

# Home assignment 1: Riemann-Roch formula in dimension 1

(Partial progress)

**Exercise 1** Prove that the ring  $\mathcal{O}_1$  of germs of holomorphic functions on  $\mathbb{C}$  is a principal ideal ring.

*Solution.* I assume that  $\mathcal{O}_1$  is the ring of germs of holomorphic functions about  $0 \in \mathbb{C}$ . I will use [Griffiths and Harris](#), Chapter 0, section *Weierstrass Theorems and Corollaries*. A *Weierstrass polynomial* in  $w$  is a function

$$w^d + a_1 w^{d-1} + \dots + a_d(z), \quad a_i(0) = 0.$$

**Theorem (Weierstrass Division Theorem)** Let  $g(z, w) \in \mathcal{O}_{n-1}[w]$  be a Weierstrass polynomial of degree  $k$  in  $w$ . Then for any  $f \in \mathcal{O}_n$  we can write

$$f = g \cdot h + r$$

with  $r(z, w)$  a polynomial of degree  $< k$  in  $w$ .

In words, we can express the germ of a holomorphic function around 0 as the product of a Weierstrass polynomial of degree  $k$  times some holomorphic function plus a polynomial of degree  $< k$ .

Then the proof is just mimicking the [proof](#) that the usual polynomial ring is a principal ideal domain.

Let  $J$  be an ideal of  $\mathcal{O}_1$  and  $g \in J$  be a Weierstrass polynomial of lowest degree. Then by Weierstrass Division Theorem we get  $h, r \in \mathcal{O}_1$  such that  $f = g \cdot h + r$  but by the choice of  $g$  we see  $r$  must be zero. This means  $J = \langle g \rangle$ .

**Remark (Based in Bruno's approach)** The former proof using Weierstrass polynomials is, though correct, a bit of an overkill, since, as explained by Griffiths & Harris, Weierstrass polynomials are used to generalize the 1-dimensional situation, where by power series expansion we know that a holomorphic function has a unique local representation ([why?](#))

$$f(z) = (z - z_0)^n u(z), \quad u(z_0) \neq 0$$

So, any function  $f \in \mathcal{O}_1$  can be expressed as

$$f(z) = z^n g(z), \quad g(0) \neq 0$$

Then we notice that if  $f$  is in some ideal  $J \subset \mathcal{O}_1$ , then the ideal  $(z^n)$  must be contained in  $J$  since any  $\phi(z)z^n \in (z^n)$  must also belong to  $J$  since

$$z^n \phi(z) = \frac{f(z)\phi(z)}{g(z)} \in J$$

(why is it ok to divide by  $g$ ? It is only non-zero at  $0 \dots$ ) Then, like in our proof, let  $n_0$  the least order of vanishing of non-zero elements of  $J$ , which cannot be zero for in that case we get  $J = \mathcal{O}_1$ . Then we get that  $J = (z^{n_0})$  since any  $f \in J$  can be written as

$$f(z) = z^n g(z) = z^{n_0} z^{n-n_0} g(z).$$

□

**Exercise 1.2 (Invariant factors theorem)** Let  $R$  be a principal ideal ring. Prove that any finitely-generated  $R$ -module is a direct sum of cyclic  $R$ -modules. Use this result to deduce the Jordan normal form theorem, and to classify the finitely-generated abelian groups.

*Solution.* I follow [Dummit and Foote](#) to prove that

**Invariant Factors Theorem** Let  $R$  be a principal ideal *domain* and let  $M$  be a finitely generated  $R$ -module. Then

$$M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \dots \oplus R/(a_m)$$

for some integer  $r \geq 0$  and nonzero elements  $a_1, a_2, \dots, a_m$  of  $R$  which are not units in  $R$  and which satisfy the divisibility relation

$$a_1 | a_2 | \dots | a_m$$

which strongly relies on

**Theorem 4** Let  $R$  be a PID,  $M$  a free  $R$ -module of finite rank  $n$  and  $N$  a submodule of  $M$ . Then  $N$  is a free module of rank  $m \leq n$  and there exists a basis  $y_1, y_2, \dots, y_n$  of  $M$  such that  $a_1 y_1, a_2 y_2, \dots, a_m y_m$  is a basis of  $N$ , where  $a_1, a_2, \dots, a_m$  are nonzero elements of  $R$  with the divisibility relations

$$a_1 | a_2 | \dots | a_m.$$

whose proof is rather involved.

