

Lista 1

Problem 1: Let V be a symplectic vector space ($\dim V = 2n$), and $\Omega \in \Lambda^2 V^*$ be a skew-symmetric bilinear form. Show that Ω is nondegenerate iff $\Omega^n \neq 0$.

Solution. I first tried to show that Ω is degenerate iff $\Omega^n = 0$. Suppose there is a vector v_0 such that $\Omega(v_0, w) = 0$ for all $w \in V$ and complete to a basis. Then for any $v_1, v_2, v_3, v_4 \in V$ we have

$$(\Omega \wedge \Omega)(v_1, v_2, v_3, v_4) = \sum_{\sigma \in S_4} \text{sgn}(\sigma) \Omega(v_{\sigma(1)}, v_{\sigma(2)}) \Omega(v_{\sigma(3)}, v_{\sigma(4)}).$$

Finally I found this proposition in Lee, Intro. Smooth Manifolds. □

Problem 2: Let (V, Ω) be a symplectic vector space, and let $W \subseteq V$ be any linear subspace.

- Show that $V_W = \frac{W}{W \cap W^\Omega}$ inherits a natural symplectic structure Ω_W uniquely determined by the condition $\pi^* \Omega_W = \Omega|_W$ (here $\pi : W \rightarrow W/(W \cap W^\Omega)$ is the quotient projection).

(The space (V_W, Ω_W) is called the **reduced space**.)

- Suppose that W is coisotropic, and let $L \subset V$ be lagrangian. Show that the image of $L \cap W$ via $\pi : W \rightarrow V_W$ is lagrangian in the reduced space.

Solution.

- Define

$$\Omega_W([w_1], [w_2]) := \Omega(w_1, w_2)$$

for any equivalence classes $[w_1], [w_2] \in V_W$. Let's check that this is well defined. Suppose $w'_1 \in [w_1]$. Then $w_1 - w'_1 \in W \cap W^\Omega$ so $\Omega(w_1 - w'_1, w_2) = 0$ since $w_2 \in W$ and $w_1 - w'_1$ is, in particular, in W^Ω . So $\Omega(w_1, w_2) = \Omega(w'_1, w_2)$. **Why not quotient only by W^Ω ? Looks like I didn't use the W part...**

Recall that $\pi^* \Omega_W(w_1, w_2) = \Omega_W([w_1], [w_2])$. It is straightforward to check that Ω_W is the only symplectic form on V_W satisfying $\pi^* \Omega_W = \Omega|_W$: if Ω'_W is another such form, then $\Omega_W([w_1], [w_2]) = \Omega|_W(w'_1, w'_2) = \Omega'_W([w_1], [w_2])$ for any $w'_1 \in [w_1]$ and $w'_2 \in [w_2]$.

- Let's first check what is $(\pi(L \cap W))^{\Omega_W}$. We have

$$\begin{aligned} (\pi(L \cap W))^{\Omega_W} &= \{[v] \in V_W : \Omega_W([v], [w]) = 0 \ \forall [w] \in \pi(L \cap W)\} \\ &= \{[v] \in V_W : \Omega(v', w) = 0 \ \forall v' \in [v] \text{ and } \forall w \text{ s.t. } [w] \in \pi(L \cap W)\} \end{aligned}$$

In words, this is the set of classes whose representatives are Ω -orthogonal to representatives of $\pi(L \cap W)$.

So if $[v]$ is any such class,

so let $[v] \in \pi(L \cap W)^{\Omega_W}$. Let's check that $[v]$ is also in $\pi(L \cap W)$, ie. that $v \in L \cap W$. Well,

If $v' \in L$, then $\Omega(v, v') = 0$ since $[v'] \in \pi(L \cap W)^{\Omega} \dots$ but what if $v' \in L \setminus W$?

Let w be such that $[w] \in \pi(L \cap W)$. Then

$$\begin{aligned}\Omega_W([v], [w]) &= 0 \\ \implies \Omega(v, w) &= 0\end{aligned}$$

so $v \in$

□

Problem 3: We saw in class that any symplectomorphism $T : V_1 \rightarrow V_2$ defines a lagrangian subspace by its graph: $\Gamma_T := \{(Tu, u) : u \in V_1\} \subset V_2 \oplus \bar{V}_1$. (Recall that if (V, Ω) is a svs, \bar{V} denotes $(V, -\Omega)$.) So we think lagrangian subspaces of $V_2 \oplus \bar{V}_1$ a generalizations of symplectomorphisms. We now see how to generalize their composition.

Consider symplectic vector spaces V_1, V_2, V_3 and $E = V_3 \oplus \bar{V}_2 \oplus V_2 \oplus \bar{V}_1$.

- Show that $\Delta := \{(v_3, v_2, v_2, v_1) \in E\}$ is coisotropic in E and its reduction E_Δ can be identified with $V_3 \oplus \bar{V}_1$.
- Given lagrangian subspaces $L_1 \subset V_2 \oplus \bar{V}_1$ and $L_2 \subset V_3 \oplus \bar{V}_2$, define the *composition* of L_2 and L_1 by

$$L_2 \circ L_1 := \{(v_3, v_1) \mid \exists v_2 \in V \text{ s.t. } (v_3, v_2) \in L_2, (v_2, v_1) \in L_1\}.$$

Show that $L_2 \circ L_1$ is a lagrangian subspace of $V_3 \oplus \bar{V}_1$. (Hint: show that the composition can be identified with the reduction of $L_2 \times L_1 \subset E$ with respect to Δ).

- Let $T_1 : V_1 \rightarrow V_2$ and $T_2 : V_2 \rightarrow V_3$ be symplectomorphisms. Show that $\Gamma_{T_2 \circ T_1} = \Gamma_{T_2} \circ \Gamma_{T_1}$.

Solution.

- Let $v := (v_3, v_2, v_2', v_1) \in \Delta^{\Omega_E} \subset E$ where $\Omega_E = \Omega_1 \oplus -\Omega_2 \oplus \Omega_2 \oplus -\Omega_1$ is the symplectic form on E . We wish to show that $v \in \Delta$, which only means that $v_2 = v_2'$. So let $\tilde{v} = (v_3, v_2, v_2, v_1)$ and $\hat{v} = (v_3, v_2', v_2', v_1)$ which are both in Δ . Then we have

$$\begin{aligned}0 &= \Omega_E(v, \tilde{v}) \\ &= \Omega_1(v_3, v_3) - \Omega_2(v_2, v_2) + \Omega_2(v_2', v_2) - \Omega_1(v_1, v_1)\end{aligned}$$

and likewise

$$\begin{aligned}0 &= \Omega_E(v, \hat{v}) \\ &= \Omega_1(v_3, v_3) - \Omega_2(v_2, v_2') + \Omega_2(v_2', v_2') - \Omega_1(v_1, v_1).\end{aligned}$$

Subtracting,

$$0 = -\Omega_2(v_2, v_2) + \Omega_2(v_2, v'_2) + \Omega_2(v'_2, v_2) - \Omega_2(v'_2, v'_2)$$

And by linearity,

$$\begin{aligned} 0 &= -\Omega_2(-v_2 + v_2, -v_2 + v'_2) + \Omega_2(v'_2 - v'_2, v_2 - v'_2) \\ \implies 0 &= -\Omega_2(0, -v_2 + v'_2) + \Omega_2(0, v_2 - v'_2) \\ \implies 0 &= \Omega_2(-v_2 + v'_2, 0) + \Omega_2(0, v_2 - v'_2) \\ \implies 0 &= \Omega_2(-v_2 + v'_2, v_2 - v'_2) \\ \implies 0 &= \Omega_2(-(v_2 - v'_2), v_2 - v'_2) \end{aligned}$$

and it follows that $v_2 = v'_2$ from nondegeneracy.

Now let's try to construct an isomorphism $E_\Delta = V_3 \oplus \bar{V}_1$. Consider

$$\begin{aligned} \varphi : E &\longrightarrow V_3 \oplus \bar{V}_1 \\ (v_3, v_2, v'_2, v_1) &\longmapsto (v_3, v_1) \end{aligned}$$

which is clearly surjective and not injective, so perhaps its kernel is $\Delta \cap \Delta^\Omega$. But $\ker \varphi = \{(0, v_2, v'_2, 0)\}$, so unfortunately no.

But perhaps we can construct some other map. Let's try

$$\begin{aligned} \varphi : E &\longrightarrow V_3 \oplus \bar{V}_1 \\ (v_3, v_2, v'_2, v_1) &\longmapsto \end{aligned}$$

- b. It looks like $L_2 \circ L_1$ is very much like E_Δ from the last exercise. $L_2 \circ L_1$ is *strictly* contained in $V_3 \oplus \bar{V}_1 \cong E_\Delta$.

Ok let's have a go at the hint. Perhaps $(L_2 \times L_1)_\Delta = L_2 \circ L_1$ and it is lagrangian. That'd be great. OK let's compute it. This is how to do it:

$$\begin{aligned} \varphi : L_2 \times L_1 &\longrightarrow V_2 \circ V_1 \\ (v_2, v_1, v_3, v_2) &\longmapsto \end{aligned}$$

□

Problem 4: Let (V, J) be a complex vector space, let Ω be a symplectic structure on V . Show that J and Ω are compatible iff there exists a hermitian inner product $h : V \times V \rightarrow \mathbb{C}$ such that Ω is its imaginary part. Show that any (complex) orthonormal basis of (V, h) can be extended to a symplectic basis of (V, Ω) .

Solution. First suppose that J and Ω are compatible, ie., $g(u, v) := \Omega(u, Jv)$ is an inner product. Define $h(u, v) = g(u, v) + i\Omega(u, v)$. Then h is the required hermitian inner product. Indeed:

1. The properties $h(u_1 + u_2, v) = h(u_1, v) + h(u_2, v)$ and $h(u, v_1 + v_2) = h(u, v_1) + h(u, v_2)$ follows easily from linearity of g and Ω .
2. $h(\lambda u, v) = \lambda h(u, v)$ follows again from linearity of g and Ω .
3. The property $h(u, \lambda v) = \bar{\lambda} h(u, v)$ follows easily from 2. and 4. since

$$\begin{aligned} h(u, \lambda v) &= \overline{h(\lambda v, u)} \\ &= \bar{\lambda} \overline{h(v, u)} \\ &= \bar{\lambda} h(u, v) \end{aligned}$$

4. $h(u, v) = \overline{h(v, u)}$ is clear by anti-symmetry of Ω :

$$\begin{aligned} h(u, v) &= g(u, v) + i\Omega(u, v) \\ &= g(v, u) - i\Omega(v, u) \\ &= \overline{h(v, u)} \end{aligned}$$

For the converse suppose that h is an hermitian inner product such that Ω is its imaginary part. Then $g(u, v) := \Omega(u, Jv)$ is an inner product:

1. Linearity of g is immediate from linearity of Ω and J .
2. Symmetry follows from

$$\begin{aligned} g(u, v) &= \Omega(u, Jv) \\ &= \Omega(-J^2 u, Jv) \\ &= -\Omega(J^2 u, Jv) \\ &= \Omega(Jv, J^2 u) \\ &= \Omega(v, Ju) \\ &= g(v, u) \end{aligned}$$

provided $\Omega(u, v) = \Omega(Ju, Jv)$. This holds since Ω is the imaginary part of h identifying J with multiplication by $i = \sqrt{-1}$:

$$\begin{aligned} \Omega(Ju, Jv) &= \text{Im}(h(Ju, Jv)) \\ &= \text{Im } h(iu, iv) \\ &= \text{Im}(i\bar{i}h(u, v)) \\ &= \text{Im}(h(u, v)) \\ &= \Omega(u, v). \end{aligned}$$

3. For positive-definiteness let $u \neq 0$. Then

$$\begin{aligned} g(u, u) &= \Omega(u, Ju) \\ &= \text{Im}(h(u, Ju)) \\ &= \text{Im}(h(u, iu)) \\ &= \text{Im}(-ih(u, u)) \end{aligned}$$

e sabemos que $h(u, u) > 0 \dots$ parece que $g(u, u)$ é negativo...

□

Problem 5: Consider the symplectic vector space $(\mathbb{R}^{2n}, \Omega_0)$, where $\Omega_0(u, v) = -u^T J_0 v$. Check that its group of linear symplectomorphisms is given by $\text{Sp}(2n) = \{A \in \text{GL}(2n) : A^T J_0 A = J_0\}$. Show that $\text{Sp}(2n)$ is a smooth submanifold of $\text{GL}(2n)$ and that its tangent space at the identity $I \in \text{GL}(2n)$ is given by $T_I \text{Sp}(2n) = \{A : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} | A^T J_0 + J_0 A = 0\}$. Conclude that $\text{Sp}(2n)$ has dimension $2n^2 + n$. Verify also that $\text{Sp}(2n)$ is not compact.

Solution. Suppose that A is a linear symplectomorphism of $(\mathbb{R}^{2n}, \Omega_0)$. Then $A^* \Omega_0 = \Omega_0$ so

$$A^* \Omega_0(u, v) = \Omega_0(Au, Av) = -(Au)^T J_0 (Av) = -u^T A^T J_0 (Av)$$

is equal to

$$\Omega_0(u, v) = -u^T J_0 v$$

In terms of usual dot product of \mathbb{R}^{2n} , which we can denote by $\langle \cdot, \cdot \rangle$ momentarily, this means that

$$\begin{aligned} \langle -u^T, A^T J_0 Av \rangle &= \langle -u^T, J_0 v \rangle \\ \iff \langle -u^T, A^T J_0 Av - J_0 v \rangle &= 0 \end{aligned}$$

for all $u \in \mathbb{R}^{2n}$, which means that $A^T J_0 Av = J_0 v$ since dot product is nondegenerate. For the converse, if $A^T J_0 A = J_0$, we have

$$\Omega(u, v) = -u^T J_0 v = -u^T A^T J_0 Av = -(Au)^T J (Av) = \Omega(Au, Av) = A^* \Omega(uv).$$

Let's try to show that $\text{Sp}(2n)$ is the inverse image of a regular value of some submersion $\text{GL}(2n) \rightarrow \mathbb{R}$. I think the determinant of J_0 is 1, so perhaps the map

$$\begin{aligned} D : \text{GL}(2n) &\longrightarrow \mathbb{R} \\ A &\longmapsto \det(A^T J_0 A) \end{aligned}$$

has inverse image of 1 equal to $\text{Sp}(2n)$. It is clear that $\text{Sp}(2n) \subset D^{-1}(1)$. Now suppose $A \in \text{GL}(2n)$ is in $D^{-1}(1)$, but then it's not necessarily true that $A^T J_0 A = J_0$ because there are so many matrices with determinant 1 that are not J_0 .

Maybe the determinant won't work. But what about

$$\begin{aligned} D : \text{GL}(2n) &\longrightarrow \text{GL}(2n) \\ A &\longmapsto A^T J_0 A \end{aligned}$$

maybe this map is a submersion at every point of $D^{-1}(J_0) = \text{Sp}(2n)$. It really looks like a submersion at every point: it's only a composition of linear isomorphisms... of course its derivative is surjective, right? But something seems to be wrong because then it would be a submanifold of dimension $2n - 2n = 0 \dots$

Let's have a look at the tangent space at the identity. Suppose that $V \in T_1 \text{Sp}(2n)$. I'm not sure how to continue here...

Following Misha's slides, it could be shown that $\text{Sp}(2n)$ is the exponent of its Lie algebra (ie. the tangent space of the identity), and that this Lie algebra is $W := \{A \in \text{GL}(2n) : A^T J_0 + J_0 A = 0\}$. This suggests that $\exp W = \text{Sp}(2n)$. But then we need a theorem saying that if the exponent of a subset of endomorphisms is a Lie group then such a subspace is its Lie algebra. This can be found in Misha's notes as follows:

Exponent is a map

$$\begin{aligned} \exp : \text{End } V &\longrightarrow \text{End } V \\ A &\longmapsto \sum_{n=0}^{\infty} \frac{A^n}{n!} \end{aligned}$$

and its differential at $0 \in \text{End } V$ is the identity. But why... we have

$$d_0 \exp = \sum_{n=1}^{\infty} \frac{A^n}{(n-1)!}$$

[I don't think](#) it's so straightforward to check that this is the identity so let's just suppose it is true.

But what does it tell us that this map is the identity? Well, that $\exp(W)$ is invertible around 0, but not only that: notice that $\exp(0) = I \in \text{GL}(2n)$ so $d_0 \exp : T_0 \text{GL}(2n) \cong \text{GL}(2n) \rightarrow T_1 \text{Sp}(2n)$ is an isomorphism as required, provided of course that $\exp(W) = \text{Sp}(2n)$. So let's check that.

But that's nowhere near obvious:

$$\exp(W) = \left\{ \sum_{n=0}^{\infty} \frac{A^n}{n!} : A^T J_0 + J_0 A = 0 \right\} \stackrel{?}{=} \{A \in \text{GL}(2n) : A^T J_0 A = J_0\} = \text{Sp}(2n)$$

Let's just go, let $A \in W$ and let's check whether $\exp(A)^T J_0 \exp(A) = J_0$. It [looks like](#) transpose of exponent is exponent of transpose, but what is $\exp(J_0)$? [This looks random...](#)

□

Problem 6: Consider the standard compatible triple (Ω_0, J_0, g_0) on \mathbb{R}^{2n} . Let $O(2n)$ be the linear orthogonal group of \mathbb{R}^{2n} (i.e., linear transformations preserving the canonical inner product g_0), and let $\text{Sp}(2n)$ be the symplectic linear group. Through the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ (as complex vector spaces), we may see $\text{GL}(n, \mathbb{C})$ (the group of linear automorphisms of \mathbb{C}^n) as a subgroup of $\text{GL}(2n, \mathbb{R})$: a complex matrix $A + iB$ is identified with the real $2n \times 2n$ matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

Let now $U(n) \subset \text{GL}(n, \mathbb{C})$ be the group of linear transformation preserving the natural hermitian inner product of \mathbb{C}^n . Show that the intersection of any two of the groups

$$\text{Sp}(2n), O(2n), \text{GL}(n, \mathbb{C}) \subset \text{GL}(2n, \mathbb{R})$$

is $U(n)$.

Solution.

Since the standard hermitian product of \mathbb{C}^n is given by $h_0 = g_0 + i\Omega_0$, it is immediate that a transformation $A \in \text{Sp}(2n) \cap O(2n)$ preserves h_0 and conversely:

$$A^*h = A^*(g + i\Omega) = A^*g + iA^*\Omega = g + i\Omega = h$$

provided that the pullback is complex-linear.

For the next item recall that

$$O(2n) = \{A \in GL(2n) : A^T A = I\}, \quad GL(n, \mathbb{C}) = \{A \in GL(2n) : AJ_0 = J_0 A\}$$

again identifying J_0 with multiplication by i . Observe that this implies that $A \in O(2n) \cap GL(n, \mathbb{C}) \implies A \in \text{Sp}(2n)$ since

$$A^T J_0 A = A^T A J_0 = J_0.$$

Likewise we see that $A \in \text{Sp}(2n) \cap GL(n, \mathbb{C}) \implies O(2n)$ since

$$A^T J_0 A = J_0 \iff A^T A J_0 = J_0 \iff A^T A = I$$

since J_0 is invertible. Going back to the initial argument for matrices in $\text{Sp}(2n) \cap A \in O(2n)$, we see that in both cases $A \in U(n)$.

For the converse notice that it is also true that $A \in \text{Sp}(2n) \cap O(2n) \implies A \in GL(n, \mathbb{C})$ since

$$J_0 = A^T J_0 A = A^{-1} J_0 A \iff J_0 A = A J_0.$$

□

Problem 7: Let (V, Ω) be a symplectic vector space, let $W \subseteq V$. Let J be a Ω -compatible complex structure and g the corresponding inner product. Verify that $J(W^\Omega) = W^{\perp_g}$.

- Use this fact to show that any coisotropic subspace of V has an isotropic complement. In particular, any lagrangian subspace $L \subset V$ has a lagrangian complement $L', V = L \oplus L'$.
- Show that there is a natural identification $L' \cong L^*$, that induces a symplectomorphism $V \cong L \oplus L^*$ (where $L \oplus L^*$ has the natural symplectic structure $((\ell, \alpha), (\ell', \alpha')) \mapsto \alpha(\ell') - \alpha'(\ell)$).

Solution. First let's check that $J(W^\Omega) = W^{\perp_g}$. Indeed,

$$\begin{aligned} J(W^\Omega) &= \{Jv : v \in W^\Omega\} \\ &= \{Jv : \Omega(v, w) = 0 \ \forall w \in W\} \\ &= \{Jv : -\Omega(w, v) \ \forall w \in W\} \\ &= \{Jv : \Omega(w, -v) \ \forall w \in W\} \\ &= \{Jv : \Omega(w, J^2 v) \ \forall w \in W\} \end{aligned}$$

re-write $Jv := \tilde{v}$ using that J is bijective:

$$\begin{aligned} J(W^\Omega) &= \{\tilde{v} \in V : \Omega(w, J\tilde{v}) = 0 \ \forall w \in W\} \\ &= \{\tilde{v} \in V : g(\tilde{v}, w) = 0 \ \forall w \in W\} \\ &= W^{\perp_g} \end{aligned}$$

□

Bonus problem: