Second Quantization

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Abstract The talk will cover a survey by Yuri Neretin of a famous eponymous book by Felix Berezin. Neretin's overview is short and clean, with no proofs (they are all exercises in functional analysis), providing an abundance of references.

The topics are: finite-dimensional and infinite-dimensional bosonic Fock space (Segal-Bargmann space (1961, 1963)) and fermionic Fock spaces, and the Weil representation of the orthogonal and symplectic groups.

The main theorem of the paper is that the representations do exist. Mostly it's about giving the right definitions. If we don't discuss the proofs, I can fit it under one hour.

Contents

1 Plan

1. Bossonic Fock space $F_n \cong L^2(\mathbb{R}^n)$. What is it, how to work with it. This will be the quantization of ordinary space $\mathbb{R}^2$.
2. Metaplectic representation (Weil representation) $\mathsf{Sp}(\mathbb{R})$ (symplectic matrices). The projective action $\mathsf{Sp}_{2n}(\mathbb{R}) \rightharpoonup F_n$. You will see how strange it is that this representation ?
3. Fermionic Fock space $\Lambda_n \cong \Lambda^\bullet(\mathbb{C}^n)$.
4. Spinor represenation. Fermionic Fock space also admits a projective representation called spinor represenation: $\mathsf{SO}(2n) \rightharpoonup \Lambda_n$.
5. $n \to \infty$. Gives rise to F_∞, Λ_∞.

Big formulas give rise to representations.

2 Fock space

Consider $\mathbb{C}^n$. define

$$F_n = \left\{ f(z)\text{-entire functions } \mathbb{C}^n \to \mathbb{C} \Big| \int_{\mathbb{C}^n} (f(x))^2 e^{-|z|^2} d\lambda(z) < \infty \right\}$$

We have an inner product:

$$\langle f, g \rangle := \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{|z|^2} d\lambda(z)$$

Proposition $z^k := z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$, where $k = (k_1, \ldots, k_n)$, form an orthogonal basis (of monomials) for F_n. Thus F_n is a Hilbert space. Moreover, $\|z^k\|^2 = k_1! \ldots k_n!$.

Question Why is this basis orthogonal— why do two different monomials integrate to zero?

Remark See *Harmonica analysis in phase space* by Folland (1988).

Definition $b_T(z) := \exp(zTz^+)$ for T symmetric.

Claim $b_T \in F_n \iff \|T\| < 1$.

Exercise $\langle b_T, b_S \rangle = \det((1 - TS^*)^{-\frac{1}{2}})$ **Hint:** Appendix 1 in Folland. This implies that

$$\|b_T\| = \det(1 - TT^*)^{-\frac{1}{4}} = \sqrt{\det(1 - TT^*)^{-\frac{1}{2}}}$$

Altan's talk exercise 2: Fock space is reproducing kernel space

Definition (Coherent states) For $a \in \mathbb{C}^n$,

$$\varphi_a(z) := \exp(z_1\overline{a_1} + \ldots + z_n\overline{a_n})$$

Claim For $f \in F_n$,

$$\langle f, \varphi_a \rangle = f(a)$$

Remark This says that for every $a \in \mathbb{C}^n$ the functional $F_n \ni f \mapsto f(z) \in \mathbb{C}$ is bounded since it is represented by φ_a (Riesz representation theorem). This is called ***Reproducing Kernel Hilbert Space***.

Proof (Exercise). Let's consider the case $n = 1$. Then $\varphi_a = \exp(z\bar{a})$ and

$$\langle f, \varphi_a \rangle = \int_{\mathbb{C}} f(z)\overline{\exp(za)}e^{-|z|^2}\,d\lambda(z)$$

But I'm not sure how to compute this integral. Upon difficulties let's suppose $f \equiv 1$. Perhaps split complex exponential into real and imaginary parts? Then we get real-valued integrals over $\mathbb{R}^2 = \mathbb{C}$:

$$\begin{aligned}\int_{\mathbb{C}} \overline{\exp(za)}e^{-|z|^2}\,d\lambda(z) &= \int_{\mathbb{R}^2} \overline{e^{\operatorname{Re}(za)}\big(\cos\operatorname{Im}(za) + i\operatorname{Im}(za)\big)}e^{-|z|^2}\,d\lambda(z) \\ &= \int_{\mathbb{R}^2} \overline{e^{\operatorname{Re}(za)}\cos\operatorname{Im}(za)}e^{-|z|^2}\,d\lambda(z) + \int_{\mathbb{R}^2} \overline{e^{\operatorname{Re}(za)}i\operatorname{Im}(za)}e^{-|z|^2}\,d\lambda(z)\end{aligned}$$

Not sure...

