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Home assignment 8: quaternionic Hermitian structures

Definition An ***almost hypercomplex structure*** on a manifold M is a triple almost complex structures (I, J, K) satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = -\mathsf{Id} \qquad \text{and} \qquad IJ = K = -JI.$$

It is called ***hypercomplex*** if I, J, K are integrable. An ***almost hypercomplex quaternionic Hermitian structure*** on M is an almost hypercomplex stricture (I, J, K) and a Riemannian metric h which is invariant under the action of I, J, K.

Exercise 8.1 Let (M, I, J, K, g) be an almost hypercomplex quaternionic Hermitian manifold, and $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$ its fundamental forms. Prove that $\omega_J + \sqrt{-1}\omega_K \in \Lambda^{2,0}(M, I)$.

Solution. Here's a proof from StackExchange using local coordinates. Recall that a $(2, 0)$-form σ is characterized by being expressible in any local holomorphic coordinate chart $(z^1, \dots, z^n)$ as

$$\sigma = \sum \sigma_{ij} dz^i \wedge dz^j$$

where σ_{ij} are functions and $\{dz^i, d\bar{z}^i\}_{i=1}^n$ is a base of the cotangent space (see Lee, lem. 4.1). In other words, a $(2, 0)$-form has no $d\bar{z}$ factors.

Let $\sigma := \omega_J + \sqrt{-1}\omega_K$ and $(z^1, \dots, z^n)$ any holomorphic local chart for I. Showing that σ has no $d\bar{z}$ factors is equivalent to showing that $\sigma(\partial_{\bar{z}^k}, \cdot) = 0$. This is indeed the case:

$$\begin{aligned}
\sigma(\partial_{\bar{z}^k}, \cdot) &= g(J\partial_{\bar{z}^k}, \cdot) + \sqrt{-1}g(K\partial_{\bar{z}^k}, \cdot) \\
&= g(J\partial_{\bar{z}^k}, \cdot) + \sqrt{-1}g(-JI\partial_{\bar{z}^k}, \cdot) \\
&= g(J\partial_{\bar{z}^k}, \cdot) + \sqrt{-1}g(-J(-\sqrt{-1})\partial_{\bar{z}^k}, \cdot) \\
&= g(J\partial_{\bar{z}^k}, \cdot) - g(J\partial_{\bar{z}^k}, \cdot) = 0
\end{aligned}$$

where in the third equality we have used that $\partial_{\bar{z}^i}{}_{i=1}^{n}$ is a local frame for $T^{0,1}(M, I)$, meaning that $I\partial_{\bar{z}^k} = -\sqrt{-1}\partial_{\bar{z}^k}$ (see Lee, prop. 1.56). □

References

Lee, J.M. *Introduction to Complex Manifolds*. Graduate Studies in Mathematics. American Mathematical Society, 2024. ISBN: 9781470476953.