Daniel González Casanova Azuela
Prof. Sergey Galkin github.com/danimalabares/ms

Minimal Surfaces

Contents

1 Class 1

1.1 Intro

Minimal surfaces started with Lagrange, at 19 years old, more than 250 years old, when he communicated with Euler. Something is minimized. For Lagrange, this was the area functional with respect to euclidean metric $(dx)^2 + (dy)^2 + (dz)^2$. He was looking for surfaces that *locally* minimize area. He wrote differential equations characterizing this property.

1.2 Minimizing area of regular surfaces in $\mathbb{R}^3$

$$\begin{tikzcd} & \mathbb{R}^3 = \mathbb{R}^3_{x,y} \perp \mathbb{R}_z & \\ \Sigma \arrow[ur, "i"] \arrow[rr, "\pi\circ j := f_L"] & & {} \arrow[ul, "\mathbb{R}^2"'] \end{tikzcd}$$

The idea is that if $L \in T_p\Sigma$, then f_L is locally a diffeorphism around p. dani: So I think that decomposition of $\mathbb{R}^3$ depends on L. By inverse function theorem, locally there exists a function $\varphi(x,y)$ wuch that $\Sigma = \Gamma_\varphi = \{(x,y,z) : z = \varphi(x,y)\}$.

Example A sphere is locally seen ass $z = \sqrt{1 - x^2 - y^2}$.

$\Omega \subset \mathbb{R}^2$ some region. Then consider a function that puts the boundary of the region in $\mathbb{R}^3$ (it's the height function): $\varphi : \overline{\Omega} \to \mathbb{R}$, $\varphi|_{\partial\Omega}$. Then minimization of area becomes a PDE on φ.

This PDE is historically the first Euler-Lagrange Equation (=equations of motion). Now there's a lot of generalizations of this in classical field theory also.hj n

2 Class 2

Lagrange's PDE - nonlinear PDE. From [Sal16]:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0$$

Laplace equation

$$\Delta x_i = 0$$

and its nonhomogeneous version Poisson equation

$$\Delta f = \rho \in \Omega^2(\Sigma)$$

Rk. you can think of laplacian on a surface as giving a 2-form by multipliyng by Vol, this will allow you to integrate. The point is that solution exists $\iff [\rho] = 0 \in H^2(\Sigma, \mathbb{R})$. If Σ is not compact then there's always a solution. So Σ compact $\iff \int_\Sigma \rho = 0$.

So when thinking of Riemann surfaces/complex analysis, if you have $\Sigma_{(u,v)} \xrightarrow{\varphi} \mathbb{R}^n$ minimal, where $x_i(u,v)$, then x_i is a harmonic function wr.t. $\varphi^* g_{\mathbb{R}^n}$.

Three ways to study minimal surfaces

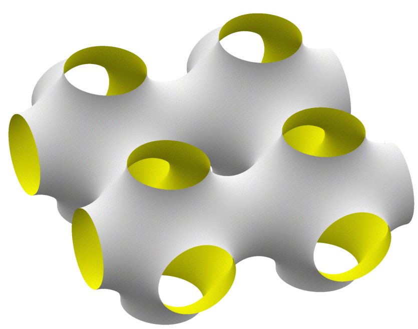
1. $\sim$ PDE. Colding-Codazzi textbook.
2. Complex analysis/geometry/Riemannsurfaces/practice. Flourished in 19$^{\text{th}}$ century.
3. Geometric measure theory. Very popular these days. Back to Lebesgue (who invented Lebesgue integral when investigating minimal surfaces).

Now we comment J. Simons who is the same from Chern-Simons and so many institutions and grants that go by the name Simons.

Result In $\mathbb{R}^3$ there are no closed (compact, without boundary) minimal surfaces.

So we look for things with boundary. So from the PDE point of view we look for some boundary condition. This is also called "plateau problem", which was solved in the 1930's.

Example Schwarz minimal surface:



You can think of this as put inside 3-torus $\mathbb{T}^3 \stackrel{\text{def}}{=} \mathbb{R}^3/\mathbb{Z}^3$.

Example Enneper surface.

We saw several other examples In conclusion, minimal surfaces exist and you can see some of them.

2.1 Basic objects

Consider
$$M^{(m)} \xhookrightarrow{f} N^{(n)} = \mathbb{R}^n, \qquad x_1, \ldots, x_n$$

It has a differential

$$T_pM \longrightarrow T_{f(p)}N = \mathbb{R}^n$$

which is a linear map. Now take local coordinates $D \subset M$, $(u_1, \dots, u_n)$. Then f is locally fiven by a functions of m variables

$$\begin{aligned} x_1(u_1, &\dots, u_m) \\ &\vdots \\ x_n(u_1, &\dots, u_m) \end{aligned}$$

while df_p is represented by a $m \times n$ matrix

$$\left(\frac{\partial x_i}{\partial u_J}\right), \qquad i = 1, \dots, n, j = 1, \dots, m$$

If $N^{(n)} \cong M^{(m)} \times \mathbb{R}^{n-m}$ and $f = \mathrm{id}_M \times \varphi$ (so $M = \Gamma_\varphi$ a graph of φ. Then we get a matrix

$$\begin{pmatrix} \mathrm{Id} & * \end{pmatrix}$$

Now look again at $M \overset{f}{\hookrightarrow} N$. We can pull back the metric of N, which is a symmetric tensor, getting a metric on M. (dani: because f is immersion=differential is injective).