In the general case we shall have

$$\begin{aligned}\langle f, \varphi_a \rangle &= \int_{\mathbb{C}^n} f(z)\overline{\varphi_a(z)}e^{-|z|^2}\,d\lambda(z) \\ &= \int_{\mathbb{C}^n} f(z)\overline{\exp(z_1\bar{a}_1 + \ldots + z_n\bar{a}_n)}e^{-|z|^2}\,d\lambda(z)\end{aligned}$$

Maybe take this somehow to a line integral around a to Cauchy integral formula...? $\square$

Remark Evaluation at a point is a bounded functional (this is not the case in general L^2 space because you can have functions with arbitrarily large values).

Question What are the operators $A \curvearrowright F_n$?

Answer $(Af)z = \int_{\mathbb{C}^n} k(z, \bar{u}) f(u) e^{-|u|^2} d\lambda(u)$, where $K(z, u)$ is holomorphic in z and antiholomorphic in u. The integral converges absolutely for all $f \in F_n$.

Proof. We need to compute the kernel. So define

$$c_k e = \left\langle A z^{\mathbf{k}}, z^{\ell} \right\rangle$$

then

$$k(z, \bar{u}) = \sum_{k,\ell} c_{k\ell} \frac{z^k}{k!} \frac{\bar{u}^{\ell}}{\ell!}$$

which means that the kernel is symmetric. So this means that restricting the kernel to diagonal is not holomorphic anymore, kind of real analytic (?).

Alternatively, $k(a, b) = \langle A\varphi_b, \varphi_a \rangle$. □

Question (Dani) Why do we need to compute the kernel? We want to show that the operator $A \curvearrowright F_n$ produces a (projective) action, right?

OK so we defined Fock space and the operators that act nicely on it.

3 Metaplectic representation

Definition

$$\mathsf{Sp}(2n, \mathbb{R} := \left(h \in \mathrm{Mat}_{2n \times 2n} : h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h^{+} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

Let

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad g := JgJ^{-1}$$

$$g = \begin{pmatrix} Q & \psi \\ \bar{\psi} & \bar{\phi} \end{pmatrix}, \qquad g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^{*} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Also

$$\mathsf{Sp}(2n, \mathbb{R}) \cong \mathsf{U}(n, n) \cap \mathsf{Sp}(2n, \mathbb{C})$$

Now

$$\left(W \begin{pmatrix} \phi & \psi \\ \bar{\psi} & \bar{\phi} \end{pmatrix} f \right)(z) = \int_{\mathbb{C}^n} \exp\left\{ \frac{1}{2} (z, \bar{u}) \begin{pmatrix} \bar{\psi}\phi^{-1} & (\phi^t)^{-1} \\ \phi^{-1} & -\phi^{-1}\psi \end{pmatrix} \begin{pmatrix} z^t \\ \bar{u}^t \end{pmatrix} \right\} f(u) e^{-|u|^2} d\lambda(u).$$

Who is this guy? It's a ***Berezin's integral***. So that is the definition of W, which is the representation of the mataplectic group $W : \mathsf{Sp}_{2n}(\mathbb{R}) \rightharpoondown \mathsf{F}_n$ we wanted.

Now let's explain what is a projective representation.

Theorem

1. $W(\cdot)$ are unitary up to scalar. More precisely,

$$\det(\phi^*\phi)^{-\frac{1}{4}} W \begin{pmatrix} \phi & \psi \\ \bar{\psi} & \bar{\phi} \end{pmatrix}$$

   are unitary.

2. $W(\cdot)$ define a projective representation of $\mathsf{Sp}_{2n}(\mathbb{R}) \rightharpoondown \mathsf{F}_n$.

   Now let's finish with this nice formula for the cocycle (how to multiply these operators?:

$$W \begin{pmatrix} \phi_1 & \psi_1 \\ \bar{\psi}_1 & \bar{\phi}_1 \end{pmatrix} W \begin{pmatrix} \phi_2 & \psi_2 \\ \bar{\psi}_2 & \bar{\phi}_2 \end{pmatrix} = \det(1 + \phi_1^{-1}\psi_1\bar{\psi}_2\phi_2^{-1})^{-\frac{1}{2}} W \left(\begin{pmatrix} \phi_1 & \psi_1 \\ \bar{\psi}_1 & \bar{\phi}_1 \end{pmatrix} \right)$$

Question (Dani) So what is projective representation?