FACTORING CREMONA TRANSFORMATIONS

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Notes at github.com/danimalabares/stack

Abstract. O grupo de Cremona em dimensão n é o grupo das transformações birracionais do espaço projetivo de dimensão n. O celebrado teorema de Noether-Castelnuovo (1871-1901) afirma que o grupo de Cremona em dimensão 2 é gerado pelos automorfismos lineares e por uma única transformação quadrática. Em dimensões superiores, não há uma descrição simples do grupo de Cremona em termos de geradores, e a situação é bem mais complicada. Por outro lado, técnicas de geometria birracional, em particular o MMP (Minimal Model Program), fornecem uma maneira de fatorar transformações de Cremona como composições de elos elementares. Essa teoria, conhecida como "Programa de Sarkisov", tem se mostrado extremamente útil no estudo do grupo de Cremona em dimensão superior. Neste minicurso, faremos uma introdução ao MMP e ao Programa de Sarkisov.

Contents

1. Cremona group

Let $\mathrm{Cr}_n(k) := \mathrm{Aut}_k k(x_1, \dots, x_n)$.

Definition 1.1. $f : X \dashrightarrow Y$ is *rational* if it is defined in an open dense set $U_X \subseteq X$. It is *birational* if it admits an inverse; equivalently, if there is $U_Y \subseteq Y$ open dense such that $f : U_X \to U_Y$ is an isomorphism.

Fix $n = 2$ and consider the map $(x_1, x_2) \mapsto \left(\frac{1}{x_1}, \frac{1}{x_2}\right)$.

Example 1.2.

$$\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$
$$(x_0, x_1, x_2) \longmapsto (x_1x_2, x_0x_2, x_0x_1).$$

Then $\sigma = \sigma^{-1}$, with

$$\{x_i = 0\} \mapsto p_i, \qquad \{x_0 = 0\} \mapsto (1:0:0).$$

σ: standard quadratic transformation.

Question (Enrique, 1895): Is $\mathrm{Cr}_n(k)$ simple? [i.e. does it have any nontrivial normal subgroups?] Do there exists any homomorphisms $\mathrm{Cr}_n(k) \to H$?
$\mathrm{PGL}_{n-1}(k) \cong \mathrm{Aut}(\mathbb{P}^n) \subseteq \mathrm{Cr}_n(k)$.

How to find homomorphism to a group H, $\mathrm{Cr}_n(k) \to H$.

(1) Find a set of generators G by $\mathrm{Cr}_n(k)$.
(2) Get a set R of relators. $\mathrm{Cr}_n(k) = \langle G | R \rangle$.
(3) Map the generators to arbitrary elements of H and check that the relators are mapped to 1_H.

Theorem 1.3 (Noether-Castelnuovo, 1872)**.** *$Cr_2(\mathbb{C})$ is generated by two automorphisms of $\mathbb{P}^2$ and σ.*

Theorem 1.4 (Gizatulin, 1983)**.** *Description of the relators with respect to two generators $\{PGL_3(k), \sigma\}$.*

Remark 1.5. Contat-Lamy, 2010. $\mathrm{Cr}_2(k)$ is not simple for $k = \bar{k}$. Lonjoi, 2017: any field. [This is the proof that uses an action on an infinite-dimensional hyperbolic space. The same technique does not work for higher dimensions.]

Theorem 1.6 (Hudson, 1927 & Pan, 1999)**.** *Any set of [nonlinear] generators for $Cr_n(k)$, $n \geq 3$, is uncountable.*

... and we don't know any relators!

2. MMP and Sarkisov theory

[MMP is an algorithm.]

$$X : {\scriptstyle \text{smooth} \atop \text{projective}} \longrightarrow \text{MMP} \longrightarrow X_{\min} : {\scriptstyle \text{mildly singular} \atop \text{projective}}$$

so that

- $X \sim_{bir} X_{\min}$
- $X_{\min}$ is the "simpler" than X
  - $K_{X_{\min}}$ is "more positive",
  - $\rho(X_{\min}) \leq \rho(X)$.
- The process is realized in "elementary"steps.

The outputs of MMP are of two types:

(1) minimal models, or
(2) Mori fiber spaces (Mfs)

Sarkisov program. An algorithm for decomposing birational maps among Mfs into "simpler" maps.

[The idea of these maps is to copy the action of the standard quadratic transformation σ from Example 1.2: we blow up three points (vertices of a triangle, the p_i), then we get a "hexagonal" arangement, and contract three lines (the other three) to get back at a triangle).]

$$\text{triangle} \rightsquigarrow \text{hexagon} \rightsquigarrow \text{triangle}$$

$$\underbrace{\mathbb{P}^2}_{\text{triangle}} \rightsquigarrow \underbrace{\mathbb{F}_1}_{\text{Hirzebruch}} \rightsquigarrow * \rightsquigarrow \mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow * \rightsquigarrow * \rightsquigarrow \underbrace{\mathbb{P}^2}_{\text{triangle}}$$

Theorem 2.1 (Corb, 1995 and Hacon-McKeman, 2011)**.** *Any birational map among Mfs can be factored as a composition of Sarkisov links if and only if the Sarkisov links are generators of the groupoid*

$$\underbrace{BirMori}_{\supseteq Cr_n(k)} = \left\{ f : X \dashrightarrow Y^{\substack{f \;\; birational \\ X,Y \;\; Mfs, \\ X,Y \sim_{bir} \mathbb{P}^n}} \right\}$$

Theorem 2.2 (Blanc-Lamy-Zimmerman)**.** *Description of the relators among the Sarkisov links.*

Theorem 2.3 (BLZ, 2021)**.** *$Cr_n(k)$ is not simple for $n \geq 3$, φ_i only appearing in relations of the form $\varphi_i \circ X \circ \varphi_i^{-1} \circ X^{-1} = id$, $\varphi_i^2 = id$.*

$$\begin{aligned} BirMor(\mathbb{P}^n) &\longrightarrow \mathbb{Z}/2\mathbb{Z} \\ \varphi &\longmapsto 1 \\ links \neq \varphi &\longmapsto 0 \end{aligned}$$

3. Review of surfaces

Definition 3.1. A *surface* is a smooth projective variety of dimension 2.

Example 3.2. (1) $S = \mathbb{P}^2$

(2)

$$S = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow z(xy - zw) \subset \mathbb{P}^3_{(x:y:z:w)}$$

(3) Blowup de $\mathbb{P}^2$ em p. [Picture showing an arbitrary line $\mathbb{P}^1$ contained in the projective plane $\mathbb{P}^2$. Given P not in the line, we can project any point Q not in the line to the line. This gives a rational map $\pi_P : \mathbb{P}^2 \setminus \{P\} \dashrightarrow \mathbb{P}^1$. Consider the graph Γ of π_P, it is contained in $\mathbb{P}^2 \setminus \{P\} \times \mathbb{P}^1$. Its closure, $\overline{\Gamma}$ is contained in $\mathbb{P}^2 \times \mathbb{P}^1$. We have projections $p : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ and $q : \mathbb{P}^2 \times P^1 \to \mathbb{P}^1$. Note that $E = p^{-1}(P) \cong \mathbb{P}^1$ and $S \setminus E \xrightarrow{\cong} \mathbb{P}^1 \setminus \{P\}$. More generally,

$$\tilde{S} = Bl_P S \xrightarrow{\pi} S$$

$$E = \pi^{-1}(P) \cong \mathbb{P}^1 \to P$$

for $P \in S$.

Definition 3.3. S surface, $\mathrm{Pic}(S) = \mathrm{Div}(S)/\sim \cong \{\text{invertible sheaves}\}/\sim$.

$$\mathrm{Div}(S) = \left\{ \sum_{\text{finite}} n_i D_i : n_i \in \mathbb{Z}, D_i \subset S \text{ irreducible curve} \right\}.$$

$$\mathrm{Div}(S) \iff D - D' = \mathrm{div}(f) = (f)_0 - (f_\infty), \qquad f \in K(S) \setminus \{0\}.$$

4. The role of the blowup in surface birational geometry

Let S and S' be two (projective nonsingular) surfaces.

- Any birational morphism $f : S \to S'$ is a composition of blowups.
- Any birational map $f : S \dashrightarrow S'$ factors as

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S \arrow[rr, dashed] & & S' \end{tikzcd}$$

  where W is a surface and p, q are compostions of blowups.

[We wish to find the simplest variety in the same birational class.]

Definition 4.1. A surface S is called *minimal* if [if whenever we have a birational map from S to another surface then that morphism is an isomorphism] if any birational morphism $S \to S'$ is an isomorphism.

5. Intersection theory for surfaces

Theorem 5.1. *Let S be a surface. There exists a unique symmetric bilinear form $Pic(S) \times Pic(S) \to \mathbb{Z}$ such that if C and C' are two curves intersecting transversally, then $[C] \cdot [C'] = \#(C \cap C')$.*

Example 5.2. (1) $S = \mathbb{P}^1$, $\mathrm{Pic}(S) = \mathbb{Z}[H]$ where H is a line (hyperplane). $H^2 = 1$.

(2) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Pic}(S) = \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2]$ there H_i is the class determined by either projection. We have

$$\begin{cases} H_1^2 = 0 & = H_2^2 \\ H_1 \cdot H_2 & = 1. \end{cases}$$

In this case $C \subset S$, $C = Z(F) \subset \mathbb{P}^1_{(x:y)} \times \mathbb{P}^1_{(z:w)}$, $F \in k[x, y, z, w]$ bi homogeneous of degree (d_1, d_2).

(3) $S = Bl_P\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$, $\mathrm{Pic}(S) = \mathbb{Z}[\pi^* H] \oplus \mathbb{Z}[E]$.

$$\begin{cases} \pi^* H \cdot \pi^* H & = H \cdot H = 1 \\ \pi^* H \cdot E & = 0 \\ E^2 & =? \end{cases}$$

$$\pi^* \ell = \tilde{\ell} + E, \qquad \ell \text{ line } \subset \mathbb{P}^2$$

$$E^2 = E \cdot (\pi^* \ell - \tilde{\ell}) = \underbrace{E \cdot \pi^* \ell}_{0} - \underbrace{E \cdot \tilde{\ell}}_{1} = -1.$$

In general [for blowups], let

$$\pi : \tilde{S} = Bl_P S \longrightarrow S$$
$$\mathbb{P}^1 \cong E \longmapsto P$$

$$\mathrm{Pic}(\tilde{S}) = \pi^*\mathrm{Pic}(S) \oplus \mathbb{Z}E$$
$$\begin{cases} \pi^*D \cdot \pi^*D' & = D \cdot D \\ \pi^*D \cdot E & = 0 \\ E^2 & = -1. \end{cases}$$

Definition 5.3. A (-1) *curve* in a surface S is a curve $C \subset S$ such the $C \cong \mathbb{P}^1$ and $C^2 = -1$.

Theorem 5.4 (Castelnuovo's contractibility theorem)**.** *Let S be a surface and $C \subset S$ a (-1)-curve. Then there exists a surface $S' \ni P$ such that $S \cong Bl_P S'$ and via this isomorphism C is an exceptional curve.*

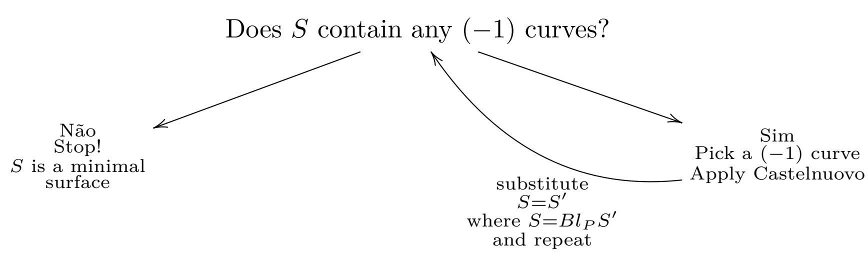
Notation. We say that $S \cong Bl_P S' \to S'$ is a contraction of C.

As a corollary,

Lemma 5.5. *A surface S is minimal if S does not contain any (-1) curves.*

6. MMP for surfaces

Let S be a (nonsingular projective) surface.



Definition 6.1. Let S be a surface and $D, D' \in \mathrm{Div}(S)$. D and D' are *numerically equivalent* if for any curve $C \subset S$ we have $D \cdot C = D' \cdot C$. $\mathrm{Num}(S) = \mathrm{Div}/\equiv = \mathrm{Pic}(S)/\equiv$.

Theorem 6.2. *$Num(S)$ is a finite-rank free abelian group. Such a rank is called the Picard number $\rho(S) := rk(Num(S))$.*

Remark 6.3. [The rank of the blowup is one larger.] $\tilde{S} = Bl_P S$ Then

$$\mathrm{Pic}(\tilde{S}) = \pi^*\mathrm{Pic}(S) \oplus \mathbb{Z} \cdot E$$
$$\mathrm{Num}(S) = \pi^*\mathrm{Num}(S) \oplus \mathbb{Z}[E].$$

[The following examples shows how we may arrive at different surface. . .]

Example 6.4. Let $S = Bl_{P,Q}\mathbb{P}^2$ and denote E_P and E_Q the exceptional divisors of P and Q. Let ℓ be the line containing P and Q. Consider a lift $\tilde{\ell} \cong \mathbb{P}^1$. Then $\pi^*\ell = \tilde{\ell} + E_P + E_Q$ and $(\tilde{\ell})^2 = -1$. [One may show that in fact these are the only (-1) curves.] Applying Castelnuovo we obtain a contraction of $\tilde{\ell}$, $S \to \mathbb{P}^1 \times \mathbb{P}^1$, where $\tilde{\ell}$ is mapped to a point $R \in \mathbb{P}^1 \times \mathbb{P}^1$. [It is AG jargon that the blowup of two points is $\mathbb{P}^1 \times \mathbb{P}^1$ with ...] Since $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q \subset \mathbb{P}^3$, [a doubly ruled surface?] [We also have a map $Q \xrightarrow{\pi_R} \mathbb{P}^2$. It looks like the two lines passing through R (since it is a doubly ruled surface) are mapped to two points under π_R.]

$$\underset{\text{i.e., } S \sim_{bir} \mathbb{P}^2}{\overset{S}{\text{rational surface}}} \overset{\text{MMP}}{\rightsquigarrow} \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{F}_n, \qquad n \geq 2.$$

Definition 6.5. A *scroll* is a surface $S \xrightarrow{\pi} \mathbb{P}^1$ such that all fibers are isomorphic to $\mathbb{P}^1$. $S = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle of rank 2.

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$$
$$\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(n)).$$

$$\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n)).$$

Exercise 6.6. $\operatorname{Pic}(\mathbb{F}_n) = \mathbb{Z} \cdot [f] \oplus \mathbb{Z} \cdot [E]$

$$\begin{cases} f^2 & = 0 \\ f \cdot E & = 1 \\ E^2 & = -n. \end{cases}$$

7. Summary

$$\text{S rational surface} \overset{\text{MMP}}{\rightsquigarrow} \begin{matrix} S = \mathbb{P}^2 \\ \text{or} \\ S = \mathbb{F}_m, \quad \begin{matrix} m = 0 \\ \text{or } m \geq 2 \end{matrix} \end{matrix}.$$

Exercise 7.1. Given $n_0 > 1$, construct S such that there exists MMP $\leftrightsquigarrow$ $\mathbb{F}_m$ for all $m = 0$ or $2 \leq m \leq n_0$.

Consider $\mathbb{F}_m$ and $\sigma^2 = -m$. If $m \geq 1 \exists!$ curve $C \subset \mathbb{F}_m$ such that $C^2 < 0$ ($C = \sigma$). If $m = 0$, $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

$$\begin{tikzcd}
& \begin{matrix}\text{variety containing} \\ \pi^* f, E, \widetilde{f_P} \cong \mathbb{P}^1 \\ \tilde{\sigma}\end{matrix} \arrow[dl, "\pi"', "Bl_P"] \arrow[d] \arrow[dr, "\text{contraction of } \widetilde{f_P}"] & \\
\begin{matrix}\mathbb{F}_m \\ \text{containing} \\ f, f_P, P\end{matrix} \arrow[r, dashed, "\text{Sarkisov link}"] \arrow[d] & \mathbb{P} \arrow[r, dashed] & \begin{matrix}\mathbb{F}_n \\ \text{containing} \\ \tilde{E}, \tilde{\tilde{\sigma}}, Q \\ n = ?\end{matrix} \arrow[d] \\
\mathbb{P}^1 & & \mathbb{P}^1.
\end{tikzcd}$$

$$(\tilde{f}_P)^2 = \tilde{f}_P \cdot (\pi^* f - E) = -1$$
$$\pi^* f_P \ldots$$

[Computations show that $n = m + 1$.]

8. Positivity of the canonical class

Now we want to reinterpret the MMP in terms of "positivity of the canonical class".

Definition 8.1. Let S be a surface and Ω^1_S the Kähler differentials on S. The canonic sheaf is $\omega_S = \Lambda^2\Omega^1_S$ [because the dimension of S is 2]. Thus, $[\omega_S] = K_S \in \mathrm{Pic}(S) = \mathrm{Div}/\sim$ [is well-defined].

More concretely, ω is a rational 2-form in S, $\mathrm{div}(\omega) = (\omega)_0 - (\omega)_\infty$.

Example 8.2.
(1) $S = \mathbb{P}^2$, $\mathrm{Pic}(S) = \mathbb{Z}\cdot[H]$. [We look for the zeroes and poles of the form.] [Choose an affine open] $U = \mathbb{A}^2_{\left(\underbrace{\frac{x}{z}}_{u},\underbrace{\frac{y}{z}}_{t}\right)} \subset \mathbb{P}^2$ where $\omega = du \wedge dt$.

Let $V = \mathbb{A}^2_{(\frac{y}{x},\frac{z}{x})}$. [We look for the coefficient function of] $\omega = ?dv \wedge ds$.

In $U \cap V$, [computations...] Thus, $\omega = -\frac{1}{s^3}dv \wedge ds$. [Thus, this form has a pole of degree 3 at infinity.]

We conclude that $K_{\mathbb{P}^2} = -3H$.

(2) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Pic}(S) = \mathbb{Z}\cdot H_1 \oplus \mathbb{Z}\cdot H_2$, [Exercise:] $K_S = -2H_1 - 2H_2$.

Lemma 8.3 (Adjunction formula)**.** *Let X be a smooth projective variety and $Y \subset X$ an irreducible hypersurface. Then $K_Y = (K_X + Y)|_Y$.*

[Then]

$$S = \mathbb{P}^1 \times \mathbb{P}^1 \cong S_2 \subset \mathbb{P}^3$$

$$K_S = (-4H + 2H)|_S = -2H|_S$$

(3) $S = \mathbb{F}_m$, $\mathrm{Pic}(\mathbb{F}_m) = \mathbb{Z}\cdot f \oplus \mathbb{Z}\cdot\sigma\ldots$ [More computations].

Definition 8.4. A divisor $D \in \mathrm{Div}(S)$ is *nef* if [it has nonegative intersection number with any curve in S] $D \cdot C \geq 0$ for any curve $C \subset S$.

Example 8.5.
(1) (A very ample divisor.) Consider

$$S \xhookrightarrow{f} \mathbb{P}^n$$

where $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}\cdot[H]$, then f^*H is very ample since $f^*H\cdot C = f_*(C)\cdot H > 0$.

(2) (Ample divisor.) Recall that D is ample if mD is very ample for some $m > 0$. Let $f : S \to \mathbb{P}^m$ be a morphism, $D = f^*H$, $C \subset S$. Then

$$D \cdot C = H \cdot f_*(C) = \begin{cases} > 0 & \text{if } f_*(C) \text{ is a curve} \\ = 0 & \text{if } f(C) = pt. \end{cases}$$

[Qualquer D que seja o pullback da classe do hiperplano sob alguma f é dito de semi-amplo.] D is called *semi-ample*.

9. Kleimann amplness theorem

"nef= limit of ample".

Recall that we defined

$$\mathrm{Num}(S) = \mathrm{Div}(S)/\equiv = \mathbb{Z}^{\rho(S)}.$$

Now let

$$N'(S) := \mathrm{Num}(S) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^{\rho(S)}.$$

[Now we consider the following cones:]

$\overline{NE(S)}$, the closed cone generated by all classes $[C]$ of curves $C \subset S$, called the *Mori cone*. $\overline{NE(S)} \subset N'(S)$.

Let $D \in \mathrm{Div}(S)$. D is nef if and only if $D \cdot \alpha \geq 0 \forall \alpha \in \overline{NE(S)}$.

$\mathrm{Nef}(S)$ is the closed convex cone generated by the nef divisors $= (\overline{NE_1(S)})^\vee \subset N^1(S)$.

Theorem 9.1 (Kleimann's amplitude theorem)**.** *Let $D \in Div(S)$. D is ample if and only if $D \cdot \alpha > 0$ for all $\alpha \in \overline{NE_1(S)} \setminus \{0\}$ if and only if $D \in Int(Nef(S))$.*

Definition 9.2. Let $C \subset \mathbb{R}^n$ a closed convex cone. A *face* $F \subset N$ is a subcone such that $u, v \in N$ and $u + v \in F$ imply that $u, v \in F$.

An *extremal ray* of N is a face of [maximal?] dimension.

Example 9.3. (1) $S = \mathbb{F}_m$, $N_1(\mathbb{R}^2)$. $K_S \cdot f = -2$, $K_S \cdot \sigma = n - 2$.

(2) (The extremal ray of the exceptional divisor is an extremal ray.)

$$\begin{tikzcd} \tilde{S} = Bl_P S \arrow[r, "\pi"] & S \\ \mathbb{P}^1 \arrow[u, hook] \arrow[r] & pt \arrow[u, hook] \end{tikzcd}$$

Then $\mathbb{R}_{\geq 0} E$ is an extremal ray of $\overline{NE_1(\tilde{S})}$.

$K_S \cdot E = -1$. [(-1)-curves always have -1 intersection number with the canonical divisor.]

Exercise 9.4. $C \subset S$, C a (-1)-curve if and only if $C^2 < 0$ and $C \cdot K_S < 0$ if and only if $C^2 = -1$ and $K_S \cdot C = -1$.

Remark 9.5. If K_S is nef, then S is a minimal surface. $\mathbb{P}^2$ is minimal but $K_{\mathbb{P}^2} = -3H$.

Definition 9.6. S is *minimal model* if K_S is nef.

Theorem 9.7 (Mori's cone theorem)**.** *Let S be a surface.*

$$\begin{aligned} \overline{NE}_1(S) &= \overline{NE}_1(S)^{K_S \geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i] \\ &= \overline{NE}_1(S)^{K_S + A \geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i], \qquad C_i \cong \mathbb{P}^1. \end{aligned}$$

[Figure of $N^1(S)$ split into two regions by K_S. One region is positive and the other negative. In the boundary we see $K_S + \varepsilon A$, ample.]

Theorem 9.8 (Classification of the C_i 's).

- *$C_i^2 < 0 \implies C_i$ is a (-1) curve.*
- *$C_i^2 = 0 \implies S \cong \mathbb{F}_m$ and $C_i = f$.*
- *$C_i^2 > 0 \implies S \cong \mathbb{P}^2$ and $C_i = \lambda H$.*

10. Summary

[Today: interperetation of the classical MMP in modern language.]

Recall:

- $D \in \mathrm{Div}(S)$ is *nef* if $D \cdot C \geq 0$ for all $C \subset S$.
- the blow up of a surface at a point: $\tilde{S} = Bl_P S \xrightarrow{\pi} S$ with exceptional divisor E, then $K_S \cdot E = -1$ and $E^2 = -1$.
- $S = \mathbb{F}_m$, $K_S \cdot f = -2$. $f^2 = 0$.
- [The Mori cone is the convex cone generated by the curves in S:] $\overline{NE}_1(S) = \overline{\langle \text{curves in } S \rangle} \subset \underbrace{N^1(S)}_{=\mathbb{R}^e} = \mathrm{Num}(S) \otimes_{\mathbb{Z}} \mathbb{R}$.
- Theorem. Let S be a surface. Then
  (1) (Cone theorem.) [Picture of the hyperplane K_S limiting two regions, a positive and a negative one. The negative side is locally polyhedral - while on the positive side it may be round. We also consider a small perturbation $K_S + \varepsilon A$.]
  (2) (Contraction theorem.) Any face on the negative side of the Mori cone admits a contraction. [See definitions below.]
  (3) (Classification of contractions of K_S-negative extrmal rays.)
    - If $C^2 < 0$, then $\mathrm{cont}_R = Bl_P$.
    - If $C^2 = 0$, then $S = \mathbb{F}_m$ and $\mathrm{cont}_R : \mathbb{F}_m \to \mathbb{P}^1$.
    - If $C^2 \geq 0$, then $S \cong \mathbb{P}^1$.
- [A face is a hyperplane that encosta no cone.] A *face* F of a cone is such that $v, u \in \overline{NE}(S)$ and $v + u \in F$ then $u, v \in F$.

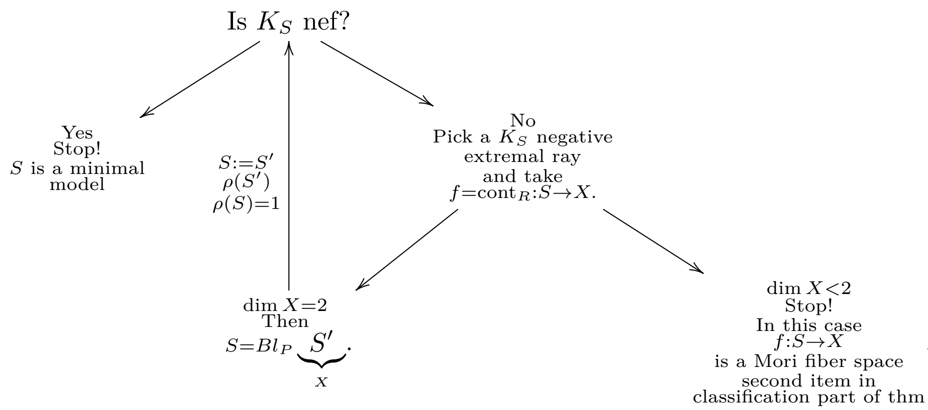
Definition 10.1. Let $F \subset \overline{NE}_1(S)$ be a face. A *contraction* of F is a morphism $\mathrm{cont}_F : S \to X$ with connected fibers, X normal such that given a curve $C \subset S$,

$$\mathrm{cont}_F(C) = pt \iff [C] \in F.$$

Remark 10.2. cont_F may or not exist, but if it exists it is unique.

11. Modern version of MMP

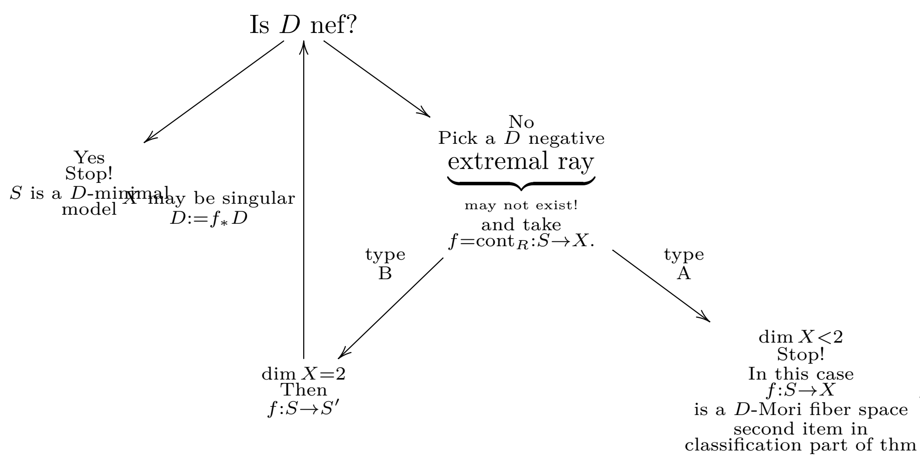
[Instead of asking whether there are any (-1) curves, now we ask if the canonical divisor is positive:]



Example 11.1. $S = \mathbb{F}_m$, we computed that $K_S = -2\sigma - (2+m)f$. $K_S \cdot f = -2$, $K_S \cdot \sigma = 2m - 2 - m = m - 2$.

[Notice the diagram is more complicated now: there are more options. Why is this formulation convenient?]

Now fix a divisor $D \in \mathrm{Pic}(S)$. We may consider now the D-MMP:



Some remarks:

- The final result of MMP (or D-MMP) may not be unique, but its type (A or B) depends only on the birational class of S (or $D = K_S$).
- When MMP ends in B (i.e. S is a [rational] ruled surface) the result is in general not unique.

Example 11.2. When S is a rational surface, the possible results are wither $\mathbb{P}^2 \to pt$ or $\mathbb{F}_m \to \mathbb{P}^1$.

When does a D-MMP exist?

- S toric. (D-MMP exists for any D.)
- S Mori dream space (MDS). (D-MMP exists for any D.) [This is the reason for the name.] If $\mathrm{Pic}(S) = \mathbb{Z}^{\oplus m}$, the *Cox ring* is $\mathrm{Cox}(S) = \bigoplus_{D \in \mathrm{Pic}(S)} H^0(S, D)$. S is a *Mori dream space* if $\mathrm{Cox}(S)$ is finitely generated.

- Let S be any surface, $D = K + \varepsilon A$, A ample. [With such a D we always guarantee that the program exists.]

  Remark 11.3. If $D \in \overline{NE}(S)$, the D-MMP always ends in a D-minimal model.

[Recall, the classical version consists of a sequence of blowdowns (or blowups) starting from S and finishing in S'.]

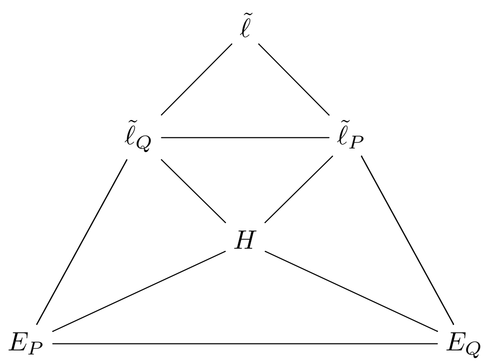
$$\begin{tikzcd} S \arrow[r] & \cdots \arrow[r] & S' \arrow[d, "g"] \\ & & \underbrace{X}_{\substack{D\text{-ample} \\ \text{model}}} \end{tikzcd}$$

$D' = g^* AX$, then D' is nef (semi-ample).

[Extra: A K3 is a MDS is it has a trivial automorphism group (for D-MMP).]

12. Geography of ample models

Example 12.1. [The following is a toric surface; we worked on this two lectures ago.] $S = Bl_{P,Q}\mathbb{P}^2$ [The cone has dimension 3, but we can make a transverse section. Imagine that the rest of the cone is behind the screen:]



[H should be in the center of the of the quadrilateral with vertices E_P, E_Q $\tilde{\ell}_P$ and $\tilde{\ell}_Q$. $Nef(S)$ is the triangle with vertices $\tilde{\ell}_Q, \tilde{\ell}_P$ and H.]

$$S \xrightarrow[\text{blowups}]{f} \underbrace{S'}_{\substack{D\text{-minimal} \\ \text{model}}} \xrightarrow{\varphi|mD|} \underbrace{X}_{\substack{D\text{-ample} \\ \text{model}}} = X(S, D).$$

Definition 12.2. Let $D, D' \in \overline{NE}(S)$. $D \sim_{\text{Mori}} D'$ if the morphisms

$$S \to X(S, D) \text{ and } S \to X(S, D')$$

are the same (up to $X(S, D) \cong X(S, D')$).

[Inside $Nef(S)$ the model is itself. In the lower edge of the triangle, the model is $\mathbb{P}^2 \to pt$. In the edge from $\tilde{\ell}_p$ to $\tilde{\ell}$, $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p_1} \mathbb{P}^1$. In the edge $\tilde{\ell}_p$ to E_Q, $\mathbb{F}_1 \to \mathbb{P}^1$.]

[This figure shows us all the Mfs that may be obtained from S. The results are codified in the faces of the cone decomposition. The strategy of the proof of Sarkisov theorem is to move around in the boundary of the cone; that is, passing from one Mfs to another.]

Theorem 12.3 (Sarkisov program)**.** *Any birational map of Mfs is a composition of Sarkisov links, which may be either of the follwing:*

(1)

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[d] & & \mathbb{P}^1 \\ pt & & \end{tikzcd}$$

(2)

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "Bl_P"'] \arrow[dr] & \\ \mathbb{F}_m \arrow[rr, dashed] \arrow[d] & & \mathbb{F}_{m\pm 1} \arrow[d] \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

(3)

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 & & \mathbb{P}^2 \arrow[d] \\ & & pt \end{tikzcd}$$

(4) $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$,

$$\begin{tikzcd} \mathbb{F}_0 \arrow[r, equal] \arrow[d, "p_1"'] & \mathbb{F}_0 \arrow[d, "p_2"] \\ \mathbb{P}^1 & \mathbb{P}^1 \end{tikzcd}$$

[In fact, the Sarkisov links appear when we pass from one chamber to another in the Mori cone.]

13. Proof of Sarkisov theorem

- Let $D \in N^1(S)_d$ be of the form $K_S + A$, A ample.
- A birational map $f : S \to T$ is called a *D-minimal model* if it is the result of a D-MMP.

- A map $g : S \to B$ is called an *ample model of D* if it factos from a D-minimal model $f : S \to T$ followed by the contraction of f_*D.
- (Geography of ample models.) We pick two ample divisors $A_1, \dots, A_k, K_S \notin \overline{NE}(S)$,

$$\mathcal{C} = \{D = A_0 K_S + \sum_{i=1} n a_i A_i : a_0, \dots, a_k \geq 0\} \cap \overline{NE}(S).$$

  D_1, D_2 are Mori equivalent if they have the same ample model.
- A *Mori chamber* is an equivalence class $(\mathcal{A}_i)_{i \in I}$ with ample model $f : S \to T_i$. $\mathcal{A}_I$ is a *big chamber* if f_i is birational.

Proposition 13.1. (1) *The chamber decomposition [of the Mori cone $\mathcal{C}$ in Mori chambers] is finite. (Cone theorem + $K_S \notin \overline{\mathrm{NE}}(S)$.)*
(2) *The chambers are convex subcones. ($\mathcal{A}_i = \langle f_i^* Nef, f_i$-exeptional divisors$\rangle$.*
(3) *$\mathcal{A}_i$ big chamber if and only if $\dim \mathcal{A}_i = \dim \mathcal{C}$.*
(4) *If $\mathcal{A}_i$ is big, $\overline{\mathcal{A}_i} = \{D \in \mathcal{C} : f_i : S \to T_i$ is a minimal D-model$\}$. (ample $\to$ nef).*
(5) *If $\overline{\mathcal{A}_i} \cap \mathcal{A}_j \neq \emptyset$ there exists $\rho_{ij} : T_i \to T_j$ making the diagram*

$$\begin{tikzcd} A \arrow[r, "f_i"] \arrow[dr, "f_j"'] & T_i \\ & T_j \end{tikzcd}$$

commute and $\rho(T_i) - \rho(T_j) = \dim(\mathcal{A}_i) - \dim(\overline{\mathcal{A}_i} - \dim(\overline{\mathcal{A}_i} \cap \mathcal{A}_j)$.

Proof of Sarkisov program. [Start with a birational map of Mfs]

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S_1 \arrow[rr, dashed, "\varphi"] \arrow[d, "\eta_1"'] & & S_2 \arrow[d, "\eta_2"] \\ B_1 & & B_2 \end{tikzcd}$$

$\mathcal{C}_w$, there exist $D_1, D_2 \in \mathcal{C}$ such that $\eta_i \circ p_i$ is an ample model of D_i. By (3), $D_i \in \partial \overline{\mathrm{NE}}(W)$.

[The idea is to show that there exists a way that joins those two points which contained completely in the boundary. After showing it exists, we show that this path is in fact a Sarkisov link decomposition.]

The visible boundary is

$$\partial^+ \mathcal{C} = \left\{ D \in \partial \mathcal{C} : \substack{D = \partial\mathcal{C} \cap [K_W, D^0] \\ \text{for some } D^0 \in \mathrm{Int}(\mathcal{C})} \right\} \subseteq \partial \mathcal{C}.$$

[Geometrically, we know that the canonical divisor is not in the cone – because it is not ample.]

Lemma 13.2. • *The visible boundary is contained in $\partial \overline{NE}(W)$.*
- $\dim \partial^+ \mathcal{C} = \dim \mathcal{C} - 1$.
- *$\partial^+ \mathcal{C}$ is connected.*

Proof.

$$
\begin{aligned}
D \in \partial^+\mathcal{C}_1 D = tK_W + (1-t)A_0, &\qquad t \in [0,1) \\
A_0 \in \mathrm{Int}\mathcal{C} \implies A_0 = \sum a_i A_i, &\qquad a_i > 0, \forall i = 1, \dots, k \\
D \notin \partial\mathcal{D} \implies D \in \partial\overline{\mathrm{NE}}(W) &
\end{aligned}
$$

where $\mathcal{D} := \{a_0 K_W + \sum a_i A_i : a_i \geq 0\}$ which appears in the definition of $\mathcal{C}$. □

Let $V_2 \subseteq N^1(W)$ be a 2-dimensional affine space. Take a slice of the cone: $\mathcal{C} \cap V_2 = \mathcal{P}$. [This space has to be general in the sense that] $\mathrm{codim}_{V_2}(\mathcal{A}_i \cap V_2) = \mathrm{codim}_{\mathcal{C}} \mathcal{A}_i$. [We obtain a polygon with a chamber decomposition. Let's see what happens at a point Θ where two (or more) chambers meet. Zooming in, we see a some lines $\mathcal{O}_0, \dots, \mathcal{O}_k$ meeting at Θ, which determine chambers $\overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2, \dots, \overline{\mathcal{A}}_k$.]

[The $\mathcal{O}_i$ are the intersections of the sheaves. The $\mathcal{O}_i$ are not chambers themselves:]

Proposition 13.3. (1) *$0 < i < k$, $\mathcal{O}_i \subseteq \mathcal{A}_i$ or $\mathcal{O}_i \subseteq \mathbb{A}_{i+1}$.*

(2) *$D_0 \in$ rel int$\mathcal{O}_0$, the ample model of D_0 is a Mfs.*

(3) *$k < 3$.*

(4) *Let $\mathcal{A}$ be the chamber such that $\Theta \in \mathcal{A}$. If there exists i such that* $\dim(\overline{\mathcal{A}}_k \cap \mathcal{A}) = 0$ *then there exists a Sarkisov link between the Mfs of Proposition 13.1.*

Proof.

$$
\begin{aligned}
\text{Type I} &\leftrightsquigarrow k = 2, \mathcal{O}_q \subseteq \mathcal{A}_2 \\
\text{Type II} &\leftrightsquigarrow k = 3 \\
\text{Type III} &\leftrightsquigarrow k = 2, \mathcal{O}_q \subseteq \mathcal{A}_1 \\
\text{Type IV} &\leftrightsquigarrow k = 1.
\end{aligned}
$$

□

□

14. About simplicity of the Cremona group

Recall that the *Cremona group* is defined by

$$\mathrm{Cr}_n(k) = \{f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n : f \text{ is birational}\}.$$

Question by Enriques (1865): is $\mathrm{Cr}_n(k)$ simple?

We know that $\mathrm{Cr}_n(k)$ is not simple in the following cases:

(1) $n = 2$, k any field. Contat-Lamy '13, Lonjou '16.
(2) $n = 2$, k perfect field such that there exists a Galois orbit of size 8. Lamy-Zimmermann '20.
(3) $n \geq 3, k = \mathbb{C}$. Blanc-Lamy-Zimmermann '21.

[The second and third item they use a different method to define the automorphisms.]

Theorem 14.1. *There is an isomorphism $Cr_3(\mathbb{C}) \simeq G * (*_J \mathbb{Z}/2\mathbb{Z})$ where J is uncountable.*

With corollaries:

Lemma 14.2. *$Aut(Cr_3(\mathbb{C}))$ is not generated by inner automorphisms nor automorphisms of the field $\mathbb{C}$.*

Déserti '06: $\mathrm{Aut}(\mathrm{Cr}_2(\mathbb{C}))$ is generated by such automorphisms.

[We shall prove Theorem 14.1.]

Define a groupoid by

$$\underbrace{\mathrm{BirMori}(\mathbb{P}^n)}_{\mathrm{Cr}_n(\mathbb{C})} = \left\{ f : X \dashrightarrow Y : {}^{X,Y \text{ Mfs}, X,Y \sim_{\mathrm{bir}} \mathbb{P}^n}_{f \text{ birational}} \right\}.$$

Definition 14.3. Let $r \geq 1$. A morphism $\eta : X \to B$ is a *rank r fibration* if

(1) $\dim X > \dim B, \rho(X) - \rho(B) = r$.
(2) X is a MDS over B
(3) $-K_X$ is η-big.
(4) The result of any D-MMP over B is $\mathbb{Q}$-factorial is terminal.
(5) $\exists \Delta_B \geq 0$ such that (B, Δ_B) is kld.

In the first case there exists a 2-ray game. In case (2) the 2-ray game ends. In the last 3 cases the 2-ray game stays in the category of MMP.

- Rank 1 fibration = Mfs.
- Rank 2 fibration = Sarkisov link.

Theorem 14.4 (BLZ)**.** *The relations among Sarkisov links are generated by by the dominated relations of rank 3 fibrations.*

$$\mathrm{BirMori} = \left\langle {}^{\text{Sarkisov}}_{\text{links}} : {}^{\text{relations given by}}_{\text{rank-3 fibrations}} \right\rangle.$$

Let $(g, d) \in \{(2, 8), (6, 9), (10, 10), (11, 14)\}$, $C \in U_{g,d} \subseteq H^{\leq}_{g,d}$, $J = \bigcup U_{g,d}$, $X = Bl_C \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

Proposition 14.5. *Let $X \to \mathbb{P}^3 \to \mathrm{Spec}(\mathbb{C})$ be a rank-2 fibration.*

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d] & & X \arrow[d] \\
\mathbb{P}^3 \arrow[rr, dashed, "\chi_C"] \arrow[dr] & & \mathbb{P}^3 \arrow[dl] \\
& \mathrm{Spec}(\mathbb{C}). &
\end{tikzcd}$$

with $\chi_C^2 = id$.

$$\begin{tikzcd}
X \arrow[dr] \arrow[dd] & & X \arrow[dl] \arrow[dd] \\
& Z \arrow[r, "t \mapsto -t"] & Z \\
\mathbb{P}^3 \arrow[rr, dashed] & & \mathbb{P}^3.
\end{tikzcd}$$

Proposition 14.6. *There are no rank 3 fibrations dominated $X \to \mathrm{Spec}(\mathbb{C})$.*