Lista 2

Problem 1 Verify (and justify) wether or not the following manifolds admit a symplectic structure: $S^1 \times S^3$, $\mathbb{R} \times S^3$ $\mathbb{R}^3 \times S^3$, $\mathbb{T}^3 \times S^3$.

Solution.

a.

b. Considere o mapa

$$\begin{aligned}(0,+\infty)\times\mathbb{S}^3 &\longrightarrow \mathbb{R}^4\\ (t,v) &\longmapsto tv\end{aligned}$$

com inversa $x \mapsto \left(|x|, \frac{x}{|x|}\right)$. Ele é um difeomorfismo, assim o pullback da forma simplética canónica em $\mathbb{R}^4$ induz uma estrutura simplética em $\mathbb{R}\times S^3 \overset{\text{dif}}{\cong} (0,+\infty)\times S^3$.

c. Considerando que S^3 é a esfera unitária nos quatérnios, os campos vetoriais $z\mapsto iz$, $z\mapsto jz$ e $z\mapsto kz$ são uma base global, de forma que S^3 é paralelizável. Sendo uma 3-variedade, temos que $TS^3\cong\mathbb{R}^3\times S^3$. Em aula vimos que todo fibrado tangente é uma variedade simplética usando o pullback da forma canónica no fibrado cotangente.

d.

□

Problem 2 Show that the tautological 1-form $\Omega^1(T^*Q)$ is *uniequely characterized* by the following property: for any 1-form $\mu\in\Omega^1(Q)$,

$$\mu^*\alpha = \mu$$

where on the left-hand side we view μ as a map $\mu : Q \to T^*Q$.

Solution. (After several tries, consulted StackExchange) Simply notice that if β is another 1-form on T^*Q such that $\mu^*\beta=\mu$ for all $\mu\in\Omega^1(Q)$, then $\mu^*(\alpha-\beta)=0$. Further, if any $\theta\in\Omega^1(T^*Q)$ satisfies $\mu^*\theta=0$ for all $\mu\in\Omega^1(Q)$, it must be identiacally zero. This follows since for any $w\in T(T^*Q)$ we can find a form μ and a vector v such that $\mu_* v = w$. □

Solution. Temos que se $v\in T_x(Q)$ é um vetor tangente a Q no ponto $x\in Q$,

$$(\mu^*\alpha)_x v = \alpha_{\mu_x}(\mu_* v)$$

onde $\mu_*\nu \in TT^*Q$, pois $\mu_* : TQ \to TT^*Q$. Lembre que $\alpha_\eta = \pi_\eta^*\eta$ para qualquer $\eta \in \Omega^1(Q)$. Assim, podemos escrever

$$\begin{aligned}\alpha_{\mu_x}(\mu_*\nu) &= \pi^*_{\mu_x}(\mu_*\nu)\\ &= \mu_x(\pi_*\mu_*\nu)\\ &= \mu_x(\nu)\end{aligned}$$

já que μ comuta com a projeção. □

Problem 3

Solution.

a. Seguindo a notação do problema anterior, note que

$$(F^*\alpha)\nu = -\alpha(F_*\nu) = -\pi^*_{\xi_{F_*\nu}}\xi_{F_*\nu}(F_*\nu) = \xi_{F_*\nu}(\pi_*F_*\nu)$$

□

Problem 6 Let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form. For $f \in C^\infty(M)$, let $X_f \in \mathfrak{X}(M)$ be defined by $i_{X_f}\omega = df$. Consider the bracket $\{f, g\} := \omega(X_g, X_f)$. Verify that $d\omega = 0$ if and only if $\{\cdot,\cdot\}$ satisfies the Jacobi identity.

Solution. (Taken from StackExhange). Using that $X_{\{g,h\}} = -[X_g, X_h]$ (Prop 22.19, Lee), we may write the Jacobi identity as

$$\begin{aligned}0 &= \{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\}\\ &= \omega(X_f,[X_h,X_g]) + \omega(X_h,[X_g,X_f]) + \omega(X_g,[X_f,X_f])\end{aligned}$$

Next we use the coordinate-free expression of the exterior derivative of a 3-form to write

$$\begin{aligned}d\omega(X_f,X_g,X_h) =&X_f\omega(X_g,X_h) - X_g\omega(X_f,X_h) + X_h\omega(X_f,X_g)\\ &- \omega([X_f,X_g],X_h) + \omega([X_f,X_h],X_g) - \omega([X_g,X_h],X_f)\end{aligned}$$

Finally we use that $X_f\omega(X_g,X_h) = \omega(X_f,[X_g,X_h])$ (why?) to see that, in the last equation, the first and second row on the right hand side are actually the same, and each of them equals the expression of the Jacobi identity. □

Problem 7 Consider symplectic manifolds (M_i, ω_i), with Poisson bracket $\{\cdot,\cdot\}_i$, $i = 1, 2$, and let $\phi : M_1 \to M_2$ be a smooth map.

a. Prove that if $\phi$is a diffeomorphism, then it is a Poisson map ($\{\phi^*f, \phi^*g\}_1 = \phi^*(\{f, g\}_2)$ for all $f, g \in C^\infty(M_2)$) if and only if $\phi^*\omega_2 = \omega_1$.

b. Find examples of M_1, M_2 and $\phi : M_1 \to M_2$ such that (1) ϕ is not a Poisson map but soes not satisfy $\phi^*\omega_2 = \phi_1$, (2) ϕ satisfies $\phi^*\omega_2 = \omega_1$ but it is not a Poisson map.

Solution.

a. Suppose that $\omega_1 = \phi^*\omega_2$. Then

$$\begin{aligned}\{\phi^*f, \phi^*g\} &= \omega_1(X_{\phi^*f}, X_{\phi^*g})\\ &= \phi^*\omega_2(X_{\phi^*f}, X_{\phi^*g})\\ &= \omega_2(\phi_*X_{\phi^*f}, \phi_*X_{\phi^*g})\\ &= \omega_2(X_f, X_g)\\ &= \{f, g\}\end{aligned}$$

For the converse I would li

b. It is clear that $\phi_1 : (q_1, p_1, q_2, p_2) \mapsto (q_1, p_1)$ does not satisfy $\phi_1^*\omega_{\mathbb{R}^2} = \omega_{\mathbb{R}^4}$ since

$$\omega_{\mathbb{R}^4}((0, 1, 0, 1),(0, 1, 0, -1)) = 2$$

but $(\phi_1)_*(0, 1, 0, 1) = (0, 1) = (\phi_1)_*(0, 1, 0, -1)$ giving

$$\phi_1^*\omega_{\mathbb{R}^2}((0, 1, 0, 1),(0, 1, 0, 1)) = \omega_{\mathbb{R}^2}((0, 1),(0, 1)) = 0.$$

However we can see that ϕ_1 is a Poisson map since for any $f, g \in \mathcal{C}^\infty(\mathbb{R}^2)$, at $x \in \mathbb{R}^4$ we have

$$\phi_1^*\{f, g\}_{\mathbb{R}^2}(x) = \left(\sum_{i=1}^{2} \frac{\partial f}{\partial x_i}\frac{\partial g}{y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right)_{\phi_1(x)}$$

and

$$\{\phi_1^*f, \phi_1^*g\}_{\mathbb{R}^4} = \left(\sum_{i=1}^{2} \frac{\partial f\circ\phi_1}{\partial x_i}\frac{\partial g\circ\phi_1}{y_i} - \frac{\partial f\circ\phi_1}{\partial y_i}\frac{\partial g\circ\phi_1}{\partial x_i}\right)_{x}$$

and they are equal by the chain rule and the fact that the derivative of ϕ_1 does not alter the first two coordinates of a vector while vanishing the last two.

It is even more clear that $\phi_2 : (q_1, p_2) \mapsto (q_1, p_1, 0, 0)$ satisfies $\phi_2\omega_{\mathbb{R}^4} = \omega_{\mathbb{R}^2}$. However, in this case we have

$$\phi_2^*\{f, g\}_{\mathbb{R}^4}(x) = \left(\sum_{i=1}^{4} \frac{\partial f}{\partial x_i}\frac{\partial g}{y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right)_{\phi_2(x)}$$

while

$$\{\phi_2^*f, \phi_2^*g\}_{\mathbb{R}^2} = \left(\sum_{i=1}^{2} \frac{\partial f\circ\phi_2}{\partial x_i}\frac{\partial g\circ\phi_2}{y_i} - \frac{\partial f\circ\phi_2}{\partial y_i}\frac{\partial g\circ\phi_2}{\partial x_i}\right)_{x}$$

meaning that the information of the last two coordinates is lost in the second expression, so they cannot be equal. $\square$

Problem 8

a. Consider $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ equipped with the are form $\omega_x(u, v) = \langle x, u \times v\rangle$ (where $x \in S^2$, $u, v \in T_x S^2$ and $\times$ is the vector product). Use cylindrical coordinates to prove Darboux's theorem directly in this example.

b. More generally: show that on a 2-dimensional manifold, any non-vanishing 1-form can be locally written as $f dg$, where f and g are smooth functions. Use this fact to give a proof of Darboux's theorem in 2 dimensions.

Solution.

a. Inspired in Da Silva, we wish to show that there are local coordinates (θ, z) of S^2 such that $\omega_x(u, v) = d\theta \wedge dz$. Since

□