Area rigidity of minimal surfaces

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Abstract We will talk about the volume spectrum of a manifold and recent results regarding its rigidity. This is joint work with Lucas Ambrozio and Fernando Marques.

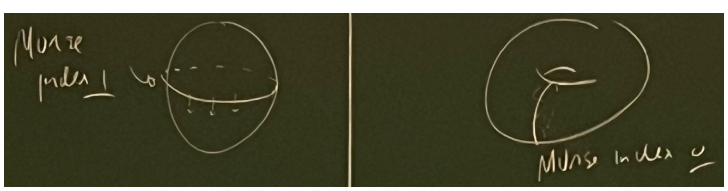
There are two different definitions for geodesics: as minimizing length functional, and as orbits of hamiltonian flow. This is a problem when studying minimal surfaces: we cannot use the latter definition.

Question (Poincaré) In 1904: every 2-sphere has a closed geodesic? Answer led to developments

Conjecture (Yau) 80's: every closed 3-manifold has infinitely many closed embedded minimal surfaces.

Theorem On every closed riemannian manifold (M^{n+1}, g) with $n \geqslant 2$ there are infinite number of distinct minimal embedded hypersurfaces $\{\Sigma_k\}$. For a generic metric the morse index of Σ_j is k and $\frac{\text{area}\,\Sigma_k}{k^{1/(n+1)}}$ obeys a Weyl-type law.

Remark Here Morse index counts the directions in which the area of the minimal surface decreases (e.g. meridian of torus has index 0, great circle of S^2 has index 1).



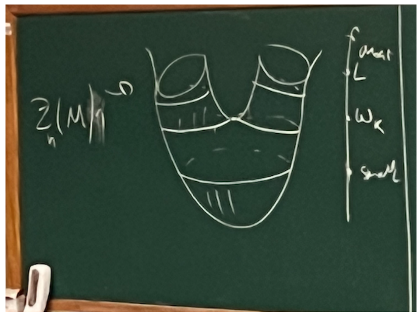
How to find Σ_k? You make an *educated guess* of what $\text{area}_g \Sigma_k$ should be.

Almgren in the 60's showed that the space of all trivial closed hypersurfaces, $\mathcal{Z}_n(M)$ is weakly homotopic to $\mathbb{RP}^\infty$. So a lot of cohomology.

For a fixed metric g, minimal hypersurfaces are critical points of

$$\mathrm{area}_g : \mathcal{Z}_n(M) \to [0, +\infty)$$

The educated guess is that the area corresponds to smallest L so that $\lambda^k \neq 0$ on $\{\Sigma \in \mathcal{Z} : \mathrm{area}(\Sigma) \leqslant L\}$



looks like some sort of Morse theory approach

0.1 Zoll metrics

A classical question in the theory of geodesics is to know to which extent the length of closed geodesics determines the ambient metric. For surfaces the answer depends on the curvature. Negative, flat, yes. Positive, no!

Question (dani) What is Zoll metric? What is volume spectrum? (I stopped following at this point.)