Degenerations of the canonical series for curves (Part 1/2)

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Abstract

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1 Introduction

What is the canonical series? Consider C a projective smooth connected curve over a closed field k. The cotangent bundle ω_C is the same as the bundle of differentials, and the canonical bundle. It has complex dimension 1. The local sections are differentials.

A differential $\alpha \in \Gamma(C, \omega_C)$ is holomorphic, it can be meromorphic. It is written as $\alpha = f dt$ for some $f \in k(C)$. This leads to the notion of a ***divisor*** of a differential, which in this case is

$$\operatorname{div}(\alpha) = \sum \operatorname{ord}_p(t_p)p$$

and it is also the ***canonical divisor***.

Now

$$\omega_C = \mathcal{O}_C(k)$$

The ***genus***, which is a topological invariant, is also $\dim_k \Gamma(C, \omega_C)$.

$$\deg(\omega_C) = \deg(K) = 2g - 2$$

And

$$\mathbb{H} = \Gamma(C, \omega_C)$$

is the ***canonical series***.

Example Smooth quartic plane curve

$$C = V(F) \subset \mathbb{P}^2$$

We have that $\deg(F) = 4$, and

$$\begin{aligned}
\omega_C &= \mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^2}(1)|_C \\
g &= \frac{(d-1)(d-L)}{2} = 3 \\
\mathcal{O}_C(1) &= \mathcal{O}_C(L \cap C) \\
\deg(\mathcal{O}_C(1)) &= 4 = 2g - 2 \\
\dim_k \Gamma(G\mathcal{O}_C(1)) &= 3 = g
\end{aligned}$$

> Think of the canonical series as a space of linear sections of a line bundle, but also as a collection of divisors parametrized by $\mathbb{P}^2$

Example (A singular quadric)

2 What we do

Given a nodal curve X (at a node there are two "branches" that intersect) which is general for its topology ($G = (V, E)$ dual graph) where

- V is the set of irreducible components). There is a correspondence of the vertices in this graph and curves in $X : v \in V \iff X_v \subset X$.
- E is the set of nodes. Here $e \in E \iff N_e \in X$. So an edge is a pair of points if the node belongs to the intersection of the corresponding curves:

  $$e = \{u, v\} \iff N_e \in X_u \cap X_v$$

- The ***genus function*** associates to every component its geometric genus:

  $$\begin{aligned}
  g : V &\longrightarrow \mathbb{Z}_{\geqslant 0} \\
  g(v) &= \text{geometric genus of } X_v
  \end{aligned}$$

  (I think the geometric genus is the genus of the normalization of the variety.)

This is the combinatorial data attached to the curve.

We We look for a general curve with respect to the geometric genus, and say it is ***general for its position*** if the nodal points are in general position.

Then we have a stratification of

$$\overline{\mathcal{M}} = \{\text{stable curves } X \text{ with finite automorphism group}\}$$

And here *stable* means nodal. So for example you can have stability if the degree of $\omega_C|_X$ is positive.

So the stratification is:

$$\overline{\mathcal{M}}_g = \bigsqcup \mathcal{M}_{(G,g)}$$

where

$$\mathcal{M}_{G,g} = \{X \text{ s.t. } G \text{ is the dual graph of } X_\omega \text{ genus function } g.\}$$

And we have that

$$\operatorname{codim} M_{G,g} = |E|,$$

the number of edges.

Remark These are graph curves. All their components are $\mathbb{P}^1$ and they intersect in prescribed way.

3 Some combinatorics

We have the ***genus formula***:

$$P_a(X) = \sum g_v + g(G)$$

$$g(G) = |E| - |V| + 1$$

$$g = 3g - 3 - (2g - 2) + 1$$

$$\deg(F) = 4$$

Remark When you have maximum number of edges, you force everything to be of a particular kind by the genus formula. (This follows from stability condition.)

Remark We may check stability looking if the canonical bundle is trivial.

Remark Stable $\iff$ every component which is $\mathbb{P}^1$ has at least 3 special points.

4 What we do

Now let's finish the statement we started before:

Given a nodal curve X which is general for its topology, we describe all limits of the canonical series in any degeneration to X, and construct a parameter space for them.

Exercise Let X be a smooth projective variety such that K_X is ample. Prove that its automorphism group is finite.

We would like to study the moduli of stable curves. So we have a parameter for the objects we want to classify. Diaz-Cutievman described the locus of curves with special Weierstrass points.

Weierstrass points are such that the line (what line?) intersect the curve in at least (some bound). So for a quartic,

$$P \text{ is a W point} \iff I(P; T_p C \cap C) \geqslant 3$$

and in fact

$$g^3 - g = 24 \text{ W points.}$$

Remark A general smooth curve of genus 3 has exactly 24 Weierstrass points.

Exercise The genus g curve has $g^3 - g$ Weierstrass points.

a. For $g = 3$ and plane quartics.

b. For hyper-elliptic curve of genus 3 (also define W point in this case).

c. For non-hyperelliptic genus 4 curve.

d. Etc.

5 The additional data (?)

Take your variety. The drawing is a bunch of blue lines. Take another line (red). How do the y intersect?

$$\lim_{t \to 0} X_t \cap L = X \cap L.$$

And intersection with another curve F?

$$\lim_{t \to 0} X_t \cap L_0 = F \cap L_0$$

Let $L = L_0$. We have

$$\begin{aligned} L_0 L_1 L_2 L_3 + tF &= 0 \\ L_0 &= 0 \end{aligned}$$

Dividing by t,

$$\begin{aligned} L L_1 L_2 L_3 + F &= 0 \\ L_0 &= 0 \end{aligned}$$

Now look at linear series generated by $L L_1 L_2 L_3 \forall L$ and F on $L_0 = 0$. $L_0 = X$.

$$\begin{aligned} (\alpha Y + \beta Z) L_1 L_2 L_3 + \gamma F = 0, &\qquad (\alpha, \beta, \gamma) \in \mathbb{P}^2 \\ L_0 = 0 & \end{aligned}$$

6 After break

Now we explain how these systems of divisor appear and how we are going to handle them.

The limit of the $\check{\mathbb{P}}^2$ is some divisors.

We are considering a smoothing

$$\begin{array}{c} \mathfrak{X} \\ \downarrow \\ B = \Delta_0 = \operatorname{Spec} k[[t]] \subset \mathbb{C} \end{array}$$

And we have

$$\begin{gathered} \omega_{\mathfrak{X}/B} \text{ is the relativa canonical bundle} \\ \omega_{\mathfrak{X}/B}\Big|_{\mathfrak{X}_\eta} = \cap \mathfrak{X}_\eta \\ \omega_{\mathfrak{X}/B}\Big|_{\mathfrak{X}_\sigma} = \omega_X \subseteq \Omega_X = \bigoplus_{\nu \in V} \Omega_\nu \end{gathered}$$

where Ω_ν is the space of meromorphic differentials over $C_\nu = \tilde{X}_\nu$.

So it's a family of bundles that is the canonical bundle on the general fibers and on the exeptional fiber it is the canonical bundle too.

7 Regular differentials (Rosenlicht)

$$(\eta_\nu)_\nu \in \bigoplus_\nu \Omega_\nu$$

$$\operatorname{res}_{p_a} \eta_\nu + \operatorname{res}_{p_{\bar{a}}} \eta_\omega = 0$$

Now η_ν can only have poles at the branches, and they should be simple.

Remark Looks like we have been computing how the bundles $\mathcal{O}_L(1)$, $\mathcal{O}_{L_i}$ look like when restricted to different subvarieties. So for example

$$\mathcal{O}_{\mathfrak{X}}(1)(-L_0)|_{L_i} = \mathcal{O}_{L_i}(1)(-1)$$

is just a skyscraper sheaf.

In the end we concluded that (L_h, W_h) has infinitely many linear series in X where

$$W_h = \Gamma(\mathfrak{X}, \mathcal{L}_h)\Big|_{\mathfrak{X}_0} \cap \Gamma(X, L_h)$$

So, importantly,

$$\{0 \neq s \in W_h \qquad Z(s) \subseteq X \qquad |Z(s)| < \infty\} = \qquad \text{lim divisors}$$

h vary.

$$\begin{aligned} \omega_X &\subseteq \Omega = \bigoplus \Omega_v \\ \omega_{\mathfrak{X}}\left(-\sum h(v) X_v\right)\Big|_X &\subseteq \Omega \\ \Omega_{X_\omega}\left(\sum p_u - \sum p_u\right) & \end{aligned}$$

Canonical case

$$W_h \subseteq \Omega = \bigoplus \Omega_v$$

where Ω_v is the space of meromorphic differentials on $C_v = \tilde{X}$. $\dim W_h = g$.

8 Kapranov

Take some k-vector spaces U_v and consider

$$U = \bigoplus_{v \in V} U_v$$

and take the grassmanian of subspaces of dimension g, and the torus action:

$$Gr(g, U) \leftharpoondown \mathbb{G}_m^{\vee} = \{\psi : V \to k^*\}$$

and the projection maps:

$$\theta_I : \bigoplus_{v \in V} U_v \longrightarrow \bigoplus_{v \in I} U_v, \qquad I \subseteq V$$

W general, $\theta_I\Big|_W$ has maximal rank.

So consider the orbit, it is a Chow variety:

$$\overline{[\mathbb{G}_m^{\vee} \cdot W} \in \mathrm{Chow}(\mathrm{Gauss}(g, U))$$

And we have the Chow quotient/Hilbert quotient/Mumford quotient (studied first by Thaddeus):

$$\overline{\{\overline{[\mathbb{G}_m^{\vee} \cdot W]}, W \text{ general}\}} \subseteq \mathrm{Chow}(\mathrm{Gauss}(g, V))$$

And then

$$\partial \overset{\{*\}}{=} \sum [\mathbb{G}_m^{\vee} \cdot W_i], \qquad \mathrm{Gauss}(g, V)$$

The polytopes asrouled to W_i form a polyhedral decomposition of a certain polytope. Now

$$W \subseteq U \implies \mu_W : 2^V \longrightarrow \mathbb{Z}$$

which is a submodular function,

$$\mu = \mu_{\mathcal{W}}(I) = \dim_k \theta_I(\mathcal{W})$$

$$\mu(I) + \mu(S) \geqslant \mu(I \cap J) + \mu(J \cup J)$$

$$P_\mu = \{q \in \mathbb{R}^\vee : q(I) \leqslant \mu(I)\ \forall I,\ q(V) = \mu(V)\}$$

What Kapranov observed

$$K : \bigcup P_{\mathcal{W}_i} = P_{\mathcal{W}\,\text{general}}$$