given a quadratic form on a lattice $V_{\mathbb{Z}} = \mathbb{Z}^n$, so

$$q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$$

is ***unimodular*** if

$$V_{\mathbb{Z}} \xrightarrow{q} \mathsf{Hom}(V_{\mathbb{Z}}, \mathbb{Z})$$

is an isomorphism.

Theorem (Universal coefficients formula)

$$H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}^{b_{n-1}(M)} \oplus T_{n-1}(M)$$

$$h^n(M, \mathbb{Z}) = \mathbb{Z}^{b_n(M)} \oplus T_{n-1}(M)$$

Corollary $H^2(X, \mathbb{Z})$ is torsion free if $\pi_1(X) = 0$ because

Definition ***Signature*** is $m - n$ if q has signature (m, n).

Theorem (Rokhlm-Wu?) Signature is divisible by 16 for simply-connected (something else).

Remark The methods used in surgery break down in smooth case because strange topological objects like infinite sums of spheres arise.

Theorem (Freedman, 1982) There are as many 4-manifolds as there are intersection forms. M simply connected 4 manifold homotopy class is uniquely determined by intersection dorm. Moreover, for every unimodular form there exists a unique M with this intersection form.

Theorem (Donaldson, 1986) M smooth compact manifold with positive definite odd intersection form q. Then

$$\begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition Bilinear symmetric form is ***indefinite*** if it is not positive definite nor negative definite.

Theorem (Classification of unimodular symmetric bilinear forms) Odd are diagonalizable, while even are related to special Lie group E_8.

Definition A ***K3 surface*** is a Kähler complex surface M with $b_1 = 0$ (simply connected) and $c_1(M, \mathbb{Z}) = 0$.

Kodaira did what André Weil couldn'g classify.

Theorem K3 surfaces have trivial canonical bundle $K_M = \Lambda^2(\Omega^1 M)$.

2 Class 2

G topological group. ***Principal*** G ***bundle*** is a space with free G-action such that the quotient E/G is Housdorff. There are several conditions that make this work. And then you have $\mathsf{Homotopy}(X, BG)$ =equivalence classes of G-bundles. Vector bundles of a manifold are the same as maps from X to $B\,\mathsf{U}(n)$.

Vector bundles up to stable equivalence are classified basically by Chern classes, so by the cohomology in $H^\bullet(B\,\mathsf{U}) = Q[c_1, c_2, \ldots, c_n$.

Now look at the loop space of X. Then $H^\bullet(\Omega X)$ is a free graded commutative algebra. Loop space has the interesting property that $\Omega\,\mathsf{U} = B\,\mathsf{U}$ and $\Omega B\,\mathsf{U} = \mathsf{U}$.

2.1 Bialgebras

Let A be a superalgebra (graded with antisymmetric product). Then we ask the axiom of coassociativity and that .

Example G group, and $C(G)$ the ring of k-valued functions $C(G \times G) = C(G) \times C(G)$ so

$$G \times G \longrightarrow G$$
$$C(G) \longmapsto C(G) \otimes C(G)$$

2.2 H-spaces

Definition H-space is a space M with a map $\mu : M \times M$ toMthat is homotopy associative,

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mu \times \mathrm{id}} & M \times M \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \mu} & & \Big\downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow{\mu} & M \end{array}$$

which is homotopy commutative. And with homotopy unit.

So it's like a homotopy algebra?

Example The loop space.

2.3 Bialgebras of finite type

Definition A bialgebra A is of ***finite type*** if it is the direct sum of $A = \bigoplus_{i \geqslant 0} A^i$ supercommutative and each A^1 is finite dimensional.

Remark Free commutative algebra is polynomial algebra

Definition $A = \mathbb{C}[x_1, \ldots, x_n, \ldots] \otimes \Lambda^\bullet(a_1, \ldots, a_n, \ldots)$ is a graded commutative free algebra. In the slides: it is $\mathsf{Sym}_{\mathsf{gr}} V^*$ where V^* is a graded vector space.

Theorem (Hopf) A graded commutative bialgebra of finite type over k of 0 characteristic is free graded commutative as a k algebra.

2.4 The cohomology algebra of U(n)

Claim The cohomology algebra $H^*(\mathsf{U}(n), \mathbb{Q})$ is a free graded commutative algebra with generators in degrees $1, 3, 5, \ldots, 2n-1$.

Demostração. Induction. $\mathsf{U}(1)$ is clear because it is a circle. Then do Serre spectral sequence. Differentials vanish on the second page because there's only nonzero groups on even degrees! And we get that $E_2^{p_1} = H^p(S^{2n-1}) \otimes H^q(\mathsf{U}(n-1))$. And then the sequence converges to that of the total space which is $\mathsf{U}(n)$. □

2.5 Grassman manifolds

Definition The ***fundamental bundle*** B_{fun} is a rank n vector bundle over $\mathsf{Gr}(n, m)$.

Claim B, B' vector bundles of rank n, $m-n$, $B \oplus B'$

$$\varphi : X \to \mathsf{Gr}(m,n)$$

$$\varphi(x) = B_x \subset B_x \oplus B'_x = \mathbb{K}^m$$

then $B = \varphi^* B_{\mathsf{fun}}$.

Theorem If you have B as a bundle on a manifold X then $B \oplus B'$ is trivial for some bundle B'.

Demostração. Embed the total space in a large enough euclidean space. □

Definition $\mathsf{Gr}(n,\infty) = \mathsf{Gr}(n)$ is $\bigcup_{m=n_1}^{\infty} \mathsf{Gr}(n,m) = \mathsf{Gr}(n)$

Corollary For every bundle B of rank n there is a function $\varphi : X \to \mathsf{Gr}(n)$ such that $B = \varphi^* B_{\mathsf{fun}}$.

Take a bundle $E \to X$ and G acts freely on E so E principal G bundle. Classifying space BG

Theorem (Atiyah-Bott) Classifying space is unique up to homotopy equivalence.

2.5.1 The fundamental bundle

In class 4 I finally understood that

Definition The ***fundamental bundle*** on the Grassmanian $\mathsf{Gr}(n)$ (the Grassmanian is this space where points are linear spaces) is the vector bundle such that the fiber of one point (which is a vector space) is the vector space that is the point. It's very tautological.

Theorem (Did we prove this?) Let B be a vector bundle of rank n on a cellular space X. Then there exists a continuous map $\varphi : X \to \mathsf{Gr}(n)$ such that B is isomorphic to the pullback $\varphi^* B_{\mathsf{fun}}$ of the fundamental bundle.

Remark In fact $\mathsf{Gr}(n)$ is the classifying space of vector bundles of rank n, in the sense that isomorphism classes of vector bundles of maps $\varphi : X \to \mathsf{Gr}(n)$.

2.5.2 The canonical bundle

When doing homework 3 I found this very nice on Hatcher, Vector bundles and K3:

Definition The ***canonical bundle*** $p : E \to \mathbb{R}P^n$ has as its total space E the subspace of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ consisting of pairs $(\ell, v$ with $v \in \ell$ and $p(\ell, v) = \ell$. There is also an infinite-dimensional projective space $\mathbb{R}P^\infty$ which is the union of the finite-dimensional projective spaces $\mathbb{R}P^n$ under the inclusions $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ coming from natural inclusions $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$. The inclusions $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ induce corresponding inclusions of canonical line bundles, and the union of all these is a canonical line bundle over $\mathbb{R}P^\infty$.

A natural generalization is the Grassmanian $\mathcal{G}r(k,n)$ along with a canonical k-dimensional vector bundle over it consisting of pairs (ℓ, v) where ℓ is a poin the Grassmanian and v is a vector in ℓ.

2.5.3 What this classification space should mean

Remember that

Definition (Representable functor) Let $\mathcal{C}$ be a category. A functor $F : \mathcal{C}^{op} \to \mathsf{Sets}$ is called ***representable*** if there exists an object $B = B_F$ in $\mathcal{C}$ with the property that there is a *natural* isomorphism of functors

$$\varphi : \mathcal{C}(-, B_F) \to F$$

where $\mathcal{C}(-, B_F)$ is the set of arrows from $-$ to B_F.

One ussually expresses the naturality condition for a map $f : X \to Y$ with the following diagram:

$$\begin{array}{ccc} \mathcal{C}(X, B) & \xrightarrow{\varphi_X} & F(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathcal{C}(Y, B) & \xrightarrow{\varphi_Y} & F(Y) \end{array}$$

And in homotopy theory I have studied that

Theorem (Brown representability theorem) Let F be a contravariant functor from the homotopy category of parallel connected CW-complexes to pointed sets. If F satisfies conditions (i) and (ii) above (for any pointed connected CW-complexes X_i, A, B, C), then F is representable.

Remark (So what is a classifying space?) The theorem says that there is a space $B = B_F$ (itself a pointed CW-complex) for which there is a natural isomorphism

$$\varphi : [X, B_F]_* \xrightarrow{\cong} F(X)$$

for any pointed CW-complex X. This space B_F is called a ***classifying space*** for F. Recall also that when such φ exists, it is completely determined by a *generic* element $\gamma \in F(B_F)$.

The classifying space together with the genereic element is unique up to homotopy.

Remark $H^n(-, G)$ is represented by $K(G, n)$ together with a chosen element in $H^n(K(G, n), G)$

But anyway. We see in wiki that for the case of homework 3 bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$ we get that the base space $BU(1) = \mathbb{C}P^\infty$, Thus, the set of isomorphism classes of circle bundles over a manifold M are in one-to-one correspondence with the homotopy classes of maps from M to $\mathbb{C}P^\infty$.

So what is the functor that we are representing? I think is K. Because the maps are isomorphic to $K(S^1)\ldots$? Circle bundles?

2.6 Stiefel spaces

Definition $\mathbb{K}^\infty$ is the direct limit of $\mathbb{K}^n$ so its just the direct sum $\bigoplus_{i=n}^{\infty}\mathbb{K}$. Stiefel space is the space of orthonormal n-frames.

If we prove that Stiefel is contractible we obtain our classifying space so let's prove that. We have a fibration

$$\mathsf{U}(n) \hookrightarrow \mathsf{St}(n,\infty) \to \mathsf{Gr}(n,\infty)$$

Theorem $\mathsf{St}(n)$ is contractible.

Demostração **Step 1** Locally trivial fibration with contractible fiber and base $Y \to X$ then Y is contractible, this is so trivial.

Step 2 Fibration $\mathsf{St}(n) \to \mathsf{St}(n-1)$ with fiber S^∞

Step 3 Show that S^∞ is contractible.

Step 4 And then some map $\mathbb{R}$ that is not surjective, and construct homotopy of identity to a constant map.

□

Exercise If $X_\infty = \bigcup X_i$ is the inductive limit of contractible cellular spaces then it is contractible. Use Whitehead theorem.

Theorem (Important) $\mathsf{Gr}(\infty) = B\,\mathsf{U}$.

2.7 Stable equivalence

Definition Vector bundles V, W are stable equivalent if $V \oplus A \cong W \cong B$ for trivial vector bundles A and B.

Homotopy classes of equivalent vector bundles are in coorespondance with...

Theorem $B\,\mathsf{U}$ is H-space.

Corollary $H^*(B\,\mathsf{U},\mathbb{Q})$ is a free supercommutative algebra.

Claim $H^*(B\,\mathsf{U}$ is a free polynomial algebra generated by classes $c_1, c_2, \ldots$ in all even degrees.

3 Class 3

3.1 Reminder

Definition ***Bialgebra*** is an algebra that is equipped with comultiplication, counit...

Remark It is when the dual space also has an algebra structure, but we prefer to use the tensor notation.

Let $\sum_{i\geqslant 0} A^i$ with $\dim A^i < \infty$. ***Free commutative algebra*** is a polynomial algebra. ***Free graded commutative algebra*** is

$$\widetilde{\mathsf{Sym}}^\bullet(W^\bullet \oplus V^\bullet) := \mathsf{Sym}^\bullet(W^\bullet) \otimes \Lambda^\bullet(V^\bullet)$$

where

$$W = \bigoplus_i W^{\text{even}} \qquad V = \bigoplus_i V^{\text{odd}}.$$

3.2 Hopf algebra

Definition A bialgebra is a ***Hopf algebra*** when it is also equipped with an antipode map (S) such that the following diagram commutes

$$\begin{tikzcd} & H \otimes H \arrow[rr, "S\otimes \mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta"] \arrow[rr, "\eta"] \arrow[dr, "\Delta"'] & & \mathbb{C} \arrow[rr, "u"] & & H \\ & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"] & & H \otimes H \arrow[ur] & \end{tikzcd}$$

[diagram from quantum group minicourse notes]

Example The cohomology of the loop space, $H^\bullet(\Omega X)$.

3.3 Primitive elements in a bialgebra

Definition An element of a bialgebra $x \in A$ is ***primitive*** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$$\begin{aligned} \Delta(xy) &= \Delta(x)\Delta(y) \\ &= (1 \otimes x + x \cdot 1)(y \otimes 1 + y \otimes y) \\ &= 1 \otimes xy + xy \otimes 1 + x \otimes y + y \otimes x. \end{aligned}$$

Remark We trying to show that Hopf algebras? bialgebras? are generated by primitive elements?

Definition $A^\bullet$ bialgebra, $\mathscr{P}^\bullet \subset A^\bullet$ space of primitive, and the natural embedding

$$\mathsf{Sym}_{gr}(\mathscr{P}^\bullet) \to A$$

We say that A is ***free up to defree*** k if

$$\bigoplus_{i\leqslant k} \mathsf{Sym}^i_{\mathsf{gr}}(P) \xrightarrow{\psi} A$$

is an embedding.

Lemma Let $A^\bullet$ be a bialgebra which is free up to degree k. Then $A^\bullet$ is free up to degree $k+1$.

Proof.

Step 1 Choose a basis of P, $\{x_i\}$. Chose a polynomial condition $Q(x_1, \dots, x_n) = 0$ of degree $k+1$. Write this as

$$Q = Q_m x_1^m + Q_{m-1} x_1^{m-1} + \dots + Q_0.$$

that is

$$Q = \sum_{i=0}^{m} Q_i x_1^i$$

with Q_i invariant somehow. Then we apply comutiplication to obtain

$$\Delta(Q) = Q \otimes 1 + 1 \otimes Q + R$$

where R is some sort of reminder with bounded degree:

$$R \in \mathfrak{U} := \bigoplus_{i \leqslant k} \mathsf{Sym}^i_{\mathsf{gr}}(P) \otimes \bigoplus_{i \leqslant k} \mathsf{Sym}^i_{\mathsf{gr}}(P)$$

which follows from a similar computation of that which we did after defining primitive elements.

Step 2 Project to drop the terms that have $Q \otimes 1 + 1 \otimes Q$:

$$\Pi : \mathfrak{U} \to x_1 \otimes \bigoplus_{i \leqslant k} \mathsf{Sym}^i_{\mathsf{gr}}(P)$$

since the x_i are primitive, i.e. $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, one has

$$\Delta(x_1^m) = (x_1 \otimes 1 + 1 \otimes x_1)^m$$

we get that

$$\Pi(\Delta(x_1^m)) = m x_1 \otimes x_1^{m-1}$$

while on the board it is written that

$$\Pi(\Delta(x_1^m)) = \Pi((x_1 \otimes 1 + 1 \otimes x_1)^m)$$

Step 3 Let $\Pi(R) := x_1 \otimes R_0$. Since $Q = 0$ in A, its component R_0 is also equal to 0. So $\Pi(\Delta(Q)) = 0$. Then

$$\begin{aligned} 0 &= \Pi\left(\Delta\left(\sum_m x_1^m \cdot Q_m\right)\right) \\ &= \sum_m x_1 \otimes x_1^{m-1} Q_m + \Pi(m x_1 \otimes x_1^{m-1} \cdot \Delta(Q_m)) \\ &= \sum_m x_1 \otimes x_1^{m-1} Q_m \end{aligned}$$

so that

$$
\begin{aligned}
& x_1 \otimes x_1^{m-1} Q_m = 0 \\
\implies & x_1^{m-1} Q_m = 0
\end{aligned}
$$

So we conclude that

$$Q_m = 0$$

□

Remark We just proved that for any subalgebra generated by finite elements, we didn't use that it is free.

3.4 Algebras with filtration

Definition A ***filtration on algebra*** is

$$A^\bullet \supset F_1 A^\bullet \supset F_2 A^\bullet \supset \dots$$

such that

$$F_i A^\bullet F_j \subset F_{i+j} A^\bullet$$

Definition ***Associated graded*** to a filtered algebra is

$$A^\bullet_{\mathrm{gr}} = \bigoplus_{i=0}^{\infty} \frac{F^1 A^\bullet}{F^{i+1} A^\bullet}$$

$$F^0 A^\bullet = A^\bullet$$

Definition $I \subset A$ ideal then I-***adic filtration*** is the filtration by the degrees of the ideal

$$A \supset I \supset I^2 \supset I^3 \dots$$

Lemma Choose an I-adic filtration. Then A_{gr} is generated by its first and second graded components $A/I \oplus I/I^2$.

Demostração. Indeed, $I^k/^{k+1}$ is generated by products of k elements in (I/I^2). □

Definition A ***augmentation ideal*** in a bialgebra is the kernel of the counit homomorphism $\varepsilon : A \to k$. We denote it by $Z = \ker A$

Remark

$$\Delta(x) = 1 \otimes x + x \otimes 1 \operatorname{mod} Z \otimes Z$$

Why? Because

$$
\begin{aligned}
x &= \varepsilon \otimes \mathrm{id}(\Delta(x)) \qquad \text{up to } Z \otimes A \\
\Delta(x) &= 1 \otimes x \qquad \text{up to } A \otimes X \\
\Delta(x) &= x \otimes 1
\end{aligned}
$$

Ok, now we can prove Hopf theorem.

Theorem (Hopf theorem) A finite type bialgebra is generated by primitive elements.

In slides: Let A be a graded bialgebra of finite type over a field k of characteristic 0. Then A is a free graded commutative k-algebra.

Proof.

Step 1 I think this is the computation above.

Step 2 A_{gr} is a bialgebra.

Step 3 A_{gr} is multiplicative generated by $\mathsf{Z}^1/\mathsf{Z}^2$. All elements $\mathsf{Z}^1/\mathsf{Z}_2$ are primitive, so this algrebra A_{gr} is generated by primitive elements.

Step 4 Let $\{x_i\}$ be a basis of primitive elements of A_{gr}. Then lifts of A have no relations because A_{gr} is already generated by primitive elements. Then there are no relations also for elements in $A^\bullet$ (I think).

$\square$

3.5 Grassmanians (Reminder)

B vector bundle of rank n on X then there exists a map (essentialy unique) $\varphi : X \to \mathsf{Gr}(n)$ such that

$$\varphi^*(B_{\mathsf{fun}} = B$$

which makes the Grassmanian a classifying space, and $\mathsf{Gr}(1) = B\,\mathsf{U}(n)$.

The infinite Grassmanian is important.

3.6 BU as an H-space (Reminder)

Bott periodicity identifies the space of loops on U and $B\,\mathsf{U}$.

Proposition Embed $\mathbb{C}^\infty \times \mathbb{C}^\infty$ into $\mathbb{C}^\infty$ taking the basis vectors of the first copy to the even basis vectors and the basis of the second copy to the odd. Then for $L_1 \subset \mathbb{C}^\infty$, $L_2 \subset \mathbb{C}^\infty$, the map

$$L, L' \mapsto S(L, L')$$

defines a structure of H-space on the infinite Grassmanian $B\,\mathsf{U}$.

Proof. Just show that H-associatity up to homotopy. $\square$

Corollary $H^\bullet(B\,\mathsf{U}, \mathbb{Q})$ is a free supercommutative algebra.

Proof. Follows from Hopf theorem. $\square$

3.7 Cohomology of BU

Claim $H^\bullet(B\,\mathsf{U}, \mathbb{Q})$ is a free polynomial algebra generated by classes $c_1, c_2, \dots$ in all even degrees.