*Proof of Invariant factors theorem.* Choose a basis  $\{x_i\}_{i=1}^n$  of  $M$ . Consider the free module  $R^n$  along with a basis  $\{b_i\}_{i=1}^n$ . The homomorphism  $\pi : R^n \rightarrow M, b_i \mapsto x_i$  is surjective, so we get  $R^n / \ker \pi \cong M$ .

Apply Theorem 4 for the module  $R^n$  and its submodule  $\ker \pi$ . We get a basis  $\{y_i\}_{i=1}^n$  of  $R^n$  such that  $\{a_i y_i\}_{i=1}^m$  is a basis of  $\ker \pi$  for  $a_i \in R$  for  $i = 1, \dots, m$  such that  $a_1 | \dots | a_m$ . Then

$$M \cong R^n / \ker \pi = (Ry_1 \oplus \dots \oplus Ry_n) / (Ra_1 y_1 \oplus \dots \oplus Ra_m y_m).$$

It remains to show that the quotient on the right-hand side of last equation is in fact the desired decomposition of  $M$ .

Consider the map

$$Ry_1 \oplus \dots \oplus Ry_n \longrightarrow R/(a_1) \oplus \dots \oplus R/(a_m) \oplus R^{n-m}$$

given by

$$(\alpha_1 y_1, \dots, \alpha_n y_n) \longmapsto (\alpha_1 + (a_1), \dots, \alpha_m + (a_m), \alpha_{m+1}, \dots, \alpha_n)$$

The kernel of this map is the set of elements which  $\alpha_i \in (a_i)$  for all  $i = 1, \dots, m$ . Thus we can write the kernel as

$$Ra_1 y_1 \oplus \dots \oplus Ra_m y_m.$$

□

**Remark** Looks like I didn't use the divisibility condition  $a_1 \mid \dots \mid a_m$ .

Now let's deduce the Jordan normal form theorem using again [Dummit and Foote](#). For this we fix a vector space  $V$  over a field  $F$  and a linear transformation  $T$ . This makes  $V$  into an  $F[x]$ -module by substitution of the variable  $x$  by  $T$ .

The *invariant factors* are the elements  $a_i$  from the last theorem, which in our present case can be shown to be monic polynomials  $a_i(x)$  satisfying  $a_1(x) \mid \dots \mid a_n(x)$ . Further, these elements are associated to the so-called *elementary divisors*, which concern another similar formulation of the Invariant factors theorem. The elementary divisors are powers of the irreducible components of the  $a_i(x)$ .

If we assume that the  $a_i(x)$  factor into linear polynomials, the elementary divisors can be written as  $(x - \lambda)^k$ , where  $\lambda$  is one of the eigenvalues of  $T$  (we consider only one of the eigenvalues at this point since we are constructing *one* of Jordan blocks of  $T$ ).

Using the Invariant factors theorem for elementary divisors, we see that  $V$  is the direct sum of finitely many cyclic modules of the form  $F[x]/(x - \lambda)^k$ . Such quotients have basis  $\bar{x}^{k-1}, \bar{x}^{k-2}, \dots, \bar{x}, 1$ . Now we observe that the polynomials

$$(\bar{x} - \lambda)^{k-1}, (\bar{x} - \lambda)^{k-2}, \dots, \bar{x} - \lambda, 1$$

are also a basis. This follows since expanding the latter in terms of the  $\bar{x}_i$  gives a triangular matrix, which makes it invertible, so it's a valid change of coordinates.

