

Equivariant K-theory of the square of an n -step partial flag variety and a " $q = 0$ " version of the affine quantum group.

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Abstract More than thirty years ago Beilinson, Lusztig, and MacPherson provided a geometric framework for quantum groups by considering a convolution product on the square of the n -step partial flag variety over finite fields. Since then there were several generalizations, in particular Ginzburg and Vasserot used equivariant K-theory of the Steinberg variety in the cotangent space of the square of the n -step flag variety to study the affine quantum group, with the quantum parameter corresponding to the dilation action in the cotangent direction. In a joint work with Sergey Arkhipov we use the convolution product on the equivariant K-theory of the square of the n -step flag variety itself to define and study a " $q=0$ " version of the affine quantum group.

In this talk, I will define our " $q=0$ " version of the affine quantum group via generators and relations, introduce the convolution algebra on the equivariant K-theory of the square of the n -step flag variety, and outline the construction of a surjective morphism from the " $q=0$ " affine quantum group onto the convolution algebra.

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1 Convolution product

1.1 Correspondence operators

Suppose we have some functor F and some space X . I want a morphism from X to itself but I don't have it and instead I have two projections from $X \times X$:

$$\begin{array}{ccc} & X \times X & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ X & \xrightarrow{\quad ? \quad} & X \end{array}$$

(The functor will be equivariant ...? $K^G(X)$).

Anyway take a class $\alpha \in F(X \times X)$. And now for $f \in F(X)$ fo

$$\varphi_\alpha(f) = \pi_{2*}(\pi_1^*(f) \times \alpha)$$

To define these correspondance operators we need

- Pullbacks
- Pushforward
- Product
- Projection formula that $f_*(\beta) \times = f_*(\beta \times f^*\alpha)$

1.2 Composition

$$\begin{array}{ccccc} & X \times X & & X \times X & \\ \pi_1 \swarrow & & \searrow \pi_2 & \pi_1 \swarrow & \searrow \pi_2 \\ X & \xrightarrow{\quad ? \quad} & X & \xrightarrow{\quad ? \quad} & X \end{array}$$

and we will do

$$\alpha, \beta \in F(X \times X), \quad f \in F(X)$$

and the composition is

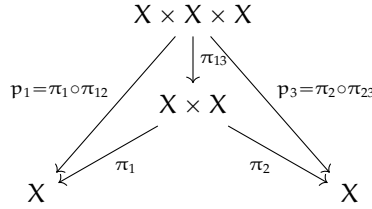
$$\varphi_\beta \circ \varphi_\alpha(f) = \pi_{2*}(\pi_1^*(\pi_{2*}(\pi_1^*(f \times \alpha)) \times \beta)$$

And now

$$\begin{array}{ccccc} & X \times X \times X & & & \\ & \pi_{12} \swarrow & & \searrow \pi_{23} & \\ & X \times X & & X \times X & \\ \pi_1 \swarrow & & \searrow \pi_2 & \pi_1 \swarrow & \searrow \pi_2 \\ X & \xrightarrow{\quad ? \quad} & X & \xrightarrow{\quad ? \quad} & X \end{array}$$

and then

$$\begin{aligned}
\varphi_\beta \circ \varphi_\alpha(f) &= \pi_{2*}(\pi_1^*(\pi_{2*}(\pi_1^*(f \times \alpha)) \times \beta)) \\
&= \pi_{2*}(\pi_{23*}(\pi_{12}^*(\pi_1^*f \times \alpha)) \times \beta) \\
&= \pi_{2*}(\pi_{23*}(\pi_{12}^*(\pi_1^*f \times \alpha) \times \pi_{23}^*\beta))
\end{aligned}$$



and then

$$\begin{aligned}
p_{3*}(\pi_{12}^*(\pi_1^*f \times \alpha) \times \pi_{23}^*\beta) &= p_{3*}(\pi_{12}^*\pi_1^*f \times \pi_{12}^*\alpha \times \pi_{23}^*\beta) \\
&= \pi_{2*}(\pi_1^*f \times \pi_{13*}(\pi_{12}^*\alpha \times \pi_{23}^*\beta)) \\
&= \varphi_{\pi_{13*}(\pi_{12}^*\alpha \times \pi_{23}^*\beta)}(f)
\end{aligned}$$

This means that the composition of two correspondances is a correspondance with respect to this class.

Remark (Altan) Like the product of matrices.

Yes, essentially this is the product of matrices.

Definition (Convolution) a star $\beta := \pi_{13*}(\pi_{12}^*\alpha \times \pi_{23}^*\beta)$

And we conclude that $F(X \times X)$ is a convolution algebra acting on $F(X)$ by corresponding operators.

2 Partial flags

$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$, $\sum \mu_i = d$, $d_k = \sum_{i=1}^k \mu_i$ and define

$$F_\mu := \{U_1 \subset U_2 \subset \dots \subset U_n = \mathbb{C}^d : \dim U_k = d_k\}$$

where the U_i are linear subspaces. Then take

$$X = F_n^d = \bigsqcup_{\mu} F_\mu$$

Concept: Convolution algebras on $F_n^d \times F_n^d$ as $d \rightarrow \infty$ should approximate the quantum group of $\mathfrak{gl}(n)$.

Beautiful paper *Geometric setting for the quantum deformation of $GL(n)$* by Beilinson, Lusztig, MacPherson.

Remark (Altan) These authors also have a paper where they use Hopf algebras to ...?

Paper *Affine quantum groups and equivariant K-theory* by Vasserot, 1998. Not the partial variety but the cotangent bundle $X = T^*F_n^d$. $X \times X$ is a Steinberg subvariety, $F = K^{GL(?) \times \mathbb{C}^*}$ and the \mathbb{C}^* gives a quantum variety.

Definition (Altan) *Steinberg variety*

$$St = T^*Fl \times_{N \times p} T^*Fl$$

where

$Fl = GL(n)/B$ the flag variety

T^*Fl the cotangent bundle

3 The algebra $U(n)$

Define $U(n)^1$ first ($\mathfrak{sl}(n)$ version).

- Generators: $E_i(p), F_i(p), 0 < i < n, p \in \mathbb{Z}$.

- Relations:

a lot of equations, description of this algebra

Our project: $X = F_n^d$ over \mathbb{C} , $F = K^{GL_d}$. $q = 0$ degeneration of Vasserot.

4 $K^6(F_\mu)$

This is a homogeneous space, a quotient. So

$$F_\mu = GL_d / P_\mu$$

where P_μ is the parabolic subgroup.

In fact,

$$K^6(F_\mu) = K^{P_\mu}(pt) = \mathbb{C}[x_1^\pm, \dots, x_d^\pm]^{S_\mu}$$

$$S_\mu = S_{\mu_1} \times \dots \times S_{\mu_n}$$