Degenerations of the canonical series for curves (Part 2/2)

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Abstract

Contents

1 Reminder on first part

We have been parametrizing canonical series.

We have X a projective connected nodal curve over a closed field of characterstic zero. We created its dual graph, where

- Vertices correspond to C_v, normalization of components of X associated to v.
- Edges correspond to p^e node of C.
- e connects u and v iff $p^e \in X_u \cap X_v$.

Associated to the dual graph there's also the arrow set $\mathbb{E}$. There's a 2-1 map from the set of arrows to the set of edges. To each arrow we associate a branch $p^a \in C_v$, where $a = uv \in \mathbb{E}$. We denote

$$\mathbb{E}_v = \{a \in \mathbb{E} : t_a = \{\text{ tail of } a\}\text{-}u\}$$

so that $\mathbb{E} = \bigsqcup_{v \in V} \mathbb{E}_v$.

We also have the ***genus function*** g mapping V to its genus, and the ***genus formula***

$$p_a(X) = g(V) + g(G) = \sum_{v \in V} g(v) + |E| + |V|$$

2 Today

We want to consider *smoothings* $\pi : \mathfrak{X} \longrightarrow B$ of X:

$$\begin{array}{c} \mathfrak{X} \\ \Big\downarrow{\scriptstyle \pi} \\ B = \begin{cases} A_0 \\ \mathrm{Spec}(k[\![E]\!]) \end{cases} \end{array}$$

So $\mathfrak{X}_{\mathbb{O}} \xrightarrow{\cong} X$ $\mathfrak{X}_\eta$ smooth (over Laurent series $k(\!(t)\!)$. $\mathfrak{X}$ is regular away from $p^e \in X$.

$$\hat{\mathcal{O}}_{\mathfrak{X},p^e} = \frac{k[\![t,u,v]\!]}{uv - t^{\ell_e}} \ell_e \in \mathbb{Z}_{>0}$$

where $\ell : E \longrightarrow Z_{>0}$ is the ***edge lenght function***. We also have $\Gamma = (G, \ell)$ the ***metric graph***, and

- $\omega_{\mathfrak{X}/B}$ the ***relative canonical bundle*** on $\mathfrak{X}$.
- $\omega_{\mathfrak{X}|B}|_{\mathfrak{X}_\eta} = \Omega_{\mathfrak{X}_\eta}$ the ***bundle of differentials***.
- $\omega_{\mathfrak{X}|B}|_X = \omega_X$ the canonical bundle of X. ω_X is the ***bundle of regular differentials*** (Rosenlicht 50's)

Notation: Ω_v is the space of meromorphic differentials of C_v,

$$\Omega = \bigoplus_{v \in V} \Omega_v$$

$\alpha = (\alpha_v)_v \in \Omega$ is regular if

1. $\forall a \in \mathbb{E}$, α_{t_a} has at most a simple pole at p^a.
2. $\forall a \in \mathbb{E}$, $\mathrm{res}_{p_a}(\alpha_{t_a} + \mathrm{res}_{pa}(\alpha_{t_a}) = 0$.

2.1 Abelian differentials

C_v, $D \in D_N(C_v)$, $D = P - N$, $P, N \geqslant 0$, $\mathrm{supp}(P) \cap \mathrm{supp}(N) = \varnothing$.

$$\mathbb{H}_v(D) = \{\alpha \in \Omega_v | \operatorname{div}_\infty(\alpha) \leqslant P, \operatorname{div}_0(\alpha) \geqslant N\}$$

$$\mathbb{H}_v(0) = \text{Abelian differentials}$$

$$W_0 = \Gamma(X, \omega_X) \subseteq \bigoplus_v \mathbb{H}_v \left(\sum_{a \in \mathbb{E}} p_a \right)$$

where W_0 is the space of global regular differnetials. $\dim W_0 = g_{\mathfrak{X}_\eta} = p_a(X)$, the arithmetic genus of X.

If D is general and characteristic of k is zero, then

$$\begin{aligned}\dim_k \mathbb{H}_\nu(D) &= \max(g_\nu + \deg D - 1, 0) \qquad \text{if } p \gneq 0 \\ &= max(g_\nu + \deg D, 0) \qquad \text{if} p = 0\end{aligned}$$

2.2 Back to our objective

$$\mathcal{L}_h = \omega_{\mathfrak{X}/B} \otimes {}''\mathcal{O}_{\mathfrak{X}}\left(-\sum h_\nu X_\nu\right)$$

where $h : V \longrightarrow \mathbb{Z}$.

$$\mathcal{L}_h|_{X_\eta} = \Omega^1_{\mathfrak{X}_\eta}$$

$$\mathcal{L}_h|_{C_\nu} = \omega_{C_\nu}\left(\sum_{a\in\mathbb{E}_\nu}(1+\partial_\ell h(a)p^a\right)$$

[∗some missing formulas ∗]

$$W_h = \mathrm{Im}(\Gamma(\mathfrak{X}, \mathcal{L}_h) \longrightarrow \Gamma(X, \mathcal{L}_h|_X)$$

$$\dim W_h = p_a(X)$$

$$W_h \subseteq \bigoplus_{\nu\in V}\mathbb{H}_\nu\left(\sum_{a\in\mathbb{E}_\nu}(1+\partial_\ell h(a)p^a\right) \subseteq \bigoplus_{u\in V}\Omega_u = \Omega$$

The goal is to describe and parametrize the collection of subspaces

$$\mathcal{C} = \{W_h \in \Omega | h \in \mathbb{Z}^\vee\}$$

when the points p^a on C_ν for $a \in \mathbb{E}$ are in general position.

3 A space that parametrizes W_h

This is work by Kapranov. First of all, W_h is not uniquely defined:

$$V \twoheadrightarrow V_h = V/\sim_h$$

where $u \sim_h \nu \iff h(u) = h(\nu)$.

We have a torus action

$$\mathbb{G}_m^{V_h} = \{\psi : V_h \longrightarrow K^*\} \subseteq \mathbb{G}_m^V$$

$$W_h \subseteq \Omega, \qquad \psi \cdot W_h = \{(\psi_\nu \alpha_\nu)_\nu | (\alpha_\nu)_\nu \in W_h\}$$

$\mathbb{G}_m^{V_h} \cdot W_h$ orbit is well-defined.

The case of $h = 0 \implies V_h = \{V\}$, so W_0 is well-defined.

So actually we want to parametrize not the W_h but their orbits, $\mathbb{G}_m^V \cdot W_h \in \mathrm{Gr}(g, \Omega)/\mathbb{G}_m^V$.

$$W \subseteq U = \bigoplus_v U_v \subseteq \bigoplus_v \Omega_v = \Omega$$
$$\mathbb{G}_m^\vee \curvearrowright \mathrm{Gr}(g, U) \subset \mathrm{Gr}(g, \Omega) \curvearrowleft \mathbb{G}_m^\vee$$

4 Polyhedral approach

Basically what you realize is that the only orbits that matter are the maximal dimension ones, and these correspond to maximal dimesional polytopes. (These appeared in the work of Kapranov and was later generalized by others.

The idea is as follows. Fix a decomposition of your space $W \subseteq \Omega = \bigoplus_{v \in N} \Omega_v$. A ***submodular function***

$$\begin{aligned} \nu_W : 2^V &\longrightarrow \mathbb{Z} \\ V \supseteq I &\longmapsto \dim_k \theta_I(W) \end{aligned}$$

$*$ some formulas $*$ Definition of base polytope. The orbit is maximal dimensional if and only if the polytope is. Definition of bricks.

Theorem The polytope associated to the W_h is a union of bricks of maximal dimension.