Next we observe that action of multiplication by  $x$ , which is the same as applying  $T$ , maps these basis elements in the quotient as follows:

$$\begin{aligned} 1 &\mapsto 1\bar{x} = \lambda \cdot 1 + \bar{x} - \lambda \\ (\bar{x} - \lambda) &\mapsto \bar{x}^2 - \lambda\bar{x} = x\lambda \cdot (\bar{x} - \lambda) + (\bar{x} - \lambda)^2 \\ &\vdots \\ (\bar{x} - \lambda)^{k-2} &\mapsto \lambda \cdot (\bar{x} - \lambda)^{k-2} + (\bar{x} - \lambda)^{k-1} \\ (\bar{x} - \lambda)^{k-1} &\mapsto \lambda \cdot (\bar{x} - \lambda)^{k-1} + (\bar{x} - \lambda)^k \end{aligned}$$

(where I'm not sure how to compute the last two). The last expression simplifies further since  $(\bar{x} - \lambda)^k = 0$  in the quotient, so that multiplication of  $x$  by the basis element  $(\bar{x} - \lambda)^{k-1}$  is barely multiplication by  $\lambda$ . In sum, we have constructed the **Jordan block**

$$J_\lambda = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \ddots & 1 \\ & & & \lambda & \\ & & & & 1 \\ & & & & & \lambda \end{pmatrix}$$

applying this procedure to the rest of the cyclic components of  $V$  we obtain the matrix representation

$$\begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_t \end{pmatrix}$$

of  $T$ .

Finally, the classification of finitely generated abelian groups follows by direct application of the Invariant factors theorem choosing  $R = \mathbb{Z}$ , since abelian groups are in correspondence with  $\mathbb{Z}$ -modules.  $\square$

**Definition 1.2** A *coherent sheaf* on a complex manifold  $M$  is a sheaf of modules over the sheaf  $\mathcal{O}_M$  of holomorphic functions on  $M$ , which is locally finitely generated and locally finitely presented (that is, the sheaf of relations between its local generators is also locally finitely generated (**I don't understand this**)).

**Exercise 1.3** Let  $C$  be a complex curve, and  $x \in C$  a smooth point. Prove that any coherent sheaf on  $C$  supported in  $x$  is isomorphic to  $\bigoplus_{i=1}^k \mathcal{O}_C/\mathfrak{m}^{d_i}$ .

*Solution.* First notice that the local ring of regular functions  $\mathcal{O}_x$  at a smooth point  $x \in C$  is a principal ideal domain. This follows from exercise 1, since any neighbourhood of  $x$  may be identified with a neighbourhood of  $0 \in \mathbb{C}$ .

**Remark** In more general algebraic geometry the statement also holds, namely, the local ring of regular functions  $\mathcal{O}_x$  at a smooth point  $x$  in a curve over a closed field  $k$  is a principal ideal domain. This is because (see Hartshorne thm I.5.1) since  $x$  is smooth,  $\mathcal{O}_x$  is regular, meaning  $\mathfrak{m}/\mathfrak{m}^2$  has the same dimension as  $\mathcal{O}_x$  as a vector space over the field  $\mathcal{O}_x/\mathfrak{m}$ . This means that any element in  $\mathfrak{m}$  will be generated by any element in  $\mathfrak{m}/\mathfrak{m}^2$ .

Now any coherent sheaf  $\mathcal{F}$  supported in  $x$  is, in particular, finitely generated over  $\mathcal{O}_x$ , so we may apply Invariant Factors Theorem at any neighbourhood  $U \ni x$  to obtain

$$\mathcal{F}(U) = \mathcal{O}_x^r \oplus \mathcal{O}_x/(a_1) \oplus \dots \mathcal{O}_x/(a_m)$$

It only remains to show that the quotients  $\mathcal{O}_x/(\mathfrak{a}_i)$  are actually  $\mathcal{O}_x/\mathfrak{m}^{d_i}$  for some positive integers  $d_i$ . This amounts to showing that every ideal in  $\mathcal{O}_x$  is a power of  $\mathfrak{m}$ . A straightforward proof of this fact (provided by Bruno) is based on exercise 1.1:

*Proof of exercise 1.1.* The idea is that  $\mathcal{O}_1$  is not only a principal ideal ring, but its ideals are in fact of the form  $(z^{n_0+k}) = (z^{n_0})^k$ .

