Symplectic and contact nature of Riemannian geometry

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Surfaces of constant extrinsic curvature k.

	$\mathbb{H}^2$	$\mathbb{R}^3$	S^3
$k>1$	S^3	S^2	S^2
$k=1$	Horospheres, horocycles		
$0<k<1$	ess. param., $\mathcal{H}(\mathbb{D}) \sqcup \mathcal{H}(\mathbb{C})\backslash\{\mathbb{C}\}$		

Remark Intrinsic curvature = extrinsic curvature + sectional curvature of the ambient space.

X a 3-manifold, take $x \in X$ and $v_x \in TX$. The Levi-Civita connection provides

$$\begin{aligned} T_v TX &\cong H_v TX \oplus V_x TX \\ &\cong \underbrace{TxX}_{\text{hor.}} \oplus \underbrace{T_x X}_{\text{vert.}} \end{aligned}$$

- Saski metric
$$\langle (\xi,\mu)(\xi',\mu')\rangle = \langle \xi,\xi'\rangle + \langle \mu,\mu'\rangle$$
- Symplectic form
$$\omega\Big((\xi,\mu),(\xi',\mu')\Big) = \langle \xi,\mu'\rangle - \langle \xi',\mu\rangle$$
which we may pull back to X via the musical isomorphism to obtain the ***Saski symplectic form*** of T^*X $\omega = \flat^*\omega_{\mathrm{st}}$.
-
$$m\Big((\xi,\mu),(\xi',\mu')\Big)$$
- There is a complex structure I
- And a quadratic form
$$m = \begin{pmatrix} 0 & I_0 \\ I_0 & 0 \end{pmatrix}$$

And a contact bundle pulling the luiville form from T^*X. And then there is a another complex structure J. So define $K := IJ$, which gives a **hyperkähler contact structure** on the unit tangent bundle.

Upshot We have created a pseudoholomorphic curve from a k-surface by taking the unit tangent normal. That is, consider the normal map as an embedding of our surface in the unit tangent bundle.

Theorem (Jurgens) $\Sigma \subset \mathbb{R}^4$

(i) complete,

(ii) J-holomorphic

(iii) $m|_{T\Sigma} \geqslant 0$

then Σ is a plane. (Morally, the graph of a linear function.)

Theorem (dim 3,4 Calabi, dim 5⩾ Pgorelov) $f : \mathbb{R}^2 \to \mathbb{R}$

(i) f convex,

(ii) $\det \mathrm{Hess}(f) = 1$ (Monge-Ampère)

Then f is quadratic.

Proof. Short, done in seminar. □

And then we want to prove a compactness property. So we take a sequence of surfaces $(\Sigma_m, p_n) \subset S^1X$ with $p_n \in \Sigma_n$. Suppose $\| \mathrm{II}_m \| \xrightarrow{m\to\infty} \infty$. Then there exists B_m and q_m such that

(i) $\| \mathrm{II}_m(q_m)\| = B_m$.

(ii) $B_m \xrightarrow{m\to\infty} \infty$ and,

(iii) by a lemma that is easy to prove, $\forall r \in B_{\frac{1}{2\sqrt{\| \mathrm{II}_m(q_m)\|}}}(q_m)$.

And then we rescale the metrics $g \to g_m = B_m^2 g$. That makes the metric be flatter and flatter, like zooming in, and also makes the shape operator of every surface have norm 1. and we get

$$(S^1X, g_m) \longrightarrow (\mathbb{R}^5,)$$
$$\Sigma_m \longrightarrow \Sigma_m \subset \mathbb{R}^5$$

with respect to the famous Cheeger Gromov topology in the limit, which is not so easily defined. And by Arzelá-Ascoli and Elliptic regularity magic (see M. Joshi Course notes) (regularity is not smoothness but some differentiability) the limit surface is compact, positive, J-homolomorphic. And by Jurgen's theorem they are flat. But that's a contradiction with the fact that the shape operators of these surfaces have norm 1.

So you have compactness. Yaaaaay!

Magnetic monopoles according to Hitchin

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Definition A ***magnetic monopole*** consists of 3 elements: (E, ∇, Φ), a bundle, a connection, and the Higgs field.

And the Bogomolny equations are:

Lemma (Bogomolny equations) ∇ is anti self dual iff $*F^{\nabla^1} = \nabla^1 \Phi_0$.

