Lista 5

Contents

Problem 1 Let G be a Lie group. Let $X : G \longrightarrow TG$ be a section of the projection $TG \longrightarrow G$, not necessarily smooth. Show that if X is left invariant (i.e., $dL_g(X) = X \circ L_g$ for all $g \in G$), then X is automatically smooth.

Conclude that an analogous result holds for differential forms: if a section $\eta : G \to \Lambda^k(T^*G)$ is left-invariant ($L_g^*\eta = \eta$), then η is a smooth k-form. Check that an analogous result holds for G-invariant forms on a homogeneous manifold.

Solution. We know that $X_g = dL_g(X_e)$. We want to show that the map $G \to TG : g \mapsto dL_g(X_e)$ is smooth. Suggestion by Ted Shiffrin at StackExchange is to consider

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (g, h) &\longmapsto L_g h = gh \end{aligned}$$

which is smooth and thus has smooth differential which turns out to have the expression

$$\begin{aligned} d\mu : TG \times TG &\longrightarrow TG \\ (u_g, v_h) &\longmapsto dR_h u_g + dL_g v_h \end{aligned}$$

Choosing g arbitrary, $h = e$, $u_g = 0$ and $v_h = X_e$, we obtain

$$d\mu(u_g, v_h) = dL_g X_e$$

and, as we have said, this differential depends smoothly on g.

The pullback map is an induced map by L_g on the Grassman algebra:

$$\begin{aligned} L_g^* : \Lambda^k(G) &\longrightarrow \Lambda^k(G) \\ \eta &\longmapsto \begin{aligned} L_g^*\eta : \mathfrak{X}^k(G) &\longrightarrow \mathbb{R} \\ (X_1, \dots, X_k) &\longmapsto \eta(dL_g X_1, \dots, dL_g X_1) \end{aligned} \end{aligned}$$

and we have by hypothesis that

$$L_g^*\eta = \eta.$$

Similarly, group multiplication μ induces a map on the Grassman algebra:

$$\begin{aligned} \mu : \Lambda^k(G) \times \Lambda^k(G) &\longrightarrow \Lambda^k(G) \\ (\alpha, \beta) &\longmapsto R_h^*\alpha + L_g^*\beta \end{aligned}$$

so again chosing $\alpha = 0$ and $\beta = \eta$ we see that η is smooth.

Now let η be a G-invariant form on a homogeneous manifold X. Following Jack Lee's suggestion, we can pull back η to G via the map $\pi : G \to X$, $g \mapsto g \cdot p$ for any fixed $p \in X$. This gives a form $\pi^*\eta$ on G that is left-invariant since η is G-invariant: pushing vectors with left-multiplication preserves the orbit of a given point. More explicitly:

$$L_g^*(\pi^*\eta) = \pi^*\eta \qquad \text{because} \qquad L_g^*(\pi^*\eta) = \eta \circ d\pi \circ dL_g$$

and η is constant along vectors on the orbit of p, which are the images of $d\pi \circ dL_g$. By the previous exercise we see that $\pi^*\eta$ is smooth.

Then we notice that π is a submersion. This follows since π is a surjective map (bacuse X is homogeneous) of constant rank. Constant rank means that $d\pi$ has the same rank at every point of G. This can be seen moving around the tangent spaces of both G and X using the differentials of both the action of G on M, and the left-translation action of G on G as follows. There is a commutative diagram:

$$\begin{array}{ccc} T_g G & \xrightarrow{d\pi} & T_{gp} X \\ {\scriptstyle d(\text{action})}\downarrow & & \downarrow{\scriptstyle d(\text{action})} \\ T_{hg} G & \xrightarrow{d\pi} & T_{hgp} X \end{array}$$

and those vertical differentials are diffeomorphisms by smothness of the actions.

Finally we conclude by taking a local inverse σ of π via inverse function theorem. Then we simply notice that $\eta = \sigma^*\pi^*\eta$, which is smooth by construction. □

Problem 2

a. Prove that any connected Lie group G is generated as a group by any open neighbourhood U of the identity element (i.e. $G = \bigcup_{n=1}^{\infty} U^n$).

b. Suppose that two Lie group homomorphisms $\varphi, \psi : G \to H$ are such that $d\varphi|_e = d\psi|_e$. Show that φ and ψ coincide on the connected component of G containing the identity e.

Solution.

a. Let U be an open neighbourhood of the identity and define

$$U^n := \{g_1 \cdot \ldots \cdot g_n : g_i \in U\}$$

for $n \in \mathbb{N}\backslash\{0\}$. We will show that $\bigcup_{n\in\mathbb{N}} U^n$ is a (non-empty, which is trivial since it contains the identity) closed and open set, implying it is G by connectedness.

We know that for every $g \in G$ right-multiplication L_g is a diffeomorphism of G (because its inverse is $L_{g^{-1}}$). So it is an open map. This means that $U \cdot g$ is open. Now

$$U^2 = U \cdot U = \bigcup_{g \in U} U \cdot g$$

must be open. Likeways U^n is open and so is $\bigcup_{n\in\mathbb{N}} U^n$.

