A TOUR THROUGH ALGEBRAIC SURFACES

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Notes at github.com/danimalabares/stack

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2. References

(1) A. Beauville. Complex algebraic surfaces.
(2) R. Miranda. Overview of algebraic surfaces.
(3) M. Reid. Chapters on algebraic surfaces.

3. Introduction

Everything will be over the complex numbers. By "surface" we mean "algebraic surface", but we drop the term "algebraic". Informally, a surface is a smooth projective algebraic variety of dimension 2. We may think of surfaces also as compact connect complex manifolds of dimension 2.

$\mathcal{M}(S)$, meromorphic functions on S, has transcendence degree 2 over $\mathbb{C}$. That is, for $p, q \in S$ there exists $f \in \mathcal{M}(S)$ such that $f(p) \neq f(q)$ and for all $p \in S$ there exists $f, g \in \mathcal{M}(S)$ such that (f, g) give local coordinates at p.

Remark 3.1. The remark on transcendence just made implies that S is smooth algebraic of dimension 2. Indeed, compactness implies that there exists meromorphic functions $\varphi_1, \ldots, \varphi_n \in \mathcal{M}(S)$ such that $S \xhookrightarrow{alg} \mathbb{P}^n$, $p \mapsto (1 : \varphi_1(p) : \ldots : \varphi_n(p))$ and Chow theorem imply that $S = V(f_1, \ldots, f_k)$ with $f_i \in \mathbb{C}[x_0, x_1, \ldots, x_n]$.

Example 3.2.
(1) $S = V(f) \subset \mathbb{P}^3$ with $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ homogeneous of degree d. For example, $f = x_0^d + x_1^d + x_2^d + x_3^d$. Take $d = 1$, then we can assume that $f = x_i$, and then $S \simeq \mathbb{P}^2$. If $d = 2$ we can assume that $f = x_0x_1 - x_2x_3$ and one can verify that $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. If $d = 3$, $S \simeq Bl_6\mathbb{P}^2$. These are all rational surfaces.
If $d = 4$, S is a K3 surface.
(2) (Complete intersections of type $(d_1, \ldots, d_{n-1})$ in $\mathbb{P}^n$.) Let $f_1, \ldots, f_{n-2} \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ homogeneous of degrees $d_1, \ldots, d_{n-2}$. Then $S = V(f_1, \ldots, f_{n-2})$.
E.g., intersections of type $(2, 3)$ in $\mathbb{P}^4$ or of type $(2, 2, 2)$ in $\mathbb{P}^5$, which are K3 surfaces.
(3) (Ruled surfaces.) Take any curve C and consider $C \times \mathbb{P}^1$ (or anything birational to it). [This is interesting because we can consider the projection to C, and we get a fibration.]

4. Curves on surfaces (divisors)

We will use: $\mathrm{Div}(S)/\sim \simeq \mathrm{Pic}(X) = H^1(S, \mathcal{O}_S^*)$ where $\sim$ is linear equivalence.

$D \subset S$ is a divisor if and only if D is one of the following:

(1) (Cartier divisor.) [I can think of D given locally as the zero locus of some function.] $D : f = 0$ for $0 \neq f \in \mathcal{M}(S)$ + gluing condition. This is equivalent to saying that $D|_f = \mathrm{div}(f)$.
(2) (Weil divisors.) $D = \sum n_i C_i$ for $n_i \in \mathbb{Z}$, $C_i \subset S$ irreducible curves, where there are only finitely many nonzero coefficients n_i.
(3) (Line bundle.) D effective (i.e. $n_i \geq 0$), we associate $\mathcal{O}_S(D)$, a holomorphic line bundle on S along with a nonzero section [whose zero locus defines D. Recall that $U \mapsto \mathcal{O}_S(D)(U)$ is "given" by $g(z, w)$ such that fg is holomorphic where $D|_U = \mathrm{div}(f)$. And we extend by linearity.

[And we say that two divisors are linearly equivalent if their difference is $div(f)$ for some f.]

Example 4.1. (1) $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$ generated by $\mathcal{O}_{\mathbb{P}^2}(1)$, the dual of the tautological bundle, or, equivalently, the class of a hyperplane H, whose sections are polynomials of degree 1 (in the appropriate number of variables).

(2) $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n}(1)$. We have $\mathcal{O}_{\mathbb{P}^n}(d) := \mathcal{O}_{\mathbb{P}^n}^{\otimes d}$. For $n = 2$, the sections of $\mathcal{O}(d)$ are plane curves of degree 2.

Suppose that $\pi : S \hookrightarrow \mathbb{P}^n$. The *hyperplane class* is $H|_S := \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. We denote $\mathcal{O}_S(1) = \mathcal{O}_S(H|_S)$. This is always a non-trivial class.

We also have the *canonical class*, which is the class of any divisor K such $\mathcal{O}_S(K) = \Omega_S^2$, the sheaf of holomorphic 2-forms, informally its elements look like $f(z, w)dx \wedge dw$. We denote this as K, K_S or ω_S.

Example 4.2 (Canonical class of projective plane)**.** Take coordinates $Z = X_1/X_0$ and $W = X_2/X_0$ on $U_0 : X_0 \neq 0$. Likewise put $U = X_0/X_1$, $V = X_2/X_1$ at U_1. Then $dZ \wedge dW = -U^{-3}dU \wedge dV$. Thus, the canonical class of $\mathbb{P}^2$ is linearly equivalent to $-3H$. In general, $K_{\mathbb{P}^n} \sim -(n + 1)H$.

In practice, we can use this to compute K_S for any $S \hookrightarrow \mathbb{P}^n$ via $\omega_S = \omega_{\mathbb{P}^n} \otimes \Lambda^{n-2}N_S/\mathbb{P}^n$.

5. Numerical invariants

- (Hodge decomposition.)

$$H^k(S, \mathbb{C}) = \bigoplus_{p+q=k} \underbrace{H^q(S, \Omega_S^p)}_{H^{p,q}(S)}.$$

Setting $h^{p,q} = \dim H^{p,q}$ + symmetries, we have the Hodge diamond

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{0,1} & & h^{1,0} & \\ h^{0,2} & & h^{1,1} & & h^{2,0} \\ & h^{1,2} & & h^{2,1} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

where we call q the *irregularity* p_g the *geometric genus* and the *Euler characteristic* is the alternating sum of Betti numbers. Notice that $h^{1,1} \geq 1$.

6. The Néron-Severi group and Lefschetz (1,1) theorem

There is always a map

$$\begin{aligned} c_1 : \mathrm{Pic}(S) &\longrightarrow H^2(S, \mathbb{Z}) \overset{\substack{\text{Poincaré} \\ \text{dual}}}{\cong} H_2(S, \mathbb{Z}) \\ \mathcal{L} &\longmapsto c_1(\mathcal{L}) \end{aligned}$$

We define $\mathrm{NS}(X) := \mathrm{Im} c_1/\text{torsion}$. It is a finitely generated abelian group isomorphic to $\mathbb{Z}^\rho$ where ρ is the Picard rank.

Theorem 6.1.

$$NS(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$$

$NS(S)$, $Div(S)/\sim$ and $Pic(S)$ have more structure! There exists a unique bilinear form

$$\begin{aligned} \cdot : Div(S) \times Div(S) &\longrightarrow \mathbb{Z} \\ (C, D) &\longmapsto C \cdot D \end{aligned}$$

such that

(1) *If C and D are smooth curves which intersect transversally, then $C.D = \# C \cap D$.*
(2) *If $C \sim C'$, then $C \cdot D = C' \sim D$.*

Definition 6.2. $C.D = \mathcal{O}_S(C).\mathcal{O}_S(D) = \langle c_1(\mathcal{O}_S(D) \cup c_1(\mathcal{O}_S(D)), [S]_{\text{fund}}\rangle$, and we say that $C.C = \deg \mathcal{N}_{C/S}$.

[An important invariant of a surface is the self-intersection number of the canonical class.] Particular case: $K^2 = c_1^2$ where c_1^2 is the first Chern number.

Example 6.3. (1) $K^2_{\mathbb{P}^2} = (-3H).(-3H) = 9$.
(2) (Degree of S.) Let $S \hookrightarrow \mathbb{P}^n$. Recall our notation above for $\mathcal{O}_S(1).\mathcal{O}_S(1)$. If $n = 3$, the degree d of S is the degree of $\mathcal{N}_{S/\mathbb{P}^3}$. Thus the adjunction becomes $\omega_S = \mathcal{O}_S(d-4$
(3) C smooth curve, then $f : S \twoheadrightarrow C$, F fiber, then $F^2 = 0$.
(4) $g = \tilde{S} \xrightarrow{d} D$, then $D_1, D_2 \in \text{Div}(S) \implies g^*(D_1) \cdot g^*(D_2) = d(D_1.D_2)$.

Definition 6.4. A divisor $D \subset S$ is *nef* if and only if for each $C \subset S$ irreducible, $D \cdot C \geq 0$. We say that D is ample if and only if $D^2 > 0$ and for each irreducible $C \subset S$ we have $D.C \geq 0$.