So let's try to see how this metric looks like. The pullback metric should of course be something of the kind

$$\sum g_{jj'} du_j du_{j'}$$

now since $g = \sum (dx_i)^2$ is euclidean metric we get

$$\sum \left(\sum \frac{\partial x^i}{\partial u_j} du_j \right)^2$$

so that

$$g_{j_1 j_2} = \sum \frac{\partial x^i}{\partial u_{j_1}} \frac{\partial x^i}{\partial u_{j_2}}$$

So if this matrix is G and J is the jacobian of f we just get

$$G = J^T J$$

Remark We could also just define a metric on M by putting those coefficient functions and create s symmetric 2-tensor.

Now consider

$$\bar{g} := \det(g_{ij})$$

if $\bar{g} \neq 0$ then matrix (g_{ij}) is invertible, so it has inverse (g^{ij}).

Definition/Proposition f is ***regular*** in a point p if any of these equivalent conditions hold:

1. $\bar{g} \neq 0$.
2. $\bar{g} > 0$.
3. Vectors $(df_p)\left(\frac{\partial}{\partial u_i}\right)$ are linearly independent.

Proof. $\square$

Definition (Cotangent and tangent space) For a local system of coordinates we can define

$$\Omega_{\mathbb{R}^n} := \left\langle dx^{(1)}, \dots, dx^{(n)} \right\rangle$$

as the space generated by the differentials of the coordinate functions. Then we define

$$T_{\mathbb{R}^n} := \left\langle \frac{\partial}{\partial x^{(1)}}, \dots, \frac{\partial}{\partial x^{(n)}} \right\rangle$$

as generated by the dual basis of the cotangent basis.

2.2 Volume of a submanifold of $\mathbb{R}^n$

Consider U, V vector spaces and

$$U \xrightarrow{T} U \xrightarrow[\cong]{\natural_g} U^\vee \xrightarrow{T^*} U^\vee$$

where g is a metric on V. Then the composition of all this is just the same as the pullback metric on U!

$$T^* \circ \natural_g \circ T = \natural_{T^* g} : U \to U^\vee$$

Now apply Λ^m to get another commutative diagram

$$\begin{tikzcd}
\Lambda^m U \arrow[r, "\Lambda^m T"] \arrow[d, equal] \arrow[rrr, bend right, "\bar{g}"] & \Lambda^m V \arrow[r, "\Lambda^m \natural g"] & \Lambda^m V^\vee \arrow[r, "\Lambda^m T^\vee"] & \Lambda^m U^\vee \arrow[d, equal] \\
\mathbb{R} & & & \mathbb{R}
\end{tikzcd}$$

and then the composition here gives us

$$\bar{g} = \sum_{1 \leqslant j_1 \leqslant \dots \leqslant j_m \leqslant n} \left(\det \left(\frac{\partial x^{j_k}}{\partial u_i} \right) \right)^2$$

Let $v_1, \dots, v_n$ be a basis of U and $e_1, \dots, e_n$ another basis. Let V be a matrix that takes one basis to another. Then

$$G = V^T V$$

$$\implies \det G = \det(V)^2 \implies \bar{g}(p) = \mathrm{Vol}_M \left((df)_p \left(\frac{\partial}{\partial u_1} \right), \dots, (df)_p \left(\frac{\partial}{\partial u_n} \right) \right)^2$$

Definition (Volume of $M \xhookrightarrow{f} \mathbb{R}^n$)

$$\operatorname{Vol} M = \int_M \sqrt{\det g}\, du_1 \dots du_m$$

Which takes us to the problem of volume minimzation.

Then followed some computation of the minors of

$$\begin{pmatrix} 1 & 0 & z_x \\ 0 & 1 & z_y \end{pmatrix} \dots$$

Extra, we discussed this at some time That ***hypersurface*** is a thing locally defined by *one* function.

3 Class 3

3.1 Isothermic coordinates

A ***Weyl transformation*** is a map of metrics

$$g \mapsto e^k g$$

where we use exponent just to mean an invertible positive number. k is a real function.

Upshot That a metric on a surface induces a complex structure given by anticlockwise 90-degree rotation. And the interesting part is that two conformal metrics induce the same conformal structure, and the map that relates these metric is the Weyl transform.

Example: euclidean coordinates Euclidean metric is $(du)^2 + (dv)^2$. Letting $dz := du + idv$ and $d\bar{z} := du - idv$ we get

$$(du)^2 + (dv)^2 = (dz) \cdot (d\bar{z})$$

Now take other coordinates w, $w = F(z)$. You'll get

$$dw \cdot d\bar{w} = \underbrace{|F^i(z)|^2}_{e^k} dz \cdot d\bar{z} = e^k((du)^2 + (dv)^2)$$

And the matrix is

$$\begin{pmatrix} e^k & 0 \\ 0 & e^k \end{pmatrix}$$

Conclusion: conformal class of metric is the same as complex structure.

