ALGEBRAIC GEOMETRY

github.com/danimalabares/stack

Contents

1. Sheaves

For a definition of presheaf, see Categories, Definition **??**.

Definition 1.1. Let X be a topological space.

(1) A *sheaf $\mathcal{F}$ of sets on X* is a presheaf of sets which satisfies the following additional property: Given any open covering $U = \bigcup_{i \in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

   there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.
(2) A *morphism of sheaves of sets* is simply a morphism of presheaves of sets.
(3) The category of sheaves of sets on X is denoted $Sh(X)$.

Let X be a topological space. Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The *stalk of $\mathcal{F}$ at x* is the set

$$\mathcal{F}_x = \operatorname{colim}_{x \in U} \mathcal{F}(U)$$

where the colimit is over the set of open neighbourhoods U of x in X. The set of open neighbourhoods is partially ordered by (reverse) inclusion: We say $U \geq U' \Leftrightarrow U \subset U'$. The transition maps in the system are given by the restriction maps of $\mathcal{F}$. See Categories, Section **??** for notation and terminology regarding (co)limits over systems. Note that the colimit is a directed colimit. Thus it is easy to describe $\mathcal{F}_x$. Namely,

$$\mathcal{F}_x = \{(U, s) \mid x \in U, s \in \mathcal{F}(U)\}/ \sim$$

with equivalence relation given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ with $x \in U''$ and $s|_{U''} = s'|_{U''}$. Given a pair (U, s) we sometimes denote s_x the element of $\mathcal{F}_x$ corresponding to the equivalence class of (U, x). We sometimes use the phrase "image of s in $\mathcal{F}_x$" to denote s_x. For example, given two pairs (U, s) and (U', s') we sometimes say "s is equal to s' in $\mathcal{F}_x$" to indicate that $s_x = s'_x$. Other authors use the terminology "germ of s at x".

2. Abelian sheaves

The following may be used to define the ideal sheaf of a variety:

Lemma 2.1. *Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ be sheaves over X with values on Grp. For every morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$,*

$$\operatorname{Ker} f : Open_X^{op} \longrightarrow Sets$$
$$U \longmapsto \operatorname{Ker} f(U)$$
$$(i : V \to U) \longmapsto \underset{x \mapsto x}{\operatorname{Ker} f(i):\operatorname{Ker} f(U) \to \operatorname{Ker} f(V)}$$

is a sheaf over X.

Proof. First observe that the correspondence on morphisms is well-defined. Indeed, $\operatorname{Ker} f(U) \subset \mathcal{F}(U) \subset \mathcal{F}(V)$ when $V \subset U$, and we just apply $f(U)$ to notice that $\operatorname{Ker} f(V) \subset \operatorname{Ker} f(U)$.

To see this is a presheaf notice it is obvious that the identity is mapped to the identity by definition of the correspondence of morphisms. It is also obvious that composition is preserved.

To see it is a sheaf consider an open cover U_i of U, and elements $x_i \in \operatorname{Ker} f(U_i)$. Then use the property of $\mathcal{F}$ being a sheaf to reconstruct an element $x \in \mathcal{F}(U)$, whose image under f will be mapped to the identity element of $\mathcal{G}(U)$ because it does so in every point of the cover of U. Thus x is in $\operatorname{Ker} f(U)$ as desired. $\qquad\square$

More formally,

Definition 2.2. Let X be a topological space.

(1) A *presheaf of abelian groups on X* or an *abelian presheaf over X* is a presheaf of sets $\mathcal{F}$ such that for each open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an abelian group, and such that all restriction maps ρ_V^U are homomorphisms of abelian groups, see Lemma **??** above.

(2) A *morphism of abelian presheaves over X $\varphi : \mathcal{F} \to \mathcal{G}$* is a morphism of presheaves of sets which induces a homomorphism of abelian groups $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$.

(3) The category of presheaves of abelian groups on X is denoted $PAb(X)$.

Definition 2.3. Let X be a topological space.

(1) An *abelian sheaf on X* or *sheaf of abelian groups on X* is an abelian presheaf on X such that the underlying presheaf of sets is a sheaf.

(2) The category of sheaves of abelian groups is denoted $Ab(X)$.

Let X be a topological space. In the case of an abelian presheaf $\mathcal{F}$ the sheaf condition with regards to an open covering $U = \bigcup U_i$ is often expressed by saying that the complex of abelian groups

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{(i_0, i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is exact. The first map is the usual one, whereas the second maps the element $(s_i)_{i \in I}$ to the element

$$(s_{i_0}|_{U_{i_0} \cap U_{i_1}} - s_{i_1}|_{U_{i_0} \cap U_{i_1}})_{(i_0, i_1)} \in \prod_{(i_0, i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

In fact, the notion of kernel of a sheaf is not really defined as I did in the beginning of this section, but in the next one, along with several other important things.

3. The abelian category of sheaves of modules

I guess that the reason to introduce coherent sheaves is not the search for an abelian category, after all. Looks like the pathologies avoided by the definition of coherence are not so obvious—something like "wildly infinitely generated".

4. Tensor product of sheaves

Here's my unexpected encounter with the definition of tensor product of sheaves. It's not the "fiber is tensor product of fibers" construction, but actually just some notion of "change of ring" sheaf that ends up being adjoint to some "restriction" sheaf. The setting is a mapping of presheaves *of rings* over a space X... (I think the usual definition is this one taking $\mathcal{O}_1$ as the other presheaf we want to tensor).

Immediately after introducing this notion there's the definition of sheaf, then stalks, abelian sheaves, some other notions like an "algebraic structure" and then

tensor product will be defined after sheafification—because the following definition is in general not a sheaf.

Furthermore, I add that Vakil leaves it as an exercise to define the tensor product of two $\mathcal{O}_X$ modules (with a hint of defining the presheaf tensor product and sheafifying), which makes me think that after all it *is* just the intuitive definition. Before diving in, also by Vakil (Exercise 26.K): the stalk of the tensor product is the tensor product of the stalks.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of* $\mathcal{F}$. We obtain the restriction functor

$$PMod(\mathcal{O}_2) \longrightarrow PMod(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$PMod(\mathcal{O}_1) \longrightarrow PMod(\mathcal{O}_2)$$

Lemma 4.1. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. $\square$

Tipologia dos feixes.

Definition 4.2. A sheaf of $\mathcal{A}$-modules $\mathcal{F}$ over a sheaf of rings $\mathcal{A}$ (on a topological space X) is called

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5. Closed immersions

Closed immersions of ringes spaces are in 01C1.

Definition 5.1. A map of ringed spaces $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a *closed immersion* if i is injective, its image is closed (I think these two are "closed immersion of topological spaces") and the induced map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective; denote its kernel by $\mathcal{I}$. To make sure that if $(X, \mathcal{O}_X)$ is a scheme then so is $(Z, \mathcal{O}_Z)$ with add the extra condition that the $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

6. Restricting sheaves to subschemes

Upshot about restricting sheaves to subschemes: the sheaves on Z correspond essentially (in the categorical sense) to sheaves on X with support on Z.

More explicitly, from 01QY: the pushforward of the inclusion (a closed immersion of schemes) $i : Z \to X$, with kernel on induced map $\mathcal{O}_Z \to \mathcal{O}_X$ denoted by I, is an exact, fully faithful functor

$$i_* : QCoh(\mathcal{O}_Z) \to QCoh(\mathcal{O}_X)$$

with essential image the quasi-coherent $\mathcal{O}_X$ modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

It makes sense to think of this operation as $\mathcal{F}|_Z = i^*\mathcal{F} = \mathcal{F} \otimes \mathcal{O}_Z$ for $\mathcal{F} \in QCoh(X)$. Indeed: tensoring by the ideal sheaf would kill the information on Z, and tensoring with the structure sheaf does the opposite — it's the sheaves on X that are no fun outside Z, so basically sheaves on Z.

Here's a more detailed discussion:

In understanding the cohomology of sheaves when we restrict them to submanifolds (see Complex Geometry Exercise 18.3, I found Stacks Project 01AW section on closed immersions and abelian sheaves. First we have a few propositions stating that sheaves on the subscheme $Z \subset X$ can be pushed to sheaves on X.

And then there's the converse: what I would like to call the restriction functor $\mathcal{H}$ but rather is called the *sub-module of sections with support in Z*. In Remark 01AY (and then in Remark 0G6N for ringed spaces; these seem to be analogue constructions...) we see that for an abelian sheaf $\mathcal{F}$ on X we can define an abelian sheaf $\mathcal{H}_Z(\mathcal{F})$ which although is a sheaf on X we can just think it's a sheaf on Z - that's beacause

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U))|\text{the support of } s \text{ is contained in } Z \cap U\}.$$

We can just think that it's the functions that vanish outside the subvariety. Which doesn't really make sense: a function vanishing outside the submanifold will most likely vanish also in the submanifold! Just by continuity, the submanifold is in the closure of the open set! I guess what I mean to say is that I don't mind if the functions do not vanish outside Z... I just want to consider them as functions on Z...

Here's another attempt in understanding how to restrict sheaves to submanifolds. First and foremost, lacking a reference this, I understand the symbol $\mathcal{F}|_Z$ for a sheaf on X and a subscheme $Z \subset X$ to mean the pullback of $\mathcal{F}$ by the inclusion. Then look at 01QY (more basically, as above, 01AW... in fact there appear to be several version of this statement) to recall that sheaves of the subscheme Z should correspond to sheaves on X with support on Z. **So essentially $\mathcal{F}|_Z$ should be a version of $\mathcal{F}$ but with support on Z.** The key point here is that this is non other than $\mathcal{F} \otimes \mathcal{O}_Z$. Why? Because when multiply with the ideal sheaf of Z we obtain $(\mathcal{F} \otimes \mathcal{O}_Z)\mathcal{I} = 0$ since sections of $\mathcal{I}$ vanish in $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. That is, the support of $\mathcal{F} \otimes \mathcal{O}_Z$ indeed contained in Z: if the sections become zero when multiplied by sections of $\mathcal{I}$ (sections of $\mathcal{I}$ are only nonzero outside Z) it means that all the action was inside Z. that outside Z

I think the upshot here is that structure sheaves act as some sort of "contour" for the geometric object: they vanish *outside* the geometric object, they tell us what is the shape of the object by describing its surroundings, rather than its interior. This is exactly how algebraic/analytic sets are defined, and this why, when we want to

restrict a sheaf to a subscheme, we tensor with the structure sheaf: tensor by the ideal sheaf would kill the information on Z, and tensoring with the structure sheaf does the opposite.

7. The category of affine schemes

This might be a cornerstone of algebraic geometry. The point is that once we take affine charts, we re in the affine category. And this category turns out to be equivalent to the category of rings. /o/ See below for some important properties of the correspondence; namely fibred products and tensor product counterparts.

8. Proj of a graded ring

(Stacks Project 01M3, Proj of a graded ring.) This section may be taken as the preamble to the question: what is projective space anyway? Yes, projective space is Proj of a graded ring S. And yes, projective space is a scheme. So in particular it's locally affine.

(Stacks Project 01MX, Functoriality of Proj.) This is reminiscent of the olden days. The point is that Proj is covariant, i.e. for rings $S' \subset S$ we have $\mathrm{Proj}S \subset \mathrm{Proj}S$ (under the right conditions, which is the point of this section), meaning that, indeed, **Projs are projective schemes**.

9. Invertible sheaves on Proj (twists!)

As for the construction of the twisted sheaves: for a projective space $\mathrm{Proj}S$, we just compute the twisted S module $S(m)$ by the formula $S(m)_d = S_{m+d}$ and turn that into a sheaf $\widetilde{S(m)}$.

Here's the construction of the twisted sheaves $\mathcal{O}_X(n)$. The point is that there is a good way to pass from an S-module M to an $\mathcal{O}_X$-module, called $\widetilde{M}$. So basically you just define the twists by $M(d)_n = M_{n+d}$ and apply that construction.

Pick an element of degree d. Then think of the module generated by this element. The least degree piece of such a module is the degree d piece of the original module, that is we have $M(-d)$.

10. Logarithmic derivations

Just to record a notion of logarithmic derivation from Vladimiro Benedetti, Danielle Faenzi and Simone.

Let F be a homogenous polynomial of degree d. The *logarithmic derivations* of F are

$$\mathrm{Der}_U(-\log F) = \{\partial \in \mathrm{Der}_U : \partial F \in \langle F \rangle_U\}$$

where $U = \mathbb{C}[x_1, \ldots, x_e]$.

11. Gonality

From a talk by Juliana Coelho at Bandoleiros 2025:

A curve C is *k-gonal* if there is a map $C \to \mathbb{P}^1$ of degree k.

A curve is *stable* if it is nodal and $\mathrm{Aut}(C)$ is finite. Equivalently, the rational components of C meet the rest of the curve in at least three points.

We may study the moduli of k-stable curves, or the space of k-gonal structures.

12. Reduced schemes

So far a reduced scheme, for me, is just a scheme where there are no "repetitions": (x^2) is not reduced because it gives the same information as (x). Algebraically this means that the coordinate ring is reduced, which is defined by asking that there are no nilpotent elements, i.e. elements such that some power vanishes. (There's no definition of reduced ring in Stacks Project since it's taken as part of the basic algebraic knowledge).

The definition is on stalks but the first lemma shows it's equivalent to define it on any open set. This is in virtue of some "injectivity" lemma from the ring at U to the product of all stalks parametrized in U; essentially that if a section vanishes when projected to all the stalks then it's zero.

Definition 12.1. Let X be a scheme. We say X is *reduced* if every local ring $\mathcal{O}_{X,x}$ is reduced.

Proof. Assume that X is reduced. Let $f \in \mathcal{O}_X(U)$ be a section such that $f^n = 0$. Then the image of f in $\mathcal{O}_{U,u}$ is zero for all $u \in U$. Hence f is zero, see Sheaves, Lemma **??**. Conversely, assume that $\mathcal{O}_X(U)$ is reduced for all opens U. Pick any nonzero element $f \in \mathcal{O}_{X,x}$. Any representative $(U, f \in \mathcal{O}(U))$ of f is nonzero and hence not nilpotent. Hence f is not nilpotent in $\mathcal{O}_{X,x}$. $\qquad\square$

13. Dominant morphisms

The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

Definition 13.1. A morphism $f : X \to S$ of schemes is called *dominant* if the image of f is a dense subset of S.

14. Morphisms of finite type

Recall that a ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, see Algebra, Definition **??**.

Definition 14.1. Let $f : X \to S$ be a morphism of schemes.
   (1) We say that f is of *finite type at* $x \in X$ if there exists an affine open neighbourhood $\operatorname{Spec}(A) = U \subset X$ of x and an affine open $\operatorname{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
   (2) We say that f is *locally of finite type* if it is of finite type at every point of X.
   (3) We say that f is of *finite type* if it is locally of finite type and quasi-compact.

15. Flat morphisms

The essential technical property for for defining flatness is the preservation of exact sequences. Right-exactness is true in general; it follows from currying in category of commutative rings. But the functor $- \otimes_R N$ does not preserve injectivity of maps— that's the point of flatness.

Lemma 15.1 (Internal Hom for R-modules)**.** *For any three R-modules M, N, P,*

$$\operatorname{Hom}_R(M \otimes_R N, P) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, P))$$

Proof. An R-linear map $\hat{f} \in \mathrm{Hom}_R(M \otimes_R N, P)$ corresponds to an R-bilinear map $f : M \times N \to P$. For each $x \in M$ the mapping $y \mapsto f(x, y)$ is R-linear by the universal property. Thus f corresponds to a map $\phi_f : M \to \mathrm{Hom}_R(N, P)$. This map is R-linear since

$$\phi_f(ax + y)(z) = f(ax + y, z) = af(x, z) + f(y, z) = (a\phi_f(x) + \phi_f(y))(z),$$

for all $a \in R$, $x \in M$, $y \in M$ and $z \in N$. Conversely, any $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ defines an R-bilinear map $M \times N \to P$, namely $(x, y) \mapsto f(x)(y)$. So this is a natural one-to-one correspondence between the two modules $\mathrm{Hom}_R(M \otimes_R N, P)$ and $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. $\qquad\square$

Lemma 15.2 (Tensor product is right exact)**.** *Let*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules and homomorphisms, and let N be any R-module. Then the sequence

$$(15.2.1) \qquad\qquad M_1 \otimes N \xrightarrow{f \otimes 1} M_2 \otimes N \xrightarrow{g \otimes 1} M_3 \otimes N \to 0$$

is exact. In other words, the functor $- \otimes_R N$ is right exact, in the sense that tensoring each term in the original right exact sequence preserves the exactness.

Proof. For every R-module P we apply the functor $\mathrm{Hom}(-, \mathrm{Hom}(N, P))$ to the first exact sequence. We obtain

$$0 \to \mathrm{Hom}(M_3, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_2, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_1, \mathrm{Hom}(N, P))$$

which is exact by Lemma **??** (1). By Lemma 15.1 this becomes the sequence

$$0 \to \mathrm{Hom}(M_3 \otimes N, P) \to \mathrm{Hom}(M_2 \otimes N, P) \to \mathrm{Hom}(M_1 \otimes N, P)$$

which is therefore also exact. Then using Lemma **??** (1) again, we arrive at the desired exact sequence. $\qquad\square$

Remark 15.3. However, tensor product does NOT preserve exact sequences in general. In other words, if $M_1 \to M_2 \to M_3$ is exact, then it is not necessarily true that $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for arbitrary R-module N.

Example 15.4. Consider the injective map $2 : \mathbf{Z} \to \mathbf{Z}$ viewed as a map of $\mathbf{Z}$-modules. Let $N = \mathbf{Z}/2$. Then the induced map $\mathbf{Z} \otimes \mathbf{Z}/2 \to \mathbf{Z} \otimes \mathbf{Z}/2$ is NOT injective. This is because for $x \otimes y \in \mathbf{Z} \otimes \mathbf{Z}/2$,

$$(2 \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

Therefore the induced map is the zero map while $\mathbf{Z} \otimes N \neq 0$.

Definition 15.5. For R-modules N, if the functor $- \otimes_R N$ is exact, i.e. tensoring with N preserves all exact sequences, then N is said to be *flat R-module*. We will discuss this later in Section **??**.

Epiphany: in the category of commutative rings pushout is tensor product. So think of Spec as a functor from *CRing*$^{\mathrm{op}}$ to *Sch*, then pushout goes to pullback, and what's an example of a pullback? Fibre! So, the coordinate ring of a fiber is essentially given by the residue field at the point that parametrizes it! (tensored with the coordinate ring of the deformation space).

There is a lot of information on Stacks Project about flatness. It looks like the heart of the concept is captured in the commutative-algebraic notion of preserving exact sequences:

Definition 15.6. Let R be a ring.

(1) An R-module M is called *flat* if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.
(2) An R-module M is called *faithfully flat* if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.
(3) A ring map $R \to S$ is called *flat* if S is flat as an R-module.
(4) A ring map $R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module.

Recall that a module M over a ring R is *flat* if the functor $- \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is exact. A ring map $R \to A$ is said to be *flat* if A is flat as an R-module.

16. Singularities

As in [Vak25], a Noetherian local ring A is *regular* if $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. The stalk $\mathcal{O}_{X,p}$ is always local so we say that p is *regular* if its stalk is regular.

17. Birational morphisms

Recall that for Kuznetsov a birational map is that is an isomorphism on two open subsets of the varieties. . .

So far I get: they are isomorphisms on a dense open set, but more formally, here in Stacks, they give **isomorphisms on the stalks of the generic points**.

You may be used to the notion of a birational map of varieties having the property that it is an isomorphism over an open subset of the target. However, in general a birational morphism may not be an isomorphism over any nonempty open, see Example **??**. Here is the formal definition.

Definition 17.1. Let X, Y be schemes. Assume X and Y have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if

(1) f induces a bijection between the set of generic points of irreducible components of X and the set of generic points of the irreducible components of Y, and
(2) for every generic point $\eta \in X$ of an irreducible component of X the local ring map $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

18. Ampleness

First is this lemma that comes from modules.tex. I think these sets X_s are the base points of the bundle. Because look: image of s just means consider the section s of the line bundle as a germ near x. Now a line bundle is a locally free rank-1 $\mathcal{O}_X$-module, so its sections, like s, may be multiplied by germs of functions in the maximal ring $\mathfrak{m}_x$, i.e. the functions that vanish at x. So X_s is the vanishing locus of the section s. If $s(x) \neq 0$, obviously $s \notin \mathfrak{m}_x \mathcal{L}_x$, so $x \in X_s$. Conversely, I would like to show that if $s(x) = 0$ then $s \in \mathfrak{m}_x \mathcal{L}_x$ but I'm not sure how. It's like: a

vector field with a zero can be multiplied by a function that vanishes at the point, sure, but what's this function?

Lemma 18.1. *From modules.tex. Let X be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set*

$$X_s = \{x \in X \mid image\ s \notin \mathfrak{m}_x \mathcal{L}_x\}$$

is open in X. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathcal{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

Proof. Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma **??**. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma **??** that s_x is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood U of x such that t_x comes from a section t of $\mathcal{L}^{\otimes -1}$ over U and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that s generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that X_s is open. Moreover, the section t constructed over U above is unique, and hence these glue to give the section s' of the lemma. $\qquad\square$

Recall from Modules, Lemma **??** that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where ss' denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

Definition 18.2. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample* if

  (1) X is quasi-compact, and
  (2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

Exercise 18.3. Let L be an ample bundle on a K3 surface M. Prove that $\mathcal{L}^{\otimes 2}$ is globally generated (that is, for each $x \in M$ there exsits a section $h \in H^0(L^{\otimes 2})$ which does not vanish in x).

Proof. This just asks that the n in Definition 18.2 is 2 for all $x \in X$. Because, again, $x \in X_s$ means that $s(x) \neq 0$ because if it was, then we could somehow write s as a product of a vanishing function on $\mathfrak{m}_x$ and a local frame of $\Gamma(X, \mathcal{L})$. But I guess for the exercise do this: a line bundle is *ample* if there is n such that the canonical embedding (cf Lemma 18.5) is an embedding, i.e. that $\mathcal{L}^{\otimes n}$ is *very ample*. (Interestingly, the notion very ampleness is defined in morphisms.tex.) $\qquad\square$

Now we pass to the part where ampleness gives you an **open immersion** to some projective space. Because, it's only very ampleness that gives an embedding, right? (Actually I think here in stacks project there are no embeddings but closed immersions.)

Definition 18.4. From modules.tex. Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X we define the *associated graded ring* to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

Lemma 18.5. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes*

$$f : X \longrightarrow Y = \mathit{Proj}(S),$$

*to the homogeneous spectrum of S (see Constructions, Section **??**). This morphism has the following properties*

(1) *$f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,*
(2) *there are $\mathcal{O}_X$-module maps $f^* \mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (**??**),*
(3) *the composition $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ is the identity map, and*
(4) *for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of x such that $f^* \mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.*

19. Adjunction formulas

There are several statements called adjunction formula in different texts. All of them concern "subvarieties", that is, closed embedded subschemes.

Exercise 19.1 (Genus formula for a curve on a surface)**.** Let $C \to X$ be a closed embedded subscheme of dimension 1 (as a topological space, i.e. pure dimension) inside a smooth surface X. Then $2p_a - 2 = (\mathcal{O}_X(C), \mathcal{O}_X(C))$.

Proof. Consider the ideal sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

This sequence splits since there is an obvious inverse morphism to the inclusion $\mathcal{O}_X(-C) \to \mathcal{O}_X$, namely mapping a function f to Then $\mathcal{O}_X \cong \mathcal{O}_X(-C) \oplus \mathcal{O}_C$. $\chi(\mathcal{O}_X) = \chi$ $\square$

20. Normalization

Definition 20.1. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_* \mathcal{O}_Y$. The *normalization of X in Y* is the scheme[1]

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over X. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

The factorization is the composition of the canonical morphism $Y \to \underline{\mathrm{Spec}}_X(f_* \mathcal{O}_Y)$ (see Constructions, Lemma **??**) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \to f_* \mathcal{O}_Y$. We can characterize the normalization as follows.

---

[1]The scheme X' need not be normal, for example if $Y = X$ and $f = \mathrm{id}_X$, then $X' = X$.

Lemma 20.2. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

(1) *the morphism ν is integral, and*

(2) *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$
\begin{array}{ccc}
Y & \xrightarrow{\ g\ } & Z \\
{\scriptstyle f'}\downarrow & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle \pi} \\
X' & \xrightarrow{\ \nu\ } & X
\end{array}
$$

for some unique morphism $h : X' \to Z$.

Moreover, the morphism $f' : Y \to X'$ is dominant and in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

21. Reflexive sheaves

Definition 21.1. Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The *reflexive hull* of $\mathcal{F}$ is the $\mathcal{O}_X$-module

$$\mathcal{F}^{**} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

We say $\mathcal{F}$ is *reflexive* if the natural map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is an isomorphism.

Lemma 21.2. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *If $\mathcal{F}$ is reflexive, then $\mathcal{F}$ is torsion free.*

(2) *The map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is injective if and only if $\mathcal{F}$ is torsion free.*

Remark 21.3 (Talk at IMPA, 11 June). Torsion could also be defined so that the sheaf can inject onto its dual. In this talk we discussed the moduli space of reflexive/torsion-free sheaves, which turned out to be parametrized by c_1, c_2 and c_3. This was denoted by $R(c_1, c_2, c_3)$. Actually I think it may have been Manolache that proved the existence of this moduli space.

Alan Muniz

Nesta palestra discutiremos a classificação de feixes reflexivos de posto dois e seus espaços de módulos. Apresentaremos algumas ferramentas básicas usadas na construção e determinação de tais feixes. Aplicaremos estas técnicas para o caso de feixes com segunda classe de Chern igual a quatro, obtido recentemente em colaboração com Marcos Jardim.

21.4. **Distributions on manifolds.** Here's the abstract from a talk by Marcos Jardim at Geometric Structures:

"I will revise the work done over the past 10 years with various collaborators on distributions and foliations on 3-folds, especially on the projective space, with a focus on properties of the tangent sheaf and singular scheme."

Here are two key ideas: if the distribution is codimension 1 we can write:

$$0 \longrightarrow F \longrightarrow TX \xrightarrow{\ \omega\ } I_Z \otimes L \longrightarrow 0$$

where L is a line bundle and $\omega \in H^0(\Omega_X \otimes L)$, and $Z = \{p : \omega(p) = 0\}$.

When codimension is 2 then $\mathcal{D}$ is given by a holomorphic vector field ν: $T_p = \langle \nu(p) \rangle$.

It can be encoded as an exact sequence

$$0 \longrightarrow L \overset{\nu}{\longrightarrow} TX \longrightarrow N \longrightarrow 0$$

where L is a line bundle and $\nu \in H^0(TX \otimes L^\vee)$; $Z = \{p | \nu(p) = 0\}$.

Remark 21.5. Saturation means that $Z \subset X$ is a union of curves and points.

And again, distributions are parametrized by Chern classes.
Two interesting open questions:

(1) **Conjeture.** if $\mathcal{D}$ is a codimension 1 foliation of degree d on $\mathbb{P}^3$, then $c_2(F) \leq d^2 - d + 1$ and bound is attained by rational foliations of type $(1, d+1)$. (True for $d \leq 2$.)

(2) **Conjecture (with Pepe Seade).** $\mathcal{D}$ is a codimension 1 foliation on a smooth projective 3-fold, then $\mathrm{Sing}\mathcal{D}$ is connected.

Theorem 21.6 (Jardim-Muniz)**.** *Conditions on Chern classes used to understand moduli space $R(c_1, c_2, c_3)$.* $c_2 = 4$ *gives (?). For $c_3 \leq 6$, possible "spectrum" exists. . .*

22. STABILITY

Question. What is stability?

(1) Stable objects in an abelian category are the "building blocks": we can reconstruct the whole category from them.

(2) An abelian subcategory (hart) $\subset$ a triangulated

(3) stability defined via stability function on $\mathcal{A}$.

(4) Q. Can we reconstruct $\mathcal{T}$ from the semistable elements of $\mathcal{A}$

(5) **Example.** $\mathcal{A} = \mathrm{Coh}X$ is heart of $D^b(X)$ w/ funny function.

(6) Stability condition is hart + stability function.

(7) Bridgeland Stabl:= the stability conditions are a complex manifold of complex codimension rkΛ:

$$\mathcal{Z} : \mathrm{Stab}(\mathcal{T}) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C})$$

(8) There's a chamber structure; moduli space changes across chambers.

(9) I think we typically think of vectors in $\mathrm{Hom}(\Lambda, \mathbb{C})$ as Chern classes, to characterize the moduli spaces.

(10) Existence: given a projective variety X, are there stability conditions on $D^b(X)$? Yes for fano 3fold pic rk 1.

(11) Moduli spaces: is $M_\sigma(v)$ a projective scheme? Cannot use usual git techniques to study. A stack!

(12) Picture: blue + black are walls. Q. What are β and α?

(13) thm: bridgeland stable = gieseker stable ?

(14) Q. slope stability = bridgeland stability? A. Not always.

(15) DT/PT correspondance: only one wall between PT and G chambers

(16) Polynomial stability function. This is an asymptotic version of BS.

(17) There are some ρ's. Arrangements of ρ_i are polynomial stability conditions on a threefold.

(18) Pata-Thomas introduced stability for rank 1 objects. Bayer compares the— wall. Q. Same for Bridge S—only one wall?

(19) Recall Gieseker stability.

(20) Def. A *stable triple*: when $\gcd(\mathrm{ch}_0, \mathrm{ch}_2, \mathrm{ch}_3) = 1$, every PT stable object comes from three conditions (missing).

(21) What happens when you cross the blue wall? Both $\mathcal{G}$ and $\mathcal{T}$ are projective. What happens at the blue wall?

(22) For X smooth threefold with rkPic $= 1$, $\mathcal{G}(v)$, $v = (r, 0, 0, -n)$, $\mathcal{G}(v)$ is a known sheaf object and $\mathcal{T} = \emptyset$.

(23) X sm 3 rkpic1, Fake wall; $\mathcal{G} = \mathcal{T}$.

(24) (Extra.) red circle is a wall for a weaker form of stability.

Definition 22.1 (Talk at impa). A rank-2 sheaf $\mathcal{F}$ is *semistable (stable)* if $H^0(\mathcal{F}(t)) = 0$ for $-t \geq (>)\frac{c_1 F}{2}$

Compare with

Definition 22.2 (moduli-curves.tex). Let $f : X \to S$ be a family of curves. We say f is a *semistable family of curves* if

(1) $X \to S$ is a prestable family of curves, and

(2) X_s has genus ≥ 1 and does not have a rational tail for all $s \in S$.

- The twistor diagram.
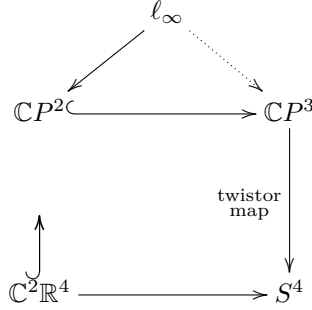
- Instantons are solutions to some Yang-Mills solution.
- Expository paper of Donaldson arXiv:2205.08639
- ADHM construction, 1978. The first appearance of Algebraic Geometry in Mathematical Physics. See Hitchin-Kobayashi correspondence.
- Donaldson: "unashamedly computational".
- Expository paper by Simon.
- Take bundle (E, ∇) with an anti-self dual (ADS) connection on S^4 and pullback to $\mathbb{C}P^3$ via the twistor map

$$\tau[x : y : z : w] = [x + jy : z + jw] \qquad \text{note: } \tau^{-1}(p) = \mathbb{C}P^1$$

  Then:
  - Restriction to fibres are trivial.
  - Invariant under anti holomorphic involution (check this formula!) $[x : y : z : w] \mapsto [-y : x : -w : z]$.
  - $\mathcal{E}$ is also an instanton bundle.
  - Penrose Transform: $H^1(\mathcal{E}(-2)) \cong \operatorname{Ker}\Delta = 0$ where Δ is a Laplacian.
  - Definition of instanton sheaf on $\mathbb{P}^3$ via $c_1(E) = 0$ and some vanishing of cohomologies.

– Passage from [differential equations? algebraic geometry?] to linear algebra: via *monads*. The point is that instanton sheaf is equivalent to "E being the cohomology of a linear monad"; theorem by Horps in the 60's, and is the main tool used by ADHM. Indeed, ADHM equations come from the cohomology sequences of the so-called monads.
– Mathamatical inst bund:= locally free instanton sheaves.

Properties.

- The only instanton of rank 1 on $\mathbb{P}^3$ is the structure sheaf.
- non-trivial rank 2 locally free instanton sheaf ois $\mu_0 stable$
- double dual is locally free and also instanton
- non-trivial rank 2 instanton sheaaf is Fieseker stable therefore it makes sance fo define moduls space of instanton sheaves as an open subset of $\mathcal{M}(c) = \mathcal{G}(k, 0, 2, 0)$.

Then studied the irreducibility (Tikhomirov) and smoothness (Jardim-Verbitsky, 2014. Uses "3rd hyperkähler quotient") of $\mathcal{I}(c)$, the moduli space of rank 2 locally gree instanton sheaves of charge c. But nobody likes this results; want new proofs.

In contrast, $\mathcal{M}(c)$ of rank 2-instanton sheaves of charge c is not irreducible in general! $\mathcal{M}(1)$ and $\mathcal{M}(2)$ are irreducible, $\mathcal{M}(3)$ has exactly 2 irreducible components of dimension 21; $\mathcal{M}(4)$ has 4 irredicuble components: the locally free is irreducible, and the other 3 that intersect the closure of the locally free, $\overline{\mathcal{I}(c)}$. 3 components of dimension 29 and one of dimension 32

Remark 22.3. In general the μ moduli space is not projective, but the Gieseker is.

Is $\mathcal{M}(c)$ connected? Use $\mathbb{C}^*$ action. True for $c \leq 4$; every component intersects $\overline{\mathcal{I}(c)}$ in this range!

Definition 22.4 (Elementary transformation)**.** F of rank 2 locally free instanton, Q of rank 0 instanton with 1 dimensional sheaf $h^p Q(-2)$ for $p = 0, 1$. So in this conditions if we have an epimorphism

$$F \overset{\varphi,\ \mathrm{surj}}{\to} \} Q$$

we get that $\operatorname{Ker} \varphi$ is an instanton.

So we might be interested in

$$E \hookrightarrow E^{**} \xrightarrow{\mathrm{surj.}} Q_E$$

Let's have a look at $\mathcal{M}(3) = \overline{\mathcal{I}(3)} \cup \overline{C(0, 1, 3}$. Consider a generic line bundle of degree 0 on a planar cubic (cwhich is encoded in that we have intersection of something of cimension 1 and something of simension 3), $Linn\mathrm{Pic}^0(C)$ so we have an epimorphism

$$0 \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \longrightarrow (i_* L) \longrightarrow 0$$

yielding a bundle E as previously outlined.

So we are studying the components via pushforwards of sheaves on complete intersection curves inside $\mathbb{P}^3$

"when we do alementary transformation of rational (not rationl?) we get something on the boundary of locally free."

Perverse instanton sheaves. Like an instnaton sheaf in monad description but with some restrictions on the cohomologies. This leads to definition of 0-*rimensional instanton*, a perverse instanton such that $\mathcal{H}^0 = 0$.

Instantons and quivers.

Framed instantons. Fix a line $j : L \to \mathbb{P}^3$ and E a perverse sheaf; a *framing* at L is an isomorphis [missing].

Apply GIT to the ADHM data to construct a moduli space

$$\mathcal{P}(r,c) = \mathcal{V}(r,c)^{\text{st}} /\!/ \text{GL}(V_c)$$

and it will follow that $\mathbb{P}(?,?)$ is connected.

Considerations on quaternionic spaces lead to generalization of what has been discussed so far to higher dimensions. I.e. an *instanton sheaf* on $\mathbb{P}^n$ is... [other cohomological conditions] sheaf

Why are instantons interesting? They are the simplest; may provide examples for Bridgeland stability.

In Kuznetsov (2012) and Faenzi (2014) introduced *rank 2 instanton bundles on Fano 3-folds*. An *instanton bundle* on X is a μ-stable ... and some Chern class is called the *charge*. An *instanton sheaf* (introduced by Marcos-Gaia) is ....

"Since we are imposing μ-stability on the defintion we can consider the moduli $\mathcal{I}_X(c) \subset \mathcal{G}_X(2, -r_X, c, 0)$".

There's also monad representations as an ingredient.

Here are two questions that invite us to join the instanton fever:

Task 1. Construct rank 2-instanton sheaves that do not deform into locally free ones, and obtain the new irreducible components of $\mathcal{G}(2, -r_X, c, 0)$.

Task 2. Nonlocally instanton sheaves that can be deformed into non-locally free ones: the *instanton boundary* $\overline{\mathcal{I}(c)}/\mathcal{I}(X)$[formula right?]

Recipe to construct your own instanton.
  (1) (Make a bunch of instantons.) Find an appropriate curve to use Serre correspondence to find some rank 2 instanton sheaf:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \underbrace{E}_{\exists} \longrightarrow \mathcal{I}_{\sqcup \text{lines}} \longrightarrow 0$$

  where E is a locally free instanton of charge = the number of lines -1.
  (2) (Is your family of instantons generic?) You look at the Exts. "Therefore, the family of instantons only defines a locally closed subset within a generically smooth irreducible component of $\mathcal{G}$".
  (3) "Find suitable rank 0 intranton sheaves to perform an elementary transformations on the examples obtained in Step 1." But they are non-locally free.
  (4) Now I have my instantons, I know they are locally free: but how to prove that the elementary transformations deform to locally free ones? Looks like the challenge is to prove that the deformation is locally free.

In the papers by the group there are several particular cases when the deformations are locally free. But they don't have a general result that would work for 3-folds.

What you need to call a thing an instanton.

- Minimal cohomology possible; try to kill as much cohomology as you can.
- Fixing c_1 (which may determine other Chern classes).
- Some stability condition like μ-stability or quiver stability. Here is an example that is not μ-semistable: $T\mathbb{P}^3(-1) \oplus \Omega_{\mathbb{P}^3}(1)$
- Whenever possible, look for a monadic representation. (The monadic representation comes from ADHM — the beginnings of this theory. And it's still here!)

23. Coherent sheaves

Lemma 23.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply Cohomology, Lemma **??**. As our basis $\mathcal{B}$ for the topology of X we are going to use the affine opens of X. As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X. Next we check that conditions (1), (2) and (3) of Cohomology, Lemma **??** hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings of any element of $\mathcal{B}$, see Schemes, Lemma **??**. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma **??**. $\square$

24. Hilbert polynomial

The following lemma will be made obsolete by the more general Lemma **??**.

Lemma 24.1. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function*
$$d \longmapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$$
is a polynomial.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and $\mathcal{F}(d) = \mathcal{F}$. Hence in this case the function is constant, i.e., a polynomial of degree 0. Assume $n > 0$. By Lemma **??** we may assume k is infinite. Apply Lemma **??**. Applying Lemma **??** to the twisted sequences $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*\mathcal{G}(d) \to 0$ we obtain
$$\chi(\mathbf{P}^n_k, \mathcal{F}(d)) - \chi(\mathbf{P}^n_k, \mathcal{F}(d-1)) = \chi(H, \mathcal{G}(d))$$
See Remark **??**. Since $H \cong \mathbf{P}^{n-1}_k$ by induction the right hand side is a polynomial. The lemma is finished by noting that any function $f : \mathbf{Z} \to \mathbf{Z}$ with the property that the map $d \mapsto f(d) - f(d-1)$ is a polynomial, is itself a polynomial. We omit the proof of this fact (hint: compare with Algebra, Lemma **??**). $\square$

Definition 24.2. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function $d \mapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$ is called the *Hilbert polynomial* of $\mathcal{F}$.

The Hilbert polynomial has coefficients in $\mathbf{Q}$ and not in general in $\mathbf{Z}$. For example the Hilbert polynomial of $\mathcal{O}_{\mathbf{P}^n_k}$ is
$$d \longmapsto \binom{d+n}{n} = \frac{d^n}{n!} + \ldots$$
This follows from the following lemma and the fact that
$$H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(d)) = k[T_0, \ldots, T_n]_d$$

(degree d part) whose dimension over k is $\binom{d+n}{n}$.

Lemma 24.3. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}_k^n$ with Hilbert polynomial $P \in \mathbf{Q}[t]$. Then*

$$P(d) = \dim_k H^0(\mathbf{P}_k^n, \mathcal{F}(d))$$

for all $d \gg 0$.

Proof. This follows from the vanishing of cohomology of high enough twists of $\mathcal{F}$. See Cohomology of Schemes, Lemma **??**. $\square$

For completeness I include earlier notes from [Har77] on the matter.

The fact that M is finitely generated is what makes the following two definitions make sense.

Definition 24.4. The *Hilbert function* of a finitely generated graded $S = k[x_0, \ldots, x_r]$ -module M is

$$H_M(d) = \dim_k M_d$$

Definition 24.5. Define F_0 to be the free S-module on the generators of M. Elements in the kernel M_1 of the inclusion are called *sysygies.* By Hilbert's basis theorem, M_1 is also finitely generated, so we may choose a set of generators and repeat this process.

Theorem 24.6 (Hilbert Syzygy Theorem). *Any finitely generated S-module M has a finite graded free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} \longrightarrow F_{m-q} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

Moreover, we may take $m \leq r + 1$, the number of variables in S.

Lemma 24.7. *Suppose that $S = k[x_0, \ldots, x_r]$ is a polynomial ring. If the graded S-module M has finite free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \cdots [r] \qquad F_1 \xrightarrow{\varphi_1} \longrightarrow F_0$$

with each F_i a finitely generated free module, $F_i = \bigoplus_j S(-a_{i,j})$, then

$$(24.7.1) \qquad\qquad H_M(d) = \sum_{i=0}(-1)^i \sum_j \binom{r + d - a_{i,j}}{r}$$

Lemma 24.8. *There is a polynomial $P_M(d)$ called the Hilbert polynomial such that, if M has free resolution as above, then $P_M(d) = H_M(d)$ for $d \geq max_{i,j}\{a_{i,j} - r\}$.*

Proof. When d satisfies this bound then the binomial coefficients in Eq. 24.7.1 are polynomials of degree r in d. $\square$

Theorem 24.9 (Hilbert-Serre). *Let M be a finitely generated graded $S = k[x_0, \ldots, x_n]$. Then there exists a unique polynomial p_M such that $p_M(\ell) = \dim S_\ell$ for large enough ℓ.*

Definition 24.10. The polynomial P_M of Hilbert-Serre Theorem [**?**] is the *Hilbert polynomial* of the finitely generated $k[x_0, \ldots, x_n]$-module M.

Definition 24.11. If $Y \subset \mathbb{P}^n$ is an algebraic set of dimension r, we define the *degree of Y* to be $r!$ times the leading coefficient of the Hilbert polynomial of the homogeneous coordinate ring $S(Y)$.

Exercise 24.12. Let H be a very ample divisor on the surface X, corresponding to a projective embedding $X \subseteq \mathbb{P}^N$. If we write the Hilbert polynomial of X as $P(z) = \frac{1}{2}az^2 + bz + c$, show that $a = H^2$, $b = \frac{1}{2}H^2 + 1 - \pi$, where π is the genus of a nonsingular curve representing H, and $c = 1 + p_a$.

25. NAKAI-MOISHEZON CRITERION

Theorem 25.1 (Nakai-Moishezon Criterion)**.** *A divisor D on the surface X is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C in X.*

Proof. The direct implication is easy: since D is ample, mD is very ample for some m, so that $m^2 D^2$ is the self-intersection number of mD. By exercise 24.12, D^2 is the leading coefficient of the Hilbert polynomial of X as a subscheme of $\mathbb{P}^n$. This means that D^2 is twice the leading coefficient of the Hilbert polynomial of a projective variety for large enough m, so that it must be a positive number (it's the dimension of one of the graded components of the coordinate ring of the surface). □

26. HILBERT SCHEME

Upshot [**?**, p. 6]**.** We wish to parametrize subschemes of a projective space (or perhaps a more general scheme?). Since there are too many such subschemes we restrict ourselves to schemes with a given Hilbert polynomial, since the latter "encodes the most important numerical invariants of schemes". The Hilbert scheme is introduced via a theorem by Grothendieck as the object that represents the functor $\mathbf{Hilb}_{P,r}$ that maps a reduced scheme B to the set of proper flat families

$$
\begin{array}{ccc}
\mathcal{X} & \overset{i}{\lhook\joinrel\longrightarrow} & \mathbb{P}^r \times B \\
 & \searrow & \downarrow \pi_B \\
 & & B
\end{array}
$$

with $\mathcal{X}$ having Hilbert polynomial P.

Theorem 26.1 (Grothendieck, '66)**.** *The functor $\mathbf{Hilb}_{P,r}$ is representable by a projective scheme $\mathcal{H}_{P,r}$.*

SEE Hilbert schemes of subschemes.

27. DEFORMATION THEORY

[**?**] has a great way of motivating deformation theory via the moduli problem. Here let's just put the following important meaning for "deformation theory" (as opposed to studying flat families!):

> "*Deformation theory* is the study of infinitesimal deformaions as a tool to understand the local structure of the moduli space. The goal is to be able to describe the restriction of the universal family to a small neighborhood of $m \in \mathcal{M}$, or, more precisely, its restriction to the germ of M at m.
>
> What is interesting here is that we can study order and infinitesimal deformations even though the functor F is not representable or simply we don't yet know it is. This is the most frequent case. Such an investigation will reveal the infinitesimal properties at $[X]$ of a yet unknown global structure on M which will be hopefully

> understood at a subsequent stage of the investigation. In order words it turns out to be possible and convenient to separate the *global moduli problem* from the *local moduli problem*, and deformation theory studies the latter, with the purpose of constructing a family of deformations of a given object parametrized by the spectrum of a local ring, and having properties as close as possible to a universal property."

I start by reading Stacks Project.

The first notion is *thickening of ringed spaces*, which I ultimately think of as a closed subscheme $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ with a nilpotent ideal sheaf, which in an imprecise way means that $\mathcal{I}_X^n = 0$ for some n, and in a precise way its given on sections.

The following definition is from [lucas-defos], which in turn comes from [Ser06]

Definition 27.1. Let X be an algebraic $\mathbb{C}$-scheme.

(1) A *deformation* of X is a Cartesian diagram ξ

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\operatorname{Spec}\mathbb{C} & \xrightarrow{\ s\ } & S
\end{array}
$$

where π is a flat surjective morphism of algebraic $\mathbb{C}$-schemes and S is connected. (Recall that flatness accounts for "continuity".)

(2) A *local deformation* of X is a deformation ξ where $S = \operatorname{Spec} A$ for A a noetherian local $\mathbb{C}$ algebra with residue field $\mathbb{C}$.

(3) An *infinitesimal deformation of X* is a local deformation with A an artinian local $\mathbb{C}$-algebra with residue field $\mathbb{C}$. X is called *rigid* if all infinitesimal deformations are trivial.

(4) An *inifinitesimal deformation of order n* is an infinitesimal deformation when $S = \operatorname{Spec}(\mathbb{C}[t]/(t^{n+1}))$.

Upshot. An interpretation of the so-called *dual numbers* $k[t]/(t^2)$ (see [Har09]) as the tangent space of something is thinking of Taylor polynomials: after quotienting by t^2 we loose the tails of the polynomials and are left with the first derivative information only - this justifies the use of the "first order deformations" term. As for why using the dual numbers as the underlying ring of the parameter space, suppose we have a series $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$
\begin{aligned}
f(x + \varepsilon) &= \sum_{n=0}^{\infty} a_n (x + \varepsilon) & &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \binom{n}{k} x^{n-k} \varepsilon^k \\
&= \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{k=0} + \varepsilon \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{k=1} \\
&= f(x) + \varepsilon f'(x),
\end{aligned}
$$

which says that

$$
f(x + \varepsilon) - f(x) = \varepsilon f'(x).
$$

(Further interpretation needed. . .)

There is the following interpretation in [continued-fractions] p. 39: the space of first order deformation classes of X is $D(\mathbb{C}[t]/(t^2))$. This is said to ""represent" the tangent space $\mathbb{T}^1_X$ of the hypothetical deformation space of X". (I put double quotations because the word "represent" is quoted in the original text.) Further, if X is nonsingular and compact, then $\mathbb{T}^1_X = H^1(X, T_X)$.

Which basically I interpret as: the dimension of the deformation space of a smooth compact variety is $H^1(X, T_X)$.

After a few months of the previous text in this section I think I have understood the construction of Kodaira-Spencer map for embedded deformations, which is [**?**, Proposition 3.2.1(ii)].

But I will actually briefly outline a particular case of this, which is easier: [**?**, Examples 3.2.4(ii)]. This the case when $X \subset Y$ is a Cartier divisor. Then X is cut out locally by a set of functions f_i defined on an affine cover U_i of Y. Then we have a line bundle defined by the transition functions $f_{ij} = f_i/f_j$, and is such that the f_i glue to a section which vanishes along X (see [Lee24, Theorem 3.39]). This is the line bundle $\mathcal{O}_Y(X)$, which by the adjunction formula is also the normal bundle $N_{X/Y}$ of X in Y.

Now consider a first order deformation

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \subset Y \times B_\varepsilon \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
0 & \longrightarrow & B_\varepsilon
\end{array}
$$

The great thing about first order deformations is that the structure sheaf of $\mathcal{X}$ on an affine open U_i of Y looks like the tensor product $\mathcal{O}_{U_i} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)$ (01I4). By definition of tensor product this just says that sections look like $f + \varepsilon g$ for $f, g \in \Gamma(\mathcal{O}_{U_i})$.

If $\mathcal{X}$ is a Cartier divisor on $Y \times B_\varepsilon$ (it is; but I'm missing a formal proof), it is given as the vanishing locus of some local sections $f_i + \varepsilon g_i$. (Perhaps the f_i coincide with the distinguished section of X.) The line bundle associated to $\mathcal{X}$ is given by the cocycle

$$F_{ij} = f_{ij} + \varepsilon + g_{ij}$$

for $g_{ij} = \frac{g_i - f_{ij} g_j}{f_j}$. Indeed, note that $\frac{1}{f_j + \varepsilon g_j} = \frac{1}{f_j} - \varepsilon \frac{g_j}{f_j^2}$ and compute $F_{ij} = (f_i + \varepsilon g_j)/(f_j + \varepsilon g_j)$.

Then

$$f_i + \varepsilon g_i = (f_{ij} + \varepsilon g_{ij})(f_j + \varepsilon g_j)$$

gives

$$g_i = f_{ij} g_j + g_{ij} f_j$$

and since f_j vanishes along X, we are left with $g_i = f_{ij} g_j$ which is how a section of the sheaf $N_{X/Y}$ is constructed by definition of sheaf.

Example 27.2. Fix $g \geq 2$. The dimension of the deformation space of a nonsingular projective curve X is $3g - 3$. This is "the dimension of the moduli space of curves of genus g".

