K3 surfaces, home assignment 1: Riemann-Roch formula in dimension 1

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Remark 1.1. The Riemann–Roch formula for the curve is $\chi(F) = \deg(F) + \chi(\mathcal{O}_C)\operatorname{\mathsf{rk}} F$. Here we deduce this formula, together with $\deg(B) = \int_C c_1(B)$ for any vector bundle B on C. However, both the degree and c_1 are defined in such a way that the Riemann–Roch formula becomes a part of their definition.

Definition 1.1. A **principal ideal** in a ring R is an ideal xR generated by an element $x \in R$. A **principal ideal ring** is a ring where all ideals are principal.

Exercise 1.1. Prove that the ring $\mathcal{O}_1$ of germs of holomorphic functions on $\mathbb{C}$ is a principal ideal ring.

Exercise 1.2. ("Invariant factors theorem"). Let R be a principal ideal ring. Prove that any finitely–generated R-module is a direct sum of cyclic R-modules. Use this result to deduce the Jordan normal form theorem, and to classify the finitely–generated abelian groups.

Definition 1.2. A **coherent sheaf** on a complex manifold M is a sheaf of modules over the sheaf $\mathcal{O}_M$ of holomorphic functions on M, which is locally finitely generated and locally finitely presented (that is, the sheaf of relations between its local generators is also locally finitely generated).

Exercise 1.3. Let C be a complex curve, and $x \in C$ a smooth point. Prove that any coherent sheaf on C supported in x is isomorphic to $\bigoplus_{i=1}^k \mathcal{O}_C/\mathfrak{m}^{d_i}$, where $\mathfrak{m}$ is the maximal ideal of x, and $d_1, ..., d_k$ is a collection of positive integers.

Hint. Use the previous exercise.

Remark 1.2. In the following exercises, you can freely assume that any compact complex curve admits a line bundle L with the following property. For any holomorphic vector bundle B, there exists $n \gg 0$ such that the tensor power $B \otimes L^{\otimes n}$ is globally generated. This result is deduced from Kodaira-Nakano vanishing theorem.

Exercise 1.4. Let C be a complex curve, and V an abelian group, freely generated by isomorphism classes of coherent sheaves on C. **The Grothendieck K-group** $K_0(C)$ is the quotient of V by its subgroup generated by relations $[F_1] + [F_3] = [F_2]$ for all exact sequences of coherent sheaves $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$.

a. Let L be a line bundle, and $0 \longrightarrow \mathcal{O}_C \longrightarrow L \longrightarrow R \longrightarrow 0$ be an exact sequence associated with a section $l \in H^0(C, L)$. Prove that $[L] - [\mathcal{O}_C] = \sum a_i[x_i]$, where $a_i \in \mathbb{Z}^{>0}$, $[x_i]$ are classes of skyscraper sheaves $\mathcal{O}_C/\mathfrak{m}_{x_i}$, and $\mathfrak{m}_{x_i}$ is the maximal ideal of a point x_i.

b. Prove that $K_0(C)$ is generated by $\mathcal{O}_C$ and the classes of skyscraper sheaves $\mathcal{O}_C/\mathfrak{m}_x$.

Exercise 1.5. Let C be a compact complex curve, and F a coherent sheaf on C. We define **the Euler characteristic of** F as $\chi(F) := \dim H^0(C,F) - \dim H^1(C,F)$. Prove that χ defines a group homomorphism $K_0(C) \longrightarrow \mathbb{Z}$.

Exercise 1.6. Consider line bundles on a compact complex curve C.

a. Let L be a line bundle, admitting a holomorphic section, and

$$0 \longrightarrow \mathcal{O}_C \longrightarrow L \longrightarrow R \longrightarrow 0$$

the corresponding exact sequence. Define the **degree** $\deg L$ as $\chi(L) - \chi(\mathcal{O}_C)$. Prove that $\deg(L) = \dim H^0(C,R)$, where R is defined above.

b. Prove that the degree is multiplicative, $\deg(L_1 \otimes L_2) = \deg(L_1) + \deg(L_2)$.

Exercise 1.7. Let X be a complex manifold, and $\operatorname{Pic}(X)$ the set of equivalence classes of vector bundles, equipped with multiplicative structure induced by the tensor product. The group $\operatorname{Pic}(X)$ is called **the Picard group of** X**.**

a. Prove that the cohomology group $H^1(X, \mathcal{O}_X^*)$ is naturally identified with $\operatorname{Pic}(X)$.

b. Consider **the exponential exact sequence** $0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$, where $\mathbb{Z}_X$ denotes the constant sheaf. The corresponding long exact sequence

$$\longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow$$

takes a line bundle $[L] \in \operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ to an element $c_1(L) \in H^2(X, \mathbb{Z})$ called **the first Chern class of** L. Prove that a non-trivial bundle L with $c_1(L) = 0$ on a compact complex curve has no holomorphic sections.

c. (*) Construct a non-trivial bundle L on a compact complex manifold such that $c_1(L) = 0$, but $H^0(L) \neq 0$.

Exercise 1.8. Let C be a complex curve and F a coherent sheaf on C.

a. Prove that the restriction of F to a certain open set $U \subset C$ is isomorphic to a vector bundle. Prove that the rank of this vector bundle is independent on the choice of U when C is irreducible. This number is called **the rank** of F.

b. Denote by $[x] \in H^2(C, \mathbb{Z})$ the fundamental class of a point, that is, the generator of the group $H^2(C, \mathbb{Z}) = \mathbb{Z}$. Define **the degree** of a coherent sheaf F as $\deg_C(F) := \chi(F) - \mathsf{rk}(F)$, and let $c_1(F) := \deg_C F \cdot [x]$ be **the first Chern class of** F. Prove that the first Chern class defines a group homomorphism $c_1 : K_0(C) \longrightarrow H^2(C, \mathbb{Z})$.

c. Prove that this definition is compatible with the definition of $c_1(L)$ for line bundles given above.

d. Prove that c_1 satisfies **the Whitney formula**: for any two vector bundles B_1, B_2 on a curve, $c_1(B_1 \oplus B_2) = c_1(B_1) + c_1(B_2)$.

e. Let B_1, B_2 be vector bundles on C. Prove that $c_1(B_1 \otimes B_2) = \mathsf{rk}\, B_1 \cdot c_1(B_2) + \mathsf{rk}\, B_2 \cdot c_1(B_1)$.