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github.com/danimalabares/riemann-surfaces

Home Assignment 8: Foliations

Definition. Let $B \subset TM$ be a involutive sub-bundle. The corresponding ***(smooth) foliation*** is a collection of immersed submanifolds $X \overset{\phi}{\hookrightarrow} M$ which are tangent to B everywhere, that is $\phi(T_x X) = B|_x$. These submanifolds are called ***leaves*** of the foliation, and B its ***tangent bundle***. Its ***rank*** is $\operatorname{rk} B$. Frobenius theorem claims that M is the union of all leaves of $\mathcal{F}$. The ***leaf space*** is the set of all leaves of $\mathcal{F}$ with the quotient topology (a set is open if its preimage in M is open).

Exercise 8.1. Construct a foliation on a simply connected compact manifold with all leaves non-compact.

Proof. Let us discard a few cases:

- By an argument related to the inexistence of nowhere vanishing vector fields, (also shown by Thurston, see StackExchange), there are no codimension-one foliations on spheres of even dimension.
- By Novikov's theorem, any codimension-one foliation on a 3-manifold with finite fundamental group must have a compact leaf.

Which leaves us with spheres of odd dimension greater than 3, or foliations that are not codimension-one.

After asking on StackExchange, a satisfactory answer was received citing Kuperberg, where a foliation on S^3 with noncompact orbits is constructed (thus giving a counterexample to a conjecture by Seifert). □

Exercise 8.2. Prove that $\mathbb{C}P^n$ does not admit a rank 1 foliation.

Proof. Suppose such foliation exists and consider the lifts of every submanifold:

$$\begin{tikzcd} & \mathbb{S}^{2n+1} \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, hook] & \mathbb{C}P^n \end{tikzcd}$$

which exist if we additionally suppose that X is contractible. □

Exercise 8.3. Let $\mathcal{F}$ be a foliation with compact leaves. Prove that its leaf space is Hausdorff.

Proof. I shall try to adapt the following familiar statement:

Proposition. If a Lie group acts continuously and properly on a manifold then the orbit space is Hausdorff. (***Properly*** means that the map $G \times M \to M \times M$ given by $(g, x) \mapsto (gx, x)$ is proper, ie. preimage of compact subset is compact.)

The conclusion follows from the general topology statement that

Proposition. Given any open quotient map $q : X \to Y$, the space Y is Hausdorff iff $X \times_Y X = \{(x_1, x_2) \in X \times X : q(x_1) = q(x_2)\}$ is closed in $X \times X$.

In our case, $M \times_{\mathcal{F}} M$ is the subset of $M \times M$ of pairs (x, y) such that x and y are in the same leaf.

Let's define a Lie group G acting properly and discontinuously on our foliated manifold M. Here might be a hint towards an answer... though it is still not clear to me what a G-structure is. $\square$

Exercise 8.4. Construct a rank 1 foliation with compact leaves on a 3-sphere M, such that the projection $M \to M/\mathcal{F}$ to its leaf space is not a smooth submersion.

Proof. The Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ induces a foliation of S^3 with fibers S^1. However, the projection onto the fiber space S^2 is indeed a smooth submersion. $\square$

Exercise 8.5. Find a foliation on a compact manifold with all leaves compact, and not all of them diffeomorphic.

Proof. (From StackExchange) Consider $S^2 \times I/\sim$ where we identify antipodal points in $S^2 \times \{0\}$ and antipodal points in $S^2 \times \{1\}$. It appears that this space is diffeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$. Then the fibers of the quotient map are a foliation with leaves S^2 on $S^2 \times (0, 1)$ and $\mathbb{R}P^2$ on the ends. $\square$

References

[1] Krystyna Kuperberg. "A Smooth Counterexample to the Seifert Conjecture". In: *Annals of Mathematics* 140.3 (1994), pp. 723–732. ISSN: 0003486X. URL: http://www.jstor.org/stable/2118623 (visited on 06/05/2024).