We can compute this number by Riemann-Roch formula on the bundle $-K_X$.
Indeed, since X is a curve, $\Omega_X^1 = K_X$ and thus $-K_X = T_X$. We get

$$h^0(-K) - h^0(K - (-K)) = \deg(-K) - g + 1$$

$$= -2g + 2 - g + 1 \qquad \substack{\text{degree additive and} \\ \deg K = 2g-2}$$

$$= -3g + 3$$

Now by Serre duality $h^0(K - (-K)) = h^1(-K) = h^1(T_X)$, and it turns out that a
Riemann surface of genus $g \geq 2$ has no holomorphic vector fields, so that $h^0(-K) = h^0(T_X) = 0$.

Exercise 27.3. Let C be a smooth genus g curve which can be embedded in a K3
surface M, and X the family of all deformations of C in M.

    (1) Prove that $\dim X \leq g$.
    (2) Let $\mathcal{X}_g$ be the space of all curves of genus g (smooth?) which can be possibly
        embedded to a K3 surface. Prove that each irreducible component Z of $\mathcal{X}_g$
        satisfies $\dim_{\mathbb{C}} \leq g + 19$. Deduce that there exists a compact complex curve
        which cannot be embedded in a K3 surface.

Exercise 27.4. Let C be a smooth curve embedded in a K3 surface X. Show that
the dimension of the deformation space of C is $\leq g$.

Proof. The deformation space of a variety is the space of isomorphism classes of
deformations as explained above. It turns out that there is a way to associate 1-
cocyles of the tangent sheaf to deformations, so that in fact the deformation space
Def_1 is isomorphic to $H^1(X, \mathcal{T}_X)$ for any variety X.

For our curve C we thus know that the dimension of the space of deformations
(deformations not necessarily contained in X) is $h^1(\mathcal{T}_C)$. The family of deformations
of C that are contained in X is the Hilbert scheme of curves with fixed Hilbert
polynomial $P(t)$ after quotienting by $\mathbb{C}s$, where s is the section whose vanishing
locus is C. This says that the number we are looking for is $h^0(X, \mathcal{O}(C)) - 1$.

Before showing that in fact $h^0(X, \mathcal{O}(C)) = 1 + g$, let's explain why this Hilbert
scheme construction actually works. Intuitively, any curve lying in the same com-
ponent as C would be deformation-equivalent to C — at least we know that it's
Hilbert polynomial is the same. But it's not clear what's the deformation involv-
ing these two (this question should be addressed!). Further, this doesn't explain
Vladimiro's suggesion of "taking quotient by $\mathbb{C}s$". To me it looks like I would just
compute the dimension of the tangent space of $\mathrm{Hilb}^P(M)$ at C.

Now I will show that $h^0(X, \mathcal{O}(C)) = 1 + g$ (see [**?**, Lemma 1.2.1, Remark 1.2.2]).
Consider the ideal sheaf exact sequence twisted by $\mathcal{O}(C)$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_C = \mathcal{O}(C)|_C \longrightarrow 0$$

By X being a K3 we know that $H^1(\mathcal{O}_X) = 0$, so that we have the short exact
sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}(C)) \longrightarrow H^0(\mathcal{O}(C)|_C) \longrightarrow 0$$

so that $h^0(\mathcal{O}(C)) = 1 + h^1(\mathcal{O}(C)|_C)$. A version of adjunction formula says $\omega_C \cong
(K_X \otimes \mathcal{O}(C)|_C$, and using that $K_X = \mathcal{O}_X$ we obtain $h^0(\mathcal{O}(C)|_C) = h^0(\omega_C) = g$.

To

For the record I put other thoughts I went through in solving this exercise.

Recall from [**?**, p. 146] that the normal bundle $\mathcal{N}$ of a hypersurface of a smooth variety X satisfies $\mathcal{N}^\vee \cong \mathcal{O}_X(-C)|_C$. Taking duals we get that $\mathcal{N} \cong \mathcal{O}_X(C)|_C$.

By adjunction formula $2g-2 = (\mathcal{O}(C), \mathcal{O}(C))$. Applying Riemann-Roch to $\mathcal{O}(C)$ (which is by definition the dual of the ideal sheaf of C, which is a line bundle on X), we obtain that $\chi(\mathcal{O}(C)) = 2 + \frac{1}{2}(\mathcal{O}(C), \mathcal{O}(C))$. Then $\chi(\mathcal{O}(C)) = g + 1$.

Now recall that $\chi(\mathcal{O}(C)) = h^0(\mathcal{O}(C)) - h^1(\mathcal{O}(C)) + h^2(\mathcal{O}(C))$. By Serre duality and X being a K3 surface we see that $h^2(\mathcal{O}(C)) \cong h^0(\mathcal{O}(-C))$, which is the ideal sheaf of C. Any section of such a sheaf would vanish along C, and since X is compact we conclude there cannot be any such section.

Now we show that also $h^1(\mathcal{O}(C)) = 0$ to conclude that $h^0(\mathcal{O}(C)) = g + 1$.

We thus conclude that $h^0(\mathcal{O}(C))$

Since C is smooth we can use the normal exact sequence

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_X|_C \longrightarrow \mathcal{N} \longrightarrow 0$$

Taking Euler characteristic we see that $\chi(\mathcal{T}_C) + \chi(\mathcal{N}) = \chi(\mathcal{T}_X|_C)$.

This means that we should be done once we compute $\chi(\mathcal{T}_X|_C)$. For this we can use Riemman-Roch formula for coherent sheaves on a curve, which tells us that

$$\deg(\mathcal{T}_X|_C) = \chi(\mathcal{T}_X|_C) - \operatorname{rk}(\mathcal{T}_X|_C) \cdot \chi(\mathcal{O}_C)$$

But of course we know that since C is a curve, by Serre duality we get that $\chi(\mathcal{O}_C) = 1 - g$. (Indeed: $h^1(\mathcal{O}_C) = h^0(\Omega_C^1) := p_a(C)$.) And now the question is what is the degree. Apparently this is just the first Chern class c_1 of the restricted tangent bundle. So what is it? And what is $h^0(\mathcal{T}_C)$ if the genus is 0 or 1, and that's it.

I know that $h^0(\mathcal{T}_X) = 0$ and $h^1(\mathcal{T}_X) = 20$ by X being a K3 surface, but I'm not sure what happens when we restrict to C.

If $g \geq 2$ we know that $h^0(\mathcal{T}_C) = 0$, so that $\boxed{\chi(\mathcal{T}_C) = h^1(\mathcal{T}_C)}$. To compute the latter Euler characteristic, which is given by definition by $\chi(\mathcal{T}_X|_C) = h^0(\mathcal{T}_X|_C) - h^1(\mathcal{T}_X|_C)$, we first note that $h^0(\mathcal{T}_X|_C) = 0$, because this is the dimension of the global holomorphic vector fields on X restricted to C, which is constant along C since X is smooth. And then this number is actually zero by Hodge numbers of a K3 surface.

and taking cohomology long exact sequence we obtain

$$\cdots \longrightarrow H^0(T_X) \longrightarrow H^0(\mathcal{N}) \longrightarrow H^1(T_C) \longrightarrow$$

$$H^1(T_X) \longrightarrow H^1(\mathcal{N}) \longrightarrow \cdots$$

Or is it $h^1(\mathcal{N})$? See [**?**]. If this was the case, then we can use adjunction formula as above to get that $2g - 2 = (\mathcal{N}, \mathcal{N}) = \deg_C \mathcal{N}$. Then we may find $h^0(\mathcal{N})$ via Riemann-Roch:

$$h^0(\mathcal{N}) - h^0(K_C - \mathcal{N}) = \deg \mathcal{N} - g + 1$$

Note that $h^0(N_C - \mathcal{N}) = h^1(-K_C + N + K_C) = h^1(\mathcal{N})$ via Serre duality, and by Riemann-Roch on a surface as above we see that

$$g + 1 = \chi(\mathcal{N}) = h^0(\mathcal{N}) - h^1(\mathcal{N}) \implies h^1(\mathcal{N}) = -g - 1 + h^0(\mathcal{N})$$

so that

$$h^0(\mathcal{N}) - (-g - 1 + h^0(\mathcal{N})) = \deg\mathcal{N} - g + 1$$

$\implies$ oops! I lost $h^0(\mathcal{N})$ in this operation...

Maybe if I just use normal exact sequence and realise that

$$h^0(\mathcal{N}) = h^0(\mathcal{T}_X|_C) - h^0(\mathcal{T}_C)$$

I know that $h^0(\mathcal{T}_C) = 0$ for $g > 1$, so the question is how to compute the restricted holomorphic vector fields. $\qquad\qquad\square$

Look for the "Perfect obstruction theory". This is used by Bruzzo for deformations of (super)stacks, work with E. Paiva and —.

28. CONTINUED FRACTIONS

Definition of HJ continued fraction. For $i > 2$ they are in bijection with $\mathbb{Q}_{>1}$.

The basic diagram of this course starts with a surface S (eg. Hirzebruch surface $S = \mathbb{F}_m$). Blowing up leads to X, and contracting Wahl chains on X leads to W, a normal projective surface that has only Wahl singularities. Then we construct $\mathbb{Q}$-Gorenstein smoothings W_t. (These $\mathbb{Q}$-Gorenstein smoothings have Milnor number $=0$.)

Continued fractions have minimal models:

- $[1, 1]$ means a 0 curve, $\mathbb{P}^1$.
- $[1]$ means a -1 curve, $\mathbb{P}^1$.
- For $\frac{m}{q} \in \mathbb{Q}_{>1}$, the continued fraction $[e_1, \ldots, e_r]$ means a chain, which is a sequence of lines that intersect transversally with $-e_1, \ldots, -e_r$. This is mapped to $\frac{1}{m}(1, q)$.

Third lecture.

Here's some slogans/recap:

(1) The most important cyclic quotient singularities (c.q.s.) are Wahl $\frac{1}{n^2}(na - 1)$. There is a model to deal with this kind o singularities using continued fractions. This is very silly but what I picked up is that "you add a 2 in the end and add $+1$ to the first number", so for example $[4] \rightsquigarrow [5, 2] \rightsquigarrow [6, 2, 2]$. But on the second step the $[5, 2]$ also goes to $[2, 5, 3]$ in a way I don't understand. This is called the Wahl algorithm.

(2) (See [KSB88]) There is a notion of M-resolution, which is a drawing of several curves Γ_i intersecting at points P_i that may be Wahl singularities or smooth points with the key property that $\Gamma_i \cdot K \geq 0$. We have "toric boundary for P_i". These M-resolutions are in 1-1 correspondence with smoothings of $\frac{1}{m}(1, q)$, and in turn in 1-1 correspondence with continued fractions $K\left(\frac{m}{m-q}\right) = \{k_1, \ldots, k_s\} : 1 \leq k_i \leq b_i \ \forall i\}$ where $\frac{m}{m-q} = [b_1, \ldots, b_s]$.

Today we consider the fibers to be $W_t = \mathbb{P}^2$ and try to find W. Set $m_1, m_2, m_3 \in \mathbb{Z}_{>0}$. Define

$$\mathbb{P}(m_1, m_2, m_3) := \mathbb{P}^2/(\mathbb{Z}/m_1 \oplus \mathbb{Z}/m_2 \oplus \mathbb{Z}/m_3) = \mathbb{C}^3\backslash\{0\}/(\lambda \in \mathbb{C}^* \lambda(x, y, z) = (\lambda^{m_1}x, \lambda^{m_2}y, \lambda^{m_3}z))$$

For $\gcd(d, m_i) = 1$ we have $\mathbb{P}(dm_1, dm_2, dm_3) = \mathbb{P}(m_1, m_2, m_3)$.

For a triangle $xyz = 0$ given by three lines Γ_i we have cqs singularities of the kind $\frac{1}{m_1}(m_2, m_3)$. In this case $K_W = -(m_1 + m_2 + m_3)\xi = -\Gamma_1 - \Gamma_2 - \Gamma_3$ for

$\xi^2 = \frac{1}{m_1 m_2 m_3}$, and $\mathrm{Cl}(W) = \mathbb{Z}\langle \xi \rangle$. Since these are Wahl singularities, we must have that the m_i are squares, i.e. $m_i = n_i^2$ for some n_i. We must have:

$$K_W^2 = (m_1 + m_2 + m_3)^2 \frac{1}{m_1 m_2 m_3} = 9 = K_{\mathbb{P}^2}^2$$
$$\implies (n_1^2 + n_2^2 + n_3^2) - 9 n_1^2 n_2^2 n_3^2 = 0$$
$$\implies (n_1^2 + n_2^2 + n_3^2 - 2 n_1 n_2 n_3) \cdot (\text{positive factor}) = 0$$
$$\implies n_1^2 + n_2^2 + n_3^2 = 3 n_1 n_2 n_3$$

The last equation is known as *Markov equation*.

Example 28.1. For $\mathbb{P}(1,1,4) = W$, a triangle with a Wahl singularity $\frac{1}{4}(1,1)$ in one vertex. Blowing up gives the Hirzebruch surface $\mathbb{F}_4$, so that a minimal resolution is the triangle. Compare with [Hacking-Prokhorov-2010]. This example satisfies the Markov equation for $n_1 = 1, n_2 = 1, n_3 = 2$.

Theorem 28.2 ([HP2010]**.** *] If $\mathbb{P}^2 \rightsquigarrow W$ with only log terminal singularities then W is a partial $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}(a^2, b^2, c^2)$ where $a^2 + b^2 + c^2 = 3abc$.*

By the Markov equation condition all the singularities must be Wahl. The triple (a, b, c) is called *Markov triple*. Any permutation of a Markov triple is another Markov triple. Is (a, b, c) is Markov then so is $(a, b, 3ab - c)$. This allows to construct a *Markov tree*. There is so-called Markov conjecture (due to Frobenius) still unsolved.

29. Fano varieties

Definition 29.1. A *Fano variety* is a projective variety with $-K_X$ ample.

The idea of the Fano index is that the larger the index, the simpler is the variety. In a talk by Jõao, it is defined as the larger integer dividing $-K_X$ in $\mathrm{Pic}(X)$.

Definition 29.2.

$$r(X) := \min\{r : \frac{c_1(X)}{r} \in H^2(X, \mathbb{Z})\}$$

Exercise 29.3. By Kodaira vanishing theorem **??**, you can show that the cohomology $H^i(X, L)$ for a Fano variety X vanishes. You just have to put $L = \mathcal{O}(k)$ with $k \geq -r$, where r is the Fano index.

Exercise 29.4. Show that $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ holds for Fano varieties.

Remark 29.5. If $H^3(X, \mathbb{Z}) = 0$ of a Fano 3-fold, then its derived category is generated by 4 elements.

Araujo-Druel em 2017 classify Fano foliations relating the Fano index (of the foliation) with its rank. João found conditions (integrability of the leaves) to describe del Pezzo foliations (a kind of Fano foliation) with Picard rank 1. Araujo (2019) classifies the leaves of a algebraically integrable del Pezzo foliation (with no restriction on the (log canonica) singularities!).

30. Quivers

Definition 30.1. A *quiver* is a set of vertices Q_0, a set of arrows Q_1 equipped with the maps of source s and target t that to each arrow they assign the point that is source or target of the arrow.

Definition 30.2. A *representation* of a quiver is a set of finite dimensional vector spaces equipped with maps between them realising a given quiver (incomplete...).

There is a notion of projective representation, which I missed to write. But it is analogous to the injective representation:

Definition 30.3. Given a quiver Q, the *injective representation* of Q_0 is given by, for $i \in Q_0$,

$$I(i)_j = \begin{cases} k & i = j \\ k^{d'} & j \neq i \end{cases}$$

where d' is the number of paths from j to i.

31. Stacks

My first definition of stack can be extracted from

Definition 31.1. A *superstack* is a stack over the étale site SSch of superschemes, i.e. it is a category fibered in groupoids over the category of superschemes, the latter equipped with the étale topology, satisfying the descent condition.

Here are some other definitions:

Definition 31.2. Let $\mathfrak{X}$ be a stack over $\mathrm{Sch}_{\text{ét}}$. An *algebraic space* is such that there exists morphism $\mathcal{U} \to \mathfrak{X}$ where $\mathcal{U}$ is a scheme, that is schematic, étale and injective (check this one).

$\mathfrak{X} \to y$ is *representable* if there exists a scheme $\mathcal{U}$ and a map $\mathcal{U} \to y$ such that the fibered product

$$\begin{array}{ccc} \mathcal{U} \times_y \mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{U} & \longrightarrow & y \end{array}$$

is an algebraic space.

Finally, a stack is *algebraic* (resp. *Deligne-Mumford*) is there exists a representable surjective morphism $\mathcal{U} \to \mathfrak{X}$ that is smooth (resp. étale).

A *stable map* over a projective variety X is an element of the first Chow group $\beta \in A_1$, where (C, g) is an algebraic curve and $f : C \to X$ with $[f(C)] = \beta$.

The curves that are points under this map (contractible) are **stable**.

32. Moduli

Pseudo-admissible covers may be the way Mumford orifinally developed stability notions.
This is a two-session minicourse by Ugo Bruzzo in LEGALzinho 2025.

Let X be a smooth complex projective variety, $X \overset{j}{\hookrightarrow} \mathbb{P}^n$. $\mathcal{O}_{\mathbb{P}^n}(1)$ is the class of hyperplanes of $\mathbb{P}^n$. $\mathcal{L} = j^* \mathcal{O}_{\mathbb{P}^n}(1)$. $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$ is a *polarization*.

Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module. For example, $\mathcal{E}$ may be a locally free $\mathcal{O}_X$-module. Recall that $H^k(X, \mathcal{E})$ are finite dimensional vector spaces and they vanish for $k > \dim X = n$. Recall that the *Euler characterstic* of $\mathcal{E}$ is given by $\chi(\mathcal{E}) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{C}} H^k(X, \mathcal{E})$.

The *reduced Hilbert polynomial* of $\mathcal{E}$ is

$$p_{\mathcal{E}}(k) = \frac{1}{\mathrm{rk}\mathcal{E}} \chi(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}^k).$$

Definition 32.1. Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module with no torsion and positive rank. $\mathcal{E}$ is *stable* if for all $f \subset \mathcal{E}$ ($0\mathrm{rk}f < \mathrm{rk}\mathcal{E}$),

$$p_f(k) < p_{\mathcal{E}}(k) \qquad \text{for } k \gg 0$$

and *semi-stable* if we put $\leq$ instead of $<$.

Lemma 32.2. *If $\mathcal{E}$ is stable then $\mathcal{E}$ is irreducible ($\mathcal{E} \neq \mathcal{E}_1 \oplus \mathcal{E}_2$) and $Aut(\mathcal{E}) \simeq \mathbb{C}^*$.*

The *functor of moduli* is

$$\mathcal{M} : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathsf{Set}$$

$$S \longmapsto \left\{ \begin{array}{c} \text{stable family} \\ \text{of sheaves with} \\ \text{Hilbert polynomial} f \\ \text{parametrized by } S \end{array} \right\} \bigg/ \sim .$$

Riemann-Roch. Let C be a smooth, projective, connected curve of genus g. Let $\mathcal{E}$ be a coherent sheaf over $H^2(C, \mathbb{Z})$ with generator of $H^2(X, \mathbb{Z})$ $w = [pt] \in H^2(C, \mathbb{Z})$. $c_1(\mathcal{E}) = \deg\mathcal{E} \cdot w$.

Theorem 32.3.

$$\chi(\mathcal{E}) = \int_C (1 + (1 - g)w)(rk\mathcal{E} + deg\mathcal{E}w)$$

$$= \int_C deg\mathcal{E} + w + (1 - g)rk\mathcal{E} \cdot w$$

$$= deg(\mathcal{E}) + 1 - grk\mathcal{E}.$$

Example 32.4.

Theorem 32.5. *Se (r, d), $r > 0$ are coprime, the functor of moduli of stable bundles over C of rank r and degree d is representable, and the moduli space $\mathcal{M}_C(\mathcal{E}, d)$ is smooth.*

In the non coprime case this does not necessarily hold.

Definition 32.6. $\mathcal{M}$ is a *grosseiro* moduli space if the funtor is not representable.

This means that although there is not a universal family, there is still a correspondence of the closed points of $\mathcal{M}$ and the equivalence classes of semistable sheaves of rank r and degree d.

Now let's fix $\dim X = 2$. Let $\mathcal{E}$ be a stable $\mathcal{O}_X$-module. Let $r = \mathrm{rk}\mathcal{E}$, and the *discriminant*

$$\Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{\mathcal{E} - 1}{2r} c_1(\mathcal{E})^2 \in H^4(X, \mathbb{Q}) \cong \mathbb{Q}.$$

Here enter $c_1(\mathcal{E}) \in H^2(X, \mathbb{Z})$ and $c_2(\mathcal{E}) \in H^4(X, \mathbb{Z})$.

