Daniel González Casanova Azuela
github.com/danimalabares/sg

Lista 5

Contents

Problem 1 Let G be a Lie group. Let $X : G \longrightarrow TG$ be a section of the projection $TG \longrightarrow G$, not necessarily smooth. Show that if X is left invariant (i.e., $dL_g(X) = X \circ L_g$ for all $g \in G$), then X is automatically smooth.

Conclude that an analogous result holds for differential forms: if a section $\eta : G \to \Lambda^k(T^*G)$ is left-invariant ($L_g^*\eta = \eta$), then η is a smooth k-form. Check that an analogous result holds for G-invariant forms on a homogeneous manifold.

Solution. (Idea by my) We know that $X_g = dL_g(X_e)$. We want to show that the map $G \to TG : g \mapsto dL_g(X_e)$ is smooth. Suggestion by Ted Shiffrin is to consider

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (g,h) &\longmapsto L_g h = gh \end{aligned}$$

which is smooth and thus has smooth differential which turns out to have the expression

$$\begin{aligned} d\mu : TG \times TG &\longrightarrow TG \\ (u_g, v_h) &\longmapsto dR_h u_g + dL_g v_h \end{aligned}$$

Choosing g arbitrary, $h = e$, $u_g = 0$ and $v_h = X_e$, we obtain

$$d\mu(u_g, v_h) = dL_g X_e$$

and, as we have said, this differential depends smoothly on g.

To generalize this result we just notice that left multiplication L_g induces smooth maps the Grassman algebra by

$$\begin{aligned} \mathbf{L_g} : \Lambda^k(G) &\longrightarrow \Lambda^k(G) \\ \eta &\longmapsto \begin{aligned} \mathbf{L_g}\eta : \mathfrak{X}^k(G) &\longrightarrow \mathbb{R} \\ (X_1, \dots, X_k) &\longmapsto (dL_g X_1, \dots, dL_g X_1) \end{aligned} \end{aligned}$$

and similarly group multiplication μ induces maps

$$\begin{aligned} \mu : \Lambda^k(G) \times \Lambda^k(G) &\longrightarrow \Lambda^k(G) \\ (\alpha, \beta) &\longmapsto \mathbf{R_h}\alpha + \mathbf{L_g}\beta \end{aligned}$$

so again chosing

$\square$

Problem 2

a. Prove that any connected Lie group G is generated as a group by any open neighbourhood U of the identity element (i.e. $G = \bigcup_{n=1}^{\infty} U^n$).

b. Suppose that two Lie group homomorphisms $\varphi, \psi : G \to H$ are such that $d\varphi|_e = d\psi|_e$. Show that φ and ψ coincide on the connected component of G containing the identity e.

Problem 3 Consider the Lie groups $\mathsf{SU}(2) = \{A \in \mathcal{M}_2(\mathbb{C}) | AA^* = \mathrm{Id}, \det A = 1\}$ and $\mathsf{SO}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) | AA^{\mathbf{T}} = \mathrm{Id}, \det A = 1\}$.

a. Show that
$$\mathsf{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$
Conclude that, as a manifold $\mathsf{SU}(2)$ is diffeomorphic to S^3 (hence it is simply connected).

Recall the definition of the quaternions $\mathbb{H}$. Show that the sphere S^3, seen as quaternions of norm 1, inherits a Lie group structure with respect to which it is isomorphic to $\mathsf{SU}(2)$.

b. Verify that
$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}. \tag{1}$$
Consider the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, that takes the element in $\mathfrak{su}(2)$ determined by α, β to the vector $(\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)$ in $\mathbb{R}^3$. Observe that, with respect to this identification, det in $\mathfrak{su}(2)$ corresponds to $\|\cdot\|^2$ in $\mathbb{R}^3$.

c. Verify that each element $A \in \mathsf{SU}(2)$ defines a linear transformation on the vector space $\mathfrak{su}(2)$ by conjugation: $B \mapsto ABA^{-1}$. Show that, with the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, we obtain a representation (i.e., a linear action) of $\mathsf{SU}(2)$ on $\mathbb{R}^3$ that is norm preserving. Conclude that we have homomorphism $\phi : \mathsf{SU}(2) \to \mathsf{O}(3)$, verifying that is image is $\mathsf{SO}(3)$ and its kernel is $\{\mathrm{Id}, -\mathrm{Id}\}$.

d. Conclude that $\mathsf{SU}(2) \cong S^3$ is a double cover of $\mathsf{SO}(3)$ *hence is its universal cover, since it's simply connected), qnd the covering map identifies antipodal points of S^3. Hence, as manifolds, $\mathsf{SO}(3)$ is identified with $\mathbb{R}P^3$.

Solution.

a. Given the computation above, it is clear that $\mathsf{SU}(2)$ is diffeomorphic to S^3 since its parameters $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, can be understood as vectors $x \in \mathbb{R}^4$ of norm 1.

The quaternions are the only 4-dimensional real division algebra. This means it is a 4-dimensional real vector space equipped with a (non-commutative) multiplication. They are also equipped with a norm that coincides with euclidean norm. With respect to this norm we define the unit sphere S^3.

To see that S^3 is a Lie subgroup we first need to check that it is closed under quaternion product. The easiest way to see that is via quaternion conjugate: if $x = x_1 + ix_2 + jx_3 + kx_4$ is a quaternion, its conjugate is $\bar{x} = x_1 - ix_2 - jx_3 - kx_4$. It may be computed that the norm is given by $|x| = x\bar{x}$. Then we see that if $x, y \in S^3$

$$|xy\overline{xy}| = |xy\bar{y}\bar{x}| = \Big|x|y|\bar{x}\Big| = 1$$

Then the fact that S^3 is a Lie group follows from the fact that the restricition (the multiplication and inverse map) of smooth maps to embedded submanifolds remains smooth ([**?**], prop ?).

An identification between $S^3 \subset \mathbb{H}$ and $\mathsf{SU}(2)$ as expressed above is given by

$$a + bi + cj + dk \longmapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

Checking that this map is a group isomorphism ammounts to checking that matrix multiplication in $\mathsf{SU}(2)$ is the same as quaternion multiplication

b. (Proof from [**?**], prop. 3.24 and coro. 3.46).

Step 1 Show that the tangent space at identity of a matrix Lie group G is the same as the matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

Step 2 Then we look for the matrices such that

$$\exp(tX)^* = (\exp(tX)^{-1} = \exp(-tX) \qquad \text{and} \qquad \det\exp(tX) = 1.$$

This means that

$$\exp(tX^*) = \exp(-tX) \qquad \text{and} \qquad \mathrm{Tr}(X) = 0.$$

The first condition is equivalent to $X^* = -X$. Thus we see that

$$\mathfrak{su}(2) = \{X \in \mathcal{M}_{2\times 2}(\mathbb{C}) : X^* = -X \text{ and } \mathrm{Tr}(X) = 0\}$$

Step 3 It is immediate that the expression in eq. (1) is contained in the set above. For the other inclusion first notice that the condition $X^* = -X$ makes the entries in the diagonal be such that

$$x + iy = -\overline{x+iy} = -(x - iy) = -x + iy \implies x = -x \implies x = 0$$

while the traceless condition implies the two entries in the diagonal must be additive inverses. For the entries in the antidiagonal we literally see the definition of conjugate transpose.

Now let's identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via $\alpha, \beta \mapsto (\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)$. We immediately see that

$$\det\begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} = i\alpha(-i\alpha) = \beta(-\bar{\beta}) = \alpha^2 + |\beta|^2 = \|(\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)\|^2$$

□