WDVV equations as bi-Hamiltonian integrable systems

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Contents

Witten–Dijkgraaf–Verlinde–Verlinde equations

(Dubrovin, Encyclopaedia of Math. Phys. 2006)
The problem: in $\mathbb{R}^N$ find a function $F = F(t^1, \ldots, t^N)$ such that

1. $F_{1\alpha\beta} := \dfrac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$ constant symmetric nondegenerate matrix
2. $c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} F_{\epsilon\alpha\beta}$ structure constants of an associative algebra
3. $F(c^{d_1} t^1, \ldots, c^{d_N} t^N) = c^{d_F} F(t^1, \ldots, t^N)$ quasihomogeneity $(d_1 = 1)$

If $e_1, \ldots, e_N$ is the basis of $\mathbb{R}^N$ then the algebra operation is

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(\mathbf{t}) e_\gamma \quad \text{with unity} \quad e_1$$

Why study WDVV?

1. The origin: Topological Quantum Field Theory
   - E. Witten. On the structure of the topological phase of two-dimensional gravity. *Nuclear Physics B*, 340(2):281–332, 1990, DOI: `https://doi.org/10.1016/0550-3213(90)90449-N`.
   - R. Dijkgraaf, H. Verlinde, and E. Verlinde. Topological strings in $d < 1$. *Nuclear Physics B*, 352(1):59–86, 1991, DOI: `https://doi.org/10.1016/0550-3213(91)90129-L`.
2. Mathematical developments: solutions of WDVV are related with Gromov–Witten invariants
   - M. Kontsevich, Yu. Manin. Gromov-Witten Classes, Quantum Cohomology, and Enumerative Geometry, Commun. Math. Phys. 164, 525-562 (1994).
   - I. Strachan. How to count curves: from 19th century problems to 21st century solutions. `https://www.maths.gla.ac.uk/~iabs/Visions.pdf`

Why study WDVV?

3. Solutions correspond to integrable hierarchies (B. Dubrovin)
   - B.A. Dubrovin. Geometry of 2D topological field theories. In *Integrable systems and quantum groups*, volume 1620 of *Lect. Notes Math.*, pages 120–348. Springer, Berlin, Heidelberg, 1996, arXiv: `https://arxiv.org/abs/hep-th/9407018`.
   - B. Dubrovin. Flat pencils of metrics and Frobenius manifolds. In M.-H. Saito, Y. Shimizu, and K. Ueno, editors, *Proceedings of 1997 Taniguchi Symposium "Integrable Systems and Algebraic Geometry"*, pages 42–72. World Scientific, 1998. `https://people.sissa.it/~dubrovin/bd_papers.html`.
   - B.A. Dubrovin. *Encyclopedia of Mathematical Physics*, volume 1 A: A-C, chapter WDVV equations and Frobenius manifolds, pages p. 438–447. SISSA, Elsevier, 2006. ISBN: 0125126611.

Recent research on WDVV

1. Supersymmetric Quantum Mechanics:
   - G. Antoniou and M. Feigin. Supersymmetric V-systems. *Journal of High Energy Physics*, 2019(2):115, Feb 2019, DOI: `https://doi.org/10.1007/JHEP02(2019)115`.
   - A. Galajinsky and O. Lechtenfeld. Superconformal $su(1, 1|n)$ mechanics. *Journal of High Energy Physics*, 2016(9):114, Sep 2016, DOI: `https://doi.org/10.1007/JHEP09(2016)114`.
   - N. Kozyrev, S. Krivonos, O. Lechtenfeld, and A. Sutulin. $Su(2|1)$ supersymmetric mechanics on curved spaces. *Journal of High Energy Physics*, 2018(5):175, May 2018, DOI: `https://doi.org/10.1007/JHEP05(2018)175`.
2. Topological quantum field theory
   - O.B. Gomez and A. Buryak. Open topological recursion relations in genus 1 and integrable systems. *Journal of High Energy Physics*, 2021(1):48, Jan 2021, DOI: `https://doi.org/10.1007/JHEP01(2021)048`.

Recent research on WDVV

3. String theory
    - A.A. Belavin and V.A. Belavin. Frobenius manifolds, integrable hierarchies and minimal Liouville gravity. *Journal of High Energy Physics*, 2014(9):151, Sep 2014, DOI: `https://doi.org/10.1007/JHEP09(2014)151`.
    - Xiang-Mao Ding, , Yuping Li, and Lingxian Meng. From r-spin intersection numbers to hodge integrals. *Journal of High Energy Physics*, 2016(1):15, Jan 2016, DOI: `https://doi.org/10.1007/JHEP01(2016)015`.
4. Supersymmetric gauge theory
    - H. Jockers and P. Mayr. Quantum K-theory of Calabi–Yau manifolds. *Journal of High Energy Physics*, 2019(11):11, Nov 2019, DOI: `https://doi.org/10.1007/JHEP11(2019)011`.

Recent research on WDVV

5. Topological Recursion
   - B. Eynard, Topological expansion for the 1-hermitian matrix model correlation functions, JHEP/024A/0904, hep-th/0407261
   - L. Chekhov, B. Eynard, N. Orantin, Free energy topological expansion for the 2-matrix model, JHEP 0612 (2006) 053, math-ph/0603003
   - B. Eynard, A short overview of the "Topological recursion", math-ph/arXiv:1412.3286.

WDVV equations

The system of associativity equations, also known as WDVV equations, follows:

$$
\begin{aligned}
S_{\alpha\beta\gamma\nu} =& \eta^{\mu\lambda}\Big(\frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta}\frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma} - \frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu}\frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma}\Big)\\
=& \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\gamma\nu} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0.
\end{aligned}
$$

The dependence of the function F on t^1 is completely specified by the requirement $F_{1\alpha\beta} = \eta_{\alpha\beta}$:

$$
F = \frac{1}{6}\eta_{11}(t^1)^3 + \frac{1}{2}\sum_{k>1}\eta_{1k}t^k(t^1)^2 + \frac{1}{2}\sum_{k,s>1}\eta_{sk}t^s t^k t^1 + f(t^2,\dots,t^N).
$$

so that the WDVV system is an overdetermined system of non-linear PDEs on one unknown function $f = f(t^2,\dots,t^N)$.

Digression: Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \dots) = 0$$

admits a Hamiltonian formulation if there exist A, $\mathcal{H} = \int h\, dx$ such that

$$u^i_t = A^{ij}\left(\frac{\delta \mathcal{H}}{\delta u^j}\right), \quad \text{with} \quad \frac{\delta \mathcal{H}}{\delta u^j} = (-1)^\sigma \partial_\sigma \frac{\partial h}{\partial u^j_\sigma}$$

where $A = (A^{ij})$ is a Hamiltonian operator (Poisson tensor), i.e. a matrix of differential operators $A^{ij} = A^{ij\sigma}\partial_\sigma$, where $\partial_\sigma = \partial_x \circ \cdots \circ \partial_x$ (total x-derivatives σ times), with further properties.

Hamiltonian operators

A is a Hamiltonian operator if and only if

$$\{F,G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij\sigma} \partial_\sigma \frac{\delta G}{\delta u^j}\, dx$$

is a Poisson bracket (skew-symmetric and Jacobi).

$\{,\}_A$ is a Poisson bracket if and only if:

- A is skew-adjoint: $A^* = -A$, where

$$A^*(\psi)^j = (-1)^\sigma \partial_\sigma \left(A^{ij\sigma} \psi_i \right)$$

- The variational Schouten bracket vanishes:

$$[A,A](\psi^1,\psi^2,\psi^3) =$$
$$2\left[\frac{\partial A^{ij\sigma}}{\partial u^l_\tau} \partial_\sigma(\psi^1_j) \partial_\tau (A^{lk\mu} \partial_\mu(\psi^2_k)) \psi^3_i + \mathrm{cyclic}(1,2,3) \right] = 0$$

(the r.h.s. is defined up to total derivatives $\partial_x(B)$).

Example: the Korteweg–de Vries equation

The equation:

$$u_t = uu_x + u_{xxx}$$

The bi-Hamiltonian formalism:

$$A_1 = \partial_x, \qquad A_2 = \frac{1}{3}u_x + \frac{2}{3}u\partial_x + \partial_{xxx}$$

with Hamiltonians:

$$H_1 = \frac{u^3}{6} + \frac{u_x^2}{2}, \qquad H_2 = \frac{u^2}{2}$$

Fundamental discoveries:

- KdV as a Hamiltonian system through A_1 (Zakharov, Faddeev '70);
- KdV as a bi-Hamiltonian system through A_1, A_2 (Magri '78);

Motivation for Hamiltonian PDEs

- A Hamiltonian operator maps *conservation laws* to *symmetries*.
- Two compatible Hamiltonian operators A_1, A_2 generate a sequence of conserved quantities (Magri, JMP 1978):

$$A_1\left(\frac{\delta H_{n+1}}{\delta u^i}\right) = A_2\left(\frac{\delta H_n}{\delta u^i}\right).$$

- Integrability: the above sequence H_1,H_2, $\ldots$,H_n,$\ldots$ is in involution:

$$\{H_i, H_j\} = 0.$$

- There is **no** analogue of Liouville theorem for PDEs, but integrable nonlinear equations usually are C-integrable or S-integrable (Calogero 1980).
- Bi-Hamiltonian systems and their hierarchy.

WDVV equations and Hamiltonian PDEs

(B. Dubrovin, '90) Let F be a solution of WDVV equations with homogeneity degrees d_1,... d_N. Let us set

$$c_{\beta}^{\delta\gamma} = \eta^{\delta\alpha}\eta^{\gamma\epsilon}\frac{\partial^3 F}{\partial t^\epsilon \partial t^\alpha \partial t^\beta}.$$

Then, the two operators

$$A_1 = \eta^{ij}\partial_x, \qquad A_2 = g^{ij}\partial_x + \Gamma_k^{ij}u_x^k$$

where, after replacing $t^k \to u^k$:

$$g^{ij} = c_k^{ij} d_k u^k$$

are Hamiltonian and compatible $[A_1, A_2] = 0$, hence they define an integrable system of PDEs of the form $u_t^i = V_j^i u_x^j$.

WDVV equations in detail

Two canonical forms by linear transformations of $(t^2,\ldots,t^N)$, if the weights d_i are distinct (Dubrovin, LNM 1996):

$d_F \neq 3$: By linear transformations preserving e_1:

$$\eta^{(1)}_{\alpha\beta} = \delta_{\alpha+\beta,N+1} = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

$$F = \tfrac{1}{2}(t^1)^2 t^N + \tfrac{1}{2}t^1 \sum_{\alpha=2}^{N-1} t^\alpha t^{N-\alpha+1} + f(t^2,\ldots,t^N);$$

$d_F = 3$: By linear transformations preserving e_1:

$$\eta^{(2)}_{\alpha\beta} = \begin{pmatrix} \mu & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \tag{1}$$

$$\mu \neq 0,\ F = \tfrac{\mu}{6}(t^1)^3 + \tfrac{1}{2}t^1 \sum_{\alpha=2}^{N} (t^\alpha)^2 + f(t^2,\ldots,t^N).$$

Simplest example: WDVV in the case $N = 3$

If $N = 3$ we have a single equation on $f = f(t^2, t^3) = f(x, t)$.
Two cases:

- $\eta^{(1)}_{\alpha\beta}$:
$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$$
- $\eta^{(2)}_{\alpha\beta}$:
$$f_{ttt} = \frac{-f_{xxt}^2 + f_{xxx} f_{xtt} + \mu f_{xtt}^2}{\mu f_{xxt} - 1}$$

Example of solution of WDVV equations

- The number of algebraic curves of degree n passing through $3n-1$ generic points in the projective plane $\mathbb{P}^2$:

$$N(n) = \sum_{i+j=n} \left(i^2 j^2 \binom{3n-4}{3i-2} - i^3 j \binom{3n-4}{3i-1} \right) N(i) N(j) \quad (2)$$

  is obtained from a solution of $f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$, the Frobenius potential F (free energy) for the projective plane.
- no applications of the other canonical form (?)

WDVV equations as quasilinear systems of first-order PDEs

Construction by O. Mokhov (1995). Let us introduce coordinates

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}.$$

Then the compatibility conditions in the two cases are

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = (b^2 - ac)_x \end{cases} \qquad \text{and} \qquad \begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = \left(\frac{ac - b^2 + \mu c^2}{\mu b - 1}\right)_x \end{cases}$$

The system on the left is bi-Hamiltonian (Ferapontov, Galvao, Mokhov, Nutku CMP'98) by a third-order and a first-order Hamiltonian operator of Dubrovin–Novikov type. **What about the system on the right?**

Higher-order homogeneous Hamiltonian operators

Higher order homogeneous operators were introduced in 1984 by Dubrovin and Novikov. We can consider the second-order and third-order homogeneous operators:

$$
\begin{aligned}
A_2^{ij} =& g_2^{ij}(\mathbf{u})\partial_x^2 + b_{2\,k}^{ij}(\mathbf{u})u_x^k\partial_x \\
&+ c_{2\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{2\,km}^{ij}(\mathbf{u})u_x^k u_x^m, \\
A_3^{ij} =& g_3^{ij}(\mathbf{u})\partial_x^3 + b_{3\,k}^{ij}(\mathbf{u})u_x^k\partial_x^2 \\
&+ [c_{3\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{3\,km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\
&+ d_{3\,k}^{ij}(\mathbf{u})u_{xxx}^k + d_{3\,km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{3\,kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n.
\end{aligned}
$$

In canonical form:

$$
\begin{aligned}
A_2^{ij} =& \partial_x \circ g_2^{ij} \circ \partial_x, \\
A_3^{ij} =& \partial_x \circ (g_3^{ij}\partial_x + c_{3\,k}^{ij}u_x^k) \circ \partial_x,
\end{aligned}
$$

bi-Hamiltonian structure of WDVV equations

$$A_1 = \begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x a & \partial_x b \\ \frac{1}{2}a\partial_x & \frac{1}{2}(\partial_x b + b\partial_x) & \frac{3}{2}c\partial_x + c_x \\ b\partial_x & \frac{3}{2}\partial_x c - c_x & (b^2 - ac)\partial_x + \partial_x(b^2 - ac) \end{pmatrix},$$

$$A_3 = \partial_x \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & \partial_x & -\partial_x a \\ \partial_x & -a\partial_x & (\partial_x b + b\partial_x + a\partial_x a) \end{pmatrix} \partial_x$$

A_1 and A_3 are completely determined by their leading coefficients:

$$g^{ij} = \begin{pmatrix} -3/2 & 1/2\,a & b \\ 1/2\,a & b & 3/2\,c \\ b & 3/2\,c & 2(b^2 - ac) \end{pmatrix}, \quad g_3^{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a \\ 1 & -a & 2b + a^2 \end{pmatrix}$$

New results: projective invariance

Theorem Reciprocal transformations of projective type

$$d\tilde{x} = \Delta dx, \quad \tilde{u}^i = T^i(u^j) = (A^i_j u^j + A^i_0)/\Delta$$

with $\Delta = c_i u^i + c_0$ preserve the canonical form of third-order homogeneous operators (Ferapontov, Pavlov, V. JGP 2014). The leading terms are transformed as

$$g_{3\,ij} \to \frac{\tilde{g}_{3\,ij}}{\Delta^4}$$

where $\tilde{g}_{3\,ij}$ is of the same type as the initial metric; g_3 is identified with a quadratic line complex.

