Classification of log Calabi-Yau pairs

Eduardo Alvez da Silva
Institut de Mathématiques d'Orsay

Contents

1 Introduction 1

2 Classification of log CY pairs in dimension 2 2

3 (Partial) Classification in dimension 3 3

4 Sketch 5

1 Introduction

Definition. A ***log Calabi-Yau*** pair is a lc pair (X, D) consisting of a normal projective variety X and a reduced Weil divisor D such that $K_X + D \sim_{\mathbb{Z}} 0$.

Remark. Let $n = \dim X$. (X, D) CY pair $\implies \exists \omega := \omega_{X,D} \in \Omega^n_x$, unique up to nonzero scaling such that $\operatorname{div}(\omega) + D = 0$. We call ω the ***volume form***.

$$(X, D)\text{minimal model} \xleftarrow[\text{log MMP}]{} (X, D) \text{ CY pair} \xrightarrow{\substack{\text{Classical}\\ \text{MMP}/X}} \text{Mori fibered space}$$

- (X, D) CY pair,

$$\begin{aligned} K_X + D \sim 0 &\implies -K_X = D \geqslant 0 \\ &\implies K_X \text{ is not pseudo effective} \\ &\overset{*}{\implies} X \text{is uniruled} \\ &\implies K(X) = -\infty \\ &\implies \text{The output of the MMP over } X \text{ is Mori fibered space} \end{aligned}$$

  where $*$ means BDPP theorem.

- (X, D) is a minimal model for the log MMP since $K_X + D$ is nef.

Example (content. . .).

Definition. Let $f : (X, D_X) \xrightarrow{\text{bir}} (Y, D_Y)$ be a birational map of CY pairs. f is ***volume preserving*** if $f^*\omega_{Y,D_Y}$, for some $\lambda \in \mathbb{C}^*$.

Remark. $\mathrm{Bir}^{\mathrm{up}}(X, D_X) \subset \mathrm{Bir}(X)$ =group of all volume-preserving maps.

Definition (Other equivalent definitions)**.**

(i) f ***preserves discrepancies***, i.e. for a divisor E over X and Y we have $a(E, X, D_X) = a(E, Y, D_Y)$.

(ii) f admits a log resolution

$$\begin{tikzcd} & (Z, D_Z) \arrow[dl, "up"'] \arrow[dr, "vq"] & \\ (X, D_X) \arrow[rr, dashed] & & (Y, D_Y) \end{tikzcd}$$

$$up^*(K_X + D_X) = q^*(K_Y + D_Y)$$

Warning D_Z does not need to be effective. Ex. look at my PhD thesis in section 5.4.

Notation. Volume preserving equivalence, or crepant birrational,

$$(X, D_X) \cong_{vp} (Y, D_Y)$$

$$(X, D_X) \cong_{cbir} (Y, D_Y)$$

Because volume-preserving maps are also called crepant maps.

Problem (Very hard!) Classification of log CY pairs up to volume-preserving equivalence.

The most important invariant to attack this problem is the following:

Definition. The ***corregularity*** of a log CY pair (X, D_X), $\mathrm{coreg}(X, D_X)$, is defined to be the dimension of a minimal lc center in a dlt modification

$$f : (X^{\mathrm{dlt}}, D_{X^{\mathrm{dlt}}}) \to (X, D_X)$$

Remark. $c := \mathrm{coreg}(X, D_X)$, $0 \leqslant c \leqslant \dim X$, $c = \dim X \iff X$ is CY and $D_X = 0$.

2 Classification of log CY pairs in dimension 2

After a minimal resolution of singularities, it follows that a surface log CY pair (X, D_X) is agiven by one of the following:

- $c = 2$: X is an abelian surface or a K3 surface, and $D_X = 0$.

- $c = 1$:
  (i) X is rational and $D_X \in |-K_X|$ is a nonsingular elliptic curve.
  (ii) $\pi : X \to E$ (not necessarily minimal) ruled surface over a nonsingular elliptic curve E, and $D_X = D_1 + D_\alpha \in |-K_X|$ is the sum of two disjoint sections of π.
- $c = 0$: X is rational and D_X is a (possible reducible) nodal curve of arithmetic genus 1.

Example. $X = \mathbb{P}^2$. Three lines, conic + line, nodal cubic, nonsingular cubic. Their corregularities are zero except for the last one, which is 1.

Definition. A log CY pair (X, D_X) has a ***toric model*** if $(X, D_X) \cong_{vp} (T, D_T)$ (where D_T is the reduced sum of all torus invariant divisors).

Theorem (Gross-Hacking-Keel)**.** Every surface log CY pair (X, D_X) of log coregularity 0 has a toric model.

Remark. Its false in dimension $\geqslant 3$.

3 (Partial) Classification in dimension 3

Theorem (Ducat, 2023)**.** Let $(\mathbb{P}^3, D)$ be a log CY pair with corregularity $c \leqslant 1$. Then there exists a volume-preserving map

$$\varphi : (\mathbb{P}^3, D) \xrightarrow{\text{bir}} (\mathbb{P}^1 \times \mathbb{P}^1, D')$$

where

$$D' = (\{0\} \times \mathbb{P}^1) + (\mathbb{P}^1 + E) + (\{\infty\} \times \mathbb{P}^2) \in |-K_{\mathbb{P}^4 \times \mathbb{P}^2}|$$

for a plane cubic $E \subset \mathbb{P}^2_{(x:y:z)}$ such that

1. $c = 1 \iff E$ is non singular.
2. If $c = 0$, then $E = \{xyz = 0\}$. In particular D' is the toric boundary of $(\mathbb{P}^1 \times \mathbb{P}^2$ and thus $(\mathbb{P}^3, D)$ has a toric model.
3. $c = 2$ (The missing case) Fact: $c = 2 \iff D$ is an irreducible normal quantic surface having canonical singularities, i.e., D is either nonsingular or has ADE singularities $\iff$ the pair is canonical

Example (Oguiso's example)**.** He constructed two nonsingular isomorphic quartic surfaces $D, D' \subset \mathbb{P}^3$ (as abstract varieties) such that there exists $\varphi \in \mathrm{Bir}(\mathbb{P}^3)$ mapping D birrationally onto $D' \implies (\mathbb{P}^3, D) \not\cong_{vp} (\mathbb{P}^3, D')$.

Thinking in terms of coarse moduli spaces, we have a natural map

$$\mathcal{M}^{c=2}_{(\mathbb{P}^3, D)} \longrightarrow \mathcal{M}^{\mathrm{can}}_{K3}$$
$$[(\mathbb{P}^3, D)] \longmapsto [D]$$

and Oguiso's example implies that this is not injective.

Conjecture (Trichotomy).

$\mathrm{coreg}(\mathbb{P}^3, D)$	0	0	?
$\dim \mathcal{M}^c_{(\mathbb{P}^3,D)}$	0	1	?
$\mathrm{Bir}^{\mathrm{vp}}$	monstruous	?	$\mathrm{Dec}(D)$
g	0	1	$\geqslant 2$
$\dim \mathcal{M}_g$	0	1	$3g-3$
$\mathrm{Bir} = \mathrm{Aut}$	$\mathrm{PGL}(2,\mathbb{C})$ infinite	$C \rtimes \mathbb{Z}_d$	$\#\mathrm{Aut}(C) \leqslant 84(g-1)$ finite

D very gen. D is nonsingular,

$$\varphi : (\mathbb{P}^3, D) \xrightarrow{\text{v.p,bir}} (\mathbb{P}^3, D)$$

$$\implies \varphi|_D D \xrightarrow{\cong} D$$

X projective variety, $Y \subset X$ irreducible subvarieties,

$$\mathrm{Bir}(Y, X) = \{f \in \mathrm{Bir}(X) | f|_Y : Y \xrightarrow{\text{bir}} Y\}$$

Conjecture (Shokuroo). Every 3-fold rational log CY pair (X, D_X) of coregularity 0 has a toric model.

Ducat's Theorem implies it is tru for $X = \mathbb{P}^3$.

Definition. (X, D_X) CY pair, $D_X = D_1 + \ldots + D_r$. The ***complexity*** of this CY pair is the non-negative number

$$c(X, D_X) := \dim X + \mathrm{rk}(\mathrm{Cl}(X)_{\mathbb{Q}}) - r$$

Fact $c(X, D_X) = 0 \implies (X, D_X)$ has a toric model (Brown, Mckenan, Lvald, Long, 2018).

Definition. (X, D_X) CY pair. The ***birrational complexity*** is

$$c_{\mathrm{bir}}(X, D_X) := \min\{c(Y, D_Y) | (Y, D_Y) \cong_{\mathrm{vp}} (X, D_X)\}$$

Theorem (Mauri, Moraga, 2023). $c_{\mathrm{bir}}(X, D_X) = 0 \iff (X, D_X)$ has a toric model.

Definition. A log CY pair (X, D_X) is ***cluster type*** if there exists a volume-preserving map

$$\varphi : (\mathbb{P}^n, H_0 + \ldots + H_{n+1} \xrightarrow{\text{bir}} (X, D_X)$$

such that $\mathrm{codim}_{\mathbb{C}^n_M}(\mathrm{Ex}(\varphi) \cap \mathbb{C}^n_m) \geqslant 2 \iff \mathbb{C}^n_m \hookrightarrow X \setminus D_X$.

Theorem (—,Figueroa, Moraga, 2024). $(\mathbb{P}^3, D)$ log CY pair of coregularity 0. Assume D general in its deformation class. Then $(\mathbb{P}^3, D)$ is cluster type unless one of the following happens:

(i) D reducible, $D = H + C$ (plane cubic, resp.) such that $H \cap C$ is a nodal plane cubic.

(ii) D irreducible and has double points along a line.

4 Sketch

$$\begin{aligned}(\mathbb{P}^3, D)\text{is cluster type} &\iff (\mathbb{P}^3, D) \text{ is cluster type over} \mathbb{P}^1 \\ &\iff \exists \text{ some dlt modification } (X, D_X) \\ &\text{of } \mathbb{P}^3 \text{ such that } \exists \text{ a crepant contraction} \\ &\text{onto } (\mathbb{P}^4, \{C\} + \{\infty\}).\end{aligned}$$