Two talks by Carolina Araujo

Contents

Automorphisms of quartic surfaces and Cremona transformations

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0.1 Motivation

X a smooth hypersurface of degree d, $X \subset \mathbb{P}^{n+1}$. We want to understand the group $\mathsf{Aut}(X)$. These are invertible polynomial maps from X to itself. X is defined by a single polynomial equation of degree d.

Theorem (Matsumura-Mousky, 1964) Except in two special cases, all automorphisms of X come from automorphisms of the ambient space:

If $(n, d) \neq (1,3), (2,4)$, there is a surjective map

$$\mathsf{Aut}(\mathbb{P}^{n+1}, X) \xrightarrow{\pi} \mathsf{Aut}(X)$$

where $\mathsf{Aut}(X, Y)$ means automorphisms of X that stabilize Y.

0.2 Exceptional cases

Let's look at the exceptionals cases.

1. $(n, d) = (1, 3)$. In this case $C = X_3 \subset \mathbb{P}^2$ is an elliptic curve and we have

   $$\mathsf{Aut}(C) \cong C \rtimes \mathbb{Z}_m \quad m = 2, 4, 6$$

   and

   $$\mathsf{Aut}(\mathbb{P}^2, C) \text{ is finite.}$$

   Elements in C are translations of the torus. The ***translation of*** x ***with respect to*** p ***and*** $(x : y : z)$ is done by intersecting the curve with the line that joins p and $(x : y : z)$ and reflecting. So we have a map

   $$t_p(x : y : z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)$$

   We have created a ***Cremona transformation*** (=biholomorphic birational map?).

   Definition (Cremona group)

   $$\mathsf{Bir}(\mathbb{P}^n) = \{\varphi : \mathbb{P}^n \xrightarrow{\mathsf{bir}} \mathbb{P}^n : \text{bimeromorphic map}\}$$

   And we then have the surjective map

   $$\mathsf{Bir}(\mathbb{P}^2, C) \xrightarrow{\pi} \mathsf{Aut}(C)$$

2. $(n, d) = (2, 4)$. Here $S = X_4 = \mathbb{P}^3$ is a smooth quartic surface.

Problem (Gizatullin) Which automorphisms of S are induced (not necesarily by automorphisms of $\mathbb{P}^3$) but at least by Cremona transformations of $\mathbb{P}^3$? ie. are restrictions of $\varphi \in \mathsf{Bir}(\mathbb{P}^3, S)$

Remark Related to K3 surface structure,

$$\mathsf{Bir}(\mathbb{P}^3, S) \xrightarrow{\pi} \mathsf{Bir}(S) \cong \mathsf{Aut}(S)$$

S is a K3 surface. $H^2(S, \mathbb{Z}) \cong \mathbb{Z}^{22}$ and the Picard group acts on this lattice.

Definition (Picard rank of S)

$$\rho(S) = \mathsf{rk}(\mathsf{Pic}(S)) \in \{1, \ldots, 20\}$$

- If S is very general, then $\rho(S) = 1$ and $\mathsf{Aut}(S) = \{1\}$.
- We are interested in $\rho(S) \geqslant 2$.

Example (Oguiso, 2012)

1. $\rho(S) = 2$. Any cremona transformation that stabilizes the quadric is the identity:

$$\mathsf{Aut}(S) = \mathbb{Z} \qquad \mathsf{Bir}(\mathbb{P}^3, S) = \{1\}$$

2. $\rho(S) = 3$,

$$\mathsf{Aut}(S) = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$$

All automorphisms are induced by Cremona transformations, ie. there is a surjective map

$$\mathsf{Bir}(\mathbb{P}^3, S) \xrightarrow{\pi} \mathsf{Aut}(S)$$

3. (Paiva, Quedo 2023) Constructed surfaces with $\rho(S) = 2$, $\mathsf{Aut}(S) = \mathbb{Z}_2$ and $\mathsf{Bir}(\mathbb{P}^3, S) = \{1\}$.

Theorem (A-Paiva-(Socrates) Zika) Solution of Giztallin S problem for $\rho(S) = 2$.

Remark Not exactly, but "there is a moduli space for K3 surfaces of dimension $20 - \rho(S)$". The generic case is $\rho(S) = 1$.

0.3 K3 surfaces

Definition A ***K3 surface*** is a smooth projective surface that is simply connected and has a nowhere vanishing symplectic form $\omega \in H^0(S, \Omega_S^2)$

0.3.1 Lattices of S

A ***lattice*** is a finitely-generated abelian group with a nondegenerate pairin.

1. $H^2(X,\mathbb{Z}) \cong \mathbb{Z} \overset{\pi}{\hookleftarrow} \mathrm{Pic}(S)$. And if we tensor this with $\mathbb{C}$ we get $H^2(C,\mathbb{C})$, which admits a Hodge decomposition.

   Let's study automorphisms of a K3 surface. Let $g \in \mathrm{Aut}(S)$. It yields an element g^* that acts on cohomology, ie $g^* \in \mathcal{O}(H^2(X,\mathbb{Z}))$ preserving the Hodge decomposition. This is called ***Hodge isometry***.

   Theorem (Global Torelli theorem)

   - The correspondence $g \mapsto g^*$ is injective
   - If $\varphi \in \mathcal{O}(H^2(X,\mathbb{Z}))$ is a Hodge isometry preserving the ample classes, then $\varphi = g^*$ for some $g \in \mathrm{Aut}(S)$.

Example $S \subset \mathbb{P}^3$ smooth quartic surface with $\rho(S) = 2$. Using the equivalence of line bundles module isomorphism and curves modulo intersection,

$$\mathrm{Pic}(S) = \langle H, C \rangle$$

where H is a hyperplane section of $\mathbb{P}^3$. Then

$$Q = \begin{pmatrix} H^2 & H \cdot C \\ H \cdot C & C^2 \end{pmatrix} = \begin{pmatrix} 4 & b \\ b & 2c \end{pmatrix}$$

using that $2g(c) - 1$, so the number in the lower-right on RHS is even.

Remark Hodge Index Them Q is rank (1,1).

Definition (Discriminant)

$$r = \mathrm{disc}(S) = -\det Q$$

Proposition S general K3 surface (not needed that it is a quartic) with $\rho(S) = 2$. Then

$$\mathrm{Aut}(S) = \begin{cases} \{1\} & \text{(finite)} \\ \mathbb{Z}_2 & \text{(finite, Dani-Ana)} \\ \mathbb{Z} & \text{(infinite, Oguiso 1)} \\ \mathbb{Z}_2 * \mathbb{Z}_2 & \text{(infinite, Dani-Ana)} \end{cases}$$

where the first two are characterized by containing a en elliptic curve or a rational? curve, ir. $\exists D \in \mathrm{Pic}(S)$ such that $D^2 = 0, -2$. On the other hand, the last two cases are distinguised by $\nexists D \in \mathrm{Pic}(S)$ such that $D^2 = 0, -2$.

$$\exists \sigma \in \mathrm{Aut}(S) \text{ of order } 2 \iff \exists \text{ ample } A \in \mathrm{Pic}(S) \text{ such that } A^2 = 2.$$

Remark In low Picard numbers there are no symplectic involutions...?

So to understand the surface we want to understand those bundles and that is all in the discriminant (not in the lattice itself!).

Remark $\exists D \in \mathsf{Pic}(S)$ s.t. $D^2 = k \iff x^2 - ry^2 = 4k$ has integer solutions.

Given the quadratic form we can find the automorphisms.

0.3.2 Main theorem

Theorem (A-Paiva-Zika) $S \subset \mathbb{P}^3$ general smooth quartic surface with $\rho(S) = 2$ and $\mathsf{disc}(S) = r$.

1. (Negative answer to Gizatulla's problem) If $r > 57$ or $r = 52$ then

$$\mathsf{Bir}(\mathbb{P}^3, S) = \{1\}$$

   So we cannot realize any automorphism as a Cremona transformation.

2. If $r \leqslant 57$ and $r \neq 52$, then we get the full automorphism group, ie. a surjective map

$$\mathsf{Bir}(\mathbb{P}^3, S) \xrightarrow{\pi} \mathsf{Aut}(S)$$

0.4 Birational geometry

Take the case of

$$\begin{tikzcd} \mathsf{Bir}(\mathbb{P}^2) = \langle \mathsf{Aut}(\mathbb{P}^2), q \rangle \\ \mathsf{Aut}(\mathbb{P}^2) \arrow[u, hook] \end{tikzcd}$$

and the map

$$\begin{aligned} q : \mathbb{P}^2 &\xrightarrow{\mathsf{bir}} \mathbb{P}^2 \\ (x : y : z) &\longmapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right) = (yz : xz : xy) \end{aligned}$$

which is well-known (Noether-Castelnuovo). So that is a decomposition of automorphisms and quadratics. Now in greater dimension, $n \geqslant 3$ we have

Theorem (Sakisov Program) (The theorem is much more general) Any $\varphi \in \mathsf{Pic}(\mathbb{P}^n)$ can be factorized with ***Sarkisov links*** φ_i:

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dashed, "\varphi_1"] \arrow[rrrr, bend left, "\varphi"] & X_1 \arrow[r, dashed, "\varphi_2"] \arrow[d] & X_2 \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, dashed] & X_k = \mathbb{P}^n \\ & T_1 & T_2 & & \end{tikzcd}$$

Now look at $n = 3$, $\mathsf{Bir}(\mathbb{P}^3, S) \subset \mathsf{Bir}(\mathbb{P}^3)$. This is a Calabi-Yau pair:

Definition (Calabi-Yau pair) A pair (X, D)

- Terminal projective variety.
- $K_X + D \sim 0$ that is, a meromorphic top form that does not vanish on hypersurface and has simple pole on D. Then (X, D) is called ***log canonical***.

Now take two Calabi-Yau pairs (X, D_X) and (Y, D_Y).

$$\operatorname{div}(\omega_{D_X}) = -D_X$$
$$\operatorname{div}(\omega_{D_Y}) = -D_Y$$

We say that a birrational map $f : X \to Y$ is ***volume preserving*** is $f_* \omega_{D_X} = \omega_{D_Y}$.

Theorem (Volume-Preserving) Everything like in the Sarkisov theorem but now maps are volume-preserving.

In our case, $(\mathbb{P}^3, S)$, we can classify the v.p. Sakisov links from $(\mathbb{P}^3, S)$. It starts by blowing up a curve $C \subset S$. But this curve has genus and degree very restricted, it's something like

$$(g(C), \operatorname{deg}(C)) \in \{(0,1), (0,2), \ldots, (11,10), (14,11)\}$$

So for the main theorem, it was checked that if $r > 57$ there are no curves from the list. And in the second item of the main theorem, there exist these curves, for instance curve $(14, 11)$ for rank 56, then produce a link that starts by blowing it up and magically gives you the Cremona transformation that restricts with automorphism with which you started.

Remark So perhaps we expect the answer to G. problem to be almost never.

Question How does that blowing-up work?

$$\begin{tikzcd}
X \arrow[r, "\text{flops}"] \arrow[d, "\mathrm{Bl}_C"] & X \arrow[d, "\text{contract?}"] \\
C \subset S \subset \mathbb{P}^3 \arrow[r, dashed] & \mathbb{P}^3 \supset S' \supset C'
\end{tikzcd}$$

So for example in case $(2, 8)$ you obtain something of the same type.

Birational geometry of Calabi-Yau pairs

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0.5 Pairs (X, D)

Itaca's program (1970's): U complex algebraic variety. We want to find invariants like Kodaira dimension. Compactify $U \rightsquigarrow X$, and consider $X \backslash U := D$ (the boundary). The $\Omega^q_X(\log D)$

Theorem (Itaca, 1977) Kodaira dimension of the pair (X, D) does not depend on the choice of compactification (and it exists ;)

Definition (Lu-Zhang, 2017) (X, D) is ***Brody-hyperbolic (Mori-hyperbolic)*** if there is no nonconstant holomorphic (analytic) morphism $f : \mathbb{C} \to X \backslash D$ and the same holds for the open strata of D.

Conjecture (X, D) is Brody-hyperbolic then $K_X + D$ is ample.

Theorem (Sraldi, 2019) (X, D) Mori-hyperbolic then $K_X + D$ is nef.

Question How to make sense of "the birational geometry of (X, D)"

0.6 Calabi-Yau pairs

Definition (Calabi-Yau pair) We relax a condition: we won't ask that X is smooth, but that it is a terminal projective variety. And also that $K_X + D \sim 0$ (I think this "would be equivalent to" Kodaira dimension 0). Also ask that (X, D) is log canonica.

Remark The condition $K_X + D \sim 0$ says that there is unique up to scaling volume form $\omega_D \in \Omega^m_{\mathbb{C}(X)}$ that does not vanish and has a simple pole along D, i.e. $\mathrm{div}(\omega_D) = -D$.

Now we study the birational geometry of CY pairs:

Definition Let $(X, D_X), (Y, D_Y)$ CY pairs and $f : X \dashrightarrow Y$ a birational map. We get a pullback $f_* : \Omega^n_{\mathbb{C}(X)} \xrightarrow{\cong} \Omega^n_{\mathbb{C}(Y)}$. Then $f : (X, D_X) \dashrightarrow (Y, D_Y)$ is ***volume preserving*** is $f_*(\omega_{D_X} = \lambda \omega_{D_Y}$.

Remark (Valuative characterization)

Definition (X, D) CY pair,

$$\mathsf{Bir}(X, D) := \{f \in \mathsf{Bir}(X) : f : (X, D) \dashrightarrow (X, D) \text{ is volume preserving}\}$$

0.7 The Sarkisov-Program

$\mathsf{Bir}(\mathbb{P}^n)$ Cremona group.

Theorem (Noether-Castelnuovo, 1870-1901) The creoma group of $\mathbb{P}^2$ has a nice set of generators: maps of degree 1 and q:

$$\mathsf{Bir}(\mathbb{P}^2) = \langle \mathsf{Aut}(\mathbb{P}^2), q \rangle$$

where q is the ***standard quadratic transformation*** given by $(x : y : z) \overset{q}{\mapsto} (yz : xz : xy)$

You might think that because it has such a nice group generators it'd be easy to study this group. It's not the case.

Now in dimension 3:

Theorem (Hilda Hudson (1927)) $\mathsf{Bir}(\mathbb{P}^n)$ does not admit a set of generators of bounded degree.

Theorem (Sarkisov Program, Corti 1995; Hazon-McKunam 2013 for $n \geqslant 4$)

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, dashed, "\varphi_1"] \arrow[d] \arrow[rrrrr, bend left, "\varphi"] & X_1 \arrow[r, dashed, "\varphi_2"] \arrow[d] & X_2 \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, dashed] & X_k = \mathbb{P}^n \arrow[d] \\
\{pt\} & T_1 & T_2 & & \{pt\}
\end{tikzcd}$$

The intermediate varieties X_i/T_i are called ***Mori-fiber spaces***, and φ_i are ***Sarkisov links***. So the theorem is that a map (what map?) can be factorized by Sarkisov links.

Theorem (Corti-Kalog(?) 2016) Any volume-preserving birrational map between Mori-fiber CY pairs is a composition of volume-preserving Sarkisov links. Now every intermidiate variety admits a divisor making it a CY pair:

$$\begin{tikzcd}
(X, D_X) \arrow[r, dashed, "\varphi_1"] \arrow[d] \arrow[rrrr, bend left, "\varphi"] & (X_1, D_1) \arrow[r, dashed, "\varphi_2"] \arrow[d] & (X_2, D_2) \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, dashed] & (Y, D_Y) \arrow[d] \\
T_X & T_1 & T_2 & & T_Y
\end{tikzcd}$$

Remark

- The Sarkisov links

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dashed] \arrow[d] & X_1 \arrow[d] \\ \mathrm{pt} & T_1 \end{tikzcd}$$

  are not classified.

- Depending on D, the volume-preserving Sarkisov links

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dashed] \arrow[d] & X_1 \arrow[d] \\ \mathrm{pt} & T_1 \end{tikzcd}$$

  can be classified.

Theorem (A-Coti-Massareti) $D \subset \mathbb{P}^n$ a *general* hypersurface of degree $n+1$. If you want to study the birational group of the pair $(\mathbb{P}^n, D)$, $\mathrm{Bir}(\mathbb{P}^n, D)$, then this group is not interesting. Because $\mathrm{Bir}(\mathbb{P}^n, D) = \mathrm{Aut}(\mathbb{P}^n, D)$.

And then also consider D smooth instead of general, and with Picard rank $\rho(D) = 1$.

Theorem (A-C-M) $D \subset \mathbb{P}^3$ general (A1 singularity and $\rho(S) = 1$) quartic with a singularity. $\mathrm{Bir}(\mathbb{P}^3, D) \cong \mathbb{G} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{G}$ is "form of $\mathbb{G}m$ over $\mathbb{C}(x, y)$".

0.8 Last application

$X = X_d \subset \mathbb{P}^{n+1}$ smooth hypersurface of degree d. I want to study the automorphism group.

Theorem (Matsumata-Monsky, 1964) Except for the two cases when $\mathcal{H}$, $(n, d) = (1, 3)$ or $(2, 4)$, any smooth automorphism is the restricition of an automorphism of the ambient space, i.e. $\mathrm{Aut}(\mathbb{P}^{n+1}, X) \twoheadrightarrow \mathrm{Aut}(X)$.

In the exceptional cases: for $(n, d) = (1, 3)$ we get $\mathrm{Bir}(\mathbb{P}^2, C) \twoheadrightarrow \mathrm{Aut}(C)$ C is a curve. For $(n, d) = (2, 4)$, we *ask*: when $\mathrm{Bir}(\mathbb{P}^3, S) \twoheadrightarrow \mathrm{Aut}(S)$?

Theorem (A-Paiva-Zika) Complete solution $\rho(S) = 2$.