

k3

1 Class 1

The most important invariant of a k3 surface is [intersection form](#).

There are three classes of manifolds

1. Smooth manifolds

$$\text{smooth manifolds} \xrightarrow{\text{forgetful functor}} \text{PL manifold} \longrightarrow \text{Topological manifolds}$$

Donaldson: countably many non-equivalent smooth structures on \mathbb{R}^4 . K3 surfaces has countably many smooth structures and only one of them is compatible with complex structure.

Definition. Intersection form. Given a quadratic form on a lattice $V_{\mathbb{Z}} = \mathbb{Z}^n$, so

$$q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \rightarrow \mathbb{Z}$$

is *unimodular* if

$$V_{\mathbb{Z}} \xrightarrow{q} \text{Hom}(V_{\mathbb{Z}}, \mathbb{Z})$$

is an isomorphism.

Theorem (Universal coefficients formula).

$$H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}^{b_{n-1}(M)} \oplus T_{n-1}(M)$$

$$h^n(M, \mathbb{Z}) = \mathbb{Z}^{b_n(M)} \oplus T_{n-1}(M)$$

Corollary. $H^2(X, \mathbb{Z})$ is torsion free if $\pi_1(X) = 0$ because

Definition. *Signature* is $m - n$ if q has signature (m, n) .

Theorem (Rokhlin-Wu?). Signature is divisible by 16 for simply-connected (something else).

Remark. The methods used in surgery break down in smooth case because strange topological objects like infinite sums of spheres arise.

Theorem (Freedman, 1982). There are as many 4-manifolds as there are intersection forms. A simply connected 4 manifold homotopy class is uniquely determined by intersection form. Moreover, for every unimodular form there exists a unique M with this intersection form.

Theorem (Donaldson, 1986). M smooth compact manifold with positive definite odd intersection form q . Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition. Bilinear symmetric form is *indefinite* if it is not positive definite nor negative definite.

Theorem (Classification of unimodular symmetric bilinear forms). Odd are diagonalizable, while even are related to special Lie group E_8 .

Definition. A **K3 surface** is a Kähler complex surface M with $b_1 = 0$ (simply connected) and $c_1(M, \mathbb{Z}) = 0$.

Kodaira did what André Weil couldn't classify.

Theorem. K3 surfaces have trivial canonical bundle $K_M = \Lambda^2(\Omega^1 M)$.

2 Class 2

G topological group. **Principal G bundle** is a space with free G -action such that the quotient E/G is Hausdorff. There are several conditions that make this work. And then you have $\text{Homotopy}(X, BG) = \text{equivalence classes of } G\text{-bundles}$. Vector bundles of a manifold are the same as maps from X to $BU(n)$.

Vector bundles up to stable equivalence are classified basically by Chern classes, so by the cohomology in $H^*(BU) = \mathbb{Q}[c_1, c_2, \dots, c_n]$.

Now look at the loop space of X . Then $H^*(\Omega X)$ is a free graded commutative algebra. Loop space has the interesting property that $\Omega U = BU$ and $\Omega BU = U$.

2.1 Bialgebras

Let A be a superalgebra (graded with antisymmetric product). Then we ask the axiom of coassociativity and that .

Example. G group, and $C(G)$ the ring of k -valued functions $C(G \times G) = C(G) \times C(G)$ so

$$\begin{aligned} G \times G &\longrightarrow G \\ C(G) &\longmapsto C(G) \otimes C(G) \end{aligned}$$

2.2 H-spaces

Definition. H-space is a space M with a map $\mu : M \times M \rightarrow M$ that is homotopy associative,

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mu \times \text{id}} & M \times M \\ \downarrow \text{id} \times \mu & & \downarrow \mu \\ M \times M & \xrightarrow{\mu} & M \end{array}$$

which is homotopy commutative. And with homotopy unit.

So it's like a homotopy algebra?

Example. The loop space.

2.3 Bialgebras of finite type

Definition. A bialgebra A is of *finite type* if it is the direct sum of $A = \bigoplus_{i \geq 0} A^i$ supercommutative and each A^1 is finite dimensional.

