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Lecture notes on K3 surfaces

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1 Class 1

The most important invariant of a k3 surface is intersection form.

There are three classes of manifolds

1. Smooth manifolds

$$\text{smooth manifolds} \xrightarrow{\text{forgetful functor}} \text{PL manifold} \longrightarrow \text{Topological manifolds}$$

Donaldson: contiually many non-equivalent smooth structures on $\mathbb{R}^4$. K3 surfaces has countably many smooth structures and only one of them is compatible with complex structure.

Definition Intersection form. Given a quadratic form on a lattice $V_{\mathbb{Z}} = \mathbb{Z}^n$, so

$$q : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$$

is ***unimodular*** if

$$V_{\mathbb{Z}} \xrightarrow{q} \mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z})$$

is an isomorphism.

Theorem (Universal coefficients formula)

$$H_{n-1}(M, \mathbb{Z}) = \mathbb{Z}^{b_{n-1}(M)} \oplus T_{n-1}(M)$$

$$h^n(M, \mathbb{Z}) = \mathbb{Z}^{b_n(M)} \oplus T_{n-1}(M)$$

Corollary $H^2(X, \mathbb{Z})$ is torsion free if $\pi_1(X) = 0$ because

Definition ***Signature*** is $m - n$ if q has signature (m, n).

Theorem (Rokhlm-Wu?) Signature is divisible by 16 for simply-connected (something else).

Remark The methods used in surgery break down in smooth case because strange topological objects like infinite sums of spheres arise.

Theorem (Freedman, 1982) There are as many 4-manifolds as there are intersection forms. M simply connected 4 manifold homotopy class is uniquely determined by intersection dorm. Moreover, for every unimodular form there exists a unique M with this intersection form.

Theorem (Donaldson, 1986) M smooth compact manifold with positive definite odd intersection form q. Then

$$\begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition Bilinear symmetric form is ***indefinite*** if it is not positive definite nor negative definite.

Theorem (Classification of unimodular symmetric bilinear forms) Odd are diagonalizable, while even are related to special Lie group E_8.

Definition A ***K3 surface*** is a Kähler complex surface M with $b_1 = 0$ (simply connected) and $c_1(M, \mathbb{Z}) = 0$.

Kodaira did what André Weil couldn'g classify.

Theorem K3 surfaces have trivial canonical bundle $K_M = \Lambda^2(\Omega^1 M)$.

2 Class 2

G topological group. ***Principal*** G ***bundle*** is a space with free G-action such that the quotient E/G is Housdorff. There are several conditions that make this work. And then you have $\text{Homotopy}(X, BG)$ =equivalence classes of G-bundles. Vector bundles of a manifold are the same as maps from X to $B\,U(n)$.

Vector bundles up to stable equivalence are classified basically by Chern classes, so by the cohomology in $H^\bullet(B\,U) = Q[c_1, c_2, \ldots, c_n$.

Now look at the loop space of X. Then $H^\bullet(\Omega X)$ is a free graded commutative algebra. Loop space has the interesting property that $\Omega\,U = B\,U$ and $\Omega B\,U = U$.

2.1 Bialgebras

Let A be a superalgebra (graded with antisymmetric product). Then we ask the axiom of coassociativity and that .

Example G group, and $C(G)$ the ring of k-valued functions $C(G \times G) = C(G) \times C(G)$ so

$$\begin{aligned} G \times G &\longrightarrow G \\ C(G) &\longmapsto C(G) \otimes C(G) \end{aligned}$$

2.2 H-spaces

Definition H-space is a space M with a map $\mu : M \times M$ to$M$that is homotopy associative,

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"] & M \end{tikzcd}$$

which is homotopy commutative. And with homotopy unit.

So it's like a homotopy algebra?

Example The loop space.

2.3 Bialgebras of finite type

Definition A bialgebra A is of ***finite type*** if it is the direct sum of $A = \bigoplus_{i \geqslant 0} A^i$ supercommutative and each A^1 is finite dimensional.

Remark Free commutative algebra is polynomial algebra

Definition $A = \mathbb{C}[x_1, \ldots, x_n, \ldots] \otimes \Lambda^\bullet(a_1, \ldots, a_n, \ldots)$ is a graded commutative free algebra. In the slides: it is $\mathrm{Sym}_{gr} V^*$ where V^* is a graded vector space.

Theorem (Hopf) A graded commutative bialgebra of finite type over k of 0 characteristic is free graded commutative as a k algebra.

2.4 The cohomology algebra of U(n)

Claim The cohomology algebra $H^*(U(n), \mathbb{Q})$ is a free graded commutative algebra with generators in degrees $1, 3, 5, \ldots, 2n - 1$.

Demostração. Induction. $U(1)$ is clear because it is a circle. Then do Serre spectral sequence. Differentials vanish on the second page because there's only nonzero groups on even degrees! And we get that $E_2^{p_1} = H^p(S^{2n-1}) \otimes H^q(U(n-1))$. And then the sequence converges to that of the total space which is $U(n)$. □

2.5 Grassman manifolds

Definition The ***fundamental bundle*** B_{fun} is a rank n vector bundle over $\mathrm{Gr}(n, m)$.

Claim B, B' vector bundles of rank n, $m - n$, $B \oplus B'$

$$\varphi : X \to \mathrm{Gr}(m, n)$$

$$\varphi(x) = B_x \subset B_x \oplus B'_x = \mathbb{K}^m$$

then $B = \varphi^* B_{\text{fun}}$.

Theorem If you have B as a bundle on a manifold X then $B \oplus B'$ is trivial for some bundle B'.

Demostração. Embed the total space in a large enough euclidean space. □

Definition $\mathrm{Gr}(n, \infty) = \mathrm{Gr}(n)$ is $\bigcup_{m=n_1}^{\infty} \mathrm{Gr}(n, m) = \mathrm{Gr}(n)$

Corollary For every bundle B of rank n there is a function $\varphi : X \to \mathrm{Gr}(n)$ such that $B = \varphi^* B_{\mathrm{fun}}$.

Take a bundle $E \to X$ and G acts freely on E so E principal G bundle. Classifying space BG

Theorem (Atiyah-Bott) Classifying space is unique up to homotopy equivalence.

2.5.1 The fundamental bundle

In class 4 I finally understood that

Definition The ***fundamental bundle*** on the Grassmanian $\mathrm{Gr}(n)$ (the Grassmanian is this space where points are linear spaces) is the vector bundle such that the fiber of one point (which is a vector space) is the vector space that is the point. It's very tautological.

Theorem (Did we prove this?) Let B be a vector bundle of rank n on a cellular space X. Then there exists a continuous map $\varphi : X \to \mathrm{Gr}(n)$ such that B is isomorphic to the pullback $\varphi^* B_{\mathrm{fun}}$ of the fundamental bundle.

Remark In fact $\mathrm{Gr}(n)$ is the classifying space of vector bundles of rank n, in the sense that isomorphism classes of vector bundles of maps $\varphi : X \to \mathrm{Gr}(n)$.

2.5.2 The canonical bundle

When doing homework 3 I found this very nice on Hatcher, Vector bundles and K3:

Definition The ***canonical bundle*** $p : E \to \mathbb{R}P^n$ has as its total space E the subspace of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ consisting of pairs $(\ell, v$ with $v \in \ell$ and $p(\ell, v) = \ell$. There is also an infinite-dimensional projective space $\mathbb{R}P^\infty$ which is the union of the finite-dimensional projective spaces $\mathbb{R}P^n$ under the inclusions $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ coming from natural inclusions $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$. The inclusions $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ induce corresponding inclusions of canonical line bundles, and the union of all these is a canonical line bundle over $\mathbb{R}P^\infty$.

A natural generalization is the Grassmanian $\mathcal{G}r(k, n)$ along with a canonical k-dimensional vector bundle over it consisting of pairs (ℓ, v) where ℓ is a poin the Grassmanian and v is a vector in ℓ.

2.5.3 What this classification space should mean

Remember that

Definition (Representable functor) Let $\mathcal{C}$ be a category. A functor $F : \mathcal{C}^{op} \to \text{Sets}$ is called ***representable*** if there exists an object $B = B_F$ in $\mathcal{C}$ with the property that there is a *natural* isomorphism of functors

$$\varphi : \mathcal{C}(-, B_F) \to F$$

where $\mathcal{C}(-, B_F)$ is the set of arrows from $-$ to B_F.

One ussually expresses the naturality condition for a map $f : X \to Y$ with the following diagram:

$$\begin{tikzcd} \mathcal{C}(X, B) \arrow[r, "\varphi_X"] \arrow[d, "f^*"] & F(X) \arrow[d, "f^*"] \\ \mathcal{C}(Y, B) \arrow[r, "\varphi_Y"] & F(Y) \end{tikzcd}$$

And in homotopy theory I have studied that

Theorem (Brown representability theorem) Let F be a contravariant functor from the homotopy category of parallel connected CW-complexes to pointed sets. If F satisfies conditions (i) and (ii) above (for any pointed connected CW-complexes X_i, A, B, C), then F is representable.

Remark (So what is a classifying space?) The theorem says that there is a space $B = B_F$ (itself a pointed CW-complex) for which there is a natural isomorphism

$$\varphi : [X, B_F]_* \xrightarrow{\cong} F(X)$$

for any pointed CW-complex X. This space B_F is called a ***classifying space*** for F. Recall also that when such φ exists, it is completely determined by a *generic* element $\gamma \in F(B_F)$.

The classifying space together with the genereic element is unique up to homotopy.

Remark $H^n(-, G)$ is represented by $K(G, n)$ together with a chosen element in $H^n(K(G, n), G)$

But anyway. We see in wiki that for the case of homework 3 bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$ we get that the base space $BU(1) = \mathbb{C}P^\infty$, Thus, the set of isomorphism classes of circle bundles over a manifold M are in one-to-one correspondence with the homotopy classes of maps from M to $\mathbb{C}P^\infty$.

So what is the functor that we are representing? I think is K. Because the maps are isomorphic to $K(S^1)$...? Circle bundles?

2.6 Stiefel spaces

Definition $\mathbb{K}^\infty$ is the direct limit of $\mathbb{K}^n$ so its just the direct sum $\bigoplus_{i=n}^{\infty} \mathbb{K}$. Stiefel space is the space of orthonormal n-frames.

If we prove that Stiefel is contractible we obtain our classifying space so let's prove that. We have a fibration

$$U(n) \hookrightarrow St(n, \infty) \to Gr(n, \infty)$$

Theorem $St(n)$ is contractible.

*Demostração***Step 1** Locally trivial fibration with contractible fiber and base $Y \to X$ then Y is contractible, this is so trivial.

Step 2 Fibration $St(n) \to St(n-1)$ with fiber S^∞

Step 3 Show that S^∞ is contractible.

Step 4 And then some map $\mathbb{R}$ that is not surjective, and construct homotopy of identity to a constant map.

□

Exercise If $X_\infty = \bigcup X_i$ is the inductive limit of contractible cellular spaces then it is contractible. Use Whitehead theorem.

Theorem (Important) $Gr(\infty) = BU$.

2.7 Stable equivalence

Definition Vector bundles V, W are stable equivalent if $V \oplus A \cong W \cong B$ for trivial vector bundles A and B.

Homotopy classes of equivalent vector bundles are in coorespondance with...

Theorem BU is H-space.

Corollary $H^*(BU, \mathbb{Q})$ is a free supercommutative algebra.

Claim $H^*(BU$ is a free polynomial algebra generated by classes $c_1, c_2, \ldots$ in all even degrees.

3 Class 3

3.1 Reminder

Definition ***Bialgebra*** is an algebra that is equipped with comultiplication, counit...

Remark It is when the dual space also has an algebra structure, but we prefer to use the tensor notation.

Let $\sum_{i\geqslant 0} A^i$ with $\dim A^i < \infty$. ***Free commutative algebra*** is a polynomial algebra. ***Free graded commutative algebra*** is

$$\widetilde{\mathrm{Sym}}^\bullet(W^\bullet \oplus V^\bullet) := \mathrm{Sym}^\bullet(W^\bullet) \otimes \Lambda^\bullet(V^\bullet)$$

where

$$W = \bigoplus_i W^{\mathrm{even}} \qquad V = \bigoplus_i V^{\mathrm{odd}}.$$

3.2 Hopf algebra

Definition A bialgebra is a ***Hopf algebra*** when it is also equipped with an antipode map (S) such that the following diagram commutes

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S\otimes \mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\eta"] \arrow[dr, "\Delta"'] & & \mathbb{C} \arrow[rr, "u"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"] & & H \otimes H \arrow[ur] &
\end{tikzcd}$$

[diagram from quantum group minicourse notes]

Example The cohomology of the loop space, $H^\bullet(\Omega X)$.

3.3 Primitive elements in a bialgebra

Definition An element of a bialgebra $x \in A$ is ***primitive*** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$$\begin{aligned}
\Delta(xy) &= \Delta(x)\Delta(y) \\
&= (1 \otimes x + x \cdot 1)(y \otimes 1 + y \otimes y) \\
&= 1 \otimes xy + xy \otimes 1 + x \otimes y + y \otimes x.
\end{aligned}$$

Remark We trying to show that Hopf algebras? bialgebras? are generated by primitive elements?

Definition $A^\bullet$ bialgebra, $\mathscr{P}^\bullet \subset A^\bullet$ space of primitive, and the natural embedding

$$\mathrm{Sym}_{gr}(\mathscr{P}^\bullet) \to A$$

We say that A is ***free up to defree*** k if

$$\bigoplus_{i\leqslant k} \mathrm{Sym}^i_{gr}(P) \xrightarrow{\psi} A$$

is an embedding.

Lemma Let $A^\bullet$ be a bialgebra which is free up to degree k. Then $A^\bullet$ is free up to degree $k+1$.

Proof.

Step 1 Choose a basis of P, $\{x_i\}$. Chose a polynomial condition $Q(x_1, \dots, x_n) = 0$ of degree $k+1$. Write this as

$$Q = Q_m x_1^m + Q_{m-1} x_1^{m-1} + \ldots + Q_0.$$

that is

$$Q = \sum_{i=0}^m Q_i x_1^i$$

with Q_i invariant somehow. Then we apply comutiplication to obtain

$$\Delta(Q) = Q \otimes 1 + 1 \otimes Q + R$$

where R is some sort of reminder with bounded degree:

$$R \in \mathfrak{U} := \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P) \otimes \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P)$$

which follows from a similar computation of that which we did after defining primitive elements.

Step 2 Project to drop the terms that have $Q \otimes 1 + 1 \otimes Q$:

$$\Pi : \mathfrak{U} \to x_1 \otimes \bigoplus_{i \leqslant k} \mathrm{Sym}^i_{gr}(P)$$

since the x_i are primitive, i.e. $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, one has

$$\Delta(x_1^m) = (x_1 \otimes 1 + 1 \otimes x_1)^m$$

we get that

$$\Pi(\Delta(x_1^m)) = m x_1 \otimes x_1^{m-1}$$

while on the board it is written that

$$\Pi(\Delta(x_1^m)) = \Pi((x_1 \otimes 1 + 1 \otimes x_1)^m)$$

Step 3 Let $\Pi(R) := x_1 \otimes R_0$. Since $Q = 0$ in A, its component R_0 is also equal to 0. So $\Pi(\Delta(Q)) = 0$. Then

$$\begin{aligned} 0 &= \Pi\left(\Delta\left(\sum_m x_1^m \cdot Q_m\right)\right) \\ &= \sum_m x_1 \otimes x_1^{m-1} Q_m + \Pi(m x_1 \otimes x_1^{m-1} \cdot \Delta(Q_m)) \\ &= \sum_m x_1 \otimes x_1^{m-1} Q_m \end{aligned}$$

so that

$$\begin{aligned} x_1 \otimes x_1^{m-1} Q_m &= 0 \\ \implies x_1^{m-1} Q_m &= 0 \end{aligned}$$

So we conclude that

$$Q_m = 0$$

□

Remark We just proved that for any subalgebra generated by finite elements, we didn't use that it is free.

3.4 Algebras with filtration

Definition A ***filtration on algebra*** is

$$A^\bullet \supset F_1 A^\bullet \supset F_2 A^\bullet \supset \dots$$

such that

$$F_i A^\bullet F_j \subset F_{i+j} A^\bullet$$

Definition ***Associated graded*** to a filtered algebra is

$$A^\bullet_{gr} = \bigoplus_{i=0}^{\infty} \frac{F^1 A^\bullet}{F^{i+1} A^\bullet}$$

$$F^0 A^\bullet = A^\bullet$$

Definition $I \subset A$ ideal then I-***adic filtration*** is the filtration by the degrees of the ideal

$$A \supset I \supset I^2 \supset I^3 \dots$$

Lemma Choose an I-adic filtration. Then A_{gr} is generated by its first and second graded components $A/I \oplus I/I^2$.

Demostração. Indeed, $I^k/^{k+1}$ is generated by products of k elements in (I/I^2). □

Definition A ***augmentation ideal*** in a bialgebra is the kernel of the counit homomorphism $\varepsilon : A \to k$. We denote it by $Z = \ker A$

Remark

$$\Delta(x) = 1 \otimes x + x \otimes 1 \bmod Z \otimes Z$$

Why? Because

$$\begin{aligned} x &= \varepsilon \otimes \mathrm{id}(\Delta(x)) \qquad \text{up to } Z \otimes A \\ \Delta(x) &= 1 \otimes x \qquad \text{up to } A \otimes X \\ \Delta(x) &= x \otimes 1 \end{aligned}$$

Ok, now we can prove Hopf theorem.

Theorem (Hopf theorem) A finite type bialgebra is generated by primitive elements.

In slides: Let A be a graded bialgebra of finite type over a field k of characteristic 0. Then A is a free graded commutative k-algebra.

Proof.

Step 1 I think this is the computation above.

Step 2 A_{gr} is a bialgebra.

Step 3 A_{gr} is multiplicative generated by Z^1/Z^2. All elements Z^1/Z_2 are primitive, so this algrebra A_{gr} is generated by primitive elements.

Step 4 Let $\{x_i\}$ be a basis of primitive elements of A_{gr}. Then lifts of A have no relations because A_{gr} is already generated by primitive elements. Then there are no relations also for elements in $A^\bullet$ (I think).

□

3.5 Grassmanians (Reminder)

B vector bundle of rank n on X then there exists a map (essentialy unique) $\varphi : X \to \mathrm{Gr}(n)$ such that

$$\varphi^*(B_{\text{fun}} = B$$

which makes the Grassmanian a classifying space, and $\mathrm{Gr}(1) = B\,U(n)$.

The infinite Grassmanian is important.

3.6 BU as an H-space (Reminder)

Bott periodicity identifies the space of loops on U and $B\,U$.

Proposition Embed $\mathbb{C}^\infty \times \mathbb{C}^\infty$ into $\mathbb{C}^\infty$ taking the basis vectors of the first copy to the even basis vectors and the basis of the second copy to the odd. Then for $L_1 \subset \mathbb{C}^\infty$, $L_2 \subset \mathbb{C}^\infty$, the map

$$L, L' \mapsto S(L, L')$$

defines a structure of H-space on the infinite Grassmanian $B\,U$.

Proof. Just show that H-associatity up to homotopy. □

Corollary $H^\bullet(B\,U, \mathbb{Q})$ is a free supercommutative algebra.

Proof. Follows from Hopf theorem. □

3.7 Cohomology of BU

Claim $H^\bullet(BU, \mathbb{Q})$ is a free polynomial algebra generated by classes $c_1, c_2, \ldots$ in all even degrees.

Demostração. Leray-Serre spectral sequence. □

3.8 Chern classes: axiomatic definition

Definition ***Chern classes*** are classes $c_i(B) \in H^{2i}(X)$ for $i = 0, 1, 2, \ldots$

Chern classes are $c_i(B) \in H^{2i}(X)$ for a complex vector bundle B over X with axioms

a. $c_0(B) = 1$

b. Functoriality (commutes with bullbacks): for $\varphi : X \to Y$ with B bundle on Y,

$$\varphi^*(c_i(B)) = c_1(\varphi^*(B))$$

c. Define ***total Chern class*** $c_* := \sum_i c_i(B)$ then

$$c_i(B) \cdot c_i(B') = c_*(B \oplus B') \qquad \text{(Whitney)}$$

d. $\mathcal{O}(1)$ on $\mathbb{C}P^n$,

$$c_i(\mathcal{O}(1) = 1 + [H]$$

where $[H]$ is the fundamental class of a hyperplane section.

Remark (Once and for all) $\mathcal{O}(-1)$ is the ***tautological line bundle*** on a Grassmanian, defined as $\{(\ell, v) \in \mathcal{G}r(k, n) \times \mathbb{C}^n : v \in \ell\}$.

$\mathcal{O}(1)$ is the ***hyperplane bundle*** which is the dual of that so $\{(\ell, v^*) \in \mathcal{G}r(k, n) \times (\mathbb{C}^n)^* : v^* \in \ell^*\}$

Suppose we have a class $a \in H^\bullet(BU)$. Then for all B on X

$$\varphi : X \to BU$$

so

$$B \cong \varphi^*(B_{\text{fun}})$$

and so

$$\varphi_B^*(c_*) = c_*(B).$$

4 Class 4

4.1 Reminder

For each rank n bundle B on X there exists $\varphi_B : X \to Gr(n, \infty) = BU(n)$ such that $\varphi_B^*(B_{\text{fun}} = B$.

The infinite grassmanian is classifying space for (?) stable bundles.

Some more review about H-space structure, primitive elements, a comment on last exercise of homework 2.

Chern classes of $\mathcal{O}(1)$ are hyperplane sections: $c_i(\mathcal{O}(1)) = 1 + [H]$.