Recall that for any ideal  $J \subset \mathcal{O}_1$  and  $f \in \mathcal{O}_1$  we can express  $f$  locally as  $f(z) = z^k g(z)$  with  $g(0) \neq 0$ . Let  $k_0 \neq 0$  be the smallest such number (if  $k_0 = 0$ , all  $f \in J$  do not vanish at 0 and we obtain that  $J = \mathcal{O}_1$ ) and  $f_0 = z^{n_0} g_0$  the corresponding element in  $J$ . Then  $J \subset (z^{k_0})$  since every  $f_1(z) = z^{k_1} g_1(z)$  can be expressed as

$$f_1(z) = z^{k_0} z^{k_1-k_0}(z) g_1(z) = z^{k_0} h(z) \in (z^{k_0})$$

This shows every ideal is of the form  $(z^{k_0})$ . Finally, every ideal must be contained in the maximal ideal  $(z^{n_0})$  (this is a straightforward consequence of Zorn's lemma). Then every ideal is of the form  $(z^{k_0}) \subset (z^{n_0})$ , so  $z^{k_0} = z^{n_0+k}$  and we get the desired assertion that  $(z^{k_0}) = (z^{n_0+k}) = (z^{n_0})^k$ .  $\square$

**Remark (by Altan)** As in the previous remark, the fact that all ideals are powers of the maximal ideal in a local ring at a smooth point in an abstract algebraic variety also holds. This follows from Nakayama's lemma. (See also [StackExchange](#): all principal ideals are powers of the maximal ideal when  $\bigcap_{n \in \mathbb{N}} \mathfrak{m}_x^n = \{0\}$ .)

**Remark (by Misha about remark by Altan)** not true

$\square$

**Exercise 1.4 (After date)** Let  $C$  be a complex curve, and  $V$  an abelian group, freely generated by isomorphism classes of coherent sheaves on  $C$ . The *Grothendieck K-group*  $K_0(C)$  is the quotient of  $V$  by its subgroup generated by relations  $[F_1] + [F_3] = [F_2]$  for all exact sequences of coherent sheaves  $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$

- Let  $L$  be a line bundle and  $0 \longrightarrow \mathcal{O}_C \xrightarrow{m} L \longrightarrow R \longrightarrow 0$  be an exact sequence associated with a section  $\ell \in H^0(C, L)$ . Prove that  $[L] - [\mathcal{O}_C] = \sum_i \alpha_i [x_i]$ , where  $\alpha_i \in \mathbb{Z}^{>0}$ ,  $[x_i]$  are classes of skyscraper sheaves  $\mathcal{O}_C/\mathfrak{m}_{x_i}$ , and  $\mathfrak{m}_{x_i}$  is the maximal ideal of a point  $x_i$ .
- Prove that  $K_0(C)$  is generated by  $\mathcal{O}_C$  and the classes of skyscraper sheaves  $\mathcal{O}_C/\mathfrak{m}_x$ .

*Proof.*

- Here I suppose that the associated exact sequence to the section  $\ell$  is given by

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_C \xrightarrow{m} L \longrightarrow R = \text{coker } m = L/\text{img } m \longrightarrow 0 \\ f \longmapsto f \cdot \ell \longmapsto f \cdot \ell + \text{img } m \end{aligned}$$

(See [Stacks Project](#).) Since  $L$  is a line bundle, it is a locally free sheaf. This makes  $R$  a finitely generated module (see [wiki](#): module is finitely generated iff it is a quotient of a free module of finite rank). Apply Invariant factors theorem to obtain  $R = \sum_i \mathcal{O}_C / \mathfrak{m}^{d_i}$ . In the Grothendieck group we get

$$[L] - [\mathcal{O}_C] = \left[ \sum_i \mathcal{O}_C / \mathfrak{m}^{d_i} \right] = \sum [\mathcal{O}_X / \mathfrak{m}^{d_i}]$$

□

**Exercise 1.5 (After date)** Let  $C$  be a compact complex curve, and  $F$  a coherent sheaf on  $C$ . We define the *Euler characteristic* of  $F$  as

$$\chi(F) := \dim H^0(C, F) - \dim H^1(C, F)$$

Prove that  $\chi$  defines a group homomorphism  $K_0(C) \rightarrow \mathbb{Z}$ .

**Exercise 1.7** Let  $X$  be a complex manifold and  $\text{Pic}(X)$  the set of equivalence classes of vector bundles, equipped with multiplicative structure induced by the tensor product. The group  $\text{Pic}(X)$  is called the *Picard group* of  $X$ .

- a. Prove that the cohomology group  $H^1(X, \mathcal{O}_X^*)$  is naturally identified with  $\text{Pic}(X)$ .

