0.0.1 first variation of a regular map to riemannian target

Consider an isometric immersion $f_0 : (M, g) \to (\tilde{M}, \tilde{g})$ and a vector field $\xi \in f^* T\tilde{M}$.

Define a variation by:

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times M &\longrightarrow \tilde{M} \\ f(s, p) &= \exp_p(s\xi) \end{aligned}$$

Then $f_s := f(s, \cdot) : M \longrightarrow \tilde{M}$ is an immersion (isometric at $s = 0$ by hypothesis). Now we define the ***volume functional*** that simply computes the volume of M_s:

$$S(s) := \int_M \mathrm{Vol}_{f_s^* \tilde{g}} = \int_M f_s^* \, \mathrm{Vol}_{\tilde{g}} \, .$$

(S is physics notation and s is Florit notation.)

Exercise Compute $S'(0)$.

TL;DR. The derivative of the determinant is a trace, that trace is the trace of the shape operator, which is the mean curvature.

Solution. How to express volume in any coordinate system?

$$\sqrt{\det(f_s^* g)_{ij}} dx^1 \wedge \ldots \wedge dx^n$$

Let's differentiate:

$$\begin{aligned} \frac{d}{ds} S(s) &= \frac{d}{ds} \int_M \mathrm{Vol}_{f_s^* \tilde{g}} = \frac{d}{ds} \int_M \sqrt{\det(f_s^* g)_{ij}} dx^1 \wedge \ldots \wedge dx^n \\ &= \int_M \frac{d}{ds} \sqrt{\det(f_s^* g)_{ij}} dx^1 \wedge \ldots \wedge dx^n \end{aligned}$$

We must differentiate the square root of the determinant of a matrix.

$$\boxed{\frac{d}{dt} \det A(t) = \det A(t) \cdot \mathrm{tr} \left(A(t)^{-1} \cdot \frac{d}{dt} A(t) \right)}$$

and using that we see that

$$\boxed{\frac{d}{dt} \sqrt{\det A(t)} = \frac{1}{2} \sqrt{\det A(t)} \cdot \mathrm{tr} \left(A(t)^{-1} \cdot \frac{d}{dt} A(t) \right)}$$

So we put $A(s) = (f_s^* \tilde{g})_{ij}$. The good news is that square root of the determinant part is exactly the local coordinate function of $\mathrm{Vol}_{f_0^* \mathrm{Vol}_{\tilde{g}}} = \mathrm{Vol}_g$. Then we only have to integrate that other function, which hopefully is related to the mean curvature.

Before computing recall the basic equations of submanifold theory: for $X, Y \in \mathfrak{X}(M)$, $\xi \in \Gamma(T^\perp M)$, $\tilde{\nabla}$ L.C. connection of $\tilde{M}$ and ∇ LC connection of M isometrically immersed in $\tilde{M}$, splitting everything into normal and tangent part we get

$$\boxed{\tilde{\nabla}_X^f f_* Y := f_* \nabla_X Y + \alpha(X, Y)}$$

$$\boxed{\tilde{\nabla}^f_X g_* \xi := -A_\xi f_* X + \nabla^\perp_X \xi}$$

and call α the second fundamental form and A the shape operator. These two equations/definitions together imply

$$\boxed{\langle \xi, \alpha(X, Y) \rangle = \langle -A_\xi X, Y \rangle}$$

Let's compute

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} (f_s^* \tilde{g})_{ij} = \frac{d}{ds}\Big|_{s=0} f_s^* \tilde{g}\left(\partial_i, \partial_j\right) = \frac{d}{ds}\Big|_{s=0} \tilde{g}(f_*^s \partial_i, f_*^s \partial_j) \\
&= \tilde{g}\left(\nabla^f_{\partial_s} f_*^s \partial_i, f_*^s \partial_j\right) + \tilde{g}\left(f_*^s \partial_i, \nabla^f_{\partial_s} f_*^s \partial_j\right)\Big|_{s=0} \\
&= \tilde{g}\left(\nabla^f_{\partial_s} f_*^0 \partial_i, f_*^0 \partial_j\right) + \tilde{g}\left(f_*^0 \partial_i, \nabla^f_{\partial_s} f_*^0 \partial_j\right) \\
&= \tilde{g}\left(\nabla^f_{\partial_i} f_*^0 \partial_s, f_*^0 \partial_j\right) + \tilde{g}\left(f_*^0 \partial_i, \nabla^f_{\partial_j} f_*^0 \partial_s\right) \qquad \text{symmetry lemma} \\
&= \tilde{g}(\nabla^f_{\partial_i} \xi, f_*^0 \partial_j) + \tilde{g}(f_*^0 \partial_i, \nabla^f_{\partial_j} \xi)
\end{aligned}$$

Where ∇^f is the pullback connection. The point is that we arrived at the variational field ξ. If we assume (for now) that ξ is normal to M then upon differentiation we get:

$$\nabla^f_{\partial_i} \xi = -A_\xi \partial_i + N$$

where the normal component N vanishes since it is orthogonal to the basic tangent vectors of M. We conclude that

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} (f_s^* \tilde{g})_{ij} &= \tilde{g}\left(-A_\xi \partial_i, \partial_j\right) + \tilde{g}\left(\partial_i, -A_\xi \partial_j\right) \\
&= \tilde{g}(\xi, \alpha(\partial_i, \partial_j)) + \tilde{g}(\alpha(\partial_j, \partial_i), \xi) \\
&= 2\tilde{g}(\xi, \alpha(\partial_i, \partial_j)) \qquad \alpha \text{ is symmetric} \\
&= -2\tilde{g}(A_\xi \partial_i, \partial_j)
\end{aligned}$$

Now we have to multiply by the inverse matrix, which is just the inverse of the metric in M because we are at $s = 0$. Then we take trace and obtain the mean curvature, defined as the trace of the shape operator:

$$\operatorname{tr}\left(g^{ij} g(A\partial_j, \partial_k)\right) = g^{ij} g(A\partial_j, \partial_i) = g^{ij} g(A_j^\ell \partial_\ell, \partial_i) = g^{ij} A_j^\ell g_{\ell i} = \delta_j^i A_i^j = A_j^j$$

We conclude

$$\frac{d}{ds}\Big|_{s=0} S(s) = -\int_M H \, \mathrm{Vol}_g$$

which says: **zeroes of the mean curvature function correspond to critical points of the volume functional**. □

Addendum. And the mean curvature vector? The mean curvature vector is $H\xi$ for unit normal ξ. And $\tilde{g}(H\xi, \xi) = H$.