Demostração. Leray-Serre spectral sequence. □

3.8 Chern classes: axiomatic definition

Definition ***Chern classes*** are classes $c_i(B) \in H^{2i}(X)$ for $i = 0, 1, 2, \dots$

Chern classes are $c_i(B) \in H^{2i}(X)$ for a complex vector bundle B over X with axioms

a. $c_0(B) = 1$

b. Functoriality (commutes with bullbacks): for $\varphi : X \to Y$ with B bundle on Y,

$$\varphi^*(c_i(B)) = c_1(\varphi^*(B))$$

c. Define ***total Chern class*** $c_* := \sum_i c_i(B)$ then

$$c_i(B) \cdot c_i(B') = c_*(B \oplus B') \qquad \text{(Whitney)}$$

d. $\mathcal{O}(1)$ on $\mathbb{C}P^n$,

$$c_i(\mathcal{O}(1) = 1 + [H]$$

where $[H]$ is the fundamental class of a hyperplane section.

Remark (Once and for all) $\mathcal{O}(-1)$ is the ***tautological line bundle*** on a Grassmanian, defined as $\{(\ell, v) \in \mathcal{G}r(k, n) \times \mathbb{C}^n : v \in \ell\}$.

$\mathcal{O}(1)$ is the ***hyperplane bundle*** which is the dual of that so $\{(\ell, v^*) \in \mathcal{G}r(k, n) \times (\mathbb{C}^n)^* : v^* \in \ell^*\}$

Suppose we have a class $a \in H^\bullet(B\,\mathsf{U})$. Then for all B on X

$$\varphi : X \to B\,\mathsf{U}$$

so

$$B \cong \varphi^*(B_{\mathsf{fun}})$$

and so

$$\varphi_B^*(c_*) = c_*(B).$$

4 Class 4

4.1 Reminder

For each rank n bundle B on X there exists $\varphi_B : X \to \mathsf{Gr}(n, \infty) = B\,\mathsf{U}(n)$ such that $\varphi_B^*(B_{\mathsf{fun}} = B$.

The infinite grassmanian is classifying space for (?) stable bundles.

Some more review about H-space structure, primitive elements, a comment on last exercise of homework 2.

Chern classes of $\mathcal{O}(1)$ are hyperplane sections: $c_i(\mathcal{O}(1)) = 1 + [H]$.

4.2 The splitting principle

Exercise Prove that $BU(1) = \mathbb{C}P^\infty$.

Solution. Hopf fibration on S^∞? It's easier, take $n = 1$, it's just by definition. □

Definition The ***fundamental bundle*** on $BU(1)^n$ has fiber

$$\ell_1 \oplus \ell_2 \oplus \dots \ell_n$$

where $\ell_i \in BU(1)$ are product $\ell_1 \times \ell_2 \times \dots \times \ell_n$.

Remark Chern classes of B_{fun} are uniquely determined by axioms, because every factor has Chern classes, and fibers are just sums, and pullbacks preserve sums. . .

$$B_{\mathsf{fun}} = \bigoplus_i \pi_i \mathcal{O}(1)$$

where

$$pi_i : BU(1)^n \to BU(1)$$

is a projection.

Remark $H^\bullet(BU(1))^n = \mathbb{Z}[z_1, \dots, zn$ Here at least I remember that the cohomology of $\mathbb{C}P^\infty$ is just polynomials so it looks reasonable that the n-th power is polynomials in more cariables.

Theorem (Splitting principle) Let $\varphi_{\mathsf{fun}} : BU(1)^n \to BU$, the ***fundamental map***, it induces embedding on cohomology up to degree $2n$. For all primer generator $\sigma_i \in H^2(BU)$, $\varphi_{\mathsf{fun}}(\sigma_1) = \lambda \sum_i z_i^k$ with $\lambda \neq 0$.

So

$$\begin{tikzcd}
BU(1)^n \arrow[r] & BU \\
X \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Remark Wiki Thus, the set of isomorphism classes of circle bundles over a manifold M are in one-to-one correspondence with the homotopy classes of maps from M to $\mathbb{C}P^\infty$

Theorem Chern classes are unique (uniquely determined by axioms).

Proof.

Step 1 Every bundle is obtained as pullback of the fundamental bundle. So for $A \in H^\bullet(BU)$ and B bundle on X, $A(B) = \varphi_B^*(A) \subset H^\bullet(X)$ so $c_i(B)$ are obtained as pullbacks of c in the fundamental bundle.

Step 2

$$BU(1)^\infty \xrightarrow{\varphi_{\mathsf{fun}}} BU$$

pullback of fundamental bundle is fundamental. (This map is defined from the former by induction).

$$\varphi_{\mathsf{fun}}^*(c_i(B_{\mathsf{fun}}) = c_i(B_{\mathsf{fun}} \text{ on } BU)$$

The Chern classes of the fundamental bundle are already known. Since φ_{fun}^* is injective by the splitting principle we are done.

□

4.3 Primitive generators of $H^*(BU)$

Recall the H-space multiplication:

$$\begin{aligned} BU \times BU &\longrightarrow BU \\ L_1 \times L_2 &\longmapsto L_1 \oplus L_2 \end{aligned}$$

and the comultiplication

$$\Delta : H^\bullet(BU) \to H^\bullet(BU)$$

Generators of $H^\bullet(BU)$ are $c_{h_1}, c_{h_2}, \ldots$ with $c_{h_i} \in H^{2i}(BU)$ and we have the comultiplication $\Delta(c_{h_i}) = c_{h_i} \otimes 1 + 1 \otimes c_{h_i}$.

Remark

$$\varphi = (\varphi_1, \varphi_2) : X \to BU \times BU$$

and we can compose so we have

$$\varphi \circ \mu : X \to BU$$

what does this map do?

$$\begin{aligned} \varphi \circ \mu : X &\longrightarrow BU \\ \varphi^*(B_{\mathsf{fun}} &\longmapsto B_1 \\ (\varphi \circ \mu)^*(B_{\mathsf{fun}}) &= B_1 \oplus B_2 \end{aligned}$$

So then we have

$$\begin{aligned} \varphi^* : H^\bullet(BU) \otimes H^\bullet(BU) &\to H^\bullet(X) \\ \Delta : H^\bullet(BU) &\to H^\bullet(BU) \otimes H^\bullet(BU) \\ \Delta \circ \varphi^* : H^\bullet(BU) &\to H^\bullet(X) \end{aligned}$$

Corollary For every $x \in H^\bullet(BU)$
$$X(B_1 \oplus B_2) = \Delta(x)(B_1, B_2)$$

Corollary If $x \in H^*(BU)$ is primitive, then $x(B_1 \oplus B_2) = x(B_1) \oplus X(B_2)$.

Proof. $\Delta(x) = x \otimes 1 + 1 \otimes x$ so $\Delta(x)$ evaluated on (B_1, B_2) □

Remark We will construct the full Chern class $c_*(B)$ as a pullback of a class $C \in H^*(BU)$.

Remark Then take exponential. Let $\chi_i \in H^{2i}(BU)$ be a primitive generator. Use Hopf theorem to see that it is unique by a constant. Snce $\chi_i(B_1 \oplus B_2) = \chi_i(B_1) + \chi_i(B_2)$, the class $C = e^{\sum_i a_i \chi_i} = 1 + \ldots + \frac{\chi_n}{n!} + \ldots$ satisfies the Whitney formula.

To construct Chern classes satisfying the axioms it remains to arrange the coefficients a_i in such a way that $C(\mathcal{O}(1)) = 1 + [H]$ I think this means hyperplane section.

Lemma An embedding
$$BU(1) \xhookrightarrow{\varphi} BU$$
with $\chi_i \in H^{2i}(BU)$ primitive generator. Then $\varphi^*(\chi_i) \neq 0$

Proof. $H^\bullet(BU) =$ symmetric polynomials in $H^i(BU(1))^n$, $\varphi_{\mathsf{fun}}(x_N) = x \sum_{i=1}^n z_i^k$ so $\varphi(x_k) = \lambda x_1^k$. □

Remark $\varphi^*(c_i(B_{\mathsf{fun}}) = c_i(\mathcal{O}(1) = 1 + [H]$

Theorem Choose generators $\chi_i \in H^2(BU)$ primitive. Then $\varphi^*(\sum_i \chi_i = \mathsf{log}(1 + [H])$ where the logarith is a formal power series, namely $\sum_{i=1}^\infty \frac{H^n}{n!}(-1)^n$.

That means $c(B_{\mathsf{fun}}) = \mathsf{exp}\left(\sum_{\chi_i}\right)$.

5 Class 5

We want to study the space of line bundles on a surface.

5.1 K-group

Definition Let V be the set of equivalence classes of vector bundles on X. Consider the free module generated by V (it's just V copies of Z):
$$\mathbb{Z}\langle V \rangle = \bigoplus_V \mathbb{Z}$$

And now consider

$$\frac{\mathbb{Z}\langle V\rangle}{[F_1]-[F_1]-[F_3]}$$

for all exact sequences

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

Equivalently, the relation is $[F_1]+[F_3]=[F_2]$.

Remark We may give an H-structure to the set of homotopy classes of maps $X \to BU$ as follows $\varphi_1, \varphi_2 : X \to B\,U$

$$B_1 = \varphi^*(B_{\mathsf{fun}})$$

define the H-product

$$\varphi := \varphi_1 \circ \varphi_2$$

such that

$$\varphi^*(B_{\mathsf{fun}} = B_1 \oplus B_2$$

And then we have an isomorphism (that we are not going to use):

$$K(X) \xrightarrow{\mathsf{hom}} \text{group of homotopy classes of maps from } X \text{ to } BU$$

This is because every bundle on X is the pullback of the fundamental bundle by some map. We need to check that the image of trivial bundle is trivial map (homotopic to constant?) and that it preserves the product.

Remark The important thing of today is that that sum corresponds to addition

Remark I guess I should first understand how is it that every bundle is the pullback of the fundamental bundle.

So for example for injectivity we need to show that if a map φ pulls back the fundamental bundle to the trivial bundle then φ is homotopic to identity. This is not obvious though.

The point is that that map is a bijection.

Claim Chern classes are defined on $K(X)$ and satisfy Whitney formula (meaning Chern classes they pass to the quotient, right?)

Proof. Let B be a bundle on X so that $B = \varphi^*(B_{\mathsf{fun}}$. We showed last time that there is a $c_\cdot \in H^0(BU)$ such that $c_\cdot(B) = \varphi^*(c_\cdot$. In fact we proved that $c_\cdot = \mathsf{exp}(\text{additive})$, but its actually Chern character, $c_\cdot = \mathsf{exp}(\mathsf{Ch}_\cdot)$, in fact $\mathsf{Ch}_\cdot(B_1 + B_2\,\mathsf{Ch}(B_1) + \mathsf{Ch}_\cdot(B_2)$. □

5.2 Coherent sheaves

Definition Let M be a complex manifold and $\mathcal{O}_M$ its structure sheaf (of holomorphic functions). A ***coherent sheaf*** is a sheaf of $\mathcal{O}_M$-modules, locally isomorphic to a quotient of a free sheaf $\mathcal{O}_M^n$ by a finitely generated $\mathcal{O}_M$-invariant subsheaf.

A ***coherent sheaf*** on a projective manifold. A ***projective manifold*** is $\mathsf{Proj}(A^\bullet)$ where $A^\bullet$ is a graded ring. ***Coherent sheafes*** are sheaves of graded $A^\bullet$-modules.

Exercise Let M be a projective manifold. Prove that any coherent sheaf F has a (projective) resolution

$$0 \longrightarrow B_n \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow F \longrightarrow 0$$

where B_i are vector bundles. This is called the ***syzygy resolution***

Solution. Every module has a projective resolution called ***Koszul resolution***. So what is Koszul resolution. First you have a resolution of a maximal ideal. For a maximal ideal it is clear since ... (Herieta? and) Eisenbud or even Bourbaki Homological algebra. □

5.3 Coherent sheaves and their Chern classes

So there's actually two K-groups. One is generated by bundles and the other by sheaves. For bundles, it is an algebra. For sheaves, it is a module over the other one. For Groethendick one was $K^\bullet$ and the other $K_\bullet$ but we don't know which is which.

Remark After this is done, it's possible to prove that the K-group of coherent sheaves on a projective manifold is equal to the K-group generated by holomorphic vector bundles.

Definition The ***Chern class*** of a coherent sheaf is the Chern class of the corresponding element in the K-group.

Remark (about singularities, see slides) Suppose we do resolution of a manifold and pullback a bundle

$$\begin{array}{ccc} \tilde{M} & & \pi^*F \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ M & & F \end{array}$$

5.4 Euler characteristic of a coherent sheaf

Definition Let F be a coherent sheaf. Its ***Euler characteristic*** is

$$\chi(F) = \sum_i (-1)^i \dim H^i(F)$$

But what is that cohomology? What is the space?

Claim For any exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

we have

$$\chi(F_2) = \chi(F_1) + \chi(F_3)$$

Proof. Should be possible... $\square$

Then

$$\chi : K(M) \to \mathbb{Z}$$

is a homomorphism.

5.5 Chern character

OK so last class we defined an homomorphism called χ that was additive. Now let's call it

$$\mathsf{c}. = \exp(\mathsf{Ch}.)$$

and it was additive

$$\mathsf{Ch}.(B_1 \oplus B_2) = \mathsf{Ch}.(B_1) + \mathsf{Ch}.(B_2)$$

So the textbook definition is that ***Chern character*** on line bundles is

$$\exp(c_\bullet(L))$$

So c_1 is additive and if you pass to the exponent it will be multiplicative:

$$\begin{gathered} c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \\ \mathsf{Ch}.(L_2 \otimes L_2) = \mathsf{Ch}.(L_1) \cdot \mathsf{Ch}.(L_2) \end{gathered}$$

5.6 Riemann-Roch-Hirzebruch theorem

Theorem (RRH) Let F be a coherent sheaf on a complex compact manifold M. Then $\chi(F)$ can be expressed through Chern classes of F and M as follows:

$$\chi(F) = \int_X \mathsf{Ch}.(F) \wedge \mathsf{Td}.(TM),$$

where $\mathsf{Td}.(TM)$ mdenotes the ***total Todd class of the tangent bundle*** TM, which is a sum of Chern classes.

$$\mathsf{Td}. = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_1}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4}{720} + \dots$$

5.7 K-group for complex curves

Lemma K-group for complex curves is generated by line bundles.

Proof.

Step 1 For each F coherent sheaf, $L^n \otimes F$ has a section. So there is a monomorphism $L^{-N} \hookrightarrow F$.

Step 2 The consider the localization to produce a short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

since $F = \bigoplus_i F_i$ for $F_i = \mathcal{O}_M/\mathfrak{m}_X^{a_i}$ so

$$0 \longrightarrow (\mathfrak{m}_X^{a_1} \longrightarrow \mathcal{O}_X \longrightarrow F_1 \longrightarrow 0$$

□

5.8 Riemann-Roch for complex curves

Theorem (Riemann-Roch for complex curves) Let F be a coherent sheaf on a compact complex curve of genus g. Then

$$\chi(F) = c_1(F) + \mathrm{rk}(F)(1-g)$$

Proof. We want to see

$$c_1(L) = \deg(L)$$

Step 1 It suffices to prove for line bundles by the lemma.

Step 2 For degree 0 its easy beacuse $c_1(k_x) = 1$. For structure sheaf $\mathcal{O}_X$ we have rank is 1.

Step 3 Now let L be a line bundle. We have

$$0 \longrightarrow \mathcal{O}_M \longrightarrow L \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow L \longrightarrow \mathcal{O}_M \longrightarrow F \otimes L = F \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_M \longrightarrow L_1^N \otimes L \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow L_1^{-N} \longrightarrow L \longrightarrow F \longrightarrow 0$$

and the point is that many things "have sections". What does it mean to have sections.

□

5.9 Riemann-Roch-Hirzebruch for line bundles on complex surfaces

Definition A ***complex surface*** is a compact complex manifold of dimension 2.

Notation

$$(L_1, L_2) = c_1(L_1) \wedge c_1(L_2)$$

and if D is a divisor we write (the ***degree of a divisor***)

$$(D, L) = \deg_D L = \int_M [D] \wedge c_1(L)$$

Theorem (RRH for surfaces) L line bundle on surface and $K_X = \Omega^2(X)$ its canonical bundle. Then

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{(L - K_X, L)}{2}$$

where (A, B) denotes the intersection form applied to cohomology classes on X.

Proof.

Step 1 Let D a smooth curve of genus g and L_1, L_2 line bundles that fit in an exact sequence

$$0 \longrightarrow L_2 \longrightarrow L_2 \longrightarrow L_2|_D \longrightarrow 0$$

Then we use Rieman-Roch for curves gives

$$\chi(L_1) = \chi(L_2) + (L_2, D) + (1 - g)$$

Step 2 Let ND denote the normal bundle on D. The adjunction formula gives $K_D = K_X|_D \otimes KD$. Since $g - 1 = \deg K_D/2$, we obtain $1 - g = -(K_X + D, D)/2$.

Step 3 The next step goes as before, with Rieman-Roch in one dimension. Let $\chi'(L)$ be the RHS of section 5.9, namely $\chi'(L) = \chi(\mathcal{O}_X) + \frac{L - K_X, L)}{2}$. In step 1 we have $c_1(L_2) = c_1(L_1) + D$. Then

$$\begin{aligned}\chi'(L_2) - \chi'(L_1) &= \frac{1}{2}\left[(L_2 - K_X, L_2) - (L_2 - K_X - D, L_2 - D)\right] \\ &= (L_2, D) - (K_X + D, D)/2\end{aligned}$$

Step 4 Comparing Step 2 and Step 3, we get

$$\chi'(L_2) - \chi'(L_1) = \chi(L_2) - \chi(L_1)$$

Therefore, section 5.9 is equivalent for L_2 and for L_1. We just need to manipulate bundles to reduce a bundle to... by building exact sequences.

Step 5 So suppose you have a smooth section of a bundle. Take an ample bundle A and do

$$0 \longrightarrow \mathcal{O}_X \longrightarrow A^N \longrightarrow A^N|_D \longrightarrow 0$$

$$0 \longrightarrow L \longrightarrow A^N \otimes L \longrightarrow A^N \otimes L|_D \longrightarrow 0$$

and then by step 4 we just need to deal with $A^N \otimes L$.

Step 6 It's very ample, it has many sections, including some that are smooth. Now we just assume L is $A^N \otimes L$. So

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L|_D \longrightarrow 0$$

so for bundles that have smooth sections the statement is free.

□

5.10 Applying the general formula to the curve case

We have

$$\mathrm{Ch.}(L) = 1 + c_1(L) + \frac{c_1^2(L)}{2}$$

$$\mathrm{Td.}(L) = 1 + \frac{c_1(TM)}{2} + \frac{c_1^2(M) + c_2}{12}$$

Now

$$\chi(L) - \chi(\mathcal{O}) = -\frac{(K_1(L), K)}{2} + \frac{c_1(L)^2}{2} = \frac{(L, K - L)}{2}$$

6 Class 6: Local Torelli theorem and its applications

6.1 Exponential exact sequence

The exponential exact sequence is

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 0$$

and it gives a long exact sequence

$$\cdots \longrightarrow H^1(\mathcal{O}_M) \longrightarrow H^1(\mathcal{O}_M^*) = \mathrm{Pic} \xrightarrow{c_1} H^2(M, \mathbb{Z}) \xrightarrow{\alpha} H^2(\mathcal{O}_M) \longrightarrow \quad \longrightarrow \cdots$$

α is just forgetful map, a projection, to the $H^{0,2}(M)$ part of a form

The group $H^2(\mathcal{O}_M)$ is identified with $H^{0,2}(M)$ which is Dolbeault cohomology, hence the kernel of α is $H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$.

Proposition c_1 holomorphic line bundle on compact Kähler manifold belongs to intersection $H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$ and every element of this group can be realised as $c_1(L)$.

6.2 K3 surfaces are holomorphically symplectic

Definition A ***complex surface*** is a compact, complex manifold of complex dimension 2.

Definition A ***K3 surface*** is a (Kähler, can drop this assumption) complex surface M with $b_1 = 0$ and $c_1(M, \mathbb{Z}) = 0$

Remark The hypothesis that $c_1 = 0$ implies that $c_1(K_M) = 0$ and thus $K_M = 0_M$ (it is trivial). This is beacuase $H^1(\mathcal{O}_M) = 0$, which follows from Hodge theory.

6.3 Hodge diamond of a K3 surface

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

since the cohomology groups

$$\text{sections of } K_X = H^{2,0} = \mathbb{C} \quad \begin{array}{ccc} & \mathbb{C} & \\ 0 & & 0 \\ & ? & \\ 0 & & 0 \\ & H^{1,1} = \mathbb{C} & \end{array} \quad \text{Hodge (Serre?) duality} \implies H^{0,2} = \mathbb{C}$$

For the missing one, we comput $\chi(\mathcal{O}_M)$ using Riemann-Roch, which gives c_2 and from that we comput b_2.