To see that $\bigcup_{n\in\mathbb{N}} U^n$ is also closed, notice that any open subgroup of a topological group is also closed. To see this in our case call $H := \bigcup_{n\in\mathbb{N}} U^n$. We'll show that $G\backslash H$ is open. Take $g \in G\backslash H$, the set $g \cdot H$ is in fact an open neighbourhood of g contained in $G\backslash H$. Indeed, if $h \in g \cdot H \cap H$, then we can take the inverse $h^{-1} \in H$ and we get that $h^{-1}hg = g \in H$ but that's not possible.

b. Let $g \in G$ be any element. Suppose we find a vector $v \in T_eG$ such that its integral curve γ passes through g at time t_0. Then we can map this vector both by $d\varphi$ and $d\psi$ to obtain a vector $w := d_e\varphi(v) = d_e\psi(v) \in T_eH$. Then the integral curve of w is the same as the integral curve of v pushed by either of φ or ψ because both are solutions to the same differential equation.

Then we get that

$$\gamma(t_0) = g \implies \varphi(g) = \varphi(\gamma(t_0)) = \psi(\gamma(t_0)) = \psi(g)$$

OK now we only need to find the vector v. The statement is that for every g in the connected component of the identity there is a maximal integral curve starting at the identity passing through g. Since g is the same connected component as the identity we can find a curve connecting them. But this curve need not be a maximal integral curve nor homotopic to one. I couldn't find such a vector!

$\square$

Problem 3 Consider the Lie groups $\mathsf{SU}(2) = \{A \in \mathcal{M}_2(\mathbb{C}) | AA^* = \mathsf{Id}, \det A = 1\}$ and $\mathsf{SO}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) | AA^{\mathbf{T}} = \mathsf{Id}, \det A = 1\}$.

a. Show that

$$\mathsf{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

Conclude that, as a manifold $\mathsf{SU}(2)$ is diffeomorphic to S^3 (hence it is simply connected).

Recall the definition of the quaternions $\mathbb{H}$. Show that the sphere S^3, seen as quaternions of norm 1, inherits a Lie group structure with respect to which it is isomorphic to $\mathsf{SU}(2)$.

b. Verify that

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}. \tag{1}$$

Consider the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, that takes the element in $\mathfrak{su}(2)$ determined by α, β to the vector $(\alpha, \mathsf{Re}\,\beta, \mathsf{Im}\,\beta)$ in $\mathbb{R}^3$. Observe that, with respect to this identification, det in $\mathfrak{su}(2)$ corresponds to $\|\cdot\|^2$ in $\mathbb{R}^3$.

c. Verify that each element $A \in \mathsf{SU}(2)$ defines a linear transformation on the vector space $\mathfrak{su}(2)$ by conjugation: $B \mapsto ABA^{-1}$. Show that, with the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, we obtain a representation (i.e., a linear action) of $\mathsf{SU}(2)$ on $\mathbb{R}^3$ that is norm preserving. Conclude that we have homomorphism $\phi : \mathsf{SU}(2) \to \mathsf{O}(3)$, verifying that is image is $\mathsf{SO}(3)$ and its kernel is $\{\mathsf{Id}, -\mathsf{Id}\}$.

d. Conclude that $\mathsf{SU}(2) \cong S^3$ is a double cover of $\mathsf{SO}(3)$ *hence is its universal cover, since it's simply connected), qnd the covering map identifies antipodal points of S^3. Hence, as manifolds, $\mathsf{SO}(3)$ is identified with $\mathbb{R}P^3$.

Solution.

a. It is very easy to see that a matrix

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Mat}_2(\mathbb{C})$$

satisfying $AA^* = \mathsf{Id} \iff A^* = A^{-1}$ must be of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

since $A^* = A^{-1}$ translates to

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which yields immediately $\bar{a} = d$ and $c = -\bar{b}$.

Conversely, if $|a|^2 + |b|^2 = 1$ the matrix

$$B := \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

has determinant

$$\mathsf{det}\, B = a\bar{a} + b\bar{b} = |a|^2 + |b|^2 = 1$$

and its inverse is

$$B^{-1} = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} = B^*.$$

Having estabilshed this expression for $\mathsf{SU}(2)$, it is clear that it is diffeomorphic to S^3 since its parameters $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, can be understood as vectors $x \in \mathbb{R}^4$ of norm 1.

The quaternions are the only 4-dimensional real division algebra. This means it is a 4-dimensional real vector space equipped with a (non-commutative) multiplication. They are also equipped with a norm that coincides with euclidean norm. With respect to this norm we define the unit sphere S^3.

To see that S^3 is a Lie subgroup we first need to check that it is closed under quaternion product. The easiest way to see that is via quaternion conjugate: if $x = x_1 + ix_2 + jx_3 + kx_4$ is a quaternion, its conjugate is $\bar{x} = x_1 - ix_2 - jx_3 - kx_4$. It may be computed that the norm is given by $|x| = x\bar{x}$. Then we see that if $x, y \in S^3$

$$|xy\overline{xy}| = |xy\bar{y}\bar{x}| = \Big|x|y|\bar{x}\Big| = 1$$

Then the fact that S^3 is a Lie group follows from the fact that the restricition (the multiplication and inverse map) of smooth maps to embedded submanifolds remains smooth.

An identification between $S^3 \subset \mathbb{H}$ and $\mathsf{SU}(2)$ as expressed above is given by

$$a + bi + cj + dk \longmapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

Checking that this map is a group isomorphism ammounts to checking that matrix multiplication in $\mathsf{SU}(2)$ is the same as quaternion multiplication.

b. (Proof from Hall, prop. 3.24.) We will show that $\mathfrak{su}(2)$ is the space of traceless anti-hermitian matrices and eq. (1) will follow.

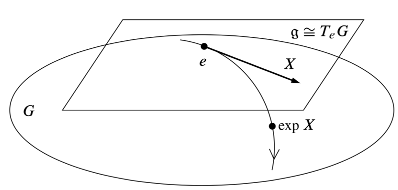
Step 1 Show that the Lie algebra of a matrix Lie group G, defined as the tangent space at identity, is the same as the matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$. (Inspired, though the proof is different because his definitions are different, in Hall, coro. 3.46.)

Proof of Step 1. First suppose that X is such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$. Then the curve $\gamma(t) = \exp(tX)$, which is contained in G, passes through the identity at $t = 0$ with velocity X, meaning that X is in the tangent space of the identity element.

For the converse suppose that $X \in T_{\mathsf{Id}}G$. Our definition of exponential map is to move along the integral curve of X, say γ_X, by 1 unit of time. Then

$$\exp(tX) = \gamma_{tX}(1) = \gamma_X(t) \in G$$

by homogenity property.



$\square$

Step 2 Then we look for the matrices such that for all $t \in \mathbb{R}$,

$$\exp(tX)^* = \exp(tX)^{-1} = \exp(-tX) \qquad \text{and} \qquad \det \exp(tX) = 1.$$

This means that

$$\exp(tX^*) = \exp(-tX) \qquad \text{and} \qquad \mathrm{Tr}(X) = 0.$$

The first implication is the result of the general facts that $(e^A)^* = e^{A^*}$ and $(e^A)^{-1} = e^{A^{-1}}$. Let's have a look at the second one:

> $(e^A)^{-1} = e^{A^{-1}}$ This is nicely shown in Lee as follows. For any $X \in \mathrm{Lie}(G)$ map $t \mapsto \exp(tX)$ is a group homomorphism $\mathbb{R} \to G$ because $\exp((t_1 + t_2)X) = \exp(t_1X)\exp(t_2X)$ (but of course that should be justified...) and from this it follows that this map preserves inverses, which what we wanted.

Now let's have a look at why determinant 1 in Lie group translates to vanishing trace in Lie algebra.

> $\forall X \in \mathcal{M}_n(\mathbb{C}),\ \det e^X = e^{\mathrm{tr}\, X}$. This is easy to see if X is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$; then the eigenvalues of e^X are $e^{\lambda_1}, \dots, e^{\lambda_n}$. So $\det e^X = \prod e^{\lambda_i} = e^{\sum \lambda_i} = e^{\mathrm{tr}\, X}$. But if X is not diagonalizable we must do Jordan decomposition and some more computations.

Finally, differentiating and evaluating at $t = 0$ the equation $\exp(tX^*) = \exp(-tX)$ gives $X^* = -X$. (See Hall, prop. 2.4. It is intuitive but not immediate.)

In conclusion, we see that

$$\mathfrak{su}(2) = \{X \in \mathcal{M}_{2\times 2}(\mathbb{C}) : X^* = -X \text{ and } \mathrm{Tr}(X) = 0\}$$

Step 3 It is immediate that the right-hand side in eq. (1) is contained in the set above. For the other inclusion first notice that the condition $X^* = -X$ makes the entries in the diagonal be such that

$$x + iy = -\overline{x + iy} = -(x - iy) = -x + iy \implies x = -x \implies x = 0$$

while the traceless condition implies the two entries in the diagonal must be additive inverses. For the entries in the antidiagonal we literally see the definition of conjugate transpose.

Now let's identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via $\alpha, \beta \mapsto (\alpha, \mathrm{Re}\,\beta, \mathrm{Im}\,\beta)$. We immediately see that

$$\det \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} = i\alpha(-i\alpha) = \beta(-\bar{\beta}) = \alpha^2 + |\beta|^2 = \|(\alpha, \mathrm{Re}\,\beta, \mathrm{Im}\,\beta)\|^2$$

c. De acordo com o exercício anterior, é suficiente mostrar que ABA^{-1} tem traço zero e $ABA^{-1} = -(ABA^{-1})^*$. A primeira propriedade é imediata dado que, em geral, $\mathsf{Tr}(XY) = \mathsf{Tr}(YX)$. Para a segunda propriedade note que

$$(ABA^{-1})^* = (A^{-1})^*B^*A^* = (A^*)^*(-B)A^{-1} = -ABA^{-1}$$

O fato de que essa ação em $\mathsf{SU}(2)$ preserva a norma é immediato do item anterior e do fato de que determinante de um produto de matrizes é o produto dos determinantes.

