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Contents

1. A primer on symplectic groupoids

Camilo Angulo, Jilin University. Geometric Structures Seminar, IMPA. February 13, 2025.

Abstract. In the late 17th century, Simeon Denis Poisson discovered an operation that helped encoding and producing conserved quantities. This operation is what we now know as a Lie bracket, an infinitesimal symmetry, but what is its global counterpart? Symplectic groupoids are one possible answer to this question. In this talk, we will introduce all the basic concepts to define symplectic groupoids, and their role in Poisson geometry. We will discuss key examples, and applications. The talk will be accessible to those familiar with differential geometry, but no prior knowledge of groupoids will be assumed.

Part 1. Poisson geometry.

Hamiltonian formalism. Recall that being a conserved quantity $f \in C^\infty(X)$ is the same thing as $\{H, f\} = 0$.

- We have seen that it is always possible to take quotient of a symplectic manifold with a group action to obtain a Poisson manifold.
- Then we have found a way to produce a symplectic foliation from a 2-vector $\pi \in \mathfrak{X}^2(M) := \Lambda^2(TM)$.
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Remark 1.1.

$$\{\text{Lie algebra on } \mathfrak{g}\} \xrightarrow{1-1} \{\text{Linear Poisson bracket on } \mathfrak{g}^*\}$$

- We saw very nice examples of foliation that have to do with Lie algebras. So $\mathfrak{b}_3^*$ which gives the "open book foliation", and $\mathfrak{e}^*$ that gives a foliation by cylinders.

Part 2. Symplectic realizations.

Consider

$$(\Sigma, \omega) \xrightarrow{\mu} (M, \pi)$$

So that

$$\pi^\sharp = d_p\mu \circ \omega^{-1} \circ (d\mu)^*$$

Lemma 1.2. $\dim(\Sigma) \geq 2\dim(M) - \mathrm{rk}(\pi_x)$ *for all* $x \in M$.

Proof. Done in seminar. □

Example 1.3. $(\mathbb{R}^2, 0)$. So the map

$$\begin{aligned} (\mathbb{R}^4, dx \wedge du + dy \wedge dv) &\longrightarrow \mathbb{R}^2 \\ (x, y, u, v) &\longmapsto (x, y) \end{aligned}$$

Exercise 1.4. Find the symplectic realization ω in $(\mathbb{R}^4, \omega) \xrightarrow{\mu} (\mathbb{R}^2, (x^2 + y^2)\partial_x \wedge \partial_y$

$$\begin{aligned} (\mathbb{R}^4, \omega) &\longrightarrow (\mathbb{R}^2, (x^2 + y^2)\partial_x \wedge \partial_y) \\ (x, y, z, w) &\longmapsto (x, y) \end{aligned}$$

Also find the symplectic realization of aff^* with $\{x, y\} = x$.

Part 3: Groupoids

Motivation

(1) Fundamental grupoid: objects are points in the manifold and arrows are paths.
(2) $S^1 \curvearrowright S^2$ by rotation does not give a nice quotient because there are two singular points. Consider the groupoid $S^1 \times S^2$ of orbits. These are the arrows. The points are just the points of S^2.
(3) Consider a foliation (like Möbius foliation of circles; where there is a singular circle, the soul). You can do the same thing as in fundamental groupid leafwise. Arrows then are equivalence classes of paths that live inside leaves. This is called monodromy of a foliation. Again objects are points.
(4) You can take a quotient of monodromy using a connection given by the foliation. This allows to identify certain paths between the leaves. This is called *holonomy (of a foliatiation)*. (So you can make this notion match the usual holonomy given by riemmanian connection.)

    Upshot. So the point is taking some sort of function space on these groupoids you can gather the information given by the non-smooth quotient (like in the case of the sphere rotating). So this groupoid motivation says how to get some structure that resembles a non smooth quotient.
(5) Last motivation: the grid of squares has a tone of symmetries. If you restrict to just a few squares you loose so many symmetries. But there's a grupoid hidden in there that tells you what you intuition knows about this finite grid of squares.

Definition 1.5. A *groupoid* is a category where all morphisms are invertible.

So there is a kind of product among the objects, given by composition but: not every two pair of objects can be multiplied!—only those whose source and target match. So that's the lance about grupoids.

Just so you make sure you understand: the groupoid G is the morphisms of the category. The objects are points (of a manifold).

Definition 1.6. *Lie grupoid* is when the following diagram is inside category of smooth manifolds and s, t (source and target maps) are submersions:

$$G^{(2)} \xrightarrow{m} G \underset{t}{\overset{s}{\rightrightarrows}} M \xrightarrow{u} G$$

Proposition 1.7 (Properties of Lie groupoids)**.**
- *m is also a submersion.*
- *i (inversion) is a diffeomorphism.*
- *u (unit=identity) is an embbeding.*

Definition 1.8. Consider $x \in M$ and the inverse image of source map: $s^{-1}(x) =$ $\{$arrows that start at $x\}$. Now if you act with t on this set you get *the orbit* of x: $\{y \in M$ such that there is an arrow from x to $y\}$.

And there also *an isotropy* $G_x = \{g \in G : g$ goes from x to $x\}$

Example 1.9.
(1) $G = M$, $M = M$.
(2) Lie groups.
(3) Lie group bundles.
(4) $G = M \times M$, $M = M$.
(5) Fundamental groupoid. Isotropy group is fundamental group! And orbit is...

universal cover!
(6) Subgroupoids.
(7) Foliations.
(8) If you have a normal group action $G \curvearrowright M$ you construct a groupoid action with groupoid $G \times M$ and objects M, with product given on the group part of the product. Orbits are orbits. Isotropy group is isotropy group.
(9) Principal bundles.

Back to Poisson.

There's also a notion of Lie algebroid. Which is strange. But the point is that to every Poisson manifold there is a Lie algebroid.

So the question is whether there is a Lie groupoid associated to that Lie algebroid. Not always.

Big question[Fernandez and ?] When a symplectic manifold is integrable?

(Remember that integrating means go from algebra(oid) to group(oid).

And the point is that:

When you *can* go back, you get a *symplectic groupoid.*

Remark 1.10. Look for Kontsevich's notes on Weinstein!

Remark 1.11. History: Weinstein did this intending to do quantization (geometric?) on Poisson manifolds. (That involves a C^*algebra coming from the symplectic groupoid.)

Definition 1.12. A *symplectic groupoid* is a groupoid G, M together with $\omega \in \Omega^2(M)$ such that ω is symplectic and multiplicative, meaning that $\partial\omega = 0$, that is, $\iff m^*\omega = \mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega \in \Omega^2(G^{(2)}) \iff$ take two vectors $X_k, Y_k \in TG$, and $\omega(X_0 \star Y_0, X_1 \star Y_1) = \omega(X_0, Y_0) + \omega(X_1, Y_1)$

Theorem 1.13. *If* (G, ω) *is a symplectic groupoid, then*
(1) $\exists!$ *poisson structure on* M
(2) *for which* $t : G \to M$ *is a symplectic realization,*
(3) *Leaves are connected components of orbits,*
(4) $\mathsf{Lie}(G) \cong T^*M$ *via* $X \mapsto -u^*(i_X\omega$.

Remark 1.14. Look for Alejandro Cabrera, Kontsevich. There are two things one is de Rham and the other… from the future: simplicial?

Upshot. The obstruction to knowing when symplectic groupoid exists is "variation of symplectic form $\omega = (1 + t^2)\omega_{S^2}$". So how does the symplectic group vary from leaf to leaf. So there are two situations in which the thing doesn't work.

2. Neutrinos

Hiroshi Nunokawas, PUC-Rio. Friday Seminar, Seminar Name. June 27, 2025.

Abstract. Hiroshi will come and tell us everything we (not) wanted to know about these mysterious particles, and are not going to be afraid to ask. In particular, about the neutrino oscillation, and the great matrices.

Protons and neutrons have very similar mass of $m_p \approx 940$ MeV, while electrons have mass of $m_e \approx 0.5$ MeV. MeV is 10^6 electronvolts, where one eV is approximately 1.6×10^{-19} J. This is standard in high energy physics, they use electronvolts instead of Joules. Recall that 2J=1N$\times$1m.

Most of the things we see are protons since they are so much larger than electrons. But protons nor neutrons are elementary particles.

Here's the standard model:

$$\begin{array}{cccc}
\text{Quarks} & \begin{bmatrix} u \\ d \end{bmatrix}_L & \begin{bmatrix} c \\ s \end{bmatrix}_L & \begin{bmatrix} t \\ b \end{bmatrix}_L \\
\text{Leptons} & \begin{bmatrix} \nu_e \\ e^- \end{bmatrix}_L & \begin{bmatrix} \nu_\mu \\ \mu^- \end{bmatrix}_L & \begin{bmatrix} \nu_\tau \\ \tau^- \end{bmatrix}_L \\
\text{Generation} & \text{1st} & \text{2nd} & \text{3rd} \\
\text{Bosons} & g,\ \gamma\ ,\ \omega^{\pm},\ z & & \\
\text{Higgs Bosson} & H & &
\end{array}$$

It is very particular that nature repeats itself three times. The L in those matrix actually means left-handed, and accounts for chirality. Only left-handed fermions have weak interaction. Right-handed have electromagnetic interaction, gravitational interaction, but not weak interaction.

And then there's neutrinos. They have negative helicity (chirality). Being left-handed, mathematically, means to have helicity -1. I think this means that the spin is left-handed. But chirality and helicity are not the same: helicity is observer-dependent, and chirality is not. Almost all neutrinos we can see (% 99.99999...) have negative helicity, but not all of them.

Consider the following:

$$n + \nu_e \leftrightarrow p + e^-$$

But it's not completely correct: we'd better put d instead of n, and u instead of p: the d and u quarks, instead of the neutrons and protons.

Now consider the following reaction: a neutron decays into a proton, an electron and an antineutrino:

$$n \to p + e^+ \overline{\nu}_e$$

Protons is very stable, that's why we are here. But neutron decays in only 15 minutes.

By experimental data, we can conclude that neutrinos' mass is consistent with zero. But if they have mass, it should be much smaller than the electron's $m_v \leq 0.5$ eV. And the electron is already the lightest fermion!

If the mass of the neutrino was zero, i.e. $m_\nu = 0$, then $v_\nu = c$ in vacuum, which would imply that

$$\begin{array}{ccccc}
\nu_e & \xrightarrow{L} & \nu_e & \longrightarrow & \nu_e \\
 & & & & \\
0:00 & & 0:00 & & 0:00
\end{array}$$

meaning: time doesn't pass! And this means the state of the particle cannot change.

3. Spheres with minimal equators

Lucas Ambrozio, IMPA. Differential Geometry Seminar, IMPA. June 24, 2025.

Abstract. We will discuss the connection between Riemannian metrics on the sphere with respect to which all equators are minimal hypersurfaces, and algebraic curvature tensors with positive sectional curvatures.

Definition 3.1. An $(n-k)$-*equator* orthogonal to Π is

$$\Sigma_\Pi := \{p \in \mathbb{S}^n : \langle p, x\rangle = 0 \forall x \in \Pi\}$$

for Π a k-dimensional linear subspace of $\mathbb{R}^{n+1}$.

Remark 3.2. Equators are totally geodesic hypersurfaces with the usual sphere metric, which implies they are minimal hypersurfaces.

Problem. Characterize the set $\mathcal{M}_k(U)$ of metrics g on an open set $U \subset \mathbb{S}^n$ such that all k-equators Σ_Π with $\Sigma \cap U \neq \emptyset$ yield are minimal hypersurfaces $\Sigma \cap U$ on (U, g).

Remark 3.3. This problem can be thought of as a problem of finding metrics on $\mathbb{R}^n$ such that k-planes are minimal. To see why project the k-equators to $T_p\mathbb{S}^n$ and pullback those metrics to the sphere.

Let $g \in \mathcal{M}_k(U)$ for $U \subset \mathbb{S}^n$ open and $n \geq 2$.

Theorem 3.4 (Beltrami, Schäfli)**.** *If $k = 1$ then g has constant sectional curvature.*

Theorem 3.5 (Hongan)**.** *If $1 < k < n-1$ then g has constant sectional curvature.*

Then Hongan also managed to produce a classification of these metrics for $k = n-1$.

Remark 3.6. If $T \in \mathrm{GL}(n+1, \mathbb{R})$, then

$$\begin{aligned} \varphi : \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ x &\longmapsto \frac{Tx}{|Tx|} \end{aligned}$$

is a diffeomorphism that maps k-equators into k-equators. Thus if $g \in \mathcal{M}_k(\mathbb{S}^n)$ then so is $\varphi(T)^* g$.

Theorem 3.7. *There exists a $GL(n+1, \mathbb{R})$ equivariant bijection*

$$\mathcal{M}_{n-1}(\mathbb{S}^n) \leftrightarrow \mathit{Curv}_+(\mathbb{R}^{n+1})$$

where the set on the right-hand-side is the set of algebraic curvature tensors (also called curvature-like, i.e. with the same symmetries as the Riemannian curvature tensor) on $\mathbb{R}^{n+1}$ with positive sectional curvature.

The group action is given as follows for $T \in GL(n+1, \mathbb{R})$:

$$(R \cdot T)(x, y, z, w) = \frac{1}{|\det(T)|^{\frac{1}{n+1}}} R(Tx, Ty, Tz, Tw)$$

The point is that $\mathrm{Curv}_+(\mathbb{R}^{n+1})$ is an open cone on a linear space. Here are two simple corollaries:

Lemma 3.8.
1. *$\mathcal{M}_{n+1}(\mathbb{S}^n)$ is in bijection with an open positive cone of an $\frac{n(n+2)(n+1)^2}{12}$-dimensional real vector space.*
2. *Every metric on $\mathcal{M}_{n-1}(\mathbb{S}^n)$ is invariant by the antipodal map.*

Algorithm. From any $R \in \mathrm{Curv}_p(\mathbb{R}^{n+1})$ we obtain a symmetric positive definite (positive-definitiness comes from the positiveness of the curvature of R) 2-tensor k_R satisfying

$$(k_R)_p(v, v) = R(pv, pv) > 0$$

Also, k_R has the *Killing property*, i.e. that $\overline{\nabla} k(X,X,X) = 0$ for all $X \in \mathfrak{X}(\mathbb{S}^n)$.

Then we define a positive function on $\mathbb{S}^n$ by

$$D_R := \left(\frac{d\mathrm{Vol}_{k_R}}{dV_g} \right)^{\frac{4}{n-1}} \tag{3.8.1}$$

and finally a Riemannian metric on $\mathbb{S}^n$ in $\mathcal{M}_{n-1}\mathbb{S}^n$ by

$$g_R = \frac{1}{D_R} k_R$$

And to go back, for $g \in \mathcal{M}_{n-1}(\mathbb{S}^n)$ define a positive function on $\mathbb{S}^n$

$$F_g := \left(\frac{dV_g}{dV_{\overline{g}}} \right)^{\frac{4}{n-1}}$$

Then let $k_g := \frac{1}{F_g} g > 0$, which is a positive definite Killing 2-tensor, from which we may define $R_g \in \mathrm{Curv}_+(\mathbb{R}^{n+1})$ with $R_g(pv, pv) = (k_g)_p(v,v)$ for all $p, v \in T\mathbb{S}^n$.

More corollaries:

Lemma 3.9. (1) *$g \in \mathcal{M}_{n-1}(\mathbb{S}^n)$ is analytic because it is a Killing tensor on $\mathbb{S}^n$, which are well-known.*

(2) *If g is left-invatiant on $\mathbb{S}^3$, seen as unit quaternions, then $g \in \mathcal{M}_2(\mathbb{S}^3$. Moreover, for $a \geq b \geq c > 0$,*

$$aL_i \odot L_i + bL_j \odot L_j + cL_k \odot L_k = k$$

is Killing, $k > 0$, D_k constant and thus $g = \frac{1}{const.} k \in \mathcal{M}_2(\mathbb{S}^3)$.

(3) *R curvature tensor of $(\mathbb{C}P^2, g_{FS})$. We may not remember what's the curvature tensor, but we know the sectional curvature is $1 \leq sec(R) \leq 4$,*

$$(k_R)_p(v,w) = \overline{g}(v,w) + 3\overline{g}(Jp, v)\overline{g}(Jp, w)$$

and $D_R = 4^{\frac{4}{3-1}} = 4$, so that by 3.8.1 we obtain $g_R = \frac{1}{4} k_R$, which is a Berger metric on $\mathbb{S}^3$ with scalar curvature 0.

Now define

$$\Sigma_V = \{p \in \mathbb{S}^n : \langle p, v \rangle = 0\} = V^{-1}(0)$$

where $V(x) := \langle x, v \rangle$ for all $x \in \mathbb{S}^n$. Then the normal vector field is $\nabla V / |\nabla V|_g$, and the second fundamental form is given by

$$A = \frac{1}{|\nabla V|_g} \mathrm{Hess}_g V$$

and its mean curvature by

$$H = \frac{1}{|\nabla V|_g} \left(\Delta_g V - \mathrm{Hess}_g V \left(\frac{\nabla V}{|\nabla V|}, \frac{\nabla V}{|\nabla V|} \right) \right) \tag{3.9.1}$$

For every $v \in \mathbb{S}^n$ and $p \in \Sigma_V$, we see that $H_{\Sigma_V} = 0$ iff

$$|\nabla V|_g^2(p) \Delta_g V(p) - \mathrm{Hess}_g V(\nabla V(p), \nabla V(p)) = 0$$

And for $\overline{g}$,

$$\mathrm{Hess}_{\overline{g}} V + V\overline{g} = 0 \implies \mathrm{Hess}_{\overline{g}} V(X,X) = 0$$

for all $X \in T_p\mathbb{S}^n$ and $p \in \Sigma_v$. Then

$$J_g(X,Y,Z) = g(\nabla_X Y - \overline{\nabla}_X Y, Z)$$

$$J_g(X, Y, \nabla V) = \mathrm{Hess}_{\overline{g}} - \mathrm{Hess}$$

Problems.

(1) Similar story for $\mathbb{C}P^n, \mathbb{H}P^n$?
(2) Complete metrics on $\mathbb{R}^n$ with minimal hyperplanes.
(3) Find geometric invariants of metrics on $\mathcal{M}_{n-1}(\mathbb{S}^n)$ (may be useful to study (M^n, g), $n \geq 4$, sec > 0.

4. Smoothable compactified Jacobians of nodal curves

Nicola Pagani, University of Liverpool and Bologna. Seminar of Algebraic Geometry UFF. August 20, 2025.

Abstract. Building from examples, we introduce an abstract notion of a 'compactified Jacobian' of a nodal curve. We then define a compactified Jacobian to be 'smoothable' whenever it arises as the limit of Jacobians of smooth curves. We give a complete combinatorial characterization of smoothable compactified Jacobians in terms of some 'vine stability conditions', which we will also introduce. This is a joint work with Fava and Viviani.

Let C be a smooth curve and $d \in \mathbb{Z}$. Define

$$J_C^d = \{L : L \text{ is a line bundle of degree } d\}/\sim$$

which is a smooth projective variety of dimension $g(C)$.

If C is nodal we still can consider J_C^d.

(1) One connected component. Then the Jacobian is $\mathbb{P}^1$ minus two points. This is not universally closed, so it is not proper.
(2) Two components intersecting at one point. The pullback of the normalization splits the degree in intintely many ways, givieng that J_C^{-1} is an infinite set of points. This is not of finite type, so it is not proper.
(3) The curve has two components intersecting at two points. This gives J_C^{-2}, which is a mixture of the two former items. (Probably not proper too.)

Now consider

$$\mathrm{TF}_C^d = \{\mathcal{F} : \text{coherent on } C, \text{ torsion-free, rank-1 on } C\}/\sim$$

This satisfies the existence oart of the valu point of properness.

Now we consider the moduli. Now we consider the ideal sheaf of the (singular?) point(s?):

(1) One component. The stack is proper!
(2) Two components intersecting once. Now we get stacky points, $x = [\bullet/\mathbb{G}_m]$. These points have generic stabilizer. The resulting stack is not separated because a morphism of a curve, say $\mathbb{P}^1$ minus a point ... there are infinitely many ways to extend a morphism from this thing to a line bundle. So you cannot include any of these stacky points. Recall that a sheaf is *simple* if its automorphism group is $\mathbb{G}_m$.
(3) The ideal sheaf of both nodes $\mathcal{I}(N_1, N_2)$ has a positive dimensional automorphism group. The stack is not proper.

Definition 4.1. A *fined compactified Jacobian* of C is an open connected substack of $\mathrm{TF}^d(C)$ that is also proper.

Remark 4.2. This thing is automatically an algebraic space.

Definition 4.3. A *compactified Jacobian* is an open connected of $\mathrm{TF}^d(C)$ that admits a proper, good moduli space.

Consider the Artin stack $\mathfrak{X} \xrightarrow{\Gamma} X$ [...] is a *good moduli space* if

(1) Every moduli factors

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[drr] & \mathcal{I} \text{ (ACC. space)} & \\ & & X \end{tikzcd}$$

(2) $\pi_* \mathcal{O}_{\mathfrak{X}} = \mathcal{O}_X$.

We expect to find a notion of stability condition to produce these things [...] $[\bullet/\mathbb{G}_m]$ would be the polystable representative.

Definition 4.4. A compactified Jacobian $\overline{J_C}$ is *smoothable* if all smoothings $\mathcal{C} \to \Delta = \{0, \eta\}$ (with $\mathcal{C}_0 = C$),

$$J^d_{\mathcal{C}_\eta} \cup C \to \overline{J_C}$$

is proper.

Definition 4.5. Let X be a curve.

$$\mathrm{BCON}(X) = \{Y \subseteq X \text{s.t. } Y, Y^c \text{ are connected}\}$$

Definition 4.6. A v-curve is a generalization of items (2) and (3) in the lists above [it looks like two long snakes $\sim$ that intersect several times, and t is the number of nodes]. A *v-condition* is a pair $n = (n_1, n_2)$ such that

$$n_1 + n_2 = \begin{cases} d + 1 - t & \text{we say the s.c. is nondegenerate} \\ d - t & \text{degenerate} \end{cases}$$

$\mathcal{F}$ on X is *n-(semi)stable* if $\deg \mathcal{F}_{X_i} > n_i$ ($\deg \mathcal{F}_{X_i} \geq n_i$) for $i = 1, 2$.

$\mathcal{F}_{X_i} = \mathcal{F}|_{X_i}$ torsion.

$$\deg(\mathcal{F}_{X_i}) + \deg(\mathcal{F}_{X_2}) = d - |\mathrm{sing}(F)|.$$

Then

$$\overline{J_C}(n) = \{\mathcal{F} \text{ is semistable}\},$$

a smooth compact Jacobian.

Definition 4.7. A *degeneratation* of v-stab. on X is $n : \mathrm{BCON}(X) \to \mathbb{Z}$ such that

(1)

$$n_Y + n_{Y^c} + |Y \cap Y^c| = \begin{cases} d + 1 & \text{we say } Y \text{ is } n\text{-nondegenerate} \\ d & Y \text{ is } n\text{-degenerate} \end{cases}$$

(2) Y_i no pa. common component $n_{Y_1} + n_{Y_2} + \dots$

$$\begin{tikzcd} & Y_1 \arrow[dl, <->] \arrow[dr, <->] & \\ Y_2 \arrow[rr, <->] & & Y_3 \end{tikzcd}$$

Theorem 4.8 (-, et al)**.** *(bijection between stability conditions and nodal curves) The map*

$$\left\{\begin{smallmatrix} sm.\ comp. \\ Jac\ of\ X \end{smallmatrix}\right\} \to \left\{\begin{smallmatrix} v\text{-}stab. \\ cond.\ of\ X \end{smallmatrix}\right\}$$
$$n \mapsto \overline{J_X}(n) = \{n\text{-}semistable\ sheaves\}$$

is a bijection. (The arrow should be from right to left!)

F. Viviani had proved it for fine compact Jacobians.

5. Equivariant spaces of matrices of constant rank

Ada Boralevi, France. Algebraic Geometry Seminar, IMPA. August 27, 2025.

Abstract. A space of matrices of constant rank is a vector subspace V, say of dimension n+1, ofthe set of matrices of size axb over a field k, such that any nonzero element of V has fixed rank r. It is a classical problem to look for different ways to construct such spaces of matrices. In this talk I will give an introduction up to the state of the art of the topic, and report on my latest joint project with D. Faenzi and D. Fratila, where we give a classification of all spaces of matrices of constant corank one associated to irreducible representation of a reductive group.

We are interested in vector spaces $U \subset \mathrm{Mat}_{m,n}(\mathbb{C})$, with $m \leq n$, of *constant rank, i.e. such that for all $f \in U$, $r := rankf$ is the same.*

Let $\ell(r, m, n) := max \dim U : U$ is of rank r.

Questions.

(1) $\ell(r, m, n) =$*? In general not known.*
(2) *Find relations among ℓ, r, m and n.*
(3) *Construction of examples and classification.*

Example 5.1. (1) $\ell(1, m, n) = n$,

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ & & & \\ & & & \end{pmatrix}$$

(2) rank = 2? There are two cases ([Atkinson '83], [Eisenbud-Haus '88])
- Compression spaces,

$$\begin{pmatrix} * & * & \cdots & * \\ * & 0 & \cdots & 0 \\ \vdots & & & \\ * & 0 & \cdots & 0 \end{pmatrix}$$

- Skew-symmetric matrices of 3×3.

(3) $\ell(r, m, n) \geq n - r + 1$. Because you can put a matrix of $m \times n$ (with the first r rows that can have nonzero entries):

$$\begin{pmatrix} x_1 & x_2 & \cdots & & x_{n-r+1} \\ & x_1 & x_2 & \cdots & \\ & & & & \\ & & & & \end{pmatrix}$$

Theorem 5.2 (Westwick '86)**.** (1) $n - r + 1 \leq \ell(r, m, n) \leq 2n - 2r + 1$.

(2) *If* $n-r+1 \not| \frac{(m-1)!}{(r-1)!} \implies \ell$

We can see these spaces as (subvarieties?) of determinantal varieties $M_n = \{f \in Mat_{m,n}(\mathbb{C}) : rank(f) \leq r\}$. *[Interpretation via secant varieties inside Segre embedding.]*

Consider a map $\varphi : U \to \mathrm{Hom}(V, W)$. *Then* $\varphi \in U^* \otimes V^* \otimes W$. *We get that* φ *is of constant rank if and only if some kernel, image and cokernel are vector bundles.*

Focus of today. *What happens when* U, V, W *are irreducible representations of a complex reductive group* G*?*

Question. *What is the natural equivariant morphism*

$$U \to \mathrm{Hom}(V, W) = V^* \otimes W$$

of constant rank?

Consider the case of $G = SL_2$. *All the irreducible representations (which are self-dual) are given by* $V(m-1) \cong \mathbb{C}[x, y]_{deg=m-1}$.

Recall the Clebsh-Gordon decomposition ($m \leq n$)

$$V(m-1) \otimes V(n-1) = \bigoplus_{i=0}^{m-1} V(n-m+2i)$$

Theorem 5.3 (B-Faenzi-Lella '22)**.**

$$V(n+m-2) \hookrightarrow \mathrm{Hom}(V(m-1), V(n-1))$$

is of constant rank (corank 1) if and only if

$$n-m+2i | m-1$$

Theorem 5.4 (B. Faenzi, Fratila '25)**.** *Let* $V(\nu)$, $V(\mu)$ *and* $V(\lambda)$ *be irreducible representations of a comple reductive group* G, *with*

$$\dim(V(\mu)) \leq \dim(V(\lambda))_-$$

If there exists a morphism of representations

$$\varphi : V(\nu) to \, \mathrm{Hom}(V(\lambda), V(\mu))$$

then φ *is of constant corank 1 if and only if there exists a simple root* α_i *such that*

(1) $\lambda = \mu + \nu - \alpha_i$,
(2) ν *is a multiple of* ν,
(3) ν *is a multple of* ω_i

6. On wrapped Floer homology barcode entropy and hyperbolic sets

Rafael Fernandes, UC Santa Cruz. Differential Geometry Seminar, IMPA. September 4, 2025.

Abstract. *In this talk, we will discuss the interplay between the wrapped Floer homology barcode and topological entropy. The concept of barcode entropy was introduced by Çineli, Ginzburg, and Gürel and has been shown to be related to the topological entropy of the underlying dynamical system in various settings. Specifically, we will explore how, in the presence of a topologically transitive, locally maximal hyperbolic set for the Reeb flow on the boundary of a Liouville domain, barcode entropy is bounded below by the topological entropy restricted to the hyperbolic set.*

Let M^n be a manifold. $\omega \in \Omega^2(M)$ is symplectic if $d\omega = 0$ and it is nondegenerate.

Example 6.1. $\mathbb{R}^n$ is symplectic with canonical Darboux form.

Recall the definition of Hamiltonian vector field associated to a function $H \in C^\infty(M)$.

Definition 6.2. A diffeomorphism $\varphi : M \to M$ is called *non-degenerate* if $\Phi(\varphi) \cap \Delta \subset M \times M$ (pitchfork, i.e. transversal intersection!).

Let M^{2n} be a closed symplectic manifold. Arnold's conjecture says

(1) *If $\varphi = \varphi_H$ (Hamiltonian flow) is nondegenerate, then*
$$\#Fix(\varphi_H) \geq \sum_{i=0}^{2n} \dim H_i(M, k) = \dim H_*(M, k)$$
(2) *If $\varphi = \varphi_H$ is degenerate, then*
$$\#Fix(\varphi) \geq Cl(M) + 1$$
where $Cl(M)$ is the maximum number of homology classes we can add before getting to zero.