6.4 Geometric structures (the story of Teichmüler space)

Definition ***Geometric structure*** on a manifold is reduction of structure group to $G \subset \mathsf{GL}()$.

6.5 Fréchet spaces

Definition A ***seminorm*** on a vector space V is a function $\nu : V \to \mathbb{R}^{\geqslant 0}$ such that

- $\nu(\lambda x) = |\lambda|\nu(x)$
- triangle inequality.

Definition Define a topology using a family of seminorms generated by the open balls of all seminorms.

Definition V infinite dimensional vector space, ν_α collection of seminorms. Sequence of vectors z_i is ***Cauchy*** if z_i is Cauchy for each ν_j. If all Cauchy sequences converge it is called ***Fréchet space***.

We can also define Fréchet space to with the distance $d(x, y) = \sum_{k=1} \frac{1}{2^k} \max(\nu_k(x-y), 1)$

Definition The ***topology*** C^k on a Riemannian manifold on the space $C_c^\infty(M)$ is

$$|\varphi|_{C^k} := \sup \sum_{i=0}^{k} |\nabla^i \varphi|$$

where ∇^i is the iterated connection $\nabla^i : \mathcal{C}^\infty(M) \to \Lambda^1(M)^{\otimes i}$

Definition Of tensor field, section of $TM^{\otimes i} \otimes T^*M^{\otimes j}$.

6.6 C^0 topology on group of diffeomorphisms

Idea To interpret diffeomorphisms as sections of a bundle.

Definition On $\mathsf{Dif}(M)$, riemannian manifold,

$$d(f_1, f_2) = \sup_{x\in M} d(f_1(x), f_2(x))$$

6.7 $\mathcal{C}^\infty$-topology on group of diffeomorphisms

It has more sets (is stronger) than the C^0 topology,

Definition Fix $\mathcal{U}$ small neightbourdhoos of id in $\mathsf{Dif}(M)$. Choose an atlas of $U_i \subset V_i$ such that U_i is relatively compact. There exists a neighbourhood of identity in Dif such that diffeomorphisms (sufficiently close to identity) they map $\tau(U_i) \subset V_i$. To find this neighbourhood use that closure of U_i is compact in V_i.

Now define the C^∞ topology on $\mathcal{U}$ as C^∞ convergence on maps from $U_i \subset \mathbb{R}^n$ to $V_i \subset \mathbb{R}^n$ using usual derivatives.

Anyways, the idea is that we only need a *uniform structure* which is a partially ordered set to define Cauchy sequences.

6.8 Teichmüler space of geometric structures

Let $\mathcal{C}$ be the set of all geometric structures of a given type equipped with C^∞ topology. The ***Teichmüller space*** is $\mathcal{C}/\mathsf{Diff}_0$, where Diff_0 is the connected component of the identity. The group $\mathsf{Diff}(M)/\mathsf{Diff}_0(M)$ is the ***mapping class group***, we are not going to use it.

6.9 Teichmüler space of symplectic structures

$\mathsf{Symp} \subset \Gamma(\Lambda^2(M))$. It is not Housdorff and we don't even know how much Housdorff it is. Maybe for four dimensional manifolds,...

6.10 Moser's theorem

Theorem (Moser, 1965) The Teichmüler space is a manifold, and the preiod map

$$\begin{aligned} \mathsf{Per} : \mathsf{Teich}_s &\longrightarrow H^2(M, \mathbb{R}) \\ w &\longmapsto [w] \end{aligned}$$

It is very beautiful but semi-elementary if you know Moser's lema.

6.11 The kernel of a differential form

If Ω is a differential form on M, its ***kernel*** is the space of all vectors $X \in TM$ such that $i_X(\Omega) = 0$.

Proposition $[\ker\Omega, \ker\Omega] \subset \ker\Omega$.

Corollary If (M, I) almost complex and $\Omega \in \Lambda^{2,0}(M)$ non-degenerate and closed, then I is integrable.

Proof. $T^{0,1} = \ker\Omega$. □

6.12 C-symplectic structures

Definition $\Omega \in \Lambda^2(M, \mathbb{C})$, M $4n$-dimensional manifold. Suppose that $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is nowhere zero. Then $\ker\Omega \oplus \overline{\ker\Omega} = TM \otimes \mathbb{C}$.

This is a ***C-symplectic*** manifold.

These manifolds have a nice Teichmüler space.

Theorem (Moser-Koebe?) $(M, I_+, \Omega_+($ family of C-symplectic forms, $[\Omega_t]$ =constant, $H^{0,1}(M_t) = 0$ then all Ω_t are related by a diffeomorphism. (This is Moser's trick!)

Notice that for $n = 1$ we have that the condition $\Omega^2 = 0$ and $\Omega \wedge \overline{\Omega}$ volume mean

Theorem CTeich Teichmüler space of C-symplectic structures on K3 surface. Consider the

$$\mathsf{Per} : \Omega \to [\Omega] \in H^2(M, \mathbb{C})$$

Then the image $\mathsf{Per}(\mathsf{Teich}) = \{pQ] : \int u \wedge u = 0, \int u \wedge \bar{u} > 0\}$ is a quadric.

This is a local diffeomorphism.

6.13 The period space of complex structures

Now take $\mathsf{CTeich}/\mathbb{C}^*$ because the Teichmüler space of complex structures has a free $\mathbb{C}^*$ action.

Proposition (Local Torelli theorem for complex structures) Teichmüler space of complex strcutres on K3.

$$\mathbb{P}\mathsf{er} = \{v \in \mathbb{P}H^2(M, \mathbb{C}) : (v, v) = 0, (v, \bar{v}) > 0\}$$

so

$$\frac{\mathsf{CTeich}}{\mathbb{C}^*} \longrightarrow \frac{Q}{\mathbb{C}^*} \subset \mathbb{P}H^2(M, \mathbb{C})$$

6.14 The period space of complex structures is a Grassmanian

Lets define

$$\mathsf{Gr}_{++}(H^2(M,\mathbb{R})) = \text{positively oriented 2-planes in } H^2(M,\mathbb{R})$$

Where positively oriented means the form is ? Then

$$\mathbb{P}\mathsf{er} = \mathsf{Gr}_{++}(H^2) = \frac{\mathsf{SO}(3,1o)}{\mathsf{SO}(2)\times\mathsf{SO}(1,1o)}$$

7 Class 7: smooth quartics

7.1 Reminder on local Torelli theorem

$$\mathsf{CTeich} = \text{Teichmüller space of hol sympl structures}$$

$$\begin{aligned} \mathsf{CTeich} &\longrightarrow H^2(M,\mathbb{C}) \\ \Omega &\longmapsto [\Omega] \end{aligned}$$

Then this map is a local differomorphism to the period space.

7.2 Hodge index theorem (without slides)

Theorem (Hodge index theorem) Consider a signature of intersection form on complex Kählerler surface is positive on real part of $\mathsf{Re}\,H^{2,0}(M)$, $(1,0)$ on $H^{1,1}(M,\mathbb{R})$, negative on

$$\begin{aligned} \ker L : H^{1,1}(M,\mathbb{R}) &\longrightarrow H^4(M) = \mathbb{R} \\ X &\longmapsto [X\wedge\omega] \end{aligned}$$

with ω Kähler form.

Proof. $\Omega^{2,0}$ 1 dimensional, $\mathsf{Re}\,\Omega^{2,0}$ 2-dimensional (at most) in $\Lambda^2(M,\mathbb{R})$.

$$\begin{aligned} \Omega &= \omega_1 + r - 1\omega_2 \\ \Omega\wedge\overline{\Omega} &= \omega_1^2+\omega_2^2 > 0 \\ \Omega\wedge\Omega &= 0 = \omega_1^2 = \omega_2^2 \\ \iff \omega_1\wedge\omega_2 &= 0 \\ \omega_1^2 &= \omega_2^2 \\ \implies \omega_1 &\perp \omega_2 \\ \omega_1^2 &= \omega_2^2 > 0 \end{aligned}$$

Then

$$\Lambda^{1,1} = \ker L \oplus \omega$$

That is a 4-dimensional bundle that is given by multiples of the Kähler form plus the primitive part. Then

$$\mathsf{ker}^{\perp} = \langle \mathsf{Re}\,\Omega, \mathsf{Im}\,\Omega, \omega \rangle$$

Then consider Hodge star operator $* : \Lambda^2 \to \Lambda^2$, $*^2 = 1$ and its complementary, this interchanges eigenvalues, negative positive, ... □

Corollary Signature of K3 surface is $(3, 19)$

7.3 The period space of complex structures is Grassmanian

Claim

$$\mathbb{P}\,\mathsf{er} = \frac{\mathsf{SO}(3, b_2 - 3)}{\mathsf{SO}(1, b_2 - 3) \times \mathsf{SO}(2)} = \mathsf{Gr}_{++}(h^2(M, \mathbb{R}))$$

Remark (V, q) real vector space signature q is (m, n), $m \geqslant 2$ then

$$\mathsf{Gr}_{++}(V, q) = \{\ell \in \mathbb{P}V_{\mathbb{C}} : q(\ell, \ell) = 0, q(\ell, \bar{\ell}) > 0\}$$

Recall that $T_p\,\mathsf{Gr}_{++} = \mathsf{Hom}(?)$

Of the claim.

Step 1 $\ell \in \mathbb{P}\,\mathsf{er}$

$$q(\mathsf{Re}(\Omega), \mathsf{Im}(\Omega)) = 0$$
$$q(\mathsf{Re}(\Omega), \mathsf{Re}(\Omega)) = q(\mathsf{Im}(\Omega), \mathsf{Im}(\Omega)) > 0$$

What is going on

$$\omega_1 = \mathsf{Re}(\Omega), \qquad \omega_2 = \mathsf{Im}(\Omega), \qquad \Omega \in \ell$$
$$\begin{aligned} q(\omega_1 + \sqrt{-1}\omega_2, \omega_1 + \sqrt{-1}\omega_2 &= 0 \\ &= q(\omega_1, \omega_1) - q(\omega_2, \omega_2) + \sqrt{-1}2q(\omega_1, \omega_2) \end{aligned}$$

and also

$$\begin{aligned} &q(\Omega, \overline{\Omega}) > 0 \\ = q(\omega_1 - \sqrt{-1}\omega_2, \omega_1 + \sqrt{-1}\omega_2) &= q(\omega_1, \omega_1) + q(\omega_2, \omega_2) > 0 \end{aligned}$$

so we have obtained from a line in Period a positive definite plane

Step 2 $p \in \mathsf{Gr}_{++}$. Project, obtain a quadric form on $\mathbb{C}^2$. There exist two lines in $P_{\mathbb{C}}$, $\ell, \bar{\ell}$ such tat

$$q(\ell, \ell) = 0, \qquad q(\bar{\ell}, \bar{\ell}) = 0$$

$$q(x, y) = xy$$

□

Corollary $U \subset \mathsf{Teich}$, $V \subset H^2(M,\mathbb{R})$ set of all nonzero $(1,1)$-classes on $H^2(M,I)$ for some $I \in U$ Then $V \subset H^2(M,\mathbb{R})$ is open.

Proof. Idea: take a 2 dimensional space and move it a bit everywhere, consider orthogonal complement. Deform P by taking a y and considering its orthogonal complement. y is chosen close to x.

$$X \in P^{\perp}\text{is a class of type (1,1)}$$

$$y\text{near } x$$

$$P\text{project to } y^{\perp}$$

$$\text{so you have an open set in Grassmanian}$$

$$U_x \xrightarrow{\phi} \mathsf{Gr}_{++} \qquad\qquad \phi^{-1}(U)$$

Step 1 Take a complex structure $I \in \mathsf{Teich}$, $P \subset H^2(M,\mathbb{R})$, then $H^{11}(M,I) = P^{\perp}$.

Step 2 Teichmüler is locally diffeomorphic to Gr_{++}. Suffices to show in a nieghbourhood $U_1 \ni P$ in Gr_{++} that $\bigcup_{P_1 \in U_1} P_1^{+}$ is open.

Step 3 $y \in H^2(M,\mathbb{R})$, $y \in U_x$, nonzero in a neighb of $x \in P^{\perp}$. P_y projection from P to $y^{\perp}$

□

7.4 Intersection form on a K3 surface

Lemma (Of linear algebra) Consider bilinear symmetric form on $V_{\mathbb{Z}}$

$$\pi : V_R \backslash 0 \longrightarrow \mathbb{P}V_Q$$
$$\longmapsto$$

where R is the set of odd vectors and Q rational vectors. Then $p(\text{odd vectors})$ is dense on $\mathbb{P}V_Q$.

Proof.

Step 1 Construct a sequence of odd vectors converging to any element $s \in V_{\mathbb{Z}} \backslash 0$.

$$\lim_n \pi(r_0 + 2ns) = \pi(s)$$

□

Theorem Intersection form of K3 is even.

Proof **Step 1** suppose it is odd. Coro 1 lema 1 imply complex structure I and odd vector $r \in H^1(M, I)$. The point is that the set of vectors of type 11 is open negithboudood of any class. But then the odd classes are dense.

Step 2 Involves Riemann-Roch. For each class $r \in H^{11}(M) \cap H^2(M, \mathbb{Z})$ we have a holomorphic line bundle by the exponential sequence, $c_1|L|$, L is hol line bundle:

$$0 \longrightarrow \mathbb{Z}_M \xrightarrow{\sqrt{-1}2\pi} \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^* \longrightarrow 0$$

and

$$\cdots \longrightarrow 0 = H^1(\mathcal{O}_M) \longrightarrow \mathsf{Pic} = H^1(\mathcal{O}_M^*) \longrightarrow H^2(M, \mathbb{Z}) \overset{\mathsf{proj}}{\longrightarrow} H^{02}(M) = H^2(\mathcal{O}_M) \longrightarrow \ \longrightarrow \cdots$$

Now Riemann-Roch:

$$\chi(L) = \chi(\mathcal{O}_M - \frac{L(K-L)}{2} = 2 - (L, L)/2$$

because canonical bundle is trivial, so that cannot be odd.

□

7.5 Smooth quartics

Definition A ***smooth quartic*** is a smooth quartic hypersurface in $\mathbb{P}^3$. So a solution of a quartic equation, ie. polynomial of degree 4.

Remark Adjunction formula. Canonical bundle of quartic is canonical bundle of $\mathbb{C}P^3$ restrictesd to quartic times normal bundle:

$$K_Q = K_{\mathbb{C}P^3}|_Q \otimes N(Q)$$

But $N(Q)$ is degree four so it is just $\mathcal{O}(4) = N(Q)$.

and canonical bundle $K_{\mathbb{C}P^3}$ of $\mathbb{C}P^3$ is $\mathcal{O}(-4)$ by Euler formula.

So $K_Q = 0_Q$—quartic has trivial canonical bundle (it is Calabi-Yau).

$$V : \mathbb{C}P^3 \hookrightarrow \mathbb{C}P^{34}$$

What is this map. It is associated to $\mathcal{O}(4)$, with the line system

$$\mathbb{C}P^{34} = \mathbb{P}H^0(\mathcal{O}4)^*$$

and it is called Veronese map.

Claim Smooth quartic is a hyperplane section of $V(\mathbb{C}P^3)$.

So any hyperplane on $\mathbb{C}P^{34}$ is a hyuperplane on $\mathbb{C}P^3$.

...So the zeroes of this restriction are quadrics.

The point is that quartics are (in correspondence with) hyperplane sections.

All quartics are sections of Veronese.

7.6 Smooth quartics and Lefschetz hyperplance section theorem

Theorem (Lefshetz hyperplane) $\pi(H \cap V) = \pi_1(V)$

8 Class 8: smooth quartics

8.1 Lefschetz again

Theorem (Lefschetz hyperplane) If $X \subset \mathbb{P}^n$ and $H = \mathbb{C}P^{n-1}$ a hyperplane in $\mathbb{C}P^n$ and $X \cap H$ (transversal justo to be safe), $X \cap H \to X$ isomorphism on homotopy group π_i for $i < \dim X$, ie. $\pi_i(Z \cap H) \xrightarrow{\cong} \pi_i(Z)$

Proof. Will discuss later but make a cellular decomposition that puts cells of certain dimension in the intersection. □

Corollary $\pi_1(\text{smooth quartic}) = \pi_1(\mathbb{C}P^3) = 0$.

Corollary Smooth quartic is K3.

8.2 Smooth submersions

Definition ***smooth submersion*** is a map $\pi : M \to M'$ such that $d\pi$ is surjective everywhere.

Remark Submersions are just products: each point has a neighbourhood that looks like a product and submersion is projection on one factor.

Theorem (Ehresemann fibration theorem) Let $\pi : M \to M'$ be a smooth submersion of compact manifolds. Then π is locally trivial fibration.

Proof. It is a vector bundle because

$$0 \longrightarrow T_\pi M \longrightarrow TM \xrightarrow{d\pi} \pi^* TM' \longrightarrow 0$$

where $T_\pi M$ is the vertical subbundle ie. $\ker \pi$

Ehresemann connection is a decomposition $T_{\text{horizontal}} \oplus T_{\text{vertica}} = TM$. Then there is a projection $d\pi : T_{\text{hor}} \to TM'$ and an associated curve. This gives the diffeomorphism that says all fibers are diffeomorphic. (see slides) □

8.3 Space of smooth quartics

Let $V = \mathbb{C}^{35} = \mathsf{Sym}^4\,\mathbb{C}^4$ be the set of homogeneous degree 4 polynomials in 4 variables. Interpret $P \in V$ as a quartic equation in $W = \mathbb{C}^4$

Claim Let $Z \subset \mathbb{P}V \times \mathbb{C}P^3$ be the set $\{(P \in \mathbb{P}V, w \in \mathbb{C}P^3 : P(w) = 0\}$. Then Z is smooth and irreducible.

Proof **Step 1** So we have a point in a hyperplane and the hyperplane is in $\mathbb{C}P^3$ Veronese (ie. embedded, it is a quartic Q). So we have $x \in \ell$, ℓ hyperplane section. And then let $\tilde{Z} \subset \mathbb{C}^4 \times \mathbb{C}^{35}$ be "the corresponding set of vectors". So $Z = \tilde{Z}/\mathbb{C}^* \times \mathbb{C}^*$. Clearly it suffices to show that $\tilde{Z}$ is smooth (?).

Step 2 Take the derivative of $(P + tQ)(w)$, it is not zero and so $\tilde{Z}$ is smooth.

Step 3 Use Sard's lemma or Bertini theorem + Lefschetz hyperplane to show F, the general fiber of the projection of Z to $\mathbb{C}P^{34}$, is connected and hence Z is irreducible.

How to use LEfschetz?

$$V(\mathbb{C}P^3) \subset \mathbb{C}P^{34}$$

$$Q = V(\mathbb{C}P^3) \cap H$$

and put all the cells in one half and then the other half just remains connected.

$\square$

Question Is there a better way to show that a general smooth quartic in $\mathbb{C}P^3$ is connected (without Lefschetz)?

Use that all quartics are equivalent (outside discriminant) and then just use X_1^4. (...?)

8.4 Smooth quartics are diffeomorphics

Corollary Smooth quartics are differomorphic.

Proof.

$$\begin{tikzcd} Z \arrow[d, "Q"] \\ \mathbb{C}P^{34} \supset \mathcal{D} \end{tikzcd}$$

We want to prove that the fibers are diffeomorphic. We need to remove the non-smooth fibers of this map. The critical values are the (...) is called ***discriminant***. So $\mathcal{D}$ is the set of al singular quartics. And $\mathbb{C}P^{34} \backslash \mathcal{D}$ is connected.

Exercise Complement to proper subvariety is connected. Take two points and try to join them. Vanya: they intersect $\mathcal{D}$ in a finite ammount of points. Misha: every path can be deformed to a path that avoids $\mathcal{D}$ by Sard's theorem because $\mathcal{D}$ has codimension 2.

Then all fibers of

$$\begin{array}{c} Z \backslash \pi^*(\mathcal{D}) \\ \big\downarrow \pi \\ \mathbb{C}P^{34} \backslash \mathcal{D} \end{array}$$

π are diffeomorphic because π is a proper smooth submersion. $\square$

Remark The same arguemtns shows that smooth hypersurfaces of degree d in $\mathbb{C}P^n$ are diffeomorphic.

8.5 Ample bundles

Definition If you have $\varphi : X \to \mathbb{C}P^n$ projective complex, then

$$\varphi^*(\mathcal{O}(1))$$

is called ***very ample*** and L is ***ample*** if $L^{\otimes n}$ is very ample for some $n > 0$.

Kähler classes are classes of Kähler forms.

Theorem (Kodaira) L is very ample iff $c_1(L)$ is a Kähler class.

Objective All K3 are diffeomorphic. Need to prove that quartics are dense in the universal family of K3 over its Teichmüler space. Then we can deform a bit complex structure, deformatino doesn't change topology.

We need to identify the quartics among all K3 surfaces M containing $x \in \mathrm{Pic}(M)$ such that $x \cap x = 4$ in other words

You have a quartic Q, the generator of picard is $\mathcal{O}(1)$. Its self intersection is 4, to see if consider $[H \cap Q]$, intersection with hyperplane.

And we want that to be very ample. $c_1(L) \cap c_1(L) = 4$.

Remark (Dani) Not every K3 is quartic but every K3 is very close to a quartic in Teichmüler.

Remark Pellisky? did it with Kummer (surfaces?), which much harder.

8.6 Very ample bundles

Interpret very ampleness as vanishing of cohomology groups.

Claim (The Following AreEquival)

(i) $\phi_L : X \to \mathbb{C}P^n = \mathbb{P}H^0(X, L)$ is injective and holomorphic.

(ii) $\forall x, y$ exists section $\gamma \in H^0(X, L)$ with $\mathcal{D} =$zero γ, $X \in \mathcal{D}, y \notin \mathcal{D}$.

This is equivalent to

$$\begin{array}{c} H^0(X, L) \\ \downarrow \\ H^0(X, \frac{L}{(\mathfrak{m}_x \cap \mathfrak{m}_y}) \otimes L) \end{array}$$

And that thing is some skyscraper things but is reall only $\mathbb{C}^2$.

Remark $\mathfrak{m}_x$ max ideal of x. The 1-jets of functions in x is $\mathcal{O}_X/\mathfrak{m}_x^2$. Then the natural map

$$\phi_L X :\to \mathbb{P}H^0(L)$$

is non-zero for all X. There exists section γ 1-jet x is non-zero $\gamma(x) = 0$.

Definition $f, g \in \mathcal{O}_{X,x}$ germs of functions at x (or even sections of line bundle in neighbourhood of a point). f has same k-jet as g if $f - g$ has zero order $k + 1$. Same Taylor series up to $k + 1$.

0jet is Taylor. 1-jet is Taylor plus differential. And the good thing is that $\mathfrak{m}/\mathfrak{m}^2 = T_x M^*$. Kernel of differential is kernel on 1-jets.

8.7 Alternative description of very ampleness

Corollary L bundle on compact complex manifold X. equivalent:

(i) L very ample.

(ii)
$$H^0(L) \to H^0(L)/(\mathfrak{m}_x \cap \mathfrak{m}_y) \otimes L)$$
(skyscraper sheaves isomorphic to $\mathbb{C}$) is surjective and also
$$H^0(L) \to H^0(L/(L \otimes \mathfrak{m}_x^2)$$
is surjective too. On right hand side of the second one in fact that it cotangent space (some comment about thinking of this like some coordinate system).

Now think of this short exact sequence of coherent sheaves

$$0 \longrightarrow L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y) \longrightarrow L \longrightarrow L/(\mathfrak{m}_x \cap \mathfrak{m}_y) \otimes L \longrightarrow 0$$

and the last sheaf is just finite dimensional space. This gives a long exact sequence.

$$\cdots \longrightarrow H^0(L) \longrightarrow H^0(L)/\mathfrak{m}_x \cap \mathfrak{m}_y \longrightarrow H^1(L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y)) \longrightarrow \quad \longrightarrow \quad \longrightarrow \cdots$$

If cohomology of that last one vanishes and also $H^1(L \otimes \mathfrak{m}_x^2)$ vanishes you are very ample. So its a vector bundle.

Remark So, for curves very amplness is very easy to check. Because every module is sum of rings by list 1 we have that the last one on the sequence is a vector bundle, finitely generated coherent, and then Kodaira says that vanishes.

Canonical bundle is very ample unless curve is elliptic?

9 Class 9: Nakai-Moishezon theorem

Today it will be mostly algebraic geometry, so no K3.

9.1 Ample bundles

Looks like here's another definition:

Definition L is ***very ample over*** X if

$$X \hookrightarrow \mathbb{P}H^0(x, L)^*$$

and L is ***ample*** if $L^{\otimes N}$ is very ample for some $N > 0$.

Remark $\overline{L}$ holomorphic line bundle on compact complex, then L is ample on a convex complex cone if and only if $\deg(L) > 0$.

9.2 Very ample bundles

Claim $x \neq y \in X$ then

$$\varphi : H^0(L) \longrightarrow H^0(L/(\mathfrak{m}_x \cap \mathfrak{m}_y)$$

is surjective, and the standard map

$$\varphi_L \longrightarrow \mathbb{P}H^0(X, L)^*$$

Two sections that have different derivative (1-jet) have different images, then the derivative is non-zero (derivative is injective). Uses inverse function theorem?

Proof. From (surjectivity?) of φ we get that

$$H^0(L) \longrightarrow H^0(L/\mathfrak{m}_x^2)$$

Then φ is isom to its image. (Because $\varphi = \varphi^*\Big|_{\mathfrak{m}_x/\mathfrak{m}^2}$) □

9.3 Very ample bundles again

Corollary If $H^1(L/(\mathfrak{m}_x \cap \mathfrak{m}_y)) = 0$ and $H^1(L \otimes \mathfrak{m}_y^2) = 0$ for all x, y then L is very ample.

Proof.

$$0 \longrightarrow H^0(L) \longrightarrow H^0(L/\mathfrak{m}_x^2) \longrightarrow H^1(\mathfrak{m}_x^2 \otimes L) \longrightarrow 0$$

□

9.4 Very ample bundles on a curve

Theorem (Kodaira-Nakano vanishing) L holomorphic line bundle on compact complex curve, $L \otimes K_M^{-1}$ ample. Then $H^i(L) = 0$ for all $i > 0$

Now we can do something about curves.

Corollary Let L be a line bundle on a compact complex curve C of genus g, and $\deg L > 2g$.

Proof.

Step 1 We need to prove that

$$H^1(L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y)) = 0 \qquad H^1(L \otimes \mathfrak{m}_x^2)$$

Slides: The sheaves $L \otimes \mathcal{O}_X \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y)$ are line bundles of $\deg L - 2$. Board: $L \otimes \mathfrak{m}^2$ and $L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y)$ are line bundles of degree $\deg L - 2$.

So what is it anyway?

$$L \otimes \mathfrak{m}_x^2 = L(-2x) \qquad L \otimes \mathfrak{m}_x \cap \mathfrak{m}_y = L(-x - y)$$

Step 2 The degree of the canonical bundle is $2g - 2$. By Kodaira, L_1 ample if and only if $\deg(L_1 \otimes K_X^{-1}) > 0$ iff $\deg L_1 > 2g - 2$. $(L \otimes \mathfrak{m}_x^2) \otimes K_X^{-1}$ ample and equals $H^1(L \otimes \mathfrak{m}_x^2) = 0$ by Kodaira-Nakano, $L_1 = L \otimes \mathfrak{m}_x$, or $L_1 = L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y$

□

9.5 Canonical map for a complex curve

Definition Let L be a line bundle on X. A point $p \in X$ is called a ***base point*** if all sections of L vanish in p.

Definition Assume K_X has no base points. Then

$$\psi_{K_X} : X \longrightarrow \mathbb{P}H^0(X, K_X)$$

is called the ***canonical map***.

Theorem C curve, $g(L) \geqslant 2$. K canonical bundle. Then $H^0(K)$ has no common zeroes, the canonical map $\psi : C \longrightarrow \mathbb{P}H^0(K)^*$ is embedded or is a two-sheeted (2-to-1) ramified covering to $\psi(C) = \mathbb{C}P^1$.

Remark In the second case C is called a ***hyperelliptic curve***. In step 3, we will prove that any curve admitting a two sheeted ramified covering to $\mathbb{C}P^1$ is hyperelliptic.

This gives you two theorems about moduli spaces. Two curves are isomorphic if these subvarietes are conjugated by linear map, so gives you a point in Hilbert scheme. So hyperelliptics are the same as $\mathbb{C}P^1\ldots$ ah, but it has to be separated, $(\mathbb{C}P^1)^n$, and without the diagonals, and

$$\Big((\mathbb{C}P^1)^n\backslash\text{diagonals}\Big)\Big/\,\mathsf{PGL}(2,\mathbb{C})\times\{2^{2g}\}$$

The power of 2 term are possible choices of ramification.

Proof.

Step 1 First we need to show there are no common zeroes. That is, sections of K have no common zeroes. Let $p\in\mathbb{C}$ and let $k_p = \mathcal{O}_C/\mathfrak{m}_p$. Consider

$$0 \longrightarrow K(-p) \longrightarrow K \longrightarrow k_p \longrightarrow 0$$

The corresponding long exact sqeuence says

$$H^0(K_C) \longrightarrow H^0(k_p) \longrightarrow H^0(K(-p))$$

so surjectivity is equivalent to $p\notin$ common zeroes.

But $H^1(K(-p)) = 0$ because $H^1(K(-p)) = H^0(\mathcal{O}(p))^*$ by Serre duality. So we only need to show that there are no section is the latter degree 1 bundle. If there exists $\gamma\in H^0(\mathcal{O}(p))$ meromorphic function with a single pole on p, we obtain a holomorphic function $f : C \longrightarrow \mathbb{C}P^1$ of degree 1, meaning $C = \mathbb{C}P^1$.

Step 2 ψ non-injective, that it glues together p and q. Then $H^1(K(-p-q)) \neq 0$.And by Serre duality $H^0(\mathcal{O}(p+q) = H^1(K(-p-q)) \neq 0$. Then there exists a meromorphic function f with poles on p, q and $f : C \longrightarrow \mathbb{C}P^1$ of degree 2.

Step 3 So you have a map with two preimages. So consider a map that exchanges the preimages.

Now we will show that $\psi(C) = \mathbb{C}P^1$ admits a two sheeted ramified covering to $\mathbb{C}P^1$. Let $\tau : C \longrightarrow C$ be the involution exchanging the sheets of the covering. It is holomorphic because it has only Riemann-extendible singularities. And it acts on $H^0(K_C)$ with eigenvalues ±1. But $H^0(\Omega^1(\mathbb{C}P^1)) = 0$ so $\tau|_{H^0(K_C)} = -\mathsf{id}$. Therefore we see that ψ glue p and q.

□

9.6 Finite morphisms

Definition Let $f : X \longrightarrow Y$ a morphism of varieties (or schemes). f is ***finite*** if for every $U\subset Y$ open, $\mathcal{O}_{f^{-1}(U)}$ is finitely generated as an $H^0(\mathcal{O}_U)$-module.

Board: if the ring $f^*\mathcal{O}_U$ is finitely generated as an $\mathcal{O}_V$-module where $V = f^{-1}(U)$.

Theorem $f : X \longrightarrow Y$ proper and the preimage of any point is finite.

Proof. Hartshorne exerice III 11.2. And also past courses and EGA. □

9.7 Amplennes and cohomology

Theorem L is ample iff $\forall$ coherent F there exists $d > 0$ such that $H^i(F \otimes L^{\otimes k}) = 0$ for all $i > 0$ and $k \geqslant$.

Proof. Hartshorne. □

Theorem f finite functor then pusforward is acyclic exact functor.

$f : X \longrightarrow Y$ finite map, F coherent sheaf on X. Then

$$H^i(f^{-1}(U), F) = H^i(U, f_* F)$$

for any open set $U \subset Y$; in other words, $R^i f_* F = 0$ for all $i > 0$.

Proof. The Rising Sea, thm 18.7.5. □

Corollary 1 L line bundle on a complex variety X such that the standard map $f : X \longrightarrow \mathbb{P}H^0(X, L)^*$ is finite. Then L is ample.

Proof. (Proof is simple but unfortunately uses complicated theorems.)

Let $Y = f(X)$ and F a coherent sheaf on X. We have that $L = f^*(\mathcal{O}(1))$. And then there is the formula of base-change (maybe) which says (and it works for any other sheaf instead of $\mathcal{O}(1)$) that $f_*(F \otimes_{\mathcal{O}_X} L^{\otimes k} = f_* F \otimes_{\mathcal{O}_Y} \mathcal{O}(k)$. Ok and then we can compute cohomology:

$$H^i(X, F \otimes L^{\otimes n}) = H^i(Y, \underbrace{F \otimes \mathcal{O}(n)}_{=0}$$

so L is ample. □

What is this for? We want to explore some invariants on K3.

9.8 Nakai-Moshezon theorem

It's a very nice criterion for ampleness. It can be generalized to Kähler, but we won't do that.

Theorem (Nakai-Moshezon) Let L be a line bundle on a projective variety X. Suppose for all subvarieties $Y \subset X$

$$\int_Y c_1(L)^d > 0, \qquad d = \dim Y$$

then L is ample.

Proof. It's seven steps.

Step 1 Let's do induction on $\dim X$. For $\dim X = 1$ it is clear. Assume that L is ample on all proper $X_1 \subsetneq X$. The next step is the most difficult step: show that $H^0(X, L^{\otimes n}) \neq 0$ for $k \gg 0$

Step 2 We need a very ample bundle. Let L_1 be a very ample bundle with a sufficiently big c_1 such that $c_1(L \otimes L_1 \otimes K_X^{-1}$ is Kähler ($\overset{?}{=}$ ample). Then $H^1(L_1 \otimes L) = 0$ for all $i > 0$. Let H be a smooth zero divisor of L_1 such that $\mathcal{O}(H) = L_n$. Now consider the short exact sequence

$$0 \longrightarrow L \longrightarrow L \otimes \mathcal{O}(H) \longrightarrow L \otimes \mathcal{O}(H)|_H \longrightarrow 0$$

I took a section of L_1, wrote this exact sequence and the one on the right is ample by assumption. Then we can replace L by a sufficiently big power $L^{\otimes d}$, we may assume that $L^{\otimes d}|_H$ is ample. (We want to show L_1? is Kähler. If you are an algebraic geometer then maybe you'd say something like any bundle is ample if you multiply be a sufficiently large power...) I think here we tensor multiply to get

$$0 \longrightarrow \mathcal{O}(-H) \longrightarrow \mathcal{O} \longrightarrow ? \longrightarrow 0$$

Anyway, we get

$$\cdots \longrightarrow H^{i-1}(L^{\otimes d} \otimes \mathcal{O}(n)|_H) \longrightarrow H^1(L^{\otimes d}) \longrightarrow H^i(L^{\otimes d} \otimes L_1) \overset{?,\text{board}}{=} H^1(L^{\otimes d} \otimes \mathcal{O}(H)) \longrightarrow \cdots$$

then there exists $d \gg 0$ such that $H^i(L^{d+j}) = 0$ for all $j \geqslant 0$ and $i > 1$.

(We didn't prove L is ample on X. The idea is that the curvature of $\mathcal{O}(n) \otimes L^{\otimes d}$ is strictly positive ($\overset{?}{=}$ has sections. Perhaps the use of d can be avoided if we choose H properly...)

Again, we have

$$\cdots \longrightarrow H^{i-1}(L^{\otimes d} \otimes \mathcal{O}(n)|_H) \longrightarrow H^1(L^{\otimes d}) \longrightarrow H^i(L^{\otimes d} \otimes \mathcal{O}(n)) \longrightarrow \cdots$$

and then

$$K_H = K_M|_H \otimes \mathcal{O}(n)$$
$$K_H^{-1} = K_M|_H \otimes \mathcal{O}(-n)$$

We proved there's only two possible non zero cohomology which is H^1 and H^0.

That is, there exists $d \gg 0$ such that $H^i(L^{\otimes j}) = 0$ for $i > 1, j > d$

Step 3 Looks like by Riemann-Roch, $\chi(kL)$ is a polynomial of k of degree n given by Todd.

$$\int_X \frac{kc_1(L)^n}{n!} = \lim_{k\to\infty} \chi(kL) = \infty$$
$$\implies \lim_{k\to\infty} \mathrm{Ch.}(H^0(kL)) = \infty$$

Step 4 Replace L by $L^{\otimes k}$. Can assume $\dim H^0(L) > d$. Then

$$0 \longrightarrow (K-L)L \longrightarrow kL \longrightarrow kL|_D \longrightarrow 0$$

Now by inductive assumption $L|_D$ ample. So $H^i(kL) = 0$ for all $k > d$ and $i > 0$. Then we get the long exact sequence

$$0 \to H^0((k-1)L) \to H^0(kL) \to H^0(kL|_D) \to H^1((k-1)L) \to H^1(kL) \to 0$$

Now the function $k \mapsto \dim H^1(kL)$ is monotonous non-increasing, so it must stabilize. Therefore for $k \gg 0$ we get an surjection $H^0(kL) \longrightarrow H^0(L|_D)$.

Step 5 The birrational map $\Phi : X \longrightarrow \mathbb{P}H^0(kL)$. Every $x \in X$ is connected in D for some section of L to every $y \in D$ (?). Slides: D can be chosen as $\Phi^*(H)$ where Φ^* is proper preiage and H a hyperplane section in $\mathbb{P}H^0(L)^*$, therefore for any two points $x, y \in X$ we may choose D containint these two points.

Then there exists a section of kL non-zero on y, implying that Φ is holomorphic. So there is a section of kL separating these points (vanishing in one and non-vanishing in the other). For each point, there is a section that is not zero.

Step 6 The bundle kL is ample by Corollary 1.

$\square$

Remark See the book *Positivity in algebraic geometry*.

10 Class 10: surfaces with Picard rank 1

10.1 Intuition and review

Today we'll see how to construct quadrics. Any K3 with rank 1 and Picard rank 4 is a quartic.

Mumford offered money to anyone who could give a K3 with Picard rank 1. Took 30 years.

So far we have:

Theorem (Kodaira) L ample $\iff$ $c_1(L)$ Kähler.

Theorem (Kodaira-Nakano vanishing) $L \otimes K_M^{-1} \implies H^i(L) = 0 \forall i > 0$.

Theorem (?) C compact complex smooth curve then K_C is globally generated $\phi : C \longrightarrow \mathbb{P}H^0(K_C)$ is 2:1 smooth cover or embedding.

Theorem (result, see last class) Is ample iff kills cohomology

Corollary 1 $X \longrightarrow \mathbb{P}H^0(X, L)^*$ is finite then L is ample.

Intuition (of what's about to happen) Consider a curve with self-intersection (a hand-drawn line/string that intersects itself). Then we will resolve that singularity by "streching the string" or "separating the branches at the intersection point".

10.2 Singular curve in a K3 surface

Claim Let $C \subset M$ be a curve in a singular complex surface. Then there exists a surface $\tilde{M} \xrightarrow{\pi} M$ obtained by succesive blow-ups of M such that the proper preimage $\tilde{C}$ of C is smooth. (The ***proper preimage*** is taking the points in the curve and not the exeptional divisord.)

Definition Multiplicity of a singular point is dimension? of cohomology.

Proof. Take the single blow-up of C in a singular point. It has smaller multiplicity; We are saying that $\tilde{C}$ has strictly smaller multiplicity. For this we have thought of

$$\begin{array}{c} \tilde{M} \\ \downarrow{\scriptstyle \pi} \\ M \subset C \end{array}$$

and the exceptional divisor is E in $\pi^{-1}(C) = \tilde{C} + E$. So to compute mutliplicity we have done

$$\tilde{L} \cap (\tilde{C} \times E) = L \cap C$$
$$\implies \tilde{L} \cap \tilde{C} < L \cap C$$

□

Remark So we can resolve singularities and that's what happens with multiplicity.

10.3 Singular curve in a K3 surface (corollary 2)

So let M be a K3 surface and C a singular curve in K3. Then take the pullback bundle of C and call it $L = \mathcal{O}(\pi^{-1}(C)) = \pi^*(\mathcal{O}(C))$. Remember that $\pi^{-1}(C) = \tilde{C} + E$. Then $\mathcal{O}(\tilde{C}) \otimes \mathcal{O}(E) = L$, so the *normal bundle* is $N(\tilde{C}) = L \otimes \mathcal{O}(-E)$ and $K_{\tilde{M}} = \mathcal{O}(E)$. So we can compute the canonical bundle: $K_{\tilde{M}} \otimes N_{\tilde{C}} = K_{\tilde{C}}, \mathcal{O}(E) \otimes L \otimes \mathcal{O}(-E) = L$.

Corollary 2 C curve of genus ¿0 in K3 surface M. Then $\mathcal{O}(C)|_C$ is globally generated.

Proof. Need to show □

10.4 Picard rank 1 is usually very ample

Theorem M K3 with $\mathrm{Pic}(M) = \mathbb{Z}$ and L a generator of $\mathrm{Pic}(M)$ such that $(L, L) \geqslant 0$. Then L or L^* is globally generated.

Proof.

Step 1 We use Riemann-Roch (like everytime when you get a surface). We get that $h^0(L) - h^1(L) + h^2(L) = 2 + \frac{(L,L)}{2}$ so $h^2(L) - h^1(L) \geqslant 2$. Now since $h^0(L) = h^2(L)^*$ we may just assume that $h^0(L) > 0$ (by interchanging L by L^*).

Step 2 Here's the only place where we use that C is a curve. Let $D \in |L| :=$ zero divisor sections of L. So dimension 1 implies (it think) that D is irreducible. No we have that $L = \mathcal{O}(D)$. We can construct the short exact sequence

$$0 \longrightarrow \mathcal{O}_M \longrightarrow L \longrightarrow L|_D \longrightarrow 0$$

Which gives

$$\cdots \longrightarrow H^0(L) \longrightarrow H^0(L|_D) \longrightarrow H^1(\mathcal{O}_M) = 0 \longrightarrow \quad \longrightarrow \quad \longrightarrow \cdots$$

so the restriction map is surjective, and every section $L|_D$ extends to M.

Step 3 $L|_D$ is globally generated. (Slides: The bundle $L|_D$ is base point free by Corollary 2.

Remark Let $\pi : (\tilde{M}, \tilde{D}) \longrightarrow (M, D)$ a resolution of singularities. Since $\pi^* L = K_{\tilde{D}}$, then the restriction $\pi^* L|_{\tilde{D}}$ is very ample if $\tilde{D}$ is not hyperelliptic.

□

10.5 The same theorem but a little different

Theorem M K3 with $\mathrm{Pic}(M) = \mathbb{Z}$ and L a generator of $\mathrm{Pic}(M)$ such that $(L, L) > 2$. Then L or L^* is ample, base point free and the map $\psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ is an embedding or a ramified covering.

Proof. Now we use Corollary 1, for ampleness. □

10.6 Hyperelliptic curves

First let's count the fixed point of the involution.

Lemma If $\tau : C \to C$ is the hyperelliptic involution, then τ has $2g$ fixed points.

Proof. Let f be the number of fixed points. So it acts on tangent send point to minus. So simple fixed points. Here e is Euler characteristic.

$$2 - 2g = e(C) = e(\mathbb{C}P^1) - f = 2 - f$$

So where do the fixed points appear? □

Proposition All curves of genus 2 are hyperelliptic.

Proof. (uses Serre duality) □

10.7 A third variation of that theorem

Theorem M K3 with $\mathrm{Pic}(M) = \mathbb{Z}$ and L a generator of $\mathrm{Pic}(M)$ such that $(L, L) > 2$. Then the map $\psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ is a two sheeted ramified cover if $(L, L) = 2$ and an emedding otherwise; M is a 2-sheeted covering of $\mathbb{C}P^2$ or a sextic (when $(L, L)^2 = 2$).

Proof.

Step 1 **The case $(L, L) > 0$, when we have an embedding.** Again, assume that $H^0(L) \neq 0$, so L is ample, globally generated and with a general smooth $D \in |L|$. Looks like we computed $(L, L) = 2$.

Step 2 **The hyperelliptic case.** $(L, L) = 2$. We use Kodaira-Nakano, so $H^1(L) = 0$, and $\chi(L) = 2 + \frac{(L,L)}{2} = 3$, so $\dim H^0(L) \overset{?}{=} 0$ and we get the map $\psi : M \to \mathbb{C}P^2$, a ramified covering.

Step 3 **It is a sextic.** ψ is ramified in a sextic. We used $R \subset M$ ramification divisor to show

$$K_M = \psi^* K_{\mathbb{C}P^2} \otimes \mathcal{O}(R) = \mathcal{O}(M)$$

Then $\psi^*(\mathcal{O}(-3)) = L^{\otimes -3}$ gives $L^{\otimes -3} \otimes \mathcal{O}(R) = \mathcal{O}_M$, $[R] = 3c_1(L)$.

$R_0 = \pi(R)$, we get

$$[R_0] \cap [\text{hyperplane section}] = [R] \cap [D] = 3(L, L) = 6$$

by computing

$$\int \pi_* x \wedge y = \int x \wedge \pi^* y$$

where $x = [R]$ and $y = [H]$.

Remark (Check!) K3 ramified over $\mathbb{C}P^2$ and sextics are in correspondence.

Corollary 3 K3, $\mathrm{Pic}(M) = \langle L \rangle$, $(L, L) > 2$, then L or L^* is very ample.

Proposition K3 surface isomorphic to quartic if and only if $\mathrm{Pic}(M)$ contains a very ample bundle $L \in \mathrm{Pic}(M)$ with $(L, L) = 4$.

Proof.

Step 1 Let $\varphi : M \hookrightarrow \mathbb{C}P^3$ be the embedding and $L := \phi^*(\mathcal{O}(1))$. We have

$$(L, L) = \int_M c_1(L) \wedge c_1(L) = \int_{\mathbb{C}P^3} [M] \wedge [H] \wedge [H] = 4$$

Step 2 L very ample $(L,L)=4$, RR $h^0(L)=\chi(L)=2+\frac{(L,L)}{2}=4$

$\square$

Corollary 4 M K3 surface with $\mathrm{Pic}(M)=\mathbb{Z}$ and L generating $\mathrm{Pic}(M)$ and $(L,L)=4$. Then M is isomorphic to a quartic.

$\square$

11 Class 11: Density of quartics deduced from Ratner theory

11.1 A result from XX century

Result M K3, there exists an integer vector $X \in H^2(M,\mathbb{Z})$ with zero intersection number, i.e., $(X,X)=0$.

This is nonelementary.

11.2 Today

Consider R, the set of all integer vectors $X \in H^2(M,\mathbb{Z})$ with $(X,X)=4$, $V(R)\subset \mathrm{Gr}_{++}$ sec of 2-planes $L\subset H^2(M,\mathbb{R})$. $L\perp x$, some $x\in R$. Then $V(R)$ is dense in Gr_{++}

Remember that

$$\mathrm{Gr}_{++}(H^2(M,\mathbb{R})) = \text{positively oriented 2-planes in } H^2(M,\mathbb{R})$$

Today we will give a proof and next lecture another. And you should find another. One proof is that Corollaries 3 and 4 imply it.

11.3 Noether-Lefschetz locus in the period space

Definition Let $D\in H^2(M,\mathbb{R})$, and consider Teich_η the set of all complex structures such that over $\eta\in H^i(M,I,\mathbb{R})$. Then

$$\mathbb{P}\mathrm{er}_\eta = \{W\in \mathrm{Gr}_{++} : W\perp\eta\}$$

is the ***Noether-Lefschetz locus***.

Remark $\mathbb{P}\mathrm{er}_\eta$ is the intersection of $\mathbb{P}\mathrm{er}$ and a complex hyperplane $\mathbb{P}\eta^\perp\subset\mathbb{P}H^2(M,\mathbb{C})$.

Claim $\mathrm{Teich}\longrightarrow\mathbb{P}\mathrm{er}_\eta$ is a local difeomorphism

Definition ($\mathbb{P}\mathrm{er}(V)$) $V\subset H^2(M,\mathbb{R})$, $\mathbb{P}\mathrm{er}\cap\mathbb{P}(V\otimes C)=\mathbb{P}\mathrm{er}(V)$ complex analytic in $\mathbb{P}\mathrm{er}$ [...] This is also called the ***Noether-Lefschetz locus***.

11.4 The set of quartics with Picard rank 1

Remark $H^{11}(M, I) = \mathbb{P}\mathrm{er}(I)^{\perp}$

Definition $\mathbb{P}\mathrm{er}^0_\eta = \{v \in \mathbb{P}\mathrm{er} : \mathrm{Pic}(M, \mathbb{Z}) = \mathbb{Z}\}$

Claim $\mathbb{P}\mathrm{er}^0_\eta$ is dense in $\mathbb{P}\mathrm{er}_\eta$, $\eta \in H^2(M, \mathbb{Z})$.

Proof. $\mathfrak{S}$ set of rank 2 subgroups in $H^2(M, \mathbb{Z})$ containing η [...See slides] □

11.5 Set of all quartics is dense

Theorem

$$\bigcup_{\eta | (\eta, \eta) = 4} \mathrm{Teich}_\eta \text{ is dense in } \mathbb{P}\mathrm{er}$$

Proof. Later today □

Corollary 2 The set $\mathfrak{D}$ of K3 with Picard group of rank 1 generated by vectors x such that $(x, x) = 4$, then $\mathfrak{D}$ is dense in Teich.

Remark By corollary 1 (?), all such x correspond to quartics, therefore, corollary 2 implies that quartics are dense in Teich.

Corollary Every K3 is diffeomorphic to a smooth quartic.

11.6 Ergodic measures

Definition Let (M, μ) be a space with a measure and G a group acting on M preserving μ. This action is ***ergodic*** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

Remark Ergodic measures are extremal rays in the cone of all G-invairiant measures.

Remark Any G-invariant measure on M is expressed as an average of a certain set of ergodic measures (Choquet's theorem). Therefore, G-invariant ergodic measures always exist.

Claim M manifold, μ Lebesgue measure, G a group acting on (M, μ) ergodically. The set of non-dense orbits has measure 0.

Proof. Done in class [see slides], should be simple. □

11.7 Ratner theory (lattices)

Definition Let G be a connected Lie group with a Haar measure. A ***lattice*** $\Gamma \subset G$ is a discrete subgroup of finite covolume, that is, G/Γ has finite volume.

Example By Borel and Harish-Chandra theorem, any integer lattice in a simple Lie group has finite covolume.

Theorem (Moore) If you have G/H, G simple, G with finite center, H is noncompact, Γ is a lattice. Then Γ acts on G/H ergodically. That is, for all Γ-invariant measurable subsets $Z \subset G/H$, either Z has measure zero or $G\backslash H/Z$ has measure 0.

Theorem (Ratner) $\Gamma = G_{\mathbb{Z}}$ integer lattice, and $H \subset G$ gnerated by unipotents.

$x \in G/H$, $\overline{\Gamma x}$ (closure of orbit) is an orbit of the smallest rational subgroup $U \subset S \subset G$. Then $S \cap \Gamma^x$ is a lattice in S.

11.8 Oppenheim conjecture

Definition An ***irrational*** quadratic form is $q : V = Z_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathbb{R}$ indefinite, non proportional to an integer.

Conjecture (Oppenheimer 29', Margulis proved in 87, and there's another proof in early 90's) q irrational quadratic form in $\mathbb{R}^n$, $S_q = q(\mathbb{Z})$, then S_q is dense in $\mathbb{R}$.

Proof. We go to Ratner theorem.

Step 1 Consider $G = \mathsf{SL}(n, \mathbb{R})$, which has a lattice, and $\mathsf{SO}(a, b)$. Let G/H be the set of all quadratic forms of sign (a, b).

[Content missing…]

□

11.9 An exercise and a theorem using Ratner theory

Exercise (Classify intermediate subgroups) Let $G = \mathsf{SO}(a, b)$ and $H \subset G$ the stabilizer of a point in $W \in \mathsf{Gr}_{++}(\mathbb{R}^{a,b})$, then there is only one type of intermediate subgroups between G and H.

Therefore, Ratner theorem implies

Proposition $G = \mathsf{SO}(3, 19)$, $H = \mathsf{SO}(1, 19) =$ stabilizer of $W \in \mathsf{Gr}_{++}$. Then $H \subseteq H_1 \subseteq G$ so H_1 is the stabilizar of a vector in W. $\mathsf{SO}(H^2(M, \mathbb{Z})) \cdot W$-dense in $W \cap H^2(M, \mathbb{Z})$

Need $\mathsf{SO}(H^2(M, \mathbb{Z})) \cdot \mathbb{P}\mathsf{er}$ is dense in $\mathbb{P}\mathsf{er}$.

Any $W \in \mathbb{P}\mathsf{er}$ with $W \notin$ (?)…

12 Class 12: density of quartics, a more elementary proof

12.1 The idea

is that complex structures are dense. Specifically

Theorem 2 The set of all vectors with $(x, x) = 4$ in $H^2(M, \mathbb{Z})$, called R. And the set of all 2-planes orthogonal to some $x \in R$, called $Z(R) \subset \mathsf{Gr}_{++}$. **Then** $Z(R)$ **is dense in** Gr_{++}**.**

12.2 Quadratic lattices

Definition ***Integer quadratic lattice*** is a lattice with integer-valued scalar product.

The lattice Λ, q ***represents*** k if there is $x \in \Lambda$ with $q(x) = k$.

Theorem 1 (About lattices) $(V_{\mathbb{Z}}, \Lambda$ lattice ... then $Z(\mathfrak{R})$ is dense

Lemma 1 Taking Z is compatible with closure: $Z(\bar{A}) = \overline{Z(A)}$.

12.3 The null quadric

Definition Null is the quadric $q(x, x) = 0$.

Remark I think that for every positive plane there is a point in the null quadric orthogonal to it. In slides: $Z(\mathsf{Null} = \mathsf{Gr}_{++}$.

Our objective of today is reduced to

Theorem (3) $\overline{\mathbb{P}R} \supset \mathsf{Null}(V_{\mathbb{R}})$

Remark (Homework hint) "They are the same sets"

12.4 Extending isometries of a lattice

Before lattices were of the same rank, now they're different ranks.

Corollary 2 One lattice inside another, then the isometris of the smaller that can be extended to the larger is of finite index I think the isometries of the smaller.

13 Class 13: limit points of orbits of $SO_{\mathbb{Z}}(p, q)$

13.1 Summary of last lecture

Running assumtions

- $(V_{\mathbb{Z}}, q)$ quadratic lattice, signature $\geqslant (1, 3)$

A great question for me is how to relate theorem 3 to theorem 1. Is is related to the orbits. But how does density follow?

13.2 Quadratic form representing 0

Definition (Λ, q), we say q ***represents*** n if there is $x \in \Lambda$ such that $q(x, x) = n$.

x is ***primitive*** if it is not divisible by a number (so product o number times vector is x?)

Remark x is primitive iff $\Lambda / \langle x \rangle$ is torsion free.

Proof. Very easy. □

Remark x is primitive iff $\exists \eta \in \Lambda^*$ such that $\langle \eta, x \rangle = 1$.

Theorem (Meyer, 1888) Is you have an indefinite lattice of rank greater o equal to five, then Λ represents 0.

Idea of proof. Go to p-adic numbers. Legandre symbols, Hilbert symbols, Hasse principle. See *Ueber eien Satz von Dirichlet* by A. Meyer. □

13.3 Quadratic forms representing 4

We saw that

- The ***hyperbolic lattice*** U_2 represents 4. (So there is an alement nameley $2x + y$ that quadatic form evaluates 4.
- Any unimodular even quadratic lattice that represents 0 contains U_2.

Which allows to prove easily that

Theorem In every K3 there is a vector such that $(v, v) = 4$. Whis means that the intersection lattice $H^2(M, \mathbb{Z})$ represents 4.

Proof. Because we know that $H^2(M, \mathbb{Z}) = 3U_2 \oplus 2E_{-8}$ which contains U_2. Also you may use Meyer to see Λ is unimodular, so represents 0 so it contains U_2 (I think). Notice we had seen that this lattice is even using Riemann-Roch. □

13.4 Reminder on discriminant of lattices

Proposition $\Lambda_1 \subset \Lambda$ quadratic lattices of the same rank. Then $\mathsf{SO}(\Lambda_1)$ and $\mathsf{SO}(\Lambda)$ are commesurable.

Proof. We will use that there only finitely many choices of intermediate lattices between Λ_1 and Λ_1^* measured by the discriminant (claim 1).

$\Gamma_2 = \mathsf{SO}(\Lambda) \cap \mathsf{SO}(\Lambda_1)$. So Γ_1 acts on the set of intermediate groups. This makes it have finite index because of the stabilizer.

Now let $\Gamma_3 = \mathsf{SO}(\Lambda)$. Then we show that Γ_2 is finite index in Γ_1: take N such that $N\Lambda \subset \Lambda_1$ so $\mathsf{SO}(N\Lambda) \cap \mathsf{SO}(\Lambda_1)$ is finite index $\mathsf{SO}(N\Lambda) = \mathsf{SO}(\Lambda)$. What is going on, the groups are isomorphic after multipliyng by a constant? □

Most important corollary in all this (A, q) nondegenerate quadratic lattice, $B \supset A$ superlattice of the smaller rank. denote $\Lambda_A \subset \mathrm{SO}(A)$ the group of isometries of A that can be extended to B. Then Γ_A is of finite index in $\mathrm{SO}(A)$.

Proof. Consider the lattice $B_1 := A \oplus A^{\perp} \subset B$ because (if you tensor it by $\mathbb{Q}$) it will have the same rank. So use previous corollary and $\mathrm{SO}(B_1) \cap \mathrm{SO}(B)$ is finite index in $\mathrm{SO}(B)$. This means every $\gamma \in \mathrm{SO}(A)$ can be extended to $\mathrm{SO}(B_1)$. Now look at stabilizers $\mathrm{St}_A\, \mathrm{SO}(B) \cap \mathrm{St}_A\, \mathrm{SO}(B_1)$ and project to $\mathrm{SO}(A)$ so the projection is of finite index since the others were already of finite index. □

What is the point of this. That Λ_2 rank 2 of signature $(1, 1)$ is a sublattice of $\mathrm{SO}(\Lambda)$, so Λ_2 ??? acts? on $\mathrm{SO}(\Lambda_2) = \mathbb{Z}$.

13.5 Pell's equation

Definition An integer is called ***square-free*** if it is not divisible by a square of some number not 1.

Remark The unit sphere of $\mathbb{Z} \oplus \mathbb{Z}\sqrt{w}$ where the norm is $N(a + b\sqrt{w} = a^2 - wb^2$ is a multiplicative group. Solution of Pell equation is norm 1.

Theorem (Pell, Dirichlet second time) Let $w \in \mathbb{Z}_{>0}$ that group is isomorphic to $\mathbb{Z}$ up to sign.

Remark Norm is a quadratic form in this ring $\mathcal{O}_K$. Solution of Pell is in $\mathrm{SO}(O_K, N)$

Pell did not solve Pell equation This is a mistake by Euler. Pell barely translated the solution by /..?

Of Pell theorem version second time. It suffices to show that Pell equation has a non trivial solution. □

Theorem (Lagrange) Pell equation has a non-trivial solution

We need a lemma to prove this theorem.

Lemma There exists infinitely many solutions x, y such that $|x - \sqrt{w}y| < 1/y$.

Proof of lemma. Partition the interval $[0, 1[$ into m little intervals starting with $[0, 1/m[$ and then $[1/m, 2/m[$ and so on. Then pigeon principle!! Because if $a, b \in [0, m]$ then their fractional parts are in the same little interval. □

Proof of Lagrange.

Step 1 By the lemma, the equation $x^2 + y^2 w = M$ has infinitely many solutions. Because there is a certain value in a certain sequence that appears infinitely many times.

Step 2 Let M be an integer for with $x^2 - wy^2 = M$ has infinitely many solutions. This means there exist two numbers $z_1, z_2 \in \mathbb{Z} + \mathbb{Z}\sqrt{w}$ such that $z_1 \equiv z_2 \bmod M$ and that $N(z_1) = N(z_2)$, why? After more computations we see that $M = M \cdot N(z)$ and $N(z) = 1$. z is $z_3\sigma(z_2) + 1$. what is σ?

□

Theorem (Pell, Dirichlet first time) Let $w \in \mathbb{Z}_{>0}$Integer solutions (a, b) of $a^2 - wb^2 = 1$ are isomorphic to $\mathbb{Z}$.

Proof. There should be an elementary proof but we can't remember. □

Theorem (The main application of Pell equation) Λ lattice of signature $(1, 1)$ which does not represent 0. Then $\mathsf{PSO}(\Lambda) = \mathbb{Z}$ (PSO *is trivial*.

14 Class 14

Today we will work on a statemnt about orbits of the symmetry group. We want to show that closure of the orbits is the Null quadric. What is the group?

Theorem 3 $\mathsf{SO}(\Lambda, 1)$ acts on $\mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ closure of each orbit contains the null quadric.

Also remember Lengendre's theorem that Pelle equation has solution. Also remember that two lattices of the same rank have commesurable isometry groups. And the other case when lattices are not of the same rank, so the group of isometries that can be extended from the small to the large is finite index in the large.

Why do we need this? $\mathsf{SO}(1, 1)$ is a Lie group that is acting on ?. But there's also the $\mathbb{Z}$ action with (limit?) points in the hyperbolas which are the orbits of the Lie group action. We will find a sublattice of the discrete one and apply corollary 3 which is the last statment in the paragraph above.

OK so looks like next step will use Pell equation. You use pell equation to produce an integer lattice via $z - wz$, which we have already shown to be a quadratic form on $\mathcal{O}_K$. "Solution of Pell gives you an isometry of this lattice"

14.1 Quadratic lattices of rank 2

Definition $\mathsf{PSO}(\Lambda)$ as the group of isometries of Λ quotient ± 1 if rank is even and no quotient if rank is odd.

Exercise Let Λ be lattice of signature (1,1) wich represent 0. Then $\mathsf{PSO}(\Lambda)$ is trivial

Solution. Consider the isotropic vectors (norm 0). This is two lines (because its like a quadric form in the plane). There are 4 interesting vectors: one positive and one negative in each of the lines; they are interesting because they are primitive. Then you have to take permutations. Then

$$\begin{array}{ccc} \mathsf{O}(\Lambda) & \hookrightarrow & (\mathbb{Z}/2)^2 \\ & & \downarrow{\scriptstyle \det} \\ & & \pm 1 \end{array}$$

□

Theorem 4 Let (Λ, q) be a integer laticce with $(1,1)$ signature wchich does not represent o. Then $\mathsf{PSO}(\Lambda) \cong \mathbb{Z}$.

Proof.

Step 1 I think you go to rationalification $\Lambda \otimes_{\mathfrak{X}} \mathbb{Q}$ and find a diagonal form of the matrix there. This is $q = ax^2 - by^2$. and then just do

$$aq = a^2x^2 - bay^2 = (ax)^2 - biy^2 \implies q = x_1^2 - wy_2^2$$

There are two lattices, multiply by denominators, get integer lattices. They are commesurable. But they are sublattices of $\mathbb{Z}$. The point is $\mathsf{PSO}(\Lambda)$ contains $\mathbb{Z}$.

Step 2 Pass to the Lie group $\mathsf{PSO}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ which is like $\mathbb{R}$. And we have put $\mathsf{PSO}(\Lambda)$ inside, and $\mathsf{PSO}(\Lambda)$ contains $\mathbb{Z}$, which is the only subgroup of $\mathbb{R}$ so it is $\mathbb{Z}$.

□

14.2 Rational lines intersecting a rational quadric in irrational points

Now I'm going to provide you with lots of sublattices not representing 0. Let's start with X a rational quadric.

Some proposition about irrational lines The set of irrtional points on a non-degenerate rational quadric in $\mathbb{R}P^2$is dense. Irrational=point not proportional to a point that has at least one coordinate irrational.

Proof.

Step 1 We notice that its enogh to find one irrational point because if we do then we act on it by reflections "centered in rational lines", which have a dense orbit because because product of *any* two reflections gives a rotation.

Step 2 We find an irrational point going to affine chart and doing some very basic computation. that felt stupid Because it should be more easy than this.

□

Idea That if you have a quadratic equation varying with some parameter there are some quadratic equat ions that don't have dsolltuions.

15 Null quadric as a limit set

Theorem 3 **The closure of $\mathbb{P}\mathfrak{R} \subset \mathbb{P}V_{\mathbb{R}}$ contains $\mathsf{Null}\, V_{\mathbb{R}}$**

Proof. We are looking for a space of signature $(2,1)$. Let η be an element of signature ¿0. Well in one case that $\langle \eta, x\rangle$ is non gegenerate so there is an element in the orthogonal complement of that non degenerate vector. Then look at the space of $\langle \eta, x\rangle + y$.

In another case $\langle \eta, x\rangle$ is degenerate so $x \perp \eta$. Then $x^{\perp} \cap \eta^{\perp}$ is codimension 1. Then $y \in (\eta^{\perp} \backslash x^{\perp})$.

So we have this subspace S connecting x to y.

Now since rational . . . $\square$

We have reducted Theorem 3 to the following lattice proposition:

Proposition $V_{\mathbb{Z}}$ quadratic lattice of signature $(2,1)$ and $\eta \in \mathbb{P}V_{\mathbb{R}}$ a point with positive square. The closure of the $\mathsf{SO}(V_{\mathbb{Z}})$-orbit of η contains $\mathsf{Null}(V_{\mathbb{R}}$.

Proof. The problem is taken to hyperbolic geometry. We see that the irrational points are limit points. And irrational points are dense. So the limit points of those isometries are dense. So the corresponding lattice doesn't represent zero.

Therefore, $\mathsf{Null}\, V_{\mathbb{R}}$ is contained in the closure of any orbit. $\square$

Remark Originally this was shown in a much harder way, a sequence of six or seven lemmas using classical hyperbolic geometry.

15.1 Summary of the past few lectures

The idea is to show that

> Quartics are dense in the Teichmüler space of K3 surfaces, so that every K3 is diffeomorphic to a quartic.

Step 1

Corollary 1 Let M be a K3 surface such that $\mathsf{Pic}(M) = \mathbb{Z}$, and L the line bundle generating $\mathsf{Pic}(M)$. Assume that $(L, L) = 4$. Then M is isomorphic to a quartic.

Step 2

Definition (Period) Let Teich be Teichmüler space of complex structures of Kähler type on a K3 surface. The corresponding period space is

$$\mathbb{P}\mathsf{er} = \left\{ v \in \mathbb{P}H^2(M, \mathbb{C}) : \int_M v \wedge v = 0, \int_M v \wedge \bar{v} > 0 \right\}.$$

Definition Let M be a K3, $\eta \in H^2(M, \mathbb{R})$ be a non-zero class, and $\mathbb{P}\mathsf{er}_{\eta}$ the set of all points such that $\eta \perp v$, or equivalently [. . .]

Step 3

Theorem $*$ Let M be a K3 and $\mathfrak{R} \subset H^2(M, \mathbb{Z})$ the set of all vectors η such that $(\eta, \eta) = 4$. Then $\bigcup_{\eta \in \mathfrak{R}} \mathbb{P}\mathrm{er}_\eta$ is dense in $\mathbb{P}\mathrm{er}$.

gives via local Torelli theorem

Corollary 2 Let $\mathfrak{D} \subset \mathsf{Teich}$ be the set of all K3 with Picard group of rank 1 generated by a vector x with $(x, x) = 4$. Then $\mathfrak{D}$ is dense in Teich.

so that

Corollary Every K3 is diffeomorphic to a smooth quartic.

where

Proposition (local Torelli theorem for complex structures) Let Teich be the space of complex structures on a K3 surface, and $\mathsf{Per} : \mathsf{Teich} \to \mathbb{P}\mathrm{er}$ the map taking (M, I) to the line $H^{2,0}(M) \subset H^2(M, \mathbb{C})$. Then Per is a local diffeomorphism.

Step 4 Using the identification $\mathbb{P}\mathrm{er} = \mathsf{Gr}_{++}(H^2(M, \mathbb{R}))$, where Gr_{++} is the space of planes positively oriented in $H^2(M, \mathbb{R})$, we can write Theorem $*$ in the following form:

Theorem 2 Let M be a Kr, $\mathfrak{R} \subset H^2(M, \mathbb{Z})$ the set of all vectors η such that $(\eta, \eta) = 4$, and $Z(\mathfrak{R})$ the set of all 2-planes orthogonal to some $\eta \in \mathfrak{R}$. Then $Z(\mathfrak{R})$ is dense in $\mathsf{Gr}_{++}(H^2(M, \mathbb{R})$.

Remark I think the identification of $\mathbb{P}\mathrm{er}$ with Gr_{++} is given by the following

Claim $\mathbb{P}\mathrm{er} = \mathsf{SO}(b_2 - 3, 3)/\mathsf{SO}(2) \times \mathsf{SO}(b_2 - 3, 1)$.

Step 5 Finally, the point is that Theorem 2 was proved using lattices by reducing it (via a discrete action, limit points, I still don't get this part) to

Theorem 3 Let $\mathfrak{R} \subset V_{\mathbb{Z}}$ be set of all vectors η such that $(\eta, \eta) = g$. Then the closure of $\mathbb{P}\mathfrak{R} \subset \mathbb{P}V_{\mathbb{Z}}$ contains $\mathsf{Null}\, V_{\mathbb{R}}$.

Here $(V_{\mathbb{Z}}, q)$ is a non-degenerate quadratic lattice (Probably cohomology lattice?) of signature (a, b) with $a \geqslant 3$ and $b \geqslant 1$, and $g \in \mathbb{Z}$ a number (maybe 4?) such that there exists $x \in V_{\mathbb{Z}}$ with $q(x, x) \neq 0$ (typo here? should be $\neq g$?). Denote $V_{\mathbb{R}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. The ***null-quadric*** is $\mathsf{Null}(V_{\mathbb{R}}) = \{\ell \in \mathbb{P}(V_{\mathbb{R}} : q(\ell, \ell) = 0\}$.

And all the lattice stuff was proved using so-called ***Pell equation***, Lagrange theorem that the equation $x^2 - y_2 w = 1$ has non-trivial integer solutions, conmesurable lattices.

Remark (Sergey) Regarding Step 1 — it is funny that if you do not assume $\mathsf{Pic} = \mathbb{Z}$, then the statement is wrong. Exercise - prove it.

Proof.

Primero recuerde el isomorfismo entre $\mathsf{Pic}(M)$ y $\mathsf{Div}(M)/\,\mathsf{PDiv}(M)$

Claim $|D| = \{E \in \mathsf{Div}(M) : E > 0, E \sim D\}$ es un espacio vectorial (finitamente generado) sobre el campo de funciones meromorfas con el producto $f \cdot E = D(f) + E$. Porque

$$E - D = \mathsf{div}(f) \iff E = D + \mathsf{div}(f) \geqslant 0$$

Question ¿Por qué finitamente generado?

El sistema lineal define un mapa (más abajo justificaremos el isomorfismo del projoyectivo)

$$\begin{aligned} \varphi_{|D|} : M &\dashrightarrow \mathbb{P}^n \cong \mathbb{P}(H^0(M, \mathcal{O}_M(D)) \\ x &\longmapsto (f_0(x) : \ldots : f_n(x)) \end{aligned}$$

donde n es la dimension de $|D|$. Note que es racional porque puede no estar definido, de hecho, estos puntos tienen un nombre:

Definition $p \in M$ es un ***punto de base*** de $|D|$ si $f_i(x) = 0$ para toda i. $\varphi_{|D|}$ incialmente es racional; es un morfismo si no hay puntos de base y en ese caso decimos que $|D|$ es ***libre de puntos de base***.

En resumen: el divisor L induce ese mapa $M \dashrightarrow \mathbb{P}^?$.

- Si L es libre de puntos de base entonces $\varphi_{|L|}$ es un morfismo.
- Si L es muy amplio entonces L es un encaje.
- Tenemos que determinar la dimensión del espacio projectivo. Resulta que es $h^0(M, \mathcal{O}_M(L)$, ¿pero qué es ese feixe? Nada más es el feixe que le corresponde al divisor mediante la identificación $\mathsf{Pic}(M) \cong \mathsf{Div}(M)/\,\mathsf{PDiv}(M)$.

Remark Bueno recuerda que $\mathsf{Pic}(M) = \{L \text{ fibrado lineal}\} \ni \mathcal{O}_X =$ fibrado tirivial=fibrado de funciones holomorfas.

Bueno, resulta que el $\varphi_{|D|}$ está construido de tal form que el condominio es $\mathbb{P}^n \cong \mathbb{P}(H^0(M, \mathcal{O}_M(D)) \cong |D|$. Básicamente es porque

$$H^0(M, \mathcal{O}_M(D)) = \{f \in \mathcal{M}_X : \mathsf{div}(f) \geqslant -D\}.$$

Por fin

- $\dim |L| = \dim_{\mathbb{C}} H^0(M, \mathcal{O}_M(L)) - 1$

Question Aún me queda la duda de qué significa $(L, L) = 4$.

Respuesta Ah pues la intersección es invariante dentro de la clase de equivalencia así que sólo tomas otro divisor en la misma clase y listo.

Criterio de Kleinmann-? D es amplio si y sólo si $D^2 > 0$ y $D \cdot C > 0$ para toda curva $C \subset M$ (eso significa ser ***nef***).

Ahora vamos a ver que L en el caso en que $\mathsf{Pic}(M) = \mathbb{Z} = \langle L \rangle$ es muy amplio usando ese criterio. Las curvas son divisores porque estamos en una superficie, así que $C \sim kL$ para alguna $k \in \mathbb{Z}$ (ya que $\mathbb{Z} = \langle L \rangle$. $L \cdot C = kL^2 = 4k$ que es positivo si k es positivo pero si es negativo entonces $-L$ es el amplio. Así que supongamos que L es amplio sin pérdida de generalidad. Eso implica que $\varphi_{|L|}$ es un morfismo.

Pero queremos muy amplio. Ver Mori p.130: H con H^2 es muy amplio si y sólo si NO

- existe una curva irreducible E con $E^2 = -2$ y $L \cdot E = 0$. Como estamos en una K3 son curvas racionales.
- existe una curva irreducible E con $E^2 = 0$ y $L \cdot E = 1$ o 2. Como estamos en una K3 son curvas elípticas.
- Existe una curva irreducible E con $E^2 = 2$ y $L \sim 2E$.

Entonces tome E una curva irreducible. Entonces está el género de E, que si E es suave es el género nomás y si no es el género aritmético. El género aritmético es $h^0(M, \mathcal{O}_M(E))$. Ahora la fórmula de Riemann-Roch dice que si $D \in \mathsf{Div}(M)$ es un divisor en una superficie, la fórmula

$$\begin{aligned}
\chi(\mathcal{O}_M(D)) &= h^0(M, \mathcal{O}_M(D)) - h^1(M, \mathcal{O}_M(D)) + h^2(M, \mathcal{O}_M(D)) \\
&= \underbrace{\chi(\mathcal{O}_M)}_{1} + \frac{D^2 - \underbrace{D \cdot K_M}_{=0 \text{ div. canónico K3}}}{2} \\
&= 1 + \frac{D^2}{2}
\end{aligned}$$

Y la fórmula del género

$$2p_a(D) - 2 = 1 + D^2 = \chi(\mathcal{O}_M(D))$$

que seguramente viene de Riemann-Roch.

OK las curvas de género cero son racionales porque son $\mathbb{P}^1$ así que $E^2 = -2$. OK eso explica lo de que estamos en K3 son racionales y elípticas en los primeros dos casos del teorema de Mori.

Entonces cada caso aquí arriba no da porque simplemente haces una cuentita y listo. L es muy amplio y $\varphi_{|D|}$ es un encaje.

Lema 2.2 Si D es efectivo en una superficie F

$$1 + \dim |L| = 2 + \frac{L}{2} + h^1(M, \mathcal{O}_M(L)$$

y si además es muy amplio la h se hace cero y qued

$$1 + \dim |L| = 2 + \frac{L}{2}$$

así que el condominio de $\varphi_{|L|}$ es $\mathbb{P}^3$.

Ahora vamos a ver que el 4 es el grado de la imagen de la superficie. hicimos un encaje

$$M \overset{\varphi_{|L|}}{\hookrightarrow} S \subseteq \mathbb{P}^3$$

así que S es una hipersufperficie así que está en el grupo de picard que resulta ser $\mathbb{Z}$ también para $\mathbb{P}^3$ así que $S \sim kH$ y esa k es el grado. Y la k se puede calcular

$$\deg(S) = k = S \cdot H \cdot H = (H|_L)^2 = L \cdot L = 4$$

porque para calcular el grado en $\mathbb{P}^3$ necesito intersectar con dos hiperplanos y para calcular el grado de una cosa encajada puedo calcularlo en el dominio, que resulta en la cuenta $M \cdot L \cdot L = L \cdot L = 4$.

Entonces en el Beauville the *Complex Algebraic Curves* vemos que si tenemos un morfismo finito $S \xrightarrow{\text{finito}} S'$ con grado $d = \dim$fibra genérica entonces la intersección del pullback es el d(intersección de los divisores en S').

Muy importante Proposición Si M K3 y existe $L \in \mathrm{Pic}(M)$ muy amplio con $L^2 = 4$ entonces $L \hookrightarrow \mathbb{P}^3$ es una cuártica porque

- L muy amplio da el encaje $\varphi_{|L|} \hookrightarrow S \subset \mathbb{P}^n$
- L^4 implica que $n = 3$ y que $\deg S = 4$ i.e. M es una cuártica.

Suponga que $M \hookrightarrow \mathbb{P}^3$ es una cuartica con $\mathrm{Pic}(M) = \mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot E$ con intesecciones

$$L^2 = 4, \qquad LE = 1, \qquad E^2 = 0$$

con $L = H|_{S'}$, $H \hookrightarrow \mathbb{P}^3$ es la sección hiperplana, i.e. es la sección que da el encaje como hicimos en todo lo anterior. Entonces es muy amplio.

Remark Si la K3 es proyectiva, la signatura del lattice de Picard es $(1, \rho - 1)$ donde $\rho = \mathrm{rk}(\mathrm{Pic}(M))$ i.e. $\mathrm{Pic}(M) \cong \mathbb{Z}^\rho$ (tal vez eso es consecuencia del índice de Hodge). Además $\mathrm{Pic}(M)$ es un lattice par i.e. la intersección de cualquier elemento consigo mismo es 2.

Theorem (Nikulin-Morrison) Si L es un lattice par de signatura $(1, \rho - 1)$ con $\rho < 10$ entonces existe M K3 tal que $\mathrm{Pic}(M) = L$.

En nuestro caso $\rho = 2$ y la lattice par porque miramos la matriz de intersección, así que sí existe esa K3.

Bueno ahora vuelve a ver la matriz:

$$\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$$

Entonces el E tiene autointersección 0 e intersección 1 con L. Ahora Riemann-Roch,

$$h^0(M, \mathcal{O}_M(E)) - h^1(M, \mathcal{O}_M(E) + h^2(M, \mathcal{O}_M(E)) = 2 + \underbrace{\frac{E^2}{2}}_{=0} = 2$$

así que quitando el de en medio porque podemos, obtenemos

$$h^0(M, \mathcal{O}_M(E)) + \underbrace{h^2(M, \mathcal{O}_M(E))}_{=0} \geqslant 2$$

y ese cero es porque $L \cdot E = 1$ y L es muy amplio. En fin,

$$h^0(M, \mathcal{O}_M(E)) \geqslant 2.$$

recuerde que ese h^0 es *casi* $|E|$, o sea que nos dice que hay muchos divisores linealmente equivalentes a E.

Para concluir usando el teorema de Mori basta mostrar que existe una curva irreducible con $E^2 = 0$ y $L \cdot E = 1$.

Proposition (Teorema de Bertin) Si E no tiene componente fijas (ni puntos de base) entonces existe $E' \sim E$ irreducible.

Proposition Si D es nef y efectivo con $D^2 \geqslant 0$ entonces $\exists D' \sim D$ curva irreducible.

Ahora existe F efectivo y linealmente equivalente a E porque $h^0(M, \mathcal{O}_M(E)) \geqslant 2$. Así que nada más falta ver que es nef i.e. $E \cdot C \geqslant ; \forall C$ curva. Tons suponga que no es nef. Entonces existe una curva C tal que $F \cdot C < 0$. Entonces $E \cdot C < 0$. Pero E es efectivo entonces $E = \sum a_i D_i$ con $a_i \geqslant 0$. Ahora

$$0 > \left(\sum a_i\right) \cdot = \sum a_i (D_i \cdot C) \implies \exists D_j \text{ tq } C = D_j$$

y $C^2 < 0$. Eso implica que $2P_a(C) - 2 = C^2 < 0$ que es par, y $P_a(C) \leqslant 0$ así que $C^2 = -2$.

