8.1 Maximum principle

Maximum principle (E. Hopf) **(Class statement.)** For any (M, g), harmonic functions satisfy ***maximum principle***: any harmonic function $f : U \to \mathbb{R}$ for U connected and open, if $\exists p \in U$ that is a local maximum or local minimum, then f is constant on U.

We shall show Do Carmo's version that f ***subharmonic***, i.e. $\Delta f \geqslant 0$, on M compact connected $\implies f$ constant. (This is Exercise 12, Chapter III, [dC79], and all steps are just other exercises from the chapter of Geodesics :))

Proof.

Step 1 **(Exercise 7, Chapter III of [dC79].)** Make sure you can pick a ***referencial geodésico*** about every point of M. This is an orthonormal frame $\{E_i\} \subset \mathfrak{X}(U)$, where $p \in U$ such that $\left(\nabla_{E_i} E_j\right)_p = 0$. This frame can be obtained by taking geodesic coordinates at the point, an orthonormal base $\{e_i\}$ of T_pM, and taking parallel transport of the vectors e_i along radial geodesics emanating from p. This immediately ensures that E_i is orthonormal since parallel transport preserves angles.

To check that Christoffel symbols vanish at p we do as follows. (This is actually a basic fact about geodesic coordinates, see [Lee19] Prop. 5.24.) Take a random vector $v \in T_pM$ and its geodesic $\gamma_v(t) = \exp_p(tv)$. I drop the subindex v for the next computations for the next computations. Then (this is Florit way of using covariant derivative along a curve; it's the ***pullback*** or ***induced connection*** ∇^γ):

$$0 = \nabla^\gamma_{\frac{d}{dt}} \gamma' = \nabla^\gamma_{\frac{d}{dt}} v^i (E_i \circ \gamma)$$

where the $v = (v^1, \dots, v^n)$. Indeed: this is very silly but, since the coordinate chart of geodesic coordinates is $\exp_p^{-1}$, the coordinate representation of γ in this chart is as simple as

$$\hat{\gamma}(t) = (\underbrace{\varphi}_{\text{chart}} \circ \gamma)(t) = \exp_p^{-1} \exp_p(tv) = tv.$$

And the composition $E_i \circ \gamma$ just means that we take our local frame *along* γ. Continue:

$$\begin{aligned} &= v^i \nabla^\gamma_{\frac{d}{dt}} E_i \circ \gamma = v^i \nabla_{\gamma_{v,*} \frac{d}{dt}} E_i \\ &= v^i \nabla_{v^j E_j} E_i = v^i v^j \nabla_{E_j} E_i \\ &= v^i v^j \Gamma^k_{ji} E_k \end{aligned}$$

along γ. Now choose $v = e_1$. You get $\Gamma^k_{11} = 0$ for all k along γ_{e_1}. Now choose $v = e_2$, then $\Gamma^k_{22} = 0$ along γ_{e_2}, so at least at p they both vanish. And now choose $v = e_1 + e_2$. You get

$$0 = (v^1)^2 \cancelto{0}{\Gamma^k_{11}} + v^1 v^2 \Gamma^k_{12} + v^2 v^1 \Gamma^k_{21} + (v^2)^2 \cancelto{0}{\Gamma^k_{22}}$$

So $\Gamma^k_{12} = 0$ since Levi-Civita is torsion-free, i.e. symmetric. And so on. So the all Christoffel symbols vanish at the same time at p.

Step 2 **(Exercise 11, Chapter III of [dC79].)** Prove that

$$di_X \mathsf{Vol} = \mathsf{div}\, X\, \mathsf{Vol}$$

To do this first recall that ***divergence*** and ***trace*** are

$$\mathsf{div}\, X := \mathsf{tr}(\nu \mapsto \nabla_\nu X)$$

$$\mathsf{tr}(T) := \sum_i \langle TE_i, E_i \rangle, \qquad T \in \mathsf{End}(V), E_i \text{ orthonormal frame}$$

Now pick a Geodesic frame E_i and its dual coframe ε^i, i.e. satisfying $\varepsilon^i(E_j) = \delta_{ij}$. Then $\mathsf{Vol} = \varepsilon^1 \wedge \ldots \wedge \varepsilon^n$. Then for any $X = X^i E_i \in \mathfrak{X}(U)$,

$$i_X \mathsf{Vol} = \varepsilon^1 \wedge \ldots \wedge \varepsilon^n (X^i E_i, \cdot, \ldots, \cdot) = X^i \varepsilon^1 \wedge \ldots \wedge \varepsilon^n (E_i, \cdot, \ldots, \cdot)$$

How to compute that? Recall that for top-forms we have

$$\varepsilon^1 \wedge \ldots \wedge \varepsilon^n (Z_1, \ldots, Z_n) = \mathsf{det}(\varepsilon^i(Z_j))$$

so for example if $n = 3$

$$\begin{aligned}
\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3 (E_1, Z_2, Z_3) &= \begin{vmatrix} \varepsilon^1(E_1) & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ \varepsilon^2(E_1) & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ \varepsilon^3(E_1) & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix} = \begin{vmatrix} 1 & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ 0 & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ 0 & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix} \\
&= \varepsilon^2(Z_2)\varepsilon^3(Z_3) - \varepsilon^2(Z_3)\varepsilon^3(Z_2) = \varepsilon^2 \wedge \varepsilon^3(Z_2, Z_3), \\
\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3 (E_2, Z_2, Z_3) &= \begin{vmatrix} 0 & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ 1 & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ 0 & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix} = -\varepsilon^1 \wedge \varepsilon^3(Z_2, Z_3)
\end{aligned}$$

and so on. When we sum over all i, we get

$$X^i \varepsilon^1 \wedge \ldots \wedge \varepsilon^n (E_i, \cdot, \ldots, \cdot) = \sum_i (-1)^{i+1} X^i \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n.$$

Meta-remark (Victor and dani discussed Koszul complex yesterday) Here! This is nervously similar to the definition of Koszul differential. Looks like: fix a vector field, contract the volume form and then just keep contracting with that vector field. We should end up with a map from 1-forms to $\mathbb{R}$... but what is that map in this case? In Koszul complex this map was the *starting point* of the construction...

Now take exterior derivative of that, we get

$$\begin{aligned}
di_X \mathsf{Vol} &= \sum_i (-1)^{i+1} (dX^i) \wedge \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n \\
&+ \sum_i (-1)^{i+1} X^i \wedge d(\varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n)
\end{aligned}$$

And then the first term actually is

$$\sum_i (-1)^{i+1} E_j X^i \varepsilon^j \wedge \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n = E_i X^i \,\mathsf{Vol}$$

while the second term vanishes because look,

$$\begin{aligned} d\varepsilon^i(E_j, E_k) &= \underbrace{E_j \varepsilon^i(E_k) - E_k \varepsilon^i(E_j)}_{\text{must vanish}} - \varepsilon^i([E_j, E_k]) \\ &= -\varepsilon^i(\nabla_{E_j} E_k - \nabla_{E_k} E_j) \qquad \text{torsion!} \end{aligned}$$

which vanishes at p because we said that this geodesic frame would have vanishing Christoffel symbols at p. So we conclude:

$$d i_X \,\mathsf{Vol} = E_i X^i \,\mathsf{Vol}$$

Now you just have to think what is divergence:

$$\begin{aligned} \operatorname{div} X = \sum_i \left\langle \nabla_{E_i} X^j E_j, E_i \right\rangle &= \sum_i \left\langle E_i X^j E_j, E_i \right\rangle + X^j \left\langle \nabla_{E_i} E_j, E_i \right\rangle \\ &= \sum_i \left\langle E_i X^j E_j, E_i \right\rangle = \sum_i E_i X^j \left\langle E_j, E_i \right\rangle = E_i X^i \end{aligned}$$

again using that we are a geodesic frame with vanishing covariant derivative at p.

Step 3 **(Exercise 9(b), Chapter III of [dC79].)** You realise that

$$\Delta(fg) = f\Delta g + g\Delta f + 2\left\langle \nabla f, \nabla g \right\rangle.$$

Recall that ***Laplacian*** and ***gradient*** are

$$\Delta f := \operatorname{div} \nabla f \qquad \left\langle \nabla f, X \right\rangle := df X = Xf.$$

So this equality is just a computation no problem, I'll say how it starts. For any $X \in \mathfrak{X}(M)$,

$$\left\langle \nabla(fg), X \right\rangle = X(fg) = fXg + gXf = f\left\langle \nabla g, X \right\rangle + g\left\langle \nabla f, X \right\rangle.$$

which says that

$$\nabla(fg) = f\nabla g + g\nabla f$$

So, for an orthonormal frame E_i

$$\Delta(fg) = \operatorname{div} \nabla(fg) = \sum_i \left\langle \nabla_{E_i}(f\nabla g + g\nabla f), E_i \right\rangle$$

Then use Leibniz rule and definition of gradient, you get there.

Step 4 **(Exercise 12, Chapter III of [dC79].)** To prove the theorem for subharmonic functions first we show that in fact they are harmonic via step 2 on $X := \nabla f$ and Stokes:

$$\int_M \Delta f \,\mathsf{Vol} = \int_M \mathsf{div}\, X \,\mathsf{Vol} = \int_M d(i_X \,\mathsf{Vol}) = \int_{\partial M} i_X \,\mathsf{Vol} = 0$$

meaning that the non-negative function Δf is in fact 0, i.e. f is harmonic. Now we do it again for $X := \nabla(f^2/2)$:

$$\int_M \Delta(f^2/2) \,\mathsf{Vol} = \int_M d(i_X \,\mathsf{Vol}) = \int_{\partial M} i_X \,\mathsf{Vol} = 0$$

And then apply step 3:

$$0 = \int_M \Delta(f^2/2) \,\mathsf{Vol} = \int_M f\Delta f \,\mathsf{Vol} + \int_M \langle \nabla f, \nabla f \rangle \,\mathsf{Vol}$$

First one vanishes because f is harmonic, so second one is zero which says f is constant!

$\square$

References

[dC79] M.P. do Carmo. *Geometria Riemanniana*. Escola de geometria diferencial. Instituto de Matemática Pura e Aplicada, 1979.

[Lee19] John M. Lee. *Introduction to Riemannian Manifolds*. Graduate Texts in Mathematics. Springer International Publishing, 2019.