4.2 The splitting principle

Exercise Prove that $BU(1) = \mathbb{C}P^\infty$.

Solution. Hopf fibration on S^∞? It's easier, take $n = 1$, it's just by definition. □

Definition The ***fundamental bundle*** on $BU(1)^n$ has fiber

$$\ell_1 \oplus \ell_2 \oplus \dots \ell_n$$

where $\ell_i \in BU(1)$ are product $\ell_1 \times \ell_2 \times \dots \times \ell_n$.

Remark Chern classes of B_{fun} are uniquely determined by axioms, because every factor has Chern classes, and fibers are just sums, and pullbacks preserve sums. . .

$$B_{\text{fun}} = \bigoplus_i \pi_i \mathcal{O}(1)$$

where

$$pi_i : BU(1)^n \to BU(1)$$

is a projection.

Remark $H^\bullet(BU(1))^n = \mathbb{Z}[z_1, \dots, zn$ Here at least I remember that the cohomology of $\mathbb{C}P^\infty$ is just polynomials so it looks reasonable that the n-th power is polynomials in more cariables.

Theorem (Splitting principle) Let $\varphi_{\text{fun}} : BU(1)^n \to BU$, the ***fundamental map***, it induces embedding on cohomology up to degree $2n$. For all primer generator $\sigma_i \in H^2(BU)$, $\varphi_{\text{fun}}(\sigma_1) = \lambda \sum_i z_i^k$ with $\lambda \neq 0$.

So

$$\begin{tikzcd}
BU(1)^n \arrow[r] & BU \\
X \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Remark Wiki Thus, the set of isomorphism classes of circle bundles over a manifold M are in one-to-one correspondence with the homotopy classes of maps from M to $\mathbb{C}P^\infty$

Theorem Chern classes are unique (uniquely determined by axioms).

Proof.

Step 1 Every bundle is obtained as pullback of the fundamental bundle. So for $A \in H^\bullet(BU)$ and B bundle on X, $A(B) = \varphi_B^*(A) \subset H^\bullet(X)$ so $c_i(B)$ are obtained as pullbacks of c in the fundamental bundle.

Step 2

$$BU(1)^\infty \xrightarrow{\varphi_{\text{fun}}} BU$$

pullback of fundamental bundle is fundamental. (This map is defined from the former by induction).

$$\varphi_{\text{fun}}^*(c_i(B_{\text{fun}}) = c_i(B_{\text{fun}} \text{ on } BU)$$

The Chern classes of the fundamental bundle are already known. Since φ_{fun}^* is injective by the splitting principle we are done.

□

4.3 Primitive generators of $H^*(BU)$

Recall the H-space multiplication:

$$\begin{aligned} BU \times BU &\longrightarrow BU \\ L_1 \times L_2 &\longmapsto L_1 \oplus L_2 \end{aligned}$$

and the comultiplication

$$\Delta : H^\bullet(BU) \to H^\bullet(BU)$$

Generators of $H^\bullet(BU)$ are $c_{h_1}, c_{h_2}, \ldots$ with $c_{h_i} \in H^{2i}(BU)$ and we have the comultiplication $\Delta(c_{h_i}) = c_{h_i} \otimes 1 + 1 \otimes c_{h_i}$.

Remark

$$\varphi = (\varphi_1, \varphi_2) : X \to BU \times BU$$

and we can compose so we have

$$\varphi \circ \mu : X \to BU$$

what does this map do?

$$\begin{aligned} \varphi \circ \mu : X &\longrightarrow BU \\ \varphi^*(B_{\text{fun}} &\longmapsto B_1 \\ (\varphi \circ \mu)^*(B_{\text{fun}}) &= B_1 \oplus B_2 \end{aligned}$$

So then we have

$$\begin{aligned} \varphi^* : H^\bullet(BU) \otimes H^\bullet(BU) &\to H^\bullet(X) \\ \Delta : H^\bullet(BU) &\to H^\bullet(BU) \otimes H^\bullet(BU) \\ \Delta \circ \varphi^* : H^\bullet(BU) &\to H^\bullet(X) \end{aligned}$$

Corollary For every $x \in H^\bullet(BU)$

$$X(B_1 \oplus B_2) = \Delta(x)(B_1, B_2)$$

Corollary If $x \in H^*(BU)$ is primitive, then $x(B_1 \oplus B_2) = x(B_1) \oplus X(B_2)$.

Proof. $\Delta(x) = x \otimes 1 + 1 \otimes x$ so $\Delta(x)$ evaluated on (B_1, B_2) ☐

Remark We will construct the full Chern class $c_*(B)$ as a pullback of a class $C \in H^*(BU)$.

Remark Then take exponential. Let $\chi_i \in H^{2i}(BU)$ be a primitive generator. Use Hopf theorem to see that it is unique by a constant. Snce $\chi_i(B_1 \oplus B_2) = \chi_i(B_1) + \chi_i(B_2)$, the class $C = e^{\sum_i a_i \chi_i} = 1 + \ldots + \frac{\chi_n}{n!} + \ldots$ satisfies the Whitney formula.

To construct Chern classes satisfying the axioms it remains to arrange the coefficients a_i in such a way that $C(\mathcal{O}(1)) = 1 + [H]$ I think this means hyperplane section.

Lemma An embedding

$$BU(1) \overset{\varphi}{\hookrightarrow} BU$$

with $\chi_i \in H^{2i}(BU)$ primitive generator. Then $\varphi^*(\chi_i) \neq 0$

Proof. $H^\bullet(BU) =$ symmetric polynomials in $H^i(BU(1))^n$, $\varphi_{fun}(x_N) = x \sum_{i=1}^n z_i^k$ so $\varphi(x_k) = \lambda x_1^k$. ☐

Remark $\varphi^*(c_i(B_{fun}) = c_i(\mathcal{O}(1) = 1 + [H]$

Theorem Choose generators $\chi_i \in H^2(BU)$ primitive. Then $\varphi^*(\sum_i \chi_i = \log(1 + [H])$ where the logarith is a formal power series, namely $\sum_{i=1}^\infty \frac{H^n}{n!}(-1)^n$.

That means $c(B_{fun}) = \exp\left(\sum_{\chi_i}\right)$.

5 Class 5

We want to study the space of line bundles on a surface.

5.1 K-group

Definition Let V be the set of equivalence classes of vector bundles on X. Consider the free module generated by V (it's just V copies of Z):

$$\mathbb{Z}\langle V \rangle = \bigoplus_V \mathbb{Z}$$

And now consider
$$\frac{\mathbb{Z}\left\langle V\right\rangle}{[F_1]-[F_1]-[F_3]}$$
for all exact sequences

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

Equivalently, the relation is $[F_1]+[F_3]=[F_2]$.

Remark We may give an H-structure to the set of homotopy classes of maps $X \to BU$ as follows $\varphi_1, \varphi_2 : X \to B\,U$
$$B_1 = \varphi^*(B_{\text{fun}})$$
define the H-product
$$\varphi := \varphi_1 \circ \varphi_2$$
such that
$$\varphi^*(B_{\text{fun}} = B_1 \oplus B_2$$

And then we have an isomorphism (that we are not going to use):

$$K\,(X) \xrightarrow{\text{hom}} \text{group of homotopy classes of maps from } X \text{ to } BU$$

This is because every bundle on X is the pullback of the fundamental bundle by some map. We need to check that the image of trivial bundle is trivial map (homotopic to constant?) and that it preserves the product.

Remark The important thing of today is that that sum corresponds to addition

Remark I guess I should first understand how is it that every bundle is the pullback of the fundamental bundle.

So for example for injectivity we need to show that if a map φ pulls back the fundamental bundle to the trivial bundle then φ is homotopic to identity. This is not obvious though.

The point is that that map is a bijection.

Claim Chern classes are defined on $K(X)$ and satisfy Whitney formula (meaning Chern classes they pass to the quotient, right?)

Proof. Let B be a bundle on X so that $B = \varphi^*(B_{\text{fun}}$. We showed last time that there is a $c_\cdot \in H^0(BU)$ such that $c_\cdot(B) = \varphi^*(c_\cdot$. In fact we proved that $c_\cdot = \exp(\text{additive})$, but its actually Chern character, $c_\cdot = \exp(\text{Ch}_\cdot)$, in fact $\text{Ch}_\cdot(B_1 + B_2\,\text{Ch}(B_1) + \text{Ch}_\cdot(B_2)$. ∎

5.2 Coherent sheaves

Definition Let M be a complex manifold and $\mathcal{O}_M$ its structure sheaf (of holomorphic functions). A ***coherent sheaf*** is a sheaf of $\mathcal{O}_M$-modules, locally isomorphic to a quotient of a free sheaf $\mathcal{O}_M^n$ by a finitely generated $\mathcal{O}_M$-invariant subsheaf.

A ***coherent sheaf*** on a projective manifold. A ***projective manifold*** is $\mathrm{Proj}(A^\bullet)$ where $A^\bullet$ is a graded ring. ***Coherent sheafes*** are sheaves of graded $A^\bullet$-modules.

Exercise Let M be a projective manifold. Prove that any coherent sheaf F has a (projective) resolution

$$0 \longrightarrow B_n \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow F \longrightarrow 0$$

where B_i are vector bundles. This is called the ***syzygy resolution***

Solution. Every module has a projective resolution called ***Koszul resolution***. So what is Koszul resolution. First you have a resolution of a maximal ideal. For a maximal ideal it is clear since ... (Herieta? and) Eisenbud or even Bourbaki Homological algebra. □

5.3 Coherent sheaves and their Chern classes

So there's actually two K-groups. One is generated by bundles and the other by sheaves. For bundles, it is an algebra. For sheaves, it is a module over the other one. For Groethendick one was $K^\bullet$ and the other $K_\bullet$ but we don't know which is which.

Remark After this is done, it's possible to prove that the K-group of coherent sheaves on a projective manifold is equal to the K-group generated by holomorphic vector bundles.

Definition The ***Chern class*** of a coherent sheaf is the Chern class of the corresponding element in the K-group.

Remark (about singularities, see slides) Suppose we do resolution of a manifold and pullback a bundle

$$\begin{array}{ccc} \tilde{M} & & \pi^*F \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ M & & F \end{array}$$

5.4 Euler characteristic of a coherent sheaf

Definition Let F be a coherent sheaf. Its ***Euler characteristic*** is

$$\chi(F) = \sum_i (-1)^i \dim H^i(F)$$

But what is that cohomology? What is the space?

Claim For any exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

we have

$$\chi(F_2) = \chi(F_1) + \chi(F_3)$$

Proof. Should be possible... □

Then

$$\chi : K(M) \to \mathbb{Z}$$

is a homomorphism.

5.5 Chern character

OK so last class we defined an homomorphism called χ that was additive. Now let's call it

$$c. = \exp(\mathrm{Ch.})$$

and it was additive

$$\mathrm{Ch.}(B_1 \oplus B_2) = \mathrm{Ch.}(B_1) + \mathrm{Ch.}(B_2)$$

So the textbook definition is that ***Chern character*** on line bundles is

$$\exp(c_\bullet(L))$$

So c_1 is additive and if you pass to the exponent it will be multiplicative:

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$
$$\mathrm{Ch.}(L_2 \otimes L_2) = \mathrm{Ch.}(L_1) \cdot \mathrm{Ch.}(L_2)$$

5.6 Riemann-Roch-Hirzebruch theorem

Theorem (RRH) Let F be a coherent sheaf on a complex compact manifold M. Then $\chi(F)$ can be expressed through Chern classes of F and M as follows:

$$\chi(F) = \int_X \mathrm{Ch.}(F) \wedge \mathrm{Td.}(TM),$$

where $\mathrm{Td.}(TM)$ mdenotes the ***total Todd class of the tangent bundle*** TM, which is a sum of Chern classes.

$$\mathrm{Td.} = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_1}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4}{720} + \dots$$

5.7 K-group for complex curves

Lemma K-group for complex curves is generated by line bundles.

Proof.

Step 1 For each F coherent sheaf, $L^n \otimes F$ has a section. So there is a monomorphism $L^{-N} \hookrightarrow F$.

Step 2 The consider the localization to produce a short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

since $F = \bigoplus_i F_i$ for $F_i = \mathcal{O}_M / \mathfrak{m}_X^{a_i}$ so

$$0 \longrightarrow (\mathfrak{m}_X^{a_1} \longrightarrow \mathcal{O}_X \longrightarrow F_1 \longrightarrow 0$$

□

5.8 Riemann-Roch for complex curves

Theorem (Riemann-Roch for complex curves) Let F be a coherent sheaf on a compact complex curve of genus g. Then

$$\chi(F) = c_1(F) + \mathrm{rk}(F)(1-g)$$

Proof. We want to see

$$c_1(L) = \deg(L)$$

Step 1 It suffices to prove for line bundles by the lemma.

Step 2 For degree 0 its easy beacuse $c_1(k_x) = 1$. For structure sheaf $\mathcal{O}_X$ we have rank is 1.

Step 3 Now let L be a line bundle. We have

$$0 \longrightarrow \mathcal{O}_M \longrightarrow L \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow L \longrightarrow \mathcal{O}_M \longrightarrow F \otimes L = F \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_M \longrightarrow L_1^N \otimes L \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow L_1^{-N} \longrightarrow L \longrightarrow F \longrightarrow 0$$

and the point is that many things "have sections". What does it mean to have sections.

□

5.9 Riemann-Roch-Hirzebruch for line bundles on complex surfaces

Definition A ***complex surface*** is a compact complex manifold of dimension 2.

Notation

$$(L_1, L_2) = c_1(L_1) \wedge c_1(L_2)$$

and if D is a divisor we write (the ***degree of a divisor***)

$$(D, L) = \deg_D L = \int_M [D] \wedge c_1(L)$$

Theorem (RRH for surfaces) L line bundle on surface and $K_X = \Omega^2(X)$ its canonical bundle. Then

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{(L - K_X, L)}{2}$$

where (A, B) denotes the intersection form applied to cohomology classes on X.

Proof.

Step 1 Let D a smooth curve of genus g and L_1, L_2 line bundles that fit in an exact sequence

$$0 \longrightarrow L_2 \longrightarrow L_2 \longrightarrow L_2|_D \longrightarrow 0$$

Then we use Rieman-Roch for curves gives

$$\chi(L_1) = \chi(L_2) + (L_2, D) + (1 - g)$$

Step 2 Let ND denote the normal bundle on D. The adjunction formula gives $K_D = K_X|_D \otimes KD$. Since $g - 1 = \deg K_D/2$, we obtain $1 - g = -(K_X + D, D)/2$.

Step 3 The next step goes as before, with Rieman-Roch in one dimension. Let $\chi'(L)$ be the RHS of section 5.9, namely $\chi'(L) = \chi(\mathcal{O}_X) + \frac{L - K_X, L)}{2}$. In step 1 we have $c_1(L_2) = c_1(L_1) + D$. Then

$$\begin{aligned}\chi'(L_2) - \chi'(L_1) &= \frac{1}{2}\left[(L_2 - K_X, L_2) - (L_2 - K_X - D, L_2 - D)\right] \\ &= (L_2, D) - (K_X + D, D)/2\end{aligned}$$

Step 4 Comparing Step 2 and Step 3, we get

$$\chi'(L_2) - \chi'(L_1) = \chi(L_2) - \chi(L_1)$$

Therefore, section 5.9 is equivalent for L_2 and for L_1. We just need to manipulate bundles to reduce a bundle to... by building exact sequences.

Step 5 So suppose you have a smooth section of a bundle. Take an ample bundle A and do

$$0 \longrightarrow \mathcal{O}_X \longrightarrow A^N \longrightarrow A^N|_D \longrightarrow 0$$

$$0 \longrightarrow L \longrightarrow A^N \otimes L \longrightarrow A^N \otimes L|_D \longrightarrow 0$$

and then by step 4 we just need to deal with $A^N \otimes L$.

Step 6 It's very ample, it has many sections, including some that are smooth. Now we just assume L is $A^N \otimes L$. So

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L|_D \longrightarrow 0$$

so for bundles that have smooth sections the statement is free.

□

5.10 Applying the general formula to the curve case

We have

$$\mathrm{Ch.}(L) = 1 + c_1(L) + \frac{c_1^2(L)}{2}$$

$$\mathrm{Td.}(L) = 1 + \frac{c_1(TM)}{2} + \frac{c_1^2(M) + c_2}{12}$$

Now

$$\chi(L) - \chi(\mathcal{O}) = -\frac{(K_1(L), K)}{2} + \frac{c_1(L)^2}{2} = \frac{(L, K - L)}{2}$$

6 Class 6: Local Torelli theorem and its applications

6.1 Exponential exact sequence

The exponential exact sequence is

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 0$$

and it gives a long exact sequence

$$\cdots \longrightarrow H^1(\mathcal{O}_M) \longrightarrow H^1(\mathcal{O}_M^*) = \mathrm{Pic} \xrightarrow{c_1} H^2(M, \mathbb{Z}) \xrightarrow{\alpha} H^2(\mathcal{O}_M) \longrightarrow \quad \longrightarrow \cdots$$

α is just forgetful map, a projection, to the $H^{0,2}(M)$ part of a form

The group $H^2(\mathcal{O}_M)$ is identified with $H^{0,2}(M)$ which is Dolbeault cohomology, hence the kernel of α is $H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$.

Proposition c_1 holomorphic line bundle on compact Kähler manifold belongs to intersection $H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$ and every element of this group can be realised as $c_1(L)$.

6.2 K3 surfaces are holomorphically symplectic

Definition A ***complex surface*** is a compact, complex manifold of complex dimension 2.

Definition A ***K3 surface*** is a (Kähler, can drop this assumption) complex surface M with $b_1 = 0$ and $c_1(M, \mathbb{Z}) = 0$

Remark The hypothesis that $c_1 = 0$ implies that $c_1(K_M) = 0$ and thus $K_M = 0_M$ (it is trivial). This is beacuase $H^1(\mathcal{O}_M) = 0$, which follows from Hodge theory.

6.3 Hodge diamond of a K3 surface

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

since the cohomology groups

$$\text{sections of } K_X = H^{2,0} = \mathbb{C} \quad \begin{array}{ccccc} & & \mathbb{C} & & \\ & 0 & & 0 & \\ & & ? & & \\ & 0 & & 0 & \\ & & H^{1,1} = \mathbb{C} & & \end{array} \quad \text{Hodge (Serre?) duality} \implies H^{0,2} = \mathbb{C}$$

For the missing one, we comput $\chi(\mathcal{O}_M)$ using Riemann-Roch, which gives c_2 and from that we comput b_2.

6.4 Geometric structures (the story of Teichmüler space)

Definition ***Geometric structure*** on a manifold is reduction of structure group to $G \subset GL()$.

6.5 Fréchet spaces

Definition A ***seminorm*** on a vector space V is a function $\nu : V \to \mathbb{R}^{\geqslant 0}$ such that

- $\nu(\lambda x) = |\lambda|\nu(x)$
- triangle inequality.

Definition Define a topology using a family of seminorms generated by the open balls of all seminorms.

Definition V infinite dimensional vector space, ν_α collection of seminorms. Sequence of vectors z_i is ***Cauchy*** if z_i is Cauchy for each ν_j. If all Cauchy sequences converge it is called ***Fréchet space***.

We can also define Fréchet space to with the distance $d(x, y) = \sum_{k=1} \frac{1}{2^k} \max(\nu_k(x-y), 1)$

Definition The ***topology*** C^k on a Riemannian manifold on the space $C_c^\infty(M)$ is

$$|\varphi|_{C^k} := \sup \sum_{i=0}^{k} |\nabla^i \varphi|$$

where ∇^i is the iterated connection $\nabla^i : \mathcal{C}^\infty(M) \to \Lambda^1(M)^{\otimes i}$

Definition Of tensor field, section of $TM^{\otimes i} \otimes T^*M^{\otimes j}$.

6.6 C^0 topology on group of diffeomorphisms

Idea To interpret diffeomorphisms as sections of a bundle.

Definition On $\mathrm{Dif}(M)$, riemannian manifold,

$$d(f_1, f_2) = \sup_{x \in M} d(f_1(x), f_2(x))$$

6.7 $\mathcal{C}^\infty$-topology on group of diffeomorphisms

It has more sets (is stronger) than the C^0 topology,

Definition Fix $\mathcal{U}$ small neightbourdhoos of id in $\mathrm{Dif}(M)$. Choose an atlas of $U_i \subset V_i$ such that U_i is relatively compact. There exists a neighbourhood of identity in Dif such that diffeomorphisms (sufficiently close to identity) they map $\tau(U_i) \subset V_i$. To find this neighbourhood use that closure of U_i is compact in V_i.

Now define the C^∞ topology on $\mathcal{U}$ as C^∞ convergence on maps from $U_i \subset \mathbb{R}^n$ to $V_i \subset \mathbb{R}^n$ using usual derivatives.

Anyways, the idea is that we only need a *uniform structure* which is a partially ordered set to define Cauchy sequences.

6.8 Teichmüler space of geometric structures

Let $\mathcal{C}$ be the set of all geometric structures of a given type equipped with C^∞ topology. The ***Teichmüller space*** is $\mathcal{C}/\,\mathrm{Diff}_0$, where Diff_0 is the connected component of the identity. The group $\mathrm{Diff}(M)/\,\mathrm{Diff}_0(M)$ is the ***mapping class group***, we are not going to use it.

6.9 Teichmüler space of symplectic structures

$\mathrm{Symp} \subset \Gamma(\Lambda^2(M)$. It is not Housdorff and we don't even know how much Housdorff it is. Maybe for four dimensional manifolds,...

