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Home assignment 2: spectral sequences

The monodromy of Gauss-Manin local system

Definition 2.1. Let $\pi : E \to B$ be a locally trivial fibration with fiber F. The family of cohomology of fibers of π is locally trivial, (what does this mean precisely?) but it might have ***the monodromy***. In other words, the group $\pi_1(B)$ naturally acts on the algebra $H^*(F)$ by autmorphisms. To obtain this action, take a loop in B and trivialize the family π along small intervals of this loop; this gives an identification of $H^*(F)$ with itself, which might be non-trivial.

Remark (Understanding the monodromy action of cohomology). (From StackExchange) Let $f : X \to U$ be a proper surjective submersion and fix $u_0 \in U$.

For any path $\gamma \subset U_j$, there is a canonical diffeomorphism $\varphi_\gamma : f^{-1}(\gamma(0)) \to f^{-1}(\gamma(1))$, using ψ_j (by a theorem of Ehresmann, all the fibers of f are diffeomorphic).

Now, for any loop γ, split γ into paths $\gamma_i \subset U_i$ and you can compose these diffeomorphisms to get a diffeomorphism

$$\phi_{\gamma_n} \circ \ldots \circ \varphi_{\gamma_1} : f^{-1}(u_0) \to f^{-1}(\gamma(u_0))$$

It induces a map on homology: you can check that it is well defined up to homotopy.

Exercise 2.1. Let $\phi^* : \mathbb{Z} \to \mathrm{Aut}(H^*(F))$ be an automorphism induced by a homeomorphism $\phi : F \to F$. Construct a locally trivial family over a circle with monodromy in cohomology induced by ϕ^*.

Interpretation Given an action $\phi^* : \mathbb{Z} = \pi_1(S^1) \to \mathrm{Aut}(H^*(F))$, construct a fibre bundle such that ϕ^* is the monodromy action on cohomology.

Proof. Consider the standard torus fibration $T^2 \to S^1$. Any path in the circle can be thought of as an number $n \in \mathbb{Z}$. Perhaps the induced automorphism on cohomology is precisely the map $\mathbb{Z} \ni a \mapsto na \in \mathbb{Z}$. But I'm not looking for an automorphism of $\mathbb{Z}$... I need an automorphism of $H^\bullet(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$... □

Leray-Serre spectral sequence

Exercise 2.4. Let $\pi : E \longrightarrow B$ be a fibration with the fiber a torus. Assume the $d_2 = 0$. Prove that all differentials vanish.

Proof. Since $d_2 = 0$, we have that $E_3^{p,q} = E_2^{p,q}$. Then

$$d_3 : H^p(B) \otimes H^q(T) \longrightarrow H^{p+3}(B) \otimes H^{q-2}(T),$$

so the only way it could be non-zero is for $q = 2$, which implies that

$$H^p(B) \otimes H^2(T) \cong H^{p+3}(B) \otimes H^0(T) \iff H^p(B) \cong H^{p+3}(B)$$

But I don't see why this couldn't happen... $\square$

Exercise 2.5. Let $\pi : E \to B$ be a fibration with the fiber a torus. Assume that the pullback map $\pi^* : H^2(B) \to H^2(E)$ is injective. Prove that all differentials d_i vanish.

Solution. I'm not sure how to use the hypothesis since I usually deal with the total space after computing the E_∞ page via the filtration... $\square$

Exercise 2.6. Let $\pi : E \to B$ be a fibration with the fiber a complex projective space. Assume that $d_2 = 0$ and $d_3 = 0$. Prove that all differentials d_i vanish.