Isso significa que cada elemento em $\mathsf{SU}(2)$ age como uma isometria linear $\mathbb{R}^3 \to \mathbb{R}^3$, i.e. temos um mapa

$$\begin{aligned} \varphi : \mathsf{SU}(2) &\longrightarrow \mathsf{O}(3) \\ A &\longmapsto I_A \end{aligned}$$

onde

$$\begin{aligned} I_A : \mathfrak{su}(2) \cong \mathbb{R}^3 &\longrightarrow \mathfrak{su}(2) \cong \mathbb{R}^3 \\ B &\longmapsto ABA^{-1} \end{aligned}$$

φ é um homomorfismo já que para $A, A' \in \mathsf{SU}(2)$ e $B \in \mathfrak{su}(2)$ temos que

$$I_{AA'}B = (AA')B(AA')^{-1} = A\Big(A'B(A')^{-1}\Big)A^{-1} = (I_A \circ I_{A'})B.$$

Para determinar que matriz corresponde com a transformação I_A mediante a identificação $\mathfrak{su}(2) \cong \mathbb{R}^3$, suponha que

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad B = \begin{pmatrix} i\alpha & \beta \\ -\overline{\beta} & -i\alpha \end{pmatrix}$$

Daí,

$$\begin{aligned} ABA^{-1} = ABA^* &= \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} i\alpha & \beta \\ -\overline{\beta} & -i\alpha \end{pmatrix} \begin{pmatrix} \overline{a} & -b \\ \bar{b} & a \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} i\alpha\overline{a} + \beta\overline{b} & -i\alpha b + \beta a \\ -\overline{\beta}\overline{a} - i\alpha\overline{b} & \overline{\beta}b - i\alpha a \end{pmatrix} \\ &= \begin{pmatrix} a(i\alpha\overline{a} + \beta\overline{b}) + b(-\overline{\beta}\overline{a} - i\alpha\overline{b}) & a(-i\alpha b + \beta a) + b(\overline{\beta}b - i\alpha a) \\ * & * \end{pmatrix} \end{aligned}$$

Daí, se (x_1, x_2, x_3) é vetor correspondente a ABA^{-1}, vemos que

$$\begin{aligned} ix_1 &= a(i\alpha\overline{a} + \beta\overline{b}) + b(-\overline{\beta}\overline{a} - i\alpha\overline{b}) \\ &= i|a|^2\alpha + a\overline{b}\beta - b\overline{a}\overline{\beta} - i\alpha|b|^2 \\ x_2 + ix_3 &= a(-i\alpha b + \beta a) + b(\overline{\beta}b - i\alpha a) \\ &= -iab\alpha \end{aligned}$$

Agora considere a base de $\mathfrak{su}(2)$ dada por

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

que corresponde à base canônica de $\mathbb{R}^3$. A matriz da transformação I_A terá nas columnas os vetores correspondentes às imagens dessas treis matrizes. Usando as fórmulas anteriores, concluimos que

$$I_A = \begin{pmatrix} |a|^2 - |b|^2 & \mathsf{Re}(-i(a\overline{b} - b\overline{a})) & \mathsf{Re}(a\overline{b} + b\overline{a}) \\ 0 & \mathsf{Re}(a^2 + b^2) & \mathsf{Re}(i(a^2 - b^2)) \\ -2ab & \mathsf{Im}(a^2 + b^2) & \mathsf{Im}(i(a^2 - b^2)) \end{pmatrix}$$

Para calcular o kernel de φ suponha que a matriz anterior é a identidade. Segue que

$$\ddot{\frown}$$

Supondo que mostramos que a imagem de φ é $\mathsf{SO}(3)$ com kernel $\{\mathsf{Id}, -\mathsf{Id}\}$, temos o isomorfismo

$$\frac{\mathsf{SU}(2)}{\{\mathsf{Id}, -\mathsf{Id}\}} \cong \mathsf{SO}(3)$$

Suponha que dois elementos $A, B \in \mathsf{SU}(2)$ estão na mesma classe de equivâlencia no quociente. Então

$$AB^{-1} \in \{\mathsf{Id}, -\mathsf{Id}\} \iff A = B \text{ ou } A = -B$$

Assim, cada elemento de $\mathsf{SO}(3)$ correponde com dois elementos de $\mathsf{SU}(2)$. Como $\{\mathsf{Id}, -\mathsf{Id}\}$ age de maneira prórpia, livre e disonctínua em $\mathsf{SU}(2)$, temos um recobrimento de $\mathsf{SO}(3)$. Também é claro que estamos identificando pontos antípodas de S^3 já que ser antípoda em S^3 corresponde com o produto com $-\mathsf{Id}$. Note que $\mathbb{RP}^3 \cong S^3/\mathbb{Z}_2$, de modo que $\mathsf{SU}(2)$ é um recobrimento de $\mathsf{SO}(3)$: cada ponto de $\mathsf{SO}(3)$ tem uma preimagem composta por dois elementos $\square$

Problem 4 Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, and let $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$be a symmetric bilinear form that is Ad-invariant (i.e. $k(\mathsf{Ad}_g(u), \mathsf{Ad}_g(v)) = k(u, v)$ for $g \in G$).

a. Show that the map

$$\begin{aligned} k^\sharp : \mathfrak{g} &\longrightarrow \mathfrak{g}^* \\ k^\sharp(u)(v) &= k(u, v) \end{aligned} \tag{2}$$

is G-equivariant:

$$k^\sharp \circ \mathsf{Ad}_g = (\mathsf{Ad}^*)_g \circ k^\sharp, \qquad \forall g \in G$$

[*Recall:* $(\mathsf{Ad}^*)_g := (\mathsf{Ad}_{g^{-1}})^*$.] In particular, when k is nondegenerate (i.e. $k^\sharp$ is an isomorphism), the adjoint and coadjoint actions are equivalent.

b. Verify that eq. (2) implies that $k([w, u], v) = -k(u, [w, v])$, $\forall u, v, w \in \mathfrak{g}$, and that both conditions are equivalent when G is connected.

Solução. a. É só abrir as definições . Fixe um elemento $x \in \mathfrak{g}$. No lado esquerdo, temos

$$\Big(k^\sharp \circ \mathsf{Ad}_g\Big)(x) = k^\sharp(\mathsf{Ad}_g\, x) = k(\mathsf{Ad}_g\, x, \cdot)$$

e no lado direito,

$$\Big((\mathsf{Ad}^*)_g \circ k^\sharp\Big)(x) = \mathsf{Ad}^*_g(k^\sharp(x)) = \mathsf{Ad}^*_g(k(x,\cdot)) = k(x, \mathsf{Ad}_{g^{-1}} \cdot)$$

mas, como k é Ad-invariante, $k(x, \mathsf{Ad}_{g^{-1}} \cdot) = k(\mathsf{Ad}_g\, x, \cdot)$.

b. Must check this later...

□

Problem 5 For a Lie algebra $\mathfrak{g}$, there is always a canonical bilinear form $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, called ***Killing form***, given by:

$$k(u, v) = \mathsf{tr}(\mathsf{ad}_u\, \mathsf{ad}_v).$$

(*Recall:* $\mathsf{ad}_u : \mathfrak{g} \to \mathfrak{g}$, $\mathsf{ad}_u(v) = [u, v]$.)

a. Note that k is symmetric, and check that it is Ad-invariant.

b. A Lie algebra is called ***semi-simple*** if k is nondegenerate. Show that $\mathfrak{so}(3)$ is semi-simple.

Solution.

a. O fato de k ser simétrica segue de que, em geral, $\mathsf{tr}(AB) = \mathsf{tr}(BA)$. Para ver que k é Ad-invariante vamos mostrar que (a idea vem de StackExchange):

$$\mathsf{ad}_{\mathsf{Ad}_g\, u} = \mathsf{Ad}_g \circ \mathsf{ad}_u \circ \mathsf{Ad}_{g^{-1}}\,. \tag{3}$$

Supondo isso, é fácil ver que

$$\begin{aligned}\mathsf{ad}_{\mathsf{Ad}_g\, u} \circ \mathsf{ad}_{\mathsf{Ad}_g\, v} &= \mathsf{Ad}_g \circ \mathsf{ad}_u \circ \mathsf{Ad}_{g^{-1}} \circ \mathsf{Ad}_g \circ \mathsf{ad}_v \circ \mathsf{Ad}_{g^{-1}} \\ &= \mathsf{Ad}_g \circ \mathsf{ad}_u \circ \mathsf{ad}_v \circ \mathsf{Ad}_{g^{-1}}\end{aligned}$$

e o resultado segue do fato de que a traça é invariante baixo mudanças de coordenadas e que $\mathsf{Ad}_g^{-1} = \mathsf{Ad}_{g^{-1}}$. Isso último segue de que o mapa inverso de I_g é $I_{g^{-1}}$, assim as derivadas deles também são inversa uma da outra.

Agora vamos mostrar eq. (3). Pegue $x \in \mathfrak{g}$. O lado esquerdo diz que

$$\mathsf{ad}_{\mathsf{Ad}_g\, u}\, x = [\mathsf{Ad}_g\, u, x]$$

enquanto o direito diz que

$$\Big(\mathsf{Ad}_g \circ \mathsf{ad}_u \circ \mathsf{Ad}_{g^{-1}}\Big)x = \mathsf{Ad}_g[u, \mathsf{Ad}_{g^{-1}}\, x] = [\mathsf{Ad}_g\, u, x$$

já que $\mathsf{Ad}_g = d_e I_g$ não é apenas um automorfismo linear, mas é um automorfismo de álgebra de Lie, i.e. $\mathsf{Ad}_g[x, y] = [\mathsf{Ad}_g\, x, \mathsf{Ad}_g\, y]$ para $x, y \in \mathfrak{g}$. Isso segue do fato de que o pushforward de colchete de Lie de dois campos vetoriais e o colchete dos pushforwards dos campos vetoriais (ver Lee, coro. 8.31).

b. Por enquanto não consegui resolver esse problema:

$\square$

Problem 6 Consider the linear isomorphism $\mathbb{R}^3 \to \mathfrak{so}(3)$, given by

$$v = (x, y, z) \longmapsto \hat{v} := \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

a. Describe the Lie bracket on $\mathbb{R}^3$ induced by the commutator in $\mathfrak{so}(3)$, and the inner product in $\mathfrak{so}(3)$ that corresponds to the canonical inner product in $\mathbb{R}^3$.

b. Describe the $\mathsf{SO}(3)$-action on $\mathbb{R}^3$ corresponding to the adjoint action, its orbits, as well as its infinitesimal generators. Find (without any calculation!) a description of the coadjoint action on $\mathbb{R}^3$ (identified with $(\mathbb{R}^3)^*$ through the canonical inner product).

Solution.

a. Lembre que

$$\mathsf{SO}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) : AA^{\mathbf{T}} = \mathsf{Id}, \det A = 1\}$$
$$\mathfrak{so}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) : A = -A^{\mathbf{T}}\}$$

Isso explica por que as matrizes em $\mathfrak{so}(3)$ tem a forma mostrada acima. O isomorfismo com $\mathbb{R}^3$ é dado por

$$\mathfrak{so}(3) \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = u$$
$$[U, V] \longmapsto u \times v$$
$$\frac{1}{2}\mathsf{tr}(UV^{\mathbf{T}}) \longmapsto \langle u, v \rangle$$

Para comprovar que de fato o commutador de $\mathfrak{so}(3)$ corresponde com o produto vetorial em $\mathbb{R}^3$, considere duas matrizes $U, V \in \mathfrak{so}(3)$ e os vetores correspondentes $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$. O commutador é

$$[U, V] = UV - VU = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$
$$- \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

a entrada $(3,2)$ da matriz resultante é a primeira coordenada do vetor correspondente a $[U,V]$; esse numero é claramente $u_1v_3 - u_3v_2$. Analogamente, a segunda coordenada é $u_3v_1 - u_1v_3$, enquanto a terceira $u_1v_2 - u_2v_1$. Essas são as coordenadas de $u \times v$.

Para ver que o produto interno de $\mathbb{R}^3$ corresponde com $\frac{1}{2}\operatorname{tr}$ note que

$$\begin{aligned} UV^{\mathbf{T}} = U(-V) &= \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} u_3v_3 + u_2v_2 & * & * \\ * & u_3v_3 + u_1v_1 & * \\ * & * & u_2v_2 + u_1v_1 \end{pmatrix} \end{aligned}$$

Agora vamos calcular a ação adjunta. Primeiro notemos que a ação adjunta, definida como a derivada do operador $I_A(X) = AXA^{-1}$, é ela mesma. Talvez esse é um fato obvio porque I_A é um mapa linear, e a sua derivada coincide com ele. Porém, tem um argumento mais explícito no StackExchange: a ação adjunta em um vetor $X \in \mathfrak{so}(3)$ que seja a derivada em $t = 0$ da curva $\gamma \subset \mathsf{SO}(3)$, i.e. $\gamma'(0) = X$, é simplesmente $\frac{d}{dt}A\gamma(t)A^{-1}\Big|_{t=0} = AXA^{-1}$. A observação chave é que como a álgebra de Lie $\mathfrak{so}(3)$ é um álgebra de matrizes, o produto do lado direito está bem definido.

Agora vamos achar uma matriz que representa esse operador linear quando identificamos $\mathfrak{so}(3)$ com $\mathbb{R}^3$. Ou seja, queremos achar uma matriz $\overline{\mathsf{Ad}_A}$ tal que $\mathsf{Ad}_A X = AXA^{-1} \rightsquigarrow \overline{\mathsf{Ad}_A}x$ onde $x \in \mathbb{R}^3$ representa a matriz $X \in \mathfrak{so}(3)$. Vamos ver que de fato essa matriz é A.

$$\begin{aligned} AXA^{\mathbf{T}} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} -za_{12} + ya_{13} & -za_{22} + ya_{23} & -za_{32} + ya_{33} \\ a_{11}z - xa_{13} & za_{21} - xa_{23} & za_{31} - xa_{33} \\ -ya_{11} + xa_{12} & -ya_{21} + xa_{22} & -ya_{31} + xa_{32} \end{pmatrix} \end{aligned}$$

Daí, se o vetor correspondente a $AXA^{\mathbf{T}}$ e (x', y', z'), sabemos que suas coordenadas estão nas entradas $(3,2)$, $(1,3)$ e $(2,1)$, respetivamente, da matriz que resulta acima. Ou seja,

$$\begin{aligned} x' &= a_{31}(-za_{22} + ya_{23}) + a_{32}(za_{21} - xa_{23}) + a_{33}(-ya_{21} + xa_{22}) \\ y' &= a_{11}(-za_{32} + ya_{33}) + a_{12}(za_{31} - xa_{33}) + a_{13}(-ya_{31} + xa_{32}) \\ z' &= a_{21}(-za_{12} + ya_{13}) + a_{22}(a_{11}z - xa_{13}) + a_{23}(-ya_{11} + xa_{12}) \end{aligned}$$

Para achar a matriz buscada simplesmente calculamos as coordenadas dos vetores canónicos de $\mathbb{R}^3$.

- Se $(x, y, z) = (1, 0, 0)$ obtemos

$$(x', y', z') = (-a_{23}a_{32} + a_{22}a_{33}, -a_{12}a_{33} + a_{13}a_{32}, -a_{22}a_{13} + a_{23}a_{12})$$

- Se $(x, y, z) = (0, 1, 0)$ obtemos

$$(x', y', z') = (a_{23}a_{31} - a_{21}a_{33}, a_{11}a_{33} - a_{13}a_{31}, a_{21}a_{13} - a_{23}a_{11})$$

- Se $(x, y, z) = (0, 0, 1)$ obtemos

$$(x', y', z') = (-a_{22}a_{31} + a_{21}a_{32}, -a_{11}a_{32} + a_{12}a_{31}, -a_{21}a_{12} + a_{22}a_{11})$$

Esses treis vetores em colunas conformam a matriz que buscamos, que podemos chamar por enquanto $\mathbf{C}$. Para mostrar que $\mathbf{C} = A$ basta ver que $\mathbf{C}A^{\mathsf{T}} = \mathsf{Id}$.

É simples ver que as coordenadas de $\mathbf{C}$ são distintos menores (determinantes de submatrizes de 2×2) de A. Quando calculamos o produto $\mathbf{C}A^{\mathsf{T}}$, vemos que nas tres entradas da diagonal aparece o determinante de A, que é 1. As entradas fora da diagonal devem ser zero. Fazendo contas não achei immediato, mas o resultado é um fato de álgebra linear: a matriz que calculamos se chama ***matriz de cofactores*** e satisfaz, em geral,

$$A^{-1} = \frac{1}{\det A}\mathbf{C}^{\mathsf{T}}$$

A menos dessa continha, isso mostra que a ação adjunta de $\mathsf{SO}(3)$ em $\mathbb{R}^3$ é por rotações com eixos que passam pela origem. Segue que a órbita da origem é trivial. As órbitas dos outros pontos são quocientes $\mathsf{SO}(3)/S^1$, já que as órbitas são difeomorfas ao quociente do grupo sobre estabilizador (pode pensar que a órbita é uma variedade homogénea). Ademais, $\mathsf{SO}(3)$ age transitivamente em S^2 com estabilizador também S^1. Segue que as órbitas são $\mathsf{SO}(3)/S^1 \overset{\mathsf{dif}}{\simeq} S^2$.

Em geral, o gerador infinitesimal da ação adjunta Ad_g em $u \in \mathfrak{g}$ é $\mathsf{ad}_u = [u, \cdot]$. No nosso caso, para $u \in \mathfrak{so}(3) \cong \mathbb{R}^3$, $\mathsf{ad}_u = [u, \cdot] \rightsquigarrow u \times \cdot$, i.e., é o campo vetorial que em cada ponto de $\mathbb{R}^3$ asoccia o produto vetorial dele com u.

Por último, a ação coadjunta em $A \in \mathsf{SO}(3)$ é um operador Ad^*_A satisfazendo para todo $\xi \in \mathfrak{so}(3)^* \cong \mathbb{R}^3$ e $Y \in \mathfrak{so}(3) \cong \mathbb{R}^3$,

$$(\mathsf{Ad}^*_A\,\xi)Y = \xi(\mathsf{Ad}_{A^{-1}}\,Y).$$

A identificação de $(\mathbb{R}^3)^*$ com $\mathbb{R}^3$ usando o produto interno canônico significa que associamos ξ com algum vetor V_ξ tal que $\xi(Y) = (V_\xi, Y)$ onde $(\cdot,\cdot)$ é o produto interno euclidiano canônico. Como ainda já sabemos que $\mathsf{Ad}_{A^{-1}}$ age em $\mathbb{R}^3$ como multiplicação por A^{-1}, vemos que

$$(\mathsf{Ad}^*_A\,\xi)Y \leftrightsquigarrow (V_\xi, A^{-1}Y).$$

Mas ainda, o vetor V_ξ é simplesmente o vetor de coordenadas (ξ^1, ξ^2, ξ^3) se escrevemos $\xi = \sum \xi^i dx^i \in (\mathbb{R}^3)^*$. Para descrever as órbitas só note que a expressão acima está determinada por A, de modo que a órbita coadjunta está em correspondência com a $\mathsf{SO}(3)$-órbita do vetor V_ξ, que já sabemos que é uma esfera S^2.

