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Lista 1

Problem 1: Let V be a symplectic vector space ($\dim V = 2n$), and $\Omega \in \Lambda^2 V^*$be a skew-symmetric bilinear form. Show that Ω is nondegenerate iff $\Omega^n \neq 0$.

Solution. I first tried to show that Ω is degenerate iff $\Omega^n = 0$. Suppose there is a vector v_0 such that $\Omega(v_0, w) = 0$ for all $w \in V$ and complete to a basis. Then for any $v_1, v_2, v_3, v_4 \in V$ we have

$$(\Omega \wedge \Omega)(v_1, v_2, v_3, v_4) = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma)\Omega(v_{\sigma(1)}, v_{\sigma(2)})\Omega(v_{\sigma(3)}, v_{\sigma(4)}).$$

Finally I found this proposition in Lee, Intro. Smooth Manifolds. $\square$

Problem 2: Let (V, Ω) be a symplectic vector space, and let $W \subseteq V$ be any linear subspace.

a. Show that $V_W = \frac{W}{W \cap W^\Omega}$ inherits a natural symplectic structure Ω_W uniquely determined by the condition $\pi^*\Omega_W = \Omega|_W$ (here $\pi : W \to W/(W \cap W^\Omega)$ is the quotient projection).

   (*The space* (V_W, Ω_W) *is called the* ***reduced space****.*)

b. Suppose that W is coisotropic, and let $L \subset V$ be lagrangian. Show that the image of $L \cap W$ via $\pi : W \to V_W$ is lagrangian in the reduced space.

Solution.

a. Define
$$\Omega_W([w_1], [w_2]) := \Omega(w_1, w_2)$$
for any equivalence classes $[w_1], [w_2] \in V_W$. Let's check that this is well defined. Suppose $w_1' \in [w_1]$. Then $w_1 - w_1' \in W \cap W^\Omega$ so $\Omega(w_1 - w_1', w_2) = 0$ since $w_2 \in W$ and $w_1 - w_1'$ is, in particular, in W^Ω. So $\Omega(w_1, w_2) = \Omega(w_1', w_2)$. Why not quotient only by W^Ω? Looks like I didn't use the W part...

   Recall that $\pi^*\Omega_W(w_1, w_2) = \Omega_W([w_1], [w_2])$. It is straightforward to check that Ω_W is the only symplectic form on V_W satisfying $\pi^*\Omega_W = \Omega|_W$: if Ω_W' is another such form, then $\Omega_W'([w_1], [w_2]) = \Omega|_W(w_1', w_2') = \Omega_W([w_1], [w_2])$ for any $w_1' \in [w_1]$ and $w_2' \in [w_2]$.

b. Since W is coisotropic, we have $V_W = W/W^\Omega$ and $L^\Omega = L$. Then
$$\begin{aligned}\pi(L \cap W)^\Omega &= \{[w] \in W/W^\Omega : \Omega(w, \ell) = 0 \ \forall \ell \in L \cap W\} \\ &= \{[w] \in W/W^\Omega : w \in (L \cap W)^\Omega\Omega\}\end{aligned}$$

Moreover,

$$(L \cap W)^{\Omega} = L^{\Omega} + W^{\Omega} = L + W^{\Omega} \subseteq L + W$$

since L is lagrangian and W coisotropic. So,

$$\begin{aligned}
\pi(L \cap W)^{\Omega} &\subseteq \{[w] \in W/W^{\Omega} : w \in (L + W) \cap W\} \\
&= \{[w] \in W/W^{\Omega} : w \in (L \cap W)\} \\
&= \pi(L \cap W)
\end{aligned}$$

□

Problem 3: We saw in class that any symplectomorphism $T : V_1 \to V_2$ defines a lagrangian subspace by its graph: $\Gamma_T := \{(Tu, u) : u \in V_1\} \subset V_2 \oplus \overline{V}_1$. (Recall that if (V, Ω) is a svs, $\overline{V}$ denotes $(V, -\Omega)$.) So we think lagrangian subspaces of $V_2 \oplus \overline{V}_1$ a generalizations of symplectomorphisms. We now see how to generalize their composition.

Consider symplectic vector spaces V_1, V_2, V_3 and $E = V_3 \oplus \overline{V}_2 \oplus V_2 \oplus \overline{V}_1$.

a. Show that $\Delta := \{(v_3, v_2, v_2, v_1) \in E\}$ is coisotropic in E and its reduction E_{Δ} can be identified with $V_3 \oplus \overline{V}_1$.

b. Given lagrangian subspaces $L_1 \subset V_2 \oplus \overline{V}_1$ and $L_2 \subset V_3 \oplus \overline{V}_2$, define the ***composition*** of L_2 and L_1 by

$$L_2 \circ L_1 := \{(v_3, v_1) | \exists v_2 \in V \text{ s.t. } (v_3, v_2) \in L_2, (v_2, v_1) \in L_1\}.$$

Show that $L_2 \circ L_1$ is a lagrangian subspace of $V_3 \oplus \overline{V}_1$. (*Hint: show that the composition can be identified with the reduction of* $L_2 \times L_1 \subset E$ *with respect to* Δ).

c. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be symplectomorphisms. Show that $\Gamma_{T_2 \circ T_1} = \Gamma_{T_2} \circ \Gamma_{T_1}$.

Solution.

a. Let $v := (v_3, v_2, v_2', v_1) \in \Delta^{\Omega_E} \subset E$ where $\Omega_E = \Omega_1 \oplus -\Omega_2 \oplus \Omega_2 \oplus -\Omega_1$ is the symplectic form on E. We wish to show that $v \in \Delta$, which only means that $v_2 = v_2'$. So let $\tilde{v} = (v_3, v_2, v_2, v_1)$ and $\hat{v} = (v_3, v_2', v_2', v_1)$ which are both in Δ. Then we have

$$\begin{aligned}
0 &= \Omega_E(v, \tilde{v}) && (1) \\
&= \Omega_1(v_3, v_3) - \Omega_2(v_2, v_2) + \Omega_2(v_2', v_2) - \Omega_1(v_1, v_1) && (2)
\end{aligned}$$

and likeways

$$\begin{aligned}
0 &= \Omega_E(v, \hat{v}) && (3) \\
&= \Omega_1(v_3, v_3) - \Omega_2(v_2, v_2') + \Omega_2(v_2', v_2') - \Omega_1(v_1, v_1). && (4)
\end{aligned}$$

Substracting (3)−(1)

$$0 = -\Omega_2(v_2, v_2) + \Omega_2(v_2, v_2') + \Omega_2(v_2', v_2) - \Omega_2(v_2', v_2')$$

And by linearity,

$$
\begin{aligned}
& 0 = -\Omega_2(v_2, -v_2 + v_2') + \Omega_2(v_2', v_2 - v_2') \\
\implies & 0 = \Omega_2(v_2,) + \Omega_2(v_2',)
\end{aligned}
$$

and it follows that $v_2 = v_2'$ from nondegeneracy.

Now let's try to construct an isomorphism $E_\Delta = V_3 \oplus \overline{V}_1$. Consider

$$
\begin{aligned}
\varphi : E &\longrightarrow V_3 \oplus \overline{V}_1 \\
(v_3, v_2, v_2', v_1) &\longmapsto (v_3, v_1)
\end{aligned}
$$

which is clearly surjective and not injective, so perhaps its kernel is $\Delta \cap \Delta^\Omega$. But $\ker \varphi = \{(0, v_2, v_2', 0)\}$, so unfortunately no.

But perhaps we can construct some other map. Let's try

$$
\begin{aligned}
\varphi : E &\longrightarrow V_3 \oplus \overline{V}_1 \\
(v_3, v_2, v_2', v_1) &\longmapsto
\end{aligned}
$$

b. It looks like $L_2 \circ L_1$ is very much like E_Δ from the last exercise. $L_2 \circ L_1$ is *strictly* contained in $V_3 \oplus \overline{V}_1 \cong E_\Delta$.

   Ok let's have a go at the hint. Perhaps $(L_2 \times L_1)_\Delta = L_2 \circ L_1$ and it is lagrangian. That'd be great. OK let's compute it. This is how to do it:

$$
\begin{aligned}
\varphi : L_2 \times L_1 &\longrightarrow V_2 \circ V_1 \\
(v_2, v_1, v_3, v_2) &\longmapsto
\end{aligned}
$$

□

Problem 4: Let (V, J) be a complex vector space, let Ω be a sympletic structure on V. Show that J and Ω are compatible iff there exists a hermitian inner product $h : V \times V \to \mathbb{C}$ such that Ω is its imaginary part. Show that any (complex) orthonormal basis of (V, h) can be extended to a symplectic basis of (V, Ω).

Solution. First suppose that J and Ω are compatible, ie., $g(u, v) := \Omega(u, Jv)$ is an inner product. Define $h(u, v) = g(u, v) + i\Omega(u, v)$. Then h is the required hermitian inner product. Indeed:

1. The properties $h(u_1 + u_2, v) = h(u_1, v) + h(u_1, v)$ and $h(u, v_1 + v_2) = h(u, v_1) + h(u, v_2)$ follows easily from linearity of g and Ω.
2. $h(\lambda u, v) = \lambda h(u, v)$ follows again from linearity of g and Ω.

3. The property $h(u, \lambda v) = \bar{\lambda} h(u, v)$ follows easily from 2. and 4. since

$$\begin{aligned} h(u, \lambda v) &= \overline{h(\lambda v, u)} \\ &= \bar{\lambda}\overline{h(v, u)} \\ &= \bar{\lambda} h(u, v) \end{aligned}$$

4. $h(u, v) = \overline{h(v, u)}$ is clear by anti-symmetry of Ω:

$$\begin{aligned} h(u, v) &= g(u, v) + i\Omega(u, v) \\ &= g(v, u) - i\Omega(v, u) \\ &= \overline{h(v, u)} \end{aligned}$$

For the converse suppose that h is an hermitian inner product such that Ω is its imaginary part. Then $g(u, v) := \Omega(u, Jv)$ is an inner product:

1. Linearity of g is immediate from linearity of Ω and J.
2. Symmetry follows from

$$\begin{aligned} g(u, v) &= \Omega(u, Jv) \\ &= \Omega(-J^2 u, Jv) \\ &= -\Omega(J^2 u, Jv) \\ &= \Omega(Jv, J^2 u) \\ &= \Omega(v, Ju) \\ &= g(v, u) \end{aligned}$$

provided $\Omega(u, v) = \Omega(Ju, Jv)$. This holds since Ω is the imaginary part of h identifying J with multiplication by i:

$$\begin{aligned} \Omega(Ju, Jv) &= \mathrm{Im}(h(Ju, Jv)) \\ &= \mathrm{Im}\, h(iu, iv) \\ &= \mathrm{Im}(i\bar{i} h(u, v)) \\ &= \mathrm{Im}(h(u, v)) \\ &= \Omega(u, v). \end{aligned}$$

3. For positive-definiteness let $u \neq 0$. Then

$$\begin{aligned} g(u, u) &= \Omega(u, Ju) \\ &= \mathrm{Im}(h(u, Ju)) \\ &= \mathrm{Im}(h(u, iu)) \\ &= \mathrm{Im}(ih(u, u)) > 0 \end{aligned}$$

since $h(u, u) > 0$.

$\square$

Problem 5: Consider the symplectic vector space $(\mathbb{R}^{2n}, \Omega_0)$, where $\Omega_0(u, v) = -u^{\mathbf{T}} J_0 v$. Check that its group of linear symplectomorphisms is given by $\mathrm{Sp}(2n) = \{A \in \mathrm{GL}(2n) : A^{\mathbf{T}} J_0 A = J_0\}$. Show that $\mathrm{Sp}(2n)$ is a smooth submanifold of $\mathrm{GL}(2n)$ and that its tangent space at the identity $I \in \mathrm{GL}(2n)$ is given by $T_I \,\mathrm{Sp}(2n) = \{A : \mathbb{R}^{2n} \to \mathbb{R}^{2n} | A^{\mathbf{T}} J_0 + J_0 A = 0\}$. Conclude that $\mathrm{Sp}(2n)$ has dimension $2n^2 + n$. Verify also that $\mathrm{Sp}(2n)$ is not compact.

Solution. Suppose that A is a linear symplectomorphism of $(\mathbb{R}^{2n}, \Omega_0)$. Then $A^*\Omega_0 = \Omega_0$ so

$$A^*\Omega_0(u, v) = \Omega_0(Au, Av) = -(Au)^{\mathbf{T}} J_0(Av) = -u^{\mathbf{T}} A^{\mathbf{T}} J_0(Av)$$

is equal to

$$\Omega_0(u, v) = -u^{\mathbf{T}} J_0 v$$

In terms of usual dot product of $\mathbb{R}^{2n}$, which we can denote by $\langle \cdot, \cdot \rangle$ momentarily, this means that

$$\begin{aligned} &\left\langle -u^{\mathbf{T}}, A^{\mathbf{T}} J_0 A v \right\rangle = \left\langle -u^{\mathbf{T}}, J_0 v \right\rangle \\ \iff &\left\langle -u^{\mathbf{T}}, A^{\mathbf{T}} J_0 A v - J_0 v \right\rangle = 0 \end{aligned}$$

for all $u \in \mathbb{R}^{2n}$, which means that $A^{\mathbf{T}} J_0 A v = J_0 v$ since dot product is nondegenerate. For the converse, if $A^{\mathbf{T}} J_0 A = J_0$, we have

$$\Omega(u, v) = -u^{\mathbf{T}} J_0 v = -u^{\mathbf{T}} A^{\mathbf{T}} J_0 A v = -(Au)^{\mathbf{T}} J(Av) = \Omega(Au, Av_0) = A^*\Omega(uv).$$

Let's try to show that $\mathrm{Sp}(2n)$ is the inverse image of a regular value of some submersion $\mathrm{GL}(2n) \to \mathbb{R}$. I think the determinant of J_0 is 1, so perhaps the map

$$\begin{aligned} D : \mathrm{GL}(2n) &\longrightarrow \mathbb{R} \\ A &\longmapsto \det(A^{\mathbf{T}} J_0 A) \end{aligned}$$

has inverse image of 1 equal to $\mathrm{Sp}(2n)$. It is clear that $\mathrm{Sp}(2n) \subset D^{-1}(1)$. Now suppose $A \in \mathrm{GL}(2n)$ is in $D^{-1}(1)$, but then it's not necessarily true that $A^{\mathbf{T}} J_0 A = J_0$ because there are so many matrices with determinant 1 that are not J_0.

Maybe the determinant won't work. But what about

$$\begin{aligned} D : \mathrm{GL}(2n) &\longrightarrow \mathrm{GL}(2n) \\ A &\longmapsto A^{\mathbf{T}} J_0 A \end{aligned}$$

maybe this map is a submersion at every point of $D^{-1}(J_0) = \mathrm{Sp}(2n)$. It really looks like a submersion at every point: it's only a composition of linear isomorphisms... of course its derivative is surjective, right? But something seems to be wrong because then it would be a sumbanifold of dimension $2n - 2n = 0$...

Let's have a look at the tangent space at the identity. Suppose that $V \in T_I \,\mathrm{Sp}(2n)$. I'm not sure how to continue here...

Following Misha's slides, it could be shown that $\mathrm{Sp}(2n)$ is the exponent of its Lie algebra (ie. the tangent space of the identity), and that this Lie algebra is $W := \{A \in \mathrm{GL}(2n) :$

$A^T J_0 + J_0 A = 0\}$. This suggests that $\exp W = \mathrm{Sp}(2n)$. But then we need a theorem saying that if the exponent of a subset of endomorphisms is a Lie group then such a subspace is its Lie algebra. This can be found in Misha's notes as follows:

Exponent is a map

$$\begin{aligned} \exp : \mathrm{End}\, V &\longrightarrow \mathrm{End}\, V \\ A &\longmapsto \sum_{n=0}^{\infty} \frac{A^n}{n!} \end{aligned}$$

and its differential at $0 \in \mathrm{End}\, V$ is the identity. But why... we have

$$d_0 \exp = \sum_{n=1}^{\infty} \frac{A^n}{(n-1)!}$$

I don't think it's so straightforward to check that this is the identity so let's just suppose it is true.

But what does it tell us that this map is the identity? Well, that $\exp(W)$ is invertible around 0, but not only that: notice that $\exp(0) = I \in \mathrm{GL}(2n)$ so $d_0 \exp : T_0 \mathrm{GL}(2n) \cong \mathrm{GL}(2n) \to T_I \mathrm{Sp}(2n)$ is an isomorphism as required, provided of course that $\exp(W) = \mathrm{Sp}(2n)$. So let's check that.

But that's nowhere near obvious:

$$\exp(W) = \left\{ \sum_{n=0}^{\infty} \frac{A^n}{n!} : A^T J_0 + J_0 A = 0 \right\} \quad \overset{?}{=} \quad \{A \in \mathrm{GL}(2n) : A^T J_0 A = J_0\} = \mathrm{Sp}(2n)$$

Let's just go, let $A \in W$ and let's check wether $\exp(A)^T J_0 \exp(A) = J_0$. It looks like transpose of exponent is exponent of transpose, but what is $\exp(J_0)$? This looks random...

□

Problem 6: Consider the standard compatible triple (Ω_0, J_0, g_0) on $\mathbb{R}^{2n}$. Let $O(2n)$ be the linear orthogonal group of $\mathbb{R}^{2n}$ (i.e., linear transformations preserving the canonical inner product g_0), and let $\mathrm{Sp}(2n)$ be the symplectic linear group. Through the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ (as complex vector spaces), we may see $\mathrm{GL}(n, \mathbb{C})$ (the group of linear automorphisms of $\mathbb{C}^n$) as a subgroup of $\mathrm{GL}(2n, \mathbb{R})$: a complex matrix $A + iB$ is identified with the real $2n \times 2n$ matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

Let now $U(n) \subset \mathrm{GL}(n, \mathbb{C})$ be the group of linear transformation preserving the natural hermitian inner product of $\mathbb{C}^n$. Show that the intersection of any two of the groups

$$\mathrm{Sp}(2n), O(2n), \mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$$

is $U(n)$.

Solution.

Since the standard hermitian product of $\mathbb{C}^n$ is given by $h_0 = g_0 + i\Omega_0$, it is immediate that a transformation $A \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$ preserves h_0 and conversely:

$$A^*h = A^*(g + i\Omega) = A^*g + iA^*\Omega = g + i\Omega = h$$

provided that the pullback is complex-linear.

For the next item recall that

$$\mathrm{O}(2n) = \{A \in \mathrm{GL}(2n) : A^{\mathbf{T}}A = I\}, \qquad \mathrm{GL}(n, \mathbb{C}) = \{A \in \mathrm{GL}(2n) : AJ_0 = J_0A\}$$

again identifying J_0 with multiplication by i. Observe that this implies that $A \in \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) \implies A \in \mathrm{Sp}(2n)$ since

$$A^{\mathbf{T}}J_0A = A^{\mathbf{T}}AJ_0 = J_0.$$

Likeways we see that $A \in \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) \implies \mathrm{O}(2n)$ since

$$A^{\mathbf{T}}J_0A = J_0 \iff A^{\mathbf{T}}AJ_0 = J_0 \iff A^{\mathbf{T}}A = I$$

since J_0 is invertible. Going back to the initial argument for matrices in $\mathrm{Sp}(2n) \cap A \in \mathrm{O}(2n)$, we see that in both cases $A \in \mathrm{U}(n)$.

For the converse notice that it is also true that $A \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n) \implies A \in \mathrm{GL}(n, \mathbb{C})$ since

$$J_0 = A^{\mathbf{T}}J_0A = A^{-1}J_0A \iff J_0A = AJ_0.$$

$\square$

Problem 7: Let (V, Ω) be a symplectic vector space, let $W \subseteq V$. Let J be a Ω-compatible complex structure and g the corresponding inner product. Verify that $J(W^\Omega) = W^{\perp_g}$.

a. Use this fact to show that any coisotropic subspace of V has an isotropic complement. In particular, any lagrangian subspace $L \subset V$ has a lagrangian complement L', $V = L \oplus L'$.

b. Show that there is a natural identification $L' \cong L^*$, that induces a symplectomorphism $V \cong L \oplus L^*$, where $L \oplus L^*$ has the natural symplectic structure

$$((\ell, \alpha), (\ell', \alpha')) \longmapsto \alpha(\ell') - \alpha'(\ell)$$

.

Solution. First let's check that $J(W^\Omega) = W^{\perp_g}$. Indeed,

$$\begin{aligned}
J(W^\Omega) &= \{Jv : v \in W^\Omega\} \\
&= \{Jv : \Omega(v, w) = 0 \ \forall w \in W\} \\
&= \{Jv : -\Omega(w, v) \ \forall w \in W\} \\
&= \{Jv : \Omega(w, -v) \ \forall w \in W\} \\
&= \{Jv : \Omega(w, J^2 v) \ \forall w \in W\}
\end{aligned}$$

re-write $Jv := \tilde{v}$ using that J is bijective:

$$\begin{aligned}
J(W^{\Omega)} &= \{\tilde{v} \in V : \Omega(w, J\tilde{v}) = 0 \ \forall w \in W\} \\
&= \{\tilde{v} \in V : g(\tilde{v}, w) = 0 \ \forall w \in W\} \\
&= W^{\perp_g}
\end{aligned}$$

a. Let W be any coisotropic subspace. We know that $V = W \oplus W^{\perp_g}$ (supposing that V is finite-dimensional), so it remains to show that $W^{\perp_g}$ is isotropic. Since W is coisotropic, we have

$$W^{\Omega} \subseteq W \implies J(W^{\Omega}) = W^{\perp_g} \subseteq JW$$

so it would be enough to show that

$$JW \subseteq \left(J(W^{\Omega})\right)^{\Omega} = (W^{\perp_g})^{\Omega}.$$

Let $w \in W$ and $w' \in W^{\Omega}$, so that $Jw \in JW$ and $Jw' \in J(W^{\Omega})$. Then

$$\Omega(Jw, Jw') = \Omega(w, w') = 0,$$

which shows that $JW \subseteq \left(J(W^{\Omega})\right)^{\Omega}$.

b.

$\square$

Bonus problem: