SYMPLECTIC GEOMETRY

github.com/danimalabares/stack

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1. Coadjoint orbits and KKS theorem

The coadjoint action induces orbits in $\mathfrak{g}^*$. The tangent space at any point of a coadjoint orbit is given by infinitesimal vector fields, that is,

$$T_\xi\mathcal{O} = \{u_{\mathfrak{g}^*}(\xi) : \forall u \in \mathfrak{g}\}.$$

We may introduce a symlpectic form $\omega \in \Lambda^2 T^*_\xi\mathcal{O}$ on the tangent space at some point of a coadjoint orbit by

$$\omega_\xi(X, Y) := \xi([u, v]) = -u_{\mathfrak{g}^*}(\xi)v = v_{\mathfrak{g}^*}(\xi)(u)$$

which does not depend on $u, v \in \mathfrak{g}$.

Theorem 1.1 (KKS). *$\xi \mapsto \omega_\xi$ is symplectic on $\mathcal{O}$.*

Proof. Non degeneracy is easy. Then show that ω is G-invariant. Closedness follows from Jacobi identity. □

2. Hamiltonian actions and moment map

To warm-up we define real symplectic and Hamiltonian actions.

Definition 2.1.
- An $\mathbb{R}$-action is *symplectic* if $\psi^*_t \iff L_X\omega = 0$, that is, X is a symplectic field.
- An $\mathbb{R}$-action is *hamiltonian* if X is Hamiltonian, that is, if there exists a function $H \in C^\infty(M)$ such that $X = X_H$.

To define moment map suppose there exists a map $\hat{\mu}$ such that

$$\begin{tikzcd}
 & C^\infty(M) \arrow[d] & f \arrow[d, mapsto] \\
\mathfrak{g} \arrow[ur, "\hat{\mu}"] \arrow[r] & \mathfrak{X}(M) & X_f \\
u \arrow[r, mapsto] & u_M \arrow[ur, equal] &
\end{tikzcd}$$

Note that if $\hat{\mu}$ exists, it's not uniquely determined, and that we may suppose that it is linear.

It's equivalente to define $\hat{\mu}$ as a map $\hat{\mu} : \mathfrak{g} \to C^\infty(M)$ or as a map

$$\mu : M \longrightarrow \mathfrak{g}^*$$
$$\hat{\mu}(u)(x) = \langle \mu(x), u \rangle .$$

Definition 2.2.

- A G-action is symplectic if $\psi_g^* \omega = \omega$ for all $g \in G$, that is, if

$$\begin{tikzcd}
\psi : G \arrow[r] \arrow[dr] & \mathrm{Dif}(M) \\
 & \mathrm{Symp}(M, \omega) \arrow[u, hook]
\end{tikzcd}$$

- A symplectic action $G \overset{\psi}{\curvearrowright} (M, \omega)$ is *weakly Hamiltonian* if there exists a map $\mu : M \to \mathfrak{g}^*$ (or equivalently, $\hat{\mu} : \mathfrak{g} \to C^\infty(M)$ linear) such that

$$i_{u_M} \omega = d \langle \mu, u \rangle , \qquad \text{(resp. } u_M = X_{\hat{\mu}(u)}\text{)}.$$

- A weakly Hamiltonian action is *Hamiltonian* if it is also μ-equivariant, that is, $\mu : G \curvearrowright M \to g^* \overset{\mathrm{Ad}^*}{\curvearrowleft} G$,

$$\mu \circ \psi_g = (\mathrm{Ad}^*)_g(\mu).$$

In this case we call μ a *moment map.*

3. Noether's principle

Theorem 3.1 (Noether's principle)**.** *Let (M, ω, μ) be a Hamiltonian G-space with Hamiltonian $H \in C^\infty(M)$. H is G-invariant if and only if μ is preserved by the Hamiltonian flow.*

Proof. one line ☐

4. Symplectic quotient

In symplectic geometry we cannot always take quotients. We need to use level-sets of the moment map.

Theorem 4.1 (Marsden-Weinstein, Megre)**.** *Let $G \curvearrowright (M, \omega)$ be a Hamiltonian action with moment map $\mu : M \to \mathfrak{g}^*$. Suppose $0 \in \mathfrak{g}^*$ is a regular value. By the equivariance of μ, we have an action $G \curvearrowright \mu^{-1}(0)$ given as follows. For $x \in \mu^{-1}(0)$, $\mu(\psi_g(x)) = Ad_g^*(\mu(x)) = 0$. Suppose that this action is regular (this will be defined later), that $\mu^{-1}(0)/G$ is smooth, and that the projection is a submersion. Then there exists a unique symplectic form ω_{red} in M_0 such that $i^* \omega = \pi_0^* \omega_{red}$.*

That is, the pullback of the symplectic form to the 0 level-set of μ passes to the quotient:

$$\begin{tikzcd} G\curvearrowright\mu^{-1}(0) \arrow[r, hook] \arrow[d, "\pi_0"] & M \\ (M_0 := \mu^{-1}(0)/G, \omega_{\mathrm{red}}) & \end{tikzcd}$$

A slightly more general setting is the following. Hamiltonian action $G\curvearrowright(M,\omega)$, and moment map $\mu : M \to \mathfrak{g}^*$. $\xi \in \mathfrak{g}^*$ regular, $G_\xi \subseteq G$ stabilizer of ξ. Then we have

$$\begin{tikzcd} G_\xi\curvearrowright\mu^{-1}(\xi) \arrow[r, hook, "i_\xi"] \arrow[d, "\pi_\xi"] & M \\ (M_\xi = \mu^{-1}(\xi)/G_\xi, \omega_{\mathrm{red}}). & \end{tikzcd}$$

5. Toric varieties

For any compact connected symplectic manifold with an action of a torus, $\mathbb{T}^n\curvearrowright(M^{2n},\omega) \xrightarrow{\mu} \mathbb{R}^n = (\mathfrak{t}^n)^*$, the image of μ is a polytope. By [Guilleimin-Sternberg] and [Atiyah], this polytope is convex. Further, if the dimension of the torus is exactly half the dimension of the manifold, by [Delzant], the moment polytope determines the manifold, i.e. there is a one-to-one correspondence between Delzant polytopes and symplectic toric varieties. Moduli space of polygons is an example of this.

In fact, this situation of having a torus action of half the dimension is what we may call a "totally integrable system".

6. (Completely) integrable systems

Let (M,ω) be a symplectic manifold. Recall that

- $\{f,g\} = \omega(X_g, X_f) = L_{X_f} g = -L_{X_g} f$.
- $X_{\{f,g\}} = [X_f, X_g]$

If $\{f,g\} = 0$ then the Hamiltonian flows of f and g commute.

Definition 6.1. Let (M^{2n},ω) and $H \in C^\infty(M)$. (M,ω,H) is a *completely integrable system* (in Liouville's sense) if there exist functions $f_1,\dots,f_n \in C^\infty(M)$ such that

(1) $H = f_1,\dots,f_n$ are linearly independent ($df_1,\dots,df_n$ are linearly independent in every point).
(2) $\{f_i, f_j\} = 0$ for all i,j.

Note that $F = (f_1,\dots,f_n) : M \to \mathbb{R}^n$ is a submersion. For all $c \in \mathbb{R}^n$, $F^{-1}(c)$ are Lagrangian submanifolds. $F^{-1}(c)$ is invariant by the flow of X_{f_i} for $i = 1,\dots,n$.

Explanation: $F^{-1}(c)$ is a level-set submanifold of M, whose tangent space is generated by the df_i. These are all in the kernel of dF by the integrability condition, namely $dF = (df_1,\dots,df_n)$, $df_i(X_{f_j}) = \{f_i, f_j\} = 0$.

The idea is that finding these "conserved quantities" we manage to describe the Hamiltonian dynamics completely. Showing that a system is integrable is a victory. The moduli space of polygons is an example, and I think that so is $(\mathbb{C}^2)^{[n]}$.

There is another method of using "bihamiltonian actions" (two Hamiltonian actions yielding the same dynamics) /Lax pairs (defining the system using eigenvalues of matrices). Examples of this are Toda, Caolengo-Moser.

The following theorem says that the connected component of a symplectic manifold with an integrable system

Theorem 6.2 (Arnold-Liouville)**.** *Let* (M^{2n},ω)*,* $H = f_1,\dots,f_n$ *be completely intgrable system and* $F : M \to \mathbb{R}^n$*,* $F = (f_1,\dots,f_n)$*. Let* M_c *be the connected component of* $F^{-1}(c)$*. Then*

1. *If* M_c *is compact, then* $M_c \simeq \mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n = \{(\theta_1,\dots,\theta_n) \ mod\ 2\pi\}$*.*
2. *In angular coordinates, the Hamiltonian flow is linear, that is,* $\frac{d}{dt}^{-1}(t) = \alpha \in \mathbb{R}^n$ *implies that* $\theta(t) = \theta_0 + \alpha t$*.*

The proof of item 1 is given by the following proposition putting $N = M$ and $X_1 = X_{f_1},\dots,X_n = X_{f_n}$.

Proposition 6.3. *Let* N *be complete, connected, of dimension* n*. Let* $X_1,\dots,X_n$ *be linearly independent vector fields on* N *such that* $[X_i,X_j] = 0$ *for all* j*. Then* $N \simeq \mathbb{T}^n$*.*

Proof. First we present the steps:

1. Flows of $X_1,\dots,X_n$ define an action $\mathbb{R}^n \curvearrowright N$.
2. This action is transitive. Then $N \simeq \mathbb{R}^n/\Gamma$ where Γ is the stabilizer.
3. Since $\dim N = n$, then Γ is a lattice of $\mathbb{R}^n$.
4. Since N is compact, then Γ is of maximum rank, that is, $\Gamma = 2\pi\mathbb{Z}^n \subset \mathbb{R}^n$.

Let φ^i_t be the flow of X_i (which are complete since N is compact). Since $[X_i,X_j] = 0$, then $\varphi^i_t\varphi^j_s = \varphi^j_s\varphi^i_t$. Thus we have an action $\mathbb{R}^n \curvearrowright N$ given by moving along the flows of the vector fields, that is

$$\underbrace{(t_1,\dots,t_n)}_{\mathbf{t}}\cdot x = \underbrace{\varphi^1_{t_1}\circ\dots\circ\varphi^n_{t_n}(x)}_{\varphi_{\mathbf{t}}(x)}$$

To prove transitivity we can show that the orbit maps of this action are surjective. Indeed, the image of the orbit map is open since it is a local diffeomorphism (it's derivative is given by the vector fields, which are linearly independent!), and it is closed by a taking a point in the boundary and using the flow at this point (and commutativity of the flows).

To check that the stabilizer Γ of this action is trivial first note that all stabilizers are equal (not only conjugate) since $\mathbb{R}^n$ is abelian. Further, since the orbit map is a local diffeomorphism, Γ is discrete. Thus, it must be a lattice of rank $k \le n$.

To show that in fact $k = n$ consider a linear map T that maps the generators of Γ to the first k canonical vectors of $\mathbb{R}^n$. By the following diagram, the arrow on the bottom is well defined as in fact a bijective local diffeomorphism, that is, a diffeomorphism.

$$\begin{tikzcd}
\mathbb{R}^k\times\mathbb{R}^{n-k}\simeq\mathbb{R}^n \arrow[r, "T"] \arrow[d, "p"'] \arrow[dr, "\psi_x\circ T"] & \mathbb{R}^n \arrow[d, "\psi_x=\substack{\text{orbit}\\ \text{map}}"] \\
\mathbb{T}^k\times\mathbb{R}^{n-k} \arrow[r, "\overline{T}"'] & N
\end{tikzcd}$$

Thus the factor $\mathbb{R}^{n-k}$ must be a point since N is compact by hypothesis. $\square$

Item 2 from Theorem 6.2 is obtained by considering the flows on the latter diagram and is left as an exercise.
In the case that N is not compact we just conclude that it M_c is a product of a torus with an Euclidean space.

Perhaps the simplest example of an integrable system is given by taking an open ball about the origin $B \subseteq \mathbb{R}^n$ and $B \times \mathbb{T}^n = \{(I_1, \ldots, I_n, \phi_1, \ldots, \phi_n)\}$ with the symplectic form $\omega = \sum_{i=1}^n dI_1 \wedge d\phi_i$ with $\{I_1, I_j\} = 0$.

The following theorem says that in fact the **symplectic form** of every level set of the kind M_c, which we just proved that is a torus, may is described locally as such a product.

Theorem 6.4. *There is a neighbourhood U of M_c and a symplectomorphism*

$$\begin{tikzcd}
\sum d\mathcal{I} \wedge d\phi_i & B \times \mathbb{T}^n \arrow[l, squiggly] \arrow[r, "r", "\simeq"'] \arrow[d, "p"'] & U \subseteq M \arrow[r, squiggly] \arrow[d, "F"] & \omega \\
& B \arrow[r, "\simeq"'] & F(U) &
\end{tikzcd}$$

The coordinates $(I_1, \ldots, I_n, \phi_1, \ldots, \phi_n)$ are called *action-angle* coordinates.

The case when $h(I)$ does not depend on the angle coordinates ϕ_i we can solve the Hamiltonian equations easily:

$$\begin{cases} \dot{I}_i = 0 \\ \dot{\phi}_i = -\frac{\partial h}{\partial I}(I) \end{cases}$$

so that $\phi_i(t) = \phi_0 + \alpha t$ for some constants α. This is why integrable systems are actually easy to solve!

7. Vertex algebras

Mônadas, álgebras sobre mônadas. Exemplo: ultrafiltros (Manes, 1976). Campo quântico, como serie de Fourier. Exemplo: Heisenberg algebra.

Problema: o producto de campos quânticos não é bem definido.

References