On the algebraic hyperbolicity of projective hypersurfaces

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Upshot Recall that last week we talked about Kobashi hyperbolicity, which means that Kobayashi pseudo-distance is non-degenerate. We saw that this implies Brody hyperbolicity, which means that X has no non-constant holomorphic maps. If X is compact this is an equivalence.

Today we shall see that this implies algebraic hyperbolicity. Demially's conjecture is that this last implication is actually iff.

Abastract A complex projective manifold X is said to be algebraically hyperbolic if every integral curve C of X satisfies the inequality $2g(C)2 > \varepsilon deg(C)$ for a fixed positive ε and ample divisor on X. This talk aims to review some techniques used to prove the algebraic hyperbolicity of very general hypersurfaces of degree $d > 2n2$ in $\mathbb{P}^n$.

Definition A complex projective variety X is ***algebraically hyperbolic*** if there exists $\varepsilon > 0$ and ample divisor H such that, for every integral curve $C \subset X$ of geometric genus $g(C)$,

$$2g(C) - 2 \geqslant \varepsilon \operatorname{deg}_H(C)$$

Remark (Vitorio) Definition of ***degree*** is $H.C$.

Remark (Misha) Is there not an associated metric here? Take the metric in all agebraic curves inside X. Perhaps it is equivalent?

Remark In particular, X does not contain any rational or elliptic curves.

In this talk: $X \subset \mathbb{P}^n$ very general projective hypersurface of degree d.

Conjecture (Kobayashi) X is algebraically hyperbolic for d sufficiently large ($d \geqslant 2n - 2$, $n \geqslant 4$ or $d \geqslant 2$, $n = 3$).

1 $n \geqslant 3$

Theorem (Clemens '86, E. '88) $n \geqslant 4$, $d \geqslant 2n$.

Theorem (Voisin, '96, 98) $n \geqslant 4$, $d \geqslant 2n-1$.

Theorem (Pocienza '04, Clemens-Ron '04) $n \geqslant 6$, $d \geqslant 2n-2$.

Theorem (Yeong '22) $n \geqslant 5$, $d \geqslant 2n-2$.

Open: $(n, d) = (4, 6)$.

2 $n = 3$

Theorem (Xu, '94) $n = 3$, $d \geqslant 6$.

Theorem (Coskan-Ried, '99) $n = 3$, $d \geqslant 5$.

And $n = 3$, $d = 4$ is a K3 so (or is it *because*?) it has rational curves.

3 Proof for $n = 4$, $d \geqslant 2n-2$

Proof. Main reference *Algebraic hyperbolicity of the very general quintic surface in* $\mathbb{P}^3$, Coskun and Reid '99.

Algebraic hyperbolicity of very general surface, Coker and Reid '22.

Algebraic hyperbolic of very general hypersurface in $\mathbb{P}^n \times \mathbb{P}^n$, Young '22. □

Open: $\mathbb{P}^2 \times \mathbb{P}^1$.

Proof. Let $S_d = \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ and let

$$\mathcal{X} = \{(p, [X]) : p \in X\} \subseteq \mathbb{P}^n \times s_d$$

which is a universal degree d hypersurface in $\mathbb{P}^n$.

We can find a generically injective map $h : Y \to X$ over $U \overset{\text{op}}{\subseteq} S_d$ of curves of geometric genus g and degree e. Y is a family of curves Y_t inside each hypersurface X_t for every parameter $t \in U$.

Define the ***vertical tangent bundles*** are the kernels

$$0 \longrightarrow T_{X/\mathbb{P}^n} \longrightarrow T_{\mathcal{X}} \longrightarrow \pi_2^* T_{\mathbb{P}^n} \longrightarrow 0$$

$$0 \longrightarrow T_{Y/\mathbb{P}^n} \longrightarrow T_Y \longrightarrow h^*\pi_2^* T_{\mathbb{P}^n} \longrightarrow 0$$

- Y dominates U under $\pi_1 \circ h$.

- We can assume Y is stable under the $\mathsf{GL}(n+1)$-action.

So $\pi_2 \circ h$ dominates $\mathbb{P}^n$ and $T_Y \longrightarrow h^*\pi_2^* T_{\mathbb{P}^2}$ is surjective.

Define the ***normal bundles*** as cokernels of

$$0 \longrightarrow T_Y \longrightarrow h^*T_{\mathcal{X}} \longrightarrow N_{h/\mathcal{X}} \longrightarrow 0$$

$$0 \longrightarrow T_{Y_t} \longrightarrow h_t^* T_{\mathcal{X}_t} \longrightarrow N_{ht/\mathcal{X}_t} \longrightarrow 0$$

Denote

$$i_t : \mathcal{X}_t \to \mathcal{X}$$

$$\begin{aligned} j_t : Y_t &\longrightarrow Y \\ p &\longmapsto (p, t) \end{aligned}$$

(technical) Lemma 1 $\quad N_{h_t/\mathcal{X}_t} \equiv j_t^* N_{h/\mathcal{X}}$

Proof. Big commutative diagram. □

Lemma 2 $\quad N_{h_t/\mathcal{X}_t} \cong j_t^* K$ where $K = \mathsf{coker}(T_{Y/\mathbb{P}^n} \to T_{Y/\mathbb{P}^n}$.

Definition $\quad$ The ***Lazarsfeld-Mukai bundle*** M_d is the kernel of

$$0 \longrightarrow M_d \longrightarrow \mathcal{O}_{\mathbb{P}^n} \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \xrightarrow{\mathsf{ev}} \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0$$

Remark $\quad$ The fiber over $p \in \mathbb{P}^n$ is M_d is the space of sections vanishing at p.

Lemma 3 $\quad T_{\mathcal{X}/\mathbb{P}^n} \cong \pi_2^* M_d$

By Lemma 2 and 3:

$$j_s^* h^* T_{\mathcal{X}/\mathbb{P}^n} \longrightarrow j_t^* K \cong N_{ht/\mathfrak{X}_t}$$

So we have a surjection

$$M_d|_{Y_t} \twoheadrightarrow N_{ht/X_t}.$$

Definition $\quad$ Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$, let L be a globally gen line bundle. We say L is ***section-dominating*** for $\mathcal{E}$ if $\mathcal{E} \otimes \mathcal{L}^\vee$ is globally gen. and

$$H^0(L \otimes I_p) \otimes H^0(\mathcal{E} \otimes \mathcal{L}^\vee) \longrightarrow H^0(\mathcal{E} \otimes I_p)$$

is surjective for all $p \in \mathbb{P}^n$.

Example $\quad$ in $\mathbb{P}^n$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(d)$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$.

Proposition $\quad$ There is a sujection

□