Symplectic and contact nature of Riemannian geometry

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Surfaces of constant extrinsic curvature k.

	$\mathbb{H}^2$	$\mathbb{R}^3$	S^3
$k > 1$	S^3	S^2	S^2
$k = 1$	Horospheres, horocycles		
$0 < k < 1$	ess. param., $\mathcal{H}(\mathbb{D}) \sqcup \mathcal{H}(\mathbb{C}) \backslash \{\mathbb{C}\}$		

Remark Intrinsic curvature = extrinsic curvature + sectional curvature of the ambient space.

X a 3-manifold, take $x \in X$ and $v_x \in TX$. The Levi-Civita connection provides

$$\begin{aligned} T_v TX &\cong H_v TX \oplus V_x TX \\ &\cong \underbrace{TxX}_{\text{hor.}} \oplus \underbrace{T_x X}_{\text{vert.}} \end{aligned}$$

- Saski metric
$$\langle (\xi, \mu)(\xi', \mu') \rangle = \langle \xi, \xi' \rangle + \langle \mu, \mu' \rangle$$
- Symplectic form
$$\omega\Big((\xi, \mu), (\xi', \mu')\Big) = \langle \xi, \mu' \rangle - \langle \xi', \mu \rangle$$
which we may pull back to X via the musical isomorphism to obtain the ***Saski symplectic form*** of T^*X $\omega = \flat^* \omega_{\text{st}}$.
-
$$m\Big((\xi, \mu), (\xi', \mu')\Big)$$
- There is a complex structure I
- And a quadratic form
$$m = \begin{pmatrix} 0 & I_0 \\ I_0 & 0 \end{pmatrix}$$

And a contact bundle pulling the luiville form from T^*X. And then there is a another complex structure J. So define $K := IJ$, which gives a **hyperkähler contact structure** on the unit tangent bundle.

Upshot We have created a pseudoholomorphic curve from a k-surface by taking the unit tangent normal. That is, consider the normal map as an embedding of our surface in the unit tangent bundle.

Theorem (Jurgens) $\Sigma \subset \mathbb{R}^4$

(i) complete,

(ii) J-holomorphic

(iii) $m|_{T\Sigma} \geqslant 0$

then Σ is a plane. (Morally, the graph of a linear function.)

Theorem (dim 3,4 Calabi, dim 5$\geqslant$ Pgorelov) $f : \mathbb{R}^2 \to \mathbb{R}$

(i) f convex,

(ii) $\det \mathrm{Hess}(f) = 1$ (Monge-Ampère)

Then f is quadratic.

Proof. Short, done in seminar. $\square$

And then we want to prove a compactness property. So we take a sequence of surfaces $(\Sigma_m, p_n) \subset S^1X$ with $p_n \in \Sigma_n$. Suppose $\| \mathrm{II}_m \| \xrightarrow{m\to\infty} \infty$. Then there exists B_m and q_m such that

(i) $\| \mathrm{II}_m(q_m) \| = B_m$.

(ii) $B_m \xrightarrow{m\to\infty} \infty$ and,

(iii) by a lemma that is easy to prove, $\forall r \in B_{\frac{1}{2\sqrt{\| \mathrm{II}_m(q_m) \|}}}(q_m)$.

And then we rescale the metrics $g \to g_m = B_m^2 g$. That makes the metric be flatter and flatter, like zooming in, and also makes the shape operator of every surface have norm 1. and we get

$$(S^1X, g_m) \longrightarrow (\mathbb{R}^5,)$$
$$\Sigma_m \longrightarrow \Sigma_m \subset \mathbb{R}^5$$

with respect to the famous Cheeger Gromov topology in the limit, which is not so easily defined. And by Arzelá-Ascoli and Elliptic regularity magic (see M. Joshi Course notes) (regularity is not smoothness but some differentiability) the limit surface is compact, positive, J-homolomorphic. And by Jurgen's theorem they are flat. But that's a contradiction with the fact that the shape operators of these surfaces have norm 1.

So you have compactness. Yaaaaay!