Bi-Hamiltonian geometry of WDVV equations: general results

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On the occasion of the 70th birthday of **P.M. Santini**
Joint work with **Stanislav Opanasenko**

Witten–Dijkgraaf–Verlinde–Verlinde equations

The problem: in $\mathbb{R}^N$ find a function $F = F(t^1, \ldots, t^N)$ such that

1. $F_{1\alpha\beta} := \dfrac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$ constant symmetric nondegenerate matrix

2. $c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} F_{\epsilon\alpha\beta}$ structure constants of an associative algebra

3. $F(c^{d_1} t^1, \ldots, c^{d_N} t^N) = c^{d_F} F(t^1, \ldots, t^N)$ quasihomogeneity $(d_1 = 1)$

If e_1,…, e_N is the basis of $\mathbb{R}^N$ then the algebra operation is

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(\mathbf{t}) e_\gamma \quad \text{with unity} \quad e_1$$

WDVV equations

The system of associativity equations, also known as WDVV equations, follows:

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\gamma\nu} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0. \tag{WDVV}$$

The dependence of the function F on t^1 is completely specified by the requirement $F_{1\alpha\beta} = \eta_{\alpha\beta}$:

$$F = \frac{1}{6}\eta_{11}(t^1)^3 + \frac{1}{2}\sum_{k>1}\eta_{1k}t^k(t^1)^2 + \frac{1}{2}\sum_{k,s>1}\eta_{sk}t^s t^k t^1 + f(t^2, \ldots, t^N).$$

so that the WDVV system is an overdetermined system of non-linear PDEs on one unknown function $f = f(t^2, \ldots, t^N)$.

Why study WDVV?

1. Solutions are related with Gromov–Witten invariants
2. Solutions define bi-Hamiltonian pairs of first-order Hamiltonian operators and yield integrable hierarchies (B. Dubrovin)
3. Applications to Quantum Field Theory (?)

Symmetries of WDVV

The WDVV system is invariant under linear change of transformations that preserves t^1,

$$\tilde{t}^i = c^i_j t^j \quad \text{with} \quad c^i_1 = \delta^i_1.$$

Dubrovin's normal form for the matrix $(\eta_{\alpha\beta})$:

$$(\eta_{\alpha\beta}) = \begin{pmatrix} \mu & 0 & & 0 & 1 \\ 0 & 0 & & 1 & 0 \\ & & \ddots & & \\ 0 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{pmatrix}$$

with two distinct cases: $\mu = 0$ and $\mu \neq 0$.

Running example, WDVV with $N = 4$

How many independent equations are in WDVV system?
If $N = 3$ there is only one equation (Ferapontov, Galvão, Mokhov, Nutku 1998).
Let $N = 4$, and set $x = t^2$, $y = t^3$, $z = t^4$.

$$\mu f_{yyz}(f_{zzz} - f_{yzz}) + 2f_{yyz}f_{xyz} - f_{yyy}f_{xzz} - f_{xyy}f_{yzz} = 0,$$

$$f_{xxy}f_{yzz} - f_{xxz}f_{yyz} - \mu f_{zzz}f_{xyz} + f_{zzz} + f_{xyy}f_{xzz} + \mu f_{xzz}f_{yzz} - f_{xyz}^2 = 0,$$

$$f_{xxy}f_{yyz} - f_{xxz}f_{yyy} + \mu f_{yyz}f_{xzz} - \mu f_{xyz}f_{yzz} + f_{yzz} = 0,$$

$$f_{xxy}f_{xzz} - \mu f_{xxz}f_{zzz} - 2f_{xxz}f_{xyz} + f_{xxx}f_{yzz} + \mu f_{xzz}^2 = 0,$$

$$f_{xxz}f_{xyy} + \mu f_{xxz}f_{yzz} - f_{yyz}f_{xxx} - \mu f_{xzz}f_{xyz} + f_{xzz} = 0,$$

$$f_{xxy}f_{xyy} + \mu f_{xxz}f_{yyz} - f_{xxx}f_{yyy} - \mu f_{xyz}^2 + 2f_{xyz} = 0.$$

Running example, WDVV with $N = 4$

Choose an independent variable, say x; it possible to find a subsystem of equations that are linear with respect to x-free derivatives:

$$f_{yyy}, \quad f_{yyz}, \quad f_{yzz}, \quad f_{zzz}.$$

This linear subsystem is overdetermined: it consists of 5 equations. They can be solved for the 4 unknowns f_{yyy}, f_{yyz}, f_{yzz}, f_{zzz}. If we introduce new field variables u^k in correspondence with every x-derivative of the third order, i.e.

$$u^1 = f_{xxx}, \quad u^2 = f_{xxy}, \quad u^3 = f_{xxz},$$
$$u^4 = f_{xyy}, \quad u^5 = f_{xyz}, \quad u^6 = f_{xzz}$$

Running example, WDVV with $N = 4$

The linear overdetermined system can be solved. For example, if $\mu = 0$ we have:

$$f_{yyy} = \frac{2u^5 + u^2u^4}{u^1}, \quad f_{yyz} = \frac{u^3u^4 + u^6}{u^1}, \quad f_{yzz} = \frac{2u^3u^5 - u^2u^6}{u^1},$$

$$f_{zzz} = (u^5)^2 - u^4u^6 + \frac{(u^3)^2u^4 + u^3u^6 - 2u^2u^3u^5 + (u^2)^2u^6}{u^1}.$$

It is remarkable that also the remaining nonlinear equation is solved by the above equations.

Reducing the WDVV system

Consider the WDVV system:

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\nu\gamma} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0.$$

we have

$$\begin{aligned}
S_{\alpha\beta\gamma\nu} &= S_{\gamma\nu\alpha\beta}, && (1)\\
S_{\alpha\beta\gamma\nu} &= S_{\beta\alpha\nu\gamma}, && (2)\\
S_{\alpha\beta\gamma\nu} &= -S_{\alpha\nu\gamma\beta}, && (3)\\
S_{\alpha\beta\gamma\nu} &= -S_{\gamma\beta\alpha\nu}, && (4)\\
S_{\alpha\beta\gamma\nu} &= S_{\alpha\beta\nu\gamma} + S_{\alpha\gamma\beta\nu}. && (5)
\end{aligned}$$

Using the above symmetries we can prove that

1. we can move any index at the first place (up to a sign);
2. $S_{1\beta\gamma\nu} = 0$ identically.

Reduced WDVV system

Let us choose $x = t^2$. Then, there are the following nontrivial cases ($3 \leqslant a, b, c \leqslant N$):

1. $S_{22ab} = 0$ with $a \leqslant b$;
2. $S_{aa2b} = 0$ with $a \neq b$;
3. $S_{2abc} = 0$ and $S_{2acb} = 0$ with $a < b < c$;
4. $S_{aabc} = 0$ with $b \leqslant c$;
5. $S_{abcd} = 0$ and $S_{abdc} = 0$ with $a < b < c < d$.

The subsystem (1), (2), (3) is linear and overdetermined with respect to t^2-free derivatives; the remaining equations are nonlinear.

Conjectures on the WDVV system

- the linear subsystem (1), (2), (3) can always be solved for t^2-free derivatives;
- the nonlinear subsystem (4), (5) vanishes identically on the solutions of the linear subsystem (1), (2), (3).

The above conjectures are **true** for $N = 4$, $N = 5$, $N = 6$ for Dubrovin's canonical forms of $(\eta_{\alpha\beta})$.

WDVV as a first-order systems of PDEs. Running example: $N = 4$

Mokhov and Ferapontov (1996) introduced new letters for third-order derivatives:

$$u^1 = f_{xxx},\ u^2 = f_{xxy},\ u^3 = f_{xxz},\ u^4 = f_{xyy},\ u^5 = f_{xyz},\ u^6 = f_{xzz},$$
$$u^7 = f_{yyy},\ u^8 = f_{yyz},\ u^9 = f_{yzz},\ u^{10} = f_{zzz}.$$

We have the following compatibility relations:

$$\begin{array}{lll}
u^1_y = u^2_x & u^1_z = u^3_x & u^2_z = u^3_y \\
u^2_y = u^4_x & u^2_z = u^5_x & u^4_z = u^5_y \\
u^3_y = u^5_x & u^3_z = u^6_x & u^5_z = u^6_y \\
u^4_y = u^7_x & u^4_z = u^8_x & u^7_z = u^8_y \\
u^5_y = u^8_x & u^5_z = u^9_x & u^8_z = u^9_y \\
u^6_y = u^9_x & u^6_z = u^{10}_x & u^9_z = u^{10}_y
\end{array}$$

