Hodge theory on Hermitian symplectic manifolds

Misha Verbitsky

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Contents

1 Introduction

Definition of complex structure, de Rham differential...

Remark. The two differentials d and d^c anticommute.

Remark. A holomorphic form is such that $\bar{\partial} = 0$, which is equivalent to having holomorphic functions as coefficients in local coordinates, and also equivalent to having its differential on the first piece of the Hodge decomposition, namely $\Omega^{k+1,0}(M)$ for a k-form.

Conjecture (Question, really). Almost complex manifold (M, I_0) of dimension $n > 2$ then there exists complex I homotopic to I_0

Definition. Compact complex manifold admitting complex structure.

Locally conformally Kähler manifold. A quotient of a Kähler manifold by a discrete group acting by (non-isometric) homotheties. Never admits a Kähler metric (Vaisman).

Example. Hopf manifold $(\mathbb{C}^n \setminus 0)/\mathbb{Z}$.

2 Hermitian structures on cm

Let (M, I, g) be a complex Hermitian n-manifold, $\omega \in \Lambda^{1,1}(M)$ its Hermitian form.

Remark. Notice that $0 = d(\omega^k) = k d\omega \wedge \omega^{k-1}$ where $\omega^{k-1} : \Lambda^3 \to \Lambda^{3+k}$ is not injective when $k = 1, n-2$. So the only interesting case is when $d(\omega^{n-1}) = 0$, in which case we say ω is ***balanced***.

All twistor spaces are balanced (Hitchin). All Moishezon manifolds are balanced (Alessandrini-Bassaneli).

Theorem (Gauduchon)**.** For each (M, I, g), there exits a unique up to a constant multiplier, ***Gauduchon metric*** in the same conformal class. Explanataion: all metrics are conformal to each other.

This is good for defining the degree of a holomorphic bundle, stability, which are concepts well-defined for the Kähler case:

Definition. Degree of a form with respect to a bundle is

$$\deg_\omega(E) = \int \omega\theta_B \wedge \omega^{\wedge 1}$$

Definition. μ-stability of a bundle is

$$\mu(B) = \frac{\deg B}{\operatorname{rk} B}$$

3 Hermitian symplectic and SKT metrics

Definition. A metric g is called ***SKT (strong Kähler torsion*** or ***pluriclosed*** is $dd^c\omega = 0$.

Definition. ω Hermitian form on (M, I) is ***hermitian symplectic*** if

$$\omega(X, IX) > 0 \qquad \forall x \neq 0$$

in this case we say ω is ***taming*** or that it ***tames*** the (tames what?)

Remark. That condition is equivalent to $\omega^{1,1}(X, IX) > 0$ because it vanishes on the 1-dimensional vector space $\langle X, IX \rangle$.. (?)

Conjecture. Hermitian symplectic implies Kähler.

Remark. This conjecture is known for almosta ll known classes of non-Kähler complex manifods: LCK manifolds (easy), twistor spaces, Moishezon and Fujiki class C (non-trivial), complex nilmanifolds (highly non-trivial).

4 What today is about

Theorem. If M hermitian symplectic dimension of M is 3. Then dd^c lemma is true on $\Lambda^{1,1}(M)$:

$$\text{if } x \in \Lambda^{1,1}(M) \text{ is exact, then } x = dd^c f.$$

Theorem. The following statements are equivalent:

1. I is integrable.
2. $\partial^2 = 0$.
3. $\bar{\partial}^2 = 0$.
4. $dd^c = -d^c d$.
5. $dd^c = 2\sqrt{-1}\partial\bar{\partial}$.

Remark. And I think also that Dolbeault cohomology is equivalent to de Rham.

5 Bott-Chern cohomology

Definition. Bott-Chern:

$$\text{in } dd^c, \quad \ker d/\Lambda^{p,q} := H^{p,q}_{BC}(M)$$

And there is a sequence with de Rham, Dolbeault and BC cohomologies.

6 Hodge-Riemann relations on (p,p)-forms

(Not that they have anything to do with Riemann.)

