Daniel González Casanova Azuela
github.com/danimalabares/k3

Home assignment 3: the splitting principle

Exercise 3.1. Let $\mathbb{C}P^\infty = \mathcal{G}r(1,\infty)$ be the union $\bigcup_i \mathbb{C}P^i$ where all maps $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3 \hookrightarrow \ldots$ are hyperplane embeddings. Prove that there exists a principal $U(n)$-bundle over $\mathbb{C}P^\infty$ with contractible total space. Prove that the cohomoogy of $\mathbb{C}P^\infty$ is a polynomial algebra with one generator in $H^2(\mathbb{C}P^\infty)$

Solution. (Based on Hatcher, *Vector Bundles and K-Theory*) The principal $U(1)$-bundle we are looking for is just the infinite case of the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$. Namely,

$$S^1 = U(1) \longhookrightarrow S^\infty \xrightarrow{p} \mathbb{C}P^\infty$$

where p sends a point in S^∞ to its equivalence class just like in the finite case. Local trivializations of this bundle are given by taking the coordinate chart $z_i \neq 0$ andgiving an homeomorphism $[z_0, z_1, \ldots] \mapsto \big([z_0, z_1, \ldots], z_i/|z_i|\big)$. This shows that the fiber is $U(1)$. S^∞ is contractible from lectures.

Given that $H^\bullet(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})$ for x of degree 2 (which may be seen by Künnet or by direct computations), to find $H^\bullet(\mathbb{C}P^\infty)$ we just notice that the inclusion $\mathbb{C}P^i \hookrightarrow \mathbb{C}P^\infty$ induces an isomorphism in cohomology (this can be seen via long exact sequence in relative cohomology, the quotient $\mathbb{C}P^\infty/\mathbb{C}P^i$ has trivial cohomology for small i). This means that the i-th cohomology group of $\mathbb{C}P^\infty$ is the degree-i polynomials. □

Definition. The ***fundamental bundle*** on $\mathbb{C}P^\infty = BU(1)$ is B_{fun}, isomorphic to $\mathcal{O}(1)$ on each $\mathbb{C}P^n \subset \mathbb{C}P^\infty$.

Exercise 3.3. Let B_{fun} be the ***fundamental vector bundle*** on $\mathcal{G}r(n)$, which has fiber W at any point of $\mathcal{G}r(n)$ corresponding to a subspace $W \subset \mathbb{C}^\infty$. Let X be a CW-space. Prove that any vector bundle on X is isomorphic to $\phi^*(B_{\text{fun}})$ for some continuous map $\phi : X \longrightarrow \mathcal{G}r(n)$.

Solution. Suppose $p : E \to X$ is a vector bundle. To define ϕ we will associate every $x \in X$ with a linear subspace of $\mathbb{C}^\infty$ using the fiber $p^{-1}(x)$.

Choose a trivializing countable open cover of X and a partition of unity φ_i. For any vector in E define a map

$$E \ni v \longmapsto (\varphi_1(p(v))g_1(v), \varphi_2(p(v))g_2(v), \ldots) \in \mathbb{C}^\infty$$

where $g_i : p^{-1}(U_i) \to \mathbb{C}^n$ gives the coordinates of the vector (it is the projection on the second factor of the trivialization $p^{-1}(U_i) \cong U_i \times \mathbb{C}^n$). We have extended the g_i to maps on all of E.

Notice that this map is injective on the fibers of p and that only finitely many of the coordinates in $\mathbb{C}^\infty$ are non-zero. Thus the image of a fiber $p^{-1}(x)$ is a n-dimensional linear subspace of $\mathbb{C}^\infty$, that is, an element of $\mathcal{G}r(n)$. Define $\phi(x)$ to be such an element of $\mathcal{G}r(n)$.

It follows by construction that $\phi^*(B_{\text{fun}}) = E$:

$$\begin{aligned} B_{\text{fun}} &= \{(\ell, v) \in \mathcal{G}r(n) \times \mathbb{C}^\infty : v \in \ell\} \\ \implies \phi^*(B_{\text{fun}}) &= \big\{\big(x, (\ell, v)\big) : \phi(x) = \ell, v \in \ell\big\} \\ &= \big\{(x, v) : v \in \ell = \phi(x) = p^{-1}(x)\big\} \\ &= E \end{aligned}$$

□

Exercise 3.4. Let $\Phi : (BU(1))^n \longrightarrow \mathcal{G}r(n)$ be a morphism such that the pullback of the fundamental bundle is the direct sum of n line bundles, obtained by lifting $\mathcal{O}(1)$ from each factor $BU(1)$. A complex vector bundle is called ***split*** if it is obtained as a direct sum of complex line bundles. Prove that a vector bundle B on X is split if and only if $\phi_B : X \longrightarrow \mathcal{G}r(n)$ is factorized through Φ.

Solution. Implication $\Longleftarrow$ is trivial. The other implication is also simple since if $B = B_1 \oplus \ldots \oplus B_k$, we have for every i a map ϕ_{B_i} and then $\phi_B = \Phi \circ (\bigoplus_i \phi_{B_i})$.

$$\begin{tikzcd} \bigoplus_i B_i \arrow[d] & \bigoplus_i \mathcal{O}(1) \arrow[d] & B_{\text{fun}} \arrow[d] \\ X \arrow[r, "\bigoplus_i \phi_{B_i}"'] & (BU(1))^n \arrow[r, "\Phi"'] & \mathcal{G}r(n) \end{tikzcd}$$

□

Exercise 3.5.

a. Let $\mathfrak{F}(V) \cong \mathbb{P}^{n-1} \times \mathbb{P}^{n-2} \times \ldots \times \mathbb{P}^1$ be the space of all orthogonal bases in $V = \mathbb{C}^{n+1}$ up to independent rescaling of each of the vectors (the ***flag space***; we will denote it by $\mathfrak{F}$). Denote by $\mathfrak{S}$ the smooth, locally trivial bundle over $\mathcal{G}r(n)$, with the fiber the flag space $\mathfrak{F}(V)$ in each subspace $V \in \mathcal{G}r(n)$. Prove that the pullback of the fundamental bundle B_{fun} to $\mathfrak{S}$ is split.

b. Prove that the induced map $H^*(\mathcal{G}r(n), \mathbb{Q}) \to H^*(\mathfrak{S}, \mathbb{Q})$ is injective.

c. Deduce that $H^*(\mathfrak{S}, \mathbb{Q})$ as $H^*(\mathcal{G}r(n), \mathbb{Q})$-module is isomorphic to $H^*(\mathcal{G}r(n), \mathbb{Q}) \otimes H^*(\mathfrak{F})$.

Solution.

a. Let $\phi : \mathfrak{S} \to \mathcal{G}r(n)$ be the bundle descibed above. We wish to show that $\phi^*(B_{\text{fun}})$ is split:

$$\begin{tikzcd} \phi^* B_{\text{fun}} \arrow[d] & B_{\text{fun}} \arrow[d] \\ \mathfrak{S} \arrow[r, "\phi"] & \mathcal{G}r(n) \end{tikzcd}$$

According to the last exercise, this ammounts to showing that there is a factorization

$$\begin{tikzcd} \mathfrak{S} \arrow[r, "\psi"] \arrow[rr, bend right, "\phi"'] & (BU(1))^n \arrow[r, "\Phi"] & \mathcal{G}r(n) \end{tikzcd}$$

I finally realized that $\mathfrak{S}$ is the Stiefel manifold $V_n(\mathbb{C}^k)$, the space of n-frames in $\mathbb{R}^k$. It is projected onto $\mathcal{G}r(n,k) \subset \mathcal{G}r(n)$ by mapping a frame to the linear space it spans, hence with fiber the space of orthonormal bases of n vectors, ie. $\mathfrak{F}$. Then we have a natural factorization:

$$\begin{array}{ccccc} \mathfrak{S} & \longrightarrow & (BU(1))^n \cong (\mathbb{C}P^\infty)^n & \longrightarrow & \mathcal{G}r(n) \\ \{v_1, \dots, v_n\} & \longmapsto & ([v_1], \dots, [v_n]) & \longmapsto & \mathrm{span}([v_1], \dots, [v_n]) \end{array}$$

b. This can be seen using a statement (similar to Exercise 3.4) called Leray-Hirsch isomorphism (see Hatcher, *Algebraic topology*, thm. 4D.1 or wikipedia), where we see $H^*(E)$ as an $H^*(B)$-module.

   More precisely, suppose that the cohomology groups of the fibers $H^n(F)$ of a fiber bundle are finitely generated modules and that there are elements $c_j \in H^*(E)$ whose pullback under the inclusion are a basis of $H^*(F)$. Then the map

$$\begin{aligned} H^*(E) \otimes H^*(F) &\longrightarrow H^*(E) \\ \sum_{ij} b_i \otimes i^*(c_j) &\longmapsto \sum_{ij} p^*(b_i) \smile c_j \end{aligned}$$

   is an isomorphism. This means that $H^*(E)$ is a $H^*(B)$-module with basis $\{c_j\}$.

   Applying this to the bundle $\mathfrak{S} \to \mathcal{G}r(n)$, we see that $H^*(\mathcal{G}r(n), \mathbb{Q})$ is just the inclusion of the coefficients in the module.

   Remark. The proof in Hatcher of this theorem is rather involved and does not use spectral sequences.

c. (Idea.) Show that all differentials in the Leray-Serre spectral sequence vanish due to the even-dimensional cell decomposition of $\mathfrak{F}$. Supposing that the monodromy action is trivial, we should obtain the desired isomorphism.

   A cell decomposition of $V_n(\mathbb{C}^k)$, of which $\mathfrak{F} = V_n(\mathbb{C}^n)$ is a prticular case, is found in Mosher and Tangora or James. It might be simpler to identify $\mathfrak{F}$ with $U(n)$, but constructing the desired decomposition was still not straightforward.

□

References

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