lie groups and lie algebras

This is just J. Lee, *Introduction to smooth manifolds*, Second Edition.

1 classical lie groups 1

2 the lie algebra of a lie group 3

3 the exponential map 5

classical lie groups

Definition. A ***Lie group*** is a smooth manifold G without boundary that is also a group in the algebraic sense, with the property that the multiplication map and the inversion map are both smooth.

Example.

1. The ***general linear group*** $GL(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries.
2. $GL^+(n, \mathbb{R})$, the subset of $GL(n, \mathbb{R})$ consisting of matrices with positive determinant.
3. If G is any Lie group and $H \subset G$ is an open subgroup, H is a Lie group (because restrictions of the operations are smooth).
4. The ***complex generar linear group*** $GL(n, \mathbb{C})$.
5. If V is any real or complex vector space, $GL(V)$ denotes que set of invertible linear maps from V to itself. If V is finite dimensional, there is an isomorphism with either of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.
6. $\mathbb{R}^n$ and $\mathbb{C}^n$.
7. The set $\mathbb{R}^*$ of nonzero real numbers. In fact, it is $GL(1, \mathbb{R})$. Also $\mathbb{C}^*$.
8. The circle $\mathbb{S}^1$.
9. Direct product of Lie groups.
10. The n-torus $\mathbb{T}^n$.
11. Any discrete group.
12. The set $SL(n, \mathbb{R})$ of $n \times n$ real matrices with determinant equal to 1 is called the ***special linear group of degree*** n. It is the kernel of the group homomorphism $\det : GL(n, \mathbb{R}) \to \mathbb{R}^*$. Because the determinan function is surjective, it is a smooth submersion by the global rank theorem so $SL(n, \mathbb{R})$ has dimension $n^2 - 1$.
13. Let n be a positive integer and define the map $\beta : GL(n, \mathbb{C}) \to GL(2n, \mathbb{R})$ by replacing each complex matrix entry by a 2×2 block matrix. Thus $GL(n, \mathbb{C})$ is isomorphic to a Lie subgroup of $GL(2n, \mathbb{R})$.

14. The subgroup $SL(n,\mathbb{C}) \subseteq GL(n,\mathbb{C})$ consisting of complex matrices of determinant 1 is called the ***complex special linear group of degree*** n. By similar arguments the real case, it is of codimension $\dim \mathbb{C}^* = 2$ and therefore of dimension $2n^2 - 2$.

15. A real $n \times n$ matrix A is said to be ***orthogonal*** if as a linear map it preserves the Euclidean dot product. The set $O(n)$ of orthogonal $n \times n$ matrices is a subgroup of $GL(n,\mathbb{R})$ called the ***orthogonal group of degree*** n. A matrix A is orthogonal if and only if it takes the standard basis of $\mathbb{R}^n$ to an orthonormal basis, which is equivalent to the columns of A being orthonormal. Since the (i,j)-entry of the matrix $A^T A$ is the dot product of the ith and jth columns of A, this condition is also equivalent to the requirement that $A^T A = I_n$. Also, $O(n)$ is the level set $\Phi^{-1}(I_n)$ of the map $\Phi : GL(n,\mathbb{R}) \to M(n,\mathbb{R})$ given by $A \mapsto A^T A$, so it is a closed set and an embedded Lie subgroup. It is also bounded because every $A \in O(n)$ has colomuns of norm 1, and therefore satisfies that $|A| = \sqrt{n}$. It has dimension $n(n-1)/2$.

16. The ***special orthogonal group of degree*** n is defined as $SO(n) = O(n) \cap SL(n,\mathbb{R}) \subseteq GL(n,\mathbb{R})$. Notice that every matrix $A \in O(n)$ satisfies

$$1 = \det I_n = \det(A^T A) = \det A \det A^T = (\det A)^2,$$

it follows that $\det A = \pm 1$. Therefore $SO(n)$ is the open subgroup of $O(n)$ consisting of matrices with positive determinant, and is therefore also an embedded Lie subgroup of dimension $n(n-1)/2$ in $GL(n,\mathbb{R})$. It is a compact group because it is a closed subset of $O(n)$.

17. For any complex matrix A, the ***adjoint of*** A is the matrix A^* formed by conjugating the entries of A and taking the transpose: $A^* = \overline{A}^T$. For any positive integer n, the ***unitary group of degree*** n is the subgroup $U(n) \subseteq GL(n,\mathbb{C})$ consisting of complex $n \times n$ matrices whose columns form an orthogonal basis for $\mathbb{C}^n$ with respect to the Hermitian dot product $z \cdot w = \sum_i z^i \overline{w^i}$. It is straightforward to check that $U(n)$ consists of those matrices A such that $A^* A = I_n$. It is a properly embedded Lie subgroup of $GL(n,\mathbb{C})$ of dimension n^2.

18. The group $SU(n) = U(n) \cap SL(n,\mathbb{C})$ is called the ***complex special unitary group of degree*** n. It is a properly embedded Lie subgroup of $U(n)$ of dimension $n^2 - 1$. It is also embedded in $GL(n,\mathbb{C})$ because composition of embedddings is an embedding.

19. The ***real symplectic group*** is the subgroup $Sp(2n,\mathbb{R}) \subseteq GL(2n,\mathbb{R})$ consisting of all $2n \times 2n$ matrices that leave the standard symplectic tensor $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ invariant, that is, the set of invertible linear maps $Z : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\omega(Zx, Zy) = \omega(x,y)$ for all $x, y \in \mathbb{R}^{2n}$.

Definition (Extra)**.** If G is any Lie group, a ***(finite-dimensional) representation of*** G is a Lie group homomorphism grom G to $GL(V)$ for some V. If a representation is injective, it is said to be ***faithful***.

the lie algebra of a lie group

Suppose G is a Lie group. Recall that G acts smoothly and transitevely on itself by left translation $L_g(h) = gh$. A vector field X on G is said to be ***left-invariant*** if it is invariant under all left translations:

$$d(L_g)_{g'}(G_{g'}) = X_{gg'} \qquad \text{for all } g, g' \in G.$$

Since L_g is a diffeomorphism, the pushforward of a vector field is well defined and we may write our condition as $(L_g)_*X = X$ for every $g \in G$.

Because $(L_g)_*(aX + bY) = a(L_g)_* + b(L_g)_*$, the set of left-invariant vector fields on G is a linear subspace of $\mathfrak{X}(G)$. But there's more

Proposition 2.1. Let G be a Lie group and suppose that X and Y are left-invariant vector fields on G. Then $[X, Y]$ is also left-invariant.

Definition. A ***Lie algebra*** (over $\mathbb{R}$) is a real vector space $\mathfrak{g}$ endowed with a map called the ***bracket*** from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ usually denoted by $(X, Y) \mapsto [X, Y]$ that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

1. BILINEARITY: For $a, b \in \mathbb{R}$,
$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$
2. ANTISYMMETRY:
$$[X, Y] = -[Y, X].$$
3. JACOBI IDENTITY:
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Notice that the Jacobi identity is a substitute for associativity, which does not hold in general for brackets in a Lie algebra. We may also define Lie algebras over $\mathbb{C}$.

Definition.

- If $\mathfrak{g}$ is a Lie algebra, a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a ***Lie subalgebra of*** $\mathfrak{g}$ if it is closed under brackets.
- If $\mathfrak{g}$ and hf are Lie algebras, a linear map $A : \mathfrak{g} \to \mathfrak{h}$ is called a ***Lie algebra homomorphism*** if it preserves brackets: $A[X, Y] = [AX, AY]$. An invertible Lie algebra homomorphism is called a ***Lie algebra isomorphism***.

Example.

1. The space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket.
2. If G is a Lie group, the set of all left-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(G)$. This is called the ***Lie algebra of*** G and it is denoted by $\mathrm{Lie}(G)$.

3. The vector space $M(n, \mathbb{R})$ of $n \times n$ real matrices is an n^2-dimensional Lie algebra under the ***commutator bracket***

$$[A, B] = AB - BA.$$

When regarding $M(n, \mathbb{R})$ as a Lie algebra with this bracket, we denote it by $\mathfrak{gl}(n, \mathbb{R})$.

4. Similarly, $\mathfrak{gl}(n, \mathbb{C})$ is the $2n^2$-dimensional (real) Lie algebra obtained by endowing $M(n, \mathbb{C})$ with the commutator bracket.

5. If V is a vector space, the vector space of all linear maps from V to itself becomes a Lie algebra which we denote $\mathfrak{gl}(V)$ with the commutator bracket:

$$[A, B] = A \circ B - B \circ A.$$

Under the usual identification of $n \times n$ matrices with linear maps from $\mathbb{R}^n$ to itself, $\mathfrak{gl}(\mathbb{R}^n)$ is the same as $\mathfrak{gl}(n, \mathbb{R})$.

6. Any vector space V becomes a lie algebra if we define all brackets to be zero. Such a Lie algebra is said to be ***abelian***.

Theorem 2.2. Let G be a Lie group. The evaluation map $\varepsilon : \mathrm{Lie}(G) \to T_e G$ given by $\varepsilon(X) = X_e$ is a vector space isomorphism. Thus $\mathrm{Lie}(G)$ is finite-dimensional with dimension equal to $\dim G$.

Corollary 2.3. Every left-invariant rough vector field on a Lie group is smooth.

Corollary 2.4. Every Lie group admits a left-invariant smooth global frame, and therefore every Lie group is parallelizable.

Example.

1. $\mathrm{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$.
2. $\mathrm{Lie}(\mathbb{S}^1) \cong \mathbb{R}$.
3. $\mathrm{Lie}(\mathbb{T}^n) \cong \mathbb{R}^n$.
4. $\mathrm{Lie}(GL(n, \mathbb{R})) \cong \mathfrak{gl}(n, \mathbb{R})$. (long proof)
5. If V is any finite-dimensional real vector space, $\mathrm{Lie}(GL(V)) \cong \mathfrak{gl}(V)$.
6. $\mathrm{Lie}(GL(n, \mathbb{C})) \cong \mathfrak{gl}(n, \mathbb{C})$.
7. $\mathfrak{o}(n) = \{\text{skew-symmetric } n \times n \text{ matrices}\} \subseteq \mathfrak{gl}(n, \mathbb{R})$ is $\mathrm{Lie}(O(n))$.
8. $\mathrm{Lie}(GL(n, \mathbb{C})) \cong \mathfrak{gl}(n, \mathbb{C})$. (long proof)

Theorem 2.5 (Ado's Theorem)**.** Every finite-dimensional real Lie algebra admits a faithful finite-dimensional representation.

Corollary 2.6. Every finite-dimensional real Lie algebra is isomorphic to a Lie subalgebra of some matrix algebra $\mathfrak{gl}(n, \mathbb{R})$ with the commutator bracket.

the exponential map

Definition. Suppose G is a Lie group. A ***one-parameter subgroup of*** G is defined to be a Lie group homomorphism $\gamma : \mathbb{R} \to G$ with R considered as a lie group under addition. Notice that by this definition, a one-parameter subgroup is not a Lie subgroup of G but rather a homomorphism into G.

Theorem 3.1. The one-parameter subgroups of G are precisely the maximal integral curves of left-invariante vector fields starting at the identity.

Definition. Given $X \in \mathrm{Lie}(G)$, the one-parameter subgroup determined by X in this way is called the ***one-parameter subgroup generated by*** X.

Because left-invariant vector fields are uniquely determined by their values at the identity, it follows that each one-parameter subgroup is unequely determined by its initial velocity in $T_e G$, and thus there are one-to-one correspondences

$$\{\text{one-parameter subgroups of } G\} \leftrightarrow \mathrm{Lie}(G) \leftrightarrow T_e G.$$

Proposition 3.2. For any $A \in \mathfrak{gl}(n, \mathbb{R})$, let

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + A + \frac{1}{2} A^2 + \dots .$$

This series converges to an invertible matrix $e^A \in GL(n, \mathbb{R})$ and the one-parameter subgroup of $GL(n, \mathbb{R})$ generated by $A \in \mathfrak{gl}(n, \mathbb{R})$ is $\gamma(t) = e^{tA}$.

Proposition 3.3. Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in $T_e H$.

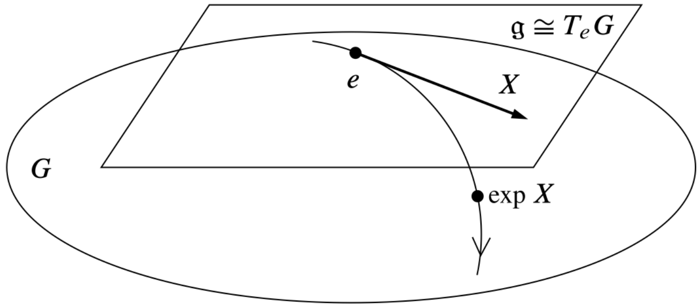
Example. If H is a Lie subgroup of $GL(n, \mathbb{R})$, the precding proposition shows that the one-parameter subgroups of H are precisely the maps of the form $\gamma(t) = e^{tA}$ for $A \in \mathfrak{h}$, where $h \subseteq \mathfrak{gl}(n, \mathbb{R})$ is the subalgebra cooresponding to $\mathrm{Lie}(H)$. For example, taking $H = O(n)$ this shows that the one-parameter subgroups of $O(n)$ are the maps of the form $\gamma(t) = e^{tA)}$ for an arbitrary skew-symmetric A. In particular, this shows that the exponential of any skew-symmetric matrix is orthogonal.

Definition. Given a Lie group G with Lie algebra $\mathfrak{g}$, we define the map $\exp : \mathfrak{g} \to G$ called the ***exponential map of*** G as follows: for any $X \in \mathfrak{g}$, we set

$$\exp X = \gamma(1),$$

where γ is the one-parameter subgroup generated by X, or equivalently the integral curve of X starting at the identity.

Proposition 3.4. Let G be a Lie group. For any $X \in \mathrm{Lie}(G)$, $\gamma(s) = \exp sX$ is the one-parameter subgroup of G generated by X.



Example.

1. $\exp A = e^A$.
2. If V is any finite-dimensional real vector space, a choice of basis for V yields isomorphisms $GL(V) \cong GL(n, \mathbb{R})$ and $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{R})$. The analysis of the $GL(n, \mathbb{R})$ case then shows that the exponential map of $GL(V)$ can be written in the form

$$\exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

when we consider $A \in \mathfrak{gl}(V)$ as a linear map from V to itself and $A^k = A \circ \ldots \circ A$.

Proposition 3.5. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra.

1. The exponential map is a smooth map from $\mathfrak{g}$ to G.
2. For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}$, $\exp(s+t)X = \exp sX \exp tX$.
3. For any $X \in \mathfrak{g}$, $(\exp X)^{-1} = \exp(-X)$.
4. For any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, $(\exp X)^n = \exp(nX)$.
5. The differential $(d\exp)_0 : T_0\mathfrak{g} \to T_eG$ is the identity map, under the canonical identifications of both $T_0\mathfrak{g}$ and T_eG with $\mathfrak{g}$ itself.
6. The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.

Notice that $\exp(X+Y) = \exp X \exp Y$ for arbitrary X, Y in the Lie algebra. In fact, for connected groups, this is only true when the group is abelian.

Theorem 3.6 (The Lie correspondence)**.** There is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups given by associating each simply connected Lie group with its Lie algebra.