Daniel González Casanova Azuela
github.com/danimalabares/k3

# Home assignment 6: cohomology and intersection forms

**Exercise 6.1** Compute the cohomology algebra of the manifold

a. $\mathbb{C}P^2 \# \mathbb{C}P^2$
b. $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$
c. $(S^2 \times S^3) \# (S^2 \times S^3)$
d. $\mathbb{H}P^2$

*Solution.*

a. First let's compute the cohomology groups. After several attempts I will use the following simple argument from StackExchange. Recall that the connected sum is obtained by removing a 4-dimensional disk in $\mathbb{C}P^2$ and glung another copy of $\mathbb{C}P^2$ along the boundary $S^3$ of the removed disk.This makes $(\mathbb{C}P^2\#\mathbb{C}P^2, S^3)$ a ***good pair*** ($S^3$ is a deformation retract some open neighbourhood in $\mathbb{C}P^2\#\mathbb{C}P^2$). This means that relative cohomology may be identified with the homology of the quotient:

$$H_\bullet(\mathbb{C}P^2\#\mathbb{C}P^2, S^3) = H_\bullet(\mathbb{C}P^2\#\mathbb{C}P^2/S^3) = H_\bullet(\mathbb{C}P^2 \vee \mathbb{C}P^2) = H_\bullet(\mathbb{C}P^2) \oplus H_\bullet(\mathbb{C}P^2)$$

And then we have the long sequence for relative homology, which may be written more succintly as

$$H_\bullet(\mathbb{C}P^2) \oplus H_\bullet(\mathbb{C}P^2) \longrightarrow H_\bullet(\mathbb{C}P^2\#\mathbb{C}P^2) \longrightarrow H_\bullet(S^3)$$

Which gives by Poincaré duality

$$H^0(\mathbb{C}P^2\#\mathbb{C}P^2) = H^4(\mathbb{C}P^2\#\mathbb{C}P^2) = \mathbb{Z}$$

This is because $\mathbb{C}P^2\#\mathbb{C}P^2$ is connected; there is no isomorphism $H_\bullet(\mathbb{C}P^2)\oplus H_\bullet(\mathbb{C}P^2) = \mathbb{C}P^2\#\mathbb{C}P^2$ because it is the first/last terms of the sequence, where $H^\bullet(S^3)$ does not vanish. For the other terms there *is* an isomorphism because the homology of the sphere vanishes:

$$H^i(\mathbb{C}P^2\#\mathbb{C}P^2) = H^i(\mathbb{C}P^2) \oplus H^i(\mathbb{C}P^2), \qquad i = 1,2,3$$

which means that

$$H^i(\mathbb{C}P^2\#\mathbb{C}P^2) = 0, \qquad i = 1,3$$
$$H^2(\mathbb{C}P^2\#\mathbb{C}P^2) = \mathbb{Z} \oplus \mathbb{Z}$$

In the StackExchange answer I'm using it is claimed that the product structure should be given by

$$\begin{aligned}\smile : \underbrace{H^2(\mathbb{C}P^2\#\mathbb{C}P^2)}_{H^2(\mathbb{C}P^2)\bigoplus H^2(\mathbb{C}P^2)} \times \underbrace{H^2(\mathbb{C}P^2\#\mathbb{C}P^2)}_{H^2(\mathbb{C}P^2)\bigoplus H^2(\mathbb{C}P^2)} &\longrightarrow H^4(\mathbb{C}P^2\#\mathbb{C}P^2) \\ \Big((\alpha,\beta),(\alpha',\beta')\Big) &\longmapsto \alpha\smile\alpha' + \beta\smile\beta'\end{aligned}$$

but I cannot see why.

**Remark** In this document it is claimed (using ***formal manifolds***) without proof that

> To obtain the cohomology algebra of a connect sum, tensor the cohomology algebras of the spaces you are connecting, and impose the relations that all products of elements coming from different connect-summands are zero, and identify the volume forms. [...] So, the cohomology algebra of the connect sum of $\mathbb{C}P^2$ is
>
> $$\begin{aligned}H^\bullet(\mathbb{C}P^2\#\mathbb{C}P^2) = \Lambda(a,b;\deg(a)=\deg(b)=2, a^3=0,\\ b^3=0, ab=0, a^2-b^2=0)\end{aligned}$$

$\square$