VERTEX ALGEBRAS

github.com/danimalabares/vertex-algebras

Contents

1. Cartan subalgebra, Cartan matrix and Serre relations

Kac-Moody algebras are Lie algebras, whose definition is motivated by the structure of finite-dimensional simple Lie algebras over $\mathbb{C}$.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ has a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ (abelian + ...). Fixing $\mathfrak{h} \subset \mathfrak{g}$ gives a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta \subset \mathfrak{h}^*$ linear dual, and, by definition

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$$

Turns out the $\mathfrak{g}_\alpha$ are all 1-dimensional, though this property is lost when we go to Kac-Moody algebras.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

The Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $\kappa(x, y) = \mathrm{Tr}_{\mathfrak{g}}\mathrm{ad}(x)\mathrm{ad}(y)$ is nondegenerate. "This is kind of the definition of semisimple." (Think of $\mathfrak{h}$ as $\mathfrak{g}_0$, btw.)

$\kappa|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\beta} \neq 0$ only when $\beta = -\alpha$. $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate. This gives a linear isomorphism $\mathfrak{h} \xrightarrow{\nu} \mathfrak{h}$ via $\nu(H)(H') = \kappa(H, H')$.

So, $\mathfrak{h}^*$ comes with a non-degenerate bilinear form.

The *reflection* $r_\alpha : \mathfrak{h} \to \mathfrak{h}^*$ in $\alpha \in \mathfrak{h}^*$ (usually a root) is $r_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \cdot \alpha$.

"Classify root systems [...] classify semisimple Lie algebras" It is a fact that $r_\alpha(\Delta) = \Delta$ for all $\alpha \in \Delta$, which motivates the definition of *root system* and permits classification.

Example 1.1. $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h} =$ diagonal matrices

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

is a basis of $\mathfrak{h}$. There are 6 roots vectors

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

E_{23}, E_{13}, etc.

Exercise 1.2. $[H_1, E_{12}] = 2E_{12}$, $[H_2, E_{12}] = -E_{12}$, $\alpha_{12} = (2, -1)$.

[Drawing of roots]

Notions of *positive roots* and *simple roots* (set of rank$\mathfrak{g}$ simple roots has ℓ elements, where $\ell = \dim(\mathfrak{h}^*)$. This will also fail for Kac-Moody algebras more generally). Next write the *Cartan matrix*

$$A = (a_{ij}), \qquad a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

for $1 \leq i, j \leq \ell$.

Example 1.3. $\mathfrak{sl}_3$. [Picture, hexagonal pattern]. $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2, (\alpha_1, \alpha_2) = -1$, so

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Example 1.4. $\mathfrak{sl}_5$. [Picture, square pattern]. $|\alpha_2| = 1$, $|\alpha_1| = 2$, $(\alpha_1, \alpha_2) = -2$, so

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

Remark 1.5. Since $\mathfrak{g}_\alpha$ is 1-dimensional, set $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$ and $E_i = E_{\alpha_i}$, $i = 1, 2, \ldots, \ell$ (simple root vectors). It turns out that

$$\mathrm{ad}(E_i)^{1-a_{ij}} E_j = 0.$$

This is called a *Serre relation*.

2. Some infinite dimensional Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, and define the *loop algebra*

$$\begin{aligned} L\mathfrak{g} &= \mathfrak{g}[t,t^{-1}], \ \text{(with basis } at^m|^{a \,\in\, \text{a basis of } \mathfrak{g}}_{m\in\mathbb{Z}}\text{)} \\ &= \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}] \end{aligned}$$

with the Lie bracket

$$[at^m, bt^n] = [a,b]t^{m+n}.$$

"This construction is absurdely general — we don't need $\mathfrak{g}$ to be semisimple [. . .]"

Take $\mathfrak{g} = \mathfrak{sl}_2$. Recall that

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

[Picture with $F, H, E, Ft, Ht, Et, Et^2 \dots$] E was a root vector, corresponding to the unique root in $\mathfrak{sl}_2$, call it α_1. We seem to have a second simple root α_0, corresponding to Ft.

This looks like it wants to have a Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

We will indeed recover (a variant of) $L\mathfrak{g}$ as a Lie algebra "built from" $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, a Kac-Moody algebra. But note first, $\mathfrak{h} = \mathbb{C}H$ is too small. "Problem with α_0 and α_1 being linearly independent . . . "

Exercise 2.1. Consider $L\mathfrak{g} \oplus \mathbb{C}d$, and set $[d, at^m] = mat^m$, $[d,d] = 0$. Check this defines a Lie algebra.

Proof. Skew-commutativity, i.e. for all $x \in L\mathfrak{g} \oplus \mathbb{C}d$,

$$[x,x] = 0, \tag{2.1.1}$$

is immediate from skew commutativity in $L\mathfrak{g}$ and the hypothesis that $[d,d] = 0$.

To confirm Jacobi identity, i.e. that for all $x,y,z \in L\mathfrak{g} \oplus \mathbb{C}d$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \tag{2.1.2}$$

notice that since this is a cyclic sum on x,y,z we only need to consider three elements in $L\mathfrak{g} \oplus \mathbb{C}d$ up to cyclic permutation. The cases in which the three elements are either in $L\mathfrak{g}$ or in $\mathbb{C}d$ are obvious, so that there are only two interesting possibilities:

$$x = d, \qquad y = at^m, \qquad z = bt^n \tag{2.1.3}$$

$$x = d, \qquad y = d, \qquad z = at^n \tag{2.1.4}$$

Case 2.1.3 gives

$$\begin{aligned} &[d,[at^m,bt^n]] + [at^m,[bt^n,d]] + [bt^n,[d,at^m]] \\ &= [d,[a,b]t^{m+n}] + [at^m, -nbt^n] + [bt^n, mat^m] \\ &= (m+n)[a,b]t^{m+n} - n[a,b]t^{m+n} + m[b,a]t^{m+n} \\ &= (m+n)[a,b]t^{m+n} - n[a,b]t^{m+n} - m[a,b]t^{m+n} \\ &= (m+n)[a,b]t^{m+n} - (m+n)[a,b]t^{m+n} = 0. \end{aligned}$$

Case 2.1.4 gives

$$\begin{aligned}&[d,[d,at^m]]+[d,[at^m,d]]+[at^m,[d,d]]\\&=[d,mat^m]+[d,-mat^m]=0.\end{aligned}$$

□

The Kac-Moody algebra turns out to be, not quite this, but slightly larger still.

Recall that an *invariant bilinear form* $(\cdot,\cdot)$ on a Lie algebra $\mathfrak{g}$ is a bilinear form such that

$$([a,b],c)=(a,[b,c]) \qquad \forall a,b,c\in\mathfrak{g}. \tag{2.1.5}$$

Exercise 2.2. Prove that an invariant bilinear form on a simple Lie algebra must in fact be symmetric.

Proof. It's enough to show that $\mathfrak{g}$ is *perfect*, i.e. that $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. In this case, let $a,b\in\mathfrak{g}$ and suppose that $b=[x,y]$. Then

$$\begin{aligned}(a,b)&=(a,[x,y])=(a,-[y,x])=(-[a,y],x)=([y,a],x)\\&=(y,[a,x])=(y,-[x,a])=(-[y,x],a)=([x,y],a)=(b,a)\end{aligned}$$

To confirm that $\mathfrak{g}$ is perfect just observe that $[\mathfrak{g},\mathfrak{g}]$ is a nontrivial ideal of $\mathfrak{g}$. □

Definition 2.3. Given $\mathfrak{g}$ simple, with $(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ invariant bilinear form, the *affine Lie algebra* is

$$\hat{\mathfrak{g}}=L\mathfrak{g}\oplus\mathbb{C}K,$$

with $[K,\hat{\mathfrak{g}}]=0$, and $[at^m,bt^n]=[a,b]t^{m+n}+m(a,b)\delta_{m,-n}K$.

"For the construction to work it doesn't actually have to be nondegenerate."

Exercise 2.4. Check that the affine Lie algebra $\hat{\mathfrak{g}}$ is a Lie algebra.

Proof. (Skew-commutativity.) Since $[K,\hat{\mathfrak{g}}]=0$ and $K\in\hat{\mathfrak{g}}$, it is immediate that $[K,K]=0$. For the case of an element in $L\mathfrak{g}$, we see that $[at^m,at^m]=0$ by skew-commutativity of the bracket in $\mathfrak{g}$ and the Kronecker delta.

(Jacobi identity.) As in Exercise 2.1, any choice of x,y,z involving K is immediate by $[K,\hat{\mathfrak{g}}]=0$. Thus the only interesting case is for Jacobi identity consider the cases

$$\begin{aligned}&[at^m,[bt^n,ct^\ell]]+[bt^n,[ct^\ell,at^m]]+[ct^\ell,[at^m,bt^n]]\\&=[at^m,[b,c]t^{n+\ell}+n(b,c)\delta_{n,-\ell}K]\\&+[bt^n,[c,a]t^{\ell+m}+\ell(c,a)\delta_{\ell,-m}K]\\&+[ct^\ell,[a,b]t^{m+n}+m(a,b)\delta_{m,-n}K]\\&=[at^m,[b,c]t^{n+\ell}]+[at^m,n(b,c)\delta_{n,-\ell}K]\\&+[bt^n,[c,a]t^{\ell+m}]+[bt^n,\ell(c,a)\delta_{\ell,-m}K]\\&+[ct^\ell,[a,b]t^{m+n}]+[ct^\ell,m(a,b)\delta_{m,-n}K]\\&=[a,[b,c]]t^{m+(n+\ell)}+m(a,[b,c])\delta_{m,-(n+\ell)}K\\&+[b,[c,a]]t^{n+(\ell+m)}+n(b,[c,a])\delta_{n,-(\ell+m)}K\\&+[c,[a,b]]t^{\ell+(m+n)}+\ell(c,[a,b])\delta_{\ell,-(m+n)}K=0\end{aligned}$$

It is clear that we obtain a Jacobi equation on $\mathfrak{g}$. To see that the remaining terms vanish, notice that the condition on the Kronecker delta in its three appearances is the same, namely, $m + n + \ell = 0$. In this case, we only need to check that $(a, [b, c]) = (b, [c, a]) = (c, [a, b])$ to conclude. This follows from the invariance of $(\cdot, \cdot)$ and the fact that $\mathfrak{g}$ simple using Exercise 2.2. □

We also have

Definition 2.5. The *extended affine Lie algebra* is

$$\tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

with $[d, at^m] = mat^m$ as before, and $[K, d] = 0$.

The extended affine Lie algebra is an example of a Kac-Moody algebra.

Exercise 2.6 (For those who like geometry)**.** Let $R = \mathbb{C}[t, t^{-1}]$. If $D \in \mathrm{Der}(R)$, then $L\mathfrak{g} \oplus \mathbb{C}d$ is a Lie algebra with $[d, a \otimes r] = a \otimes D(r)$. Is $(\mathfrak{g} \otimes R) \oplus \mathrm{Der}(R)$ a Lie algebra? (The Lie alegra $L\mathfrak{g} \oplus \mathbb{C}d$ from Exercise 2.1 is a particular case, for $D = t\frac{d}{dt}$.)

Proof. Checking that $L\mathfrak{g} \oplus \mathbb{C}d$ is a Lie algebra with $[d, a \otimes r] = a \otimes D(r)$ is similar to Exercise 2.1: skew-commutativity is immediate from skew-commutativity in each of the components, while Jacobi identity is verified in two cases. For $x = y = d$ and $z = a \otimes r$ we quickly obtain

$$\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
&= [d, [d, a \otimes r]] + [d, [a \otimes r, d]] + [a \otimes r, [d, d]] \\
&= [d, a \otimes D(r)] + [d, -a \otimes D(r)] = 0.
\end{aligned}$$

And for $x = d$, $y = a \otimes r$ and $z = b \otimes s$, we get

$$\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
&= [d, [a \otimes r, b \otimes s]] + [a \otimes r, [b \otimes s, d]] + [b \otimes s, [d, a \otimes r]] \\
&= [d, [a, b] \otimes rs] + [a \otimes r, -b \otimes D(s)] + [b \otimes s, a \otimes D(r)] \\
&= [a, b] \otimes D(rs) - [a, b] \otimes rD(s) + [b, a] \otimes sD(r) = 0.
\end{aligned} \tag{2.6.1}$$

To check whether $(\mathfrak{g} \otimes R) \oplus \mathrm{Der}(R)$ is a Lie algebra first put the Lie bracket on $\mathrm{Der}(R)$ as $[D, D_1] = DD_1 - D_1D$. It is clear that this bracket is skew-commutative. Jacobi identity reads

$$\begin{aligned}
&[D, [D_1, D_2]] + [D_1, [D_2, D]] + [D_2, [D, D_1]] \\
&= [D, D_1D_2 - D_2D_1] + [D_1, D_2D - DD_2] + [D_2, DD_1 - D_1D] \\
&= D(D_1D_2 - D_2D_1) - (D_1D_2 - D_2D_1)D + D_1(D_2D - DD_2) \\
&- (D_2D - DD_2)D_1 + D_2(DD_1 - D_1D) - (DD_1 - D_1D)D_2 \\
&= DD_1D_2 - DD_2D_1 - D_1D_2D + D_2D_1D + D_1D_2D - D_1DD_2 \\
&- D_2DD_1 + DD_2D_1 + D_2DD_1 - D_2D_1D - DD_1D_2 + D_1DD_2 = 0.
\end{aligned}$$

Now put the bracket on $(\mathfrak{g} \otimes R) \oplus \mathrm{Der}(R)$ as $[D, a \otimes r] = a \otimes D(r)$. Skew-commutativity is immediate. Jacobi identity for $x = D, y = a \otimes r$ and $z = b \otimes s$ is identical to the computation 2.6.1. In the case $x = D$, $y = D_1$ and $z = a \otimes r$, we get

$$\begin{aligned}
&[D, [D_1, a \otimes r]] + [D_1, [a \otimes r, D]] + [a \otimes r, [D, D_1]] \\
&= [D, a \otimes D_1(r)] + [D_1, -a \otimes D(r)] + [a \otimes r, [D, D_1]] \\
&= a \otimes DD_1(r) - a \otimes D_1 D(r) - a \otimes [D, D_1](r) = 0
\end{aligned}$$

□

3. Kac-Moody algebras

Recall the notion of the free Lie algebra on a vector space V of generators (or a set X, think of V as a vector space with basis X):

Definition 3.1. The *free Lie algebra* on V is characterized by the universal property

$$\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathfrak{g} \\
& F(V) \arrow[ur, "\exists! \tilde{f}"'] &
\end{tikzcd}$$

That is, for any linear map $f : V \to \mathfrak{g}$ with $\mathfrak{g}$ Lie algebra, there exists a unique $\tilde{f}$ homomorphism of Lie algebras $F(V) \to \mathfrak{g}$ such that $\tilde{f} \circ i = f$.

$$\mathrm{Hom}_{\mathrm{Lie}}(F(V), \mathfrak{g}) = \mathrm{Hom}_{\mathrm{Vec}}(V, \mathfrak{g})$$

naturally.

That is, F and the forgetful functor $G : \underline{\mathrm{Lie}} \to \underline{\mathrm{Vec}}$ are adjoint:

$$\mathrm{Hom}_{\underline{\mathrm{Lie}}}(F(V), \mathfrak{g}) \xrightarrow{\simeq} \mathrm{Hom}_{\underline{\mathrm{Vec}}}(V, G(\mathfrak{g}))$$

A realisation of $F(V)$. Let

$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

be the tensor algebra of V.

Then inside $T(V)$ consider $F(V)$ the span of iterated commutators of elements of V.

Proposition 3.2. *This realises the free Lie algebra.*

Proof. In online notes. □

In the finite dimensional simple case, we had

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$

which we think also as $\alpha_i, \alpha_j \in \mathfrak{h}^*$, and $\alpha_i^\vee = \frac{2}{(\alpha_i, \alpha_i)} \nu^{-1}(\alpha_i) \in \mathfrak{h}$.

Clearly, $\alpha_{ii} = 2$ for all i. a_{ij} might not equal a_{ji}, but certainly $a_{ij} = 0 \iff a_{ji} = 0$. And $\forall i \neq j$, $a_{ij} \leq 0$.

One further property. Set

$$\varepsilon_i = \frac{2}{(\alpha_i, \alpha_i)}, \quad \text{and} \quad D = \begin{smallmatrix}\text{diagonal matrix} \\ \text{with entries } \varepsilon_i\end{smallmatrix}$$

Then $A = DB$, where $B = ((\alpha_i, \alpha_j))$ is symmetric. If a matrix A is equal to (diag)(symm), we call it *symmetrizable*.

Definition 3.3. A *generalized Cartan matrix* is an integer matrix $A = (a_{ij})$ which is

- symmetrizable,
- $a_{ii} = 2$ for all i,
- $a_{ij} = 0 \iff a_{ji} = 0$,
- $a_{ij} \leq 0$ for $i \neq j$.

Definition 3.4. A *realisation* of a generalized Cartan matrix is a complex vector space $\mathfrak{h}$, and two sets

$$\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\}, \quad \text{and,}$$
$$\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$, $1 \leq i, j \leq n$.

Exercise 3.5. $\dim(\mathfrak{h}) \geq 2n - \operatorname{rank}(A)$.

Proof. For $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, a realisation is given by

$$\Pi^\vee = \{H_1, H_0\}, \qquad \Pi = \{\alpha_0, \alpha_1\}$$

$$\mathfrak{h} = \mathbb{C}H \oplus \mathbb{C}d \oplus \mathbb{C}K,$$
$$\mathfrak{h}^* = \mathbb{C}\alpha_1 \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$$

(Canonical dual, $\langle \alpha_1, H \rangle = 2$, $\langle \delta, d \rangle = 1 = \langle \Lambda_0, K \rangle$, every other pairing 0.)

Then

$$\begin{cases} \alpha_1 = \alpha_1 \\ \alpha_0 = \delta - \alpha_1 \end{cases} \qquad \begin{cases} \alpha_1^\vee = H \\ \alpha_0^\vee = K - H \end{cases}$$

So we obtain

$$\langle \alpha_0^\vee, \alpha_1 \rangle = \langle K - H, \alpha_1 \rangle = 2$$
$$\langle \alpha_1^\vee, \alpha_0 \rangle = \langle H, \delta - \alpha_1 \rangle = -2$$
$$\langle \alpha_0^\vee, \alpha_0 \rangle = \langle K - H, \delta - \alpha_1 \rangle = +2$$

□

Finally let's define Kac-Moody algebras.

Let A be a generalized Cartan matrix. Let

$$\tilde{\mathfrak{n}}_+ = F(e_1, \ldots, e_n),$$

the free Lie algebra on n generators, and similarly

$$\tilde{\mathfrak{n}}_- = F(f_1, \ldots, f_n).$$

Let $\mathfrak{h}$ be a realisation of A. Set $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.

Make $\tilde{\mathfrak{g}}(A)$ a Lie algebra by defining

- $[\mathfrak{h}, \mathfrak{h}] = 0$,
- $\forall H \in \mathfrak{h}$, $[H, e_i] = \langle \alpha_i, H \rangle \, e_i = \alpha_i(H) e_i$. And similarly, $[H, f_i] = -\alpha_i(H) f_i$.
- $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$.

Then $\tilde{\mathfrak{g}}(A)$ is a Lie algebra (though not yet the Kac-Moody algebra). See Kac, [Kac90, Thorem 1.2].

Remark 3.6. In $\mathfrak{h}$ we have a lattice

$$\begin{aligned} Q^\vee &= \mathbb{Z}\alpha_1^\vee + \ldots + \mathbb{Z}\alpha_n^\vee, \quad \text{and} \\ Q &= \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_n \text{ in } \mathfrak{h}^* \end{aligned}$$

(root and coroot lattices). $\tilde{\mathfrak{g}}(A)$ is naturally Q-graded, with

$$\tilde{\mathfrak{g}}(A)_\beta = \mathrm{span}\{\text{commutators of } e_i \text{ with } \sum \alpha_i = \beta\}.$$

$\tilde{g}(A) = \mathfrak{h}$.

Theorem 3.7 (Gabber-Kac). *Denote by $I \subset \tilde{\mathfrak{g}}(A)$ the maximal Q-graded ideal, such that $I \cap \mathfrak{h} = \{0\}$. Then I is generated by the Serre relations*

$$ad(e_i)^{1-a_{ij}} e_j \qquad \text{and} \qquad ad(f_i)^{1-a_{ij}} f_j,\ i \neq j.$$

Proof. [Kac90, Theorem 9.11]. □

(The existence of the ideal I does not need the theorem; the importance of the theorem is providing an expression for the generators.)

Definition 3.8. The *Kac-Moody algebra* $\mathfrak{g}(A)$ is $\tilde{\mathfrak{g}}(A)/I$.

4. Affine Kac-Moody algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, with $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ invariant bilinear form,

$$([x, y], z) = (z, [y, z]) \qquad \forall x, y, z \in \mathfrak{g}$$

(Eg. the Killing form $\kappa(x, y) = \mathrm{Tr}_{\mathfrak{g}}\mathrm{ad}(x)\mathrm{ad}(y)$ is invariant.)

Typically we normalise $(\cdot,\cdot)$ so that $(\alpha, \alpha) = 2$ for the long roots of $\mathfrak{g}$.

Then $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ (affine Lie algebra),

$$[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a, b)K, \qquad [K, \hat{\mathfrak{g}}] = 0$$

and $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d$, $[d, K] = 0$, $[d, at^m] = mat^m$, (affine Kac-Moody algebra or "extended affine Lie algebra")

Theorem 4.1. *$\tilde{\mathfrak{g}}$ is a Kac-Moody algebra.*

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, ($\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$.)

The simple roots and coroots. $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. We identify $\tilde{\mathfrak{h}}^*$ with $\mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ where

$$\begin{aligned} \Lambda_0(\mathfrak{h}) &= \delta(\mathfrak{h}) = 0 \\ \Lambda_0(d) &= \delta(K) = 0 \\ \Lambda_0(K) &= \delta(d) = 1 \end{aligned}$$

The *real coroots* are

$$\hat{\Delta}^{V,re} = \{E_\alpha t^m | \alpha \in \Delta, m \in \mathbb{Z}\}$$

and there are also imaginary roots and coroots

$$\hat{\Delta}^{V,im} = \{Ht^m | H \in \mathfrak{h}, m \in \mathbb{Z} \setminus \{0\}\}$$

Roots:

$$\hat{\Delta}^{re} = \{\alpha + m\delta | \alpha \in \Delta, m \in \mathbb{Z}\}$$
$$\hat{\Delta}^{im} = \{m\delta | m \neq 0\}$$

Xt^m:

$$[H, Xt^m] = [H, x]t^m, \qquad H \in \mathfrak{h}$$
$$[K, xt^m] = 0$$
$$[d, xt^m] = mxt^m$$

so it $x \in \mathfrak{g}_\alpha$, $xt^m \in \tilde{\mathfrak{g}}_{\alpha+m\delta}$.

