Daniel González Casanova Azuela
github.com/danimalabares/sg

Lista 5

Contents

Problem 1 Let G be a Lie group. Let $X : G \longrightarrow TG$ be a section of the projection $TG \longrightarrow G$, not necessarily smooth. Show that if X is left invariant (i.e., $dL_g(X) = X \circ L_g$ for all $g \in G$), then X is automatically smooth.

Conclude that an analogous result holds for differential forms: if a section $\eta : G \to \Lambda^k(T^*G)$ is left-invariant ($L_g^*\eta = \eta$), then η is a smooth k-form. Check that an analogous result holds for G-invariant forms on a homogeneous manifold.

Solution. (Idea by my) We know that $X_g = dL_g(X_e)$. We want to show that the map $G \to TG : g \mapsto dL_g(X_e)$ is smooth. Suggestion by Ted Shiffrin is to consider

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (g, h) &\longmapsto L_g h = gh \end{aligned}$$

which is smooth and thus has smooth differential which turns out to have the expression

$$\begin{aligned} d\mu : TG \times TG &\longrightarrow TG \\ (u_g, v_h) &\longmapsto dR_h u_g + dL_g v_h \end{aligned}$$

Choosing g arbitrary, $h = e$, $u_g = 0$ and $v_h = X_e$, we obtain

$$d\mu(u_g, v_h) = dL_g X_e$$

and, as we have said, this differential depends smoothly on g.

To generalize this result we just notice that left multiplication L_g induces smooth maps the Grassman algebra by

$$\begin{aligned} \mathbf{L_g} : \Lambda^k(G) &\longrightarrow \Lambda^k(G) \\ \eta &\longmapsto \begin{array}{l} \mathbf{L_g}\eta : \mathfrak{X}^k(G) \longrightarrow \mathbb{R} \\ (X_1, \dots, X_k) \longmapsto (dL_g X_1, \dots, dL_g X_1) \end{array} \end{aligned}$$

and similarly group multiplication μ induces maps

$$\begin{aligned} \mu : \Lambda^k(G) \times \Lambda^k(G) &\longrightarrow \Lambda^k(G) \\ (\alpha, \beta) &\longmapsto \mathbf{R_h}\alpha + \mathbf{L_g}\beta \end{aligned}$$

so again chosing

□

Problem 2

a. Prove that any connected Lie group G is generated as a group by any open neighbourhood U of the identity element (i.e. $G = \bigcup_{n=1}^{\infty} U^n$).

b. Suppose that two Lie group homomorphisms $\varphi, \psi : G \to H$ are such that $d\varphi|_e = d\psi|_e$. Show that φ and ψ coincide on the connected component of G containing the identity e.

Problem 3 Consider the Lie groups $\mathsf{SU}(2) = \{A \in \mathcal{M}_2(\mathbb{C}) | AA^* = \mathrm{Id}, \det A = 1\}$ and $\mathsf{SO}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) | AA^{\mathbf{T}} = \mathrm{Id}, \det A = 1\}$.

a. Show that

$$\mathsf{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

Conclude that, as a manifold $\mathsf{SU}(2)$ is diffeomorphic to S^3 (hence it is simply connected).

Recall the definition of the quaternions $\mathbb{H}$. Show that the sphere S^3, seen as quaternions of norm 1, inherits a Lie group structure with respect to which it is isomorphic to $\mathsf{SU}(2)$.

b. Verify that

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}. \tag{1}$$

Consider the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, that takes the element in $\mathfrak{su}(2)$ determined by α, β to the vector $(\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)$ in $\mathbb{R}^3$. Observe that, with respect to this identification, det in $\mathfrak{su}(2)$ corresponds to $\|\cdot\|^2$ in $\mathbb{R}^3$.

c. Verify that each element $A \in \mathsf{SU}(2)$ defines a linear transformation on the vector space $\mathfrak{su}(2)$ by conjugation: $B \mapsto ABA^{-1}$. Show that, with the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, we obtain a representation (i.e., a linear action) of $\mathsf{SU}(2)$ on $\mathbb{R}^3$ that is norm preserving. Conclude that we have homomorphism $\phi : \mathsf{SU}(2) \to \mathsf{O}(3)$, verifying that is image is $\mathsf{SO}(3)$ and its kernel is $\{\mathrm{Id}, -\mathrm{Id}\}$.

d. Conclude that $\mathsf{SU}(2) \cong S^3$ is a double cover of $\mathsf{SO}(3)$ *hence is its universal cover, since it's simply connected), qnd the covering map identifies antipodal points of S^3. Hence, as manifolds, $\mathsf{SO}(3)$ is identified with $\mathbb{R}P^3$.

Solution.

a. Given the computation above, it is clear that $\mathsf{SU}(2)$ is diffeomorphic to S^3 since its parameters $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, can be understood as vectors $x \in \mathbb{R}^4$ of norm 1.

   The quaternions are the only 4-dimensional real division algebra. This means it is a 4-dimensional real vector space equipped with a (non-commutative) multiplication. They are also equipped with a norm that coincides with euclidean norm. With respect to this norm we define the unit sphere S^3.

   To see that S^3 is a Lie subgroup we first need to check that it is closed under quaternion product. The easiest way to see that is via quaternion conjugate: if $x = x_1 + ix_2 + jx_3 + kx_4$ is a quaternion, its conjugate is $\bar{x} = x_1 - ix_2 - jx_3 - kx_4$. It may be computed that the norm is given by $|x| = x\bar{x}$. Then we see that if $x, y \in S^3$

$$|xy\overline{xy}| = |xy\bar{y}\bar{x}| = \Big|x|y|\bar{x}\Big| = 1$$

   Then the fact that S^3 is a Lie group follows from the fact that the restricition (the multiplication and inverse map) of smooth maps to embedded submanifolds remains smooth ([**?**], prop ?).

   An identification between $S^3 \subset \mathbb{H}$ and $\mathsf{SU}(2)$ as expressed above is given by

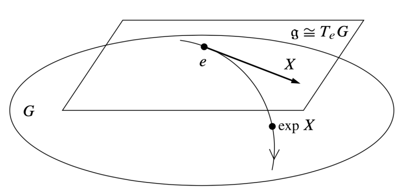
$$a + bi + cj + dk \longmapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

   Checking that this map is a group isomorphism ammounts to checking that matrix multiplication in $\mathsf{SU}(2)$ is the same as quaternion multiplication

b. (Proof from [**?**], prop. 3.24 and coro. 3.46).

Step 1 Show that the tangent space at identity of a matrix Lie group G (definition in lectures) is the same as the matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

Proof of Step 1. ([**?**], coro. 3.46) If $X \in T_{\mathrm{Id}}G$, by definition of exponential map we have $\exp(tX) \in G$.



[**?**], fig. ?

Now suppose that X is such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

I got stuck with the proof in Hall so I will use prop 20.3 from [?]. The idea is to show see $\mathsf{SU}(2)$ as a subgroup of, say $\mathsf{GL}(2, \mathbb{C})$. Then a ☐

Step 2 Then we look for the matrices such that for all $t \in \mathbb{R}$,

$$\exp(tX)^* = (\exp(tX)^{-1} = \exp(-tX) \qquad \text{and} \qquad \det \exp(tX) = 1.$$

This means that

$$\exp(tX^*) = \exp(-tX) \qquad \text{and} \qquad \operatorname{Tr}(X) = 0.$$

The first condition is equivalent to $X^* = -X$. Thus we see that

$$\mathfrak{su}(2) = \{X \in \mathcal{M}_{2\times 2}(\mathbb{C}) : X^* = -X \text{ and } \operatorname{Tr}(X) = 0\}$$

Step 3 It is immediate that the expression in eq. (1) is contained in the set above. For the other inclusion first notice that the condition $X^* = -X$ makes the entries in the diagonal be such that

$$x + iy = -\overline{x + iy} = -(x - iy) = -x + iy \implies x = -x \implies x = 0$$

while the traceless condition implies the two entries in the diagonal must be additive inverses. For the entries in the antidiagonal we literally see the definition of conjugate transpose.

Now let's identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via $\alpha, \beta \mapsto (\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)$. We immediately see that

$$\det\begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} = i\alpha(-i\alpha) = \beta(-\bar{\beta}) = \alpha^2 + |\beta|^2 = \|(\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)\|^2$$

c. De acordo como exercício anterior, é suficiente mostrar que ABA^{-1} tem traza zero é $ABA^{-1} = -(ABA^{-1})^*$. A primeira propriedade é imediata dado que, em geral, $\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$. Para a segunda propriedade note que

$$(ABA^{-1})^* = (A^{-1})^* B^* A^* = (A^*)^*(-B)A^{-1} = -ABA^{-1}$$

O fato de que essa ação em $\mathsf{SU}(2)$ preserva a norma é immediato do item anterior e do fato de que determinante de um produto de matrizes é o produto dos determinantes.

Isso significa que cada elemento em $\mathsf{SU}(2)$ é uma transformação linear $\mathbb{R}^3 \to \mathbb{R}^3$, i.e. um mapa $\mathsf{SU}(2) \to \mathsf{O}(3)$. Esse mapa é um homomorfismo já que

☐

Problem 4 Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, and let $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$be a symmetric bilinear form that is Ad-invariant (i.e. $k(\mathrm{Ad}_g(u), \mathrm{Ad}_g(v)) = k(u, v)$ for $g \in G$).

a. Show that the map

$$\begin{gathered} k^\sharp : \mathfrak{g} \longrightarrow \mathfrak{g}^* \\ k^\sharp(u)(v) = k(u, v) \end{gathered} \tag{2}$$

is G-equivariant:

$$k^\sharp \circ \mathrm{Ad}_g = (\mathrm{Ad}^*)_g \circ k^\sharp, \qquad \forall g \in G$$

[*Recall:* $(\mathrm{Ad}^*)_g := (\mathrm{Ad}_{g^{-1}})^*$.] In particular, when k is nondegenerate (i.e. $k^\sharp$ is an isomorphism), the adjoint and coadjoint actions are equivalent.

b. Verify that eq. (2) implies that $k([w, u], v) = -k(u, [w, v])$, $\forall u, v, w \in \mathfrak{g}$, and that both conditions are equivalent when G is connected.

Solução. a. É só abrir as definições . Fixe um elemento $x \in \mathfrak{g}$. No lado esquerdo, temos

$$k^\sharp \circ \mathrm{Ad}_g(x) = k^\sharp(\mathrm{Ad}_g x) = k(\mathrm{Ad}_g x, \cdot)$$

e no lado direito,

$$(\mathrm{Ad}^*)_g \circ k^\sharp(x) = \mathrm{Ad}_g^*(k^\sharp(x)) = \mathrm{Ad}_g^*(k(x, \cdot)) = k(x, \mathrm{Ad}_{g^{-1}} \cdot)$$

mas, como k é Ad-invariante, $k(x, \mathrm{Ad}_{g^{-1}}, \cdot) = k(\mathrm{Ad}_g x, \cdot)$.

b. Must check this later...

□

Problem 5 For a Lie algebra $\mathfrak{g}$, there is always a canonical bilinear form $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, called ***Killing form***, given by:

$$k(u, v) = \mathrm{tr}(\mathrm{ad}_u \, \mathrm{ad}_v).$$

(*Recall:* $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_u(v) = [u, v]$.)

a. Note that k is symmetric, and check that it is Ad-invariant.

b. A Lie algebra is called ***semi-simple*** if k is nondegenerate. Show that $\mathfrak{so}(3)$ is semi-simple.