Digression: Plücker's line geometry

Two infinitesimally close points $V, V + dV \in \mathbb{P}(\mathbb{C}^{n+1})$,

$$V = [v^1, \ldots, v^{n+1}], \quad V + dV = [v^1 + dv^1, \ldots, v^{n+1} + dv^{n+1}]$$

define a line with coordinates

$$p^{\lambda\mu} = v^\lambda dv^\mu - v^\mu dv^\lambda = \det \begin{pmatrix} v^\lambda & v^\mu \\ v^\lambda + dv^\lambda & v^\mu + dv^\mu \end{pmatrix}$$

inside the projective space: $\mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$ (S. Lie coordinates for Plücker embedding).
We regard (u^i), $i = 1, \ldots, n$ as an affine chart on $\mathbb{P}(\mathbb{C}^{n+1})$, so that $u^{n+1} = 1$, $du^{n+1} = 0$ and

$$p^{ij} = u^i du^j - u^j du^i, \quad p^{(n+1)i} = du^i.$$

The algebraic variety of A_3

(Ferapontov, Pavlov, V., JGP 2014, IMRN 2016) The third-order operator A_3 fulfills the condition:

$$\partial_i (g_3)_{jk} + \partial_k (g_3)_{ij} + \partial_j (g_3)_{ki} = 0.$$

It implies that g_3 is a Monge metric: a quadratic form in Plücker's coordinates

$$g_3 = X^T Q X = f_{\lambda\mu,\rho\sigma} p^{\lambda\mu} p^{\rho\sigma}.$$

Intersecting g_3 with the Grassmannian

$$\mathbb{G}(2, \mathbb{C}^{n+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$$

we obtain, in the generic case, a quadratic line complex.

Third-order operators and systems of conservation laws

Following a construction of Agafonov and Ferapontov (1996-2001) we associate to each system $u^i_t = (V^i)_{,j} u^j_x$ a congruence of lines in $\mathbb{P}^{n+1}$ with coordinates $[y^1, \ldots, y^{n+2}]$

$$y^i = u^i y^{n+1} + V^i y^{n+2}$$

A method introduced by Kersten, Krasil'shchik, Verbovetsky (JGP 2004) to characterize Hamiltonian operators yields the following compatibility conditions between the operator and quasilinear first-order systems:

$$g_{3\,im} V^m_{,j} = g_{3\,jm} V^m_{,i}, \tag{3}$$

$$g_{3\,ks} V^k_{,ij} = c_{smj} V^m_{,i} + c_{smi} V^m_{,j}. \tag{4}$$

WDVV: new results

When applied to the WDVV systems, the above equations allow to determine the third-order operators (Vašiček, V, Journal of High Energy Physics 2021):

- In the cases $N = 3$, $N = 4$, $N = 5$ both canonical forms of WDVV equations as quasilinear first-order systems of PDEs admit a third-order homogeneous Hamiltonian operator in canonical form.
- In the case $N = 3$ also the canonical form $\eta^{(2)}$ of WDVV equations as quasilinear first-order systems of PDEs admits a compatible first-order homogeneous Hamiltonian operator. The operator is nonlocal of Ferapontov type.
- In the case $N = 3$ the bi-Hamiltonian pair is invariant with respect to $\partial/\partial t^1$-preserving affine coordinate changes in the WDVV space $(t^1, \ldots, t^N)$.

WDVV systems: new results

WDVV systems themselves turn out to have interesting projective goemetric properties:

Theorem. Every WDVV system (for $N = 3, 4, 5$), interpreted as a linear line congruence, has the following properties:

- The congruence is linear: there are n linear relations between u^i, V^i, $u^iV^j - u^jV^i$.
- The system is linearly degenerate, and non diagonalizable.
- The system admits non-local Hamiltonian, momentum and Casimirs.

WDVV, $N = 3$, $\eta = \eta^{(2)}$, third-order A_3:

The system of PDEs has a third-order homogeneous Hamiltonian operator defined by the Monge metric

$$g_{3\,ij} = \begin{pmatrix} b(\mu b - 2) & (a + \mu c)(1 - \mu b) & (\mu b - 1)^2 \\ (a + \mu c)(1 - \mu b) & \mu(a + \mu c)^2 + 1 & \mu(a + \mu c)(1 - \mu b) \\ (\mu b - 1)^2 & \mu(a + \mu c)(1 - \mu b) & \mu(\mu b - 1)^2 \end{pmatrix}, \tag{5}$$

and has the following form:

$$A_3 = \begin{pmatrix} -\mu\partial_x^3 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & \partial_x^2 \frac{a+\mu c}{\mu b-1}\partial_x \\ \partial_x^3 & \partial_x \frac{a+\mu c}{\mu b-1}\partial_x^2 & \frac{1}{2}(\partial_x^2 K \partial_x + \partial_x K \partial_x^2) \end{pmatrix}, \tag{6}$$

where $K = \frac{(a+\mu c)^2 + b(2-\mu b)}{(\mu b-1)^2}$.

