

lie groups and lie algebras

This is just Lee, *Introduction to smooth manifolds*, Second Edition.

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classical lie groups

Definition. A *Lie group* is a smooth manifold G without boundary that is also a group in the algebraic sense, with the property that the multiplication map and the inversion map are both smooth.

Example.

1. The *general linear group* $GL(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries.
2. $GL^+(n, \mathbb{R})$, the subset of $GL(n, \mathbb{R})$ consisting of matrices with positive determinant.
3. If G is any Lie group and $H \subset G$ is an open subgroup, H is a Lie group (because restrictions of the operations are smooth).
4. The *complex general linear group* $GL(n, \mathbb{C})$.
5. If V is any real or complex vector space, $GL(V)$ denotes the set of invertible linear maps from V to itself. If V is finite dimensional, there is an isomorphism with either of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.
6. \mathbb{R}^n and \mathbb{C}^n .
7. The set \mathbb{R}^* of nonzero real numbers. In fact, it is $GL(1, \mathbb{R})$. Also \mathbb{C}^* .
8. The circle \mathbb{S}^1 .
9. Direct product of Lie groups.
10. The n -torus \mathbb{T}^n .
11. Any discrete group.
12. The set $SL(n, \mathbb{R})$ of $n \times n$ real matrices with determinant equal to 1 is called the *special linear group of degree n* . It is the kernel of the group homomorphism $\det : GL(n, \mathbb{R}) \rightarrow \mathbb{R}^*$. Because the determinant function is surjective, it is a smooth submersion by the global rank theorem so $SL(n, \mathbb{R})$ has dimension $n^2 - 1$.
13. Let n be a positive integer and define the map $\beta : GL(n, \mathbb{C}) \rightarrow GL(2n, \mathbb{R})$ by replacing each complex matrix entry by a 2×2 block matrix. Thus $GL(n, \mathbb{C})$ is isomorphic to a Lie subgroup of $GL(2n, \mathbb{R})$.

14. The subgroup $SL(n, \mathbb{C}) \subseteq GL(n, \mathbb{C})$ consisting of complex matrices of determinant 1 is called the **complex special linear group of degree n** . By similar arguments the real case, it is of codimension $\dim \mathbb{C}^* = 2$ and therefore of dimension $2n^2 - 2$.
15. A real $n \times n$ matrix A is said to be **orthogonal** if as a linear map it preserves the Euclidean dot product. The set $O(n)$ of orthogonal $n \times n$ matrices is a subgroup of $GL(n, \mathbb{R})$ called the **orthogonal group of degree n** . A matrix A is orthogonal if and only if it takes the standard basis of \mathbb{R}^n to an orthonormal basis, which is equivalent to the columns of A being orthonormal. Since the (i, j) -entry of the matrix $A^T A$ is the dot product of the i th and j th columns of A , this condition is also equivalent to the requirement that $A^T A = I_n$. Also, $O(n)$ is the level set $\Phi^{-1}(I_n)$ of the map $\Phi : GL(n, \mathbb{R}) \rightarrow M(n, \mathbb{R})$ given by $A \mapsto A^T A$, so it is a closed set and an embedded Lie subgroup. It is also bounded because every $A \in O(n)$ has columns of norm 1, and therefore satisfies that $|A| = \sqrt{n}$. It has dimension $n(n - 1)/2$.
16. The **special orthogonal group of degree n** is defined as $SO(n) = O(n) \cap SL(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$. Notice that every matrix $A \in O(n)$ satisfies

$$1 = \det I_n = \det(A^T A) = \det A \det A^T = (\det A)^2,$$

it follows that $\det A = \pm 1$. Therefore $SO(n)$ is the open subgroup of $O(n)$ consisting of matrices with positive determinant, and is therefore also an embedded Lie subgroup of dimension $n(n - 1)/2$ in $GL(n, \mathbb{R})$. It is a compact group because it is a closed subset of $O(n)$.

17. For any complex matrix A , the **adjoint of A** is the matrix A^* formed by conjugating the entries of A and taking the transpose: $A^* = \overline{A}^T$. For any positive integer n , the **unitary group of degree n** is the subgroup $U(n) \subseteq GL(n, \mathbb{C})$ consisting of complex $n \times n$ matrices whose columns form an orthogonal basis for \mathbb{C}^n with respect to the Hermitian dot product $z \cdot w = \sum_i z^i \overline{w^i}$. It is straightforward to check that $U(n)$ consists of those matrices A such that $A^* A = I_n$. It is a properly embedded Lie subgroup of $GL(n, \mathbb{C})$ of dimension n^2 .
18. The group $SU(n) = U(n) \cap SL(n, \mathbb{C})$ is called the **complex special unitary group of degree n** . It is a properly embedded Lie subgroup of $U(n)$ of dimension $n^2 - 1$. It is also embedded in $GL(n, \mathbb{C})$ because composition of embeddings is an embedding.
19. The **real symplectic group** is the subgroup $Sp(2n, \mathbb{R}) \subseteq GL(2n, \mathbb{R})$ consisting of all $2n \times 2n$ matrices that leave the standard symplectic tensor $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ invariant, that is, the set of invertible linear maps $Z : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $\omega(Zx, Zy) = \omega(x, y)$ for all $x, y \in \mathbb{R}^{2n}$.

Definition (Extra). If G is any Lie group, a **(finite-dimensional) representation of G** is a Lie group homomorphism from G to $GL(V)$ for some V . If a representation is injective, it is said to be **faithful**.

the lie algebra of a lie group

Suppose G is a Lie group. Recall that G acts smoothly and transitively on itself by left translation $L_g(h) = gh$. A vector field X on G is said to be *left-invariant* if it is invariant under all left translations:

$$d(L_g)_{g'}(X_{g'}) = X_{gg'} \quad \text{for all } g, g' \in G.$$

Since L_g is a diffeomorphism, the pushforward of a vector field is well defined and we may write our condition as $(L_g)_*X = X$ for every $g \in G$.

Because $(L_g)_*(aX + bY) = a(L_g)_*X + b(L_g)_*Y$, the set of left-invariant vector fields on G is a linear subspace of $\mathfrak{X}(G)$. But there's more

Proposition 2.1. Let G be a Lie group and suppose that X and Y are left-invariant vector fields on G . Then $[X, Y]$ is also left-invariant.

Definition. A *Lie algebra* (over \mathbb{R}) is a real vector space \mathfrak{g} endowed with a map called the *bracket* from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} usually denoted by $(X, Y) \mapsto [X, Y]$ that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

1. BILINEARITY: For $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

2. ANTISYMMETRY:

$$[X, Y] = -[Y, X].$$

3. JACOBI IDENTITY:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Notice that the Jacobi identity is a substitute for associativity, which does not hold in general for brackets in a Lie algebra. We may also define Lie algebras over \mathbb{C} .

Definition.

- If \mathfrak{g} is a Lie algebra, a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a *Lie subalgebra* of \mathfrak{g} if it is closed under brackets.
- If \mathfrak{g} and \mathfrak{h} are Lie algebras, a linear map $A : \mathfrak{g} \rightarrow \mathfrak{h}$ is called a *Lie algebra homomorphism* if it preserves brackets: $A[X, Y] = [AX, AY]$. An invertible Lie algebra homomorphism is called a *Lie algebra isomorphism*.

Example.

1. The space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket.
2. If G is a Lie group, the set of all left-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(G)$. This is called the *Lie algebra of G* and it is denoted by $\text{Lie}(G)$.

3. The vector space $M(n, \mathbb{R})$ of $n \times n$ real matrices is an n^2 -dimensional Lie algebra under the *commutator bracket*

$$[A, B] = AB - BA.$$

When regarding $M(n, \mathbb{R})$ as a Lie algebra with this bracket, we denote it by $\mathfrak{gl}(n, \mathbb{R})$.

4. Similarly, $\mathfrak{gl}(n, \mathbb{C})$ is the $2n^2$ -dimensional (real) Lie algebra obtained by endowing $M(n, \mathbb{C})$ with the commutator bracket.
5. If V is a vector space, the vector space of all linear maps from V to itself becomes a Lie algebra which we denote $\mathfrak{gl}(V)$ with the commutator bracket:

$$[A, B] = A \circ B - B \circ A.$$

Under the usual identification of $n \times n$ matrices with linear maps from \mathbb{R}^n to itself, $\mathfrak{gl}(\mathbb{R}^n)$ is the same as $\mathfrak{gl}(n, \mathbb{R})$.

6. Any vector space V becomes a Lie algebra if we define all brackets to be zero. Such a Lie algebra is said to be *abelian*.

Theorem 2.2. Let G be a Lie group. The evaluation map $\varepsilon : \text{Lie}(G) \rightarrow T_e G$ given by $\varepsilon(X) = X_e$ is a vector space isomorphism. Thus $\text{Lie}(G)$ is finite-dimensional with dimension equal to $\dim G$.

Corollary 2.3. Every left-invariant rough vector field on a Lie group is smooth.

Corollary 2.4. Every Lie group admits a left-invariant smooth global frame, and therefore every Lie group is parallelizable.

Example.

1. $\text{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$.
2. $\text{Lie}(S^1) \cong \mathbb{R}$.
3. $\text{Lie}(T^n) \cong \mathbb{R}^n$.
4. $\text{Lie}(\text{GL}(n, \mathbb{R})) \cong \mathfrak{gl}(n, \mathbb{R})$. (long proof)
5. If V is any finite-dimensional real vector space, $\text{Lie}(\text{GL}(V)) \cong \mathfrak{gl}(V)$.
6. $\text{Lie}(\text{GL}(n, \mathbb{C})) \cong \mathfrak{gl}(n, \mathbb{C})$.
7. $\mathfrak{o}(n) = \{\text{skew-symmetric } n \times n \text{ matrices}\} \subseteq \mathfrak{gl}(n, \mathbb{R})$ is $\text{Lie}(\text{O}(n))$.
8. $\text{Lie}(\text{GL}(n, \mathbb{C})) \cong \mathfrak{gl}(n, \mathbb{C})$. (long proof)

Theorem 2.5 (Ado's Theorem). Every finite-dimensional real Lie algebra admits a faithful finite-dimensional representation.

Corollary 2.6. Every finite-dimensional real Lie algebra is isomorphic to a Lie subalgebra of some matrix algebra $\mathfrak{gl}(n, \mathbb{R})$ with the commutator bracket.

the exponential map

Definition. Suppose G is a Lie group. A *one-parameter subgroup of G* is defined to be a Lie group homomorphism $\gamma : \mathbb{R} \rightarrow G$ with \mathbb{R} considered as a Lie group under addition. Notice that by this definition, a one-parameter subgroup is not a Lie subgroup of G but rather a homomorphism into G .

Theorem 3.1. The one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector fields starting at the identity.

Definition. Given $X \in \text{Lie}(G)$, the one-parameter subgroup determined by X in this way is called the *one-parameter subgroup generated by X* .

Because left-invariant vector fields are uniquely determined by their values at the identity, it follows that each one-parameter subgroup is uniquely determined by its initial velocity in $T_e G$, and thus there are one-to-one correspondences

$$\{\text{one-parameter subgroups of } G\} \leftrightarrow \text{Lie}(G) \leftrightarrow T_e G.$$

Proposition 3.2. For any $A \in \mathfrak{gl}(n, \mathbb{R})$, let

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + A + \frac{1}{2} A^2 + \dots$$

This series converges to an invertible matrix $e^A \in GL(n, \mathbb{R})$ and the one-parameter subgroup of $GL(n, \mathbb{R})$ generated by $A \in \mathfrak{gl}(n, \mathbb{R})$ is $\gamma(t) = e^{tA}$.

Proposition 3.3. Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in $T_e H$.

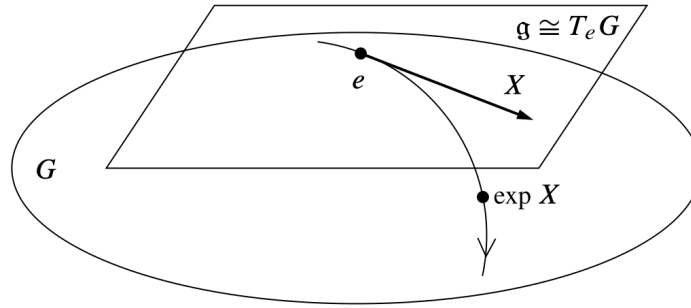
Example. If H is a Lie subgroup of $GL(n, \mathbb{R})$, the preceding proposition shows that the one-parameter subgroups of H are precisely the maps of the form $\gamma(t) = e^{tA}$ for $A \in \mathfrak{h}$, where $\mathfrak{h} \subseteq \mathfrak{gl}(n, \mathbb{R})$ is the subalgebra corresponding to $\text{Lie}(H)$. For example, taking $H = O(n)$ this shows that the one-parameter subgroups of $O(n)$ are the maps of the form $\gamma(t) = e^{tA}$ for an arbitrary skew-symmetric A . In particular, this shows that the exponential of any skew-symmetric matrix is orthogonal.

Definition. Given a Lie group G with Lie algebra \mathfrak{g} , we define the map $\exp : \mathfrak{g} \rightarrow G$ called the *exponential map of G* as follows: for any $X \in \mathfrak{g}$, we set

$$\exp X = \gamma(1),$$

where γ is the one-parameter subgroup generated by X , or equivalently the integral curve of X starting at the identity.

Proposition 3.4. Let G be a Lie group. For any $X \in \text{Lie}(G)$, $\gamma(s) = \exp sX$ is the one-parameter subgroup of G generated by X .



Example.

1. $\exp A = e^A$.
2. If V is any finite-dimensional real vector space, a choice of basis for V yields isomorphisms $GL(V) \cong GL(n, \mathbb{R})$ and $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{R})$. The analysis of the $GL(n, \mathbb{R})$ case then shows that the exponential map of $GL(V)$ can be written in the form

$$\exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

when we consider $A \in \mathfrak{gl}(V)$ as a linear map from V to itself and $A^k = A \circ \dots \circ A$.

Proposition 3.5. Let G be a Lie group and \mathfrak{g} be its Lie algebra.

1. The exponential map is a smooth map from \mathfrak{g} to G .
2. For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}$, $\exp(s + t)X = \exp sX \exp tX$.
3. For any $X \in \mathfrak{g}$, $(\exp X)^{-1} = \exp(-X)$.
4. For any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, $(\exp X)^n = \exp(nX)$.
5. The differential $(d\exp)_0 : T_0\mathfrak{g} \rightarrow T_e G$ is the identity map, under the canonical identifications of both $T_0\mathfrak{g}$ and $T_e G$ with \mathfrak{g} itself.
6. The exponential map restricts to a diffeomorphism from some neighborhood of 0 in \mathfrak{g} to a neighborhood of e in G .

Notice that $\exp(X + Y) = \exp X \exp Y$ for arbitrary X, Y in the Lie algebra. In fact, for connected groups, this is only true when the group is abelian.

Theorem 3.6 (The Lie correspondence). There is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups given by associating each simply connected Lie group with its Lie algebra.