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Home assignment 1: Riemann-Roch formula in dimension 1

(Partial progress)

Exercise 1. Prove that the ring $\mathcal{O}_1$ of germs of holomorphic functions on $\mathbb{C}$is a principal ideal ring.

Solution. I assume that $\mathcal{O}_1$ is the ring of germs of holomorphic functions about $0 \in \mathbb{C}$. I will use Griffiths and Harris, Chapter 0, section *Weierstrass Theorems and Corollaries*. A ***Weierstrass polynomial*** in w is a function

$$w^d + a_1 w^{d-1} + \ldots + a_d(z), \qquad a_i(0) = 0.$$

Theorem (Weierstrass Division Theorem)**.** Let $g(z, w) \in \mathcal{O}_{n-1}[w]$ be a Weierstrass polynomial of degree k in w. Then for any $f \in \mathcal{O}_n$ we can write

$$f = g \cdot h + r$$

with $r(z, w)$ a polynomial of degree $< k$ in w.

In words, we can express the germ of a holomorphic function around 0 as the the product of a Weierstrass polynomial of degree k times some holomorphic function plus a polynomial of degree $< k$.

Then the proof is just mimicking the proof that the usual polynomial ring is a principal ideal domian.

Let J be an ideal of $\mathcal{O}_1$ and $g \in J$ be a Weierstrass polynomial of lowest degree. Then by Weierstrass Division Theorem we get $h, r \in \mathcal{O}_1$ such that $f = g \cdot h + r$ but by the choice of g we see r must be zero. This means $J = \langle g \rangle$. □

Exercise 1.2 (Invariant factors theorem)**.** Let R be a principal ideal ring. Prove that any finitely-generated R-module is a direct sum of cyclic R-modules. Use this result to deduce the Jordan normal form theorem, and to classify the finitely-generated abelian groups.

Solution. I follow Dummit and Foote to prove that

Theorem. Let R be a principal ideal *domain* and let M be a finitely generated R-module. Then

$$M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \ldots \oplus R/(a_m)$$

for some integer $r \geqslant 0$ and nonzero elements $a_1, a_2, \ldots, a_m$ of R which are not units in R and which satisfy the divisibility relation

$$a_1 | a_2 | \ldots | a_m$$

which strongly relies on

Theorem (4). Let R be a PID, M a a free R-module of finite rank n and N a submodule of M. Then N is a free module of rank $m \leqslant n$ and there exists a basis $y_1, y_2, \dots, y_n$ of M such that $a_1y_1, a_2y_2, \dots, a_my_m$ is a basis of N, where $a_1, a_2, \dots, a_m$ are nonzero elements of R with the divisibility relations

$$a_1|a_2|\dots|a_m.$$

whose proof is rather involved.

Proof of Invariant factors theorem. Choose a basis $\{x_i\}_{i=1}^n$ of M. Consider the free module R^n along with a basis $\{b_i\}_{i=1}^n$. The homomorphism $\pi : R^n \to M$, $b_i \mapsto x_i$ is surjective, so we get $R^n / \ker\pi \cong M$.

Apply Theorem 4 for the module R^n and its submodule $\ker\pi$. We get a basis $\{y_i\}_{i=1}^n$ of R^n such that $\{a_iy_i\}_{i=1}^m$ is a basis of $\ker\pi$ for $a_i \in R$ for $i = 1, \dots, m$ such that $a_1|\dots|a_m$. Then

$$M \cong R^n / \ker\pi = (Ry_1 \oplus \dots \oplus Ry_n \Big/ (Ra_1y_1 \oplus \dots \oplus Ra_my_m).$$

It remains to show that the quotient on the right-hand side of last equation is in fact the desired decomposition of M.

Consider the map

$$Ry_1 \oplus \dots \oplus Ry_n \quad \longrightarrow \quad R/(a_1) \oplus \dots \oplus R/(a_m) \oplus R^{n-m}$$

given by

$$(\alpha_1y_1, \dots, \alpha_ny_n) \quad \longmapsto \quad (\alpha_1 + (a_1), \dots, \alpha_m + (a_m), \alpha_{m+1}, \dots, a_n)$$

The kernel of this map is the set of elements which $\alpha_i \in (a_i)$ for all $i = 1, \dots, m$. Thus we can write the kernel as

$$Ra_1y_1 \oplus \dots \oplus Ra_my_m.$$

□

Remark. Looks like I didn't use the divisibility condition $a_1|\dots|a_m$.

Now let's deduce the Jordan normal form theorem using again Dummit and Foote. For this we fix a vector space V over a field F and a linear transformation T. This makes V into an $F[x]$-module by substitution of the variable x by T.

The ***invariant factors*** are the elements a_i from the last theorem, which in our present case can be shown to be monic polynomials $a_i(x)$ satisfying $a_1(x)|\dots|a_n(x)$. Further, these elements are associated to the so-called ***elementary divisors***, which concern another similar formulation of the Invariant factors theorem. The elementary divisors are powers of the irreducible components of the $a_i(x)$.

If we assume that the $a_i(x)$ factor into linear polynomials, the elementary divisors can be written as $(x-\lambda)^k$, where λ is one of the eigenvalues of T (we consider only one of the eigenvalues at this point since we are constructing *one* of Jordan blocks of T).

Using the Invariant factors theorem for elementary divisors, we see that V is the direct sum of finitely many cyclic modules of the form $F[x]/(x-\lambda)^k$. Such quotients have basis $\bar{x}^{k-1}, \bar{x}^{k-2}, \dots, \bar{x}, 1$. Now we observe that the polynomials

$$(\bar{x}-\lambda)^{k-1}, (\bar{x}-\lambda)^{k-2}, \dots, \bar{x}-\lambda, 1$$

are also a basis. This follows since expanding the latter in terms of the $\bar{x}_i$ gives a triangular matrix, which makes it invertible, so it's a valid change of coordinates.

Nex we observe that action of multiplication by x, which is the same as applying T, maps these basis elements in the quotient as follows:

$$\begin{aligned}
1 &\mapsto 1\bar{x} = \lambda \cdot 1 + \bar{x} - \lambda \\
(\bar{x}-\lambda) &\mapsto \bar{x}^2 - \lambda\bar{x} = x\lambda \cdot (\bar{x}-\lambda) + (\bar{x}-\lambda)^2 \\
&\vdots \\
(\bar{x}-\lambda)^{k-2} &\mapsto \lambda \cdot (\bar{x}-\lambda)^{k-2} + (\bar{x}-\lambda)^{k-1} \\
(\bar{x}-\lambda)^{k-1} &\mapsto \lambda \cdot (\bar{x}-\lambda)^{k-1} + (\bar{x}-\lambda)^{k}
\end{aligned}$$

(where I'm not sure how to compute the last two). The last expression simplifies further since $(\bar{x}-\lambda)^k = 0$ in the quotient, so that multiplication of x by the basis element $(\bar{x}-\lambda)^{k-1}$ is barely multiplication by λ. In sum, we have constructed the ***Jordan block***

$$J_\lambda = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & & & \\ & & \ddots & 1 & \\ & & & \lambda & \\ & & & & 1 \\ & & & & \lambda \end{pmatrix}$$

applying this procedure to the rest of the cyclic components of V we obtain the matrix representation

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_t \end{pmatrix}$$

of T.

Finally, the classification of finitely generated abelian groups follows by direct application of the Invariant factors theorem choosing $R = \mathbb{Z}$, since abelian groups are in correspondence with $\mathbb{Z}$-modules.

$\square$

Exercise 1.7. Let X be a complex manifold and $\mathrm{Pic}(X)$ the set of equivalence classes of vector bundles, equipped with multiplicative structure induced by the tensor product. The group $\mathrm{Pic}(X)$ is called the ***Picard group***of X.

a. Prove that the cohomology group $H^1(X, \mathcal{O}_X^*)$ is naturally identified with $\mathrm{Pic}(X)$.

Solution.

a. Recall that a line bundle L may be reconstructed from the gluing functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, which can be thought as changes of coordinates on each fiber. These functions satisfy the consistency condition that $g_{\gamma\beta} g_{\beta\gamma} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

   On the other hand, a cochain in Čech cohomology is an assigment of an element $g_{\alpha\beta} \in \mathcal{O}_X^*$ for every intersection $U_\alpha \cap U_\beta$. Elements of $H^1(X, \mathcal{O}_X^*)$ are cocycles, meaning that coboundary operator vanishes. For 1-cocycles this is typically written as $(g_{\beta\gamma} - g_{\alpha\gamma} + \gamma_{\alpha\beta})|_{U_\alpha \cap U_\beta \cap U_\gamma} = 0$. In multiplicative notation this just says that $g_{\beta\gamma} g_{\alpha\gamma}^{-1} \gamma_{\alpha\beta} = 1$ which equivalent to the consistency condition for gluing functions.

□

Exercise 1.8. Let $\mathbb{C}$ be a complex curve and F a coherent sheaf on $\mathbb{C}$.

a. Prove that the restriction of F to a certain open set $U \subset C$ is isomorphic to a vector bundle. Prove that the rank of this vector bundle is independant on the choice of U when $\mathbb{C}$ is irreducible. This number is called the ***rank*** of F.

Solution.

a. (From StackExchange) I show how to construct a vector bundle from a *locally free* sheaf. Given an open cover U_i such that $\mathcal{F}(U_i)$ is free for all i, we just need to find gluing functions $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$.

   From the definition of locally free, we have isomorphisms $f_i : \mathcal{F}(U_i) \to \mathcal{O}_{U_i}^n$. Restricting to the intersection $U_i \cap U_j$ we may define the functions

$$f_{ij} = f_j|_{U_i \cap U_j} \circ f_i|_{U_i \cap U_j}^{-1} : \mathcal{O}_{U_i}^n|_{U_i \cap U_j} \to \mathcal{O}_{U_j}^n|_{U_i \cap U_j}.$$

   The claim in StackExchange is that every such map is induced by a gluing function, but I still cannot see why.

□

References

Dummit, D.S. and R.M. Foote. *Abstract Algebra*. Wiley, 2003.

Griffiths, Phillip and Joseph Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. A Wiley-Interscience Publication. New York: John Wiley & Sons, 1978.