Lista 1

Geometria simplética

Problem 1: Let V be a symplectic vector space ($\dim V = 2n$), and $\Omega \in \Lambda^2 V^*$be a skew-symmetric bilinear form. Show that Ω is nondegenerate iff $\Omega^n \neq 0$.

Solution. I first tried to show that Ω is degenerate iff $\Omega^n = 0$. Suppose there is a vector v_0 such that $\Omega(v_0, w) = 0$ for all $w \in V$ and complete to a basis. Then for any $v_1, v_2, v_3, v_4 \in V$ we have

$$(\Omega \wedge \Omega)(v_1, v_2, v_3, v_4) = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma)\Omega(v_{\sigma(1)}, v_{\sigma(2)})\Omega(v_{\sigma(3)}, v_{\sigma(4)}).$$

Finally I found this proposition in Lee, Intro. Smooth Manifolds. □

Problem 2: Let (V, Ω) be a symplectic vector space, and let $W \subseteq V$ be any linear subspace.

a. Show that $V_W = \frac{W}{W \cap W^\Omega}$ inherits a natural symplectic structure Ω_W uniquely determined by the condition $\pi^*\Omega_W = \Omega|_W$ (here $\pi : W \to W/(W \cap W^\Omega)$ is the quotient projection).

   (*The space* (V_W, Ω_W) *is called the* ***reduced space***.)

b. Suppose that W is coisotropic, and let $L \subset V$ be lagrangian. Show that the image of $L \cap W$ via $\pi : W \to V_W$ is lagrangian in the reduced space.

Solution.

a. Define
$$\Omega_W([w_1], [w_2]) := \Omega(w_1, w_2)$$
for any equivalence classes $[w_1], [w_2] \in V_W$. Let's check that this is well defined. Suppose $w_1' \in [w_1]$. Then $w_1 - w_1' \in W \cap W^\Omega$ so $\Omega(w_1 - w_1', w_2) = 0$ since $w_2 \in W$ and $w_1 - w_1'$ is, in particular, in W^Ω. So $\Omega(w_1, w_2) = \Omega(w_1', w_2)$. Why not quotient only by W^Ω? Looks like I didn't use the W part…

   Recall that $\pi^*\Omega_W(w_1, w_2) = \Omega_W([w_1], [w_2])$. It is straightforward to check that Ω_W is the only symplectic form on V_W satisfying $\pi^*\Omega_W = \Omega|_W$: if Ω_W' is another such form, then $\Omega_W([w_1], [w_2]) = \Omega|_W(w_1', w_2') = \Omega_W'([w_1], [w_2])$ for any $w_1' \in [w_1]$ and $w_2' \in [w_2]$.

b. Let's first check what is $(\pi(L \cap W))^{\Omega_W}$. We have

$$\begin{aligned}(\pi(L \cap W))^{\Omega_W} &= \{[v] \in V_W : \Omega_W([v], [w]) = 0 \ \forall [w] \in \pi(L \cap W)\} \\ &= \{[v] \in V_W : \Omega(v', w) = 0 \ \forall v' \in [v] \text{ and } \forall w \text{ s.t. } [w] \in \pi(L \cap W)\}\end{aligned}$$

   In words, this is the set of classes whose representatives are Ω-orthogonal to representatives of $\pi(L \cap W)$.

So if $[v]$ is any such class,

so let $[v] \in \pi(L \cap W)^{\Omega_W}$. Let's check that $[v]$ is also in $\pi(L \cap W)$, ie. that $v \in L \cap W$. Well,

If $v' \in L$, then $\Omega(v, v') = 0$ since $[v'] \in \pi(L \cap W)^{\Omega}$... but what if $v' \in L \backslash W$? .

Let w be such that $[w] \in \pi(L \cap W)$. Then

$$\begin{aligned} &\Omega_W([v], [w]) = 0 \\ \implies \ &\Omega(v, w) = 0 \end{aligned}$$

so $v \in$

$\square$

Problem 3: We saw in class that any symplectomorphism $T : V_1 \to V_2$ defines a lagrangian subspace by its graph: $\Gamma_T := \{(Tu, u) : u \in V_1\} \subset V_2 \oplus \overline{V}_1$. (Recall that if (V, Ω) is a svs, $\overline{V}$ denotes $(V, -\Omega)$.) So we think lagrangian subspaces of $V_2 \oplus \overline{V}_1$ a generalizations of symplectomorphisms. We now see how to generalize their composition.

Consider symplectic vector spaces V_1, V_2, V_3 and $E = V_3 \oplus \overline{V}_2 \oplus V_2 \oplus \overline{V}_1$.

a. Show that $\Delta := \{(v_3, v_2, v_2, v_1) \in E\}$ is coisotropic in E and its reduction E_Δ can be identified with $V_3 \oplus \overline{V}_1$.

b. Given lagrangian subspaces $L_1 \subset V_2 \oplus \overline{V}_1$ and $L_2 \subset V_3 \oplus \overline{V}_2$, define the ***composition*** of L_2 and L_1 by

$$L_2 \circ L_1 := \{(v_3, v_1) | \exists v_2 \in V \text{ s.t. } (v_3, v_2) \in L_2, (v_2, v_1) \in L_1\}.$$

Show that $L_2 \circ L_1$ is a lagrangian subspace of $V_3 \oplus \overline{V}_1$. (*Hint: show that the composition can be identified with the reduction of* $L_2 \times L_1 \subset E$ *with respect to* Δ).

c. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be symplectomorphisms. Show that $\Gamma_{T_2 \circ T_1} = \Gamma_{T_2} \circ \Gamma_{T_1}$.ad

Solution.

a. Let $v := (v_3, v_2, v_2', v_1) \in \Delta^{\Omega_E} \subset E$ where $\Omega_E = \Omega_1 \oplus -\Omega_2 \oplus \Omega_2 \oplus -\Omega_1$ is the symplectic form on E. We wish to show that $v \in \Delta$, which only means that $v_2 = v_2'$. So let $\tilde{v} = (v_3, v_2, v_2, v_1)$ and $\hat{v} = (v_3, v_2', v_2', v_1)$ which are both in Δ. Then we have

$$\begin{aligned} 0 &= \Omega_E(v, \tilde{v}) \\ &= \Omega_1(v_3, v_3) - \Omega_2(v_2, v_2) + \Omega_2(v_2', v_2) - \Omega_1(v_1, v_1) \end{aligned}$$

and likeways

$$\begin{aligned} 0 &= \Omega_E(v, \hat{v}) \\ &= \Omega_1(v_3, v_3) - \Omega_2(v_2, v_2') + \Omega_2(v_2', v_2') - \Omega_1(v_1, v_1). \end{aligned}$$

Substracting,

$$0 = -\Omega_2(v_2, v_2) + \Omega_2(v_2, v_2') + \Omega_2(v_2', v_2) - \Omega_2(v_2', v_2')$$

And by linearity,

$$\begin{aligned}
& 0 = -\Omega_2(-v_2 + v_2, -v_2 + v_2') + \Omega_2(v_2' - v_2', v_2 - v_2') \\
\implies & 0 = -\Omega_2(0, -v_2 + v_2') + \Omega_2(0, v_2 - v_2') \\
\implies & 0 = \Omega_2(-v_2 + v_2', 0) + \Omega_2(0, v_2 - v_2') \\
\implies & 0 = \Omega_2(-v_2 + v_2', v_2 - v_2') \\
\implies & 0 = \Omega_2(-(v_2 - v_2'), v_2 - v_2')
\end{aligned}$$

and it follows that $v_2 = v_2'$ from nondegeneracy.

Now let's try to construct an isomorphism $E_\Delta = V_3 \oplus \overline{V}_1$. Consider

$$\begin{aligned}
\varphi : E &\longrightarrow V_3 \oplus \overline{V}_1 \\
(v_3, v_2, v_2', v_1) &\longmapsto (v_3, v_1)
\end{aligned}$$

which is clearly surjective and not injective, so perhaps its kernel is $\Delta \cap \Delta^\Omega$. But $\ker \varphi = \{(0, v_2, v_2', 0)\}$, so unfortunately no.

But perhaps we can construct some other map. Let's try

$$\begin{aligned}
\varphi : E &\longrightarrow V_3 \oplus \overline{V}_1 \\
(v_3, v_2, v_2', v_1) &\longmapsto
\end{aligned}$$

b. It looks like $L_2 \circ L_1$ is very much like E_Δ from the last exercise. $L_2 \circ L_1$ is *strictly* contained in $V_3 \oplus \overline{V}_1 \cong E_\Delta$.

   Ok let's have a go at the hint. Perhaps $(L_2 \times L_1)_\Delta = L_2 \circ L_1$ and it is lagrangian. That'd be great. OK let's compute it. This is how to do it:

$$\begin{aligned}
\varphi : L_2 \times L_1 &\longrightarrow V_2 \circ V_1 \\
(v_2, v_1, v_3, v_2) &\longmapsto
\end{aligned}$$

□

Problem 4: Let (V, J) be a complex vector space, let Ω be a sympletic structure on V. Show that J and Ω are compatible iff there exists a hermitian inner product $h : V \times V \to \mathbb{C}$ such that Ω is its imaginary part. Show that any (complex) orthonormal basis of (V, h) can be extended to a symplectic basis of (V, Ω).

Solution. First suppose that J and Ω are compatible, ie., $g(u, v) := \Omega(u, Jv)$ is an inner product. Define $h(u, v) = g(u, v) + i\Omega(u, v)$. Then h is the required hermitian inner product. Indeed:

1. The properties $h(u_1 + u_2, v) = h(u_1, v) + h(u_1, v)$ and $h(u, v_1 + v_2) = h(u, v_1) + h(u, v_2)$ follows easily from linearity of g and Ω.
2. $h(\lambda u, v) = \lambda h(u, v)$ follows again from linearity of g and Ω.
3. The property $h(u, \lambda v) = \bar{\lambda} h(u, v)$ follows easily from 2. and 4. since

$$\begin{aligned} h(u, \lambda v) &= \overline{h(\lambda v, u)} \\ &= \bar{\lambda}\overline{h(v, u)} \\ &= \bar{\lambda} h(u, v) \end{aligned}$$

4. $h(u, v) = \overline{h(v, u)}$ is clear by anti-symmetry of Ω:

$$\begin{aligned} h(u, v) &= g(u, v) + i\Omega(u, v) \\ &= g(v, u) - i\Omega(v, u) \\ &= \overline{h(v, u)} \end{aligned}$$

For the converse suppose that h is an hermitian inner product such that Ω is its imaginary part. Then $g(u, v) := \Omega(u, Jv)$ is an inner product:

1. Linearity of g is immediate from linearity of Ω and J.
2. Symmetry follows from

$$\begin{aligned} g(u, v) = \Omega(u, Jv) & \\ &= \Omega(-J^2 u, Jv) \\ &= -\Omega(J^2 u, Jv) \\ &= \Omega(Jv, J^2 u) \\ &= \Omega(\qquad\qquad = -\Omega(Jv, u) \\ &= \Omega(-v, g(v, u) \qquad\qquad = \Omega(v, Ju) \\ &= -\Omega(Ju, v) \end{aligned}$$

3. For positive-definiteness let $u \neq 0$. Then

$$g(u, u) = \Omega(u, Ju)$$

□

Problem 5: Consider the symplectic vector space $(\mathbb{R}^{2n}, \Omega_0)$, where $\Omega_0(u,v) = -u^T J_0 v$. Check that its group of linear symplectomorphisms is given by $\mathrm{Sp}(2n) = \{A \in \mathrm{GL}(2n) : A^T J_0 A = J_0\}$. Show that $\mathrm{Sp}(2n)$ is a smooth submanifold of $\mathrm{GL}(2n)$ and that its tangent space at the identity $I \in \mathrm{GL}$ is given by $T_I\,\mathrm{Sp}(2n) = \{A : \mathbb{R}^{2n} \to \mathbb{R}^{2n} | A^T J_0 + J_0 A = 0\}$. Conclude that $\mathrm{Sp}(2n)$ has dimension $2n^2 + n$. Verify also that $\mathrm{Sp}(2n)$ is not compact.

Problem 6: Consider the standard compatible triple (Ω_0, J_0, g_0) on $\mathbb{R}^{2n}$. Let $O(2n)$ be the linear orthogonal group of $\mathbb{R}^{2n}$ (i.e., linear transformations preserving the canonical inner product g_0), and let $\mathrm{Sp}(2n)$ be the symplectic linear group. Through the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ (as complex vector spaces), we may se e somente se $\mathrm{GL}(n,\mathbb{C})$ (the group of linear automorphisms of $\mathbb{C}^n$ as a subgroup of $\mathrm{GL}(2n,\mathbb{R})$: a complex matrix $A + iB$ is identified with the real $2n \times 2n$ matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

Let now $U(n) \subset \mathrm{GL}(n,\mathbb{C})$ be the group of linear transformation preserving the natural hermitian inner product of $\mathbb{C}^n$. Show that the intersection of any two of the groups

$$\mathrm{Sp}(2n), O(2n), \mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$$

is $U(n)$.

Problem 7: Let (V,Ω) be a symplectic vector space, let $W \subseteq V$. Let J be a Ω-compatible complex structure and g the corresponding inner product. Verify that $J(W^\Omega) = W^{\perp_g}$.

a. Use this fact to show that any coisotropic subspace of V has an isotropic complement. In particular, any lagrangian subspace $L \subset V$ has a lagrangian complement L', $V = L \oplus L'$.

b. Show that there is a natural identification $L' \cong L^*$, that induces a symplectomorphism $V \cong L \oplus L^*$ (where $L \oplus L^*$ has the natural symplectic structure $((ell, \alpha), (\ell', \alpha')) \mapsto \alpha(\ell') - \alpha'(\ell)$.

Solution. First let's check that $J(W^\Omega) = W^{\perp_g}$. Indeed,

$$\begin{aligned} J(W^\Omega) &= \{Jv : v \in W^\Omega\} \\ &= \{Jv : \Omega(v,w) = 0\ \forall w \in W\} \\ &= \{Jv : -\Omega(w,v)\ \forall w \in W\} \\ &= \{Jv : \Omega(w,-v)\ \forall w \in W\} \\ &= \{Jv : \Omega(w,J^2 v)\ \forall w \in W\} \end{aligned}$$

re-write $Jv := \tilde{v}$ using that J is bijective:

$$\begin{aligned} J(W^{\Omega)} &= \{\tilde{v} \in V : \Omega(w, J\tilde{v}) = 0\ \forall w \in W\} \\ &= \{\tilde{v} \in V : g(\tilde{v}, w) = 0\ \forall w \in W\} \\ &= W^{\perp_g} \end{aligned}$$

□

Bonus problem: