ALGEBRA

github.com/danimalabares/stack

Contents

1. Rings

So elementary that this is not in Stacks Project.

Theorem 1.1. *Let R be a ring and $I \subset R$ an ideal. There is a correspondence between ideals of R containing I and ideals of R/I.*

Proof. We show that there are two maps between the set of ideals of R containing I and the ideals of R/I whose compositions are the identities in the corresponding set. One map is the projection to the quotient, and the other map is taking the elements whose equivalence class lies in a given ideal.

Suppose that $\tilde{J}$ is an ideal of R/I. Define $J = \{j \in R : j + I \in \tilde{J}\}$, the set of elements that project to $\tilde{J}$. Then $I \subset J$, because $i + I = I$ is the zero of R/I which is contained in any ideal of R/I. Also notice that J is an ideal of R since for any $r \in R$, $rj + I \in \tilde{J}$ by $\tilde{J}$ being an ideal. Notice that projecting J back to R/I gives J by definition.

Conversely, we project to the quotient. Let J be any ideal of R containing I Let $\tilde{J}$ be the projection to R/I. Then $\tilde{J}$ is an ideal of R/I by J being an ideal of R. Consider the set J' of all elements whose projection lies in $\tilde{J}$. Then $J' = J$ by definition. □

This is the essential tool in proving that R/I is a field when I is a maximal ideal

Lemma 1.2. *Let R be a ring and I a maximal ideal of R. Then R/I is a field.*

Proof. It's easy to prove that a ring is a field if and only if its only ideals are $\{0\}$ and R. By Theorem **??**, the ideals of R/I are in correspondence with ideals of R containing I, but only R is such. □

2. Fields

Just for the record, any simple field extension, i.e. the smallest field in F containing a single element, is isomorphic to either que "rational function field $k(t)/k$" or to one of the field extensions $k[t]/(P)$ with $P \in k[t]$ irreducible. (Stacks Project tag 09G1.)

3. Modules

It looks like most books (at least [**?**] and [**?**]) define a module to be an abelian group along with a certain operation with elements of the ring.

Which secretly just says that

Definition 3.1. Let R be a ring. An *(left) R-module* M is an abelian group such that there exists a (left) ring morphism

$$R \to \mathrm{End}(M).$$

Indeed, the three usual requirements correspond to:

(1) The endomorphism corresponding to $a \in R$ respect the group structure of the group M:
$$a(x + y) = ax + ay,$$
(2) The representation map is a map of rings
$$(a + b)x = ax + by, \qquad (ab)x = a(bx).$$

4. Algebras

There appears to be no definition of "algebra" in Stacks Project. But there is one in [**?**]:

Definition 4.1. If R is a commutative ring, then a *commutative algebra* over R is a commutative ring S together with a ring morphism $R \to S$.

But actually I was expecting that S would be defined as a ring that is also an R-module. So let us note that a morphism of rings $R \to S$ gives a representation $R \to \mathrm{End}(S)$ via left multiplication. But the other way around, given an endomorphism associated to some element in R, how do we assign an element of S so as to produce a map $R \to S$?

5. Finitely-generated and finitely presented algebras

See Stacks Project tag 00F2.

Upshot: a finitely-generated R-algebra S is such that $S \simeq R[x_1, \dots, x_n]/I$ for some ideal $I \subset R$. Finitely-presentedness is when $I = (s_1, \dots, s_k)$.

Fun fact: The second item in the following definition is the algebraic counterpart to an affine algebraic set (variety); i.e. the reason why we hear that "affine varieties are in correspondence with finitely presented k-algebras" (wasn't there a notion of reduced algebra in that phrase…?) And the difference with that and finite type, I think, is that the kernel is finitely generated.

Definition 5.1. Let $R \to S$ be a ring map.

(1) We say $R \to S$ is of *finite type*, or that S *is a finite type R-algebra* if there exist an $n \in \mathbf{N}$ and an surjection of R-algebras $R[x_1, \dots, x_n] \to S$.

(2) We say $R \to S$ is of *finite presentation* if there exist integers $n, m \in \mathbf{N}$ and polynomials $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ and an isomorphism of R-algebras $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \cong S$.

Informally, M is a finitely presented R-module if and only if it is finitely generated and the module of relations among these generators is finitely generated as well. A choice of an exact sequence as in the definition is called a *presentation* of M.

So, probably an "algebra" A over a ring R is when A contains (or, is isomorphic to?) $R[X]$ for some possibly very arbitrary set $X \subset R$.

6. Finite and integral ring extensions

The upshot is: let $\varphi : R \to S$ be a ring map. An element $s \in S$ is integral over R if there is a monic polynomial with coefficients in R that gives zero when you put s instead of the variable.

But it might be convenient to think about finite (or, for humans, finitely-generated) R-modules: x is integral over R if and only if $R[x]$ is a finitely-generated R-module (i.e., not only finitely generated as an algebra but also as a module). See Zvi Rosen's video. The forward implication is: suppose x is integral, then $x^n + r_1 x^{n-1} + \ldots + r_n = 0$, so $x^n = -r_1 x^{n-1} + \ldots + r_n$, which "says" $R[x] \cong \bigoplus_{k=0}^{n-1} R x^k$.

So maybe the details of that are not entirely trivial (Zvi uses some theorem, and Stacks Project does lots of things). But the point is: I think that after applying Spec, the fibers of an integral extension are **finite**.

Ok, now I go to Stacks Project:

Trivial lemmas concerning finite and integral ring maps. We recall the definition.

Definition 6.1. Let $\varphi : R \to S$ be a ring map.

(1) An element $s \in S$ is *integral over* R if there exists a monic polynomial $P(x) \in R[x]$ such that $P^\varphi(s) = 0$, where $P^\varphi(x) \in S[x]$ is the image of P under $\varphi : R[x] \to S[x]$.
(2) The ring map φ is *integral* if every $s \in S$ is integral over R.

Lemma 6.2. *Let $\varphi : R \to S$ be a ring map. Let $y \in S$. If there exists a finite R-submodule M of S such that $1 \in M$ and $yM \subset M$, then y is integral over R.*

Proof. Consider the map $\varphi : M \to M$, $x \mapsto y \cdot x$. By Lemma **??** there exists a monic polynomial $P \in R[T]$ with $P(\varphi) = 0$. In the ring S we get $P(y) = P(y) \cdot 1 = P(\varphi)(1) = 0$. $\square$

Lemma 6.3. *A finite ring map is integral.*

Proof. Let $R \to S$ be finite. Let $y \in S$. Apply Lemma 6.2 to $M = S$ to see that y is integral over R. $\square$

Lemma 6.4. *Let $\varphi : R \to S$ be a ring map. Let $s_1, \ldots, s_n$ be a finite set of elements of S. In this case s_i is integral over R for all $i = 1, \ldots, n$ if and only if there exists an R-subalgebra $S' \subset S$ finite over R containing all of the s_i.*

Proof. If each s_i is integral, then the subalgebra generated by $\varphi(R)$ and the s_i is finite over R. Namely, if s_i satisfies a monic equation of degree d_i over R, then this subalgebra is generated as an R-module by the elements $s_1^{e_1} \ldots s_n^{e_n}$ with

$0 \le e_i \le d_i - 1$. Conversely, suppose given a finite R-subalgebra S' containing all the s_i. Then all of the s_i are integral by Lemma 6.3. □

Lemma 6.5. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is finite,*
(2) *$R \to S$ is integral and of finite type, and*
(3) *there exist $x_1, \ldots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R.*

Proof. Clear from Lemma 6.4. □

Lemma 6.6. *Suppose that $R \to S$ and $S \to T$ are integral ring maps. Then $R \to T$ is integral.*

Proof. Let $t \in T$. Let $P(x) \in S[x]$ be a monic polynomial such that $P(t) = 0$. Apply Lemma 6.4 to the finite set of coefficients of P. Hence t is integral over some subalgebra $S' \subset S$ finite over R. Apply Lemma 6.4 again to find a subalgebra $T' \subset T$ finite over S' and containing t. Lemma **??** applied to $R \to S' \to T'$ shows that T' is finite over R. The integrality of t over R now follows from Lemma 6.3. □

Lemma 6.7. *Let $R \to S$ be a ring homomorphism. The set*

$$S' = \{s \in S \mid s \text{ is integral over } R\}$$

is an R-subalgebra of S.

Proof. This is clear from Lemmas 6.4 and 6.3. □

Lemma 6.8. *Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Let R and S denote the product of the R_i and S_i respectively. Then an element $s = (s_1, \ldots, s_n) \in S$ is integral over R if and only if each s_i is integral over R_i.*

Proof. Omitted. □

Definition 6.9. Let $R \to S$ be a ring map. The ring $S' \subset S$ of elements integral over R, see Lemma 6.7, is called the *integral closure* of R in S. If $R \subset S$ we say that R is *integrally closed* in S if $R = S'$.

In particular, we see that $R \to S$ is integral if and only if the integral closure of R in S is all of S.

Lemma 6.10. *Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Denote the integral closure of R_i in S_i by S_i'. Further let R and S denote the product of the R_i and S_i respectively. Then the integral closure of R in S is the product of the S_i'. In particular $R \to S$ is integrally closed if and only if each $R_i \to S_i$ is integrally closed.*

Proof. This follows immediately from Lemma 6.8. □

Lemma 6.11. *Integral closure commutes with localization: If $A \to B$ is a ring map, and $S \subset A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subset B$ is the integral closure of A in B.*

Proof. Since localization is exact we see that $S^{-1}B' \subset S^{-1}B$. Suppose $x \in B'$ and $f \in S$. Then $x^d + \sum_{i=1,\ldots,d} a_i x^{d-i} = 0$ in B for some $a_i \in A$. Hence also

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} a_i/f^i (x/f)^{d-i} = 0$$

in $S^{-1}B$. In this way we see that $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} (a_i/f_i)(x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f' f_1 \ldots f_d x)^d + \sum\nolimits_{i=1,\ldots,d} f^i (f')^i f_1^i \ldots f_i^{i-1} \ldots f_d^i a_i (f' f_1 \ldots f_d x)^{d-i} = 0$$

for a suitable $f' \in S$. Hence $f' f_1 \ldots f_d x \in B'$ and thus $x/f \in S^{-1}B'$ as desired. □

Lemma 6.12. *Let $\varphi : R \to S$ be a ring map. Let $x \in S$. The following are equivalent:*

(1) *x is integral over R, and*
(2) *for every prime ideal $\mathfrak{p} \subset R$ the element $x \in S_{\mathfrak{p}}$ is integral over $R_{\mathfrak{p}}$.*

Proof. It is clear that (1) implies (2). Assume (2). Consider the R-algebra $S' \subset S$ generated by $\varphi(R)$ and x. Let $\mathfrak{p}$ be a prime ideal of R. Then we know that $x^d + \sum_{i=1,\ldots,d} \varphi(a_i) x^{d-i} = 0$ in $S_{\mathfrak{p}}$ for some $a_i \in R_{\mathfrak{p}}$. Hence we see, by looking at which denominators occur, that for some $f \in R$, $f \notin \mathfrak{p}$ we have $a_i \in R_f$ and $x^d + \sum_{i=1,\ldots,d} \varphi(a_i) x^{d-i} = 0$ in S_f. This implies that S'_f is finite over R_f. Since $\mathfrak{p}$ was arbitrary and Spec(R) is quasi-compact (Lemma **??**) we can find finitely many elements $f_1, \ldots, f_n \in R$ which generate the unit ideal of R such that S'_{f_i} is finite over R_{f_i}. Hence we conclude from Lemma **??** that S' is finite over R. Hence x is integral over R by Lemma 6.4. □

Lemma 6.13. *Let $R \to S$ and $R \to R'$ be ring maps. Set $S' = R' \otimes_R S$.*

(1) *If $R \to S$ is integral so is $R' \to S'$.*
(2) *If $R \to S$ is finite so is $R' \to S'$.*

Proof. We prove (1). Let $s_i \in S$ be generators for S over R. Each of these satisfies a monic polynomial equation P_i over R. Hence the elements $1 \otimes s_i \in S'$ generate S' over R' and satisfy the corresponding polynomial P'_i over R'. Since these elements generate S' over R' we see that S' is integral over R'. Proof of (2) omitted. □

Lemma 6.14. *Let $R \to S$ be a ring map. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*

(1) *If each $R_{f_i} \to S_{f_i}$ is integral, so is $R \to S$.*
(2) *If each $R_{f_i} \to S_{f_i}$ is finite, so is $R \to S$.*

Proof. Proof of (1). Let $s \in S$. Consider the ideal $I \subset R[x]$ of polynomials P such that $P(s) = 0$. Let $J \subset R$ denote the ideal (!) of leading coefficients of elements of I. By assumption and clearing denominators we see that $f_i^{n_i} \in J$ for all i and certain $n_i \geq 0$. Hence J contains 1 and we see s is integral over R. Proof of (2) omitted. □

Lemma 6.15. *Let $A \to B \to C$ be ring maps.*

(1) *If $A \to C$ is integral so is $B \to C$.*
(2) *If $A \to C$ is finite so is $B \to C$.*

Proof. Omitted. □

Lemma 6.16. *Let $A \to B \to C$ be ring maps. Let B' be the integral closure of A in B, let C' be the integral closure of B' in C. Then C' is the integral closure of A in C.*

Proof. Omitted. □

Lemma 6.17. *Suppose that $R \to S$ is an integral ring extension with $R \subset S$. Then $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. We have to show $\mathfrak{p}S_{\mathfrak{p}} \neq S_{\mathfrak{p}}$, see Lemma **??**. The localization $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is injective (as localization is exact) and integral by Lemma 6.11 or 6.13. Hence we may replace R, S by $R_{\mathfrak{p}}$, $S_{\mathfrak{p}}$ and we may assume R is local with maximal ideal $\mathfrak{m}$ and it suffices to show that $\mathfrak{m}S \neq S$. Suppose $1 = \sum f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$ in order to get a contradiction. Let $R \subset S' \subset S$ be such that $R \to S'$ is finite and $s_i \in S'$, see Lemma 6.4. The equation $1 = \sum f_i s_i$ implies that the finite R-module S' satisfies $S' = \mathfrak{m}S'$. Hence by Nakayama's Lemma **??** we see $S' = 0$. Contradiction. □

Lemma 6.18. *Let R be a ring. Let K be a field. If $R \subset K$ and K is integral over R, then R is a field and K is an algebraic extension. If $R \subset K$ and K is finite over R, then R is a field and K is a finite algebraic extension.*

Proof. Assume that $R \subset K$ is integral. By Lemma 6.17 we see that $\mathrm{Spec}(R)$ has 1 point. Since clearly R is a domain we see that $R = R_{(0)}$ is a field (Lemma **??**). The other assertions are immediate from this. □

Lemma 6.19. *Let k be a field. Let S be a k-algebra over k.*

(1) *If S is a domain and finite dimensional over k, then S is a field.*
(2) *If S is integral over k and a domain, then S is a field.*
(3) *If S is integral over k then every prime of S is a maximal ideal (see Lemma **??** for more consequences).*

Proof. The statement on primes follows from the statement "integral + domain $\Rightarrow$ field". Let S integral over k and assume S is a domain, Take $s \in S$. By Lemma 6.4 we may find a finite dimensional k-subalgebra $k \subset S' \subset S$ containing s. Hence S is a field if we can prove the first statement. Assume S finite dimensional over k and a domain. Pick $s \in S$. Since S is a domain the multiplication map $s : S \to S$ is surjective by dimension reasons. Hence there exists an element $s_1 \in S$ such that $ss_1 = 1$. So S is a field. □

Lemma 6.20. *Suppose $R \to S$ is integral. Let $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}(S)$ be distinct primes having the same image in $\mathrm{Spec}(R)$. Then neither $\mathfrak{q} \subset \mathfrak{q}'$ nor $\mathfrak{q}' \subset \mathfrak{q}$.*

Proof. Let $\mathfrak{p} \subset R$ be the image. By Remark **??** the primes $\mathfrak{q}, \mathfrak{q}'$ correspond to ideals in $S \otimes_R \kappa(\mathfrak{p})$. Thus the lemma follows from Lemma 6.19. □

Lemma 6.21. *Suppose $R \to S$ is finite. Then the fibres of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ are finite.*

Proof. By the discussion in Remark **??** the fibres are the spectra of the rings $S \otimes_R \kappa(\mathfrak{p})$. As $R \to S$ is finite, these fibre rings are finite over $\kappa(\mathfrak{p})$ hence Noetherian by Lemma **??**. By Lemma 6.20 every prime of $S \otimes_R \kappa(\mathfrak{p})$ is a minimal prime. Hence by Lemma **??** there are at most finitely many. □

Lemma 6.22. *Let $R \to S$ be a ring map such that S is integral over R. Let $\mathfrak p \subset \mathfrak p' \subset R$ be primes. Let $\mathfrak q$ be a prime of S mapping to $\mathfrak p$. Then there exists a prime $\mathfrak q'$ with $\mathfrak q \subset \mathfrak q'$ mapping to $\mathfrak p'$.*

Proof. We may replace R by $R/\mathfrak p$ and S by $S/\mathfrak q$. This reduces us to the situation of having an integral extension of domains $R \subset S$ and a prime $\mathfrak p' \subset R$. By Lemma 6.17 we win. $\square$

The property expressed in the lemma above is called the "going up property" for the ring map $R \to S$, see Definition **??**.

Lemma 6.23. *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. Then M is finitely presented as an R-module if and only if M is finitely presented as an S-module.*

Proof. One of the implications follows from Lemma **??**. To see the other assume that M is finitely presented as an S-module. Pick a presentation

$$S^{\oplus m} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

As S is finite as an R-module, the kernel of $S^{\oplus n} \to M$ is a finite R-module. Thus from Lemma **??** we see that it suffices to prove that S is finitely presented as an R-module.

Pick $y_1, \ldots, y_n \in S$ such that $y_1, \ldots, y_n$ generate S as an R-module. By Lemma 6.2 each y_i is integral over R. Choose monic polynomials $P_i(x) \in R[x]$ with $P_i(y_i) = 0$. Consider the ring

$$S' = R[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n))$$

Then we see that S is of finite presentation as an S'-algebra by Lemma **??**. Since $S' \to S$ is surjective, the kernel $J = \mathrm{Ker}(S' \to S)$ is finitely generated as an ideal by Lemma **??**. Hence J is a finite S'-module (immediate from the definitions). Thus $S = \mathrm{Coker}(J \to S')$ is of finite presentation as an S'-module by Lemma **??**. Hence, arguing as in the first paragraph, it suffices to show that S' is of finite presentation as an R-module. Actually, S' is free as an R-module with basis the monomials $x_1^{e_1} \ldots x_n^{e_n}$ for $0 \leq e_i < \deg(P_i)$. Namely, write $R \to S'$ as the composition

$$R \to R[x_1]/(P_1(x_1)) \to R[x_1, x_2]/(P_1(x_1), P_2(x_2)) \to \ldots \to S'$$

This shows that the ith ring in this sequence is free as a module over the $(i-1)$st one with basis $1, x_i, \ldots, x_i^{\deg(P_i)-1}$. The result follows easily from this by induction. Some details omitted. $\square$

Lemma 6.24. *Let R be a ring. Let $x, y \in R$ be nonzerodivisors. Let $R[x/y] \subset R_{xy}$ be the R-subalgebra generated by x/y, and similarly for the subalgebras $R[y/x]$ and $R[x/y, y/x]$. If R is integrally closed in R_x or R_y, then the sequence*

$$0 \to R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \to 0$$

is a short exact sequence of R-modules.

Proof. Since $x/y \cdot y/x = 1$ it is clear that the map $R[x/y] \oplus R[y/x] \to R[x/y, y/x]$ is surjective. Let $\alpha \in R[x/y] \cap R[y/x]$. To show exactness in the middle we have to prove that $\alpha \in R$. By assumption we may write

$$\alpha = a_0 + a_1 x/y + \ldots + a_n (x/y)^n = b_0 + b_1 y/x + \ldots + b_m (y/x)^m$$

for some $n, m \geq 0$ and $a_i, b_j \in R$. Pick some $N > \max(n, m)$. Consider the finite R-submodule M of R_{xy} generated by the elements

$$(x/y)^N, (x/y)^{N-1}, \ldots, x/y, 1, y/x, \ldots, (y/x)^{N-1}, (y/x)^N$$

We claim that $\alpha M \subset M$. Namely, it is clear that $(x/y)^i(b_0 + b_1 y/x + \ldots + b_m(y/x)^m) \in M$ for $0 \leq i \leq N$ and that $(y/x)^i(a_0 + a_1 x/y + \ldots + a_n(x/y)^n) \in M$ for $0 \leq i \leq N$. Hence α is integral over R by Lemma 6.2. Note that $\alpha \in R_x$, so if R is integrally closed in R_x then $\alpha \in R$ as desired. $\square$

References