6.10 Moser's theorem

Theorem (Moser, 1965) The Teichmüler space is a manifold, and the preiod map

$$\begin{aligned} \mathrm{Per} : \mathrm{Teich}_s &\longrightarrow H^2(M, \mathbb{R}) \\ w &\longmapsto [w] \end{aligned}$$

It is very beautiful but semi-elementary if you know Moser's lema.

6.11 The kernel of a differential form

If Ω is a differential form on M, its ***kernel*** is the space of all vectors $X \in TM$ such that $i_X(\Omega) = 0$.

Proposition $[\ker \Omega, \ker \Omega] \subset \ker \Omega$.

Corollary If (M, I) almost complex and $\Omega \in \Lambda^{2,0}(M)$ non-degenerate and closed, then I is integrable.

Proof. $T^{0,1} = \ker \Omega$. □

6.12 C-symplectic structures

Definition $\Omega \in \Lambda^2(M, \mathbb{C})$, M $4n$-dimensional manifold. Suppose that $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is nowhere zero. Then $\ker \Omega \oplus \overline{\ker \Omega} = TM \otimes \mathbb{C}$.

This is a ***C-symplectic*** manifold.

These manifolds have a nice Teichmüler space.

Theorem (Moser-Koebe?) $(M, I_+, \Omega_+($ family of C-symplectic forms, $[\Omega_t]$ =constant, $H^{0,1}(M_t) = 0$ then all Ω_t are related by a diffeomorphism. (This is Moser's trick!)

Notice that for $n = 1$ we have that the condition $\Omega^2 = 0$ and $\Omega \wedge \overline{\Omega}$ volume mean

Theorem CTeich Teichmüler space of C-symplectic structures on K3 surface. Consider the

$$\mathrm{Per} : \Omega \to [\Omega] \in H^2(M, \mathbb{C})$$

Then the image $\mathrm{Per}(\mathrm{Teich}) = \{pQ] : \int u \wedge u = 0, \int u \wedge \bar{u} > 0\}$ is a quadric.

This is a local diffeomorphism.

6.13 The period space of complex structures

Now take CTeich $/\mathbb{C}^*$ because the Teichmüler space of complex structures has a free $\mathbb{C}^*$ action.

Proposition (Local Torelli theorem for complex structures) Teichmüler space of complex strcutres on K3.

$$\mathbb{P}\mathrm{er} = \{v \in \mathbb{P}H^2(M, \mathbb{C}) : (v, v) = 0, (v, \bar{v}) > 0\}$$

so

$$\frac{\mathrm{CTeich}}{\mathbb{C}^*} \longrightarrow \frac{Q}{\mathbb{C}^*} \subset \mathbb{P}H^2(M, \mathbb{C})$$

6.14 The period space of complex structures is a Grassmanian

Lets define

$$\mathrm{Gr}_{++}(H^2(M, \mathbb{R})) = \text{positively oriented 2-planes in } H^2(M, \mathbb{R})$$

Where positively oriented means the form is ? Then

$$\mathbb{P}\mathrm{er} = \mathrm{Gr}_{++}(H^2) = \frac{SO(3, 1o)}{SO(2) \times SO(1, 1o)}$$

7 Class 7: smooth quartics

7.1 Reminder on local Torelli theorem

$$\mathrm{CTeich} = \text{Teichmüller space of hol sympl structures}$$

$$\begin{aligned} \mathrm{CTeich} &\longrightarrow H^2(M, \mathbb{C}) \\ \Omega &\longmapsto [\Omega] \end{aligned}$$

Then this map is a local differomorphism to the period space.

7.2 Hodge index theorem (without slides)

Theorem (Hodge index theorem) Consider a signature of intersection form on complex Kählerler surface is positive on real part of $\mathrm{Re}\, H^{2,0}(M)$, $(1, 0)$ on $H^{1,1}(M, \mathbb{R})$, negative on

$$\begin{aligned} \ker L : H^{1,1}(M, \mathbb{R}) &\longrightarrow H^4(M) = \mathbb{R} \\ X &\longmapsto [X \wedge \omega] \end{aligned}$$

with ω Kähler form.

Proof. $\Omega^{2,0}$ 1 dimensional, $\mathrm{Re}\, \Omega^{2,0}$ 2-dimensional (at most) in $\Lambda^2(M, \mathbb{R})$.

$$\begin{aligned} \Omega &= \omega_1 + r - 1\omega_2 \\ \Omega \wedge \overline{\Omega} &= \omega_1^2 + \omega_2^2 > 0 \\ \Omega \wedge \Omega &= 0 = \omega_1^2 = \omega_2^2 \\ \iff \omega_1 \wedge \omega_2 &= 0 \\ \omega_1^2 &= \omega_2^2 \\ \implies \omega_1 &\perp \omega_2 \\ \omega_1^2 &= \omega_2^2 > 0 \end{aligned}$$

Then

$$\Lambda^{1,1} = \ker L \oplus \omega$$

That is a 4-dimensional bundle that is given by multiples of the Kähler form plus the primitive part. Then

$$\ker^{\perp} = \langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega \rangle$$

Then consider Hodge star operator $* : \Lambda^2 \to \Lambda^2$, $*^2 = 1$ and its complementary, this interchanges eigenvalues, negative positive, ... ☐

Corollary Signature of K3 surface is $(3, 19)$

7.3 The period space of complex structures is Grassmanian

Claim

$$\mathbb{P}\mathrm{er} = \frac{SO(3, b_2 - 3)}{SO(1, b_2 - 3) \times SO(2)} = \mathrm{Gr}_{++}(h^2(M, \mathbb{R})$$

Remark (V, q) real vector space signature q is (m, n), $m \geqslant 2$ then

$$\mathrm{Gr}_{++}(V, q) = \{\ell \in \mathbb{P}V_{\mathbb{C}} : q(\ell, \ell) = 0, q(\ell, \bar{\ell}) > 0\}$$

Recall that $T_p \mathrm{Gr}_{++} = \mathrm{Hom}(?)$

Of the claim.

Step 1 $\ell \in \mathbb{P}\mathrm{er}$

$$q(\operatorname{Re}(\Omega), \operatorname{Im}(\Omega)) = 0$$
$$q(\operatorname{Re}(\Omega), \operatorname{Re}(\Omega)) = q(\operatorname{Im}(\Omega), \operatorname{Im}(\Omega)) > 0$$

What is going on

$$\omega_1 = \operatorname{Re}(\Omega), \qquad \omega_2 = \operatorname{Im}(\Omega), \qquad \Omega \in \ell$$
$$\begin{aligned} q(\omega_1 + \sqrt{-1}\omega_2, \omega_1 + \sqrt{-1}\omega_2 &= 0 \\ &= q(\omega_1, \omega_1) - q(\omega_2, \omega_2) + \sqrt{-1}2q(\omega_1, \omega_2) \end{aligned}$$

and also

$$q(\Omega, \overline{\Omega}) > 0$$
$$= q(\omega_1 - \sqrt{-1}\omega_2, \omega_1 + \sqrt{-1}\omega_2) = q(\omega_1, \omega_1) + q(\omega_2, \omega_2) > 0$$

so we have obtained from a line in Period a positive definite plane

Step 2 $p \in \mathrm{Gr}_{++}$. Project, obtain a quadric form on $\mathbb{C}^2$. There exist two lines in $P_{\mathbb{C}}$, $\ell, \bar{\ell}$ such tat

$$q(\ell, \ell) = 0, \qquad q(\bar{\ell}, \bar{\ell}) = 0$$

$$q(x, y) = xy$$

☐

Corollary $U \subset \text{Teich}$, $V \subset H^2(M,\mathbb{R})$ set of all nonzero $(1,1)$-classes on $H^2(M,I)$ for some $I \in U$ Then $V \subset H^2(M,\mathbb{R})$ is open.

Proof. Idea: take a 2 dimensional space and move it a bit everywhere, consider orthogonal complement. Deform P by taking a y and considering its orthogonal complement. y is chosen close to x.

$$X \in P^{\perp} \text{is a class of type (1,1)}$$

$$y \text{near } x$$

$$P \text{project to } y^{\perp}$$

$$\text{so you have an open set in Grassmanian}$$

$$U_x \xrightarrow{\phi} Gr_{++} \qquad\qquad \phi^{-1}(U)$$

Step 1 Take a complex structure $I \in \text{Teich}$, $P \subset H^2(M,\mathbb{R})$, then $H^{11}(M,I) = P^{\perp}$.

Step 2 Teichmüler is locally diffeomorphic to Gr_{++}. Suffices to show in a nieghbourhood $U_1 \ni P$ in Gr_{++} that $\bigcup_{P_1 \in U_1} P_1^+$ is open.

Step 3 $y \in H^2(M,\mathbb{R})$, $y \in U_x$, nonzero in a neighb of $x \in P^{\perp}$. P_y projection from P to $y^{\perp}$

□

7.4 Intersection form on a K3 surface

Lemma (Of linear algebra) Consider bilinear symmetric form on $V_{\mathbb{Z}}$

$$\pi : V_R \backslash 0 \longrightarrow \mathbb{P}V_Q$$
$$\longmapsto$$

where R is the set of odd vectors and Q rational vectors. Then $p(\text{odd vectors})$ is dense on $\mathbb{P}V_Q$.

Proof.

Step 1 Construct a sequence of odd vectors converging to any element $s \in V_{\mathbb{Z}} \backslash 0$.

$$\lim_n \pi(r_0 + 2ns) = \pi(s)$$

□

Theorem Intersection form of K3 is even.

Proof. **Step 1** suppose it is odd. Coro 1 lema 1 imply complex structure I and odd vector $r \in H^1(M, I)$. The point is that the set of vectors of type 11 is open negithboudood of any class. But then the odd classes are dense.

Step 2 Involves Riemann-Roch. For each class $r \in H^{11}(M) \cap H^2(M, \mathbb{Z})$ we have a holomorphic line bundle by the exponential sequence, $c_1|L|$, L is hol line bundle:

$$0 \longrightarrow \mathbb{Z}_M \xrightarrow{\sqrt{-1}2\pi} \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^* \longrightarrow 0$$

and

$$\cdots \longrightarrow 0 = H^1(\mathcal{O}_M) \longrightarrow \mathrm{Pic} = H^1(\mathcal{O}_M^*) \longrightarrow H^2(M, \mathbb{Z}) \stackrel{\text{proj}}{\longrightarrow} H^{02}(M) = H^2(\mathcal{O}_M) \longrightarrow \quad \longrightarrow \cdots$$

Now Riemann-Roch:

$$\chi(L) = \chi(\mathcal{O}_M - \frac{L(K-L)}{2} = 2 - (L, L)/2$$

because canonical bundle is trivial, so that cannot be odd.

□

7.5 Smooth quartics

Definition A ***smooth quartic*** is a smooth quartic hypersurface in $\mathbb{P}^3$. So a solution of a quartic equation, ie. polynomial of degree 4.

Remark Adjunction formula. Canonical bundle of quartic is canonical bundle of $\mathbb{C}P^3$ restrictesd to quartic times normal bundle:

$$K_Q = K_{\mathbb{C}P^3}|_Q \otimes N(Q)$$

But $N(Q)$ is degree four so it is just $\mathcal{O}(4) = N(Q)$.

and canonical bundle $K_{\mathbb{C}P^3}$ of $\mathbb{C}P^3$ is $\mathcal{O}(-4)$ by Euler formula.

So $K_Q = 0_Q$—quartic has trivial canonical bundle (it is Calabi-Yau).

$$V : \mathbb{C}P^3 \hookrightarrow \mathbb{C}P^{34}$$

What is this map. It is associated to $\mathcal{O}(4)$, with the line system

$$\mathbb{C}P^{34} = \mathbb{P}H^0(\mathcal{O}4)^*$$

and it is called Veronese map.

Claim Smooth quartic is a hyperplane section of $V(\mathbb{C}P^3)$.

So any hyperplane on $\mathbb{C}P^{34}$ is a hyuperplane on $\mathbb{C}P^3$.

...So the zeroes of this restriction are quadrics.

The point is that quartics are (in correspondence with) hyperplane sections.

All quartics are sections of Veronese.

7.6 Smooth quartics and Lefschetz hyperplance section theorem

Theorem (Lefshetz hyperplane) $\pi(H \cap V) = \pi_1(V)$

8 Class 8: smooth quartics

8.1 Lefschetz again

Theorem (Lefschetz hyperplane) If $X \subset \mathbb{P}^n$ and $H = \mathbb{C}P^{n-1}$ a hyperplane in $\mathbb{C}P^n$ and $X \cap H$ (transversal justo to be safe), $X \cap H \to X$ isomorphism on homotopy group π_i for $i < \dim X$, ie. $\pi_i(Z \cap H) \xrightarrow{\cong} \pi_i(Z)$

Proof. Will discuss later but make a cellular decomposition that puts cells of certain dimension in the intersection. □

Corollary $\pi_1(\text{smooth quartic}) = \pi_1(\mathbb{C}P^3) = 0$.

Corollary Smooth quartic is K3.

8.2 Smooth submersions

Definition ***smooth submersion*** is a map $\pi : M \to M'$ such that $d\pi$ is surjective everywhere.

Remark Submersions are just products: each point has a neighbourhood that looks like a product and submersion is projection on one factor.

Theorem (Ehresemann fibration theorem) Let $\pi : M \to M'$ be a smooth submersion of compact manifolds. Then π is locally trivial fibration.

Proof. It is a vector bundle because

$$0 \longrightarrow T_\pi M \longrightarrow TM \xrightarrow{d\pi} \pi^* TM' \longrightarrow 0$$

where $T_\pi M$ is the vertical subbundle ie. $\ker \pi$

Ehresemann connection is a decomposition $T_{\text{horizontal}} \oplus T_{\text{vertica}} = TM$. Then there is a projection $d\pi : T_{\text{hor}} \to TM'$ and an associated curve. This gives the diffeomorphism that says all fibers are diffeomorphic. (see slides) □

8.3 Space of smooth quartics

Let $V = \mathbb{C}^{35} = \mathrm{Sym}^4\,\mathbb{C}^4$ be the set of homogeneous degree 4 polynomials in 4 variables. Interpret $P \in V$ as a quartic equation in $W = \mathbb{C}^4$

Claim Let $Z \subset \mathbb{P}V \times \mathbb{C}P^3$ be the set $\{(P \in \mathbb{P}V, w \in \mathbb{C}P^3 : P(w) = 0\}$. Then Z is smooth and irreducible.

Proof **Step 1** So we have a point in a hyperplane and the hyperplane is in $\mathbb{C}P^3$ Veronese (ie. embedded, it is a quartic Q). So we have $x \in \ell$, ℓ hyperplane section. And then let $\tilde{Z} \subset \mathbb{C}^4 \times \mathbb{C}^{35}$ be "the corresponding set of vectors". So $Z = \tilde{Z}/\mathbb{C}^* \times \mathbb{C}^*$. Clearly it suffices to show that $\tilde{Z}$ is smooth (?).

Step 2 Take the derivative of $(P + tQ)(w)$, it is not zero and so $\tilde{Z}$ is smooth.

Step 3 Use Sard's lemma or Bertini theorem + Lefschetz hyperplane to show F, the general fiber of the projection of Z to $\mathbb{C}P^{34}$, is connected and hence Z is irreducible.

How to use LEfschetz?

$$V(\mathbb{C}P^3) \subset \mathbb{C}P^{34}$$

$$Q = V(\mathbb{C}P^3) \cap H$$

and put all the cells in one half and then the other half just remains connected.

$\square$

Question Is there a better way to show that a general smooth quartic in $\mathbb{C}P^3$ is connected (without Lefschetz)?

Use that all quartics are equivalent (outside discriminant) and then just use X_1^4. (…?)

8.4 Smooth quartics are diffeomorphics

Corollary Smooth quartics are differomorphic.

Proof.

$$\begin{array}{c} Z \\ \big\downarrow{\scriptstyle Q} \\ \mathbb{C}P^{34} \supset \mathcal{D} \end{array}$$

We want to prove that the fibers are diffeomorphic. We need to remove the non-smooth fibers of this map. The critical values are the (…) is called ***discriminant***. So $\mathcal{D}$ is the set of al singular quartics. And $\mathbb{C}P^{34} \backslash \mathcal{D}$ is connected.

Exercise Complement to proper subvariety is connected. Take two points and try to join them. Vanya: they intersect $\mathcal{D}$ in a finite ammount of points. Misha: every path can be deformed to a path that avoids $\mathcal{D}$ by Sard's theorem because $\mathcal{D}$ has codimension 2.

Then all fibers of

$$\begin{array}{c} Z\backslash\pi^*(\mathcal{D}) \\ \Big\downarrow{\scriptstyle \pi} \\ \mathbb{C}P^{34}\backslash\mathcal{D} \end{array}$$

π are diffeomorphic because π is a proper smooth submersion. □

Remark The same arguemtns shows that smooth hypersurfaces of degree d in $\mathbb{C}P^n$ are diffeomorphic.

8.5 Ample bundles

Definition If you have $\varphi : X \to \mathbb{C}P^n$ projective complex, then

$$\varphi^*(\mathcal{O}(1))$$

is called ***very ample*** and L is ***ample*** if $L^{\otimes n}$ is very ample for some $n > 0$.

Kähler classes are classes of Kähler forms.

Theorem (Kodaira) L is very ample iff $c_1(L)$ is a Kähler class.

Objective All K3 are diffeomorphic. Need to prove that quartics are dense in the universal family of K3 over its Teichmüler space. Then we can deform a bit complex structure, deformatino doesn't change topology.

We need to identify the quartics among all K3 surfaces M containing $x \in \mathrm{Pic}(M)$ such that $x \cap x = 4$ in other words

You have a quartic Q, the generator of picard is $\mathcal{O}(1)$. Its self intersection is 4, to see if consider $[H \cap Q]$, intersection with hyperplane.

And we want that to be very ample. $c_1(L) \cap c_1(L) = 4$.

Remark (Dani) Not every K3 is quartic but every K3 is very close to a quartic in Teichmüler.

Remark Pellisky? did it with Kummer (surfaces?), which much harder.

8.6 Very ample bundles

Interpret very ampleness as vanishing of cohomology groups.

Claim (The Following AreEquival)

(i) $\phi_L : X \to \mathbb{C}P^n = \mathbb{P}H^0(X, L)$ is injective and holomorphic.

(ii) $\forall x, y$ exists section $\gamma \in H^0(X, L)$ with $\mathcal{D}$ =zero γ, $X \in \mathcal{D}, y \notin \mathcal{D}$.

This is equivalent to

$$\begin{array}{c} H^0(X, L) \\ \downarrow \\ H^0(X, \frac{L}{(\mathfrak{m}_x \cap \mathfrak{m}_y}) \otimes L) \end{array}$$

And that thing is some skyscraper things but is reall only $\mathbb{C}^2$.

Remark $\mathfrak{m}_x$ max ideal of x. The 1-jets of functions in x is $\mathcal{O}_X/\mathfrak{m}_x^2$. Then the natural map

$$\phi_L X :\to \mathbb{P}H^0(L)$$

is non-zero for all X. There exists section γ 1-jet x is non-zero $\gamma(x) = 0$.

Definition $f, g \in \mathcal{O}_{X,x}$ germs of functions at x (or even sections of line bundle in neighbourhood of a point). f has same k-jet as g if $f - g$ has zero order $k + 1$. Same Taylor series up to $k + 1$.

0jet is Taylor. 1-jet is Taylor plus differential. And the good thing is that $\mathfrak{m}/\mathfrak{m}^2 = T_x M^*$. Kernel of differential is kernel on 1-jets.

8.7 Alternative description of very ampleness

Corollary L bundle on compact complex manifold X. equivalent:

(i) L very ample.

(ii)
$$H^0(L) \to H^0(L)/(\mathfrak{m}_x \cap \mathfrak{m}_y) \otimes L)$$
(skyscraper sheaves isomorphic to $\mathbb{C}$) is surjective and also
$$H^0(L) \to H^0(L/(L \otimes \mathfrak{m}_x^2)$$
is surjective too. On right hand side of the second one in fact that it cotangent space (some comment about thinking of this like some coordinate system).

Now think of this short exact sequence of coherent sheaves

$$0 \longrightarrow L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y) \longrightarrow L \longrightarrow L/(\mathfrak{m}_x \cap \mathfrak{m}_y) \otimes L \longrightarrow 0$$

and the last sheaf is just finite dimensional space. This gives a long exact sequence.

$$\cdots \longrightarrow H^0(L) \longrightarrow H^0(L)/\mathfrak{m}_x \cap \mathfrak{m}_y \longrightarrow H^1(L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y)) \longrightarrow \ \longrightarrow \ \longrightarrow \cdots$$

If cohomology of that last one vanishes and also $H^1(L \otimes \mathfrak{m}_x^2)$ vanishes you are very ample. So its a vector bundle.

Remark So, for curves very amplness is very easy to check. Because every module is sum of rings by list 1 we have that the last one on the sequence is a vector bundle, finitely generated coherent, and then Kodaira says that vanishes.

Canonical bundle is very ample unless curve is elliptic?

9 Class 9: Nakai-Moishezon theorem

Today it will be mostly algebraic geometry, so no K3.

9.1 Ample bundles

Looks like here's another definition:

Definition L is ***very ample over*** X if

$$X \hookrightarrow \mathbb{P}H^0(x, L)^*$$

and L is ***ample*** if $L^{\otimes N}$ is very ample for some $N > 0$.

Remark $\bar{L}$ holomorphic line bundle on compact complex, then L is ample on a convex complex cone if and only if $\deg(L) > 0$.

9.2 Very ample bundles

Claim $x \neq y \in X$ then

$$\varphi : H^0(L) \longrightarrow H^0(L/(\mathfrak{m}_x \cap \mathfrak{m}_y)$$

is surjective, and the standard map

$$\varphi_L \longrightarrow \mathbb{P}H^0(X, L)^*$$

Two sections that have different derivative (1-jet) have different images, then the derivative is non-zero (derivative is injective). Uses inverse function theorem?

Proof. From (surjectivity?) of φ we get that

$$H^0(L) \longrightarrow H^0(L/\mathfrak{m}_x^2)$$

Then φ is isom to its image. (Because $\varphi = \varphi^*\Big|_{\mathfrak{m}_x/\mathfrak{m}^2}$) □

9.3 Very ample bundles again

Corollary If $H^1(L/(\mathfrak{m}_x \cap \mathfrak{m}_y)) = 0$ and $H^1(L \otimes \mathfrak{m}_y^2) = 0$ for all x, y then L is very ample.

Proof.

$$0 \longrightarrow H^0(L) \longrightarrow H^0(L/\mathfrak{m}_x^2) \longrightarrow H^1(\mathfrak{m}_x^2 \otimes L) \longrightarrow 0$$

□

9.4 Very ample bundles on a curve

Theorem (Kodaira-Nakano vanishing) L holomorphic line bundle on compact complex curve, $L \otimes K_M^{-1}$ ample. Then $H^i(L) = 0$ for all $i > 0$

Now we can do something about curves.

Corollary Let L be a line bundle on a compact complex curve C of genus g, and $\deg L > 2g$.

Proof.

Step 1 We need to prove that

$$H^1(L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y)) = 0 \qquad H^1(L \otimes \mathfrak{m}_x^2)$$

Slides: The sheaves $L \otimes \mathcal{O}_X \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y)$ are line bundles of $\deg L - 2$. Board: $L \otimes \mathfrak{m}^2$ and $L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y)$ are line bundles of degree $\deg L - 2$.

So what is it anyway?

$$L \otimes \mathfrak{m}_x^2 = L(-2x) \qquad L \otimes \mathfrak{m}_x \cap \mathfrak{m}_y = L(-x-y)$$

Step 2 The degree of the canonical bundle is $2g - 2$. By Kodaira, L_1 ample if and only if $\deg(L_1 \otimes K_X^{-1}) > 0$ iff $\deg L_1 > 2g-2$. $(L \otimes \mathfrak{m}_x^2) \otimes K_X^{-1}$ ample and equals $H^1(L \otimes \mathfrak{m}_x^2) = 0$ by Kodaira-Nakano, $L_1 = L \otimes \mathfrak{m}_x$, or $L_1 = L \otimes (\mathfrak{m}_x \cap \mathfrak{m}_y$

□

9.5 Canonical map for a complex curve

Definition Let L be a line bundle on X. A point $p \in X$ is called a ***base point*** if all sections of L vanish in p.

Definition Assume K_X has no base points. Then

$$\psi_{K_X} : X \longrightarrow \mathbb{P}H^0(X, K_X)$$

is called the ***canonical map***.

Theorem C curve, $g(L) \geqslant 2$. K canonical bundle. Then $H^0(K)$ has no common zeroes, the canonical map $\psi : C \longrightarrow \mathbb{P}H^0(K)^*$ is embedded or is a two-sheeted (2-to-1) ramified covering to $\psi(C) = \mathbb{C}P^1$.

Remark In the second case C is called a ***hyperelliptic curve***. In step 3, we will prove that any curve admitting a two sheeted ramified covering to $\mathbb{C}P^1$ is hyperelliptic.

This gives you two theorems about moduli spaces. Two curves are isomorphic if these subvarietes are conjugated by linear map, so gives you a point in Hilbert scheme. So hyperelliptics are the same as $\mathbb{C}P^1\ldots$ ah, but it has to be separated, $(\mathbb{C}P^1)^n$, and without the diagonals, and

$$\left((\mathbb{C}P^1)^n\backslash\text{diagonals}\right)\Big/\operatorname{PGL}(2,\mathbb{C})\times\{2^{2g}\}$$

The power of 2 term are possible choices of ramification.

Proof.

Step 1 First we need to show there are no common zeroes. That is, sections of K have no common zeroes. Let $p\in\mathbb{C}$ and let $k_p=\mathcal{O}_C/\mathfrak{m}_p$. Consider

$$0\longrightarrow K(-p)\longrightarrow K\longrightarrow k_p\longrightarrow 0$$

The corresponding long exact sqeuence says

$$H^0(K_C)\longrightarrow H^0(k_p)\longrightarrow H^0(K(-p))$$

so surjectivity is equivalent to $p\notin$ common zeroes.

But $H^1(K(-p))=0$ because $H^1(K(-p))=H^0(\mathcal{O}(p))^*$ by Serre duality. So we only need to show that there are no section is the latter degree 1 bundle. If there exists $\gamma\in H^0(\mathcal{O}(p))$ meromorphic function with a single pole on p, we obtain a holomorphic function $f:C\longrightarrow\mathbb{C}P^1$ of degree 1, meaning $C=\mathbb{C}P^1$.

Step 2 ψ non-injective, that it glues together p and q. Then $H^1(K(-p-q))\neq 0$.And by Serre duality $H^0(\mathcal{O}(p+q)=H^1(K(-p-q))\neq 0$. Then there exists a meromorphic function f with poles on p,q and $f:C\longrightarrow\mathbb{C}P^1$ of degree 2.

Step 3 So you have a map with two preimages. So consider a map that exchanges the preimages.

Now we will show that $\psi(C)=\mathbb{C}P^1$ admits a two sheeted ramified covering to $\mathbb{C}P^1$. Let $\tau:C\longrightarrow C$ be the involution exchanging the sheets of the covering. It is holomorphic because it has only Riemann-extendible singularities. And it acts on $H^0(K_C)$ with eigenvalues ±1. But $H^0(\Omega^1(\mathbb{C}P^1))=0$ so $\tau|_{H^0(K_C)}=-\mathrm{id}$. Therefore we see that ψ glue p and q.

□

9.6 Finite morphisms

Definition Let $f:X\longrightarrow Y$ a morphism of varieties (or schemes). f is ***finite*** if for every $U\subset Y$ open, $\mathcal{O}_{f^{-1}(U)}$ is finitely generated as an $H^0(\mathcal{O}_U)$-module.

Board: if the ring $f^*\mathcal{O}_U$ is finitely generated as an $\mathcal{O}_V$-module where $V=f^{-1}(U)$.

Theorem $f:X\longrightarrow Y$ proper and the preimage of any point is finite.

Proof. Hartshorne exerice III 11.2. And also past courses and EGA. □

9.7 Amplennes and cohomology

Theorem L is ample iff $\forall$ coherent F there exists $d > 0$ such that $H^i(F \otimes L^{\otimes k}) = 0$ for all $i > 0$ and $k \geqslant$.

Proof. Hartshorne. $\square$

Theorem f finite functor then pusforward is acyclic exact functor.

$f : X \longrightarrow Y$ finite map, F coherent sheaf on X. Then

$$H^i(f^{-1}(U), F) = H^i(U, f_* F)$$

for any open set $U \subset Y$; in other words, $R^i f_* F = 0$ for all $i > 0$.

Proof. The Rising Sea, thm 18.7.5. $\square$

Corollary 1 L line bundle on a complex variety X such that the standard map $f : X \longrightarrow \mathbb{P}H^0(X, L)^*$ is finite. Then L is ample.

Proof. (Proof is simple but unfortunately uses complicated theorems.)

Let $Y = f(X)$ and F a coherent sheaf on X. We have that $L = f^*(\mathcal{O}(1))$. And then there is the formula of base-change (maybe) which says (and it works for any other sheaf instead of $\mathcal{O}(1)$) that $f_*(F \otimes_{\mathcal{O}_X} L^{\otimes k} = f_* F \otimes_{\mathcal{O}_Y} \mathcal{O}(k)$. Ok and then we can compute cohomology:

$$H^i(X, F \otimes L^{\otimes n}) = H^i(Y, \underbrace{F \otimes \mathcal{O}(n)}_{=0}$$

so L is ample. $\square$

What is this for? We want to explore some invariants on K3.

9.8 Nakai-Moshezon theorem

It's a very nice criterion for ampleness. It can be generalized to Kähler, but we won't do that.

Theorem (Nakai-Moshezon) Let L be a line bundle on a projective variety X. Suppose for all subvarieties $Y \subset X$

$$\int_Y c_1(L)^d > 0, \qquad d = \dim Y$$

then L is ample.

Proof. It's seven steps.

Step 1 Let's do induction on $\dim X$. For $\dim X = 1$ it is clear. Assume that L is ample on all proper $X_1 \subsetneq X$. The next step is the most difficult step: show that $H^0(X, L^{\otimes n}) \neq 0$ for $k \gg 0$

Step 2 We need a very ample bundle. Let L_1 be a very ample bundle with a sufficiently big c_1 such that $c_1(L \otimes L_1 \otimes K_X^{-1}$ is Kähler ($\overset{?}{=}$ ample). Then $H^1(L_1 \otimes L) = 0$ for all $i > 0$. Let H be a smooth zero divisor of L_1 such that $\mathcal{O}(H) = L_n$. Now consider the short exact sequence

$$0 \longrightarrow L \longrightarrow L \otimes \mathcal{O}(H) \longrightarrow L \otimes \mathcal{O}(H)|_H \longrightarrow 0$$

I took a section of L_1, wrote this exact sequence and the one on the right is ample by assumption. Then we can replace L by a sufficiently big power $L^{\otimes d}$, we may assume that $L^{\otimes d}|_H$ is ample. (We want to show L_1? is Kähler. If you are an algebraic geometer then maybe you'd say something like any bundle is ample if you multiply be a sufficiently large power...) I think here we tensor multiply to get

$$0 \longrightarrow \mathcal{O}(-H) \longrightarrow \mathcal{O} \longrightarrow ? \longrightarrow 0$$

Anyway, we get

$$\cdots \longrightarrow H^{i-1}(L^{\otimes d} \otimes \mathcal{O}(n)|_H) \longrightarrow H^1(L^{\otimes d}) \longrightarrow H^i(L^{\otimes d} \otimes L_1) \overset{?,\text{board}}{=} H^1(L^{\otimes d} \otimes \mathcal{O}(H)) \longrightarrow \cdots$$

then there exists $d \gg 0$ such that $H^i(L^{d+j}) = 0$ for all $j \geqslant 0$ and $i > 1$.

(We didn't prove L is ample on X. The idea is that the curvature of $\mathcal{O}(n) \otimes L^{\otimes d}$ is strictly positive ($\overset{?}{=}$ has sections. Perhaps the use of d can be avoided if we choose H properly...)

Again, we have

$$\cdots \longrightarrow H^{i-1}(L^{\otimes d} \otimes \mathcal{O}(n)|_H) \longrightarrow H^1(L^{\otimes d}) \longrightarrow H^i(L^{\otimes d} \otimes \mathcal{O}(n)) \longrightarrow \cdots$$

and then

$$K_H = K_M|_H \otimes \mathcal{O}(n)$$
$$K_H^{-1} = K_M|_H \otimes \mathcal{O}(-n)$$

We proved there's only two possible non zero cohomology which is H^1 and H^0.

That is, there exists $d \gg 0$ such that $H^i(L^{\otimes j}) = 0$ for $i > 1, j > d$

Step 3 Looks like by Riemann-Roch, $\chi(kL)$ is a polynomial of k of degree n given by Todd.

$$\int_X \frac{kc_1(L)^n}{n!} = \lim_{k\to\infty} \chi(kL) = \infty$$
$$\Longrightarrow \lim_{k\to\infty} \mathrm{Ch.}(H^0(kL)) = \infty$$

Step 4 Replace L by $L^{\otimes k}$. Can assume $\dim H^0(L) > d$. Then

$$0 \longrightarrow (K-L)L \longrightarrow kL \longrightarrow kL|_D \longrightarrow 0$$

Now by inductive assumption $L|_D$ ample. So $H^i(kL) = 0$ for all $k > d$ and $i > 0$. Then we get the long exact sequence

$$0 \to H^0((k-1)L) \to H^0(kL) \to H^0(kL|_D) \to H^1((k-1)L) \to H^1(kL) \to 0$$

Now the function $k \mapsto \dim H^1(kL)$ is monotonous non-increasing, so it must stabilize. Therefore for $k \gg 0$ we get an surjection $H^0(kL) \longrightarrow H^0(L|_D)$.

Step 5 The birrational map $\Phi : X \longrightarrow \mathbb{P}H^0(kL)$. Every $x \in X$ is connected in D for some section of L to every $y \in D$ (?). Slides: D can be chosen as $\Phi^*(H)$ where Φ^* is proper preiage and H a hyperplane section in $\mathbb{P}H^0(L)^*$, therefore for any two points $x, y \in X$ we may choose D containint these two points.

Then there exists a section of kL non-zero on y, implying that Φ is holomorphic. So there is a section of kL separating these points (vanishing in one and non-vanishing in the other). For each point, there is a section that is not zero.

Step 6 The bundle kL is ample by Corollary 1.

□

Remark See the book *Positivity in algebraic geometry*.

10 Class 10: surfaces with Picard rank 1

10.1 Intuition and review

Today we'll see how to construct quadrics. Any K3 with rank 1 and Picard rank 4 is a quartic.

Mumford offered money to anyone who could give a K3 with Picard rank 1. Took 30 years.

So far we have:

Theorem (Kodaira) L ample $\iff c_1(L)$ Kähler.

Theorem (Kodaira-Nakano vanishing) $L \otimes K_M^{-1} \implies H^i(L) = 0 \forall i > 0$.

Theorem (?) C compact complex smooth curve then K_C is globally generated $\phi : C \longrightarrow \mathbb{P}H^0(K_C)$ is 2:1 smooth cover or embedding.

Theorem (result, see last class) Is ample iff kills cohomology

Corollary 1 $X \longrightarrow \mathbb{P}H^0(X, L)^*$ is finite then L is ample.

Intuition (of what's about to happen) Consider a curve with self-intersection (a hand-drawn line/string that intersects itself). Then we will resolve that singularity by "streching the string" or "separating the branches at the intersection point".

10.2 Singular curve in a K3 surface

Claim Let $C \subset M$ be a curve in a singular complex surface. Then there exists a surface $\tilde{M} \xrightarrow{\pi} M$ obtained by succesive blow-ups of M such that the proper preimage $\tilde{C}$ of C is smooth. (The ***proper preimage*** is taking the points in the curve and not the exeptional divisord.)

Definition Multiplicity of a singular point is dimension? of cohomology.

Proof. Take the single blow-up of C in a singular point. It has smaller multiplicity; We are saying that $\tilde{C}$ has strictly smaller multiplicity. For this we have thought of

$$\begin{array}{c} \tilde{M} \\ \downarrow \pi \\ M \subset C \end{array}$$

and the exceptional divisor is E in $\pi^{-1}(C) = \tilde{C} + E$. So to compute mutliplicity we have done

$$\tilde{L} \cap (\tilde{C} \times E) = L \cap C$$
$$\implies \tilde{L} \cap \tilde{C} < L \cap C$$

$\square$

Remark So we can resolve singularities and that's what happens with multiplicity.

10.3 Singular curve in a K3 surface (corollary 2)

So let M be a K3 surface and C a singular curve in K3. Then take the pullback bundle of C and call it $L = \mathcal{O}(\pi^{-1}(C)) = \pi^*(\mathcal{O}(C))$. Remember that $\pi^{-1}(C) = \tilde{C} + E$. Then $\mathcal{O}(\tilde{C}) \otimes \mathcal{O}(E) = L$, so the *normal bundle* is $N(\tilde{C}) = L \otimes \mathcal{O}(-E)$ and $K_{\tilde{M}} = \mathcal{O}(E)$. So we can compute the canonical bundle: $K_{\tilde{M}} \otimes N_{\tilde{C}} = K_{\tilde{C}}, \mathcal{O}(E) \otimes L \otimes \mathcal{O}(-E) = L$.

Corollary 2 C curve of genus ¿0 in K3 surface M. Then $\mathcal{O}(C)|_C$ is globally generated.

Proof. Need to show $\square$

10.4 Picard rank 1 is usually very ample

Theorem M K3 with $\mathrm{Pic}(M) = \mathbb{Z}$ and L a generator of $\mathrm{Pic}(M)$ such that $(L, L) \geqslant 0$. Then L or L^* is globally generated.

Proof.

Step 1 We use Riemann-Roch (like everytime when you get a surface). We get that $h^0(L) - h^1(L) + h^2(L) = 2 + \frac{(L,L)}{2}$ so $h^2(L) - h^1(L) \geqslant 2$. Now since $h^0(L) = h^2(L)^*$ we may just assume that $h^0(L) > 0$ (by interchanging L by L^*).

Step 2 Here's the only place where we use that C is a curve. Let $D \in |L| :=$ zero divisor sections of L. So dimension 1 implies (it think) that D is irreducible. No we have that $L = \mathcal{O}(D)$. We can construct the short exact sequence

$$0 \longrightarrow \mathcal{O}_M \longrightarrow L \longrightarrow L|_D \longrightarrow 0$$

Which gives

$$\cdots \longrightarrow H^0(L) \longrightarrow H^0(L|_D) \longrightarrow H^1(\mathcal{O}_M) = 0 \longrightarrow \quad \longrightarrow \quad \longrightarrow \cdots$$

so the restriction map is surjective, and every section $L|_D$ extends to M.

Step 3 $L|_D$ is globally generated. (Slides: The bundle $L|_D$ is base point free by Corollary 2.

Remark Let $\pi : (\tilde{M}, \tilde{D}) \longrightarrow (M, D)$ a resolution of singularities. Since $\pi^* L = K_{\tilde{D}}$, then the restriction $\pi^* L|_{\tilde{D}}$ is very ample if $\tilde{D}$ is not hyperelliptic.

□

10.5 The same theorem but a little different

Theorem M K3 with $\mathrm{Pic}(M) = \mathbb{Z}$ and L a generator of $\mathrm{Pic}(M)$ such that $(L, L) > 2$. Then L or L^* is ample, base point free and the map $\psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ is an embedding or a ramified covering.

Proof. Now we use Corollary 1, for ampleness. □

10.6 Hyperelliptic curves

First let's count the fixed point of the involution.

Lemma If $\tau : C \to C$ is the hyperelliptic involution, then τ has $2g$ fixed points.

Proof. Let f be the number of fixed points. So it acts on tangent send point to minus. So simple fixed points. Here e is Euler characteristic.

$$2 - 2g = e(C) = e(\mathbb{C}P^1) - f = 2 - f$$

So where do the fixed points appear? □

Proposition All curves of genus 2 are hyperelliptic.

Proof. (uses Serre duality) □

10.7 A third variation of that theorem

Theorem M K3 with $\mathrm{Pic}(M) = \mathbb{Z}$ and L a generator of $\mathrm{Pic}(M)$ such that $(L, L) > 2$. Then the map $\psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ is a two sheeted ramified cover if $(L, L) = 2$ and an emedding otherwise; M is a 2-sheeted covering of $\mathbb{C}P^2$ or a sextic (when $(L, L)^2 = 2$).

Proof.

Step 1 **The case** $(L, L) > 0$**, when we have an embedding.** Again, assume that $H^0(L) \neq 0$, so L is ample, globally generated and with a general smooth $D \in |L|$. Looks like we computed $(L, L) = 2$.

Step 2 **The hyperelliptic case.** $(L, L) = 2$. We use Kodaira-Nakano, so $H^1(L) = 0$, and $\chi(L) = 2 + \frac{(L,L)}{2} = 3$, so $\dim H^0(L) \stackrel{?}{=} 0$ and we get the map $\psi : M \to \mathbb{C}P^2$, a ramified covering.

Step 3 **It is a sextic.** ψ is ramified in a sextic. We used $R \subset M$ ramification divisor to show

$$K_M = \psi^* K_{\mathbb{C}P^2} \otimes \mathcal{O}(R) = \mathcal{O}(M)$$

Then $\psi^*(\mathcal{O}(-3)) = L^{\otimes -3}$ gives $L^{\otimes -3} \otimes \mathcal{O}(R) = \mathcal{O}_M$, $[R] = 3c_1(L)$.

$R_0 = \pi(R)$, we get

$$[R_0] \cap [\text{hyperplane section}] = [R] \cap [D] = 3(L, L) = 6$$

by computing

$$\int \pi_* x \wedge y = \int x \wedge \pi^* y$$

where $x = [R]$ and $y = [H]$.

Remark (Check!) K3 ramified over $\mathbb{C}P^2$ and sextics are in correspondence.

Corollary 3 K3, $\mathrm{Pic}(M) = \langle L \rangle$, $(L, L) > 2$, then L is very ample.

Proposition K3 surface isomorphic to quartic if and only if $\mathrm{Pic}(M)$ contains a very ample bundle $L \in \mathrm{Pic}(M)$ with $(L, L) = 4$.

Proof.

Step 1 Let $\varphi : M \hookrightarrow \mathbb{C}P^3$ be the embedding and $L := \phi^*(\mathcal{O}(1))$. We have

$$(L, L) = \int_M c_1(L) \wedge c_1(L) = \int_{\mathbb{C}P^3} [M] \wedge [H] \wedge [H] = 4$$

Step 2 L very ample $(L, L) = 4$, RR $h^0(L) = \chi(L) = 2 + \frac{(L,L)}{2} = 4$

$\square$

Corollary 4 M K3 surface with $\mathrm{Pic}(M) = \mathbb{Z}$ and L generating $\mathrm{Pic}(M)$ and $(L, L) = 4$. Then M is isomorphic to a quartic.

$\square$

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