k3

1 Class 1

The most important invariant of a k3 surface is intersection form.

There are three classes of manifolds

1. Smooth manifolds

$$\text{smooth manifolds} \xrightarrow{\text{forgetful functor}} \text{PL manifold} \longrightarrow \text{Topological manifolds}$$

Donaldson: contiunally many non-equivalent smooth structures on $\mathbb{R}^4$. K3 surfaces has countably many smooth structures and only one of them is compatible with complex structure.

Definition. Intersection form. Given a quadratic form on a lattice $V_{\mathbb{Z}} = \mathbb{Z}^n$, so

$$q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$$

is ***unimodular*** if

$$V_{\mathbb{Z}} \xrightarrow{q} \mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z})$$

is an isomorphism.

Theorem (Universal coefficients formula)**.**

$$H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}^{b_{n-1}(M)} \oplus T_{n-1}(M)$$

$$h^n(M, \mathbb{Z}) = \mathbb{Z}^{b_n(M)} \oplus T_{n-1}(M)$$

Corollary. $H^2(X, \mathbb{Z})$ is torsion free if $\pi_1(X) = 0$ because

Definition. ***Signature*** is $m - n$ if q has signature (m, n).

Theorem (Rokhlm-Wu?)**.** Signature is divisible by 16 for simply-connected (something else).

Remark. The methods used in surgery break down in smooth case because strange topological objects like infinite sums of spheres arise.

Theorem (Freedman, 1982)**.** There are as many 4-manifolds as there are intersection forms. M simply connected 4 manifold homotopy class is uniquely determined by intersection dorm. Moreover, for every unimodular form there exists a unique M with this intersection form.

Theorem (Donaldson, 1986)**.** M smooth compact manifold with positive definite odd intersection form q. Then

$$\begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition. Bilinear symmetric form is ***indefinite*** if it is not positive definite nor negative definite.

Theorem (Classification of unimodular symmetric bilinear forms)**.** Odd are diagonalizable, while even are related to special Lie group E_8.

Definition. A ***K3 surface*** is a Kähler complex surface M with $b_1 = 0$ (simply connected) and $c_1(M, \mathbb{Z}) = 0$.

Kodaira did what André Weil couldn'g classify.

Theorem. K3 surfaces have trivial canonical bundle $K_M = \Lambda^2(\Omega^1 M)$.

2 Class 2

G topological group. ***Principal*** G ***bundle*** is a space with free G-action such that the quotient E/G is Housdorff. There are several conditions that make this work. And then you have $\text{Homotopy}(X, BG)$ =equivalence classes of G-bundles. Vector bundles of a manifold are the same as maps from X to $B\,U(n)$.

Vector bundles up to stable equivalence are classified basically by Chern classes, so by the cohomology in $H^\bullet(B\,U) = Q[c_1, c_2, \ldots, c_n$.

Now look at the loop space of X. Then $H^\bullet(\Omega X)$ is a free graded commutative algebra. Loop space has the interesting property that $\Omega\,U = B\,U$ and $\Omega B\,U = U$.

2.1 Bialgebras

Let A be a superalgebra (graded with antisymmetric product). Then we ask the axiom of coassociativity and that .

Example. G group, and $C(G)$ the ring of k-valued functions $C(G \times G) = C(G) \times C(G)$ so

$$\begin{aligned} G \times G &\longrightarrow G \\ C(G) &\longmapsto C(G) \otimes C(G) \end{aligned}$$

2.2 H-spaces

Definition. H-space is a space M with a map $\mu : M \times M$ to$M$that is homotopy associative,

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mu \times \mathrm{id}} & M \times M \\ \downarrow{\scriptstyle \mathrm{id} \times \mu} & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow{\quad \mu \quad} & M \end{array}$$

which is homotopy commutative. And with homotopy unit.

So it's like a homotopy algebra?

Example. The loop space.

2.3 Bialgebras of finite type

Definition. A bialgebra A is of ***finite type*** if it is the direct sum of $A = \bigoplus_{i \geqslant 0} A^i$ super-commutative and each A^1 is finite dimensional.

Remark. Free commutative algebra is polynomial algebra

Definition. $A = \mathbb{C}[x_1, \dots, x_n, \dots] \otimes \Lambda^\bullet(a_1, \dots, a_n, \dots)$ is a graded commutative free algebra. In the slides: it is $\mathrm{Sym}_{gr} V^*$ where V^* is a graded vector space.

Theorem (Hopf). A graded commutative bialgebra of finite type over k of 0 characteristic is free graded commutative as a k algebra.

2.4 The cohomology algebra of U(n)

Claim. The cohomology algebra $H^*(U(n), \mathbb{Q})$ is a free graded commutative algebra with generators in degrees $1, 3, 5, \dots, 2n-1$.

Demostração. Induction. $U(1)$ is clear because it is a circle. Then do Serre spectral sequence. Differentials vanish on the second page because there's only nonzero groups on even degrees! And we get that $E_2^{p,} = H^p(S^{2n-1}) \otimes H^q(U(n-1))$. And then the sequence converges to that of the total space which is $U(n)$. $\square$

2.5 Grassman manifolds

Definition. The ***fundamental bundle*** B_{fun} is a rank n vector bundle over $\mathrm{Gr}(n, m)$.

Claim. B, B' vector bundles of rank n, $m-n$, $B \oplus B'$

$$\varphi : X \to \mathrm{Gr}(m, n)$$

$$\varphi(x) = B_x \subset B_x \oplus B'_x = \mathbb{K}^m$$

then $B = \varphi^* B_{\text{fun}}$.

Theorem. If you have B as a bundle on a manifold X then $B \oplus B'$ is trivial for some bundle B'.

Demostração. Embed the total space in a large enough euclidean space. $\square$

Definition. $\mathrm{Gr}(n, \infty) = \mathrm{Gr}(n)$ is $\bigcup_{m=n_1}^{\infty} \mathrm{Gr}(n, m) = \mathrm{Gr}(n)$

Corollary. For every bundle B of rank n there is a function $\varphi : X \to \mathrm{Gr}(n)$ such that $B = \varphi^* B_{\mathrm{fun}}$.

Take a bundle $E \to X$ and G acts freely on E so E principal G bundle. Classifying space BG

Theorem (Atiyah-Bott)**.** Classifying space is unique up to homotopy equivalence.

2.6 Stiefel spaces

Definition. $\mathbb{K}^\infty$ is the direct limit of $\mathbb{K}^n$ so its just the direct sum $\bigoplus_{i=n}^{\infty} \mathbb{K}$. Stiefel space is the space of orthonormal n-frames.

If we prove that Stiefel is contractible we obtain our classifying space so let's prove that. We have a fibration

$$U(n) \hookrightarrow \mathrm{St}(n, \infty) \to \mathrm{Gr}(n, \infty)$$

Theorem. $\mathrm{St}(n)$ is contractible.

*Demostração***Step 1** Locally trivial fibration with contractible fiber and base $Y \to X$ then Y is contractible, this is so trivial.

Step 2 Fibration $\mathrm{St}(n) \to \mathrm{St}(n-1)$ with fiber S^∞

Step 3 Show that S^∞ is contractible.

Step 4 And then some map $\mathbb{R}$ that is not surjective, and construct homotopy of identity to a constant map.

$\square$

Exercise. If $X_\infty = \bigcup X_i$ is the inductive limit of contractible cellular spaces then it is contractible. Use Whitehead theorem.

Theorem (Important)**.** $\mathrm{Gr}(\infty) = B\,U$.

2.7 Stable equivalence

Definition. Vector bundles V, W are stable equivalent if $V \oplus A \cong W \cong B$ for trivial vector bundles A and B.

Homotopy classes of equivalent vector bundles are in coorespondance with...

Theorem. $B\,U$ is H-space.

Corollary. $H^*(B\,U, \mathbb{Q})$ is a free supercommutative algebra.

Claim. $H^*(B\,U$ is a free polynomial algebra generated by classes $c_1, c_2, \ldots$ in all even degrees.

3 Class 3

3.1 Reminder

Definition. ***Bialgebra*** is an algebra that is equipped with comultiplication, counit...

Remark. It is when the dual space also has an algebra structure, but we prefer to use the tensor notation.

Let $\sum_{i \geqslant 0} A^i$ with $\dim A^i < \infty$. ***Free commutative algebra*** is a polynomial algebra. ***Free graded commutative algebra*** is

$$\widetilde{\mathrm{Sym}}^\bullet(W^\bullet \oplus V^\bullet) := \mathrm{Sym}^\bullet(W^\bullet) \otimes \Lambda^\bullet(V^\bullet)$$

where

$$W = \bigoplus_i W^{\mathrm{even}} \qquad V = \bigoplus_i V^{\mathrm{odd}}.$$

3.2 Hopf algebra

Definition. A bialgebra is a ***Hopf algebra*** when it is also equipped with an antipode map such that the following diagram commutes

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"] & H \otimes H \arrow[d, "m"] \\ H \otimes H \arrow[r, "m"] & H \end{tikzcd}$$

A2 Unitality.

$$\begin{tikzcd} H \cong H \otimes \mathbb{C} \arrow[r, "id \otimes u"] \arrow[rr, bend right, "id"'] & H \otimes H \arrow[r, "m"] & H \end{tikzcd}$$

Example. The cohomology of the loop space, $H^\bullet(\Omega X)$.

3.3 Primitive elements in a bialgebra

Definition. An element of a bialgebra $x \in A$ is ***primitive*** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$$\begin{aligned}\Delta(xy) &= \Delta(x)\Delta(y) \\ &= (1 \otimes x + x \cdot 1)(y \otimes 1 + y \otimes y) \\ &= 1 \otimes xy + xy \otimes 1 + x \otimes y + y \otimes x.\end{aligned}$$

Remark. We trying to show that Hopf algebras? bialgebras? are generated by primitive elements?

Definition. $A^\bullet$ bialgebra, $\mathscr{P}^\bullet \subset A^\bullet$ space of primitive, and the natural embedding

$$\mathrm{Sym}_{gr}(\mathscr{P}^\bullet) \to A$$

We say that A is ***free up to defree*** k if

$$\bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P) \xrightarrow{\psi} A$$

is an embedding.

Lemma. Let $A^\bullet$ be a bialgebra which is free up to degree k. Then $A^\bullet$ is free up to degree $k+1$.

Proof.

Step 1 Choose a basis of P, $\{x_i\}$. Chose a polynomial condition $Q(x_1, \dots, x_n) = 0$ of degree $k+1$. Write this as

$$Q = Q_m x_1^m + Q_{m-1} x_1^{m-1} + \ldots + Q_0.$$

that is

$$Q = \sum_{i=0}^{m} Q_i x_1^i$$

with Q_i invariant somehow. Then we apply comutiplication to obtain

$$\Delta(Q) = Q \otimes 1 + 1 \otimes Q + R$$

where R is some sort of reminder with bounded degree:

$$R \in \mathfrak{U} := \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P) \otimes \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P)$$

which follows from a similar computation of that which we did after defining primitive elements.

Step 2 Project to drop the terms that have $Q \otimes 1 + 1 \otimes Q$:

$$\Pi : \mathfrak{U} \to x_1 \otimes \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P)$$

since the x_i are primitive, i.e. $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, one has

$$\Delta(x_1^m) = (x_1 \otimes 1 + 1 \otimes x_1)^m$$

we get that

$$\Pi(\Delta(x_1^m)) = mx_1 \otimes x_1^{m-1}$$

while on the board it is written that

$$\Pi(\Delta(x_1^m)) = \Pi((x_1 \otimes 1 + 1 \otimes x_1)^m)$$

Step 3 Let $\Pi(R) := x_1 \otimes R_0$. Since $Q = 0$ in A, its component R_0 is also equal to 0. So $\Pi(\Delta(Q)) = 0$. Then

$$\Pi\left(\Delta\left(\sum_m x_1^m \cdot Q_m\right)\right) = \sum_m x_1 \otimes x_1^{m-1} Q_m + \Pi(mx_1 \otimes x_1^{m-1} \cdot \Delta(Q_m))$$

and second summand is zero

$\square$

4 Class 4