WDVV as a first-order systems of PDEs. Running example: $N = 4$

If we express the coordinates $u^7 = f_{yyy}$, $u^8 = f_{yyz}$, $u^9 = f_{yzz}$, $u^{10} = f_{zzz}$ by means of (u^k), $k = 1, \ldots, 6$ using *all* WDVV equations, we have two *commuting* quasilinear systems of first-order PDEs (systems of PDEs of hydrodynamic type) and a third set of trivial identities:

$$\begin{cases} u^1_y = u^2_x, \\ u^2_y = u^4_x, \\ u^3_y = u^5_x, \\ u^4_y = \left(\frac{2u^5+u^2u^4}{u^1}\right)_x \\ u^5_y = \left(\frac{u^3u^4+u^6}{u^1}\right)_x \\ u^6_y = \left(\frac{2u^3u^5-u^2u^6}{u^1}\right)_x \end{cases}$$

$$\begin{cases} u^1_z = u^3_x, \\ u^2_z = u^5_x, \\ u^3_z = u^6_x, \\ u^4_z = \left(\frac{u^3u^4+u^6}{u^1}\right)_x \\ u^5_z = \left(\frac{2u^3u^5-u^2u^6}{u^1}\right)_x \\ u^6_z = \Big((u^5)^2 - u^4u^6 + \\ \qquad \frac{(u^3)^2u^4+u^3u^6-2u^2u^3u^5+(u^2)^2u^6}{u^1}\Big)_x \end{cases}$$

WDVV as a first-order systems of PDEs. Running example: $N = 4$

What about the residual compatibility conditions?
It can be proved that the system

$$\begin{array}{ll} u^2_z = u^3_y & u^4_z = u^5_y \\ u^5_z = u^6_y & u^7_z = u^8_y \\ u^8_z = u^9_y & u^9_z = u^{10}_y \end{array}$$

is identically verified when you restrict it to the two commuting systems on the previous slide.

WDVV as first-order systems of PDEs

1. Let $\sigma \in \mathbb{N}^{N-1}$, and introduce new variables $u^i = f_{(3,0,\dots,0)}$, $u^2 = f_{(2,1,0,\dots 0)}$, $\dots$, $u^n = f_{(1,0,\dots,2)}$, $n = N(N-1)/2$.
2. For any other t^h, $h > 2$, find $u^i_{t^h}$ as the t^2-derivative of an expression V^i:
$$u^i_{t^h} = V^i(\mathbf{u})_{t^2}. \tag{6}$$
There are two possibilities:
   - 2.1 either $V^i(\mathbf{u})$ is one of the coordinates u^j, with $j \neq 2$;
   - 2.2 V^i is a third-order derivative of f which is not one of the u^j. In this case, according with the conjecture, V^i must be expressed by means of one of the equations of the WDVV system.

WDVV as first-order systems of PDEs

Conjecture. Let us choose t^h and t^h, $h, k \geqslant 2$, $h \neq k$.
Then, WDVV equations are equivalent to $N-2$ commuting hydrodynamic-type systems; all other compatibility conditions vanish identically.

The conjecture has been verified in dimensions $N = 4$, $N = 5$.

Bi-Hamiltonian structures

When WDVV equations are written as first-order systems we can look for Hamiltonian formulations:

$$u^i_t = (V^i(u^j))_x = A^{ik}\frac{\delta \mathcal{H}}{\delta u^k},$$

where $\mathcal{H}$ is a Hamiltonian density and A is a matrix of differential operators. A defines a *Poisson bracket*:

$$\{\mathcal{F},\mathcal{G}\}_A = \int \frac{\delta \mathcal{F}}{\delta u^i} A^{ij} \frac{\delta \mathcal{G}}{\delta u^j}\, dx$$

Two Poisson brackets defined by A_1, A_2 (Hamiltonian operators) are compatible if any linear combination is a Poisson bracket. Equivalently, the Schouten bracket $[A_1, A_2] = 0$.

