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Home assignment 4: quadratic lattices

Definition 4.1 A ***lattice*** is a finitely generated torsion-free $\mathbb{Z}$-module. ***Quadratic form*** on a lattice is a function $q : L \to \mathbb{Z}$, $q(\ell) = B(\ell, \ell)$ where B is a bilinear symmetric pairing $B : L \otimes_{\mathbb{Z}} L \longrightarrow \mathbb{Z}$. ***Quadratic lattice*** is a lattice equipped with a quadratic form. A quadratic form is ***indefinite*** if it takes positive and negative values, and ***unimodular*** if B is non-degenerate and defines an isomorphism $L \xrightarrow{\sim} L^*$.

Exercise 4.1 Let (L, q) be a quadratic lattice, $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$, and L^* the set of all $x \in L_{\mathbb{Q}}$ such that $q(x, L) \subset \mathbb{Z}$.

a. Prove that L^* is a lattice of the same rank as L and $L \subset L^*$.

b. The ***dscriminant group*** of L is $\operatorname{Disc} L := L^*/L$. Prove that L is unimodular if and only if $\operatorname{Disc}(L) = \{0\}$.

c. Let G be an abelian group. Construct a lattice (L, q) such that $\operatorname{Disc}(L) = G$.

Solution.

a. Let $\{a_i\}$ be a basis of L. Recall that the space of linear functionals on L is identified with L^* via the map $x \mapsto q(x, \cdot)$. Then the functionals given by $a_i^\vee(a_j) = \delta_{ij}$ are a basis of L^*.

   Attempt to identify functionals with L^*. We want to show that there is only one element $a_i^\vee \in L_{\mathbb{Q}}$ such that $q(a_i^\vee, a_j) = \delta_{ij}$. Then suppose $\tilde{a}_i^\vee$ is another such element. Then $q(a_i^\vee - \tilde{a}_i^\vee, a_j) = 0$ so if q is nondegerate I'm done, but if q is not nondegenerate, how to prove $a_i^\vee = \tilde{a}_i^\vee$?

   Remark In Dolgachev, the definition of L^* is $\operatorname{Hom}(L, \mathbb{Z})$. The discriminant is $L^*/\operatorname{img} i$ where i denotes interior multiplication, that is, $i_x q = q(x, \cdot)$.

b. Implication $\Longrightarrow$ is trivial. Implication $\Longleftarrow$ is also straightforward since $L^*/L = \{0\}$ means that every element of L^* is in L, so $L = L^*$.

c. (*Idea.*) I think L is the free part of G. I want to show that there is an exact sequence $L \xrightarrow{i} \operatorname{Hom}(L, \mathbb{Z}) \xrightarrow{\psi} L \oplus T = G \longrightarrow 0$ because this way I get that G is the cokernel of the interior multiplication i, which is the discriminant by definition.

$\square$

Exercise 4.2 Let (L, q) be a quadratic lattice, and $L_1 \subset L$ a sublattice.

a. Prove that $L_1{}^{*} \supset L^*$. Prove that any isometry $\alpha \in O(L)$ takes L_1 to another lattice $L^* \subset \alpha(L_1) \subset L$.

b. Denote by $\delta(L_1)$ the image of L_1 in $\mathrm{Disc}(L)$. Prove that an isometry $\alpha \in \mathsf{O}(L)$ which satisfies $\delta(L_1) = \delta(\alpha(L_1))$ preserves L_1.

Solution.

a. (Perhaps there is a typo in the question because $L^* \subset L_1 \subset L$ would imply $L^* = L$.) However, the contention $L_1^* \supset L^*$ holds; it is immediate from definition: if $x^* \in L^*$ then $q(x^*, L) \subseteq \mathbb{Z} \implies q(x^*, L_1) \subset \mathbb{Z}$.

   Since $\alpha \in \mathsf{O}(L)$, we know that $\alpha(L_1) \subset L$.

   To see $\alpha(L_1)$ is lattice notice that a basis is $\alpha(e_1), \dots, \alpha(e_n)$ where $e_1, \dots, e_n$ is a basis of L_1. This is immediate from the definition of ***orthogonal group*** as found in Dolgachev, where isometries are taken to be (bijective) *homomorphisms* of the abelian groups. This means that for any $x \in L_1$

$$\alpha(x) = \alpha\left(\sum x^i e_i\right) = \sum x^i \alpha(x_i)$$

b. We interpret δ as the map

$$\begin{tikzcd} L \arrow[r, hook] \arrow[rr, bend left, "\delta"] & L^* \arrow[r, two heads] & L^*/L = \mathrm{Disc}(L) \end{tikzcd}$$

   But since $\alpha(L_1) \subset L$, we always have that $\delta(L_1) = \delta(\alpha(L_1))$ since both are the equivalence class of the identity in $L^*/L = \mathrm{Disc}(L)$.

   If instead we consider the projection onto $\mathrm{Disc}(L_1)$ we get

$$\begin{tikzcd} L_1 \arrow[r, hook] \arrow[rr, bend left, "\delta_1"] & L_1^* \arrow[r, two heads] & L_1^*/L_1 = \mathrm{Disc}(L_1) \end{tikzcd}$$

   and in this case the condition $\delta_1(L_1) = \delta_1(\alpha(L_1))$ means that the equivalence class of $\alpha(L_1)$ in $\mathrm{Disc}(L_1)$ is the identity class, meaning that $\alpha(\ell_1) \in L_1$ for every $\ell_1 \in L_1$. This is very tautological... maybe I didn't understand the question correctly...

$\square$

Definition 4.2 Two subgroups $G_1, G_2 \subset \mathsf{GL}(n, \mathbb{R})$ are called ***commensurable*** if $G_1 \cap G_2$ has finite index in G_1 and in G_2.

Exercise 4.3 Let (L, q) be a quadratic lattice, and $L_1 \subset L$ a sublattice. Prove that $\mathsf{O}(L_1, q) \cap \mathsf{O}(L, q)$ has finite index in $\mathsf{O}(L, q)$.

Solution. We are looking at the equivalence classes of $\mathsf{O}(L_1)$ within $\mathsf{O}(L)$. $\mathsf{O}(L_1)$ is the equivalence class of the identity. Two isometries $b, c \in \mathsf{O}(L)$ are in the same equivalence class when $bc^{-1} \in \mathsf{O}(L_1)$. I'd like to use the criterion of last exercise but I don't see how... $\square$

Exercise 4.4 Let $nL := \bigcup_{x \in L} nx$. Prove that $nL_1 \subset L$ for any integer lattices L, L_1 and n sufficiently big. Prove that $\mathsf{O}(nL, q) = \mathsf{O}(L, q)$.

Solution. It is not clear what $\bigcup_{x \in L} nx$ means. Perhaps it is the set $\{nx : x \in L_1\}$, in which case it is immediate that $nL_1 \subset L$ if $L_1 \subset L$ as in the previous exercises. For arbitrary lattices L and L_1, the statement $nL_1 \subset L$ might not make sense ($L = \mathbb{Z}, L_1 = \mathbb{Z} \oplus \mathbb{Z}$). $\square$