Definition ***Isothermic coordinates*** are local coordinates (u, v) such that $g_{11} = g_{22}$ and $g_{12} = 0$. So, **scalar** multiple of euclidean metric.

Claim Any Riemannian surface (Σ, g) has local isothermic coordinates. (In fact, for every power series we get another isothermic coordinates. There is a group action of the ***local biholomorphisms***. In higher dimension this is only finite dimensional, generated by inversions (this is a group, like inversion about circle in $\mathbb{C}$) and isometries (i.e. $\mathsf{Isom}(\mathbb{R}^n) \supset \mathsf{O}(n), \mathbb{R}^n$). So conclusion: in dimension 2 it has to do with complex analysis and in higher dimension it's just different.)

We have learnt that there are some thing of complex analysis like maximum principle and probably also Schwarz principle that generalize to higher dimension, but not everything.

Remark Recall that conformal maps are either holomorphic or antiholomorphic.

3.2 Gauss map

3.2.1 Grassman and normal bundles

Recall what is grassmanian $\mathsf{Gr}(m, n)$ with $m < n$, the variety of m-dimensional vector subspaces of an n-dimensional vector space V, and Plucker embedding $\mathsf{Gr}(m, n) \hookrightarrow \mathbb{P}\Lambda^m V$.

Definition V has ***vectors***. $V^\vee$ has ***covectors***. $\Lambda^2 V$ has ***bivectors***. $\Lambda^3 V$ ***trivectors***. ***Polyvectors***. For bundles: $\Lambda^2 T_M$ ***bivector fields***. $\Lambda^m T_M$ ***polyvector (or multivector) fields***, they are dual to $\Omega^m = \Lambda^m \Omega$, differential m-forms.

For a subspace $U \subset V$ we get corresponding $\underbrace{\Lambda^m U}_{\substack{\text{1-dim case} \\ \cong \mathbb{R}}} \xrightarrow{\Lambda^m j} \Lambda^m V$ and corresponidingly

$$\underbrace{\mathbb{P}\Lambda^m U}_{\substack{\text{1 dim} \\ = pt}} \xrightarrow{\mathbb{P}(\Lambda^m j)} \mathbb{P}\Lambda^m U$$

Next. If $V = \mathbb{R}^n$ is equipped with a metric g you have for two orthonormal bases

$$e_1 \wedge \ldots \wedge e_m = \det C f_1 \wedge \ldots \wedge f_m$$

For a map $f_i = C_i^j e_j$. If they are orthonormal you get $\det C = \pm 1$ and of you orient it all you get $\det C = 1$.

Correspondingly define the space of *oriented* vector subspaces of euclidean space by $\mathsf{Gr}^+(m, \mathbb{R}^n)$. For a basis $U = \langle u_1, \ldots, u_m \rangle$, we get a basis $\Lambda^m U = \langle u_1 \wedge u_2 \wedge \ldots \wedge u_m \rangle$.

Example

- $\mathsf{Gr}(1, 2) =$ circle.
- $\mathsf{Gr}^+(1, 2)$ space of rays.
- $\mathsf{Gr}^+(1, \mathbb{R}^n) = S^{n-1}$.
- $\mathsf{Gr}(1, \mathbb{R}^n) = \mathbb{R}P^{n-1}$.

Now do

$$0 \longrightarrow U^{(m)} \longrightarrow V^{(n)} \longrightarrow Q^{(n-m)} \longrightarrow 0$$

dualize

$$0 \longrightarrow Q^{\vee} \longrightarrow V^{\vee} \longrightarrow U^{\vee} \longrightarrow 0$$

So you automatically get isomorphisms

$$\mathsf{Gr}(m, V) = \mathsf{Gr}(V, n-m) = \mathsf{Gr}(n-m, V^{\vee}) = \mathsf{Gr}(V^{\vee}, m)$$

Now apply Λ:

Exercise $\Lambda^n V \cong \Lambda^m U \otimes \Lambda^{n-m} Q$

Attempt of solution. The exact sequence above gives $V = U \oplus Q$, so that $\Lambda^n V = \Lambda^n(U \oplus Q)$. So I think well $\Lambda^n V$ is just tensor of V n times quotiented by the antisymmetry relation $x \otimes x = 0$. And that's close but not exactly... □

Example $\mathsf{Gr}^+(2,3) = S^2 =$ conic $\subset \mathbb{C}P^2$, more exactly $\{x^2 + y^2 + z^2 = 0\}$.

Exercise $\mathsf{Gr}^+(2, \mathbb{R}^n) \cong Q^{n-2} \subset \mathbb{C}P^{n-1}$ of the form $\left\{\sum_{i=1}^n z_i^2 = 0\right\}$

Solution. When is a complex subspace of a complex vector space expressed with real coordinates?

Here's the general statement. For any $U_{\mathbb{C}} \subset V_{\mathbb{C}}$, the following are equivalent

- There exists $U_{\mathbb{R}} \subset V_{\mathbb{R}}$ such that $U_{\mathbb{C}} = U_{\mathbb{R}} \otimes \mathbb{C}$ (equations are real)
- $\overline{U_{\mathbb{C}}} = U_{\mathbb{C}}$ where bar denotes all the complex conjugate vectors inside the space, i.e. bar : $V_{\mathbb{C}} \to V_{\mathbb{C}}$, $(z_1, \dots, z_n) \mapsto (\overline{z_1}, \dots, \overline{z_n}$. In conclusion, $\sum a_i \cdot z_i = 0 \iff \sum \overline{a_i z_i} = 0$.
- $\overline{\Lambda^m U_{\mathbb{C}}} = \Lambda^m U_{\mathbb{C}} \in \Lambda^m V_{\mathbb{C}}$.
- $\exists \pi \in \Lambda^m V_{\mathbb{C}}$ s.t. $\Lambda^m U = \pi \otimes_{\mathbb{R}} \mathbb{C}$.

the point is that these points have **real Plücker coordinates**.

"for complex isotropic vector construct a real bivector (the line is the same)" And every multiple of this vector gives you the same line. So you can projectivize. So for any point in the projectivized quadric we get a line. And also we can conjugate and get the same line. So that's from quadric to grasmannian. Choice of orientation is a choice of the two points. □

Remark (dani) Lo que yo entendí: te pegas la métrica euclidiana real en el subespacio. Es una forma quadráitca. Como es positiva no degenerada no hay zeros. Pero complejificas y obtienes que sí hay zeros, de hecho, tienen que ser dos zeros. Esos zeros creo que son zeros en el plano. Pero orientas, entonces hay uno nomás. E é isso, esse aí é um mapa bijetivo entre a quadrica e a grasmanniana orientada.

Remark And this has all to do with Weierstrass representation.

3.2.2 Gauss map

Consider

$$M^{(m)} \xrightarrow{f} N^{(2)}$$

and do

$$\mathbb{R}^m \cong T_p^m M \xrightarrow{d_p f} T_{f(p)} N \cong \mathbb{R}^n$$

so for every regular (=rank is m=the map is an embedding=the thing is a subvariety) p you get a point in grassmanian

$$\mathrm{Gr}(m, T_{f(p)} N)$$

Consider $f^* T_N$ which is a vector bundle over M, then we can consider

$$(df) : T_M^{(m)} \overset{\text{by regularity,}}{\hookrightarrow} f^* T_N^{(n)} \to \underbrace{f^*(T_N / df(T_M))^{(n-m)}}_{\textit{normal bundle}}$$

and then consider the ***Grassmanization*** of $f^* T_N$, which is a bundle

$$\begin{array}{c} \mathrm{Gr}(m, f^* T_N) \\ \downarrow \\ M \end{array}$$

Now consider sections of this thing.

Important observation (Gauss map definition) Given a trivialization of $f^* T_N$, a section of the thing becomes just a map

$$M \to \mathrm{Gr}(m, n)$$

(Because a section of a trivial bundle is just a map to the fiber.)

Now fix $N := \mathbb{R}^n$ and put a connection, the Levi-Civita connection. Actually now better take the oriented Grassmanian for the whole bussiness. And we have the map

$$M \to \mathrm{Gr}^+(m, n)$$

which is a map that to every point associates its tangent space. And it is called the ***Gauss map***.

Recall that

Claim A surface is minimal $\iff$ Gauss map is conformal (anti-holomorphic).

Put a surface in $\mathbb{R}^3$. Basis for tangent space is

$$(x_u, y_u, z_u), (x_v, y_v, z_v)$$

So basis for projectivization is

$$(y_u z_u - y_v z_u, x_u z_v - x_v z_u, x_u y_v, x_v y_u)$$

and in general, when you get

$$\Sigma \to \mathbb{R}^n$$
$$\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \rightsquigarrow \frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}$$

Lesson Plücker coordinates of oriented Grassmanian are the same as coordinates of normal bundle!

But also notice that you can write

$$\{x^i, x^j\}_{du \wedge dv}$$

And the point of all this is that

Claim Minimal surfaces are equivalently defined as the ciritical points of the ***Schild action***

$$\int \left(\sum_{i<j} \{x_i, x_j\}^2 \right) \omega$$

Now put a curve $I \subset \Sigma \to \mathbb{R}^n \ldots$ [what was the curve for…?]

Hint From 1 to 2 you "vary" the volume and look for critical points.

Important exercise If $x = u, y = v$ and $z = \varphi(u, v)$, you get two derivatives

$$\partial_u \to (1, 0, \varphi_u) \qquad \partial_v \to (0, 1, \varphi_v)$$

Notice that

$$df[\partial_u] = \partial_x + \varphi_u \cdot \partial_z$$

And now

$$\partial_u \wedge \partial_v = \partial_u \wedge \partial_y - \varphi_v \partial_x \wedge \partial_z - \varphi_v \partial_y \wedge \partial_z$$

Then you have proportionality

$$N \sim (-\varphi_u, -\varphi_v, 1) := \tilde{N}$$

So

$$\|\tilde{N}\|^2 = \varphi_u^2 + \varphi_v^2 + 1 = g$$
$$N = \frac{\tilde{N}}{\sqrt{g}}$$

So ***Gauss map for non-parametric surface*** is

$$G : (u, v) \longrightarrow \left(\frac{-\varphi_u}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}}, \frac{-\varphi_v}{\sqrt{1 - \varphi_u^2 + \varphi_v^2}}, \frac{1}{\sqrt{1 - \varphi_u^2 + \varphi_v^2}} \right)$$

(and you can just write $\sqrt{1 + \varphi_u^2 + \varphi_v^2} = \sqrt{g}$ but Ok). Similarly…

3.2.3 Second fundmental form

And differentiate! At every point we get

$$d_p G : T_p M \to T_{G(p)} \mathsf{Gr}(m, n) = \mathsf{Hom}(T_p M, N)$$

(where N is the normal bundle). So globally

$$dG \in \mathsf{Hom}(T_p M, \mathsf{Hom}(T_p M, N))$$

So it's a map aa $T_p M \times T_p M \to N$ But N is trivialized! So you can think of *numbers*. So it's a form.

Claim (Exercise) That map is symmetric.

So you can define it to be the ***second fundamental form***.

Notations

- I fundamental form $\rightsquigarrow g_{ij}$.
- II fundamental form $\rightsquigarrow B_{ij}$ for a vector in the normal bundle, and b_{ij} when you are in $\mathbb{R}^3$ and you have trivialized it to get numbers

Claim

$$B_{ij} = \left(\frac{\partial^2 \vec{x}}{\partial u_i \partial u_j} \right)^N$$

where exponent N means projection to the normal bundle which can mean algebraic projection when you think of a quotient bundle or orthogonal projection when you think geometric.

3.3 Reminder: variational approach

Reminder of the end of last lecture (Variation of the tangent direction)

$$\begin{aligned} V'(t=0)[\underbrace{\delta\varphi}_{:=w}] &= \int_\Sigma \frac{z_x \cdot w_x + z_y \cdot w_y}{\sqrt{g}} du dv \\ &= \int \langle \delta\varphi, \text{something} \rangle \frac{\sqrt{g} du dv}{\text{volume form}} \end{aligned}$$

where $\delta\varphi$ is a section of $f^* T_{\mathbb{R}^n} = df(T_M) \oplus^\perp N$. And **something** is a unique thing that exists making the equality hold. This is a possible definition of ***mean curvature vector*** $\vec{H} \in N$.

And you can also write

$$\int dv \wedge \left(\frac{z_u}{\sqrt{g} \cdot w_u du} \right) = \int dv \wedge \left(\frac{z_u}{\sqrt{g}} dw \right)$$

where $dw = w_u du + \ldots + dv$. And now we want to integrate by parts to get

$$= - \int w \cdot \partial \left(\frac{z_u}{v_g} dv \right)$$

So the integral above (with the **something** inside) becomes

$$\int_\Sigma \frac{z_x \cdot w_x + z_y \cdot w_y}{\sqrt{g}} dudv = -\int \left(\partial_u \frac{z_u}{\sqrt{g}} + \partial_v \frac{z_v}{\sqrt{g}}\right) \cdot w dudv$$

so if that is zero for all w, then the thing in parenthesis must be zero by nondegeneracy of metric.

Right so what is the ***first variation of volume***? It's

$$V'|_{t=0} = -\int_\Sigma (\sqrt{g}du) \langle H, W \rangle$$

And it has all to do with the expression $\Sigma g^{ij} B_{ij}$.

4 Class 4

5 Class 5

5.1 Classical Field theory: fields, lagrangian and action

1. ***Field***: the map $\varphi : \Sigma \to X$.
2. ***Action***: $S[\varphi] = \int_\Sigma \underbrace{\mathcal{L}[\varphi] d \,\mathsf{Vol}_\Sigma}_{\text{Lagrangian density}}$.

   You give me a field, I give you a top-form, i.e. something you can integrate in Σ. That's the idea of the Lagrangian. So $\mathcal{L}(\varphi)$ is a "rule" to produce densities from fields. $\mathcal{L}[\varphi]$ is a local operator. $\mathcal{L}[\varphi]$ depends on the fields and their derivatives. Classical version is $\mathcal{L} = E - U$ i.e. kinetic $-$ potential energy.
3. ***Equation of motion (EOM):*** are the critical points of S

5.2 Area functional aka Nambu-Goto action

$$S_{\mathsf{NG}}[\varphi] = \int_\Sigma d \,\mathsf{Vol}_{\varphi * g}$$

where $d \,\mathsf{Vol}_G = \sqrt{-\det G_{ij}}$ where $\varphi : \Sigma \to X$ and X might have lorenzian metric, so that's why there's sign.

5.3 Polyakov action

It's **a thing that depends both on** h **and** φ:

$$S_{\mathrm{pol}}[\varphi; h] = \int_\Sigma d \,\mathsf{Vol}_h \,\|d\varphi\|_g^2$$

where $\|d\varphi\|^2$ might as well be explained in the context of:

5.4 Dirichlet energy/energy functional

$$\mathrm{tr}\left(\begin{tikzcd} T_p\Sigma \arrow[r, "df"] & T_{f(p)}X \arrow[d, "\sharp_g"] \\ T_p^\vee\Sigma \arrow[u, "(\sharp_h)^{-1}"] & T_{f(p)}X \arrow[l, "(df)^\vee"] \end{tikzcd}\right)$$

which gives

$$\frac{\partial X^\mu}{\partial u_i} g_{\mu\nu}(\varphi(u)) \frac{\partial X^\nu}{\partial u_j} h^{ij}$$

So you give me a map, I give you a number. OK but definition of of ***Dirichlet energy/energy functional*** it's actually the same thing as Polyakov action but h **is fixed**:

$$E_h(\varphi) = \int_\Sigma d\,\mathrm{Vol}_h \|d\varphi\|_g^2$$

Important example If we take usual euclidean metrics we get usual laplacian operator and usual harmonic functions.

Nice example If you had a line instead of a surface, i.e. $\Sigma = \mathbb{R}$ then you'd get that the it is a geodesic (in X) iff... the map is harmonic!

Upshot That's what's going on: harmonicity characterizes minimizing these functionals. So minimal surfaces, geodesics, appear along with geodesics.

5.5 Harmonic function

Definition A map $\varphi : (\Sigma, h) \to (X, g)$ is called (h, g)-***harmonic*** if it is a critical point of the energy functional

So in terms of Polyakov action you have some equations of motion (=critical points of P. action):

1. $\frac{\delta S}{\delta \varphi} = 0$ i.e. φ is h-harmonic.
2. $\frac{\delta S}{\delta h} = 0$ some other thing we'll study up front.

5.6 Symmetries of Dirichlet energy

Weil transformations (conformal transformations= take the metric and multiply by a function) preserve Dirichlet energy. Which is a complex structure (the conformal class of the metric). We mean— E depends on the complex structure of Σ.

5.7 Feynman quantization

$$Z(\hbar) = \int_{\text{Fields}} e^{\frac{iS[\varphi]}{\hbar}} \mathcal{D}\varphi$$

5.8 The other critical point of Polyakov

Recall that Polyakov depends both on h and φ. So there must be some interpretation of

$$\frac{\delta S}{\delta h} = 0.$$

First we need to differentiate determinant. See wiki, cofactor matrix appears. So the inverse matrix appears.

more computations

> "So 2nd EOM says that h and G (I fundamental form of $\varphi^* g$) lie in the same conformal class (and also shows the exact solution)"

So they give the same complex structure. And it goes back to Nambu-Goto action—we get rid of the h.

5.9 Mean curvature vector (again)

$\vec{H}(u)$ is a normal vector such that

1. $$\delta S_{\mathsf{NG}} = \int \delta\vec{X} \cdot \vec{H}\, \mathsf{Vol}\, g$$

   for any variation $\delta\vec{X}$.

2. $\vec{H} = \Delta\vec{X}$.

3. $\vec{H} = B_{ij}G^{ij}$ where

   $$B_{ij} = \left(\frac{\partial^2\vec{X}}{\partial u_i \partial u_j}\right) \leftsquigarrow \text{II fundamental form}$$

   so the projection to normal space.

4. $\vec{H} = \mathsf{tr}(S)$ (mean curvature is the trace of shape operator.) So ***shape operator*** is

   $$S := \sharp_G^{-1} \circ \sharp_B,$$

   an operator on $T\Sigma$ valued in f^*N.

Exercise

1. Prove 1 $\iff$ 2 $\iff$ 3. **Hint.** You write the variation formula and integrate by parts and that's it.

2. Write them down in case $n = 3$. φ grpo off $u, v \to (u, v, f(u, v))$, G_{ij}, G^{ij}, B^{ij}.

Definition ***Third fundamental form*** is...

6 Class ?

6.1 Vier bein

If you happen to read the paper by Shild (?) (or is it "The relativistic string" you may find the concept of ***vier bein*** which is equivalently a choice of local orthonormal base of a vector bundle and a filtration $F^1 \subset F^2 \subset F^3 \subset \ldots F^n = E$. (So you just choose the first i of the orthonormal sections to produce F^i.) What's nice about this is that you allow the sections to be nonconmutative, unlike the ∂_i which satisfy $[\partial_i, \partial_j] = \delta_{ij}$.

6.2 Schild action

1. Length: $\delta\left(m\int_{u*}^{u**}[(\dot{x})^2]^{\frac{1}{2}}du\right)$
2. Energy: $\delta\left(_{u*}^{u**}\frac{1}{2}(\dot{x})^2\right)$

Question

1. why variation is defined in terms of velocity why delta s2 = only the mid term in the xapsion

Remark First variation formula shows: critical point of length functional $\implies$ geodesic $\dddot{x} = 0$

Definition ***Mean curvature flow*** is such that its derivative is $\vec{H}$.

7 Class

Claim Locally any harmonic function

8 Class

1. Two definitions of hodge star.
2. Second definition gives an isomorphism $\det V \otimes \Lambda^k \cong \Lambda^{n-k}V$
3. $d^* = * \circ d \circ *$

   Exercise Notice that in the case of $n = 2k$ we have that $*_g : \Omega^k(V) \to \Omega^k(V)$. For $n = 2$, $k = 1$, show that $*_g = J_g$, where J_g is the almost complex structure defined by g (discussed in earlier sessions).

4. Laplace operator on Riemannian manifolds: $\Delta^g := [d, d^*] = dd^* - d^*d = d * d * - * d * d$.
5. ***Harmonic*** k***-forms***: $\ker \Delta^g : \Omega^k(M) \to \Omega^k(M)$.

6. Notice that $\Delta^g = \underbrace{*d*}_{\text{div}}\underbrace{d}_{\text{grad}}$ matches the other definition for laplacian in riemannian manifolds.

7. **Thm (E. Hopf).** For any (M, g), harmonic functions satisfy ***maximum principle***. That is, any harmonic function $f : U \to \mathbb{R}$ for U connected and open, if $\exists p \in U$ that is a local maximum or local minimum, then f is constant on U. (So if f was nonconstant the point should be in the boundary somehow.)

8. Another definition of laplacian: $\Delta^g = \sum(\xi_k)^2 + \nabla_{\xi_k}\xi_k$for an orthonormal frame ξ_k, where ξ_k^2 means apply ξ as a differential operator twice. So this is a second order differential operator. So if you are euclidean the ∇ part vanishes (so maybe think of Christoffel symbols) and you are left with the other part, which is usual laplacian.

9. **Prop.** On euclidean space, $\Delta = \sum \partial_i^2$.

10. **Prop (mean value).** $\Delta f = 0 \iff \forall B(P, R)$ ball,
$$f(P) = \frac{1}{\mathrm{Vol}\, B}\int f(Q)\, \mathrm{Vol}_g$$
which is the mean value of f in B. (Comment: you can derive mean value property from Cauchy integral formula.)

11. Another remark about mean value property: holds in general for manifolds. For surface case it's easier because we have isothermal coordinates.

12. Recall that for $X : \Sigma \to \mathbb{R}^n$ we have a laplacian $\Delta_g\vec{X} = (\Delta X^1, \ldots, \Delta X^n)$ because vector fields on $\mathbb{R}^n$ are vectors in $\mathbb{R}^n$. So we showed at some point that $\Delta_g\vec{X} = \vec{H}$.

13. Then a remark on isothermal coordinates: basic vectors are orthogonal and of the same size.

14. Now consider
$$\phi^\alpha(u, v) = \frac{\partial X^\alpha}{\partial u} - i\frac{\partial X^\alpha}{\partial v}$$
(I think ϕ is a map $\mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$.) So **Claim.** X^α is harmonic $\iff$ ϕ^α is holomorphic.

    Recall from complex analysis that $\Delta f = 0 \iff \partial_z f$ is holomorphic. So choose for f one of the component functions of the map X.

15. So notice: when you have a minimal surface you have X is harmonic so ϕ^α will be holomorphic for all α. Then, in conformal coordinates (cf. isothermal coordinates above), we get
$$\mathrm{Re}\,\vec{\phi} \perp \mathrm{Im}\,\vec{\phi}, \qquad |\mathrm{Re}\,\vec{\phi}| = |\mathrm{Im}\,\vec{\phi}|.$$

16. Now the other direction: if you get the data from the last equation, can you construct a minimal surface? So first notice that
$$dX^\alpha = \frac{\partial X^\alpha}{\partial u}du + \frac{\partial X^\alpha}{\partial v}dv.$$

Then some computation followed and we conclude that

$$X^{\alpha}(p) = \mathrm{Re}\int_{p_0}^{p} Q(z)dz, \qquad z = u + iv$$

And then

Claim If $Q^{\alpha}(z)dz$ are holomorphic differentials on (Σ, J) + connection, then the last equation gives a minimal surface in $\mathbb{R}^n$. And then some more computations and we conlcude that

$$\text{conformal coordinates} \iff \sum \phi_k(z)^2 = 0$$

So you have a collection of holomorphic functions that their sum of squares equals zero.

17. ***Weierstrass data***:
    (a) A complex structure J on Σ.
    (b) $\alpha_1, \dots, \alpha_n \in \Gamma(\Sigma_J, \omega_\Sigma)$ such that
    $$\sum_{k=1}^{n} (\alpha_k)^2 = 0 \in \Gamma(\Sigma_J, \omega_\Sigma^{\otimes 2})$$
    (c) $\forall \gamma \in H_1(\Sigma, \mathbb{R})$ s.t. $\forall k = 1, \dots, n$,
    $$\mathrm{Re}\int_\gamma \alpha_k = 0$$
    "Global conditions on periods"

Claim Weierstrass data $\iff$ Minimal surfaces with

$$\vec{X}(p) = \mathrm{Re}\int_{p_0}^{p} \vec{\alpha}.$$

8.1 Outline of the proof of E. Hopf. theorem

Theorem (E. Hopf) For any (M, g), harmonic functions satisfy ***maximum principle***. That is, any harmonic function $f : U \to \mathbb{R}$ for U connected and open, if $\exists p \in U$ that is a local maximum or local minimum, then f is constant on U.

Proof.

Step 1 Make sure you can pick a "Referencial geodésico" about every point of M. This is an orthonormal frame $\{E_i\} \subset \mathfrak{X}(U)$, where $p \in U$ such that $\left(\nabla_{E_i} E_j\right)_p = 0$. This frame can be obtained by taking geodesic coordinates at the point, an orthonormal base $\{e_i\}$ of T_pM, and taking the parallel transport of the vectors e_i along radial geodesics emanating from p. ([dC79], Cap. III, Ex. 7.)

Step 2 Prove that

$$d i_X \mathsf{Vol} = \mathsf{div}\, X\, \mathsf{Vol}$$

([dC79] Cap. III, Ex. 11.) To do this pick a Geodesic frame E_i and its dual coframe ε^i, i.e. satisfying $\varepsilon^i(E_j) = \delta_{ij}$. Then $\mathsf{Vol} = \varepsilon^1 \wedge \ldots \wedge \varepsilon^n$. Then for any $X = X^i E_i \in \mathfrak{X}(U)$,

$$i_X \mathsf{Vol} = \varepsilon^1 \wedge \ldots \wedge \varepsilon^n(X^i E_i, \cdot, \ldots, \cdot) = X^i \varepsilon^1 \wedge \ldots \wedge \varepsilon^n(E_i, \cdot, \ldots, \cdot)$$

How to compute that? Recall that for top-forms we have

$$\varepsilon^1 \wedge \ldots \wedge \varepsilon^n(Z_1, \ldots, Z_n) = \mathsf{det}(\varepsilon^i(Z_j))$$

so for example if $n = 3$

$$\begin{aligned}
\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3(E_1, Z_2, Z_3) &= \begin{vmatrix} \varepsilon^1(E_1) & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ \varepsilon^2(E_1) & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ \varepsilon^3(E_1) & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix} = \begin{vmatrix} 1 & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ 0 & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ 0 & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix} \\
&= \varepsilon^2(Z_2)\varepsilon^3(Z_3) - \varepsilon^2(Z_3)\varepsilon^3(Z_2) = \varepsilon^2 \wedge \varepsilon^3(Z_2, Z_3), \\
\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3(E_2, Z_2, Z_3) &= \begin{vmatrix} 0 & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ 1 & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ 0 & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix} = -\varepsilon^1 \wedge \varepsilon^3(Z_2, Z_3)
\end{aligned}$$

and so on. When we sum over all i, we get

$$X^i \varepsilon^1 \wedge \ldots \wedge \varepsilon^n(E_i, \cdot, \ldots, \cdot) = \sum_i (-1)^{i+1} X^i \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n.$$

Now take exterior derivative of that, we get

$$\begin{aligned}
d i_X \mathsf{Vol} &= \sum_i (-1)^{i+1} (dX^i) \wedge \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n \\
&\quad + \sum_i (-1)^{i+1} X^i \wedge d(\varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n)
\end{aligned}$$

And then the first term actually is

$$\sum_i (-1)^{i+1} E_j X^i \varepsilon^j \wedge \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n = E_i X^i \mathsf{Vol}$$

while the second term vanishes because look,

$$\begin{aligned}
d\varepsilon^i(E_j, E_k) &= E_j \varepsilon^i(E_k) - E_k \varepsilon^i(E_j) - \varepsilon^i([E_j, E_k]) \\
&= -\varepsilon^i(\nabla_{E_j} E_k - \nabla_{E_k} E_j) \qquad \text{torsion!}
\end{aligned}$$

which vanishes at p because we said that this geodesic frame would have vanishing Christoffel symbols at p. So we conclude:

$$d i_X \mathsf{Vol} = E_i X^i \mathsf{Vol}$$

Now you just have to think what is divergence: divergence is $\operatorname{div} X \stackrel{\text{def}}{=} \operatorname{tr}(v \mapsto \nabla_v X)$. So

$$\begin{aligned}
\operatorname{div} X &= \sum_i \left\langle \nabla_{E_i} X^j E_j, E_i \right\rangle = \sum_i \left\langle E_i X^j E_j, E_i \right\rangle + X^j \left\langle \nabla_{E_i} E_j, E_i \right\rangle \\
&= \sum_i \left\langle E_i X^j E_j, E_i \right\rangle = \sum_i E_i X^j \left\langle E_j, E_i \right\rangle = E_i X^i
\end{aligned}$$

again using that we are a geodesic frame with vanishing covariant derivative at p.

Step 3 (This is the next exercise in [dC79]... *in progress*)

$\square$

References

[dC79] M.P. do Carmo. *Geometria Riemanniana*. Escola de geometria diferencial. Instituto de Matemática Pura e Aplicada, 1979.

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