What's going on (I arrived in late) ChatGPT says F^{∇} is the curvature of this connection on this principal bundle. So Bogomolny equations are equations on the curvature. Like what? **Like Maxwell's equations!**

Here's some more ChatGPT on what's about to come in this talk:

Magnetic Monopoles and Twistors. A magnetic monopole consists of a vector bundle E, a connection ∇, and a Higgs field Φ. The Bogomolny equation, $*F^{\nabla} = \nabla\Phi$, describes monopole solutions balancing curvature and the Higgs field. The Higgs field Φ can be interpreted as a difference between two connections or as a section of $\mathrm{End}(E)$.

Twistor theory provides a complex-geometric approach to gauge fields. The twistor space of $\mathbb{R}^3$ can be described in terms of oriented lines, isotropic $\mathbb{C}$-lines, or null $\mathbb{C}$-planes in $\mathbb{C}^3$. For a $U(2)$-connection, one constructs a holomorphic vector bundle $\tilde{E} \to X$ over the complex manifold $X = TS^2$, encoding monopoles in terms of holomorphic structures.

Remark (Misha) Φ is sections of endomorphisms of the group which is G.

Question What is the Higgs field? It is a *difference* between two connections.

Question What are twistors?

0.1 Twistors in $\mathbb{R}^3$

Embed $\mathbb{R}^3 \hookrightarrow \mathbb{C}^3$. Extender the inner product of $\mathbb{R}^3$ $\langle \cdot, \cdot \rangle$ *blinearly* to $\mathbb{C}^3$. (It's not hermitian, it's a bilinear form.)

Twistor space of $\mathbb{R}^3$:

1. Oriented lines in $\mathbb{R}^3$.
2. Null $\mathbb{C}$-lines in $\mathbb{C}^3$ (isotropic).
3. Null $\mathbb{C}$-planes in $\mathbb{C}^3$.

The three are equivalent (we have shown that in the seminar).

It turns out that this space can be parametrized by TS^3 as follows:

$$L^+ \to \underbrace{x}_{\substack{\text{unit}\\\text{tangent}}} \qquad \underbrace{y}_{\substack{\text{closest}\\\text{point}}}$$

$$x \in S^2 \qquad y \perp x \iff y \in T_x S^2.$$

Now we have a complex integrable structre becuase $S^2 = \hat{\mathbb{C}}$

Now

suppose (E^2, ∇, Φ). Fix the group to be $U(2)$ (it's a $U(2)$-connection. Define a bundle $\tilde{E}$ over X by

$$\begin{aligned}\tilde{E}_x &= \{\sigma : L_x \to R : \nabla_T \sigma - u\Phi\sigma = 0\}\\ &= \{\sigma : L_x \to E : \tilde{\nabla}_{(T+ie_0)}\sigma = 0\}\end{aligned}$$

where $\tilde{\nabla}$ is the Yang-Mills connection.

Now take the complex manifold $X = TS^2$, so $(E^2, \nabla, \Phi$ and produce the bundle $\tilde{E} \to X$. Then you may actually put a holomorphic structure on $\tilde{E}$, making into a holomorphic vector bundle over X.

Now we write:

$$\begin{aligned}X &= \{\text{oriented lines in } \mathbb{R}^3\}\\ T_{(x,y)}TS^2 &= \{(u,v) : \langle u,v\rangle = 0, \langle u,y\rangle + \langle x,v\rangle = 0\}\\ V_x &= \{(0,v) : \langle x,v\rangle = 0\}\\ J(0,v) &= (0, x \times v)\\ H_{(x,y)}TS^2 &= \{(u, -\langle u,y\rangle x : \langle u,x\rangle = 0\}\\ \hat{u} := (u, -\langle u,y\rangle x)\\ J\hat{u} &= \widehat{x \times u}\end{aligned}$$

And then: Jac fields orthogonal to L^+ is 4 dimensional.

$$J_\xi = T \times \xi$$

Constant curvature $\to$ integrable.

$$\begin{aligned}&\gamma_{x,y}(t) = y + tx\\ &(u, v - \langle u,y\rangle x\\ &\quad \xi(t) = v - \langle u,y\rangle x + tu\end{aligned}$$

G lie group, $H \subseteq G$

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$
$$[\mathfrak{g}, \mathfrak{p}] \subseteq \mathfrak{p} \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{g} \qquad \text{Polarization}$$

$\mathfrak{p}$ id. with $T(G/H)$ fwy $\mathsf{Ad}_{\mathfrak{g}}$-invariant form on $\mathfrak{p}$ generates a parallel form on G/H.

$$G = \mathsf{SO}(3) \ltimes \mathbb{R}^3$$
$$H = \mathsf{SO}(2) \times \mathbb{R}$$
$$\mathfrak{p} = 4, J, \nabla J = 0$$

Here's ChatGPT:

Twistor Space and Structure. The twistor space of $\mathbb{R}^3$ is given by $X = TS^2$, which parametrizes oriented lines in $\mathbb{R}^3$. The tangent space at $(x, y) \in TS^2$ decomposes as

$$T_{(x,y)}TS^2 = \{(u, v) : \langle u, v\rangle = 0, \langle u, y\rangle + \langle x, v\rangle = 0\}.$$

The vertical and horizontal subspaces are defined as

$$V_x = \{(0, v) : \langle x, v\rangle = 0\}, \quad H_{(x,y)}TS^2 = \{(u, -\langle u, y\rangle x) : \langle u, x\rangle = 0\}.$$

An almost complex structure J is introduced by

$$J(0, v) = (0, x \times v), \quad J\hat{u} = \widehat{x \times u}.$$

Jacobi Fields and Integrability. The Jacobi fields orthogonal to an oriented line L^+ form a 4-dimensional space, where

$$J_\xi = T \times \xi.$$

Constant curvature ensures integrability.

The geodesic equation for a curve $\gamma_{x,y}(t) = y + tx$ leads to

$$\xi(t) = v - \langle u, y\rangle x + tu.$$

Lie Groups and Homogeneous Structure. Consider a Lie group G with a subgroup $H \subseteq G$, and the Lie algebra decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \quad [\mathfrak{g}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{g}.$$

Since $\mathfrak{p}$ is identified with $T(G/H)$, an $\mathsf{Ad}_{\mathfrak{g}}$-invariant form on $\mathfrak{p}$ induces a parallel form on G/H.

For the twistor space,

$$G = \mathsf{SO}(3) \ltimes \mathbb{R}^3, \quad H = \mathsf{SO}(2) \times \mathbb{R}.$$

The space $\mathfrak{p}$ has dimension 4, and the almost complex structure J satisfies $\nabla J = 0$, ensuring integrability.

0.2 Scattering transform

Suppose you have a Lie group G, and two Lie subgroups $H_1, H_2 \subseteq G$. Now take

$$X := G/H_1, \qquad X_2 = G/H_2$$

$$\begin{tikzcd} & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Radon transform:

$$R : C^\infty(X_1) \longrightarrow C^\infty(X_2)$$

So

$$\begin{aligned} R &= \pi_{2*}\pi_1^* \\ G &= \mathsf{SO}(3) \ltimes \mathbb{R}^3 \qquad \text{isometries of } \mathbb{R}^3 \\ H_1 &= \mathsf{Stab}\ \text{point} = \mathsf{SO}(3) \\ H_2 &= \text{Stable line} = S^1 \times \mathbb{R} \end{aligned}$$

Now we can put a little zero on the continuous functions (that means they tend to zero?):

$$R : C_0^\infty(X_1) \longrightarrow C^\infty(X_2)$$

$$\begin{aligned} R &= \pi_{2*}\pi_1^* \\ \pi_1^*(f) &= f \circ \pi_1 \\ \pi_{2*} f &= \int_H f dV \frac{1}{2\pi} \\ \hat{f}(L) &= \int_L f de \\ F &= \pi_{2*}\mu_\phi \pi_1^* \end{aligned}$$

$$\begin{tikzcd} & G = \mathsf{SO}(3) \ltimes \mathbb{R}^3 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{R}^3 & & X \\ & (\pi_1^* E, \pi_1^* \nabla, \pi_1^* \phi) \arrow[uu] & \end{tikzcd}$$

So a global section

$$(\pi_1^* E_1|_F, \pi_1^* \nabla|_F, \pi_1^* \phi|_F$$

Summary of the Scattering Transform. Given a Lie group G and two subgroups $H_1, H_2 \subseteq G$, we define the homogeneous spaces

$$X_1 = G/H_1, \quad X_2 = G/H_2.$$

The *Radon transform* is a mapping

$$R : C_0^\infty(X_1) \to C^\infty(X_2),$$

which factors through the pullback π_1^* and pushforward π_{2*}:

$$R = \pi_{2*}\pi_1^*.$$

For $G = \mathsf{SO}(3) \ltimes \mathbb{R}^3$ (the isometry group of $\mathbb{R}^3$), the subgroups correspond to:

$$H_1 = \mathsf{SO}(3) \quad \text{(stabilizing a point)}, \quad H_2 = S^1 \times \mathbb{R} \quad \text{(stabilizing a line)}.$$

The transform integrates functions along fibers:

$$\pi_1^* f = f \circ \pi_1, \quad \pi_{2*} f = \int_H f dV.$$

A section of a lifted bundle $(\pi_1^* E, \pi_1^* \nabla, \pi_1^* \phi)$ over G descends through π_2, encoding the scattering transform in terms of differential geometric data.