The invariant bilinear form $(\cdot,\cdot)$ from $\mathfrak{g} \times \mathfrak{g}$ extends uniquely to $(\cdot,\cdot) : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \mathbb{C}$. $(d, d) = (K, K) = 0$, $(d, K) = 1$ and $(d, \mathfrak{h}) = (K, \mathfrak{h}) = 0$.

So, in $\tilde{\mathfrak{h}}^*$:

$$(\Lambda_0, \Lambda_0) = (\delta, \delta) = 0$$
$$(\Lambda_0, \mathfrak{h}^*) = (\delta, \mathfrak{h}^*) = 0$$
$$(\Lambda_0, \delta) = 1.$$

Hence, $|\alpha + m\delta|^2 = |\alpha|^2$, $|m\delta|^2 = 0$.

Example 4.2. $\widetilde{\mathfrak{sl}_2}, \tilde{\mathfrak{h}}^* = \mathrm{span}\{\alpha, \Lambda_0, \delta\}$ with Gram matrix ...

We can make a choice of positive roots,

$$\hat{\Delta}_+ = \{\alpha + m\delta | \alpha \in \Delta, m > 0\} \cup \{m\delta | m > 0\} \cup \Delta_+$$

Obviously, if $\alpha \in \Delta_+$ is simple, $\alpha \in \hat{\Delta}_+$ is simple.

Notation. Let $\theta \in \Delta_+$ be a the highest root. ($\nexists \alpha \in \Delta_+$ such that $\alpha - \theta \in \mathbb{Z}_+\Delta_+$.) and $\alpha = \delta - \theta$.

Then $\alpha_0 \in \hat{\Delta}_+$ is simple and the set of simple roots is $\hat{\Pi} = \{\alpha_0, \underbrace{\alpha_1, \ldots, \alpha_\ell}_{\text{the finite simple roots}}\}$.

where $\ell = \mathrm{rank}(\mathfrak{g})$.

The coroot corresponding to α_0 is

$$\alpha_0^\vee = K - \theta^\vee, \qquad \theta^\vee = \frac{2}{(\theta, \theta)}\nu^{-1}(\theta) \in \mathfrak{h}$$
$$\text{and} \qquad E_{\alpha_0} = E_{-\theta}t.$$

5. Weyl group

Upshot. The Weyl group is a semidirect product of pseudoreflections and translations.

In any Kac-Moody algebra, we have

$$\text{roots} \qquad \Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset \mathfrak{h}^*$$
$$\text{coroots} \qquad \Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset \mathfrak{h},$$

and *reflections* $r_i \in \mathrm{GL}(\mathfrak{h}^*)$, defined by

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i.$$

One can check that

$$(r_i\lambda, r_i\mu) = (\lambda, \mu) \qquad \forall \lambda, \mu \in \mathfrak{h}^*$$

The *Weyl group* W is $\langle r_i | i = 1, \dots, \ell \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$.

Example 5.1. For $\widetilde{\mathfrak{sl}_2}$, r_1 is easy,

$$\begin{aligned} r_1(\alpha) &= -\alpha \qquad (\text{as in } \mathfrak{sl}_2) \\ r_1(\delta) &= \delta, \qquad r_1(\Lambda_0) = \Lambda_0. \end{aligned}$$

To compute r_0 take an arbitrary element $m\alpha_1 + k\Lambda_0 + f\delta$ and do:

$$r_0(m\alpha_1 + k\Lambda_0 + f\delta) = m\alpha_1 + k\Lambda_0 + f\delta - \langle \alpha_0^\vee, m\alpha_1 + k\Lambda_0 + f\delta \rangle \, \alpha_0$$

$$\alpha_0 = \delta - \alpha_1, \qquad \alpha_0^\vee = K - \alpha^\vee$$

so we obtain

$$\begin{aligned} &= m\alpha_1 + k\Lambda_0 + f\delta - (k - 2m)(\delta - \alpha_1) \\ &(k - m)\alpha_1 + k\Lambda_0 + (f - k + 2m)\delta. \end{aligned}$$

Relative to basis $\{\alpha_1, \Lambda_0, \delta\}$.

$$r_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} m \\ k \\ f \end{pmatrix}$$

$$t = r_1 r_0 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

Notice that δ is fixed by all r_i. Also $m\alpha + k\Lambda_0 + f\delta$, the *coefficient* of Λ_0 is fixed by all r_i.

Then

$$t(m\alpha_1 + k\Lambda_0 + f\delta) = (m - k)\alpha_1 + k\Lambda_0 + (f - k + 2m)\delta.$$

Think of t as a translation.

The number k in

$$\mathfrak{h}^* \ni \hat{\lambda} = \lambda + k\Lambda_0 + f\delta$$

is called the *level* of $\hat{\lambda}$.

$\hat{\mathfrak{h}}$ = union of (hyper)planes of constant level which are stable under W. The roots α are all of level 0.

[Picture] "r_1 changes the sign of the finite path". And $t = r_1 r_0$ is a sort of translation. Indeed, in general we can consider $t_{\alpha_i} = r_{\alpha_i} \circ r_0 \in W$,

$$t_\alpha(\beta + m\delta) = \beta + (m + (\beta, \alpha_i))\delta$$

One can describe the action of t_α on $\hat{\lambda}$ in general (e.g. see [Kac90, Chapter 6])

Proposition 5.2. *For the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ with $\hat{W} = \langle r_0, r_1, \dots, r_\ell \rangle$ its Weyl group (and $W = \langle r_1, \dots, r_\ell \rangle \subset \hat{W}$ the Weyl group of $\mathfrak{g}$), then $\hat{W} \simeq W \times t_{Q^\vee}$ (where it should be semidirect product instead of $\times \dots$) where $Q^\vee$ is the coroot lattice of $\mathfrak{g}$.*

Remark 5.3. For general Kac-Moody algebras, the Weyl groups are much larger, hyperbolic reflection groups.

In the affine case, $\hat{W}$ fixes level k, and $|\hat{\lambda}|$. One gets, in the intersection, paraboloids [Picture of section of hyperboloid that is a parabola].

6. Weyl character formula

Highest weight representations of Kac-Moody algebras. Let $\lambda \in \mathfrak{h}^*$, where $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a Kac-Moody algebra. We define a *Verma module*

$$M(\Lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n}_+)} \mathbb{C}v_\Lambda$$

where $\mathfrak{h} + \mathfrak{n}_+$ acts on V_Λ by:

$$\begin{aligned} Xv_\Lambda &= 0, && \forall x \in \mathfrak{n}_+, \\ Hv_\Lambda &= \Lambda(H)v_\Lambda, && \forall H \in \mathfrak{h} \end{aligned}$$

So $\mathbb{C}v_\Lambda$ is a $U(\mathfrak{h}+\mathfrak{n}_+)$-module,

$$\begin{array}{c} U(\mathfrak{h}+\mathfrak{n}_+) \\ \big\downarrow \\ U(\mathfrak{g}) \end{array}$$

By the PBW theorem, $M(\Lambda)$ has a linear $\mathbb{C}$-basis.

Let $\{F_{\alpha,i} : i = 1, \dots, \dim \mathfrak{g}_\alpha\}$ be a basis of $\mathfrak{g}_{-\alpha}$, $\forall \alpha \in \Delta_+$. Also choose a total order on Δ_+. (Some sort of lexicographical order that takes longer to write than to say.)

$$F_{\alpha_1,i_1}, F_{\alpha_2,i_2}, \dots, F_{\alpha_s,i_s}, v_\Lambda$$
$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_2 \text{ and if } \alpha_p = \alpha_{p+1}, i_p \leq i_{p+1}$$

We have $M(\Lambda)_\lambda = \{m | Hm = \lambda(H)m\}$ weight spaces.

$$M(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} M(\Lambda)_\lambda$$

The vector v_Λ is in $M(\Lambda)_\Lambda$ by definition,

$$F_{\alpha,i}V_\Lambda \in M(\Lambda)_{\Lambda-\alpha}$$
$$H(Fv_\Lambda) = \underbrace{[H,F]v_\Lambda}_{=-\alpha(H)Fv_\Lambda} + \underbrace{FHv_\Lambda}_{=\Lambda(H)FV_\Lambda}$$

So $\chi_{M(\Lambda)} = \sum_{\lambda \in \mathfrak{h}^*} \dim M(\Lambda)_\lambda e^\lambda$ is computed by counting monomials y with fixed $\sum_i \alpha_i$.

$$\chi_{M(\Lambda)} = e^\Lambda \prod_{\alpha \in \Delta_+} \frac{1}{(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \tag{6.0.1}$$

The product on Eq. 6.0.1 is called *Weyl denominator.*

Exercise 6.1. Convince yourself of this.

Example 6.2. $\mathfrak{g} = \mathfrak{sl}_2$, [Picture]

$$\begin{aligned} \chi_{M(\Lambda)} &= e^\Lambda + e^{\Lambda-\alpha} + e^{\Lambda-2\alpha} + \dots \\ &= e^\Lambda(1 + e^{-\alpha} + e^{-2\alpha} + \dots \\ &= e^\Lambda \frac{1}{1-e^{-\alpha}}. \end{aligned}$$

For certain Λ, $M(\Lambda)$ is *reducible* (i.e. there exists a submodule $0 \neq N \subset M(\Lambda)$ (with proper contention).

Lemma 6.3. *For any submodule N,*

$$N = \bigoplus_{\mu \in \mathfrak{h}^*} N \cap M(\Lambda)_\mu.$$

Corollary. The sum of all proper submodules of $M(\Lambda)$ is proper, in particular there is a maximal proper submodule.

Notation. $L(\Lambda) = M(\Lambda) \Big/ \left(\substack{\text{max. proper} \\ \text{submodule}}\right)$

Example 6.4. $\mathfrak{sl}_2$. $\Lambda = 3\omega$ (ω: fundamental weight, $\alpha = 2\omega$.) $L(3\omega) = \mathbb{C}\left\langle e^{3\omega}, e^{-\omega}, e^{-3\omega}\right\rangle$.
[Picture]

Definition 6.5. A $\mathfrak{g}$-module is *integrable* if
- $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ (weight module).
- For all simple roots α_i; e_i and f_i are locally nilpotent on V (i.e. for all $v \in V$ there exists N such that $e_i^N v = f_i^N v = 0$.)

Remark 6.6.
- Vermas are not integrable.
- $\dim V < \infty \implies V$ integrable.
- $\mathfrak{g}$ itself (Kac-Moody) is integrable.

Dominant integrable weights. Let $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ be the simple coroots.

Definition 6.7. The *dominant integral weights* are the weights that pair with the coroots to give integers:

$$P_+ = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0,1}, i = 1, \dots, \ell\}.$$

For $L(\Lambda)$ to be integrable, it is necessary that $\Lambda \in P_+$.
Indeed, suppose $L(\Lambda)$ is integrable. Then $f_i^N v_\Lambda = 0$ in $L(\Lambda)$, or rather

$$\underbrace{e_i f_i^{N+1} v_\Lambda}_{K f_i^N = 0} \in M(\Lambda),$$

and K can only be zero if $\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$. Applying for all i, we find $\Lambda \in P_+$ is necessary.

Proposition 6.8. *$L(\Lambda)$ is integrable if and only if $\Lambda \in P_+$.*

Proof. For the converse, use induction and Serre relations (we know the result for the highest weight, and want to prove for others). □

Example 6.9. $\mathfrak{sl}_3$. [Picture]

Example 6.10. $\widehat{\mathfrak{sl}_2}$. [Picture, P_+ looks like diagonal lines.]

$$\alpha_0^\vee = K - H, \qquad \alpha_1^\vee = H \in \mathfrak{sl}_2, \qquad \langle \delta, \alpha_i^\vee \rangle = 0, \quad i = 0, 1$$

Remark 6.11. For affine Kac-Moody algebras, almost nothing about the structure of $M(\Lambda)$ depends on the coefficient of δ in Λ. So it's common to consider

$$M(\Lambda) = M_k(\lambda) = M(k\Lambda_0 + \lambda), \qquad \lambda \in \mathfrak{h}^*$$

where k, the level of Λ, is super important.

Then

$$\underbrace{\hat{P}_+}_{\substack{\delta\text{-coef.}\\=0}} = \bigcup_{k\in\mathbb{Z}_{\geq 0}} \{k\Lambda_0 + \lambda | \lambda \in P_+^k\}\{k\Lambda_0 + \lambda | \lambda \in P_+^k\}.$$

$$P_+^k = \{\lambda \in P_+ | \langle \lambda, \theta\rangle \leq k\} \subset P_+ \text{ for } \mathfrak{g}.$$

Consider $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$ integrable, and $V_\lambda \neq 0$. Let $i \in \{1, \ldots, \ell\}$. Consider $U = \bigoplus_{n\in\mathbb{Z}} V_{\lambda+n\alpha_i} \subset V$, and the action of e_i, f_i and $h_i = [e_i, f_i]$.

$\mathfrak{sl}_2 \curvearrowright U$, locally integrable. By structure of $\mathfrak{sl}_2$-represenations, U must be finite-dimensional with "symmetrical" weight space multiplicities, i.e.,

$$\{\lambda + n\alpha_i | n \in \mathbb{Z}\} \cap \{\text{weights of } V\} = \{\lambda + n\alpha_i | -p \leq n \leq q\}.$$

and

$$\langle \lambda - p\alpha_i, h_i\rangle = -\langle \lambda + q\alpha_i, h_i\rangle .$$

Consequently, the reflection $r_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee\rangle \alpha_i$ has the same multiplicity as λ.

[Picture]

Now consider $M(\lambda)$ and $L(\Lambda)$ for $\Lambda \in P_+$. Actually, for general Λ, $M(\Lambda)$, while not necessarilly irreducible, has an (Ω-)composition series* by irreducibles $L(\lambda)$.

$$M(\Lambda) = V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_n = 0,$$

such that V_i/V_{i+1} is $\simeq L(\lambda_i)$ (i.e. an irreducible highest weight module) for some $\lambda_i = \Lambda - \beta_i$, $\beta_i \in Q$. For Kac-Moody algebras we also consider the case that $(V_i/V_{i+1}) = 0$ for all $\mu \in \Omega + Q_+$, (which is the Ω-composition series).

[Picture.]

For an Ω-composition series, as above, we have

$$\mathrm{ch}_{M(\Lambda)} - \sum_{\lambda\geq\Omega} \underbrace{[M(\Lambda) : L(\lambda)]}_{\substack{\#\text{ of times } L(\lambda)\\ \text{appears in the compos. series}}} \mathrm{ch}_{L(\lambda)} \in \langle e^\mu : \mu \not\geq \Omega\rangle$$

Sending "$\Omega \to -\infty$", the identity

$$\mathrm{ch}_{M(\Lambda)} = \sum_{\lambda\leq\Lambda} [M(\Lambda) : L(\lambda)]\mathrm{ch}_{L(\lambda)}$$

makes sense.

Notation. $b_{\Lambda,\lambda} = [M(\Lambda) : L(\lambda)]$.

Remark 6.12. Recall the partial order on weights that $\lambda \leq \Lambda$ if $\Lambda - \lambda \in Q = \sum \mathbb{Z}_+\alpha_i$. $b_{\Lambda,\lambda} = 1$ if $\lambda = \Lambda$ and $b_{\Lambda,\lambda} = 0$ if not $(\lambda \leq \Lambda)$.

If we choose a total order on $\mathfrak{h}^*$, compatible with $\leq$. Then $\{b_{\Lambda,\lambda}\}$ is a lower triangular matrix with 1 on the diagonal. We an define $\{m_{\Lambda,\lambda}\}$ the *inverse matrix*. It's again lower triangular, 1 on the diagonal, and all $m_{\Lambda,\lambda}$ are <u>integers</u> (maybe negative now). And we have

$$\mathrm{ch}_{L(\Lambda)} = \sum_{\lambda\leq\Lambda} m_{\Lambda,\lambda}\mathrm{ch}_{M(\lambda)}. \tag{6.12.1}$$

Example 6.13. $\mathfrak{sl}_2$. $M(3) \underset{L(3)}{\supset} M(-5) \underset{L(-5)}{\supset} 0$. Since $M(-5)$ is already irreducible.

[Missing. . .]

We want to discover $m_{\Lambda,\lambda}$. In general massively difficult. For $\Lambda \in P_+$, $m_{\Lambda,\lambda}$ easy. Multiply Eq. 6.12.1 by R

$$R\mathrm{ch}_{L(\Lambda)} = \sum_{\lambda \geq \Lambda} m_{\Lambda,\lambda} e^{\lambda}.e^{\rho} R\mathrm{ch}_{L(\Lambda)} = \sum_{\lambda \leq \Lambda} m_{\Lambda,\lambda} e^{\lambda+\rho}.$$

What's ρ? It's $\rho \in \mathfrak{h}^*$ chosen so that $\langle \rho, \alpha_i^\vee \rangle = 1$, and it's called the *Weyl vector*.

Remark 6.14. For $\mathfrak{g}$ finite-dimensional, $\rho = \sum_{i=1}^{\ell} \omega_i$ necessarily. (And equals $\frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha$.

For $\mathfrak{g}$ finite dimensional and the affine $\hat{\mathfrak{g}}$, $\hat{\rho} = h^\vee \Lambda_0 + \rho$ works.

For $w \in W = \langle r_i | i = 1, \ldots, \ell \rangle$, define $w(\mathrm{ch}_V) = \sum \dim V_\mu e^{W(\mu)}$. We saw $w(\mathrm{ch}_V) = \mathrm{ch}_V$ if V integrable. In particular $w(\mathrm{ch}_{L(\Lambda)} = \mathrm{ch}_{L(\Lambda)}$, $\Lambda \in P_+$.

Lemma 6.15. $m_{\Lambda,\lambda} = 0$ *unless* $\lambda + \rho = w(\Lambda + \rho)$ *for some* $w \in W$

Claim. $r_i(e^\rho R) = -e^\rho R$. So $w(e^\rho R) = \det(w) e^\rho R$ for all $w \in W$.

Proof.

$$R = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}.$$

Note that

(1) $\mathrm{mult}(r_i(\alpha)) = \mathrm{mult}(\alpha)$ for all $\alpha \in \Delta$. (Since $\mathfrak{g}$ is integrable!)
(2) $r_i(\Delta_+) = \{-\alpha_i\} \cup (\Delta_+ \setminus \{\alpha_i\}$.

Any $\alpha \in \Delta_+$ is of the form $\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$. If $\alpha \neq \alpha_i$, some $k_{j_0} \neq 0$, $j_0 \neq i$ and

$$\begin{aligned} r_i(\alpha) &= \sum_j k_j \alpha_j \, \langle \alpha, \alpha_i^\vee \rangle \, \alpha_i \\ &= \sum k'_j \alpha_j \\ &= e^\rho (e^{-\alpha_i} - 1) \left(\prod_{\alpha \in \Delta_+ \setminus \alpha_i} \right) \\ &= -e^\rho R. \end{aligned}$$

for $k'_{j_0} = k_{j_0} > 0$. Can't have a mixture of signs, so $r_i(\alpha) \in \Delta_\perp$.

So

$$r_i(e^\rho R) = \prod_{\alpha \in \Delta_+ \setminus \alpha_i} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)} \cdot (1 - e^{t\alpha_i} \cdot e^{\rho - \alpha_i}$$

$$r_i = \rho - \langle \alpha_i^\vee, \rho \rangle \, \alpha_i = \rho - \alpha_i$$

Hence

$$\sum_{\lambda \leq \Lambda} m_{\Lambda,\lambda} e^{\lambda+\rho}$$

is W-skew-invariant. (Lemma 6.15.)

Hence

$$\sum_{\lambda \leq \Lambda} m_{\Lambda,\lambda} e^{\lambda+\rho} = \sum_{w \in W} \det(w) \cdot e^{w(\Lambda+\rho)}$$

In conclusion, the *Weyl character formula*

$$\mathrm{ch}_{L(\Lambda)} = \sum_{w \in W} \det(w) \cdot \frac{e^{w(\Lambda+\rho)-\rho}}{R}$$

$\square$

Corollary (Weyl denominator formula).

$$e^{\rho} R = \sum_{w \in W} \det(w) e^{w(\rho)}.$$

Next time: Affine case, θ-functions. Modular forms (Poisson summation.)

7. Characters of integrable highest weight for affine Kac-Moody algebras

Can be calculated using the Weyl character formula.
Recall: $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$.
Dual Cartan $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$.
We consider weights $\Lambda = k\Lambda_0 + \lambda$, $\lambda \in \mathfrak{h}^*$, $k \in \mathbb{Z}_+$, $\lambda \in P_+^k$.
Simple roots: $\alpha_0 = \delta - \theta$, $\{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^*$.
The affine Weyl group is (the *semidirect* product!)

$$\widehat{W} = \langle r_0, \dots, r_\ell \rangle \cong W \times T$$

where $T = \{t_\alpha | \alpha \in Q\}$, where t_α is the translation

$$t_\alpha(\Lambda) = \Lambda + k\alpha - \left((\lambda, a) + k\frac{|\alpha|^2}{2} \right) \delta$$

$\Lambda = k\Lambda_0 + \lambda$.

Let's compute $\chi_{L(\Lambda_0)}$ using the Weyl character formula

$$\chi_{L(\Lambda)} = \frac{\sum_{w \in \tilde{W}} \varepsilon(w) e^{W(\Lambda+\rho)-\rho}}{R}$$

Firstly $R = \prod_{\alpha \in \hat{\Delta}} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}$
Recall

$$\hat{\Delta}_+ = \{m\delta | m \in \mathbb{Z}_{\geq 1}\} \cup \{\alpha_1 + m\delta | m \in \mathbb{Z}_{\geq 0}\} \cup \{-\alpha_1 + m\delta | m \in \mathbb{Z}_{\geq 1}\}$$

So let's write $e^{m\delta} = q^m$ (i.e. $q = e^{-\delta}$, is a symbol) and $y = e^{\alpha_1}$.
So

$$R = \prod_{n=1}^{\infty} \underbrace{(1 - y^{-1} q^{n-1})}_{\alpha+(n-1)\delta} \underbrace{(1 - q^n)}_{n\delta} \underbrace{(1 - yq^n)}_{-\alpha+n\delta}$$

Now let's express numerator also in terms of q and y

$$\hat{\rho} = h^\vee \Lambda_0 + \rho$$

$h^\vee$ dual Coxeter number for $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}_2$, $h^\vee = 2$. (For $\mathfrak{g} = \mathfrak{sl}_n$, $h^\vee = n$ and $\mathfrak{g} = E_8$, $h^\vee = 30$.)
For $\mathfrak{sl}_2$, $\rho = \frac{1}{2}\alpha_1 = \omega_1$.
$\Lambda = \Lambda_0$, $\Lambda + \rho = 3\Lambda_0 + \omega_1$.

Using the formula, we find

$$t_{m\alpha_1}(3\Lambda_0+\omega_1)-(\Lambda_0+\omega_1)= \\ \dots,\Lambda_0-3\alpha,-2\delta,\Lambda_0,\Lambda_0+3\alpha_1-4\delta,\dots$$

[Picture]
$\hat{W}=T\cup T\sigma$, $\sigma=r_1$ finite reflection.

Notation. $w(\Lambda+\rho):=w\circ\Lambda$.

One finds

$$\sum_{w\in\widehat{W}}\varepsilon(w)e^{w\circ\Lambda_0}=e^{\Lambda_0}\left(\underbrace{1+y^3q^4+y^{-3}q^2+\dots}_{+\text{ signs because } w\in T}\ \underbrace{-y^{-1}-y^2q^2-\dots}_{-\text{ signs because } w\in T_\sigma}\right)$$

We find explicitly

$$\chi_{L(\Lambda_0)}=e^{\Lambda_0}\frac{\sum_{m\in\mathbb{Z}}y^{3m}q^{3m^2+m}-\sum_{m\in\mathbb{Z}}y^{3m-1}q^{3m^2-m}}{\prod_{n=1}^\infty(1-y^{-1}q^{n-1})(1-q^n)(1-yq^n)}$$

Exercise 7.1. Put this formula in mathematica and confirm that [Picture]

$$\chi_{L(\Lambda_0)}=e^{\Lambda_0}\left(1+q(y^{-1}+1+y)+q^2(y^{-1}+2+y)+(q^3(2y^{-1}+3+2y)+\dots\right)$$

Appears that the central column here are *partitions*, i.e. $p(n)=\#$ of partitions of n. To see why recall the generating function $\sum_{n=0}^\infty q^np(n)=\prod_{k=1}^\infty\frac{1}{1-q^k}$, and expand.

It would appear that

$$\chi_{L(\Lambda_0)}=\prod_{k=1}^\infty\frac{1}{1-q^k}\cdot\sum_{m\in\mathbb{Z}}y^mq^{m^2}$$

This identity is true, and we will see it has a vertex-algebra interpretation.

Remark 7.2. If we compute $L(0)=1$ using the formula, we obtain

$$\prod_{n=1}^\infty(1-yq^{n-1})(1-q^n)(1-y^{-1}q^n)=\sum(\text{exercise})$$

This identity is called the *Jacobi triple product identity.*

These functions

$$\sum_{n\in\mathbb{Z}}q^{n^2},\sum_{n\in\mathbb{Z}}y^nq^{n^2},\sum_{n\in\mathbb{Z}}y^{3n}q^{3n^2+n},\text{ etc}\dots$$

are all examples of *θ-functions.*

Remark 7.3. In the formula for $\chi_{L(\Lambda_0)}(y,q)$, we could put $y=1$ to get

$$\chi_{L(\Lambda_0)}(q)=1+3q+4q^2+7q^3+\dots \\ =\prod\frac{1}{1-q^k}\cdot\sum_{m\in\mathbb{Z}}q^{m^2}$$

If one looks at $L(\Lambda_0+\omega_1)$ (the other $\Lambda\in P_+^1$),

$$\chi_{L(\Lambda+\omega_1)}=\prod\frac{1}{1-q^k}\sum_{m\in\mathbb{Z}}q^{(m+1/2)^2-1/4}$$

8. θ-FUNCTIONS

Let's consider

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}, \qquad q = e^{2\pi i\tau}$$

This converges (absolutely on compact regions in) the comain $\mathrm{Im}(\tau) > 0$.

Consider the Fourier transform

$$\hat{g}(y) = \int_{-\infty}^{\infty} g(x)e^{2\pi ixy}dx$$

Theorem 8.1 (Poisson summation)**.**

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \hat{g}(n)$$

Let's take $g(x,t) = e^{-\pi tx^2}$. Then $\theta(it) = \sum_{n\in\mathbb{Z}} g(n,t)$.

In this case

$$\hat{g}(y) = \sqrt{t}e^{-\pi y^2/t}$$

(integral of Gaussian).

So we conclude that

$$\theta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\theta(\tau)$$

Note also that

$$\theta(\tau+2) = \theta(\tau)$$

because $q^{\frac{1}{2}} = e^{\pi i\tau}$.

So $\theta(\tau)$ is an example of a modular for.

What is a modular form? Let

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{Z}, \det = 1 \right\},$$

which acts on $\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau+b}{c\tau+d}.$$

Definition 8.2. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ or some subgroup. A *(weak) modular form (of weight k)* is $f : \mathbb{H} \to \mathbb{C}$ holomorphic such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau). \tag{8.2.1}$$

If we demand that Γ contains $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ for some $N \geq 1$, (so $f(\tau+N) = f(\tau)$, and so $f(\tau) = \sum_{n=-\infty}^{\infty} a(n)q^{2\pi in/2}$), for $f : \mathbb{H} \to \mathbb{C}$ satisfying Eq. 8.2.1 and such that $f(\tau)$ is "meromorphic at cusps", i.e. $f(\tau) = \sum_{n\geq N_0} a(n)q^{n/N}$, etc.

Finally we would add a factor

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(A)(c\tau+d)^k f(\tau).$$

So in particular a weight-0 modular form is a modular function.

Denote $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So $S\tau = -\frac{1}{\tau}$, and denote $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so $T\tau = \tau + 1$.

Then $\Gamma = \langle S, T^2 \rangle \subset \mathrm{SL}_2(\mathbb{Z})$.

We saw

$$\begin{aligned} \theta(T^2\tau) &= \theta(\tau) \\ \theta(S\tau) &= \sqrt{\frac{\tau}{i}}\theta(\tau) = i^{-1/2}(c\tau + d)^{1/2}\theta(\tau). \end{aligned}$$

So $\theta(\tau)$ is a modular form of weight 1/2, for the group $\Gamma = \langle S, T^2 \rangle$ with multiplier system $\varepsilon : \Gamma \to \mathbb{C}^\times$ defined by

$$\begin{aligned} \varepsilon(S) &= i^{-1/2} \\ \varepsilon(T^2) &= 1 \end{aligned}$$

Theorem 8.3. *The Dedekind eta function*

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = e^{2\pi i \tau}$$

is also a modular form of weight $1/2$ *for* $SL_2(\mathbb{Z}) = \langle S, T \rangle$.

$$\begin{aligned} \eta\left(-\frac{1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}\eta(\tau) \\ \eta(T\tau) &= e^{2\pi i/24}\eta(\tau). \end{aligned}$$

9. Vertex algebras

(1) Recall $\hat{\mathfrak{a}} = \mathbb{C}K \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C}h_n$ the *Heisenberg Lie algebra*, or *oscillator Lie algebra*,

$$[h_m, h_n] = m\delta_{m,-n}K, \qquad [K, \hat{\mathfrak{a}}] = 0.$$

Remark 9.1. "It's the simplest case of an affine Lie algebra: a 1-dimensional Lie algebra with a bilinear form". It's an example of the affine Lie algebra construction $\mathfrak{g} \rightsquigarrow \hat{\mathfrak{g}}$, where now $\mathfrak{a} = \mathbb{C}$, and

$$\begin{aligned} (\cdot, \cdot) : \mathfrak{a} \times \mathfrak{a} &\longrightarrow \mathbb{C} \\ (1, 1) &\longmapsto 1 \end{aligned}$$

(2) Witt Lie algebra

$$\begin{aligned} W &= \bigoplus_{n \in \mathbb{Z}} \mathbb{C}D_n, \\ [D_m, D_n] &= (m - n)D_{m+n}. \end{aligned}$$

The oscillator algebra $\hat{\mathfrak{a}}$ has a representation which we have seen already:

$$H = \mathbb{C}[x_1, x_2, \ldots]$$

$$h_n \mapsto \begin{cases} n\frac{\partial}{\partial x_n} & \text{if } n > 0 \\ x_{-n} & \text{if } n < 0 \\ 0 & \text{if } n = 0 \end{cases}, \qquad K \mapsto \mathrm{Id}$$

A picture of H [Picture].

Here I am introducing a grading of H:

$$H = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} H_n, \qquad H_n = \operatorname{span}\{x_{m_1}, \ldots, x_{m_s} | \sum m_s = n\}.$$

Remark 9.2. $\dim(H_n) = \#\{\text{integer partitions of } n\} = p(n)$ and $\sum_{n=0}^{\infty} \dim(H_n) q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$.

Remark 9.3 (Verma module style). H can be presented alternatively as

$$H = U(\hat{\mathfrak{a}}) \otimes_{U(\hat{\mathfrak{a}}_+)} \mathbb{C}1$$

where $\mathbb{C}1$ is a 1-dimensional representation of

$$\hat{\mathfrak{a}}_+ = \bigoplus_{n \geq 0} \mathbb{C}h_n \oplus \mathbb{C}K.$$

$$h_n \cdot 1 = 0 \qquad \forall n \geq 0, \qquad K \cdot 1 = 1$$

Exercise 9.4. H is irreducible.

Proof. We need to prove that there is no vector subspace $V \subset \mathbb{C}[x_1, x_2, \ldots] = H$ such that $xV \subset V$ for all $x \in \hat{\mathfrak{a}}$. And then the proof is basically noticing that the orbit of the action of $\hat{\mathfrak{a}}$ on H is all of H. That is, if we had proper subspace of H, we can always find an element of $\hat{\mathfrak{a}}$ that takes some element in V to the one of the elements that is not in V. Indeed, by applying h_n for different values of positive n we can take any element of V to a constant. Then we apply h_n for different values of negative n to obtain any monomial. Then we add these monomials and obtain any polynomial in H. □

Notation. Instead of 1 let's write $|0\rangle$.

Remark 9.5. We can easily generalise H to

$$H^\mu = U(\hat{\mathfrak{a}}) \otimes_{U(\hat{\mathfrak{a}}_+)} \mathbb{C}|\mu\rangle,$$

($\mu \in \mathbb{C}$), where $\hat{\mathfrak{a}}_+$ is the same, but now

$$h_n|\mu\rangle = \begin{cases} 0 & \text{if } n > 0 \\ \mu \cdot |\mu\rangle & \text{if } n = 0 \end{cases} \qquad K|\mu\rangle = |\mu\rangle.$$

H^μ is again a (irreducible) $\hat{\mathfrak{a}}$-module.

Generalise even more: Let $\mathfrak{h}$ be a finite-dimensional vector space, and $(\cdot, \cdot) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ a symmetric bilinear form. $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$,

$$[at^m, bt^n] = m(a, b)\delta_{m,-n} K, \qquad [K, \hat{\mathfrak{h}}] = 0.$$

Let $\mu \in \mathfrak{h}$. Define $H^\mu = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}})} \mathbb{C}|\mu\rangle$, $\hat{\mathfrak{h}}_+ = \mathfrak{h}[t] \oplus \mathbb{C}K$, and

$$at^m \cdot |\mu\rangle = \begin{cases} (\mu, a)|\mu\rangle & \text{if } m = 0 \\ 0 & \text{if } m > 0. \end{cases}$$

Returning to $\hat{\mathfrak{a}} \curvearrowright H = H^0$. Introduce

$$L_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} h_{m-k} h_k.$$

If $m \neq 0$, this sum is well-defined in $\operatorname{End}(H)$.

Indeed, h_{m-k} and h_k commute ($m \neq 0$), and $\forall p(\underline{x}) \in H$, there exists N such that $p(\underline{x}) = p(x_1, \ldots, x_{N-1})$, then $h_k \cdot p(x) = 0$ $\forall k \geq N$, and $h_{m-k} \cdot p(x) = 0$ $\forall m-k \geq N$.

So $(\sum h_{m-k}h_k)p(\underline{x})$ is a <u>finite</u> sum. Of course, the number of terms in the sum depends on $p(x)$ and can be arbitrarily large.

Exercise 9.6. If m, n and $m+n$ are not zero, then

$$[L_m, L_n] = (m-n)L_{m+n}$$

holds in $\mathrm{End}(H)$.

(Trying to be a representation of W

$$D_n \mapsto L_n \in \mathrm{End}(H).)$$

But $\sum_{k \in \mathbb{Z}} h_{-k}h_k$ is not well-defined. Indeed,

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}} h_{-k}h_k\right)1 &= \sum_{k\geq 1} k\frac{\partial}{\partial x_k}(x_k 1) \\ &= (1+2+3+4+\ldots)1\end{aligned}$$

!

Normal ordering idea ("that physicist do"). Let's cheat and redefine the product as

$$:h_{-k}h_k: = \begin{cases} h_{-k}h_k & \text{if } k \geq 0 \\ h_k h_{-k} & \text{if } k < 0. \end{cases}$$

Now $\sum_{k\in\mathbb{Z}} :h_{-k}h_k:$ is well-defined!

Notation. Let us consider series of the form

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}, \qquad a_{(n)} \in \mathrm{End}(V),$$

where V is a vector space.

Definition 9.7 (Most important in the course)**.** We say $a(z)$ is a *quantum field* on V if, for every $v \in V$ there exists $N \in \mathbb{Z}$ such that $a_{(n)}v = 0$ $\forall n \geq N$.

Example 9.8. $V = H = \mathbb{C}[x_1, x_2, \ldots]$,

$$a(z) = h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}.$$

$$h(z) = \ldots + 3z^{-4}\frac{\partial}{\partial x_3} + 2z^{-3}\frac{\partial}{\partial x_1} + x_1 + x_2 z + x_3 z^2 + \ldots$$

Any fixed $v \in H$ is $v = p(x_1, \ldots, x_{N-1})$ for some N. Then $h_N v = N\frac{\partial}{\partial x_N}v = 0$.

So $h(z) \curvearrowright H$ is a quantum field.

Definition 9.9. For a quantum field (or any series in fact) $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$, define the *creation* and *annihilating operators*

$$a(z)_+ = \sum_{n\leq -1} a_{(n)}z^{-n-1}, \qquad a(z)_- = \sum_{n\geq 0} a_{(n)}z^{-n-1}.$$

The quantum field condition solves the problem of the infinite series...

The normally order product of quantum fields $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$, and $b(z) = \sum_{n\in\mathbb{Z}} b_{(n)}z^{-n-1}$ is.

$$:a(z)b(z): = a(z)_+b(z) + b(z)a(z)_-.$$

Exercise 9.10 (Most important of the course)**.** Show that if $a(z)$ and $b(z)$ are quantum fields, then $:a(z)b(z):$ is well-defined, and is a quantum field.

Next idea: in our example of $h(z)\curvearrowright H$, the coefficients h_n were coming from a Lie algebra, so we had relations $[h_m, h_n] = (\ldots)$. (Indeed, $[h_m, h_n] = m\delta_{m,-n}\mathrm{Id}_H$ in this case.) We sould like to interpret such relations at the level of $h(z)$.

$$\begin{aligned}
[h(z), h(z)] &= h(z)h(z) - h(z)h(z) \\
&= \sum_p \sum_{m+n=p} h_m h_n z^{-(m+n)-2} - \sum_{m+n=p} h_n h_m z^{-(m+n)-2} \\
&= \sum_p z^{-p-2}\left(\underbrace{\sum_{m+n=p} [h_m, h_n]}_{\text{infinite?}}\right)
\end{aligned}$$

which is a bad idea, since that sum can be infinite. Better idea: change a variable:

$$[h(z), h(w)] = \sum_{m,n\in\mathbb{Z}} [h_m, h_n] z^{-m-1}w^{-n-1}.$$

In this example,

$$\begin{aligned}
[h(z), h(w)] &= \sum_{m,n\in\mathbb{Z}} m\delta_{m,-n} z^{-m-1}w^{-n-1} I_H \\
&= \sum_{m\in\mathbb{Z}} z^{-m-1}w^{m-1} I_H.
\end{aligned}$$

Observe (geometric series):

$$\begin{aligned}
\frac{1}{z-w} &= \sum_{k\geq 0} z^{-k-1}w^k \qquad \text{(convergent for } |z| > |w|) \\
\frac{1}{z-w} &= -\sum_{k\geq 0} w^{-k-1}z^k \\
&= -\sum_{k<0} z^{-k-1}w^k \qquad \text{(convergent for } |z| < |w|)
\end{aligned}$$

So, in a sense

$$\sum_{k\in\mathbb{Z}} z^{-k-1}w^k \text{“}=\text{”} \frac{1}{z-w} - \frac{1}{z-w}.$$

This motivates us to introduce the expression

Definition 9.11. The *formal delta function* is

$$\delta(z, w) = \sum_{k\in\mathbb{Z}} z^{-k-1}w^k \in \mathbb{C}[\![z, z^{-1}, w, w^{-1}]\!].$$

Why delta function? It behaves like Dirac delta. Recall that $\delta(x)$ is the *distribution* on $\mathbb{R}$ defined by $\delta[f(x)] = f(0)$ for all test functions $f(x)$. Every function $k(x)$ gives a distribution D_k.

$$D_k[f] = \int_{-\infty}^{\infty} k(x)f(x)dx,$$

$f \in C_c^\infty(\mathbb{R})$.

If δ were of the form D_k (it's not) it would have to look like [Picture, positive part of y axis, all x axis.

One can show that (it's a theorem by Plamelj)

$$\delta(x) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left(\frac{1}{x - i\varepsilon} - \frac{1}{x - i\varepsilon} \right)$$

as distributions (i.e. limit taken in "distributional sense".)

(So that's why we called the delta function that way.)

Denote by $i_{z,w}$ the "expansion in positive powers of w". E.g.

$$i_{z,w} \frac{1}{z-w} = \sum_{k \geq 0} z^{-k-1} w^k,$$

similarly,

$$i_{z,w} \frac{1}{(z-w)^2} = \sum_{k \geq 0} k z^{-k-1} w^{k-1},$$

$$i_{z,w}(w^{-2}) = w^{-2}.$$

Exercise 9.12. So, in fact,

$$[h(z), h(w)] = i_{z,w} \frac{1}{(z-w)^2} - i_{w,z} \frac{1}{(z-w)^2} = \partial_w \delta(z, w).$$

What exactly is $i_{z,w}$?

Notation.

$\mathbb{C}[z]$	ring of polynomials
$\mathbb{C}[z, z^{-1}]$	ring of Laurent polynomials
$\mathbb{C}[\![z]\!]$	ring of power series
$\mathbb{C}[\![z, z^{-1}]\!]$	vector space (not ring!) of formal distributions
$\mathbb{C}(\!(z)\!)$	field of Laurent series

where the last is $\sum_{n \geq N} f a_n z^n$ for some N.

Since $\mathbb{C}(\!(z)\!)$ is a field, $\mathbb{C}(\!(z)\!)(\!(w)\!)$ is also a field.

There are natural inclusions

$$\mathbb{C}[z, w] \to \mathbb{C}(\!(z)\!)(\!(w)\!) \to \mathbb{C}[\![z^{\pm 1}, w^{\pm 1}]\!].$$

Let $\mathbb{C}(z, w)$ denote the fraction field of the domain $\mathbb{C}[z, w]$. By the property of $\mathbb{C}(z, w)$, there exists an embedding

$$\begin{tikzcd} \mathbb{C}[z,w] \arrow[rr] \arrow[dr] & & \mathbb{C}(\!(z)\!)(\!(w)\!) \\ & \mathbb{C}(z,w) \arrow[ur, "i_{z,w}"] & \end{tikzcd}$$

The following diagram does not commute: [diagram]

We computed

$$[h(z), h(w)] = \partial_w \delta(z, w)$$

We can similarly compute

$$h(z)h(w) = :h(z)h(w): + i_{z,w}\frac{1}{(z-w)^2}$$
$$h(w)h(z) = :h(z)h(w): + i_{w,z}\frac{1}{(z-w)^2}.$$

Notation. We write $\partial_w = \frac{\partial}{\partial w}$ and $\partial_w^{(j)} = \frac{1}{j!}\partial_w^j$.

Lemma 9.13. (1) *If we multiply the delta function with $(z-w)$ we get zero, that is,*

$$(z-w)\delta(z, w) = 0.$$

(2)

$$i_{z,w}\frac{1}{(z-w)^{j+1}} - i_{w,z}\frac{1}{(z-w)^{j+1}} = \partial_w^{(j)}\delta(z, w).$$

(3) $\forall j \geq 0$,

$$(z-w)^{j+1}\partial_w^{(j)}\delta(z, w) = 0.$$

(4) *Whenever $m \leq j$,*

$$(z-w)^m \partial^{(j)}(z, w) = \partial_w^{(j-m)}\delta(z, w).$$

Proof. □

Remark 9.14. All this is completely parallel to the Dirac δ-distribution $\delta(x)$.

$$x\delta(x) = 0, \qquad x\delta'(x) = \delta(x), \text{ etc.}, x^2\delta'(x) = 0.$$

10. The residue pairing

Let $f(z) \in \mathbb{C}[z, z^{-1}]$, which is a vector space with basis $\{z^n : n \in \mathbb{Z}\}$.

An element of the dual vector space is a formal linear combination

$$\sum_{n\in\mathbb{Z}} c_n\varphi_n, \qquad \text{where } \varphi_n(z^k) = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

(no restriction on the $c_n \in \mathbb{C}$).

Identify $\sum c_n\varphi_n$ with

$$c(z) = \sum c_n z^{-n-1} \in \mathbb{C}[z^{\pm 1}$$

so that φ acts on $f(z) = \sum f_n z^n \in \mathbb{C}[z^{\pm 1}]$, as

$$\varphi(f) = \sum_n c_n f_n = \mathrm{Res}_z c(z) f(z) dz,$$

where

Definition 10.1. If U is a vector space and $a(z) = \sum_n a_n z^n \in U[\![z^{\pm 1}]\!]$,

$$\mathrm{Res}_z a(z) dz = a_{-1}$$

Let's record a few properties of $\mathrm{Res}_z(\cdot)dz := \mathrm{Res}_z(\cdot)$.

Lemma 10.2. (1) *$Res_z f(z)\delta(z, w) = f(w)$.*

(2) $Res_z f(z)(\partial_z g(z)) = -Res_(\partial_z f(z)) f g(z)$.

Proof. (1) Straightforward computation.

(2) Product rule, and uses that derivatives have no residues.

□

So, in our example, $H, h(z)$ we see that

$$(z-w)^2[h(z), h(w)] = 0,$$

which is a nontrivial example: we really need to take the square, and also the bracket is not zero.

Definition 10.3. Two quantum fields $a(z)$ and $b(z)$ on a vector space V are *mutually local* if there exists N such that

$$(z-w)^N[a(z), b(w)] = 0.$$

Remark 10.4. See [**?**, Chapter 1] for physics motivation. More generally, if $D = \sum_{m,n} D_{m,n} z^m w^n \in U[\![z^{\pm 1}, w^{\pm 1}]\!]$, we call D *local* if $(z-w)^N D = 0$ for some N.

Proposition 10.5. *Let $a(z), b(z)$ be a local pair of quantum fields on V. Then*

$$a(z)b(w) = :a(z)b(z): + \sum_{j=0}^{N-1} c_j(w) i_{z,w} \frac{1}{(z-w)^{j+1}} \tag{10.5.1}$$

and

$$b(w)a(z) = :a(z)b(z): + \sum_{j=0}^{N-1} c_j(w) i_{w,z} \frac{1}{(z-w)^{j+1}} \tag{10.5.2}$$

In particular,

$$[a(z), b(w)] = \sum_{j=0}^{N-1} c_j(w) \partial_w^{(j)} \delta(z,w). \tag{10.5.3}$$

Furthermore

$$c_j(w) = Res_z (z-w)^j [a(z), b(w)]. \tag{10.5.4}$$

Proof. First we prove 10.5.3 using 10.5.4. □

Exercise 10.6. Deduce 10.5.1 and 10.5.2 from 10.5.3.

Notation. If $a(w)$ and $b(w)$ are quantum fields on V, we denote

$$a(w)_{(j)} b(w) = \mathrm{Res}_z (z-w)^j [a(z), b(w)]$$

for $j \in \mathbb{Z}_{\geq 0}$.

Example 10.7. On H, $[h(z), h(w)] = \partial_w \delta(z,w) I_H$. So $\mathrm{Res}_z \partial_w \delta(z,w) I_H = 0$ and $\mathrm{Res}_z (z-w) \partial_w \delta(z,w) I_H = \mathrm{Res}_z \delta(z,w) I_H = I_H$.

Note that I_H is a (very simple) quantum field.

The following definition generalizes the j-product to any integer, possible negative.

Definition 10.8. For $n \in \mathbb{Z}$, and $a(w)$, $b(w)$ quantum fields, we define the n^{th} *product* $a(w)_{(n)} b(w)$ as

$$a(w)_{(n)} b(w) = \mathrm{Res}_z [i_{z,w} (z-w)^n a(z) b(w) - i_{w,z} (z-w)^n b(w) a(z)].$$

Remark 10.9. (1) We recover the prior definition: if $n \geq 0$,

$$i_{z,w}(z-w)^n = i_{w,z}(z-w)^n = \sum_{r=0}^{n} \binom{n}{r} z^{n-r}(-w)^r,$$

a finite sum.

(2) (Exercise.) If $n = 1$, then

$$\begin{aligned} a(w)_{(-1)}b(w) &= a(w)_+ b(w) + b(w)a(w)_- \\ &=: a(w)b(w): \end{aligned}$$

(3) (Exercise.) The more negative products are not something new: for $k \geq 0$,

$$a(w)_{(-k-1)}b(w) =: (\partial_w^{(k)} a(w))b(w):$$

We have actually already used the following proposition:

Proposition 10.10. *If $a(w), b(w)$ are quantum fields, then $a(w)_{(n)}b(w)$ is also a quantum field for all $n \in \mathbb{Z}$.*

Definition 10.11. A *vertex algebra* consists of a vector space V, a set $\mathcal{F}$ of quantum fields on V, a nonzero vector $|0\rangle \in V$, and a linear map $T : V \to V$, such that

(1) $T|0\rangle = 0$, and $[T, a(z)] = \partial_z a(z)\ \forall a(z) \in \mathcal{F}$.
(2) V is spanned by $a^{i_1}_{(n_1)}, \ldots, a^{i_s}_{(n_s)}|0\rangle$, where $a^{i_j}(z) \in \mathcal{F}$ (and $s(z) = \sum a_{(n)} z^{-n-1}$ always).
(3) All pairs $a(z), b(z) \in \mathcal{F}$ are mutually local. (We saw (ref?) that this is equivalent to $[a(z), b(w)] = \sum_{j=0}^{N-1} c_j(w)\partial_w^{(j)}\delta(z,w)$ and described the coefficients $c_j, \ldots$)

We call $|0\rangle$ the *vacuum vector* and T the *translation operator*.

In fact, our Heisenberg example that we've been discussing so far is an example:

Example 10.12. $V = H = \mathbb{C}[x_1, x_2, \ldots]$, $\mathcal{F} = \{h(z)\}$, $|0\rangle = 1$, $T = ?$, is a vertex algebra.
Answer 1. Recall $L(z) = \frac{1}{2} :h(z)h(z): = \frac{1}{2}h(z)_{(-1)}h(z)$, $L(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-z}$. In particular, $L_{-1} = \frac{1}{2}\sum_{k\in Z} h_k h_{-1-k}$. Well, $T = L_{-1}$.
Answer 2. Given that T must satisfy $T1 = 0$, and $[T, h(z)] = \partial_z h(z)$, i.e.

$$[T, h_n] = -nh_{n-1},$$

and H is generated from 1 by $\{h_n | n \leq -1\}$, the action of T on H (if well-defined) is completely determined.

Exercise 10.13. Write a formula for $T(x_1^{m_1} x_2^{m_2} \ldots x_s^{m_2})$.

11. A second definition of vertex algebra

Now we aim for a second definition of vertex algebras.

Let $(V, |0\rangle, T, \mathcal{F})$ be a vertex algebra. Define

$$\begin{aligned}
\tilde{\mathcal{F}} &= \{\text{quantum fields } a(z) \text{ on } V | [T, a(Z)] = \partial_z a(z)\} \\
\overline{\mathcal{F}} &= \bigcup_{k \geq 0} \mathcal{F}_k, \\
\mathcal{F}_0 &= \{I_V\}, \\
\mathcal{F}_k &= \{a(z)_{(n)} b(z) | a(z) \in \mathcal{F}, b(z), \mathcal{F}_{k-1}\} \\
\mathcal{F}' &= \{b(z) \in \tilde{\mathcal{F}} | a(z), b(z) \text{ is local}, \forall a(z) \in \mathcal{F}\}
\end{aligned}$$

Where I_V is the identity of V considered as a quantum field. The idea is that this is like taking a subalgebra and taking the commutator over and over again.

Lemma 11.1. *$a(z)_{(-1)} I_V = a(z)$.*

Proof. Direct calculation. $\square$

Lemma 11.2. *$a(z)_{(-n-1)} I_V = \partial_z^{(n)} a(z)$.*

Proof. Similar. $\square$

Lemma 11.3 (Dong)**.** *Suppose $a(z), b(z)$ and $c(z)$ are mutually local in pairs. Let $n \in \mathbb{Z}$. Then $a(z)_{(n)} b(z)$ and $c(z)$ is a local pair.*

Proof. Done in lecture. $\square$

Therefore

$$\mathcal{F} \subset \overline{\mathcal{F}} \overset{\text{Dong}}{\subset} \mathcal{F}' \subset \tilde{\mathcal{F}}$$

Remark 11.4. To be sure, we should check $\overline{\mathcal{F}} \subset \tilde{\mathcal{F}}$.

F tilde is the translation invariant

Exercise 11.5. If $[T, a(z)] = \partial_z a(z)$, $[T, b(z)] = \partial_z b(z)$, then

$$[T, a(z)_{(n)} b(z)] = \partial_z (a(z)_{(n)} b(z)).$$

Theorem 11.6. (1) *For any $a(z) \in \tilde{\mathcal{F}}$,*

$$\begin{aligned}
a(z)|0\rangle &= v + z \cdot Tv + \frac{z^2}{2} T^2 v + \dots \\
&= e^{2T} \qquad (v \in V)
\end{aligned}$$

Thus we define

$$\begin{aligned}
s : \tilde{\mathcal{F}} &\longrightarrow V \\
a(z) &\longmapsto a(z)|0\rangle|_{z=0}
\end{aligned}$$

(2)

$$s(z(z)_{(n)} b(z) = a_{(n)} s(b(z)).$$

(3) *Let $\mathcal{G} \subset \tilde{\mathcal{F}}$ be such that $s(\mathcal{G}) = V$, and suppose $a(z)$ is local with all $b(z) \in \mathcal{G}$. "If you commute with a large enough bunch of guys, you are close enough to being zero." If $s(a(z)) = 0$, then $a(z) = 0$.*

Proof. (1) We need to prove $a(z)|0\rangle$ does not have negative powers of z. So suppose it does, i.e. there is $n \geq 0$ such that $a_{(n)}|0\rangle z^{-n-1} \neq 0$.

Idea: use translation invariance to pair n and $-n$, and then the fact that $a(z)$ is a quantum field to get some vanishing.

So first we translate and find that:

$$Tv(z) = Ta((z)|0\rangle = (Ta(z) - a(z)T)|0\rangle = \partial_z a(z)|0\rangle$$

so that

$$v_n = \frac{1}{n} T v_{n-1}, \qquad (n \neq 0). \tag{11.6.1}$$

So, $v_n \neq 0$ for some $n < 0 \implies v_{n-1}$ also not zero. But we also know by $a(z)$ being a quantum field that there exists N such that $a_{(n)}|0\rangle = 0$ for all $n \geq N$. So, we'd have a contradiction. So $v(z) = \sum_{n\geq 0} v_n z^n$.

Now Eq. 11.6.1 also implies

$$v_1 = Tv_0, v_0 = \frac{1}{2} T v_1, \ldots, v_n = \frac{1}{n!} T^n v_0.$$

So $a(z)|0\rangle = e^{zT} v_0$ ($v_0 \in V$).

(2) Consider

$$a(w)_{(n)} b(w)|0\rangle = \mathrm{Res}_z(a(z)b(w)i_{z,w}(z-w)^n - \underbrace{b(w)a(z)i_{w,z}(z-w)^n}_{=0}$$

where that vanishing is because the $i_{w,z}$ expands the argument in positive powers, and $b(w)a(z)$ also has only positive powers since $a(z)|0\rangle$ also has only positive powers (and the residue picks the coefficient of the power -1). So we obtain

$$= \mathrm{Res}_z a(z) b(w) i_{z,w}(z-w)^n.$$

But

$$\begin{aligned} s(a(w)_{(n)} b(w)) &= (\mathrm{Res}_z s(z) b(w) i_{z,w}(z-w)^n |0\rangle)\,|_{w=0} \\ &= \mathrm{Res}_z a(z) b(w) z^n |0\rangle|_{z=0} \\ &= \mathrm{Res}_z z^n a(z)(s(b(w))) \\ &= \mathrm{Res}_z z^n \sum_k a_{(k)} z^{-k-1} s(b) \\ &= a_{(n)} s(b). \end{aligned}$$

(3) By locality, there exists N such that

$$\begin{aligned} (z-w)^N a(z) b(w)|0\rangle &= (z-w)^N b(w) a(z)|0\rangle \\ &= (z-w)^N b(w) e^{2T} s(a) = 0 \end{aligned}$$

$$\begin{aligned} (z-w)^N a(z) b(w)|0\rangle|_{w=0} &= 0 \\ \implies z^N a(z) s(b) &= 0 \\ \implies a(z) v &= 0 \\ \implies a(z) &= 0 \end{aligned}$$

$\square$

Corollary.

(1) $S|_{\mathcal{F}'}$ is injective.
(2) $S|_{\overline{\mathcal{F}}}$ is surjective.
(3) $S|_{\mathcal{F}}$ is isomorphism $\overline{\mathcal{F}} \to V$.
So we define the inverse (linear) map $Y : V \to \overline{\mathcal{F}}$, i.e. for all $a \in V$, $Y(a,z)$ denotes $s^{-1}(a) \in \overline{\mathcal{F}}$.
(4) $(V, |0\rangle, T, \overline{\mathcal{F}})$ is a vertex algebra, and

$$Y(a_{(n)}b, z) = Y(a,z)_{(n)}Y(b,z) \qquad \forall a, b \in V.$$

So we have

$$\begin{tikzcd} \mathcal{F} \arrow[r] & \overline{\mathcal{F}} \arrow[r] \arrow[d, two heads] & \mathcal{F}' \arrow[r] \arrow[d, hook] & \tilde{\mathcal{F}} \arrow[d, "s"] \\ & V \arrow[r, "\mathrm{id}"] & V \arrow[r, "\mathrm{id}"] & V \end{tikzcd}$$

Proof. (1) Now we apply part 3 in the theorem to $a(z) \in \mathcal{F}'$, $\mathcal{G} = \overline{\mathcal{F}}$. If $s(a(z)) = 0$, then $a(z) = 0$. So $s : \mathcal{F}' \to V$ is injective (as in the diagram).
(2) By iterating the property in the theorem that $a_{(n)}s(b(z)) = s(a(z)_{(n)}b(z))$ so we can take out "one by one" in the following product to obtain

$$a^1_{(n_1)} \dots a^s_{(n_s)}|0\rangle = s(a^1(z)_{(n)} \dots a^s(z)_{(n_s)}I_V$$

This proves the double headed arrow in the diagram.
(3) Now define $Y(-,z) : V \to \overline{\mathcal{F}}$.

$$s(Y(a,z)_{(n)}Y(b,z)) = a_{(n)}s(Y(b,z)) = a_{(n)}b.$$

(4) Then all the axioms in our definition of vertex algebra are satisfied!

□

Remark 11.7. Let $(V, |0\rangle, T, \mathcal{F})$ be a vertex algebra. $\overline{\mathcal{F}}$ and Y as above. $\forall a, b \in V$,

$$[Y(a,z), Y(b,w)] = \sum_{j=0}^{N-1} c_j(w)\partial_w^{(j)}\delta(z,w). \tag{11.7.1}$$

$c_j(w) = Y(a,w)_{(j)}Y(b,w) = Y(a_{(j)}b, w)$.

Which is like something we had written before. But now we know more. Let us extract the z^{-m-1}, w^{-n-1} coefficient of Eq. 11.7.1. We obtain

$$LHS = \sum_{m,n}[a_{(m)}z^{-m-1}, b_{(n)}w^{-n-1}] = [a_{(m)}, b_{(n)}]$$

$$RHS = \sum_j \left(\sum_k (a_{(j)}b)_{(k)}w^{-k-1}\right)\left(\sum_s \binom{s}{j} z^{-s-1}w^{s-j}\right)$$

The coefficient of $z^{-m-1}w^{-n-1}$ of RHS is, $s = m$,

$$\sum_j \binom{m}{j} w^{m-j}(a_{(j)}b)_{(k)}w^{-k-1} = \sum_j \binom{m}{j}(a_{(j)}b)_{(m+n-j)}$$

So we put this as a proposition:

Proposition 11.8 (Commutator formula)**.** *In a vertex algebra* $(V, |0\rangle, T, Y)$*, we have*

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0}\binom{m}{j}(a_{(j)}b)_{(m+n-j)} \qquad \forall a, b \in V, \quad m, n \in \mathbb{Z}$$

Exercise 11.9. Prove that in a vertex algebra
(11.9.1)
$$[Y(a,z)Y(b,w)i_{z,w} - Y(b,w)Y(a,z)i_{w,z}](z-w)^n = \sum_{j\geq 0} Y(a_{(n+j)}b,w)\partial_w^{(j)}\delta(z,w).$$

Hint. Prove that he left hand side is local.

By extracting coefficients of Eq. 11.9.1, we obtain for all $a,b,c \in V$ and $m,n,k \in \mathbb{Z}$:

(11.9.2)
$$\begin{aligned}&\sum_{j\geq 0}\binom{m}{j}(a_{(n+j)}b)_{(m+k-j)}c\\ &= \sum_{j\geq 0}(-1)^j\binom{n}{j}\left(a_{(m+n-j)}b_{(k+j)}c - (-1)^n b_{(n+k-j)}a_{(m+j)}c\right)\end{aligned}$$

This is called *Borcherd's identity*, and is the key ingredient in our second definition of vertex algebra (see [Kac01, Proposition 4.8(b)]):

Definition 11.10. A *vertex algebra* is a vector space V, a nonzero vector $|0\rangle \in V$, and a set of bilinear products $V \times V \to V$, $a,b \mapsto a_{(n)}b$, $n \in \mathbb{Z}$ such that

(1) $\forall a,b \in V$ $\exists N$ such that $a_{(n)}b = 0$ $\forall n \geq N$.
(2) $\forall a \in V$, $|0\rangle_{(n)}a = \delta_{n,-1}a$.
(3) $\forall a \in V$, $a_{(-1)}|0\rangle = a$ and $a_{(\geq 0)}|0\rangle = 0$.
(4) Equation 11.9.2 holds $\forall a,b,c \in V$, $m,n,k \in \mathbb{Z}$.

12. Calculating vertex algebras

We begin with some notation.

Definition 12.1. Let V be a vertex algebra and $a,b \in V$. We package the elements $a_{(0)}b, a_{(1)}b, \ldots$ into a series
$$[a_\lambda b] = \sum_{j\geq 0} a_{(j)}b\frac{\lambda^j}{j!}$$
called the *λ-bracket* or *operator product expansion (OPE)* of a and b.

Recall that elements in V correspond to fields $Y(a,z)$ and $Y(b,z)$. They are local, and the coefficients of their bracket are $c_j(w) = Y(a_{(j)}b,w)$. So we put that information in the λ-bracket.

We also denote by $:ab:$ the vector correponding to the normally ordered product $:Y(a,z)Y(b,z):$. Again, by last time's theorem, $:ab: = a_{(-1)}b$.

Example 12.2. $V = H = \mathbb{C}[x_1,x_2,\ldots]$, $\mathcal{F} = \{h(z)\}$ where $h(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}$ (so in this example we are denoting $h_n \equiv h_{(n)}$), where, as before,
$$h_n = \begin{cases} n\frac{\partial}{\partial x_n} & n>0\\ x_n & n<0\end{cases}$$

Recall we completed $\mathcal{F}$ to $\overline{\mathcal{F}}$ and discovered
$$\begin{array}{ccc} V & \xrightarrow{\simeq} & \overline{\mathcal{F}} \\ a(z)|0\rangle|_{z=0} & \xleftarrow[s]{} & a(z) \\ a & \xmapsto[Y]{} & Y(a,z)\end{array}$$

For $I_V \in \overline{\mathcal{F}}$, where

$$I_V = \sum I_{V(n)} z^{-n-1}$$
$$I_{V(n)} = \begin{cases} I_V & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$

we get $I_V|0\rangle|_{z=0} = |0\rangle$. So,

$$Y(|0\rangle, z) = I_V$$

This shows that the vacuum $|0\rangle$ corresponds to the identity.

Next $h(z)$ in our example. We have

$$\begin{aligned} h(z)|0\rangle &= \sum_{n\in\mathbb{Z}} h_n 1 z^{-n-1} \\ &= \sum_{k\geq 1} x^k z^{k-1} + \sum_{k\geq 1} k\frac{\partial}{\partial x_k} 1 \\ \implies h(z)|0\rangle|_{z=0} &= x_1 \\ \implies S(h(z)) &= x_1 \\ \implies Y(x, z) &= h(z). \end{aligned}$$

So, x_1 corresponds to the sort of "generating field" $h(z)$.

What about x_2? Well, $h(z) = h(z)_{(-1)} I_V$. There was a lemma (maybe Remark 10.9?) that

$$a(z)_{(-2)} I_V = \partial_z a(z).$$

So $\partial_z h(z) \in \overline{\mathcal{F}}$,

$$\begin{aligned} a(\partial_z h(z)) &= \partial \sum_{k\geq 1} x_k z^{k-1}\Big|_{z=0} \\ &= \sum_{k\geq 1} (k-1) x_k z^{k-2}\Big|_{z=0} \\ &= x_2. \end{aligned}$$

So

$$Y(x_{n+1}, z) = \partial_z^{(n)} h(z).$$

Continuing:

$$Y(x_3, z) = \frac{1}{2}\partial_z^2 h(z),$$

and in general

$$Y(x_{n+1}, z) = \partial_z^{(n)} h(z).$$

The next question is: what about x_1^2? The trick here is that x_1 is identified with h, so

$$\begin{aligned} Y(x_1^2, z) &= Y(x_1 \cdot x_1, z) \\ &= Y(h_{(-1)} x_1, z) \\ &= Y(x_1, z)_{(-1)} Y(x, z) \\ &= {:}h(z)h(z){:}\ . \end{aligned}$$

Recall that $L(z) = \frac{1}{2} :h(z)h(z):$. We have claimed (Exercise 9.6) that $[L_m, L_n] = (m-n)L_{m+n}$ when $m, n, m+n \neq 0$ where

$$\begin{aligned} L(z) &= \sum_{m \in \mathbb{Z}} L_m z^{-m-2} \\ &= \sum_{n \in \mathbb{Z}} L_{(n)} z^{-n-1}. \end{aligned}$$

What about the λ-bracket? Let's denote $h = x_1$, (so $Y(h,z) = h(z)$, $L = \frac{1}{2}x_1^2$ and $Y(L,z) = L(z)$).

Well, long ago we computed that

$$\begin{aligned} [h(z), h(w)] &= \partial_w \delta(z,w) I_V \\ &= \sum_{j \geq 0} Y(h_{(j)}h, w) \partial_w^{(j)} \delta(z,w). \end{aligned}$$

Then

$$\begin{aligned} Y(h_{(0)}h, w) &= 0 \\ Y(h_{(1)}h, w) &= I_V \\ Y(h_{\geq 2)}h, w) &= 0 \\ \implies h_{(1)}h &= |0\rangle, \text{ and} \\ h_{(j)}h = 0, \qquad j &= 0 \text{ and } j \geq 2. \end{aligned}$$

So we see that

$$[h_\lambda h] = \lambda |0\rangle.$$

We often omit "$|0\rangle$" from these computations, as if it were "1".

So

$$[h_\lambda h] = \lambda.$$

$[h_\lambda |0\rangle] = 0 = [|0\rangle_\lambda h]$ because $[h(z), I_V] = 0$. In fact $[x_\lambda |0\rangle] = [|0\rangle_\lambda x] = 0$ for all $x \in V$.

What about $h_{(0)}h$? Well, $h_{(0)} = 0$ by definition of $h(z)$. So $h_{(0)}h = 0$. Next, $h_{(j)} = j\frac{\partial}{\partial x_j}$ for $j > 0$, so

$$h_{(1)}h = \frac{\partial}{\partial x_1}(x_1) = 1 = |0\rangle.$$

So $[h_\lambda h] = \lambda$ again.

What I really want is $[L_\lambda L] =$? Let's use Borcherd's formula 11.9.2.

First let's compute $(h_{(-1)}h)_{(0)}L$, i.e. we are putting $n = -1$, $m = 0$ and $k = 0$.

[missing computations]

Exercise 12.3. Check $(h_{(-1)}h)_{(1)}h = 2h$ and $(h_{(-1)}h)_{\geq 2}h = 0$. Thus $[L_\lambda h] = Th + \lambda h$.

If we wanted, we could continue to calculate $[L_\lambda L]$ by putting $c = \frac{1}{2}x_1^2$ above instead of $c = x_1$.

We find

$$[L_\lambda L] = \underbrace{TL}_{L_{(0)}L} + \underbrace{2\lambda L}_{L_{(1)}L} + \underbrace{\frac{\lambda^3}{12}|0\rangle}_{L_{(3)}L}.$$

This means

$$[L(z), L(w)] = (\partial_w L(w))\delta(z,w) + 2L(w)\partial_w\delta(z,w) + \frac{1}{12}\partial_w^3\delta(z,w)I_V. \tag{12.3.1}$$

So we can extract coefficients:

$$LHS = \sum_{m,n}[L_m, L_n]w^{-m-2}z^{-n-2}$$

Coefficient of $w^{-m-2}z^{-n-2}$ on RHS is:

$$\begin{aligned}&\sum_k(-k-2)L_k w^{-k-3}\sum z^{-a-1}w^a\\ &+2\sum_k L_k w^{-k-2}\sum_a az^{-a-1}w^{a-1} + \frac{1}{12}\sum_a a(a-1)(a-2)z^{-a-1}w^{a-3}.\end{aligned}$$

So actually we found

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}I_V.$$

The moral is: one tries to define $L = \frac{1}{2}h^2$ naively, expecting $[L_m, L_n] = (m-n)L_{m+n}$. Need to "normally order/normalise" to make L well-defined, now $L = \frac{1}{2} :hh:$. The "cost" is that the expected relation $[L_m, L_n] = (m-n)L_{m+n}$ gets altered by an "anomaly" $\frac{m^3-m}{12}I_V$.

Right, so we showed how to compute this using Bochner's formula, but there's actually another way using λ-bracket:

Theorem 12.4. *Let V be a vertex algebra, $a, b, c \in V$. Then*

$$[Ta_\lambda b] = -\lambda[a_\lambda b] \tag{12.4.1}$$

$$[a_\lambda Tb] = (T+\lambda)[a_\lambda b] \tag{12.4.2}$$

$$T(:ab:) = :(Ta)b: + :a(Tb): \tag{12.4.3}$$

$$[b_\lambda a] = -[a_{-\lambda-T}b] \tag{12.4.4}$$

$$[a_\lambda :bc:] = :[a_\lambda b]c: + :b[a_\lambda c]: + \int_0^\lambda [[a_\lambda b]_\mu c]d\mu \tag{12.4.5}$$

$$[a_\lambda[b_\mu c]] - [b_\mu[a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu}c]. \tag{12.4.6}$$

Proof. For the first one use $\partial_z\delta(z,w) = -\partial_z\delta(z,w)$.

For the last one notice that

$$-[a_{-\lambda-T}b] = \sum_j \frac{1}{j!}(-\lambda-T)^j(a_{(j)}b)$$

if $[a_\lambda b] = \sum_j \frac{\lambda^j}{j!}b$. □

Remark 12.5. By noting that $[a_\lambda b] = \mathrm{Res}_z e^{\lambda z}Y(a,z)b$, these identities can be proved pretty efficiently.

That is,

Exercise 12.6. Use the fifth equation to prove what we proved with Bochner's formula.

And another one:

Exercise 12.7. Let $V = H = \mathbb{C}[x_1, x_2, \ldots]$ again. Define $B = \frac{1}{2}\ {:}hh{:} + \beta Th$ with $\beta \in \mathbb{C}$. Confirm that

$$[B_\lambda B] = TB + 2\lambda B = \frac{1}{12}(1 - 12\beta^2)|0\rangle.$$

Definition 12.8. The *Virasoro Lie algebra* is

$$\mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$$

with Lie bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}C, \qquad [C, \mathrm{Vir}] = 0.$$

By the discussion above (all we've said so far!), there is a representation ρ of Vir in H given by

$$\rho(\underbrace{L_n}_{\text{as an abstract object in Vir}}) = \underbrace{L_n}_{\text{as a complicated operator}}, \qquad \rho(C) = I_H.$$

Also H carries a representation ρ_β of Vir,

$$\rho_\beta(L_n) = B_n, \qquad \rho_\beta(C) = (1 - 12\beta^2)I_V$$

If (M, ρ) is a representation of Vir, in which $\rho(C) = c \cdot I_M$ for some scalar c, we say M has a *central charge c*.

Looks like we have a second example of a vertex algebra. Consider $V = H$, $|0\rangle = 1$, $T = T$ from before, but this time put $F = \{B(z)\}$. Then all the axioms of the first definition of vertex algebra are satisfied, except the second: $B_{(n_1)}B_{(n_2)}\ldots B_{(n_s)}|0\rangle$ might not span all of $V = H$. So take $V = \mathrm{span}\{\text{these monomials}\} \subset H$. One can check that $T(V) \subset V$, and $B(z)$ is a quantum field on V. So $(V, 1, T, \{B(z)\})$ is a vertex algebra, called the *Virasoro vertex algebra*.

13. The charged free fermions: a vertex superalgebra

Also known as:

- The Clifford (vertex) algebra.
- The Dirac sea.
- The bc system.

First we consider a vector superspace with basis φ and φ^*

$$\mathfrak{a} = \mathbb{C}\varphi + \mathbb{C}\varphi^*$$

(where φ and φ^* are odd, so this is a O,2-dimensional superspace).

(A vector superspace is a $\mathbb{Z}/2$-graded vector space, i.e. a vector space split in two pieces, $V = V_0 \oplus V_1$, which we call even and odd.)

We consider a bilinear form

$$\begin{aligned}\langle\cdot,\cdot\rangle : \mathfrak{a}\times\mathfrak{a} &\longrightarrow \mathbb{C}\\ \langle\varphi^*, \varphi\rangle &= 1\\ \langle\varphi, \varphi^*\rangle &= 1 \text{ (why? see below)}\end{aligned}$$

Now consider

$$\tilde{\mathfrak{a}} = \underbrace{t^{1/2}\mathfrak{a}[t,t^{-1}]}_{\text{odd}} \oplus \underbrace{\mathbb{C}K}_{\text{even}}.$$

with Lie bracket

$$[at^m, b^n] = \delta_{m,-n}\,\langle a,b\rangle\,K,$$
$$[K,\tilde{\mathfrak{a}}] = 0.$$

(Notice the small difference from the Heisenberg case: there is no m before the δ!)

To see why $\langle\varphi,\varphi^*\rangle = 1$, notice that in a Lie superalgebra we always want $[x,y] = (-1)(-1)^{p(x)p(y)}[y,x]$ where p denotes the parity. Since both at^m and bt^n are odd, we have $[at^m,bt^n] = [bt^n,at^m]$. Then apply the definition of bracket.

Since a,b odd here, we say $\langle a,b\rangle = \langle b,a\rangle$ means $\langle\cdot,\cdot\rangle$ is skew-super symmetric.

Definition 13.1. A bilinear form

$$(\cdot,\cdot) : U\times U \to \mathbb{C}$$

is *supersymmetric* if $(b,a) = (-1)^{p(a)p(b)}(a,b)$, and *skew-supersymmetric* if $(b,a) = -(-1)^{p(a)p(b)}(a,b)$.

Exercise 13.2. Invent the definition of Lie superalgebra, and confirm that if $U = U_0\oplus U_1$ is a vector superspace, $\mathrm{End}(U)$ is a Lie superalgebra with $[X,Y] := XY - (-1)^{p(X)p(Y)}YX$.

Let's build a Fock-type representation of $\tilde{\mathfrak{a}}$.

Construction 1.

$$\tilde{\mathfrak{a}_+} = \bigoplus_{n>0} t^n\mathfrak{a}\oplus\mathbb{C}K, \qquad (n\in\frac{1}{2}+\mathbb{Z},\ \text{recall})$$

$\tilde{\mathfrak{a}_+}\subset\tilde{\mathfrak{a}}$ is a superalgebra.

Consider $U(\tilde{\mathfrak{a}})$ and $U(\tilde{\mathfrak{a}})\subset U(\tilde{\mathfrak{a}})$.

Recall that the PWB theorem says that $U(\mathfrak{g})$ "looks like" polynomials on $\mathfrak{g}$.

When $\mathfrak{g}$ is pure even, this says: take $\{a_1,a_1,\ldots\}$ a basis of $\mathfrak{g}$, then

$$\{a_{i_1}^{m_1}, a_{i_2}^{m_2}\ldots a_{i_s}^{m_s} : i_1\le i_2\le\ldots\le i_s, m_j\ge 1\}.$$

For $\mathfrak{g} = \mathfrak{g}_0\oplus\mathfrak{g}_1$ super Lie algebra, the statement is similar, only now, we take each a^i homogeneous ($\in\mathfrak{g}_0$ or $\mathfrak{g}_1$):

$$\{a_{i_1}^{m_1}a_{i_2}^{m_2}\ldots a_{i_s}^{m_s}, i_1\le i_2\le\ldots\le i_s, m_j\ge 1,\ \text{if } a_{i_j}\ \text{odd},\ m_j = 1\}.$$

The point is that if we put an odd guy twice it becomes one.

Why? Because in $U(\mathfrak{g})$, $X\in\mathfrak{g}_1$ should satisfy $XX+XX = [X,X] = 0$. So $X^2 = 0$. So we only allow odd basis vectors to appear 0 or 1 times each.

Dfine $F = U(\tilde{\mathfrak{a}})\otimes_{U(\tilde{\mathfrak{a}}_+)}\mathbb{C}|0\rangle$, where $at^n\cdot|0\rangle$ for all $n>\frac{1}{2}$, and $K|0\rangle = |0\rangle$.

[Picture of F]

We can introduce a bi-grading of F:

$$F = \bigoplus_{\substack{c\in\mathbb{Z}\\ \Delta\in\frac{1}{2}\mathbb{Z}_+}} F_\Delta^c$$

where, for a monomial

$$\varphi^*_{-n}\varphi^*_{-n_2}\cdots\varphi^*_{-n_s}\varphi_{-m},\varphi_{-m_2}\cdots\varphi_{-m_t}|0\rangle$$

we define its *energy* by $\Delta = \sum_i n_i + \sum_j m_j$ and its *charge* by $c = s - t$.

The point is that F is an infinite-dimensional vector space, but we get a reasonable mental picture of what it looks like.

What is $\dim F^c_\Delta$? Let's write...

So we have a vector superspace F, and an action on F of $\tilde{\mathfrak{a}}$. Let's build some quantum fields!

There are going to be two: $\varphi(z)$ and $\varphi^*(z)$.

$$\varphi(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \varphi_n z^{-n-\frac{1}{2}}$$

(Notice the exponent on z is an integer!)

$$\varphi^*(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \varphi^*_n z^{-n-\frac{1}{2}}.$$

Now let's compute $[\varphi^*(z), \varphi(w)]$.

$$\begin{aligned}
[\varphi^*(z), \varphi(w)] &= \sum_{m,n \in \frac{1}{2}+\mathbb{Z}} [\varphi^*_m, \varphi_n] z^{-m-\frac{1}{2}} w^{-n-\frac{1}{2}} \\
&= \sum_{n \in \frac{1}{2}+\mathbb{Z}} K z^{n-\frac{1}{2}} w^{-n-\frac{1}{2}} \\
&= \sum_{\substack{r \in \mathbb{Z} \\ r = n-\frac{1}{2}}} z^r w^{-r-1} \\
&= \delta(z, w).
\end{aligned}$$

So $F, |0\rangle, \mathcal{F} = \{\varphi(z), \varphi^*(z)\}$, $T : F \to F$, is a vertex superalgebra. Here $T : F \to F$ has to satisfy $T|0\rangle = 0$ and $[T, \varphi(z)] = \partial_z \varphi(z)$,

$$\begin{aligned}
\sum [T, \varphi_n] z^{-n-\frac{1}{2}} &= \sum (-n - \frac{1}{2}) \varphi_n z^{-n-\frac{3}{2}} \\
&= \sum (-k + \frac{1}{2}) \varphi_{k-1} z^{-k-\frac{1}{2}}
\end{aligned}$$

after putting $n = k - 1$. So

$$[T, \varphi_n] = -(n - \frac{1}{2}) \varphi_{n-1}.$$

Inductively, this determines how T acts on F.

In λ-bracket notation, this just says

$$[\varphi^*_\lambda \varphi] = |0\rangle.$$

Skew-symmetry says $[b_\lambda a] = -(-1)^{p(a)p(b)} [a_{-\lambda-T} b]$.

For example,

$$\begin{aligned}
[h_\lambda h] &= \lambda |0\rangle, \\
[h_\lambda h] &= -[h_{-\lambda-T} h] = -(-\lambda - T)|0\rangle \\
&= T\lambda |0\rangle.
\end{aligned}$$

In this case $[\varphi_\lambda \varphi^*] = +|0\rangle$ too.

Notice $\sum \varphi_n z^{-n-\frac{1}{2}} = \sum \varphi_{(m)} z^{-m-1}$.

So

$$\varphi_{(m)} = \varphi_{m-\frac{1}{2}} \text{ and}$$
$$\varphi^*_{(m)} = \varphi_{m-\frac{1}{2}}.$$

So

$$:\varphi^*\varphi: = \varphi^*_{(-1)}\varphi_{(-1)}|0\rangle$$
$$= \varphi^*_{-\frac{1}{2}}\varphi_{-\frac{1}{2}}|0\rangle.$$

I.e.

$$\varphi = \varphi_{(-1)}|0\rangle = \varphi_{-\frac{1}{2}}|0\rangle$$

Now we can put the former picture but with the associated operators using the second definition of vertex algebra: [Picture]

Let's compute some brackets: [lots of computations]

So,

$$[\alpha_\lambda \alpha] = \lambda|0\rangle.$$

Just like $[h_\lambda h] = \lambda|0\rangle$. This relation is the basis of the "boson-fermion correspondence", i.e. consider the subspace $U \subset F$ defined as

$$U = \text{span}\{\alpha_{n_1}\alpha_{n_2}, \dots, \alpha_{n_s}|0\rangle : n_i \in \mathbb{Z}\}.$$

We claim that since $[T, \alpha(z)] = \partial_z \alpha(z)$, $[T, \alpha_n] = -n\alpha_{n-1}$.

So $T(U) \subset U$. Put $\mathcal{F}_1 = \{\alpha(z)\}$. Then $(U, |0\rangle, T|_U, \{\alpha(z)\})$ is a vertex algebra (an honest one, not super).

Proposition 13.3. *U is isomorphic to the Heisenberg vertex algebra. (In particular, $U \simeq \mathbb{C}[x_1, x_2, \dots]$ as a vector space.)*

Proof. Next time. □

Since $[\alpha_0, \varphi_n] = -\varphi_n$, $[\alpha_0, \varphi^*_m] = \varphi^*_m$, and $\alpha_0|0\rangle = 0$, in fact the charge of a monomial is exactly the eigenvalue of α_0 acting on it.

$$\alpha_0 \cdot \varphi^*_{-m}\varphi^*_{-m_2}\varphi_{-n}|0\rangle = [\alpha_0, \varphi^*_{-m}, \varphi^*_{-m_2}, \varphi_{-n}]|0\rangle$$
$$= \underbrace{(+1+1-1)}_{\text{charge}}\varphi^*_{-m_1}\varphi_{-m_2}\varphi_{-n}|0\rangle.$$

From $[\alpha_\lambda \alpha] = \lambda|0\rangle$, we get

$$[\alpha_m, \alpha_n] = m\delta_{m,-n}I_F \text{ (as for } h)$$

So $[\alpha_0, \alpha_n] = 0$ for all $n \in \mathbb{Z}$. Thus $\alpha_0|_U = 0$ and thus $U \subset F^{(0)}$.

Proposition 13.4. *$U = F^{(0)}$.*

Proof. Next time. □

Let's take two copies of F, i.e. $F_2 = F \otimes F$ with fields $\varphi^1(z), \varphi^2(z), \varphi^{1*}(z), \varphi^{2*}(z)$

$$[\varphi^{i*}_\lambda \varphi^j] = \delta_{ij}.$$

Basis of F_2 is

$$\varphi^1_{-n_1} \dots \varphi^1_{-n_2}\varphi^2_{-m_1} \dots \varphi^2_{-m_t}\varphi^{*-1}_{-p_1} \dots \varphi^{*-1}_{-p_u}\varphi^{*2}_{-q_1} \dots \varphi^{*2}_{-q_v}|0\rangle.$$

Let $\alpha^{ij} = :\varphi^{j*}\varphi^i:$.

$$[\alpha^{ij}_\lambda \alpha^{k\ell}] = ?$$

(certainly $[\alpha^{ii}_\lambda \alpha^{ii}] = \lambda|0\rangle$.) In general

$$[\varphi^k_\lambda \alpha^{ij}] = \text{computation} = \delta_{ki}\varphi^j.$$

Similarly

$$[\varphi^{k*}_\lambda \alpha^{ij} = -\delta_{kj}\varphi^{i*}.$$

$$[\alpha^{ij}_\lambda \varphi^k] = -\delta_{ik}\varphi^j$$

$$[\alpha^{ij}_\lambda \varphi^{k*}] = +\delta_{jk}\varphi^{i*}.$$

We would like to consider

$$F_n = F^{\otimes n} = F \otimes \ldots \otimes F$$

where on the i-th factor we have φ_i, φ_i^*.

Then we have relations among the generators:

$$[\varphi^i_\lambda \varphi_j^*] = [\varphi_j^* \lambda \varphi_i] = \delta_{ij}|0\rangle$$

Today let's define

$$\alpha_{ij} = :\varphi_i\varphi_j^*:$$

We notice that

- the φ_i behave like e_i,
- the α_{ij} behave like E_{ij},
- the φ_i^* behave like e_i^*.

[Computations of some λ-brackets]

Recall, the E_{ij} span a Lie algebra: $\mathfrak{gl}_n$. It has a representation on $\mathbb{C}^n = \langle e_1, \ldots, e_n\rangle$ by $E_{ij}e_k = \delta_{ik}e_i$, and on $(\mathbb{C}^n)^*$ we have

[computations]

$$E_{ij}e_k^* = -\delta_{ki}e_j^*.$$

Let $A \in \mathfrak{gl}_n$. Let's write

$$\alpha^A = \sum_{i,j} a_{ij}\alpha_{ij} \in F_n$$

We'd like to compute $[\alpha^A_\lambda \alpha^B]$.

Some computations done in lecture are the proof of

Theorem 13.5. *For $\alpha_{ij} = :\varphi_i\varphi_j^*: \in F_n$, and α^A as above,*

$$[\alpha^A_\lambda \alpha^B] = \alpha^{[A,B]} + \lambda\, Tr(AB)|0\rangle.$$

Which says that α^A behaves like matrices, but with that correction term.

Nex, let $\mathfrak{g} \subset \mathfrak{gl}_n$ be a Lie subalgebra. For $A, B \in \mathfrak{g}$, $[A,B] \in \mathfrak{g}$ also, and so the set

$$\mathcal{F} = \{\alpha^A : A \in \mathfrak{g} \cup \{|0\rangle\}\}$$

is "closed under λ-brackets".

More precisely, for

$$F_n \supset V = \text{span}\{\alpha^{A_1}_{(n_1)} \ldots \alpha^{A_s}_{(n_s)} : n_i \in \mathbb{Z}, A_i \in \mathfrak{g}\},$$

with $|0\rangle = |0\rangle$, $T = T$ and $\mathcal{F} = \{\alpha^A(z) : A \in \mathfrak{g}\} \cup \{I\}$ is a vertex algebra.

14. Universal affine vertex algebra

Recall the construction of the affine vertex algebra, Definition 2.3.

Let (M, ρ) be a representation of $\hat{\mathfrak{g}}$.

For $a \in \mathfrak{g}$, define

$$a^M(z) = \sum_{n\in\mathbb{Z}} \rho(at^n) z^{-n-1} \in \mathrm{End}(M)[\![z^{\pm 1}]\!]. \tag{14.0.1}$$

Definition 14.1. A *smooth* $\hat{\mathfrak{g}}$*-module* is a $\hat{\mathfrak{g}}$-module (M, ρ) such that for each $m \in M$ there is $N \in \mathbb{Z}$ such that $\rho(at^n)m = 0$ for all $a \in \mathfrak{g}$ and $n \geq N$.

Notice that on a smooth module M, the fields $a^M(z)$ are quantum fields.

We may calculate:

$$\begin{aligned}
[a^m(z), b^m(w)] &= \sum_{m,n} [\rho(at^m), \rho(bt^n)] z^{-m-1} w^{-n-1} \\
&= \sum_{m,n} z^{-m-1} w^{-n-1} ([a,b]t^{m+n} + m\delta_{m,-n}(a,b)\rho(K)) \\
&= \sum_{m,n} ([a,b]t^{m+n} w^{-(m+n)-1}) z^{-m-1} w^m + \sum_{m,n} z^{-m-1} w^{-n-1} \delta_{m,-n}(a,b)\rho(K) \\
&= [a,b]^m(w)\delta(z,w) + (a,b)\partial_w \delta(z,w)\rho(K).
\end{aligned}$$

Now suppose the K acts by some constant, i.e. $\rho(K) = kI_M$. Then our quantum fields $a^M(z)$ are mutually local, with exponent 2, i.e. $(z-w)^2[a^M(z), b^M(w)] = 0$.

As a special case, we may take M to be a sort of "Fock space":

$$\hat{\mathfrak{g}}_+ = \mathfrak{g}[t] \oplus \mathbb{C}K \subset \hat{\mathfrak{g}}.$$

Let $\hat{\mathfrak{g}}_+$ act on $\mathbb{C}|0\rangle$ by

$$\begin{aligned}
at^m|0\rangle &= 0 \qquad \forall m \geq 0, \\
K \cdot |0\rangle &= k|0\rangle \qquad (k \in \mathbb{C} \text{ called } \textit{level}).
\end{aligned}$$

Consider

$$V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+)} \mathbb{C}|0\rangle.$$

Remark 14.2. $H = \mathbb{C}[x_1, x_2, \ldots]$ is an example of this construction with $\mathfrak{g}$ one-dimensional and $(h,h) = 1$.

Exercise 14.3. $V^k(\mathfrak{g})$ is a smooth $\hat{\mathfrak{g}}$-module, in which $K \mapsto k\mathrm{Id}$.

Proof. To prove $V^k(\mathfrak{g})$ is smooth we need to show that for every formal power series of endomorphisms of $V^k(\mathfrak{g})$ of the form Equation 14.0.1, the coefficient operators vanish at every v for sufficiently large n. But to define these power series we need to find a copy of $\hat{\mathfrak{g}}$ inside $V^k(\mathfrak{g})$. □

The set $\mathcal{F} = \left\{ a(z) = \sum_{n\in\mathbb{Z}} (at)^n z^{-n-1} : a \in \mathfrak{g} \right\}$ are mutually local quantum fields.

Define $T : V^k(\mathfrak{g}) \curvearrowright V^k(\mathfrak{g})$ by the relation $T|0\rangle = 0$ and $[T, at^m] = -mat^{m-1}$. $V^k(\mathfrak{g})$ is a vertex algebra called the *universal affine vertex algebra* of level k associated with $\mathfrak{g}$.

[Picture of $V^k(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_2$.]

We have a bigrading, vertical grading Δ (energy) is

$$\Delta(a^1_{(-n_1)} a^2_{(-n_2)} \ldots a^s_{(-n_s)}|0\rangle = n_1 + n_2 + \ldots + n_s.$$

Notice that for $a \in \mathfrak{g}$, we have

$$\begin{aligned}\mathcal{F} \ni a(z) \overset{s}{\mapsto} a(z)|0\rangle|_{z=0} \\ &= \sum_m at^m|0\rangle z^{-m-1}\Big|_{z=0} \\ &= (at^{-1})|0\rangle + (at^{-2})z|0\rangle + \dots \\ &= at^{-1}|0\rangle.\end{aligned}$$

Thus we can think of $\{at^{-1}|0\rangle : a \in \mathfrak{g}\} \subset V^k(\mathfrak{g})$ as a "copy" of $\mathfrak{g}$ inside $V^k(\mathfrak{g})$. Let's **abuse notation** and write a for $at^{-1}|0\rangle$.

Then $at^{-2}|0\rangle = Ta$, also $(at^{-1})(bt^{-1})|0\rangle = :ab:$, etc. $e = et^{-1}|0\rangle = e_{(-1)}|0\rangle$, $\Delta(e) = 1$.

In fact $\Delta(a) = 1$ for all $a \in \mathfrak{g}$, $\Delta(Ta) = 2$, $\Delta(:ab:) = 2$, etc.

In this example, i.e. $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{sl}_2$ is itself a $\mathbb{Z}$-graded Lie algebra,

$$\begin{aligned} w(E) &= 2 \qquad & [H,E] &= 2E \\ W(H) &= 0 \qquad & [H,H] &= 0 \\ w(F) &= -2 \qquad & [H,F] &= -2F. \end{aligned}$$

(That is, $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, $[\mathfrak{g}_m, \mathfrak{g}_n] = \mathfrak{g}_{m+n}$.) w = eigenvalue of $\mathrm{ad}(H)$. This induces a $\mathbb{Z}$-grading on $V^k(\mathfrak{sl}_2)$, compatible, i.e.

$$w(a^1_{(-n_1)} \dots a^s_{(-n_s)}|0\rangle = 2\#(E) - 2\#(F).$$

As for the character,

$$\begin{aligned}\chi(q,u) &= \sum_{\Delta,w} \dim V^k(\mathfrak{g})^w_\Delta q^\Delta u^w \\ &= 1 + q(u^2 + 1 + u^{-2}) + q^2(u^4 + 2u^2 + 2 + 2u^{-2} + u^{-4} + \dots\end{aligned}$$

Then by the generating function argument we have discussed before,

$$\chi(q,u) = \prod_{n=1}^{\infty} \frac{1}{(1-u^2q^n)(1-q^n)(1-u^{-2}q^n)}.$$

Can discard u to get vertical grading

$$\chi(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^3}.$$

[Missing: we changed a little the definition of $V^k(M)$; now it also depends on a bilinear form B.]

Here's another example:

Example 14.4. $\mathfrak{g} = \mathfrak{gl}_n$, with $B(X,Y) = \mathrm{Tr}(XY)$, and $k = 1$.

Remark 14.5. In general $V^1(\mathfrak{g}, kB) = V^k(\mathfrak{g}, B)$. If $\mathfrak{g}$ is finite-dimensional simple, we typically take $B(a,b) = \frac{1}{2h^\vee}\kappa(a,b)$ where κ is the Killing form. (This B has the property that $B(\theta,\theta) = 2$ for **long** roots $\theta \in \Delta \subset \mathfrak{h}^*$.) (In fact, in a simple f.d. there's all invariant forms are proportional.) Then we write this vertex algebra as $V^k(\mathfrak{g})$ without specifying B.

So perhaps that's why last time we didn't put B.

Back to the Fermion algebra F_n, and now denoting J instead of α, we built fields $\{J^A : A \in \mathfrak{gl}_n\}$ with λ-bracket $[J^A_\lambda J^B] = J^{[A,B]} + \lambda \mathrm{Tr}(AB)|0\rangle$. This suggests a relation with $V^1(\mathfrak{gl}_n, B)$, where $B(X,Y) = \mathrm{Tr}(X,Y)$.

Consider $\mathcal{F} = \{J^A(z) : A \in \mathfrak{gl}_n\} \subset \mathbb{F}_{F_n} \cong F_n$. Let $V = \mathrm{span}\{J^{A_1}_{(n_1)} \dots J^{A_s}_{(n_s)}|0\rangle : A_i \in \mathfrak{gl}_n, n_i \in \mathbb{Z}\} \subset F_n$.

Then $(V, \mathcal{F}, T|_V, |0\rangle)$ is a vertex algebra. $V \subset F_n$. Notice that $V \neq F^n$ since V consists only of even fields, indeed, $V = \left\langle :\varphi_i \varphi_j^*: \right\rangle$.

Here $[T, \varphi_{(n)}] = -n\varphi_{(n-1)}$.

$$T(\varphi^{i_1}_{(n_1)} \dots \varphi^{i_s}_{(n_s)}|0\rangle) = -\sum_{j=1}^{s} n_j \varphi^{i_1}_{(n_1)} \dots vo^{i_j}_{(n_j-1)} \dots \varphi^{i_s}_{(n_s)}|0\rangle.$$

Does $V = F_n^{(0)}$ then?

Let's examine $n = 1$ first. $\mathfrak{g} = \mathbb{C}1$, $B(1,1) = 1$, $k = 1$. We just remarked that $V^1(\mathfrak{g}, B) = H$ is Heisenberg. And $F_1 = F = \langle \varphi, \varphi^* \rangle$ which we have drawn previously. We also computed the character as an infinite product.

We have $F^{(0)} \ni J = :\varphi\varphi^*:$ with λ-bracket relation $[J_\lambda J] = \lambda|0\rangle$. Recall $J(z) = \sum_{n\in\mathbb{Z}} J_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$. The λ-bracket relation implies

$$[J(z), J(w)] = \partial_w \delta(z,w) I$$
$$\implies [J_m, J_n] = m\delta_{m,-n} I.$$

This implies that $F^{(0)}$ is a representation of the oscillator Lie algebra $\hat{\mathfrak{a}} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}h_n \oplus \mathbb{C}K$, in which $h_m \mapsto J_m$, $k \mapsto I$ and $h_m|0\rangle$ for all $m \geq 0$.

Proposition 14.6. *As representations of* $\mathfrak{a}$, $F^{(0)} \simeq H = \mathbb{C}[x_1, x_2, \dots]$.

Proof. Using $H \simeq U(\hat{\mathfrak{a}}) \otimes_{U(\hat{\mathfrak{a}_+})} \mathbb{C}1$, by its universal property, there exists a morphism of $\hat{\mathfrak{a}}$-representations $f : H \to F^{(0)}$, such that $f(1) = |0\rangle$.

By Exercise 9.4 we know H is irreducible, so $f : H \to F^{(0)}$ is injective. Is it an isomorphism? Consider $\tilde{F} = F^{(0)}/f(H)$. We want to prove $\tilde{F} = \{0\}$. The trick is to consider $L = \frac{1}{2} :JJ:$.

We have seen that

$$F^{(m)} = \{v \in F : J_0 v = mv\}$$

(since $[J_\lambda \varphi] = -\varphi$ and $[J_\lambda \varphi^*] = +\varphi^*$, $[J_0, \varphi_n] = -\varphi_n$).

Write $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, then $L_0 = \frac{1}{2}J_0^2 + \sum_{m>0} J_{-m}J_m$. This implies (for example) that $L_0|0\rangle = 0$.

One may compute

$$[L_\lambda \varphi] = T\varphi + \frac{1}{2}\lambda\varphi$$
$$[L_\lambda \varphi^*] = T\varphi^* + \frac{1}{2}\lambda\varphi^*.$$

If you expand these in terms of coefficients, you find $[L_0, \varphi_n] = -n\varphi_n$, $[L_0, \varphi_n^*] = -n\varphi_n^*$.

From these relations we conclude that $\Delta(\text{monomial}) = L_0$-eigenvalue (monomial), i.e., the (charge, energy)-grading is the eigenspace grading by (J_0, L_0). Suppose $\tilde{F} \neq 0$. Then $\exists \overline{v} \in \tilde{F}$ for which $J_m \overline{v} = 0$ for all $m > 0$. Indeed, by Lemma

(?) the $\mathbb{Z}_+$-grading of $F^{(0)}$ is inherited by $\tilde{F}$. Since $|0\rangle \in f(H)$, $\tilde{F} = \bigoplus_{\Delta \geq N} \tilde{F}_\Delta$ for some $N > 0$. Let $\overline{v} \in \tilde{F}_N$, $\overline{v} \neq 0$. For $m > 0$, $J_m\overline{v} \in \tilde{F}_{N-m} = 0$.

Key point: $J_0\overline{v}$, because $\tilde{F}$ is quotient of $F^{(0)}$. Hence $L_0\overline{v} = 0$. But $L_0 = \Delta$, and we know the only element of $F^{(0)}$ with $\Delta = 0$ is $|0\rangle$. This contradiction implies $\tilde{F} = 0$, hence $F^{(0)} = f(H) \simeq H$. $\square$

In the process we saw that Δ coincides with L_0, where $L = \frac{1}{2}\ :JJ:$, and $J = f(h)$. So consider $L = \frac{1}{2}\ :hh_i:$.

We have $[L_\lambda h] = Th + \lambda h$ so $[L_0, h_n] = -nh_n$ in particular, and

$$L_0(h_{-n_1}, h_{-n_2}, \ldots, h_{-n_s}|0\rangle = L_0(x_{n_1}x_{n_2}\ldots x_{n_s}|0\rangle) \quad = \left(\sum_j n_j x_{n_1}\ldots x_{n_s}|0\rangle\right)$$

Using this we obtain

$$\chi_{F^{(0)}}(q) = \sum_\Delta \dim(F^{(0)}_\Delta)q^\Delta = \prod_{m=1}^\infty \frac{1}{1-q^m}.$$

In $F^{(m)}$, let us denote

$$|m\rangle = \begin{cases} \varphi_{-1/2}\varphi_{-3/2}\varphi_{-5/2}\cdots\varphi_{-\frac{2(-m)-1}{2}}|\rangle & \text{if } m<0 \\ \varphi^*_{-1/2}\varphi^*_{3/2}\cdots\varphi^*_{-\frac{2m-1}{2}} & \text{if } m>0 \end{cases}$$

We observe that $\Delta(|m\rangle) = \frac{m^2}{2}$ and $F^{(m)}_{m^2/2} = \mathbb{C}|m\rangle$.

Proposition 14.7. *As an $\hat{\mathfrak{a}}$-module, $F^{(m)} \simeq H^m$, and is irreducible.*

Proof. Same as above. $\square$

This gives a formula

$$\chi_{F^{(m)}}(q) = \sum_\Delta \dim F^{(m)}_\Delta q^\Delta = q^{m/2}\prod_{k=1}^\infty \frac{1}{1-q^k}.$$

As a corollary, we obtain again Jacobi's triple product formula. (Again!)

$$\prod_{m=1}^\infty (1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}) = \prod_{k=1}^\infty \frac{1}{1-q^k}\left(\sum_{n\in\mathbb{Z}} y^n q^{n^2/2}\right).$$

15. Another presentation the charged free Fermions

The idea is to consider $X = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}e_i$, a countably infinite dimensional space, and define

$$\Lambda^{\infty/2} = \left\{ \begin{matrix} \text{span of symbols of the form} \\ e_{i_0}\wedge e_{i_1}\wedge e_{i_1}\wedge e_{i_3}\wedge\ldots \\ \text{where } \exists N \text{ s.t. } \forall n\geq N, i_{n+1}=i_n+1 \end{matrix} \right\} \Big/ \begin{matrix} \text{relation that} \\ e_i\wedge e_j = -e_j\wedge e_i \\ \text{“wherever it occurs”} \end{matrix}. \tag{15.0.1}$$

That is, the indices can dance around as they like but at some point they will become consecutive. Clearly a basis of $\Lambda^{\infty/2}$ is given by those “semi-infinite monomials” for which $i_0 < i_1 < i_2 < \ldots$.

For any semi infinite monomial $\underline{e} = e_{i_0}\wedge e_{i_1}\wedge\ldots$ there exists a number m such that $i_j = -m + j$ for all $j \gg 0$. This number is called the *charge* of $\underline{e}$.

We have a decomposition in vector spaces

$$\Lambda^{\infty/2} = \bigoplus_{m\in\mathbb{Z}} \Lambda^{\infty/2,(m)}$$

called *charge decomposition.*

Let's introduce some operators in $\Lambda^{\infty/2}$:

$$\varphi_{(n)} : \Lambda^{\infty/2} \longrightarrow \Lambda^{\infty/2}$$
$$\varphi_{(n)}(\underline{e}) = e_n \wedge \underline{e}$$

i.e. we just put e_n at the beginning of the monomial.

Next

$$\varphi^*_{(n)} : \Lambda^{\infty/2} \longrightarrow \Lambda^{\infty/2}$$
$$\varphi^*_{(n)} = \sum_{k\geq 0}(-1)^k \delta_{n,i_k} e_{i_0} \wedge e_{i_1} \wedge \ldots \wedge \underbrace{\widehat{e_{i_k}}}_{\text{remove this}} \wedge \ldots$$

so in analogy, we remove this term. The $(-1)^k$ accounts for moving around the unwanted term to the beginning of the monomial.

Exercise 15.1. Something like

$$\varphi_{(m)}\varphi^*_{(n)} + \varphi^*_{(n)}\varphi_{(m)} = \delta_{m,+n} I_{\Lambda^{\infty/2}}$$

Let's introduce some quantum fields now:

$$\varphi(w) = \sum_{n\in\mathbb{Z}} \varphi_{(n)} w^{-n-1}$$

which vanishes for very large n because we eventually get to the "consecutive" region, and

$$\varphi^*(w) = \sum_{n\in\mathbb{Z}} \varphi^*_{(n)} w^n$$

which vanishes for very negative n since the star operators are defined to give zero if the e_n is not found in the monomial.

Notice that the convention $\varphi^*(w) = \sum \varphi^*_{(n)} w^n$, instead of $\sum \varphi^*_{(n)} w^{-n}$ is so that $\varphi^*(w)$ is a quantum field (indeed, by our convention on how we write quantum fields, see Definition 9.7). The convention $\varphi^*(w) = \sum \varphi^*_{(n)} w^n$ instead of, say, $\sum \varphi^*_{(n)} w^{n+1}$ is so that the equation in Exercise 15.1 turns into a nice relation

$$[\varphi(z), \varphi^*(z)] = \delta(z,w)I,$$

(i.e. $[\varphi_\lambda \varphi^*] = |0\rangle$, just like in the charged fermions F^{ch}!).

Exercise 15.2. These fields will match with our previous φ, φ^* as

$$\sum \varphi_{(n)} z^{-n-1} = \sum \varphi_n z^{-n-1/2} : \varphi_n = \varphi_{n-1/2}$$
$$\sum \varphi^*_{(n)} z^n = \sum \varphi^*_n z^{-n-1/2} : \varphi^*_n = \varphi_i$$

The space $\Lambda^{\infty/2}$ has a super-structure with parity $\Lambda^{\infty/2,(m)}$ = the parity of m. Setting $|0\rangle = e_0 \wedge e_1 \wedge e_2 \ldots$, $\mathcal{F} = \{\varphi(z), \varphi^*(z)\}$, T what it needs to be, we get a vertex superalgebra. In fact,

$$F^{\text{ch}} = U(\hat{\mathfrak{a}}) \otimes_{U(\hat{\mathfrak{a}_+})} \mathbb{C}|0\rangle \xrightarrow{\simeq} \Lambda^{\infty/2}$$

is an isomorphism of vertex algebras. Indeed, if $n_j, m_k \geq 1$,

$$\varphi_{(n-1)}\varphi_{(-n_2)}\cdots\varphi_{(-n_s)}\varphi^*_{(m_1)}\cdots\varphi^*_{(m_t)}|0\rangle \quad = \pm e_{-n_1}\wedge e\ldots\wedge e_{-n_s}\wedge \underbrace{(e_0\wedge e_1\wedge\ldots)}_{\substack{\text{but with}\\ e_{m_1}\ldots e_{m_t}\\ \text{missing}}}$$

But these two pictures are bases of F^{ch} and $\Lambda^{\infty/2}$, so we have our linear isomorphism $F^{\mathrm{ch}} \to \Lambda^{\infty/2}$. Just need to check carefully φ and φ^* are compatible with it.

Pauli exclusion principle. In an ensemble state of fermions, two cannot occupy the same state, (so they are modelled by odd variables like "e_n" such that $e_n \wedge e_n = 0$.)

Electron waves should obey "quantized" wave-equation, which is something like

$$\partial_t^2\psi = (-\overline{h}^2\nabla + m)\psi$$
$$\partial_t\psi = \sqrt{-\overline{h}\nabla + m}\psi.$$

A solution to this is, instead of

$$\sqrt{\partial_{xx} + m},$$

try to define

$$\sqrt{(\partial_{xx} + m)I_4}$$

where I_4 is the identity 4×4 matrix. Then the "square root" of $-\overline{h}^2\nabla + m$ becomes a matrix valued differential operator, denoted sometimes as $\partial\!\!\!/ + \gamma$, where γ is an explicit 4×4 matrix. In $\partial_t\psi = \ldots$, ψ has become a vector-valued function on space, with a splitting in spin up and spin down parts of the electron wavefunction, and also an "unphysical" part. The latter can be interpreted as the positrons.

Problem: the Dirac equation has positive and negative energy solutions.

Let's examine the neutral part of $\Lambda^{\infty/2(0)}$. Consider the element

$$\underline{e} = e_{-4}\wedge e_{-2}\wedge e_{-1}\wedge e_0\wedge e_1\wedge e_4\wedge e_5\wedge \underbrace{e_7\wedge e_8\wedge e_9\wedge\ldots}_{\text{consecutive part}}$$

The successive differences are

$$2 \quad 1 \quad 1 \quad 1 \quad 3 \quad 1 \quad 2 \quad 1 \quad 1 \quad \ldots$$

Since the sequence will stabilize at 1, we might as well subtract 1 and obtain the finite sequence

$$(1 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 1)$$

In fact, you could reconstruct $\underline{e}$ from this list, and knowing that $\mathrm{charge}(\underline{e}) = 0$.

Form the partial sums

$$1000201 \to 4333311$$

these numbers are non-increasing, so define uniquely a partition of some integer (in this case 18), and if we compare with

$$\underline{e} = \pm\varphi_{-4}\varphi_{-2}\varphi_{-1}\varphi_2^*\varphi_3^*\varphi_6^*|0\rangle$$

which has energy

$$\Delta = (4-1/2) + (2-1/2) + (1-1/2) + (2+1/2) + (3+1/2) + (6+1/2) = 18.$$

We denote by $\underline{e}_\lambda$ the semi-infinite monomial of charge 0 given by this procedure.

So we have a basis $\{\underline{e}_\lambda | \lambda \text{ integer partitions}\}$ of $F^{(0)} \simeq H = \mathbb{C}[x_1, x_2, x_3, \ldots]$, which also has the basis $\{\underline{x}_\lambda = x_1^{\lambda_1} \ldots x_5^{\lambda_5} | \lambda \text{ integer partitions}\}$ where $\lambda = (1^{\lambda_1}, 2^{\lambda_2}, \ldots, s^{\lambda_s})$, that is, 1 appears λ_1 times and so on. We do **not** have $\underline{e}_\lambda = \underline{x}_\lambda$!

16. Schur polynomials

The Schur polynomials are given by the exponential generating function of $\sum z^k x_k$. More precisely,

Definition 16.1. We set $S_k(\underline{x})$ by

$$\sum_{k\geq 0} z^k S_k(\underline{x}) = \exp\left(\sum_{k\geq 1} z^k x_k\right).$$

For a partition $\lambda = (1^{\lambda_1} 2^{\lambda_2} \ldots k^{\lambda_k})$, define

$$S_\lambda(\underline{x}) = \det\begin{pmatrix} S_{\lambda_1} & S_{\lambda_1+1} & \cdots & S_{\lambda_1+k-1} \\ S_{\lambda_2-1} & S_{\lambda_2} & \cdots & \\ \vdots & & & \vdots \\ S_{\lambda_k-k+1} & & & S_{\lambda_k} \end{pmatrix}$$

Recall we defined $X = \bigoplus_{i\in\mathbb{Z}} \mathbb{C} e_i$ and $\Lambda^{\infty/2} = \langle e_{i_0} \wedge \ldots \rangle$. Let $g : X \to X$ be an invertible endomorphism. We would like to define an endomorphism

$$R(g) : \Lambda^{\infty/2} \longrightarrow \Lambda^{\infty/2}$$
$$R(g)(e_{i_0} \wedge e_{i_1} \wedge \ldots) = (ge_{i_0}) \wedge (ge_{i_1}) \wedge \ldots$$

For some g, εg, $g(e_n) = e_{-n}$, $R(g)$ does not make sense (or does not send $\Lambda^{\infty/2} \to \Lambda^{\infty/2}$).

As a note, if X were finite-dimensional, say $X = \langle e_1, \ldots, e_n \rangle$, then $R(g) : \Lambda^{k+1} X \to \Lambda^{k+1} X$ has matrix entry

$$(e_{i_0} \wedge e_{i_1} \wedge \ldots \wedge e_{i_k}) \to (e_{j_0} \wedge e_{j_1} \wedge \ldots \wedge e_{j_k}),$$

(with strictly increasing indices on both sides, up to a sign maybe) given by the determinant of the $(k+1) \times (k+1)$ matrix given by selecting the rows $i_0, i_1, \ldots, i_k$, and the columns $j_0, j_1, \ldots, j_k$ of the matrix of g.

For our X and $\Lambda^{\infty/2}$, $R(g)$ makes sense if g is such that

$$g(e_i) - e_i \in \operatorname{span}\{e_{>i}\} \qquad \forall i.$$

Proposition 16.2. *$\underline{e}_\lambda = S_\lambda(\underline{x})$ where*

Proof. Write $\underline{e}_\lambda = P(x)$. We wish to show $P(x) = S_\lambda(x)$. Introduce new variables $y_1, y_2, \ldots$ and consider

$$F(y) = \exp\left(\sum_{k>0} y_k \frac{\partial}{\partial x_k}\right) P(x)\Big|_{x=0}$$

First we observe that

$$F(y) = P(x_1 + y_1, x_2 + y_2, \ldots)|_{x=0} = P(x+y)|_{x=0} = P(y)$$

since in general exponentiaing a differential operator gives shifting by the coefficient, i.e. $e^{a\frac{\partial}{\partial t}} f(t) = f(t+a)$.

Considering

$$P(x) \in \mathbb{C}[x_1, x_2, \ldots] = H \simeq F^{(0)},$$

observe that for $k > 0$,

$$\frac{\partial}{\partial x_k} = h_k = (:\varphi\varphi^*:)_k = \sum_{i \in \mathbb{Z}} \varphi_{k+1}\varphi_i^*$$

i.e. it's a sum of "remove e_i" and "insert e_{k+i}".

Denote

$$\Lambda_k : X \longrightarrow X$$
$$\Lambda_k(e_n) = e_{n+k} \qquad \forall n \in \mathbb{Z}$$

$R(\Lambda_k) = \frac{\partial}{\partial x_k}$, since $\Lambda^{\infty/2(0)} \xrightarrow{\simeq} H$.

Therefore

$$F(y) = R\exp\left(\sum_{k>0} y_k \Lambda_k\right) \underline{e}_\lambda \Big|_{\text{coef. in } |0\rangle}.$$

Notice that $R\exp(\Lambda_k)$ is of the form $g(e_i) - e_i$.

Finally, notice that $\Lambda_k = \Lambda_1^k$, so

$$\begin{aligned} &R\exp\left(\sum_{k>0} y_k \Lambda_k\right) \underline{e}_\lambda \Big|_{\text{coef. in } |0\rangle} \\ &= R\exp\left(\sum y_k \Lambda_1^k\right) \underline{e}_\lambda \\ &= R((S_k\Lambda_k)\, \underline{e}_\lambda. \end{aligned}$$

□

17. The Tate extension and the Japanese cocycle

Recall $X = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}e_n$ and $\Lambda^{\infty/2}$. (You can be floppy and think that $\Lambda^{\infty/2}$ is the exterior power of X, though that's not completely right.) Define

$$\mathfrak{gl}_\infty = \left\{(a_{ij}) : {}^{\;i,j\in\mathbb{Z}, a_{ij}\in\mathbb{C}}_{\text{all but finitely many of the } a_{ij} \text{ vanish}}\right\},$$

add and multiply as usual.

$\mathrm{GL}_\infty = \{I + (a_{ij}) : a_{ij} \in \mathfrak{gl}_\infty\}$, with $I = (I_{ij})$ and $I_{ij} = \delta_{ij}$. A typical element of $\mathfrak{gl}_\infty$ is E_{ij} with $E_{ij}(e_k) = \delta_{jk} e_i$.

Last time we used shift operators $\Lambda_k : e_n \mapsto e_{n+k}$ for all n. These are **not** in $\mathfrak{gl}_\infty$, but **are** in

$$\widetilde{\mathfrak{gl}_\infty} = \left\{(a_{ij}) : {}^{\;\exists N s.t. a_{ij}=0}_{\text{whenever } |i-j|>N}\right\}.$$

Then $\Lambda_k = \sum E_{i+k,k} \in \widetilde{\mathfrak{gl}_\infty}$.

Remark 17.1. $\widetilde{\mathfrak{gl}_\infty}$ is an associative Lie algebra (hence a Lie algebra with commutator). This is because the condition of finiteness in the definition excludes the possibility of infinite sums.

We have a represenation r of $\mathfrak{gl}_\infty$ on $\Lambda^{\infty/2}$ by

$$r : E_{ij} \mapsto \varphi_i \varphi_j^*$$

But this doesn't extend to $\widetilde{\mathfrak{gl}_\infty}$. For instance, $r(\Lambda_0) = \sum_{j\in\mathbb{Z}} \varphi_{-j}\varphi_j^*$ diverges.

A "solution" is to use normal order, so $r(\Lambda_0) = \sum_{j\in\mathbb{Z}} :\varphi_{-j}\varphi_j^*:$, is now well-defined, but r is no longer a representation.

Today: more conceptual point of view. Consider the vector space $\mathbb{C}(\!(t)\!) = X$. (Which is *uncountably* infinite-dimensional.) Give X a linear topology with a base of open neighbourhoods of 0 being $t^N\mathbb{C}[\![t]\!]$ for $N \in \mathbb{Z}$. The idea is that $t, t^2, t^3, \dots$ "tends to 0" in this topology.

Definition 17.2. A *Tate vector space* is a linearly topologised vector space X and a set $\mathcal{L}$ of linear subspaces $L \subset X$ (called *lattices*) such that

(1) (Separated.) Any neighbourhood of 0 contains a lattice.
(2) (Exhaustive.) Every $x \in X$ is contained in some lattice.
(3) (Commensurable.) For all $L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2$ has finite codimension in L_1 (and in L_2). (This says that any two lattices cannot be "infinitely far apart".)
(4) (Complete.) For all $L_1, L_2 \in \mathcal{L}$, all vector subspaces $L_1 \cap L_2 \subset S \subset L_1 + L_2$ then $S \in \mathcal{L}$.
(5) (Another completeness.) By the universal property of limit, we know there exists a map $X \to \lim_{\substack{L\in\mathcal{L}\\ L\to 0}} X/L$. We ask this is an isomorphism.

Example 17.3. (1) Laurent series is an example of a Tate vector space. Let $X = \mathbb{C}(\!(t)\!)$ with

$$\mathcal{L} = \left\{ L \subset X : \exists N \text{ s.t. } t^N\mathbb{C}[\![t]\!] \subset L \subset t^{-N}\mathbb{C}[\![t]\!] \right\}.$$

We can also say that $\mathcal{L}$ is the unique structure such that $\mathcal{L} \ni \mathbb{C}[\![t]\!]$.
(2) If $\mathcal{L} = \{0\}$, then $\mathcal{L} = \{L \subset X \mid \dim(L) < \infty\}$.
(3) If $\mathcal{L} \ni X$, then $\mathcal{L} = \{L \subset X \mid \dim(X/L) < \infty\}$.

Since we introduced a topology on X, we can consider the continuous endomorphisms $\mathrm{End}_{\mathrm{cont}}(X)$. Let's denote throughout this section

$$\mathrm{End}(X) = \{f \in \mathrm{End}_{\mathrm{cont}}(X) : \exists U, V \in \mathcal{L}, f(U) \subset V\}.$$

Relative to the "basis" $\{t^n | n \in \mathbb{Z}\}$, the matrix of $f \in \mathrm{End}(X)$ looks like

$$\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \begin{pmatrix} 0 \\ * \end{pmatrix}$$

since the vectors in $V \subset t^m\mathbb{C}[\![t]\!]$ have zeroes before a certain index n.

$$\mathrm{End}_c(X) = \{f \in \mathrm{End}(X) | \exists V \in \mathcal{L} \text{ s.t. } f(X) \subset V\} = \left\{ \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \right\}$$

$$\mathrm{End}_d(X) = \{f \in \mathrm{End}(X) | \exists U \in \mathcal{L} \text{ s.t. } f(U) = 0\} = \left\{ \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \right\}$$

$$\mathrm{End}_f(X) = \mathrm{End}_c(X) \cap \mathrm{End}_d(X) = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\}.$$

Remark 17.4. For $f \in \mathrm{End}_f(X)$, the trace $\mathrm{Tr}(f) \in \mathbb{C}$ is well-defined because the intersection of the diagonal with the lower-left part of any matrix in $\mathrm{End}_f(X)$ is finite. For End_c and End_d trace is not well-defined.

We introduce the map

$$\begin{aligned} p : \mathrm{End}_c(X) \oplus \mathrm{End}_d(X) &\longrightarrow \mathrm{End}(X) \to 0 \\ (f,g) &\longmapsto f+g \end{aligned}$$

which is clearly surjective since we can put

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

We can complete this to a short exact sequence (of vector spaces) by putting

$$\begin{aligned} i : \mathrm{End}_f(X) &\longrightarrow \mathrm{End}_c \oplus \mathrm{End}_d \\ h &\longmapsto (h, -h). \end{aligned}$$

Considering the trace $\mathrm{Tr} : \mathrm{End}_f(X) \to \mathbb{C}$, we can form the pushout L in the category of **vector spaces**

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{End}_f \arrow[r] \arrow[d, "\mathrm{Tr}"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{End}_c(X) \oplus \mathrm{End}_d(X) \arrow[r] \arrow[d] & \mathrm{End}(X) \arrow[r] & 0 \\ & \mathbb{C} \arrow[r] & L & & \end{tikzcd}$$

Concretely,

$$L = \frac{(\mathrm{End}_c(X) \oplus \mathrm{End}_d(X)) \oplus \mathbb{C}}{\langle (h,-h) = (0,0,\mathrm{Tr}(h)) | \forall h \in \mathrm{End}_f(X) \rangle}.$$

Notice that the map i is **not** a Lie algebra map: for this it would have to be a Lie algebra map in both entries, which is true in the first entry but not on the second. Meanwhile, p *is* a morphism of Lie algebras. We could correct i to map $h \mapsto (h,h)$ and then change p to be $\beta - \gamma$, but then p would stop being a Lie algebra morphism. So this is not a short exact sequence of Lie algebras; indeed it's not sensible to think of short exact sequences of Lie algebras since Lie algebras do not form an Abelian category.

But it is a short exact sequence of vector spaces and we can form its exact sequence pushout by the following exercise. (Also see Stacks Project tag 010I.)

Exercise 17.5. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of vector spaces and $t : A \to K$ a linear map of vector spaces. Let X be the pushout

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "t"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ K \arrow[r] & X. \end{tikzcd}$$

Explicitly,

$$X = \frac{K \oplus B}{\langle (t(a), 0) - (0, i(a)) | a \in A \rangle}.$$

Show that there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow C \longrightarrow 0$$

such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "t"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & K \arrow[r] & X \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

commutes.

Proof. To define a map $X \to C$ we first notice that all we have to do is define the map on $K \oplus B$ at points not coming from A, i.e. whose entries are not of the form $(i(a), t(a))$ for $a \in A$. Indeed, those are cancelled by the pushout definition as a quotient. Further, we already have a map $\varphi : B \to C$ defined on elements not of the form $i(a)$ for $a \in A$ by exactness of the given short exact sequence. Then we must define $(k, b) \mapsto \varphi(b)$ for the diagram to commute.

The resulting exact sequence is exact once we know that $K \to B$ in the pushout is given by $k \mapsto (k, 0) \mathrm{mod} \ \sim$. ☐

Then we obtain

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{End}_f \arrow[r] \arrow[d, "\mathrm{Tr}"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{End}_c(X) \oplus \mathrm{End}_d(X) \arrow[r] \arrow[d] & \mathrm{End}(X) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & L \arrow[r] & \mathrm{End}(X) \arrow[r] & 0 \end{tikzcd}$$

Notice that the exact sequence

$$0 \longrightarrow \mathrm{End}_f \xrightarrow{i} \mathrm{End}_d \oplus \mathrm{End}_f \xrightarrow{p} \mathrm{End} \longrightarrow 0$$

is a bit more than just a short exact sequence of vector spaces. Notice that $\mathrm{End}_d, \mathrm{End}_c \subset \mathrm{End}$ are ideals. Indeed, if $f \in \mathrm{End}_c$ and $g \in \mathrm{End}$, then $f \circ g \in \mathrm{End}_c$ obviously, $f(X) \subset V$ so $g \circ f(X) \subset g(V)$, continuity of g (plus axioms of Tate vector space) implies that there exists $\widetilde{V} \in \mathcal{L}$ such that $g(V) \subset \widetilde{V}$.

Exercise 17.6. Spell this out.

So $g \circ f \in \mathrm{End}_c$.

Exercise 17.7. $\mathrm{End}_d \subset \mathrm{End}$ is an ideal too.

So $\mathrm{End}_c, \mathrm{End}_d$ are End-modules (as associative algebras and as Lie algebras), and

$$0 \longrightarrow \mathrm{End}_f \longrightarrow \mathrm{End}_d \oplus \mathrm{End}_c \longrightarrow \mathrm{End} \longrightarrow 0$$

is a short exact sequence of End-modules.

Let $\mathfrak{g}$ be a Lie algebra, E a $\mathfrak{g}$-module with $E \xrightarrow{p} \mathfrak{g}$ surjective $\mathfrak{g}$-module morphism. Consider

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} E \xrightarrow{p} \mathfrak{g} \longrightarrow 0$$

i.e. $\mathfrak{a} = \mathrm{Ker}(p)$.

Then one obtains a symmetric bilinear form defined by

$$(a, b) = p(a)b + p(b)a.$$

Exercise 17.8. This form is (1) symmetric and (2) $\mathfrak{g}$-invariant.

Exercise 17.9. Check that the data above, satisfying $(\cdot,\cdot) = 0$ is the same as a central extension of $\mathfrak{g}$ by $\mathfrak{a}$, i.e. Lie bracket $[\cdot,\cdot] : E \times E \to E$ compatible with $[\cdot,\cdot]$ on $\mathfrak{g}$ such that $\mathfrak{a} \subset E$ is central. **Hint.** Set $[a,b]^E = p(a)b$.

As a particular case of this, if we have

- $\mathfrak{g}$ a Lie algebra,
- $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{g}$ two ideals such that $\mathfrak{g} = \mathfrak{a}_1 + \mathfrak{a}_2$,
- setting $\mathfrak{a}_0 = \mathfrak{a}_1 \cap \mathfrak{a}_2$, a linear map $\mathfrak{a}_0 \xrightarrow{T} K$ such that $T([x_1, x_2]) = 0$ for all $x_1 \in \mathfrak{a}_1$ and $x_2 \in \mathfrak{a}_2$.

Then we can form

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a}_0 \arrow[r, "i"] \arrow[d, "T"'] & \mathfrak{a}_1 \oplus \mathfrak{a}_2 \arrow[r, "p"] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & K \arrow[r] & X \arrow[r] & \mathfrak{g} \arrow[r] & 0 \end{tikzcd}$$

with pushout at level of vector spaces and $i : \alpha \mapsto (\alpha, -\alpha)$ and $p : (\beta, \gamma) \mapsto \beta + \gamma$.

Exercise 17.10. Then X will become a Lie algebra with K central and $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \to X$ a map of Lie algebras.

The main point of this construction is to use $T([x_1, x_2])$ to confirm that

$$T(p(a)b + p(b)a) = 0 \qquad \forall a, b \in \mathfrak{a}_1 \oplus \mathfrak{a}_2.$$

Indeed, write $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Then

$$\begin{aligned} &T(([a_1, a_2, b_1], [a_1 + a_2, b_2]) + ([b_1 + b_2, b_1], [b_1 + b_2, a_2]) \\ &= T([a_1, b_1] + [a_2, b_1] + [b_1, a_1] + [b_2, a_1], [a_1, b_2] + [a_2, b_2] + [b_1, a_2] + [b_2, a_1]) \\ &= T(i(\underbrace{[a_2, b_1]}_{\in [\mathfrak{a}_1, \mathfrak{a}_2]} + \underbrace{[b_2, a_1]}_{\in [\mathfrak{a}_2, \mathfrak{a}_1]})) \\ &= 0 \qquad \text{by hypothesis.} \end{aligned}$$

We apply this, in particular, to

$$\mathfrak{g} = \mathrm{End}(X) \qquad \mathfrak{a}_1 = \mathrm{End}_d(X) \qquad \mathfrak{a}_2 = \mathrm{End}_c(X) \qquad \mathfrak{a}_0 = \mathrm{End}_f(X)$$

$$X \text{ our Tate vector space} \qquad T \text{ trace.}$$

For the construction above to work we still have to check that

$$\mathrm{Tr}([A_d, A_c]) = 0$$

whenever $\mathrm{Ad} \in \mathrm{End}_d$ and $A_c \in \mathrm{End}_c$. You would think this is obvious, but it isn't. (I.e., $\mathrm{Tr}(AB - BA) = 0$ for $A, B \in \mathrm{End}$(finite-dimensional vector space).)

Let us sketch the proof. Let $f \in \mathrm{End}_c$ and $f(X) \subset V$ ($V \in \mathcal{L}$), and $g \in \mathrm{End}_d$ and $g(U) = 0$ ($U \in \mathcal{L}$). Then

$$X \xrightarrow{f} \underbrace{f(X)}_{\subset V} \xrightarrow{g} \underbrace{gf(X)}_{\subset g(V)}.$$

And

$$X \xrightarrow{g} g(X) \to \underbrace{fg(X)}_{\subset V}.$$

So $\mathrm{Im}[f,g] \subset V + g(V)$ So $[f,g](U \cap f^{-1}(U)) = 0$.

$$\underbrace{f^{-1}g^{-1}(0)}_{\subset f^{-1}(U)} \xrightarrow{f} \underbrace{g^{-1}(0)}_{\subset U} \xrightarrow{g} 0$$

and

$$\underbrace{g^{-1}f^{-1}(U)}_{\subset U} \xrightarrow{g} f^{-1}(0) \xrightarrow{f} 0.$$

Now the idea is to argue that

$$\mathrm{Tr}[f,g] = \mathrm{Tr}_Q[f,g],$$

where

$$Q = (V + g(V))/(U \cap f^{-1}(U))$$

is a finite-dimensional vector space. But on finite dimensional vector spaces the trace of commutator vanishes.

You might imagine that $\mathrm{Tr}[f,g] = 0$ whenever $[f,g] \in \mathrm{End}_f(X)$, but this is false.

Example 17.11. Let $A \subset \mathrm{End}(X)$ be a commutative subalgebra. (For instance, if $X = \mathbb{C}((t))$, then $f(t) \in \mathbb{C}((t))$ we have $\mu_f \in \mathrm{End}(X)$ and $\mu_f(g) = fg$.) Then I claim, if $f, g \in A$, $f = f_c + f_d$, $g = g_c + g_d$, that $[f_c, g_c] \in \mathrm{End}_f$. Obviously $[f_c, g_c] \in \mathrm{End}_c$, but also

$$\begin{aligned} [f_c, g_c] &= [f - f_d, g - g_d] \\ &= \underbrace{[f,g]}_{=0} \underbrace{-[f, g_d] - [f_d, g] + [f_d, g_d]}_{\substack{\in \mathrm{End}_d \text{ because} \\ \mathrm{End}_d \overset{\text{ideal}}{\subset} \mathrm{End}}}. \end{aligned}$$

I claim that $\mathrm{Tr}[f_c, g_c]$ might not vanish.

In fact let $X = \mathbb{C}((t))$, $f = \mu_{t^{-2}}$, $g = \mu_{t^2}$ (or with any $N \geq 0$ in place of 2).

To fix f_c, g_c, lwt's make the following choice:

$$\begin{aligned} \pi &: X \longrightarrow X \\ \pi(t^n) &= \delta_{n \geq 0} t^n. \end{aligned}$$

Then set $f_c = \pi \circ f$, etc. Let's compute $[f_c, g_c]$. [Picture]. We obtain $\mathrm{Tr}[f_c, g_c] = 2$.

The next proposition is "Tate's definition of the residue". Tate was trying to generalize the Residue Theorem, i.e. that $\sum_{p \in C} \mathrm{Res}_p \omega = 0$ for points on a curve C and a differential form ω.

Proposition 17.12. *For any choice of splittings, $f = f_c + f_d$, etc, and for all $f, g \in \mathbb{C}((t))$,*

$$Tr[f_c, g_c] = Res_t f \cdot dg.$$

18. Representing the endomorphisms algebra on the charged fermions

We want to represent $\mathrm{End}(X)$ on $\Lambda^{\infty/2}$. We start presenting the naive idea. Let $x \in X$ and $\varphi \in X^*$. We already have

$$\rho(x) = x \wedge (-) \in \mathrm{End}(\Lambda^{\infty/2})$$

and

$$\rho(\varphi) = \sum_{j=0}^{\infty} (-1)^j \varphi(x_{i_j}) x_{i_0} \wedge x_{i_1} \wedge \ldots \wedge \widehat{x_{i_j}} \wedge \ldots.$$

We already saw that $\rho(\hat{x}_i)\rho(\hat{\varphi}_j) + \rho(\hat{\varphi}_j)\rho(\hat{x}_i) = \delta_{ij} I$ (here I'm identifying $\hat{x}_i \equiv t^i \in \mathbb{C}((t)) = X$, $\varphi_i \in X^*$ is $\hat{\varphi}_i(\sum c_j t^j) = c_j$). See Exercise 15.1.

If $f \in \mathrm{End}_d$, we can think of f as

$$f = \sum_{i=0}^{a} x_i \otimes \varphi_i,$$

where $\varphi_i \to 0$ as $i \to \infty$.

Here $x_i \to 0$ means $\forall N \geq 0$ $\exists n_0$ such that $x_n \in t^N \mathbb{C}[[t]]$ for all $n \geq n_0$. And $\varphi_i \to 0$ means $\forall N \geq 0$ $\exists n_0$ such that $\varphi_n(t^{-N}\mathbb{C}[[t]]) = 0$ for all $n \geq n_0$.

One can confirm that for $f \in \mathrm{End}_d$,

$$\rho_d(f) := \sum_i \rho(x_i)\rho(\varphi_i)$$

acts a finite sum, when applied to any fixed vector of $\Lambda^{\infty/2}$. So

$$\rho_d : \mathrm{End}_d \to \mathrm{End}(\Lambda^{\infty/2} X)$$

is well defined.

For $\mathrm{End}_c \ni f$, we can write

$$f = \sum_i x_i \otimes \varphi_i,$$

where $x_i \to 0$. Now $\rho_d(f)$ makes no sense, but

$$\rho_c(f) := \sum_{i=0}^{\infty} \rho(\varphi_i)\rho(x_i)$$

does.

Exercise 18.1. Confirm that $\rho_d : \mathrm{End}_d \to \mathrm{End}(\Lambda^{\infty/2})$ and $\rho_c : \mathrm{End}_c \to \mathrm{End}(\Lambda^{\infty/2} X)$ are morphisms of Lie algebras.

(Neither of these morphisms work at the level of associative algebras.)

We notice that, for $f \in \mathrm{End}_f(X)$,

$$\rho_d(f) - \rho_c(f) = \mathrm{Tr}(f).$$

This means

$$\mathrm{End}_d \oplus \mathrm{End}_c \xrightarrow{(\rho_d, \rho_c)} \mathrm{End}(\Lambda^{\infty/2} X)$$

descends to a morphism

$$\begin{tikzcd} \mathrm{End}_d \oplus \mathrm{End}_c \arrow[r, "(\rho_d,\rho_c)"] \arrow[d] & \mathrm{End}(\Lambda^{\infty/2} X) \\ \mathfrak{gl}^{\flat}(X). \arrow[ur, "\rho"'] & \end{tikzcd}$$

Theorem 18.2. *Let X be a Tate vector space. There exists a natural representation of $\mathfrak{gl}^{\flat}(X)$ on $\Lambda^{\infty/2}(X)$.*

Next time we'll apply this to the case X itself is already a Lie algebra!

19. The 26-dimensionality of the universe

Last time: X a Tate vector space (for us $X = \mathbb{C}((t))$). We say that $\mathrm{End}(X) = \mathrm{End}_c(X) + \mathrm{End}_d(X)$ comes with a canonical central extension (as a Lie algebra

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathfrak{gl}(X)^\flat \longrightarrow \mathrm{End}(X) \longrightarrow 0$$

with

$$\mathfrak{gl}(X)^\flat = \frac{\mathrm{End}_c \oplus \mathrm{End}_d \oplus \mathbb{C}}{\langle (h, -h, 0) = (0, 0, \mathrm{Tr}(h)) | h \in \mathrm{End}_f(X) \rangle},$$

and there is a canonical representation of $fl(X)^\flat$ on the semi-infinite wedge space $\Lambda^{\infty/2}(X) = \langle i_{i_0} \wedge e_{i_1} \wedge \dots \rangle$ defined by

$$\begin{aligned} &\rho_d(\underbrace{\sum_i x_i \varphi_i}_{\in \mathrm{End}_d?}) = \sum_i \rho(x_i)\rho(\varphi_i) \\ &\rho_c(\underbrace{\sum_i x_i \varphi_i}_{\in \mathrm{End}_c}) = -\sum_i \rho(\varphi_i)\rho(x_i), \end{aligned} \tag{19.0.1}$$

where

$$\rho(x) = x \wedge (-), \qquad \rho(\varphi) = \sum_{j=0}^{\infty} (-1)^j \varphi(e_{i_j}) e_{i_0} \wedge e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots$$

Now suppose X itself were a Lie algebra.

Example 19.1. $X = \mathbb{C}((t))$ as above, but we identify $t^m \rightsquigarrow L_m$ in Vir @ $c = 0$. So $[t^m, t^n] = (m-n)t^{m+n}$. To avoid confusion, let's write t^m as L_m in fact. In such situation, we have a linear map $\mathrm{ad} : X \to \mathrm{End}(X)$.

We can pull back $\mathfrak{gl}(X)^\flat$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] & \mathfrak{gl}(X)^\flat \arrow[r, "\pi"] & \mathrm{End}(X) \arrow[r] & 0 \\ & & \hat{X} \arrow[u] \arrow[r] & X. \arrow[u, "\mathrm{ad}"'] & \end{tikzcd}$$

Explicitly,

$$\hat{X} = \{(x, A) \in X \oplus \mathfrak{gl}(X)^\flat | \mathrm{ad}(X) = \pi(A) \text{ in } \mathrm{End}(X)\}.$$

Exercise 19.2. Similarly to Exercise 17.5, show $\hat{X}$ fits into a sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \hat{X} \longrightarrow X \longrightarrow 0.$$

So we get a (canonical!) central extension of X:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathfrak{gl}(X)^\flat \arrow[r, "\pi"] & \mathrm{End}(X) \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & \hat{X} \arrow[u] \arrow[r] & X \arrow[u, "\mathrm{ad}"'] \arrow[r] & 0 \end{tikzcd}$$

For $X = \mathrm{Der}(\mathbb{C}[t^{\pm 1}] =$ centerless Vir, we will find the canonical central extension is Vir @ $c = -26$. (i.e. charge is -26.)

What might be amazing here is that any infinite-dimensional Lie algebra has a central extension — just by being infinite-dimensional.

We'll exploit

$$\Lambda^{\infty/2}(X) = \langle L_{m_0} \wedge L_{m_1} \wedge L_{m_2} \wedge \ldots \rangle .$$

We have quantum fields

$$\varphi(w) = \sum_n \varphi_n w^{-n-1}, \qquad \varphi = L_n \wedge (-),$$
$$\varphi^*(w) = \sum_n \varphi_n^* w^n, \qquad \varphi_n^*(\underline{L}) = \sum (-1)^j \varphi(L_{i_j}) L_{i_0} \wedge L_{i_1} \wedge \ldots$$

Now for

$$L_m \in X, \qquad L_m : L_n \mapsto (m-n)L_{m+n},$$
$$\mathrm{ad}(L_m) = \underbrace{\sum_{n\in\mathbb{Z}} (m-n) L_{m+n} L_n^*}_{\in \mathrm{End}_c + \mathrm{End}_d} .$$

To represent $\mathrm{ad}(L_m)$ on $\Lambda^{\infty/2}$ we have to split $\mathrm{ad}(L_m)$ into pieces in End_d and End_c and send each part to their corresponding pieces according to Equation 19.0.1.

Let's work at the level of fields.

$$\begin{aligned}
L(w) &= \sum_m \rho(\mathrm{ad}(L_m)) w^{-m-2} \\
&= \sum_{m,n} (m-n) \rho(L_{m+n} L_n^*) w^{-m-2} \\
&= \sum_{m,n} (m-n) \ :\varphi_{m+n}\varphi_n^*: \ w^{-m-2} \\
&= \sum_{m,n} (m-n) \ :(\varphi_{m+n} w^{-(m+n)-?})(\varphi_n^* w^{+n+?}:
\end{aligned}$$

where

$$\rho(L_{m+n}L_n^* = \begin{cases} \rho(L_{m+n})\rho(L_n^*) & \text{when } n \ll 0 \\ -\rho(L_n^*)\rho(L_{m+n}) & \text{when } n \gg 0. \end{cases}$$

Notice that

$$\begin{aligned}
\partial_w \varphi(w) &= \partial_w \sum \varphi_{m+n} w^{-m+nj} \\
&= -\sum (m+n) \varphi_{m+n} w^{-(m+n)-1} \\
&= \sum n \varphi_n^* w^{n-1}
\end{aligned}$$

since $m-n = (m+n) - 2n$, so it seems we should consider

$$- \ :(\partial\varphi)\varphi^*: \ -2 \ :\varphi(\partial\varphi^*): \ k$$

So, we have a vertex superalgebra $V = \Lambda^{\infty/2}$, with quantum fields φ and φ^*, OPE relation $[\varphi_\lambda \varphi^*] = 1$.

We are defining a field

$$L = - \ :(\partial\varphi)\varphi^* : \ -2 \ :\varphi(\partial\varphi^*): ,$$

(which is a quantum field, $L(w) = \sum_m L_m w^{-m-2}$), adn we want to know/check

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}CI_{\Lambda^{\infty/2}}.$$

So, what's c? Let's do it.

- (Method 1.) Brute force.
- (Method 2.) Use mathematica (Thielman's package).
- (Method 3.) Recall $\alpha = :\varphi\varphi^*:$, satisfies $[\alpha_\lambda\alpha] = \lambda$ and $L^0 = \frac{1}{2}\ :\alpha\alpha:$ is Virasoro @ $c = 1$.

$$[L^0_\lambda L^0] - TL^0 + 2\lambda L^0 + \frac{\lambda^3}{12}.$$

$$\begin{aligned} L^0 &= \frac{1}{2}\ :\alpha\alpha: \\ &= \frac{1}{2}\ :(:\varphi\varphi^*:)(:\varphi\varphi^*:): \\ &= \dots \\ &= \frac{1}{2}(:(T\varphi)\varphi^*: + :\varphi(T\varphi^*):\ . \end{aligned}$$

How to see this? Recall Borcherds identity (Equation 11.9.2):

$$\begin{aligned} &\sum_{j\geq 0}\binom{m}{j}(a_{(n+j)}b)_{(m+k-j)}c \\ &= \sum_{j\geq 0}(-1)^j\binom{n}{j}(a_{(a+m-j)}b_{(k+j)}c - (-1)^n b_{(n+k-j)}a_{(m+j)}c. \end{aligned}$$

So put

$$\begin{aligned} a &= \varphi & m &= 0 \\ b &= \varphi^* & n &= -1 \\ c &= :\varphi\varphi^*: & k &= -1. \end{aligned}$$

Then

$$LHS = (a_{(-1)}b)_{(-1)}c = :(:\varphi\varphi^*:)(:\varphi\varphi^*:):$$

and

$$RHS = \sum_{j\geq 0}(\varphi_{(-i-j)}\varphi^*_{(-1+j)}\varphi_{(-1)}\varphi^* + \varphi^*_{(-2-j)}\varphi_{(j)}\varphi_{(-1)}\varphi^*).$$

[Picture]

$$RHS = \varphi_{(-1)}\varphi^*_{(-1)}\alpha + \varphi_{(-2)}\varphi^*_{(0)}\alpha + \varphi^*_{(-2)}\varphi_{(0)}\alpha.$$

Note:

$$\varphi_{(0)}\alpha = -\alpha_{(0)}\varphi = +\varphi.$$

In general

$$b_{(0)}a = -\sum_{j\geq 0}\frac{1}{j!}T^j(a_{(j)}b) = -a_{(0)}b + T(\text{stuff}).$$

Remark 19.3. Recall (ref?) that if V is any vertex algebra, V/TV, $[\overline{a}, \overline{b}] = a_{(0)}b$, is a Lie algebra.

So,

$$RHS = \varphi_{(-1)}\varphi^*_{(-1)}\alpha - \varphi_{(-2)}\varphi^* + \varphi^*_{(-2)}\varphi.$$

Now, since in general

$$b_{(n)}a = -(-1)^{p(a)p(b)} \sum_{j\geq 0} \frac{(-1)^j}{j!} T^j (a_{(n+j)}b,$$

then

$$\varphi^*_{(-2)}\varphi = \sum_{j\geq 0} \frac{(-1)^j}{j!} T^j (\varphi_{(-2+j)}\varphi^*.$$

But this led to mistaken calculations. I was trying to do the following: we know that $\alpha = :\varphi\varphi^*:$ satisfies $[\alpha_\lambda \alpha] = \lambda$, and $L^0 = \frac{1}{2} :\alpha\alpha:$ is $\mathrm{Vir}[L^0_\lambda L^0] = TL^0 + 2\lambda L^0 + \frac{\lambda^3}{12}$ $(c = 1)$. Also, if $B = L^0 + kT_\alpha$ then

$$[B_\lambda B] = TB + 2\lambda B + \frac{\lambda^3}{12} c_k, \qquad c_k = 1 - 12k^2.$$

Writing B in terms of vp, φ^*, one gets <u>something</u> like

$$B = b\ :(T\varphi)\varphi^*:\ + (1-b)\ :\varphi(T\varphi^*):\ .$$

The correct answer is: if $L = -\ :(T\varphi)\varphi^*:\ -2\ :\varphi(T\varphi^*):$, we may verify that

$$[L_\lambda L] = TL + 2\lambda L - \frac{26}{12}\lambda^3.$$

In summary, today we did the following. For φ, φ^* odd, $[\varphi_\lambda \varphi^*] = 1$, we define the operator

$$L = -\ :(T\varphi)\varphi^* := 2\ :\varphi(T\varphi^*):\ .$$

Then compute and find out that

$$[L_\lambda L] = TL + 2\lambda L + \frac{c}{12}\lambda^3,$$

where $c = -26$.

That is, after computing the central extension of the Virasoro algebra, we turn to the vertex algebra language to find that the central charge is -26.

Even more explicitly: by the Tate vector space construction we have the central extension

$$0 \longrightarrow \mathbb{C}C \longrightarrow \mathfrak{gl}(X)^\flat \longrightarrow \mathfrak{gl}(X) \longrightarrow 0$$

where $W = \mathbb{C}((t))$ is the Witt algebra, with bracket $[L_m, L_n] = (m-n)L_{m+n}$.

And then we do the pullback in the following way:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}K \arrow[r] \arrow[d, "\mathrm{id}"] & \mathfrak{gl}(W)^\flat \arrow[r, "\pi"] & \mathrm{End}(X) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C}K \arrow[r] & W \oplus \mathbb{C}K \arrow[r] \arrow[u] & W \arrow[r] \arrow[u, "\mathrm{ad}"'] & 0
\end{tikzcd}$$

Then we get a representation in which K goes to the identity, namely

$$\mathfrak{gl}(X)^\flat \curvearrowright \Lambda^{\infty/2}, \qquad K \mapsto \mathrm{Id}.$$

So, the vertex algebra language allows us to compute the central charge of the new algebra we obtained, $W \oplus \mathbb{C}K$, and realise it's -26. So the bracket in the new algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}(-26)K.$$

20. Lattice Vertex algebras

Definition 20.1. A *lattice* (for us) is a discrete subgroup $L \subset \mathbb{R}^n$, $L \simeq \mathbb{Z}^n$, and such that $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in L$, where $(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the standard bilinear form $((u_1, \dots, u_n), (v_1, \dots, v_n)) = \sum u_i v_i$.

In particular, L contains a basis of the ambient space $\mathbb{R}^n$.

Example 20.2. (1) $\mathbb{Z}^n \subset \mathbb{R}^n$ is a lattice.
(2) A_2, the root lattice of $\mathfrak{sl}_3$, is a lattice. Recall this looks like a triangular tiling with $(\alpha_1, \alpha_1) = 2$, $(\alpha_2, \alpha_2) = 2$ and $(\alpha_1, \alpha_2) = -1$, which in says that the angle between α_1 and α_2 is 120 degrees.

Definition 20.3. A lattice is called *even* if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$.

(That is, the product of every element with *itself* is even, not that $(\alpha, \beta) \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$.)

So A_2 is even. In fact, whenever the basis elements are even, say $(\alpha, \alpha), (\beta, \beta) \in 2\mathbb{Z}$, then the lattice is even since $(\alpha \pm \beta, \alpha \pm \beta) = (\alpha, \alpha) \pm 2(\alpha, \beta) + (\beta, \beta) \in 2\mathbb{Z}$ too.

Definition 20.4. The *dual* of L is

$$L^\vee = \{x \in \mathbb{R}^n | (x, \alpha) \in \mathbb{Z} \forall \alpha \in L\}.$$

Clearly $L \subset L^\vee$. But $L^\vee$ might be strictly larger.

Exercise 20.5. $L^\vee / L$ is a finite group.

Notice that for $\gamma, \delta \in L^\vee$ it might happen that $(\gamma, \delta) \notin \mathbb{Z}$.

For $L = A_2$ we find that

$$L^\vee = A_2 \cup (\omega + A_2) \cup (2\omega + A_2).$$

So $L^\vee / L \simeq \mathbb{Z}/3$ as groups.

For $L = \mathbb{Z}^n$, we have $L^\vee = \mathbb{Z}^n = L$, that is, $\mathbb{Z}^n$ is self dual. On the other hand $\mathbb{Z}^n$ is not even.

Are there any other even self-dual lattices (other than $\{0\}$)?

The simplest nontrivial example is

$$E_8 = \{(x_1, x_2, \dots, x_8) \in \mathbb{Z}^8 \cup (\frac{1}{2} + \mathbb{Z})^8 | \sum_{i=1}^{8} \equiv 0 \text{mod } 2\}.$$

Lemma 20.6. *E_8 is even and self-dual.*

Theorem 20.7. (1) *If $L \subset \mathbb{R}^n$ is even and self dual, then $8|n$*
(2) *If, also, $n = 8$, then $L \simeq E_8$.*

Remark 20.8. Let L be an even lattice. On the group $D = L^\vee / L$ we define

$$q : D \longrightarrow \mathbb{Q}/\mathbb{Z}$$
$$q(a) = (\alpha, \alpha)/2 \text{ mod } \mathbb{Z}.$$

Then q is a well-defined quadratic form.

Indeed, for $\beta \in L$,

$$q(\alpha+\beta) = \frac{(\alpha+\beta,\alpha+\beta)}{2} = \underbrace{\frac{(\alpha,\alpha)}{2}}_{=q(\alpha)} + \underbrace{\frac{2(\alpha,\beta)}{2}}_{\in\mathbb{Z}} + \underbrace{\frac{(\beta,\beta)}{2}}_{\in\mathbb{Z}}.$$

We call (D, q) the *discriminant form* of L.

Now we explain the neighbour construction/orbifold. Let L be a self-dual even lattice. Let $\phi : L \to \mathbb{Z}/2$ a homomorphism (i.e. if $\{e_1, \ldots, e_n\}$ is a basis of L, set $\phi(e_i) = \varepsilon_i \in \{0, 1\}$ and $\phi(\sum m_i e_i) = \sum m_i\varepsilon_i \bmod 2$.

Assume ϕ is nontrivial and surjective, so that its kernel $L_0 = \operatorname{Ker}\phi \subset L$ has index 2. That is, $L/L_0 \simeq \mathbb{Z}/2$.

Now we have $L_0 \underbrace{\subset}_{\text{index } 2} L = L^\vee \underbrace{\subset}_{\substack{\text{index } 2\\ \text{exercise}}} L_0^\vee$.

Remark 20.9. $D = L_0^\vee/L_0 \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

Why? If $\omega \in L_0^\vee$, then $2\omega\ldots$ Exercise.

What about (D, q)? I claim there are two possibilities (up to $\simeq$):

$$\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1/4 \\ 0 & 3/4 \end{pmatrix} \tag{20.9.1}$$

Exercise.

Suppose (by choice of ϕ) we are in the first case. Then $L = L_0 \cup (\alpha + L_0)$, $q(\beta) = 0$, i.e. $(\beta, \beta) \equiv 0 \bmod 2$,

$$\begin{pmatrix} L_0 & \beta + L_0 \\ \alpha + L_0 & \gamma + L_0 \end{pmatrix}.$$

Define $L^{\mathrm{orb}(\phi)} = L_0 \cup (\beta + L_0)$.

Lemma 20.10. *$L^{orb(\phi)}$ is an even lattice.*

Proof. Exercise. Idea: $(L_0, L_0) \subset \mathbb{Z}$ and $(L_0, \beta + L_0) \subset \mathbb{Z}$ by definition. ☐

For $\beta, \beta' \in \beta + L_0$,

$$\begin{aligned} (\beta', \beta) &= (\beta + \alpha, \beta), \qquad \alpha \in L_0 \\ &= \underbrace{(\beta, \beta)}_{\in\mathbb{Z}} + \underbrace{(\alpha, \beta)}_{\in\mathbb{Z}} \in \mathbb{Z} \end{aligned}$$

Also, $(L^{\mathrm{orb}(\phi)})^\vee = L^{\mathrm{orb}(\phi)}$. Indeed

$$L_0 \underbrace{\subset}_{\text{index } 2} L^{\mathrm{orb}(\phi)} \underbrace{\subset}_{\text{by above}} (L^{\mathrm{orb}(\phi)})^\vee \underbrace{\subset}_{\text{index } 2} L_0^\vee.$$

Theorem 20.11 (Niemer)**.** *If we consider the set Γ_n of all even self-dual lattices of rank n ($8|n$) as a graph, with edge whenever there exists an operation $L \to L^{orb(\phi)}$, then Γ_n is connected for all n.*

Remark 20.12. There is a reverse orbifold construction.

Let L be even self dual, $L = L_0 \cup L_1$, and

$$\begin{pmatrix} L_0 & L_+ \\ L_1 & L_- \end{pmatrix}$$

$L_0^\vee = L_0 \cup L_1 \cup L_+ \cup L_-$, say L_+ is $q(\alpha) = 0$ for all $alp \in L_+$ and $q(\beta) = 1/2$ for all $\beta \in L_-$ since we are fixed in the first case of Equation 20.9.1.

Define

$$\psi : L^{\mathrm{orb}(\phi)} \longrightarrow \mathbb{Z}/2$$
$$\psi(\alpha) = \begin{cases} 0 & \text{for } \alpha \in L_0 \\ 1 & \qquad\qquad \text{for } \alpha \in L_+. \end{cases}$$

By definition, $\mathrm{Ker}(\psi) = (L^{\mathrm{orb}(\phi)})_0 = L_0 \subset L^{\mathrm{orb}(\phi)}$. So $(L^{\mathrm{orb}(\phi)})_0^\vee = L_0^\vee$ and the discriminant form $(L^{\mathrm{orb}(\phi)})_0^\vee / L_0^{\mathrm{orb}(\phi)}$ brecovers the same picture.

$$(L^{\mathrm{orb}(\phi)})^{\mathrm{orb}(\phi)} = L.$$

$$\begin{array}{cccc} L & \underbrace{0}_{L_0^{\mathrm{orb}(\phi)} = L_0} & 0 & M \\ & 0 & 1/2 & \end{array}$$

Definition 20.13. Let $L \subset \mathbb{R}^n$ be an even lattice. The *root system* of L is

$$\Delta(L) = \{\alpha \in L | (\alpha, \alpha) = 2\}.$$

Proposition 20.14. *The root system of an even lattice is a root systems.*

Example 20.15. The root system of E_8 (the lattice) is E_8 (the root system).

If $\Delta(L_1) \not\cong \Delta(L_2)$ then $L_1 \not\cong L_2$.

In fact, in $\mathbb{R}^{16}$ there exists two even self-dual lattices, with root systems $E_8 \oplus E_8$ and D_{16}.

For rank 24, there exist (coincidentally) 24 even self-dual lattices called *Neimeier lattices*. They are constructed as a graph going from one another by applying the orbifold construction on different homomorphisms ϕ. Eventually the graph looks like this:

$$\begin{tikzcd}[arrows=dash] \bullet \arrow[dr] & & \\ & A_1^{24} \arrow[r] & \Lambda \\ \bullet \arrow[ur] & & \end{tikzcd}$$

where Λ is the *Leech lattice*, which has *empty* root system, that is, Λ contains 0, and no vectors of norm 2, and 196560 (?) vectors of squared norm 4.

There is something called the Siegel mass formula. Consider the orthogonal group of a lattice, namely

$$\begin{aligned} \mathrm{Aut}(L) &= \{g \in \mathrm{GL}_n(\mathbb{R}) | (g\alpha, g\beta) = (\alpha, \beta) \forall \alpha, \beta \in L \text{ and } g(L) \subset L\} \\ &= \{g \in O_n(\mathbb{R}) | g(L) \subset L\}, \end{aligned}$$

which is a finite group.

Then

$$\sum_{L\in\Gamma_n} \frac{1}{\#\mathrm{Aut}(L)} = \frac{|B_{n/2}|}{n} \prod_{1\le j\le n/2} \frac{|B_{2j}|}{4j}$$

where B_k is a Bernoulli number. (See Wikipedia page for Neimeier lattice.)

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