Geometria Simplética 2024, Lista 2

Professor: H. Bursztyn

Entrega dia 10/09

Problem 1: Verify (and justify) whether or not the following manifolds admit a symplectic structure: $S^1 \times S^3$, $\mathbb{R} \times S^3$, $\mathbb{R}^3 \times S^3$, $\mathbb{T}^3 \times S^3$.

Problem 2: Show that the tautological 1-form $\alpha \in \Omega^1(T^*Q)$ is *uniquely characterized* by the following property: for any 1-form $\mu \in \Omega^1(Q)$,

$$\mu^*\alpha = \mu,$$

where on the left-hand side we view μ as a map $\mu : Q \to T^*Q$.

Problem 3: We will characterize symplectomorphisms $T^*Q \to T^*Q$ which are cotangent lifts of diffeomorphisms $\phi : Q \to Q$. Let α be the tautological 1-form on $M = T^*Q$ and $\omega = -d\alpha$. We saw in class that cotangent lifts preserve α. We willl show the converse of this fact.

Let $F : M \to M$ be a symplectomorphism such that $F^*\alpha = \alpha$.

(a) Let $v \in \mathfrak{X}(M)$ be the unique vector field such that $i_v\omega = -\alpha$; note that, locally, it is given by $\sum_i \xi_i \frac{\partial}{\partial \xi_i}$ (v is known as the *Euler vector field*) . Show that $F_*v = v$.

(b) Let φ_t^v denote the flow of v. Show that $\varphi_t^v \circ F = F \circ \varphi_t^v$. Check that, in coordinates, $\varphi_t^v(x, \xi) = (x, e^t\xi)$, $-\infty < t < \infty$.

(c) Verify that, for $p \in T_x^*Q$, $F(\lambda p) = \lambda F(p)$, $\forall \lambda \in \mathbb{R}$. Conclude that there exists $\phi : Q \to Q$ such that $\phi \circ \pi = \pi \circ F$ (here $\pi : T^*Q \to Q$ is the projection). Finally, show that $F = \widehat{\phi}$ (the cotangent lift of ϕ).

Problem 4: Let $\alpha \in \Omega^1(T^*Q)$ be the tautological 1-form. We will now see examples of symplectomorphisms of T^*Q which are not cotangent lifts. Let $A \in \Omega^1(Q)$ and consider the associated "fiber-translation" map $\varphi_A : T^*Q \to T^*Q$, $(x, \xi) \mapsto (x, \xi + A_x)$.

(a) Show that

$$\varphi_A^*\alpha - \alpha = \pi^*A,$$

where $\pi : T^*Q \to Q$ is the projection. It follows that φ_A is a symplectomorphism iff A is *closed*.

(b) Consider functions that are constant along the fibers of T^*Q (i.e., of the form $H = \pi^*f$, for $f \in C^\infty(Q)$). Describe their hamiltonian vector fields in local cotangent coordinates, as well as their flows.

Problem 5: Let $\omega = -d\alpha$ be the canonical symplectic form on T^*Q. Prove that, if $B \in \Omega^2(Q)$ is closed, then

$$\omega_B := \omega - \pi^* B$$

is symplectic and that, if $B, B' \in \Omega^2(Q)$ are closed and such that $B - B' = dA$, then φ_A (defined in the previous problem) is a symplectomorphism from (T^*Q, ω_B) to $(T^*Q, \omega_{B'})$.

Problem 6: Let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form. For $f \in C^\infty(M)$, let $X_f \in \mathfrak{X}(M)$ be defined by $i_{X_f}\omega = df$. Consider the bracket $\{f, g\} := \omega(X_g, X_f)$. Verify that $d\omega = 0$ if and only if $\{\cdot, \cdot\}$ satisfies the Jacobi identity.

Problem 7: (1) Consider symplectic manifolds (M_i, ω_i), with Poisson bracket $\{\cdot, \cdot\}_i$, $i = 1, 2$, and let $\phi : M_1 \to M_2$ be a smooth map.

(a) Prove that, if ϕ is a diffeomorphism, then it is a Poisson map ($\{\phi^* f, \phi^* g\}_1 = \phi^*(\{f, g\}_2)$ for all $f, g \in C^\infty(M_2)$) if and only if $\phi^*\omega_2 = \omega_1$.

(b) Find examples of M_1, M_2 and $\phi : M_1 \to M_2$ such that (1) ϕ is a Poisson map but does not satisfy $\phi^*\omega_2 = \omega_1$; (2) ϕ satisfies $\phi^*\omega_2 = \omega_1$ but is not a Poisson map.

Hint: Consider $\mathbb{R}^2$ and $\mathbb{R}^4$ with their canonical symplectic structures and Poisson brackets, and the maps $\mathbb{R}^2 \to \mathbb{R}^4$, $(q_1, p_1) \mapsto (q_1, p_1, 0, 0)$, and $\mathbb{R}^4 \to \mathbb{R}^2$, $(q_1, p_1, q_2, p_2) \mapsto (q_1, p_1)$.

Problem 8:

(a) Consider $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ equipped with the area form $\omega_x(u, v) = \langle x, u \times v\rangle$ (where $x \in S^2$, $u, v \in T_x S^2$, and $\times$ is the vector product). Use cylindrical coordinates to prove Darboux's theorem directly in this example.

(b) More generally: show that on a 2-dimensional manifold, any non-vanishing 1-form can be locally written as $f dg$, where f and g are smooth functions. Use this fact to give a direct proof of Darboux's theorem in 2 dimensions.