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Home assignment 1: Riemann-Roch formula in dimension 1

(Partial progress)

Exercise 1. Prove that the ring $\mathcal{O}_1$ of germs of holomorphic functions on $\mathbb{C}$is a principal ideal ring.

Solution. I assume that $\mathcal{O}_1$ is the ring of germs of holomorphic functions about $0 \in \mathbb{C}$. I will use Griffiths and Harris, Chapter 0, section *Weierstrass Theorems and Corollaries*. A ***Weierstrass polynomial*** in w is a function

$$w^d + a_1 w^{d-1} + \ldots + a_d(z), \qquad a_i(0) = 0.$$

Theorem (Weierstrass Division Theorem)**.** Let $g(z, w) \in \mathcal{O}_{n-1}[w]$ be a Weierstrass polynomial of degree k in w. Then for any $f \in \mathcal{O}_n$ we can write

$$f = g \cdot h + r$$

with $r(z, w)$ a polynomial of degree $< k$ in w.

In words, we can express the germ of a holomorphic function around 0 as the the product of a Weierstrass polynomial of degree k times some holomorphic function plus a polynomial of degree $< k$.

Then the proof is just mimicking the proof that the usual polynomial ring is a principal ideal domian.

Let J be an ideal of $\mathcal{O}_1$ and $g \in J$ be a Weierstrass polynomial of lowest degree. Then by Weierstrass Division Theorem we get $h, r \in \mathcal{O}_1$ such that $f = g \cdot h + r$ but by the choice of g we see r must be zero. This means $J = \langle g \rangle$. □

Exercise 1.7. Let X be a complex manifold and $\operatorname{Pic}(X)$ the set of equivalence classes of vector bundles, equipped with multiplicative structure induced by the tensor product. The group $\operatorname{Pic}(X)$ is called the ***Picard group***of X.

a. Prove that the cohomology group $H^1(X, \mathcal{O}_X^*)$ is naturally identified with $\operatorname{Pic}(X)$.

Solution.

a. Recall that a line bundle L may be reconstructed from the gluing functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, which can be thought as changes of coordinates on each fiber. These functions satisfy the consistency condition that $g_{\gamma\beta} g_{\beta\gamma} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

   On the other hand, a cochain in Čech cohomology is an assigment of an element $g_{\alpha\beta} \in \mathcal{O}_X^*$ for every intersection $U_\alpha \cap U_\beta$. Elements of $H^1(X, \mathcal{O}_X^*)$ are cocycles, meaning that coboundary operator vanishes. For 1-cocycles this is typically written as $(g_{\beta\gamma} - g_{\alpha\gamma} + \gamma_{\alpha\beta})|_{U_\alpha \cap U_\beta \cap U_\gamma} = 0$. In multiplicative notation this just says that $g_{\beta\gamma} g_{\alpha\gamma}^{-1} \gamma_{\alpha\beta} = 1$ which equivalent to the consistency condition for gluing functions.

□

Exercise 1.8. Let $\mathbb{C}$ be a complex curve and F a coherent sheaf on $\mathbb{C}$.

a. Prove that the restriction of F to a certain open set $U \subset C$ is isomorphic to a vector bundle. Prove that the rank of this vector bundle is independant on the choice of U when $\mathbb{C}$ is irreducible. This number is called the ***rank*** of F.

Solution.

a. (From StackExchange) I show how to construct a vector bundle from a *locally free* sheaf. Given an open cover U_i such that $\mathcal{F}(U_i)$ is free for all i, we just need to find gluing functions $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$.

   From the definition of locally free, we have isomorphisms $f_i : \mathcal{F}(U_i) \to \mathcal{O}^n_{U_i}$. Restricting to the intersection $U_i \cap U_j$ we may define the functions

   $$f_{ij} = f_j|_{U_i \cap U_j} \circ f_i|^{-1}_{U_i \cap U_j} : \mathcal{O}^n_{U_i}|_{U_i \cap U_j} \to \mathcal{O}^n_{U_j}|_{U_i \cap U_j}.$$

   The claim in StackExchange is that every such map is induced by a gluing function, but I still cannot see why.

□