Remark. Free commutative algebra is polynomial algebra

Definition. $A = \mathbb{C}[x_1, \dots, x_n, \dots] \otimes \Lambda^\bullet(a_1, \dots, a_n, \dots)$ is a graded commutative free algebra. In the slides: it is $\text{Sym}_{\text{gr}} V^*$ where V^* is a graded vector space.

Theorem (Hopf). A graded commutative bialgebra of finite type over k of 0 characteristic is free graded commutative as a k algebra.

2.4 The cohomology algebra of $U(n)$

Claim. The cohomology algebra $H^*(U(n), \mathbb{Q})$ is a free graded commutative algebra with generators in degrees $1, 3, 5, \dots, 2n-1$.

Demonstração. Induction. $U(1)$ is clear because it is a circle. Then do Serre spectral sequence. Differentials vanish on the second page because there's only nonzero groups on even degrees! And we get that $E_2^{p,1} = H^p(S^{2n-1}) \otimes H^q(U(n-1))$. And then the sequence converges to that of the total space which is $U(n)$. \square

2.5 Grassman manifolds

Definition. The *fundamental bundle* B_{fun} is a rank n vector bundle over $\text{Gr}(n, m)$.

Claim. B, B' vector bundles of rank $n, m-n, B \oplus B'$

$$\varphi : X \rightarrow \text{Gr}(m, n)$$

$$\varphi(x) = B_x \subset B_x \oplus B'_x = \mathbb{K}^m$$

then $B = \varphi^* B_{\text{fun}}$.

Theorem. If you have B as a bundle on a manifold X then $B \oplus B'$ is trivial for some bundle B' .

Demonstração. Embed the total space in a large enough euclidean space. □

Definition. $\text{Gr}(n, \infty) = \text{Gr}(n)$ is $\bigcup_{m=n_1}^{\infty} \text{Gr}(n, m) = \text{Gr}(n)$

Corollary. For every bundle B of rank n there is a function $\varphi : X \rightarrow \text{Gr}(n)$ such that $B = \varphi^* B_{\text{fun}}$.

Take a bundle $E \rightarrow X$ and G acts freely on E so E principal G bundle. Classifying space BG

Theorem (Atiyah-Bott). Classifying space is unique up to homotopy equivalence.

2.6 Stiefel spaces

Definition. \mathbb{K}^{∞} is the direct limit of \mathbb{K}^n so its just the direct sum $\bigoplus_{i=n}^{\infty} \mathbb{K}$. Stiefel space is the space of orthonormal n -frames.

If we prove that Stiefel is contractible we obtain our classifying space so let's prove that. We have a fibration

$$U(n) \hookrightarrow \text{St}(n, \infty) \rightarrow \text{Gr}(n, \infty)$$

Theorem. $\text{St}(n)$ is contractible.

Demonstração **Step 1** Locally trivial fibration with contractible fiber and base $Y \rightarrow X$ then Y is contractible, this is so trivial.

Step 2 Fibration $\text{St}(n) \rightarrow \text{St}(n-1)$ with fiber S^{∞}

Step 3 Show that S^{∞} is contractible.

Step 4 And then some map \mathbb{R} that is not surjective, and construct homotopy of identity to a constant map. □

Exercise. If $X_{\infty} = \bigcup X_i$ is the inductive limit of contractible cellular spaces then it is contractible. Use Whitehead theorem.

Theorem (Important). $\text{Gr}(\infty) = BU$.

2.7 Stable equivalence

Definition. Vector bundles V, W are stable equivalent if $V \oplus A \cong W \oplus B$ for trivial vector bundles A and B .

Homotopy classes of equivalent vector bundles are in corespondance with...

Theorem. BU is H-space.

Corollary. $H^*(BU, \mathbb{Q})$ is a free supercommutative algebra.

Claim. $H^*(BU)$ is a free polynomial algebra generated by classes c_1, c_2, \dots in all even degrees.