

# Lista 1

## Geometria simplética

**Problem 1:** Let  $V$  be a symplectic vector space ( $\dim V = 2n$ ), and  $\Omega \in \Lambda^2 V^*$  be a skew-symmetric bilinear form. Show that  $\Omega$  is nondegenerate iff  $\Omega^n \neq 0$ .

*Solution.* I first tried to show that  $\Omega$  is degenerate iff  $\Omega^n = 0$ . Suppose there is a vector  $v_0$  such that  $\Omega(v_0, w) = 0$  for all  $w \in V$  and complete to a basis. Then for any  $v_1, v_2, v_3, v_4 \in V$  we have

$$(\Omega \wedge \Omega)(v_1, v_2, v_3, v_4) = \sum_{\sigma \in S_4} \text{sgn}(\sigma) \Omega(v_{\sigma(1)}, v_{\sigma(2)}) \Omega(v_{\sigma(3)}, v_{\sigma(4)}).$$

□

**Problem 2:** Let  $(V, \Omega)$  be a symplectic vector space, and let  $W \subseteq V$  be any linear subspace.

- Show that  $V_W = \frac{W}{W \cap W^\Omega}$  inherits a natural symplectic structure  $\Omega_W$  uniquely determined by the condition  $\pi^* \Omega_W = \Omega|_W$  (here  $\pi : W \rightarrow W/(W \cap W^\Omega)$  is the quotient projection).

(The space  $(V_W, \Omega_W)$  is called the **reduced space**.)

- Suppose that  $W$  is coisotropic, and let  $L \subset V$  be lagrangian. Show that the image of  $L \cap W$  via  $\pi : W \rightarrow V_W$  is lagrangian in the reduced space.

*Solution.*

- Define

$$\Omega_W([w_1], [w_2]) := \Omega(w_1, w_2)$$

for any equivalence classes  $[w_1], [w_2] \in V_W$ . Let's check that this is well defined. Suppose  $w'_1 \in [w_1]$ . Then  $w_1 - w'_1 \in W \cap W^\Omega$  so  $\Omega(w_1 - w'_1, w_2) = 0$  since  $w_2 \in W$  and  $w_1 - w'_1$  is, in particular, in  $W^\Omega$ . So  $\Omega(w_1, w_2) = \Omega(w'_1, w_2)$ . **Why not quotient only by  $W^\Omega$ ? Looks like I didn't use the  $W$  part...**

Recall that  $\pi^* \Omega_W(w_1, w_2) = \Omega_W([w_1], [w_2])$ . It is straightforward to check that  $\Omega_W$  is the only symplectic form on  $V_W$  satisfying  $\pi^* \Omega_W = \Omega|_W$ : if  $\Omega'_W$  is another such form, then  $\Omega_W([w_1], [w_2]) = \Omega|_W(w'_1, w'_2) = \Omega'_W([w_1], [w_2])$  for any  $w'_1 \in [w_1]$  and  $w'_2 \in [w_2]$ .

- Let's first check what is  $(\pi(L \cap W))^{\Omega_W}$ . We have

$$\begin{aligned} (\pi(L \cap W))^{\Omega_W} &= \{[v] \in V_W : \Omega_W([v], [w]) = 0 \ \forall [w] \in \pi(L \cap W)\} \\ &= \{[v] \in V_W : \Omega(v', w) = 0 \ \forall v' \in [v] \text{ and } \forall w \text{ s.t. } [w] \in \pi(L \cap W)\} \end{aligned}$$

In words, this is the set of classes whose representatives are  $\Omega$ -orthogonal to representatives of  $\pi(L \cap W)$ .

So if  $[v]$  is any such class,

so let  $[v] \in \pi(L \cap W)^{\Omega_W}$ . Let's check that  $[v]$  is also in  $\pi(L \cap W)$ , ie. that  $v \in L \cap W$ . Well,

If  $v' \in L$ , then  $\Omega(v, v') = 0$  since  $[v'] \in \pi(L \cap W)^{\Omega} \dots$  but what if  $v' \in L \setminus W$ ?

Let  $w$  be such that  $[w] \in \pi(L \cap W)$ . Then

$$\begin{aligned}\Omega_W([v], [w]) &= 0 \\ \implies \Omega(v, w) &= 0\end{aligned}$$

so  $v \in$

□

**Problem 3:** We saw in class that any symplectomorphism  $T : V_1 \rightarrow V_2$  defines a lagrangian subspace by its graph:  $\Gamma_T := \{(Tu, u) : u \in V_1\} \subset V_2 \oplus \bar{V}_1$ . (Recall that if  $(V, \Omega)$  is a svs,  $\bar{V}$  denotes  $(V, -\Omega)$ .) So we think lagrangian subspaces of  $V_2 \oplus \bar{V}_1$  a generalizations of symplectomorphisms. We now see how to generalize their composition.

Consider symplectic vector spaces  $V_1, V_2, V_3$  and  $E = V_3 \oplus \bar{V}_2 \oplus V_2 \oplus \bar{V}_1$ .

- Show that  $\Delta := \{(v_3, v_2, v_2, v_1) \in E\}$  is coisotropic in  $E$  and its reduction  $E_\Delta$  can be identified with  $V_3 \oplus \bar{V}_1$ .
- Given lagrangian subspaces  $L_1 \subset V_2 \oplus \bar{V}_1$  and  $L_2 \subset V_3 \oplus \bar{V}_2$ , define the **composition** of  $L_2$  and  $L_1$  by

$$L_2 \circ L_1 := \{(v_3, v_1) | \exists v_2 \in V \text{ s.t. } (v_3, v_2) \in L_2, (v_2, v_1) \in L_1\}.$$

Show that  $L_2 \circ L_1$  is a lagrangian subspace of  $V_3 \oplus \bar{V}_1$ . (Hint: show that the composition can be identified with the reduction of  $L_2 \times L_1 \subset E$  with respect to  $\Delta$ ).

- Let  $T_1 : V_1 \rightarrow V_2$  and  $T_2 : V_2 \rightarrow V_3$  be symplectomorphisms. Show that  $\Gamma_{T_2 \circ T_1} = \Gamma_{T_2} \circ \Gamma_{T_1}$ .

*Solution.*

- Let  $v := (v_3, v_2, v_2', v_1) \in \Delta^{\Omega_E} \subset E$  where  $\Omega_E = \Omega_1 \oplus -\Omega_2 \oplus \Omega_2 \oplus -\Omega_1$  is the symplectic form on  $E$ . We wish to show that  $v \in \Delta$ , which only means that  $v_2 = v_2'$ . So let  $\tilde{v} = (v_3, v_2, v_2, v_1)$  and  $\hat{v} = (v_3, v_2', v_2', v_1)$  which are both in  $\Delta$ . Then we have

$$\begin{aligned}0 &= \Omega_E(v, \tilde{v}) \\ &= \Omega_1(v_3, v_3) - \Omega_2(v_2, v_2) + \Omega_2(v_2', v_2) - \Omega_1(v_1, v_1)\end{aligned}$$

and likewise

$$\begin{aligned}0 &= \Omega_E(v, \hat{v}) \\ &= \Omega_1(v_3, v_3) - \Omega_2(v_2, v_2') + \Omega_2(v_2', v_2') - \Omega_1(v_1, v_1).\end{aligned}$$

Subtracting,

$$0 = -\Omega_2(v_2, v_2) + \Omega_2(v_2, v'_2) + \Omega_2(v'_2, v_2) - \Omega_2(v'_2, v'_2)$$

And by linearity,

$$\begin{aligned} 0 &= -\Omega_2(-v_2 + v_2, -v_2 + v'_2) + \Omega_2(v'_2 - v'_2, v_2 - v'_2) \\ &\implies \Omega_2(0, -v_2 + v'_2) = \Omega_2(0, v_2 - v'_2) \end{aligned}$$

and the fact that  $v_2 = v'_2$  just follows from nondegeneracy.

□

**Problem 4:** Let  $(V, J)$  be a complex vector space, let  $\Omega$  be a symplectic structure on  $V$ . Show that  $J$  and  $\Omega$  are compatible iff there exists a hermitian inner product  $h : V \times V \rightarrow \mathbb{C}$  such that  $\Omega$  is its imaginary part. Show that any (complex) orthonormal basis of  $(V, h)$  can be extended to a symplectic basis of  $(V, \Omega)$ .

*Solution.* First suppose that  $J$  and  $\Omega$  are compatible, ie.,  $g(u, v) := \Omega(u, Jv)$  is an inner product. Define  $h(u, v) = g(u, v) + i\Omega(u, v)$ . Then  $h$  is the required hermitian inner product. Indeed:

1. The properties  $h(u_1 + u_2, v) = h(u_1, v) + h(u_2, v)$  and  $h(u, v_1 + v_2) = h(u, v_1) + h(u, v_2)$  follows easily from linearity of  $g$  and  $\Omega$ .
2.  $h(\lambda u, v) = \lambda h(u, v)$  follows again from linearity of  $g$  and  $\Omega$ .
3. The property  $h(u, \lambda v) = \bar{\lambda} h(u, v)$  follows easily from 2. and 4. since

$$\begin{aligned} h(u, \lambda v) &= \overline{h(\lambda v, u)} \\ &= \bar{\lambda} \overline{h(v, u)} \\ &= \bar{\lambda} h(u, v) \end{aligned}$$

4.  $h(u, v) = \overline{h(v, u)}$  is clear by anti-symmetry of  $\Omega$ :

$$\begin{aligned} h(u, v) &= g(u, v) + i\Omega(u, v) \\ &= g(v, u) - i\Omega(v, u) \\ &= \overline{h(v, u)} \end{aligned}$$

For the converse suppose that  $h$  is an hermitian inner product such that  $\Omega$  is its imaginary part. Then  $g(u, v) := \Omega(u, Jv)$  is an inner product:

1. Linearity of  $g$  is immediate from linearity of  $\Omega$  and  $J$ .

2. Symmetry follows from

$$\begin{aligned}
g(u, v) &= \Omega(u, Jv) \\
&= \Omega(-J^2u, Jv) \\
&= -\Omega(J^2u, Jv) \\
&= \Omega(Jv, J^2u) \\
&= \Omega(Jv, u) \\
&= \Omega(-v, g(v, u)) &= -\Omega(Jv, u) \\
&= -\Omega(Ju, v) &= \Omega(v, Ju)
\end{aligned}$$

3. For positive-definiteness let  $u \neq 0$ . Then

$$g(u, u) = \Omega(u, Ju)$$

□

**Problem 5:** Consider the symplectic vector space  $(\mathbb{R}^{2n}, \Omega_0)$ , where  $\Omega_0(u, v) = -u^T J_0 v$ . Check that its group of linear symplectomorphisms is given by  $\text{Sp}(2n) = \{A \in \text{GL}(2n) : A^T J_0 A = J_0\}$ . Show that  $\text{Sp}(2n)$  is a smooth submanifold of  $\text{GL}(2n)$  and that its tangent space at the identity  $I \in \text{GL}$  is given by  $T_I \text{Sp}(2n) = \{A : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} | A^T J_0 + J_0 A = 0\}$ . Conclude that  $\text{Sp}(2n)$  has dimension  $2n^2 + n$ . Verify also that  $\text{Sp}(2n)$  is not compact.

**Problem 6:** Consider the standard compatible triple  $(\Omega_0, J_0, g_0)$  on  $\mathbb{R}^{2n}$ . Let  $O(2n)$  be the linear orthogonal group of  $\mathbb{R}^{2n}$  (i.e., linear transformations preserving the canonical inner product  $g_0$ ), and let  $\text{Sp}(2n)$  be the symplectic linear group. Through the identification  $\mathbb{R}^{2n} \cong \mathbb{C}^n$  (as complex vector spaces), we may see  $\text{Sp}(2n)$  as a subgroup of  $\text{GL}(n, \mathbb{C})$  (the group of linear automorphisms of  $\mathbb{C}^n$ ) as a subgroup of  $\text{GL}(2n, \mathbb{R})$ : a complex matrix  $A + iB$  is identified with the real  $2n \times 2n$  matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

Let now  $U(n) \subset \text{GL}(n, \mathbb{C})$  be the group of linear transformation preserving the natural hermitian inner product of  $\mathbb{C}^n$ . Show that the intersection of any two of the groups

$$\text{Sp}(2n), O(2n), \text{GL}(n, \mathbb{C}) \subset \text{GL}(2n, \mathbb{R})$$

is  $U(n)$ .

**Problem 7:** Let  $(V, \Omega)$  be a symplectic vector space, let  $W \subseteq V$ . Let  $J$  be a  $\Omega$ -compatible complex structure and  $g$  the corresponding inner product. Verify that  $J(W^\Omega) = W^{\perp_g}$ .

- Use this fact to show that any coisotropic subspace of  $V$  has an isotropic complement. In particular, any lagrangian subspace  $L \subset V$  has a lagrangian complement  $L'$ ,  $V = L \oplus L'$ .
- Show that there is a natural identification  $L' \cong L^*$ , that induces a symplectomorphism  $V \cong L \oplus L^*$  (where  $L \oplus L^*$  has the natural symplectic structure  $((\ell, \alpha), (\ell', \alpha')) \mapsto \alpha(\ell') - \alpha'(\ell)$ ).

*Solution.* First let's check that  $J(W^\Omega) = W^{\perp_g}$ . Indeed,

$$\begin{aligned} J(W^\Omega) &= \{Jv : v \in W^\Omega\} \\ &= \{Jv : \Omega(v, w) = 0 \ \forall w \in W\} \\ &= \{Jv : -\Omega(w, v) \ \forall w \in W\} \\ &= \{Jv : \Omega(w, -v) \ \forall w \in W\} \\ &= \{Jv : \Omega(w, J^2v) \ \forall w \in W\} \end{aligned}$$

re-write  $Jv := \tilde{v}$  using that  $J$  is bijective:

$$\begin{aligned} J(W^\Omega) &= \{\tilde{v} \in V : \Omega(w, J\tilde{v}) = 0 \ \forall w \in W\} \\ &= \{\tilde{v} \in V : g(\tilde{v}, w) = 0 \ \forall w \in W\} \\ &= W^{\perp_g} \end{aligned}$$

□

**Bonus problem:**