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Notes on complex geometry

Contents

1 basic math

1.1 tensor product

Definition (Tensor product by [?]) If V and W are R-modules and R is a ring, I think of the tensor product as

$$V \otimes W := \left\{ \sum_{\text{finite}} \lambda_i (v \otimes w)_i, v \in V, w \in W \right\} \Big/ \begin{matrix} v \otimes (w_1 + w_2) \sim v \otimes w_1 + v \otimes w_2 \\ (v_1 + v_2) \otimes w \sim (v_1 \otimes w) + (v_2 \otimes w) \\ \lambda(v \otimes w) \sim (\lambda v) \otimes w \sim v \otimes (\lambda v) \end{matrix}$$

More formally, the free R-module generated by $V \times W$ (we say this so that things of the kind $\lambda(v, w)$ make sense) quotiented by the ideal generated by these guys:

$$\begin{gathered} (v, w_1 + w_2) - (v, w_1) - (v, w_2), \\ (v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ (\lambda v, w) - \lambda(v, w), \\ (v, \lambda w) - \lambda(v, w) \end{gathered}$$

So that we denote the projection of (v, w) by $v \otimes w$. Also never forget that the product was defined when you have a product and Vakil has a product…

and I have a product… (Universal property of product) It's actually nice to recall that any two objects satisfying the "every map of such kind factors through a map involving the product"-property, you can construct an isomorphism between them. So the only thing satisfying the product property is the product.

And of course you must understand the construction of tensor product using universal property. Suppose you have a tensor product…

Remark Maybe when I think of span of some set of symbols I mean the free R-module generated by the (as many symbols I need)-th product of the place where the symbols live, quotiented by the obvious linearlity relations.

Remark The difference between tensor product and cartesian product is that the latter is not bilinear in addition and scalar multiplication, look:

$$\begin{gathered} (v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \\ \lambda(v, w) = (\lambda v, \lambda w) \end{gathered}$$

14.3.6, [?] (Tensor algebra constructions) Let M be an A-module. The tensor algebra $T^\bullet(M)$ is just (the direct sum, probably) of $T^0(M) := A$, $T^n(M) := \underbrace{M \otimes_A \ldots \otimes_A M}_{n \text{ times}}$.

The symmetric algebra $\mathrm{Sym}^\bullet(M)$ is the quotient of $T^\bullet(M)$ by the (two-sided) ideal generated by all the elements of the form $x \otimes y - y \otimes x$. So,

$$\mathrm{Sym}^n(M) = M \otimes \ldots \otimes M \Big/ m_1 \otimes \ldots \otimes m_n - m_1' \otimes \ldots \otimes m_n'$$

where $(m_1', \ldots, m_n')$ is a rearrangement of $(m_1, \ldots, m_n)$.

Finally the exterior algebra $\Lambda^\bullet(M)$ is defined to be the quotient of $T^\bullet M$ by the (two-sided) ideal generated by all the elements of the form $x \otimes x$ for all $x \in M$. Which implies that $a \otimes b = -b \otimes a$ (Pf. do $(a+b) \otimes (a+b)$, but only if char $\neq 2$ right?) Apparently this gives you that

$$\Lambda^n M = \text{quotient of no sé que with a permutation}$$

Anyway I can finally say that if $\mathcal{F}$ is a locally free rank-m sheaf, I can define $T^n\mathcal{F}$, $\mathrm{Sym}^n \mathcal{F}$ and $\Lambda^n\mathcal{F}$, that will be locally free (exercise, and find their ranks, exercise). And just to conclude for today, the determinant (line) bundle is $\Lambda^{\mathrm{rk}\,\mathcal{F}}\mathcal{F} := \det \mathcal{F}$.

So why all this. Because adjunction formula gives you that the canonical bundle of the submanifold is $K_{\text{ambient mfd}} \otimes_{\mathbb{C}?} \det \mathcal{N}$ where $\mathcal{N}$ is the normal bundle of the submanifold which is the cokernel sheaf/bundle of the inclusion. So that's that.

Plücker embedding It's a way to put Grassman inside a projective space:

$$\mathrm{Gr}(V, k) \hookrightarrow \mathbb{P}\Lambda^k V = \mathrm{Gr}(\Lambda^k V, 1)$$

evidently given by

$$\text{basis } e_1, \ldots, e_k \text{ of } U \in \mathrm{Gr}(V, k) \longmapsto \mathrm{span}\{e_1 \wedge \ldots \wedge e_k\}$$

1.2 sheaves

$\mathfrak{m}_x \subset C^\infty(\mathbb{R}^n)$ is the ideal of smooth functions vanishing at $x \in \mathbb{R}^n$. In [?] definition of local ring of P in Y, which is the ring of germs of regular functions of the variety Y at the point P, that it is a local ring with maximal ideal $\mathfrak{m}$, the set of germs of regular functions which vanish at P. The residue field is k.

Exercise Show that indeed $\mathfrak{m}_x$, the ideal of functions vanishing at x, is maximal.

Solution. Suppose that $\mathfrak{n} \supseteq \mathfrak{m}_x$ is another ideal contained in $\mathcal{O}_{X,x}$. If $[f] \in \mathfrak{n}$ does not vanish at x, then we can do $[1/f]$ very near x, giving $[f][1/f] \in \mathfrak{n}$, so that $\mathfrak{n} = \mathcal{O}_{X,x}$. And if all $[f] \in \mathfrak{n}$ vanish at x, we get $\mathfrak{n} = \mathfrak{m}_x$. □

Little discussion with GPT In differential geometry we have the sheaf $\mathcal{F}$ of smooth functions. That's like the scalars of some other structure that we build over it: the tangent sheaf, which is defined as derivations on $\mathcal{F}(U)$. Remember that for any sheaf

$\mathcal{F}$, the elements of $\mathcal{F}(U)$ are called sections. Yes, the sections of the tangent bundle are vector fields.

(The proof that $\mathcal{T}_M$, the tangent sheaf, is a locally free $\mathcal{F}$-module (and thus a vector bundle) is just showing that ∂_i are basis of $\mathcal{F}(U)$.)

The tangent bundle TM of a smooth manifold M is a vector bundle whose dual, the cotangent bundle T^*M, is locally generated by differentials dx^i.

In algebraic geometry, we have an analogous structure:

- The structure sheaf $\mathcal{O}_X$ assigns to each open set $U \subseteq X$ the ring of regular functions on U.
- The sheaf of Kähler differentials Ω^1_X is the analogue of the cotangent bundle: it is an $\mathcal{O}_X$-module generated by formal differentials df.
- The tangent sheaf $\mathcal{T}_X$ is the dual of Ω^1_X, i.e.,

$$\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X),$$

  and it corresponds to derivations of the structure sheaf:

$$\mathcal{T}_X(U) = \mathrm{Der}_{\mathbb{K}}(\mathcal{O}_X(U), \mathcal{O}_X(U)).$$

- These sheaves are quasicoherent sheaves, which generalize vector bundles to the algebraic setting.

So, quasicoherent sheaves play the role of vector bundles in algebraic geometry, and in particular, $\mathcal{T}_X$ is the algebraic analogue of the tangent bundle.

1.3 Normalization

From [?] 10.7 “Normalization”.

> “Normalization is a means of turning a reduced scheme into a normal scheme. (Unimportant remark: reducedness is not a necessary hypothesis. [...].) A normalization of a reduced scheme X is a morphism $\nu : \widetilde{X} \to X$ from a normal scheme, where ν induces a bijection of irreducible components of $\widetilde{X}$ and X [...]. It will satisfy a universal property, and hence it will be unique up to unique isomorphism.”

So basically that’s it—another universal construction. Here’s the diagram Vakil puts:

$$\begin{tikzcd}
\text{normal} & Y \arrow[rr, dashed, "\exists!"] \arrow[dr, "\text{dominant}"'] & & \widetilde{X} \arrow[dl, "\nu\ \text{dominant}"] & \text{normal} \\
& & X & &
\end{tikzcd}$$

Question In what cases is this just a blow-up?

Question So what is a normal variety anyways? A variety such that the local ring at every point is an integrally closed domain over the field of regular functions. Which means that "the only zeros in $K(A)$ [=field of fractions of the ring, in this case fraction function field] to any monic polynomial in $A[x]$ must lie in A [A is the local ring in this case of course] itself".

Right but once again that's just some abstract algebra the point is

> (before defining normal scheme…) "We can now define property of schemes that says that they are "not too far from smooth", called normality […] See §13.8.7 and §26.3.5 for the fact that "smoothness" (really "regularity") implies normality."

So smooth implies normal but what about the inverse? Probably more algebra lurking behind the answer.

2 basic complex geometry

2.1 complex and almost complex structures

Remark That $\Omega^{1,0}(M)$ is the annihilator of $T^{0,1}(M)$. So this addresses the question OK, $T^{0,1}(M)$ is the $-\sqrt{-1}$-eigenspace but what is $\Omega^{0,1}(M)$ and $\Omega^{1,0}(M)$?

Facts form Sergey class

- A local system of coordinates for a 1-complex-dimensional manifold is given by solution of $\bar{\partial} f = 0$ for nonconstant f.
- In dimension 1 there's no almost complex structures that are not complex: all are integrable! Why right? There's several equivalences for being integrable:
  - Neghjouse tensor vanish
  - involutive $T^{1,0}$
  - Holomorphic coordinate, so previous item which is solution to CR eq.
- Given an almost complex structure on a 1 dimensional complex manifold and define the Laplacian to be $\partial\bar{\partial}$. Exercise If J and g are compatible them $\Delta_J = \Delta_g \overset{\text{def}}{=} \operatorname{tr}(\nabla f)$.

  So maybe you want to use the following Claim: that for $\lambda \in C^\infty(M, \mathbb{R}^+$ we have $\Delta_{\lambda g}(f) = \lambda^{-1}\Delta_g(f)$ because $\sqrt{\lambda g} = \lambda\sqrt{g}$, and I think this last one only holds in dimension 1 so all this only works in dimension 1.

2.2 ∂ and ∂

2.3 Volume form

dani: It's

$$\text{constant} \prod \partial z_i \wedge \partial z_i = \text{const.} \sum dx_i \wedge dy_i$$

voi:It's

$$\frac{\omega^n}{n!} = \prod dz_i \wedge d\bar{z}_i$$

where $dz_i = dx_i + \sqrt{-1}dy_i$. It is the volume form of the hermitian manifold, i.e. the unique nowhere-vanishing section of the determinant bundle that gives 1 to the volume of the real unit cube $e_1 \wedge Ie_1 \wedge \ldots \wedge e_n \wedge Ie_n$ obtained from an h-orthonormal complex basis $\{e_i\}$.

3 Line bundles and divisors

Skimming —because understanding the details looks... difficult— [?] chpt. 1, sec. 1 "Divisors and line bundles"

The idea is that meromorphic functions are abstract things which may not be functions at all. They are given locally, so w.r.t. some covering of the manifold, and in each open set they are a quotient of relatively prime (in $\mathcal{O}(U_\alpha)$) functions.

But these data are just very closely related to the Čech cohomology and the trivializing atlas of vector bundles. And it turns out it's all very related.

So a divisor is a formal sum of hypersurfaces. A hypersurface is, by [?] slice theorem, defined locally with a slice chart, which just says that one of the coordinates vanishes. So you see: there's an open cover and some functions. So in the first place it turns out that divisors are sections of a certain sheaf of meromorphic functions ($\mathcal{M}^*/\mathcal{O}^*$):

$$\text{divisors} \overset{\text{complicated details}}{\leftrightsquigarrow} \text{meromorphic functions}$$

And in the second place you will have line bundles also. And I guess "meromorphic functions" is also "sheaves". There's these three things.

Here's a quote from [?] p.145:

> "Suppose V is given locally by functions $f_\alpha \in \mathcal{O}(U_\alpha)$; the line bundle $[V]$ on M is then given by transition functions $\{g_{\alpha\beta} = f_\alpha/f_\beta\}$."

4 Exponential exact sequence

It's a short exact sequence of sheaves:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

where X is a complex manifold, $\mathcal{O}(X)$ the sheaf of holomorphic functions and $\mathcal{O}^*(X)$ the sheaf of nowhere-vanishing holomorphic functions. Neither of these two sheaves have cohomologies that match either of $H^\bullet(X,\mathbb{C})$ nor $H^\bullet(X,\mathbb{R})$; however the constant sheaf $\underline{\mathbb{Z}}$ does give the same cohomology as singular integer cohomology.

