PDE without tears for geometry students

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1 Part 1

Conceptualization: S. Lie. Formalization: Eheresmann.

P.D.E. is

$$\begin{array}{ll} x^k & \lambda = 1, \ldots, \text{independent variables} \\ u^i & i = 1, \ldots, m \text{depdendent variables} \end{array}$$

$$\begin{aligned} F^1(x^k, u^i, u^i_J) &= 0 \\ &\vdots \\ F^k(x^k, u^i, u^i_J) &= 0 \end{aligned}$$

where

$$u^i_J = \frac{\partial^{|\sigma|} u^i}{(\partial u^i)^{\sigma_1} \ldots (\partial x^m)^{\sigma_m}}$$

for some multi index $\sigma \in \mathbb{N}^n \sigma = (\sigma_1, \ldots, \sigma_m)$.

In general the u^i are functions; here we may think they are coordinates of some space since we want a geometric introduction to PDEs.

Let N be the spaces of independent variables, $\dim N = n$, and M the space of dependent variables of dimension m. Fields-unknowns are $u : U \subset N \to M$, $u = (u^i) = (u^1, \ldots, u^m)$.

Now we can consider the field as a section of the trivial bundle: $s : N \to N \times M$, $s(x^k) = (x^k, u^i(x^k))$

Now we introduce the notion of contact, which has to do with the derivatives.

1-contact of sections $s_1, s_2 : N \to N \times M$ at $x_0 \in N$:

(0) $s_1(x_0) = s_2(x_0)$.

1. (The sections have two graphs that share the tangent plane) $\frac{\partial u^i i_1}{\partial x^k}(x_0) = \frac{\partial u_2^i}{\partial x^k}(x_0)$.

Higher contact:

$$\frac{\partial^{|\sigma|} u_1^i}{(\partial x^1)^{\sigma_1}} \dots (\partial x^m)^{\sigma_m}(x_0) = \frac{\partial^{|\sigma|} u_2^i}{(\partial x^1)^{\sigma_1} \dots (\partial x^u)^{\sigma_n}}(x_0)$$

$|\sigma| = \sum_{i=1}^m \sigma_i$, ***order of derivative***, $\forall \sigma \in \mathbb{N}^m$, $0 \leqslant |\sigma| \leqslant r$. This is r***-th contact***.

(if the Taylor polynomial unit a certain point is the same)

General notion of contact. E an $(n+m)$-dimensional manifold and $L_1, L_2 \subset E$ two n-dimensional submanifolds. They have a contact of order r at x_0 if there exists a chat $(\underbrace{u^k}_{n}, \underbrace{u^i}_{m})$ such that

- L_1 is the graph $(x^1) \to (x^k, u_1^i(x^k))$ and
- L_2 is the graph $(x^1) \to (x^k, u_1^i(x^k))$

have a contact of order r. (If the manifolds can be seen as graphs in the same chart.)

Now consider the class of all submanifolds touching at the point x_0 of order r, $[L]_{x_0,r}$. Now, the ***jet space of order*** r ***at*** x_0 ***of*** (E, n)***,*** n ***independant variables*** is

$$\bigcup_{x_0 \in E} [L]_{x_0,r} := J_r(E, n)$$

That's a bundle. The fiber at each point is all the equivalence classes of manifolds that touch at contact etc.

$$\cdots \longrightarrow J_2(E, u) \longrightarrow J_1(E, n) \longrightarrow E$$

$$\cdots \longrightarrow [L]_{x_0,2} \longmapsto [L]_{x_0,1} \longmapsto x_0$$

Remark This bundle is in general a Grassmannian.

Coordinates on $J_2(E, u) : (x^1, u^i, J)$

1. $E \to N$ is a further bundle (e.g. $N \times M \to N$).
2. E is just a manifold.

Each has to be dealt with in a way:

1. For the trivial bundle $N \times M \to N$: Transformations of the tyupe (u^μ, v^j),

$$\begin{cases} y^\mu = Y^\mu(x^k) \\ v^j = U^j(x^k) \end{cases}$$

For a bundle $E \to N$

$$\begin{cases} y^\mu = Y^\mu(x^k) \\ v^j = U^j(x^k, u^i) \end{cases}$$

2.

$$\begin{cases} y^\mu = Y^\mu(x^k, u^i) \\ v^j = U^j(x^k, u^i) \end{cases}$$

Definition A ***partial differential equation of order*** r is a subbundle $\mathcal{E} \subset J_r(E, m)$.

Example $n = m = 1$, $u_x = 1$. $\mathcal{E} \subset J_1(E, 1)$, $E = \mathbb{R} \times \mathbb{R}$, so $J_1(E, 1) = \mathbb{R}^3$.

Now take $u(x) = u + c$. We get $j_1 u : N \to J_1(E, 1)$, $u_0 \mapsto [u]_{x_0,1}$.

Example Riemann equation, or Hopf equation, or Inviscid Burgers equation.

$$u_t + \underbrace{u}_{\text{velocity of wave}} u_x = 0$$

u depends on t and x. $n = 2$, $m = 1$, $r = 1$.

…the speed is proportional to the ammount of cars…

Let's introduce a variable

$$\xi := x - tu.$$

We can complement this variable with two other variables that are unchanged:

$$\begin{cases} \tilde{t} = t \\ \xi = x - tu \\ v = u \end{cases}$$

now let's take derivatives so that we find the protagonists of Riemann equation:

$$\begin{cases} u_x = v_{\tilde{t}} \tilde{t}_x + v_\xi \xi_x = v_\xi (1 - t u_x) \\ u_t = v_{\tilde{t}} \tilde{t}_t + v_\xi \xi_t = v_{\tilde{t}} + v_\xi(-u - t u_t) \end{cases}$$

Now we want to express some in terms of others:

$$u_x = \frac{v_\xi}{1 + \tilde{t} v_\xi}, \qquad u_t = \frac{v_{\tilde{t}} - v v_i}{1 + \tilde{t} + v_3}$$

Now condier

$$v \frac{v_\xi}{1 + \tilde{t} v_\xi} + \frac{(v_{\tilde{t}} - v v_\xi}{1 + \tilde{t} v_\xi} = 0$$

But things cancel (which?) and we get

$$\frac{v_{\tilde{t}}}{1 + \tilde{t} v_\xi} = 0$$

so we have a singularity given by

$$1 + \tilde{t}v\xi = 0 \iff v_\xi = -\frac{1}{\tilde{t}}$$

and outside the singularity we have an implicit solution of the Riemann equation:

$$v = f(\xi), \qquad u = f(x - tu).$$

Upshot Look for symmetries and conservations laws, then a linearizing procedure that allows you to find a solution.

2 Part 2

Suppose we are given a submanifold $\mathcal{E} \subset J_r(E, u)$ of a jet space. Given $L \subset E$ an n-dimensional submanifold, we denote by $j_r L : L \to J_r(E, n)$

$$\begin{aligned} j_r L : L &\longrightarrow J_2(E, u) \\ x_0 &\longmapsto [L]_{x_0, r} \end{aligned}$$

$$L : (x^k) \mapsto (x^k, u^i(x^k))$$
$$j_2 L : (x^k) \mapsto (x^k, u^i_\sigma(x^k))$$

L is a soution of $\mathcal{E}$ if and only if $j_r L : L \to \mathcal{E} \subset J_r(E, n)$.

$$\mathcal{C} = \{T j_r L(TL) : L \subset E \text{ an } n\text{-dimensional submanifold}\} \subset TJ_r(E, u)$$

So a subbundle of the tangent bundle of the jet-space. Its a distribution of prolonged...

Coordinates:

$$s(x^k) = (x^k, u^i(x^k))$$
$$T_{j_r} s\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial u^k} + \frac{\partial u^i}{\partial u^k}\frac{\partial}{\partial u^i} + \frac{\partial^2 u^i}{\partial x^k \partial x^\mu}\frac{\partial}{\partial u^i_\mu} + \dots$$

One can prove that $\mathcal{C}$ is generated by:

$$\mathcal{C} = \mathsf{span}\left\{D_\lambda, \frac{\partial}{\partial u^i_2}\right\}$$

Where

$$D_\lambda = \frac{\partial}{\partial x^k} + u^i_{\sigma+k}\frac{\partial}{\partial u^i_J}, |\sigma| \leqslant r - 1$$

are called ***total derivatives***. They are vector fields in the jet space.

Now if $|\tau| = r$,

$$[D_\lambda, D_\mu] = 0, \qquad \left[\frac{\partial}{\partial u^{i_1}_{\tau_1}}, \frac{\partial}{\partial u^{i_2}_{\tau_2}}\right] = 0$$

and

$$\left[D_\mu, \frac{\partial}{\partial u^i_x}\right] \neq 0.$$

3 Part 3

Now we introduce ***Lie maps*** which are bundle morphisms $\phi : J_r(E,n) \to J_r(E,n)$ that preserve the contact distribution of each bundle, i.e.

$$T\phi(\mathcal{C}_2) \subset \mathcal{C}_r.$$

Theorem (Lie-Backlund) $m > 1$. All the morphisms have the form $\phi = \bar{\phi}^{(r)}$, $\bar{\phi} = E \to E$ defined as follows

$$\bar{\phi}^{(r)}(j_2 L) = j_r(\phi \circ L)$$

See Kraslkschik-Vigradov: symmetries and conservation laws of PDE of mathematical physics.

Now

$$\bar{\phi}^{(r)}(j_2 L) = j_r(\phi \circ L)$$

For $m = 1$, $\phi = \phi_1$. We have a contact transformation

$$\bar{\phi}_1 : J_1(E,n) \longrightarrow J_1(E,n), \qquad \dim E = u + 1$$

> Contact transformation help us linearize equations, like the Monge-Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = 1$, which is a Hessian, using Euler transformation. See Kushner-Lychagin-Roubtsov

Now lets define a ***symmetry*** of $\mathcal{E}$. It is a map $\phi : \mathcal{E} \to \mathcal{E}$ such that $T\phi(\mathbb{C}|_{\mathcal{E}}) = \mathbb{C}_{\mathcal{E}}$.

Now take a vector field tangent to the $\mathcal{E}$quation: $X : \mathcal{E} \to T\mathcal{E}$ and $L_X \mathcal{C}|_{\mathcal{E}} \subset \mathcal{C}_{\mathcal{E}}$.

How to find symmetries of a PDE? Simplest approach: point symmetries. $X = \bar{X}^{(r)}$.

$$\begin{cases} TF(X) = 0 \\ \mathcal{E} : F = 0 \end{cases}$$

So X_1, X_2 symmetries, then $[X_1, X_2]$ is a symmetry.

Why symmetries? for $n = 1$ ODE,

Theorem (Lie-Bianchi) If there exists a solvable lie algebra o symmetries of an ODE, then the ODE is solvavle by quadratures.

This is basically reduction of order.

Upshot Finding symmetries simplifies equation, e.g. less variables or turning PDE to ODE.

4 Part 4

In order to continue with symmetries, we want to look at vector fields on jets and prolonged vector fields of jets. This is a vector field on jets

$$X = X^k \frac{\partial}{\partial x^k} + x^i_\sigma \frac{\partial}{\partial u^i_\sigma} : J_r(E,u) \to TJ_r(E,u) \qquad = x^k D_\lambda + (X^i_\sigma - u^i_{J+\lambda} X^\lambda) \frac{\partial}{\partial u^i_J}$$

$$X = \bar{X} \iff (X^i_\sigma - u^i_{J+\lambda} X^k) = D_\sigma (X^i - u^i_\lambda X^k) \tag{1}$$

$$\bar{X} = X^\lambda \frac{\partial}{\partial X^\lambda} + X^i \frac{\partial}{\partial u^i} = X^\lambda \left(\frac{\partial}{\partial u^i} + u^i_\lambda \frac{\partial}{\partial u^i} \right) + (X^i - u^i_\lambda X^\lambda) \frac{\partial}{\partial u^i}$$

You can split it in a part that uses total derivatives (vertical, I guess) and a part that uses horizontal derivatives. So it's not a actual connection, it's a partial connection. More explicitly,

$$TE \times_E J_1(E,u) = H_1 \oplus_{J_1(E,u)} V_1$$

And if the vector field is a prolongation of a vector field, then *the vertical part must preserve eq.* (1).

A ***generlized symmetry*** is

$$TF(X_v) = \frac{\partial F^k}{\partial u^i_J} \underbrace{D_\sigma (X^i - u^i_\lambda X^\lambda)}_{\ell_F(\bar{X}_V)} = 0$$

In other words, a generalized vector field $\varphi : J_R(E,u) \to VE$, $\varphi = \varphi^i \frac{\partial u^i}{\partial}$.

Now let's do

$$F = u_t + u u_x + u_{xxu} = 0 \tag{2}$$

So

$$\frac{\partial F}{\partial u} = u_x, \frac{\partial F}{\partial u_t} = 1, \qquad \frac{\partial F}{\partial u_x} = u, \qquad \frac{\partial F}{\partial u_{xxx}} = 1$$

Let us compute the linearization operator ℓ_F:

$$\frac{\partial F}{\partial u^i_J} D_J(\varphi) = u_x \varphi + D_t \varphi + u D_x \varphi + D_{xxx} \varphi = 0.$$

And then

$$\varphi = \varphi^u(t,x,u) - u_t \varphi^t(t,u,u) - u_x \varphi^x(t,x,u) \tag{3}$$

$$\varphi = \varphi(t,x,u,u_x).$$

Now suppose that $\varphi = u_t$. Let's insert it in eq. (3) and see that goes to zero. Of course, we must the eq. (2). OK:

$$\ell_F(u_t) = u_x u_t + \dots\dots\dots = 0$$

usually you do this with computer.

5 Conservation laws

They are differential forms that are closed on the equation.

So for example, for KDV, α_x is ***density*** and α_t the ***flux***.

$$\alpha = \alpha_x \, dx + \alpha_x \, dt$$

$$\alpha_x = \alpha_x(t, x, u, u_t, u_x, \dots)$$
$$\alpha_t = \alpha_t(t, x, u, u_t, u_x, \dots)$$

So we want

$$d_{tt}\alpha = D_t \alpha_x \, dt \wedge dx + D_x \alpha_t \, dx \wedge dt\Big|_{F=0} = 0$$

$$D_t \alpha_x - D_x \alpha_t\Big|_{F=0} = 0$$

$$d_H \rho = D_x \rho dx - D_t \rho dt$$

$$d_H^2 \rho = 0$$

We integrate:

$$D_t \int_a^b \alpha_x \, dx = \int_a^b D_t \alpha_x \, dx = \int_e^b D_x \alpha_t \, dx = \alpha_t|_a^b$$

Upshot This is the higher-dimensional version of Poisson brackets, first integrals...