Why do we care? Because
$$\#Fix(\varphi_H) \leftrightarrow \{\text{1-periodic orbits of } X_H\}$$

Idea by Floer. Construct an invariant that would say something about periodic orbits.

Question. *Can Floer theory capture other "dynamical information"? (Other than the periodic orbits.)*

A persistence module is a pair (V, Π), where $V = \{V_t\}_{t \in \mathbb{R}}$ is a family of $\mathbb{F}$-vector spaces and $\Pi = \{\Pi_{st}\}_{s \leq t}$ is a family of maps such that

(1) $\Pi_{ss} =, \Pi_{ts} \circ \Pi_{rt} = \Pi_{rs}$.
(2) *$\exists s \subset \mathbb{R}$ such that Π_{st} is an isomorphism for s, t in the same connected component of $R \setminus S$.*
(3) *Π_{st} have finite rank.*
(4) *$\exists s_0$ $V_s = \{0\}$, $s \leq s_0$.*
(5) *$V_t = \lim_{s \to t} V_s$ (lower limit!!)*

Theorem 6.3. *Any persistence module is a sum of integral persistence modules,*
$$(V, \Pi) \cong \bigoplus_{I \in B(V)} F(I).$$

Example 6.4. Heart and sphere. There is a noise in the persistence module of the heart due to an unnecessary critical point.

(M^{2n}, ω) a Liouville domain is a compact symplectic manifold and $X \in \mathfrak{X}(M)$ with $X \cap \partial M$ (pitchfork, i.e. transversal intersection!) pointing outwards and preserved by the symplectic form, i.e. $\mathcal{L}_X \omega = \omega$ ($\omega = d\alpha$).

When we restrict ω to the boundary, we obtain a contact form and get some interesting dynamics.

A Lagrangian $(L, \partial L) \subset (M, \partial M)$ is asympotically conical if

(1) *$\partial L \subset \partial M$ is Legendrian.*

(2) $L \cap [1-\varepsilon, 1] \times \partial M = [1-\varepsilon, 1] \times \partial L$.

Remark 6.5. Take a Hamiltonian $H : \hat{M} \to \mathbb{R}$ such that

$$\begin{cases} H(r,x) = h(r) & r = 1 \\ H(r,x) = rT - B \end{cases}$$

then $X_H = h'(r)R_\alpha$.

For $L_0, L, A \subset$ Lagrangians, H linear at infinite, then $A_H^{L_0 \to L_1}$,

$$\begin{aligned} A_H^{L_0 \to L_1} : P_{L_0 \to L_1} &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \int_0^1 \gamma^* \alpha - \int_0^1 H(x(t))dt \end{aligned}$$

where $P_{L_0 \to L} = \{\gamma : [0,1] \to \hat{M} : \gamma(0) \in L_0, \gamma(1) \in L_1\}$ is the set of chords.

Remark 6.6. $\text{crit}(A_H^{L_0 \to L_1}) = \{$1-chords of X_H from L_0 to $L_1\}$.

Putting a metric on $P_{L_0 \to L_1}$ we can consider $\varphi : \mathbb{R} \times [0,1] \to \hat{M}$, solutions of some PDE which is some kind of generalization of a gradient, $-\nabla A_H^{L_0 \to L_1}$. These solutions can be put in a moduli space

$$\tilde{\mathcal{M}}(x_-, x_+, H, J) = \{\varphi \textit{ solutions s.t. } \dots\}$$

Then we define a boundary operator ∂.

Theorem 6.7. $\partial^2 = 0$

So that we have a homology, called wrapped Floer homology $HW^t(H, L_0, L_1, J)$

Remark 6.8. We have $H \leq K \rightsquigarrow HW^t(H, L_0, L_1, J) \to HW^t(K, L_0, L_1, K)$.

Definition 6.9. For $t \geq 0$

$$HW^t(M, L_0, L_1) = \underline{\lim}_H HW^t(H, L_0, L_1, J)$$

(Where we have taken direct limit.)

Taking direct limit of the homology, we make sure the homology theory is independent of the choice of objects (I think, complex structure and Hamiltonian) we used to construct it.

Proposition 6.10. *$t \to HW^t(M, L_0, L_1)$ is a peristence module $B(M, L_0, L_1)$.*

Finally we can define barcode entropy. Fix $\varepsilon > 0$, $t \geq 0$,

$$b_\varepsilon(M, L_0, L_1, t) = \#\{ \textit{ of bars in } B(M, L_0, L_1) \textit{with length} \geq \varepsilon \textit{ and start before } t\}$$

Then

$$\bar{h}^{HW}(M_0, L_0, L_1) = \lim_{\varepsilon \to 0} \textit{lim sup}_{t \to \infty} \frac{\textit{log}^+(b_\varepsilon(M, L_0, L_1, t)}{t}$$

Consider a contact manifold $(\Sigma, \lambda, L_0, L_1)$ and A the Lagrangians. (Example raising question of filling.)

Theorem 6.11 (M '24)**.** *$\bar{h}^{HW}$ is independent of the filling.*

Theorem 6.12 (M '24)**.** *$\bar{h}^{HW}(M, L_0, L_1) \leq h_{top}(\alpha)$.*

Theorem 6.13 (M '25)**.** *Consider (M, L_0, L_1). Let K be a compact topologically transitive hyperbolic set for the Reeb flow α. Assume $W^s_\delta(q) \subset \partial L_0$, $W^s_\delta(p) \subset \partial L_1$. Then*

$$\bar{h}^{HW}(M, L_0, L_1) \geq h_{top}(\alpha|_K) > 0.$$

Which says it captures dynamics beyond unconditional phenomena. In lower dimensions these tend to coincide, but in higher dimension we don't know. This is related to "the sup over hyperbolic sets [. . .]"

Here's a conjecture:

$$sup_{L_0,L_1}\bar{h}^{HW}(M, L_0, L_1) = h_{top}(\alpha).$$

Extra comments. *One of the aims is to describe topological entropy h_{top} using Floer theory. Theorems by Çineli-Ginzburg-Gürel show bounds of topological entropy and barcode entropy (one of which is for hyperbolic sets).*

There is a notion of admissible Hamiltonian and Reeb vector fields which is related to some asymptotical behaviour "linear at infinity". I understand that admissible vector fields give the interesting chords for the Floer homology construction.

7. Revisiting cotangent bundles

Mieugel Cueca, KU Leuven. Symplectic Geometry Joint Seminar, IMPA. September 5, 2025.

Abstract. *Cotangent bundles provide key examples of symplectic manifolds. On the other hand, one can think of Lie groupoids as generalizations of manifolds. In this context, Alan Weinstein constructed their cotangent bundles and proved that they are so-called symplectic groupoids. In this talk, I will recall this construction and explain what happens when one replaces a Lie groupoid with a Lie 2 (or n)-groupoid. If time permits, I will exhibit some of their main applications. This is joint work with Stefano Ronchi.*

Recall the basic properties of the cotangent bundle T^*M for a symplectic manifold:

1. *It's a vector bundle.*
2. $\langle\cdot,\cdot\rangle : TM \otimes T^*M \to \mathbb{R}_M$ *is the dual pairing.*
3. $\omega_{can} \in \Omega^2(T^*M)$ *is symplectic.*
4. $\mathcal{L}_\varepsilon\omega_{can} = \omega_{can}$*,* ε *Euler vector field.*

Main goal. *Reproduce the above for Lie n-groupoids.*

For $n = 0$ we get the above situation. For $n = 1$ [Duzud-Weinstein], [Prezun], for $n \geq 2$? and we care about $n = 2$.

Definition 7.1. A *Lie n-groupoid* $\mathcal{G} : \Delta^{\text{op}} \to \mathsf{Man}$ such that

$$P_{e,j} : \mathcal{G}_\ell \to \Lambda^\ell_j\mathcal{G}$$

are surjective submersions $\forall \ell, j$ are differomorphisms for $\ell > n$.

Remark 7.2. This sort of manifolds-valued presheaf category is generated by

$$d^\ell_i : \mathcal{G}_\ell \to \mathcal{G}_{\ell-1}\text{face maps}0 \leq i, j \leq \ell$$
$$s^\ell_j : \mathcal{G}_\ell \to \mathcal{G}_{\ell+1} \qquad \text{degeneracies}$$

The tangent is a functor, it satisfies

$$T_\bullet(\mathcal{G}) = T_k(\mathcal{G}) = T\mathcal{G}_k$$

(It looks like T preserves diagrams.)

Dold-Kan. *The category* SVect *of simplicial vector spaces has objects*

$$\mathbb{V}_\bullet \qquad \mathbb{V}_n \longrightarrow \cdots \longrightarrow \mathbb{V}_2 \xrightarrow{\textit{3 arrows}} \mathbb{V}_1 \xrightarrow{\textit{2 arrows}} \mathbb{V}_0$$

where the $\mathbb{V}_i$ are vector spaces.

There is a functor

$$\mathsf{SVect} \xrightarrow{N} \{\textit{chain complexes} \geq 0\}$$
$$\mathbb{V}_\bullet \to N\mathbb{V} = (N_\ell \mathbb{V}\operatorname{Ker} P_{\ell,\ell}, \partial = d_\ell)$$

Theorem 7.3 (Dold-Kan)**.** *That's an equivalence of categories. [Confirm this!]*

Those categories are monodial:

$$(\mathsf{SVect}, \otimes), \qquad (\mathbb{V}_\bullet \otimes \mathbb{W})_\ell = \mathbb{V}_\ell \otimes \mathbb{W}_\ell$$
$$(\mathsf{ch}_{\geq 0}, \otimes), \qquad (V \otimes W)_i = \bigoplus_{\ell+k=1} V_\ell \otimes W_k$$

And N is Lax monoidal with Lax structure given by the Eilenberg-Zilber map, though we won't explain the details of this.

There are duals given by internal Hom:

$$\mathbb{V}^{n*} = \underline{\mathrm{Hom}}(\mathbb{V}, B^n\mathbb{R})$$

Where the internal Hom is given by

$$\underline{\mathrm{Hom}}(\mathbb{V}, B^n\mathbb{R})_\ell = \mathrm{Hom}_{\mathsf{SVect}}(\mathbb{V} \otimes \Delta_n[\ell], B^n\mathbb{R})$$

for an object $\Delta[\ell] = \mathbb{R}[\Delta[\ell]]$.

Properties.

1. $\mathbb{V}^{n*}$ *isa simplicial vector bundle.*
2. $N(V^{n*})$ *and* $N(\mathbb{V})^*[n]$ *is a quasi isomorphism.*
3. $\langle \cdot, \cdot \rangle : \mathbb{V} \otimes \mathbb{V}^{n*} \to B^n\mathbb{R}$ *is non-degenerate on homology.*
4. $\mathbb{V} \hookrightarrow (\mathbb{V}^{n*})^{n*}$ *Mont. a eq.*

The vector bundle case.

$$\mathit{Maps}(\Delta[i], \mathcal{G})_k = \mathrm{Hom}_{\mathsf{SSet}}(\Delta[i] \times \Delta[k], \mathcal{G})$$

Proposition 7.4. *Let $\mathcal{G}$ be a Lie n-groupoid.*

1. $\mathit{Maps}(\Delta[i], \mathcal{G})$ *Lie n-groupoids ME $\mathcal{G}$.*
2. $\mathit{Maps}(\Delta[i], \mathcal{G})_0 = \mathcal{G}_i$.
3. $ev : \Delta[i] \times \mathit{Maps}(\Delta[i], \mathcal{G}) \to \mathcal{G}$.

[Staircase looking diagram.]

Definition 7.5. $\mathcal{G}_\bullet$.

$$T_i^{n*}\mathcal{G} = \mathrm{Hom}_{\mathsf{SVect}}(1^*\Pi_{\Delta[i]}(T\mathcal{G}), B^n\mathbb{R}_{\mathcal{G}_i})$$
$$(d_j, F)_{K|d_j\mathcal{G}}(x^a) = (F_k)|_{\mathcal{G}}(x^{\delta_j a}).$$

Proposition 7.6. *$\mathcal{G}$ Lie n-groupoid, then $T^{n*}\mathcal{G}$ satisfy*

1. *is a vector bundle n-groupoid*

(2) *dual to* $T\mathcal{G}$

$$\langle \cdot, \cdot \rangle : T\mathcal{G} \otimes T^{n*}\mathcal{G} \to B^n\mathbb{R}_{\mathcal{G}}$$

non-degenerate on homology.

(3) *n-shifted symplectic*

$$T_n^{n*}\mathcal{G} \xrightarrow{p} T^*\mathcal{G}_n$$

and $p^*\omega_{can}$.

[More computations I missed]

8. A theorem on complexifications of Lie groups

9. Holomorphic extensions of s-proper Lie groupoids

Rui L. Fernandes, ?. Symplectic Geometry Seminar, IMPa. September 17, 2025.

Abstract. *Every smooth manifold admits a compatible analytic structure, and a classical result of Whitney–Bruhat states that any analytic manifold has a holomorphic extension. Lie groups also admit compatible analytic structures, and another classical result, due to C. Chevalley, shows that any compact Lie group has a holomorphic extension to a complex Lie group. D. Martínez Torres has shown that any proper Lie groupoid admits a compatible analytic structure. I will discuss an extension of the classical results of Whitney–Bruhat and Chevalley, establishing that any s-proper Lie groupoid has a holomorphic extension. This talk is based on recent joint work with Ning Jiang (arXiv:2508.18036).*

Theorem 9.1 (Whitney-Beuhat)**.** *If M is analytic there exists a complex manifold $M_{\mathbb{C}}$ together with an analytic map $i : M \to M_{\mathbb{C}}$ which is totally real ($T_H M_{\mathbb{C}} = TM \oplus J(TM)$) and*

(1) *For every complex manifold X and every analytic map $\phi : N \to X$ there exists $\phi^* : U \to X$ holomorphic contained open $M \subset U \subset M_{\mathbb{C}}$.*
(2) *If $\psi : V \to X$ holomorphic on $M \subset V \subset M_{\mathbb{C}}$ then $\psi = \phi^*$ where $\phi = \psi \circ i$ on a possibly smaller open contained in $U \cap V$.*

Theorem 9.2 (Chevalley)**.** *If G is a Lie group, there exists a complex Lie group $G_{\mathbb{C}}$ and a morphism $i : G \to G_{\mathbb{C}}$ satisfying the following universal property. For every complex Lie group H and morphism $\phi : G \to H$ there exists a unique holomorphic map $\phi^* : G_{\mathbb{C}} \to H$ such that $\phi = \phi^* \circ i$.*

Here's the construction of $G_{\mathbb{C}}$:

$$\begin{tikzcd}
N \subset \tilde{G} \arrow[r] \arrow[d] & G^* \arrow[d] \\
\tilde{G}/N \simeq G \arrow[r, "i"'] & G_{\mathbb{C}} = G^*/\overline{i^*N}
\end{tikzcd}$$

where $\tilde{G}$ is the universal cover group of G, G^* is the group that integrates to the complexification of the Lie algebra of G, i.e. $Lie(G^*) = \mathfrak{g}_{\mathbb{C}}$ and $\overline{i^*(N)}$ is the smallest closed normal complex Lie subgroup of G^* containing $i^*(N)$.

Example 9.3.

$$\begin{tikzcd}
\widehat{\mathrm{SL}_2(\mathbb{R})} \arrow[d] \arrow[dr] & \\
\mathrm{SL}_2(\mathbb{R}) \arrow[r, hook] & \mathrm{SL}_2(\mathbb{C})
\end{tikzcd}$$

which is a simple case, but consider instead $\tilde{\mathrm{SL}}_2(\mathbb{R}) \times \mathbb{R}$ and the normal subgroup $N = \langle a \rangle \times \langle \lambda \rangle$ for irrational λ. Then we obtain

$$\begin{tikzcd}
\widehat{\mathrm{SL}_2(\mathbb{R})} \times \mathbb{R} \arrow[r] \arrow[d] & \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C} \arrow[d] \\
\widehat{\mathrm{SL}_2(\mathbb{R})} \times \mathbb{R}/N \simeq G \arrow[r] & G_{\mathbb{C}}
\end{tikzcd}$$

and $G_{\mathbb{C}}$ is 3 complex dimensions! It is not $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}$.

10. Everything you always wanted to know about polygons but were too afraid to ask

Alessia Mandini, UFF. GAAG, IMPA. September 22 and 23, 2025.

Abstract. *Moduli spaces of polygons form a family of Kähler manifolds that can be constructed as a Kähler reduction of coadjoint orbits. These spaces have deep connections to various areas of mathematics, including symplectic and algebraic geometry as well as representation theory. In this talk, I will define these spaces and explore their connections. I will then discuss wall-crossing phenomena in these spaces and demonstrate how it can be used to determine their cohomology rings. Finally, I will introduce the hyperkähler analogue of these spaces, known as hyperpolygon spaces, and describe some of their generalizations.*

Plan of the talk

(1) *Polygons in $\mathbb{R}^3$ and relations with other moduli spaces.*
(2) *Polygons in other spaces.*

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{>0}$. Consider S^2 with its usual symplectic, Kähler form. Also consider the product of several spheres. There's a Hamiltonian action of $SO(3)$ by rotations. The moduli space of polygons is the symplectic reduction obtained with this Hamilatonian action,

$$M(\alpha) = \prod S^2_{\alpha_i} // SO(3).$$

Why is it called the space of polygons? We think that

$$[v_1, \dots, v_n] \in M(\alpha) \iff \sum_{i=1}^{n} v_i = 0$$

are polygons.

When smooth, $M(\alpha)$ is a $(n-3)$-complex-dimensional Kähler manifold. Consider

$$\varepsilon_I(\alpha) = \sum_{i \in I} \alpha_i - \sum_{j \in I^c} \alpha_j$$

for some index set $I \subseteq \{1, \dots, n\}$. $M(\alpha)$ is smooth if $\varepsilon_I(\alpha) \neq 0$ for all $I \subseteq \{1, \dots, n\}$. If so, we say α is generic.

Theorem 10.1 (Kapovich-Millson)**.** *$M(\alpha)$ is a complex analytic space with (eventually) isolated singularity (homogeneous quadratic cones).*

Example 10.2. (1) ($n = 3$.) Then $M(\alpha)$ is either empty or a point.
(2) ($n = 4$.) $M(\alpha)$ is either empty or a sphere.

Remark 10.3 (Hausmann-Knutson). Let M_n be the space of all n-gons modulo rigid motions. Then M_n can be equipped with a Poisson structure for which $M(\alpha)$ are the symplectic leaves.

Polygons as quiver varieties. Consider the star-shaped quiver, which is a distinguished point with some points around it, and arrows from every point to the distinguished one. Let the distinguished point be $V_0 = \mathbb{C}^2$ and the rest $\mathbb{C}$. Then a representation of this quiver is

$$\mathrm{Rep}Q = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) = \mathbb{C}^{2n}.$$

Now put

$$K = (U(2) \times U(1)^n)/\Delta$$

where Δ is the diagonal S^1. This gives a Hamiltonian action on $\mathbb{C}^{2n}$ as follows. For

$$(A, \lambda_1, \ldots, \lambda_n) \cdot (q_1, \ldots, q_n)$$

we map $q_i \mapsto A^{-1} q_i \lambda_i$. That is

$$\begin{aligned} \mu : \mathbb{C}^{2n} &\longrightarrow K^* \\ (q_1, \ldots, q_n) &\longmapsto \left(\sum_{n=1}^{n} (q_i q_i^*), \ldots, \frac{1}{2}|q_i|^2 \ldots \right) \end{aligned}$$

Theorem 10.4 (Hausmann-Knutson).

$$\mathbb{C}^{2n} /\!/_{(0,\alpha)} K = M(\alpha)$$

Proof. Uses

$$\begin{aligned} \mathbb{R}^3 &\xrightarrow{\simeq} \mathfrak{su}(2)^* \\ v_i &\longmapsto (q_i q_i^*)_0 \end{aligned}$$

□

$$\begin{tikzcd} & \mathbb{C}^{2n} \arrow[dl, "/\!/_\alpha U(1)^n"'] \arrow[dr, "/\!/U(2)"] & \\ \prod S^2 \arrow[dr, "/\!/_\alpha SO(3)"'] & & Gr(2, \mathbb{C}^n) \arrow[dl, "/\!/_\alpha U(1)^n"] \\ & M(\alpha). & \end{tikzcd}$$

So we have (I think former symplectic reduction)

$$\mu_{U(1)^n} : Gr(2, \mathbb{C}^n) \to \mathbb{R}^n$$

Walls and wall crossing. *The walls are*

$$W_I = \{\alpha \in \mathbb{R}^n_{>0} : \varepsilon_I(\alpha) = 0\}$$

for $I \subseteq \{1, \dots, n\}$.

To understand wall crossing suppose we have a wall W_I, with a^c in the wall, α^+ on one side and α^- on the other.

Not that

$$\varepsilon_I(\alpha^+) > 0 \iff \sum_{k \in I} \alpha_i^+ > \sum_{j \in I^c} \alpha_j^+$$

Define

$$\begin{aligned} M_{I^c}(\alpha^+) &= \{(v_1, \dots, v_n) \in M(\alpha^+) : v_i = \lambda v_j \forall i, j \in I^c, \lambda > 0\} \\ &= M(\tilde{\alpha}), \qquad \tilde{\alpha} = \left(\alpha_{i_1}, \dots, \alpha_{i_k}, \sum_{j \in I^c} \alpha_j\right) \end{aligned}$$

Then $\varepsilon_I(\alpha^+) < 0$ implies $M_I(\alpha^-) \subseteq M(\alpha^-)$.

$$\begin{tikzcd} & \tilde{M} \subseteq E = M_I \times M_{I^c} \arrow[d, "blow\ up"] & \\ M(\alpha^+) \arrow[r] & M(\alpha^c) & M(\alpha^-) \arrow[l] \end{tikzcd}$$

Remark 10.5.
- $M(\alpha) \cong M(\sigma(\alpha))$ for σ a permutation on the order of sets.
- $M(\alpha)$ is conformally symplectomorphic to $M(\lambda\alpha)$ for all $\lambda > 0$.

Definition 10.6. Let $\alpha \in \mathbb{R}^n_{>0}$. $I \subseteq \{1, \dots, n\}$ is *short* if $\varepsilon_I(\alpha) < 0$ and *long* if $\varepsilon_I(\alpha) > 0$.

Example 10.7. We did an example of wall crossing. The relevant manifolds $M_{I^c}(\alpha)$ and $M_I(\alpha)$ were projective spaces.

Now recall our first quotient

$$\begin{array}{c} \mu^{-1}_{U(1)}(\alpha) \subseteq Gr(2, \mathbb{C}^n) \\ \big\downarrow {\scriptstyle //U(1)^n} \\ M(\alpha) \end{array}$$

Let c_i be the first Chern classes associated to the n S^1-bundles above (by taking reduction in stages).

Theorem 10.8 (Haussmann-Knutson,M.)**.** *The c_i generate the cohomology of $M(\alpha)$.*

Theorem 10.9 (Guillemin-Stenberg)**.** *In this setup, for α generic,*

$$H(M) = \mathbb{C}[c_1, \dots, c_n]/Ann(Vol(M(\alpha)))$$

i.e., $Q(c_1, \dots, c_n) \in Ann(VolM(\alpha)) \iff Q\left(\frac{\partial}{\partial \alpha_i}, \dots, \frac{\partial}{\partial \alpha_1}\right) Vol(M(\alpha)) = 0$

Theorem 10.10 (Takakura, The Koi)**.**

$$Vol(M(\alpha)) = -\frac{(2\pi)^{n-3}}{(n-3)!} \sum_{I\ long} (-1)^{n-|I|} \varepsilon_I(\alpha)^{n-3}.$$

Example 10.11. According to Example 10.7 (which I did not copy) we find that $\alpha_1 \in \Delta_1$ gives $\mathrm{Vol}(M(\alpha_1)) = 2\pi^3(1-2\alpha_3)^2$ and

$$H(M(\alpha_1)) = \mathbb{C}[c_3]/(c_3^3)$$

Polygon game. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then the co-adjoint orbits $\mathfrak{g}^* \supseteq \mathcal{O}_{\xi_i}$ are symplectic manifolds with the KKS form. Then*

$$\begin{aligned} G \curvearrowright \Pi\mathcal{O}_{\xi_i} &\longrightarrow \mathfrak{g}^* \\ (A_i, \dots \alpha_n) &\longmapsto \sum A_i \end{aligned}$$

Then the quotient

$$M(\xi) = \Pi\mathcal{O}_{\xi_i}//_0 G$$

generalizes the previous construction, which we get with $G = SU(2)$.

It could be interesting to investigate which of the next constructions can be generalized to other Lie groups.

Theorem 10.12 (Sotillo-Floentins-Gadihl)**.** *Wall-crossing for $SU(m)$.*

Bending action. *We consider a polygon of n sides and put some diagonals that don't intersect. We introduce some notion of "bending" that allows to define a Hamiltonian function. In turn, this defines a torus action on $M(\alpha)$ and a moment map.*

For $n = 2$ we obtain the moment map

$$\begin{aligned} \mu : M(\alpha) &\longrightarrow \mathbb{R}^2 \\ p &\longmapsto (\ell_1(p), \ell(p)) \end{aligned}$$

and we can see the moment polytope in $\mathbb{R}^2$.

Remark 10.13. Any system of $(n-3)$ non-vanishing and non-intersecting diagonals determine a torus action $T^{n-3} \curvearrowright M(\alpha)$.

Relations of polygon spaces to other moduli spaces. *Representations of the fundamental group of the punctured sphere in $SU(2)$, i.e.*

$$Rep(\pi_1(S^2 \setminus \{p_1, \dots, p_n\}, SU(2))/SU(2) \cong M(\alpha)$$

This can be put very explicitly:

$$\begin{aligned} &Rep(\pi_1(S^2 \setminus \{p_1, \dots, p_n\}, SU(2)) \\ &= \{(g_1, \dots, g_n) \in SU(2)^n : g_1 \cdot \dots \cdot g_n = Id, t_r g_i = 2\cos\pi\alpha_i\} \end{aligned}$$

See [Agapito-Godinho]. That's all we will say about this example.

Now consider the moduli space of parabolic bundles. Consider a holomorphic bundle of rank 2 over $\mathbb{C}P^2$. Choose some points $D = \{x_1, \dots, x_n\}$ in $\mathbb{C}P^2$ and a flag $E_{x_i} = E^{x_i,1} \supseteq E^{x_i,2} \supseteq \{0\}$ for every $x_i \in D$. Each of the $E_{x_i,j}$ is isomorphic to $\mathbb{C}$ (1-dimensional). A quasi-parabolic bundle is an holomorphic bundle E with such a choice of flag. A parabolic bundle is a quasi-parabolic bundle with a choice of parabolic weights $0 < \beta_1(x_i) < \beta_2(x_i) < 1$.

There is also a notion of stability:

$$pdegE = degE + \sum_{i=1}^{n} (\beta_1(x_i) + \beta_2(x_i))$$

$$\mu(E) = \frac{pdeg(E)}{rank(E)}$$

We say E is (semi)stable if ţ$(E) > \mu(L)$ *($\mu(E) \geq \mu(L)$) for any $L \subseteq E$ parabolic subbundle.*

Then we obtain that $\mathcal{M}_{\pi,d}(\beta)$ is the moduli space of (semi)-stable parabolic bundles on $\mathbb{P}^1$ of rank r and degree d. In particular $\mathcal{M}_{2,0}(\beta)$ is the moduli space of (semi)-stable parabolic bundles on $\mathbb{P}^1$ of rank 2 and degree 0 holomorphically trivial.

Theorem 10.14 (Jeffrey, Godinho, M.)**.** *For generic α, $M(\alpha)$ is diffeomorphic to $\mathcal{M}_{2,0}(\beta)$ whenever $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$.*

Idea of proof. The correspondence can be made quite explicit as a map

$$M(\alpha) \longrightarrow \mathcal{M}_{2,0}(\beta)$$

$$[q_1, \ldots, q_n] \longmapsto {}^{E=\mathbb{C}P^1 \times \mathbb{C}^2}_{E_{x_i} \supset E_{x_i,1} \supset X_{x_i,2} \supset \{0\}}$$

$\square$

Now consider the quiver variety we described above:

$$RepQ = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) = \mathbb{C}^{2n}.$$

Now put

$$K = (U(2) \times U(1)^n)/\Delta$$

where Δ is the diagonal S^1.

But this time put

$$Rep\tilde{Q} = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}^2, \mathbb{C})$$

We also have an action of K, and we get

$$K \curvearrowright Rep\tilde{Q} = T^*\mathbb{C}^{2n},$$

a so-called hyperHamiltonian action. The quotient

$$X(\alpha) = /\!/\!/_{\substack{(0,\alpha)\\(0,0)}} K = \frac{\mu_{\mathbb{R}} 1(0,\alpha) \wedge \mu_{\mathbb{C}}^{-1}(0,0)}{K}$$

called hyperpolygon space.

Here

$$\mu_{\mathbb{R}} : T^*\mathbb{C}^{2n} \to \mathcal{K}^*$$

$$\mu_{\mathbb{C}} : T^*\mathbb{C}^{2n} \to \mathcal{K}_{\mathbb{C}}^*$$

$$(q_1, \ldots, q_n, p_1, \ldots, p_n) \mapsto (\sum_{i=1}^{n} (p_i, q_i)_0, \ldots)$$

It turns out that $X(\alpha)$ is smooth if and only if $\varepsilon_1(\alpha) = 0$ for all $I \subseteq \{1, \dots, n\}$. When smooth, $X(\alpha)$ is a hyperkähler manifold (non-compact, unfortunately), $M(\alpha) = \{[0_{1i}, 0] \in X(\alpha)\}$.

Theorem 10.15 (Boalch)**.** *$X(\alpha)$ is the moduli space of polygons for $GL(2, \mathbb{C})$.*

Parabolic Higgs bundles. *Let E be a parabolic bundle as before, i.e. $E \in \mathcal{M}_{0,2}(\beta)$. A Higgs field on E is*

$$\phi \in H^0(\mathbb{P}^1, SPEnd(E) \otimes K_{\mathbb{P}^1}(D)))$$

where a strongly parabolic endomorphism is $f : E \to E$ such that $f(E_{x_i}) \subseteq E_{x_{i+1}}$ for all i. (Fix details!)

A parabolic Higgs bundle is (probably the pair (E, ϕ)). A parabolic Higgs bundle (E, ϕ) is (semi)stable if $\mu(E) > \mu(L)$ ($\mu(E) \geq \mu(L)$) for all L parabolic Higgs subbundle.

Then $\mathcal{N}^{0,1}_{r,a}(\beta)$ is the moduli space of parabolic Higgs bundles over $\mathbb{P}^1$ with rank r and degree d (with fixed determinant, and traceless; two notions that we will not define here).

Theorem 10.16 (Goldinho, M.; Biswar, Florentino, Godinho, M.)**.** *Let α be generic, let β be such that $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$. Then the hyperpolygon space $X(\alpha)$ is symplectomorphic to the moduli space of parabolic HIggs bundles over $\mathbb{P}^1$, β-stable, holomorphically trivial (with fixed determinant and trace-free).*

Idea of proof. We can define a correspondence

$$\begin{aligned} X(\alpha) &\longrightarrow \mathcal{H}(\beta) \\ [p, q] &\longmapsto {}^{E = \mathbb{C}P^1 \times \mathbb{C}^2}_{E_{x_i} \simeq E_{x_i,1} \supset E_{x_i,2} \supset \{0\}} \end{aligned}$$

No we can use the moment map condition that the sum of the residues is zero, where $\mathrm{Res}_{x_i}\phi = (p_i, q_i)$ where ϕ is the unique function that satisfies the last equality by the residue theorem. ☐

Wall-crossing for moduli spaces of parabolic bundles. *See "Translation"from Thaddeus. What happens is that there is an S^1-action on $X(\alpha)$ given by*

$$\lambda \cdot [p, q] = [\lambda p, q].$$

(This can be paralleled with the S^1-action on $\mathcal{H}(\beta)$, given by $\lambda \cdot [E, \phi] = [E, \lambda\phi]$.)

This action has a moment map

$$\mu_{S^1}([p, q]) = \frac{1}{2} \sum_i |p_i|^2$$

Fixed points are [Konno]

$$X_S = \{[p, q] \in X(\alpha) : S, S^c \text{ are straight, } p_j = 0 \forall j \in S\}$$

for all $|S| \geq 2$ short. Here $S \subseteq \{1, \dots, n\}$ is straight if $q_i = \lambda_j q_j$ for $\lambda_j > 0$ $\forall i, j \in S$ (which in the case of polygons means the edges are aligned).

Let U_S be the flow down from X_S Then, crossing a wall W_I as defined above, i.e. $W_I = \{\alpha \in \mathbb{R}^n_{>0} : \varepsilon_S(\alpha) = 0\}$, "replaces" U_S by U_{S^c}.

Let α and $\tilde{\alpha}$ be generic, then $X(\alpha)$ is diffeomorphic to $X(\tilde{\alpha})$. The wall crossing for $M(\alpha)$ involves

$$M_S(\alpha^+) = U_S \cap M(\alpha^+)$$
$$M_{S^c}(\alpha^-) = U_{S^c} \cap M(\alpha)$$

This is thought of as a Mukai transform.

Generalization. *See [Florentino, Godinho, Sotillo] for wall crossing. See also [Fisher] PhD thesis, [Fisher, Rayan]. [Hausel et al], [Rayan-Schopnick].*

11. Vertex algebras and special holonomy on quadratic Lie algebras

Mario García Fernandez, ICMAT. GAAG, IMPA. September 24, 2025.

Abstract.
The chiral de Rham complex (CDR) is a sheaf of vertex algebras on any smooth manifold, introduced by Malikov, Schechtman and Vaintrob, which provides a formal quantization of the non-linear sigma model in mathematical physics. Motivated by the algebra of chiral symmetries in two-dimensional superconformal field theories, vertex algebra embeddings on the CDR have been studied for special holonomy Riemannian manifolds, thanks mainly to the work of Heluani, Zabzine, and collaborators, with interesting applications to the elliptic genus. In these lectures, we will discuss extensions of some of these results to the case of special holonomy manifolds with skew-torsion. The presence of torsion typically allows for continuous symmetries in the geometry, with an enhanced interplay with Lie theory and algebra, as well as the application of techniques from generalized geometry.

Based on joint work with Luis Álvarez Cónsul, Andoni De Arriba de la Hera. arXiv:2012.01851 (IMRN '24).

Motivation. *Let (M^n, g) be a Riemannian spin manifold with parallel spinor $\nabla^g \varphi = 0$. Then hol$(g) \subset G_\varphi \subseteq SO(n)$, and Ric$(g) = 0$.*

There is a construction by Markov-S-V that puts a sheaf of vertex algebras $\mathcal{V} \to M^n$. How to construct special embeddings of trivial vertex algebras in the cohomology of $\mathcal{V}$, i.e. $\mathcal{V}_\varphi \hookrightarrow H^*(\mathcal{V})$ by Heluani et al.

Applications. Construction of topological invariants. Elliptic genus, [Borisov-L]. How to understand mirror symmetry using vertex algebras [Borisov]. Holography [Witten].

Geometry in algebra. *We shall do geometry in quadratic Lie algebras. Recall that a quadratic Lie algebra is $(\mathfrak{g}, [\cdot,\cdot], (\cdot,\cdot))$ where $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra (in this course we use $\mathbb{R}$ as base field) and $(\cdot,\cdot)$ is a symmetric bilinear invariant form.*

Example 11.1. Let R be a Lie algebra with $\langle\cdot,\cdot\rangle_R : R \otimes R \to \mathbb{R}$. Take $\mathfrak{g} = R \oplus R^*$, and pick $H \in \Lambda^3 R^*$. Put $H(a,b,c) = \langle [a,b], c\rangle$, and define

$$[v+\alpha, w+\varphi] = [v,w] - \varphi([v,-]) + \alpha([v,-]) + H(v,w,-)$$

for $v, w \in R$ and $\varphi, \alpha \in R^*$. This turns out to be a bracket.

If you like geometry you can pick K compact, Lie$K = R$, this is "Generalized geometry on $TK \oplus T^*K$" à la Hitchin.

QLA:

- *Courant algebroids $/\{*\}$.*

- *Symplectic supermanifolds.*

Definition 11.2. (1) A *(generalized) metric* on $(\mathfrak{g}, (\cdot,\cdot))$ is $G \in \mathrm{End}(\mathfrak{g})$ with $G^2 = \mathrm{Id}$, $(G, G) = (\cdot,\cdot)$,

This gives $\mathfrak{g} = V_+ \oplus V_-$ where G acts as identity on the first term and as $-\mathrm{Id}$ on the second one. $(G_-, -)$ is a non-degenerate pairing. $(-,-)_{V_I}$ non-degenerate tensor.

(2) A divergence $\varphi \in \mathfrak{g}^*$.

(3) A *connection* is

$$D : \mathfrak{g} \to \mathfrak{g}^* \otimes \mathfrak{g}$$

such that $(D_a b, c) + (b, D_a c) = 0$. So that $D \in \mathfrak{g}^* \otimes \Lambda^2\mathfrak{g}$ (where we identify $\mathfrak{g}$ with its dual using the pairing).

D is *compatible* with G if $[D_a, G] = 0$. This condition says that D splits into four operators:

$$\begin{tikzcd}
D^+_+ & & D^-_+ \\
& D \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
D^+_- & & D^-_-
\end{tikzcd}$$

(4) Let $a \in \mathfrak{g}$, $a = a_+ + a_-$. A *generalized connection* satisfies $D^+_a = [a_-, b_+]_+ + [a_+, b_-]_-$. (This will happen to things living in $\mathcal{D}^0$, see below.)

Definition 11.3. Given D define

(1) $\varphi_D(a) = -\mathrm{tr} Da$.

(2) Torsion$(AX) : T_D \in \Lambda^2\mathfrak{g}^*$,

$$T_D(a, b, c) = (D_a b - D_b a - [a, b], c) + (D_c a, b).$$

(Just copying the definition of torsion.)

Lemma 11.4. *Define $\mathcal{D}^0(G, \varphi) = \{D : G\text{-compatible}\ , \varphi_D = \varphi\}$. Then this is non-empty, but is not a point. Furthermore, $\forall D \in \mathcal{D}^0(G, \varphi)$,*

$$D_{a_-} b_+ = [a_-, b_+]_+, \qquad D_{a_+} b_- = [a_+, b_-]_-.$$

(Checking non-emptyness is just writing out the equations.)

Given G consider $Cl(V_+)$, $a_+ \cdot a_+ = (a_+, a_+)$ for $a_+ \in V_+$.

Fix an irreducible representation for $Cl(V_+)$: S_+. Given $D \in \mathcal{D}^0(G, \varphi)$:

(1) $D^{\delta_+}_- \in V_-^* \otimes End((S_+) \supset V_-^* \otimes \Lambda^2 V_+ \ni D^+_-$.

(2) *An operator like Dirac operator:*

$$\begin{aligned} \underline{D}^+ : S_+ &\longrightarrow S_+ \\ \zeta &\longmapsto \sum_{j=1}^{\dim V} e_j^+ \cdot D_{e_j^+}\zeta. \end{aligned}$$

Lemma 11.5. *D^{S_+} and $\underline{D}^+$ are independnt of $D \in \mathcal{D}^0(G, \varphi)$.*

Definition 11.6. (G, φ, ζ), $\varphi \in S_+$ satisfies *Killing spinor equations* if

$$KSE \qquad \underbrace{D_-^{S_+}\zeta = 0,}_{\text{Gravitino Eq.}} \qquad \underbrace{\underline{D}^+\varphi = 0}_{\text{Dilatino Eq.}}$$

Expectation.

(1) $\mathcal{M} = \{(G, \varphi, \zeta) : KSE\}/\mathfrak{g}$ *special metric.*
(2) *Given a solution* (G, φ, ζ) *then*

$$V_{(g,\varphi,\zeta)} \hookrightarrow V^k(\mathfrak{g}).$$

(Where I think $V^k(\mathfrak{g})$ *is the Kac-Moody affinization.)*

Remark 11.7. In the case $TK \oplus T^*K : \mathcal{M}$ 2-stack. See [Bursztyn].

Lemma 11.8. *Associated to* (G, φ) *there are well-defined "Ricci tensors"* $Ric^+ \in V_- \otimes V_+$, $Ric^- \in V_+ \otimes V_-$.

$$Ric^+(a_-, b_+) = Tr(C_+ \to R_D(c_+, a_-)b_+)$$

$$R_D(a, b) = [D_a, D_b]_c - D_{[a,b]}c, \qquad D \in \mathcal{D}^0(G, \varphi).$$

Remark 11.9. In geometric setup, the vanishing of the Ricci corresponds to the motion equations of some physical supersymmetry theory.

Proposition 11.10. *If* (G, φ, ζ) *is solution of KSE, then* $Ric^+_{G,\phi} = 0$. *If* $[\varphi, G] = 0 \implies Ric^-_{G,\varphi} = 0$.

Proof. $\mathrm{Ric}^+(a-, -)\cdot\varphi = [\underline{D}^+, D^{S_+}_{a_-}\zeta - D^{S_+}_{D_+a_-}\zeta = 0$. $[\varphi, G] = 0 \implies \mathrm{Ric}^+(a_-, b_+) = \mathrm{Ric}^-(b_+, a_-)$. ☐

Generalized Ricci flow. *Finding solutions to these equations. Evaluating by*

$$G_t^{-1}\partial_t G_t = -2(Ric^+ - Ric^-)$$

$\mathrm{Hom}(V_+, V_-) \oplus \mathrm{Hom}(V_-, V_+)$. $GG + G - G = 0$, $G = Id_{V_+} - Id_{V_-}$.

Exercise 11.11. (1) Prove STE for GRF.
(2) Assuming there exists a solution of KSE, prove long time existence and convergence. (See theorem by Streets, Jordan, GF; and a book by Streets, GF.)

Remark 11.12. For physicists the generalized Ricci flow (GRF) is the GRF of a $2d$ σ-model with target a compact Lie group K, $\mathfrak{g} = R \oplus R^*$

Let's explain something in this situation. consider a compact Lie group K *and* $\mathfrak{g} = R \oplus R^*$. *A generalized metric:* $\mathfrak{g} = V_+ \oplus V_-$, $(-,-)|_{V_\pm}$ *non-degenerate,* $(-,-)|_{V_+} > 0$. $V_+ = \mathcal{C}^b\{X + g(X)\}$, $X \in R$, $g \in S^2(R^*)$, $b \in \Lambda^2 R^*$, $H = H_0 + \overline{\partial}$.

Case $\dim V_+ = 2n$**.** *Complex pure spinor* φ *on* V_+ *is equivalent* $V_+ \otimes \mathbb{C} = \ell \oplus \overline{\ell}$, *where* $\ell = \{0 \in V_+ \oplus \mathbb{C} : 0 \cdot \zeta = 0\}$. $(\ell, \ell) = 0$, $(\overline{\ell}, \overline{\ell}) = 0$.

This gives a complex structure $J_\zeta : V_+ \to V_+$, $J_\zeta = iId_\ell - iId_{\overline{\ell}}$.

Lemma 11.13. (G, φ, ζ) *with* ζ *pure satisfies KSE if and only if*

(1) $[\ell, \ell] \subset \ell$.
(2) *Take bars* $\{\varepsilon_j, \overline{\varepsilon}_j\}_{j=0}^n$, $\varepsilon_j \in \ell$, $\overline{\varepsilon}_j \in \overline{\ell}$ $(\varepsilon_j, \overline{\varepsilon}_j) = s_{j,k}$. $\sum_{j=1}^n [\varepsilon_j, \overline{\varepsilon}_j = -J_\zeta \varphi_+$.

(This sum is a moment map condition.)

Consider $\mathcal{L} = \{\ell \subset \mathfrak{g} \otimes \mathbb{C} : \dim \ell = n, \ell \cap \overline{0}, (-,-)|_{\ell \oplus \overline{\ell}} \text{non-degenerate}\}$. *This is a complex manifold.* $T_\ell \mathcal{L} \cong \mathrm{Hom}(\ell, \overline{\ell} \oplus V_- \oplus \mathbb{C})$, $V_- \otimes \mathbb{C} = (\ell \oplus \overline{\ell})^\perp$ *Pick a vector* $\dot{\ell} = \dot{J} + \dot{G} \in \mathrm{Hom}(\ell, \overline{\ell} \oplus V_- \oplus \mathbb{C})$ *That is,* $\dot{J} \in \mathrm{Hom}(\ell, \overline{\ell})$, $\dot{J} : V_+ \to V_+$, $\dot{J}J + J\dot{J} = 0$. *(Deformations of the complex structure.)* $G \in \mathrm{Hom}(V_+, V_-) \oplus \mathrm{Hom}(V_-, V_+)$.

Proposition 11.14 (Romero)**.** *$\mathcal{L}$ has a pseudo-Kähler structure preserved by the $\mathfrak{g}$-action and there exists a moment map*

$$\mu : \mathcal{L} \longrightarrow \mathfrak{g}^*$$
$$\ell \longmapsto \frac{i}{2} \sum_{j=1}^{n} [\varepsilon_j, \overline{\varepsilon}_j], -),$$

which is the quantity we mentioned in Lemma above.

Proof. Use the natural complex structure on the space of complex structures (found by Fujiki),

$$\mathrm{tr}_{V_+}(J\dot{G}_2, \dot{G}_1) - \mathrm{tr}_{V_+}(J\dot{J}_\varepsilon \dot{J}_1).$$

□

Problem: prove that $\mathcal{M} = \{(G, \varphi, \zeta) : \zeta \text{ puse}\}/\mathfrak{g}$ *is a pseudo-Kähler manifold.*

(1) *Pseudo-Kähler.*
(2) *Shifter symplectic stuff.*

Example 11.15. Take $K = \mathrm{SU}(2) \times \mathrm{U}(1) \cong S^3 \times S^1$ and $\mathfrak{g} = R \oplus R^*$. Take generators v_1, v_2, v_3 of SU(2) and v_4 of U(1). We have $[v_2, v_3] = -v_1$, $[v_3, v_1] = -v_2$, $[v_1, v_2] = -v_3$. Put $H_\ell = \ell v^{123}$. $\ell \in \mathbb{R}$. $x, a \in \mathbb{R}_{>0}$.

$$g_{x,a} = \frac{a}{x}\left(\sum_{i=1}^{3} \omega^{\otimes 2} + x^2 (v^4)^{\otimes 2}\right)$$
$$V_+ = \{x + g_{xa}\} \subset \mathfrak{g}$$
$$I_X v_4 = x v_1, \qquad U_X v_2 = v_3, \qquad \varphi = -x v_4$$

Exercise 11.16. If $\ell = \frac{a}{x} \implies$ solution of KSE.

Furthermore $[\varphi_+, \ell] \subset \ell$ *(holomorphic divergence).*

The a parameter is naturally complexified by $b = yv^{23}$. $y + ia = z$.

$$V = log\left(\frac{\omega_{x,n}^2}{v^{1234}}\right) = loga.$$

KSV hyperbolic metric, metric on $\mathbb{H}$.

$\dim V_+ = 7$. *A real spinor on* V_+ *is equivalent to* $\phi \in (\Lambda^3 V_+^*)_{>0}$. *The space* $GL(\mathbb{R}^7) \curvearrowright \Lambda^3(V_+^*)$. *The space of spinors* $S_+ = \mathbb{R}^8 = \mathbb{R}^7 \oplus \mathbb{R}\langle\varphi\rangle$. $\phi(x, y, z) = \langle x \cdot y \cdot z \cdot \zeta, \zeta\rangle$.

Remark 11.17. $v^1, \ldots, v^7 \cong \mathbb{R}^6 \oplus \mathbb{R}$. $\phi = (v^{12} + v^{34} + v^{56}) \wedge v^7 + \mathrm{Re}((v^1 + iv^2) \wedge (v^3 + iv^4) \wedge (v^5 + iv^6))$.

Example 11.18. Take $R = \mathrm{SU}(2) \oplus \mathrm{SU}(2) \oplus \mathbb{R}$. $\mathfrak{g} = R \oplus R^*$. $e, s \in \mathbb{R}$, $H = sv^{123} + \ell v^{456}$. $\phi = \omega \wedge \zeta + \Omega^+$, $\zeta = \sqrt{\varepsilon/\ell} v^7$, $\omega = \sqrt{s\ell}(v^{14} + v^{25} - v^{36})$, $\Omega^+ = \sqrt{s^3} v^{123} + \ell\sqrt{s} v^{156} - \ell\sqrt{s} v^{345}$.

References