SEMINARS

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Contents

1. A primer on symplectic groupoids

Camilo Angulo, Jilin University. Geometric Structures Seminar, IMPA. February 13, 2025.

Abstract. In the late 17th century, Simeon Denis Poisson discovered an operation that helped encoding and producing conserved quantities. This operation is what we now know as a Lie bracket, an infinitesimal symmetry, but what is its global counterpart? Symplectic groupoids are one possible answer to this question. In this talk, we will introduce all the basic concepts to define symplectic groupoids, and their role in Poisson geometry. We will discuss key examples, and applications.

The talk will be accessible to those familiar with differential geometry, but no prior knowledge of groupoids will be assumed.

Part 1. Poisson geometry.

Hamiltonian formalism. Recall that being a conserved quantity $f \in C^\infty(X)$ is the same thing as $\{H, f\} = 0$.

- We have seen that it is always possible to take quotient of a symplectic manifold with a group action to obtain a Poisson manifold.
- Then we have found a way to produce a symplectic foliation from a 2-vector $\pi \in \mathfrak{X}^2(M) := \Lambda^2(TM)$.
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  Remark 1.1.

  $$\{\text{Lie algebra on } \mathfrak{g}\} \xrightarrow{1-1} \{\text{Linear Poisson bracket on } \mathfrak{g}^*\}$$
- We saw very nice examples of foliation that have to do with Lie algebras. So $\mathfrak{b}_3^*$ which gives the "open book foliation", and $\mathfrak{e}^*$ that gives a foliation by cylinders.

Part 2. Symplectic realizations.

Consider

$$(\Sigma, \omega) \xrightarrow{\mu} (M, \pi)$$

So that

$$\pi^\sharp = d_p\mu \circ \omega^{-1} \circ (d\mu)^*$$

Lemma 1.2. $\dim(\Sigma) \geq 2\dim(M) - \mathrm{rk}(\pi_x)$ *for all* $x \in M$.

Proof. Done in seminar. □

Example 1.3. $(\mathbb{R}^2, 0)$. So the map

$$\begin{aligned}(\mathbb{R}^4, dx \wedge du + dy \wedge dv) &\longrightarrow \mathbb{R}^2\\ (x, y, u, v) &\longmapsto (x, y)\end{aligned}$$

Exercise 1.4. Find the symplectic realization ω in $(\mathbb{R}^4, \omega) \xrightarrow{\mu} (\mathbb{R}^2, (x^2+y^2)\partial_x \wedge \partial_y$

$$\begin{aligned}(\mathbb{R}^4, \omega) &\longrightarrow (\mathbb{R}^2, (x^2 + y^2)\partial_x \wedge \partial_y)\\ (x, y, z, w) &\longmapsto (x, y)\end{aligned}$$

Also find the symplectic realization of aff^* with $\{x, y\} = x$.

Part 3: Groupoids

Motivation

(1) Fundamental grupoid: objects are points in the manifold and arrows are paths.
(2) $S^1 \curvearrowright S^2$ by rotation does not give a nice quotient because there are two singular points. Consider the groupoid $S^1 \times S^2$ of orbits. These are the arrows. The points are just the points of S^2.
(3) Consider a foliation (like Möbius foliation of circles; where there is a singular circle, the soul). You can do the same thing as in fundamental groupid leafwise. Arrows then are equivalence classes of paths that live inside leaves. This is called monodromy of a foliation. Again objects are points.

(4) You can take a quotient of monodromy using a connection given by the foliation. This allows to identify certain paths between the leaves. This is called *holonomy (of a foliatiation)*. (So you can make this notion match the usual holonomy given by riemmanian connection.)

Upshot. So the point is taking some sort of function space on these groupoids you can gather the information given by the non-smooth quotient (like in the case of the sphere rotating). So this groupoid motivation says how to get some structure that resembles a non smooth quotient.

(5) Last motivation: the grid of squares has a tone of symmetries. If you restrict to just a few squares you loose so many symmetries. But there's a grupoid hidden in there that tells you what you intuition knows about this finite grid of squares.

Definition 1.5. A *groupoid* is a category where all morphisms are invertible.

So there is a kind of product among the objects, given by composition but: not every two pair of objects can be multiplied!—only those whose source and target match. So that's the lance about grupoids.

Just so you make sure you understand: the groupoid G is the morphisms of the category. The objects are points (of a manifold).

Definition 1.6. *Lie grupoid* is when the following diagram is inside category of smooth manifolds and s, t (source and target maps) are submersions:

$$G^{(2)} \xrightarrow{m} G \xrightarrow[t]{s} M \xrightarrow{u} G$$

Proposition 1.7 (Properties of Lie groupoids)**.**

- *m is also a submersion.*
- *i (inversion) is a diffeomorphism.*
- *u (unit=identity) is an embbeding.*

Definition 1.8. Consider $x \in M$ and the inverse image of source map: $s^{-1}(x) = \{$arrows that start at $x\}$. Now if you act with t on this set you get *the orbit* of x: $\{y \in M$ such that there is an arrow from x to $y\}$.

And there also *an isotropy* $G_x = \{g \in G : g$ goes from x to $x\}$

Example 1.9.

(1) $G = M$, $M = M$.
(2) Lie groups.
(3) Lie group bundles.
(4) $G = M \times M$, $M = M$.
(5) Fundamental groupoid. Isotropy group is fundamental group! And orbit is...

universal cover!
(6) Subgroupoids.
(7) Foliations.
(8) If you have a normal group action $G \curvearrowright M$ you construct a groupoid action with groupoid $G \times M$ and objects M, with product given on the group part of the product. Orbits are orbits. Isotropy group is isotropy group.
(9) Principal bundles.

Back to Poisson.

There's also a notion of Lie algebroid. Which is strange. But the point is that to every Poisson manifold there is a Lie algebroid.

So the question is whether there is a Lie groupoid associated to that Lie algebroid. Not always.

Big question[Fernandez and ?] When a symplectic manifold is integrable?

(Remember that integrating means go from algebra(oid) to group(oid).

And the point is that:

When you *can* go back, you get a *symplectic groupoid.*

Remark 1.10. Look for Kontsevich's notes on Weinstein!

Remark 1.11. History: Weinstein did this intending to do quantization (geometric?) on Poisson manifolds. (That involves a C^*algebra coming from the symplectic groupoid.)

Definition 1.12. A *symplectic groupoid* is a groupoid G, M together with $\omega \in \Omega^2(M)$ such that ω is symplectic and multiplicative, meaning that $\partial\omega = 0$, that is, $\iff m^*\omega = \mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega \in \Omega^2(G^{(2)}) \iff$ take two vectors $X_k, Y_k \in TG$, and $\omega(X_0 \star Y_0, X_1 \star Y_1) = \omega(X_0, Y_0) + \omega(X_1, Y_1)$

Theorem 1.13. *If (G, ω) is a symplectic groupoid, then*
(1) $\exists!$ *poisson structure on M*
(2) *for which $t : G \to M$ is a symplectic realization,*
(3) *Leaves are connected components of orbits,*
(4) $\mathsf{Lie}(G) \cong T^*M$ *via* $X \mapsto -u^*(i_X\omega$.

Remark 1.14. Look for Alejandro Cabrera, Kontsevich. There are two things one is de Rham and the other... from the future: simplicial?

Upshot. The obstruction to knowing when symplectic groupoid exists is "variation of symplectic form $\omega = (1 + t^2)\omega_{S^2}$". So how does the symplectic group vary from leaf to leaf. So there are two situations in which the thing doesn't work.

2. Neutrinos

Hiroshi Nunokawas, PUC-Rio. Friday Seminar, Seminar Name. June 27, 2025.

Abstract. Hiroshi will come and tell us everything we (not) wanted to know about these mysterious particles, and are not going to be afraid to ask. In particular, about the neutrino oscillation, and the great matrices.

Protons and neutrons have very similar mass of $m_p \approx 940$ MeV, while electrons have mass of $m_e \approx 0.5$ MeV. MeV is 10^6 electronvolts, where one eV is approximately 1.6×10^{-19} J. This is standard in high energy physics, they use electronvolts instead of Joules. Recall that 2J=1N$\times$1m.

Most of the things we see are protons since they are so much larger than electrons. But protons nor neutrons are elementary particles.

Here's the standard model:

$$\begin{array}{cccc}
\text{Quarks} & \begin{bmatrix} u \\ d \end{bmatrix}_L & \begin{bmatrix} c \\ s \end{bmatrix}_L & \begin{bmatrix} t \\ b \end{bmatrix}_L \\
\text{Leptons} & \begin{bmatrix} \nu_e \\ e^- \end{bmatrix}_L & \begin{bmatrix} \nu_\mu \\ \mu^- \end{bmatrix}_L & \begin{bmatrix} \nu_\tau \\ \tau^- \end{bmatrix}_L \\
\text{Generation} & \text{1st} & \text{2nd} & \text{3rd} \\
\text{Bosons} & g,\, \gamma\ ,\, \omega^{\pm},\, z & & \\
\text{Higgs Bosson} & H & &
\end{array}$$

It is very particular that nature repeats itself three times. The L in those matrix actually means left-handed, and accounts for chirality. Only left-handed fermions have weak interaction. Right-handed have electromagnetic interaction, gravitational interaction, but not weak interaction.

And then there's neutrinos. They have negative helicity (chirality). Being left-handed, mathematically, means to have helicity -1. I think this means that the spin is left-handed. But chirality and helicity are not the same: helicity is observer-dependent, and chirality is not. Almost all neutrinos we can see (% 99.99999...) have negative helicity, but not all of them.

Consider the following:

$$n + \nu_e \leftrightarrow p + e^-$$

But it's not completely correct: we'd better put d instead of n, and u instead of p: the d and u quarks, instead of the neutrons and protons.

Now consider the following reaction: a neutron decays into a proton, an electron and an antineutrino:

$$n \to p + e^+ \overline{\nu}_e$$

Protons is very stable, that's why we are here. But neutron decays in only 15 minutes.

By experimental data, we can conclude that neutrinos' mass is consistent with zero. But if they have mass, it should be much smaller than the electron's $m_v \leq 0.5$ eV. And the electron is already the lightest fermion!

If the mass of the neutrino was zero, i.e. $m_\nu = 0$, then $v_\nu = c$ in vacuum, which would imply that

$$\begin{array}{ccccc}
\nu_e & \xrightarrow{\ L\ } & \nu_e & \xrightarrow{\qquad} & \nu_e \\
 & & & & \\
0:00 & & 0:00 & & 0:00
\end{array}$$

meaning: time doesn't pass! And this means the state of the particle cannot change.

3. Spheres with minimal equators

Lucas Ambrozio, IMPA. Differential Geometry Seminar, IMPA. June 24, 2025.

Abstract. We will discuss the connection between Riemannian metrics on the sphere with respect to which all equators are minimal hypersurfaces, and algebraic curvature tensors with positive sectional curvatures.

Definition 3.1. An $(n-k)$-*equator* orthogonal to Π is

$$\Sigma_\Pi := \{p \in \mathbb{S}^n : \langle p, x\rangle = 0 \forall x \in \Pi\}$$

for Π a k-dimensional linear subspace of $\mathbb{R}^{n+1}$.

Remark 3.2. Equators are totally geodesic hypersurfaces with the usual sphere metric, which implies they are minimal hypersurfaces.

Problem. Characterize the set $\mathcal{M}_k(U)$ of metrics g on an open set $U \subset \mathbb{S}^n$ such that all k-equators Σ_Π with $\Sigma \cap U \neq \emptyset$ yield are minimal hypersurfaces $\Sigma \cap U$ on (U, g).

Remark 3.3. This problem can be thought of as a problem of finding metrics on $\mathbb{R}^n$ such that k-planes are minimal. To see why project the k-equators to $T_p\mathbb{S}^n$ and pullback those metrics to the sphere.

Let $g \in \mathcal{M}_k(U)$ for $U \subset \mathbb{S}^n$ open and $n \geq 2$.

Theorem 3.4 (Beltrami, Schäfli)**.** *If $k = 1$ then g has constant sectional curvature.*

Theorem 3.5 (Hongan)**.** *If $1 < k < n-1$ then g has constant sectional curvature.*

Then Hongan also managed to produce a classification of these metrics for $k = n-1$.

Remark 3.6. If $T \in \mathrm{GL}(n+1, \mathbb{R})$, then

$$\begin{aligned} \varphi : \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ x &\longmapsto \frac{Tx}{|Tx|} \end{aligned}$$

is a diffeomorphism that maps k-equators into k-equators. Thus if $g \in \mathcal{M}_k(\mathbb{S}^n)$ then so is $\varphi(T)^* g$.

Theorem 3.7. *There exists a $GL(n+1, \mathbb{R})$ equivariant bijection*

$$\mathcal{M}_{n-1}(\mathbb{S}^n) \leftrightarrow Curv_+(\mathbb{R}^{n+1})$$

where the set on the right-hand-side is the set of algebraic curvature tensors (also called curvature-like, i.e. with the same symmetries as the Riemannian curvature tensor) on $\mathbb{R}^{n+1}$ with positive sectional curvature.

The group action is given as follows for $T \in GL(n+1, \mathbb{R})$:

$$(R \cdot T)(x, y, z, w) = \frac{1}{|\det(T)|^{\frac{1}{n+1}}} R(Tx, Ty, Tz, Tw)$$

The point is that $\mathrm{Curv}_+(\mathbb{R}^{n+1})$ is an open cone on a linear space. Here are two simple corollaries:

Lemma 3.8.
1. *$\mathcal{M}_{n+1}(\mathbb{S}^n)$ is in bijection with an open positive cone of an $\frac{n(n+2)(n+1)^2}{12}$-dimensional real vector space.*
2. *Every metric on $\mathcal{M}_{n-1}(\mathbb{S}^n)$ is invariant by the antipodal map.*

Algorithm. From any $R \in \mathrm{Curv}_p(\mathbb{R}^{n+1})$ we obtain a symmetric positive definite (positive-definitiness comes from the positiveness of the curvature of R) 2-tensor k_R satisfying

$$(k_R)_p(v, v) = R(pv, pv) > 0$$

Also, k_R has the *Killing property*, i.e. that $\overline{\nabla}k(X,X,X)=0$ for all $X\in\mathfrak{X}(\mathbb{S}^n)$.

Then we define a positive function on $\mathbb{S}^n$ by

$$D_R := \left(\frac{d\mathrm{Vol}_{k_R}}{dV_g}\right)^{\frac{4}{n-1}} \tag{3.8.1}$$

and finally a Riemannian metric on $\mathbb{S}^n$ in $\mathcal{M}_{n-1}\mathbb{S}^n$ by

$$g_R=\frac{1}{D_R}k_R$$

And to go back, for $g\in\mathcal{M}_{n-1}(\mathbb{S}^n)$ define a positive function on $\mathbb{S}^n$

$$F_g:=\left(\frac{dV_g}{dV_{\overline{g}}}\right)^{\frac{4}{n-1}}$$

Then let $k_g:=\frac{1}{F_g}g>0$, which is a positive definite Killing 2-tensor, from which we may define $R_g\in\mathrm{Curv}_+(\mathbb{R}^{n+1})$ with $R_g(pv,pv)=(k_g)_p(v,v)$ for all $p,v\in T\mathbb{S}^n$.

More corollaries:

Lemma 3.9. (1) *$g\in\mathcal{M}_{n-1}(\mathbb{S}^n)$ is analytic because it is a Killing tensor on $\mathbb{S}^n$, which are well-known.*

(2) *If g is left-invatiant on $\mathbb{S}^3$, seen as unit quaternions, then $g\in\mathcal{M}_2(\mathbb{S}^3$. Moreover, for $a\geq b\geq c>0$,*

$$aL_i\odot L_i+bL_j\odot L_j+cL_k\odot L_k=k$$

is Killing, $k>0$, D_k constant and thus $g=\frac{1}{const.}k\in\mathcal{M}_2(\mathbb{S}^3)$.

(3) *R curvature tensor of $(\mathbb{C}P^2,g_{FS})$. We may not remember what's the curvature tensor, but we know the sectional curvature is $1\leq sec(R)\leq 4$,*

$$(k_R)_p(v,w)=\overline{g}(v,w)+3\overline{g}(Jp,v)\overline{g}(Jp,w)$$

and $D_R=4^{\frac{4}{3-1}}=4$, so that by 3.8.1 we obtain $g_R=\frac{1}{4}k_R$, which is a Berger metric on $\mathbb{S}^3$ with scalar curvature 0.

Now define

$$\Sigma_V=\{p\in\mathbb{S}^n:\langle p,v\rangle=0\}=V^{-1}(0)$$

where $V(x):=\langle x,v\rangle$ for all $x\in\mathbb{S}^n$. Then the normal vector field is $\nabla V/|\nabla V|_g$, and the second fundamental form is given by

$$A=\frac{1}{|\nabla V|_g}\mathrm{Hess}_gV$$

and its mean curvature by

$$H=\frac{1}{|\nabla V|_g}\left(\Delta_gV-\mathrm{Hess}_gV\left(\frac{\nabla V}{|\nabla V|},\frac{\nabla V}{|\nabla V|}\right)\right) \tag{3.9.1}$$

For every $v\in\mathbb{S}^n$ and $p\in\Sigma_V$, we see that $H_{\Sigma_V}=0$ iff

$$|\nabla V|_g^2(p)\Delta_gV(p)-\mathrm{Hess}_gV(\nabla V(p),\nabla V(p))=0$$

And for $\overline{g}$,

$$\mathrm{Hess}_{\overline{g}}V+V\overline{g}=0\implies\mathrm{Hess}_{\overline{g}}V(X,X)=0$$

for all $X\in T_p\mathbb{S}^n$ and $p\in\Sigma_v$. Then

$$J_g(X,Y,Z)=g(\nabla_XY-\overline{\nabla}_XY,Z)$$

$$J_g(X, Y, \nabla V) = \text{Hess}_{\overline{g}} - \text{Hess}$$

Problems.

(1) Similar story for $\mathbb{C}P^n, \mathbb{H}P^n$?
(2) Complete metrics on $\mathbb{R}^n$ with minimal hyperplanes.
(3) Find geometric invariants of metrics on $\mathcal{M}_{n-1}(\mathbb{S}^n)$ (may be useful to study (M^n, g), $n \geq 4$, sec > 0.

4. Smoothable compactified Jacobians of nodal curves

Nicola Pagani, University of Liverpool and Bologna. Seminar of Algebraic Geometry UFF. August 20, 2025.

Abstract. Building from examples, we introduce an abstract notion of a 'compactified Jacobian' of a nodal curve. We then define a compactified Jacobian to be 'smoothable' whenever it arises as the limit of Jacobians of smooth curves. We give a complete combinatorial characterization of smoothable compactified Jacobians in terms of some 'vine stability conditions', which we will also introduce. This is a joint work with Fava and Viviani.

Let C be a smooth curve and $d \in \mathbb{Z}$. Define

$$J_C^d = \{L : L \text{ is a line bundle of degree } d\}/\sim$$

which is a smooth projective variety of dimension $g(C)$.

If C is nodal we still can consider J_C^d.

(1) One connected component. Then the Jacobian is $\mathbb{P}^1$ minus two points. This is not universally closed, so it is not proper.
(2) Two components intersecting at one point. The pullback of the normalization splits the degree in intintely many ways, givieng that J_C^{-1} is an infinite set of points. This is not of finite type, so it is not proper.
(3) The curve has two components intersecting at two points. This gives J_C^{-2}, which is a mixture of the two former items. (Probably not proper too.)

Now consider

$$\text{TF}_C^d = \{\mathcal{F} : \text{coherent on } C, \text{ torsion-free, rank-1 on } C\}/\sim$$

This satisfies the existence oart of the valu point of properness.

Now we consider the moduli. Now we consider the ideal sheaf of the (singular?) point(s?):

(1) One component. The stack is proper!
(2) Two components intersecting once. Now we get stacky points, $x = [\bullet/\mathbb{G}_m]$. These points have generic stabilizer. The resulting stack is not separated because a morphism of a curve, say $\mathbb{P}^1$ minus a point ... there are infinitely many ways to extend a morphism from this thing to a line bundle. So you cannot include any of these stacky points. Recall that a sheaf is *simple* if its automorphism group is $\mathbb{G}_m$.
(3) The ideal sheaf of both nodes $\mathcal{I}(N_1, N_2)$ has a positive dimensional automorphism group. The stack is not proper.

Definition 4.1. A *fined compactified Jacobian* of C is an open connected substack of $\text{TF}^d(C)$ that is also proper.

Remark 4.2. This thing is automatically an algebraic space.

Definition 4.3. A *compactified Jacobian* is an open connected of $\mathrm{TF}^d(C)$ that admits a proper, good moduli space.

Consider the Artin stack $\mathfrak{X} \xrightarrow{\Gamma} X$ [...] is a *good moduli space* if

(1) Every moduli factors

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[drr] & \mathcal{I} \text{ (ACC. space)} & \\ & & X \end{tikzcd}$$

(2) $\pi_* \mathcal{O}_{\mathfrak{X}} = \mathcal{O}_X$.

We expect to find a notion of stability condition to produce these things [...] $[\bullet/\mathbb{G}_m]$ would be the polystable representative.

Definition 4.4. A compactified Jacobian $\overline{J_C}$ is *smoothable* if all smoothings $\mathcal{C} \to \Delta = \{0, \eta\}$ (with $\mathcal{C}_0 = C$),

$$J^d_{\mathcal{C}_\eta} \cup C \to \overline{J_C}$$

is proper.

Definition 4.5. Let X be a curve.

$$\mathrm{BCON}(X) = \{Y \subseteq X \text{s.t. } Y, Y^c \text{ are connected}\}$$

Definition 4.6. A v-curve is a generalization of items (2) and (3) in the lists above [it looks like two long snakes $\sim$ that intersect several times, and t is the number of nodes]. A *v-condition* is a pair $n = (n_1, n_2)$ such that

$$n_1 + n_2 = \begin{cases} d + 1 - t & \text{we say the s.c. is nondegenerate} \\ d - t & \text{degenerate} \end{cases}$$

$\mathcal{F}$ on X is *n-(semi)stable* if $\deg \mathcal{F}_{X_i} > n_i$ ($\deg \mathcal{F}_{X_i} \geq n_i$) for $i = 1, 2$.

$\mathcal{F}_{X_i} = \mathcal{F}|_{X_i}$ torsion.

$$\deg(\mathcal{F}_{X_i}) + \deg(\mathcal{F}_{X_2}) = d - |\mathrm{sing}(F)|.$$

Then

$$\overline{J_C}(n) = \{\mathcal{F} \text{ is semistable}\},$$

a smooth compact Jacobian.

Definition 4.7. A *degeneratation* of v-stab. on X is $n : \mathrm{BCON}(X) \to \mathbb{Z}$ such that

(1)

$$n_Y + n_{Y^c} + |Y \cap Y^c| = \begin{cases} d + 1 & \text{we say } Y \text{ is } n\text{-nondegenerate} \\ d & Y \text{ is } n\text{-degenerate} \end{cases}$$

(2) Y_i no pa. common component $n_{Y_1} + n_{Y_2} + \dots$

$$\begin{tikzcd} & Y_1 \arrow[dl, <->] \arrow[dr, <->] & \\ Y_2 \arrow[rr, <->] & & Y_3 \end{tikzcd}$$

Theorem 4.8 (-, et al)**.** *(bijection between stability conditions and nodal curves) The map*

$$\left\{\begin{smallmatrix} \textit{sm. comp.} \\ \textit{Jac of } X \end{smallmatrix}\right\} \to \left\{\begin{smallmatrix} \textit{v-stab.} \\ \textit{cond. of } X \end{smallmatrix}\right\}$$
$$n \mapsto \overline{J_X}(n) = \{n\text{-semistable sheaves}\}$$

is a bijection. (The arrow should be from right to left!)

F. Viviani had proved it for fine compact Jacobians.

5. Equivariant spaces of matrices of constant rank

Ada Boralevi, France. Algebraic Geometry Seminar, IMPA. August 27, 2025.

Abstract. A space of matrices of constant rank is a vector subspace V, say of dimension n+1, ofthe set of matrices of size axb over a field k, such that any nonzero element of V has fixed rank r. It is a classical problem to look for different ways to construct such spaces of matrices. In this talk I will give an introduction up to the state of the art of the topic, and report on my latest joint project with D. Faenzi and D. Fratila, where we give a classification of all spaces of matrices of constant corank one associated to irreducible representation of a reductive group.

We are interested in vector spaces $U \subset \mathrm{Mat}_{m,n}(\mathbb{C})$, with $m \leq n$, of *constant rank*, i.e. such that for all $f \in U$, $r := \mathrm{rank} f$ is the same.

Let $\ell(r, m, n) := \max \dim U : U$ is of rank r.

Questions.

(1) $\ell(r, m, n) =$? In general not known.
(2) Find relations among ℓ, r, m and n.
(3) Construction of examples and classification.

Example 5.1. (1) $\ell(1, m, n) = n$,

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

(2) rank = 2? There are two cases ([Atkinson '83], [Eisenbud-Haus '88])
- Compression spaces,

$$\begin{pmatrix} * & * & \cdots & * \\ * & 0 & \cdots & 0 \\ \vdots & & & \\ * & 0 & \cdots & 0 \end{pmatrix}$$

- Skew-symmetric matrices of 3×3.

(3) $\ell(r, m, n) \geq n - r + 1$. Because you can put a matrix of $m \times n$ (with the first r rows that can have nonzero entries):

$$\begin{pmatrix} x_1 & x_2 & \cdots & & x_{n-r+1} \\ & x_1 & x_2 & \cdots & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

Theorem 5.2 (Westwick '86)**.** (1) $n - r + 1 \leq \ell(r, m, n) \leq 2n - 2r + 1$.

(2) *If* $n - r + 1 \not| \frac{(m-1)!}{(r-1)!} \implies \ell$

We can see these spaces as (subvarieties?) of determinantal varieties $M_n = \{f \in \mathrm{Mat}_{m,n}(\mathbb{C}) : \mathrm{rank}(f) \leq r\}$. [Interpretation via secant varieties inside Segre embedding.]

Consider a map $\varphi : U \to \mathrm{Hom}(V, W)$. Then $\varphi \in U^* \otimes V^* \otimes W$. We get that φ is of constant rank if and only if some kernel, image and cokernel are vector bundles.

Focus of today. What happens when U, V, W are irreducible representations of a complex reductive group G?

Question. What is the natural equivariant morphism

$$U \to \mathrm{Hom}(V, W) = V^* \otimes W$$

of constant rank?

Consider the case of $G = \mathrm{SL}_2$. All the irreducible representations (which are self-dual) are given by $V(m-1) \cong \mathbb{C}[x, y]_{\deg = m-1}$.

Recall the Clebsh-Gordon decomposition ($m \leq n$)

$$V(m-1) \otimes V(n-1) = \bigoplus_{i=0}^{m-1} V(n - m + 2i)$$

Theorem 5.3 (B-Faenzi-Lella '22)**.**

$$V(n + m - 2) \hookrightarrow \mathrm{Hom}(V(m-1), V(n-1))$$

is of constant rank (corank 1) if and only if

$$n - m + 2i | m - 1$$

Theorem 5.4 (B. Faenzi, Fratila '25)**.** *Let $V(\nu)$, $V(\mu)$ and $V(\lambda)$ be irreducible representations of a comple reductive group G, with*

$$\dim(V(\mu)) \leq \dim(V(\lambda))_-$$

If there exists a morphism of representations

$$\varphi : V(\nu) to \, \mathrm{Hom}(V(\lambda), V(\mu))$$

then φ is of constant corank 1 if and only if there exists a simple root α_i such that

(1) $\lambda = \mu + \nu - \alpha_i$,
(2) *ν is a multiple of ν,*
(3) *ν is a multple of ω_i*

6. On wrapped Floer homology barcode entropy and hyperbolic sets

Rafael Fernandes, UC Santa Cruz. Differential Geometry Seminar, IMPA. September 4, 2025.

Abstract. In this talk, we will discuss the interplay between the wrapped Floer homology barcode and topological entropy. The concept of barcode entropy was introduced by Çineli, Ginzburg, and Gürel and has been shown to be related to the topological entropy of the underlying dynamical system in various settings. Specifically, we will explore how, in the presence of a topologically transitive, locally maximal hyperbolic set for the Reeb flow on the boundary of a Liouville domain, barcode entropy is bounded below by the topological entropy restricted to the hyperbolic set.

Let M^n be a manifold. $\omega \in \Omega^2(M)$ is *symplectic* if $d\omega = 0$ and it is nondegenerate.

Example 6.1. $\mathbb{R}^n$ is symplectic with canonical Darboux form.

Recall the definition of Hamiltonian vector field associated to a function $H \in C^\infty(M)$.

Definition 6.2. A diffeomorphism $\varphi : M \to M$ is called *non-degenerate* if $\Phi(\varphi) \cap \Delta \subset M \times M$ (pitchfork, i.e. transversal intersection!).

Let M^{2n} be a closed symplectic manifold. Arnold's conjecture says

(1) If $\varphi = \varphi_H$ (Hamiltonian flow) is nondegenerate, then
$$\#\mathrm{Fix}(\varphi_H) \geq \sum_{i=0}^{2n} \dim H_i(M,k) = \dim H_*(M,k)$$
(2) If $\varphi = \varphi_H$ is degenerate, then
$$\#\mathrm{Fix}(\varphi) \geq \mathrm{Cl}(M) + 1$$
where $\mathrm{Cl}(M)$ is the maximum number of homology classes we can add before getting to zero.

Why do we care? Because
$$\#\mathrm{Fix}(\varphi_H) \leftrightarrow \{\text{1-periodic orbits of } X_H\}$$

Idea by Floer. Construct an invariant that would say something about periodic orbits.

Question. Can Floer theory capture other "dynamical information"? (Other than the periodic orbits.)

A *persistence module* is a pair (V, Π), where $V = \{V_t\}_{t\in\mathbb{R}}$ is a family of $\mathbb{F}$-vector spaces and $\Pi = \{\Pi_{st}\}_{s\leq t}$ is a family of maps such that

(1) $\Pi_{ss} =, \Pi_{ts} \circ \Pi_{rt} = \Pi_{rs}$.
(2) $\exists s \subset \mathbb{R}$ such that Π_{st} is an isomorphism for s, t in the same connected component of $R \setminus S$.
(3) Π_{st} have finite rank.
(4) $\exists s_0$ $V_s = \{0\}$, $s \leq s_0$.
(5) $V_t = \lim_{s\to t} V_s$ (lower limit!!)

Theorem 6.3. *Any persistence module is a sum of integral persistence modules,*
$$(V, \Pi) \cong \bigoplus_{I\in B(V)} F(I).$$

Example 6.4. Heart and sphere. There is a noise in the persistence module of the heart due to an unnecessary critical point.

(M^{2n}, ω) a *Liouville domain* is a compact symplectic manifold and $X \in \mathfrak{X}(M)$ with $X \cap \partial M$ (pitchfork, i.e. transversal intersection!) pointing outwards and preserved by the symplectic form, i.e. $\mathcal{L}_X\omega = \omega$ ($\omega = d\alpha$).

When we restrict ω to the boundary, we obtain a contact form and get some interesting dynamics.

A Lagrangian $(L, \partial L) \subset (M, \partial M)$ is *asympotically conical* if

(1) $\partial L \subset \partial M$ is Legendrian.

(2) $L \cap [1-\varepsilon, 1] \times \partial M = [1-\varepsilon, 1] \times \partial L$.

Remark 6.5. Take a Hamiltonian $H : \hat{M} \to \mathbb{R}$ such that

$$\begin{cases} H(r,x) = h(r) & r = 1 \\ H(r,x) = rT - B \end{cases}$$

then $X_H = h'(r) R_\alpha$.

For $L_0, L, A \subset$ Lagrangians, H linear at infinite, then $A_H^{L_0 \to L_1}$,

$$\begin{aligned} A_H^{L_0 \to L_1} : P_{L_0 \to L_1} &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \int_0^1 \gamma^* \alpha - \int_0^1 H(x(t)) dt \end{aligned}$$

where $P_{L_0 \to L} = \{\gamma : [0,1] \to \hat{M} : \gamma(0) \in L_0, \gamma(1) \in L_1\}$ is the set of chords.

Remark 6.6. $\operatorname{crit}(A_H^{L_0 \to L_1}) = \{$1-chords of X_H from L_0 to $L_1\}$.

Putting a metric on $P_{L_0 \to L_1}$ we can consider $\varphi : \mathbb{R} \times [0,1] \to \hat{M}$, solutions of some PDE which is some kind of generalization of a gradient, $-\nabla A_H^{L_0 \to L_1}$. These solutions can be put in a moduli space

$$\tilde{\mathcal{M}}(x_-, x_+, H, J) = \{\varphi \text{ solutions s.t. } \dots\}$$

Then we define a boundary operator ∂.

Theorem 6.7. $\partial^2 = 0$

So that we have a homology, called *wrapped Floer homology* $HW^t(H, L_0, L_1, J)$

Remark 6.8. We have $H \leq K \rightsquigarrow HW^t(H, L_0, L_1, J) \to HW^t(K, L_0, L_1, K)$.

Definition 6.9. For $t \geq 0$

$$HW^t(M, L_0, L_1) = \underline{\lim}_H HW^t(H, L_0, L_1, J)$$

(Where we have taken direct limit.)

Taking direct limit of the homology, we make sure the homology theory is independent of the choice of objects (I think, complex structure and Hamiltonian) we used to construct it.

Proposition 6.10. *$t \to HW^t(M, L_0, L_1)$ is a peristence module $B(M, L_0, L_1)$.*

Finally we can define *barcode entropy*. Fix $\varepsilon > 0$, $t \geq 0$,

$$b_\varepsilon(M, L_0, L_1, t) = \#\{ \text{ of bars in } B(M, L_0, L_1) \text{with length} \geq \varepsilon \text{ and start before } t\}$$

Then

$$\bar{h}^{HW}(M_0, L_0, L_1) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{\log^+(b_\varepsilon(M, L_0, L_1, t)}{t}$$

Consider a contact manifold $(\Sigma, \lambda, L_0, L_1)$ and A the Lagrangians. (Example raising question of filling.)

Theorem 6.11 (M '24)**.** *$\bar{h}^{HW}$ is independent of the filling.*

Theorem 6.12 (M '24)**.** *$\bar{h}^{HW}(M, L_0, L_1) \leq h_{top}(\alpha)$.*

Theorem 6.13 (M '25)**.** *Consider (M, L_0, L_1). Let K be a compact topologically transitive hyperbolic set for the Reeb flow α. Assume $W^s_\delta(q) \subset \partial L_0$, $W^s_\delta(p) \subset \partial L_1$. Then*

$$\bar{h}^{HW}(M, L_0, L_1) \geq h_{top}(\alpha|_K) > 0.$$

Which says it captures dynamics beyond unconditional phenomena. In lower dimensions these tend to coincide, but in higher dimension we don't know. This is related to "the sup over hyperbolic sets [...]"

Here's a conjecture:

$$\sup_{L_0, L_1} \bar{h}^{HW}(M, L_0, L_1) = h_{\text{top}}(\alpha).$$

Extra comments. One of the aims is to describe topological entropy h_{top} using Floer theory. Theorems by Çineli-Ginzburg-Gürel show bounds of topological entropy and barcode entropy (one of which is for hyperbolic sets).

There is a notion of *admissible Hamiltonian and Reeb vector fields* which is related to some asymptotical behaviour "linear at infinity". I understand that admissible vector fields give the interesting chords for the Floer homology construction.

7. Revisiting cotangent bundles

Mieugel Cueca, KU Leuven. Symplectic Geometry Joint Seminar, IMPA. September 5, 2025.

Abstract. Cotangent bundles provide key examples of symplectic manifolds. On the other hand, one can think of Lie groupoids as generalizations of manifolds. In this context, Alan Weinstein constructed their cotangent bundles and proved that they are so-called symplectic groupoids. In this talk, I will recall this construction and explain what happens when one replaces a Lie groupoid with a Lie 2 (or n)-groupoid. If time permits, I will exhibit some of their main applications. This is joint work with Stefano Ronchi.

Recall the basic properties of the cotangent bundle T^*M for a symplectic manifold:

(1) It's a vector bundle.
(2) $\langle \cdot, \cdot \rangle : TM \otimes T^*M \to \mathbb{R}_M$ is the dual pairing.
(3) $\omega_{\text{can}} \in \Omega^2(T^*M)$ is symplectic.
(4) $\mathcal{L}_\varepsilon \omega_{\text{can}} = \omega_{\text{can}}$, ε Euler vector field.

Main goal. Reproduce the above for Lie n-groupoids.

For $n = 0$ we get the above situation. For $n = 1$ [Duzud-Weinstein], [Prezun], for $n \geq 2$? and we care about $n = 2$.

Definition 7.1. A *Lie n-groupoid* $\mathcal{G} : \Delta^{\text{op}} \to \mathsf{Man}$ such that

$$P_{e,j} : \mathcal{G}_\ell \to \Lambda^\ell_j \mathcal{G}$$

are surjective submersions $\forall \ell, j$ are differomorphisms for $\ell > n$.

Remark 7.2. This sort of manifolds-valued presheaf category is generated by

$$d^\ell_i : \mathcal{G}_\ell \to \mathcal{G}_{\ell-1} \text{face maps} 0 \leq i, j \leq \ell$$
$$s^\ell_j : \mathcal{G}_\ell \to \mathcal{G}_{\ell+1} \qquad \text{degeneracies}$$

The tangent is a functor, it satisfies

$$T_\bullet(\mathcal{G}) = T_k(\mathcal{G}) = T\mathcal{G}_k$$

(It looks like T preserves diagrams.)

Dold-Kan. The category SVect of simplicial vector spaces has objects

$$\mathbb{V}_\bullet \qquad \mathbb{V}_n \longrightarrow \cdots \longrightarrow \mathbb{V}_2 \xrightarrow{\text{3 arrows}} \mathbb{V}_1 \xrightarrow{\text{2 arrows}} \mathbb{V}_0$$

where the $\mathbb{V}_i$ are vector spaces.

There is a functor

$$\mathsf{SVect} \xrightarrow{N} \{\text{chain complexes} \geq 0\}$$
$$\mathbb{V}_\bullet \to N\mathbb{V} = (N_\ell \mathbb{V}\operatorname{Ker} P_{\ell,\ell}, \partial = d_\ell)$$

Theorem 7.3 (Dold-Kan)**.** *That's an equivalence of categories. [Confirm this!]*

Those categories are monodial:

$$(\mathsf{SVect}, \otimes), \qquad (\mathbb{V}_\bullet \otimes \mathbb{W})_\ell = \mathbb{V}_\ell \otimes \mathbb{W}_\ell$$
$$(\mathsf{ch}_{\geq 0}, \otimes), \qquad (V \otimes W)_i = \bigoplus_{\ell+k=1} V_\ell \otimes W_k$$

And N is Lax monoidal with Lax structure given by the Eilenberg-Zilber map, though we won't explain the details of this.

There are duals given by internal Hom:

$$\mathbb{V}^{n*} = \underline{\mathrm{Hom}}(\mathbb{V}, B^n\mathbb{R})$$

Where the internal Hom is given by

$$\underline{\mathrm{Hom}}(\mathbb{V}, B^n\mathbb{R})_\ell = \mathrm{Hom}_{\mathsf{SVect}}(\mathbb{V} \otimes \Delta_n[\ell], B^n\mathbb{R})$$

for an object $\Delta[\ell] = \mathbb{R}[\Delta[\ell]]$.

Properties.

(1) $\mathbb{V}^{n*}$ isa simplicial vector bundle.
(2) $N(V^{n*})$ and $N(\mathbb{V})^*[n]$ is a quasi isomorphism.
(3) $\langle \cdot, \cdot \rangle : \mathbb{V} \otimes \mathbb{V}^{n*} \to B^n\mathbb{R}$ is non-degenerate on homology.
(4) $\mathbb{V} \hookrightarrow (\mathbb{V}^{n*})^{n*}$ Mont. a eq.

The vector bundle case.

$$\mathrm{Maps}(\Delta[i], \mathcal{G})_k = \mathrm{Hom}_{\mathsf{SSet}}(\Delta[i] \times \Delta[k], \mathcal{G})$$

Proposition 7.4. *Let $\mathcal{G}$ be a Lie n-groupoid.*

(1) *Maps$(\Delta[i], \mathcal{G})$ Lie n-groupoids ME $\mathcal{G}$.*
(2) *Maps$(\Delta[i], \mathcal{G})_0 = \mathcal{G}_i$.*
(3) *$ev : \Delta[i] \times$ Maps$(\Delta[i], \mathcal{G}) \to \mathcal{G}$.*

[Staircase looking diagram.]

Definition 7.5. $\mathcal{G}_\bullet$.

$$T_i^{n*}\mathcal{G} = \mathrm{Hom}_{\mathsf{SVect}}(1^*\Pi_{\Delta[i]}(T\mathcal{G}), B^n\mathbb{R}_{\mathcal{G}_i})$$
$$(d_j, F)_{K|d_j\mathcal{G}}(x^a) = (F_k)|_{\mathcal{G}}(x^{\delta_j a}).$$

Proposition 7.6. *$\mathcal{G}$ Lie n-groupoid, then $T^{n*}\mathcal{G}$ satisfy*

(1) *is a vector bundle n-groupoid*

(2) *dual to $T\mathcal{G}$*

$$\langle\cdot,\cdot\rangle : T\mathcal{G}\otimes T^{n*}\mathcal{G}\to B^n\mathbb{R}_{\mathcal{G}}$$

non-degenerate on homology.

(3) *n-shifted symplectic*

$$T^{n*}_n\mathcal{G}\xrightarrow{p} T^*\mathcal{G}_n$$

and $p^*\omega_{can}$.

[More computations I missed]

8. Holomorphic extensions of s-proper Lie groupoids

Rui L. Fernandes, Instituto Superior Técnico (Lisboa)/University of Illinois at Urbana-Champaign. Symplectic Geometry Seminar, IMPA. September 17, 2025.

Abstract. Every smooth manifold admits a compatible analytic structure, and a classical result of Whitney–Bruhat states that any analytic manifold has a holomorphic extension. Lie groups also admit compatible analytic structures, and another classical result, due to C. Chevalley, shows that any compact Lie group has a holomorphic extension to a complex Lie group. D. Martínez Torres has shown that any proper Lie groupoid admits a compatible analytic structure. I will discuss an extension of the classical results of Whitney–Bruhat and Chevalley, establishing that any s-proper Lie groupoid has a holomorphic extension. This talk is based on recent joint work with Ning Jiang (arXiv:2508.18036).

Theorem 8.1 (Whitney-Beuhat)**.** *If M is analytic there exists a complex manifold $M_{\mathbb{C}}$ together with an analytic map $i : M\to M_{\mathbb{C}}$ which is totally real ($T_HM_{\mathbb{C}} = TM\oplus J(TM)$) and*

(1) *For every complex manifold X and every analytic map $\phi : N\to X$ there exists $\phi^* : U\to X$ holomorphic contained open $M\subset U\subset M_{\mathbb{C}}$.*
(2) *If $\psi : V\to X$ holomorphic on $M\subset V\subset M_{\mathbb{C}}$ then $\psi=\phi^*$ where $\phi=\psi\circ i$ on a possibly smaller open contained in $U\cap V$.*

Theorem 8.2 (Chevalley)**.** *If G is a Lie group, there exists a complex Lie group $G_{\mathbb{C}}$ and a morphism $i : G\to G_{\mathbb{C}}$ satisfying the following universal property. For every complex Lie group H and morphism $\phi : G\to H$ there exists a unique holomorphic map $\phi^* : G_{\mathbb{C}}\to H$ such that $\phi=\phi^*\circ i$.*

Here's the construction of $G_{\mathbb{C}}$:

$$\begin{tikzcd}
N\subset\tilde{G} \arrow[r] \arrow[d] & G^* \arrow[d] \\
\tilde{G}/N\simeq G \arrow[r, "i"'] & G_{\mathbb{C}} = G^*/\overline{i^*N}
\end{tikzcd}$$

where $\tilde{G}$ is the universal cover group of G, G^* is the group that integrates to the complexification of the Lie algebra of G, i.e. $\mathrm{Lie}(G^*)=\mathfrak{g}_{\mathbb{C}}$ and $\overline{i^*(N)}$ is the smallest closed normal complex Lie subgroup of G^* containing $i^*(N)$.

Example 8.3.

$$\begin{tikzcd}
\widehat{\mathrm{SL}_2(\mathbb{R})} \arrow[d] \arrow[dr] & \\
\mathrm{SL}_2(\mathbb{R}) \arrow[r, hook] & \mathrm{SL}_2(\mathbb{C})
\end{tikzcd}$$

which is a simple case, but consider instead $\mathrm{S\tilde{L}}_2(\mathbb{R}) \times \mathbb{R}$ and the normal subgroup $N = \langle a \rangle \times \langle \lambda \rangle$ for irrational λ. Then we obtain

$$\begin{tikzcd}
\widehat{\mathrm{SL}_2(\mathbb{R})} \times \mathbb{R} \arrow[r] \arrow[d] & \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C} \arrow[d] \\
\widehat{\mathrm{SL}_2(\mathbb{R})} \times \mathbb{R}/N \simeq G \arrow[r] & G_{\mathbb{C}}
\end{tikzcd}$$

and $G_{\mathbb{C}}$ is 3 complex dimensions! It is not $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}$.

9. Everything you always wanted to know about polygons but were too afraid to ask

Alessia Mandini, UFF. GAAG, IMPA. September 22 and 23, 2025.

Abstract. Moduli spaces of polygons form a family of Kähler manifolds that can be constructed as a Kähler reduction of coadjoint orbits. These spaces have deep connections to various areas of mathematics, including symplectic and algebraic geometry as well as representation theory. In this talk, I will define these spaces and explore their connections. I will then discuss wall-crossing phenomena in these spaces and demonstrate how it can be used to determine their cohomology rings. Finally, I will introduce the hyperkähler analogue of these spaces, known as hyperpolygon spaces, and describe some of their generalizations.

Plan of the talk

(1) Polygons in $\mathbb{R}^3$ and relations with other moduli spaces.
(2) Polygons in other spaces.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{>0}$. Consider S^2 with its usual symplectic, Kähler form. Also consider the product of several spheres. There's a Hamiltonian action of $\mathrm{SO}(3)$ by rotations. The moduli space of polygons is the symplectic reduction obtained with this Hamilatonian action,

$$M(\alpha) = \prod S^2_{\alpha_i} // \mathrm{SO}(3).$$

Why is it called the space of polygons? We think that

$$[v_1, \dots, v_n] \in M(\alpha) \iff \sum_{i=1}^n v_i = 0$$

are polygons.

When smooth, $M(\alpha)$ is a $(n-3)$-complex-dimensional Kähler manifold. Consider

$$\varepsilon_I(\alpha) = \sum_{i \in I} \alpha_i - \sum_{j \in I^c} \alpha_j$$

for some index set $I \subseteq \{1, \dots, n\}$. $M(\alpha)$ is *smooth* if $\varepsilon_I(\alpha) \neq 0$ for all $I \subseteq \{1, \dots, n\}$. If so, we say α is *generic*.

Theorem 9.1 (Kapovich-Millson)**.** *$M(\alpha)$ is a complex analytic space with (eventually) isolated singularity (homogeneous quadratic cones).*

Example 9.2. (1) ($n = 3$.) Then $M(\alpha)$ is either empty or a point.
(2) ($n = 4$.) $M(\alpha)$ is either empty or a sphere.

Remark 9.3 (Hausmann-Knutson). Let M_n be the space of all n-gons modulo rigid motions. Then M_n can be equipped with a Poisson structure for which $M(\alpha)$ are the symplectic leaves.

Polygons as quiver varieties. Consider the star-shaped quiver, which is a distinguished point with some points around it, and arrows from every point to the distinguished one. Let the distinguished point be $V_0 = \mathbb{C}^2$ and the rest $\mathbb{C}$. Then a representation of this quiver is

$$\mathrm{Rep}Q = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) = \mathbb{C}^{2n}.$$

Now put

$$K = (U(2) \times U(1)^n)/\Delta$$

where Δ is the diagonal S^1. This gives a Hamiltonian action on $\mathbb{C}^{2n}$ as follows. For

$$(A, \lambda_1, \dots, \lambda_n) \cdot (q_1, \dots, q_n)$$

we map $q_i \mapsto A^{-1} q_i \lambda_i$. That is

$$\mu : \mathbb{C}^{2n} \longrightarrow K^*$$
$$(q_1, \dots, q_n) \longmapsto \left(\sum_{n=1}^{n} (q_i q_i^*), \dots, \frac{1}{2}|q_i|^2 \dots \right)$$

Theorem 9.4 (Hausmann-Knutson).

$$\mathbb{C}^{2n} /\!/_{(0,\alpha)} K = M(\alpha)$$

Proof. Uses

$$\mathbb{R}^3 \xrightarrow{\simeq} \mathfrak{su}(2)^*$$
$$v_i \longmapsto (q_i q_i^*)_0$$

□

$$\begin{tikzcd} & \mathbb{C}^{2n} \arrow[dl, "/\!/_{\alpha} U(1)^n"'] \arrow[dr, "/\!/U(2)"] & \\ \prod S^2 \arrow[dr, "/\!/_{\alpha} \mathrm{SO}(3)"'] & & \mathrm{Gr}(2, \mathbb{C}^n) \arrow[dl, "/\!/_{\alpha} U(1)^n"] \\ & M(\alpha). & \end{tikzcd}$$

So we have (I think former symplectic reduction)

$$\mu_{U(1)^n} : \mathrm{Gr}(2, \mathbb{C}^n) \to \mathbb{R}^n$$

Walls and wall crossing. The walls are

$$W_I = \{\alpha \in \mathbb{R}^n_{>0} : \varepsilon_I(\alpha) = 0\}$$

for $I \subseteq \{1, \ldots, n\}$.

To understand wall crossing suppose we have a wall W_I, with a^c in the wall, α^+ on one side and α^- on the other.

Not that

$$\varepsilon_I(\alpha^+) > 0 \iff \sum_{k \in I} \alpha_i^+ > \sum_{j \in I^c} \alpha_j^+$$

Define

$$\begin{aligned} M_{I^c}(\alpha^+) &= \{(v_1, \ldots, v_n) \in M(\alpha^+) : v_i = \lambda v_j \forall i, j \in I^c, \lambda > 0\} \\ &= M(\tilde{\alpha}), \qquad \tilde{\alpha} = \left(\alpha_{i_1}, \ldots, \alpha_{i_k}, \sum_{j \in I^c} \alpha_j\right) \end{aligned}$$

Then $\varepsilon_I(\alpha^+) < 0$ implies $M_I(\alpha^-) \subseteq M(\alpha^-)$.

$$\begin{tikzcd} & \tilde{M} \subseteq E = M_I \times M_{I^c} \arrow[d, "\text{blow up}"] & \\ M(\alpha^+) \arrow[r] & M(\alpha^c) & M(\alpha^-) \arrow[l] \end{tikzcd}$$

Remark 9.5.
- $M(\alpha) \cong M(\sigma(\alpha))$ for σ a permutation on the order of sets.
- $M(\alpha)$ is conformally symplectomorphic to $M(\lambda\alpha)$ for all $\lambda > 0$.

Definition 9.6. Let $\alpha \in \mathbb{R}^n_{>0}$. $I \subseteq \{1, \ldots, n\}$ is *short* if $\varepsilon_I(\alpha) < 0$ and *long* if $\varepsilon_I(\alpha) > 0$.

Example 9.7. We did an example of wall crossing. The relevant manifolds $M_{I^c}(\alpha)$ and $M_I(\alpha)$ were projective spaces.

Now recall our first quotient

$$\begin{tikzcd} \mu^{-1}_{U(1)}(\alpha) \subseteq \mathrm{Gr}(2, \mathbb{C}^n) \arrow[d, "//U(1)^n"] \\ M(\alpha) \end{tikzcd}$$

Let c_i be the first Chern classes associated to the n S^1-bundles above (by taking reduction in stages).

Theorem 9.8 (Haussmann-Knutson,M.)**.** *The c_i generate the cohomology of $M(\alpha)$.*

Theorem 9.9 (Guillemin-Stenberg)**.** *In this setup, for α generic,*

$$H(M) = \mathbb{C}[c_1, \ldots, c_n]/\mathit{Ann}(\mathit{Vol}(M(\alpha)))$$

i.e., $Q(c_1, \ldots, c_n) \in \mathit{Ann}(\mathit{Vol} M(\alpha)) \iff Q\left(\frac{\partial}{\partial \alpha_i}, \ldots, \frac{\partial}{\partial \alpha_1}\right) \mathit{Vol}(M(\alpha)) = 0$

Theorem 9.10 (Takakura, The Koi)**.**

$$\mathit{Vol}(M(\alpha)) = -\frac{(2\pi)^{n-3}}{(n-3)!} \sum_{I\ long} (-1)^{n-|I|} \varepsilon_I(\alpha)^{n-3}.$$

Example 9.11. According to Example 9.7 (which I did not copy) we find that $\alpha_1 \in \Delta_1$ gives $\mathrm{Vol}(M(\alpha_1)) = 2\pi^3(1-2\alpha_3)^2$ and

$$H(M(\alpha_1)) = \mathbb{C}[c_3]/(c_3^3)$$

Polygon game. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then the co-adjoint orbits $\mathfrak{g}^* \supseteq \mathcal{O}_{\xi_i}$ are symplectic manifolds with the KKS form. Then

$$\begin{aligned} G\curvearrowright \Pi\mathcal{O}_{\xi_i} &\longrightarrow \mathfrak{g}^* \\ (A_i, \dots \alpha_n) &\longmapsto \sum A_i \end{aligned}$$

Then the quotient

$$M(\xi) = \Pi\mathcal{O}_{\xi_i}//_0 G$$

generalizes the previous construction, which we get with $G = \mathrm{SU}(2)$.

It could be interesting to investigate which of the next constructions can be generalized to other Lie groups.

Theorem 9.12 (Sotillo-Floentins-Gadihl)**.** *Wall-crossing for $SU(m)$.*

Bending action. We consider a polygon of n sides and put some diagonals that don't intersect. We introduce some notion of "bending" that allows to define a Hamiltonian function. In turn, this defines a torus action on $M(\alpha)$ and a moment map.

For $n = 2$ we obtain the moment map

$$\begin{aligned} \mu : M(\alpha) &\longrightarrow \mathbb{R}^2 \\ p &\longmapsto (\ell_1(p), \ell(p)) \end{aligned}$$

and we can see the moment polytope in $\mathbb{R}^2$.

Remark 9.13. Any system of $(n-3)$ non-vanishing and non-intersecting diagonals determine a torus action $T^{n-3}\curvearrowright M(\alpha)$.

Relations of polygon spaces to other moduli spaces. Representations of the fundamental group of the punctured sphere in $SU(2)$, i.e.

$$\mathrm{Rep}(\pi_1(S^2 \setminus \{p_1, \dots, p_n\}, \mathrm{SU}(2))/\mathrm{SU}(2) \cong M(\alpha)$$

This can be put very explicitly:

$$\begin{aligned} &\mathrm{Rep}(\pi_1(S^2 \setminus \{p_1, \dots, p_n\}, \mathrm{SU}(2)) \\ &= \{(g_1, \dots, g_n) \in \mathrm{SU}(2)^n : g_1 \cdot \dots \cdot g_n = \mathrm{Id}, t_r g_i = 2\cos\pi\alpha_i\} \end{aligned}$$

See [Agapito-Godinho]. That's all we will say about this example.

Now consider the moduli space of parabolic bundles. Consider a holomorphic bundle of rank 2 over $\mathbb{C}P^2$. Choose some points $D = \{x_1, \dots, x_n\}$ in $\mathbb{C}P^2$ and a flag $E_{x_i} = E^{x_i,1} \supseteq E^{x_i,2} \supseteq \{0\}$ for every $x_i \in D$. Each of the $E_{x_i,j}$ is isomorphic to $\mathbb{C}$ (1-dimensional). A *quasi-parabolic bundle* is an holomorphic bundle E with such a choice of flag. A *parabolic bundle* is a quasi-parabolic bundle with a choice of parabolic weights $0 < \beta_1(x_i) < \beta_2(x_i) < 1$.

There is also a notion of stability:

$$\mathrm{pdeg}E = \deg E + \sum_{i=1}^{n}(\beta_1(x_i) + \beta_2(x_i))$$
$$\mu(E) = \frac{\mathrm{pdeg}(E)}{\mathrm{rank}(E)}$$

We say E is *(semi)stable* if $\mathfrak{z}(E) > \mu(L)$ ($\mu(E) \geq \mu(L)$) for any $L \subseteq E$ parabolic subbundle.

Then we obtain that $\mathcal{M}_{\pi,d}(\beta)$ is the moduli space of (semi)-stable parabolic bundles on $\mathbb{P}^1$ of rank r and degree d. In particular $\mathcal{M}_{2,0}(\beta)$ is the moduli space of (semi)-stable parabolic bundles on $\mathbb{P}^1$ of rank 2 and degree 0 holomorphically trivial.

Theorem 9.14 (Jeffrey, Godinho, M.)**.** *For generic α, $M(\alpha)$ is diffeomorphic to $\mathcal{M}_{2,0}(\beta)$ whenever $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$.*

Idea of proof. The correspondence can be made quite explicit as a map

$$M(\alpha) \longrightarrow \mathcal{M}_{2,0}(\beta)$$
$$[q_1, \ldots, q_n] \longmapsto {}^{E=\mathbb{C}P^1\times\mathbb{C}^2}_{E_{x_i}\supset E_{x_i,1}\supset X_{x_i,2}\supset\{0\}}$$

□

Now consider the quiver variety we described above:

$$\mathrm{Rep}Q = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) = \mathbb{C}^{2n}.$$

Now put

$$K = (U(2) \times U(1)^n)/\Delta$$

where Δ is the diagonal S^1.

But this time put

$$\mathrm{Rep}\tilde{Q} = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}^2, \mathbb{C})$$

We also have an action of K, and we get

$$K \curvearrowright \mathrm{Rep}\tilde{Q} = T^*\mathbb{C}^{2n},$$

a so-called *hyperHamiltonian action.* The quotient

$$X(\alpha) = ///_{\substack{(0,\alpha)\\(0,0)}} K = \frac{\mu_{\mathbb{R}}1(0,\alpha) \wedge \mu_{\mathbb{C}}^{-1}(0,0)}{K}$$

called *hyperpolygon space.*

Here

$$\mu_{\mathbb{R}} : T^*\mathbb{C}^{2n} \to \mathcal{K}^*$$
$$\mu_{\mathbb{C}} : T^*\mathbb{C}^{2n} \to \mathcal{K}^*_{\mathbb{C}}$$
$$(q_1, \ldots, q_n, p_1, \ldots, p_n) \mapsto (\sum_{i=1}^{n}(p_i, q_i)_0, \ldots)$$

It turns out that $X(\alpha)$ is smooth if and only if $\varepsilon_1(\alpha) = 0$ for all $I \subseteq \{1, \ldots, n\}$. When smooth, $X(\alpha)$ is a hyperkähler manifold (non-compact, unfortunately), $M(\alpha) = \{[0_{1i}, 0] \in X(\alpha)\}$.

Theorem 9.15 (Boalch)**.** *$X(\alpha)$ is the moduli space of polygons for $GL(2, \mathbb{C})$.*

Parabolic Higgs bundles. Let E be a parabolic bundle as before, i.e. $E \in \mathcal{M}_{0,2}(\beta)$. A *Higgs field on E* is

$$\phi \in H^0(\mathbb{P}^1, \mathrm{SPEnd}(E) \otimes K_{\mathbb{P}^1}(D)))$$

where a *strongly parabolic endomorphism* is $f : E \to E$ such that $f(E_{x_i}) \subseteq E_{x_{i+1}}$ for all i. (Fix details!)

A *parabolic Higgs bundle* is (probably the pair (E, ϕ)). A parabolic Higgs bundle (E, ϕ) is *(semi)stable* if $\mu(E) > \mu(L)$ $(\mu(E) \geq \mu(L))$ for all L parabolic Higgs subbundle.

Then $\mathcal{N}^{0,1}_{r,a}(\beta)$ is the moduli space of parabolic Higgs bundles over $\mathbb{P}^1$ with rank r and degree d (with fixed determinant, and traceless; two notions that we will not define here).

Theorem 9.16 (Goldinho, M.; Biswar, Florentino, Godinho, M.)**.** *Let α be generic, let β be such that $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$. Then the hyperpolygon space $X(\alpha)$ is symplectomorphic to the moduli space of parabolic HIggs bundles over $\mathbb{P}^1$, β-stable, holomorphically trivial (with fixed determinant and trace-free).*

Idea of proof. We can define a correspondence

$$\begin{aligned} X(\alpha) &\longrightarrow \mathcal{H}(\beta) \\ [p, q] &\longmapsto {}^{E = \mathbb{C}P^1 \times \mathbb{C}^2}_{E_{x_i} \simeq E_{x_i,1} \supset E_{x_i,2} \supset \{0\}} \end{aligned}$$

No we can use the moment map condition that the sum of the residues is zero, where $\mathrm{Res}_{x_i} \phi = (p_i, q_i)$ where ϕ is the unique function that satisfies the last equality by the residue theorem. ☐

Wall-crossing for moduli spaces of parabolic bundles. See "Translation"from Thaddeus. What happens is that there is an S^1-action on $X(\alpha)$ given by

$$\lambda \cdot [p, q] = [\lambda p, q].$$

(This can be paralleled with the S^1-action on $\mathcal{H}(\beta)$, given by $\lambda \cdot [E, \phi] = [E, \lambda \phi]$.)

This action has a moment map

$$\mu_{S^1}([p, q]) = \frac{1}{2} \sum_i |p_i|^2$$

Fixed points are [Konno]

$$X_S = \{[p, q] \in X(\alpha) : S, S^c \text{ are straight}, \ p_j = 0 \forall j \in S\}$$

for all $|S| \geq 2$ short. Here $S \subseteq \{1, \ldots, n\}$ is *straight* if $q_i = \lambda_j q_j$ for $\lambda_j > 0$ $\forall i, j \in S$ (which in the case of polygons means the edges are aligned).

Let U_S be the flow down from X_S Then, crossing a wall W_I as defined above, i.e. $W_I = \{\alpha \in \mathbb{R}^n_{>0} : \varepsilon_S(\alpha) = 0\}$, "replaces" U_S by U_{S^c}.

Let α and $\tilde{\alpha}$ be generic, then $X(\alpha)$ is diffeomorphic to $X(\tilde{\alpha})$. The wall crossing for $M(\alpha)$ involves

$$M_S(\alpha^+) = U_S \cap M(\alpha^+)$$
$$M_{S^c}(\alpha^-) = U_{S^c} \cap M(\alpha)$$

This is thought of as a Mukai transform.

Generalization. See [Florentino, Godinho, Sotillo] for wall crossing. See also [Fisher] PhD thesis, [Fisher, Rayan]. [Hausel et al], [Rayan-Schopnick].

10. Vertex algebras and special holonomy on quadratic Lie algebras

Mario García Fernandez, ICMAT. GAAG, IMPA. September 24, 2025.

Abstract. The chiral de Rham complex (CDR) is a sheaf of vertex algebras on any smooth manifold, introduced by Malikov, Schechtman and Vaintrob, which provides a formal quantization of the non-linear sigma model in mathematical physics. Motivated by the algebra of chiral symmetries in two-dimensional superconformal field theories, vertex algebra embeddings on the CDR have been studied for special holonomy Riemannian manifolds, thanks mainly to the work of Heluani, Zabzine, and collaborators, with interesting applications to the elliptic genus. In these lectures, we will discuss extensions of some of these results to the case of special holonomy manifolds with skew-torsion. The presence of torsion typically allows for continuous symmetries in the geometry, with an enhanced interplay with Lie theory and algebra, as well as the application of techniques from generalized geometry.

Based on joint work with Luis Álvarez Cónsul, Andoni De Arriba de la Hera. arXiv:2012.01851 (IMRN '24).

Motivation. Let (M^n, g) be a Riemannian spin manifold with parallel spinor $\nabla^g\varphi = 0$. Then $\mathrm{hol}(g) \subset G_\varphi \subseteq \mathrm{SO}(n)$, and $\mathrm{Ric}(g) = 0$.

There is a construction by Markov-S-V that puts a sheaf of vertex algebras $\mathcal{V} \to M^n$. How to construct special embeddings of trivial vertex algebras in the cohomology of $\mathcal{V}$, i.e. $\mathcal{V}_\varphi \hookrightarrow H^*(\mathcal{V})$ by Heluani et al.

Applications. Construction of topological invariants. Elliptic genus, [Borisov-L]. How to understand mirror symmetry using vertex algebras [Borisov]. Holography [Witten].

Geometry in algebra. We shall do geometry in quadratic Lie algebras. Recall that a *quadratic Lie algebra* is $(\mathfrak{g}, [\cdot,\cdot], (\cdot,\cdot))$ where $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra (in this course we use $\mathbb{R}$ as base field) and $(\cdot,\cdot)$ is a symmetric bilinear invariant form.

Example 10.1. Let R be a Lie algebra with $\langle\cdot,\cdot\rangle_R : R\otimes R \to \mathbb{R}$. Take $\mathfrak{g} = R \oplus R^*$, and pick $H \in \Lambda^3 R^*$. Put $H(a,b,c) = \langle [a,b], c\rangle$, and define

$$[v+\alpha, w+\varphi] = [v,w] - \varphi([v,-]) + \alpha([v,-]) + H(v,w,-)$$

for $v, w \in R$ and $\varphi, \alpha \in R^*$. This turns out to be a bracket.

If you like geometry you can pick K compact, $\mathrm{Lie}K = R$, this is "Generalized geometry on $TK \oplus T^*K$" à la Hitchin.

QLA:

- Courant algebroids $/\{*\}$.
- Symplectic supermanifolds.

Definition 10.2. (1) A *(generalized) metric* on $(\mathfrak{g}, (\cdot,\cdot))$ is $G \in \mathrm{End}(\mathfrak{g})$ with $G^2 = \mathrm{Id}$, $(G, G) = (\cdot,\cdot)$,

This gives $\mathfrak{g} = V_+ \oplus V_-$ where G acts as identity on the first term and as $-\mathrm{Id}$ on the second one. $(G_-, -)$ is a non-degenerate pairing. $(-,-)_{V_I}$ non-degenerate tensor.

(2) A divergence $\varphi \in \mathfrak{g}^*$.

(3) A *connection* is

$$D : \mathfrak{g} \to \mathfrak{g}^* \otimes \mathfrak{g}$$

such that $(D_a b, c) + (b, D_a c) = 0$. So that $D \in \mathfrak{g}^* \otimes \Lambda^2 \mathfrak{g}$ (where we identify $\mathfrak{g}$ with its dual using the pairing).

D is *compatible* with G if $[D_a, G] = 0$. This condition says that D splits into four operators:

$$\begin{tikzcd}
D_+^+ & & D_+^- \\
& D \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
D_-^+ & & D_-^-
\end{tikzcd}$$

(4) Let $a \in \mathfrak{g}$, $a = a_+ + a_-$. A *generalized connection* satisfies $D_a^+ = [a_-, b_+]_+ + [a_+, b_-]_-$. (This will happen to things living in $\mathcal{D}^0$, see below.)

Definition 10.3. Given D define

(1) $\varphi_D(a) = -\mathrm{tr} Da$.

(2) Torsion$(AX) : T_D \in \Lambda^2 \mathfrak{g}^*$,

$$T_D(a, b, c) = (D_a b - D_b a - [a, b], c) + (D_c a, b).$$

(Just copying the definition of torsion.)

Lemma 10.4. *Define $\mathcal{D}^0(G, \varphi) = \{D : G\text{-compatible }, \varphi_D = \varphi\}$. Then this is non-empty, but is not a point. Furthermore, $\forall D \in \mathcal{D}^0(G, \varphi)$,*

$$D_{a_-} b_+ = [a_-, b_+]_+, \qquad D_{a_+} b_- = [a_+, b_-]_-.$$

(Checking non-emptyness is just writing out the equations.)

Given G consider $Cl(V_+)$, $a_+ \cdot a_+ = (a_+, a_+)$ for $a_+ \in V_+$.

Fix an irreducible representation for $Cl(V_+)$: S_+. Given $D \in \mathcal{D}^0(G, \varphi)$:

(1) $D_-^{\delta_+} \in V_-^* \otimes \mathrm{End}((S_+) \supset V_-^* \otimes \Lambda^2 V_+ \ni D_-^+$.

(2) An operator like Dirac operator:

$$\begin{aligned} \underline{D}^+ : S_+ &\longrightarrow S_+ \\ \zeta &\longmapsto \sum_{j=1}^{\dim V} e_j^+ \cdot D_{e_j^+} \zeta. \end{aligned}$$

Lemma 10.5. *D^{S_+} and $\underline{D}^+$ are independnt of $D \in \mathcal{D}^0(G, \varphi)$.*

Definition 10.6. (G, φ, ζ), $\varphi \in S_+$ satisfies *Killing spinor equations* if

$$KSE \qquad \underbrace{D_-^{S_+} \zeta = 0,}_{\text{Gravitino Eq.}} \qquad \underbrace{\underline{D}^+ \varphi = 0}_{\text{Dilatino Eq.}}$$

Expectation.

(1) $\mathcal{M} = \{(G, \varphi, \zeta) : KSE\}/\mathfrak{g}$ special metric.
(2) Given a solution (G, φ, ζ) then
$$V_{(g,\varphi,\zeta)} \hookrightarrow V^k(\mathfrak{g}).$$
(Where I think $V^k(\mathfrak{g})$ is the Kac-Moody affinization.)

Remark 10.7. In the case $TK \oplus T^*K : \mathcal{M}$ 2-stack. See [Bursztyn].

Lemma 10.8. *Associated to (G, φ) there are well-defined "Ricci tensors" $Ric^+ \in V_- \otimes V_+$, $Ric^- \in V_+ \otimes V_-$.*
$$Ric^+(a_-, b_+) = Tr(C_+ \to R_D(c_+, a_-)b_+)$$
$$R_D(a, b) = [D_a, D_b]_c - D_{[a,b]}c, \qquad D \in \mathcal{D}^0(G, \varphi).$$

Remark 10.9. In geometric setup, the vanishing of the Ricci corresponds to the motion equations of some physical supersymmetry theory.

Proposition 10.10. *If (G, φ, ζ) is solution of KSE, then $Ric^+_{G,\phi} = 0$. If $[\varphi, G] = 0 \implies Ric^-_{G,\varphi} = 0$.*

Proof. $\mathrm{Ric}^+(a-, -)\cdot\varphi = [\underline{D}^+, D^{S_+}_{a_-}\zeta - D^{S_+}_{D_+a_-}\zeta = 0$. $[\varphi, G] = 0 \implies \mathrm{Ric}^+(a_-, b_+) = \mathrm{Ric}^-(b_+, a_-)$. ☐

Generalized Ricci flow. Finding solutions to these equations. Evaluating by
$$G_t^{-1}\partial_t G_t = -2(\mathrm{Ric}^+ - \mathrm{Ric}^-)$$
$\mathrm{Hom}(V_+, V_-) \oplus \mathrm{Hom}(V_-, V_+)$. $GG + G - G = 0$, $G = \mathrm{Id}_{V_+} - \mathrm{Id}_{V_-}$.

Exercise 10.11. (1) Prove STE for GRF.
(2) Assuming there exists a solution of KSE, prove long time existence and convergence. (See theorem by Streets, Jordan, GF; and a book by Streets, GF.)

Remark 10.12. For physicists the generalized Ricci flow (GRF) is the GRF of a $2d$ σ-model with target a compact Lie group K, $\mathfrak{g} = R \oplus R^*$

Let's explain something in this situation. consider a compact Lie group K and $\mathfrak{g} = R \oplus R^*$. A generalized metric: $\mathfrak{g} = V_+ \oplus V_-$, $(-,-)|_{V_\pm}$ non-degenerate, $(-,-)|_{V_+} > 0$. $V_+ = \mathcal{C}^b\{X + g(X)\}$, $X \in R$, $g \in S^2(R^*)$, $b \in \Lambda^2 R^*$, $H = H_0 + \overline{\partial}$.

Case $\dim V_+ = 2n$**.** Complex pure spinor φ on V_+ is equivalent $V_+ \otimes \mathbb{C} = \ell \oplus \overline{\ell}$, where $\ell = \{0 \in V_+ \oplus \mathbb{C} : 0 \cdot \zeta = 0\}$. $(\ell, \ell) = 0$, $(\overline{\ell}, \overline{\ell}) = 0$.

This gives a complex structure $J_\zeta : V_+ \to V_+$, $J_\zeta = i\mathrm{Id}_\ell - i\mathrm{Id}_{\overline{\ell}}$.

Lemma 10.13. (G, φ, ζ) *with ζ pure satisfies KSE if and only if*

(1) $[\ell, \ell] \subset \ell$.
(2) *Take bars* $\{\varepsilon_j, \overline{\varepsilon}_j\}_{j=0}^n$, $\varepsilon_j \in \ell$, $\overline{\varepsilon}_j \in \overline{\ell}$ $(\varepsilon_j, \overline{\varepsilon}_j) = s_{j,k}$. $\sum_{j=1}^n [\varepsilon_j, \overline{\varepsilon}_j = -J_\zeta\varphi_+$.

(This sum is a moment map condition.)

Consider $\mathcal{L} = \{\ell \subset \mathfrak{g} \otimes \mathbb{C} : \dim \ell = n, \ell \cap \overline{0}, (-,-)|_{\ell\oplus\overline{\ell}}$non-degenerate$\}$. This is a complex manifold. $T_\ell\mathcal{L} \cong \mathrm{Hom}(\ell, \overline{\ell} \oplus V_- \oplus \mathbb{C})$, $V_- \otimes \mathbb{C} = (\ell \oplus \overline{\ell})^\perp$ Pick a vector $\dot{\ell} = \dot{J} + \dot{G} \in \mathrm{Hom}(\ell, \overline{\ell} \oplus V_- \oplus \mathbb{C})$ That is, $\dot{J} \in \mathrm{Hom}(\ell, \overline{\ell})$, $\dot{J} : V_+ \to V_+$, $\dot{J}J + J\dot{J} = 0$. (Deformations of the complex structure.) $G \in \mathrm{Hom}(V_+, V_-) \oplus \mathrm{Hom}(V_-, V_+)$.

Proposition 10.14 (Romero)**.** *$\mathcal{L}$ has a pseudo-Kähler structure preserved by the $\mathfrak{g}$-action and there exists a moment map*

$$\begin{aligned}\mu : \mathcal{L} &\longrightarrow \mathfrak{g}^* \\ \ell &\longmapsto \frac{i}{2}\sum_{j=1}^{n}[\varepsilon_j, \overline{\varepsilon}_j], -),\end{aligned}$$

which is the quantity we mentioned in Lemma above.

Proof. Use the natural complex structure on the space of complex structures (found by Fujiki),

$$\mathrm{tr}_{V_+}(J\dot{G}_2, \dot{G}_1) - \mathrm{tr}_{V_+}(J\dot{J}_\varepsilon \dot{J}_1).$$

□

Problem: prove that $\mathcal{M} = \{(G, \varphi, \zeta) : \zeta \text{ puse}\}/\mathfrak{g}$ is a pseudo-Kähler manifold.

(1) Pseudo-Kähler.
(2) Shifter symplectic stuff.

Example 10.15. Take $K = \mathrm{SU}(2) \times \mathrm{U}(1) \cong S^3 \times S^1$ and $\mathfrak{g} = R \oplus R^*$. Take generators v_1, v_2, v_3 of SU(2) and v_4 of U(1). We have $[v_2, v_3] = -v_1$, $[v_3, v_1] = -v_2$, $[v_1, v_2] = -v_3$. Put $H_\ell = \ell v^{123}$. $\ell \in \mathbb{R}$. $x, a \in \mathbb{R}_{>0}$.

$$g_{x,a} = \frac{a}{x}\left(\sum_{i=1}^{3}\omega^{\otimes 2} + x^2(v^4)^{\otimes 2}\right)$$

$$V_+ = \{x + g_{xa}\} \subset \mathfrak{g}$$

$$I_X v_4 = xv_1, \qquad U_X v_2 = v_3, \qquad \varphi = -xv_4$$

Exercise 10.16. If $\ell = \frac{a}{x} \implies$ solution of KSE.

Furthermore $[\varphi_+, \ell] \subset \ell$ (holomorphic divergence).
The a parameter is naturally complexified by $b = yv^{23}$. $y + ia = z$.

$$V = \log\left(\frac{\omega_{x,n}^2}{v^{1234}}\right) = \log a.$$

KSV hyperbolic metric, metric on $\mathbb{H}$.

$\dim V_+ = 7$. A real spinor on V_+ is equivalent to $\phi \in (\Lambda^3 V_+^*)_{>0}$. The space $\mathrm{GL}(\mathbb{R}^7) \curvearrowright \Lambda^3(V_+^*)$. The space of spinors $S_+ = \mathbb{R}^8 = \mathbb{R}^7 \oplus \mathbb{R}\langle\varphi\rangle$. $\phi(x, y, z) = \langle x \cdot y \cdot z \cdot \zeta, \zeta\rangle$.

Remark 10.17. $v^1, \dots, v^7 \cong \mathbb{R}^6 \oplus \mathbb{R}$. $\phi = (v^{12} + v^{34} + v^{56}) \wedge v^7 + \mathrm{Re}((v^1 + iv^2) \wedge (v^3 + iv^4) \wedge (v^5 + iv^6))$.

Example 10.18. Take $R = \mathrm{SU}(2) \oplus \mathrm{SU}(2) \oplus \mathbb{R}$. $\mathfrak{g} = R \oplus R^*$. $e, s \in \mathbb{R}$, $H = sv^{123} + \ell v^{456}$. $\phi = \omega \wedge \zeta + \Omega^+$, $\zeta = \sqrt{\varepsilon/\ell}v^7$, $\omega = \sqrt{s\ell}(v^{14} + v^{25} - v^{36})$, $\Omega^+ = \sqrt{s^3}v^{123} + \ell\sqrt{s}v^{156} - \ell\sqrt{s}v^{345}$.

11. Non-Kähler Hodge Lefschetz theory and the Bianchi identity

Arpan Saha, UNICAMP. Geometric Structures Seminar, IMPA. September 25, 2025.

Abstract.
Being Kähler imposes severe constraints on the cohomology of compact complex manifolds such as the Hard Lefschetz property, and the question of how far this generalises beyond the class of Kähler manifolds has been of great interest for a while. In this talk, I shall report on ongoing joint work with Mario García Fernández and Raúl González Molina that abstracts out the definition of a variation of Hodge–Lefschetz structure and provides evidence that, under certain natural assumptions, such a structure exists more generally on distinguished subspaces within moduli spaces of Bismut–Ricci-flat metrics that are pluriclosed up to source terms. In particular, these distinguished subspaces may be regarded as replacements for the Kähler cone, with affine structure modelled on a subspace of the (1,1) Aeppli cohomology of the compact complex manifold.

Broad motivation. As we move on the parameter space associated to a CY manifold we encounter walls. The Kähler cone is contained in this parameter space, and we may cross its walls. When we cross a wall we obtain a birational transformation (I think this means that the moduli on either side are birational). But the Kähler condition is not a birational invariant.

Hodge-Lefschetz theory. For (X, J, ω) Kähler, the cohomology $H^\bullet(X)$ comes with:

- the Lefschetz operator L, given by $[\omega] \wedge -$.
- Hodge star operator.
- $\Lambda = *L*$ satisfying the $\mathfrak{sl}_2$ relations:
$$[L, \Lambda] = H, \qquad [H, L] = 2L, \qquad [H, \Lambda] = -2\Lambda.$$
These relations are equivalent to the Hard Lefschetz property that
$$L^q : H^{d-q} \to H^{d+q}.$$
- The Poincaré pairing (which is a different pairing from Lefschetz) of the form
$$H^{d-q} \cong (H^{d+q})$$
given by integration.
- Hodge-Riemann bilinear relations which is a positive definite form given by $\int_X \cdot \wedge *\cdot$.

Moving about on the Kähler cone gives a variation of Hodge-Lefschetz structure.

Definition 11.1. A *variation of Hodge Lefschetz structure (VHLS)* of weight d over a manifold K consists of

- a graded real vector bundle $E = \bigoplus_{q=0}^d E^q \to K$ with $\mathrm{rk}_{\mathbb{R}} E^0 = 1$.
- An isomorphism $E^0 \otimes TK \xrightarrow{\simeq} E^1$ (this is reminiscent of the usual definition of variations of Hodge structures, and we will not use it in this talk).
- Endomorphism fields $L, \Lambda, H \in \mathcal{A}^0(K, \mathrm{End}(E))$ satisfying the $\mathfrak{sl}_2$ relations that
$$[L, \Lambda] = H, \qquad [H, L] = 2L, \qquad [H, \Lambda] = -2\Lambda.$$

and that

$$H|_{E_q} = (2q-d)|_{\mathrm{id}_{E^q}}.$$

- An involution $* \in \mathcal{A}^0(K, \mathrm{End}(E))$ such that $*L* = \Lambda$.
- A nondegenerate symmetric pairing $P \in \mathcal{A}^0(\mathcal{K}, E^* \otimes E^*)$ w.r.t. L and $*$ are self-adjoint.
- (like the Gauss-Manin connection) a flat connection $D : \mathcal{A}^0(K,E) \to \mathcal{A}^1(K,E)$ such that $DH = 0 = DP$.

The VHLS is said to be *positive* if $P(*-,-)$ is positive definite.

Calabi's dream beyond Kähler manifolds. Let (X,Ω) be a compact Calabi-Yau manifold, where Ω is the volume form. In general such a manifold is not Kähler.

Example 11.2 (Non-Kähler manifolds)**.** These manifolds are not Kähler: Heisenberg algebra quotiented by some lattice, torus bundle, Hopf surface.

By [Yau], the Kähler cone is the moduli space of Kähler Ricci-flat metrics.

We look for PDEs on X such that

- Assumption C (Calabi). Moduli space of alutions is an open set K of an affine space modelled on a subspace $\mathbb{V}$ of $(1,1)$-Aeppli cohomology group, which is given by
$$H_A^{1,1}(X) = \frac{\mathrm{Ker}\,\partial\overline{\partial}|_{\Omega^{1,1}}}{\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}}$$
- Assumtion D (dilaton). A volume function $v : K \to \mathbb{R}_{>0}$ such that $\underline{d}\log V$ is nowhere zero (where $\underline{d}$ denotes Aeppli differential), and the bilinear form $g_K := -Dd\log V$ is nondegenerate (basically the Hessian metric). (Ideally this would be positive definite.)
- Assumption V (vector field). $Z = g_K^{-1} d\log V$ is such that DZ is invertible (when thought of as an endomorphism) $g_K(Z,Z) = d \in \mathbb{Z}_{>0}$,
$$(D_{(DZ)^{-1}})^{d+1} V = 0,$$

Remark 11.3.
$$(D_{(DZ)^{-1}})^{d} V = \int_X \wedge \wedge \wedge.$$

Theorem 11.4 (García Fernández-González Molina-AS)**.** *If assumptions C, D and V all hold with $d \leq 3$, then K admits a VHLS of weight d.*

Idea of proof for $d = 3$.

$$E = \underbrace{\mathbb{R}_K}_{E^0} \oplus \underbrace{TK}_{E^1} \oplus \underbrace{T^*K}_{E^2} \oplus \underbrace{\mathbb{R}_K^*}_{E^3}.$$

Define L in each of the terms of the direct sum as

$$L1 = Z, \qquad Lv = D_v^2 V, \qquad L_\alpha = i_Z \alpha$$

and $*$ as

$$*1 = V1^\vee, \qquad *v = VD_0^2\log V = Vg_K(v,-).$$

$\square$

Recall the Hull-Strominger system, which is given by two conditions on the connection and the metric

$$d\left(\|\Omega\|_\omega \frac{\omega^{n-1}}{(n-1)!}\right) = 0, \qquad F_\theta \wedge \frac{\omega^{n-1}}{(n-1)!} = 0.$$

This determines $dd^c\omega = \alpha \langle F_\theta \wedge F_\theta \rangle$.

12. Irreducibility of the Hilbert scheme of points and the class of 2-step ideals

Michele Graffeo, SISSA. Algebraic Geometry Seminar, UFF. September 29, 2025.

Abstract. Hilbert schemes of points on a quasi-projective variety X are classical objects in algebraic geometry. Roughly speaking, they parametrise ideals of a polynomial ring with complex coefficients having finite colength. Although Hilbert schemes always have a distinguished component called the smoothable component, their geometry is quite pathological and one of the main open problems around them concerns their irreducibility. In a joint work with Giovenzana, Giovenzana, Lella we introduce the notion of 2-step ideals. We show that the loci parametrising these ideals are in general not contained in the smoothable component of the Hilbert scheme, thus providing new examples of extra-components. In my seminar I will discuss our class of ideals and relate them to the compressed algebras considered by Iarrobbino in the eighties. Finally, I will show how to extend our result to the nested setting.

As usual, you start with a functor and prove representativity.

Definition 12.1. Let X be a smooth quasi-projective variety and a nonegative integer d. Define

$$\begin{aligned} \underline{Hilb}^d : \mathrm{Sch}_{\mathbb{C}} &\longrightarrow \mathsf{Set} \\ B &\longmapsto \left\{ Z \hookrightarrow B \times X : \substack{B\text{-flat} \\ B\text{-finite} \\ \text{length } d} \right\} \end{aligned}$$

Theorem 12.2 (Grothendieck)**.** *$\underline{Hilb}^d(X)$ is represented by a quasi-projective scheme $Hilb^d(X)$*

Idea: $\mathbb{C}$-points of $Hilb^d(X)$ are in correspondence with $Z \overset{\subset}{\text{finite}} X$ of length.
A fat point $Z = \mathrm{Spec}(A)$ $(A, \mathfrak{m})$ is a local artinian $\mathbb{C}$-algebra of finite type.
If X smooth can consider $X = \mathbb{A}^n$, and it's the same to consider

- $Z \hookrightarrow \mathbb{A}^n$ of length d.
- $I \subset \mathbb{C}[x_1, \ldots, x_n]$ of colength d.
- $A = \mathbb{C}[x_1, \ldots, x_n]/I$ with $\dim_{\mathbb{C}} = d$.

Definition 12.3. X, $0 \le d_1 \le \ldots d_r = \underline{d}$.

$$Hilb^{\underline{d}}(X) = \{Z_1 \not\hookrightarrow \ldots \not\hookrightarrow Z_r \not\hookrightarrow X : \mathrm{len}(Z_i) = d_i\}$$

Theorem 12.4 (Hartshorne (r=0), Fogarty, Kalpan (r>1))**.** *$Hilb^d(X)$ is connected.*

For $r = 1$, $Hilb^d(\mathbb{A}^n)$ is smooth iff $n \le 2$, $d \le 3$. If $r > 1$, $n = 2$, it is smooth iff $r = 2$ and $d_1 = d_2 = -1$. If $r > 1$ and $n > 2$, it is smooth iff $r = 2$, $(d_1, d_2) = \{(1, 2), (2, 3)\}$.

Iror components.

- $(r=1)$ $n \geq 4$, $Hilb^d(\mathbb{A}^3)$ is irreducible iff $d \leq 7$ ($\Longleftarrow$ [E-I], $\Longrightarrow$ [Mozzola]. $n=3$ irredicuble if $d \leq 11$ (8,9 Sivic; 10 Jardim et al; 11, Jelisievv et al.)
- $(r=2)$, $n=2$, irreducible [G-Rosul-Sebastian].
- $(r \geq 3)$ $n=2$, there exist $d_1 \leq \ldots \leq d_5$ such that $Hilb^{\underline{d}}(\mathbb{A}^i)$ is irreducible [S-R].

There is an analogy between classical in dimension 3 and nested in dimension 2. Schematic structure.

I $Hilb^{21}(\mathbb{A}^4)$ has at least a generically non-reduced component.
- Problem: what about $n=3$?
- Theorem. If $Hilb^{d_1,\ldots d_r}(\mathbb{A}^i)$ is irreducible $\Longrightarrow$ $Hilb^{(1,d_1,\ldots,d_r)}(\mathbb{A}^n$ has a generically non reduced component.

Other results.

- ...

Hilbert scheme function. For a local artinian $\mathbb{C}$-algebra of finite type there is an associated graded $\mathrm{Gr}_{\mathfrak{m}}(A) = \bigoplus_{i\geq 0} \mathfrak{m}^1/\mathfrak{m}^{i+1}$ which allows us to define the Hilbert function

$$\begin{aligned} h_A : \mathbb{Z} &\longrightarrow \mathbb{N} \\ i &\longmapsto \dim_{\mathbb{C}} \mathfrak{m}_i/\mathfrak{m}_{i+1} \end{aligned}$$

sistability for graded A-modules. The socol is the annihilator of the ideal which gives the algebra e.g. $\mathbb{C}[x,y](x^2,xy,y^2)$ then $\mathrm{Soc}(A) = \langle x, y^2 \rangle_{\mathbb{C}}$.

Definition 12.5. The *smoothable component* is

$$V_{sm} = \overline{\{[(z_i)_{i=1}^n \in Hilb^{\underline{d}}(\mathbb{A}^n) : Z_r \text{ is reduced}\}}$$

Definition 12.6. $V \subset Hilb^{\underline{d}}(\mathbb{A}^n)$ is an *elementary component* if $\forall [Z_1,\ldots,Z_r] \in V$ then Z_r is a fat point.

Instead of looking for all components of the Hilbert scheme we look for elementary components. This may not be some simple; sometimes there's infinitely many. There is no method to find them.

Theorem 12.7 (Irrabaro)**.** *Every irreducible component $V \subset Hilb^d(\mathbb{A}^n)$ is generically étale locally product of elementary components.*

Definition 12.8. Let $h : \mathbb{Z} \to \mathbb{N}$ be a function with finite support, $|h| = \sum_{i\in\mathbb{Z}} h(i) = d$. Define

- $H_n = \{[A] \in Hilb^d(\mathbb{A}^n) : h_A \equiv h\}$. This is closed by semicontinuity of the function; but can be expressed a by a representable functor.
- The following also has a canonical schematic structure

$$\begin{aligned} \pi_n : H_n &\longrightarrow \mathcal{H}_n = \{[A] \in H_n : A \text{ is graded}\} \\ A &\longmapsto A = \mathrm{Gr}_{\mathfrak{m}}(A) \end{aligned}$$

Where the first is $\not\to$ in the second.

Recall that

$$T_{[I]} Hilb^d(\mathbb{A}^n) = \mathrm{Hom}_R(I, R/I)$$

Theorem 12.9 (B-B,J,G G G L)**.** *There is a graded $[I] \in \mathcal{H}_n$. Then*

$$\begin{aligned} T_{[I]}\mathcal{H}_n &= \mathrm{Hom}_R(I, R/I)_{=0} \\ T_{[I]}\pi_n([I]) &=' > 0 \\ T_{[I]}H_n &=' \geq 0. \end{aligned}$$

Definition 12.10. $(A, \mathfrak{m}_A)$ is *compressed* is for all $(A', \mathfrak{m}')$ with $e(A) = e(A')$ and $len(A) \geq len(A')$.

Theorem 12.11. *$E_h \subset H_n$ is H-compressed $E_h \neq \emptyset$*

13. Normal and Restricted Tangent Bundles of Rational Curves in Hypersurfaces

Lucas Mioranci, IMPA. Algebra Seminar, IMPA. October 8, 2025.

Abstract. Let $X \subset \mathbb{P}^n$ be a degree d hypersurface containing a smooth rational curve C of degree e. The normal bundle $N_{C/X}$ and the restricted tangent bundle $T_X|_C$ split as direct sums of line bundles of the form $\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ called their splitting type. The splitting type of $N_{C/X}$ and $T_X|_C$ controls the local structure of the space of rational curves and determines how many general points of X we can interpolate by deforming the curve C. By combining explicit computations of $T_X|_C$ and an induction argument, I classify all triples (e, d, n) such that a general degree d hypersurface $X \subset \mathbb{P}^n$ contains a rational curve C of degree e whose restricted tangent bundle $T_X|_C$ is balanced. The case of quadrics is particular, in which we show that odd-degree rational curves do not interpolate the expected number of points on quadric hypersurfaces. For the normal bundle, I compute explicit examples of hypersurfaces X for all possible splitting types of $N_{C/X}$ when C is the rational normal curve. Additionally, for $d \geq 3$, we compute the dimension of the space of hypersurfaces X such that $N_{C/X}$ has a given splitting type.

$\mathcal{O}$ with no subindex means $\mathcal{O}_{\mathbb{P}^1}$.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ & T_{\mathbb{P}^1} = \mathcal{O}(2) \arrow[r, "="] \arrow[d] & T_{\mathbb{P}^1} = \mathcal{O}(2) \arrow[d] & & \\ 0 \arrow[r] & T_X|_C \arrow[r] \arrow[d] & T_{\mathbb{P}^n}|_C \arrow[r] \arrow[d] & N_{X/\mathbb{P}^n}|_C \cong \mathcal{O}(de) \arrow[r] \arrow[d, "="'] & 0 \\ 0 \arrow[r] & N_{C/X} \arrow[r] \arrow[d] & N_{C/\mathbb{P}^n} \arrow[r] \arrow[d] & N_{X/\mathbb{P}^n}|_C \cong \mathcal{O}(de) \arrow[r] & 0 \\ & 0 & 0 & & \end{tikzcd}$$

Throughout this talk X is a degree d hypersurface in $\mathbb{P}^n$ and $C \subset X$ is a smooth rational curve of degree e.

Since C is rational we can think of it as a map $f : \mathbb{P}^1 \to X$. Then we can pullback any bundle over C back to $\mathbb{P}^1$, and by Birkhoff-? Theorem we know that vector bundles over $\mathbb{P}^1$ split as

$$E \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i), \qquad a_1 \leq \ldots \leq a_n.$$

This decomposition is called the *splitting type* of E. We say E is *balanced* if $|a_i - a_j| \leq 1$ for all i, j.

Balancedness is an open condition, i.e. if you have a family of bundles over $\mathbb{P}^1$ and one of them is balanced, then all of them are, (also you can think it's a generic condition).

We know the normal bundle $N_{C/X}$ has rank $n-2$ and degree $e(n-d+1)-2$. And the tangent bundle $T_{X/C}$ has rank $n-1$ and degree $e(n-d+1)$.

Example 13.1. A conic $C \subset \mathbb{P}^3$. Since a conic in $\mathbb{P}^3$ is contained in a plane, this obliges to have a "distinguished normal direction", $\mathcal{O}(2)$. Indeed,

$$\begin{aligned} N_{C/\mathbb{P}^3} &\cong (N_{Q/\mathbb{P}^3} \oplus N_{\mathbb{P}^2/\mathbb{P}^3}|_C \\ &\cong (\mathcal{O}(2) \oplus \mathcal{O}(1))|_C \\ &\cong \mathcal{O}(1) \oplus \mathcal{O}(2). \end{aligned}$$

Now we discuss interpolation of general points. Let $p_1, \ldots, p_n \in \mathbb{P}^1$ and $x_1, \ldots, x_n \in X$ be general points. Let $f : \mathbb{P}^1 \to X$ be such that $f(p_i) = x_i$.

Consider the space of morphisms mapping the p_i to x_i. It's known that it's tangent space at f is given by

$$T_{[f]} \operatorname{Mor}(\mathbb{P}^1, X, p_i \mapsto x_i) \cong H^0(\mathbb{P}^1, f^* T_X(-n)).$$

Deformations of f (mapping $p_i \mapsto x_i$) dominate X if $a_i - n_i \geq 0$. Also, deformations of f interpolate up to $a_1 + f$ general points in X.

How many points can deformations of the curve interpolate? This means (I think) that the deformations of the curve are still such that $p_i \mapsto x_i$. "And for the normal bundle, interpolations means that the curves contain the points".

If $T_X|_C$ is balanced, C can interpolate up to

$$\text{floor function}\left(\frac{e(n+1-d)}{n_1}\right) + 1$$

general points. The case of $N_{C/X}$ is similar.

Next we survey some relevant results.

Theorem 13.2 (Coskun-Riedl, 2018)**.** *A general Fano X of degree $d \geq 2$ contains a degree c rational smooth curve C with balanced $N_{C/X}$ for every $1 \leq e \leq n$.*

The idea is to observe that the space of curves has balanced normal bundle.

Theorem 13.3 (Ran, 2021)**.** *Extends the above for $1 \leq e \leq 2n-2$ and $d \geq 4$.*

Idea: general hypersurface contains a curve of balanced normal bundle.

Theorem 13.4 (Ran, 2024)**.** *X general Fano hypersurface. There exist smooth rational curve with C with balanced tangent bundle $T_X|_C$ for e in some arithmetic progressions.*

e is in general very large. They show there are curves of arbitrarily large degrees with balanced tangent bundle.

Now we describe the new results.

Theorem 13.5. *Let $X \subseteq \mathbb{P}^n$ be Fano hypersurface of degree $3 \leq d \leq n$.*

(1) *If $C \subset X$ is a rational smooth curve of degree e with $e \leq \frac{n-1}{n+1-d}$, then $T_X|_C$ is not balanced.*
(2) *A general hypersurface X contains rational curves C of degree e with balanced $T_X|_C$ for every $e > \frac{n-1}{n+1-d}$.*

Theorem 13.6. *Let $X \subseteq \mathbb{P}^n$ be a smooth quadric hypersurface.*

(1) *If $e \leq 2$ is even, then exists a curve C of degree e with $T_X|_C \cong \mathcal{O}(e)^{n-1}$. (I.e. we can interpolate $e+2$ points if e is even.)*
(2) *If $e \geq 1$ is odd, then there is no curve C of degree e with balanced $T_X|_C$. The best we can do is the most balanced bundle before balancedness, namely $T_X|_C \cong \mathcal{O}(e-1) \oplus \mathcal{O}(e) \oplus \mathcal{O}(e+1)$. (I.e. we can only interpolate e points if e is odd.)*

Next we give the main ideas in the proof. Let C be the rational normal curve of degree e in $\mathbb{P}^1$,

$$
\begin{aligned}
C : \mathbb{P}^1 &\longrightarrow \mathbb{P}^n \\
(s:t) &\longmapsto (s^2 : s^{e-1}t : \ldots : t^e : 0 : \ldots : 0).
\end{aligned}
$$

$X = V(F)$.

Then find some examples. Then,

Proposition 13.7. *If $\mu(T_X|_C) = \frac{e(n+1-d)}{n-1} \leq 1$ then $T_X|_C$ is not balanced.*

Use the proposition to show that $T_X|_C$ is not balanced if $d \leq n$ and $ele\frac{n-1}{n+1-d}$.

This implies Theorem 13.5.

Lemma 13.8. *If $N_{C/X} \cong \bigoplus_i \mathcal{O}(a_i)$ with $a_i < 4$ for all i, then $T_X|_C \cong N_{C/X} \oplus \mathcal{O}(2)$.*

This implies that $Ext^1(N_{C/X}, \mathcal{O}(2)) = 0$.

The most important step is the following, used as the induction step:

Proposition 13.9. *If $T_Y|_C$ is balanced for some degree d hypersurface $Y \subseteq \mathbb{P}^{n-1}$, then there exists a degree d hypersurface $C \subseteq \mathbb{P}^n$ with balanced $T_X|_C$.*

Combining the examples, Lemma 13.8 and Proposition 13.9, we settle that for $e \leq max\{2d-2, n\}$.

The following argument is used to show that actually what we showed so far is enough for all degrees.

Lemma 13.10. *$C = C_1 \cup C_2$, $C_1 \cong \mathbb{P}^1$. E vector bundle on C such that $E|_{C_1}$ is balanced, E_{C_2} is perfectly balanced (numerical condition that works when $e = n-1$). Then if E is the specialization of a family of vector bundles E' on $\mathbb{P}^1$, then E' is balanced.*

Basically, glue and smooth low degree curves (i.e. $e \leq max\{2d-2, n\}$) to get all degrees

Now we prove the result for quadrics, i.e. X is a quadraic. For even e compute examples and conclude $T_X|_C \cong \mathcal{O}(e)^{n-1}$.

For odd e, compute examples, $T_X|_C \cong \mathcal{O}(c-1) \oplus \mathcal{O}(2)^3 \oplus \mathcal{O}(-1)$.

Proposition 13.11. *$n \geq 3$, $m \leq e+1$. There is a degree e curve ϕ_e interpolating m general points if and only if there is a degree $e-2$ curve ψ_{e-2} interpolating $m-2$ general points in X.*

The idea is to do induction. Suppose you can interpolate e points, then show you can interpolate $e+2$.

Chose $y_1, \ldots, y_{m-2}$ on the quadric. Choose a line in $\mathbb{P}^{m-1}$ intersecting the quadric, and $x_1, \ldots, x_{m-1}$ points on the line. Join the points on the line to the points on the quadric, Then you get somem points of intersection with the quadric. Those points and the two points of intersection of the line and the quadric let us construct a scroll, which intersects the quadric. The intersection of the scroll and the quadric (I guess the scroll contains the lines, and so the points y_i) gives the curve passing through the y_i.

14. Regular but non smooth curves of genus 3

Cesar Hilario, . Algebra Seminar, IMPA. October 15, 2025.

Abstract. Regular but non-smooth curves are a unique feature of geometry in positive characteristic, that results from the fact that over an imperfect field the notion of regularity is weaker than the notion of smoothness. In the setting of algebraic geometry over an algebraically closed field, these curves correspond to fibrations by singular curves, which are fibrations of relative dimension 1 whose fibers are singular. The most famous examples are arguably the so-called quasi-elliptic curves and quasi-elliptic fibrations, which play a key role in the Bombieri-Mumford classification of algebraic surfaces in characteristics 2 and 3. In this talk I will discuss the case of genus 3 in characteristic 2, with an eye towards the classification of regular plane projective quartic curves that become rational after base change.

arXiv 2409.05464.

C will denote a projective geometrically integral curve over a not necessaily closed field K. Geometricallt integral means that $C \otimes_K \overline{K}$ is integra. Suppose that $p = \mathrm{char}(K) \geq 0$ and $g = h_1(\mathcal{O}_C)$ is of genus C.

We define

(1) C regular iff local rings of C are regular (DVR).
(2) C smooth iff C regular and $K \otimes_K \overline{K}$ regular.

Thus it's obvious that smoothness implies regularity. In this talk we explain why the converse is not true.

Note that if K is perfect (e.g. $p = 0$ or $K = \overline{K}$) implies smooth = regular. Recall that K is imperfect iff $p > 0$ the powers of K, K^p, is a proper subfield of K (it's always a subfield but we ask it's proper).

Assume that C is regular and $p > 0$. The genus of rthe normalization is less than or equal to the genus of the curve, i.e.

$$\tilde{g} = h^1(\mathcal{O}_{\widetilde{C \otimes_K \overline{K}}}) \leq g.$$

We know that C is smooth iff $\overline{g} = g$. So $g - \overline{g}$ is a measure of non-smoothness.

Theorem 14.1 (Tate)**.** $\frac{p-1}{2}$ *divides* $g - \overline{g}$*.*

As a corollary, we get that if C is non-smooth, $g - \overline{g} > 0$ and so $p \leq 2g + 1$.

Example 14.2. There are cases in which the bound is sharp, namely $C : y^2 = x^p + t$, with $t \in K \setminus K^p$. Then $g = \frac{p-1}{2}$ and $\overline{g} = 0$. Over $\overline{K}$ rhere exists a singular point, $(x, y) = (-t^{1/p}, 0)$ which **is not visible over** K, that is we get that $t^{1/p} \notin K$.

Example 14.3. $\{$elliptic curves$\} = \{g = \overline{g} = 1, \exists K$-rational point $\} = \{$smooth cubic curves with K-rational po
$y^2 = x^3 + 4g_i x + 27$.

Example 14.4. $g = 1$, $\overline{g} = 0$, $p = 2, 3$, quasielliptic curves. (Important in the classification of surfaces in positive characteristic.)

Now we make a digression to describe Kodaira-Enriques (over $\mathbb{C}$) in positive characteristic. Bombieri-Mumford: **new objects** appear, quasielliptic surfaces.

Let $k = \overline{k}$, S smooth elliptic (*quasielliptic*) surface over k. Suppose we have a fibration where the general fiber is an elliptic curve an elliptic curve (resp. a plane cuspidal cubic) and the generic fiber an elliptic (resp. quasielliptic) curve over $k(B)$.

$$\begin{tikzcd} S \arrow[d] \\ B \end{tikzcd}$$

Back to our main content, let's assume that $g = 3$.

$$\{g = 1, \exists K\text{-rat. pt}\} \longleftrightarrow \left\{ \begin{array}{c} \text{reg. cubic curves} \\ \text{with } K\text{-rat pt} \end{array} \right\}$$

Recall that genus-degree formula

$$g = \frac{(d-1)(d-2)}{2} = 1,$$

So then $d = 3, 4$ gives $g = 3$ (right?).

No we present our main theorem, which is a classification result:

Theorem 14.5.
- *C regular non-hyperelliptic curve over K.*
- *$g = 3$, $\overline{g} = 0$.*
- *$p = 2$.*

(Not that non-hyperllipticity of C and $g = 3$ imply that C is a quartic over K, and that $\overline{g} = 0$ implies C is geometrically rational.) Then C is isomorphic to one of the following quartics.

1. *$y^4 + az^4 + xz^3 + bx^2z^2 + cx^4 = 0$ where $a, b, c \in K$, $c \notin K^2$. (Notice the imperfectness of K is crucial.)*
2. *$y^4 + az^4 + bx^2y^2 + cx^2z^2 + bx^3z + dx^4 = 0$ for $a, b, c, d, \in K$, $a \notin K^2$, $b \neq 0$.*
3.
4.
5.

Now we enumerate properties of these families.

- C is a purely inseparable double cover of a quasielliptic curve. To understand this consider the induced map from Frobenius map $K \to K$, $a \mapsto a^p$ and do base change:

$$\begin{tikzcd} C^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d] \\ \operatorname{Spec} K \arrow[r, "\text{induced from Frobenius}"] & \operatorname{Spec} K \end{tikzcd}$$

This gives a map $C \to C^{(p)}$ by universal property of pullback.

$$\begin{tikzcd} C \arrow[dr, dashed] \arrow[drr, "\text{Frobenius}"] \arrow[ddr] & & \\ & C^{(p)} \arrow[r] \arrow[d] & C \arrow[d] \\ & \operatorname{Spec} K \arrow[r] & \operatorname{Spec} K \end{tikzcd}$$

Then consider the normalization of $C^{(p)} = C_1$ which turns out to be quasielliptic; by universal property of normalization we can lift:

$$\begin{tikzcd} & \widehat{C^{(p)}} \arrow[d] \\ C \arrow[ur, dotted] \arrow[r] & C^{(p)}. \end{tikzcd}$$

Here the lower arrow is purely inseparable of degree 2.
- There exists a unique non smooth point $x \in C$.
- Now we iterate the construction of the first item:

$$\underbrace{C}_{g=3} \longrightarrow \underbrace{C_1}_{g=1} \longrightarrow \underbrace{C_2}_{g_2-9} \longrightarrow \underbrace{C_3}_{g_3=0} \longrightarrow \underbrace{C_4}_{g_4=0} \longrightarrow \cdots$$

Let x be the non-smooth (non-rational) point of C. Map it under these maps to x_1(non-smooth,rational), then x_2 (smooth, rational?), then x_3 (smooth, rational.), then x_4 (smooth, rational), etc.

We have

	x_2 rational	x canoical divisor	$E = K(c_2)$
(i)	Y	Y	Y
(ii)	N	Y	N
(iii)	Y	N	N
(iv)	N	N	Y
(v)	N	N	N

Where the last column is explained next.
- Canonical field of C = subfield of $K(C)$ generated by the quotients of all non-zero holomorphic differentials = $K(C)$ (C non-hyperelliptic).
- Pseudocanonical field of $C = \dots$ non-zero <u>exact</u> holomorphic differential $:= E$. So one has $[K(c), E] = 4 = p^2$.

$$
\begin{tikzcd}
 & K(C) \arrow[dl, "4"'] \arrow[dr, "4"] & \\
E & & K(C_2).
\end{tikzcd}
$$

In conclusion, we manage to characterize these families.

Example 14.6. In case 1, let $a = b = 0$. We get

$$C : y^4 + xz^3 + \underbrace{c}_{\in K \setminus K^2} x^4 = 0 \tag{14.6.1}$$

$$\underbrace{C_1}_{\text{quasiell.}} : xy^2 + z^3 + cx^3 = 0.$$

Further

$$C \longrightarrow C_1 \longrightarrow C_2 \cong \mathbb{P}^1 \longrightarrow C_3 \cong \mathbb{P}^1 \longrightarrow C_4 \cong \mathbb{P}^1 \longrightarrow \cdots .$$

With C one can construct a pencil (fibration over $\mathbb{P}^1$) of quartics. Let $k = \overline{k}$ (new base field from what we fixed at the beginning!). Idea: replace c for t Equation 14.6.1. $K = k(\mathbb{P}^1)$, $S = V(t_0(y^4 + xz^3) + t_1x^4) \subseteq \mathbb{P}^2_{(x:y:z)} \times \mathbb{P}_{(t_0:t_1)}$. This has a singular point, so we blow up:

$$
\begin{tikzcd}
\tilde{S} \arrow[d] \arrow[dd, bend left, "f"] \\
S \arrow[d] \\
\mathbb{P}^1
\end{tikzcd}
$$

The generic fiber is the curve $y^4 + xz^3 + tx^4 = 0$, so that $t_1/t_0 \in k(\mathbb{P}^1) = K$. Fibers are plane rational quartics, and the singular fiber $f_1(0:1)$ is a configuration of lines given by a Dynkin diagram

$$
\begin{tikzcd}
E_1 \arrow[r, dash] & E_2 \arrow[r, dash] & E_3 \arrow[r, dash] & E_4 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & E_{15} \\
 & & & E & &
\end{tikzcd}
$$

We do the same with C_1: define $S^1 = V(t_0(xy^2 + z^3) + tx^3) \subseteq \mathbb{P}^2 \times \mathbb{P}^1$. Then S_1 is also singular, and we blow up to obtain a smooth quasielliptic fibration ($\tilde{S^1}$ is quasielliptic). The fibers are plane cuspidal cubics. The singular fiber an arrangement of curves

$$
\begin{tikzcd}
F_1 \arrow[r, dash] & F_2 \arrow[r, dash] & F_3 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & F_8 \\
 & & F & &
\end{tikzcd}
$$

In fact, there is a correspondence between the singular fibers of the two fibrations.

15. Lagrangian blow-up and blow-down for 4-dimensional symplectic manifolds

Misha Verbitsky, IMPA. Geoemtric Structures Seminar, IMPA. October 23, 2025.

Abstract. The usual (complex geometric) blow-up has a symplectic version, called the symplectic cut. I will introduce a variant of this construction, which is valid only in symplectic category, called "Lagrangian blow-up", and its inverse, called Lagrangian blow-down. Lagrangian blow-down takes a Lagrangian sphere in a 4-dimensional symplectic manifold and contracts it to a symplectic orbifold with a double point. Given a symplectic orbifold with a double point, Lagrangian blow-up produces a symplectic manifold, and this construction is inverse to the Lagrangian blow-down. I will explain why these constructions are functorial, that is, defined on the corresponding symplectic Teichmuller spaces. This is used to prove an orbifold counterexample to a famous conjecture, sometimes attributed to Donaldson, who asked whether any symplectic form on a K3 surface is compatible with a Kahler structure. I will prove that this is false for orbifolds: some K3 orbifolds admit symplectic forms not compatible with any Kahler structure. The same argument is used to produce a countable family of Lagrangian spheres in a K3 surface which are not Lagrangian isotopic to special Lagrangian spheres. This is a joint work with Michael Entov.

Consider the total space of the cotangent bundle of $\mathbb{C}P^1$. It's $\mathcal{O}(-2)$. Let $\gamma \in H^0(\mathcal{O}(2))$. γ defines a fiberwise linear function on $T^*\mathbb{C}P$.

The blow-up of $\mathbb{C}^2/\pm 1$, which is the orbifold $\mathbb{C}^2$ with a doble point, is the cotangent bundle $T^*\mathbb{C}P^1$. That is, $T^*\mathbb{C}P^1 \to \mathbb{C}^2/\pm 1$. This is called *crepant resolution.* Consider the $dx \wedge dy$, which is holomorphically symplectic on $\mathbb{C}^2$ and descends to the quotient $\mathbb{C}^2/\pm 1$ as Ω_0.

Proposition 15.1. $\pi : T^*\mathbb{C}P^1 \to \mathbb{C}^2/\pm 1$. *Then* $\pi^*\Omega_0$ *is holomorphically symplectic.*

Proof. Actually, it gives the standard symplectic form on $T^*\mathbb{C}^2$. $\square$

Now we shall define the Langrangian blow-up. It's a procedure that starts with a symplectic manifold with double points and outputs a symplectic manifold without double points.

Definition 15.2 (Lagrangian blow up)**.** Let (B, ω) be the standard ball in $\mathbb{R}^n$, $dz_1 \wedge dz_2 + dz_3 \wedge dz_4$. Consider the symplectic orbifold $(B, \omega)/\pm 1$. Consider M be a symplectic orbifold with double point singularities. Darboux coordinates in a neighbourhood of 0 with $(B, \omega)/\pm 1$ resolve singularities.

This definition has the caveat that there ar e lots of choices involved. We shall later that these choices do not alter the result, namely Theorem **??**.

To define the Lagrangian blow-down we first recall

Theorem 15.3 (Weinstein's Lagrangian neighbourhood)**.** *Let* $X \subset M$ *be a compact Lagrangian submanifold in* (M, ω)*. Then there exists a neighbourhood* U *of* $X \subset M$ *which is symplectomorphic to a neighbourhood of* X *in* $X \subset T^*M$*.*

Let $S \subset M$ be a Lagrangian sphere (diffeomorphic to sphere and Lagrangian).

Definition 15.4. The *Lagrangian blow-down* is obtained from M by removing $W \subset M$ and gluing W_0 in its place, where W_0 is such that

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & T^*\mathbb{C}P^1 \arrow[d] \\ W_0 \arrow[r, hook] & \mathbb{C}^2/\pm 1. \end{tikzcd}$$

Now we recall Teichmüller spaces.

Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M. The space of symplectic forms is a subspace of the closed 2-forms on M, which in turn is a subspace of $\Gamma(\Lambda^2 M)$. Equip this space with the C^∞ topology. The *Teichmüller space* is $\mathrm{Symp}/\mathrm{Diff}_0$.

Two symplectic structures are called *isotopic* if they lie in the same orbit of Diff_0. The *period map* $\mathrm{Per} : \mathrm{Teich}_s \to H^2(M, \mathbb{R})$ maps a symplectic structure to its cohomology class.

Theorem 15.5 (Moser, 1965)**.** *The period map is a local diffeomorphism.*

This implies that Teich_s is smooth.

Theorem 15.6 (Moser's trick)**.** *Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t]$ is constant. Then all ω_t are diffeomorphic.*

This means that the fibers of $\mathrm{Symp} \to H^2(M)$ are the orbits (of the action of Diff_0, I suppose).

Consider a Lagrangian submanifold $L \subset (M, \omega)$. Let us assume for simplicity (though the results can also be obtained without this) that $b_1(L) = 0$, i.e. first cohomology group is zero.

Let Symp_L be the set of symplectic forms vanishing on L, and Diff_0^L the connected component of diffeomorphisms preserving L. Define $\mathrm{Teich}_L = \mathrm{Symp}_L/\mathrm{Diff}_0^L$.

Theorem 15.7. *Let $V \subset H^2(M, \mathbb{R})$ be the space of all cohomology classes on M which vanish on L, and $Per_L : Symp_L/Diff_0^L \to V$ take (M, ω) to the cohomology class of ω. Then Per is a local diffeomorphism.*

Proof. We used a version of Moser's trick: for ω_t a family of symplectic forms with $\omega_t|_L = 0$, with $[\omega_t]$ constant, then ω_t are Diff_0^L-isotropic. □

As a corollary we obtain that this Teichmüller space classifies Lagrangian submanifolds:

Lemma 15.8. *Assuma that L and L' are Lagrangian submanifolds in (M, ω), and $\varphi \in Diff_0(M)$ a smooth isotopy such that $\varphi(L) = L'$. Then (ω, L) and $(\varphi^*\omega, L)$ represent the same point in $Teich_L$ if and only if L and L' are Lagrangian isotopic in (M, ω).*

Again: points of this Teichmüller space are isotopy classes of Lagrangian submanifolds.

Theorem 15.9. *Let L be a Lagrangian 2-sphere in a K3 surface M and $\hat{M}$ the Lagrangian blow-down of M. The blow-up/blow-down is a diffeomorphism $Teich_L(M) \to Teich(\hat{M})$.*

Now we prove the theorem which asserts that the Lagrangian blow-up is independent of choices.

Theorem 15.10. *Let $(\hat{M}, \hat{\omega})$ be a orbifold. Then its blow-up (M, L) defines a point in $Teich_L$ independent of the choices.*

Proof. By taking local Darboux coordinates, and then Moser's trick. □

Actually, the same happens for blow-downs

Theorem 15.11. *Let (M, L, ω) be a 4-manifold with a Lagrangian sphere L, and $(\hat{M}, \hat{\omega})$ be its Lagrangian blow-down.*

Then the corresponding point in $Teich_{\hat{M}}$ is independent from the choices made.

Proof. Similar. □

A conjecture by Donaldson ways that all symplectic structures on a K3 surface are compatible with a Kähler structure. This conjecture (although it's commonly believed to be true) is still open. When trying to prove that this would hold for orbifolds, Misha and collaborators realised it's actually false.

Note: Seiberg-Witten invariants may be used to tell whether a symplectic form is compatible with a complex structure.

Theorem 15.12. *There exists an orbifold symplectic K3 surface M with a single double point not admitting an orbifold Kähler structures.*

Proof. Step 1. Consider

$$\mathrm{Teich}_{\text{Kähler-symplectic}}(\hat{M}) = \{\eta \in H^2(\hat{M}) : \eta^2 > 0\}.$$

As follows from [Amerik-V., 2005], the Teichmüller space of symplectic structures compatible with a hyperkähler structure is Hausdorff and connected (the same argument works for Kähler K3 orbifolds).

Essentially, if there wasn't any orbifold as required, the Teichmüller space classifying Lagrangian submanifolds would be connected, which is impossible by [Seidel 2000]. □

In fact, Seidel's result is essentially mirror symmetry.

Now we discuss special Lagrangian submanifolds.

Definition 15.13. Let (M, I, ω, Ω) be a Calabi-Yau manifold, where $\Omega \in \Lambda_I^{n,0}(M)$ is nondegenerate and ω is Kähler. Let L be a Lagrangian submanifold. Define the *phase* as

$$\begin{aligned} \text{phase} : L &\longrightarrow \Lambda^n L \otimes \mathbb{C} \\ x &\longmapsto \Omega|_{T_x L}. \end{aligned}$$

A better definition is:

$$\begin{aligned} \text{phase} : L &\longrightarrow S^1 = \mathrm{U}(1) \\ x &\longmapsto \frac{\Omega|_{T_x L}}{|\Omega T_x L|}. \end{aligned}$$

Definition 15.14. A special Lagrangian submanifold is the phase is constant.

Special Lagrangian submanifolds have deformation space of dimension 1, it's unobstructed. They are calibrated, so minimal.

Theorem 15.15. *$L \subset K3$ is Lagrangian is isotopic to a special Lagrangian if and only if its blow-down is Kähler type.*

One of the basic results about special Lagrangian submanifolds is

Lemma 15.16 (Hitchin)**.** *If (M, Ω) is holomorphically symplectic and $X \subset M$ is Lagrangian with respect to $Re\Omega$ and $Im\Omega$ then X is complex.*

As corollaries:

Lemma 15.17. *Let (M, I, J, K, g) be a hyperkähler manifold of real dimension $4n$, considered as a Kähler manifold (M, I, ω_I) and $\phi : \Omega^n \in \Lambda^{2n,0}(M, I)$ the corresponding holomorphic volume form. Consider a Lagrangian submaniold $S \subset (M, \omega_I)$ such that $a\omega_J + b\omega_K|_S = 0$ for some nonzero real numbers $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. Then S is special Lagrangian, and, moreover, it is holomorphic with respect to the complex structure $-aK + bJ$.*

Lemma 15.18. *Let ω_I, ω_J and ω_K on M, with $\dim M = 4$. Let $S \subset (M, \omega_I)$ be Lagrangian. Then S is special Lagrangian if and only if S is complex analytic on $aJ + bK$ for two nonzero numbers $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$.*

Now an auxiliary claim:

Lemma 15.19. *Ω is holomorphic symplectic*

Theorem 15.20. *Let $S \subset M$ be a Lagrangian sphere in a K3 surface. Then S is isotropic to a special Lagrangian if and only if M is of Kähler type.*

16. Systolic inequality for tori

Dimitri Korshunov, IMPA. Minimal Surfaces end of course presentation. October 24, 2025.

This is due to L. Guth.

Definition 16.1. Given a Riemannian manifold, a *systole* $\mathrm{Sys}(M, g)$ is the infimum of the lengths of noncontractible paths.

Theorem 16.2 (Gromov)**.** *For any metric on a torus T^n,*

$$Sys(T^n, g) \leq C_n \, Vol(T^n, g)^{\frac{1}{n}},$$

where C_n is a constant depending only on the dimension of the torus n.

This also holds for $\mathbb{R}P^n$. We will use induction, for which we need the following stronger statement: given a number bounded by half the systole, there exists a ball which is *big*:

Theorem 16.3 (Guth)**.** *For (T^n, g), given $R < \frac{1}{2} Sys(T^n, g)$, then there exists a point p on T^n such that the volume of ball of radius r with center in p satisfies*

$$VolB(p, R) \geq C_n R^n.$$

It was conjectured by Gromov that $C_n = 1$.

Before stating the theorem that we will prove, we introduce the following definition that will allow us to measure cohomological cycles:

Definition 16.4. Let $\alpha \in H^1(M_g, \mathbb{Z}_2)$.

$$L(\alpha) = \inf\{|\lambda| : \lambda \text{ 1-cycles}, \alpha(\lambda) \neq 0\}.$$

The theorem that we will prove, which implies Theorem 16.3 is the following

Theorem 16.5. *There exists a constant $\varepsilon_n > 0$ such that given a manifold (M, g), and $\alpha_1, \ldots, \alpha_n \in H^1(M, \mathbb{Z}_2)$,*

(1) $L(\alpha_i) > 2R$

(2) $\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_n \neq 0 \in H^n(M, \mathbb{Z})$.

Then there exists $p \in M$ such that

$$VolB(p, R) \geq \varepsilon_n R^n.$$

In particular T^n and $\mathbb{R}P^n$ satisfy (2).

Before we proceed to the proof we need the following definition. Essentially we ask that the surface minimizes volume in its homology class up to some constant δ.

Definition 16.6. A *δ-minimal surface* is an embedded hypersurface Z such that for any Z' such that $[Z' - Z] = 0$ in $H_{n-1}(M, \mathbb{Z}_2)$, then $\mathrm{Vol}Z < \mathrm{Vol}Z' + \delta$.

Moreover, we need the following

Lemma 16.7 (Stability)**.** *Let (M, g) de closed Riemannian manifold and $\alpha \in H^1(M, \mathbb{Z}_2)$. Suppose Z is Poincaré dual to α and that Z is δ-minimal. If $R < \frac{1}{2}L(\alpha)$ then the following holds for any $p \in M$:*

$$|Z \cap VolB(p, \frac{R}{2}| \leq \frac{2}{R}|\,VolB(p, R)| + \delta.$$

Proof of the stability lemma. Step 1. Using commas formula, namely

$$\mathrm{Vol}(B_R) = \int_0^R S_t dt,$$

we see that there exists t such that $\frac{R}{2} < t < R$ and

$$|S(p, t)|_R < \frac{2}{R}|\mathrm{Vol}B(p, R)|.$$

Step 2. Consider $[Z \cap B(p, t)] \in H^{n-1}(B(p, t), S(p, t), \mathbb{Z})$. This can be interpreted geometrically since we are using relative homology.

Step 3. We claim that $[Z \cap B(p, t)] = 0$. By contradiction, if it weren't trivial, then there would exist a cycle $H_1(B, \mathbb{Z})$ such that $Z' \cap \gamma \neq 0$ since Poincaré pairing is nondegenerate — namely any cocycle that is not zero pairs nontrivially with some cycle. Then we will just split the cocycle in very small parts, and we will reach a contradiction with the fact that $R < \frac{1}{2}L(\alpha)$.

We will state this as another lemma. Essentially it says that any cycle in homology can be decomposed as a sum of cycles with small lengths.

Lemma 16.8 (Curve factoring, Gromov)**.** *Let (M, g) be a Riemannian manifold and γ a cycle in $B(p, t)$. Then $[\gamma] = \sum \sigma_i$ is such that $L(\sigma_i) < 2t + a$ for all i.*

Proof. We spit a cycle (a curve) into some quantity of intervals such that each has length smaller than a, then add paths joining the center point of length t, so we count t twice: to reach the point and to get away from it. $\square$

We apply this to γ and reach a contradiction with the hypothesis that $R < \frac{1}{2}L(\alpha)$ making a smaller and smaller.

Step 4. Finally we use the minimality of Z and the property derived from Commas lemma to arrive at a contradiction. $\square$

Now we prove Theorem 16.5:

Proof. By induction. The case $n = 1$ is trivial. Now take the last one of our classes, α_n. Let Z be Poincaré dual of α_n and δ-minimal. Notice that $\alpha_1 \cup \ldots \cup \alpha_{n-1} \neq 0$ since if it were, then $\alpha_1 \cup \ldots \cup \alpha_n$ would also vanish.

By induction hypothesis we get a point $p \in Z$ such that $\mathrm{Vol}B_Z(p, \frac{R}{2}) > R^{n-1}$. This means that , $|Z \cap B(p, \frac{R}{2})| > \varepsilon_n (R/2)^{n-1}$.

By the Stability Lemma 16.7,

$$\frac{2}{R}|B(p,R)| > \varepsilon_n \frac{R^{n-1}}{2^{n-1}} - \delta$$
$$\implies |B(p,B)| > \frac{\varepsilon_n}{2^n} R^n - \frac{\delta R}{2} > \frac{\varepsilon_n}{2^n} R^n.$$

$\square$

17. From higher Lie groupoids to higher Lie algebroids

Matias del Hoyo, UFRJ. Symplectic Geometry Seminar, IMPA. October 29, 2025.

Abstract. This is a report of an ongoing collaboration with A. Cabrera, where we develop a higher Lie theory. Starting with simple, concrete examples, we present the differentiation functor in a geometric, explicit way, using simplicial methods and differential graded algebras. We describe how our construction extends the classical Lie functors, relate our project with previous works, and show a van Est theorem linking the cohomology at the global and infinitesimal levels.

Upshot: Integration-differentiation relation between Lie groups and Lie algebras in higher versions. Groupoids are the counterparts of Poisson manifolds, and in the next level there are 2-groupoids and Courant algebroids. How to integrate a 2-groupoid to a Courant algebroid? This is an open question which motivated this work. See arXiv 2309.14105.

Let M be a smooth manifold.

Remark 17.1. $M \mapsto C^\infty M$ is fully faithful.

Which allows us to think of manifolds as algebraic objects. Notice that $\Omega^k M = \{k\text{-forms}\} = \Lambda^k_{C^\infty M}\mathrm{Der}(C^\infty M, C^\infty M)$, which is (algebraically) locally spanned by the exact 1-forms.

Recall that the exterior differential is the only operator $d : \Omega^k M \to \Omega^{k+1} M$ such that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$ and that $df(X) = Xf$.

This makes $(\Omega M, \wedge, d)$ a differential graded algebra. Recall that $\mathrm{dom}(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$. We also have de Rham cohomology.

Consider a submanifold $S \subset M$. Recall that $\mathcal{I}_S/\mathcal{I}_S^2$ is essentially the sections of the normal bundle. In particular, for $M \subset M \times M$ we can see that the embedding vanishes on the diagonal, so there is no linear part in the Taylor expansion, and the second term is in fact the de Rham differential $df = \pi_1^* f - \pi_2^* f \bmod \mathcal{I}^2_{\Delta(M)}$.

In fact, this construction can be extended to a simplicial manifold (which I think is just a presheaf valued on category of manifolds) using projections and diagonals,

$$\cdots \qquad M \times M \times M \Rrightarrow M \times M \rightrightarrows M.$$

To this we can apply the C^∞ functor and obtain a cosimplicial ring (not a dga?). The idea here is that de Rham differential is an alternating sum of pullbacks; as other differentials will also be.

Now observe that for a finite-dimensional vector space V,

$$\left\{ \substack{\text{Lie bracket} \\ V \times V \xrightarrow{[\cdot,\cdot]} V} \right\} \quad \leftrightsquigarrow \quad \left\{ \substack{\text{differential on} \\ \Lambda V^*} \right\},$$

making ΛV^* into a differential graded algebra.

Recall that given a Lie algebra $\mathfrak{g}$, we have $(\Lambda^\bullet \mathfrak{g}^*, d) = \mathrm{CE}(\mathfrak{g})$, the Chevalley Eilenberg algebra, which gives Lie algebra cohomology.

[Missing part on Simplicial approach, exponential map, Simplicial manifold nerve]

Lie algebroid: $\Delta \to M$ vector bundle with $[\cdot,\cdot]$ on ΓA, $A \xrightarrow{\rho} TM$ Leibniz.

Remark 17.2.

$$\left\{ \substack{\text{Lie algebroids on} \\ E \to M} \right\} \quad \leftrightsquigarrow \quad \left\{ \substack{\text{differentials} \\ \text{on } \Lambda \Gamma E^*} \right\}.$$

This gives a differential graded algebra

$$\Delta \longrightarrow M \longrightarrow \mathrm{CE}(A)$$

and

$$C^\infty M \xrightarrow{\tilde{\rho}} \Gamma E^* \xrightarrow{\widetilde{[\cdot,\cdot]}} \Gamma E^* \wedge \Gamma E^* \longrightarrow \cdots.$$

From Lie groupoids to Lie algebroids: Lie functor/differentiation.

$$\Omega(G) = \Gamma \Lambda^\bullet \underbrace{\supset}_{\text{sub dga}} \Gamma \Lambda^\bullet T^s \xrightarrow{-/G} \mathrm{CE}(A_G).$$

There's the following ecosystem around Lie groupoids:

$$\begin{tikzcd}
 & \text{Lie groupoids} \arrow[dl] \arrow[d] \arrow[dr, "\text{nerve}"] & \\
\text{Lie algebroids} & \substack{\text{Differentiable stacks} \\ \supset \text{orbifolds}} & \text{simplicial manifolds.}
\end{tikzcd}$$

Lie groupoids are models for stacks, which are geometric objects solving classification problems with objects identified up to isomorphism. Relaxing the identification to a weaker notion of equivalence we must introduce higher versions of stacks.

Moving higher,

$$\begin{tikzcd}
\text{Lie groupoid} \arrow[r] \arrow[d, "\text{Lie}"'] & \text{Simplicial manifolds} \arrow[d, "?"] \\
\text{Lie algebroids} \arrow[r] & \text{dg-manifolds}
\end{tikzcd}$$

dg algebra is a dga ($A = \bigoplus_{k=1}^n A_k \to M, \wedge, d$) starting at $C^\infty M$, and continuing until you get to the sections, namely

$$C^\infty M \xrightarrow[d]{} A_2 \longrightarrow \cdots \qquad \cong \Gamma A.$$

[dH-Cabrera] provide the solution to this problem, namely what to put in the right vertical arrow:

Theorem 17.3 (dH-Cabrera)**.**

$$\frac{(C^\infty_N(G), \cup, \sum(-1)A_i^*)}{J + \delta J} = CE(A_G)$$

where A_G is a higher Lie algebroid and J is defined by: f is in J if it has higher vanishing.

Groupoids are the counterparts of Poisson manifolds, and in the next level there are 2-groupoids and Courant algebroids. How to integrate a 2-groupoid to a Courant algebroids? This is an open question which motivated this work.

18. Formality of Dolbeault cohomology

Misha Verbitsky, IMPA. Geometric Structures Seminar, IMPA. October 29, 2025.

Abstract. A quasi-isomorphism of differential graded algebras (DGA) is a multiplicative map inducing an isomorphism on cohomology. A DGA is called formal if it can be connected by a chain of quasi-isomorphisms to its cohomology algebra. For the de Rham algebra of a compact manifold, formality is an important topological property which has many implications (such as vanishing of Massey products). For the Dolbeault DGA of a compact complex manifold, the situation is not well understood, unless it is Kahler. The Dolbeault DGA of a torus, symmetric spaces and hyperkahler manifolds are known to be formal. We prove that the Dolbeault DGA of a complex nilmanifold is formal only if it is a torus, and the Dolbeault algebra of (0,p)-forms is formal if and only if the complex structure is abelian. This is a joint work with Tommaso Sferruzza.

Upshot: the Dolbeault dga of a nilmanifold M is never formal.

If a space is formal, Massey products vanish. Unfortunately, Massey products are not well-defined. Let $a, b, c \in \Lambda^*(M)$ be closed forms such that their cohomology classes satisfy $[a][b] = [b][c] = 0$. Let $\alpha, \gamma \in \Lambda^*(M)$ be such that $d(\alpha) = a \wedge b$ and $d(\gamma) b \wedge c$. Then the *Massey product* of a, b, c is the class $[\alpha \wedge c - (-1)^{\tilde{a}} a \wedge \gamma]$ which is only defined up to $\langle \mathrm{img} L_a + \mathrm{img} L_c \rangle$ where L_a and L_c are multiplication by a and c operators.

The *Heisenberg nilmanifold* is the quotient of the Heisenberg group G of upper triangular matrices with 1 on the diagonal over $G_{\mathbb{Z}}$ of the same group with integer entries. The Heisenberg nilmanifold fibers over a torus with fiber a circle:

$$0 \longrightarrow S^1 \longrightarrow G/G_{\mathbb{Z}} \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow 0$$

which in fact comes from the central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow G \longrightarrow \mathbb{R}^2 \longrightarrow 0.$$

Lemma 18.1. *Massey products on $G/G_{\mathbb{Z}}$ are non-zero.*

Proof. Step 1. There is an action of G in the cohomology of G. It's not hard to see that the cohomology classes on $G/G_{\mathbb{Z}}$ can be represented by right G-invariant forms, and the cohomology of $G/G_{\mathbb{Z}}$ is equal to the cohomology of the complex of right-G-invariant forms on G.

Step 2. In fact, this complex is the same as the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$ of G, and the cohomology is the Lie algebra cohomology.

Step 3. From a basis a, b, t of $\mathfrak{g}$ with the only nontrivial commutator $[a, b] = t$ and α, β, τ dual basis in $\mathfrak{g}^*$ with the only nontrivial differential $d\tau = \alpha \wedge \tau$, we obtain a basis $\alpha \wedge \beta, \alpha \wedge \tau, \beta \wedge \tau$ in $\Lambda^2(\mathfrak{g}^*)$ with $d|_{\Lambda^2 \mathfrak{g}^*} = 0$. This gives $\mathrm{rk} H^1(G/G_{\mathbb{Z}}) = 2$ and $\mathrm{rk} H^2(G/G_{\mathbb{Z}}) = 2$.

Step 4. Compute the Massey product of α, β, τ: it does not vanish. □

Digression: this is Sullivan's motivation for working on dgas. Consider the class of spaces whose homotopy and homology groups are vectors spaces over $\mathbb{Q}$ (and it's better to also assume that π_1 is nilpotent). There is a equivalence of categories between old category and homotopy category of rational homotopy spaces. And in fact it's also equivalent to the (homotopy?) category of dga algebras.

Now we move to differential graded algebras (dga). A morphism of dgas (which I think only means that commutes with the differentials) is called *quasi-isomorphism* if it induces isomorphisms on all $H^i(A^\bullet)$.

A dga is *formal* if it is connected to $H^\bullet(A^\bullet)$ by a chain of quasi-isomorphisms.

Recall the dd^c-lemma. It states that for the twisted differential $d^c = JdJ^{-1}$, if a form α is d-closed and d^c-exact, (or ∂-exact and $\overline{\partial}$-exact), then $\alpha \in \mathrm{img} dd^c$.

Lemma 18.2. *The map* $(\Lambda^\bullet(M)_{d^c}, d) \to (\Lambda^\bullet(M), d)$ *is a quasi-isomorphism.*

Proof. Uses Hodge decomposition of a form to prove induced maps on cohomology are injective. Then prove surjectivity. □

Lemma 18.3. *A compact Kähler manifold is formal (i.e. the usual cohomology algebra is formal).*

Theorem 18.4. *M Kähler then* $(\Lambda^\bullet(M), \overline{\partial})$ *is formal.*

A manifold M is *nilmanifold* if it has an action of aa nilpotent Lie group. All nilmanifolds are obtained as qutient spaces $M = G/\Gamma$ for a cocompact lattice in a nilpotent Lie group.

I gather there are two things: one is an *integrable complex structure on a Lie algebra*, which is a subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathfrak{g}^{1,0} \oplus \overline{g^{1,0}} = \mathfrak{g} \otimes \mathbb{C}$. Integrability is $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$.

And then there's a *complex nilmanifold*, which is $M = G/\Gamma$ with a complex structure I such that G has a right-invariant complex structure and $G \to M$ is holomorphic. In this case we say I is *invariant*. Not all complex structures on a nilmanifold are invariant: a compact torus admits a non-Kähler complex structure that is not invariant. A Kodaira surface, a locally trivial holomorphic fibration over T with fiber T' with non-trivial Chern class, for T and T' elliptic curves, is a nilmanifold.

Recall the Chevalley-Eilenberg cohomology on $\Lambda^\bullet \mathfrak{g}^*$ where the Lie bracket becomes a differential by Cartan formula. More generally,

Proposition 18.5. *Let $w \in \mathrm{Hom}(\Lambda^2 V, V)$. Consider the dual map $d_w : V^* \to \Lambda^2 V^*$. Extend this map to $d_w : \Lambda^k V^* \to \Lambda^{k+1} V^*$ using Leinbiz rule. Then $d_w^2 = 0$ if and only if w defines the Lie algebra structure on V.*

We may identify the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$ with the complex of left-invariant differential forms in its Lie group G. This defines a natural homomorphism of dgas $(\Lambda^\bullet \mathfrak{g}^*, d) \to (\Lambda^\bullet(G/\Gamma), d)$ for any discrete group $\Gamma \subset G$.

Theorem 18.6 (Nomizu)**.** *Let $M = G/\Gamma$ be a nilmanifold. Then the map above is a quasi-isomorphism.*

As a corollary, we see that nilmanifolds, with exception of tori, are never formal.

Proposition 18.7. *Let $(\mathfrak{g}, I)$ be a complex structure on a Lie algebra. Then we may consider the Dolbeault cohomology of $\partial_w : \Lambda^\bullet(\mathfrak{g}^*) \to \Lambda^2(\mathfrak{g}^*)$. Then $[\cdot,\cdot]_{\partial_w}$ is defined as follows: let $x, y \in \mathfrak{g}^{1,0}$, $x', y' \in \mathfrak{g}^{0,1}$. Then*

$$\begin{aligned} [x,y]_{\partial_w} &= [x,y] \\ [x,x']_{\partial_w} &= [x,x']^{1,0} \\ [x',y']_{\partial_w} &= 0. \end{aligned}$$

This implies that $(CE, \overline{\partial})$ is formal if and only if $[\cdot,\cdot] = 0$.

We intend to apply the construction made before for the usual Lie algebra cohomology but for Dolbeault cohomology. Unfortunately there is no analogue of Nomizu's theorem Theorem 18.6.
[Missing slides]

We say a dga has *Poincaré duality* if the product map $H^i(A^*) \times H^{n-i}(A^*) \to H^n(A^*) = \mathbb{C}$ is a non-degenerate pairing in cohomology. A morphism of dgas $(B^*, d) \to (A^*, d)$, where (B^*, d) admits Poincaré duality, that induces injective maps on cohomology is called a *domination*.

Theorem 18.8 (Milivojevic,Stelzig,Zoller arXiv 2306.12364)**.** *Let $\psi : (B^*, d) \to (A^*, d)$ be a domination of dga. If (B^*, d) is not formal, then (A^*, d) is not formal.*

Theorem 18.9 (Barberis-Dotti-V., 2007)**.** *A complex nilmanifold has trivial canonical bundle, trivialized by an invariant holomorphic top form.*

As corollaries we get that the natural dga morphism $(\Lambda^\bullet \mathfrak{g}^*, \overline{\partial}_w) \to (\Lambda^\bullet G/\Gamma, \overline{\partial})$ is a domination, and that the Dolbeault dga of a nilmanifold M is never formal.

19. Hyperbolic surfaces and Mizakhani's curve counting

Viveka Erlandson, University of Bristol, UK. 8th Brazilian School of Dynamical Systems, IMPA. November 4, 2025.

Abstract. It's a classical problem to study the growth of the number of periodic orbits of bounded length L under a geodesic flow on a manifold. When it comes to (closed) hyperbolic surfaces Huber proved that the number is asymptotic to $e^L L$ and this results has since been generalized in many directions, maybe most famously by Margulis to negatively curved manifolds as well as more general flows. However, a fundamentally different result is due to Mirzakhani who counted the subset of those closed geodesics on hyperbolic surfaces which have no self-intersections (or more generally, those inside a fixed mapping class group orbit) and showed that in

this setting the growth is asymptotic to a polynomial in L of degree only depending on the topology of the surface. In this mini-course we will give the necessary background on hyperbolic surfaces to understand these results, give an idea of the proof of Mirzakhani's theorem and introduce some tools used such as measured laminations and train tracks. Time allowing we present some generalizations (for example, to non-simple closed geodesics or to various metrics) and applications to the distribution of closed geodesics.

Our goal is to count certain closed geodesics on hyperbolic surfaces. By "certain" we main, for example, simple or fixed topological type; we will define these properly. Also, we may treat more general types of curves than only hyperbolic, e.g. combinatorial, algebraic. As "closed geodesics", we mean free homotopy classes of curves.

As historical note, results on coarse polynomial growth for simple curves were are due to Birman-Serres, Rees in the 80's. In 2001, McShane-Runn studied asymptotic growth for simple curves on punctured torus. Asymptoric growth on any hyperbolic surface was proved by Mirzakhani for simple curves in 2008 and then in 2016 for any topological type of curve.

Here's an outline of this minicourse:

- Introduction, motivation.
- Measured laminations, train tracks, Thurston measure.
- (Geodesic) currents.
- Curve counting theorem.

Throughout S will be a topological (connected, orientable of finite type, i.e. finite genus). Let $g < \infty$ be the genus and $r < \infty$ the number of punctures. We assume that $\chi(S) = 2 - 2g - r < 0$, which is equivalent to saying that S admits hyperbolic metrics. We will not consider the pair of pants, since there's only three simple closed geodesics on it, making it uninteresting for our means.

Let X be a hyperbolic surface homeomorphic to S. Consider the universal cover $\tilde{X} = \mathbb{H}^2$ of X, and $\pi_1(X) = \Gamma \underbrace{<}_{\substack{\text{discrete}\\ \text{free}}} \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$. Recall that a $4g$-gon yields a genus g surface S_g by adequate identification of its sides.

For us, a *curve* γ is a free homotopy class of a closed immersion $\mathbb{S}^1 \hookrightarrow S$, which are closed X-geodesics. In turn, a *multi-curve* is a formal sum $\sum_{i=1}^n a_i\gamma_i$ for γ_i distinct curves and $a_i > 0$, $a_i \in \mathbb{Q}$. The *length* of a curve γ is $\ell_X(\gamma)$ is the length of X-geodesic representing γ, and the length of a multi-curve is the sum of the lengths of the curves in the formal sum.

We won't use this: we can think of X as an element of Teichmüller space $\mathcal{T}(S) \cong \mathbb{R}^{6g-6+2r}$ parametrizing complex structures on S.

Notice that given $X \in \mathcal{T}(S)$, the number of curves with length bounded by a given number L, $\#\{\gamma | \ell_X(\gamma) \leq L\}$, is finite. This can be checked going to the universal cover. However, as $L \to \infty$, the number also tends to infinity.

Huber, Hekhal (59' or later) showed that if X is hyperbolic on $S_{g,r}$, then $\#\{\gamma | \ell_X(\gamma) \leq L\} \sim \frac{e^L}{L}$ asymptotically, that is,

$$\lim_{L\to\infty} \frac{\#\{\gamma | \ell_X(\gamma) \leq L\}}{e^L/L} = 1.$$

What is we restrict to a fixed topological type?

Recall that the *mapping class group* of S is $\mathrm{Map}(S) := \mathrm{Homeo}^+(S)/\text{homotopy}$. Then γ and γ_0 are of the same (topological) type if $\gamma \in \mathrm{Map}(S)\gamma_0$.

We define the *(geometric) intersection* of α, β distinct curves by

$$i(\alpha,\beta) = \min\{\#(\alpha' \cap \beta'|_{\beta' \text{ homotopic to } \beta}^{\alpha' \text{ homotopic to } \alpha}\}.$$

(In fact, this minimum is always realized by geodesics, but it's more a topological definition.) The *self-intersection* of a curve is

$$i(\alpha,\alpha) = \frac{1}{2}\min\{\#(\alpha' \cap \alpha''|\alpha', \alpha'' \text{ homotopic to } \alpha\}.$$

We say γ is *simple* if $i(\gamma,\gamma) = 0$.

It is known that $i(\cdot,\cdot)$ is $\mathrm{Map}(S)$-invariant, and that $\{\gamma : i(\gamma,\gamma) = k\}$ is a finite union of topological types.

Theorem 19.1 (Mirzakhani, '08, '16)**.** *Let X be a hyperbolic surface, $X \cong S_{g,r}$ and γ_0 any (multi-)curve. Then*

$$\#\{\gamma \textit{ type } \gamma_0|\ell_X(\gamma) \leq L\} \sim CL^{6g-6+2r},$$

where $C > 0$, $C = \frac{c(\gamma_0)}{b_\gamma}B(X)$.

As a corollary, we obtain

$$\#\{\gamma \text{ simple}|\ell_X(\gamma) \leq L\} \sim \text{const.}L^{6g-6+2r}$$
$$\#\{\gamma|i(\gamma,\gamma) \leq k, \ell_X(\gamma) \leq L\} \sim (\text{const. depending on } k)L^{6g-6+2r}.$$

As an example, consider the torus with simple (multi-)curves. If we put a puncture it becomes a hyperbolic curve. Forgetting the puncture, we have an Euclidean flat torus, of which we know the geodesics: they are straight lines with rational slope in the plane (before quotienting). We see that the amount of geodesics of length bounded by L is $\frac{1}{2}\mathbb{Z}^2 \cap B((0,0),L)$, which is in fact $\sim \pi L^2$. A result by Mc-Shane-Rivn, 2001 shows that

$$\{(\text{primitive}) \text{ simple curves}\} \sim \frac{\pi}{6}L^2.$$

Where primitive (most likely) excludes counting a curve twice as a multicurve.

Let us introduce a measure on $\mathbb{R}^2$. Let $L > 0$. Then

$$m^L = \frac{1}{L^2}\sum_{p\in\mathbb{Z}^2}\delta_{\frac{1}{L}p} \xrightarrow{L\to\infty} \mathrm{Leb}_{\mathbb{R}^2}.$$

For $p = (a,b)$, $\frac{1}{L}p = (\frac{1}{L}a, \frac{1}{L}b)$.

$$\begin{aligned}
U_{\mathbb{E}} &= \{x \in \mathbb{R}^2 : |x| \leq 1\}\\
m^L(U_{\mathbb{E})} &= \frac{1}{L^2}\#\{p \in \mathbb{Z}^2 \left|\frac{1}{L}p\right| \leq 1\}\\
&= \frac{1}{L^2}\#\{p \in \mathbb{Z}^2 : |p| \leq L\}\\
&= \frac{1}{L^2}\#\mathbb{Z}^2 \cap B_{\mathbb{R}^2}((0,0),L)\\
&\to \mathrm{Leb}_{\mathbb{R}^2}(U_{\mathbb{E})} = \pi.
\end{aligned}$$

Let's go back to study the geodesics on a torus. First we recall a fact, which is that on any hyperbolic surface S there exists a compact $K \subset S$ such that all simple closed geodesics stay in K.

Now consider a geodesic given by a curve of irrational slope, which is dense in the torus. However, it is not dense on a hyperbolic torus $S_{1,1}$. We may approximate this geodesic by $q_n \in \mathbb{Q}$ with $q_n \to \alpha$. Each $q_n \rightsquigarrow \gamma_n$ is a closed geodesic on $S_{1,1}$. The lamination in this case is the Hausdorff limit of the γ_n.

Definition 19.2. A *lamination* L on a (hyperbolic) surface S is a compact subset of S consisting of simple, pairwise disjoint complete geodesics.

A *measured lamination* λ is $\lambda = (L, \lambda)$ is a lamination L together with a *transverse measure* λ: a measure is assigned to each arc I transverse to L.

We write $ML(S)$ for the space of all measured laminations on S.

While the definition of lamination depends on a choice of hyperbolic metric on the surface, the different $ML(S)$ that we obtain with different metrics are actually homeomorphic, and we thus do not distinguish between them.

As an example, every simple multi-curve is a measured lamination.

Morally we think of measured laminations as limits of simple multicurves: simple multi-curves are dense in $ML(S)$.

Now we will discuss train tracks, which will give us a combinatorial model for measured laminations.

Definition 19.3. A *train track* τ on S is an embedded (trivalent) graph such that at every vertex the three half edges meet with the same tangent line and such that the complementary region (cc. $S \setminus \tau$) are not disk (with a certain caveat: it can be a contractible region as long as the vertices cannot preserve it's smooth incidence structure along the homotopy), a cyllinder (ring) nor a punctured disk.

A *set of weights* on τ is an assignment of a non-negative real number w_e to each edge e. They satisfy the *switch equations* if at every vertex we have $w_a = w_b + w_c$ (where b and c "stem" from a).

We denote $w(\tau)$ the solutions to the switch equations and $w_{\mathbb{Z}}(\tau)$ the integral solutions.

We say a measured lamination $\lambda \in ML(S)$ is *carried* by τ ($\lambda < \tau$) if its support can be homotopied into τ. We denote

$$\begin{aligned} ML(\tau) &= \{\lambda \in ML(S) : \lambda < \tau\} \\ ML_{\mathbb{Z}}(\tau) &= \{\lambda \in ML_{\mathbb{Z}}(S) : \lambda < \tau\} \\ ML_{\mathbb{Z}}(S) &= \{\lambda \in ML(S) : {\scriptsize\begin{matrix}\text{simple multi-curve}\\ \text{with integral coefficients}\end{matrix}}\}. \end{aligned}$$

Now we will explain the fact that there are one-to-one correspondences

$$\begin{aligned} ML(\tau) &\longleftrightarrow w(\tau) \\ ML_{\mathbb{Z}}(\tau) &\longleftrightarrow w_{\mathbb{Z}}(\tau). \end{aligned}$$

In a simple case where we pick a multicurve $\lambda \in ML_{\mathbb{Z}}(\tau)$, the associated measure is just counting the intersection number of the curve with a perpendicular segment near the train track edge.

The following argument appears to be a converse construction. Given a solution to the switch equations, i.e. an element in $\omega(\tau)$, we can produce a multicurve $\lambda \in ML_{\mathbb{Z}}(\tau)$. (The method is essentially putting as many strands as the weights say near the vertex; this yields formal sums of curves.) This also works for formal sums with rational coefficients. And for non-negative real coefficients, we take a thickening of τ.

Theorem 19.4 (Thurston)**.** *The weights on τ uniquely determine a transverse measure on L.*

Facts:

(1) Every $\lambda \in MS(S)$ is carried by some train track τ and determines a unique point in $\omega(\tau)$.
(2) There are finitely many train tracks τ_i such that every $\lambda \in ML(S)$ is carried by at least τ_i. From this it follows that we can write $ML(S) = ML(\tau_1) \cup ML(\tau_2) \cup \ldots \cup ML(\tau_n)$ and $ML(\tau) \approx \omega(\tau) \approx \mathbb{R}^{E-V}$ (under very mild conditions on the train tracks).

    Here is an exercise: suppose S is closed. Show that $E - V \leq 6g - 6$, and $E - V = 6g - 6$ if τ is maximal (which means that all complementary regions are triangles).
(3) $ML(\tau_i) \cap ML(\tau_j)$ for $i \neq j$ has dimension $< 6g - 6$.

Given those facts,

$$\begin{aligned} ML(S) &= ML(\tau_1)\dot{\cup}\ldots\dot{\cup}ML(\tau_n) \\ &= \omega(\tau_1)\dot{\cup}\ldots\dot{\cup}\omega(\tau_n). \end{aligned}$$

In particular, $ML(S) \approx \mathcal{E}^{6g+6L+2r}$ $(\omega(\tau_1) \approx \mathbb{R}^{6g+6L-2r})$. We may now introduce a measure on the space of laminations:

Definition 19.5. The *Thurston measure* on $ML(S)$ is

$$\lim_{L\to\infty} \frac{1}{L^{6g-6+2r}} \sum_{\lambda \in ML_{\mathbb{Z}}(S)} \delta_{\frac{1}{L}\lambda}$$

One checks that this is a Lebesgue measure (in the limit) as follows:

$$\begin{aligned} ML_{\mathbb{Z}}(S) &= \bigcup_{i=1}^{n} ML_{\mathbb{Z}}(\tau_i) \\ &\frac{1}{L^{6g-6+2r}} \sum_{\lambda \in ML_{\mathbb{Z}}\tau_1} \delta_{\frac{1}{L}\lambda} \\ &\frac{1}{L^{6g-6+2r}} \sum_{(\omega) \in \omega_{\mathbb{Z}}(\tau_i)} \delta_{\frac{1}{L}\omega} \\ &\frac{1}{L^{6g-6+2r}} \sum_{p \in \mathbb{Z}^{6g-6+2r}} \delta_{\frac{1}{L}p} \end{aligned}$$

on $\mathbb{R}^{6g-6+2r}$.

Notice that there is an action of the mapping class group on the space of laminations: a multicurve is taken to another multicurve.

The following theorem was used for proving the speaker's results:

Theorem 19.6 (Masur '85)**.** *m_{Thu} is Map(S)-invariant and is ergodic with respect to Map(S).*

Now we discuss (geodesic) currents. Consider de space $\mathcal{G}(\tilde{S})$ of unoriented geodesics on $\tilde{S} = \mathbb{H}^2$ as $(S^1 \times S^1 \setminus \Delta)/(a,b) \sim (b,a)$. This is well defined (even topologically, i.e. independently of the choice of metric) since two geodesics on Poincaré plane are determined by their endpoints.

Definition 19.7. A *(geodesic) current* is a $\pi_1(S)$-invariant Radon measure on $\mathcal{G}(S)$.

Example 19.8. Any (multi-)curve is a current.

We shall see that not only curves, but also measured laminations and in fact the Teichmüller space are contained in currents:

$$\begin{aligned}\{\text{curves}\} &\subset C(S)\\ ML(S) &\subset C(S)\\ \mathcal{T}(S) &\subset C(S)\end{aligned}$$

20. Variations on a theorem of Baum and Bott

Stéphane Druel, Université Claude Bernard Lyon. Algebra Seminar, IMPA. November 5, 2025.

Abstract. In this talk, I will discuss an analogue of the Baum-Bott vansihing theorem for the tangent bundle of the space of leaves of regular foliations together with applications to transversely projectively flat foliations of dimension one. This is work in progress.

Baum-Bott 70'.

Let E be a vector bundle on X complex manifold is a sheaf of 1-jets of E:

$$J'_X(E) = E \oplus \Omega^1_X \otimes E$$

where equality is the category of sheavs of abelian groups. Where the $\mathcal{O}_X$ structure is given by

$$\underbrace{f}_{\substack{\text{local function}\\ \text{on } U}} \Big(\underbrace{e}_{\substack{\text{local section}\\ \text{of } E \text{ on } U}} , \underbrace{\alpha}_{\substack{\text{local section}\\ \text{of } \Omega^1_X \otimes E}} \Big) = (fe, f\alpha - df \otimes e)$$

The *Atiyah class* of E is the element

$$\mathrm{at}_X(E) \in H^1(X, \Omega^1_X \otimes \mathrm{End}(E))$$

corresponding to the exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \Omega^1_X \otimes E \longrightarrow J^1_X(E) \longrightarrow E \longrightarrow 0.$$

Remark 20.1. $\mathrm{at}(E) = 0$ if and only if there exists $\nabla : E \to \Omega^1_X \otimes E$ map of abelian sheaves such that $e \mapsto (e - \nabla e)$ is $\mathcal{O}_X$-linear

$$-\nabla fe = -f\nabla e - df \otimes e$$

if and only if E admits a holomorphic connextion.

Example 20.2. If E is a line bundle, at $(E) = (d\log f_{ij})_{ij} \in H^1(X, \Omega^1_X)$, $E = (f_{ij})$.

Theorem 20.3 (Atiyah, '57)**.** *Let $r = rankE$, k an integer, X be compact Kähler and Σ_k a polynomial of degree k on M_r such that $\Sigma_k(g)$ is the k-th elementary* *symmetric function on the characteristic zeros of* $g \in M_r$*.* S_k = *polynomial given by polarization.*

$$M_r^{\otimes k} \to \mathbb{C}$$

$$EndE^{\otimes k} \to \mathcal{O}_X$$

$$Ex.,\ k = 1 \qquad EndE \xrightarrow{Tr} \mathcal{O}_X.$$

$$\begin{tikzcd} H^1(X, \Omega_X^1 \otimes End(E))^{\otimes k} \arrow[r, "\cup"] & H^k((\Omega_X^1)^{\otimes k} \otimes EndE^{\otimes k} \arrow[r] & H^k(X, \Omega_X^k) \arrow[r] & H^{2k}(X, \mathbb{C}) \end{tikzcd}$$

$$at(E)^{\otimes k} \longrightarrow (2\pi i)^k c_{2k}(E).$$

(Double check the term $(\Omega_X^1)^{\otimes k}$*... I think there's a typo.)*

Partial connection. $\mathcal{F}$ *regular distribution.* $\mathcal{F} \subseteq T_X$ *subbundle. An* $\mathcal{F}$*-connection on* E *is* $\nabla : E \to \mathcal{F}^\vee \otimes E$ *such that*

$$\nabla(fe) = f\nabla e + f_{\mathcal{F}} f \otimes e$$

where $d_{\mathcal{F}} : \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \to \mathcal{F}^\vee$*.*

Proposition 20.4 (Bott-Baum, '70)**.** *If* E *admits an* $\mathcal{F}$*-connection then all characteristic classes of degree* $>$ *codim*$\mathcal{F}$ *vanish.*

Proof. There exists an $\mathcal{F}$-connection if and only if

$$\begin{tikzcd} 0 \arrow[r] & \Omega_X^1 \otimes X \arrow[r] \arrow[d] & J_X^1 \arrow[r] & E \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}^\vee \otimes E \arrow[r] & \mathcal{F}^\vee \otimes E \oplus E \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

if and only if $\mathrm{at}(E) \in H^1(X, \Omega_X^1 \otimes \mathrm{End}E)$ maps to 0 in $H^1(X, \mathcal{F}^\vee \otimes \mathrm{End}E)$ is and only if $\mathrm{at}(E)$ comes from

$$H^1(X, N^\vee \otimes \mathrm{End}E) \to H^1(X, \Omega_X^1 \otimes \mathrm{End}E)$$

where $N = T_{X/\mathcal{F}}$. □

Example 20.5 (Bott's partial connection)**.** Suppose $\mathcal{F}$ is a regular foliation and $N = T_{X/\mathcal{F}}$ has a $\mathcal{F}$-connection. Take n a local section of N, v a local section of $\mathcal{F}$ (over the same open) and suppose exists a section t of T_X that is a lift of t, i.e. $p(t) = n$ for $p : T_X \to N$. Then $\nabla_v^B n = p([v, t])$ is a section of N. This defines an $\mathcal{F}$-connection on N.

Lemma 20.6 (Beauville)**.** *Let* X *be a complex manifold and* E *is a direct submanifold of* T_X*. Then* $at(E) \in H^1(X, \Omega_X^1 \otimes EndE)$ *comes from* $H^1(X, E^\vee \otimes End(E))$*.*

Proof. Consider

$$T_X = E \oplus F \xrightarrow{p} E.$$

We claim that there exists an $\mathcal{F}$-connection on E. Let v be a local section of F, e a local section of E. Define $\nabla_v e = p([v, e])$. One checks that this satisfies the required properties. □

Notice that no involutivity is imposed on E - it's just a distribution.

Now we will discuss some variations of Baum-Bott's theorem **??**.

Let $\mathcal{F}$ be regular foliation, E a vector bundle on X equipped with a flat $\mathcal{F}$-connection. For example, ∇^B flat.

Recall that (E, ∇) can be thought of as $\mathrm{Mon}(\mathcal{F})$-equivariant vector bundles where $\mathrm{Mon}(\mathcal{F})$ is the monodromy groupoid.

Let $X/\mathcal{F}$ be the space of leaves, and $\mathcal{O}_{X/\mathcal{F}} = \mathcal{O}_X^{d_\mathcal{F}} = \{\text{functions } f \text{ such that } d_\mathcal{F} f = 0\}$.

It is a fact that E^∇ is a locally free $\mathcal{O}_{X/\mathcal{F}}$ sheaf of rank the rank of E and such that $E = E^\nabla \otimes_{F_{X/\mathcal{F}}} \mathcal{O}_X$.

Now one defines bundles of 1-forms and tangent bundle as follows. $\Omega^1_{X/\mathcal{F}} = (N^*_\mathcal{F})^{\nabla B}$ and $T_{X/\mathcal{F}} = N_\mathcal{F}^{\nabla B}$, which are locally free sheaves of $\mathcal{O}_{X/\mathcal{F}}$-modules of rank $q = \mathrm{codim}\mathcal{F}$.

The Atiyah class of (E, ∇) is

$$J^1_{X/\mathcal{F}}(E, \nabla) = E^\nabla \oplus \Omega^1_{X/\mathcal{F}} \otimes_{\mathcal{O}_{X/\mathcal{F}}} E^\nabla$$

with $\mathcal{O}_{X/\mathcal{F}}$-module structure given by

$$(e, \alpha) = (f\alpha, f\alpha - \underbrace{df}_{\in \Omega^1_{X/\mathcal{F}}} \otimes e.$$

Then

$$\mathrm{at}_{X/\mathcal{F}}(E, \nabla) \in H^1(X, \Omega^1_{X/\mathcal{F}} \otimes_{\mathcal{O}_{X/\mathcal{F}}} \mathrm{End}_{\mathcal{O}_{X/\mathcal{F}}} E^\nabla)$$

corresponding to

$$0 \longrightarrow \Omega^1_{X/\mathcal{F}} \otimes_{\mathcal{O}_{X/\mathcal{F}}} \longrightarrow J^1_{X/\mathcal{F}}(E) \longrightarrow E^\nabla \longrightarrow 0.$$

Lemma 20.7. *$at_{X/\mathcal{F}}(E, \nabla)$ maps to $at_X(E)$ under*

$$H^1(X, \Omega^1_{X/\mathcal{F}} \otimes_{\mathcal{O}_{X/\mathcal{F}}} End_{\mathcal{O}_{X/\mathcal{F}}} E^\nabla \to H^1(X, \Omega_X \otimes End E).$$

Partial connections. Let $G \subseteq T_{X/\mathcal{F}}$ be a subbundle a *G-connection* (E, ∇) is

$$D : E^\nabla \longrightarrow G^\vee \otimes E^\nabla$$
$$\text{s.t.} D(fe) = fDe + d_G f \otimes e$$

where $d_G : \mathcal{O}_{X/\mathcal{F}} \to \Omega^1_{X/\mathcal{F}} \to G^\vee$.

It's a fact that a G-connection (E, ∇) exists if and only if $\mathrm{at}_{X/\mathcal{F}}(E, \nabla)$ maps to 0 in $H^1(X, G^\vee \otimes_{G_{X/\mathcal{F}}} \mathrm{End}_{\mathcal{O}_{X/\mathcal{F}}} E^\nabla$.

The analogue of Beauville's lemma 20.6 is

Lemma 20.8 (Beauville)**.** *Suppose $N_\mathcal{F} = \mathcal{A} \oplus B$ such that $\nabla^B \mathcal{A} \subseteq \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{A}$ and $\nabla^B \mathcal{B} \subseteq \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{B}$. Let $\nabla_\mathcal{A}$ and $\nabla_\mathcal{B}$ be a flat $\mathcal{F}$-connection on $\mathcal{A}$ and $\mathcal{B}$.*

Then $at_{X/\mathcal{F}}(\mathcal{A}, \nabla_\mathcal{A}) \in H^1(X, \Omega^1_{X/\mathcal{F}} \otimes_{\mathcal{O}_{X/\mathcal{F}}} End_{\mathcal{O}_{X/\mathcal{F}}} \mathcal{A}^{\nabla_\mathcal{A}}$ comes from $H^1(X, (\mathcal{A}^{\nabla_\mathcal{A}})^\vee \otimes_{X/\mathcal{F}} End_{\mathcal{O}_{X/\mathcal{F}}} \mathcal{A}^{\nabla_\mathcal{A}}$.

Proof. Same proof but we need a Lie bracket on $T_{X/\mathcal{F}} = N_\mathcal{F} \subseteq N_\mathcal{F} \stackrel{p}{\leftarrow} T_X$. Let v_1, v_2 be local sections on $T_{X/\mathcal{F}}$ with $v_1 = p(t_1)$ and $v_2 = p(t_2)$. Define

$$[v_1, v_2] = p([t_1, t_2]).$$

One can check that this a Lie bracket on $T_{X/\mathcal{F}}$. Suppose that $t_1 \in \mathcal{F}$. Then

$$p([t_1, t_2]) \stackrel{\text{def}}{=} \nabla^\mathcal{B}_{t_1} v_2 = 0.$$

For $v_3 \in \mathcal{F}$,

$$\nabla^{\mathcal{B}}_{v_3}[v_1, v_2] = p([v_3, [t_1, t_2]]) = \pm\nabla^{\mathcal{B}}_{v_3} v_1 \pm \nabla^{\mathcal{B}}_{v_3} v_2 = 0.$$

$\square$

Lemma 20.9. *Suppose that $\mathcal{F}$ is a regular foliation on X. Let $N_{\mathcal{F}} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_q$ be line bundles on X, $\nabla^{\mathcal{B}}\mathcal{L}_i \subseteq \mathcal{F}^\vee \otimes \mathcal{L}_i \rightsquigarrow \nabla_i$. Suppose $(\mathcal{L}_i, \nabla_i) \cong (\mathcal{L}_j, \nabla_j)$ for all i, j. Then $c_1(\mathcal{L}_i) = 0$.*

Proof. By the Comparison Lemma,

$$\begin{aligned} H^1(X, \Omega^1_{X/\mathcal{F}} &\longrightarrow H^1(X, \Omega^1_X) \\ \mathrm{at}_{X/\mathcal{F}}(\mathcal{L}, \nabla_i) &\longmapsto \mathrm{at}(\mathcal{L}_i), \end{aligned}$$

it suffices to prove that $\mathrm{at}_{X/\mathcal{F}}(\mathcal{L}_i, \nabla_i)$ vanishes.

$$\mathrm{at}_{X/\mathcal{F}}(\mathcal{L}_i, \nabla_i) \in H^1(X, \Omega^1_{X/\mathcal{F}}) = \bigoplus_{j=1}^{9} H^1(X, (\mathcal{L}_j^{\nabla_j})^\vee)$$

comes from $H^1(X, (\mathcal{L}_i^{\nabla_i})^\vee)$. $\square$

Example 20.10. $A = E^1 \times B^{n-1}$, an abelian variety. Let L be the pullback of a degree 2 divisor on E. Then there exists $T_X \twoheadrightarrow L^{\oplus n-1}$.

Proposition 20.11. *$\mathcal{F}$ regular foliation of dimension 1 on X complex projective of dimension ≥ 3. Suppose $\mathcal{F}$ is transversally projective flat and that X contains a rational curve. Then $\mathcal{F}$ is a $\mathbb{P}^1$-bundle and it is tangent to $\mathcal{F}$.*

Now we explain what it means to be projectively flat. It is when the projectivization of the normal bundle $\mathbb{P}(N)$ is induced by a representation of $\pi_1(X) \to \mathrm{PGL}_g$ it comes with an Ehresmann connection on

$$\begin{array}{c} \mathbb{P}(N) = \mathbb{P} \\ \big\downarrow f \\ X \end{array}$$

$T_{\mathbb{P}} = T_f \oplus g$.

Now, it is also called transversally projectively flat if ∇^B is a partial connection on N and there exists a foliation $\mathcal{H}$ on

$$\begin{array}{c} \mathbb{P}(N) \\ \big\downarrow \\ X \end{array}$$

such that $\mathcal{H}$ projects onto $\mathcal{F}$. Transversality means that $\mathcal{H} \subseteq \mathcal{G}$.

Finally, transversally projectively flat means that it is induced on a $\mathrm{Mon}(\mathcal{F})$-module by a representation of $\pi_1(\mathrm{Mon}(\mathcal{F}) \to \mathrm{PGL}_g$.

21. Dolbeault cohomology of nilmanifolds

Misha Verbitsky, IMPA. Geometric Structures Seminar, IMPA. November 6, 2025.

Abstract. A nilmanifold is a (left) quotient of a nilpotent Lie group by a cocompact lattice. If this Lie group is equipped with a left-invariant complex structure, the quotient is called a complex nilmanifold. Examples of complex nilmanifolds include Kodaira-Thurston surface and many other non-Kahler complex manifolds. The differential graded algebra of invariant differential forms on a nilmanifold has the same cohomology as a nilmanifold, as shown by Nomizu. I am going to explain a Dolbeault cohomology version of this theorem. A Lie algebra of a nilmanifold admits a natural rational structure, induced by the cocompact lattice. If the complex structure operator, acting on the Lie algebra, has rational coefficients, Console and Fino proved that the Dolbeault cohomology of a nilmanifold is equal to the Dolbeault cohomology of its invariant forms. I will describe an attempt to prove this result for a general nilpotent Lie algebra, based on number-theoretic arguments.

Let G be a nilpotent Lie group. [For me this would mean that the Lie algebra of G is nilpotent as in Lie Algebras Definition 2.1, but according to Misha this would be the case for G connected only; so there's must some other notion of nilpotent Lie groups.] A *nilmanifold* M is a manifold with a transitive action of a nilpotent Lie group. This means that $M = G/\Gamma$.

Theorem 21.1.

Let G be a real Lie group and $I \in \mathrm{End}(TG)$ with $I^2 = -\mathrm{Id}$. Suppose I is right-invariant. Then I is integrable iff $[T^{1,0}G, T^{1,0}G] \subset T^{1,0}G$ iff $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes \mathbb{C}$ where $\mathfrak{g}^{1,0}$ is the $\sqrt{-1}$-eigenspace of I.

A *complex nilmanifold* is $(G, I)/\Gamma$.

Example 21.2. An *Iwasawa manifold* is a quotient of the set of upper triangular matrices with 1 on the diagonal (this group is $U_3(\mathbb{C})$) by a lattice Γ. For example, we can take $\Gamma = U_3(\mathbb{Z}[\sqrt{-1}])$.

The complex structure I is bi-invariant iff it is paralelizable.

Let E be an elliptic curve with an ample line bundle L. We put an action on $\mathrm{Tot}L$ given by multiplication by $\lambda \in \mathbb{C}$ for $|\lambda| > 0$.

Theorem 21.3. *De Rham cohomology of a nilmanifold coincides with the CE cohomology of* $\mathfrak{g}$*.*

Theorem 21.4 (Console-Fino)**.** *For complex nilmanifolds,* $H^{\bullet,\bullet}(\mathfrak{g}^\bullet, \overline{\partial}$ *and* $H^{\bullet,\bullet}_{\overline{\partial}}(M)$ *are isomorphic.*

There's several results addressing a converse situation (when the algebras are isomorphic) and other situations involving fibrations.

Now we make a sudden change to deformation theory.

Lemma 21.5. *Complex spaces on* $V = \mathbb{R}^{2n}$ *are uniquely determined by* $W \in V \otimes_{\mathbb{R}} \mathbb{C}$ *such that* $W \oplus \overline{W} = V \otimes \mathbb{C}$ *which is an element in* $Gr(n, V \otimes \mathbb{C})$*.*

Definition 21.6. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. The set of complex structures on $\mathfrak{g}$ is $\mathrm{Comp}(\mathfrak{g})$ which is $\{W \in \mathrm{Gr}_0(\mathfrak{g}_{\mathbb{C}} : W \subset W \subset \mathfrak{g}_{\mathbb{C}}$ is a subalgebra$\}$ since being a subalgebra means $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$.

We can define a variety out of this:

$$\overline{\mathrm{Comp}(\mathfrak{g}_{\mathbb{C}}} = \{W \in \mathrm{Gr}(n, \mathfrak{g}_{\mathbb{C}} : W \text{ subalgebra}\}$$

is a complex subvareity of $\mathrm{Gr}(n, \mathfrak{g}_{\mathbb{C}})$.

Definition 21.7. Let $X \subset \mathbb{C}P^n$ be a projective subvariety. Consider the smallest field $k \supset \mathbb{Q}$ such that any ideal I of X [ideal in the coordinate algebra of X?] can be generated by polynomials with coefficients in $k[k^n]$ is called a *field of definition.*

Apparently there's some discussion about this notion not being easily defined.

Lemma 21.8. *$Gal_{\mathbb{Q}}(k)$ acts on $\mathbb{P}^n_k$. Let $X \subset \mathbb{P}^n$ be a projective subvariety and $\Gamma \subset Gal_{\mathbb{Q}}(k)$ the stabilizer of X. Then the field of definition of $X = \{\lambda \in k : \Gamma(\lambda) = \lambda\}$.*

We have a simpler definition of field of definition for a point (which is apparently the former definition when the variety is a point)

Definition 21.9. For a point $(z_1, \ldots, z_n) \in \mathbb{C}^n$, its *field of definition* is the field generated by $z_1, \ldots, z_n$.

So, for instance, a rational point has transcendence degree 0.

Remark 21.10. Let X be an n-dimensional algebraic variety defined over $\mathbb{Q}$ or any finite extension of $\mathbb{Q}$, and let $\pi : X \to \mathbb{C}^n$ be finite dominant over $\mathbb{Q}$.

We have a fun theorem:

Theorem 21.11. *Let X be an irreducible complex algebraic variety defined over $\mathbb{Q}$. Then the group $Aut_{\mathbb{Q}}(\mathbb{C})$ acts transitively on points of maximal transcendence degree in X.*

Definition 21.12. We say that a point $x \in X$ is *algebraic* if its transcendence degree over k_X is zero.

It's not very clear whether the variety X is projective or affine.

Lemma 21.13. *The set of algebraic points in $X \subset \mathbb{C}P^n$ is dense in the analytic topology.*

Theorem 21.14. *Let X be a complex (affine) algebraic variety defined over $\mathbb{Q}$. Consider an $Aut_{\mathbb{Q}}\mathbb{C}$-orbit of a point $p \in X$. Then the closure of its orbit is a minimal subvariety $Z \subset X$ defined over $\mathbb{Q}$ and containing p.*

Now we introduce a Galois action on Dolbeault cohomology.

Theorem 21.15. *$\mathfrak{g}$ nilpotent Lie algebra, M corresponding nilmanifold, $\sigma \in Aut_{\mathbb{Q}}(\mathbb{C})$, and let $J = \sigma(I) \in Comp(G)$. Then $\dim H^{p,q}_{\bar\partial}(M, I) = \dim H^{p,q}_{\bar\partial}(M, J)$ for all p, q.*

From this it follows that

Lemma 21.16. *For a nilmanifold with complex structure I, if the field of definition of I is $\mathbb{Q}$, then the Console-Fino map $H^{p,q}_{\bar\partial}(\mathfrak{g}) \to H^{p,q}_{\bar\partial}(M)$ is an isomorphism.*

The idea is that if we know the result for algebraic points then we'll have it for transcendental points as well. Namely,

Lemma 21.17. *If Console-Fino is an isomorphism for all algebraic complex structures, then it holds for all transcendental complex structures too.*

Proof. Let $I \in \mathrm{Comp}(\mathfrak{g})$ be trascendental. Consider the variety $\overline{\mathrm{Aut}_{\mathbb{Q}}\mathbb{C}}$, which is algebraic and thus contains algebraic points. Then Console-Fino is an isomorphism on such points. Then it is so in a neighbourhood of this point. Then it is so in the orbit of this neighbourhood. □

Consider $(\mathfrak{g}, I)$ with I with number field k. Consider $\tilde{\mathfrak{g}} = \bigoplus_{\sigma \in \mathrm{Gal}(k:\mathbb{Q})}(\mathfrak{g}, \sigma(I))\dots$ this constructions leads to a Lie algebra $(\tilde{\mathfrak{g}}, \tilde{I})$ equipped with a complex structure defined over $\mathbb{Q}$. This satisfies Console-Fino. We can lift $(\tilde{M}, \tilde{I})$ since it remains Dolbeault harmonic — which is the main tool used by Colsole-Fino in their theorem. This is how to proof Console-Fino for complex structures over a closed field.

References