Definition. Let $\{L, \Lambda := *L*, H\} \subset \mathrm{End}(\Lambda^* M)$ be the standard Lefschetz triple acting on differential forms on a Hermitian n-manifold. $(L(\eta) = \eta \wedge \omega$ and $H|_{\Lambda^p(M)} = p - n$A form $\eta \in \Lambda^p(M)$ is called ***primitive*** if $\Lambda(\eta) = 0$.

Proposition (Special case of Hodge-Riemann relations)**.** $\eta \in \Lambda^{1,1}(M, \mathbb{R})$, η primitive then

$$\frac{\eta\eta\omega^{n-2}}{\mathrm{Vol}} \leqslant 0$$

and < 0 when $\eta \neq 0$ (positive *negative*? Because it's hyperbolic…)

7 Michelson's theorem

She proved that twistor spaces are balanced.

$\eta \in \Lambda^{n-1,n-1}(M)$, it is a bivector (a form on the dual space, a symmetric scalar product on $\Lambda^1(M)$) as follows:

$$\begin{aligned} \Lambda^1(M) \times \Lambda^1(M) &\longrightarrow \mathcal{C}^\infty(M) \\ \alpha, \beta &\overset{\eta(\alpha,\beta)}{\longrightarrow} \frac{\alpha \wedge \beta \wedge \eta}{\mathrm{Vol}} \end{aligned}$$

η positive if $\eta(\alpha, I\alpha) \geqslant 0$ for all $\alpha \in \Lambda^1(M, \mathbb{R})$.

Theorem (Michelson)**.** For any strictly positive $(n-1, n-1)$-form φ there exists a Hermitian form ω such that $\omega^{n-1} = \varphi$.

Exercise. Prove this theorem.

Proof. Take a basis of $\Lambda^{n-1,0}(M)$:

$$\zeta_i^* = \zeta_1 \wedge \zeta_2 \wedge \ldots \hat{\zeta}_i \wedge \ldots \wedge \zeta_n$$

and write

$$\left(\sum_i \alpha_i \zeta_i \wedge \bar{\zeta}_i \right)^{n-1}$$

(I guess for ω^{n-1}) and then solve for the coefficients. $\square$

Lemma (That I missed)**.**

Lemma (3)**.** $\alpha \in H^{1,1}_{BC}(M)$, ω Gauduchon metric (so it's a Gauduchon Hermitian complex manifold). Assume that the degree of α with respect to ω is zero, ie.

$$\deg_\omega \alpha := \int \alpha \wedge \omega^{n-1} = 0.$$

Then $[\alpha]$ can be represented by a closed, primitive $(1,1)$-form.

Remark. The next day after this talk Lucas defined the ***degree*** of a surface to be integral of the first Chern class...

The point is that Chern class are in BC cohomology.

Proof. Define a differential operator, ...(?) $\square$

8 Dependence on ω

Proposition. Let ω, ω_1 be Hermitian forms on M, and α a $(1,1)$-form which is ω-primitive. Then

$$\frac{\alpha \wedge \alpha \wedge \omega_1^{n-1}}{\mathrm{Vol}} < 0$$

in all points where $\omega\alpha \neq 0$.

Remark. Primitive means: to be in the orthogonal complement to ω (or to q_ω, the bivector above...?).

Proof. ...? $\square$

Corollary. Let (M, I) be a Hermitian symplectic manifold. Then

(i) The natural map $H^{1,1}_{BC} \to H^2(M)$ is injective (map from Bott-Chern to de Rham).

(ii) All holomorphic 1-forms on M are closed, and represent non-zero classes in cohomology.

(iii) $H^1(M)$ is even-dimensional, and satisfies

$$H^1(M) = H^0(\Omega^1(M)) \oplus \overline{H^0(\Omega^1(M))}$$

Remark. The last item is what we expect in a Kähler manifold: in other words, the first cohomology, like for Kähler manifolds, is isomorphic to the sum of holomorphic and anti-holomorphic forms.