Bi-Hamiltonian structures on WDVV first-order systems.
Known results

$\underline{N = 3}$

- 1st Dubrovin normal form ($\mu = 0$): local 3rd order + compatible local 1st order Hamiltonian operator [Ferapontov, Galvao, Mokhov, Nutku, 1997]
- 2nd Dubrovin normal form ($\mu \neq 0$): local 3rd order + compatible nonlocal 1st order HO [Vašiček, Vitolo, 2021].
- Mokhov–Pavlenko normal forms: local 3rd order + compatible nonlocal 1st order HO [Vašiček, Vitolo, 2021].

Bi-Hamiltonian structures on WDVV first-order systems.
Known results

$\underline{N = 4}$

- 1st Dubrovin normal form ($\mu = 0$): local 1st order HO [Ferapontov, Mokhov, 1996] + compatible local 3rd order HO [Pavlov, Vitolo, 2015]
- 2nd Dubrovin normal form ($\mu \neq 0$): local 3rd order [Vašiček, Vitolo, 2021].

$\underline{N = 5}$
Dubrovin normal forms ($\mu \in \{0, 1\}$): local 3rd order [Vašiček, Vitolo, 2021].

Third-order Hamiltonian operator for WDVV

Third-order homogeneous Hamiltonian operators in a canonical Doyle–Potemin form is

$$A_3^{ij} = \mathrm{D}_x \circ (h^{ij}\mathrm{D}_x + c_k^{ij} u_x^k) \circ \mathrm{D}_x.$$

Given $c_{ijk} = h_{iq}h_{jp}c_k^{pq}$, the skew-symmetry conditions and the Jacobi identities for the operator above are equivalent to

$$\begin{gathered} c_{skm} = \frac{1}{3}(h_{sm,k} - h_{sk,m}), \\ h_{mk,p} + h_{kp,m} + h_{mp,k} = 0 \\ c_{msk,l} = -h^{pq} c_{pml} c_{qsk}, \end{gathered}$$

which imply that h_{ij} defines an algebraic variety in Plücker's space of lines [Ferapontov, Pavlov, V., 2014].

Third-order Hamiltonian operator for WDVV

The metric h_{ij} can be factorized [Balandin, Potemin, 2001] as

$$h_{ij} = \varphi_{\alpha\beta}\psi_i^\alpha\psi_j^\beta, \quad \Big(\text{or, in a matrix form,} \quad h = \Psi\Phi\Psi^\top\Big) \tag{7}$$

where φ is a constant non-degenerate symmetric matrix of dimension n, and

$$\psi_k^\gamma = \psi_{ks}^\gamma u^s + \omega_k^\gamma$$

is a non-degenerate square matrix of dimension n.

For the conservative system $\mathbf{u}_t = (V(\mathbf{u}))_x$, the necessary and sufficient conditions to admit the above Hamiltonian operator are

$$h_{im}V_j^m = h_{jm}V_i^m,$$
$$V_{ij}^k = h^{ks}c_{smj}V_i^m + h^{ks}c_{smi}V_j^m.$$

Running example, WDVV $N = 4$
3rd order Hamiltonian operator

$$h_{11} = u_4^2, \quad h_{12} = (\mu u_5 - 2)u_5, \quad h_{13} = 2u_4(1 - \mu u_5),$$
$$h_{14} = \mu u_3 u_5 - u_1 u_4 - u_3,$$
$$h_{15} = -\mu^2 u_5 u_6 - \mu(u_2 u_5 - u_3 u_4 - u_6) + u_2,$$
$$h_{16} = (\mu u_5 - 1)^2, \quad h_{22} = 2u_3(\mu u_5 - 1),$$
$$h_{23} = -\mu^2 u_5 u_6 - \mu(u_2 u_5 + u_3 u_4 - u_6) + u_2, \quad h_{24} = \mu u_3^2,$$
$$h_{25} = -\mu^2 u_3 u_6 - \mu(u_1 u_5 + u_2 u_3) + u_1, \quad h_{26} = 2\mu u_3(\mu u_5 - 1),$$
$$h_{33} = \mu^2(2u_4 u_6 + u_5^2) + 2\mu(u_2 u_4 - u_5) + 2,$$
$$h_{34} = -\mu^2 u_3 u_6 + \mu(u_1 u_5 - u_2 u_3) - u_1, \quad h_{35} = \mu((\mu u_6 + u_2)^2 - h_{14}),$$
$$h_{36} = \mu h_{23}, \quad h_{44} = u_1^2, \quad h_{45} = -2\mu u_1 u_3,$$
$$h_{46} = \mu^2 u_3^2, \quad h_{55} = \mu^2(2u_1 u_6 + u_3^2) + 2\mu u_1 u_2,$$
$$h_{56} = \mu h_{25}, \quad h_{66} = 2\mu^2 u_3(u_5 \mu - 1).$$

