

Lista 2

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Problem 1 Verify (and justify) whether or not the following manifolds admit a symplectic structure: $S^1 \times S^3$, $\mathbb{R} \times S^3$, $\mathbb{R}^3 \times S^3$, $\mathbb{T}^3 \times S^3$.

Solution.

a. By Künneth formula for cohomology, we see that

$$\begin{aligned} H^2(S^1 \times S^3) &= H^2(S^1) \otimes H^0(S^3) \oplus H^1(S^1) \otimes H^1(S^3) \oplus H^0(S^1) \otimes H^1(S^3) \\ &= (0 \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes 0) \oplus (\mathbb{R} \otimes 0) \\ &= 0 \end{aligned}$$

which means $S^1 \times S^3$ is not symplectic since it is compact.

b. Considere o mapa

$$\begin{aligned} (0, +\infty) \times S^3 &\longrightarrow \mathbb{R}^4 \\ (t, v) &\longmapsto tv \end{aligned}$$

com inversa $x \mapsto \left(|x|, \frac{x}{|x|}\right)$. Ele é um difeomorfismo, assim o pullback da forma simplética canônica em \mathbb{R}^4 induz uma estrutura simplética em $\mathbb{R} \times S^3 \stackrel{\text{dif}}{\cong} (0, +\infty) \times S^3$.

c. Considerando que S^3 é a esfera unitária nos quatérnios, os campos vetoriais $z \mapsto iz$, $z \mapsto jz$ e $z \mapsto kz$ são uma base global, de forma que S^3 é paralelizável. Sendo uma 3-variedade, temos que $TS^3 \cong \mathbb{R}^3 \times S^3$. Em aula vimos que todo fibrado tangente é uma variedade simplética usando o pullback da forma canônica no fibrado cotangente.

d. (Idea from [StackExchange](#)) Künneth formula gives us a ring isomorphism

$$H^\bullet(T^3 \times S^3) \stackrel{\varphi}{\cong} H^\bullet(T^3) \otimes H^\bullet(S^3).$$

Now the second cohomology group reads

$$\begin{aligned} H^2(T^3 \times S^3) &= H^0(T^3) \otimes H^2(S^3) \oplus H^1(T^3) \otimes H^1(S^2) \oplus H^0(T^3) \otimes H^2(S^3) \\ &= H^2(T^3). \end{aligned}$$

If ω was a symplectic form on $T^3 \times S^3$, its cohomology class would correspond to a cohomology class $\varphi[\omega]$ in $H^2(T^3) \subset H^\bullet(T^3) \otimes H^\bullet(S^3)$. But since T^3 is only a 3-manifold, the 6-form $(\varphi[\omega])^3$ must be zero. Since φ preserves product, we see that $[\omega^3]$ itself must be zero, which is a contradiction since it must be a volume form.

□

Problem 2 Show that the tautological 1-form $\Omega^1(T^*Q)$ is *uniquely characterized* by the following property: for any 1-form $\mu \in \Omega^1(Q)$,

$$\mu^* \alpha = \mu$$

where on the left-hand side we view μ as a map $\mu : Q \rightarrow T^*Q$.

Solution. (After several tries, consulted [StackExchange](#)) Simply notice that if β is another 1-form on T^*Q such that $\mu^* \beta = \mu$ for all $\mu \in \Omega^1(Q)$, then $\mu^*(\alpha - \beta) = 0$. Further, if any $\theta \in \Omega^1(T^*Q)$ satisfies $\mu^* \theta = 0$ for all $\mu \in \Omega^1(Q)$, it must be identically zero. This follows since for any $w \in T(T^*Q)$ we can find a form μ and a vector $v \in TQ$ such that $\mu_* v = w$. (I'm not sure how to prove the last statement.)

□

Prova de que $\mu^ \alpha = \mu$.* Temos que se $v \in T_x(Q)$ é um vetor tangente a Q no ponto $x \in Q$,

$$(\mu^* \alpha)_x v = \alpha_{\mu_* v} (\mu_* v)$$

onde $\mu_* v \in TT^*Q$, pois $\mu_* : TQ \rightarrow TT^*Q$. Lembre que $\alpha_\eta = \pi_\eta^* \eta$ para qualquer $\eta \in \Omega^1(Q)$. Assim, podemos escrever

$$\begin{aligned} \alpha_{\mu_* v} (\mu_* v) &= \pi_{\mu_* v}^* (\mu_* v) \\ &= \mu_x (\pi_* \mu_* v) \\ &= \mu_x (v) \end{aligned}$$

já que μ comuta com a projeção.

□

Problem 3 We will characterize symplectomorphisms $T^*Q \rightarrow T^*Q$ which are cotangent lifts of diffeomorphisms $\phi : Q \rightarrow Q$. Let α be the tautological 1-form on $M := T^*Q$ and $\omega = -d\alpha$. We saw in class that cotangent lifts preserve α . We will show the converse of this fact.

Let $F : M \rightarrow M$ be a symplectomorphism such that $F^*\alpha = \alpha$.

- Let $v \in \mathfrak{X}(M)$ be the unique vector field such that $i_v\omega = -\alpha$; note that, locally, it is given by $\sum_i \xi_i \frac{\partial}{\partial \xi_i}$ (v is known as the *Euler vector field*). Show that $F_*v = v$.
- Let φ_t^v denote the flow of v . Show that $\varphi_t^v \circ F = F \circ \varphi_t^v$. Check that, in coordinates,

$$\varphi_t^v(x, \xi) = \varphi_t^v(x, e^t \xi), \quad -\infty < t < \infty.$$

- Verify that, for $p \in T_x^*Q$, $F(\lambda p) = \lambda F(p)$, $\forall \lambda \in \mathbb{R}$. Conclude that there exists $\phi : Q \rightarrow Q$ such that $\phi \circ \pi = \pi \circ F$ (here $\pi : T^*Q \rightarrow Q$ is the projection). Finally, show that $F = \hat{\phi}$ (the cotangent lift of ϕ).

Solution.

- Usando que $F^*\alpha = \alpha$, $i_v\omega = -\alpha$ e que $F^*\omega = \omega$ obtemos que para $u \in T(T^*Q)$,

$$\omega(F_*v, F_*u) = \omega(v, u) = -\alpha(u) = -\alpha(F_*u) = \omega(v, F_*u)$$

i.e., $\omega(F_*v - v, F_*u) = 0$, o que implica que $F_*v = v$ já que ω é não degenerada (mas, por que F_* é surjetiva?).

Out of time!

□

Problem 4 Let $\alpha \in \Omega^1(T^*Q)$ be the tautological 1-form. We will now see examples of symplectomorphisms of T^*Q which are not cotangent lifts. Let $A \in \Omega^1(Q)$ and consider the associated "fiber-translation" map $\varphi_A : T^*Q \rightarrow T^*Q$, $(x, \xi) \mapsto (x, \xi + A_x)$.

- Show that

$$\varphi_A^*\alpha - \alpha = \pi^*A,$$

where $\pi : T^*Q \rightarrow Q$ is the projections. It follows that φ_A is symplectomorphism iff A is closed.

- Consider functions that are constant along the fibers of T^*Q (i.e., of the form $H = \pi^*f$, for $f \in C^\infty(Q)$). Describe their hamiltonian vector fields in local cotangent coordinates, as well as their flows.

Solution.

- a. Notice that if $v \in T_{(x,\xi)}(T^*Q)$ is a tangent vector at (x, ξ) , then the differential of φ_A pushes it to a tangent vector at $(x, \xi + A_x)$, i.e., $d\varphi_A v \in T_{(x,\xi+A_x)}(T^*Q)$ using d notation instead of lower star. Moreover, since φ_A preserves fibers,

$$\begin{array}{ccc} T^*Q & \xrightarrow{\varphi_A} & T^*Q \\ & \searrow \pi \quad \swarrow \pi & \\ & Q & \end{array}$$

its differential satisfies $\pi_* d\varphi_A = \pi_*$. Unrolling the definitions, we see that

$$\begin{aligned} (\varphi_A^* \alpha)(v) &= \alpha(d\varphi_A(v)) = (\xi + A_x)(\pi_* d\varphi_A v) = \xi(\pi_* v) + A_x(\pi_* v), \\ -\alpha(v) &= \xi(\pi_* v), \\ \pi^* A(v) &= A(\pi_* v). \end{aligned}$$

- b. In local cotangent coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, the Hamiltonian vector field of $H = f \circ \pi$ according to [Lee](#) Eq. (22.9) is

$$X_f = \sum_i \left(\frac{\partial f \circ \pi}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial f \circ \pi}{\partial x_i} \frac{\partial}{\partial \xi_i} \right)$$