$\square$

Problem 7 Let (V, Ω) be a symplectic vector space, and consider $H := V \times \mathbb{R} = \{(v, t)\}$. This space H with the multiplication

$$(v_1, t_1) \cdot (v_2, t_2) = \left(v_1 + v_2, \frac{1}{2}\Omega(v_1, v_2) + t_1 + t_2\right)$$

is a Lie group called the ***Heisenberg group*** (find the identity elements and inverses in H).

a. Show (directly from the conjugation formula in H) that $\mathsf{Ad}_{(v,t)}(X, r) = (X, r + \Omega(v, X))$, for $(X, r) \in \mathfrak{h} = \mathsf{Lie}(H) = V \times \mathbb{R}$. Describe the adjoint orbits, verifying that their possible dimensions are zero and one.

b. Verify that $\mathsf{ad}_{(Y,s)}(X, r) = (0, \Omega(Y, X))$. [Recalling that $\mathsf{ad}_{(Y,s)}(X, r) = [(Y, s), (X, r)]$, we obtain a formula for the Lie bracket in $\mathfrak{h}$.]

c. Describe the coadjoint action of H on $\mathfrak{h}^* = V^* \times \mathbb{R}^*$ and its orbits, analyzing the possible dimensions.

Solution. É imediato que a identidade em H é $(0, 0) \in V \times \mathbb{R}$, e o elemento inverso de (v, t) é $(-v, -t)$.

a. Primeiro note que $\mathsf{Ad}_{(v,t)} X = (v, t) \cdot X \cdot (-v, -t)$ onde $\cdot$ denota o produto en $H \cong \mathfrak{h}$ (esse isomorfismo é claro já que H é um espaço vetorial, assim o espaço tangente em $(0, 0)$ é isomofo a ele). Para justificar isso (como no exercício anterior), considere uma curva $\gamma \subset H$ tal que $\gamma(0) = (0, 0)$ e $\gamma'(0) = (X, r)$. Então

$$\begin{aligned}\mathsf{Ad}_{(v,t)}(X, r) &= d_{(0,0)} I_{(v,t)}(X, r) = \frac{d}{d\tau} I_{(v,t)} \circ \gamma\Big|_{\tau=0} \\ &= \frac{d}{d\tau}(v, t) \cdot \gamma(\tau) \cdot (-v, -t)\Big|_{\tau=0} = (v, t) \cdot (X, r) \cdot (-v, -t).\end{aligned}$$

daí é só calcular

$$\begin{aligned}\mathsf{Ad}_{(v,t)}(X, r) &= (v, t) \cdot (X, r) \cdot (-v, -t) \\ &= (v, t) \cdot \left(X - v, \frac{1}{2}\Omega(X - r) + r - t\right) \\ &= \left(X, \frac{1}{2}\Omega(v, X - v) + t + \frac{1}{2}\Omega(X, -v) + r - t\right) \\ &= \left(X, \frac{1}{2}\Omega(v, X) + \frac{1}{2}\Omega(v, -v) + \frac{1}{2}\Omega(X, -v) + s\right) \\ &= (X, \Omega(v, X) + r)\end{aligned}$$

A órbita adjunta de (X, r) é $X \times \mathbb{R}$ sempre que $X \neq 0$ já que Ω é não degenerada. Caso $X = 0$, a órbita é um ponto, $(0, r)$.

b. Para calcular $\mathsf{ad}_{(Y,s)}(X, r)$ devemos calcular o generador infinitesimal da ação adjunta:

$$\begin{aligned}
\mathsf{ad}_{(Y,s)}(X,r) &= \frac{d}{dt}\mathsf{Ad}_{\mathsf{exp}(tY,ts)}(X,r) \\
&= \frac{d}{dt}(X, r + \Omega(\mathsf{exp}(tY), X) \\
&= \left(0, \frac{d}{dt}\Omega(\mathsf{exp}(tY,X))\right)
\end{aligned}$$

supondo que a exponencial age em H entrada a entrada. Para calcular essa derivada lembre que a derivada de uma forma bilinear $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ em (a, b) é

$$Df_{(x,y)}(a,b) = f(x,b) + f(a,y)$$

Agora considere que a expressão $\mathsf{exp}(tY, X)$ é a composição de Ω com a curva $\gamma(t) = (\mathsf{exp}(tY), X)$. A equação acima vira

$$\begin{aligned}
&= (0, D_{\gamma(t)}\Omega \cdot \gamma'(t) \\
&= \left(0, \Omega(tY, 0) + \Omega(\mathsf{exp}(tY)Y, X)\right) \\
&= \left(0, \Omega(\mathsf{exp}(tY)Y, X)\right)
\end{aligned}$$

em $t = 0$ obtemos o resultado.

c.

□