7. Big theorems

Theorem 7.1. *Let S be a surface.*

(1) *(Noether's formula.)*

$$12\chi = K^2 + e = c_1^2 + c_2$$

where e is the Euler number.
(2) *(Riemann-Roch.) Given $D \in Div(S)$,*

$$\chi(\mathcal{O}_S(D)) = \frac{D.(D-K)}{2} + \chi(\mathcal{O}_S).$$

(3) *(Genus formula.) Given $C \subset S$ smooth,*

$$2g - 2 = C.(C + K).$$

Example 7.2. (1) For $\mathbb{P}^2$, $K^2 = 9$ and $e = 3$. Thus $12\chi = 12 \implies \chi = 1$. The irregularity is $q = H^0(\mathbb{P}^2, \Omega^1) = 0$. The arithmetic genus is $p_g =$

$\dim H^0(\mathbb{P}^2, \Omega^2) = 0$. Thus, the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(2) Any rational surface has a Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & \rho & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The Noether formula gives $12 = K_S^2 + \rho + 2 \implies K_S^2 = 10 - \rho$.

(3) $C \subset \mathbb{P}^2$ smooth of degree d, then $2g - 2 = d^2 - 3d$ and $g = \frac{d^2-3d+2}{2} = \frac{(d-1)(d-2)}{2} = \binom{d-1}{2}$. In particular, if $d = 1$ or 2, then $g = g(C) = 0$ if $d = 3$ then $g(C) = 1$.

8. Birational maps and classification theorem

Question: given two surfaces S and S', when are S and S' birational/biregular or isomorphic?

- Rules surfaces (all birational to $C \times \mathbb{P}^1$ for some C.
- Classify minimal (models of) surfaces by looking at the positive of K (and related invariants q, p_g and κ).

K	κ	p_g	q	Structure
$K^2 > 0$	2			General type
$K^2 = 0$	1			Elliptic*
$K = 0$	0	1	2	abelian
		0	1	hyperelliptic
		1	0	K3
		0	0	Enriques
	$-\infty$	0	≥ 1	ruled surfaces
		0	0	rational

The following proposition is also a definition:

Proposition 8.1. *Let S and S' be two surfaces, then the following statements about S and S' are equivalent:*

(1) *S and S' are birational*
(2) *$\mathcal{M}(S) \simeq \mathcal{M}(S')$ as algebras (over $\mathbb{C}$).*
(3) *There exists $U \subset S$, $V \subset S'$ opens such that $U \simeq V$.*

9. The blow up of the projective plane at a point

Consider $p = (0:0:1)$ and

$$\begin{aligned} \varphi_p : \mathbb{P}^2 \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\longmapsto (x_0 : x_1) \\ q &\longmapsto L_{pq} \end{aligned}$$

Consider also

$$\Gamma := \left\{ ((x_0 : x_1 : x_2), (y_0 : y_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 : \det \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} = 0 \right\}.$$

By construction,

$$\pi : \Gamma \to \mathbb{P}^2$$

is an isomorphism over $U = \mathbb{P}^2 \setminus \{p\}$ and $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1 \simeq \mathbb{P}^1 := E$.

Definition 9.1. $\pi : \Gamma \to \mathbb{P}^2$ as above is called the *blowup* of $\mathbb{P}^2$ at p (also Γ). The curve E is called the *exceptional divisor.*

Note that we have the following local description (chart $x_2 = 1$): $\{(x, \ell) \in \mathbb{C}^2 \times \mathbb{P}^1 : x \in \ell\} \to \mathbb{C}^2$. That is, the exceptional divisor may be identified with the zero section of the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, which means that the self intersection of E is 1, i.e. $E^2 = 1$.

Remark 9.2. (1) The surface $Bl_p\mathbb{P}^2 := \Gamma$ is the Hirzebruch surface $\mathbb{F}_1$ (or Σ_1).
(2) Choosing local coordinates, we can talk about Bl_pS for $p \in S$.
(3) In fact, $\pi : Bl_pS \to S$ is characterized by the following universal property: if $f : \tilde{S} \to S$ is birational such that f^{-1} is not defined at $p \in S$, then f factorizes as $f : \underbrace{\tilde{S} \xrightarrow{g} \hat{S}}_{\text{bir}} := Bl_pS \xrightarrow{\varepsilon} S$ where $\varepsilon^{-1}(S \setminus \{p\}) \simeq S \setminus \{p\}$ and $\varepsilon^{-1}(p) \simeq \mathbb{P}^1$.

Proposition 9.3. *Consider $\pi : \tilde{S} = Bl_pS \to S$ as above, with exceptional divisor E,*

1. $E^2 = -1$,
2. $E \cdot \pi^* D = 0$ *for all* $D \in Div(S)$,
3. *If $C \subset S$ has mutliplicity m at p, then $\tilde{C} := \overline{\pi^{-1}(C) \setminus E} \subset \tilde{S}$ satisfies $\tilde{C} \cdot E = m$ and $\tilde{C}^2 = C^2 - m^2$*
4. $K_{\tilde{S}} = \pi^* K_S + E \implies K_{\tilde{S}}^2 = K_S^2 - 1$
5. *q, p_g and χ are the same for $\tilde{S}$ as for S*
6. *e, $h^{1,1}$ and b_2 all go up by 1*
7. $Pic(\tilde{S}) = \pi^*(Pic(S)) \oplus \mathbb{Z}E$.

10. Castelnuovo's criterion

[It turns out that if you find a surface with a -1 curve, then that surface is the blow up of another!]

Theorem 10.1. *If on a surface $\tilde{S}$ one finds a curve $E \simeq \mathbb{P}^1$ with $E^2 = -1$, then $\tilde{S} = Bl_pS$ for some $p \in S$.*

Definition 10.2. A surface S is *minimal* if it has no (-1)-curves (i.e. a curve $\simeq \mathbb{P}^1$ with self-intersection -1).

(If and only if $f : S \dashrightarrow \tilde{S}$ and S minimal, then f is an isomorphism.)

11. Mori's point of view

If S is a surface such that K_S is not nef [so, by definition of nef there must exist an irreducible curve that intersects the canonical divisor negatively, and this curve in fact is a (-1) curve:] then there exists a (-1)-curve on S.

12. Kodaira dimension

Let S be a surface.

Definition 12.1. For each $n \geq 0$, we define the *pluri-genera*

$$P_n := \dim H^0(S, \mathcal{O}_S(nK))$$

and the *Kodaira dimension*

$$\kappa := \begin{cases} -\infty & \text{if } P_n = 0 \forall n \\ \text{smallest } k \text{ such that } P_n = O(n^k) & \text{otherwise} \end{cases}$$

Remark 12.2. (By Daniel.) In one of Misha's course we have that the function $\dim H^0(nK)$ is known to be polynomial for all projective varieties (Birkar, Cascini, Hacon, McKernan). Conjecturally it is always polynomial. Thus, the definition of Kodaira dimension is nothing more than the degree of this polynomial.

Further, in [**?**, Definition 2.2.26] we see that this should be equivalent to defining it as the transcendence degree over $\mathbb{C}$ of the fraction field of the ring $\bigoplus_{m\geq 0} H^0(X, K_X^{\oplus m})$, which is endowed with a natural product which, in general for vector bundles E and F, maps $H^0(X,E) \otimes H^0(X,F) \to H^0(X, E \otimes F)$.

13. Results of Mori's theory

Theorem 13.1. *Given a surface S, there exists a chain*

$$S = S^0 \xrightarrow{\sigma_1} S^1 \to \cdots \xrightarrow{\sigma_N} S_N = S'$$

where each σ_i is the contraction of a (-1) curve E_i and S' satisfies

(1) *$K_{S'}$ is nef or*
(2) *S' is a $\mathbb{P}^1$-bundle over a curve or*
(3) *$S' \simeq \mathbb{P}^2$.*

Theorem 13.2. *If a surface S is such that K_S is not nef then there exists $\varphi : S \to X$, with $\dim X = 0, 1$ or 2, contracting at least one curve to a point, such that $-K_S \cdot C > 0$ for every curve C contained in a fiber of φ.*

Note:

$$\dim X = \begin{cases} 0 \rightsquigarrow & -K \text{ is ample} \\ 1 \rightsquigarrow & S \text{ is a conic bundle} \\ 2 \rightsquigarrow & S = Bl_{p_1, \ldots, p_n}. \end{cases}$$

14. Examples and the cubic surface revisited

(1) Slogan: "blowups resolve singularities". Consider the plane cubic

$$C = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : \underbrace{x_1^2 x_2 - x_0^2(x_0 + x_2) =}_{:= f_C} 0\},$$

the point $p = (0:0:1)$ and $\pi : Bl_p\mathbb{P}^2 \to \mathbb{P}^2$, then

$$\pi^{-1}(C) = \begin{cases} f_C = 0 \\ x_0y_0 = y_0x_1 \end{cases}$$

[This curve has a simple node. In the blowup, the curve will intersect the exceptional divisor in two points — we have "separated" the singularity into two points]

(2) $Bl_1\mathbb{P}^1 \times \mathbb{P}^1 \simeq Bl_1\mathbb{F}_1 \simeq Bl_2\mathbb{P}^2$. [Consider two curves L_1 and L_2 on $\mathbb{P}^1 \times \mathbb{P}^1$. Their strict transforms along with the exceptional divisor become (-1) curves. Then contract $\tilde{L}_1$, this gives $\mathbb{F}_1$, the Hirzebruch surface, which is just the blowup at a point of $\mathbb{P}^2$. $\mu(\tilde{L}_2) = -1$, $\mu(E \setminus \tilde{L}_1) = 0$.]

(3) Consider two cubics C_1 and C_2. They intersect in 9 points. Do

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ p &\longmapsto (C_1(p) : C_2(p)) \end{aligned}$$

Let $S = Bl_{p_1,...,p_9}\mathbb{P}^2$. Then

  (a) $K_S^2 = 0$ [Because K^2 of $\mathbb{P}^2$ is 9, and for every blowup we substract 1!]
  (b) $\mathrm{Pic}(S)$ "=" $\pi^*\mathrm{Pic}(\mathbb{P}^2) \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_q$.
  (c) $-K_S = -\pi^*(K_{\mathbb{P}^2}) - E_1 - \ldots - E_q$ is nef. (In fact, this is the class of a fiber of $S \to \mathbb{P}^1$.)

(4) (Continuation of previous item.) $-K_S$ nef + genus formula $\implies C^2 \geq -2$ for every $C \subset S$ irreducible and rational. This is an elliptic surface!

[We have taken a 1-dimensional linear system. But why not take a higher-dimensional linear system?]

(5) Fix $p_1, \ldots, p_6$ in $\mathbb{P}^2$ in general position (not three in a line, not six in a conic). Consider $f_1, f_2, f_3, f_4 \in \mathbb{C}[x_0, x_1, x_2]$ homogeneous of degree 3 vanishing at p_i and consider

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\ p &\longmapsto (f_1(p) : f_2(p) : f_3(p) : f_4(p)). \end{aligned}$$

The resolved morphism $Bl_6\mathbb{P}^2 \to \mathbb{P}^3$ is an embedding with image of degree 3 [a cubic surface on $\mathbb{P}^3$].

15. Elliptic surfaces

Definition 15.1. An *elliptic surface* is a surface S together with $f : S \to C$ holomorphic (with connected fibers) from S to a smooth curve C such that the generic fiber is a smooth curve of genus 1.

Moreover, we assume that there do not exist (-1)-curves on fibers of f.

Warning: I don't assume the existence of a section [a distinguished point on the fibers], and I'm not saying that all fibers are smooth.

Singular fibers have been classified by Kodaira: $I_0, I_1, (m)I_n, II, IV, IV^* = \tilde{E}_6, III^* = \tilde{E}_7, II^* = \tilde{E}_8$ and I_n.

Example 15.2.

$$\left\{ \begin{matrix} \text{pencils} \\ \text{of cubics} \\ \text{w/ smooth} \\ \text{member} \end{matrix} \right\} \leftrightsquigarrow \left\{ \begin{matrix} \text{RES} \\ \text{w/ section} \end{matrix} \right\}.$$

$$\mathcal{P} \mapsto S_{\mathcal{P}}.$$

Question: given $\mathcal{P}$ and $S_{\mathcal{P}}$ how many singular fibers does $S_{\mathcal{P}}$ have?

Lemma 15.3. *The discriminant locus in $\mathbb{P}^9$ [this $\mathbb{P}^9$ parametrizes cubics in $\mathbb{P}^2$] has degree 12.*

Note that the Hodge diamond $S_{\mathcal{P}}$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(Notice that adding up the center column gives 12.)

Example 15.4. [We take a conic C and a triple line $3L$ (it has to be triple so that it is a pencil of conics) which is tangent to C at an inflection point. Then we blow up the point of intersection 9 times, each time picking a point in the intersection of the strict transform of the original cubic and the triple line.]

Blowing up 9 times give a surface $S_{\mathcal{P}}$ with a II^* fiber.

Question: How is Kodaira's classification obtained?

(1) If $F = \sum n_i C_i$ is singular fiber, then the intersection form restricts to $\operatorname{span}(\{C_i\})$. This restriction is negative semi-definite with a one-dimensional kernel spanned by F.
(2) If F is irreducible and $p_a(F) = 1$, then F =smooth,nodal or cuspidal. If F is reducible, then each C_i is a (-2)-curve.

This two items combined imply that the dual graph [vertices are curves,

two vertices joined if they intersect] is a Dynkin diagram (ADE type) [by a result of graph theory].

16. The Weistrass model

Let $S \to C$ be an elliptic surface with a section. A choice of a section defines a divisor of degree 1 on the generic fiber S_η, i.e. a point p.

Looking at $H^0(S_\eta, np)$ for every n, we get:

$$H^0(p) = \langle 1 \rangle, \qquad H^0(2p) = \langle 1, x \rangle, \qquad H^0(3p) = \langle 1, x, y \rangle,$$
$$\ldots, \qquad H^0(6p) = \left\langle 1, x, y, x^2, xy, x^3, y^2 \right\rangle,$$

so there must exist a relation. Up to further tricks, we get to $y^2 = x^3 + ax + b$.

Making the construction global gives the Weierstrass model.

- Any $S \to C$ is birational to a Weierstrass fibration $W \to C$ (where W can be singular, while C is smooth) where all fibers are irreducible and have $p_a = 1$.

Geometrically, W is obtained from S by contracting all fiber components that do not meet the chosen section

[Once we show that we can do this, it will suffice to study Weierstrass fibrations.]

[From now on W is a Weierstrass fibration.] Given $\pi_i : W \to C$ (with a section σ) consider the fundamental line bundle $\mathcal{L} = (\pi_* N_{\sigma/K})^\vee = (R^1\pi_*\mathcal{O}_W)^\vee$. Then for all $n \geq 2$ we have a splitting $\pi_*\mathcal{O}_W(n\sigma) \simeq \mathcal{O}_C \oplus \mathcal{L}^{-2} \oplus \ldots \oplus \mathcal{L}^{-n}$. In particular, we have a section

$$\pi^*\pi_*\mathcal{O}_W(3\sigma) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}_W(3\sigma)$$

gives a map $f : W \to \mathbb{P}(\pi_*\mathcal{O}_W(3\sigma))$ that exhibits W as a relative cubic in a $\mathbb{P}^2$-bundle over C.

A global equation is given by

$$Y^2Z = X^3 + AXZ^2 + BZ^3$$

where (Z, X, Y, A, B) are interpreted as sections of $(\mathcal{O}_C, \mathcal{L}^2, \mathcal{L}^3, \mathcal{L}^4, \mathcal{L}^6)$. σ is given by $Z = X = 0$.

Remark 16.1. The singular fibers of a minimal resolution of W are determined by the order of vanishing of A, B and $\Delta = 4A^3 + 27B^2$.

Consequences:

(1) $\omega_W = \pi^*(\omega_C \otimes \mathcal{L})$ implies that $K_W^2 = 0$. $e = R\cdot\deg\mathcal{L} =$ number of singular fibers.
(2) $\kappa(W) \leq 1$.
(3) $\deg\mathcal{L} = \chi(W)$, $e = R \cdot \deg\mathcal{L} =$ number of singular fibers.
(4) If $C = \mathbb{P}^1$, then $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(d)$ for some d.

$$\begin{cases} d = 1 \;\leftrightsquigarrow & \text{(general) RES w/section} \\ d = 2 \;\leftrightsquigarrow & \text{K3s, } K_W = 0 \\ 3 \geq 3 \;\leftrightsquigarrow & \kappa = 1 \text{ and } K_W = \alpha F \text{for some } \alpha > 0. \end{cases}$$

Invariants. $\pi : W \to C$. Set $g = g(C)$.

	q	p_g
W not a product	g	$g + \deg\mathcal{L} - 1$
product	$g + 1$	$g + \deg\mathcal{L}$.

We can also compute the pluri-genera P_n and deduce:

- $g = 0$: done.
- $g = 1$:

$$\begin{cases} \mathcal{L} = \mathcal{O}_C & E \times E \\ \mathcal{L} \text{ is torsion of order 2,3,4 or 6 hyp} & \\ \deg(\mathcal{L}) \geq 1 & \kappa = 1 \end{cases}$$

- $g \geq 2$: $\kappa = 1$.

Proposition 16.2. *Let S be a minimal surface with $\kappa = 1$. Then*

(1) $K_S^2 = 0$ *and*
(2) *there exists $f : S \longrightarrow\!\!\!\!\!\rightarrow C$ elliptic fibration.*

Proof.
- S contains a curve $E = \sum n_i C_i$ ($n_i \in \{0, 1\}, \text{supp}(E)$ is connected) with $E \overset{num}{\not\sim} 0$ and $E^2 = K_S E = 0$.
  In fact $K_S \overset{num}{\sim} rE$ for some $r \in \mathbb{Q}$.
- Some multiple of E moves in an elliptic pencil. Some multiple of E is the movable part of $\tilde{r}K_S$ for some $\tilde{r} \in \mathbb{Q}$.

□

17. Examples

- (The Fermat quartic.) Consider $S \subset \mathbb{P}^3$ given by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ and for each $(\lambda : \mu) \in \mathbb{P}^1$, let

$$C_{(\lambda:\mu)} = \begin{cases} \lambda(x_0^2 + \zeta^2 x_1^2 - \mu(x_2^2 - \zeta^2 x_3^2 = 0 \\ \mu(x_0^2 - \zeta^2 x_1) + \lambda(x_2^2 + \zeta^2 x_3^2) = 0 \end{cases}$$

  where $\zeta \in \mathcal{M}_8$ (an eight root of unity), primitive. These curves live in S and

$$\begin{aligned} \pi : S &\longrightarrow \mathbb{P}^1 \\ (x_0 : x_1 : x_2 : x_3) &\longmapsto (x_0^2 - \zeta^2 x_1^2 : -x_2^2 - \zeta x_3^2) \end{aligned}$$

  is an elliptic surface.
- Consider a smooth plane quartic Q and choose a point $p \notin Q$.

$$\begin{tikzcd} S \arrow[d, "Bl_{q_1,q_2}Y \text{ where } \pi^{-1}(p) = \{q_1, q_2\}"] \\ Y \simeq Bl_7\mathbb{P}^2 \arrow[d, "\pi"', "2:1"] \\ \mathbb{P}^2 \supset Q. \end{tikzcd}$$

  where $\pi^{-1}(\ell)$ has $g = 1$.

  [This produces a rational elliptic surface with a section]

Notice that there exist elliptic surfaces with multiple fibers (hence without a section).

Example 17.1. Start with three lines in the plane intersecting pairwise, and a conic intersecting tangentially once every line [Fano plane]. We blow up these three points and obtain a type $2I_3$ fiber.

In general, we take pencils of sextics with nodes at 9 points. These are in correspondence with Halpen surfaces.

18. Halphen surfaces

[These are a class of rational surfaces.]

Proposition 18.1. *Let $\pi : S \to C$ be an elliptic surface with S rational. Then*

(1) $C = \mathbb{P}^1$, $b_2(S) = 10$.
(2) *π has at most one multiple fibre mF (necessarily type mI_n for some n and some m.*
(3) *either π has no multiple fiber and $-K_S \sim F$ (any fiber), or π has one muliple fiber of multiplicity m and $-mK_S \sim S_p$*
(4) *if π has no multiple fibers, there exists a section.*

Proof of 2. and 3. Canonical bundle formula:

$$\omega_S = \pi^*(\omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_S\left(\sum_p (m(p) - 1)S_p)\right). \tag{18.1.1}$$

Then, given any smooth fiber F, for any $n \in \mathbb{N}$, we have $nK_S \sim n\left(\sum_p \frac{m(p)-1}{m(p)} - 1\right)^\alpha F$.

Note that $\alpha < 0$ because no multiple of K_S can be effective. Thus, $m(p) = 1$ except for at most 1 at p.

Why is 18.1.1 true? In general, for any relatively minimal elliptic surface $\pi : S \to C$ with multiple fibers $m_1F_1, \dots, m_kF_k$:

$$\omega_S = \pi^*(\omega_C \otimes \underbrace{R^1\pi_*\mathcal{O}_S)^\vee}_{\deg\chi}) \otimes \mathcal{O}_S\left(\sum m(p) - 1S_p\right).$$

Why? Because $\pi_*\omega_S$ is locally free, thus $\omega_S = \pi^*(\omega_C \otimes \mathcal{L}) \otimes \mathcal{O}_S(D)$ where D is the zero divisor of $\pi^*\pi_*\omega_S \to \omega_S$. Argue that $D = \sum n_iF_i$ with $n_i < m_i$ using adjunction:

$$\mathcal{O}_{F_i} \simeq \omega_{F_i} = \omega_S \otimes \underbrace{\mathcal{O}_{F_i}(F_i)}_{\text{torsion}} = \mathcal{O}_{F_i}(F_i)^{\otimes(n_i+1)}$$

thus $n_i + 1$ has to divide m_i. □

Remark 18.2. Let $\pi : S \to \mathbb{P}^1$ be a rational elliptic surface (RES) and note that $K_S^2 = 0$. So, S is not minimal and there exists a (-1)-curve. By Proposition 18.1, there are two possibilities:

(1) There are no multiple fibers, $M.S_n = 1$.
(2) There exists a multiple fiber mF and $MS_n = m$,

Definition 18.3. A RES as in Item 2 is called a *Halpen surface* of index m. Alternatively, a rational surface S is a *Halphen surface* if there exists m such that $|-mK_S|$ is one dimensionsional and has no fixed component and no base points. The smallest such m is the *index*. (We include $m = 1 \iff$ there exists a section.)

Proposition 18.4. *If $\pi : S \to \mathbb{P}^1$ is a Halphen surface of index m, there exists $f : S \to \mathbb{P}^2$ birational such tat $\pi \circ f^{-1}$ is a Halpen pencil of index m (i.e. a pencil of plane curves with degree $3M$ with 9 base points, each of multiplicity m). Conversely, the resolution of any such pencil gives a Halpen surface.*

Example 18.5. $Q : y^2 + xz = 0$, $L : y = 0$, $C : (y^2 + xz)(\alpha y + z) + \beta yx^2$, $\alpha, \beta \neq 0$.

The pencil $\lambda(4Q + L) + \mu(3C) = 0$ is a Halphen pencil of index 3.

[The intersection of C, Q and L consists of two points, $p_1 = (0 : 0 : 1)$ and $p_2 = (1 : 0 : 0)$.] Blow up at p_1 seven times and at p_2 two times. [We obtain one the exceptional Dynkin diagram fibers.]

Example 18.6. Consider any smooth cubic C. Take $p_1, \dots, p_9 \in C$ such that $p_1 \oplus \dots \oplus p_9$ is m-torsion. Then there exists a curve D of degree $3m$ passing though $p_1, \dots, p_9$ with multiplicity m, thus $\lambda(D) + \mu(mC) = 0$ is a Halpen pencil.

19. Singular fibers

Recall that

$$\left\{\begin{matrix}\text{Halphen pencils} \\ \lambda C_{3m} + \mu(mC) = 0\end{matrix}\right\} \leftrightsquigarrow \left\{\begin{matrix}\text{Halphen surfaces} \\ \pi : S \xrightarrow{|-mK_S|} \mathbb{P}^1\end{matrix}\right\}$$
$$C_{3m} \leftrightsquigarrow F \in |-mK_S|.$$

Proposition 19.1 (Z)**.** *If F is of type II^*, III^* or IV^*, then C_{3m} is non-reduced.*

Proof. $R \subset S$ rational smooth $\implies R^2 \geq -2$, $\implies$ exceptional curves in S can only appear in disjoint chains. Now, if F is as in the statement, then at least one curve in [picture of fiber] is not exceptional. Here are the fibers: □

Proposition 19.2 (Z)**.** *Let $S \to \mathbb{P}^1$ be a Halphen surface of index 2 and assume that π contains a fiber F of type II^*. Then there exists $\varphi : S \to \mathbb{P}^2$ such that the sextic $\varphi(F)$ is one of the following: [Picture of conic of multiplicity 3, line of multiplicity 3 intersecting cubic, two lines of multiplicity 3, quadric and line of multiplicity 4, a line and a line with multiplicity 5].*

Example 19.3. $C : y^2z = x(x-2)(x-\alpha z)$, $\alpha \neq 0, 1$. $L : z = 0$, $\tilde{L} : x = 0$, $\mathcal{P} : \lambda(L + 5\tilde{L}) + \mu(2C) = 0$.

[A point of intersection is $p_1 = (0 : 1 : 0)$, where the line is tangent at the inflection point of C. Another point of intersection is $p_2 = (0 : 0 : 1)$.] Blow up p_1 4 times and p_2 5 times.

20. Constructions

(1)

$$\begin{tikzcd} X \arrow[d, "m:1"] \\ S \arrow[d, "|-mK_S| \ni F \text{ smooth}"] \\ \mathbb{P}^1 \end{tikzcd}$$

For $m = 2$, X is a K3. For $m \geq 3$, $\kappa(X) = 1$ and contains a $(-m)$ P^1 section.

(2) Type II degenerations of K3 surfaces and degenerations of Enriques surfaces (see Alexeev and Engel).

(3) It $\pi : S \to \mathbb{P}^1$ is a general Halphen surface of index 2 and we blow up we get a Coble surface.

(4) The Halpen transform

$$\begin{tikzcd} S \arrow[d, "|-mK_S|"] \\ \mathbb{P}^1 \end{tikzcd}$$

$|F + M|$ curves of $g = 1$ through q. