Geometria Simplética 2024, Lista 3

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Problem 1: Consider a symplectic manifold (M^{2n}, ω) with hamiltonian $H \in C^\infty(M)$. Suppose c is a regular value of H. We will show that $M_c = H^{-1}(c)$ inherits a natural volume form, invariant by the hamiltonian flow. We will actually show something more general.

Let $f_1, \ldots, f_k \in C^\infty(M)$ be first integrals of the flow of H (i.e., $\{H, f_i\} = 0$). Let $F = (f_1, \ldots, f_k) : M \to \mathbb{R}^k$, and let $c \in \mathbb{R}^k$ be a regular value. Note that $M_c := F^{-1}(c)$ in invariant by the flow of H. We will show that M_c carries a natural invariant volume form.

a) Take a neighborhood $\mathcal{U}$ of M_c where $df_1, \ldots, df_k$ are linearly independent pointwise. Show that the Liouville volume form ($\Lambda_\omega = \omega^n/n!$) can be written in $\mathcal{U}$ as $\Lambda_\omega = df_1 \wedge \ldots \wedge df_k \wedge \sigma$, for some $\sigma \in \Omega^{2n-k}(M)$. We then define a volume form $\Lambda_c := \iota^*\sigma \in \Omega^{2n-k}(M_c)$, were $\iota : M_c \hookrightarrow M$ is the inclusion.

   Hint: find σ locally and use partition of unity.

b) Show that $df_1 \wedge \ldots \wedge df_k \wedge \mathcal{L}_{X_H}\sigma = 0$, and use this fact to see that we can write $\mathcal{L}_{X_H}\sigma = \sum_{i=1}^k df_i \wedge \rho_i$. Conclude that Λ_c is invariant by the flow of H.

c) Show that Λ_c does not depend on the choice σ.

Problem 2: Let M be a symplectic manifold, $\Psi = (\psi^1, \ldots, \psi^k) : M \to \mathbb{R}^k$ a smooth map, and c a regular value. Consider a submanifold $N = \Psi^{-1}(c) \hookrightarrow M$.

(a) Show that N is coisotropic if and only if $\{\psi^i, \psi^j\}|_N = 0$ for all $i, j = 1, \ldots, k$.

(b) Show that N is symplectic if and only if the matrix (c^{ij}), with $c^{ij} = \{\psi^i, \psi^j\}$, is invertible for all $x \in N$. In this case, verify that we have the following expression for the Poisson bracket $\{\cdot, \cdot\}_N$ on N (known *Dirac's bracket*):

$$\{f, g\}_N = (\{\tilde{f}, \tilde{g}\} - \sum_{ij} \{\tilde{f}, \psi^i\} c_{ij} \{\psi^j, \tilde{g}\})|_N,$$

where $(c_{ij}) = (c^{ij})^{-1}$, $f, g \in C^\infty(N)$, e $\tilde{f}, \tilde{g} \in C^\infty(M)$ are arbitrary extensions of f, g, respectively. *[Hint: we have $TM|_N = TN \oplus TN^\omega$, and projections $q_1 : TM|_N \to TN$ and $q_2 : TM|_N \to TN^\omega$; show that $X_f = q_1(X_{\tilde{f}})$, and verify that $q_2(Y) = \sum_{i,j} d\psi^i(Y) c_{ij} X_{\psi^j}$.]*

Problem 3: Let $D \subseteq TM$ be a vector subbundle, and let $\mathrm{Ann}(D) \subseteq T^*M$ be its annihilator. Show that D is involutive iff $\mathrm{Ann}(D)$ is a coisotropic submanifold of T^*M.

Problem 4: Consider a smooth map $\phi : Q_1 \to Q_2$, and let

$$R_\phi := \{((x, \xi), (y, \eta)) \mid y = \phi(x),\ \xi = (T\phi)^*\eta\} \subset T^*Q_1 \times T^*Q_2.$$

Verify that R_ϕ is a lagrangian submanifold of $T^*Q_1 \times \overline{T^*Q_2}$. Whenever ϕ is a diffeo, what is the relation between R_ϕ and the cotangent lift $\widehat{\phi}$?

Denote by $\Gamma_\phi \subset Q_1 \times Q_2$ the graph of ϕ. What is the relation between $N^*\Gamma_\phi$ (the conormal bundle of Γ_ϕ) and R_ϕ?

Problem 5: Let M be a manifold and $\omega \in \Omega^k(M)$. Suppose that $\pi : M \to B$ is a surjective submersion with connected fibers. We say that ω is *basic* (with respect to π) if there exists a form $\overline{\omega} \in \Omega^k(B)$ such that $\pi^*\overline{\omega} = \omega$.

(a) Show that ω is basic iff $i_X\omega = 0$ and $\mathcal{L}_X\omega = 0$ for all vector fields X tangent to the fibers of π. In particular, if ω is closed, show that it is basic if $\ker(T\pi) \subseteq \ker(\omega)$ (pointwise in M).

(b) Suppose that ω is a closed 2-form on M and $\ker(T\pi) = \ker(\omega)$. Show that $\omega = \pi^*\overline{\omega}$ and $\overline{\omega} \in \Omega^2(B)$ is symplectic.

(c) (Application to reduction) Let (M, ω) be a symplectic manifold and $\iota : N \hookrightarrow M$ a submanifold such that $D = TN \cap TN^\omega \subset TN$ has constant rank (e.g., N could be coisotropic). We saw in class that D is an integrable distribution (by Frobenius); suppose that the leafspace $B := N/\sim$ is smooth so that the natural projection $\pi : N \to B$ is a submersion. Show that B inherits a unique symplectic form ω_{red} with the property that $\pi^*\omega_{red} = \iota^*\omega$.