WDVV, $N=3$, $\eta=\eta^{(2)}$, first-order A_1:

The system of PDEs has a non-local first-order homogeneous Hamiltonian operator of Feraportov type

$$A_1^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + \alpha V^i_q u^q_x \partial_x^{-1} V^j_p u^p_x \\ + \beta\left(V^i_q u^q_x \partial_x^{-1} u^j_x + u^i_x \partial_x^{-1} V^j_q u^q_x\right) + \gamma u^i_x \partial_x^{-1} u^j_x, \quad (7)$$

defined by the metric (in upper indices)

$$g^{ij} = \begin{pmatrix} b^2\mu^2 - a^2\mu - 2b\mu - 3 & a - ab\mu + bc\mu^2 - c\mu & 2b - b^2\mu + c^2\mu^2 \\ a - ab\mu + bc\mu^2 - c\mu & 2b - b^2\mu + c^2\mu^2 & \frac{c(ac\mu^2 - 2b^2\mu^2 + 4b\mu + c^2\mu^3 - 3)}{b\mu - 1} \\ 2b - b^2\mu + c^2\mu^2 & \frac{c(ac\mu^2 - 2b^2\mu^2 + 4b\mu + c^2\mu^3 - 3)}{b\mu - 1} & \frac{\delta}{(b\mu - 1)^2} \end{pmatrix}, \quad (8)$$

where

$$\delta = a^2c^2\mu^2 - 2ab^2c\mu^2 + 4abc\mu + 2ac^3\mu^3 - 4ac + b^4\mu^2 - 4b^3\mu - 3b^2c^2\mu^3 \\ + 4b^2 + 6bc^2\mu^2 + c^4\mu^4 - 5c^2\mu$$

and $\alpha = -\mu^2, \beta = 0, \gamma = \mu$.

WDVV, new results with $N = 4$

How many independent equations are in WDVV system?
If $N = 3$ there is only one equation.

Let $N = 4$, and set $x = t^2$, $y = t^3$, $z = t^4$.

$$\mu f_{yyz}(f_{zzz} - f_{yzz}) + 2f_{yyz}f_{xyz} - f_{yyy}f_{xzz} - f_{xyy}f_{yzz} = 0,$$
$$f_{xxy}f_{yzz} - f_{xxz}f_{yyz} - \mu f_{zzz}f_{xyz} + f_{zzz} + f_{xyy}f_{xzz} + \mu f_{xzz}f_{yzz} - f_{xyz}^2 = 0,$$
$$f_{xxy}f_{yyz} - f_{xxz}f_{yyy} + \mu f_{yyz}f_{xzz} - \mu f_{xyz}f_{yzz} + f_{yzz} = 0,$$
$$f_{xxy}f_{xzz} - \mu f_{xxz}f_{zzz} - 2f_{xxz}f_{xyz} + f_{xxx}f_{yzz} + \mu f_{xzz}^2 = 0,$$
$$f_{xxz}f_{xyy} + \mu f_{xxz}f_{yzz} - f_{yyz}f_{xxx} - \mu f_{xzz}f_{xyz} + f_{xzz} = 0,$$
$$f_{xxy}f_{xyy} + \mu f_{xxz}f_{yyz} - f_{xxx}f_{yyy} - \mu f_{xyz}^2 + 2f_{xyz} = 0.$$

First ‘generic’ case: WDVV with $N = 4$

Choose an independent variable, say x; it possible to find a subsystem of equations that are linear with respect to x-free derivatives:

$$f_{yyy}, \quad f_{yyz}, \quad f_{yzz}, \quad f_{zzz}.$$

This linear subsystem is overdetermined: it consists of 5 equations. They can be solved for the 4 unknowns f_{yyy}, f_{yyz}, f_{yzz}, f_{zzz}. If we introduce new field variables u^k in correspondence with every x-derivative of the third order, i.e.

$$u^1 = f_{xxx}, \quad u^2 = f_{xxy}, \quad u^3 = f_{xxz},$$
$$u^4 = f_{xyy}, \quad u^5 = f_{xyz}, \quad u^6 = f_{xzz}$$

First 'generic' case: WDVV with $N = 4$

The linear overdetermined system can be solved. For example, if $\mu = 0$ we have:

$$f_{yyy} = \frac{2u^5 + u^2u^4}{u^1}, \quad f_{yyz} = \frac{u^3u^4 + u^6}{u^1}, \quad f_{yzz} = \frac{2u^3u^5 - u^2u^6}{u^1},$$

$$f_{zzz} = (u^5)^2 - u^4u^6 + \frac{(u^3)^2u^4 + u^3u^6 - 2u^2u^3u^5 + (u^2)^2u^6}{u^1}.$$

It is remarkable that also the remaining nonlinear equation is solved by the above equations.

Reducing the WDVV system

Consider the WDVV system:

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\nu\gamma} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0.$$

we have

$$S_{\alpha\beta\gamma\nu} = S_{\gamma\nu\alpha\beta}, \tag{9}$$

$$S_{\alpha\beta\gamma\nu} = S_{\beta\alpha\nu\gamma}, \tag{10}$$

$$S_{\alpha\beta\gamma\nu} = -S_{\alpha\nu\gamma\beta}, \tag{11}$$

$$S_{\alpha\beta\gamma\nu} = -S_{\gamma\beta\alpha\nu}, \tag{12}$$

$$S_{\alpha\beta\gamma\nu} = S_{\alpha\beta\nu\gamma} + S_{\alpha\gamma\beta\nu}. \tag{13}$$

Using the above symmetries we can prove that

1. we can move any index at the first place (up to a sign);
2. $S_{1\beta\gamma\nu} = 0$ identically.

