Home assignment 4: quadratic lattices

Definition 4.1 A ***lattice*** is a finitely generated torsion-free $\mathbb{Z}$-module. ***Quadratic form*** on a lattice is a function $q : L \to \mathbb{Z}$, $q(\ell) = B(\ell, \ell)$ where B is a bilinear symmetric pairing $B : L \otimes_{\mathbb{Z}} L \longrightarrow \mathbb{Z}$. ***Quadratic lattice*** is a lattice equipped with a quadratic form. A quadratic form is ***indefinite*** if it takes positive and negative values, and ***unimodular*** if B is non-degenerate and defines an isomorphism $L \tilde{\to} L^*$.

Exercise 4.1 Let (L, q) be a quadratic lattice, $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$, and L^* the set of all $x \in L_{\mathbb{Q}}$ such that $q(x, L) \subset \mathbb{Z}$.

a. Prove that L^* is a lattice of the same rank as L and $L \subset L^*$.

b. The ***dscriminant group*** of L is $\mathrm{Disc}_L := L^*/L$. Prove that L is unimodular if and only if $\mathrm{Disc}(L) = \{0\}$.

c. Let G be an abelian group. Construct a lattice (L, q) such that $\mathrm{Disc}(L) = G$.

Solution.

a. Consider the canonical identification of

$$\begin{aligned} L &\longrightarrow L^\star \\ x &\longmapsto q(x, \cdot) \end{aligned}$$

where $L^\star$ is the set of linear forms on L. I expect to find an identification $L^* \cong L^\star$. What exactly is $q(x, \ell)$?. Since $x = \sum_i q_i e_i$ with $q_i \in \mathbb{Q}$,

$\square$