Invent. math. 123, 35–59 (1996)

Inventiones mathematicae

A Kähler structure on the moduli space of isometric maps of a circle into Euclidean space

John J. Millson★**, Brett Zombro**★★

Department of Mathematics, University of Maryland, College Park, MD 20742, USA; e-mail: jjm@math.umd.edu. and zombro@math.umd.edu

Oblatum 15-X-1994 & 5-VII-1995

Abstract. We study the spaces $\mathscr{M}$ and $\mathscr{M}'_{\mathrm{Lip}}$ of smooth (resp. non-degenerate Lipschitz) isometric maps of a circle into Euclidean space modulo orientation preserving Euclidean motions. We prove that $\mathscr{M}$ and $\mathscr{M}'_{\mathrm{Lip}}$ are infinite dimensional Kähler manifolds. In particular, they are complex Fréchet (resp. Banach) manifolds. This is proved by an infinite dimensional version of the Kirwan, Kempf–Ness Theorem [Kir84], [KN78], [Nes84] relating symplectic quotients to holomorphic quotients, applied to the action of $PSL_2(\mathbb{C})$ on the free loop space of S^2.

Introduction

In [KM], the authors studied the moduli space M_r of n-gons in Euclidean space $\mathbb{E}^3$ with fixed side lengths $r_1, r_2, \ldots, r_n$ modulo proper Euclidean motions. They identified M_r with the weighted quotient Q_{sst} of n points on S^2 constructed by Deligne and Mostow in [DM86]. This was accomplished by an extension of the theorem of Kirwan, Kempf and Ness, [Kir84], [KN78], [Nes84] relating symplectic quotients to "Mumford quotients". This extension used the notion of conformal center of mass of Douady and Earle [DE86]. The authors constructed a Hamiltonian action of $\mathbb{R}^{n-3}$ on M_r with Lagrangian orbits by "bending" along diagonals and used this action to construct action-angle variables on a dense open subset $M_r^O \subset M_r$. They also proved that such bendings act transitively on M_r, for generic r. The identification of M_r and Q_{sst} was obtained independently (and earlier) by a different method in [Kly92].

The purpose of this paper and its successor [MZ] is to extend these results to "∞-gons with fixed side lengths," that is, to smooth isometric maps

★ This research was supported in part by NSF grant DMS-92-05154.
★★ This research was partially supported by AFOSR grant F49620-92-J-0093.

(necessarily immersions) of S^1 into $\mathbb{E}^3$ modulo proper Euclidean motions. This quotient will be denoted $\mathscr{M}$. It is remarkable that most of the results (and their proofs) of [KM] generalize.

In this paper the geometric invariant theory of $PSL_2(\mathbb{C})$ acting on $(S^2)^n$ of [DM86] is replaced by $PSL_2(\mathbb{C})$ acting on the free loop space LS^2. We define the notions of stable and semi-stable loops $(LS^2)^{\mathrm{s}}$ and $(LS^2)^{\mathrm{ss}}$ by analogy with $(S^2)^n$ (a stable loop on S^2 spends less than half its time at any point). We then prove the corresponding Kirwan, Kempf–Ness Theorem identifying the symplectic quotient of LS^2 by $SO(3)$ to the holomorphic quotient of $(LS^2)^{\mathrm{s}}$ by $PSL_2(\mathbb{C})$. A key role is played by the conformal center of mass of Douady and Earle [DE86]. We give a self-contained treatment of this notion in Sect. 4. We conjecture that there is an infinite dimensional representation of $PSL_2(\mathbb{C})$ and a corresponding equivariant projective embedding of LS^2 such that $(LS^2)^{\mathrm{s}}$ and $(LS^2)^{\mathrm{ss}}$ coincide with those given by a suitable infinite dimensional geometric invariant theory. It is possible that such a representation can be obtained by "quantizing" LS^2, i.e., by considering the L^2 holomorphic sections of a holomorphic Hermitian line bundle over LS^2. To this end in Sect. 6, we construct a holomorphic line bundle $\mathscr{L}$ with connection ∇ such that the curvature of ∇ is the Kähler form of LS^2. Jean–Luc Brylinski has observed that $\mathscr{L}$ is the "average pull-back" of the hyperplane section bundle over S^2 via the evaluation map $S^1 \times LS^2 \to S^2$ in the sense of [Bry]. As a first step toward the conjecture, one needs to construct holomorphic sections of $\mathscr{L}^k$, for suitable k. Our main theorem relating symplectic quotients and holomorphic quotients is analogous to the results of [AB82], [Cor88] and [Don83] for the gauge group acting on connections.

It is interesting to relate our results to those of Brylinski in [Bry92]. Brylinski studies the space of unparametrized loops $\widehat{Y}$ in a three-manifold M^3 and obtains a symplectic structure and an almost complex structure J with vanishing Nijenhuis tensor. The J-operator of Brylinski is local and can be described as follows: Let g be a Riemannian metric on M and $r \in \widehat{Y}$. Let $\eta \in T_r(\widehat{Y})$. We may identify η with a vector field along r which is normal to r; if γ is the unit tangent to r, then $g(\gamma, \eta) = 0$. Then $J_r(\eta)$ is obtained by rotating η by 90° in the normal bundle to r. Thus, $J_r(\eta) = \gamma \times \eta$. In [Bry92], Brylinski proved that the Nijenhuis tensor of J was zero. However, in [Lem93], Lempert proved that $\widehat{Y}$ is never a complex Fréchet manifold.

It is the main point of this paper that in the case $M = \mathbb{R}^3$ there is a non-local almost complex structure, which we again denote J, on the above moduli space $\mathscr{M}$ *which is integrable*. Let $r \in \mathscr{M}$, $\eta \in T_r(\mathscr{M})$ and $\gamma = r_s$ be the unit tangent to r. Then η may be represented by a map $\eta : S^1 \to \mathbb{R}^3$ such that $\eta(0) = 0$ (a cross-section to the translations) and $\gamma \cdot \eta_s = 0$ where $\cdot$ is the Euclidean dot-product (this equation is the infinitesimal version of $\gamma \cdot \gamma = 1$). Then our J-operator is given by the formula

$$J_r(\eta) = \int_0^s r_s(t) \times \eta_s(t)\, dt .$$

Roughly speaking, J is the pull-back of the integrable almost complex structure on the complex manifold LS^2 by the map sending r to γ. However, this map is not onto. We deal with this by using our analogue of the theorem of Kirwan, Kempf and Ness to identify $\mathcal{M}$ with a "Mumford quotient" of LS^2 by $PSL_2(\mathbb{C})$. It follows that J is integrable and $\mathcal{M}$ is a complex manifold. Our space $\mathcal{M}$ is acted on by S^1 and the quotient $\mathcal{M}/S^1$ is the hypersurface in Brylinski's space corresponding to loops of length 2π.

Since our space $\mathcal{M}$ is an infinite-dimensional analogue of the space Q_{sst}, it is natural to ask whether there is a variation of Hodge structure parametrized by $\mathcal{M}$ such that the period map maps to an infinite-dimensional complex ball. In particular, does $\mathcal{M}$ admit a natural complex hyperbolic cone structure [Thu]?

1 The moduli space of smooth isometric maps as a smooth Fréchet manifold

Fix a smooth Riemannian metric g on the circle S^1. Let $\mathrm{Imm}(S^1, \mathbb{R}^3)$ denote the space of C^∞ immersions from S^1 into $\mathbb{R}^3$. Let $(\,\cdot\,,\,\cdot\,)$ denote the usual inner-product on $\mathbb{R}^3$ – we will often write $u \cdot v$ instead of (u, v). Let $\mathrm{Iso}((S^1, g), \mathbb{R}^3) \subset \mathrm{Imm}(S^1, \mathbb{R}^3)$ denote the subspace of isometric immersions. The group $E(3)$ of orientation preserving Euclidean motions acts properly and freely on $\mathrm{Iso}((S^1, g), \mathbb{R}^3)$. The group of dilations $\{a_\lambda : \lambda \in \mathbb{R}_+\}$ acts on $\mathrm{Imm}(S^1, \mathbb{R}^3)$ by post-composition and carries $\mathrm{Iso}((S^1, g), \mathbb{R}^3)$ homeomorphically onto $\mathrm{Iso}((S^1, \lambda^2 g), \mathbb{R}^3)$. Also, a_λ normalizes $E(3)$ in $\mathrm{Diff}(\mathbb{R}^3)$, hence, the moduli spaces associated to g and $\lambda^2 g$ are canonically homeomorphic and we lose no information in assuming that the length of S^1 is 2π. Since any two smooth metrics which assign length 2π to S^1 are isometric we will assume henceforth that $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ where $\mathbb{R}$ has the usual metric ds^2. Thus we will restrict our attention henceforth to

$$\mathcal{M} = \{r \in \mathrm{Imm}(S^1, \mathbb{R}^3) \colon (r_s, r_s) = 1\}/E(3)\,.$$

In other words, $\mathcal{M}$ is the space of smoothly immersed closed curves in $\mathbb{R}^3$ of length 2π parametrized by arc-length s.

Remark 1.1. The spaces $\mathrm{Iso}((S^1, g), \mathbb{R}^3)/E(3)$ for g as above, are symplectic leaves for a Poisson structure on $C^\infty(S^1, \mathbb{R}^3)$, [MZ]. We may regard $\mathrm{Iso}((S^1, g), \mathbb{R}^3)$ as the configuration space of an inhomogeneous wire loop with linear density $\sqrt{g}$.

In this section, we show that $\mathcal{M}$ is naturally a C^∞ Fréchet manifold. Let $\mathcal{F} \subset \mathrm{Iso}(\mathbb{R}, \mathbb{R}^3)$ consist of those $r : \mathbb{R} \to \mathbb{R}^3$ such that $r(0) = 0$ and r_s is smoothly periodic with period 2π. We do not assume that $r(2\pi) = r(0) = 0$. Since $(r_s, r_s) = 1$, the map $\gamma : S^1 \to \mathbb{R}^3$ defined by $\gamma(s) = r_s(s)$ takes values in S^2 – it is the Gauss map for r. Thus, we obtain a natural $SO(3)$-equivariant homeomorphism $\varepsilon : \mathcal{F} \to LS^2$ defined by $\varepsilon(r) = \gamma$. By [Ham82, Part II, Corollary 2.3.2], LS^2 is a (tame) Fréchet manifold, see also Proposition 5.4 below.