First-order Hamiltonian operator for WDVV

They are nonlocal Ferapontov operators of the type

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + \sum_{\alpha,\beta} c^{\alpha\beta} w^i_{\alpha k} u^k_x \mathrm{D}_x^{-1} \circ w^j_{\beta h} u^h_x,$$

where $(c^{\alpha\beta})$ is a real symmetric matrix

The operator A_1 is Hamiltonian if and only if the following conditions hold:

$$\begin{gathered}
g^{ij} = g^{ji}, \\
g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k, \\
g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s, \\
g_{ik} w^k_{\alpha j} = g_{jk} w^k_{\alpha i}, \\
\nabla_k w^i_{\alpha j} = \nabla_j w^i_{\alpha k}, \\
[w_\alpha, w_\beta] = 0, \\
R^{ij}_{hl} = c^{\alpha\beta}\left(w^i_{\alpha l} w^j_{\beta h} - w^i_{\alpha h} w^j_{\beta l}\right).
\end{gathered}$$

A conjecture on A_1

Conjecture

Let a system of first-order conservation laws admit a third-order Hamiltonian operator as above parameterised by a metric h with decomposition

$$h = \Psi\Phi\Psi^{\top},$$

where Φ is a constant matrix, and the entries of Ψ are linear in u_k's. Then the metric g defining a compatible Ferapontov-type first-order Hamiltonian operator is of the form

$$g = \Psi^{-1}Q(\Psi^{-1})^{\top}, \quad (g^{ij} = \psi^i_\alpha Q^{\alpha\beta}\psi^j_\beta),$$

where Q is a matrix whose entries are polynomials in u_k of order at most 2.

Valid for all known examples [Opanasenko, V., Proc. R. Soc. A (2024)].

Running example, WDVV $N = 4$ first-order Hamiltonian operator

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + \sum_{\alpha,\beta=0}^{3} c^{\alpha\beta} w^i_{\alpha k}(\mathbf{u}) u^k_x \mathrm{D}_x^{-1} \circ w^j_{\beta h}(\mathbf{u}) u^h_x,$$

where $(g^{ij}) = (\Psi^{-1})Q(\Psi^{-1})^\top$, Φ is a constant symmetric matrix, and the entries of Ψ are linear in u_k's,

$$\Psi = \begin{pmatrix} \frac{u_4}{\mu} & \frac{u_5}{\mu} & 1 & 0 & 0 & 0 \\ 0 & \frac{u_3}{\mu} & 0 & -u_5 & 1 & 0 \\ -u_5 & -\frac{u_2}{\mu} - u_6 & 0 & u_4 & 0 & 1 \\ -\frac{u_1}{\mu} & 0 & 0 & -u_3 & 0 & 0 \\ u_3 & -\frac{u_1}{\mu} & 0 & \mu u_6 + u_2 & 0 & 0 \\ 0 & u_3 & 0 & -\mu u_5 + 1 & 0 & 0 \end{pmatrix},$$

Running example, WDVV $N = 4$ first-order Hamiltonian operator

$$Q^{11} = -\frac{4}{\mu}u_3u_5 + \frac{4}{\mu^2}u_1u_4 + u_6^2, \quad Q^{12} = -\frac{2}{\mu}u_3u_6 + \frac{4}{\mu^2}u_1u_5,$$

$$Q^{13} = u_1u_5 - \frac{1}{\mu}u_3u_6 + u_2u_3 + \frac{2}{\mu}u_1, \quad Q^{14} = -\frac{2}{\mu}(u_2u_5 - u_4u_3 + u_6),$$

$$Q^{15} = -\mu u_5u_6 + u_2u_5 + u_3u_4 + u_6, \quad Q^{16} = \mu u_6^2 + 2u_3u_5,$$

$$Q^{22} = \frac{2}{\mu^2}(u_1u_6 - u_3^2),$$

$$Q^{23} = -\frac{2}{\mu}u_1u_2 + u_3^2, \quad Q^{24} = \frac{4}{\mu}u_3u_5 - \frac{2}{\mu}u_2u_6 - u_6^2,$$

$$Q^{25} = u_3u_5 - \frac{1}{\mu}u_1u_4 - \frac{2}{\mu}u_3 - \frac{1}{\mu}u_2^2,$$

$$Q^{26} = -\frac{1}{\mu}u_1u_5 + u_3u_6 - \frac{1}{\mu}u_2u_3,$$

Running example, WDVV $N = 4$ first-order Hamiltonian operator

$$Q^{33} = \mu^2 u_3^2 - 2\mu u_1 u_2, \quad Q^{34} = -\mu u_3 u_5 + u_1 u_4 + u_2^2 + 4u_3,$$
$$Q^{35} = \mu^2 u_3 u_5 - \mu u_1 u_4 - \mu u_2^2 - \mu u_3,$$
$$Q^{36} = \mu^2 u_3 u_6 - \mu u_1 u_5 - \mu u_2 u_3 + u_1,$$
$$Q^{44} = 2u_4 u_6 - 2u_5^2, \quad Q^{45} = -\mu u_5^2 + 2u_2 u_4 + 4u_5,$$
$$Q^{46} = -\mu u_5 u_6 + u_2 u_5 + u_3 u_4 + 3u_6,$$
$$Q^{55} = \mu^2 u_5^2 - 2\mu u_2 u_4 - 2\mu u_5 - 2,$$
$$Q^{56} = \mu^2 u_5 u_6 - \mu u_2 u_5 - \mu u_3 u_4 - \mu u_6 + u_2,$$
$$Q^{66} = \mu^2 u_6^2 - 2\mu u_3 u_5 + 2u_3,$$

Running example, WDVV $N = 4$ first-order Hamiltonian operator

The nonlocal part of the operator

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + c^{\alpha\beta} w^i_{\alpha k}(\mathbf{u}) u^k_x \mathrm{D}_x^{-1} \circ w^j_{\beta h}(\mathbf{u}) u^h_x,$$

is defined by the matrix

$$\begin{pmatrix} c^{11} & c^{12} & c^{13} \\ c^{21} & c^{22} & c^{23} \\ c^{31} & c^{32} & c^{33} \end{pmatrix} = \begin{pmatrix} 0 & -\mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & \mu^2 \end{pmatrix}.$$

and by the commuting symmetries

$$w^i_{1j} = \delta^i_j, \quad w^j_{2i} = V^i_j, \quad w^j_{3i} = W^i_j,$$

where

$$u^i_y = (V^i)_x = V^i_j u^j_x, \qquad u^i_z = (W^i)_x = W^i_j u^j_x,$$

are the WDVV first-order systems.

Results on bi-Hamiltonian structures for WDVV

Theorem Let $u^i_{t^h} = (V^i)_{t^k}$ be a family of commuting first-order WDVV systems, $h = 2,\dots,N$, $h \neq k$. If there is one value of h such that the first-order system is bi-Hamiltonian with a pair of compatible Hamiltonian operators A_1, A_3, then all first-order WDVV systems corresponding to all other values h are endowed with exactly the same bi-Hamiltonian pair.

Proof Compatibility of the operators A_1 and A_3 gives

$$h_{im}w^m_{\alpha j} = h_{jm}w^m_{\alpha i}, \quad w^k_{\alpha i,j} = h^{ks}c_{smj}w^m_{\alpha i} + h^{ks}c_{smi}w^m_{\alpha j}.$$

These are the conditions under which $w^m_{\alpha i}$ define Hamiltonian systems for the third-order operator defined by h_{ij}.

Results on bi-Hamiltonian structures for WDVV

Theorem An invariance transformation of the WDVV equation preserves the form of the Hamiltonian operators in a bi-Hamiltonian first-order WDVV system.

Proof The symmetry group of a third-order WDVV projects to the symmetry group $\mathrm{GL}(N-1,\mathbb{C})$ of a first-order WDVV.

Invariance transformations that involve only two independent variables preserve the form of the Hamiltonian operators [Vašiček, V., 2021].

Any matrix in $\mathrm{GL}(\mathbb{C}^{N-1})$ can be generated by means of 2×2 Gauss' elementary matrices (up to permutations).

Future research

- Complete the reduction of the WDVV system
- Show that any first-order WDVV system is bi-Hamiltonian.

Thank you very much!

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