Now at $(p, \xi) \in T^*Q$, writing $\pi = (\pi_1, \dots, \pi_n)$,

$$\begin{aligned} \left. \frac{\partial f \circ \pi}{\partial \xi_i} \right|_{(p,\xi)} &= \sum_j \left. \frac{\partial f}{\partial x_j} \right|_p \left. \frac{\partial \pi_j}{\partial \xi_i} \right|_{(p,\xi)} = \sum_j \left. \frac{\partial f}{\partial x_j} \right|_p \cdot 0 = 0 \\ \left. \frac{\partial f \circ \pi}{\partial x_i} \right|_{(p,\xi)} &= \sum_j \left. \frac{\partial f}{\partial x_j} \right|_p \left. \frac{\partial \pi_j}{\partial x_i} \right|_{(p,\xi)} = \sum_j \left. \frac{\partial f}{\partial x_j} \right|_p \cdot 1 = \sum_j \left. \frac{\partial f}{\partial x_j} \right|_p \end{aligned}$$

so that

$$X_f = - \sum_j \left. \frac{\partial f}{\partial x_j} \right|_p \sum_i \frac{\partial}{\partial \xi_i}$$

back to [Lee](#), Eq. 22.11 shows that the integral curves of H are $\gamma(t) = (x_i(t), \xi_i(t))$ such that

$$\begin{aligned} \dot{x}_i(t) &= \frac{\partial H}{\partial \xi_i}(x(t), \xi(t)) = 0 \\ \dot{\xi}_i(t) &= -\frac{\partial H}{\partial x_i}(x(t), \xi(t)) = - \sum_j \frac{\partial f}{\partial x_j}(x(t), \xi(t)). \end{aligned}$$

but I am unsure of what this means.

□

Problem 5 Let $\omega = -d\alpha$ be the canonical symplectic form on T^*Q . Prove that, if $B \in \Omega^2(Q)$ is closed, then

$$\omega_B := \omega - \pi^*B$$

is symplectic and that, if $B, B' \in \Omega^2(Q)$ are closed and such that $B - B' = dA$, then φ_A (defined in the previous problem) is a symplectomorphism from (T^*Q, ω_B) to $(T^*Q, \omega_{B'})$.

Solution. Showing ω_B is symplectic is equivalent to showing that for any non-zero $v \in T(T^*Q)$ there is $w \in T(T^*Q)$ such that $\omega(v, w) \neq 0$. However, if this equality was true, varying the last coordinates of w would modify the value of ω while leaving unchanged that of π^*B .

Using all we learnt in the previous exercise,

$$\begin{aligned} \varphi_A^* \omega_{B'} &= \varphi_A^* (\omega - \pi^* B') \\ &= \varphi_A^* \omega - \varphi_A^* \pi^* B' \\ &= \varphi_A^* \omega - \pi^* B \\ &= \varphi_A^* \omega - \pi^* dA - \pi^* B \\ &= \varphi_A^* \omega - d\pi^* A - \pi^* B \\ &= \varphi_A^* \omega - d(\varphi_A^* \alpha - \alpha) - \pi^* B \\ &= \varphi_A^* \omega - d\varphi_A^* \alpha + d\alpha - \pi^* B \\ &= \varphi_A^* \omega - \varphi_A^* \omega + \omega - \pi^* B \\ &= \omega_B. \end{aligned}$$

□

Problem 6 Let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form. For $f \in C^\infty(M)$, let $X_f \in \mathfrak{X}(M)$ be defined by $i_{X_f} \omega = df$. Consider the bracket $\{f, g\} := \omega(X_g, X_f)$. Verify that $d\omega = 0$ if and only if $\{\cdot, \cdot\}$ satisfies the Jacobi identity.

Solution. (Taken from [StackExchange](#)). Using that $X_{\{g, h\}} = -[X_g, X_h]$ (Prop 22.19, [Lee](#)), we may write the Jacobi identity as

$$\begin{aligned} 0 &= \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} \\ &= \omega(X_f, [X_h, X_g]) + \omega(X_h, [X_g, X_f]) + \omega(X_g, [X_f, X_h]) \end{aligned}$$

Next we use the coordinate-free expression of the exterior derivative of a 3-form to write

$$\begin{aligned} d\omega(X_f, X_g, X_h) &= X_f \omega(X_g, X_h) - X_g \omega(X_f, X_h) + X_h \omega(X_f, X_g) \\ &\quad - \omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g) - \omega([X_g, X_h], X_f) \end{aligned}$$

Finally we use that $X_f \omega(X_g, X_h) = \omega(X_f, [X_g, X_h])$ (I believe this to be derived too from the coordinate-free expression of the exterior derivative) to note that, in the last equation, the first and second row on the right hand side are actually the same, and each of them equals the expression of the Jacobi identity. □

Problem 7 Consider symplectic manifolds (M_i, ω_i) , with Poisson bracket $\{\cdot, \cdot\}_i$, $i = 1, 2$, and let $\phi : M_1 \rightarrow M_2$ be a smooth map.

- Prove that if ϕ is a diffeomorphism, then it is a Poisson map ($\{\phi^*f, \phi^*g\}_1 = \phi^*\{f, g\}_2$) for all $f, g \in C^\infty(M_2)$) if and only if $\phi^*\omega_2 = \omega_1$.
- Find examples of M_1, M_2 and $\phi : M_1 \rightarrow M_2$ such that (1) ϕ is not a Poisson map but does not satisfy $\phi^*\omega_2 = \omega_1$, (2) ϕ satisfies $\phi^*\omega_2 = \omega_1$ but it is not a Poisson map.

Solution.

- (Asked by me on [StackExchange](#))

For \implies , fix a point $p \in M_1$ and two vectors $u, v \in T_p M$. Then we can find two functions $f, g \in C^\infty(M)$ such that $(X_f)_p = u$ and $(X_g)_p = v$. I'm not sure how to justify this globally, but since $\omega(u, \cdot)$ is a 1-form, it can be expressed *locally* as $df \dots$

It turns out that $\phi^*X_f = X_{\phi^*f}$ and $\phi_*X_f = X_{\phi_*f}$, where we define pushforward as the inverse of the pullback using that ϕ is a diffeomorphism. Then

$$\begin{aligned}
 (\phi^*\omega_2)_p(u, v) &= (\phi^*\omega_2)(X_f, X_g)(p) \\
 &= \omega_2(\phi_*X_f, \phi_*X_g)(\phi p) \\
 &= \omega_2(X_{\phi_*f}, X_{\phi_*g})(\phi p) \\
 &= \{\phi_*f, \phi_*g\}_2(\phi p) \\
 &= \phi_*\{f, g\}_1(\phi p) \\
 &= \{f, g\}_1((\phi^{-1} \circ \phi)p) \\
 &= \{f, g\}_1(p) \\
 &= \omega_1(X_f, X_g)(p) \\
 &= (\omega_1)_p(u, v)
 \end{aligned}$$

For \impliedby ,

$$\begin{aligned}
 \{\phi^*f, \phi^*g\}_1(p) &= (\omega_1)_p(X_{\phi^*f}, X_{\phi^*g}) \\
 &= (\phi^*\omega_2)_p(X_{\phi^*f}, X_{\phi^*g}) \\
 &= (\omega_2)_{\phi p}(\phi_*X_{\phi^*f}, \phi_*X_{\phi^*g}) \\
 &= (\omega_2)_{\phi p}(\phi_*\phi^*X_f, \phi_*\phi^*X_g) \\
 &= (\omega_2)_{\phi p}(X_f, X_g) \\
 &= \{f, g\}_2(\phi p) \\
 &= \phi^*\{f, g\}_2
 \end{aligned}$$

- It is clear that $\phi_1 : (q_1, p_1, q_2, p_2) \mapsto (q_1, p_1)$ does not satisfy $\phi_1^*\omega_{\mathbb{R}^2} = \omega_{\mathbb{R}^4}$ since

$$\omega_{\mathbb{R}^4}((0, 1, 0, 1), (0, 1, 0, -1)) = 2$$

but $(\phi_1)_*(0, 1, 0, 1) = (0, 1) = (\phi_1)_*(0, 1, 0, -1)$ giving

$$\phi_1^* \omega_{\mathbb{R}^2}((0, 1, 0, 1), (0, 1, 0, -1)) = \omega_{\mathbb{R}^2}((0, 1), (0, 1)) = 0.$$

However we can see that ϕ_1 is a Poisson map since for any $f, g \in C^\infty(\mathbb{R}^2)$, at $x \in \mathbb{R}^4$ we have

$$\phi_1^* \{f, g\}_{\mathbb{R}^2}(x) = \left(\sum_{i=1}^2 \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right)_{\phi_1(x)}$$

and

$$\{\phi_1^* f, \phi_1^* g\}_{\mathbb{R}^4} = \left(\sum_{i=1}^2 \frac{\partial f \circ \phi_1}{\partial x_i} \frac{\partial g \circ \phi_1}{\partial y_i} - \frac{\partial f \circ \phi_1}{\partial y_i} \frac{\partial g \circ \phi_1}{\partial x_i} \right)_x$$

and they are equal by the chain rule and the fact that the derivative of ϕ_1 does not alter the first two coordinates of a vector while vanishing the last two.

It is even more clear that $\phi_2 : (q_1, p_2) \mapsto (q_1, p_1, 0, 0)$ satisfies $\phi_2^* \omega_{\mathbb{R}^4} = \omega_{\mathbb{R}^2}$. However, in this case we have

$$\phi_2^* \{f, g\}_{\mathbb{R}^4}(x) = \left(\sum_{i=1}^4 \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right)_{\phi_2(x)}$$

while

$$\{\phi_2^* f, \phi_2^* g\}_{\mathbb{R}^2} = \left(\sum_{i=1}^2 \frac{\partial f \circ \phi_2}{\partial x_i} \frac{\partial g \circ \phi_2}{\partial y_i} - \frac{\partial f \circ \phi_2}{\partial y_i} \frac{\partial g \circ \phi_2}{\partial x_i} \right)_x$$

meaning that the information of the last two coordinates is lost in the second expression, so they cannot be equal. \square

Problem 8

- Consider $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ equipped with the area form $\omega_x(u, v) = \langle x, u \times v \rangle$ (where $x \in S^2$, $u, v \in T_x S^2$ and \times is the vector product). Use cylindrical coordinates to prove Darboux's theorem directly in this example.
- More generally: show that on a 2-dimensional manifold, any non-vanishing 1-form can be locally written as $f dg$, where f and g are smooth functions. Use this fact to give a proof of Darboux's theorem in 2 dimensions.

Solution.

- Inspired in Da Silva, we wish to show that the local cylindrical coordinates (θ, z) of S^2 are such that $\omega_x(u, v) = d\theta \wedge dz$.

Converting ω to cylindrical coordinates amounts to computing its pullback by the parametrization

$$\begin{aligned} \varphi : \mathcal{U} \xrightarrow{\text{open}} \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (\theta, z) &\longmapsto (x, y, z) \end{aligned}$$

given by

$$\begin{aligned}x &= \sqrt{1 - z^2} \cos \theta \\y &= \sqrt{1 - z^2} \sin \theta \\z &= z\end{aligned}$$

where the first term in the first two variables is the radio of the cylinder given by $\sqrt{x^2 + y^2}$, and since we are on the sphere we have $x^2 + y^2 + z^2 = 1$.

In conclusion, we wish to show that

$$\varphi^* \omega = d\theta \wedge dz$$

so we wish to compute the pullback of any vectors $u, v \in T_p \mathbb{R}^2$ by φ at a point $p = (\theta, z)$. The differential of $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ at $u = u^\theta \partial_\theta + u^z \partial_z$ is given by

$$\begin{aligned}\varphi_* u &= u^\theta \left(\frac{\partial \varphi^1}{\partial \theta} \Big|_p \frac{\partial}{\partial x} + \frac{\partial \varphi^2}{\partial \theta} \Big|_p \frac{\partial}{\partial y} + \frac{\partial \varphi^3}{\partial \theta} \Big|_p \frac{\partial}{\partial z} \right) \\&= u^z \left(\frac{\partial \varphi^1}{\partial z} \Big|_p \frac{\partial}{\partial x} + \frac{\partial \varphi^2}{\partial z} \Big|_p \frac{\partial}{\partial y} + \frac{\partial \varphi^3}{\partial z} \Big|_p \frac{\partial}{\partial z} \right)\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \varphi^1}{\partial \theta} \Big|_p &= -\sqrt{1 - z^2} \sin \theta & \frac{\partial \varphi^1}{\partial z} \Big|_p &= \frac{1}{2\sqrt{1 - z^2}} \cos \theta \\ \frac{\partial \varphi^2}{\partial \theta} \Big|_p &= \sqrt{1 - z^2} \cos \theta & \frac{\partial \varphi^2}{\partial z} \Big|_p &= \frac{1}{2\sqrt{1 - z^2}} \sin \theta \\ \frac{\partial \varphi^3}{\partial \theta} \Big|_p &= 0 & \frac{\partial \varphi^3}{\partial z} \Big|_p &= 1\end{aligned}$$