Solution. Since a complex projective space hascohomology equal to the coefficients in even dimensions and 0 in odd dimensions, we have the following second page of the spectral sequence:

$$\begin{array}{c|ccccccc}
\vdots & & & & & & & \\
6 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & \\
5 & & & & & & & \\
4 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & \\
3 & & & & & & & \\
2 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & \\
1 & & & & & & & \\
0 & H^0(B) & H^1(B) & H^2(B) & H^3(B) & H^4(B) & H^5(B) & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots
\end{array}$$

It is immediate that d_4 is also zero, meaning that $E_2 = E_3 = E_4 = E_5$. However the case of d_5 is not so obvious since we get a map

$$d_5 : H^0(B) \to H^5(B)$$

that could be non-zero. The same will happen for all odd-index diferentials. $\square$

Exercise 2.7. Let $\tau : F \to E$ be the standard embedding map. Prove that the sequence

$$0 \longrightarrow H^1(B) \xrightarrow{\pi^*} H^1(E) \xrightarrow{\tau^*} H^1(F) \xrightarrow{d_2} H^2(B) \xrightarrow{\pi^*} H^2(E)$$

is exact.

Exercise 2.8. Let $F = S^k$, that is, $\pi : E \to B$is a sphere bundle. Prove that all differentials d_{k+1} vanish. Construct the ***Gysin exact sequence***

$$\cdots \longrightarrow H^p(B) \longrightarrow H^{p+k+1}(B) \xrightarrow{\pi^*} H^{p+k+1}(E) \longrightarrow H^{p+1}(B) \longrightarrow \cdots$$

Solution. (This argument is adapted from the construction of Wang exact sequence found in Wikipedia). We have that $E_2^{p,q} = H^p(B) \otimes H^q(S^k)$ can only be non-zero for $q = 0, k$. This means that the only non-zero differentials are of the form

$$d_{k+1} : E_2^{p,k} \cong H^p(B) \longrightarrow E_2^{p+k+1,0} \cong H^{p+k+1}(B)$$

$$\begin{tikzcd}
H^0(B) \otimes H^k(S^k) \arrow[ddrr, "d_{k+1}"] & H^1(B) \otimes H^k(S^k) \arrow[ddrr, "d_{k+1}"] & \cdots & H^k(B) \otimes H^k(S^k) \\
\vdots & & & \vdots \\
H^0(B) \otimes H^0(S^k) & \cdots & H^{k+1}(B) \otimes H^0(S^k) & H^{k+2}(B) \otimes H^0(S^k)
\end{tikzcd}$$

which means that $E^{k+1} = E^{\infty}$. Since $E^{k+1} = \ker d_{k+1} / \operatorname{img} d_{k+1}$, we can write

$$\begin{tikzcd}
0 \arrow[r] & \ker d_{k+1} \arrow[r] \arrow[d, equal] & H^p(B) \arrow[r, "d_{k+1}"] & H^{p+k+1}(B) \arrow[r] & \operatorname{coker} d_{k+1} \arrow[r] \arrow[d, equal] & 0 \\
& \frac{\ker d_{k+1}}{\operatorname{img} d_{k+1}} \arrow[d, equal] & & & \frac{E^{p+k+1,0}}{\operatorname{img} d_{k+1}} \arrow[d, equal] & \\
& E_{k+1}^{p,k} \arrow[d, equal] & & & E_{k+1}^{p+k+1,0} \arrow[d, equal] & \\
& E_{\infty}^{p,k} & & & E_{\infty}^{p+k+1,0} &
\end{tikzcd}$$

This is the "first half" of the Gysin sequence. For the other half we must compute the E_∞ terms. We use the filtration

$$H^n(E) = F^0H^n \supset F^1H^n \supset \ldots \supset F^nH^n$$

that we know to satisfy

$$E_{\infty}^{p,q} \cong \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}.$$

We may write (I'm not completely sure why this works)

$$\begin{tikzcd}
0 \arrow[r] & E_{\infty}^{p+k+1,0} \arrow[r] \arrow[d, equal] & H^{p+k+1}(E) \arrow[r] & E_{\infty}^{p+1,k} \arrow[r] \arrow[d, equal] & 0 \\
& \frac{F^{p+k+1}H^{p+k+1}}{F^{p+k+2}H^{p+k+1}} & & \frac{F^{p+1}H^{p+k+1}}{F^{p+2}H^{p+k+1}} &
\end{tikzcd}$$

Putting this together with the first sequence we computed, we get that

$$\longrightarrow E_{\infty}^{p,k} \longrightarrow H^p(B) \xrightarrow{d_{k+1}} H^{p+k+1}(B) \longrightarrow E_{\infty}^{p+k+1,0} \longrightarrow H^{p+k+1}(E) \longrightarrow E_{\infty}^{p+1,k} \longrightarrow$$

and we simply remove the E_∞ terms to get the Gysin sequence

$$\longrightarrow H^p(B) \xrightarrow{d_{k+1}} H^{p+k+1}(B) \longrightarrow H^{p+k+1}(E) \longrightarrow H^{p+1}(B) \longrightarrow$$

Remark. I still cannot see why the map $H^{p+k+1}(B) \to H^{p+k+1}(E)$ is the map induced by the projection.

$\square$

Exercise 2.11. Let $\pi : E \to B$ be a fibration with $B = S^k$. Prove that all differentials except d_k vanish. Construct an exact sequence

$$\cdots \longrightarrow H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F) \xrightarrow{\mu} H^{p+k}(E) \longrightarrow H^{p+k+1}(F) \longrightarrow \cdots$$

where μ is multiplication by $\pi^* \operatorname{Vol}_{S^k}$ and $\tilde{d}_k$ is equal to d_k after the identification $H^p(F) = H^k(S^k) \otimes H^p(F) = E_2^{k,p}$

Solution. Like in Exercise 2.8 we see that the only non-zero differentials are

$$d_k : H^0(S^k) \otimes H^{k+p} \longrightarrow H^k(S^k) \otimes H^{p+1}(F)$$

since $E_2 = E_k$ is of the form

$$\begin{tikzcd}
H^0(S^k) \otimes H^{k+1}(F) \arrow[rrddd, "d_k"] & \cdots & H^k(S^k) \otimes H^{k+1}(F) \\
H^0(S^k) \otimes H^k(F) \arrow[rrddd, "d_k"] & \cdots & H^k(S^k) \otimes H^k(F) \\
\vdots & & \vdots \\
H^0(S^k) \otimes H^2(F) & \cdots & H^k(S^k) \otimes H^2(F) \\
H^0(S^k) \otimes H^1(F) & \cdots & H^k(S^k) \otimes H^1(F) \\
H^0(S^k) \otimes H^0(F) & \cdots & H^k(S^k) \otimes H^0(F)
\end{tikzcd}$$

Again like in Exercise 2.8 we obtain a sequence

$$0 \longrightarrow E_\infty^{0,q} \longrightarrow H^q(F) \xrightarrow{d_k} H^{q-k+1}(F) \longrightarrow E_\infty^{k,q-k+1} \longrightarrow 0$$

Remark. The exercise has the map $H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F)$, but my computations suggest it should be $H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^{p+1}(F)$.

Then we compute the E_∞ terms using a filtration

$$H^n(E) = F^0H^n \supset F^1H^n \supset \ldots \supset F^nH^n, \qquad E_\infty^{p,q} = \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}$$

which yields

$$0 \longrightarrow E_\infty^{k-1,q-k+1} \longrightarrow H^q(E) \longrightarrow E_\infty^{0,q} \longrightarrow 0$$

and then we get

$$\cdots \longrightarrow H^q(E) \longrightarrow H^q(F) \longrightarrow H^{q-k+1}(F) \longrightarrow H^{q+1}(E) \longrightarrow H^{q+1}(F) \longrightarrow \cdots$$

Remark. As in Exercise 2.8, I don't know why the map $H^{q-k+1}(F) \to H^{q+1}(E)$ should be multiplication by the volume form of S^k.

$\square$

Exercise last. Generators here (horizontal), generators there (vertical, 1,3,5), "Extend generators by Leibniz rule, and then they just kill everyting"