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Home assignment 2: spectral sequences

The monodromy of Gauss-Manin local system

Definition 2.1. Let $\pi : E \to B$ be a locally trivial fibration with fiber F. The family of cohomology of fibers of π is locally trivial, (this means that the cohomology of the fibers is locally the same as the cohomology of the base?) but it might have ***the monodromy***. In other words, the group $\pi_1(B)$ naturally acts on the algebra $H^*(F)$ by autmorphisms. To obtain this action, take a loop in B and trivialize the family π along small intervals of this loop; this gives an identification of $H^*(F)$ with itself, which might be non-trivial.

Remark (Understanding the monodromy action of cohomology). (From StackExchange) Let $f : X \to U$ be a proper surjective submersion and fix $u_0 \in U$.

For any path $\gamma \subset U_j$, there is a canonical diffeomorphism $\varphi_\gamma : f^{-1}(\gamma(0)) \to f^{-1}(\gamma(1))$, using ψ_j (by a theorem of Ehresmann, all the fibers of f are diffeomorphic).

Now, for any loop γ, split γ into paths $\gamma_i \subset U_i$ and you can compose these diffeomorphisms to get a diffeomorphism

$$\phi_{\gamma_n} \circ \ldots \circ \varphi_{\gamma_1} : f^{-1}(u_0) \to f^{-1}(\gamma(u_0))$$

It induces a map on homology: you can check that it is well defined up to homotopy.

Exercise 2.1. Let $\phi^* : \mathbb{Z} \to \mathrm{Aut}(H^*(F))$ be an automorphism induced by a homeomorphism $\phi : F \to F$. Construct a locally trivial family over a circle with monodromy in cohomology induced by ϕ^*.

Attempt of proof. As I understand the exercise, I must show what could be the total space E of a locally trivial fibration $F \to E \to S^1$ such that the monodromy action is ϕ^*.

An automorphism of $H^\bullet(S^1) = \mathbb{Z} \oplus \mathbb{Z}$ must be of the form $(n, m) \mapsto (an, bn)$ for some $a, b \in \mathbb{Z}$. (However since it is determined by a single number it looks like there will be less options.)

I think the possibility $b = 0$ corresponds to $E = S^1 \times \mathbb{R}$ since a loop in the base is just a translation. My guess is that $a = b$ corresponds to $S^1 \times S^1$, and $a = -b$ to a Klein bottle. □

Leray-Serre spectral sequence

Exercise 2.4. Let $\pi : E \longrightarrow B$ be a fibration with the fiber a torus. Assume the $d_2 = 0$. Prove that all differentials vanish.

Proof. Since $d_2 = 0$, we have that $E_3^{p,q} = E_2^{p,q}$. Then

$$d_3 : H^p(B) \otimes H^q(T) \longrightarrow H^{p+3}(B) \otimes H^{q-2}(T),$$

so the only way it could be non-zero is for $q = 2$, which implies that

$$H^p(B) \otimes H^2(T) \cong H^{p+3}(B) \otimes H^0(T) \iff H^p(B) \cong H^{p+3}(B)$$

But I don't see why this couldn't happen... $\square$

Exercise 2.5. Let $\pi : E \to B$ be a fibration with the fiber a torus. Assume that the pullback map $\pi^* : H^2(B) \to H^2(E)$ is injective. Prove that all differentials d_i vanish.

Solution. I'm not sure how to use the hypothesis since I usually deal with the total space after computing the E_∞ page via the filtration... $\square$

Exercise 2.6. Let $\pi : E \to B$ be a fibration with the fiber a complex projective space. Assume that $d_2 = 0$ and $d_3 = 0$. Prove that all differentials d_i vanish.

Solution. Since a complex projective space hascohomology equal to the coefficients in even dimensions and 0 in odd dimensions, we have the following second page of the spectral sequence:

$$\begin{array}{c|ccccccc}
\vdots & & & & & & & \\
6 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & \\
5 & & & & & & & \\
4 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & \\
3 & & & & & & & \\
2 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & \\
1 & & & & & & & \\
0 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots
\end{array}$$

It is immediate that d_4 is also zero, meaning that $E_2 = E_3 = E_4 = E_5$. However the case of d_5 is not so obvious since we get a map

$$d_5 : H^0(B) \to H^5(B)$$

that could be non-zero. The same will happen for all odd-index diferentials. $\square$

Exercise 2.7. Let $\tau : F \to E$ be the standard embedding map. Prove that the sequence

$$0 \longrightarrow H^1(B) \xrightarrow{\pi^*} H^1(E) \xrightarrow{\tau^*} H^1(F) \xrightarrow{d_2} H^2(B) \xrightarrow{\pi^*} H^2(E)$$

is exact.