yielding

$$\begin{aligned}\varphi_* u &= \left(-u^\theta \sqrt{1 - z^2} \sin \theta + u^z \frac{1}{2\sqrt{1 - z^2}} \cos \theta \right) \frac{\partial}{\partial x} \\&\quad + \left(u^\theta \sqrt{1 - z^2} \cos \theta + u^z \frac{1}{2\sqrt{1 - z^2}} \sin \theta \right) \frac{\partial}{\partial y} \\&\quad + u^z \frac{\partial}{\partial z}\end{aligned}$$

so for $v = v^\theta \partial_\theta + v^z \partial_z$,

$$\begin{aligned} u \times v = & \left(\left(u^\theta \sqrt{1-z^2} \cos \theta + u^z \frac{1}{2\sqrt{1-z^2}} \sin \theta \right) v^z \right. \\ & \left. - u^z \left(v^\theta \sqrt{1-z^2} \cos \theta + v^z \frac{1}{2\sqrt{1-z^2}} \sin \theta \right) \right) \frac{\partial}{\partial x} \\ & + \left(u^z \left(-v^\theta \sqrt{1-z^2} \sin \theta + v^z \frac{1}{2\sqrt{1-z^2}} \cos \theta \right) \right. \\ & \left. - \left(-u^\theta \sqrt{1-z^2} \sin \theta + u^z \frac{1}{2\sqrt{1-z^2}} \cos \theta \right) v^z \right) \frac{\partial}{\partial y} \\ & + \left(\left(-u^\theta \sqrt{1-z^2} \sin \theta + u^z \frac{1}{2\sqrt{1-z^2}} \cos \theta \right) \right. \\ & \cdot \left(v^\theta \sqrt{1-z^2} \cos \theta + v^z \frac{1}{2\sqrt{1-z^2}} \sin \theta \right) \\ & - \left(u^\theta \sqrt{1-z^2} \cos \theta + u^z \frac{1}{2\sqrt{1-z^2}} \sin \theta \right) \\ & \cdot \left. \left(-v^\theta \sqrt{1-z^2} \sin \theta + v^z \frac{1}{2\sqrt{1-z^2}} \cos \theta \right) \right) \frac{\partial}{\partial z} \end{aligned}$$

We expect that for $\varphi(p) = (\sqrt{1-z^2} \cos \theta, \sqrt{1-z^2} \sin \theta, z)$,

$$\langle \varphi(p), \varphi_* u \times \varphi_* v \rangle = d\theta \wedge dz(u, v) = u^\theta v^z - u^z v^\theta.$$

Which seems likely... Perhaps doing the computations with the basis vectors like $u = \frac{\partial}{\partial \theta}$ would have helped... Was there another way to do this?

- b. My initial approach was to prove a version of the straightening lemma for 1-forms. Upon failure I discovered the following proof by Jack Lee (=John M. Lee) on [StackExchange](#):

Let α be any 1-form. Fix a point in the manifold and choose another 1-form β such that (α, β) is a local frame for the space of 1-forms at that point. This can be done since α is nowhere-vanishing. The dual frame consists is a pair of vector fields (W, V) such that locally, in particular, $\alpha(V) \equiv 0$. Then by the straightening lemma we can find a smaller neighbourhood of the point such that $V = \partial_x$.

Now if $\alpha = \alpha_1 dx + \alpha_2 dy$ in these coordinates, we see that $\alpha(\partial_x) = 0$ implies $\alpha_1 = 0$, so that $\alpha = \alpha_2 dy$.

Remark. This doesn't work in higher dimensions since we don't have the straightening lemma for several vector fields.

(This part is based on [StackExchange](#)) To prove Darboux theorem on a symplectic 2-manifold (M, ω) we need to show that there every point $x \in M$ has a coordinate

chart $\varphi : U \rightarrow \mathbb{R}^2$ such that $\varphi^* \omega_0 = \omega$ where $\omega_0 = dx \wedge dy$. Choose a contractible neighbourhood of x so that $\omega = d\eta$.

Since we are only interested in the derivative of η , we may assume that it is non-zero by adding if necessary a closed 1-form on some smaller compact neighbourhood of x where the coefficients of η are bounded i.e. a constant closed form larger than the bound. Then we can apply the result to write $\eta = fdg$. We get that

$$\omega = d\eta = d(fdg) = df \wedge dg.$$

Define $\varphi = (f, g)$ to get

$$\varphi^*(dx \wedge dy) = d\varphi^*x \wedge d\varphi^*y = d(x \circ \varphi) \wedge d(y \circ \varphi) = df \wedge dg = \omega.$$

I'm only left with the doubt on why φ is a diffeomorphism.

□

References

Lee, John M. *Introduction to Smooth Manifolds*. Second Edition. 2013.