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Lista 5

Contents

Problem 1 Let G be a Lie group. Let $X : G \longrightarrow TG$ be a section of the projection $TG \longrightarrow G$, not necessarily smooth. Show that if X is left invariant (i.e., $dL_g(X) = X \circ L_g$ for all $g \in G$), then X is automatically smooth.

Conclude that an analogous result holds for differential forms: if a section $\eta : G \to \Lambda^k(T^*G)$ is left-invariant ($L_g^*\eta = \eta$), then η is a smooth k-form. Check that an analogous result holds for G-invariant forms on a homogeneous manifold.

Solution. We know that $X_g = dL_g(X_e)$. We want to show that the map $G \to TG : g \mapsto dL_g(X_e)$ is smooth. Suggestion by Ted Shiffrin at StackExchange is to consider

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (g, h) &\longmapsto L_g h = gh \end{aligned}$$

which is smooth and thus has smooth differential which turns out to have the expression

$$\begin{aligned} d\mu : TG \times TG &\longrightarrow TG \\ (u_g, v_h) &\longmapsto dR_h u_g + dL_g v_h \end{aligned}$$

Choosing g arbitrary, $h = e$, $u_g = 0$ and $v_h = X_e$, we obtain

$$d\mu(u_g, v_h) = dL_g X_e$$

and, as we have said, this differential depends smoothly on g.

The pullback map is an induced map by L_g on the Grassman algebra:

$$L_g^* : \Lambda^k(G) \longrightarrow \Lambda^k(G)$$
$$\eta \longmapsto \begin{array}{l} L_g^*\eta : \mathfrak{X}^k(G) \longrightarrow \mathbb{R} \\ (X_1, \dots, X_k) \longmapsto \eta(dL_g X_1, \dots, dL_g X_1) \end{array}$$

and we have by hypothesis that

$$L_g^*\eta = \eta.$$

Similarly, group multiplication μ induces a map on the Grassman algebra:

$$\mu : \Lambda^k(G) \times \Lambda^k(G) \longrightarrow \Lambda^k(G)$$
$$(\alpha, \beta) \longmapsto R_h^*\alpha + L_g^*\beta$$

so again chosing $\alpha = 0$ and $\beta = \eta$ we see that η is smooth.

Now let η be a G-invariant form on a homogeneous manifold X. Following Jack Lee's suggestion, we can pull back η to G via the map $\pi : G \to X$, $g \mapsto g \cdot p$ for any fixed $p \in X$. This gives a form $\pi^*\eta$ on G that is left-invariant since η is G-invariant: pushing vectors with left-multiplication preserves the orbit of a given point. More explicitly:

$$L_g^*(\pi^*\eta) = \pi^*\eta \qquad \text{because} \qquad L_g^*(\pi^*\eta) = \eta \circ d\pi \circ dL_g$$

and η is constant along vectors on the orbit of p, which are the images of $d\pi \circ dL_g$. By the previous exercise we see that $\pi^*\eta$ is smooth.

Then we notice that π is a submersion. This follows since π is a surjective map (bacuse X is homogeneous) of constant rank. Constant rank means that $d\pi$ has the same rank at every point of G. This can be seen moving around the tangent spaces of both G and X using the differentials of both the action of G on M, and the left-translation action of G on G as follows. There is a commutative diagram:

$$\begin{tikzcd} T_g G \arrow[r, "d\pi"] \arrow[d, "d(\text{action})"'] & T_{gp} X \arrow[d, "d(\text{action})"] \\ T_{hg} G \arrow[r, "d\pi"] & T_{hgp} X \end{tikzcd}$$

and those vertical differentials are diffeomorphisms by smothness of the actions.

Finally we conclude by taking a local inverse σ of π via inverse function theorem. Then we simply notice that $\eta = \sigma^*\pi^*\eta$, which is smooth by construction. $\square$

Problem 2

a. Prove that any connected Lie group G is generated as a group by any open neighbourhood U of the identity element (i.e. $G = \bigcup_{n=1}^{\infty} U^n$).

b. Suppose that two Lie group homomorphisms $\varphi, \psi : G \to H$ are such that $d\varphi|_e = d\psi|_e$. Show that φ and ψ coincide on the connected component of G containing the identity e.

Solution.

a.

b. Let $g \in G$ be any element. Suppose we find a vector $v \in T_e G$ such that its integral curve γ passes through g at time t_0. Then we can map this vector both by $d\varphi$ and $d\psi$ to obtain a vector $w := d_e\varphi(v) = d_e\psi(v) \in T_e H$. Then the integral curve η of w is the same as the integral curve of v pushed by either of φ or ψ. Let me show this explicitly for self-conviction:

> $\varphi \circ \gamma(t) = \eta(t)$ for all t because differentaition at $t = 0$ yields $\frac{d}{dt}(\varphi \circ \gamma)\Big|_{t=0} = d_e\varphi\gamma'(0) = d_e\varphi v = \eta'(0)$ so both $\varphi \circ \gamma$ and η are solutions to the same differential equation thus coincide.

Of course the same goes for ψ. Then we get that

$$\gamma(t_0) = g \implies \varphi(g) = \varphi(\gamma(t_0)) = \eta(t_0) = \psi(\gamma(t_0)) = \psi(g)$$

OK now we only need to find the vector v. The statement is that for every $g \in G$ there is a maximal integral curve starting at the identity passing through g. Since g is the same connected component as the identity we can find a curve connecting them. But this curve need not be a maximal integral curve nor homotopic to one. So I'm stuck and I suppose that there is no maximal integral curve. Then all the vectors in the unit ball... I'm pretty sure here is the answer.

□

Problem 3 Consider the Lie groups $\mathsf{SU}(2) = \{A \in \mathcal{M}_2(\mathbb{C}) | AA^* = \mathrm{Id}, \det A = 1\}$ and $\mathsf{SO}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) | AA^{\mathsf{T}} = \mathrm{Id}, \det A = 1\}$.

a. Show that

$$\mathsf{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

Conclude that, as a manifold $\mathsf{SU}(2)$ is diffeomorphic to S^3 (hence it is simply connected).

Recall the definition of the quaternions $\mathbb{H}$. Show that the sphere S^3, seen as quaternions of norm 1, inherits a Lie group structure with respect to which it is isomorphic to $\mathsf{SU}(2)$.

b. Verify that

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}. \tag{1}$$

Consider the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, that takes the element in $\mathfrak{su}(2)$ determined by α, β to the vector $(\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)$ in $\mathbb{R}^3$. Observe that, with respect to this identification, det in $\mathfrak{su}(2)$ corresponds to $\|\cdot\|^2$ in $\mathbb{R}^3$.

c. Verify that each element $A \in \mathsf{SU}(2)$ defines a linear transformation on the vector space $\mathfrak{su}(2)$ by conjugation: $B \mapsto ABA^{-1}$. Show that, with the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$, we obtain a representation (i.e., a linear action) of $\mathsf{SU}(2)$ on $\mathbb{R}^3$ that is norm preserving. Conclude that we have homomorphism $\phi : \mathsf{SU}(2) \to \mathsf{O}(3)$, verifying that is image is $\mathsf{SO}(3)$ and its kernel is $\{\mathrm{Id}, -\mathrm{Id}\}$.

d. Conclude that $\mathsf{SU}(2) \cong S^3$ is a double cover of $\mathsf{SO}(3)$ *hence is its universal cover, since it's simply connected), qnd the covering map identifies antipodal points of S^3. Hence, as manifolds, $\mathsf{SO}(3)$ is identified with $\mathbb{RP}^3$.

Solution.

a. Given the computation above, it is clear that $\mathsf{SU}(2)$ is diffeomorphic to S^3 since its parameters $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, can be understood as vectors $x \in \mathbb{R}^4$ of norm 1.

   The quaternions are the only 4-dimensional real division algebra. This means it is a 4-dimensional real vector space equipped with a (non-commutative) multiplication. They are also equipped with a norm that coincides with euclidean norm. With respect to this norm we define the unit sphere S^3.