Entonces tenemos

$$-2 = C^2 = (aL + bE)^2 + 4a^2 + 2ab \implies -1 = a(2a + b)$$

entonces $a = 1$ y $2a + b = -1$ o $a = -1$ y $2a + b = 1$. Entonces $C = L - 3E$ o $C = -L + 3E$.

Ahora intersectamos con L para ver que $C \cdot L = (L - 2E) \cdot L = 4 - 3 = 1$. Y el otro da $(-L + 3E) \cdot L = -4 + 3 = -1$. Así que tendría que ser la primera. Pero no es, porque $0 > C \cdot E = (L - 3E)E = 1$.

Así que F es nef, E es nef. Entonces aplicamos la prop, existe F' curva irreducible tal que $F' \sim F \sim E$. Entonces ya contradijimos Mori. ¡Bravo!

$\square$

16 Class 15: Lefschetz hyperplane section theorem

16.1 Lefschetz hyperplane section theorem

Theorem (Lefschetz hyperplane section throem) If $X \subset \mathbb{C}P^m$ is a projective manifold and $H \subset \mathbb{C}P^n$ a hyperplane (so its $\mathbb{C}P^{n-1}$) is transversal to X then

$$\pi_1(X \cap H) l \to \pi_1(X)$$

is an isomorphism for all $i < \dim X$ and surjective for $1 = \dim X$.

Corollary If X is a complex surface then

$$\pi_1(X \cap H) \twoheadrightarrow \pi_1(X)$$

Corollary If X complete intersection (it's an intersection of divisors in $\mathbb{C}P^n$) then $H_2(X) = \mathbb{Z}$ (like $\mathbb{C}P^n$). $H_{2i-1}(X) = 0$ only $H_{\dim X}$ has rank ¿1.

In $\mathbb{C}P^n$ divisors are sections of $\mathcal{O}(1)$ (follows by Gauss lema that all codiemnsion 1 ideals in this graded ring are principal). This means that $X \cap H$ is a hyperplane section of $\mathsf{Ver}^1(X)$.

Important corollary Lefschetz implies that $\pi_1(\text{Quartic}) = 1$.

16.2 Graded vector spaces and algebras

Definition A ***graded algebra*** is a direct sum $\bigoplus_{i \in \mathbb{Z}} A^i$ with the product compatible with the grading in the sense that $A^i \cdot A^j \subset A^{i+j}$.

- There is an action of $U(1)$ on A^i

16.3 Supercommutator

Definition An operator on a graded vector space is called ***even (odd)*** if it shifts the grading by even (odd) number. The ***parity*** if $\tilde{a}$ is 0 if it is even and 1 if it is odd. A commutaror is ***pure*** if it is even or odd.

Definition A ***supercommutator*** of pure operators on a graded vector space is defined by a formula $\{a, b\} = ab - (-1)^{\tilde{a}\tilde{b}} ba$

Definition A graded associative algebra is ***graded commutative*** or ***supercommutative*** if its supercommutator vanishes.

Example The Grassman algebra is supercommutative.

Definition A ***graded Lie algebra*** is a graded vector space with agraded vector spaces with a bilinear graded map $\{\cdot, \cdot\}$ which is graded anticommutative: $\{a, b\} = -(-1)^{\tilde{a}\tilde{b}}\{b, a\}$ and satisfies the ***super Jacobi identity***.

Lemma Let d be an odd element of a Lie superalgebra, satisfying $\{d, d\} = 0$ and L an even or odd element. Then $\{L, \{d, d\}\} = 0$.

16.4 The twisted differential d^c

Definition The ***twisted differential*** is $d^c := IdI^{-1}$.

Claim
$$\partial = \frac{d + \sqrt{-1}d^c}{2} \qquad \bar{\partial} = \frac{d - \sqrt{-1}d^c}{2}$$
are the ***Hodge components*** of d, namely $\partial = d^{1,0}$, $\bar{\partial} = d^{0,1}$.

Definition The ***Weil operator*** is

$$W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}(p-q)$$

Claim
$$d^c = [W, d]$$

16.5 Plurilaplacian

Theorem Let (M, I) be a complex manifold. Then

1. $\partial^2 = 0$.
2. $\bar{\partial}^2 = 0$.
3. $dd^c = -d^c d$
4. $dd^c = 2\sqrt{-1}\partial\bar{\partial}$.

Definition The operator dd^c is called the ***pluri-Laplacian***.

Remark The pluri-Laplacian takes real functions to $(1,1)$-forms.

Exercise On a Riemann surface $(M, I, \omega) dd^c f = \Delta(f)\omega$.

17 Positive $(1,1)$-forms

Claim A real $(1,1)$-form $\eta \in \Lambda^{1,1}(M) \cap \Lambda^2(M, \mathbb{R})$. Then the bilinear form $g_\eta(x,y) := \eta(x, Iy)$ is symmetric.

Remark (This might be a big result) The above construction (there is) a bijective correspondance between Hermitian $(1,1)$-forms and $U(1)$-invariant Riemannian metric tensors on M.

Definition Recall from handout 5 that a real $(1,1)$-form is called ***Hermitian*** if $(x, Ix) > 0$ for all $x \neq 0$. It is called ***positive*** if $(x, Ix) \geqslant 0$.

Remark (Very basic) $U(1)$-invariant Riemannian forms are in 1-1 correspondance with Hermiatan invariant forms.

Example $\xi \in \Lambda^{1,0}(M)$ the form $\pm\sqrt{-1}\xi \wedge \bar{\xi}$.

17.1 Pluri-harmonic functions

Definition A function $f \in C^\infty(M)$ is ***plurisubharmonic (psh)*** is $dd^c f$ is positive.

Definition f is ***pluri-harmonic*** if $dd^c f = 0$.

Theorem (Exercise) f is pluri-harmonic if f is a sum of holomorphic and antiholomorphic.

18 Morse functions

Definition Let $f \in C^\infty(M)$ and $x \in M$ its critical point. Choose a coordinate system in a neighbourhood of x. The ***hessian*** of f is a symmetric matrix

$$\sum_i \frac{d^2 f}{dx_i dx_j} dx_i \otimes dx_j \in \mathsf{Sym}^2(M)$$

Claim (Second year undergrad) $\mathsf{Hess}(f)$ is coordinate independant and defines a symmetric 2-form on $T_x M$.

Proof. Do this. □

Definition $f \in C^\infty(M)$ is called ***Morse*** if it is proper (preimage of closed interval is compact), its critical points are isolated, and for each of these critical point, the form $\mathsf{Hess}\, f$ is non-degenerate.

Claim Every manifold admits a Morse function. Moreover, the set of Morse functions is dense and open in the space of all proper smooth functions taken with C^2 or C^∞-topology.

18.1 The Hessian and torsion-free connections

Definition Let $\mathsf{Alt} : \Lambda^1(M) \otimes \Lambda^1(M) \to \Lambda^2(M)$ be antisymmetrization of tensor product. A connection $\nabla : \Lambda^1(M) \to \Lambda^1(M) \otimes \Lambda^1(M)$ is called ***torsion free*** if $\mathsf{Alt}(\nabla\theta) = d\theta$ for any 1-form θ on M.

Claim The 2-form $\nabla(df)$ is symmetric.

Claim $\nabla(df) = \mathsf{Hess}\, f$ on $T_x M$.

18.2 Torsion-free connections preserving the complex structure

Exercise Let (M, I) be an almost complex manifold and ∇ a torsion-free connection on TM preserving I, that is, satisfying $\nabla(I) = 0$. Prove that I is integrable.

Exercise Let (M, I) be a complex manifold. Then there exists a connection ∇ preserving I.

Remark In complex coordinates $z_1, \dots, z_n$ with $x_i = \operatorname{Re} z_i$ and $y_i = \operatorname{Im} z_i$.

$$\nabla\theta = \sum \frac{d}{dx_i}\theta \otimes dx_i + \sum \frac{d}{dy_i}\theta \otimes dy$$

then $\nabla I = 0$

18.3 The pluri-Laplacian and the Hessian

Definition ***psh*** is when $dd^c f \geqslant 0$.

Remark If ∇ is torsion-free connection preserving I we have

$$dd^c f = d(Id(f)) = \mathsf{Alt}(\nabla I(df)) = \mathsf{Alt}(I\dot{t}I(\mathsf{Hess}\, f))$$

So the tensor product with

Corollary $f \in C^\infty(M)$ with torsion free connection. Then

$$dd^c f(x, Ix) = \frac{1}{2}\big(\mathsf{Hess}(f)(x,x) + \mathsf{Hess}(f)(Ix, Ix)\big)$$

Idea Average hessian with I obtain ddc

18.4 Morse index of a plurisubharmonic function

Definition Let x be a Morse critical point of $f \in C^\infty(M)$ and (u, v) is signature of Hessian. The ***Morse index*** of f in x is v.

Theorem Morse index of psh function f is $\leqslant \dim_{\mathbb{C}} M$.

The idea is that two forms whose negative subspaces intersect will add up to a form that still has a negative subspace. If instead they have half the dimension? then they the there is no negative subspace in the sum of forms.

18.5 Stable manifold of a critical point

Definition Let f be a Morse function on a smooth manifold M and $\mathsf{grad}\, f$ its gradient vector field. The ***stable manifold*** of a critical point m is all points $z \in M$ such that $\lim_{t\to\infty} e^{t\,\mathsf{grad}\, f}(z) = m$

Proposition Let Z_m be a stable manifold of a critical point $m \in M$ of index p. Then Z_m is a smooth p-dimensional sumbanifold in M.

Proof. Uses Morse lemma that there is a coordinate system around m such that □

18.6 Lefschetz Hyperplane Section Theorem

Proof.

Step 1 Take f on $\mathbb{CP}^n \backslash H = \mathbb{C}^n$. $f_1 := \sum |z_i|^2$.

Our form is the following:

$$dd^c f_1 = \sum 2dx_i \wedge dy_i$$

it is strictly plurisubharmonic. Strictly plurisubharmonic are open in C^2 topology and Morse are dense in C^2 topology, hence there exists a small deformation f of f_1 which is strictly plurisubharmonic and Morse on $Z \cap \mathbb{CP}^n \backslash H$.

So we have a Morse strictly psh.

Step 2 Let $\{V_i \subset Z \backslash (H \cap Z)\}$ be all stable sets of all critical points of f on Z. Then the intersection $Z \cap H$ is a deformation retract of $Z_0 := Z \backslash \bigcup_i V_i$.

□

19 Class 16: C-symplectic Moser lemma and the local Torelli theorem

First step toward proving local Torelli. Typically this lasts 3-lectures.

I'll start by putting here again what we had before on C-symplectic structures.

19.1 C-symplectic structures

Definition M smooth manifold. Ω complex valued closed form is called ***C-symplectic*** if

1. $d\Omega = 0$
2. $\Omega^{n+1} = 0$ its rank is half of maximal
3. $4n = \dim_{\mathbb{R}} M$.
4. $\Omega^n \wedge \overline{\Omega}^n$ non-degenerate (volume form).

Proposition The fourth condition gives

$$\ker \Omega \oplus \ker \overline{\Omega} = TM \otimes \mathbb{C}$$

Remark (Dani) The point is that you can decompose the tangent bundle as $\ker \Omega \oplus \overline{\Omega}$

20 Closed forms and integrable distributions

So $\ker \Omega$ is a distribution and we wish to see it is integrable.

Theorem Take a closed form Ω, $d\Omega = 0$. Then $[\ker\Omega, \ker\Omega] \subset \ker\Omega$.

Proof. The idea is this: take $X, X_1 \in \ker\Omega$, compute

$$\mathcal{L}_X \Omega(X_1, \ldots, X_p)$$

for $X_2, \ldots, X_p$ any other vector fields. Only the term that survives is when the commutator is $[X, X_1]$, and since everything else is zero, we have put it in the kernel. □

Definition $\frac{\mathsf{CSymp}}{\mathsf{Diff}_0}$ is the space of holomorphic symplectic forms on a manifold M.

21 Local Torell

The period map takes a form in CSymp and maps it to its cohomology class.

Theorem (Local Torelli) This map is a local diffeomorphism.

22 C-symplectic structures on surfaces

Claim On a surface, a C-symplectic form $\Omega = \omega_1 + 1\sqrt{-1}\omega_2$ is C-symplectic if and only if $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$ is non-degenerate.

Proof. So because we are in a surface we have $\Omega^2 = 0$. Then $\operatorname{Im}\Omega = 0$ and $\operatorname{Re}\Omega = 2$. Finally non-degeneracy gives $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2 = 2\omega_1 = 0$ □

Moser's trick will be used to show that

$$\begin{array}{c} \mathsf{Teich} = \mathsf{Symp}\,/\,\mathsf{Diff}_0 \\ \Big\downarrow{\scriptstyle \mathbb{P}\mathsf{er}} \\ H^2(M) \end{array}$$

is a local diffeo.

23 Moser lemma

Lemma (Moser's trick) ω_t is a smooth family of symmetric forms parametrized by $t \in [0,1]$, M compact. Assume that the cohomology class of the ω_t is constant. Then all ω_t are related by a flow of diffeomorphisms.

OK so $\mathbb{P}\mathsf{er}$ is a smooth submersion. So the fibers are families of forms, which by Moser lemma we can pullback to one an another. So? When we collapse the identity component… a local bijection?

Proof of Moser's trick.

Step 1 Since ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact So probably just writing it down. The harder part is to show that this dereivative $\frac{d\omega_t}{dt} = d\eta_t$ is "depends smoothly in t".

Step 2 I think in the next part we suppose that ψ_t already exists, and efine $\nu_t = \psi_t^{-1}\frac{d\psi_t}{dt}$. Looks like it's very similar to how Henrique did it...

□

24 C-symplectic Moser lemma

Theorem (C-symplectic Moser lemma) Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class of them is constant and $H^{0,1}(M, I_t) = 0$ which is a weird assumption and it is the same as saying that first cohomology of the holomorphic function sheaf $\mathcal{O}_M$ vanishes. Then all Ω_t are diffeomorphic.

Proof.

Step 1 X_t vector field such that

$$\mathcal{L}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$$

so what is the flow of X_t? Looks like by definition

$$V_{t_1}^*\Omega_0 - \Omega_0 = \int_0^{t_1} \mathcal{L}_{X_t}\Omega_{(t)}$$

but that's only

$$= \int_0^1 \frac{d\Omega_t}{dt}dt = \Omega_t - \Omega_0$$

But it is the exactly the same reasoning like in usual Moser lemma. I guess this is just solving the equation

$$\mathcal{L}_{X_t}\Omega_t = \frac{d\Omega_t}{dt}$$

Step 2 There is an isomorphism

$$T_{\mathbb{R}}Ml \to \Lambda^{1,0}(M, I)$$

$$\begin{aligned} T_{\mathbb{R}}M &\longrightarrow \Lambda^{1,0}(M, I) \\ X &\longmapsto i_X\Omega \end{aligned}$$

which is just linear algebra.

Step 3 Again $\frac{d}{dt}\Omega_t$ is exact, so suppose it is $d\alpha_t$, $\alpha \in \Lambda^1_{\mathbb{C}}(M)$. If we found $\alpha_t \in \Lambda^{1,0}(M)$ we could obtain it as $i_{X_t}\Omega_t$, and we could solve the green equation.

Step 4 Now this part is when C-symplectic properties come in. We define $\Omega_t' = \frac{d}{dt}\Omega_t$ and multiply by Ω_t^n. This multiplication gives isomorphism

$$\Lambda^{0,2}(M) \cong \Lambda^{2,2}(M)$$

and

$$\Lambda^{1,1}(M) = 0 = \Lambda^{2,0}(M)$$

which means that

$$\Omega_t \wedge \Omega_t' = 0 \iff \Omega_t' \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$$

Step 5 The end of the proof is a lemma that grants the existence of α_t. So its funny because it's just like changing the names of the things we have but OK the lemma is

Lemma M complex manifold with $H^{1,0}(M) = 0$ and $\eta \in \Lambda^{2,0}(M) + \Lambda^{1,1}(M)$ exact. Then $\eta d\alpha$ $\alpha \in \Lambda^{1,0}(M)$.

Proof of lemma. Looks like computations on exterior algebra. □

□

Corollary (Local Torelli for C-symplectic) The period map for C-symplect structures is a local diffeomorphism.

25 Class 17: local Torelli theorem: local surjectivity of the period map

25.1 dd^c-lemma

Theorem (dd^c-lemma) $\eta \in \Lambda^{p,q}(M)$ on compact Kähler manifold which is either

a. exact

b. ∂-exact, $\bar\partial$-exact.

c. $\bar\partial$-closed, ∂-exact

Then $\eta \in \operatorname{img} dd^c = \operatorname{img} \partial\bar\partial$ (the latter equality because these are proportional, namely $dd^c = \pm 2\sqrt{-1}\partial\bar\partial$).

Proof.

Claim $\eta \perp \ker \Delta$.

Proof of claim.

$$\Delta = dd^* + d^*d = 2(\partial\partial^* + \partial^*\partial) = 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

So

$$X \in \ker d \iff = 0 = (\Delta X, X) = (dx, dx) + (d^*x, d^*x) \iff dx = d^*x = 0$$

and

$$X \in \operatorname{ker} d^* \iff$$

□

□

25.2 Massey producs

Some algebraic geometry ignored by Griffiths-Harris.

Definition $\alpha, \beta, \gamma \in \Lambda^\bullet(M)$ closed, $[\alpha \wedge \beta] = [\beta \wedge \gamma] = 0$ cohomology classes

$$\alpha \wedge \beta = da \qquad \beta \wedge \gamma = db$$

This one?

$$d(\alpha \wedge \gamma - (-1)^{\tilde{\alpha}} \alpha \wedge \,) = \alpha \wedge \beta\gamma - \alpha \wedge \beta \wedge \gamma = 0$$

Then

$$a \wedge \gamma - (-1)^{\tilde{\alpha}} \alpha \wedge \beta]$$

is called ***Massey product*** $M_{\alpha\beta\gamma|}$ and is well-defined up to modulo $\operatorname{img} L_{[\alpha]} + \operatorname{img} L_{[\gamma]}$ where these are the operators of multiplication by the cohomology classes of $[\alpha], [\gamma]$.

Theorem On a compact Kähler manifold, Massey products vanish on Hodge pure classes.

Proof. **Implied by** dd^c **lemma** Actually looks like very simple computations...

α, β, γ pure hodge classes, $\alpha \wedge \beta$ pure, d-ecact $\implies$ dd^c exact dd^c-exact.

$$=$$

□

25.3 Heisenberg group

Definition

$$G = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

$G_{\mathbb{Z}}$ is that group with integers. $G/G_{\mathbb{Z}}$ is a torus.

Theorem $G/G_{\mathbb{Z}}$ has nontrivial massey products.

25.4 Local Torelli: surjectivity

We had already shown that period map is injective.

Theorem A small deformation of C-symplectic structure remains nondegenerate.

(M, Ω) C-symplectic K3 surface, and $\eta \in H^{1,1}(M)$ "sufficiently small". Then there exists $\rho \in \Lambda^{11}(M) + \Lambda^{02}(M)$ such that $\Omega_\eta := \Omega + \rho$ is C-symplectic. So ρ is $\bar{\partial}$-cohomologous to η and also $\rho^{11} \wedge \rho^{11} = -\Omega \wedge \rho^{02}$

So that second assumption says $\Omega + \rho$ is closed because it's literally expanding $(\Omega + \rho)^2$.

Remark Some computations show that

$$[\Omega_\eta] = [\Omega + \eta + \overline{\Omega}\frac{[\eta]^2}{[\Omega \wedge \overline{\Omega}}$$

so that class is determine dby η.

Remark This set of equations:

$$\rho^{11} \wedge \rho^{11} = -\Omega \wedge \rho^{02} \qquad d\rho = 0$$

give a hyperplane that ultimately gives the surjectivity of period map.

26 Class 18: kummer

26.1 The instinct developed with k3

Corollary If C is a smooth curve on a K3 surface,

$$g(C) = \frac{(C, C)}{2} + 1$$

So for example elliptic curve has genus 1 because it does not autointersect.

27 Class 19: more kummer and (-2)-curves

27.1 More Kummer

We began with the proof that Kummer surfaces are K3. So what is a Kummer surface: it is the blow up of a complex torus $S^1 \times S^1 \times S^1 \times S^1$ on the 16 singular points quotient by the action of $\{\pm 1\}$. So the proof was mainly proving that they are simply-connected. This used a covering argument to construct an exact sequence and then express the fundamental group as a semidirect product $\mathbb{Z}^4\{\pm 1\}$, and then use an elliptic curve, and finish with Van Kampen.

