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Home assignment 4: quadratic lattices

Definition 4.1 A ***lattice*** is a finitely generated torsion-free $\mathbb{Z}$-module. ***Quadratic form*** on a lattice is a function $q : L \to \mathbb{Z}$, $q(\ell) = B(\ell, \ell)$ where B is a bilinear symmetric pairing $B : L \otimes_{\mathbb{Z}} L \longrightarrow \mathbb{Z}$. ***Quadratic lattice*** is a lattice equipped with a quadratic form. A quadratic form is ***indefinite*** if it takes positive and negative values, and ***unimodular*** if B is non-degenerate and defines an isomorphism $L \xrightarrow{\sim} L^*$.

Exercise 4.1 Let (L, q) be a quadratic lattice, $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$, and L^* the set of all $x \in L_{\mathbb{Q}}$ such that $q(x, L) \subset \mathbb{Z}$.

a. Prove that L^* is a lattice of the same rank as L and $L \subset L^*$.

b. The ***dscriminant group*** of L is $\operatorname{Disc} L := L^*/L$. Prove that L is unimodular if and only if $\operatorname{Disc}(L) = \{0\}$.

c. Let G be an abelian group. Construct a lattice (L, q) such that $\operatorname{Disc}(L) = G$.

Solution.

a.

Answer Let $\{a_i\}$ be a basis of L. Recall that the space of linear functionals on L is identified with L^* via the map $x \mapsto q(x, \cdot)$. Then the functionals given by $a_i^\vee(a_j) = \delta_{ij}$ are a basis of L^*.

OK its rather stupid but I would like to make the following identification explicit:

$$\begin{aligned} L^* &\longrightarrow L^\star \\ x &\longmapsto q(x, \cdot) \end{aligned}$$

where $L^\star$ is the set of linear forms on L. But the kernel of this map is trivial only when q is nondegenerate.

So let us proceed as follows: we will show that there is only one element $a_i^\vee \in L_{\mathbb{Q}}$ such that $q(a_i^\vee, a_j) = \delta_{ij}$. Then suppose $\tilde{a}_i^\vee$ is another such element. Then $q(a_i^\vee - \tilde{a}_i^\vee, a_j) = 0$ so if q is nondegerate I'm done, but if q is not nondegenerate How to prove $a_i^\vee = \tilde{a}_i^\vee$?

b. Implication $\Longrightarrow$ is trivial. Implication $\Longleftarrow$ is also straightforward since $L^*/L = \{0\}$ means that every element of L^* is in L, so $L = L^*$.

c.

□

Exercise 4.2 Let (L, q) be a quadratic lattice, and $L_1 \subset L$ a sublattice.