Loop spaces: conformal center of mass and infinite dimensional GIT for the space of isometric loops

Dmitrii Korshunov
IMPA

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Abstract We show, following Millson and Zombro, how to construct a Kähler structure on the moduli space of isometric maps of a circle into Euclidean space. The proof upgrades the Kirwan/Kempf–Ness theorem relating symplectic reduction and GIT to the infinite dimensional setting. The construction crucially uses the Douady–Earle's notion of conformal center of mass, which is of independent interest.

1 Three spaces

Definition ***Millson-Zombro*** space is

$$\left\{(S^1, g) \xrightarrow{\text{isometric}} \mathbb{R}^3\right\} \Big/ \mathsf{SO}(3) \ltimes \mathbb{R}^3$$

1. It is a Kähler manifold.
2. J is integrable. This means there are holomorphic coordinates, *not* algebraic integrability (N tensor vanishes).

The upshot is that there is no Newlander-Niremerg theorem in infinite dimension.

Definition ***Brylinskly loop space*** is: for M 3-dimensional Riemannian manifold,

$$\left\{S^1 \to M\right\} \Big/ \mathsf{Diff}^+(S^1)$$

(Without isometry condition.)

Take a normal vector field V_1 to a curve $\gamma : S^1 \to M$,

1. Define a complex structure $Jv_1 := v_1 \times \frac{\dot\gamma}{|\dot\gamma|}$. This is formally integrable (N tensor vanishes) but never has a holomorphic structure (infinite dimensional!)
2. $\int_{S^1} \langle V_1, V_2 \rangle \, dt$ where V_2 is another vector field along γ, and perhaps we must choose arclength parametrization. A Riemannian structure.
3. Now consider the volume form on M. The form $V(\cdot, \cdot, \frac{\dot\gamma}{|\dot\gamma|}$ is skew symmetric. So, a symplectic structure.

Definition ***Kapovich-Millson space of polygons***. First, a ***polygon*** is all the embeddings of n points joined by edges of lengths $\ell_1, \ldots, \ell_n$ in that order and possibly intersecting each other. Now fix a point ...? The polygon space is

$$P = \text{polygons} \Big/ \mathsf{SO}(3) \times \mathbb{R}.$$

Question Why $\times\mathbb{R}$?

$$P \subset S^2_{\ell_1} \times \ldots \times S^2_{\ell_n} \curvearrowleft \mathsf{SO}(3)$$

where action is diagonal action.

In fact, this action is hamiltonian. Because the action $\mathsf{SO}(3) \curvearrowright S^2$ is hamiltonian with moment map the identity. Then the moment map of the product of spheres is just the sum of moment maps. Also it's simple because $\mathfrak{so}(3)^* \cong \mathbb{R}^3$. So we may perform symplectic reduction and obtain a symplectic manifold $\mu^{-1}(0)/\mathsf{SO}(3)$. *And the resulting space is the space of polygons!* That's the symplectic structure.

Question what about complex and Riemannian strutures? are they more straightforward to show?

1.1 Back to MZ space = the loop space

Consider a loop that does not necessarily close up, so a map $\gamma : \mathbb{R} \to \mathbb{R}^3$. And suppose it is ***smoothly periodic*** so unit veolicity and derivative $\dot{\gamma} : S^1 \to S^2$ periodic. Now Gauss map gives us $\mathbb{R} \to S^2$, but since veolicity is periodic it's actually $S^1 \to S^2$. Now let's put some structures on the space of all $S^1 \to S^2$:

1. (Almost complex structure) Tangent vectors to this space are vector fields tangent to S^2—tangent to the loop! Again: loop is point, vector at this point is a vector field tangent to the loop. Anyway we can act with the complex structure of S^2 on that vector and that's the complex structure.
2. The riemannian structure is inherited from the sphere: pair them and integrate. The question is wether that depends on parametrization. Last time we just declared that we use unit speed parametrization.
3. Symplectic structure:
$$\omega(v_1, v_2) = \int \dot{\gamma} \cdot (v_1 \times v_2) dt$$
where $(v_1 \times v_2)$ is the triple product presented before.

   Proposition ω is closed.

   Proof. Consider
$$\mathsf{ev} : LS^2 \times S^1 \to S^2$$
$$\pi : LS^2 \times S^2 \to LS^2.$$

Then

$$\omega_{LS^2} = \pi_* \mathsf{ev}^*(\omega_{S^2}) \wedge dt$$

where ω is the symplectic structure on S^2 and π is "fiber integration over ?" . $\square$

Corollary $\frac{\omega}{8\pi^2}$ is integral. (There will be a prequantization)

And there's an action $LS^2 \curvearrowleft \mathsf{SO}(3)$. And it's hamiltonian with moment map $\mu : LS^2 \to \mathbb{R}^3 \cong \mathfrak{so}(3)^*$. So the fundamental vector field is $\xi \times \beta$. And the moment map is, for $\beta \in LS^2$

$$\mu(\beta) = \int_0^{2\pi} \beta(t)dt$$

$$\int \nu \cdot \xi ds = \int_0^{2\pi} \beta \cdot \Big((\xi \times \beta) \times \nu\Big) ds$$