Outline of solution. In nLab we see how to construct such an exact sequence using certain connectedness assumptions on the base and the fiber. The idea is similar to Gysin and Wang exact sequences below: connectedness and Hurewicz theorem make the first homology groups (except the 0-th) vanish, just like in the case of the sphere.

More explicitly, if the base is $(n_1 - 1)$ connected and the fiber is $(n_2 - 1)$-connected,

$$\begin{aligned} H_k(B) &= 0, \qquad 0 < k < n_1 \\ H_k(F) &= 0, \qquad 0 < k < n_2 \end{aligned}$$

This could then be taken to cohomology via Poincaré duality if both B and F are manifolds (this is necessary because Hurewicz theorem is for homology).

Then we find that the only possible non-vanishing differential is on the k-th page and of the form (asterisk represents some index I have not computed)

$$d_k : E_k^* = H^*(B) \longrightarrow E_k^* = H^*(F)$$

As in my proofs below, to extend this to an exact sequence involving the cohomology of the total space we use the convergence of the spectral sequence (the E_∞ terms) and the associated filtration. $\square$

Exercise 2.8. Let $F = S^k$, that is, $\pi : E \to B$is a sphere bundle. Prove that all differentials d_{k+1} vanish. Construct the ***Gysin exact sequence***

$$\cdots \longrightarrow H^p(B) \longrightarrow H^{p+k+1}(B) \xrightarrow{\pi^*} H^{p+k+1}(E) \longrightarrow H^{p+1}(B) \longrightarrow \cdots$$

Solution. (This argument is adapted from the construction of Wang exact sequence found in Wikipedia). We have that $E_2^{p,q} = H^p(B) \otimes H^q(S^k)$ can only be non-zero for $q = 0, k$. This means that the only non-zero differentials are of the form

$$d_{k+1} : E_2^{p,k} \cong H^p(B) \longrightarrow E_2^{p+k+1,0} \cong H^{p+k+1}(B)$$

$$\begin{tikzcd}
H^0(B) \otimes H^k(S^k) \arrow[ddrr, "d_{k+1}"] & H^1(B) \otimes H^k(S^k) \arrow[ddrr, "d_{k+1}"] & \cdots & H^k(B) \otimes H^k(S^k) \\
\vdots & & & \vdots \\
H^0(B) \otimes H^0(S^k) & \cdots & H^{k+1}(B) \otimes H^0(S^k) & H^{k+2}(B) \otimes H^0(S^k)
\end{tikzcd}$$

which means that $E^{k+1} = E^{\infty}$. Since $E^{k+1} = \ker d_{k+1} / \operatorname{img} d_{k+1}$, we can write

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \ker d_{k+1} & \longrightarrow & H^p(B) & \xrightarrow{d_{k+1}} & H^{p+k+1}(B) & \longrightarrow & \operatorname{coker} d_{k+1} & \longrightarrow & 0 \\
 & & \| & & & & & & \| & & \\
 & & \frac{\ker d_{k+1}}{\operatorname{img} d_{k+1}} & & & & & & \frac{E^{p+k+1,0}}{\operatorname{img} d_{k+1}} & & \\
 & & \| & & & & & & \| & & \\
 & & E^{p,k}_{k+1} & & & & & & E^{p+k+1,0}_{k+1} & & \\
 & & \| & & & & & & \| & & \\
 & & E^{p,k}_{\infty} & & & & & & E^{p+k+1,0}_{\infty} & &
\end{array}$$

This is the "first half" of the Gysin sequence. For the other half we must compute the E_∞ terms. We use the filtration

$$H^n(E) = F^0H^n \supset F^1H^n \supset \ldots \supset F^nH^n$$

that we know to satisfy

$$E^{p,q}_{\infty} \cong \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}.$$

We may write (I'm not completely sure why this works)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^{p+k+1,0}_{\infty} & \longrightarrow & H^{p+k+1}(E) & \longrightarrow & E^{p+1,k}_{\infty} & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
 & & \frac{F^{p+k+1}H^{p+k+1}}{F^{p+k+2}H^{p+k+1}} & & & & \frac{F^{p+1}H^{p+k+1}}{F^{p+2}H^{p+k+1}} & &
\end{array}$$

Putting this together with the first sequence we computed, we get that

$$\longrightarrow E^{p,k}_{\infty} \longrightarrow H^p(B) \xrightarrow{d_{k+1}} H^{p+k+1}(B) \longrightarrow E^{p+k+1,0}_{\infty} \longrightarrow H^{p+k+1}(E) \longrightarrow E^{p+1,k}_{\infty} \longrightarrow$$

and we simply remove the E_∞ terms to get the Gysin sequence

$$\longrightarrow H^p(B) \xrightarrow{d_{k+1}} H^{p+k+1}(B) \longrightarrow H^{p+k+1}(E) \longrightarrow H^{p+1}(B) \longrightarrow$$

Remark. I still cannot see why the map $H^{p+k+1}(B) \to H^{p+k+1}(E)$ is the map induced by the projection.

