BIRATIONAL MAPS FROM THE COMMUTATIVE ALGEBRA VIEWPOINT

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Notes at github.com/danimalabares/stack

O estudo da equivalência de variedades algébricas a partir da biracionalidade é um dos temas centrais da Geometria Algébrica. Nessa direção, a classificação de variedades pode ser vista como parte da chamada geometria birracional. De modo análogo, podemos concentrar-nos na estrutura dos mapas racionais que definem biracionalidades entre variedades — este será o ponto de partida do minicurso. Apresentarei aplicações de métodos de Álgebra Comutativa, como o uso de syzygies no estudo dos ideais de base de mapas racionais, e discutirei critérios algébricos que fornecem ferramentas computacionais para verificar a biracionalidade de tais mapas. Introduziremos também alguns métodos no Macaulay2 para testar biracionalidade. Além disso, veremos como diferentes teorias de multiplicidades — como mixed multiplicity e j-multiplicity — desempenham um papel importante na abordagem algébrica dos mapas racionais.

Contents

1. Plan

- A. Simis, A. Doria, - 2012 (Adv. Math) Criterio de Biracionalidade
- Simis, - 2012: Plane Cremona.
- Simis, -, 2017. Bounds for degrees of Birational maps with CM graph.
- Simis, Chardin 2021: degree of rational map versus syzygy.
- Mostafazadeh, 2024: loci de Bir(X).
- `RationalMaps.m2`

2. Introdução

Let R be a ring. We are interested in Mod_R and R-algebras categories. The first one is abelian, while the second one isn't. There are other diferences.

We are interested in discussing flat R-algebras and flat R-modules.

Theorem 2.1 (Grothendieck Generic freeness theorem)**.** *Let R be noetherian domain, $\varphi : R \to S$, S a finitely generated R-algebra. Then there exists $f \in R$ such that $\varphi : R_f \to S_{\varphi(f)}$, $S_{\varphi(f)}$ is a free R_f-module. $S_{\varphi(f)}$ is a flat R_f-module.*

Theorem 2.2 (Dimension of fibers theorem (Matsumura 15.1))**.** *$\varphi : R \to S$, R Noetherian, S Noetherian, $Q \in \operatorname{Spec}(S)$, $p = \varphi^{-1}(Q) \in \operatorname{Spec}(R)$,*

(1) *$htQ \le htp + \dim \frac{S_Q}{pS_Q}$ [Shafarevich section 3].*
(2) *If φ is flat then $htQ = htp + \dim \frac{S_Q}{pS_Q}$.*

If $Q \not\ni \varphi(f) \implies S_Q$ is a free R_p-module (by Grothendieck's theorem). $\dim S_Q = \dim R_p + \dim \frac{S_Q}{pS_Q}$.

Topologia de Zariski.

$$\operatorname{Spec}(S) = \{Q : Q \text{ primo}\}$$
$$\text{fechado } = V(I) = \{Q : Q \supseteq I\}.$$

S is a domain, $\operatorname{Spec}(S)$ is an irreducible space, $\implies$ open nonempty sets are dense.

k field. An algebraic set is $V(I) = Z(I)$ for an ideal $I \subset k[x_1, \ldots, x_n]$. I is prime if and only if $X := V(I)$ is irreducible. Recall that a polynomial map is given by polynomial functions in the entries. Any such map yields in the category of algebras a map

$$k[Y] = A[Y] = \mathcal{O}_Y = \frac{k[y_1, \ldots, y_m]}{J} \xrightarrow{\varphi^*} \frac{k[x_1, \ldots, x_n]}{I}$$
$$y_i \mapsto \overline{f_i(x)}$$

where $Y := V(J)$. $\varphi^*(g) = 0\ \forall g \in J$, $g \in J \iff g(f_1, \ldots, f_m) \in I$ where f_i are the coordinate polynomial functions of φ.

φ is an isomorphism if it has a right and left inverse morphism. $\varphi : X \to X$ is an automorphism if it is an isomorphism.

Question: determine the automorphism group $\operatorname{Aut}(X)$.

If $X = \mathbb{A}^1$, a morphism

$$\varphi : k[x] \longrightarrow k[x]$$
$$x \longmapsto f(x)$$

has an inverse if and only if there exists $f(x)$ such that $f(g(x)) = x$. [This forces f to have degree 1 since composition $f \circ g$ must have degree $\deg f \cdot \deg g = \deg x = 1$.] So $f(x) = ax + b$, $a \neq 0$. Thus, we can parametrize the automorphism by two numbers a, b with $a \neq 0$. This is a quasi-affine variety.

For $X = \mathbb{A}^2$ the situation is a little more involved. de Jonequeres, Triangular group. Hierzch Jung 1942, chre. Wout Van der Kulk, 1953, chr arbitrary.

$$\operatorname{Aut}(\mathbb{A}^2) = \langle \text{linear, triangular} \rangle$$

For $\operatorname{Aut}(\mathbb{A}^3)$, the question if whether $\operatorname{Aut}(\mathbb{A}^3) = \langle \text{linear, triangular} \rangle$. This is false in general

Definition 2.3. An auotomorphism $\sigma \in \mathrm{Aut}(\mathbb{A}^3)$ is called *tame* if $\sigma \in \langle \text{linear, triangular}\rangle$ and *wild* otherwise.

Nagata's conjecture is that a $\sigma : k[x,y,z] \to k[x,y,z]$, $(x,y,z) \mapsto (x+(x^2-yz)z, y+2(x^2yz)x+(x^2-yz)^2z, z)$ is wild.

Shiguero Kuroda 2014. Introduce a monomial order $\leq$ in $k[x,y,z]$. [The following is not a criterion; this just says "if this happens, then the morphism is wild", but not an iff.] Let

$$\begin{aligned} \sigma : k[x,y,z] &\longrightarrow k[x,y,z] \\ &\longmapsto (f_1, f_2, f_3) \end{aligned}$$

σ is wild if

(1) $\mathrm{int}(f_1), \mathrm{int}(f_2), \mathrm{int}(f_3)$ are linearly dependent over $\mathbb{Z}$, pairwise linear over $\mathbb{Z}$.
(2) $\mathrm{int}(f_{i_1}) \neq p\mathrm{int}(f_{i_2}) + q(\mathrm{int}(f_{i_3})$ for $p, q \in \mathbb{Z}_{\geq 0}$.

Jacobian conjecture: a polynomial map $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ [with coordinate functions] $(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ is an automorphism if and only if $\det \mathrm{Jacob}(\varphi) = \text{constant} \neq 0$.

The forward implication is clear: if $\varphi\psi = \mathrm{id}$ then $\mathrm{Jac}(\varphi)(\psi)\mathrm{Jac}(\psi) = \mathrm{Jac}(\mathrm{id}) = 1$.

[Bass-Conell-Wright, 1982] Let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ a polynomial map with $\det \mathrm{Jac}\varphi =$ constant. Then φ is invertible (its inverse is a polynomial map) [note that by the holomorphic inverse function theorem we already know that there exists a local inverse that is a series, but here we see it's actually polynomial] if and only if φ is injective if and only if φ is proper ($\varphi^{-1}(\text{compact}) = \text{compact}$).

Theorem 2.4. *If the Jacobian conjecture is valid for all n and for any polynomial of degree ≤ 3, then it's valid.*

Theorem 2.5. *If the Jacobian conjecture is valid over $\mathbb{C}$ then it is valid over any domain.*

3. Rational maps of projective varieties

$X = V(I) \subseteq \mathbb{P}^n$, $I \subseteq k[x_0, \ldots, x_n]$ homogeneous.

Definition 3.1. A rational map is $\varphi : X \dashrightarrow Y$ for $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ given by $(f_0 : \ldots : f_m)$ is *presented* by $f_i \in R = k[\mathbf{x}]/I_X$, $\deg f_i = \deg f_j$, f_i homogeneous.

A map can have another presentation:

$$\begin{aligned} \varphi : X &\dashrightarrow Y \\ &(g_0 : \ldots : g_m) \end{aligned}$$

For all $p \in \mathrm{Dom}\varphi$,

$$(f_0(p) : \ldots : f_m(p)) = (g_0(p) : \ldots : g_m(p)).$$

If

$$I_2 \begin{pmatrix} f_0(\mathbf{x}) & f_m(\mathbf{x}) \\ g_0(\mathbf{x}) & g_m(\mathbf{x}) \end{pmatrix} \in I_X$$

[then the presentations are equivalent].

Exercise 3.2. $X = V(y^3 - x^2z) \subseteq \mathbb{P}^2$, $\sigma : X \dashrightarrow X$, $(yz : xz : xy)$. [Show that] $(xz : y^2 : x^2)$ [is an equivalent presentation, that is,]

$$I_2 \begin{pmatrix} yz & xz & xy \\ xz & y^2 & x^2 \end{pmatrix} \in (y^3 - x^2z).$$

What are the possible presentations of a given rational map?

$$f_0 g_j = f_j g_0 \qquad \text{in } I_X.$$

If $f_0 \neq 0 \implies g_j = \left(\frac{g_0}{f_0}\right) f_j$.

[We see that there is a 1-1 correspondence between]

$$\text{Presentations of } \varphi \longleftrightarrow \operatorname{Hom}_R(I, R).$$

From

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

we obtain

$$0 \longrightarrow \underbrace{\operatorname{Hom}(R/I, R)}_{=0} \longrightarrow \operatorname{Hom}(R, R) \longrightarrow \operatorname{Hom}(I, R) \longrightarrow \operatorname{Ext}^1(R/I, R) \longrightarrow \underbrace{\operatorname{Ext}^1(R, R)}_{=0}$$

And $\operatorname{Hom}(R/I, R) = * \iff \operatorname{grade}(I) \geq 1$.

If $\operatorname{Ext}^1(R/I, R) = 0 \implies R \simeq \operatorname{Hom}(I, R)$. Note that $\operatorname{Ext}^1(R/I, R) \iff \operatorname{grade}(I, R) \geq 2$.

[This would help us understand $\operatorname{Bir}(X)_d$.

$$\operatorname{Domin}\varphi = \bigcup \operatorname{Domin}(\text{presentation}) = X.$$

4. Image and graph of a rational map

[Given a rational map] $\varphi : X \dashrightarrow Y$, $(f_0 : \dots : f_m)$, the ideal of definition of the image of φ is

$$\mathcal{K} = \{h(y) \in k[y_0, \dots, y_m] : h(f_0(p), \dots, f_m(p)), \forall p \in \operatorname{Dom}\varphi\}.$$

[The coordinate ring of the image has a special name:]

$$\frac{k[y_0 : \dots : y_m]}{\mathcal{K}} \simeq k[f_0, \dots, f_m] \subseteq \frac{k[x_0, \dots, x_n]}{I_X}$$

is called *special fiber $\mathcal{I}_I$ of the ideal $I \subseteq R$.*

The *analytic spread* is

$$\dim(\operatorname{Im}\varphi) = \dim \mathcal{I}_I - 1 = \ell_R(I) - 1.$$

Para um anel em geral $f_0, \dots, f_m \in R$, $\mathcal{I}_I = k[f_0 t, \dots, f_m t] \subseteq R[t]$.

$$\operatorname{ht}(I) \leq \ell(I) \leq \dim R.$$

One can also prove via Dedekind-Mertens lemma, which says that $c(f)^n c(fg) = c(g)c(f)^{n+1}$, where c is the content of a polynomial as in Gauss lemma, that

$$\ell(I_j) \leq \ell(I) + \ell(J) - 1.$$

The *graph* of φ is

$$\Gamma_\varphi = \overline{\{(p_0 : \dots : p_n) \times (f_0(p) : \dots : f_m(p)) \subseteq X \times \mathbb{P}^m\} : p \in \operatorname{Dom}(\varphi)\}}$$

$$\{h(\mathbf{x}, \mathbf{y}) \in k[x_0, \dots, x_n, y_0, \dots, y_m] : h(\mathbf{p}, \mathbf{f}(\mathbf{x})) = 0 \forall p \in \operatorname{Dom}(\varphi)\}.$$

The *ideal of definition of the graph* is given by homogeneous polynomial equations of a presentation of φ.

The *Rees algebra* is

$$\frac{R[y_0, \dots, y_m]}{J} \simeq R[f_0 t, \dots, f_{\shortmid} \to] \subseteq R[t]$$

where $J =$ polynomial equations of $f_0, \dots, f_m$.

5. Finite maps

[Another concept before we get to birational maps.]

Definition 5.1. A rational map $\varphi : X \dashrightarrow \mathbb{P}^m$ presented by $(f_0 : \dots : f_m)$ is called a *finite map* if and only if $\ell(I) = \dim R$. I maximal analytic spread.

[Keep in mind that dimension of image is analytic spread minus 1, and dimension of domain is $\dim R - 1$.]

[The following is another version of Hilbert's Nullstellensatz in commutative algebra:]

Theorem 5.2 (Matsumura 5.6)**.** *R Noetherian domain containing a field k,*

$$\dim(R) = \mathit{trdeg}_k Q(R).$$

[Equations involving fraction fields of coordinate rings]

We obtain that $\varphi : X \dashrightarrow Y$ is finite if and only if $\dim(\varphi^{-1}(q)) = 0$ for generic $q \in Y$. $\deg(\varphi) = \#\{p : \varphi(p) = q\}$.

How can we compute (algebraically) $\deg(\varphi)$?

$$\begin{aligned} \deg Z &= \sum_{\substack{z \text{ closed} \\ \text{in } Z}} \lambda(\mathcal{O}_{Z,z} \deg(Z) \\ &= \sum_{p \not\supseteq I} \lambda\left(\left(\frac{R}{I_q}\right)_p\right) e\left(\frac{R}{p}\right) \end{aligned}$$

since

$$\{p : (f_0(p) : \dots : f_m(p)) = (q_0 : \dots : q)m) : p \in \mathrm{Domin}(\varphi)\},$$

$$I_q = I_2 \begin{pmatrix} f_0(\mathbf{x}) & f_m(\mathbf{x}) \\ q_0 & q_m \end{pmatrix}, \qquad V(I_q).$$

$$I_q = Q_1 \cap \dots \cap Q_t \cap \underbrace{\dots \cap Q_s}_{\sqrt{Q_i} \supseteq I}.$$

Exercise 5.3. $(I_q : I^\infty) = Q_1 \cap \dots \cap Q_t$.

$$\deg(\varphi) = e\left(\frac{R}{I_q : I^\infty}\right).$$

Integral closure (1,82).

$$
\begin{tikzcd}
 & V \arrow[dr, "\pi_L"] & \\
X \arrow[ur, dashed, "{(g_0,\dots,g_N)}"] \arrow[rr, dashed, "{\varphi=(f_0,\dots,f_m)}"] & & Y
\end{tikzcd}
$$

where π_L is a linear map.

$$\underbrace{(f_0,\dots,f_m)}_{J} \subseteq \underbrace{(g_0,\dots,g_N)}_{I}.$$

The center of the projection π_L does not intersect V if and only if J is reduction of I, that is, $I \subseteq \overline{J}$.

Another way of stating this result is: given $J \subseteq I$,

$$J \subseteq \mathcal{F}_I = k[g] = \frac{k[y_0,\dots,y_m]}{k}, \ell_1,\dots,\ell_m.$$

$J \subseteq I$ reduction if and only if $(\ell_0,\dots,\ell_m)$ is m-primary.

6. Summary

Let $x \subset \mathbb{P}^n$ and $y \subset \mathbb{P}^m$. $\varphi : x \dashrightarrow y$, $(f_0 : \dots : f_m)$, $r = k[x] = k[x_0,\dots,x_n]/I_x$, $I = (f_0,\dots,f_n) \subseteq R$.

We defined φ to be *finite* if $\ell(I) = \dim r$. $[q(x), q(y)] < \infty$. $\#\{p : p \in x : \varphi(p) = q, q \text{ general in } y\}$.

$$i_q = i_2 \begin{pmatrix} f_0(\mathbf{x} & \cdots & f_n(\mathbf{x}) \\ q_0 & \cdots & q_m \end{pmatrix} \subseteq r$$

$$e\left(\frac{R}{I_q : I^\infty}\right) = \deg(\varphi)$$

[is the topological degree of φ.]

Eisenbud-Ulrich 2007: raw ideal. $I_q : I^\infty$ = morphism fiber ideal. Correspondence fiber ideal

$$
\begin{tikzcd}
 & \underbrace{\Gamma}_{\substack{\text{graph}\\ \text{of } X}} \subseteq X \times \mathbb{P}^m \arrow[dl, "{\substack{\pi_1\\ \text{(birational)}}}"'] \arrow[dr, "\pi_2"] & \\
X \arrow[rr, dashed] & & Y \ni q
\end{tikzcd}
$$

$$\pi_2^* : \frac{k[Y]}{I_Y} \to R_I = \frac{y_0,\dots,y_m]}{J} y_i \qquad \longmapsto \mathbf{y_i} = f_i t \in R[\mathbf{f}t].$$

$$C_q = (I_q t R_I :_{R_I} I^\infty t) \cap R, \qquad R \subseteq R_I.$$

Exercise 6.1. $C_q = (I_q t R_I : R_i^\infty t) = \bigcup_{n=1}^\infty (I_q I^{n-1} :_R I^n) \subseteq (I_q : I^\infty)$.

$$v(q) = \{p : (I_q)_p : \text{ is not a reduction of } I_p\}.$$

$$I_q(q:\psi) = (\underbrace{I_q : I}_{\text{raw ideal}} \subseteq C_q \subseteq (I_q : I^\infty)$$

Question: when to $(I_q : I)$ and C_q have the same saturation?

If I has a [crossed: linear generalized] raw and I is locally of linear type $\implies$ $(I_q : I)^{\text{sat}} = C_q^{\text{sat}}$.

[Simis Chardin, 2021.] $Q(X) = R_i\underline{0} = \left\{\frac{a}{b} : b \neq 0\right\}$. $R_{(\underline{0})} = \left\{\frac{a}{b} : b \neq 0, b \text{ homogeneous}\right\}$.

$$k[x]_x = k[x, x^{-1}], R_{((0))} = k,$$
$$k(X) = R_{((0))}.$$

Lemma 6.2. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded domain over a field $A_0 = F$ with $\dim A = 1$. Then $e(A) = [K(A) : F]$.*

Proof. There exists d such that $d_F A_i = e$ for all $i \geq d$, $A_d = F\langle a_1, \dots, a_e\rangle$. Show that $K(A) = F\left\langle \frac{a_1}{a_1}, \dots, \frac{a_e}{a_1}\right\rangle$ (exercise). $\square$

[Then]

$$\begin{aligned} e(S/J \otimes_B Q(Y)) &= [K(S/J \otimes Q(Y)) : Q(Y)] \\ &= [K(S/J)(K_0) : K(Y)(K_0)] \\ &= [K(\Gamma) : K(Y)] \\ &= \deg \varphi. \end{aligned}$$

[The ring $S/J \otimes_B Q(Y)$ is always Cohen-Macaulay of rank 1.]

Theorem 6.3. *Suppose that J is prime minimal over an ideal I generated by elements of degrees $d_1 \geq d_2 \geq \dots$.*

$$deg\varphi \leq d_1 \dots d_t e(R)$$

where $t = \dim X$.

We have the following corollary:

Lemma 6.4. *$deg(\varphi) \leq d_1 \dots d_{d_{max}} e(R)$.*

7. Eisenbud-Goto conjecture

1982. Let $X \subseteq \mathbb{P}^n$ irreducible and "non-degenerated".

$$\begin{aligned} reg(X) &\leq \deg(X) - \operatorname{codim}(X) \\ reg(X) &= \min\{r : H^1(X, \mathcal{O}_X(r-i)) = 0 \forall i > 0\} \\ &= \min\{r : H^2_{\underline{m}}(R)_{r-i} = 0 \forall i > 0\}. \end{aligned}$$

[Regularity controls properties of the ideal such as the degree of the generators in a Gröbner base.]

$$reg(R/I_X) = \max\{t_i + i : t_i = \max\{d_{ji}\}\}$$
$$reg(R/f) = \deg f - (n+1) + n = \deg f - 1.$$

[Notice the term "non-degenerate". It means that the ideal I_X in $R = \frac{k[x_0, \dots, x_n]}{I_X}$ does not contain any linear terms.]

$$e\left(\frac{S}{J} \otimes Q(Y)\right) \geq n+1-\dim_{Q(Y)}(\tilde{J}\underbrace{1}_{\deg X} + reg(\tilde{J})-2.$$

$\deg\varphi \geq n+q-\dim_{Q(Y)}(\tilde{J}_1) =$ Jacobian dual rank.

$$reg(\tilde{J}) \geq \deg(\text{gens}) \geq 2$$

$$\deg\varphi \geq n+1-r\text{jdrank}(\varphi).$$

[The conjecture is false in general, but in some cases holds, such as in the Cohen-Macaulay case.]

8. Summary

Recall our notations:

- $\varphi : X \dashrightarrow Y$, $(f_1, \dots, f_n)$.
- $I = (f_0, \dots, f_n)$.
- $R = k[x_0, \dots, x_n]/I_X$.
- $\mathcal{R}_I(R) = R[y_0, \dots, y_n]/J = S/J$.
- $B = k[y_0, \dots, y_m]$.
- $k[Y] = k[y_0, \dots, y_m]/J \cap B = B/q/$
- $S/J \otimes_B Q(Y)$, $Q(Y) = Q(B/q)$. CM dim 1, domain.
- $S/J\otimes)BQ(Y) = \frac{Q(Y)[x_0,\dots,x_n]}{\underbrace{\tilde{J}}_{=\text{Im}\theta}}$,

$$0 \longrightarrow J \longrightarrow S \longrightarrow S/J \longrightarrow 0$$

$$J \otimes Q(Y) \xrightarrow{\theta} S \otimes Q(Y) \longrightarrow S/J \otimes_B Q(Y) \longrightarrow 0$$

$S = k[\mathbf{x}, \mathbf{y}]/I_X$.

- $e\left(\frac{Q(Y)[\mathbf{x}]}{\tilde{J}}\right) = \deg\varphi$.
- $\text{djrank} = \dim_{Q(Y)}(\tilde{J})_1$
- $\deg(\varphi) \geq (n - \text{jdrank}(\varphi)) + \text{reg}(\tilde{J}) - 1)$.

Theorem 8.1. *$n = jdrank(\varphi)$ if and only if φ is birational.*

Proof. $\deg(\varphi) = e(Q(Y)[x_0, \dots, x_n]/\tilde{J})$.

($\Longrightarrow$) If $\text{jdrank}(\varphi) = n \implies \tilde{J}_1 = (\ell_1, \dots, \ell_n) \implies e(A) = e(Q(Y)[x]/\tilde{J})$ where $A := S/J \otimes_B Q(Y)$. A domain, dim $= 1 \implies \tilde{J}' = 0 \implies e(A) = 1 = \deg(\varphi)$.

($\Longleftarrow$) $e(A) = 1 \implies$ ACM $\implies$ quasi-unmixed $\implies \hat{A}$ is equidimensional and A is analytically unmixed $\implies A$ is a regular ring. $A = \frac{Q(Y)[\mathbf{x}]}{\tilde{J}}$ (See [Bruns-Herzog §2]). [Thus $\tilde{J}$ is linear,] $\tilde{J} = (\ell_1, \dots, \ell_n)$ since $\text{codim}\tilde{J} = n$. □

Proposition 8.2. *Let $(A, \mathfrak{m})$ a regular local ring and $I \subseteq \mathfrak{m}$. If A/I is regular then I is generated by a part of a system of parameters.*

9. How to compute the Jacobian dual rank

$$R = \frac{k[x]}{I_X} \supseteq I = (f_0, \dots, f_m),$$

$$\mathcal{I}_I(R) = \frac{R[\mathbf{y}]}{J}.$$

$$\langle J_{(1,*)} \rangle = \left\langle \underbrace{g_1}_{\in R[\mathbf{y}]}, \dots, \underbrace{g_t}_{R[\mathbf{y}]} \right\rangle$$

lifts $g_1(\overline{Q}_i)$, $Q_i \in k[\mathbf{x}, \mathbf{y}]_{i,*)}$.

$$k[\mathbf{y}] \ni \begin{pmatrix} \frac{\partial Q_1}{\partial x_0} & \cdots & \frac{\partial Q_t}{\partial x_0} \\ \vdots & & \vdots \\ \frac{\partial Q_1}{\partial x_n} & \cdots & \frac{\partial Q_t}{\partial x_n} \end{pmatrix} = M$$

Jacobian dual matrix.

$$- : k[\mathbf{y}] \to \frac{k[t]}{I_Y} /$$

$$\psi = \begin{pmatrix} \overline{\frac{\partial Q_1}{\partial x_0}} & \cdots & \overline{\frac{\partial Q_1}{\partial x_0}} \\ \vdots & & \vdots \\ \overline{\frac{\partial Q_1}{\partial x_n}} & \cdots & \overline{\frac{\partial Q_t}{\partial x_n}} \end{pmatrix} \in M_{(n+1)\times t}(k[Y]).$$

$$\tilde{\psi} = \begin{pmatrix} \frac{\partial Q_1}{\partial x_0} & & \\ & \ddots & \\ & & \frac{\partial Q_t}{\partial x_n} \end{pmatrix} \Big|_{(f_0(\mathbf{x}), \dots, f_m(\mathbf{x})} \in M_{(n+1)t}(R).$$

Proposition 9.1. *$rk\psi = rk\tilde{\psi} = jrank(\varphi) = \dim_{Q(Y)} \tilde{J}_1$, $rk\psi = n$ if and only if φ is birational.*

Compute minors, add a row that is missing... this way we obtain a candidate for the inverse map... □

10. Extra

Consider

$$\mathrm{Bir}(X)_d = \{\varphi : X \dashrightarrow X, (f_0, \dots, f_n) : \deg f_i = d\}.$$

$$f_0 = d_0 x^d + a_1 x_0^{d-1} x_1 + a_2 x_0^{d-1} x_1 x_2 \dots$$

Let $H_d \subseteq \mathbb{A}^{(n+1)\times\binom{n+d}{d}}$ be the points whose corresponding map is birational. We have a map $H_d \to \mathrm{Bir}(X)_d$. [We have proved that] H_d is a quasi-projective variety and $\mathrm{Bir}(X)_d$ is a constructible set.

[If the Cremona map has degree d, what's the degree of the inverse map?] $\varphi^{-1} \leq d^{n-1}$. A proof of this is due to Gabber.

11. Extra 2

$$D = \underbrace{\sum a_i P_i}_{E} - \sum \underbrace{b_j}_{\in \mathbb{Z}_{\geq 0}} \underbrace{q_j}_{\text{proj}}, \qquad \text{proj}(R) = X.$$

$$\mathcal{O}_X(D) = \tilde{M}$$

$$\begin{aligned} \mathcal{O}_X(D) &= \tilde{M} \\ \text{prod}E &= \prod p_i^{[a_i]} = (f^{a_i} : \dots : f_k^{a_i}) \\ \text{prod}F &= \prod q_j^{[b_j]} \\ \text{prod}E^{**} &= \text{Hom}(\text{Hom}(\text{prod}F, R', R)) \\ \text{dual} &= \text{prod}E^{**} \otimes \text{Hom}(\text{prod}(D^{**}, R) \\ M &= \text{Hom}(\text{dual}, R). \end{aligned}$$

D. OO(D). divisorToModule(D)

Divisor (Karl Schewede,...) [is the name of this Macaulay2 package]. M rk 1, torsion-free. embedAsIdeal(M).

$$\begin{tikzcd} \bigoplus R(-e_j) \arrow[r, "\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}", "\psi"'] & R^n(-d_i) \arrow[r] & M \arrow[r] \arrow[d, "m_i \mapsto f_i"] & 0 \\ & & R & \end{tikzcd}$$

$(\underline{f}) \in \text{Ker}\, \psi^t$.

$I =$.

$$\psi^t : R^n(td_i) \to R(fe_j)$$

$I = (\underline{f}) \in R^n(td_i)_a$.

mapToProjectiveSpace(D), Map defined by base of I_a.