# Equivariant K-theory of the square of an n-step partial flag variety and a "q = 0" version of the affine quantum group.

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**Abstract** More than thirty years ago Beilinson, Lusztig, and MacPherson provided a geometric framework for quantum groups by considering a convolution product on the square of the n-step partial flag variety over finite fields. Since then there were several generalizations, in particular Ginzburg and Vasserot used equivariant K-theory of the Steinberg variety in the cotangent space of the square of the n-step flag variety to study the affine quantum group, with the quantum parameter corresponding to the dilation action in the cotangent direction. In a joint work with Sergey Arkhipov we use the convolution product on the equivariant K-theory of the square of the n-step flag variety itself to define and study a "q=0" version of the affine quantum group.

In this talk, I will define our "q=0" version of the affine quantum group via generators and relations, introduce the convolution algebra on the equivariant K-theory of the square of the n-step flag variety, and outline the construction of a surjective morphism from the "q=0" affine quantum group onto the convolution algebra.

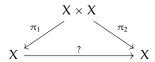
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# 1 Convolution product

## 1.1 Correspondence operators

Suppose we have some functor F and some space X. I want a morphism from X to itself but I dont have it and instead I have two projections from  $X \times X$ :



(The functor will be equivariant ...?  $K^G(X)$ .

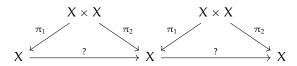
Anyway take a class  $\alpha \in F(X \times X)$ . And now for  $f \in F(X)$  fo

$$\phi_{\alpha}(f) = \pi_{2*}(\pi_1^*(f) \times \alpha$$

To define these correspondance operators we need

- Pullbacks
- Pushforward
- Product
- Projection formula that  $f_*(\beta) \times = f_*(\beta \times f^*\alpha)$

## 1.2 Composition



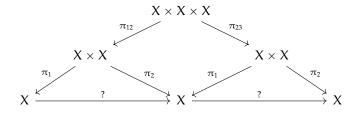
and we will do

$$\alpha$$
,  $\beta \in F(X \times X)$ ,  $f \in F(X)$ 

and the composition is

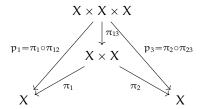
$$\phi_{\beta}\circ\phi_{\alpha})(f)=\pi_{2*}(\pi_1^*(\pi_{2*}(\pi_1^*(f\times\alpha))\times\beta)$$

And now



and then

$$\begin{split} \phi_{\beta} \circ \phi_{\alpha})(f) &= \pi_{2*}(\pi_{1}^{*}(\pi_{2*}(\pi_{1}^{*}(f \times \alpha)) \times \beta) \\ &= \pi_{2*}(\pi_{23*}(\pi_{12}^{*}(\pi_{1}^{*}f \times \alpha)) \times \beta) \\ &= \pi_{2*}(\pi_{23*}(\pi_{12}^{*}(\pi_{1}^{*}f \times \alpha) \times \pi_{23}^{*}\beta)) \end{split}$$



and then

$$\begin{split} p_{3*}(\pi_{12}^*(\pi_1^*f \times \alpha) \times \pi_{23}^*\beta) &= p_{3*}(\pi_{12}^*\pi_1^*f \times \pi_{12}^*\alpha \times \pi_{23}^*\beta) \\ &= \pi_{2*}(\pi_1^*f \times \pi_{13*}(\pi_{12}^*\alpha \times \pi_{23}^*\beta) \\ &= \phi_{\pi_{13*}(\pi_{12}^*\times\pi_{23}^*\beta)}(f) \end{split}$$

This means that the composition of two correspondances is a correspondance with repsect to this clsss.

Remark (Altan) Like the product of matrices.

Yes, essentially this is the product of matrices.

**Definition** (Convolution) a star
$$\beta := \pi_{13*}(\pi_{12}^* \alpha \times \pi_{23}^* \beta)$$

And we conclude that  $F(X \times X)$  is a convolution algebra acting on F(X) by corresponding operators.

# 2 Partial flags

$$\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}_{\geqslant 0}^n, \sum \mu_i=d,$$
  $d_k=\sum_{i=1}^k \mu_i$  and define

$$\mathsf{F}_{\mu} := \{\mathsf{U}_1 \subset \mathsf{U}_2 \subset \ldots \subset \mathsf{U}_n = \mathbb{C}^d : \mathsf{dim}\,\mathsf{U}_k = \mathsf{d}_k\}$$

where the U<sub>i</sub> are linear subspaces. Then take

$$X = F_n^d = \bigsqcup_{\mathfrak{u}} F_{\mathfrak{u}}$$

**Concept:** Convolution algebras on  $F_n^d \times F_n^d$  as  $d \to \infty$  should approximate the quantum group of  $\mathfrak{gl}(n)$ .

**Beautiful paper** *Geometric setting for the quantum deformation of* GL(n) by Beilinson, Lusztig, MacPherson.

Remark (Altan) These authors also have a paper where they use Hopf algebras to ...?

**Paper** Affine quantum groups and equivariant K-theory by Vasserot, 1998. Not the partial variety but the cotangent bundle  $X = T^*F_n^d$ .  $X \times X$  is a Steinberg subvariety,  $F = K^{GL(?) \times \mathbb{C}^*}$  and the  $\mathbb{C}^*$  gives a quantum variety.

**Definition** (Altan) Steinberg variety

$$St = T^* Fl \times_{N \times p} T^* Fl$$

where

$$Fl = GL(n)/B$$
 the flag variety  $T^*Fl$  the cotangent bundle

# 3 The algebra U(n)

Define  $U(n)^1$  first ( $\mathfrak{sl}(n)$  version).

- Generators:  $E_i(p)$ ,  $F_i(p)$ , 0 < i < n,  $p \in \mathbb{Z}$ .
- Realtions:

\*a lot of equations, descriptiono of this algebra\*

Our project:  $X = F_n^d$  over  $\mathbb{C}$ ,  $F = K^{Gl_d}$ . q = 0 degeneratorion of Vasserot.

Remark One cannot simply plug

**4** 
$$K^6(F_{\mu})$$

This is a homogeneous space, a quotient. So

$$F_{\mu} = Gl_d/P_{\mu}$$

where  $P_{\mu}$  is the parabolic subgroup.

In fact,

$$\begin{split} K^6(F_\mu) &= K^{P_\mu}(pt) = \mathbb{C}[x_1^\pm, \dots, x_d^\pm]^{S\,\mu} \\ S_\mu &= S_{\mu_1} \times \dots \times S_{\mu_n} \end{split}$$