Automorphisms of algbraically hyperbolic manifolds

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Upshot using dynamic argument, hyperkähler is non hyperbolic. In the end it is beacuse of automorphism group. We shall see how to prove it is finite.

Past upshot

Recall that last week we talked about Kobashi hyperbolicity, which means that Kobayashi pseudo-distance is non-degenerate. We saw that this implies Brody hyperbolicity, which means that X has no non-constant holomorphic maps. If X is compact this is an equivalence.

Today we shall see that this implies algebraic hyperbolicity. Demially's conjecture is that this last implication is actually iff.

Abstract Kobayashi hyperbolic manifold is a compact Hermitian manifold M such that any holomorphic map from the Poincare disk to M is C-Lipschitz, with a fixed constant C. Such manifolds admit a metric, called "Kobayashi metric", which is invariant under all holomorphic automorphisms; this immediately implies that the group of holomorphic automorphisms is compact. It is not hard to see that compactness implies that this group is finite. Algebraic hyperbolicity is a seemingly weaker algebraic version of this notion which was first defined by J.-P. Demailly. Conjecturally, it is equivalent to Kobayashi hyperbolicity. I will explain why algebraically hyperbolic manifolds have finite fundamental group. This is a result obtained jointly with F. Bogomolov and L. Kamenova. I will explain how it implies that hyperkahler manifolds with Picard rank $\geqslant 3$ are not algebraically hyperbolic.

1 Introduction

Question What is up with Picard rank $\geqslant 3$ hypothesis?

Another characterizarion of Koba hyp (M, I, ω) compact, Hermitian manifold such that there exists $C > 0$ and for each holomorphic map from (Δ,Poincaré metric) to M is C-Lipschitz.

Proof. Because we bound kobayashi pseudo distance from below with the hermitian/riemannian metric, making it into a distance. I think I read this sometime ago in some Kobayashi book. □

Claim If kobayashi sectional curvature is bounded from below by $-\varepsilon$ with $\varepsilon \gg 0$ then any map $\Delta \to M$ is C-Lipschitz with $C = \frac{c}{\varepsilon}$

Theorem (Brody) if M is compact Kobayahi hyperbolic is equivalent to not having a copy of $\mathbb{C}$.

2 Review/other perspective of algebraically hyperbolic

Definition M projective manifold. (makes 0 sense for Kähler). M is called ***kobayashi hyperbolic*** if for each compact curve $S \subset M$,

$$\int_S M\omega \leqslant (g - q)C$$

for some $C > 0$

Remark So Lucas just put degree, but the integral is the intersection number with hyperplane sections. (the two definitions *are* equivalent)

Question H o o o ow!? To pass from the integral to the intersection form.

Theorem (Who?) Kobayashi hyperbolic implies algebraically hyperbolic.

conjecture (Demailly) converse implication holds. He proposed a scheme to prove this that was later proved wrong by his students.

now we prove

Theorem (Who?) Kobayashi hyperbolic implies algebraically hyperbolic.

Proof.

Step 1 Start with a curve C of genus > 1. Then $C = \Delta/\Gamma$. Then $\mathsf{Vol}\, C = -\int_C c_1(c) = (g-1)\alpha$. Where α is probably 2π—just a constant. *Because curvature is Chern class!* ⍨. So that proves it.

Step 2 Suppose M is Kobayashi hyperbolic compact Hermitian manifold. "By compactness" there exists a constant ε such that $h \leqslant \varepsilon h_K$, where h is normal hermitian metric and h_K is Kobayashi metric.

Step 3 Let $j : S \hookrightarrow M$ be a curve in a projective manifold, its genus is $\geqslant 2$ by hyperbolicity. Notice j is 1-Lipschitz with respect to Kobayashi hyperbolic metric because it is holomorphic. Now

$$(g-1)\alpha = \mathsf{Vol}_K(S) \geqslant \mathsf{Vol}_{KM}(S) \geqslant \varepsilon\, \mathsf{Vol}_{\text{Fubini-Study}}\, S$$

This gives $g - 1 \geqslant \frac{\varepsilon}{2\alpha} \deg S$ proving hyperbolicity.

∎

3 The automorphism group of an algebraically hyperbolic manifold is discrete

Claim Any hyperbolic manifold M has discrete automorphism group.

Proof.

Step 1 The group of automorphisms of a projective manifold is a complex Lie group. Its connected component of identity G_0 Couldn't follow

∎

Theorem (Not in slides) X projective admits a self map $f : X \to X$ of degree $\deg f > 1$ then X it not algebraically hyperbolic.

Proof. Degree is the number o preimages (or the image of the fundamental class in cohomology) so we have $\deg S \deg f$ and "the same for genus" ∎

Claim The automorphism group of a compact Kobayashi hyperbolic manifold is finite.

Proof. Compactness and using the claim whose prof I didn't follow. ∎

Theorem (with F. Bogomolov and L. Kamenova) The automorphism group of an algebraically hyperbolic manifold is finite.

Proof is in three statements.

Proposition If there is an automorphism of M not preserving the rational Kähler class then M is not algebraically hyperbolic. (If the image of $\mathsf{Aut}(M)$ in $\mathsf{GL}(H^{1,1}(M,\mathbb{R}))$ does not preserve any rational Kähler class, then M is not algebraically hyperbolic.)

Proposition 2 (The trickyiest) The image of the automorphism group in $\mathsf{GL}(H^{1,1}(M))$ is finite and image of automorphism group in $\mathsf{Aut}(\mathsf{Pic}(M))$ infinite. Then M is noa algebraically hyperbolic.

Proposition 3 $\mathsf{Aut}(M)$ infinite, image in $\mathsf{Aut}(\mathsf{Pic}(M))$ finite. Then M is not algebraically hyperbolic.

Upshot Torus dynamics breaks hyperbolicity.

4 HyperKähler case

Theorem (Them?) M hyperkähler with $\mathsf{rk}(\mathsf{Pic}(M)) \geqslant 3$ then M is not algebraically hyperbolic.

Proof. Quadraitc lattice has infinietly many automorphisms (infinite automorphism group) if $(1, n)$ fr $n \geqslant 2$. And in thie case we have that the picard lattice is such a lattice; on board:

$$\mathsf{Sym}(M) \cong \mathsf{O}(H^{1,1}(M, \mathbb{Z}), q)$$

where $\mathsf{Sym}(M)$ is holomorphic symplectic automorphisms.

□