   To see that S^3 is a Lie subgroup we first need to check that it is closed under quaternion product. The easiest way to see that is via quaternion conjugate: if $x = x_1 + ix_2 + jx_3 + kx_4$ is a quaternion, its conjugate is $\bar{x} = x_1 - ix_2 - jx_3 - kx_4$. It may be computed that the norm is given by $|x| = x\bar{x}$. Then we see that if $x, y \in S^3$

$$|xy\overline{xy}| = |xy\bar{y}\bar{x}| = \Big|x|y|\bar{x}\Big| = 1$$

   Then the fact that S^3 is a Lie group follows from the fact that the restricition (the multiplication and inverse map) of smooth maps to embedded submanifolds remains smooth ([**?**], prop ?).

   An identification between $S^3 \subset \mathbb{H}$ and $\mathsf{SU}(2)$ as expressed above is given by

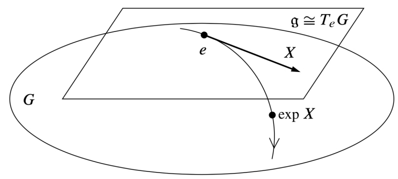
$$a + bi + cj + dk \longmapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

   Checking that this map is a group isomorphism ammounts to checking that matrix multiplication in $\mathsf{SU}(2)$ is the same as quaternion multiplication

b. (Proof from [**?**], prop. 3.24 and coro. 3.46). We will show that $\mathfrak{su}(3)$ is the space of traceless anti-hermitian matrices and eq. (1) will follow.

Step 1 Show that the Lie algebra of a matrix Lie group G, defined as the tangent space at identity, is the same as the matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

Proof of Step 1. ([**?**], coro. 3.46) If $X \in T_{\mathrm{Id}}G$, by definition of exponential map we have $\exp(tX) \in G$.



[?], fig. ?

Now suppose that X is such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

I got stuck with the proof in Hall so I will use prop 20.3 from [?]. The idea is to show see $\mathsf{SU}(2)$ as a subgroup of, say $\mathsf{GL}(2, \mathbb{C})$. Then a □

Step 2 Then we look for the matrices such that for all $t \in \mathbb{R}$,

$$\exp(tX)^* = \exp(tX)^{-1} = \exp(-tX) \qquad \text{and} \qquad \det \exp(tX) = 1.$$

This means that

$$\exp(tX^*) = \exp(-tX) \qquad \text{and} \qquad \mathrm{Tr}(X) = 0.$$

The first implication is the result of the general facts that $e^{A*} = e^{A^*}$ and $(e^A)^{-1} = e^{A^{-1}}$. Let's have a look at the second one:

> $(e^A)^{-1} = e^{A^{-1}}$ This is nicely shown in [?] as follows. For any $X \in \mathsf{Lie}(G)$ map $t \mapsto \exp(tX)$ is a group homomorphism $\mathbb{R} \to G$ because $\exp((t_1+t_2)X) = \exp(t_1X)\exp(t_2X)$ (but of course that could be justified...) anyway from this it follows that this map preserves inverses, which what we wanted.

Now let's have a look at why determinant 1 in Lie group translates to vanishing trace in Lie algebra.

> $\forall X \in \mathcal{M}_n(\mathbb{C}),\ \det e^X = e^{\mathrm{tr}\, X}$. This is easy to see if X is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$; then the eigenvalues of e^X are $e^{\lambda_1}, \ldots, e^{\lambda_n}$. So $\det e^X = \prod e^{\lambda_i} = e^{\sum \lambda_i} = e^{\mathrm{tr}\, X}$. But if X is not diagonalizable we must do Jordan decomposition and some more computations.

Finally, differentiating and evaluating at $t = 0$ the equation $\exp(tX^*) = \exp(-tX)$ gives $X^* = -X$. (See [?], prop. 2.4. It is intuitive but not immediate.)

In conclusion, we see that

$$\mathfrak{su}(2) = \{X \in \mathcal{M}_{2\times 2}(\mathbb{C}) : X^* = -X \text{ and } \mathrm{Tr}(X) = 0\}$$

Step 3 It is immediate that the right-hand side in eq. (1) is contained in the set above. For the other inclusion first notice that the condition $X^* = -X$ makes the entries in the diagonal be such that

$$x + iy = -\overline{x + iy} = -(x - iy) = -x + iy \implies x = -x \implies x = 0$$

while the traceless condition implies the two entries in the diagonal must be additive inverses. For the entries in the antidiagonal we literally see the definition of conjugate transpose.

Now let's identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via $\alpha, \beta \mapsto (\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)$. We immediately see that

$$\det\begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} = i\alpha(-i\alpha) = \beta(-\bar{\beta}) = \alpha^2 + |\beta|^2 = \|(\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)\|^2$$

c. De acordo com o exercício anterior, é suficiente mostrar que ABA^{-1} tem traza zero e $ABA^{-1} = -(ABA^{-1})^*$. A primeira propriedade é imediata dado que, em geral, $\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$. Para a segunda propriedade note que

$$(ABA^{-1})^* = (A^{-1})^* B^* A^* = (A^*)^*(-B)A^{-1} = -ABA^{-1}$$

O fato de que essa ação em $\mathsf{SU}(2)$ preserva a norma é immediato do item anterior e do fato de que determinante de um produto de matrizes é o produto dos determinantes.

Isso significa que cada elemento em $\mathsf{SU}(2)$ é uma transformação linear $\mathbb{R}^3 \to \mathbb{R}^3$, i.e. um mapa $\mathsf{SU}(2) \to \mathsf{O}(3)$. Esse mapa é um homomorfismo já que

$\square$

Problem 4 Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, and let $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$be a symmetric bilinear form that is Ad-invariant (i.e. $k(\operatorname{Ad}_g(u), \operatorname{Ad}_g(v)) = k(u, v)$ for $g \in G$).

a. Show that the map
$$\begin{aligned} k^\sharp &: \mathfrak{g} \longrightarrow \mathfrak{g}^* \\ k^\sharp(u)(v) &= k(u, v) \end{aligned} \tag{2}$$
is G-equivariant:
$$k^\sharp \circ \operatorname{Ad}_g = (\operatorname{Ad}^*)_g \circ k^\sharp, \qquad \forall g \in G$$
[*Recall:* $(\operatorname{Ad}^*)_g := (\operatorname{Ad}_{g^{-1}})^*$.] In particular, when k is nondegenerate (i.e. $k^\sharp$ is an isomorphism), the adjoint and coadjoint actions are equivalent.

b. Verify that eq. (2) implies that $k([w, u], v) = -k(u, [w, v])$, $\forall u, v, w \in \mathfrak{g}$, and that both conditions are equivalent when G is connected.

Solução. a. É só abrir as definições . Fixe um elemento $x \in \mathfrak{g}$. No lado esquerdo, temos

$$k^\sharp \circ \operatorname{Ad}_g(x) = k^\sharp(\operatorname{Ad}_g x) = k(\operatorname{Ad}_g x, \cdot)$$

e no lado direito,

$$(\operatorname{Ad}^*)_g \circ k^\sharp(x) = \operatorname{Ad}^*_g(k^\sharp(x)) = \operatorname{Ad}^*_g(k(x, \cdot)) = k(x, \operatorname{Ad}_{g^{-1}} \cdot)$$

mas, como k é Ad-invariante, $k(x, \operatorname{Ad}_{g^{-1}}, \cdot) = k(\operatorname{Ad}_g x, \cdot)$.

b. Must check this later...

$\square$

Problem 5 For a Lie algebra $\mathfrak{g}$, there is always a canonical bilinear form $k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, called ***Killing form***, given by:

$$k(u, v) = \operatorname{tr}(\operatorname{ad}_u \operatorname{ad}_v).$$

(*Recall:* $\operatorname{ad}_u : \mathfrak{g} \to \mathfrak{g}$, $\operatorname{ad}_u(v) = [u, v]$.)

a. Note that k is symmetric, and check that it is Ad-invariant.

b. A Lie algebra is called ***semi-simple*** if k is nondegenerate. Show that $\mathfrak{so}(3)$ is semi-simple.

Solution.

a. O fato de k ser simétrica segue de que, em geral, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Para ver que k é Ad-invariante, note que, para $g \in G$,

$$k(\operatorname{Ad}_g u, \operatorname{Ad}_g v) = \operatorname{tr}(\operatorname{ad}_{\operatorname{Ad} u} \operatorname{ad}_{\operatorname{Ad} v}) = \operatorname{tr}([\operatorname{Ad} u, [\operatorname{Ad} v, \cdot]]) =$$

$\square$

Problem 6 Consider the linear isomorphism $\mathbb{R}^3 \to \mathfrak{so}(3)$, given by

$$v = (x, y, z) \longmapsto \hat{v} := \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

a. Describe the Lie bracket on $\mathbb{R}^3$ induced by the commutator in $\mathfrak{so}(3)$, and the inner product in $\mathfrak{so}(3)$ that corresponds to the canonical inner product in $\mathbb{R}^3$.

b. Describe the $\mathsf{SO}(3)$-action on $\mathbb{R}^3$ corresponding to the adjoint action, its orbits, as well as its infinitesimal generators. Find (without any calculation!) a description of the coadjoint action on $\mathbb{R}^3$ (identified with $(\mathbb{R}^3)^*$ through the canonical inner product).

Solution.

a. Lembre que

$$\mathsf{SO}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) : AA^{\mathsf{T}} = \operatorname{Id}, \det A = 1\}$$
$$\mathfrak{so}(3) = \{A \in \mathcal{M}_3(\mathbb{R}) : A = -A^{\mathsf{T}}\}$$

Isso explica por que as matrizes em $\mathfrak{so}(3)$ tem a forma mostrada acima. O isomorfismo com $\mathbb{R}^3$ é dado por

$$\begin{aligned} \mathfrak{so}(3) &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} &\longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = u \\ [U, V] &\longmapsto u \times v \\ \frac{1}{2}\operatorname{tr}(UV^{\mathsf{T}}) &\longmapsto \langle u, v \rangle \end{aligned}$$

Para comprovar que de fato o commutador de $\mathfrak{so}(3)$ corresponde com o produto vetorial em $\mathbb{R}^3$, considere duas matrizes $U, V \in \mathfrak{so}(3)$ e os vetores correspondentes $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$. O commutador é

$$\begin{aligned} [U, V] = UV - VU &= \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \\ &- \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \end{aligned}$$

a entrada $(3, 2)$ da matriz resultante é a primeira coordenada do vetor correspondente a $[U, V]$; esse numero é claramente $u_1v_3 - u_3v_2$. Analogamente, a segunda coordenada é $u_3v_1 - u_1v_3$, enquanto a terceira $u_1v_2 - u_2v_1$. Essas são as coordenadas de $u \times v$.

Para ver que o produto interno de $\mathbb{R}^3$ corresponde com $\frac{1}{2}$ tr note que

$$\begin{aligned} UV^{\mathsf{T}} = U(-V) &= \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} u_3v_3 + u_2v_2 & * & * \\ * & u_3v_3 + u_1v_1 & * \\ * & * & u_2v_2 + u_1v_1 \end{pmatrix} \end{aligned}$$

Agora vamos calcular a ação adjunta. Primeiro notemos que a ação adjunta, definida como a derivada do operador $I_A(X) = AXA^{-1}$, é ela mesma. Talvez esse é um fato obvio porque I_A é um mapa linear, e a sua derivada coincide com ele. Porém, tem um argumento mais explícito no StackExchange: a ação adjunta em um vetor $X \in \mathfrak{so}(3)$ que seja a derivada em $t = 0$ da curva $c \subset \mathsf{SO}(3)$, i.e. $c'(0) = X$, é simplesmente $\frac{d}{dt}Ac(t)A^{-1}\Big|_{t=0} = AXA^{-1}$. A observação chave é que como a álgebra de Lie $\mathfrak{so}(3)$ é um álgebra de matrizes, o produto do lado direito está bem definido.

Em fim, agora vou seguir este documento para achar uma matriz que representa esse operador linear cuando identificamos $\mathfrak{so}(3)$ com $\mathbb{R}^3$. Mais precisamente, queremos achar uma matriz $\overline{\mathrm{Ad}_A}$ tal que $\mathrm{Ad}_A X = AXA^{-1} \rightsquigarrow \overline{\mathrm{Ad}_A} x$ onde $x \in \mathbb{R}^3$ representa a matriz $X \in \mathfrak{so}(3)$.

O truque é usar que as matrizes em $\mathsf{SO}(3)$, sendo isometrías de $\mathbb{R}^3$, preservam o produto vetorial, i.e. $Ax \times Ay = A(x \times y)$ para qualquer $x, y \in \mathbb{R}^3$. Temos que

$$\mathrm{Ad}_A XAy =$$

□

Problem 7 Let (V, Ω) be a symplectic vector space, and consider $H := V \times \mathbb{R} = \{(v, t)\}$. This space H with the multiplication

$$(v_1, t_1) \cdot (v_2, t_2) = \left(v_1 + v_2, \frac{1}{2}\Omega(v_1, v_2) + t_1 + t_2\right)$$

is a Lie group called the ***Heisenberg group*** (find the identity elements and inverses in H).

a. Show (directly from the conjugation formula in H) that $\mathrm{Ad}_{(v,t)}(X, r) = (X, r + \Omega(v, X))$, for $(X, r) \in \mathfrak{h} = \mathsf{Lie}(H) = V \times \mathbb{R}$. Describe the adjoint orbits, verifying that their possible dimensions are zero and one.

b. Verify that $\mathrm{ad}_{(Y,s)}(X, r) = (0, \Omega(Y, X))$. [Recalling that $\mathrm{ad}_{(Y,s)}(X, r) = [(Y, s), (X, r)]$, we obtain a formula for the Lie bracket in $\mathfrak{h}$.]

c. Describe the coadjoint action of H on $\mathfrak{h}^* = V^* \times \mathbb{R}^*$ and its orbits, analyzing the possible dimensions.

Solution. É imediato que a identidade em H é $(0, 0) \in V \times \mathbb{R}$, e o elemento inverso de (v, t) é $(-v, -t)$.

a. Primeiro note que $\mathrm{Ad}_{(v,t)} X = (v, t) \cdot X \cdot (-v, -t)$ onde $\cdot$ denota o produto en $H \cong \mathfrak{h}$ (esse isomorfismo é claro já que H é um espaço vetorial, assim o espaço tangente em $(0, 0)$ é isomofo a ele). Para justificar isso (como no exercício anterior), considere uma curva $\gamma \subset H$ tal que $\gamma(0) = (0, 0)$ e $\gamma'(0) = (X, r)$. Então

$$\begin{aligned}\mathrm{Ad}_{(v,t)}(X, r) = d_{(0,0)} I_{(v,t)}(X, r) &= \frac{d}{d\tau} I_{(v,t)} \circ \gamma \Big|_{\tau=0} \\ &= \frac{d}{d\tau}(v, t) \cdot \gamma(\tau) \cdot (-v, -t) \Big|_{\tau=0} = (v, t) \cdot (X, r) \cdot (-v, -t).\end{aligned}$$

daí é só calcular

$$\begin{aligned}\mathrm{Ad}_{(v,t)}(X, r) &= (v, t) \cdot (X, r) \cdot (-v, -t) \\ &= (v, t) \cdot \left(X - v, \frac{1}{2}\Omega(X - r) + r - t\right) \\ &= \left(X, \frac{1}{2}\Omega(v, X - v) + t + \frac{1}{2}\Omega(X, -v) + r - t\right) \\ &= \left(X, \frac{1}{2}\Omega(v, X) + \frac{1}{2}\Omega(v, -v) + \frac{1}{2}\Omega(X, -v) + s\right) \\ &= (X, \Omega(v, X) + r)\end{aligned}$$

A órbita adjunta de (X, r) é $X \times \mathbb{R}$ sempre que $X \neq 0$ já que Ω é não degenerada. Caso $X = 0$, a órbita é um ponto, $(0, r)$.

b. Em fim... o problema aqui é que não sei como calcular a derivada de Ad em $(0, 0)$. Porque Ad é um mapa de G em $\mathrm{Aut}(\mathfrak{g})$.

$\square$