Geometria Simplética 2024, Lista 5

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Entrega dia 11/10

Problem 1: Let G be a Lie group. Let $X : G \to TG$ be a section of the projection $TG \to G$, not necessarily smooth. Show that if X is left invariant (i.e., $dL_g(X) = X \circ L_g$ for all $g \in G$), then X is automatically smooth.
Conclude that an analogous result holds for differential forms: if a section $\eta : G \to \wedge^k T^*G$ is left invariant ($L_g^*\eta = \eta$), then η is a smooth k-form. Check that an analogous result holds for G-invariant forms on a homogeneous manifold.

Problem 2: (a) Prove that any connected Lie group G is generated (as a group) by any open neighborhood U of the identity element (i.e., $G = \cup_{n=1}^{\infty} U^n$). (b) Suppose that two Lie group homomorphisms $\varphi, \psi : G \to H$ are such that $d\varphi|_e = d\psi|_e$. Show that φ and ψ coincide on the connected component of G containing the identity e.

Problem 3: Consider the Lie groups $SU(2) = \{A \in M_2(\mathbb{C}) \,|\, AA^* = \mathrm{Id},\, \det(A) = 1\}$ and $SO(3) = \{A \in M_3(\mathbb{R}) \,|\, AA^t = \mathrm{Id},\, \det(A) = 1\}$.

a) Show that
$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$
Conclude that, as a manifold, $SU(2)$ is diffeomorphic to S^3 (hence it is simply connected).

Recall the definition of the quaternions $\mathbb{H}$. Show that the sphere S^3, seen as quaternions of norm 1, inherits a Lie group structure, with respect to which it is isomorphic to $SU(2)$.

b) Verify that
$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}.$$
Consider the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, that takes the element in $\mathfrak{su}(2)$ determined by α, β to the vector $(\alpha, \mathrm{Re}\beta, \mathrm{Im}\beta)$ in $\mathbb{R}^3$. Observe that, with respect to this identification, det in $\mathfrak{su}(2)$ corresponds to $\|\cdot\|^2$ in $\mathbb{R}^3$.

c) Verify that each element $A \in SU(2)$ defines a linear transformation on the vector space $\mathfrak{su}(2)$ by conjugation: $B \mapsto ABA^{-1}$. Show that, with the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, we obtain a representation (i.e., a linear action) of $SU(2)$ on $\mathbb{R}^3$ that is norm preserving. Conclude that we have a homomorphism $\phi : SU(2) \to O(3)$, verifying that its image is $SO(3)$ and its kernel is $\{\mathrm{Id}, -\mathrm{Id}\}$.

d) Conclude that $SU(2) \cong S^3$ is a double cover of $SO(3)$ (hence it is its universal cover, since it's simply connected), and the covering map identifies antipodal points of S^3. Hence, as manifolds, $SO(3)$ is identified with $\mathbb{R}P^3$.

Problem 4: Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, and let $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a symmetric bilinear form that is Ad-invariant (i.e., $k(\mathrm{Ad}_g(u), \mathrm{Ad}_g(v)) = k(u, v)$ for $g \in G$).

a) Show that the map $k^\sharp : \mathfrak{g} \to \mathfrak{g}^*$, $k^\sharp(u)(v) = k(u, v)$, is G-equivariant:

$$k^\sharp \circ \mathrm{Ad}_g = (\mathrm{Ad}^*)_g \circ k^\sharp, \quad \forall g \in G. \tag{1}$$

[*recall:* $(\mathrm{Ad}^*)_g := (\mathrm{Ad}_{g^{-1}})^*$]. In particular, when k is nondegenerate (i.e., $k^\sharp$ is an isomorphism), the adjoint and coadjoint actions are equivalent.

b) Verify that (1) implies that $k([w, u], v) = -k(u, [w, v])$, $\forall u, v, w \in \mathfrak{g}$, and that both conditions are equivalent when G is connected.

Problem 5: For a Lie algebra $\mathfrak{g}$, there is always a canonical bilinear form $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, called *Killing form*, given by:

$$k(u, v) = \mathrm{tr}(\mathrm{ad}_u \mathrm{ad}_v).$$

(recall: $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_u(v) = [u, v]$.)

a) Note that k is symmetric, and check that it is Ad-invariant.

b) A Lie algebra is called *semi-simple* if k is nondegenerate. Show that $\mathfrak{so}(3)$ is semi-simple.

Problem 6: Consider the linear isomorphism $\mathbb{R}^3 \to \mathfrak{so}(3)$, given by

$$v = (x, y, z) \mapsto \widehat{v} := \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

a) Describe the Lie bracket on $\mathbb{R}^3$ induced by the commutator in $\mathfrak{so}(3)$, and the inner product in $\mathfrak{so}(3)$ that corresponds to the canonical inner product in $\mathbb{R}^3$.

b) Describe the $SO(3)$-action on $\mathbb{R}^3$ corresponding to the adjoint action, its orbits, as well as its infinitesimal generators. Find (without any calculation!) a description of the coadjoint action on $\mathbb{R}^3$ (identified with $(\mathbb{R}^3)^*$ through the canonical inner product).

Problem 7: Let (V, Ω) be a symplectic vector space, and consider $H := V \times \mathbb{R} = \{(v, t)\}$. This space H, with the multiplication

$$(v_1, t_1) \cdot (v_2, t_2) = (v_1 + v_2, \frac{1}{2}\Omega(v_1, v_2) + t_1 + t_2),$$

is a Lie group, called the *Heisenberg group* (find the identity elements and inverses in H).

(a) Show (directly from the conjugation formula in H) that $\mathrm{Ad}_{(v,t)}(X, r) = (X, r + \Omega(v, X))$, for $(X, r) \in \mathfrak{h} = \mathrm{Lie}(H) = V \times \mathbb{R}$. Describe the adjoint orbits, verifying that their possible dimensions are zero and one.

(b) Verify that $\mathrm{ad}_{(Y,s)}(X, r) = (0, \Omega(Y, X))$. [Recalling that $\mathrm{ad}_{(Y,s)}(X, r) = [(Y, s), (X, r)]$, we obtain a formula for the Lie bracket in $\mathfrak{h}$.]

(c) Describe the coadjoint action of H on $\mathfrak{h}^* = V^* \times \mathbb{R}^*$ and its orbits, analyzing the possible dimensions.