COMPLEX GEOMETRY

github.com/danimalabares/stack

Contents

1. Complex analysis in several variables

Lemma 1.1. [Lee24], *Theorem 1.21. Let $U \subseteq \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$. The following are equivalent:*

(1) *f is holomorphic (i.e. it is continuous and has a complex partial derivative with respect to each variable at each point of U)*
(2) *f is smooth and satisfies the following Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x^j} = \frac{\partial v}{\partial y^j}, \qquad \frac{\partial u}{\partial y^j} = -\frac{\partial v}{\partial x^j} \tag{1.1.1}$$

where $z^j = x^j + \sqrt{-1}y^j$ and $f(s) = u(z) + \sqrt{-1}v(x)$.
(3) *For each $p = (p^1, \dots, p^n) \in U$ there exists a neighbourhood of p in U on which f is equal to the sum of an absolutely convergent power series of the form*

$$f(z) = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n}(z^1 - p^1)\dots(z^n - p^n) \tag{1.1.2}$$

Proof. I will prove that if f is holomorphic then it has a Taylor series for $n = 2$. First apply Cauchy integral formula on each variable to obtain

$$f(z^1, z^2) = \frac{1}{(2\pi\sqrt{-1})^2} \int_{\substack{|z^1-w^1|=r \\ |z^2-w^2|=r}} \frac{f(w^1, w^2)}{(w^1 - z^1)(w^2 - z^2)} dw^1 dw^2$$

Now observe:

$$\frac{1}{w^1 - z^1} = \frac{1}{w^1 - p^1 + p^1 - z^1} = \frac{1}{w^1 - p^1} \frac{1}{1 - \frac{p^1 - z^1}{w^1 - p^1}} \tag{1.1.3}$$

And on the right-hand-side we have a geometric series so that we may write

$$\frac{1}{w^1 - z^1} = \frac{1}{w^1 - p^1} \sum_{k=0}^{\infty} \left(\frac{p^1 - z^1}{w^1 - p^1} \right)^k$$

finally substituting this into (1.1.3) we may take the products $(p^1 - z^1)^{k_1}(p^2 - z^2)^{k_2}$ out of the integral and define the remaining term as $a_{k_1 k_2}$. □

2. Weierstrass preparation theorem

Definition 2.1. A *germ* of a function near a point is a function defined on some open neighbourhood of the point.

Definition 2.2. A *Weierstrass polynomial* is a polynomial whose coefficients are holomorphic functions.

Theorem 2.3 (Weierstrass preparation theorem)**.** *If $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic and f is not identically zero in the coordinate axis $z_n := w$, there is a unique germ of a monic Weierstrass polynomial g whose coefficients are holomorphic functions on the first $n-1$ variables and a germ of a holomorphic function h with $h(0) \neq 0$ such that $f = gh$.*

Proof. Since f is not identically zero near 0, then there is a point that we may suppose is in the z_n-axis where f is not zero.

Consider the function of one complex variable $f(0, \ldots, 0, w)$. Since it has a zero at 0, is holomorphic, and is not identically zero, the zero 0 must be of finite order m (cf. Lemma **??**). Recall that m is the smallest integer m such that $f^{(m)}(0)$.

We want to apply the Argument Principle **??**.

We want to count the number of zeros that $f(0, \ldots, 0, w)$.

□

Lemma 2.4. *If R is a UDF, then $R[x]$ is a UFD.*

Lemma 2.5. *The stalk $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.*

Proof. By induction on n. For $n = 0$ it is trivial. Suppose $\mathcal{O}_{n-1}$ is a UFD. Then by Gauss' Lemma **??**, $\mathcal{O}_{n-1}[w]$ is a UFD too. Thus we may express any Weierstrass polynomial g as a product of irreducible elements (uniquely up to multiplication by units).

Let $f \in \mathcal{O}_n$. We want to express f as a product of (unique up to multiplication by units) of irreducible elements. By Weierstrass Preparation Theorem 2.3 there is a Weierstrass polynomial $g \in \mathcal{O}_n[w]$ and a holomorphic function not vanishing on 0 (i.e. a unit of $\mathcal{O}_n$) such that $f = gh$. By the previous remark g is factored uniquely up to multiplication by units as $g = g_1 \dots g_m$. This shows existence of the factorization.

To prove uniqueness suppose that $f = f_1 \dots f_k$ for some irreducible $f_1, \dots, f_k \in \mathcal{O}_n$. Since f does not vanish in the w axis, neither can each f_i, so that we may decompose each of them as $f_i = g_i' h_i$ by Weierstrass Preparation Theorem. Since f_i is irreducible, it follows that g_i' is irreducible. Then we have that

$$f = gh = \prod g_i' \prod h_i$$

so by uniqueness in Weierstrass Preparation Theorem we conclude that $g = \prod g_i'$, and by uniqueness from the fact that $\mathcal{O}_n[w]$ is a UFD we conclude that g coincides with $\prod g_i'$ up to multiplication by units. □

3. Complex manifolds

Definition 3.1. A *complex manifold* M is a smooth manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all α, β.

Definition 3.2. A function on an open set $U \subset M$ is *holomorphic* if for all α, $f\varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n$.

Lemma 3.3. *The sheaf $\mathcal{O}_M$ of holomorphic functions is a sheaf.*

4. Riemann surfaces

Definition 4.1. A *meromorphic function on a Riemann surface* is a holomorphic map to the Riemann sphere which is not identically equal to ∞.

Definition 4.2. A *pole* of a meromorphic function on a Riemann surface is any point x such that $F(x) = \infty$.

Definition 4.3. The *order* of a pole of a meromorphic function on a Riemann surface is the integer k_x such that the coordinate representation of f is $f(x) = z^{k_x}$ near x.

Proposition 4.4. *Let $F : X \to Y$ be a proper, non-constant holomorphic map between connected Riemann surfaces. Then the integer $d(y) = \# f^{-1}(y)$ does not depend on $y \in Y$.*

Proof. Using local representation of f as z^k. But see also **??**: this number is locally constant because we can find a neighbourhood V of any point y such that any other y' in this neighbourhood has the same number of preimages. Using compactness fix neighbourhoods U_i of all the preimages of y, using that y is a regular value we may suppose the f is a diffeomorphism at the U_i, and then define $V := \cap V_i \setminus f(M \setminus \cup U_i)$. □

Lemma 4.5. *Let X be a compact Riemann surface. If there is a meromorphic function on X having exactly one pole, and that pole has order 1, then X is equivalent to the Riemann sphere.*

Proof. A meromorphic function has no critical points! That is, ∞ is a regular value, having one preimage by hypothesis. Then the degree of f is one, so that it must be a bijection (why?). And it has no crticial points… (why?) So by Lemma something, its inverse must be holomorphic. □

Exercise 4.6. Show that the canonical bundle K of a Riemann surface S has no base points.

Proof. Suppose p is a base point of K. We want to construct a meromorphic function with exactly one pole and use Lemma 4.5 to arrive at a contradiction. Any such function is an element of $H^0(\mathcal{O}(p))$. Note that by Serre duality we have:

$$H^0(\mathcal{O}(p)) = H^0(\mathcal{O}(p) \otimes K^* \otimes K) = H^1(K(-p))^*$$

So it's enough to show that $H^1(K(-p))$ is not zero. So consider the sheaf exact sequence twisted by the ideal sheaf $\mathcal{I}_p = \mathcal{O}_S(-p)$ of p:

$$0 \longrightarrow K(-p) \longrightarrow K \longrightarrow K(p) \longrightarrow 0$$

Then we use the exact sequence in cohomology to prove that $H^1(K(-p))$ is not zero. First, $H^1(K(p)) = 0$ because by Serre duality it has the same dimension as the space of holomorphic functions vanishing at p, which is only the zero function since M is compact. Next, $H^1(K)$ is not zero because by Dolbeault theorem it is $H^{1,1}(M, \mathbb{R})$ which contains real 2-forms. Then $H^1(K(-p))$ cannot be zero because it surjects onto a nontrivial space. □

5. Belyi functions

To understand Belyi functions we start by considering meromorphic functions $f : X \to \mathbb{C}P^1$ as ramified coverings:

Proposition 5.1. *A nonconstant meromorphic function $f : X \to \mathbb{C}P^1$, considered as a mapping of the underlying topological space, is a ramified covering of the sphere $S^2 \cong \mathbb{C}P^1$.*

Be careful: isomorphic complex ramified coverings produce isomorphic Riemann surfaces (by definition), but the converse is certainly false. The same Riemann surface may be obtained by many different and pairwise non-isomorphic coverings. Just consider different meromorphic functions on the same surface.

The following paragraph is an approximate quote:

Proposition 5.1, and the fact that every Riemann surface admits a meromorphic function (which can be seen by Riemann-Roch, cf. [**?**, Fact 1.8.6]) show that every Riemann surface may be represented by a ramified covering of $\mathbb{C}P^1$. The following theorem, which affirms the converse statement, is one of the most fundamantal:

Theorem 5.2 (Riemann's existence theorem)**.** *Suppose a base star is fixed in $\mathbb{C}P^1$, and the squence of its terminal vertices is $R = [y_1, \dots, y_k]$. Then for any constellation $[g_1, \dots, g_k]$, $g_i \in S_n$, there exists a compact Riemann surface X and a meromorphic function $f : X \to \mathbb{C}P^1$ such that $y_1, \dots, y_k$ are the critical values of f (i.e. f' vanishes at these points) and $g_1, \dots, g_k$ are the corresponding monodromy permutations. The ramified covering $f : X \to \mathbb{C}P^1$ is independent of the choice of the base star in a given homotopy type and is unique up to isomorphism.*

Proof. The heart of this theorem is the correspondence between constellations, which are abstract sets of permutations whose product is the identity and act transitively on the set of n elements (and are also maps on surfaces), and the monodromy group of a covering. Indeed: for a regular point y_0 of a degree-n convering we get an action of the symmetric group of n elements on the fiber

$E := \pi^{-1}(y_0)$; taking each generator of the fundamental group to be any of these permutations we obtain a constallation (cf [**?**, Construction 1.2.13].

The other way around, [**?**, Proposition 1.2.15], we can define a group homomorphism $\pi(S^2 \setminus \{p_i\}, y_0) \to G$ where G is the group of the constellation; the fact that the it is indeed a group homomorphism is due to the fact that both sets of generators satisfy the property that their product is identity. Then for any point x in the set of n elements we can consider its stabilizer. This corresponds to a subgroup M_x of $\pi(S^2 \setminus \{p_i\}, y_0)$. Such a subgroup determines a finite-sheeted covering of $S^2 \setminus \{p_i\}$. The covering is connected since G acts transitively on E. *Habemus superficie.* $\square$

The surprising result by Belyi is that for the case $k = 3$ it will happen that the corresponding Riemann surfaces will be defined over $\overline{\mathbb{Q}}$, the field of algebraic numbers. Therefore, the absolut Galois group $\mathrm{Aut}(\overline{\mathbb{Q}}|\mathbb{Q})$ (that is, the automorphism group of the field $\overline{\mathbb{Q}}$) acts on them, and thus on 3-constellations as well. The mysterious nature of the group and the simplicity of the objects on which it acts, gave rise to the following term which may look a bit strange: *theory of dessins d'enfants*.

Definition 5.3. If it is possible to realize a Riemann surface X by a system of equations with coefficients in a subfield $K \subseteq \mathbb{C}$, then we say that X is *defined over* K.

Theorem 5.4 (Belyi)**.** *A Riemann surface X admits a model over the field $\overline{\mathbb{Q}}$ of algebraic numbers if and only if there exists a covering $X \to \overline{\mathbb{C}}$ unramified outside $\{0, 1, \infty\}$. In such a case, the meromorphic function f can also be chosen in such a way that it will be defined over $\overline{\mathbb{Q}}$.*

6. Analytic varieties

Definition 6.1. An *analytic variety* is a subset V of an open set $U \subset \mathbb{C}^n$ such that for any $p \in V$ there is a neighbourhood $U' \ni p$ such that $V \cap U'$ is given as the zero locus of a finite set of holomorphic functions $f_1, \dots, f_k$ defined on U'.

Definition 6.2. An analytic variety is a *hypersurface* if it is given as the vanishing locus of a single holomorphic function.

Definition 6.3. An analytic variety $V \subset U \subset \mathbb{C}^n$ is *irreducible* if V cannot be written as the union of two distinct analytic varieties $V_1, V_2 \subset U$, both distinct to V.

Lemma 6.4. *If V is an irreducible analytic hypersurface given locally as $V = \{f = 0\}$, then f is irreducible in $\mathcal{O}_p$.*

Proof. If $f = gh$ and neither of g and h are units and they are distinct, we could express V as the union of two distinct varieties: $V(g)$ and $V(h)$. (If one of them, was a unit then the vanishing set would be all of U.) $\square$

Definition 6.5. The *germ* of a set in the origin $0 \in \mathbb{C}^n$ is given by a subset $X \subset \mathbb{C}^n$. To subsets X, Y define the same germ if there exists an open neighbourhood $0 \in U \subset \mathbb{C}^n$ with $U \cap X = U \cap Y$. A germ is called *analytic* if there are functions $f_1, \dots, f_k \in \mathcal{O}_n$ such that X and $Z(f_1, \dots, f_k)$ define the same germ.

Lemma 6.6. [Huy05, Lemma 1.1.28] *An analytic germ X is irreducible if and only iff $I(X)$ is a prime ideal.*

Proof. If X is irreducible let $fg \in I(X)$. Then $V(I(X)) \subset V(f) \cup V(g)$. But since X is irreducible we cannot express $X \cap (V(f) \cup V(g))$ unless either of $V(f)$ or $V(g)$ are trivial or equal to $X \cap V(f)$ or $X \cap V(g)$.

The converse I won't need right now. □

7. Sheaf of holomorphic functions

For a definition of sheaf see Algebraic Geometry Definition **??**.

Lemma 7.1. *Let X be a complex manifold of complex dimension n.*

The functor

$$\begin{aligned} \mathcal{O}_X : Open_X^{op} &\longrightarrow Set \\ U &\longmapsto \mathcal{O}_X(U) := \{f : U \to \mathbb{C}^n : f \text{ is holomorphic}\} \\ i : V \hookrightarrow V &\longmapsto \mathcal{O}_X(i) : \mathcal{O}_X(U) \to \mathcal{O}_X(V) \end{aligned}$$

where $\mathcal{O}_X(i)$ is given by restriction, is a sheaf.

Proof. $\mathcal{O}_X$ is a presheaf, i.e. a functor, by definition: restriction maps the identity to the identity and preserves compositions.

To check the condition of being a sheaf let U_i be an open cover of some open set U of X and $f_i : U_i \to X$ sections of $\mathcal{O}_X(U_i)$, that is, holomorphic functions, which coincide pairwise in intersections. Then it is obvious that the function $f : U \to \mathbb{C}$, $x \mapsto f_i(x)$ for any $U_i \ni x$ is well defined. It is also holomorphic since holomorphicity is a condition defined on open sets. □

8. Analytic subvarieties

Definition 8.1. An *analytic subvariety* V of a complex manifold M is a subset given locally as the zeros of a finite collection of holomorphic functions.

Lemma 8.2. *Let $V \subset X$ be an analytic subvariety of a complex manifold X. Then*

$$\begin{aligned} \mathcal{I}_V : Open_X^{op} &\longrightarrow Set \\ U &\longmapsto \mathcal{I}_V(U) := \{f|_{U \cap V} : f \in \mathcal{O}_X(U)\} \\ i : U' \hookrightarrow U &\longmapsto \mathcal{I}_V(i) : \mathcal{I}_V(U) \to \mathcal{I}_V(U') \end{aligned}$$

where $\mathcal{I}_V(i)$ is given by restriction of functions, is a sheaf.

Proof. The proof the $\mathcal{I}_V$ is a presheaf (i.e. a functor) is immediate by definition, see Lemma **??**.

To check it is also a sheaf take a cover U_i of some open set U of X and functions $f_i \in \mathcal{I}_V(U_i)$. By definition, for every such f_i we have a function $\tilde{f}_i \in \mathcal{O}_X(U_i)$ such that $f_i = \tilde{f}_i$. We then glue the $\tilde{f}_i$ to obtain a function $\tilde{f} \in \mathcal{O}_X(U)$ and restrict it to V. □

Definition 8.3. Let $V \subset X$ be an analytic subvariety of a complex manifold X. The *ideal sheaf* is $\mathcal{I}_V$ as above.

For a smooth submanifold M of a complex manifold N we have the *normal short exact sequence*

$$0 \longrightarrow TM \longrightarrow TN \longrightarrow T^{\perp}M \longrightarrow 0$$

If M is a singular subscheme we have the *ideal short exact sequence*

$$0 \longrightarrow \mathcal{I}_M \longrightarrow \mathcal{O}_N \longrightarrow \mathcal{O}_M \longrightarrow 0$$

Here the ideal sheaf is defined as the kernel of the induced map $i_* : \mathcal{O}_{\widetilde{M}} \to i_*\mathcal{O}_M$. That is, it is the sheaf that to every open set assigns the ring of functions vanishing along M.

If M is a singular analytic subvariety we also have an ideal short exact sequence, this time using the holomorphic function sheaves.

Notice that for any open set $U \subset X$, $\mathcal{I}_V(U)$ is an ideal of $\mathcal{O}_X(U)$. (Indeed, if a function vanishes at V then its product with any other function will also vanish.)

In the case that V is an irreducible subvariety, this ideal is prime: if $fg \in \mathcal{I}_V(U)$, then $V(f) \cup V(g) \supset V \cap U$ so that $(V(f) \cup V(g)) \cap V = V \cap U$, and then it must be true that $V(f) \cap V \subseteq V(g) \cap V$ or that $V(f) \cap V \supseteq V(g) \cap V \cap U$ since V is irreducible, and thus, say, $V(f) \supseteq V \cap U$, that is, $I(V(f)) \subseteq V \cap U$ so that $f \in I(V \cap U)$.

If we further ask that V is a hypersurface, then $\mathcal{I}_V$ is locally principal. Indeed, if $f \in \mathcal{I}_V(U)$ is the defining function of some hypersurface V of a variety X on an open set $U \subset X$, we shall have that $\mathcal{I}_V(U) = (f)$. That is, if g also vanishes at $V \cap U$, we want to see that $g = fh$ for some $h \in \mathcal{O}_X(U)$.

Lemma 8.4. *The ideal sheaf of an irreducible codimension-1 closed subscheme of a smooth scheme is a line bundle.*

Proof. Since M is codimension-1 irreducible, the ideal sheaf at every affine chart is a codimension-1 prime ideal. This means that there aren't any nontrivial prime ideals of $\mathcal{I}_X(\mathrm{Spec}A) := I$. Let $f \in I$. Since X is smooth, $O_X\mathrm{Spec}A = A$ is a UFD (why?). Then there exists an irreducible element $g \in I$ such that $gh = f$. The ideal generated by g is prime (again because A is a UFD) and contained in I, so that I is principal. This shows that $\mathcal{I}_M$ is locally principal, i.e. it is a line bundle. □

In the case of complex manifolds and analytic subvarieties, we can imitate this proof as long as we prove that $\mathcal{O}_n$ is a UFD and the follwing lemma:

Lemma 8.5. *The ideal of an irreducible analytic hypersurface is a height-1 prime ideal.*

Proof. The fact that it is prime comes from the fact that X is irreducible as explained above.

Now suppose that $0 \subset \mathfrak{p} \subseteq I$. Then $V(\mathfrak{p})$ is an analytic variety that contains X. At regular points of both $V(\mathfrak{p})$ and X, both are smooth manifolds, but since X is of codimension 1 and $V(\mathfrak{p})$ does not equal all of M, we conclude that they coincide. Thus, in a neighbourhood of a regular point we have $\mathfrak{p} = I(V(\mathfrak{p})) = I$ by the Nullstellensatz **??**. □

Now we shall use the Gauss Lemma 6.3 to prove:

Lemma 8.6. *The stalk $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.*

Proof. By induction on n. For $n = 0$ it is trivial. Suppose $\mathcal{O}_{n-1}$ is a UFD. Then by the Gauss Lemma 6.3, $\mathcal{O}_{n-1}[w]$ is a UFD too. Thus we may express any Weierstrass polynomial g as a product of irreducible elements (uniquely up to multiplication by units).

Let $f \in \mathcal{O}_n$. We want to express f as a product of (unique up to multiplication by units) of irreducible elements. By Weierstrass Preparation Theorem 2.3 there is a Weierstrass polynomial $g \in \mathcal{O}_n[w]$ and a holomorphic function not vanishing on 0 (i.e. a unit of $\mathcal{O}_n$) such that $f = gh$. By the previous remark g is factored uniquely up to multiplication by units as $g = g_1 \dots g_m$. This shows existence of the factorization.

To prove uniqueness suppose that $f = f_1 \dots f_k$ for some irreducible $f_1, \dots, f_k \in \mathcal{O}_n$. Since f does not vanish in the w axis, neither can each f_i, so that we may decompose each of them as $f_i = g_i' h_i$ by Weierstrass Preparation Theorem. Since f_i is irreducible, it follows that g_i' is irreducible. Then we have that

$$f = gh = \prod g_i' \prod h_i$$

so by uniqueness in Weierstrass Preparation Theorem we conclude that $g = \prod g_i'$, and by uniqueness from the fact that $\mathcal{O}_n[w]$ is a UFD we conclude that g coincides with $\prod g_i'$ up to multiplication by units. □

Lemma 8.7. *The ideal sheaf of an irreducible codimension-1 analytic subvariety of a smooth complex manifold is a line bundle.*

Proof. By definition, our subvariety M is locally defined by the zeroes of a single holomorphic function defined on the ambient manifold X. That is, at every $p \in M$ there is an open set $U \subset X$ such that $U \cap M = f^{-1}(0)$ for some $f \in \mathcal{O}_X(U)$. Since U is an open set of a complex manifold, the ring $\mathcal{O}_X(U)$ is isomorphic to the ring $\mathcal{O}_n$ of holomorphic functions on $\mathbb{C}^n$ in a neighbourhood of the origin; that's by Definition 1.1 since a holomorphic function on a manifold is defined as such if its composition with the coordinate chart is a holomorphic function on $\mathbb{C}^n$.

By Lemma 6.4, since M is irreducible it follows that f is irreducible and so is the ideal of functions vanishing on M.

By Lemma 7.4, we prove identically as in Lemma 7.2 that the ideal of functions vanishing on M is principal. □

Definition 8.8. Let $D := S$ be an effective Cartier divisor, that is, an analytic hypersurface of a complex manifold. The ideal sheaf $\mathcal{I}_S := \mathcal{O}_X(-S)$ of S is the dual line bundle of the *line bundle associated to the divisor* D, which is denoted by $\mathcal{O}_X(D)$.

9. Riemann-Roch formulas

Theorem 9.1 (Riemann-Roch for curves)**.** *Let D be a divisor on a compact Riemann surface X, that is, D is a collection of d points on X.*

$$h^0(D) - h^0(K - D) = d - g + 1 \tag{9.1.1}$$

Exercise 9.2. Prove that if X is a complex smooth curve with $g \geq 2$ then $h^0(T_X) = 0$.

Proof. First you notice that the dimension of the holomorphic 1-forms is the genus. This is just because the genus p_a is defined as $h^1(\mathcal{O}_X)$, which is just $h^0(K_X)$ by

Serre duality. Then you say well if you have a holomorphic vector field, pair it with the nonzero holomorphic 1-form. This gives a nonzero function, but it must be constant because X is compact. Then it is actually nowhere vanishing. This says the vector field cannot vanish anywhere, which means the tangent bundle of the curve is trivial. Apparently elliptic curves, i.e. $g = 1$ are the □

Theorem 9.3 (Riemann-Roch for line bundles on surfaces)**.** *Let L be a line bundle on a complex surface X, and $K_X = \Omega^2 X$ the canonical bundle of X. Then*

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}(L, L - K_X) \tag{9.3.1}$$

10. Adjunction formula

Perhaps the most accessible statement that could be interpreted as "adjunction formula" is that the line bundle associated to a smooth hypersurface is the normal bundle.

Theorem 10.1 (Adjunction formula I)**.** *If Y is a smooth hypersurface of a smooth complex manifold X, then $\mathcal{O}_X(Y) \simeq \mathcal{N}_{Y/X}$.*

Lemma 10.2. *Let S be any curve on a surface X. Then*

$$2p_a(S) - 2 = (S.S - K_X) \tag{10.2.1}$$

Proof. If S is nonsingular we may use the normal bundle short exact sequence. If S is singular we need to use the ideal sheaf short exact sequence, which means we think of S either as a closed embedded subscheme of X or as an analytic subvariety. Taking Euler class we obtain

$$\chi(\mathcal{O}(-S)) + \chi(\mathcal{O}_S) = \chi(\mathcal{O}_X)$$

Since $p_a(S) := 1 - \chi(\mathcal{O}_S)$, we obtain that $p_a(S) = 1 - \chi(\mathcal{O}_X) - \chi(\mathcal{O}(-S))$. To compute $\chi(\mathcal{O}(-S))$ we use Riemann-Roch formula **??** for curves on surfaces, which gives

$$\chi(\mathcal{O}(-S)) = \chi(\mathcal{O}_X) + \frac{(\mathcal{O}(-S).\mathcal{O}(-S) + \omega_X^*)}{2}$$

Taking duals on both entries of the intersection form and substituting in our previous expression for $p_a(S)$ we obtain the result. □

Exercise 10.3. Let S be a singular, irreducible complex curve on a K3 surface. Prove that $(S.S) \geq 0$.

Proof. This is just an application of Eq. 9.2.1. Since $K_X = \mathcal{O}_X$ it suffices to show that $p_a(S)$ is not zero. This follows fact that S is singular, since its arithmetic genus is defined as the genus of its normalization, which must be strictly positive since there exists at least one singularity. □

11. Chow's theorem

Proposition 11.1. *Every codimension-p submanifold of a complex n-manifold has slice charts, i.e. for any $p \in M$ there's a coordinate chart $U \ni p$ such that points of M have coordinates $(z_1, \ldots, z_n, 0, \ldots 0)$.*

Proof. This may be taken as the definition of smooth (real or complex) submanifold, or as a proposition using inverse function theorem (real or complex). □

Proposition 11.2. *Stalk of holomorphic functions is a PID.*

Proof. This can be proved via Weierstrass Preparation theorem 2.3 which implies Lemma **??**, and this in turn implies Lemma **??**.

Also, it can be shown directly for smooth hypersurfaces as in [Lee24] Lemma 3.38. □

Theorem 11.3 (Chow for hypersurfaces)**.** *Every complex codimension-1 submanifold of $\mathbb{C}P^n$ is algebraic.*

12. Ampleness

Definition 12.1. Let L be a holomorphic bundle on a complex manifold X. A point $x \in X$ is a *base point* of L if $s(x) = 0$ for all $s \in H^0(X, L)$.

Definition 12.2. Let L be a holomorphic bundle on a complex manifold X. The *base locus* $\mathrm{Bs}(L)$ is the sate of all base points of L.

Proposition 12.3. *Let L be a holomorphic line bundle on a complex manifold X and suppose that $s_0, \dots, s_N \in H^0(X, L)$ is a basis. Then*

$$\begin{aligned} \varphi_L : X \setminus Bs(L) &\longrightarrow \mathbb{P}^N \\ x &\longmapsto (s_0(x) : \ldots : s_N(x)) \end{aligned}$$

defines a holomorphic map such that $\varphi_L^* \mathcal{O}_{\mathbb{P}^N}(1) \cong L|_{X \setminus Bs(L)}$.

Definition 12.4. The map φ_L in Proposition 11.3 is said to be associated to the *complete linear system* $H^0(X, L)$ (i.e. the sections of L are what we call a complete linear system) whereas a subspace of $H^0(X, L)$ is called a *linear system* of L.

Definition 12.5. L is *globally generated* by the sections $s_0, \dots, s_N$ if $\mathrm{Bs}(L, s_0, \dots, s_N) = \emptyset$.

Definition 12.6. A line bundle L on a complex manifold is called *ample* if for some $k > 0$ and some linear system in $H^0(X, L^k)$ the associated map φ is an embedding.

Exercise 12.7. Let L be an ample bundle on a K3 surface M. Prove that $L^{\otimes 2}$ is globally generated (that is, for each $x \in M$ there exists a section $h \in H^0(L^{\otimes 2})$ which does not vanish in x).

Proof. As an outline for exposition:

(1) Show that L has sections (by ampleness+Kodaira Vanishing+Riemann-Roch), pick a section C and note that if $L^{\otimes 2}$ had base points they would be in C.
(2) Show that restriction map is surjective (Kodaira Vanishing).
(3) Show that inclusion map is not surjective. For this it's enough to show that $\dim |D| = \dim |D - p| + 1$. This follows by writing the formula of Riemann-Roch for both of these Weil divisors (have to pass to Weil divisors). But you will need to make sure that $h^0(K - D)$ and $h^0(K - (D - p))$ are zero.

First we need to show that L has sections. To see this I will shot that since L is ample, $L^2 > 0$. Indeed, if mL is very ample we apply by Riemann-Roch, Kodaira Vanishing and the fact that $\chi(\mathcal{O}_X) = 2$ on a K3 to obtain $h^0(mL) = 2 + \frac{1}{2}m^2L^2$ which must be positive since mL has sections by being very ample. Notice L^2 cannot be zero as $h^0(mL) = 2$ would imply we have an embedding of a K3 surface into $\mathbb{P}^1$.

Applying again Riemann-Roch and Kodaira vanishing we see that $h^0(L) > 0$. Let C be a section of L.

Notice that if $L^{\otimes 2}$ had any base points, they would have to be on C. Indeed, since C is the vanishing set of a section $s \in H^0(L)$, we must have $s \otimes s \in H^0(L^{\otimes 2})$ vanishing in any base point of $L^{\otimes 2}$.

Then we consider the ideal exact sequence for C and tensor by $L^{\otimes 2}$ to obtain

$$0 \longrightarrow L^{\otimes 2}(-C) \longrightarrow L^{\otimes 2} \longrightarrow L^{\otimes 2} \otimes \mathcal{O}_C \longrightarrow 0$$

Notice that the term on the left is actually L since the ideal sheaf $\mathcal{O}_M(-C) = \mathcal{I}_C$ is dual to L because C is defined as the vanishing set of a section of L. Passing to cohomology we obtain

$$H^0(L^{\otimes 2}) \longrightarrow H^0(L^{\otimes 2} \otimes \mathcal{O}_C) \longrightarrow H^1(L)$$

where the latter term vanishes by Kodaira Vanishing because L is ample. This says that every section of $L^{\otimes 2}|_C$ is the restriction to C of a section of $L^{\otimes 2}$. In turn this shows that it's enough to show that the bundle $L^{\otimes 2}$ has no base points along C.

By adjunction formula for (possibly singular) curves on smooth surfaces and by the fact that M is smooth, we see that $2p_a(C) - 2 = (L.L)$ so that

$$4p_a(C) - 4 = (2L.L) = \deg_C(L^{\otimes 2})$$

Notice that since $L^2 > 0$ we exclude the cases that $p_a(C) = 0, 1$.

To conclude pick a point $p \in C$. Saying that p is not a base point of $L^{\otimes 2}|_C$ is the same as saying that not every section of $L^{\otimes 2}|_C$ vanishes at p, that is, that $h^0(L^{\otimes 2}|_C(-p)) < h^0(L^{\otimes 2}|_C)$.

Now I will prove that since the degree of $L^{\otimes 2}|_C$ and $L^{\otimes 2}|_C(p)$ is greater than $2p_a(C)$ we have that $h^0(\omega_C \otimes (L^{\otimes 2}|_C)^\vee) = 0 = h^0(\omega_C \otimes (L^{\otimes 2}|_C(-p))^\vee)$. For this we need to see these line bundles as Weil divisors by simply taking sections, whose vanishing sets are finite sets of points with multiplicities. Then their degree is the sum of the multiplicities. This definition makes degree additive with respect to tensor product (we may take local sections and sum degrees). By Riemann-Roch on the curve C, we know that $\deg \omega_C = 2p_a - 2$. Then

$$\deg_C(\omega_C \otimes L^{\otimes 2}) = \deg_C(\omega_C) + \deg_C(L^{\otimes 2}) = 2p_a - 2 - (4p_a(C) - 4) < 0$$

and a similar computation works for $L^{\otimes 2}(-p)$ as it has the same degree of $L^{\otimes 2}$ minus 1. This implies that neither of these bundles can have sections since any section would provide a linearly equivalent effective divisor. This would be a contradiction since both $L^{\otimes 2}$ and $L^{\otimes 2}(-p)$ have sections. Therefore $h^0(\omega_C \otimes (L^{\otimes 2}|_C)^\vee) = 0 = h^0(\omega_C \otimes (L^{\otimes 2}|_C(-p))^\vee)$ as claimed.

Finally we apply Riemann-Roch to $L^{\otimes 2}$ and $L^{\otimes 2}(-p)$ to obtain that $h^0(L^{\otimes 2}) > h^0(L^{\otimes 2}(-p))$. □

13. Bertini's theorem

14. Serre duality

Theorem 14.1 (Serre duality)**.** [Voi02] *II.5.32. The pairing*

$$H^q(X, \mathcal{E}) \otimes H^{n-q}(X, \mathcal{E}^* \otimes K_X) \to H^n(X, K_X) \cong \mathbb{C}$$

is perfect.

So when you put the dual $\vee$ on one of these you get isomorphism.

Our course version says:

$$H^k(X, \mathcal{L})^\vee = H^{n-k}(X, \omega_X \otimes \mathcal{L}^*)$$

15. Kodaira Vanishing theorem

Theorem 15.1 (Kodaira Vanishing)**.** *Suppose k is a field of characteristic 0, and X is a smooth projective k-variety. Then for any ample invertible sheaf L, $H^i(X, K_X \otimes L) = 0$ for $i > 0$.*

Proof. No proof in [Vak25]. ☐

16. Kodaira Embedding theorem

The forward implication is easy and sometimes not considered as part of the theorem.

Compare this theorem with Nakai-Moishezon Criterion **??**.

Theorem 16.1 (Kodaira Embedding)**.** *Suppose M is a compact complex manifold. A holomorphic line bundle $L \to M$ is ample if and only if it is positive. Thus M is projective if and only if it admits a holomorphic line bundle.*

Proof. The easy implication as follows. If L is ample then there is N such that $L^{\otimes N}$ is very ample, meaning by Misha's definition that the canonical map is an embedding such that $L^{\otimes N} = \varphi^* \mathcal{O}(1)$, which has degree 1 by definition as in [Vak25, 15.4.14]. ☐

17. Hypercomplex manifolds

Exercise 17.1. Let M be a compact hypercomplex manifold of real dimension 4, equipped with a quaternionic Hermitian structure, and V the space of closed SU(2)-invariant 2-forms. Prove that V is finite-dimensional.

Proof by Arpan. Consider the Hodge star operator with respect to the Riemannian metric on M.

(1) A closed anti-self-dual form is harmonic. Indeed, by some characterization a form is harmonic $\iff$ it is d-closed and d^*-closed. So if α is self-dual and d-closed we get $d^*\alpha = (-1)^{-\bullet} * d * \alpha = 0$ since $d\alpha = 0$.

(2) The space of harmonic forms is finite-dimensional (analysis, cf. Fredholm theory).

(3) I should be able to prove that SU(2)-invariant closed form is self-dual.
   (a) Define Λ^+ :=self-dual 2-forms and Λ^- :=anti-self dual 2-forms. **Claim.** $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$.
   (b) **Claim.** $\Lambda^+ = \mathrm{span}(\omega_I, \omega_J, \omega_K)$.
   (c) **Claim.** ω_I, ω_J and ω_K are SU(2)-invariant.
   The last two items imply that $\mathrm{SU}(2)\Lambda^+ = \Lambda^+$.
   (d) **Claim.** $\mathrm{SU}(2) \curvearrowright \Lambda^+$ has no fixed points.
   We conclude that if α is SU(2)-invariant (i.e. a fixed point of $\mathrm{SU}(2) \curvearrowright \Lambda^2$) it's positive part will vanish, so that α is anti-self-dual.

☐

Proof. Idea: show that Hodge star $*\omega$ is in the SU(2)-orbit of ω and conclude that $\int \omega \wedge *\omega = 0$, implying that $\|\omega\| = 0$. ☐

Exercise 17.2.

Proof. Idea: find a counterexample. The easiest should be a Kummer surface. It looks possible to find an almost hypercomplex structure on $\mathbb{C}^2$ passing to the torus and then to the $\mathbb{Z}_2$ quotient, but not clear what will happen after blowing up. $\square$

References

[Huy05] D. Huybrechts, *Complex geometry: An introduction*, Universitext (Berlin. Print), Springer, 2005.

[Lee24] J.M. Lee, *Introduction to complex manifolds*, Graduate Studies in Mathematics, American Mathematical Society, 2024.

[Vak25] R. Vakil, *The rising sea: Foundations of algebraic geometry*, Princeton University Press, 2025.

[Voi02] Claire Voisin, *Hodge theory and complex algebraic geometry i*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2002.