Equivariant K-theory of the square of an n-step partial flag variety and a "$q = 0$" version of the affine quantum group.

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October 18 2024

Abstract More than thirty years ago Beilinson, Lusztig, and MacPherson provided a geometric framework for quantum groups by considering a convolution product on the square of the n-step partial flag variety over finite fields. Since then there were several generalizations, in particular Ginzburg and Vasserot used equivariant K-theory of the Steinberg variety in the cotangent space of the square of the n-step flag variety to study the affine quantum group, with the quantum parameter corresponding to the dilation action in the cotangent direction. In a joint work with Sergey Arkhipov we use the convolution product on the equivariant K-theory of the square of the n-step flag variety itself to define and study a "q=0" version of the affine quantum group.

In this talk, I will define our "q=0" version of the affine quantum group via generators and relations, introduce the convolution algebra on the equivariant K-theory of the square of the n-step flag variety, and outline the construction of a surjective morphism from the "q=0" affine quantum group onto the convolution algebra.

Contents

1 Convolution product

1.1 Correspondence operators

Suppose we have some functor F and some space X. I want a morphism from X to itself but I dont have it and instead I have two projections from $X \times X$:

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, "?"] & & X \end{tikzcd}$$

(The functor will be equivariant . . . ? $K^G(X)$.

Anyway take a class $\alpha \in F(X \times X)$. And now for $f \in F(X)$ fo

$$\varphi_\alpha(f) = \pi_{2*}(\pi_1^*(f) \times \alpha$$

To define these correspondance operators we need

- Pullbacks
- Pushforward
- Product
- Projection formula that $f_*(\beta) \times = f_*(\beta \times f^*\alpha)$

1.2 Composition

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, "?"] & & X \arrow[rr, "?"] & & X \end{tikzcd}$$

and we will do

$$\alpha, \beta \in F(X \times X), \qquad f \in F(X)$$

and the composition is

$$\varphi_\beta \circ \varphi_\alpha)(f) = \pi_{2*}(\pi_1^*(\pi_{2*}(\pi_1^*(f \times \alpha)) \times \beta)$$

And now

$$\begin{tikzcd} & & X \times X \times X \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{23}"] & & \\ & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, "?"] & & X \arrow[rr, "?"] & & X \end{tikzcd}$$

and then

$$\begin{aligned}
\varphi_\beta \circ \varphi_\alpha)(f) &= \pi_{2*}(\pi_1^*(\pi_{2*}(\pi_1^*(f \times \alpha)) \times \beta) \\
&= \pi_{2*}(\pi_{23*}(\pi_{12}^*(\pi_1^* f \times \alpha)) \times \beta) \\
&= \pi_{2*}(\pi_{23*}(\pi_{12}^*(\pi_1^* f \times \alpha) \times \pi_{23}^*\beta))
\end{aligned}$$

$$\begin{tikzcd}
& X \times X \times X \arrow[dl, "p_1=\pi_1\circ\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "p_3=\pi_2\circ\pi_{23}"] & \\
X & X \times X \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X
\end{tikzcd}$$

and then

$$\begin{aligned}
p_{3*}(\pi_{12}^*(\pi_1^* f \times \alpha) \times \pi_{23}^*\beta) &= p_{3*}(\pi_{12}^*\pi_1^* f \times \pi_{12}^*\alpha \times \pi_{23}^*\beta) \\
&= \pi_{2*}(\pi_1^* f \times \pi_{13*}(\pi_{12}^*\alpha \times \pi_{23}^*\beta) \\
&= \varphi_{\pi_{13*}(\pi_{12}^* \times \pi_{23}^*\beta)}(f)
\end{aligned}$$

This means that the compostiton of two correspondances is a correspondance with repsect to this clsss.

Remark (Altan) Like the product of matrices.

Yes, essentially this is the product of matrices.

Definition (Convolution) $a\ star \beta := \pi_{13*}(\pi_{12}^*\alpha \times \pi_{23}^*\beta)$

And we conclude that $F(X \times X)$ is a convolution algebra acting on $F(X)$ by corresponding operators.

2 Partial flags

$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geqslant 0}^n$, $\sum \mu_i = d$, $d_k = \sum_{i=1}^k \mu_i$ and define

$$F_\mu := \{U_1 \subset U_2 \subset \ldots \subset U_n = \mathbb{C}^d : \dim U_k = d_k\}$$

where the U_i are linear subspaces. Then take

$$X = F_n^d = \bigsqcup_\mu F_\mu$$

Concept: Convolution algebras on $F_n^d \times F_n^d$ as $d \to \infty$ should approximate the quantum group of $\mathfrak{gl}(n)$.

Beautiful paper *Geometric setting for the quantum deformation of* $\mathsf{GL}(n)$ by Beilinson, Lusztig, MacPherson.

Remark (Altan) These authors also have a paper where they use Hopf algebras to ...?

Paper *Affine quantum groups and equivariant K-theory* by Vasserot, 1998. Not the partial variety but the cotangent bundle $X = T^*F_n^d$. $X \times X$ is a Steinberg subvariety, $F = K^{\mathsf{GL}(?) \times \mathbb{C}^*}$ and the $\mathbb{C}^*$ gives a quantum variety.

Definition (Altan) ***Steinberg variety***

$$\mathrm{St} = T^* \mathrm{Fl} \times_{N \times p} T^* \mathrm{Fl}$$

where

$$\mathrm{Fl} = \mathsf{GL}(n)/B \qquad \text{the flag variety}$$
$$T^* \mathrm{Fl} \qquad \text{the cotangent bundle}$$

3 The algebra $\mathsf{U}(n)$

Define $\mathsf{U}(n)$[1] first ($\mathfrak{sl}(n)$ version).

- Generators: $E_i(p), F_i(p), 0 < i < n, p \in \mathbb{Z}$.
- Realtions:

a lot of equations, descriptiono of this algebra

Our project: $X = F_n^d$ over $\mathbb{C}$, $F = K^{\mathrm{Gl}_d}$. $q = 0$ degeneratorion of Vasserot.

Remark One cannot simply plug

4 $K^6(F_\mu)$

This is a homogeneous space, a quotient. So

$$F_\mu = \mathrm{Gl}_d / P_\mu$$

where P_μ is the parabolic subgroup.

In fact,

$$K^6(F_\mu) = K^{P_\mu}(\mathrm{pt}) = \mathbb{C}[x_1^{\pm}, \dots, x_d^{\pm}]^{S_\mu}$$
$$S_\mu = S_{\mu_1} \times \dots \times S_{\mu_n}$$