Theorem 32.7 (Bogomolov). $\Delta(\mathcal{E}) \geq 0$.

It could be interesting to study the proof of this theorem.

Miyaoka (1987). Consider a curve C and a bundle E. We can for any fibre, which is a vector space V, consider its projectivization $V \setminus \{0\}/\mathbb{C}^* = \mathbb{P}(V)$. Then we can bundlize this construction ad obtain $\mathbb{P}E \to C$. Consider $\mathcal{O}_{\mathbb{P}E}(1)$. We define

$$\lambda = rc_1(\mathcal{O}_{\mathbb{P}E}(1)) - \pi^* c_1(E) \in H^2(\mathbb{P}E, \mathbb{Z}).$$

Theorem 32.8. *E is semistable if λ is semicorrente efectiva.*

Definition 32.9. Consider again a projective smooth connected curve Γ and let $f : \Gamma \to \mathbb{P}E$. We say λ is *nef* if

$$\int_\Gamma f^* \lambda \geq 0.$$

The following theorem was proved by Nakayama (who is a different author than that of Nakayama's lemma) in 1999 but was unkown to Bruzo and Daniel HR.

Theorem 32.10 (Nakayama '99, Bruzzo-HR '04)**.** *The following are equivalent:*

(1) *λ is nef.*
(2) *For all $f : \Gamma \to X$, f^*E is semi-stable.*
(3) *E is semistable with respect to the polarization, and $\Delta(E) = 0$.*

The polarization says how the surface is embedded in projective space, it is $H = c_1(\mathcal{L})$. The stability notion is

$$\mu(E) = \frac{c_1(E) \cdot H^{n-1}}{\mathrm{rk}E}.$$

They have proved that for Higgs bundles $1 \iff 2$, $3 \implies 1, 2$. And also for $1, 2 \implies 3$ but only for $r = 2$ using the Higgs Grassmannian, but the problem is very complicated for general rank.

Second lecture. I think this might be a more general definition of the Hilbert polynomial:

$$\sum_{k=0}^{n} (-1)^k \dim H^k(X, \mathcal{E}) = \chi(\mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E}) \mathrm{td}X$$

And we also have:

$$\chi(\mathcal{E} \otimes \mathcal{L}^*) = \int_X (1 + (1 - g)w)(r + kr \underbrace{c_1(\mathcal{L})}_{\deg \mathcal{L} \cdot w} + \deg \mathcal{E} \cdot w.$$

And then

$$p_{\mathcal{E}} = k\deg\mathcal{L} + 1 - g + \mu(\mathcal{E}).$$

Now we discuss Higgs bundles. Let X be ac onnected complex projective variety.

Definition 32.11. A *Higgs bundle* on X is a pair $\mathfrak{C} = (E, \phi)$ with

- E a vector bundle on X
- ϕ : a morphism (connection-like. . .)

For the following see Nitsuze, FGA explained. There following scheme is also representable (I think this is the "universal quotient")

We have also consider the Grassmanization of a vector bundle. We obtained a short exact sequence and tried to compute the tensor product with the holomorphic differentials. The idea was to pullback the grassman bundle given a map $f : Y \to X$. The result is not satisfactory, the pulled back thing may be strange. This is called the *Higgs Grassmannian.*

Here's some notion also studied by Campana, Demailly: numerically flat bnef bundle E. Then we have that

Lemma 32.12. *If $c_1 = 0$ then numerically flat bundle iff semistable.*

This is the beginning of a talk by Alan Muniz at Bandoleros 2025 titled "Rank 2-bundles on $\mathbb{P}^3$ with odd determinant". (Work in progress with A. Fontes.)

Problem: Classify rank-2 bundles on $\mathbb{P}^3$.

We consider bundles E with chern classes $c_1 = -1$, $c_2 = 2n$ and $c_3 = 0$. Let $\mathcal{B}(-1, 2n)$ be the moduli space of μ-stable rank 2-bundles.

- $\mathcal{B}(-1, 2)$ is irreducible of dimension 11 [Hartshorne-Sols '82, Manolache '81].
- $\mathcal{B}(-1, 4)$ has two components [Barnica-Manolache '85].
- $c_2 \geq 6$, Tikhomirov (2014) Fontes-Jardim (2023, 2024, 2025). $\mathcal{B}(-1, 6)$ has ≥ 4 components.

Serre correspondence. How can we describe these spaces? Let E be a vector bundle of rank 2 on $\mathbb{P}^3$ and $\sigma \in H^0(E(r))$. Then $C = (\sigma_0) \subset \mathbb{P}^3$ is a curve * equipped with an "extension class" $\zeta \in H^0(\omega_C(4 - 2r - c_1))$. In fact, this correspondence is 1-1, and we call it Serre correspondence. We have that $\deg C = c_2(E(r))$ and $p_a(C) = 1 = \frac{c_2(E(r))(c_1(E(r)) - 4)}{2}$ To pass from the curve to the bundle and viceversa we use

$$0 \longrightarrow \mathcal{O}(-r) \longrightarrow E \longrightarrow \mathcal{I}_C(c_1 + r) \longrightarrow 0$$

Upshot about Deligne-Mumford stacks: the sit below a stack via a map that is étale.

32.13. Framed moduli. Framed instantons on the blow-up of $\mathbb{P}^3$ at a point.

Let X be a projective variety and fix a polarization H. Let $\mathcal{E}$ be a sheaf on X. A *framed pair* is a $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a coherent sheaf on X and α is a *framing datum*, i.e. a map $\alpha : \mathcal{F} \to \mathcal{E}$.

Set

$$\mathcal{E}(\alpha) = \begin{cases} 1 & \alpha \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define the Hilbert polynomial by

$$P(k) = \chi(\mathcal{F} \otimes \mathcal{O}(kH)).$$

For fied $\delta \in \mathbb{Q}[t]$ rational positive, define

$$P_{(\mathcal{F}, \alpha)} = P_{\mathcal{F}}(k) - \varepsilon(\alpha)\delta.$$

We can also define induced framings from subsheaves $\mathcal{F}' \subset \mathcal{F}$, by asking that the inclusion commutes with the framing datum α. Then we put

$$\varepsilon(\alpha') = \begin{cases} 1 & \mathcal{F}' \subseteq Ker\alpha \\ 0 & \text{otherwise.} \end{cases}$$

We say that $\mathcal{F}$ is δ-*(semi)stable* if for any subpair $(\mathcal{F}', \alpha')$ with $\mathrm{rk}\mathcal{F}' = r' < r = \mathrm{rk}\mathcal{F}$ one has

$$\frac{1}{r'}P_{(\mathcal{F}',\alpha')}(k) \leq \frac{1}{r}P_{(\mathcal{F},\alpha)}(k).$$

If $\deg(\delta) \leq \dim X$, there are moduli functors $\mathcal{M}^{st}(X, \mathcal{E}, \delta) \subseteq \mathcal{M}^{ss}(X, \mathcal{E}, \delta)$.

Remark 32.14. If $\deg\delta > \dim X$ then α is injective or zero, and $\mathcal{F}$ is semistable. This gives quot schemes.

Huybrechts-L. In $\dim X \leq 2$ there exists a quasi-projective fine moduli space for $\delta \in \mathbb{Q}[t]$ and $p \in \mathbb{Q}[k]$.

Then there is a notion of good framing by Bruzz-M, depending on D, an effective divisor on X. They proved that having good framing is enough to obtain a fine quasi-projective moduli space.

Definition 32.15. A coherent sheaf $\mathcal{E}$ on $X = \tilde{\mathbb{P}^3}$ is an *instanton* if it is $\mu_{\mathcal{L}}$-semistable, $c_1(\mathcal{E}) = 0$ and $H^0(\mathcal{E}) = 0 = H^0(Ec(-3,2))$, $H^1(\mathcal{E}(-2,1)) = 0 = H^2(\mathcal{E}(-2,1))$ and $H^3(\mathcal{E}(0,-1)) = H^2(\mathcal{E}(-1,1))$.

Some examples are locally free sheaves of rank 2 (via Harrshorne-Serre correspondence), torsion-free rank 2 constructed via elementary transformations.

General result: for rank 2 instantons on Fano threefolds that have a plane E, $\mathcal{E}|_E$ is μ-semistable if and only if $h^1(\mathcal{E}(-2,1) \otimes \mathcal{O}(-E)) = 0$.

33. CAMPANA

A *special* variaty should be thought of as the farthest notion possible from being of general type.

Definition 33.1 (Campana)**.** X complex projective. X is *special* if $\forall i \leq p \leq \dim X$, for all line bundle $L \subset \Omega_X^P$, $\kappa(L) < P$ (Kodaira dimension).

A conjecture by Campana says that X is special if and only if there exists $f : \mathbb{C} \to X$ Zariski dense. Compare this with Kobayashi hyperbolicity! There are no lines on ...?

The main tool for proving this kind of "Bloch-Ochai" statements (i.e. something of the kind: if $h^0(\Omega_X) > \dim X$ then there are no nonconstant $f : \mathbb{C} \to X$ Zariski dense) is by passing through the Albanese variety.

In a talk [fill details] we study a theorem of this kind but with the bound given by $q^+(X)$ defined as the supremum of the $h^0(\hat{X}, \pi^*\Omega_X$ for $\hat{X} \to X$ some (étale) cover. Can we prove a BO stement with the condition $q^+(X) > \dim X$?

Notion of special comes in: indeed

$$q^+(X) > \dim X \implies X \text{ not special} \implies \nexists f : \mathbb{C} \to X \text{Zariski dense.}$$

The following theorem is not at all easy to prove:

Theorem 33.2 (Campana)**.** $\hat{X} \xrightarrow{\text{étale}} X$, *then* $\hat{X}$ *is special iff* X *is special.*

My question: why does the abelianess of the Albanese variety is important in this construction?

References

[Har77] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977.

[Har09] R. Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, Springer New York, 2009.

[Lee24] J.M. Lee, *Introduction to complex manifolds*, Graduate Studies in Mathematics, American Mathematical Society, 2024.

[Ser06] Edoardo Sernesi, *Deformations of algebraic schemes*, 2006.

[Vak25] R. Vakil, *The rising sea: Foundations of algebraic geometry*, Princeton University Press, 2025.