Geometria Simplética 2024, Lista 6

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Entrega dia 23/10

Problem 1:

(a) Consider hamiltonian actions of G on two symplectic manifolds (M_i, ω_i), $i = 1, 2$, with moment maps $\mu_i : M_i \to \mathfrak{g}^*$, $i = 1, 2$. Show that the diagonal action of G on $M_1 \times M_2$ $(g(x_1, x_2) \mapsto (gx_1, gx_2))$ is hamiltonian, with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$, $\mu(x_1, x_2) = \mu_1(x_1) + \mu_2(x_2)$.

(b) Suppose that $G \curvearrowright M$ is a hamiltonian action with moment map μ, and let $H \subseteq G$ be a Lie subgroup. Show that the restriction of the action to H, $H \curvearrowright M$, is hamiltonian with moment map $\iota^* \circ \mu$, where $\iota : \mathfrak{h} \to \mathfrak{g}$ is the inclusion.

Problem 2: Consider the group $SO(3)$ acting on $T^*\mathbb{R}^3$ by the cotangent lift of the usual action of $SO(3)$ on $\mathbb{R}^3$.

a) For $u \in \mathfrak{so}(3)$, compute the corresponding infinitesimal generator $u_{T^*\mathbb{R}^3} \in \mathfrak{X}(T^*\mathbb{R}^3)$.

b) Identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ (as in Lista 6). Show that, with this identification, we have $u_{T^*\mathbb{R}^3}(q, p) = (u \times q, u \times p)$.

c) Identifying $\mathfrak{so}(3)^* \cong (\mathbb{R}^3)^* \cong \mathbb{R}^3$ using the usual inner product, show that the moment map for the action of $SO(3)$ on $T^*\mathbb{R}^3$ is $\mu(q, p) = q \times p$. Conclude (by Noether's theorem) that if $V \in C^\infty(\mathbb{R}^3)$ is $SO(3)$-invariant, then the flow of the hamiltonian $H(q, p) = \frac{p^2}{2m} + V(q)$ preserves "angular momentum" $q \times p$.

Problem 3: Consider $G = \mathbb{R}^2$ acting on $\mathbb{R}^2$ by $g \cdot (x, y) = (x + a, y + b)$, where $g = (a, b)$. Show that this action is *weakly* hamiltonian (i.e., there exists $\mu : M \to \mathfrak{g}^*$ such that $i_{u_M}\omega = d\langle \mu, u \rangle$) but it does *not* admit an equivariant moment map.

Problem 4: Consider a weakly hamiltonian G-action on (M, ω), with moment map $\mu : M \to \mathfrak{g}^*$ (i.e., not necessarily equivariant). We now see two independent cases in which we can always find a moment map which is equivariant.

a) For each $g \in G$, define $g \cdot \mu := (\mathrm{Ad}^*)_g(\mu \circ g^{-1})$ (in such a way that μ is equivariant if and only if $g \cdot \mu = \mu$ for all g). Show that $g \cdot \mu$ is also a moment map (not necessarily equivariant) for the action.

b) Suppose that G is compact. In this case, we can take a left-invariant volume form Λ on G (i.e., $L_g^*\Lambda = \Lambda$) satisfying $\int_G \Lambda = 1$ (why?). Consider the "average" $\overline{\mu} := \int_G g \cdot \mu$ (integral with respect to Λ). Show that $\overline{\mu}$ is an equivariant moment map.

c) Suppose that M is compact and connected. Then there is an equivariant moment map.

(*Hint: Note that we can take μ normalized so that $\int_M \mu = 0$ (integral with respect to the Liouville volume Λ_ω). Verify, using that M is connected, that this normalization uniquely characterizes the moment map. Conclude that μ is equivariant by showing that $\int_M g \cdot \mu = 0$, for all $g \in G$.*)

Problem 5: Consider the torus $\mathbb{T}^n$ acting on $\mathbb{C}^n$ (the canonical symplectic forms reads $\frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$) by:

$$(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (z_1, \dots, z_n) = (e^{ik_1\theta_1} z_1, \dots, e^{ik_n\theta_n} z_n),$$

where $k_1, \dots, k_n \in \mathbb{Z}$ are fixed.

a) Show that this action is hamiltonian, with moment map $\mu : \mathbb{C}^n \to (\mathfrak{t}^n)^* \cong \mathbb{R}^n$,

$$\mu(z_1, \dots, z_n) = -\frac{1}{2}(k_1|z_1|^2, \dots, k_n|z_n|^2).$$

b) Conclude (see Problem 1) that the action of S^1 on $\mathbb{C}^n$ given by multiplication by $e^{i\theta}$ on each coordinate is hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathbb{R}$, $\mu(z) = -\frac{1}{2}|z|^2$.

Problem 6: Consider the usual action of $U(n)$ on $\mathbb{C}^n$.

a) Writing elements $U \in U(n)$ in the form $A + iB$, check that the action of U on $\mathbb{R}^{2n}$ ($\cong \mathbb{C}^n$) is given by the linear symplectomorphism

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

The Lie algebra $\mathfrak{u}(n)$ consists of anti-hermitian matrices $u = \xi + i\eta$, with $\xi = -\xi^t \in M_n(\mathbb{R})$, $\eta = \eta^t \in M_n(\mathbb{R})$. Show that the infinitesimal generator of $u \in \mathfrak{u}(n)$ is hamiltonian with respect to

$$\mu^u(z) = -\frac{1}{2}\langle x, \eta x\rangle + \langle y, \xi x\rangle - \frac{1}{2}\langle y, \eta y\rangle,$$

where $z = x + iy$, $x, y \in \mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$ is the usual inner product.

b) Show that $\mu^u(z) = \frac{1}{2} i z^* u z = \frac{1}{2} i \operatorname{tr}(z z^* u)$.

c) Identify $\mathfrak{u}(n)$ with $\mathfrak{u}(n)^*$ through the inner product $(A, B) = \operatorname{tr}(A^* B)$. Let $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$,

$$\mu(z) = \frac{i}{2} z z^*.$$

Here $z \in \mathbb{C}^n$ is viewed as a $n \times 1$ matrix. Show that μ is equivariant (recall what the adjoint and coadjoint actions are) and conclude that μ is a moment map for the $U(n)$-action.

d) Consider the action of $U(k)$ on the space $\mathbb{C}^{k\times n}$ (with the canonical symplectic form), viewed as $k \times n$ matrices. Identify $\mathfrak{u}(k)$ with its dual as in item c). Show that a moment map for this action is

$$\mu(A) = \frac{i}{2} A A^* - \frac{i\mathrm{Id}}{2}.$$

(*Hint: combine problem 1(a) and the previous items, the constant factor is just for convenience.*)

Verify that $\mu^{-1}(0)/U(k)$ is naturally identified with the Grassmannian of k-planes in $\mathbb{C}^n$ (which hence acquires a symplectic form from symplectic reduction).

Problem 7: Consider a hamiltonian action $\psi : G \curvearrowright (M, \omega)$, with moment map $\mu : M \to \mathfrak{g}^*$. Consider the co-moment map $\hat{\mu} : \mathfrak{g} \to C^\infty(M)$, $\hat{\mu}(u) = \langle \mu, u \rangle$, and consider $C^\infty(M)$ equipped with the Poisson bracket.

(a) We saw in class that the equivariance of μ implies that $\hat{\mu}$ is an anti-homomorphism of Lie algebras. Show that the converse holds when G is connected.

(b) Show that for G connected, the fact that $\psi_g^* \omega = \omega$ follows from the condition $i_{u_M}\omega = d\langle \mu, u \rangle$.

Problem 8: Let G be a Lie group and consider the action by multiplication on the left: $G \times G \to G$, $(g, a) \mapsto L_g(a) = g.a$. Take the G-action on T^*G by cotangent lift, $\psi : G \times T^*G \to T^*G$.

(a) Consider the action of G_ξ on G by left multiplication, and the map $q : G \to \mathcal{O}_\xi$, $g \mapsto \mathrm{Ad}^*_{g^{-1}}\xi$ (where $\mathcal{O}_\xi$ is the coadjoint orbit thru ξ). Note that we have an induced bijection $G/G_\xi \to \mathcal{O}_\xi$. (It is a general fact that the quotient G/G_ξ is naturally a smooth manifold, and we equip $\mathcal{O}_\xi$ with the smooth structure for which this bijection is a diffeomorphism.)

Verify that $q(R_h(g)) = \mathrm{Ad}^*_{h^{-1}}(q(g))$, and conclude that $dq(u^L|_g) = u_{\mathfrak{g}^*}|_{q(g)}$ and $dq(u^R|_g) = (\mathrm{Ad}_{g^{-1}}(u))_{\mathfrak{g}^*}|_{q(g)}$.

(b) Verify that $\mu : T^*G \to \mathfrak{g}^*$, $\mu(\zeta_g) = (d_e R_g)^* \zeta_g$, is a moment map for ψ. Note that any $\xi \in \mathfrak{g}^*$ is a regular value for μ and that the natural projection $\pi : \mu^{-1}(\xi) \to G$ is a diffeomorphism. Show also that the projection induces a diffeomorphism:

$$(\mu^{-1}(\xi)/G_\xi) \xrightarrow{\sim} (G/G_\xi).$$

Using the identification $G/G_\xi \cong \mathcal{O}_\xi$ of (a), find an expression for the resulting diffeomorphism $\varphi : (\mu^{-1}(\xi)/G_\xi) \xrightarrow{\sim} \mathcal{O}_\xi$.

(c) Show that $-\varphi^* \omega_{kks} = \omega_{red}$, that is, the reduced space $(\mu^{-1}(\xi)/G_\xi, \omega_{red})$ is symplectomorphic to $(\mathcal{O}_\xi, -\omega_{kks})$.

Hints: recall that, for the projection $\pi_\xi : \mu^{-1}(\xi) \to \mathcal{O}_\xi$ and inclusion $i_\xi : \mu^{-1}(\xi) \to T^*G$, we must show that $\pi_\xi^* \omega_{kks} = i_\xi^* d\alpha_{tau}$; note that it suffices to check this equality on vectors X tangent to $\mu^{-1}(\xi)$ satisfying $\pi_* X = u^R$, for $u \in \mathfrak{g}$ (why?).

Problem 9: ("Shift trick") Let (M, ω, μ) be a hamiltonian G-space. Take a coadjoint orbit $\overline{\mathcal{O}_\xi}$ (with symplectic form $-\omega_{kks}$). Verify that the diagonal G-action on $M \times \overline{\mathcal{O}_\xi}$ is hamiltonian, with moment map

$$\hat{\mu} : M \times \overline{\mathcal{O}_\xi} \to \mathfrak{g}^*, \quad \hat{\mu}(x, \eta) = \mu(x) - \eta,$$

and that ξ is a regular value for μ if and only if 0 is a regular value for $\hat{\mu}$.
Note that we have a natural inclusion $j : \mu^{-1}(\xi) \hookrightarrow \hat{\mu}^{-1}(0)$, $x \mapsto (x, \xi)$. Show that this inclusion induces a diffeomorphism $\mu^{-1}(\xi)/G_\xi \xrightarrow{\sim} \hat{\mu}^{-1}(0)/G$ preserving the reduced symplectic forms.