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Lista 3

Contents

Problem 1

Solution.

a. Note that a basis of T_x^*M is $df_1, \dots, df_k, x_{k+1}, \dots, x_{2n}$. Then any $2n$-form is a multiple of $df_1 \wedge \dots \wedge df_k \wedge dx_{k+1} \wedge \dots \wedge d_{2n}$.

$\square$

Problem 2 Let M be a symplectic manifold, $\Psi = (\psi^1, \dots, \psi^k) : M \to \mathbb{R}^k$ a smooth map, and c a regular value. Consider a submanifold $N = \Psi^{-1}(c) \hookrightarrow M$.

a. Show that N is coisotropic if and only if $\{\psi^i, \psi^j\}|_N = 0$ for all $i, j = 1, \dots, k$.

b. Show that N is symplect if and only if the matrix (c^{ij}), with $c^{ij} = \{\psi^i, \psi^j\}$, is invertible for all $x \in N$. In this case, verify that we have the following expression for the Poisson bracket $\{\cdot, \cdot\}_N$ on N (known as ***Dirac's bracket***):

$$\{f, g\}_N = \left(\{\tilde{f}, \tilde{g}\} = \sum_{ij} \{\tilde{f}, \tilde{g}\} c_{ij} \{\psi^j, \tilde{g}\}\right)\Bigg|_N$$

Solution.

a. Since M is symplectic we have a bundle isomorphism

$$\begin{aligned} \omega^\flat : TM &\longrightarrow T^*M \\ v &\longmapsto i_v \omega \end{aligned}$$

Then
$$TN^\omega = (\omega^\flat)^{-1}(\mathrm{Ann}(TN)).$$

Since M is the level set of a regular value, there are local coordinates of the form $(\psi^1, \dots, \psi^k, x^{k+1}, \dots, x^{2n})$. Vectors tangent to N are expressed only in terms of the vectors $\partial_{k+1}, \dots, \partial_{2n}$ and thus covectors that vanish on TN are those which vanish on $\partial_{k+1}, \dots, \partial_{2n}$. This means that a basis for $\mathrm{Ann}(TN)$ is given by $d\psi^1, \dots, d\psi^k$

(indeed, the canonical basic covectors for the coordinates ψ^i are the differentials $d\psi^i$—this can be checked using a change of coordinates matrix). These generators map to their hamiltonian vector fields under $(\omega^\flat)^{-1}$:

$$\left(\omega^\flat\right)^{-1}(d\psi^i) = X_{\psi^i}$$

So TN^ω is generated by the X_{ψ^i}. Notice that any vector $\nu \in TM$ is actually in TN iff $\alpha(\nu) = 0\ \forall \alpha \in \mathrm{Ann}(TN)$. Then we see that

$$\begin{aligned} TN^\omega \subset TM &\iff X_{\psi^i} \in TN \quad i = 1, \dots, k \\ &\iff d\psi^j(X_{\psi^i})|_N = 0 \quad i, j = 1, \dots, k \\ &\iff \omega(X_{\psi^i}, X_{\psi^j})|_N = 0 \quad i, j = 1, \dots, k \\ &\iff \{\psi^i, \psi^j\}|_N = 0 \quad i, j = 1, \dots, k \end{aligned}$$

b. The matrix (c^{ij}) determines the bundle isomorphism $\omega^\sharp$:

□

Problem 3

Solution. By an analogue argument to Problem 2a we know that $\mathrm{Ann}(D)$ is a coisotropic submanifold iff $\{f, g\} = 0$ for all $f, g \in I_{\mathrm{Ann}(D)}$. Now a vector field X corresponds naturally to an element of the double dual $\xi \in T^*(T^*(M))$ given by $\xi\eta(X) = \eta(X)$ for $\eta \in T^*M$. Notice that if $X \in D$ then $\xi \in \mathrm{Ann}(\mathrm{Ann}(D))$. □