

# Automorphisms of quartic surfaces and Cremona transformations

Carolina Araujo

IMPA

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## 1 Motivation

$X$  a smooth hypersurface of degree  $d$ ,  $X \subset \mathbb{P}^{n+1}$ . We want to understand the group  $\text{Aut}(X)$ . These are invertible polynomial maps from  $X$  to itself.  $X$  is defined by a single polynomial equation of degree  $d$ .

**Theorem** (Matsumura-Mousky, 1964). Except in two special cases, all automorphisms of  $X$  come from automorphisms of the ambient space:

If  $(n, d) \neq (1, 3), (2, 4)$ , there is a surjective map

$$\text{Aut}(\mathbb{P}^{n+1}, X) \xrightarrow{\pi} \text{Aut}(X)$$

where  $\text{Aut}(X, Y)$  means automorphisms of  $X$  that stabilize  $Y$ .

## 2 Exceptional cases

Let's look at the exceptional cases.

1.  $(n, d) = (1, 3)$ . In this case  $C = X_3 \subset \mathbb{P}^2$  is an elliptic curve and we have

$$\text{Aut}(C) \cong C \rtimes \mathbb{Z}_m \quad m = 2, 4, 6$$

and

$$\text{Aut}(\mathbb{P}^2, C) \text{ is finite.}$$

Elements in  $C$  are translations of the torus. The *translation of  $x$  with respect to  $p$*  **and**  $(x : y : z)$  is done by intersecting the curve with the line that joins  $p$  and  $(x : y : z)$  and reflecting. So we have a map

$$t_p(x : y : z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$$

We have created a *Cremona transformation* (=biholomorphic birational map?).

**Definition (Cremona group).**

$$\text{Bir}(\mathbb{P}^n) = \{\varphi : \mathbb{P}^n \xrightarrow{\text{bir}} \mathbb{P}^n : \text{bimeromorphic map}\}$$

And we then have the surjective map

$$\text{Bir}(\mathbb{P}^2, C) \xrightarrow{\pi} \text{Aut}(C)$$

2.  $(n, d) = (2, 4)$ . Here  $S = X_4 = \mathbb{P}^3$  is a smooth quartic surface.

**Problem** (Gizatullin) Which automorphisms of  $S$  are induced (not necessarily by automorphisms of  $\mathbb{P}^3$ ) but at least by Cremona transformations of  $\mathbb{P}^3$ ? ie. are restrictions of  $\varphi \in \text{Bir}(\mathbb{P}^3, S)$

**Remark.** Related to K3 surface structure,

$$\text{Bir}(\mathbb{P}^3, S) \xrightarrow{\pi} \text{Bir}(S) \cong \text{Aut}(S)$$

$S$  is a K3 surface.  $H^2(S, \mathbb{Z}) \cong \mathbb{Z}^{22}$  and the Picard group acts on this lattice.

**Definition (Picard rank of  $S$ ).**

$$\rho(S) = \text{rk}(\text{Pic}(S)) \in \{1, \dots, 20\}$$

- If  $S$  is very general, then  $\rho(S) = 1$  and  $\text{Aut}(S) = \{1\}$ .
- We are interested in  $\rho(S) \geq 2$ .

**Example (Oguiso, 2012).**

1.  $\rho(S) = 2$ . Any cremona transformation that stabilizes the quadric is the identity:

$$\text{Aut}(S) = \mathbb{Z} \quad \text{Bir}(\mathbb{P}^3, S) = \{1\}$$

2.  $\rho(S) = 3$ ,

$$\text{Aut}(S) = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$$

All automorphisms are induced by Cremona transformations, ie. there is a surjective map

$$\text{Bir}(\mathbb{P}^3, S) \xrightarrow{\pi} \text{Aut}(S)$$

3. (Paiva, Quedo 2023) Constructed surfaces with  $\rho(S) = 2$ ,  $\text{Aut}(S) = \mathbb{Z}_2$  and  $\text{Bir}(\mathbb{P}^3, S) = \{1\}$ .

**Theorem (A-Paiva-(Socrates) Zika).** Solution of Giztallin S problem for  $\rho(S) = 2$ .

**Remark.** Not exactly, but "there is a moduli space for K3 surfaces of dimension  $20 - \rho(S)$ ". The generic case is  $\rho(S) = 1$ .

### 3 K3 surfaces

**Definition.** A *K3 surface* is a smooth projective surface that is simply connected and has a nowhere vanishing symplectic form  $\omega \in H^0(S, \Omega_S^2)$

#### 3.1 Lattices of S

A *lattice* is a finitely-generated abelian group with a nondegenerate pairing.

1.  $H^2(X, \mathbb{Z}) \cong \mathbb{Z} \xleftarrow{\pi} \text{Pic}(S)$ . And if we tensor this with  $\mathbb{C}$  we get  $H^2(C, \mathbb{C})$ , which admits a Hodge decomposition.

Let's study automorphisms of a K3 surface. Let  $g \in \text{Aut}(S)$ . It yields an element  $g^*$  that acts on cohomology, ie  $g^* \in \mathcal{O}(H^2(X, \mathbb{Z}))$  preserving the Hodge decomposition. This is called *Hodge isometry*.

**Theorem (Global Torelli theorem).**

- The correspondence  $g \mapsto g^*$  is injective
- If  $\varphi \in \mathcal{O}(H^2(X, \mathbb{Z}))$  is a Hodge isometry preserving the ample classes, then  $\varphi = g^*$  for some  $g \in \text{Aut}(S)$ .

**Example.**  $S \subset \mathbb{P}^3$  smooth quartic surface with  $\rho(S) = 2$ . Using the equivalence of line bundles modulo isomorphism and curves modulo intersection,

$$\text{Pic}(S) = \langle H, C \rangle$$

where  $H$  is a hyperplane section of  $\mathbb{P}^3$ . Then

$$Q = \begin{pmatrix} H^2 & H \cdot C \\ H \cdot C & C^2 \end{pmatrix} = \begin{pmatrix} 4 & b \\ b & 2c \end{pmatrix}$$

using that  $2g(c) - 1$ , so the number in the lower-right on RHS is even.

**Remark.** Hodge Index Then  $Q$  is rank  $(1,1)$ .

**Definition (Discriminant).**

$$r = \text{disc}(S) = -\det Q$$

**Proposition.**  $S$  general K3 surface (not needed that it is a quartic) with  $\rho(S) = 2$ . Then

$$\text{Aut}(S) = \begin{cases} \{1\} & (\text{finite}) \\ \mathbb{Z}_2 & (\text{finite, Dani-Ana}) \\ \mathbb{Z} & (\text{infinite, Oguiso 1}) \\ \mathbb{Z}_2 * \mathbb{Z}_2 & (\text{infinite, Dani-Ana}) \end{cases}$$

where the first two are characterized by containing an elliptic curve or a rational curve, i.e.  $\exists D \in \text{Pic}(S)$  such that  $D^2 = 0, -2$ . On the other hand, the last two cases are distinguished by  $\nexists D \in \text{Pic}(S)$  such that  $D^2 = 0, -2$ .

$$\exists \sigma \in \text{Aut}(S) \text{ of order } 2 \iff \exists \text{ ample } A \in \text{Pic}(S) \text{ such that } A^2 = 2.$$

**Remark.** In low Picard numbers there are no symplectic involutions...?

So to understand the surface we want to understand those bundles and that is all in the discriminant (not in the lattice itself!).

**Remark.**  $\exists D \in \text{Pic}(S)$  s.t.  $D^2 = k \iff x^2 - ry^2 = 4k$  has integer solutions.

Given the quadratic form we can find the automorphisms.

### 3.2 Main theorem

**Theorem (A-Paiva-Zika).**  $S \subset \mathbb{P}^3$  general smooth quartic surface with  $\rho(S) = 2$  and  $\text{disc}(S) = r$ .

1. (Negative answer to Gizatulla's problem) If  $r > 57$  or  $r = 52$  then

$$\text{Bir}(\mathbb{P}^3, S) = \{1\}$$

So we cannot realize any automorphism as a Cremona transformation.

2. If  $r \leq 57$  and  $r \neq 52$ , then we get the full automorphism group, i.e. a surjective map

$$\text{Bir}(\mathbb{P}^3, S) \xrightarrow{\pi} \text{Aut}(S)$$

## 4 Birational geometry

Take the case of

$$\begin{array}{c} \text{Bir}(\mathbb{P}^2) = \langle \text{Aut}(\mathbb{P}^2), q \rangle \\ \uparrow \\ \text{Aut}(\mathbb{P}^2) \end{array}$$

and the map

$$q : \mathbb{P}^2 \xrightarrow{\text{bir}} \mathbb{P}^2$$

$$(x : y : z) \longmapsto \left( \frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right) = (yz : xz : xy)$$

which is well-known (Noether-Castelnuovo). So that is a decomposition of automorphisms and quadratics. Now in greater dimension,  $n \geq 3$  we have

**Theorem (Sarkisov Program).** (The theorem is much more general) Any  $\varphi \in \text{Pic}(\mathbb{P}^n)$  can be factorized with *Sarkisov links*  $\varphi_i$ :

$$\begin{array}{ccccccc} & & & & \varphi & & \\ & & & & \curvearrowright & & \\ \mathbb{P}^n & \xrightarrow{\varphi_1} & X_1 & \xrightarrow{\varphi_2} & X_2 & \xrightarrow{\dots} & X_k = \mathbb{P}^n \\ & & \downarrow & & \downarrow & & \\ & & T_1 & & T_2 & & \end{array}$$

Now look at  $n = 3$ ,  $\text{Bir}(\mathbb{P}^3, S) \subset \text{Bir}(\mathbb{P}^3)$ . This is a Calabi-Yau pair:

**Definition (Calabi-Yau pair).** A pair  $(X, D)$

- Terminal projective variety.
- $K_X + D \sim 0$  that is, a meromorphic top form that does not vanish on hypersurface and has simple pole on  $D$ . Then  $(X, D)$  is called *log canonical*.

Now take two Calabi-Yau pairs  $(X, D_X)$  and  $(Y, D_Y)$ .

$$\text{div}(\omega_{D_X}) = -D_X$$

$$\text{div}(\omega_{D_Y}) = -D_Y$$

We say that a birrational map  $f : X \rightarrow Y$  is *volume preserving* is  $f_* \omega_{D_X} = \omega_{D_Y}$ .

**Theorem (Volume-Preserving).** Everything like in the Sarkisov theorem but now maps are volume-preserving.

In our case,  $(\mathbb{P}^3, S)$ , we can classify the v.p. Sarkisov links from  $(\mathbb{P}^3, S)$ . It starts by blowing up a curve  $C \subset S$ . But this curve has genus and degree very restricted, it's something like

$$(g(C), \deg(C)) \in \{(0, 1), (0, 2), \dots, (11, 10), (14, 11)\}$$

So for the main theorem, it was checked that if  $r > 57$  there are no curves from the list. And in the second item of the main theorem, there exist these curves, for instance curve  $(14, 11)$  for rank 56, then produce a link that starts by blowing it up and magically gives you the Cremona transformation that restricts with automorphism with which you started.

**Remark.** So perhaps we expect the answer to G. problem to be almost never.

**Question.** How does that blowing-up work?

$$\begin{array}{ccc}
 X & \xrightarrow{\text{flops}} & X \\
 \downarrow \text{Bl}_C & & \downarrow \text{contract?} \\
 C \subset S \subset \mathbb{P}^3 & \dashrightarrow & \mathbb{P}^3 \supset S' \supset C'
 \end{array}$$

So for example in case  $(2,8)$  you obtain something of the same type.