$\square$

Exercise 2.11. Let $\pi : E \to B$ be a fibration with $B = S^k$. Prove that all differentials except d_k vanish. Construct an exact sequence

$$\cdots \longrightarrow H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F) \xrightarrow{\mu} H^{p+k}(E) \longrightarrow H^{p+k+1}(F) \longrightarrow \cdots$$

where μ is multiplication by $\pi^* \operatorname{Vol}_{S^k}$ and $\tilde{d}_k$ is equal to d_k after the identification $H^p(F) = H^k(S^k) \otimes H^p(F) = E_2^{k,p}$

Solution. Like in Exercise 2.8 we see that the only non-zero differentials are

$$d_k : H^0(S^k) \otimes H^{k+p} \longrightarrow H^k(S^k) \otimes H^{p+1}(F)$$

because $E_2 = E_k$ looks like this:

$$\begin{tikzcd}
H^0(S^k) \otimes H^{k+1}(F) \arrow[dddrr, "d_k"] & \cdots & H^k(S^k) \otimes H^{k+1}(F) \\
H^0(S^k) \otimes H^k(F) \arrow[dddrr, "d_k"] & \cdots & H^k(S^k) \otimes H^k(F) \\
\vdots & & \vdots \\
H^0(S^k) \otimes H^2(F) & \cdots & H^k(S^k) \otimes H^2(F) \\
H^0(S^k) \otimes H^1(F) & \cdots & H^k(S^k) \otimes H^1(F) \\
H^0(S^k) \otimes H^0(F) & \cdots & H^k(S^k) \otimes H^0(F)
\end{tikzcd}$$

Again like in Exercise 2.8 we obtain a sequence

$$0 \longrightarrow E_\infty^{0,q} \longrightarrow H^q(F) \xrightarrow{d_k} H^{q-k+1}(F) \longrightarrow E_\infty^{k,q-k+1} \longrightarrow 0$$

Remark. The exercise has the map $H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F)$, but my computations suggest it should be $H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^{p+1}(F)$.

Then we compute the E_∞ terms using a filtration

$$H^n(E) = F^0H^n \supset F^1H^n \supset \ldots \supset F^nH^n, \qquad E_\infty^{p,q} = \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}$$

which yields

$$0 \longrightarrow E_\infty^{k-1,q-k+1} \longrightarrow H^q(E) \longrightarrow E_\infty^{0,q} \longrightarrow 0$$

and then we get

$$\cdots \longrightarrow H^q(E) \longrightarrow H^q(F) \longrightarrow H^{q-k+1}(F) \longrightarrow H^{q+1}(E) \longrightarrow H^{q+1}(F) \longrightarrow \cdots$$

Remark. As in Exercise 2.8, I don't know why the map $H^{q-k+1}(F) \to H^{q+1}(E)$ should be multiplication by the volume form of S^k.

$\square$

Exercise 2.11. Let $\pi : E \to B$ be a fibration with the fiber F an odd-dimensional sphere. Assume that the rings $H^*(B)$ and $H^*(E)$ are freely generated by generators in odd degrees. Prove that all d_i vanish.

Idea. Suppose $F = S^{2k+1}$. Like in exercise 2.8, we have the following situation on the second page of the associated spectral sequence:

$$
\begin{array}{c|cccc}
2k+1 & H^0(B) & H^1(B) & \cdots & H^{2k+1}(B) \\
\vdots & & & & \\
 & & & & \\
2 & & & & \\
1 & & & & \\
0 & H^0(B) & H^1(B) & \cdots & H^{2k+1}(B) \\
\hline
 & 0 & 1 & \cdots & 2k+1
\end{array}
$$

so that the only non-zero differentials are d_{k+1}. Now since $H^*(B)$ is freely generated over $\mathbb{K}$, we may write

$$
\begin{array}{c|cccc}
2k+1 & a_{1,2k+1}\mathbb{K} & a_{2,2k+1}\mathbb{K} & \cdots & a_{2k+1,2k+1}\mathbb{K} \\
\vdots & & & & \\
 & & & & \\
2 & & & & \\
1 & & & & \\
0 & a_{1,1}\mathbb{K} & a_{1,2}\mathbb{K} & & a_{2k+1,1}\mathbb{K} \\
\hline
 & 0 & 1 & \cdots & 2k+1
\end{array}
$$

where the relations between the $a_{i,j}$ is determined by the differentials and Leibniz rule... but I'm uncertain on how to continue. $\square$