Symplectic Geometry 02/2024, Homework 7

Professor: Leonardo Macarini

Deadline: 22/11

Problem 1:

(a) Let $\alpha \in \Omega^1(N)$ be a contact form. Consider $N \times \mathbb{R}$ equipped with the 2-form $d(e^t\alpha)$ (where t is the coordinate on $\mathbb{R}$). Verify that this 2-form is symplectic, and conclude that any (N, α) can be viewed as a hypersurface of contact type of a symplectic manifold.

(b) On the other hand: Let (M, ω) be symplectic and $\iota : S \hookrightarrow M$ a hypersurface of contact type, with contact form α. Suppose that S is compact. Show that there is a neighborhood U of S in M that is symplectomorphic to a neighborhood of S in its symplectication, $(S \times (-\epsilon, \epsilon), d(e^t\alpha))$, for an $\epsilon > 0$.

Hints:

- *Consider the conformally symplectic vector field X in a neighborhood of S (such that $\alpha = \iota^*(i_X\omega)$); use the tubular neighborhood theorem to obtain an identification $\psi : U \xrightarrow{\sim} S \times (-\epsilon, \epsilon)$, in such a way that X corresponds to $\frac{\partial}{\partial t}$.*
- *Let Y be the Reeb vector field on S. For each $x \in S$, write $T_xM = W_1 \oplus W_2$, where $W_1 = \ker(\alpha) \subset T_xM$ and $W_2 \subset T_xM$ is the subspace generated by $X(x)$ and $Y(x)$. Verify that W_1 and W_2 are symplectically orthogonal with respect to ω, and also with respect to $\psi^*d(e^t\alpha)$.*
- *Use the decomposition $T_xM = W_1 \oplus W_2$ to verify that ω and $\psi^*d(e^t\alpha)$ coincide on $TM|_S$. Use the Darboux-Weinstein theorem.*

(c) Let $\xi \in \Omega^1(N)$ be a contact form on N and $D = \ker(\xi) \subset TN$. Let $L \subset N$ be a submanifold such that $TL \subset D|_L$. (1) Check that T_xL is an isotropic subspace of the symplectic vector space $(D_x, d\xi|_x)$ for all $x \in L$, so $\dim(L) \leq \frac{1}{2}\mathrm{rank}(D)$. In case of equality, we call L **legendrian**. (2) Verify that L is legendrian iff $L \times \mathbb{R}$ is a lagrangian submanifold of the symplectization $N \times \mathbb{R}$.

(d) We saw that S^{2n-1} has contact structure $\alpha = \iota^*(\frac{1}{2}\sum_{i=1}^n q^i dp_i - p_i dq^i)$, where $\iota : S^{2n-1} \hookrightarrow \mathbb{R}^{2n}$ is the inclusion. Show that its symplectization is symplectomorphic to $\mathbb{R}^{2n} - \{0\}$ (with the canonical symplectic form).

Problem 2: Show the Darboux theorem for contact manifolds: Given contact manifold (N^{2n-1}, α), around any point there exist local coordinates $q^1, \ldots, q^{n-1}, p_1, \ldots, p_{n-1}, z$, such that $\alpha = \sum_i q^i dp_i + dz$.

Hint: One can, for example, use symplectization and adapt the symplectic Darboux theorem

Problem 3: Let N be a closed sympletic manifold (e.g. S^2), and let $M = N \times \mathbb{R}^2$, equipped with the product symplectic structure. Show that the hypersurface $C = N \times \{z \in \mathbb{R}^2 \mid \|z\| = 1\} \hookrightarrow M$ is not of contact type.

Problem 4: The *manifold of contact elements* of an n-dimensional manifold X is $\mathcal{C} = \{(x, \chi_x);\ x \in X \text{ and } \chi_x \text{ is a hyperplane in } T_xX\}$. On the other hand, the projectivization of the cotangent bundle of X is $\mathbb{P}^*X = (T^*X \setminus \text{zero section})/\simeq$, where $(x,\xi) \simeq (x,\xi')$ whenever $\xi = \lambda\xi'$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

(a) Show that $\mathcal{C}$ is naturally isomorphic to $\mathbb{P}^*X$ as a bundle over X.

(b) There is on $\mathcal{C}$ a canonical field of hyperplanes $\mathcal{H}$: $\mathcal{H}$ at the point $p = (x, \chi_x) \in \mathcal{C}$ is the hyperplane $\mathcal{H}_p = (d\pi_p)^{-1}\chi_x$, where $\pi : \mathcal{C} \to X$ is the projection. Therefore, by item (a), $\mathcal{H}$ induces a field of hyperplanes $\mathbb{H}$ on $\mathbb{P}^*X$. Describe $\mathbb{H}$.

(c) Check that $(\mathbb{P}^*X, \mathbb{H})$ is a contact manifold, and therefore $(\mathcal{C}, \mathcal{H})$ is a contact manifold.

(d) What is the symplectization of $\mathcal{C}$?

Problem 5: Let (M,α) be a contact manifold with contact structure $\xi = \ker\alpha$. A *contact vector field* X on M is a vector field whose (linearized) flow preserves ξ.

(a) Let R_α be the Reeb vector field of α. Prove that R_α is a contact vector field.

(b) Suppose that X is a contact vector field transverse to ξ. Show that it can be written as a Reeb vector field for some 1-form α_X defining the contact structure ξ.

Problem 6: Consider, in $\mathbb{R}^{2n}$, the ellipsoid

$$S_{(r_1,\ldots,r_n)} = \{(q_1,\ldots,q_n,p_1,\ldots,p_n) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} \frac{q_j^2 + p_j^2}{r_j^2} = 1\},$$

where $r_1,\ldots,r_n$ are positive. We saw that S is a hypersurface of contact type. Compute its Reeb vector field and describe its flow. Describe general conditions on $r_1,\ldots,r_n$ so that: (1) there are only n periodic orbits; (2) there are infinitely many periodic orbits.

Problem 7: Consider the unit sphere $S^3 \subset \mathbb{R}^4$ endowed with the contact form induced by the Liouville form. Show that its Reeb flow has a global surface of section given by a disk, that is, there is an embedding $\psi : D^2 \to S^3$ such that $\psi(\partial D^2)$ is a closed Reeb orbit, the Reeb vector field is transverse to $\psi(\operatorname{Int} D^2)$ and every Reeb orbit not contained in $\psi(\partial D^2)$ intersects $\psi(\operatorname{Int} D^2)$ infinitely many times in the future and the past.