As exact sequences often do, this ones gives a cohomology long exact sequence

$$H^0(X,\mathbb{Z}) \longrightarrow H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^*) \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow$$

$$\to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \to \cdots$$

To see it probably you'd have to delve back into Čech, but $H^1(X,\mathcal{O}_X^*)$ is the same as $\mathrm{Pic}(X)$ (see [?], p. 133, you take transition functions of the line bundle, this is where the cocycle condition comes in with twofold meaning). So in the end a line bundle gets assigned the cohomology class of its curvature w.r.t the Chern connection, or what

And not only that but actually the intersection form is cup product of Chern classes

5 Intersection form on surfaces

See [?] defs. I.2, I.3.

6 Riemann-Roch

6.1 Genus

Definition ([?], p. 234) The arithmetic genus of a compact complex manifold X of dimension n is

$$p_a(X) := (-1)^n(\chi(\mathcal{O}_X) - 1)$$

Following

Definition ([?] chpt. 1, sec. 1, p.18) "Consider a curve $C \subset X$ in a smooth surface X. A priori, C is allowed to be singular, reducible, non-reduced, etc. The arithmetic genus of C is by definition

$$p_a(C) := 1 - \chi(C, \mathcal{O}_C).$$

And then [?] says:

> "For an arbitrary [so, possibly singular] reduced curve C the geometric genus is by definition the genus of the normalization $\nu : \tilde{C} \to C$, i.e. $g(C) := g(\tilde{C}$ [I wonder: is $g(\tilde{C}) = p_a(\tilde{C})$ i.e. arithmetic=geometric since $\tilde{C}$ is normal ($\approx$smooth)]. Thus, $p_a(C) = g(C) + h^0(\delta)$ [in which case we could also write $p_a(C) = p_a(\tilde{C}) + h^0(\delta)$]."

Upshot For singular curves arithmetic and geometric genus may not coincide:

$$\underbrace{p_a(C)}_{\text{arithmetic}} = \underbrace{g(C)}_{\text{geometric}} + h^0(\delta)$$

where $\delta = \nu_*(\mathcal{O}_{\tilde{C}}/\mathcal{O}_C$ "is concentrated in the singular points of C" (what is δ right?).

6.2 Genus in Harshorne

Definition ([?], p.180) X nonsingular projective variety, geometric genus of X is $p_g := \dim_k \Gamma(X, \omega_X)$, where $\omega_X = \Lambda^n(\Omega_{X/k})$ is the canonical sheaf/bundle.

Remark I think these sections $\Gamma(X, \omega_X)$ can also be written as $H^1(X, \mathcal{O}_X)$. (To me it's not obvious why.)

Definition (p. 54) The hilbert polynomial P_Y of a projective variety Y is the polynomial whose coefficients are the dimensions of every summand in the graded decomposition $\bigoplus S^i$ of $\mathcal{O}_X$ (using that Y is projective).

The arithmetic genus of Y is $p_a(Y) := (-1)^r(P_Y(0) - 1)$.

Remark In the case of a projective nonsingular curve, the arithmetic genus and the geometric genus coincide (by Serre duality). This may not be true in dimension $\geqslant 2$.

Proposition IV.1.1 If X is a curve, then

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X),$$

so we call this number simply the genus of X and denote it by g.

6.3 Euler characteristic

Definition ([?], p. 360) For any coherent sheaf $\mathcal{F}$,

$$\chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Remark Once upon a time, after a long discussion with Donaldson I convinced myself that $H^0(X, \mathcal{F})$ is the set of sections of the sheaf $\mathcal{F}$. (Not in person.)

6.4 For curves

6.4.1 with Lee

Recall (?) that the fundamental class of a manifold is the homology class of a triangulation of the manifold (triangulation exists), which is a top-dimensional simplicial chain. No really, for compact oriented manifolds the top homology group is a cyclic group and the fundamental class is an element that generates $H_n(M|x, R) \cong R$ for all x (thm. 3.26 [?] as indicated in [?], p. 165).

Then for any line bundle on M you can think of its Chern class, which is a 2-cochain and, if you are a Riemann surface, a real-2-dimension manifold, your fundamental class is a 2-chain and you can do

$$\langle c(L), [M] \rangle \stackrel{\text{def}}{=} c(L)([M]) := \deg L.$$

(Which is probably just the integral of the first Chern class of L!)

Riemann-Roch ([?]) Suppose M is a connected compact Riemann surface of genus g, and $L \to M$ is a holomorphic line bundle. Then

$$\dim \mathcal{O}(M; L) = \deg L + 1 - g + \dim \mathcal{O}(M; K \otimes L^*)$$

where $\mathcal{O}(M; E)$ is just the holomorphic sections of the vector bundle E.

Remark On the proof we get that

> "[...] and the Riemann-Roch theorem is equivalent to the claim that
>
> $$\chi(\mathcal{O}(L)) = \deg L + 1 - g."$$

which might be useful if you compare with Riemann-Roch for surfaces as in Misha's course.

6.4.2 with Hartshorne/wiki

According to wikipedia, $\ell(D)$ for a divisor D on a Riemann surface (D is a sum of points with some coefficients) is the set of all meromorphic functions h such that the coefficients of $(h) + D$ are non-negative.

Theorem IV.1.3 (Riemann-Roch) Let D be a divisor on a curve X of genus g. Them

$$\ell(D) - \ell(K - D) = \deg D - g + 1.$$

6.5 For surfaces

Theorem (K3 course) L line bundle on a surface and $K_X = \Omega^2(X)$ its canonical bundle. Then

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{(L - K_X, L)}{2}.$$

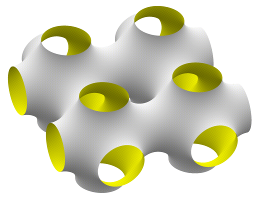
7 Noether's formula

From [?], I.14.

Noether's formula

8 Ampleness

> "In $\mathbb{C}^n$ many complex hypersurfaces can be written as regular level sets of globally defined holomorphic functions. But in a compact complex manifold, this is never possible, because all global holomorphic functions are constants. Instead, we can use sections of line bundles."



Riemann surfaces case

> A line bundle on a riemann surface is like its tangent bundle–a bunch of planes put somehow over the points of the surface. And a section is like a vector field. And the zeroes of the section are a bunch points. So the submanifold is a bunch of points.

8.1 The hyperplane bundle

Upshot (About the hyperplane bundle) Because the dual bundle is, of course, the bundle whose fiber at each point is the dual vector space of the original one. So hyperplane is because every linear functional of the dual space (an element of the fiber of the dual) is just a hyperplane.

And if you take a lot of tensor powers? You'll end up with the homogeneous degree d functions (which you should not think as hyperplanes, that's only for $d = 1$). (But most likely they are irreducible varieties... so is $\mathcal{O}(d)$ the "degree-d-variety bundle"?)

And that's why [?] Example 3.37 works: a homogeneous polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ gives me a section of $\mathcal{O}(d)$: at each point $\xi \in \mathbb{C}P^n$ I have the homogeneous degree d function that f is when restricted to ξ. (This is also very tautological.) (I would call this the tautological section.)

Notice that the tautological section is a section. That is, first you construct the hyperplane bundle saying that at every point you have the functionals on the line that the point is (so, at every point you put all the hyperplanes on the line that the point is). But also there's the tautological section that is itself a hyperplane. But only one hyperplane, you see? Because the section, which actually gives at every point a hyperplane, oh, just so many hyperplanes, can be tautologically and confusingly thought as a single hyperplane only, oh, just a single one, as this hyperplane isn't but a linear function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ which, when restricted to any line passing through the origin, oh, and lines are points, would give me the hyperplane that the section would give at the point. So yes, sections of the hyperplane bundle are degree 1-polynomials. And degree 1 polynomials are in fact hyperplanes of $\mathbb{C}P^n$: it's a hyperplane of $\mathbb{C}^{n+1}$, and then projectivizing you get a hyperplane of $\mathbb{C}P^n$. So a section of the hyperplane bundle is (also) just a hyperplane.

And a section of $\mathcal{O}(d)$ is just a degree d variety (=homogeneous degree d polynomial) I'm pretty sure by now.

Here's what I wrote when I first understood the hyperplane bundle:

In lem. 3.30 [?] we see that the d-th tensor power of the dual bundle of a line bundle L, a thing denoted by $(L^*)^d$, is naturally isomorphic to the bundle whose fiber at a point $p \in M$ is the space of functions $\varphi : L_p \to \mathbb{C}$ that are homogeneous of degree d, meaning that $\varphi(\lambda v) = \lambda^d \varphi(v)$ for all $\lambda \in \mathbb{C}$ and $v \in L_p$.

> which basically makes me understand that $\mathcal{O}(1) = \mathcal{O}(-1)^* =$ dual of the tautological bundle, is naturally isomorphic to the bundle whose fiber is the space of homogeneous functionals of degree 1. Which are hyperplanes. So that's why $\mathcal{O}(1)$ is called the hyperplane bundle.

8.2 Line bundle associated with a hypersurface?=?normal bundle

Theorem 3.39, [?] (Line bundle associated with a hypersurface) Save techincalities, we have that if $S \subseteq M$ is a closed complex hypersurface, there exists a line bundle $L_S \to M$ and a section $\sigma \in \mathcal{O}(M; L_S)$ that vanishes (simply) on S and nowhere else.

Nice example So the associated line bundle of a hypersurface determined by a homogeneous degree d polynomial is... the tautological section of this polynomial! (and the bundle is $\mathcal{O}(d)$).

8.3 Ampleness

Objective: when do global sections appear?

Story goes, the space of global sections of a holomorphic vector bundle on a compact complex manifold is finite dimensional. This is essentialy due to complex analysis. Montel's theorem is used to prove this in Lee. For a down-to-earth-illustrative point of view consider the line bundle of $\mathcal{O}_X$ of holomorphic functions on the the manifold (which is a bundle since it is a free rank-1 $\mathcal{O}_X$ module—the trivial bundle), which, if we remember, it's only constant functions because M is compact, so it's actually $\mathbb{C}$, which is finite-dimensional over $\mathbb{C}$. (But the corresponding thing in smooth function world is very infinite-dimensional.)

Anyway you can choose a basis $(s_0, \dots, s_m)$ of $\mathcal{O}(L)$. And then. you. construct. a. map. to. $\mathbb{P}^m$. As follows: choose a point $x \in M$, and take a local trivializaing open set of the bundle, where you have a local frame consisting of one local section (because it's a line bundle) called $s : U \to E(U)$. Then each of the global sections s_i is represented locally as $f_i s = s_i$. So you get this $\mathbb{C}$-valued functions, right? And then you get the map $p \mapsto (f_0(p), \dots, f_n(p))$. But that would depend on the trivializing open set. But if you think projectively you can show that it does not depend on the choice of frame.

Exercise You have 13 minutes to prove that.

Solution. The section s_i is a map $X \to W \times \mathbb{C}$. The transition function is an endomorphism of W, giving $s_i(p) = (p, v_i) \mapsto (p, \tau v_i$. Now if $s_i(p) = f_i(p)s(p)$ I get $s_i(p) \mapsto \tau(p)$, substitute...

Now I'm at home and I prove it like this. Just notice that $s_i = f_i s : W \to \pi^{-1}(W)$ (where of course $W \stackrel{\text{def}}{=} U \cap V$ is the intersection of two different trivializing charts and f_i is the function corresponding to the base $s \stackrel{\text{def}}{=} s_U$ of the chart U). Right so there is a transition function mapping $\pi^{-1}(W) \to \pi^{-1}(W)$ that is the identity on the base and a linear map on the fibers. Now, as I said above, since s_i is a function from W to the bundle, the transition function maps the vector part of the fiber w.r.t U to the vector part w.r.t. V by a linear transformation. Which is a number. So the equation we all want is

$$s = \xi s'$$

which gives

$$s_i = f_i s, \qquad s_i = g_i s' = g_i \xi s' \implies f_i = \xi g_i$$

and then the coordinates of this map we are defining become

$$[f_0, \dots, f_n] = [\xi g_0, \dots, \xi g_n] = [g_0, \dots, g_n]$$

so that the thing when defined projectively does not depend on the charts. □

Nice! So we see, that whenever $H^0(X, L)$ is not empty, we can construct a map from X to projective space. And the next question is whether this map is an embedding. And the answer is ampleness. L is very ample when it is an embedding.

8.4 Ampleness in Hartshorne

Here's the upshot about ampleness:

From II.7, subsection Ample Invertible Sheaves, p. 153:

> Now that we have seen that a morphism of a scheme X to a projective space can be characterized by giving an invertible sheaf on X and a suitable set of its global sections, [...]
>
> Recall that in §5 we defined a sheaf $\mathcal{L}$ on X to be very ample relative to Y if there is an immersion $i : X \to \mathbb{P}^n_Y$ for some n such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. In case $Y = \operatorname{Spec} A$, this is the same thing as saying that $\mathcal{L}$ admits a set of global sections $s_0, \dots, s_n$ such that the corresponding morphism $X \to \mathbb{P}^n_A$ is an immersion.
>
> We have also seen (5.17) that if $\mathcal{L}$ is a very ample invertible sheaf on a projective scheme X over a noetherian ring A, then for any coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ such that for all $n \geqslant n-$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. [...]
>
> Definition An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is said to be ample if for every coherent sheaf $\mathcal{F}$ on X there is an integer $n_0 > 0$ (depending on $\mathcal{F}$) such that for every $n \geqslant n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections. (Here $\mathcal{L}^n := \mathcal{L}^{\otimes n}$.)
>
> [...]
>
> Remark II.7.4.3 [...] we will see below (7.6) that if $\mathcal{L}$ is ample, then some tensor power $\mathcal{L}^m$ of $\mathcal{L}$ is very ample.

So, being very ample is just a term to hide the possibility of embedding the variety in a projective space and pulling back the hyperplane bundle to that line bundle. But it turns out that this is equivalent to being generated by global sections in some sense (maybe taking tensor product).

Example II.6.4 We will see later (IV, 3.3) that if D is a divisor on a complete nonsingular curve X, them $\mathscr{L}(\mathsf{D})$ is ample iff $\deg \mathsf{D} > 0$. This is a consequence of the Rimeann-Roch theorem.

Question What's up with the base-points?

To answer we move along to subsection Linear systems in Hartshorne.

> [...] global sections of an invertible sheaf correspond to effective divisors on a variety. Thus giving an invertibnle sheaf and a set of its global sections (which is related to finding the embedding of the variety into projective space pulling back the hyperplane bundle!) is the same as giving a certain set of effective divisors, all linearly equivalent to each other.

Idea (dani) That we can associate divisors to sections. Then we consider all linearly equivalent divisors to a given one (a linear system), which corresponds to a set of sections $\mathsf{V} \subseteq \Gamma(\mathsf{X}, \mathscr{L})$.

And then

Lemma II.7.8 [...] In particular, [a linear system] $\mathfrak{d}$ is base-point-free iff $\mathscr{L}$ is generated by the global sections in V.

9 Adjunction formula

9.1 Sergey

Today evening (17h) we discussed the adjunction formula, written sometimes as

$$\mathsf{K}_\mathsf{D} = \mathsf{K}_\mathsf{X} + \mathsf{D} \quad \text{or} \quad \mathsf{K}_\mathsf{Y} = (\mathsf{K}_\mathsf{X} + \mathsf{Y})|_\mathsf{Y}.$$

The formulation and proof are relatively straightforward for smooth hypersurfaces in smooth total spaces (manifold case). Some objects featured in the discussion include: tangent or cotangent bundles, conormal sheaf or normal bundle, and determinants of vector bundles.

We also discussed how to define determinants for complexes of vector bundles, and how to define determinants of coherent sheaves on smooth projective varieties by taking their resolution. More generally, the determinant is well-defined for objects of the bounded derived category of vector bundles, and behaves well in triangles.

This all follows from the formula for determinants of finite-dimensional vector spaces in a short exact sequence:

$$0 \longrightarrow \mathsf{U} \longrightarrow \mathsf{V} \longrightarrow \mathsf{W} \longrightarrow 0$$

In this case, we have a canonical isomorphism:

$$\det(V) \cong \det(U) \otimes \det(W),$$

and there is a quite explicit canonical isomorphism. Thanks to its canonicity, it globalizes to trivial vector bundles, and locally trivial vector bundles (so to all vector bundles), and then by standard homological algebra and K-theory to complexes of vector bundles.

In fact, the same standard K-theory argument shows that the isomorphism class of $\det(C)$ in $\mathrm{Pic}(X)$ depends only on the class $[C]$ of the complex in the K-group $\mathsf{K}(X)$.

One thing we have not (yet) discussed is how to prove that $N \cong \mathcal{O}(D)$.

More generally, we looked at the relation between the (co)normal sheaf $\mathcal{N}_{X/Y}$ and the ideal sheaf $\mathcal{I}_Y$.

Hint: the conormal sheaf is isomorphic to

$$\mathcal{I}_Y/\mathcal{I}_Y^2.$$

From this relation, the formula $N = \mathcal{O}_X(D)$ follows easily.

Exercise (dani puts this on April 8 2025) Show that canonical isomorphism of determinant bundles given an exact sequence. Solution at MSE: here.

9.2 Huybrechts

So what is the determinant bundle? See basic math section, but it is the top exterior power of the bundle, i.e. at every point you just put the top exterior power of the corresponding vector space.

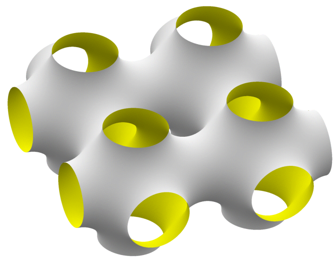
Proposition 2.2.17, [?] (Adjunction formula) Let Y be a submanifold of a complex manifold X. Then the canonical bundle K_Y of Y is naturally isomorphic to the line bundle $K_X|_Y \otimes \det(\mathcal{N}_{Y/X})$.

Proof. So you see, the proof is that exercise of the last section. □

There's more. Today I asked Bruno well how do you associate a divisor to a section of a line bundle? It's the zeros of the section! Now think of the normal bundle of the divisor. What is it?

Proposition 2.4.7 ([?]) Let X be a smooth hypersurface of a complex manifold Y defined by a section $s \in H^0(Y, L)$ of some holomorphic line bundle L on Y. Then $\mathcal{N}_{X/Y} \cong L_X$ and thus

How to think this?



Riemann surfaces case

> So take that Riemann surface and that line bundle and that section of that line bundle and that zero locus of that section of that line bundle on that Riemann surface, which is your divisor/submanifold. It's some points, just some points. The tangent space of some points is totally boring, it's some points again, so the normal space is all the space of the ambient manifold, so the line bundle again. Whoa.

proposition finishes: and thus $K_X \cong (K_Y \otimes L)|_X$.

Upshot I think the proof is essentially the canonical isomorphism of determinant sheaves of the normal exact sequence.

9.3 a formula from Hartshorne

This is really an application of Riemann-Roch for surfaces:

Proposition V.1.5 ([?]) If C is a nonsingular curve of genus g on the surface X and K is the canonical divisor on X, then

$$2g - 2 = C.(C + K).$$

9.4 Kodaira ampleness criterion

In K3 lecture 9 we have

Theorem (Kodaira) A bundle L is very ample iff $c_1(L)$ is a Kähler class.

Recall that a line bundle is prequantizable if its curvature is symplectic and integral.

9.5 Serre duality

Serre duality (dani notes Complex geometry course 2024.1)

$$H^k(X, \mathcal{L})^\vee = H^{n-k}(X, \omega_X \otimes \mathcal{L}^*)$$

Theorem II.5.32[?] (Serre duality) The pairing

$$H^q(X,\mathcal{E}) \otimes H^{n-q}(X,\mathcal{E}^* \otimes K_X) \to H^n(X,K_X) \cong \mathbb{C}$$

(probably given by some integral) between

$$H^q(X,\mathcal{E}) \qquad \text{and} \qquad H^{n-q}(X,\mathcal{E}^* \otimes K_X)$$

is perfect.

So when you put the dual $\vee$ on one of these you get isomorphism.

10 Positive $(1,1)$-forms

Any positive $(1,1)$ form looks like this: $\sum \alpha I x_i \wedge x_i$ for some positive functions $\alpha_i \geqslant 0$.

11 Hodge Index Theorem

Theorem V.1.9 (Hodge Index Theorem) Let

12 Kähler metric

The Kähler form is the differential of a plurisubharmonic function ψ. that is $\omega = dd^c\psi = \sqrt{-1}\partial\partial\psi$.

13 Picard group, Neron-Severi group

Definition (dani) Picard group is the group of line bundles with tensor product.

Remark See [?] p. 151 for the quick comment "We have seen (6.17) that $\operatorname{Pic}\mathbb{P}^n_k \cong \mathbb{Z}$ and is generated by $\mathcal{O}(1)$".

Definition ([?], p.5) Néron-Severy group of an algebraic surface X is the quotient

$$\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}_0(X)$$

by the connected component of the Picard variety $\mathrm{Pic}(X)$, i.e. by the subgroup of line bundles that are algebraically equivalent to zero (?).

Definition ([?], p.5)

$$\mathrm{Num}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$$

where $\mathrm{Pic}^\tau(X)$ is the subgroup of numerically trivial line bundles, i.e. line bundles L such that $(L.L') = 0$ for all line bundles L'. (E.g. any $L \in \mathrm{Pic}^0$ is numerically trivial.)

14 K3 surfaces

Definition (dani) A K3 surface is a complex surface with trivial canonical bundle and vanishing first (co)homology. (It's dimension 2 so 1st cohomology is first homology by Poincaré.)

Definition (K3 course) A K3 surface is a complex surface M with $b_1 = 0$ (Betti number is dimension of homology) and $c_1(M, \mathbb{Z}) = 0$. Recall that the Chern class coming from the exponential sequence is $\mathrm{Pic}(M) \xrightarrow{c_1} H^2(M, \mathbb{Z})$, so in particular it means that the first Chern class of the canonical bundle $K_X \in \mathrm{Pic}(M)$ is trivial, which in turn makes it trivial via Hodge theory.

Remark [?] shows that K3 surfaces have trivial (algebraic) fundamental group (what is that?) in remark 2.3.

Definition ([?]) A K3 surface over k is a complete non-singular variety X of dimension two such that

$$\Omega^2_{X/k} \cong \mathcal{O}_X \qquad \text{and} \qquad H^1(X, \mathcal{O}_X) = 0$$

Proposition 1.2.1 ([?]) NS(X) and its quotient Num(X) are finitely generated.

The rank of NS(X) is called the Picard number $\rho(X) := \mathrm{rk}\, \mathrm{NS}(X)$.

Proposition 1.2.4 ([?]) For a K3 surface X the natural surjections are isomorphisms (warning: the second isomorphism might not hold for general complex K3 surfaces):

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{Num}(X)$$

and the intersection pairing on Pic(X) is even, non-degenerate, and of signature $(1, \rho(X) - 1)$.

Remark ($\chi(K3, \mathcal{O}_X) = 2$) From [?] I.2.3, p.6:

> "For a K3 surface X one has by definition $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, O_X) = 0$. Moreover, by Serre duality $H^2(X, \mathcal{O}_X) \cong H^0(X, \omega_X)^*$ and hence $h^2(X, \mathcal{O}_X) = 1$. Therefore,
>
> $$\chi(X, \mathcal{O}_X) = 2."$$

So here's that computation using Serre duality:

$$H^0(X, \mathcal{O}_X) \overset{\substack{\text{Serre}\\\text{duality}}}{=} H^2(X, \mathcal{O}_X^* \otimes K_X)^* \overset{\substack{\text{dual}\\\text{of trivial}\\\text{is the}\\\text{trivial}}}{=} H^2(X, \mathcal{O}_X \otimes K_X)^* \overset{\substack{\text{you are}\\\text{a K3}}}{=} H^2(X, \mathcal{O}_X)^* = \mathbb{C}^* = \mathbb{C}$$

because the dual.

don't get confused Here's three things that are not the same:

1. $\chi(X, \mathcal{O}_X)$, the Euler characteristic of the structure sheaf $\mathcal{O}_X$.

2. $\chi(X)$, the Euler characteristic of a manifold. Probably $\sum(-1)^i \dim H_i(X, \mathbb{Z})$.
3. $\chi(X, K_X)$, the Euler characteristic of the line bundle K_X, because bundles have Euler characterstic too.

14.1 Kummer

Take a complex 2-torus A and consider the involution $\tau : [z_1, z_2] \mapsto -[z_1, z_2]$. It has 16 fixed points, so the quotient A/τ has singularities. Smooth it to obtain $\widehat{A}$ and consider the involution $\widehat{\tau}$. Then $\widehat{A}/\widehat{\tau}$ is a K3.

15 Fubini-Study

It's a metric, it's a symplectic form. Fubini-Study (symplectic) form is a closed 2-form defined on $\mathbb{C}P^n$ as the exterior differential of the logarithm of the length functions $\ell = \sum |z_i|^2$, i.e. $\omega = dd^c \log \ell$.

This also has a local expression in coordinates $(z_1, \ldots, z_n)$ that might be interesting.

The Fubini-Study metric is $g(\cdot, \cdot) = \omega(\cdot, I\cdot)$.

16 hypercomplex manifolds

Definition A manifold M is hypercomplex if it has three integrable almost complex structures I, J, K satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\mathrm{Id}$ and $IJ = -JI = K$.

16.1 Obata connection

Remark (Obata Connection, GPT) Given a hypercomplex manifold (M, I, J, K), there exists a unique torsion-free connection ∇^{ob} such that

$$\nabla^{\mathrm{ob}} I = \nabla^{\mathrm{ob}} J = \nabla^{\mathrm{ob}} K = 0.$$

This is called the Obata connection. Unlike the Levi-Civita connection, it is not necessarily compatible with a metric. Instead, it preserves the entire hypercomplex structure and serves as the natural connection in hypercomplex geometry.

16.2 SU(n)

How to think of SU(2) Take a real vector space V equipped with a hypercomplex structure and think of SU(2) as

$$\mathrm{SU}(2) = \{a + bI + cJ + dK : a^2 + b^2 + c^2 + d^2 = 1, \qquad a, b, c, d \in \mathbb{R}\}$$

Just because that's the way to go. (But at some point of terrenal mathematics you should prove that indeed, $\mathrm{SU}(2) \cong$ unit quaternions $\cong S^3$.)

But what's the lance?

"We are using these three complex structures (quaternionic is the term right? (it's not: this is called a hypercomplex structure) for their nice relations IJ=-JI=K) to model the division algebra H that is god-given. Then this thing which is a group acts on the space of endomorphisms of the quaternionic vector space by conjugation"

Yes—you're getting it beautifully now.

17 Fano

Exercise By Kodaira vanishing theorem, you can show that the cohomology $H^i(X, L)$ of a Fano variety X vanishes. You just have to put $L = \mathcal{O}(k)$ with $k \geqslant -r$ for the Fano index

$$r(X) = \min\{r : \frac{c_1(X)}{r} \in H^2(X, \mathbb{Z})\}.$$

Anyway the point is that $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ for Fano.

18 Appendices

18.1 basic smooth manifold theory

18.1.1 Slice charts and what is AG

At some point [?] defines analytic subvariety of a complex manifold to be $V \subset M$ such that $\forall p \in V$ there is a neighbourhood of M containing p where V looks like the zeros of a single holomorphic function. So that reminds us of:

Theorem (Slice charts for immersion, [?]) Suppose $F : M \to N$ is an immersion. Then there are charts $(\varphi, U), (\psi, V)$ such that

$$(\psi \circ F\varphi^{-1})(\vec{x}) = (x_1, \ldots, x_m, 0, \ldots, 0)$$

Proof. Just to record that the proof is not just inverse function theorem. You do this: pick any random charts $(\overline{\varphi}, \overline{U})$ and $(\overline{\psi}, \overline{V})$ and consider the Jacobian matrix of $\widehat{F} = \overline{\psi} \circ F \circ \overline{\varphi}^{-1}$. It must have a nonsingular submatrix of rank m (or of rank r if you go full constant tank theorem, but that would need some other manipulations...). Then you do:

1. Reorder the coordinates of the random charts we picked so that the $\widehat{F}$ looks like

$$\widehat{F} = (Q, R)$$

   so that dQ is nonsingular in the new chart. So you use inverse function theorem to put Q as a nice local diffeomorphism i.e. a chart. But you are not done:

2. Figure out how to turn Q into the identity.

3. Figure out how to turn R into zero.

And all that is some manipulations that you'd have to do if you had to teach a course on smooth manifolds, or doing such course (but at this point of the adventure most likely I won't be doing this course right?).

□

18.1.2 super technical: how to show vector fields are smooth

Sometimes we need to show something is smooth and we just say well, everything is smooth so the new thing is also smooth. Here's the formal way to show smoothness for vector fields:

Proposition 8.14 ([?]) A rough vector field $X :\to TM$ is smooth iff Xf is smooth for every $f \in C^\infty(M)$.

18.2 some riemannian geometry

18.2.1 why I can't understand riemannian geometry

Unrelated to complex geometry but a breakthrough in my long-standing philosophical search to understand why I can't understand riemannian geometry.

Definition 7.14 ([?]) Let E and F be vector bundles over a manifold M. An $\mathbb{R}$-linear map $T : \Gamma(E) \longrightarrow \Gamma(F)$ is a point operator if whenever two sections $s, s' \in \Gamma(E)$ coincide at a point p in M, $T(s)$ and $T(s')$ also coincide at p. So:

$$\forall s \in \Gamma(E) \text{ and } \forall p \in M, \quad s_p = s'_p \implies T(s)_p = T(s')_p$$

and actually (...exercise) that's equivalent to being a $\mathcal{F}(M)$ linear map, i.e. a tensor.

T is called a local operator if whenever two sections coincide in an open set U, so do $T(s)$ and $T(s')$ in that open set.

What Tensors let you do linear algebra globally, and local operators are derivations. Remember that you can take F to be trivial rank-1 vector bundle and you get $\Gamma(F)$ =functions so here's all the riemannian geometry.

Exercise Point operator iff $\mathcal{F}(M)$-linear.

Solution. ($\implies$) You pick a local trivialization of E, so that $s = \sum s^i e_i$, and then $T\left(\sum s^i e_i\right) = \sum s^i T(e_i)$ but since $s_p = 0$, you know that $s^i = 0$ for all i. (Rk. you have secretly used that point operator $\implies$ local operator, see [?].)

($\impliedby$) You wish that $T(fs) - fT(s)$ is the zero section of $\Gamma(F)$ i.e. that gives the zero vector at all p. Fix p. Define $s' := fs - f(p)s$. Gives zero at p. So, $T(s')_p = 0$. Then

$$0 = T(s')_p = T(fs - f(p)s)_p = T(fs)_p - f(p)T(s)_p = \Big(T(fs) - fT(s)\Big)_p .$$

□

Superpower Now you can define tensors pointwise.

Opinion we should work with bundles

18.2.2 $dx^i dx^j$ finally explained

The question is why is $dx^i \otimes dx^j$ not defined as an element of any tensor product of vector spaces. But that's OK because it is: it's an element of $T_p^*M \otimes T_p^*M$. But why is not defined like that.

It's defined as the bilinear map $T_pM \times T_pM \to \mathbb{R}$, $(v, w) \mapsto dx^i(v)dx^j(w)$. And what that has to do with $T_p^*M \otimes T_p^*M$. That by the universal product of tensor product any bilinear map $T_pM \times T_pM \to \mathbb{R}$ factors uniquely through a map $T_pM \otimes T_pM \to \mathbb{R}$, i.e. an element of $\mathrm{Hom}(T_pM \otimes T_pM, \mathbb{R})$. And you probably can prove that $\mathrm{Hom}(T_pM \otimes T_pM, \mathbb{R}) \cong \mathrm{Hom}(T_pM, \mathbb{R}) \otimes \mathrm{Hom}(T_pM, \mathbb{R}) \stackrel{\text{def}}{=} T_p^*M \otimes T_p^*M$.

So: there's a unique guy in the tensor product $T_p^*M \otimes T_p^*M$ that corresponds to "$dx^i \otimes dx^j$" defined as a bilinear map $T_pM \otimes T_pM \to \mathbb{R}$, $(v, w) \mapsto dx^i(v)dx^j(w)$.

And to conclude you just denote the symmetrization of $dx^i \otimes dx^j$, which was previously defined by $\mathrm{Sym}(dx^i \otimes dx^j) \stackrel{\text{def}}{=} \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$, by the symbol $dx^i dx^j$. And to conclude more you would prove that those guys are a basis for $\mathrm{Sym}^2 T_p^*M \stackrel{\text{def}}{=} (T_p^*M \otimes T_p^*M)/\langle x \otimes y - y \otimes x \rangle$. To conclude further bundlize this construction to get $\mathrm{Sym}^2 T^*M$, whose sections (that are pointwise nondegenerate) are the metrics and pseudometrics of M.

Trip what if that bundle is trivial? You get a global basis for the metric. GPT says: $\mathrm{Sym}^2 T^*M$ is trivial iff TM is. So in general you don't get global functions g_{ij} and you compute them locally every time.

18.2.3 never forget how to extend riemannian metric to other bundles

You pick an orthonormal base e_i of the tangent space and declare that the metric on forms will be the only one that makes the dual base ε_i orthonormal. And you check this is well defined. And then you do that for other vector bundles… hmm maybe take one or two more minutes to make sure this is right

18.2.4 volume form at once

First some facts about top-forms.

Fact 1 about top-forms Top-forms are determinants.

$$\varepsilon_1 \wedge \ldots \wedge \varepsilon_n(E_1, \ldots, E_n) = \det \varepsilon_i(E_j)$$

where ε_i are any 1-forms and E_i are any vector fields. (Notation is reminiscent of orthonormal frames but this is very general.) Note that for lower-degree forms there is also determinant-like formulas of the kind sum over permutations some products, they are not very nice and hopefully you don't need to use them anytime soon.

Now you need to know what is the pullback of top forms.

Pullback of top-forms A diffeomorphism $\varphi : M \to \widetilde{M}$, which is a diffeomorphism, it must be a diffeomorphism for these things to exist. (But eventually you want to integrate the surface, the world-sheet, in the ambient space X but easy.) So it's a diffeomorphism and you take local charts $(x_1, \dots, x_n)$ of M and $(y_1, \dots, y_n$ of $\widetilde{M}$.

You get a top-form on the chart $\widetilde{U}$ of $\widetilde{M}$ called $dy_1 \wedge \dots \wedge dy_n$. If you pull it back, you get a new top-form on U called $\varphi^*(dy_1 \wedge \dots \wedge dy_n$. Being top-forms, you know there's a function so that the new guy is that function times $dx_1 \wedge \dots \wedge dx_n$, which is also a top-form on U. Who's the function? The function is the determinand of the jacobian matrix of your map:

$$\varphi^*(dy_1 \wedge \dots \wedge dy_n) = \det D\varphi \, dx_1 \wedge \dots \wedge dx_n.$$

Now you want to know what happens in the case of the Volume forms. The volume forms can be exactly the guys from the past section if you happen to have orthonormal coordinates. So you get:

$$\varphi^* \operatorname{Vol}_{\widetilde{M}} = \det D\varphi \operatorname{Vol}_M .$$

Which is great. So you can integrate:

$$\int_M \varphi^*$$

OK out of time but if φ is an isometry the determinant of its differential is… 1. So volumes coincide. Yiipa!

18.2.5 jacobi field

Just because I finally understood something about Jacobi field. A geodesic is given by $\gamma_v(t) = \exp_p(tv)$ where $p = \gamma(0)$ and $v = \gamma'(0)$. Then choose a vector $w \in T_pM$ and consider these lines:

Then taking exp maps them to some geodesics. Then the Jacobi field in this case is given by differentiating with respect to s along the curve tv:

$$\begin{aligned} \nabla_s \gamma_{v+sw}(t) = \nabla_s \exp_p(t(v + sw)) &= d_{tv} \exp_p \cdot \frac{d}{ds}\Big(t(v + sw)\Big) \\ &= d_{tv} \exp_p(tw). \end{aligned}$$

And that's it.

18.3 some Lie group theory

Adjoint representation The adjoint representation of a Lie group G is a map $\mathrm{Ad} : G \to GL(\mathfrak{g})$ that represents G as a group of linear automorphisms of its Lie algebra. It is given by $G \ni a \mapsto (dC_a)_e$, the differential of the conjugation map C_a.

18.3.1 flow of left-invariant is right multiplication

First you notice that the flow φ of a left invariant vector field $u \in \mathfrak{g}$ commutes with left multiplication: $h \cdot \varphi_t(e) = \varphi_t(e) \cdot h$. Why? Because u is left invariant so $L_{h,*}u = u_h$. So when you multiply the flow by h on either side and differentiate you need to get to the vector u_h, because you'll be standing at h and you are the flow—the derivative of the flow is u (and at h it's u_h).

It's super confusing but the point is that

$$\varphi_t(h) = R_{\varphi_t(e)}(h)$$

18.4 some harmonic analysis

Claim Locally any harmonic function $u(x, y)$ is $u = \operatorname{Re} f$ for some holomorphic function f.

Proof. Idea: you need to construct $f = u + iv$ so really you want v. The point is that laplace and Cauchy riemann give you the way to construct v integrating something... □