Two talks on groupoids

Contents

A primer on symplectic grupoids

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13 February, 2025
Geometric Structures Seminar

Abstract In the late 17th century, Simeon Denis Poisson discovered an operation that helped encoding and producing conserved quantities. This operation is what we now know as a Lie bracket, an infinitesimal symmetry, but what is its global counterpart? Symplectic groupoids are one possible answer to this question. In this talk, we will introduce all the basic concepts to define symplectic groupoids, and their role in Poisson geometry. We will discuss key examples, and applications. The talk will be accessible to those familiar with differential geometry, but no prior knowledge of groupoids will be assumed.

1.1 Part 1: Poisson geometry

Hamiltonian formalism. Recall that being a conserved quantity $f \in C^\infty(X)$ is the same thing as $\{H, f\} = 0$.

- We have seen that it is always possible to take quotient of a symplectic manifold with a group action to obtain a **Poisson manifold**.
- Then we have found a way to produce a symplectic foliation from a 2-vector $\pi \in \mathfrak{X}^2(M) := \Lambda^2(TM)$.
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  Remark
  $$\{\text{Lie algebra on } \mathfrak{g}\} \xrightarrow{1-1} \{\text{Linear Poisson bracket on } \mathfrak{g}^*\}$$

- We saw very nice examples of foliation that have to do with Lie algebras. So $\mathfrak{b}_3^*$ which gives the "open book foliation", and $\mathfrak{e}^*$ that gives a foliation by cilinders.

1.2 Part 2: symplectic realizations

Consider
$$(\Sigma, \omega) \xrightarrow{\mu} (M, \pi)$$

$$\begin{tikzcd} T_p\Sigma \arrow[r, "d_p\mu"] \arrow[d, "\omega^\flat"] & T_xM \\ T_p^*\Sigma & T_p^*M \arrow[l, "(d\mu)^*"'] \arrow[u, "\pi^\sharp"'] \end{tikzcd}$$

So that

$$\pi^\sharp = d_p\mu \circ \omega^{-1} \circ (d\mu)^*$$

Lemma $\dim(\Sigma) \geqslant 2\dim(M) - \mathrm{rk}(\pi_x)$ for all $x \in M$.

Proof. Done in seminar. □

Example $(\mathbb{R}^2, 0)$. So the map

$$\begin{aligned}(\mathbb{R}^4, dx \wedge du + dy \wedge dv) &\longrightarrow \mathbb{R}^2 \\ (x, y, u, v) &\longmapsto (x, y)\end{aligned}$$

Exercise Find the symplectic realization ω in $(\mathbb{R}^4, \omega) \xrightarrow{\mu} (\mathbb{R}^2, (x^2 + y^2)\partial_x \wedge \partial_y$

$$\begin{aligned}(\mathbb{R}^4, \omega) &\longrightarrow (\mathbb{R}^2, (x^2 + y^2)\partial_x \wedge \partial_y) \\ (x, y, z, w) &\longmapsto (x, y)\end{aligned}$$

Also find the symplectic realization of aff^* with $\{x, y\} = x$.

1.3 Part 3: Grupoids

1.3.1 Motivation

1. Fundamental grupoid: objects are points in the manifold and arrows are paths.
2. $S^1 \curvearrowright S^2$ by rotation does not give a nice quotient because there are two singular points. Consider the groupoid $S^1 \times S^2$ of orbits. These are the arrows. The points are just the points of S^2.
3. Consider a foliation (like Möbius foliation of circles; where there is a singular circle, the soul). You can do the same thing as in fundamental groupid **leafwise**. Arrows then are equivalence classes of paths that live inside leaves. This is called monodromy of a foliation. Again objects are points.
4. You can take a quotient of monodromy using a connection given by the foliation. This allows to identify certain paths between the leaves. This is called ***holonomy (of a foliatiation)***. (So you can make this notion match the usual holonomy given by riemmanian connection.)

Upshot So the point is taking some sort of function space on these groupoids you can gather the information given by the non-smooth quotient (like in the case of the sphere rotating). So this groupoid motivation says how to get some structure that resembles a non smooth quotient.

5. Last motivation: the grid of squares has a tone of symmetries. If you restrict to just a few squares you loose so many symmetries. But there's a grupoid hidden in there that tells you what you intuition knows about this finite grid of squares.

Definition A ***grupoid*** is a category where all morphisms are invertible.

So there is a kind of product among the objects, given by composition but: **not every two pair of objects can be multiplied!**—only those whose source and target match. So that's the lance about grupoids.

Just so you make sure you understand: the groupoid G is the morphisms of the category. The objects are points (of a manifold).

Definition ***Lie grupoid*** is when the following diagram is inside category of smooth manifolds and s, t (source and target maps) are submersions:

$$G^{(2)} \xrightarrow{m} G \xrightarrow[t]{s} M \xrightarrow{u} G$$

Properties of Lie groupoids

- m is also a submersion.
- i (inversion) is a diffeomorphism.
- u (unit=identity) is an embbeding.

Definition Consider $x \in M$ and the inverse image of source map: $s^{-1}(x) = \{\text{arrows that start at } x\}$. Now if you act with t on this set you get ***the orbit*** of x: $\{y \in M$ such that there is an arrow from x to $y\}$.

And there also ***an isotropy*** $G_x = \{g \in G : g \text{ goes from } x \text{ to } x\}$

Example

1. $G = M, M = M$.
2. Lie groups.
3. Lie group bundles.
4. $G = M \times M, M = M$.
5. Fundamental groupoid. Isotropy group is fundamental group! And orbit is...

universal cover!

6. Subgroupoids.
7. Foliations.
8. If you have a normal group action $G \curvearrowright M$ you construct a groupoid action with groupoid $G \times M$ and objects M, with product given on the group part of the product. Orbits are orbits. Isotropy group is isotropy group.
9. Principal bundles.

1.4 Back to Poisson

There's also a notion of Lie algebroid. Which is strange. But the point is that to every Poisson manifold there is a Lie algebroid.

So the question is whether there is a Lie groupoid associated to that Lie algebroid. Not always.

Big question (Fernandez and ?) When a symplectic manifold is integrable?

(Remember that integrating means go from algebra(oid) to group(oid).

And the point is that

The point When you *can* go back, you get a ***symplectic groupoid***.

Remark Look for Kontsevich's notes on Weinstein!

Remark History: Weinstein did this intending to do quantization (geometric?) on Poisson manifolds. (That involves a C^*algebra coming from the symplectic groupoid.)

Definition A ***symplectic groupoid*** is a groupoid G, M together with $\omega \in \Omega^2(M)$ such that ω is symplectic and multiplicative, meaning that $\partial\omega = 0$, that is, $\iff$ $m^*\omega = \mathsf{pr}_1^*\,\omega + \mathsf{pr}_2^*\,\omega \in \Omega^2(G^{(2)})$ $\iff$ take two vectors $X_k, Y_k \in TG$, and $\omega(X_0 \star Y_0, X_1 \star Y_1) = \omega(X_0, Y_0) + \omega(X_1, Y_1)$

Theorem If (G, ω) is a symplectic groupoid, then

1. $\exists!$ poisson structure on M
2. for which $t : G \to M$ is a symplectic realization,
3. Leaves are connected components of orbits,
4. $\mathsf{Lie}(G) \cong T^*M$ via $X \mapsto -u^*(i_X\omega$.

Remark Look for Alejandro Cabrera, Kontsevich. There are two things one is de Rham and the other. . .

Upshot The obstruction to knowing when symplectic groupoid exists is "variation of symplectic form $\omega = (1 + t^2)\omega_{S^2}$". So how does the symplectic group vary from leaf to leaf. So there are two situations in which the thing doesn't work.

Lie Groupoids: Foundations, Advances, and Future Directions

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March 21, 2025

Seminário das Sextas

Abstract Lie groupoids are central to modern geometry and mathematical physics, connecting differential geometry, topology, and dynamical systems. Based on a recent chapter co-authored with H. Bursztyn for the Encyclopedia of Mathematical Physics this talk will explore the foundations of Lie groupoids, their interactions with related structures like Lie algebroids and Poisson manifolds, and highlight recent advances and emerging research directions in the field.

2.1 Lie groupoids

Definition A ***Lie groupoid*** $G \rightrightarrows M$ consists of manifolds G, M, submersions $s, t : G \to M$ and a multiplication m with unit u and inverse i

$$
\begin{aligned}
m : G \times_M G \to G \qquad & (z \xleftarrow{g} y, y \xleftarrow{g} x) \mapsto (z \xleftarrow{hg} x) \\
u : M \to G \qquad & x \mapsto (x \xleftarrow{1_x} x) \\
i : G \to G \qquad & (y \xleftarrow{g} x) \mapsto (x \xleftarrow{g^{-1}} y)
\end{aligned}
$$

A ***morphism*** $\phi : (\tilde{G} \rightrightarrows M) \to (G \rightrightarrows M)$ consists of maps $\phi_1 : \tilde{G} \to G$, $\phi_2 : \tilde{M} \to M$ preserving source, target, multiplication and units.

2.2 Examples

Example

- For M manifold:
  - $M \rightrightarrows M$ unit groupoid.
  - $M \times M \rightrightarrows M$ pair groupoid.
  - $\pi_1(M) \rightrightarrows M$ fundamental groupoid.

- G Lie group: $G \rightrightarrows *$ Lie group
- $q : M \to N$ submersion: $M \times_M \rightrightarrows M$ submersion groupoid. Ex: $\mathcal{U} = \{U_i\}$ open cover $\rightsquigarrow \coprod U_{ji} \rightrightarrows \coprod U_i$ Čhech groupoid.
- $G \curvearrowright M$ action: $G \times M \rightrightarrows M$ action groupoid.
- $X \in \Gamma(TM)$: $U_X \subset \mathbb{R} \times M \rightrightarrows M$ flow gruopoid.
- $P \to M$ principal bundle: $(P \times P)/G \rightrightarrows M$ gauge groupoid.
- $F \subset M$ foliation:
  - $\mathsf{Mon}\, F \rightrightarrows M$ monodromy groupoid
  - $\mathsf{Hol}\, M \rightrightarrows M$ holonomy groupoid.
  - $V \to M$ vector bundle: $\mathsf{GL}(V) \rightrightarrows M$ bigeneral linear groupoid.

2.3 Actions and representations

Here's one of the key ideas: relation between group and groupoid. Groups in geometry serve to model symmetries of a space:

$$\text{group } G \rightrightarrows * \leftrightsquigarrow \text{symmetries of a space } E \to *$$

And groupoids model symmetries of a *family* parametrized by M:

$$\text{groupoid } G \rightrightarrows M \leftrightsquigarrow \text{symmetries of a family } E \to M$$

Definition Given $G \rightrightarrows M$ a Lie grouoid and $E \to M$ a surjective submersion, a ***groupoid action*** $\rho : G \curvearrowright E$ is a map such that $\rho_{1_x} = \mathsf{id}_{E_x}$ and $\rho_h \rho_g = \rho_{gh}$:

$$\rho : G \times_M E \to E \qquad (y \leftarrow x, e/x) \mapsto \rho_g(e)/y$$

It is a ***representation*** if $E \to M$ vector bundle and $\rho_g E_x \to E_y$ linear.

Example

- The ***parallel action***. Take a vector bundle with a flat connection. Then the fundamental groupoid *acts by parallel transport*, i.e. $(E \to M, \nabla)$ is a groupoid action $(\pi_1(M) \rightrightarrows M \curvearrowright (E \to M)$. Notice this is *not* a group action, it's naturally a groupoid action.
- A ***Hamiltonian action*** $G \curvearrowright (M, \omega)$ on a symplectic manifold is an action of the action groupoid $(G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^* \curvearrowright (M \to \mathfrak{g}^*)$.
- The ***(linear) holonomy*** of a foliation $F \subset TM$ is a representation of the monodromy groupoid $(\mathsf{Mon}(F) \rightrightarrows M \curvearrowright (TM/F \to M)$.

2.4 Groupoid fibrations

A representation $(G \rightrightarrows M) \curvearrowright (E \to M)$ gives rise to a groupoid morphism with a (unique) lift of arrows:

$$\begin{tikzcd} G \times_M E \rightrightarrows E \arrow[d] & & \cdot & e \arrow[l, "\rho(g,e)"'] \\ G \rightrightarrows M & & y & x \arrow[l, "g"'] \end{tikzcd}$$

A ***fibration*** $\Gamma \rightrightarrows E) \to (G \rightrightarrows M)$ is a morphism with (non-necessarily unique) lift of arrows. A ***VB-groupoid*** is a linear fibration (called like this for historical reasons).

Example

- A usual representation $(G \ltimes_E \rightrightarrows E) \to (G \rightrightarrows M)$
- The tangent and the cotangent groupoid $(TG \rightrightarrows TM) \to (G \rightrightarrows M)$ and $(T^*G \rightrightarrows A^*) \to (G \rightrightarrows M)$

Theorem (Grothendieck;Gacria-Saz,Mehta;dH,Ortiz) There is a 1-1 correspondence between VB-groupoids $\Gamma \to E$ over $G \rightrightarrows M$ and representations up to homotopy $G \curvearrowright (C \oplus E \to M)$.