Solution.

a. O fato de k ser simétrica segue de que, em geral, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Para ver que k é Ad-invariante, note que, para $g \in G$,

$$k(\mathrm{Ad}_g u, \mathrm{Ad}_g v) = \mathrm{tr}(\mathrm{ad}_{\mathrm{Ad}\, u} \, \mathrm{ad}_{\mathrm{Ad}\, v}) = \mathrm{tr}([\mathrm{Ad}\, u, [\mathrm{Ad}\, v, \cdot]]) =$$

□

Problem 6 Consider the linear isomorphism $\mathbb{R}^3 \to \mathfrak{so}(3)$, given by

$$v = (x, y, z) \longmapsto \hat{v} := \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

a. Describe the Lie bracket on $\mathbb{R}^3$ induced by the commutator in $\mathfrak{so}(3)$, and the inner product in $\mathfrak{so}(3)$ that corresponds to the canonical inner product in $\mathbb{R}^3$.

b. Describe the $\mathsf{SO}(3)$-action on $\mathbb{R}^3$ corresponding to the adjoint action, its orbits, as well as its infinitesimal generators. Find (without any calculation!) a description of the coadjoint action on $\mathbb{R}^3$ (identified with $(\mathbb{R}^3)^*$ through the canonical inner product).

Solution.

a. Lembre que

$$\mathsf{SO}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) : AA^{\mathbf{T}} = \mathrm{Id}, \det A = 1\}$$
$$\mathfrak{so}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) : A = -A^{\mathbf{T}}\}$$

Isso explica por que as matrizes em $\mathfrak{so}(3)$ tem a forma mostrada acima. O isomorfismo com $\mathbb{R}^3$ é dado por

$$\begin{aligned} \mathfrak{so}(3) &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} &\longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = u \\ [U, V] &\longmapsto u \times v \\ \frac{1}{2}\operatorname{tr}(UV^{\mathbf{T}}) &\longmapsto \langle u, v\rangle \end{aligned}$$

Para comprovar que de fato o commutador de $\mathfrak{so}(3)$ corresponde com o produto vetorial em $\mathbb{R}^3$, considere duas matrizes $U, V \in \mathfrak{so}(3)$ e os vetores correspondentes $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$. O commutador é

$$\begin{aligned} [U, V] = UV - VU = &\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \\ &- \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \end{aligned}$$

a entrada $(3, 2)$ da matriz resultante é a primeira coordenada do vetor correspondente a $[U, V]$; esse numero é claramente $u_1v_3 - u_3v_2$. Analogamente, a segunda coordenada é $u_3v_1 - u_1v_3$, enquanto a terceira $u_1v_2 - u_2v_1$. Essas são as coordenadas de $u \times v$.

Para ver que o produto interno de $\mathbb{R}^3$ corresponde com $\frac{1}{2}\operatorname{tr}$ note que

$$\begin{aligned} UV^{\mathbf{T}} = U(-V) &= \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} u_3v_3 + u_2v_2 & * & * \\ * & u_3v_3 + u_1v_1 & * \\ * & * & u_2v_2 + u_1v_1 \end{pmatrix} \end{aligned}$$

Agora vamos calcular a ação coadjunta. Primeiro notemos que a ação coadjunta, definida como a derivada do operador $I_A(X) = AXA^{-1}$, é ela mesma. Talvez esse é um fato obvio porque I_A é um mapa linear, e a sua derivada coincide com ele. Porém, tem um argumento mais explícito no StackExchange: a ação adjunta em um vetor $X \in \mathfrak{so}(3)$ que seja a derivada em $t = 0$ da curva $c \subset \mathsf{SO}(3)$, i.e. $c'(0) = X$, é simplesmente $\frac{d}{dt}Ac(t)A^{-1}\Big|_{t=0} = AXA^{-1}$. A observação chave é que como a álgebra de Lie $\mathfrak{so}(3)$ é um álgebra de matrizes, o produto do lado direito está bem definido.

Em fim, agora vou seguir este documento para achar uma matriz que representa esse operador linear cuando identificamos $\mathfrak{so}(3)$ com $\mathbb{R}^3$. Mais precisamente, queremos achar uma matriz $\overline{\mathrm{Ad}_A}$ tal que $\mathrm{Ad}_A X = AXA^{-1} \rightsquigarrow \overline{\mathrm{Ad}_A}x$ onde $x \in \mathbb{R}^3$ representa a matriz $X \in \mathfrak{so}(3)$.

O truque é usar que as matrizes em $\mathsf{SO}(3)$, sendo isometrías de $\mathbb{R}^3$, preservam o produto vetorial, i.e. $Ax \times Ay = A(x \times y)$ para qualquer $x, y \in \mathbb{R}^3$. Temos que

$$\mathrm{Ad}_A XAy =$$

$\square$

Problem 7 Let (V, Ω) be a symplectic vector space, and consider $H := V \times \mathbb{R} = \{(v, t)\}$. This space H with the multiplication

$$(v_1, t_1) \cdot (v_2, t_2) = \left(v_1 + v_2, \frac{1}{2}\Omega(v_1, v_2) + t_1 + t_2\right)$$

is a Lie group called the ***Heisenberg group*** (find the identity elements and inverses in H).

a. Show (directly from the conjugation formula in H) that $\mathrm{Ad}_{(v,t)}(X, r) = (X, r + \Omega(v, X))$, for $(X, r) \in \mathfrak{h} = \mathsf{Lie}(H) = V \times \mathbb{R}$. Describe the adjoint orbits, verifying that their possible dimensions are zero and one.

b. Verify that $\mathrm{ad}_{(Y,s)}(X, r) = (0, \Omega(Y, X))$. [Recalling that $\mathrm{ad}_{(Y,s)}(X, r) = [(Y, s), (X, r)]$, we obtain a formula for the Lie bracket in $\mathfrak{h}$.]

c. Describe the coadjoint action of H on $\mathfrak{h}^* = V^* \times \mathbb{R}^*$ and its orbits, analyzing the possible dimensions.

Solution. É imediato que a identidade em H é $(0, 0) \in V \times \mathbb{R}$, e o elemento inverso de (v, t) é $(-v, -t)$.

a. Primeiro note que $\mathrm{Ad}_{(v,t)} X = (v, t) \cdot X \cdot (-v, -t)$ onde $\cdot$ denota o produto en $H \cong \mathfrak{h}$ (esse isomorfismo é claro já que H é um espaço vetorial, assim o espaço tangente em $(0, 0)$ é isomofo a ele). Para justificar isso (como no exercício anterior), considere uma curva $\gamma \subset H$ tal que $\gamma(0) = (0, 0)$ e $\gamma'(0) = (X, r)$. Então

$$\begin{aligned}\mathrm{Ad}_{(v,t)}(X, r) = d_{(0,0)} I_{(v,t)}(X, r) &= \frac{d}{d\tau} I_{(v,t)} \circ \gamma\Big|_{\tau=0} \\ &= \frac{d}{d\tau}(v, t) \cdot \gamma(\tau) \cdot (-v, -t)\Big|_{\tau=0} = (v, t) \cdot (X, r) \cdot (-v, -t).\end{aligned}$$

daí é só calcular

$$
\begin{aligned}
\mathrm{Ad}_{(\nu,t)}(X,r) &= (\nu,t)\cdot(X,r)\cdot(-\nu,-t)\\
&= (\nu,t)\cdot\left(X-\nu,\frac{1}{2}\Omega(X-r)+r-t\right)\\
&= \left(X,\frac{1}{2}\Omega(\nu,X-\nu)+t+\frac{1}{2}\Omega(X,-\nu)+r-t\right)\\
&= \left(X,\frac{1}{2}\Omega(\nu,X)+\frac{1}{2}\Omega(\nu,-\nu)+\frac{1}{2}\Omega(X,-\nu)+s\right)\\
&= (X,\Omega(\nu,X)+r)
\end{aligned}
$$

A órbita adjunta de (X,r) é $X\times\mathbb{R}$ sempre que $X\neq 0$ já que Ω é não degenerada. Caso $X=0$, a órbita é um ponto, $(0,r)$.

$\square$