We now observe that LS^2 inherits from S^2 a weak Riemannian metric $(\cdot\,,\,\cdot)$, an almost complex structure J and a weak symplectic form Ω. Indeed let $\alpha, \beta \in T_\gamma(LS^2) = \Gamma(S^1, \gamma^* T(S^2))$. Then

$$(\alpha, \beta) = \int_0^{2\pi} \alpha(s) \cdot \beta(s)\, ds \tag{1.1}$$

$$J_\gamma \alpha(s) = \gamma(s) \times \alpha(s) \tag{1.2}$$

$$\Omega_\gamma(\alpha, \beta) = \int_0^{2\pi} \gamma(s) \cdot (\alpha(s) \times \beta(s))\, ds\,. \tag{1.3}$$

Here $\times$ denotes the cross-product of vectors in $\mathbb{R}^3$.

We first verify that Ω is closed. For the definition of the exterior differential of a differential form on a smooth Fréchet manifold, see [Bry92, Sect. 1.4]. We will give two proofs that Ω is closed, the first computational, the second functorial.

Lemma 1.1. *Ω is closed.*

Proof (Computational). We observe that Ω is the restriction of a 2-form Ω on $C^\infty(S^1, \mathbb{R}^3)$ defined by the same formula. Let vol be the volume form on $\mathbb{R}^3$. Then vol induces a 3-form Ψ on $C^\infty(S^1, \mathbb{R}^3)$ by

$$\Psi(\xi, \eta, \nu) = \int_0^{2\pi} \mathrm{vol}(\xi(s), \eta(s), \nu(s))\, ds = \int_0^{2\pi} \xi \cdot (\eta \times \nu)\, ds$$

for $\xi, \eta, \nu \in C^\infty(S^1, \mathbb{R}^3)$.

We claim $d\Omega = 3\Psi$. Indeed let η, ξ, ν be parallel vector fields on $C^\infty(S^1, \mathbb{R}^3)$. Then

$$d\Omega_\gamma(\xi, \eta, \nu) = \sum \xi(\Omega_\gamma(\eta, \nu))$$

where the sum is over the cyclic permutations of ξ, η, ν. But

$$\xi\left(\int_0^{2\pi} \gamma \cdot (\eta \times \nu)\, ds\right) = \int_0^{2\pi} \xi \cdot (\eta \times \nu)\, ds\,. \tag{1.4}$$

Since any tangent vector to $C^\infty(S^1, \mathbb{R}^3)$ extends to a parallel vector field, the claim follows. But it is clear that the restriction of Ψ to $C^\infty(S^1, S^2)$ is zero. This concludes the first proof. □

Proof (Functorial). Let $\mathrm{ev} : LS^2 \times S^1 \to LS^2$ be evaluation and $\pi : LS^2 \times S^1 \to LS^2$ be the projection. Let ω be the Riemannian volume form of S^2. Then

$$\Omega = \pi_* \mathrm{ev}^*(\omega) \wedge ds\,;$$

where π_* is integration over the fiber. □

Corollary 1.2. *$\frac{1}{8\pi^2}\Omega$ is an integral class.*

Proof. $\frac{1}{4\pi}\omega$ and $\frac{1}{2\pi}ds$ are integral. But ev^* and π_* carry integral classes to integral classes. □

Remark 1.2. We will prove in Sect. 5 that LS^2 is a complex manifold with associated almost complex structure J. Thus LS^2 is a Kähler manifold. In Sect. 6, we will prove that $\frac{1}{8\pi^2}\Omega$ is the Chern form of a holomorphic Hermitian line bundle over $\mathscr{M}$.

We now record the induced Riemannian and symplectic structures $(\,\cdot\,,\,\cdot\,)$ and Ω on $\mathscr{F}$. Note first that $\mathscr{F} \subset C^\infty(\mathbb{R}, \mathbb{R}^3)$. Consequently, we may identify $T_r(\mathscr{F})$ with the subspace of $C^\infty(\mathbb{R}, \mathbb{R}^3)$ consisting of those a satisfying

(i) a_s is smoothly periodic;
(ii) $a_s(s) \cdot r_s(s) = 0, \forall s \in \mathbb{R}$;
(iii) $a_s(0) = 0$.

Then for $a, b \in T_r(\mathscr{F})$ we have

$$(a,b) = \int_0^{2\pi} a_s(s) \cdot b_s(s)\, ds \tag{1.5}$$

$$\Omega_r(a,b) = \int_0^{2\pi} r_s(s) \cdot (a_s(s) \times b_s(s))\, ds\,. \tag{1.6}$$

The group $SO(3)$ acts on LS^2 by post-composition and this action preserves Ω. The next lemma and its corollary are critical for all that follows. We will henceforth identify the Lie algebra $so(3)$ of $SO(3)$ with $(\mathbb{R}^3, \times)$.

Lemma 1.3. *The above action of $SO(3)$ is Hamiltonian and has associated momentum map $\mu : LS^2 \to \mathbb{R}^3 = so(3)$ given by*

$$\mu(\gamma) = \int_0^{2\pi} \gamma(s)\, ds\,.$$

Proof. Let $v \in \mathbb{R}^3 = so(3)$. Then the induced vector field v_γ on LS^2 is given by $v_\gamma = v \times \gamma$. Thus we are required to prove for $\alpha \in T_\gamma(LS^2)$

$$\langle d\mu(\gamma)(\alpha), v\rangle = \Omega_\gamma(v_\gamma, \alpha)\,.$$

This equation translates into

$$\int_0^{2\pi} \alpha \cdot v\, ds = \int_0^{2\pi} \gamma \cdot ((v \times \gamma) \times \alpha)\, ds\,.$$

But $(v \times \gamma) \times \alpha = (v \cdot \alpha)\gamma - (\gamma \cdot \alpha)v = (v \cdot \alpha)\gamma$ since $\gamma \cdot \alpha = 0$. □

Corollary 1.4. *Let $r \in \mathscr{F}$. Then $r(2\pi) = 0$ if and only if $\mu(\varepsilon(r)) = 0$.*

Remark 1.3. Thus, the closing condition $r(2\pi) = 0$ on $r \in \mathscr{F}$ is equivalent to the vanishing of a momentum map. This was a main point of [KM] and will be exploited in an analogous fashion here.

The rest of this section will be devoted to proving that $\mathcal{M}$ is a smooth Fréchet manifold. We will use an easy special case of the Nash–Moser Theorem – [Ham82, Theorem 2.3.1 of Part III]. This theorem states roughly that the usual implicit function theorem is valid for maps from tame Fréchet spaces to finite dimensional spaces. We note LS^2 is tame by [Ham82, Part II, Corollary 2.3.2].

Let $\mathcal{N} \subset LS^2$ be the zero level set of the above momentum map μ. Then

$$\mathcal{N} = \left\{ \gamma \in LS^2 : \int_0^{2\pi} \gamma(s)\, ds = 0 \right\} .$$

Lemma 1.5. *$\mathcal{N}$ is a smooth Fréchet submanifold of LS^2.*

Proof. It suffices to prove that 0 is a regular value of $\mu : LS^2 \to \mathbb{R}^3$. Let $\gamma \in \mathcal{N}$. The image of $d\mu(\gamma)$ is a linear subspace $W \subseteq \mathbb{R}^3$. Suppose W is a proper subspace. Hence, there exists $v \in W^{\perp}$ with $\|v\| = 1$. Then for all $\alpha \in \Gamma(S^1, \gamma^* T(S^2))$ we have

$$d\mu(\gamma)(\alpha) \cdot v = \int_0^{2\pi} \alpha(s) \cdot v\, ds = 0 .$$

Define $\widehat{v} \in \Gamma(S^1, \gamma^* T(S^2))$ by $\widehat{v}(s) = v - (v \cdot \gamma(s))\gamma(s)$. Then

$$\int_0^{2\pi} \widehat{v}(s) \cdot v = 2\pi - \int_0^{2\pi} (v \cdot \gamma(s))^2\, ds .$$

But $(v \cdot \gamma(s))^2 \leqq 1$ and $(v \cdot \gamma(s))^2 = 1$ if and only if $\gamma(s) = \pm v$. Suppose $\gamma(s) \in \{\pm v\}$ for all s. Since γ is continuous, either $\gamma(s) = v$ for all s or $\gamma(s) = -v$ for all s contradicting $\int_0^{2\pi} \gamma(s)\, ds = 0$. Thus if $U \subset S^1$ is the open set given by $U = \{s \in S^1 : (\gamma(s) \cdot v)^2 < 1\}$, then U is non-empty. Hence, $\int_0^{2\pi} (v \cdot \gamma(s))^2\, ds < 2\pi$ and $\int_0^{2\pi} \widehat{v}(s) \cdot v\, ds \neq 0$. This is a contradiction. □

Now $\mathcal{M} = \mathcal{N}/SO(3)$. It is easy to see that $SO(3)$ acts properly and freely on $\mathcal{N}$. In order to prove that $\mathcal{M}$ is a smooth manifold and that the quotient map $\pi : \mathcal{N} \to \mathcal{M}$ is a principal bundle it suffices to find smooth local cross-sections to the $SO(3)$ orbits.

Let $s_0 \in S^1$. Define $\mathcal{U}_{s_0} \subset \mathcal{N}$ by

$$\mathcal{U}_{s_0} = \{ r \in \mathcal{N} : \kappa(r)(s_0) \neq 0 \} .$$

Here $\kappa(r)$ is the curvature of r. We note that $\mathcal{U}_{s_0}$ is $SO(3)$-invariant. We note also that $\{\mathcal{U}_s : s \in S^1\}$ is an open cover of $\mathcal{N}$. Thus, it suffices to construct a smooth cross-section Σ_s to the $SO(3)$-orbits in $\mathcal{U}_s$.

We now construct $\Sigma_{s_0} \subset \mathcal{U}_{s_0}$. Let $B(\mathbb{R}^3)$ be the manifold of properly oriented orthonormal bases for $\mathbb{R}^3$. Let $F : \mathcal{U}_{s_0} \to B(\mathbb{R}^3)$ be the smooth map that to $r \in \mathcal{U}_{s_0}$ assigns the Frenet frame $F(r)$ for $T_{r(s_0)}(\mathbb{R}^3) = \mathbb{R}^3$. Let b be the standard basis for $\mathbb{R}^3$. We then define

$$\Sigma_{s_0} = \{r \in \mathscr{U}_{s_0} : F(r) = b\} .$$

Lemma 1.6. *The space Σ_{s_0} is a smooth cross-section to the $SO(3)$-orbits in $\mathscr{U}_{s_0}$.*

Proof. Clearly Σ_{s_0} is a cross-section. It remains to prove Σ_{s_0} is smooth. By the Implicit Function Theorem, it suffices to prove that b is a regular value of F. Let $r \in \Sigma_{s_0} = F^{-1}(b)$ and define $\sigma_r : SO(3) \to \mathscr{U}_{s_0}$ by $\sigma_r(g) = gr$. Then $F \circ \sigma_r : SO(3) \to B(\mathbb{R}^3)$ is a diffeomorphism. □

We obtain the required result

Theorem 1.7. *The spaces $\mathscr{M}$ and $\mathscr{N}$ are smooth Fréchet manifolds and the quotient map $p : \mathscr{N} \to \mathscr{M}$ is a smooth $SO(3)$ principal bundle.*

Proof. Define $A : SO(3) \times \Sigma_{s_0} \to \mathscr{U}_{s_0}$ by $A(g,r) = g \circ r$. Then A is a smooth bijection. We claim A^{-1} is smooth. Indeed define $g : \mathscr{U}_{s_0} \to SO(3)$ by the equation $F(r) = g(r)^{-1} \cdot b$. Clearly $g(r)$ is smooth. But $A^{-1}(r) = (g(r)^{-1}, g(r) \cdot r)$.

□

We conclude this section by describing the tangent spaces and Riemannian, symplectic and almost complex structures on $\mathscr{M}$. We begin by describing the horizontal distribution of a connection on the principal bundle $p : \mathscr{N} \to \mathscr{M}$. We define

$$T_\gamma^{\mathrm{hor}}(\mathscr{N}) = \left\{ \alpha \in T_\gamma(\mathscr{N}) : \int_0^{2\pi} \gamma(s) \cdot (\alpha(s) \times v)\, ds = 0 \ \forall v \in \mathbb{R}^3 \right\} .$$

Remark 1.4. The defining equation for $T_\gamma^{\mathrm{hor}}(\mathscr{N})$ may be rewritten

$$\int_0^{2\pi} \alpha(s) \cdot (v \times \gamma(s))\, ds = 0 .$$

Since $v_\gamma(s) = v \times \gamma(s)$ is the vector field on $\mathscr{N}$ induced by $v \in \mathbb{R}^3 = so(3)$ we see that $T_\gamma^{\mathrm{hor}}(\mathscr{N})$ is the orthogonal complement to the tangent space to the $SO(3)$-orbit through γ.

We identify $T_{p(\gamma)}(\mathscr{M})$ with $T_\gamma^{\mathrm{hor}}(\mathscr{N})$ and obtain formulae for the symplectic form Ω, Riemannian metric $(\cdot\, , \cdot)$ and almost complex structure J on $T_{p(\gamma)}(\mathscr{M})$ by restricting the previous ones for $T_\gamma(\mathscr{F})$. We need to observe that $T_\gamma^{\mathrm{hor}}(\mathscr{N})$ is J-stable. This follows immediately from the "closing condition" defining $\alpha \in T_\gamma(\mathscr{N})$

$$\int_0^{2\pi} \alpha(s)\, ds = 0$$

and the "coclosing condition" defining $\alpha \in T_\gamma^{\mathrm{hor}}(\mathscr{N})$, namely

$$\int_0^{2\pi} \gamma(s) \times \alpha(s)\, ds = \int_0^{2\pi} J\alpha(s)\, ds = 0 .$$

2 The moduli space of non-degenerate Lipschitz isometric maps as a smooth Banach manifold

In this section, we prove results analogous to those of the previous section with smooth isometric maps replaced by Lipschitz isometric maps. There is an important new phenomenon that enters.

Definition 2.1. *A Lipschitz isometric map $r : S^1 \to \mathbb{R}^3$ is degenerate if its image is contained in a line.*

Remark 2.1. We will see that the singularities of the moduli space of Lipschitz isometric maps occur at the degenerate maps.

Let $\mathrm{Lip}(S^1, \mathbb{R}^3)$ denote the space of Lipschitz maps from S^1 to $\mathbb{R}^3$ and $W^{k,p}(S^1, \mathbb{R}^3)$ denote the Sobolev space of maps from S^1 to $\mathbb{R}^3$ with k derivatives of class L^p. Then it is standard that

$$\mathrm{Lip}(S^1, \mathbb{R}^3) = W^{1,\infty}(S^1, \mathbb{R}^3) .$$

We let $\mathrm{Iso\,Lip}(S^1, \mathbb{R}^3)$ denote the isometric Lipschitz from S^1 to $\mathbb{R}^3$. More precisely,

$$\mathrm{Iso\,Lip}(S^1, \mathbb{R}^3) = \{r \in \mathrm{Lip}(S^1, \mathbb{R}^3) \colon (r_s, r_s) = 1 \text{ a.e.}\} .$$

We recall that a Lipschitz map is differentiable a.e.. We define $\mathscr{M}_{\mathrm{Lip}}$ to be the quotient of $\mathrm{Iso\,Lip}(S^1, \mathbb{R}^3)$ by $E(3)$.

We define $\mathscr{F}_{\mathrm{Lip}} \subset \mathrm{Lip}(S^1, \mathbb{R}^3)$ by

$$\mathscr{F}_{\mathrm{Lip}} = \{r \in \mathrm{Lip}([0, 2\pi], \mathbb{R}^3) \colon r_s \cdot r_s = 1 \text{ a.e. and } r(0) = 0\} .$$

We obtain an $SO(3)$-equivariant homeomorphism $\varepsilon : \mathscr{F}_{\mathrm{Lip}} \to (LS)^2_{(\infty)}$ given by

$$\varepsilon(r) = r_s .$$

Here, $(LS)^2_{(\infty)}$ denotes the measurable loops on S^2. Thus, in order to prove that $\mathscr{F}_{\mathrm{Lip}}$ is a smooth Banach manifold, it suffices to prove $(LS)^2_{(\infty)}$ is.

Proposition 2.1. *The space $(LS)^2_{(\infty)}$ is a smooth submanifold of the Banach space $L^\infty(S^1, \mathbb{R}^3)$.*

The proposition will follow from the next two lemmas. We define $q : L^\infty(S^1, \mathbb{R}^3) \to L^\infty(S^1, \mathbb{R})$ by

$$q(F) = (F, F) .$$

Then q is smooth and $(LS)^2_{(\infty)} = q^{-1}(1)$.

We choose $F_0 \in q^{-1}(1)$ and define subspaces R and T of $L^\infty(S^1, \mathbb{R}^3)$ by

$$R = \{G \in L^\infty(S^1, \mathbb{R}^3) : G \times F_0 = 0\}$$

and

$$T = \{G \in L^\infty(S^1, \mathbb{R}^3) \colon (G, F_0) = 0\} .$$

We define $p_1 : L^\infty(S^1, \mathbb{R}^3) \to T$ and $p_2 : L^\infty(S^1, \mathbb{R}^3) \to R$ by

$$p_1(G) = G - (G, F_0)F_0$$
$$p_2(G) = (G, F_0)F_0 \, .$$

Clearly, p_1 and p_2 are norm-decreasing. It is clear that $p_1 + p_2 = \mathrm{id}$ whence

$$L^\infty(S^1, \mathbb{R}^3) = R + T \, .$$

Lemma 2.2. *The above sum is direct.*

Proof. Suppose $G \in R \cap T$. There is a subset $U \subset S^1$ of full measure such that if $x \in U$ then $F_0(x) \in S^2$. For all such x we have

$$G(x) \times F_0(x) = 0 \quad \text{and} \quad (G(x), F_0(x)) = 0$$

whence $G(x) = 0$. Thus $G = 0$. □

Lemma 2.3. *The map q is a submersion of F_0.*

Proof. Let $G \in L^\infty(S^1, \mathbb{R}^3)$. Then

$$dq(F_0)(G) = 2(F_0, G) \, .$$

Thus, if $f \in L^\infty(S^1, \mathbb{R})$, then

$$dq(F_0)(\tfrac{1}{2} f F_0) = f \, .$$

□

Corollary 2.4. *$\mathscr{F}_{\text{Lip}}$ is a smooth Banach manifold.*

We now consider the momentum map $\mu : \mathscr{F}_{\text{Lip}} \to \mathbb{R}^3$ for the natural action of $SO(3)$ on $\mathscr{F}_{\text{Lip}}$. The argument of Lemma 1.1 generalizes to give

$$\mu(r) = \int_0^{2\pi} r_s(s)\, ds = r(2\pi)$$

whence

$$d\mu(r)(X) = \int_0^{2\pi} X_s(s)\, ds = \int_0^{2\pi} n(s)\, ds \, .$$

Here we have set $n = X_s$.

We let $\widetilde{\mathscr{U}} \subset \mathscr{F}_{\text{Lip}}$ be the set of non-degenerate maps. Thus $\widetilde{\Sigma} = \mathscr{F}_{\text{Lip}} \backslash \widetilde{\mathscr{U}}$ is the set of degenerate maps.

Lemma 2.5. *$\widetilde{\mathscr{U}}$ is open.*

Proof. The set of degenerate maps is the saturation under $SO(3)$ of the space of maps contained in the x-axis. This latter set is closed since it is the set of r satisfying $r \times e_1 = 0$. It is immediate that the saturation of a closed subset by a compact group is a closed subset. □

We will need the following criterion for a map r to be non-degenerate. In what follows, we assume r is fixed and that r_s exists on a subset $U \subset S^1$ of full measure. Let $u \in \mathbb{R}^3$. Then we define $u^N \in L^\infty(S^1, \mathbb{R}^3)$ by

$$u^N(s) = u - (u, r_s) r_s(s), \quad s \in U .$$

Lemma 2.6. *Suppose $u \in \mathbb{R}^3$ and $u^N = 0$ on a set $V \subset U$ of full measure. Then r is contained in the line through u.*

Proof. By assumption, we have $(r \times u)_s = r_s \times u = 0$ on V. But $r \times u$ is absolutely continuous and V has full measure whence $r \times u$ is constant. But $r(0) = 0$. □

Lemma 2.7. *Suppose r is non-degenerate. Then e_1^N, e_2^N and e_3^N are independent elements of $L^\infty(U, \mathbb{R}^3)$.*

Proof. Suppose not. Then there is a dependence relation

$$c_1 e_1^N + c_2 e_2^N + c_3 e_3^N = 0$$

in $L^\infty(U, \mathbb{R}^3)$. Define $u \in \mathbb{R}^3$ by $u = c_1 e_1 + c_2 e_2 + c_3 e_3$. Then $u^N = 0$ in U whence r is degenerate by the previous lemma. □

Lemma 2.8. *Suppose r is non-degenerate. Then μ is a submersion near r.*

Proof. Let $v = ae_1 + be_2 + ce_3 \in \mathbb{R}^3$. We want to find $n \in L^\infty(S^1, \mathbb{R}^3)$ with $(n, r_s) = 0$ a.e. such that $\int_0^{2\pi} n(s)\, ds = v$.

Since r is non-degenerate, e_1^N, e_2^N and e_3^N are independent elements of $L^\infty(U, \mathbb{R}^3)$ hence of $L^1(U, \mathbb{R}^3)$. Hence, there exist elements n_1, n_2 and $n_3 \in L^\infty(U, \mathbb{R}^3)$ such that

$$\int_U (n_i, e_j^N)\, ds = \delta_{ij} .$$

Hence $\int_U (n_i^N, e_j^N)\, ds = \delta_{ij}$. We extend n_i^N, $1 \leqq i \leqq 3$, to S^1 by zero. We obtain

$$\int_0^{2\pi} (n_i^N, e_j^N)\, ds = \int_0^{2\pi} (n_i^N, e_j)\, ds = \delta_{ij} .$$

Hence, if we define $n = an_1^N + bn_2^N + cn_3^N$, then $\int_0^{2\pi} n(s)\, ds = v$. □

Let $\mathcal{N}_{\mathrm{Lip}}$ be the inverse image of 0 under μ. We have proved the following proposition.

Proposition 2.9. *$\mathcal{N}_{\mathrm{Lip}} \cap \widetilde{\mathcal{U}}$ is a smooth submanifold of $\mathcal{F}_{\mathrm{Lip}}$ of codimension 3.*

Finally, we have to pass to the quotient $\mathcal{N}_{\mathrm{Lip}} \cap \widetilde{\mathcal{U}}$ by $SO(3)$. To show that the quotient is a Banach manifold, it suffices to exhibit smooth local cross-sections to the $SO(3)$ orbits on $\widetilde{\mathcal{U}}$.

Let $s_1, s_2 \in S^1$ and define

$$\mathcal{U}_{s_1, s_2} = \{r \in \widetilde{\mathcal{U}} \colon r(s_1) \text{ and } r(s_2) \text{ are independent}\} .$$

We observe that $\mathscr{U}_{s_1,s_2}$ is $SO(3)$ invariant. Clearly $\{\mathscr{U}_{s_1,s_2}: s_1,s_2 \in S^1\}$ covers $\widetilde{\mathscr{U}}$. We now construct a cross-section Σ_{s_1,s_2} to the orbits of $SO(3)$ acting on $\mathscr{U}_{s_1,s_2}$. We define a map $F : \mathscr{U}_{s_1,s_2} \to SO(3)$ by

$$F(r) = \mathrm{Gram}(r(s_1), r(s_2), r(s_1) \times r(s_2)) .$$

Here Gram($\cdot$) denotes the Gram–Schmidt orthogonalization procedure. Clearly, F is an equivariant map; that is

$$F(gr) = gF(r), \quad r \in \mathscr{U}_{s_1,s_2}, \ g \in SO(3) .$$

We define $\Sigma_{s_1,s_2} \subset \mathscr{U}_{s_1,s_2}$ by

$$\Sigma_{s_1,s_2} = F^{-1}(I) .$$

Clearly, Σ_{s_1,s_2} is a cross-section to the orbits of $SO(3)$ acting on $\mathscr{U}_{s_1,s_2}$. It remains to prove that Σ_{s_1,s_2} is smooth. It suffices to check that I is a regular value for F. To this end, let $\sigma_r : SO(3) \to \mathscr{U}_{s_1,s_2}$ be defined by

$$\sigma_r(g) = g \cdot r .$$

Then $F \circ \sigma_r : SO(3) \to SO(3)$ is the identity map whence $dF(r)$ is surjective and I is a regular value. We have obtained the main result of this section (see the proof of Theorem 1.7).

Theorem 2.10. *$\mathscr{N}_{\mathrm{Lip}} \cap \widetilde{\mathscr{U}}/SO(3)$ is a smooth Banach manifold.*

Remark 2.2. Let $\mathscr{M}'_{\mathrm{Lip}}$ be the space of classes modulo $E(3)$ of non-degenerate Lipschitz isometric maps. Then $\mathscr{M}'_{\mathrm{Lip}}$ is open in $\mathscr{M}_{\mathrm{Lip}}$ and $\mathscr{M}'_{\mathrm{Lip}}$ is isomorphic (via ε) to the above manifold. Thus $\mathscr{M}'_{\mathrm{Lip}}$ is a smooth Banach manifold. We let $\Sigma = \mathscr{M}_{\mathrm{Lip}} - \mathscr{M}'_{\mathrm{Lip}}$ whence Σ is the set of degenerate Lipschitz isometric maps.

Remark 2.3. Note that the (weak) Riemannian metric ($\cdot$, $\cdot$), the symplectic form Ω and the almost complex structure J on $\mathscr{M}$ all extend to $\mathscr{M}'_{\mathrm{Lip}}$ using the formulae in Sect. 1. We will see in Sect. 5 that $\mathscr{M}'_{\mathrm{Lip}}$ is a complex Banach manifold and that the complex structure is compatible with J. Thus $\mathscr{M}'_{\mathrm{Lip}}$ is a (Banach) Kähler manifold.

3 Stable measures on S^2

We let $\mathscr{P}(S^2)$ denote the space of probability measures on S^2. We define $\nu \in \mathscr{P}(S^2)$ to be stable if ν has no atom of mass greater than or equal to 1/2. The measure ν is defined to be semi-stable if ν has no atom of mass greater than 1/2 and nice semi-stable if it has exactly two atoms each of mass 1/2. We let $\mathscr{P}^s(S^2)$ denote the space of stable measures. If $i : S^2 \to \mathbb{R}^3$ is the canonical inclusion then we define the center of mass $B(\nu)$ for $\nu \in \mathscr{P}(S^2)$ by

$$B(\nu) = \int_{S^2} i(x)\, d\nu(x) .$$

If $g \in PSL_2(\mathbb{C})$ and $\nu \in \mathscr{P}(S^2)$ we define $g_*\nu$ to be the push-forward of ν by g. Thus, if $f \in C(S^2)$ then $g_*\nu(f) = \nu(f \circ g)$ and if $X \subset S^2$ is a Borel set, then $g_*\nu(X) = \nu(g^{-1}X)$. We observe that if $k \in SO(3)$, then

$$B(k_*\nu) = kB(\nu) .$$

We note that $\mathscr{P}^s(S^2)$ is invariant under $PSL_2(\mathbb{C})$. The main theorem of this section is the following

Theorem 3.1. *If $\nu \in \mathscr{P}(S^2)$ and ν is stable, then there exists $g \in PSL_2(\mathbb{C})$ such that $B(g_*\nu) = 0$.*

Proof. Douady and Earle in [DE86] define the *conformal* center of mass $C(\nu) \in B^3$ for any stable probability measure ν on S^2. The assignment $C(\nu)$ has the following properties:

(a) For any $g \in PSL_2(\mathbb{C})$,

$$C(g_*\nu) = gC(\nu) ;$$

(b) $B(\nu) = 0$ if and only if $C(\nu) = 0$.

Note that in (a) the group $PSL_2(\mathbb{C})$ acts on B^3 as isometries of hyperbolic 3-space. The theorem follows from the transitivity of this action. □

Remark 3.1. We will give the definition of C and prove properties (a) and (b) in Sect. 4.

We will now study $\mathscr{P}^s(S^2)$. We will need some simple observations concerning the weak $*$ topology on $\mathscr{P}(S^2)$. We recall that the weak $*$ topology is defined by the family of seminorms $\|\cdot\|_{f_1,\dots,f_L}$ with

$$\|\mu\|_{f_1,\dots,f_k} = \max(\mu(f_1),\dots,\mu(f_k))$$

and $f_1, f_2, \dots, f_k$ an arbitrary k-tuple of continuous functions. If $\{\mu_n\} \subset \mathscr{P}(S^2)$ and $\mu \in \mathscr{P}(S^2)$, then $\mu_n \to \mu$ if and only if for every open subset $U \subset S^2$ with $\mu(\partial U) = 0$ we have $\mu_n(U) \to \mu(U)$. This theorem may be found in [DS88, p. 316]. We will need the following lemma on the limiting behavior of point masses. Point masses cannot disappear (they can appear) under weak $*$ limits. We will use constantly the fact that given $\mu \in \mathscr{P}(S^2)$, any point $P \in S^2$ has a fundamental system of disk neighborhoods $\{D_i\}$ with $\mu(\partial D_i) = 0$.

Lemma 3.2. *Suppose $\mu_n \to \mu$. Then for any $P \in S^2$ we have*

$$\limsup_{n\to\infty} \mu_n(P) \leqq \mu(P) .$$

Proof. Let $\varepsilon > 0$ be given. Choose a disk D around P with $\mu(\partial D) = 0$ and $\mu(D) \leqq \mu(P)+\varepsilon$. Then $\lim_{n\to\infty} \mu_n(D) = \mu(D) \leqq \mu(P)+\varepsilon$. But $\mu_n(P) \leqq \mu_n(D)$ whence

$$\limsup_{n\to\infty} \mu_n(P) \leqq \limsup_{n\to\infty} \mu_n(D) \leqq \mu(P) + \varepsilon .$$

Since ε is arbitrary, the lemma follows. □

We can now prove a basic property of $\mathscr{P}^s(S^2)$.

Proposition 3.3. *$\mathscr{P}^s(S^2)$ is an open, dense, convex subset of $\mathscr{P}(S^2)$.*

Proof. We first show that the complement of $\mathscr{P}^s(S^2)$ in $\mathscr{P}(S^2)$ is closed. Choose a point $x_0 \in S^2$. Then

$$\mathscr{P}(x_0, S^2) = \{\mu \in \mathscr{P}(S^2)\colon\ \mu(x_0) \geqq \tfrac{1}{2}\}$$

is closed by the previous lemma. But the complement of $\mathscr{P}^s(S^2)$ in $\mathscr{P}(S^2)$ is $SO(3) \cdot \mathscr{P}(x_0, S^2)$ and consequently it too is closed.

In order to see that $\mathscr{P}^s(S^2)$ is dense, it suffices to prove that the smooth measures are dense since a smooth measure has no atoms. Given $P \in S^2$, there exists a sequence of smooth measures $\{\mu_n\}$ such that $\mu_n \longrightarrow \delta(x - P)$. Hence, any convex combination of Dirac measures is a weak $*$ limit of smooth measures. But it is easily seen (by triangulating S^2 and concentrating the mass of each triangle at its barycenter) that such convex combinations are dense. □

The next lemma is obvious but has important consequences.

Lemma 3.4. *Suppose $\nu \in \mathscr{P}(S^2)$ and that $B(\nu) = 0$. Suppose further that there exists $P \in S^2$ with $\nu(P) \geqq \frac{1}{2}$. Then, in fact $\nu(P) = \frac{1}{2}$ and $\nu = \frac{1}{2}\delta(x - P) + \frac{1}{2}\delta(x + P)$. Hence, ν is nice semi-stable.*

Corollary 3.5. *Suppose $\gamma \in LS^2$. Put $\nu = \gamma_* \, ds$ where ds is the Riemannian measure on S^1. Then $B(\gamma_* \, ds) = 0$ implies $\gamma_* \, ds$ is stable.*

Proof. Since γ is smooth $\gamma_* \, ds$ cannot be a sum of atoms unless γ is constant. □

We now prove another important property of $\mathscr{P}^s(S^2)$. In the following proposition, we abbreviate $PSL_2(\mathbb{C})$ to G and let $A \subset G$ be the subgroup of diagonal matrices with real positive diagonal entries (λ, λ^{-1}). We define $A^+ \subset A$ by $A^+ = \{a = (\lambda, \lambda^{-1}) \in A : \lambda > 1\}$. We let N be the north pole of S^2 and $S = -N$ be the south pole. We assume that $PSL_2(\mathbb{C})$ operates on S^2 such that N is the attracting fixed-point of A^+ so S is the repelling fixed-point of A^+.

Proposition 3.6. *G acts properly on $\mathscr{P}^s(S^2)$.*

Proof. It is immediate that G acts properly if and only if A acts properly. Suppose A does not act properly. Then there exists $a_n \longrightarrow \infty$ in A^+, $\mu_n \longrightarrow \mu$ in $\mathscr{P}^s(S^2)$ with $(a_n)_* \mu_n \longrightarrow \nu$ in $\mathscr{P}^s(S^2)$. We will derive a contradiction from this. Indeed, let D be a small disk centered at S with $\mu(\partial D) = 0$. We will show $\mu(D) \geqq \frac{1}{2}$. This contradicts the stability of μ.

Since ν is stable, there exists $\alpha > 0$ such that $\nu(N) = \frac{1}{2} - \alpha < \frac{1}{2}$. Consequently, $\nu(S^2 - N) = \frac{1}{2} + \alpha$. Hence, there exists a disk D' centered at N with

$\mu(\partial D') = 0$ and $\nu(S^2 - D') \geqq \frac{1}{2} + \frac{\alpha}{2}$. Now,

$$\lim_{n\to\infty} (a_n)_* \mu_n(S^2 - D') = \nu(S^2 - D') .$$

Hence, there exists N_1 such that if $n \geqq N_1$ then $\mu_n(a_n^{-1}(S^2 - D')) \geqq \frac{1}{2} + \frac{\alpha}{4}$. Next, we may choose N_2 such that if $n \geqq N_2$ then $a_n^{-1}(S^2 - D') \subset D$. Finally, since $\lim_{n\to\infty} \mu_n(D) = \mu(D)$, we may choose N_3 such that $n \geqq N_3$ implies $\mu(D) \geqq \mu_n(D) - \alpha/8$. Put $M = \max(N_1, N_2, N_3)$. If $n \geqq M$, then

$$\mu(D) \geqq \mu_n(D) - \frac{\alpha}{8} \geqq \mu_n(a_n^{-1}(S^2 - D')) - \frac{\alpha}{8} \geqq \frac{1}{2} + \frac{\alpha}{8} .$$

The proposition follows. □

Corollary 3.7. *The quotient $\mathscr{P}^s(S^2)/PSL_2(\mathbb{C})$ is Hausdorff.*

Proof. See [Kos65], Chapter I, Sect. 2. □

4 The conformal center of mass

In this section, we give a self-contained treatment of the conformal center of mass, proving its existence and properties (a) and (b) used in the proof of Theorem 3.1. Our treatment is based on [DE86]. The results and proofs of the main results of this section (Propositions 4.5, 4.8, and 4.14) generalize immediately to the other rank 1 symmetric spaces of noncompact type. Their generalizations to higher rank are to be found in [LM].

We will use the ball model B^3 for hyperbolic 3-space. Let $u \in S^2 = \partial B^3$. Let $\{c(t) : t \geqq 0\}$ be the geodesic ray in B^3 from 0 to u. We define the Buseman function $b_u : B^3 \to \mathbb{R}$ by $b_u(x) = \lim_{t\to\infty}(d(x, c(t)) - t)$. See [BGS85, Sect. 3.3] for properties of $b_u(x)$. We note that $b_u(c(t)) = -t$. We will sometimes write $b(x, u)$ for $b_u(x)$. Here $d(x, y)$ is the hyperbolic distance function. In fact, it is easily checked that $b_u(x) = -\frac{1}{2} \log \frac{1-\|x\|^2}{\|x-u\|^2}$, where $\|\cdot\|$ is the Euclidean norm. Now let $\mu \in \mathscr{P}(S^2)$. Following Douady and Earle in [DE86], we define $h_\mu(x)$, "the average distance from x to ∂B^3" (the description is due to Thurston) by

$$h_\mu(x) = \int_{S^2} b_u(x)\, d\mu(u) ,$$

and $\xi_\mu(x) : B^3 \to \mathbb{R}^3$ by

$$\xi_\mu(x) = \operatorname{grad} h_\mu(x) .$$

Here, the gradient is taken using the hyperbolic metric.

We first determine how ξ_μ and h_μ transform under $PSL_2(\mathbb{C})$.

Lemma 4.1. *There is a constant $c(g, \mu)$ such that for $g \in PSL_2(\mathbb{C})$ we have*

$$h_{g\mu}(gx) = h_\mu(x) + c(g, \mu) .$$

Proof. By [Hel81, p. 83, (46)], we have

$$b(gx, gu) = b(x, u) + b(g0, gu)\,.$$

Then

$$\begin{aligned} h_{g\mu}(gx) &= \int_{S^2} b(gx, u) g_*\, d\mu(u) = \int_{S^2} g^* b(gx, u)\, d\mu(u) \\ &= \int_{S^2} b(gx, gu)\, d\mu(u) = \int_{S^2} b(x, u)\, d\mu(u) + \int_{S^2} b(g0, gu)\, d\mu(u)\,. \end{aligned}$$

□

Corollary 4.2. $dg\xi_\mu(x) = \xi_{g\mu}(gx)$.

Proof. From the lemma, we have $g^* dh_{g\mu} = dh_\mu$. Let $\Phi : T^*(B^3) \to T(B^3)$ be the isomorphism given by the hyperbolic metric and $f \in C^\infty(B^3)$. Then $\Phi(df) = \operatorname{grad} f$ and $\Phi(g^* df) = g \cdot \operatorname{grad} f$. Here $(g \cdot V)(x) = dg^{-1}(V(gx))$ for V a smooth vector field on B^3. □

For the next computation, we will use the upper half space model for hyperbolic 3-space. Thus, our underlying manifold is the upper-half space $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ equipped with the Riemannian metric $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$. For any C^2-function f on a Riemannian manifold, the Hessian of f will be denoted $D^2 f$.

Lemma 4.3. $D^2 b_u = ds^2 - db_u \otimes db_u$.

Proof. Since both sides are invariant under $SO(3,1)$ we may assume $u = \infty$. In this case, we have $b_u(x, y, z) = -\log z + c$ with c a constant (since grad b_u is the unit normal to the horospheres associated to $u = \infty$ [BGS85, Lemma 3.4]). We use the formula for X and Y vector fields

$$D^2 b(X, Y) = (XY)b - (\nabla_X Y)b\,.$$

Here ∇ is the Riemannian covariant derivative. Now we have $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \frac{1}{z}\frac{\partial}{\partial z}$, $\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{1}{z}\frac{\partial}{\partial z}$ and $\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = -\frac{1}{z}\frac{\partial}{\partial z}$, with all other covariant derivatives equal to zero. We find that $D^2 b_u$ is diagonal relative to $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ with diagonal entries $(\frac{1}{z^2}, \frac{1}{z^2}, 0)$. □

We now switch back to the ball model for hyperbolic 3-space.

Corollary 4.4. *Let $v \in T_0(B^3)$ and $u \in S^2$. Then*

$$D^2 b_u(0)(v, v) = (v - (v, u)u, v - (v, u)u)\,.$$

Here $(\cdot\,,\,\cdot)$ is the Euclidean metric on $\mathbb{R}^3$ and we identify $T_0(B^3)$ with $\mathbb{R}^3$ in the usual way.

We first give necessary and sufficient conditions for h_μ to be strictly convex.

Proposition 4.5. *The function h_μ is strictly convex if and only if μ is not supported on a pair of antipodal points $\{\pm u\}$ in S^2.*

Proof. Assume first that μ is not supported on a pair of antipodal points. By conformal invariance, it suffices to prove that the Hessian $D^2 h_\mu(0)$ is positive definite for all such measures μ. By Corollary 4.4, we have

$$D^2 b(0,u)(v,v) = (v - (v,u)u, v - (v,u)u) .$$

Hence

$$D^2 h_\mu(0)(v,v) = (v,v) - \int_{S^2} (u,v)^2 \, d\mu(u)$$

We claim that if μ is not supported on $\{\pm v\}$, then

$$\int_{S^2} (u,v)^2 \, d\mu(u) < 1 .$$

Indeed, since μ is not supported at $\{\pm v\}$, we have $\mu(S^2 - \{\pm v\}) = \varepsilon > 0$. Then since $(v,v)^2 = 1$ and $(u,v)^2 < 1$ on $S^2 - \{\pm v\}$, we have

$$\int_{\{\pm v\}} (u,v)^2 \, d\mu(u) = 1 - \varepsilon , \qquad \int_{S^2 - \{\pm v\}} (u,v)^2 \, d\mu(u) < \varepsilon$$

and the claim follows. Hence, $D^2 h_\mu(0)$ is positive definite as asserted in the lemma. The converse is clear. □

Corollary 4.6. *Suppose μ is stable, then h_μ is strictly convex.*

We owe Lemma 4.7 and Proposition 4.8 to Bernhard Leeb.

Lemma 4.7. $\operatorname{grad} h_\mu(0) = -B(\mu)$.

Proof.

$$\operatorname{grad} h_\mu(0) = \int_{S^2} \operatorname{grad} b_u(0) \, d\mu(u) .$$

Now $\operatorname{grad} b_u(0)$ is the unit tangent vector to 0 pointing from 0 to $-u$. Thus, in the usual identification between $T_0(\mathbb{R}^3)$ and $\mathbb{R}^3$ we have $\operatorname{grad} b_u(0) = -u$. Thus

$$-\operatorname{grad} h_\mu(0) = \int_{S^2} u \, d\mu(u) = B(\mu) .$$

□

Proposition 4.8. *Suppose μ is stable. Then h_μ attains its minimum in B^3.*

The proposition is a consequence of the next two lemmas. The first lemma is Proposition 9.7 of [BO69].

Lemma 4.9. *Suppose f is a convex function on a complete simply-connected Riemannian manifold X of non-positive curvature and $x_0 \in X$. Suppose f does not attain its infimum on X. Then there is a complete ray γ starting at x_0 such that f is nonincreasing on γ.*

We now return to the special case $X = B^3$.

Lemma 4.10. *Suppose $u \in S^2$ satisfies $\mu(u) < \frac{1}{2}$. Let γ be the ray starting at 0 and ending at u. Then h_μ is increasing on γ.*

Proof. Let $x \in \gamma$. Let a_λ be the dilation with repelling fixed point u and attracting fixed point $-u$ satisfying $a_\lambda x = 0$. Put $v = (a_\lambda)_* \mu$. Then $v(u) < \frac{1}{2}$ since a_λ fixes u. By Corollary 4.2 and Lemma 4.7 we have

$$da_\lambda(x)(\operatorname{grad} h_\mu(x)) = \operatorname{grad} h_v(0) = -B(v)\,.$$

But $B(v)$ clearly lies in the half-ball defined by $u \cdot y < 0$ whence $\operatorname{grad} h_v(0)$ lies in the half-ball $u \cdot y > 0$. In particular, $\operatorname{grad} h_v(0) \cdot u > 0$, so $\operatorname{grad} h_\mu(x) \cdot u > 0$ and h_μ is increasing on γ. □

We can now prove that the conformal center of mass is well-defined for stable measures and is conformally natural.

Lemma 4.11. *ξ_μ has a unique zero $C(\mu)$ in B^3. Moreover $C(g\mu) = gC(\mu)$.*

Proof. The first statement follows immediately from Proposition 4.8 and Corollary 4.6. The second follows from Corollary 4.2 and uniqueness of the zero of $\xi_{g\mu}$. Indeed, by Corollary 4.2, we have

$$dg^{-1} \cdot \xi_{g\mu}(gC(\mu)) = \xi_\mu(C(\mu)) = 0\,.$$

Hence $gC(\mu) = C(g\mu)$. □

Property (ii) of the conformal center of mass now follows from Lemma 4.7.

Lemma 4.12. *$C(\mu) = 0$ if and only if $B(\mu) = 0$.*

Proof. If $B(\mu) = 0$ then $\xi_\mu(0) = 0$, so $C(\mu) = 0$. Conversely, if $C(\mu) = 0$, then $\xi_\mu(0) = 0$, so $B(\mu) = 0$. □

Finally, we consider the function $\widetilde{C} : LS^2 \to \mathbb{R}^3$ given by $\widetilde{C}(\gamma) = C(\gamma_*\, ds)$. We then have the following proposition; the space of stable smooth loops $(LS^2)^s$ is defined in Definition 5.2.

Proposition 4.13. *The restriction of $\widetilde{C}$ to $(LS^2)^s$ is smooth.*

Proof. We consider the map $F : (LS^2)^s \times B^3 \to \mathbb{R}^3$ given by $F(\gamma, x) = \xi_\mu(x)$ with $\mu = \gamma_*\, ds$. By Corollary 4.6, $\frac{\partial F}{\partial x}(\gamma, x)$ is invertible, hence the equation

$$F(\gamma, x) = 0$$

defines x locally as a smooth function $f(\gamma)$ of γ. But $f = \widetilde{C}$ by definition. □

Remark 4.1. Although we will not need it in what follows, we now state and prove a proposition which will allow us to define C on the complement of the nice semistable measures provided we allow it to take values in the closed ball $\overline{B^3}$. The existence of a unique ray of "fastest decrease" for unstable measures is a special case of a theorem of Kempf, [Kem78]. The relation is to be found in [LM].

Proposition 4.14. *Suppose μ is neither stable nor nice semistable. Then there is a unique ray γ starting at zero such that h_μ is decreasing along γ.*

Proof. The proof of Lemma 4.10 is easily modified to yield the result that if $u \in S^2$ satisfies $\mu(u) > \frac{1}{2}$ then h_μ is decreasing along γ. Thus the proposition is proved in the case that μ is unstable. Suppose now that μ is semistable and that there exists $u \in S^2$ such that $\mu(u) = \frac{1}{2}$ and $\mu(-u) < \frac{1}{2}$ (so μ is not nice semistable). Let $v \in S^2$, $v \neq u$. Then $\mu(v) < \frac{1}{2}$, consequently by Lemma 4.10, the function h_μ increases along the ray joining 0 to v. Since $\mu(-u) < \frac{1}{2}$, the argument of Lemma 4.10 proves that $\operatorname{grad} h_\mu(x) \cdot u < 0$ for any x on the ray γ joining 0 to u. Hence h_μ is decreasing along γ and the lemma follows. □

Remark 4.2. In the semistable case treated above, the rate of decrease $\operatorname{grad} h_\mu(x) \cdot u < 0$ of h_μ along γ goes to zero.

5 The moduli spaces as complex manifolds

In this section, we prove $\mathcal{M}$ is a complex Fréchet manifold and $\mathcal{M}'_{\mathrm{Lip}}$ is a complex Banach manifold. We will prove the following theorem – after first establishing that LS^2 is a complex Fréchet manifold and that the action of $PSL_2(\mathbb{C})$ on LS^2 by post-composition is holomorphic.

Theorem 5.1. *There is a $PSL_2(\mathbb{C})$-invariant open subset $(LS^2)^s \subset LS^2$, the set of stable smooth loops, containing $\mu^{-1}(0)$ such that the induced map $\mu^{-1}(0)/SO(3) \to (LS^2)^s/PSL_2(\mathbb{C})$ is a diffeomorphism. The quotient space $(LS^2)^s/PSL_2(\mathbb{C})$ is a complex Fréchet manifold. Thus, $\mathcal{M} = \mu^{-1}(0)/SO(3)$ is a complex Fréchet manifold by transport of structure.*

A similar theorem holds in the Lipschitz case provided we exclude the degenerate loops.

Theorem 5.2. *There is a $PSL_2(\mathbb{C})$-invariant open subset $(LS^2_{\mathrm{Lip}})^s$, the set of stable Lipschitz loops, containing the intersection $\mu^{-1}(0)'$ of $\mu^{-1}(0)$ with the non-degenerate Lipschitz loops such that the induced map $\mu^{-1}(0)'/SO(3) \to (LS^2_{\mathrm{Lip}})^s/PSL_2(\mathbb{C})$ is a diffeomorphism. The quotient space $(LS^2_{\mathrm{Lip}})^s/PSL_2(\mathbb{C})$ is a complex Banach manifold. Thus, $\mathcal{M}'_{\mathrm{Lip}} = \mu^{-1}(0)'/SO(3)$ is a complex Banach manifold by transport of structure.*

Remark 5.1. In the smooth and Lipschitz cases, the almost complex structure induced by the complex structure agrees with those of Sects. 1 and 2, respectively. Thus, $\mathcal{M}$ and $\mathcal{M}'_{\mathrm{Lip}}$ are infinite dimensional Kähler manifolds.

We now begin the proof of Theorem 5.1. The proof of Theorem 5.2 is analogous and is left to the reader.

Definition 5.1. *Let $\mathcal{U}$ be an open subset of a complex Fréchet space $\mathcal{E}$ and $\mathcal{F}$ be complex Fréchet spaces. Let $F : \mathcal{U} \to \mathcal{F}$ be a map. Then F will be* *said to be holomorphic if it admits a complex-linear first derivative at every point of $\mathcal{U}$.*

We now give some elementary examples of holomorphic maps on open subsets of $L\mathbb{C}$ which are all that we need in what follows. Let U and V be open subsets of $\mathbb{C}$ and $\phi : U \to V$ be a holomorphic map. Define $\Phi : LU \to LV$ by $\Phi(\gamma) = \phi \circ \gamma$. Then if $\eta \in L\mathbb{C} = T_\gamma(LU)$, we have

$$d\Phi(\gamma)(\eta)|_s = d\phi(\gamma(s))(\eta(s)) .$$

Since, $J\eta|_s = i\eta(s)$, it is immediate that Φ is holomorphic. Thus, we have proved the following:

Lemma 5.3. *Φ is holomorphic.*

We can now prove that LS^2 is a complex manifold.

Proposition 5.4. *LS^2 is a complex Fréchet manifold modelled on $L\mathbb{C}$. The group $PSL_2(\mathbb{C})$ acts holomorphically on LS^2 by post-composition.*

Proof. For $p \in S^2$, we define $U_p = S^2 - \{p\}$, and $\mathcal{U}_p$ to be the open subset of LS^2 consisting of the loops that lie in U_p. Then $\{\mathcal{U}_p : p \in S^2\}$ covers LS^2. Let $\phi_p : U_p \to \mathbb{C}$ be stereographic projection and $\Phi_p : LU_p \to L\mathbb{C}$ be the map induced by ϕ_p as above. Then Φ_p is a homeomorphism. We take $\{\mathcal{U}_p, \Phi_p) : p \in S^2\}$ as our atlas for LS^2. The transition functions for this atlas are of the form considered in Lemma 5.3 and consequently are biholomorphic. Similarly, an element $g \in PSL_2(\mathbb{C})$, when expressed in charts of the above form around p and gp is of the form considered in Lemma 5.3. The proposition follows.

□

We now define an embedding $\iota : LS^2 \to \mathcal{P}(S^2)$ by $\iota(\gamma) = \gamma_* ds$ where ds is the Riemannian measure on S^1. We observe that ι is continuous and $PSL_2(\mathbb{C})$-equivariant.

Definition 5.2. *γ is stable if $\iota(\gamma) = \gamma_* ds \in \mathcal{P}^s(S^2)$.*

Since ι is continuous, it follows from Proposition 3.3 that $(LS^2)^s$ is open in LS^2. It follows from Proposition 3.6 that $PSL_2(\mathbb{C})$ acts properly on $(LS^2)^s$. We put $Z = \mu^{-1}(0)$, whence Z is the subset of those $\gamma \in LS^2$ such that $B(\gamma_* ds) = C(\gamma_* ds) = 0$. Then $Z \subset (LS^2)^s$ by Corollary 3.5. We have an induced map

$$\tau : Z/SO(3) \to (LS^2)^s/PSL_2(\mathbb{C}) .$$

We begin by proving the following proposition.

Proposition 5.5. *$(LS^2)^s/PSL_2(\mathbb{C})$ is a Hausdorff complex Fréchet manifold.*

Proof. To show $(LS^2)^s/PSL_2(\mathbb{C})$ is a manifold, it suffices to find a cover of $(LS^2)^s$ by $PSL_2(\mathbb{C})$-stable open sets such that each open set admits a smooth holomorphic orbit cross-section. For each triple of distinct points $(s_1, s_2, s_3) \in S^1 \times S^1 \times S^1$, we define $\mathscr{U}_{s_1,s_2,s_3} \in (LS^2)^s$ by

$$\mathscr{U}_{s_1,s_2,s_3} = \{\gamma \in (LS^2)^s \colon \gamma(s_1) \neq \gamma(s_2) \neq \gamma(s_3)\} .$$

Since a stable loop assumes infinitely many distinct values, it follows that $\{\mathscr{U}_{s_1,s_2,s_3}\}$ covers $(LS^2)^s$. We define $\Sigma_{s_1,s_2,s_3} \subset \mathscr{U}_{s_1,s_2,s_3}$ by

$$\Sigma_{s_1,s_2,s_3} = \{\gamma \in \mathscr{U}_{s_1,s_2,s_3} \colon \gamma(s_1) = 0,\ \gamma(s_2) = 1,\ \gamma(s_3) = \infty\} .$$

Clearly, Σ_{s_1,s_2,s_3} is a cross-section to the $PSL_2(\mathbb{C})$ orbits in $\mathscr{U}_{s_1,s_2,s_3}$. We claim Σ_{s_1,s_2,s_3} is a submanifold of $\mathscr{U}_{s_1,s_2,s_3}$. Let $\Phi : \mathscr{U}_{s_1,s_2,s_3} \to S^2 \times S^2 \times S^2$ be given by $\Phi(\gamma) = (\gamma(s_1), \gamma(s_2), \gamma(s_3))$. Then, $\Phi(g\gamma) = g\Phi(\gamma)$ for $g \in PSL_2(\mathbb{C})$. Hence, the restriction of Φ to any $PSL_2(\mathbb{C})$-orbit in $\mathscr{U}_{s_1,s_2,s_3}$ is a submersion and *a fortiori* Φ is a submersion. Consequently, Σ_{s_1,s_2,s_3} is smooth by the Implicit Function Theorem. It is immediate that Σ_{s_1,s_2,s_3} is a *complex* submanifold since Φ is clearly a holomorphic mapping.

We conclude by proving $(LS^2)^s/PSL_2(\mathbb{C})$ is Hausdorff. Indeed, by Proposition 3.6, $PSL_2(\mathbb{C})$ acts properly on $\mathscr{P}^s(S^2)$. Hence, it acts properly on $(LS^2)^s$ and the quotient $(LS^2)^s/PSL_2(\mathbb{C})$ is Hausdorff. □

Theorem 5.1 will follow from the next proposition.

Proposition 5.6. *The map τ is a diffeomorphism.*

The proposition will be a consequence of the next three lemmas.

Lemma 5.7. *τ is a bijection.*

Proof. Let $v = \gamma_* ds$. Then there exists $g \in PSL_2(\mathbb{C})$ such that $gC(v) = 0$ where C is the conformal center of mass (see Sect. 4). Hence $C(gv) = 0$ and consequently $B(gv) = 0$. Hence, $g \circ \gamma \in Z$ and τ is onto. Now suppose there exist $\gamma_1, \gamma_2 \in Z$ and $g \in PSL_2(\mathbb{C})$ with $g\gamma_1 = \gamma_2$. Then $gC(\gamma_1) = C(\gamma_2)$ whence g fixes $0 \in B^3$ and $g \in SO(3)$. □

Since the Inverse Function Theorem does not hold in Fréchet space, we will need to prove that τ^{-1} is smooth by hand. This we now do.

Let B be the Borel subgroup of $PSL_2(\mathbb{C})$ corresponding to the upper-triangular matrices in $SL_2(\mathbb{C})$ with positive real diagonal elements. Then $PSL_2(\mathbb{C}) = B.\ SO(3)$ is the Iwasawa decomposition and B is a solvable subgroup of $PSL_2(\mathbb{C})$ acting simply-transitively on B^3. We may define a map $b : (LS^2)^s \to B$ by defining $b(\gamma)$ to be the unique element in B such that $b(\gamma)^{-1}\gamma \in Z$. We note $b(\gamma) = I$ if and only if $\gamma \in Z$.

Lemma 5.8. *The map b is smooth.*

Proof. Let $x \in B^3$. Then there is a unique element $\beta(x) \in B$ such that $\beta(x) \cdot x = 0$. Clearly $\beta : B^3 \to B$ is smooth. But $b(\gamma) = \beta(\widetilde{C}(\gamma))$ and $\widetilde{C}$ is smooth by Proposition 4.13. □

Lemma 5.9. *Let $g \in PSL_2(\mathbb{C})$, $\gamma \in (LS^2)^s$. Then there exists $k(g,\gamma) \in SO(3)$ such that*

$$b(g\gamma) = gb(\gamma)k(g,\gamma)\,.$$

Proof. By definition $b(\gamma)$ is determined by the equation

$$b(\gamma)^{-1}C(\gamma) = 0\,.$$

Then $b(g\gamma)^{-1}C(g\gamma) = 0$, hence $b(g\gamma)^{-1}gC(\gamma) = 0$ and

$$C(\gamma) = g^{-1}b(g\gamma) \cdot 0\,.$$

Since $C(\gamma) = b(\gamma) \cdot 0$ we have $b(\gamma)^{-1}g^{-1}b(g\gamma) \cdot 0 = 0$ whence $b(\gamma)^{-1}g^{-1}b(g\gamma) = k(g,\gamma) \in SO(3)$. □

We now define $z : (LS^2)^s \to Z$ by $z(\gamma) = b(\gamma)^{-1}\gamma$. Then z is smooth and

$$\begin{aligned} z(g\gamma) &= b(g\gamma)^{-1}g\gamma = k(g,\gamma)^{-1}b(\gamma)^{-1}g^{-1}g\gamma \\ &= k(g,\gamma)^{-1}b(\gamma)^{-1}\gamma = k(g,\gamma)^{-1}z(\gamma)\,. \end{aligned}$$

Hence, z induces a smooth map

$$\bar{z} : (LS^2)^s/PSL_2(\mathbb{C}) \to Z/SO(3)\,.$$

If $\gamma \in Z/SO(3)$, then $\bar{z}(\gamma) = \gamma$ whence $\bar{z} \circ \tau = \mathrm{id}$ and $\bar{z} = \tau^{-1}$. Hence τ^{-1} is smooth and Theorem 5.1 is proved.

6 Holomorphic line bundles over $\mathcal{M}$ and the Kähler potential

In this section we construct holomorphic line bundles with connection $(\mathcal{L}_n, \nabla)$ over $\mathcal{M}$ with curvature $\frac{n}{8\pi^2}\Omega$, $n \in \mathbb{Z}$. We recall that Ω was defined in Sect. 1.

Let $E = \mathbb{C}^2 \backslash \{0\}$ and $p : E \to \mathbb{P}^1$ be the tautological principal $\mathbb{C}^*$-bundle. We obtain a tautological principal $L\mathbb{C}^*$-bundle over $L\mathbb{P}^1$ by applying the functor L. Thus, we have $Lp : LE \to L\mathbb{P}^1$. Now $E|U_p$ has a section σ whence $LE|\mathcal{U}_p$ has a section $L\sigma$. Consequently LE is locally trivial.

We now review the formulas for the curvature and connection forms of the $SU(2)$-invariant connection on the tautological bundle E over $\mathbb{P}^1$. We note that the flat Hermitian metric $(\,\cdot\,,\,\cdot\,)$ on $\mathbb{C}^2$ induces a Hermitian metric on E since the fibres of E are subspaces of $\mathbb{C}^2$. We note that we have a real-linear isomorphism $T_e(E) \cong \mathbb{C}^2$ and a complex-linear isomorphism $T_e^{1,0}(E) \cong \mathbb{C}^2$. We will make these identifications henceforth. The (1,0)-part ϕ' of the Hermitian connection ϕ on E is given for $v \in T_e^{1,0}(E) = \mathbb{C}^2$ by

$$\phi_e'(v) = \frac{(v,e)}{\|e\|^2} = \partial \log \|e\|^2(v)$$

and the curvature ω_e is given for $v, w \in T_e^{1,0}(E) = \mathbb{C}^2$ by

$$\omega_e(v, \bar{w}) = \operatorname{Im} \frac{\left(v - \frac{(v,e)e}{\|e\|^2}, w - \frac{(w,e)e}{\|e\|^2}\right)}{\|e\|^2}$$

$$\omega_e(v, \bar{w}) = \operatorname{Im} \frac{(v,w)}{\|e\|^2} - \operatorname{Im} \frac{(v,e)(e,w)}{\|e\|^4} .$$

We now give the corresponding formulas for the $L\mathbb{C}^*$-principal bundle of LE. Since LE is open in $L\mathbb{C}^2$, we have a real-linear isomorphism $T_\gamma(LE) \cong L\mathbb{C}^2$ and a complex-linear isomorphism $T_\gamma^{1,0}(LE) \cong L\mathbb{C}^2$. We define a distribution $T^{\mathrm{hor}}(LE)$ on LE by

$$T_\gamma^{\mathrm{hor}}(LE) = \{\eta \in L\mathbb{C}^2 \mid (\eta(s), \gamma(s)) = 0, \text{ all } s\} .$$

The reader will check that the distribution $T^{\mathrm{hor}}(LE)$ defines an $L\mathbb{C}^*$-connection which is invariant under $LSU(2)$. We will use Φ and $\widetilde{\Omega}$ to denote the associated connection and curvature forms on LE. We note that the $(1,0)$ part Φ' of the connection form Φ is given for $\eta \in L\mathbb{C}^2$ by

$$\Phi'_\gamma(\eta) = \frac{(\eta, \gamma)}{\|\gamma\|^2} .$$

Thus, Φ_γ takes values in $L\mathbb{C}$.

We leave the following lemma to the reader.

Lemma 6.1. $\widetilde{\Omega}_\gamma(\zeta, \bar{\eta}) = \operatorname{Im} \frac{(\zeta, \eta)}{\|\gamma\|^2} - \operatorname{Im} \frac{(\zeta, \gamma)(\gamma, \eta)}{\|\gamma\|^4}$.

Remark 6.1. We may interpret the previous formulas for the connection and curvature as follows: Let τ be a p-form on E. Then we may interpret τ as a function $\tau : E \times (\mathbb{C}^2)^p \to \mathbb{C}$ which is skew multilinear in the last p variables. By applying the free loop functor, we obtain a function $L\tau : LE \times (L\mathbb{C}^2)^p \to L\mathbb{C}$ which corresponds to an $L\mathbb{C}$-valued p-form $L\tau$ on LE.

Using this notation, we have the following

Lemma 6.2. *The connection Φ and curvature $\widetilde{\Omega}$ of LE are given by*
(i) $\Phi = L\phi$;
(ii) $\widetilde{\Omega} = L\omega$.

We observe that $L\mathbb{C}^*$ acts holomorphically on LE by pointwise multiplication. Now we have a single-valued homomorphism $\lambda : L\mathbb{C}^* \to \mathbb{C}^*$ of complex Lie groups given by

$$\lambda(\gamma) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \gamma(s)\, ds\right) .$$

Here, $\log \gamma(s)$ is any branch of the logarithm on $[0, 2\pi]$. We may accordingly form the associated principal $\mathbb{C}^*$-bundle $\mathscr{P}$ over $\mathscr{M}$ with total space $\mathscr{E}$

given by

$$\mathscr{E} = LE \times_{L\mathbb{C}^*} \mathbb{C}^* \quad \text{or} \quad \mathscr{E} = LE/\ker\lambda .$$

The connection Φ induces the connection $d\lambda \circ \Phi$ with curvature $d\lambda \circ \widetilde{\Omega}$ on $\mathscr{P}$. We obtain the following

Lemma 6.3. *The induced connection $d\lambda \circ L\phi$ and curvature $d\lambda \circ L\omega$ on $\mathscr{P}$ are given by*

(i) $d\lambda \circ L\phi_\gamma(\eta) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Im} \frac{(\eta(s),\gamma(s))}{\|\gamma(s)\|^2}\, ds$; *and*

(ii) $d\lambda \circ L\omega_\gamma(\zeta,\bar\eta) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Im} \frac{(\zeta(s),\eta(s))}{\|\gamma(s)\|^2}\, ds - \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Im} \frac{(\zeta(s),\gamma(s))(\gamma(s),\eta(s))}{\|\gamma(s)\|^4}\, ds$.

We obtain the associated Hermitian line bundle $\mathscr{L}$ over LS^2 with total space $\mathscr{E} \times_{\mathbb{C}^*} \mathbb{C}$ where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication. Now the action of $PSL_2(\mathbb{C})$ on LS^2 lifts to an action of $SL_2(\mathbb{C})$ on LE which commutes with the action of $L\mathbb{C}^*$. Hence, by taking quotients, we obtain an induced bundle again denoted $\mathscr{L}$ over $\mathscr{M}$. Clearly the Hermitian structure and the connection on $\mathscr{E}$ descend, also. We let $\mathscr{L}_n$, $n \in \mathbb{Z}$ be the line bundle over $\mathscr{M}$ given by the n-th tensor power of $\mathscr{L}$ (so $\mathscr{L}^{-1}$ is the dual of $\mathscr{L}$). Then $\mathscr{L}_n$ has an induced Hermitian structure with curvature n times that of $\mathscr{L}$.

Remark 6.2. With our conventions, the Chern class $c_1(L)$ of a line bundle with connection (L,∇) is represented by $-\frac{1}{2\pi}K$; where K is the curvature.

We obtain the following:

Theorem 6.4. *The normalized Kähler form $\frac{1}{8\pi^2}\Omega$ is the Chern form $c_1(\mathscr{L}^{-1},\nabla)$ of the dual of $\mathscr{L}$ (equipped with the induced connection ∇).*

Proof. Let $\mathrm{ev} : L\mathbb{P}^1 \times S^1 \to L\mathbb{P}^1$ be evaluation and $\pi : L\mathbb{P}^1 \times S^1 \to L\mathbb{P}^1$ be projection. We will use the notation of Proposition 5.4 and compare $c_1(\mathscr{L}^{-1},\nabla)$ with Ω on $\mathscr{U}_\infty$. Let v be the 2-form on U_∞ given by $v = \frac{-1}{2\pi i}\frac{dz\wedge d\bar z}{(1+|z|^2)^2}$. Then it is immediate from Lemma 6.3(ii) that $c_1(\mathscr{L}^{-1},\nabla) = \frac{1}{2\pi}\pi_*\mathrm{ev}^* v$ (notation of Sect. 1). But v is $SO(3)$-invariant with $\int_{\mathbb{P}^1} v = 1$. Hence $v = \frac{1}{4\pi}\mathrm{vol}$ where vol is the Riemannian volume element of S^2 with the metric induced from that of $\mathbb{R}^3$. Since $\Omega = \pi_*\mathrm{ev}^*\mathrm{vol}$, the theorem follows. □

We can now describe the Kähler potential. We recall that a Kähler potential ρ for the Kähler form Ω is a locally defined real-valued function defined up to the addition of a pluriharmonic function such that

$$\omega = \frac{i}{2}\partial\bar\partial\rho .$$

Define $\widetilde{\rho}$ on LE by

$$\widetilde{\rho}(\gamma) = \int_0^{2\pi} \log\|\gamma(s)\|^2\, ds .$$

We claim $\widetilde{\rho}$ is invariant under $\ker\lambda$ and hence descends to $\mathscr{P}$. Indeed suppose $\gamma_0 \in L\mathbb{C}^*$ satisfies $\lambda(\gamma_0) = 1$. Then

$$\mathrm{Re} \int_0^{2\pi} \log \gamma_0(s)\, ds = \int_0^{2\pi} \log |\gamma_0(s)|\, ds = 0 .$$

Accordingly, we have

$$\begin{aligned}\widetilde{\rho}(\gamma\gamma_0) &= \int_0^{2\pi} \log \|\gamma(s)\gamma_0(s)\|^2\, ds \\ &= \int_0^{2\pi} \log \|\gamma(s)\|^2\, ds + \int_0^{2\pi} \log |\gamma_0(s)|^2\, ds = \widetilde{\rho}(\gamma) .\end{aligned}$$

The claim is proved. We let ρ denote the induced function on $\mathscr{P}$ and $\pi : \mathscr{P} \to \mathscr{M}$ be the bundle projection.

Lemma 6.5. $\pi^*\Omega = \frac{i}{2}\partial\overline{\partial}\rho$.

Proof. Pull both sides back to LE. Let $\gamma \in LE$, ζ, $\eta \in L\mathbb{C}^2$. Identify ζ, η with constant vector fields on LE. The lemma now follows from the easy calculation of $\partial\overline{\partial}\rho(\zeta, \overline{\eta})$. □

Thus, we obtain a Kähler potential, which we again denote ρ, for Ω on each open subset $\mathscr{U}_p$. For example, on $\mathscr{U}_\infty$ we have affine coordinates $(1, z(s))$ with $z(s) \in L\mathbb{C}$ and we obtain

$$\rho(z(s)) = \int_0^{2\pi} \log(1 + |z(s)|^2)\, ds .$$

We obtain the following local coordinate formula. Let $z(s) \in L\mathbb{C}$, $\zeta(s)$, $\eta(s) \in L\mathbb{C}$. Then

$$\Omega_{z(s)}(\zeta(s), \overline{\eta}(s)) = \int_0^{2\pi} \frac{\mathrm{Im}(\zeta(s)\overline{\eta}(s))}{(1 + |z(s)|^2)^2}\, ds .$$

Acknowledgements. We would like to thank Simon Donaldson who suggested our measure-theoretic approach to stability on LS^2. We would also like to thank David Hamilton who brought [DE86] to our attention and Bernhard Leeb for showing us Lemma 4.7 and Proposition 4.8. We thank Jean–Luc Brylinski for sending us [Bry]. Finally, we would like to thank Michael Kapovich for numerous helpful conversations.

References

[AB82] M.F. Atiyah, R. Bott: The Yang–Mills equations over Riemann surfaces. Phil. Trans. R. Soc. Lond. A **308**, 523–615 (1982)

[BGS85] W. Ballman, M. Gromov, V. Schroeder: Manifolds of Nonpositive Curvature. Volume 61 of Progress in Math. Birkhäuser, Basel Boston, 1985

[BO69] R.L. Bishop, B. O'Neill: Manifolds of negative curvature. Trans. of the A.M.S. **143**, 1–49 (1969)

[Bry] J.-L. Brylinski (Private communication)
[Bry92] J.-L. Brylinski: Loop Spaces, Characteristic Classes and Geometric Quantization. Volume 107 of Progress in Math. Birkhäuser, Basel Boston, 1992
[Cor88] K. Corlette: Flat G-bundles with canonical metrics. J. Differ. Geom. **28**, 361–382 (1988)
[DE86] A. Douady, C. Earle: Conformally natural extensions of homeomorphisms of the circle. Acta Math. **157**, 23–48 (1986)
[DM86] P. Deligne, G.D. Mostow: Monodromy of hypergeometric functions and non-lattice integral monodromy. Publ. Math. I.H.E.S. **63**, 5–90 (1986)
[Don83] S.K. Donaldson: A new proof of a theorem of Narasimhan and Seshadri. J. Differ. Geom. **18**, 269–277 (1983)
[DS88] N. Dunford, J.T. Schwartz: Linear Operators Part I: General Theory. Wiley, New York, 1988
[Ham82] R.S. Hamilton: The inverse function theorem of Nash and Mosher. Bull. Am. Math. Soc. **7**, 65–221 (1982)
[Hel81] S. Helgason: Topics in Harmonic Analysis on Homogeneous Spaces. Volume 13 of Progress in Math. Birkhäuser, Basel Boston, 1981
[Kem78] G. Kempf: Instability in invariant theory. Ann. Math. **108**, 299–316 (1978)
[Kir84] F. Kirwan: Cohomology of Quotients in Symplectic and Algebraic Geometry. Volume 31 of Mathematical Notes. Princeton University Press, 1984
[Kly92] A. Klyachko: Spatial polygons and stable configurations of points in the projective line, Algebraic Geometry and its Applications. pp. 67–84. Yaroslavl, 1992
[KM] M. Kapovich, J.J. Millson: The symplectic geometry of polygons in Euclidean space (Pre-print)
[KN78] G. Kempf, L. Ness: The length of vectors in representation spaces. Vol. 732, pp. 233–244. Springer Lecture Notes in Math., 1978
[Kos65] J.L. Koszul: Lectures on Groups of Transformations, Lectures on Mathematics 32. Tata Institute of Fundamental Research, 1965
[Lem93] L. Lempert: Loop spaces as complex manifolds. J. Differ. Geom. **38**, 519–543 (1993)
[LM] B. Leeb, J.J. Millson: The non-euclidean center of mass and geometric invariant theory (In preparation)
[LP] J. Langer, R. Perline: The Poisson geometry of the filament equation (Pre-print)
[MZ] J.J. Millson, B. Zombro (In preparation)
[Nes84] L. Ness: A stratification of the null cone via the moment map. Am. J. Math. **106**, 1281–1329 (1984)
[Thu] W. Thurston: Shapes of polyhedra (Pre-print)