Reduced WDVV system

Let us choose $x = t^2$. Then, there are the following nontrivial cases ($3 \leqslant a, b, c \leqslant N$):

1. $S_{22ab} = 0$ with $a \leqslant b$;
2. $S_{aa2b} = 0$ with $a \neq b$;
3. $S_{2abc} = 0$ and $S_{2acb} = 0$ with $a < b < c$;
4. $S_{aabc} = 0$ with $b \leqslant c$;
5. $S_{abcd} = 0$ and $S_{abdc} = 0$ with $a < b < c < d$.

The subsystem (1), (2), (3) is linear and overdetermined with respect to t^2-free derivatives; the remaining equations are nonlinear.

Conjectures on the WDVV system

- The linear subsystem (1), (2), (3) can always be solved for t^2-free derivatives;
- the nonlinear subsystem (4), (5) vanishes identically on the solutions of the linear subsystem (1), (2), (3).

The above conjectures are **true** for $N = 4$, $N = 5$, $N = 6$ for Dubrovin's canonical forms of $(\eta_{\alpha\beta})$.

WDVV as a first-order systems of PDEs. Running example: $N = 4$

Mokhov and Ferapontov (1996) introduced new letters for third-order derivatives:

$$u^1 = f_{xxx},\ u^2 = f_{xxy},\ u^3 = f_{xxz},\ u^4 = f_{xyy},\ u^5 = f_{xyz},\ u^6 = f_{xzz},$$
$$u^7 = f_{yyy},\ u^8 = f_{yyz},\ u^9 = f_{yzz},\ u^{10} = f_{zzz}.$$

We have the following compatibility relations:

$$\begin{array}{ccc}
u^1_y = u^2_x & u^1_z = u^3_x & u^2_z = u^3_y \\
u^2_y = u^4_x & u^2_z = u^5_x & u^4_z = u^5_y \\
u^3_y = u^5_x & u^3_z = u^6_x & u^5_z = u^6_y \\
u^4_y = u^7_x & u^4_z = u^8_x & u^7_z = u^8_y \\
u^5_y = u^8_x & u^5_z = u^9_x & u^8_z = u^9_y \\
u^6_y = u^9_x & u^6_z = u^{10}_x & u^9_z = u^{10}_y
\end{array}$$

WDVV as a first-order systems of PDEs. Running example: $N = 4$

If we express the coordinates $u^7 = f_{yyy}$, $u^8 = f_{yyz}$, $u^9 = f_{yzz}$, $u^{10} = f_{zzz}$ by means of (u^k), $k = 1, \ldots, 6$ using *all* WDVV equations, we have two *commuting* quasilinear systems of first-order PDEs and a third set of trivial identities:

$$\begin{cases} u^1_y = u^2_x, \\ u^2_y = u^4_x, \\ u^3_y = u^5_x, \\ u^4_y = \left(\frac{2u^5 + u^2u^4}{u^1}\right)_x \\ u^5_y = \left(\frac{u^3u^4 + u^6}{u^1}\right)_x \\ u^6_y = \left(\frac{2u^3u^5 - u^2u^6}{u^1}\right)_x \end{cases}$$

$$\begin{cases} u^1_z = u^3_x, \\ u^2_z = u^5_x, \\ u^3_z = u^6_x, \\ u^4_z = \left(\frac{u^3u^4 + u^6}{u^1}\right)_x \\ u^5_z = \left(\frac{2u^3u^5 - u^2u^6}{u^1}\right)_x \\ u^6_z = \Big((u^5)^2 - u^4u^6 + \\ \qquad \frac{(u^3)^2u^4 + u^3u^6 - 2u^2u^3u^5 + (u^2)^2u^6}{u^1}\Big)_x \end{cases}$$

WDVV as a first-order systems of PDEs. Running example: $N = 4$

What about the residual compatibility conditions?
It can be proved that the system

$$
\begin{array}{ll}
u^2_z = u^3_y & u^4_z = u^5_y \\
u^5_z = u^6_y & u^7_z = u^8_y \\
u^8_z = u^9_y & u^9_z = u^{10}_y
\end{array}
$$

is identically verified when you restrict it to the two commuting systems on the previous slide.

WDVV as first-order systems of PDEs

1. Let $\sigma \in \mathbb{N}^{N-1}$, and introduce new variables $u^i = f_{(3,0,\ldots,0)}$, $u^2 = f_{(2,1,0,\ldots 0)}$, $\ldots$, $u^n = f_{(1,0,\ldots,2)}$, $n = N(N-1)/2$.
2. For any other t^h, $h > 2$, find $u^i_{t^h}$ as the t^2-derivative of an expression V^i:
   $$u^i_{t^h} = V^i(\mathbf{u})_{t^2}. \tag{14}$$
   There are two possibilities:
   - 2.1 either $V^i(\mathbf{u})$ is one of the coordinates u^j, with $j \neq 2$;
   - 2.2 V^i is a third-order derivative of f which is not one of the u^j. In this case, according with the conjecture, V^i must be expressed by means of one of the equations of the WDVV system.

WDVV as first-order systems of PDEs

Conjecture. Let us choose t^h and t^h, $h, k \geqslant 2$, $h \neq k$.
Then,

- WDVV equations are equivalent to $N-2$ commuting quasilinear systems of first-order PDEs;
- the above systems of PDEs are bi-Hamiltonian by a pair of a third-order homogeneous Hamiltonian operator in canonical form and a first-order nonlocal homogeneous Hamiltonian operator of Ferapontov type.

The conjecture has been verified in dimensions $N = 4$, $N = 5$.

Third-order Hamiltonian operator for WDVV

The metric $g_{3\,ij}$ can be factorized [Balandin, Potemin, 2001] as

$$g_{3\,ij} = \varphi_{\alpha\beta}\psi_i^\alpha \psi_j^\beta, \quad \Big(\text{or, in a matrix form,} \quad g_3 = \Psi\Phi\Psi^\top\Big) \quad (15)$$

where φ is a constant non-degenerate symmetric matrix of dimension n, and

$$\psi_k^\gamma = \psi_{ks}^\gamma u^s + \omega_k^\gamma$$

is a non-degenerate square matrix of dimension n.

For the conservative system $\mathbf{u}_t = (V(\mathbf{u}))_x$, the necessary and sufficient conditions to admit the above Hamiltonian operator are

$$g_{3\,im}V_j^m = g_{3\,jm}V_i^m,$$
$$V_{ij}^k = g_3^{ks}c_{smj}V_i^m + g_3^{ks}c_{smi}V_j^m.$$

Running example, WDVV $N = 4$
3rd order Hamiltonian operator

$$g_{3\,11} = u_4^2, \quad g_{3\,12} = (\mu u_5 - 2)u_5, \quad g_{3\,13} = 2u_4(1 - \mu u_5),$$
$$g_{3\,14} = \mu u_3 u_5 - u_1 u_4 - u_3,$$
$$g_{3\,15} = -\mu^2 u_5 u_6 - \mu(u_2 u_5 - u_3 u_4 - u_6) + u_2,$$
$$g_{3\,16} = (\mu u_5 - 1)^2, \quad g_{3\,22} = 2u_3(\mu u_5 - 1),$$
$$g_{3\,23} = -\mu^2 u_5 u_6 - \mu(u_2 u_5 + u_3 u_4 - u_6) + u_2, \quad g_{3\,24} = \mu u_3^2,$$
$$g_{3\,25} = -\mu^2 u_3 u_6 - \mu(u_1 u_5 + u_2 u_3) + u_1, \quad g_{3\,26} = 2\mu u_3(\mu u_5 - 1),$$
$$g_{3\,33} = \mu^2(2u_4 u_6 + u_5^2) + 2\mu(u_2 u_4 - u_5) + 2,$$
$$g_{3\,34} = -\mu^2 u_3 u_6 + \mu(u_1 u_5 - u_2 u_3) - u_1,$$
$$g_{3\,35} = \mu((\mu u_6 + u_2)^2 - g_{3\,14}),$$
$$g_{3\,36} = \mu g_{3\,23}, \quad g_{3\,44} = u_1^2, \quad g_{3\,45} = -2\mu u_1 u_3,$$
$$g_{3\,46} = \mu^2 u_3^2, \quad g_{3\,55} = \mu^2(2u_1 u_6 + u_3^2) + 2\mu u_1 u_2,$$
$$g_{3\,56} = \mu g_{3\,25}, \quad g_{3\,66} = 2\mu^2 u_3(u_5 \mu - 1).$$

Running example, WDVV $N = 4$ first-order Hamiltonian operator

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma_k^{ij} u_x^k + \sum_{\alpha,\beta=0}^{3} c^{\alpha\beta} w_{\alpha k}^i(\mathbf{u}) u_x^k \mathrm{D}_x^{-1} \circ w_{\beta h}^j(\mathbf{u}) u_x^h,$$

where $(g^{ij}) = (\Psi^{-1})Q(\Psi^{-1})^\top$, Φ is a constant symmetric matrix, and the entries of Ψ are linear in u_k's,

$$\Psi = \begin{pmatrix} \dfrac{u_4}{\mu} & \dfrac{u_5}{\mu} & 1 & 0 & 0 & 0 \\ 0 & \dfrac{u_3}{\mu} & 0 & -u_5 & 1 & 0 \\ -u_5 & -\dfrac{u_2}{\mu} - u_6 & 0 & u_4 & 0 & 1 \\ -\dfrac{u_1}{\mu} & 0 & 0 & -u_3 & 0 & 0 \\ u_3 & -\dfrac{u_1}{\mu} & 0 & \mu u_6 + u_2 & 0 & 0 \\ 0 & u_3 & 0 & -\mu u_5 + 1 & 0 & 0 \end{pmatrix},$$

Running example, WDVV $N = 4$ first-order Hamiltonian operator

$$Q^{11} = -\frac{4}{\mu}u_3u_5 + \frac{4}{\mu^2}u_1u_4 + u_6^2, \quad Q^{12} = -\frac{2}{\mu}u_3u_6 + \frac{4}{\mu^2}u_1u_5,$$
$$Q^{13} = u_1u_5 - \frac{1}{\mu}u_3u_6 + u_2u_3 + \frac{2}{\mu}u_1, \quad Q^{14} = -\frac{2}{\mu}(u_2u_5 - u_4u_3 + u_6),$$
$$Q^{15} = -\mu u_5u_6 + u_2u_5 + u_3u_4 + u_6, \quad Q^{16} = \mu u_6^2 + 2u_3u_5,$$
$$Q^{22} = \frac{2}{\mu^2}(u_1u_6 - u_3^2),$$
$$Q^{23} = -\frac{2}{\mu}u_1u_2 + u_3^2, \quad Q^{24} = \frac{4}{\mu}u_3u_5 - \frac{2}{\mu}u_2u_6 - u_6^2,$$
$$Q^{25} = u_3u_5 - \frac{1}{\mu}u_1u_4 - \frac{2}{\mu}u_3 - \frac{1}{\mu}u_2^2,$$
$$Q^{26} = -\frac{1}{\mu}u_1u_5 + u_3u_6 - \frac{1}{\mu}u_2u_3,$$

Running example, WDVV $N = 4$ first-order Hamiltonian operator

$$Q^{33} = \mu^2 u_3^2 - 2\mu u_1 u_2, \quad Q^{34} = -\mu u_3 u_5 + u_1 u_4 + u_2^2 + 4u_3,$$
$$Q^{35} = \mu^2 u_3 u_5 - \mu u_1 u_4 - \mu u_2^2 - \mu u_3,$$
$$Q^{36} = \mu^2 u_3 u_6 - \mu u_1 u_5 - \mu u_2 u_3 + u_1,$$
$$Q^{44} = 2u_4 u_6 - 2u_5^2, \quad Q^{45} = -\mu u_5^2 + 2u_2 u_4 + 4u_5,$$
$$Q^{46} = -\mu u_5 u_6 + u_2 u_5 + u_3 u_4 + 3u_6,$$
$$Q^{55} = \mu^2 u_5^2 - 2\mu u_2 u_4 - 2\mu u_5 - 2,$$
$$Q^{56} = \mu^2 u_5 u_6 - \mu u_2 u_5 - \mu u_3 u_4 - \mu u_6 + u_2,$$
$$Q^{66} = \mu^2 u_6^2 - 2\mu u_3 u_5 + 2u_3,$$

Running example, WDVV $N = 4$ first-order Hamiltonian operator

The nonlocal part of the operator

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + c^{\alpha\beta} w^i_{\alpha k}(\mathbf{u}) u^k_x \mathrm{D}_x^{-1} \circ w^j_{\beta h}(\mathbf{u}) u^h_x,$$

is defined by the matrix

$$\begin{pmatrix} c^{11} & c^{12} & c^{13} \\ c^{21} & c^{22} & c^{23} \\ c^{31} & c^{32} & c^{33} \end{pmatrix} = \begin{pmatrix} 0 & -\mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & \mu^2 \end{pmatrix}.$$

and by the commuting symmetries

$$w^i_{1j} = \delta^i_j, \quad w^j_{2i} = V^i_j, \quad w^j_{3i} = W^i_j,$$

where

$$u^i_y = (V^i)_x = V^i_j u^j_x, \qquad u^i_z = (W^i)_x = W^i_j u^j_x,$$

are the WDVV first-order systems.

Results on bi-Hamiltonian structures for WDVV

Theorem Let $u^i_{t^h} = (V^i)_{t^k}$ be a family of commuting first-order WDVV systems, $h = 2,\dots,N$, $h \neq k$. If there is one value of h such that the first-order system is bi-Hamiltonian with a pair of compatible Hamiltonian operators A_1, A_3, then all first-order WDVV systems corresponding to all other values h are endowed with exactly the same bi-Hamiltonian pair.

Proof Compatibility of the operators A_1 and A_3 gives

$$g_{3\,im} w^m_{\alpha j} = g_{3\,jm} w^m_{\alpha i}, \quad w^k_{\alpha i,j} = g_3^{ks} c_{smj} w^m_{\alpha i} + g_3^{ks} c_{smi} w^m_{\alpha j}.$$

These are the conditions under which $w^m_{\alpha i}$ define Hamiltonian systems for the third-order operator defined by $g_{3\,ij}$.

Results on bi-Hamiltonian structures for WDVV

Theorem An invariance transformation of the WDVV equation preserves the form of the Hamiltonian operators in a bi-Hamiltonian first-order WDVV system.

Proof The symmetry group of a third-order WDVV projects to the symmetry group $\mathrm{GL}(N-1,\mathbb{C})$ of a first-order WDVV.

Invariance transformations that involve only two independent variables preserve the form of the Hamiltonian operators [Vašiček, V., 2021].

Any matrix in $\mathrm{GL}(\mathbb{C}^{N-1})$ can be generated by means of 2×2 Gauss' elementary matrices (up to permutations).

Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `https://reduce-algebra.sourceforge.io/`.

CDE (by RV) can compute symmetries and conservation laws, local and nonlocal Hamiltonian operators, Schouten brackets of local multivectors, Fréchet derivatives (or linearization of a system of PDEs), formal adjoints, Lie derivatives of Hamiltonian operators, anticommuting variables and super-PDEs.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

A book, in cooperation with JS Krasil'shchik and AM Verbovetsky: *The symbolic computation of integrability structures for partial differential equations*, is published in the series Texts and Monographs in Symbolic Computation, Springer, 2018.

Thank you!

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