*Solution.*

- a. Recall that a line bundle  $L$  may be reconstructed from the gluing functions  $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{C}^*$ , which can be thought as changes of coordinates on each fiber. These functions satisfy the consistency condition that  $g_{\gamma\beta} g_{\beta\gamma} = g_{\gamma\alpha}$  on  $U_\alpha \cap U_\beta \cap U_\gamma$ .

On the other hand, a cochain in Čech cohomology is an assignment of an element  $g_{\alpha\beta} \in \mathcal{O}_X^*$  for every intersection  $U_\alpha \cap U_\beta$ . Elements of  $H^1(X, \mathcal{O}_X^*)$  are cocycles, meaning that coboundary operator vanishes. For 1-cocycles this is typically written as  $(g_{\beta\gamma} - g_{\alpha\gamma} + g_{\alpha\beta})|_{U_\alpha \cap U_\beta \cap U_\gamma} = 0$ . In multiplicative notation this just says that  $g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1$  which equivalent to the consistency condition for gluing functions.

- b.  
c. Hopf surfaces.

$$\begin{array}{c} \mathbb{C}^2 \setminus \{0\} = \langle \alpha \rangle \\ \downarrow \mathbb{C}^* / \langle \alpha \rangle \\ \mathbb{CP}^1 \end{array}$$

now  $H^2(\text{Hopf surface}) = 0$ . Topologically it is  $S^1 \times S^3$ .

Here's why it won't work on K3 surfaces:

$$\begin{aligned} \alpha &\in H^0(L) \\ c_1(L) &= [\mathcal{D}] \in H^2(M, \mathbb{Z}) \\ \mathcal{D} &\text{ zero divisor of } de? \\ \int_{\mathcal{D}} \omega^{n-1} &= \langle [\mathcal{D}], \omega^{n-1} \rangle = 0 \end{aligned}$$

So this statement tells you that Euler number is the same as its first Chern class.

That

$$e(L_1 \otimes L_2) = e(L_1) + e(L_2)$$

follows from taking zero sections on  $L_2$ , on  $L_2$  and then tensor product goes to addition.

□

**Exercise 1.8** Let  $\mathbb{C}$  be a complex curve and  $F$  a coherent sheaf on  $\mathbb{C}$ .

- Prove that the restriction of  $F$  to a certain open set  $U \subset \mathbb{C}$  is isomorphic to a vector bundle. Prove that the rank of this vector bundle is independent on the choice of  $U$  when  $\mathbb{C}$  is irreducible. This number is called the *rank* of  $F$ .

*Solution.*

- (From [StackExchange](#)) I show how to construct a vector bundle from a *locally free* sheaf. Given an open cover  $U_i$  such that  $\mathcal{F}(U_i)$  is free for all  $i$ , we just need to find gluing functions  $g_{ij} : U_i \cap U_j \rightarrow GL(n, \mathbb{C})$ .

From the definition of locally free, we have isomorphisms  $f_i : \mathcal{F}(U_i) \rightarrow \mathcal{O}_{U_i}^n$ . Restricting to the intersection  $U_i \cap U_j$  we may define the functions

$$f_{ij} = f_j|_{U_i \cap U_j} \circ f_i^{-1}|_{U_i \cap U_j} : \mathcal{O}_{U_i}^n|_{U_i \cap U_j} \rightarrow \mathcal{O}_{U_j}^n|_{U_i \cap U_j}.$$

The claim in [StackExchange](#) is that every such map is induced by a gluing function, but I still cannot see why.

The question indeed is to show why a coherent sheaf can be locally expressed as a locally free sheaf. We cover the manifold with sheafs that look like exercise 1.3. And that basically says that the sheaf is locally a direct sum of free and a finite number of torsion quotients  $\mathcal{O}_X/\mathfrak{m}^{d_i}$ .

So

$$0 \longrightarrow \text{Torsion} \longrightarrow \mathcal{F} \longrightarrow (\mathcal{F}^{**}) \longrightarrow 0$$

where  $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, 0)$ .

□

## References

Dummit, D.S. and R.M. Foote. *Abstract Algebra*. Wiley, 2003.

Griffiths, Phillip and Joseph Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. A Wiley-Interscience Publication. New York: John Wiley & Sons, 1978.