27.2 What are the (-2)-curves?

Theorem Every connected curve S on a K3 surface has intersection number $(S, S) \geqslant 2$.

Proposition 1 If $S \subset K3$ then $(S, S) \geqslant -2$ and equality when the genus of S is zero $\iff$ S is a rational curve (so ***rational*** means birrational to $\mathbb{P}^1$).

Proof here involves Euler characteristic and genus. Interestingly, we used that $\chi(\mathcal{O}_S) = 1 - g(S)$ (usually we have $\chi = 2 - 2g$).

Finally,

Definition A (-2)-***curve*** on a K3 surface is a connected curve with $(S, S) = 2$.

27.3 Why are (-2)-curves nice?

Theorem Any (-2)-curve is a collection of p smooth (-2)-curves that intersect transversally in $p - 1$ points. (This means their incidence graph is a tree).

Proof. So here we first discussed that the incidence matrix of these curves is negative definite, and then the proof. □

(-2)-curves are contractible This is related to Grauert theorem, also Akira ?. The proof in a simple case is given by Moser trick.

The statement says that we can put the K3 M bimeromorphically holomorphically in another complex variety M_1 that puts S to a point.

Proof. This proof was very hard for me—it involves lots of ample and very ample bundles in exact sequences. □

We need (-2)-curves for the Kähler cone—next class.

28 Lecture 20: Calabi-Yau structures and HyperKähler structures

28.1 Holomorphic vector bundle

Definition Definition of $\bar{\partial}$ as the usual connection but using

$$\bar{\partial} : B \to B \otimes \Lambda^{0,1}(M)$$

and satisfying Leibniz. Extending this operator to all $\Lambda^{0,i}(M)$ usine Leibniz. A ***holomorphic vector bundle*** is a bundle with such an operator satisfying $\bar{\partial}^2 = 0$

Exercise A function is holomorphic iff it vanishes under $\bar{\partial}$.

28.2 Chern connection, curvature and first Chern class

Review of the Chern connection which is the Levi-Civita complex version. And curvature Θ that is composition of this connection with itself. And then a proof of Bianchi identity real quick.

Remark We can consider the curvature as an $\mathsf{End}(B)$-valued 2-form.

Definition The ***real first Chern class of a line bundle*** B is

$$c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$$

Then the interesting formula that

Claim L holomorphic hermitian line bundle, b holomorphic non degenerate section then

$$\nabla b/b = 2\partial(\log|b|) = 2\frac{\partial|b|}{|b|}$$

Corollary L holomorphic line bundle h Hermitian form on L b non-vanishin section then

$$\Theta_\nabla = 2\partial\bar{\partial}\log|b|$$

Remark From now on we loose $\sqrt{-1}\dots$?

28.3 Ricci curvature

This is a Ricci curvature defined without any metric whatsoever.

Claim That $dd^c\log|f| = 0$ because $\log|f|$ is holomorphic plus antiholomorphic and somebody did an exercise showing antiholomorphic plus holomorphic are the kernel of dd^c.

Definition (M, Vol) complex n manifold with volume form $\mathsf{Vol} \in \Lambda^{n,n}_{\mathbb{R}}(M)$ then

$$\mathsf{Ric}(M, \mathsf{Vol}) = dd^c\log\frac{\ell\wedge\bar{\ell}}{\mathsf{Vol}}$$

for ℓ local section of $K_M = \Lambda^{n,0}(M)$ holomorphic non-vanishing.

Remark It is independent of the choice of section.

Remark A remark that says that two symplectic volume forms $\omega_1 = \omega_2 + dd^c\varphi$.

Corollary (M, ω) a connected Kähler $\Omega \in \Lambda^{n,0}(M)$ holomorphic non-degenerate. Define f by $\Omega \wedge \overline{\Omega} = e^f\omega^n$. Assume that $\int_M \omega^n = \int_M e^f\omega^n$ consider a Kähler form $\omega + dd^c\varphi$ wuch that $(\omega + dd^c\varphi)^n = e^f\omega^n$. Then $\omega + dd^c$ is Kähler and Ricci-flat.

Upshot To find a Ricci-flat metric it remains to solve an equation

$$(\omega + dd^c\varphi)^n = \omega^n e^f$$

28.4 Monge-Ampère equation

Definition (Complex Monge-Ampère equation)

$$\omega + dd^c\varphi)^n = Ae^f\omega^n.$$

φ unkown and $A \in \mathbb{R}$ is determined by

$$A = \frac{\int \omega^n}{\int e^f \omega^n}.$$

Theorem (Calabi-Yau) M compact Kähler then $\mathsf{MA}(f) = Ae^f\omega^n$ has a unique (up to a constant) solution $\varphi \forall f$.

Proof. Very complicated, involving estimates. □

Proposition (Calabi) A complex Monge-Ampère equation has at most one solution up a constant.

Proof. Looks like here we used all the lemmas and results done in this lecture. Some computations on exterior algebra using star operator to finally conclude that

$$\int_M |d\varphi|^2 \omega^n = 0$$

where $\omega_2 = \omega_1 + dd^c\varphi$ are the two solutions of MA and we needed to see that φ is constant. So that's that. □

Maybe this is interesting from Yulia's talk:

Start with M a riemannian oriented 4 manifold and you have the ***Hodge star operator***:

$$\begin{aligned} * : \Lambda^2(M) &\longrightarrow \Lambda^2(M) \\ \alpha \wedge *\beta &g(\alpha, \beta)\, \mathsf{Vol}_M \end{aligned}$$

and it happens that

$$*^2 = 1 \implies \text{exist } \pm 1 \text{ eigenspaces}$$

Remark

$$\Lambda^2(M) \cong \mathsf{SO}(TM)$$

so $s \in \Lambda^+(M)$ is seen as an endomorphism of TM.

28.5 Bochner's vanishing

Theorem (Bochner vanishing) (M, g, I) complex Ricci flat Kähler then holomorphic p-forms are parallel with respect to the Levi-Civita connection.

Corollary (M, Ω, g) is holomorphic symplectic Kähler type then (M, Ω) admits a Ricci flat Kähler Ω Levi-Civita parallel.

And will be used but we first need to define

28.6 Hyperkähler

Definition A manifold M is ***hypercomplex*** if it has three integrable almost complex structures I, J, K satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\mathsf{Id}$ and $IJ = K = -JI$.

Definition (Calabi, 1978) Let (M, g) be a Riemannian manifold equipped with three complex structure operators I, J, K satisfying the quaternionic relations above. Suppose that g is Kähler with respect to I, J, K. Then (M, I, J, K, g) is called ***hyperkähler***.

Now remember that

$$\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$$

where $\Lambda^{\pm}(M)$ are the $\pm$-eigenspeces of the Hodge star operator acting on $\Lambda^2(M)$.

Exercise Prove that $\Lambda^+(M)$ is generated by $\mathsf{Re}\,\Omega$, $\mathsf{Im}\,\Omega$ and ω where ω is the Kähler form and Ω is a holomorphic symplectic form

and a proposition that

Proposition M a K3 surface, g a Kähler metric. Then the following are equivalent:

(a) g is hyperkähler

(b) g is Ricci-flat

(c) The bundle $\Lambda^+(M)$ is trivialized by parallel sections.

29 Class 21: Kähler cone and hyperkähler period space

Definition M Kähler manifold. The ***Kähler cone*** is $\mathsf{Kah}(M) \subset H^{1,1}(M)$ the set of all cohomology classes of Kähler forms

Theorem (Demailly-Paň) M compact Kähler,

$$K = \left\{ \eta \in H^{1,1}(M, \mathbb{R}) : \int_Z \eta^{\dim Z} > 0 \ \forall Z \subset M \text{ subvariety} \right\}$$

Then the Kähler cone is in one of the connected components of K.

Exercise We did an exercise that if signature $(1, n-1)$ form, then if $q(x,x) > 0$ and $q(y,y) \geqslant 0$ then $q(x,y) > 0$.

Corollary That statement applied to curves S_1 and S_2 on a surface.

Theorem M Kähler surface, $S \subset M$ curve with $(S,S) \geqslant 0$. Then

$$\mathsf{Kah}(M) = \left\{ \omega \in h^{1,1} : \int_C \omega > 0, \int_C \omega \wedge \omega > 0 \ \forall C \text{ curve} \right\}$$

Theorem M K3, $\mathfrak{R}$ the set of all (-2)-classes on M. Let $\mathsf{Pos}(M)$ be the connected component of the Kähler class in $\{\omega : \int_M \omega \wedge \omega > 0\}$. Then $\mathsf{Kah}(M)$ is the connected component of $\mathsf{Pos} \setminus \bigcup_{\omega \in \mathfrak{R}} \omega^{\perp}$.

We split the Kähler cone into two polyhedra by taking orthogonal complements of (-2)-classes.

29.1 Gromov-Hausdorff distance

Definition The ***Gromov-Hausdorff distance*** of $X, Y \subset M$ for a metric space M is the infinimum ε such that $X(\varepsilon) \supset Y$ and $Y(\varepsilon) \supset X$ where these are the ε-neighbourhoods of X and Y.

Example The distance between $\mathbb{Z}^n$ and $\mathbb{R}^n$ is something like $\sqrt{2}$, because it is half the distance of the center of a square to one vertex.

Remark Gromov-Hausdorff distance is the infinimum $d_H(\varphi(X), \psi(Y))$ over all isometric embeddings $\varphi : X \to Z$ and $\psi : Y \to Z$. So if GH distance is zero then the metric spaces are isometric.

Definition Let $\varphi : X \longrightarrow Y$, X Y metric spaces. ***Defect*** is how far the distance is disturbed:

$$\delta_{\psi} = \mathsf{sup}_{a,b \in X} |d(a,b) - d(\varphi(a), \varphi(b))|$$

Now define

$$\hat{d}_{GH}(x,y) = \mathsf{inf}_{\substack{\varphi : X \to Y \\ \psi : Y \to X}} \mathsf{max}\left(\delta(\psi \circ \varphi, \delta(\psi \circ \varphi)\right)$$

Remark There is another definition: there exist C_1, C_2 such that

$$C_1 \hat{d}(x,y) \leqslant d_{GH}(x,y) \leqslant C_2 \hat{d}(x,y)$$

Why do we care? This is the topology in hyperkähler Teichmüler space—and thus it is Hausdorff.

29.2 Hyperkähler period map

$$\mathbb{P}\mathsf{er}(\text{Riemannian metrics}) \longrightarrow \frac{\mathsf{O}(h^2(M))}{\mathsf{O}(h^+(M)) \times \mathsf{O}(h^-(M))}$$

it's a symmetric space.

30 Class 22: hyperkähler period space

30.1 Hyperkähler metrics, induced complex structure

Claim If a K3 has a metric of hyperkähler-type then there is a unique up to isotopy hk structure (I, J, K).

Remark Which basically says that it's enough for us to study metrics of hyperkähler type

Theorem (Calabi-Yau) (M, Ω) compact holomorphically symplectic of Kähler type with $[\omega]$ Kähler class. Then there exists a unique Ricci flat metric g that its Kähler hclass is ω.

Definition An ***induced complex structure*** is a quaternion $L \in \mathbb{H}$ such that $L^2 = -1$.

30.2 Hyperkähler Teichmüler space

Definition ***HK Teichmüler space*** is the space of all metrics quotiented by Dif_0.

Remark It's possible to define a Riemannian version of Albanese map.

30.3 Period of hyperkähler

Definition (M, g) a K3 with metric of hyperkähler type, then $\mathbb{P}\mathsf{er}(g)$ is the 3-subspace in $H^2(M, \mathbb{R})$ given by $\omega_i, \omega_j, \omega_k$.

Definition ***Hyperkähler period space*** is the subset $\mathbb{P}\mathsf{er}_h \subset \mathsf{Gr}_{+++}$ (which is the grassmanian of positively oriented 3-spaces in $H^2(M, \mathbb{R})$) consisting of W such that $W^\perp$ does not contain any (-2)-classes.

Claim 1

Consider the hyperkähler period map

$$\mathsf{Per}_h : \mathsf{Teich} \longrightarrow \mathsf{Gr}_{+++}.$$

Then $\mathsf{Per}_h(\mathsf{Teich}_h) \subset \mathbb{P}\mathsf{er}_h$.

Claim 2 The peroid map is locally a homeomorphism.

Proof. First extending Teich to Teich' which is the space of triples of symplectic forms that are compatible with the metric. Also extend $\mathbb{P}\mathsf{er}$ to the space of $a, b, c \in H^2(M, \mathbb{R})$ such that $a^2 = b^2 = c^2$ and orthogonality i.e. $a \wedge b = a \wedge c = b \wedge c = 0$. We get a principal bundle for each. These can be visualized using also the extendion of Per map to

$$\omega_i, \omega_j, \omega_k \longmapsto [\omega_j], [\omega_j], [\omega_k]$$

in this diagram:

$$\begin{tikzcd}
\mathbb{P}\mathsf{er}'_h \arrow[r, "\mathsf{Per}"] \arrow[d, "\mathsf{SO}(3)\times\mathbb{R}_{>0}"'] & \mathsf{Teich}' \arrow[d, "\mathsf{SO}(3)\times\mathbb{R}_{>0}"] \\
\mathbb{P}\mathsf{er} \arrow[r, "\mathsf{Per}"'] & \mathrm{Teich}
\end{tikzcd}$$

and principal bundle map is local homeomorphism iff the bottom map is as well. $\square$

31 Class 23: Metric structures associated with twistor curves

31.1 Sub-Riemannian structures

It's like a contact manifold

M Riemannianan, $B \subset TM$, for every $x, y \in M$ do

$$d_B(x, y) = \inf L(\gamma)$$

where the infinimum ranges over paths from x to y tangent on B. It's not very interesting because it might not be a metric, for example in $X \times Y$ because projection on one factor gives a vanishing metric.

Definition $B \subset TM$ subbundle satisfies ***Chow-Rashevskii condition*** if by defining $B_0 = B$ and $B_{i+1} := [B_i, B_i]$ we get that $B_s = TM$ for s sufficiently large. So it's something like

$$[[[X_1, X_2], X_3], X_4], \ldots, X_s]$$

with X_i section of B.

Theorem (Chow-Raschevskii, 1938, 1939) If B satisfies Chow-Raschevskii condition, d_B is finite. In this case we say d_B is ***sub-Riemannian***.

31.2 Metric notions

Definition If (M, d) is a metric space and $\gamma : [a, b] \to M$ is a continuos path then the ***length*** of γ is the infimum of the sums of lengths over all partitions of the interval.

Definition The ***intrinsic metric*** or ***path metric*** on the metric space (M, d) is the infinimum over all lengths of paths joining two given points.

Remark The intrinsic metric of the intrinsic metric is just the intrinsic metric.

Definition (Gromov? *Structures on Riemannian and non-Riemannian. .. ?)* ***Class of admissible paths*** a topological space M is a set of continious maps $[a, b] \to M$ such that

1. γ_1, γ_2 admissible then the concatenation $\gamma_1 \cdot \gamma_2$ is admissible.
2. To concatenate we need to reparametrize so we impose that the reparametrization of admissible paths is admissible.
3. We had gluing of paths, now we cut them. Restriction of admissible paths is admissible (to closed sub-intervals of $[a, b]$).

Definition ***A length structure*** is a class of admissible paths along with a ***length function*** $L(\gamma)$ satisfying all these axioms and that the length functions depends on $b \in [a, c]$ continuously.

Definition The ***path metric*** associated with a length structure is the infimimum of the lengths given by the length function.

31.3 Finsler metric

Definition M manifold and for each $x \in M$ define a norm that depends continuously on x, the norm is ν_x. Now define

$$L_\nu(\gamma) = \int_0^1 \nu_{\gamma(t)}(\gamma'(t))dt.$$

This defines a length functional on the class of piece-wisely smooth paThis defines a length functional on the class of piece-wisely smooth paths. The corresponding metric is called ***Finsler***.

Definition Now suppose that $B \subset TM$ is a sub bundle satisfying CR. Then take the following class of admissible paths: piecewise smooth and tangent to B. Then the same $L(\gamma)$ as above works if ν_x are norms in $B_x \subset T_x M$ varying continuously with $x \in M$.

31.4 Pencils of 3-spaces

Definition Let (H, q) be a real quadratic space of signature $(3, n)$. If $W_1, W_2 \in \mathsf{Gr}_{+++}$, i.e. they are three dimensional positive spaces, and $\dim W_1 \cap W_2 = 2$. That makes $U := W_1 + W_2$ a 4-dimensional space. Then the ***pencil*** of 3-spaces in U is a 1-dimensional family of 3-dimensional spaces obtained from a 2-dimensional subfamily in U^*.

Note that 3-dimensional subspaces in U are 1-dimensional subspaces in U^*, so points in $\mathbb{P}U^*$. And inside there there is a projective line: $\mathbb{P}U^* \supset \mathbb{R}P^1$.

So a ***pencil*** is made of 3-dimensional subspaces that intersect in a 2-dimensional subspace. Not all of them are positive but the contain one.

Remark U^* is a q-quadric space of signature $(3, 1)$.

Claim The quadratic form q on ℓ has signature $(1, 1)$.

31.5 Subtwistor chains

Definition Gr_{+++} positive oriented 3-spaces in $H^2(M,\mathbb{R}) \cong \mathbb{R}^{3,19}$, M K3, $\mathbb{P}\mathsf{er}_h \subset \mathsf{Gr}_{+++}$ all W such athat $W^\perp \not\ni (-2)$-vectors. A ***sub-twistor*** segment is a positive segment γ contained in $\mathbb{P}\mathsf{er}$ connecting 3-spaces W_1 and W_2 such that $W_1 \cap W_2$ is 2-dimensional and its complement $(W_1 \cap W_2)^\perp$ does not contain any (-2)-vectors. So probably this (-2)-vector thing has some algebraic-geometric background...

Definition A ***subtwistor chain*** is a swquence of subtwistor segments connecting points

Exercise

$$\mathsf{Gr}_{+++} = \frac{\mathsf{SO}^+(H^2(M,\mathbb{R}))}{\mathsf{SO}(3) \times \mathsf{SO}(19)}$$

admits a unique, up to a constant, $\mathsf{SO}^+(H^2(M,\mathbb{R}))$-invariant Riemannian metric.

Solution. It's enough to show that the representation is irreducible and then applying Schur lemma, which says that there exists a unique metric as desired. But why is is irreducible?

A key observation is that

$$T_W \mathsf{Gr}(V) = \mathsf{Hom}(W, V/W)$$

And we continue it

$$T_W \mathsf{Gr}(V) = \mathsf{Hom}(W, V/W) = W \otimes (W^\perp)^*.$$

and *the tensor product of irreducible is irreducible* (argument given by Vanya). □

Definition A ***subtwistor metric*** is the infimum of lengths of all subtwistor chains.

31.6 Cone structures

Definition $X = G/H$ homogeneous and $C \subset \mathsf{Tot}(TX)$ a G-invariant and $\mathbb{R}$-invariant subset. $C \subset \overline{C}$ is called a ***cone structure*** if

$$\bigcup_{x \in M} \overline{C}_x = \overline{C}$$

where $C_x = C \cap T_x M$.

Definition Define a ***cone metric*** on the cone as the infimimum of the Riemannian length (fix a Riemannian metric on X) of all paths joining x and y tangent to C.

Theorem (Mitia) Cone metric is sub-finsler.

Upshot We are taking the space of 3-dimensional subspaces (we are thinking of the space generated by the three Kähler forms) and fixing one and rotating the other, and this takes the problem back to the situation when we only had one form. This is what the segments represent? Fixing one and moving the other...

31.7 A theorem on cone structures and subtwistor metric

Take C to be the set of all vectors of Gr_{+++} of a M K3 that are tangent to subtwistor segments.

Remark (dani) So it looks like subtwistor segments are paths in the Period space, which is perhaps a Grasmannian. So a path there is also a subbundle.

Claim $C_W \cap T_W \mathsf{Gr}_{+++}$ is the set of all $A \in \mathsf{Hom}(W, W^{\perp})$ of rank 1, moreover $\bigcup \overline{C}_W = \overline{C}$.

Corollary Subtwistor metric on Gr_{+++} is Finsler.

Proof. By Mitia's theorem, and a bit more. □