Geometric quantization on Kähler manifolds, Berezin-Toeplitz, coherent states

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The idea of geometric quantization is to start with a symplectic manifold (M, ω) and for some geometric structure that you have on M, like a line bundle with a hermitian metric, which is the geometric quantization data.

So on every point $x \in M$ you have a copy of $\mathbb{C}$, and you put a hermitian metric $h_x : \mathcal{L}_x \otimes \bar{\mathcal{L}}_x \to \mathbb{C}$.

Now the second part of the geometric quantization data is a connection ∇ on $\mathcal{L}$ that preserves h. Recall that this means that

$$\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1(M)$$

such that

- $\forall v \in \mathcal{C}^\infty(TM)$, $\nabla_v : \mathcal{L} \to \mathcal{L}$ is $\mathbb{C}$-linear
- Leibniz in the sense that $\nabla_v(fs) = df(v)s + f\nabla_v s$.

Property (Bohr-Sommerfield) $\frac{2\pi}{i} R_\nabla = \omega$, where (E, ∇) is a vector bundle with connection we define $R_\nabla \in \Omega^2_M \otimes \mathrm{End}(E)$ (the curvature).

Definition M is ***quantizable*** when it admits the data above satisfyint the property Bohr-Sommerfield.

We call $L^2(M, \mathcal{L})$ a ***pre-quantum Hilbert space***.

Question What is it that makes it better in Kähler manifolds? (Dani: I thought in general the function space didn't have to be the sections of a bundle)

Remark (Sergey) When $\mathbb{R}^{2n}$ is the phase space and $\mathcal{L}$ is the trivial bundle then $L^2(\mathbb{R}^2)$ is *too big!*

Definition A ***Kähler manifold*** is (M, ω, g, I) where ω is a symplectic structure, g a riemannian structure and I a complex structure, and they are all compatible and I is integrable. So remember that compatibility is for example that $g(u, v) = \omega(Iu, v)$ and g is

I-invariant. On the other hand, integrability is a non-trivial PDE and gives holomorphic local coordinates.

Starting again, take a Kähler manifold M and define a pre-quantum line bundle so $\mathcal{L}$ holomorphic with the Chern connection ∇. Also suppose that ∇ satisfies the Bohr-Sommerfield property.

Remark (Kodaira theorem) The Kähler class (cohomology class of symplectic form) is the first Chern class, i.e., $[\omega] = c_1(\ell) \in H^2(M, \mathbb{Z})$ iff $\mathcal{L}$ is ample.

Definition

$$\mathcal{H}_m = H^0(X, \mathcal{L}^{\otimes m})$$

so the sections of that bundle.

You see hare making the space of sections smaller:

$$\mathcal{H}_m = H^0(X, \mathcal{L}^{\otimes m}) \hookrightarrow L^2(M, \mathcal{L}^{\otimes m})$$

And if M is not compact, take

$$H^0(M, \mathcal{L}^{\otimes m}) \cap L_2$$

Question (Altan) What about the measure here?

Answer (Bruno) Related to Liouville measure; a clever choice of h. So maybe

$$\langle s_1, s_2 \rangle = \int_M h\left(s_1(x), s_2(x)\right) \frac{\omega^d}{d!}$$

And now d is what in Altan's talk was n… or was it m?

Remark So in the case of Altan's talk

$$\mathcal{H} = \left\{ f : \mathbb{C}^n \xrightarrow{\text{entire}} \mathbb{C} : \int_{\mathbb{C}^n} |f|^2 \exp(-|z|^2) d\lambda(z) \right\}$$

so notice that the exp term is stopping the space to be trivial because bounded holomorphic functions are constant right?

OK back to the quantization. Given $f \in \mathcal{C}^\infty(M)$ define $A_f \in \mathrm{End}(\mathcal{H}_m)$…

Berezin-Toeplitz

$$T_f(S) = \Pi_m(fS)$$

Π_m orthogonal projection to $H^0(M, \mathcal{L}^{\otimes m})$.

Kostant-Sorian pre-quantum operators f quantizable means $Q_f(s) := \nabla_{X_f} s - 2\pi i f s$. Here X_f is the hamiltonian vector field of f with respect to ω.

So that's an operator associated to a function—quantization!

And we *wish* that if s is holomorphic then so is $Q_f(s)$.

Remark (Sergey) Also in the wish list is some property of brackets right?

Right so we are doing

$$\mathcal{C}^\infty(M) \longrightarrow \mathrm{End}(V)$$
$$f \longmapsto \hat{f}$$

and we want

$$\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}} \qquad \hat{1} = \mathrm{Id}, \qquad \widehat{\varphi \circ f} = \varphi(\hat{f})$$

Question (Dani) So why is it impossible that the wishes became true? Is that a difficult theorem?

Now

$$L_f(s) = \Pi_m(Q_f(s))$$

and

Tuyman's lemma $iQ_f = T_f - \perp_{2m} \Delta f$

Coherent state quantization $p \in \mathcal{L}_x^{\otimes m} \backslash \{0\} \rightsquigarrow e_p \in \mathcal{H}_m$. So

$$s \in H^0(X, \mathcal{L}^{\otimes m}) \longmapsto s(x) \in \mathcal{L}_x = \lambda_p(s) \cdot p$$

Now a ***coherent state*** is defined via Riesz' representation theorem so $\lambda_p \in \mathcal{H}_m^*$ such that

$$\lambda_p(s) = \langle e_p, s \rangle$$

Question (Dani) Why are coherent states so important?

They give us a rational map which is essentially Kodaira map (the map defined by any line bundle)

$$\mathrm{coh} : M \dashrightarrow \mathbb{P}(\overline{\mathcal{H}}_m$$

So given a bounded operator $A \in B(\mathcal{H}_m)$ we define

$$\hat{A}(x) = \frac{\langle Ae_p, e_p \rangle}{\|e_p\|^2} \qquad x \in M$$

called the ***covariant symbol***. (That is not the same hat than the hat of the quantization!) We have a map $B(\mathcal{H}_m) \to \mathcal{C}^\infty(M)$.

Now we try to invert this map: which functions are the covariant symbols of some bounded operators?

Remark (Sergey) This is like in altan talk, look for the kernel!

Here's two formulas to go in the other direction:

$$(As)(x) = \int_M h_y(s(y), s(y))\hat{A}(x, y)\frac{\omega^m}{n!}$$
$$\mathrm{Tr}(A) = \int_M \hat{A}(x)\theta(x)\frac{\omega^n}{n!}$$

and that's a nice formula wich has to do with Rawsley, $\theta(x) = |q|^2 \cdot \|e_q\|^2$.

Remark (Dani) So it looks like we want an equivalence of operators and functions.

Bondermann-Meinken-Schlihenmeier $\|f\|_\infty - \frac{c}{m} \leqslant \|T_f^m\| \leqslant \|f\|_\infty$

$$\|m[T_f^{(m)}, T_y^{(m)}]iT_{\{f,g\}}^{(m)}\| = \mathcal{O}(m^{-1}) \qquad \text{as } n \to \infty$$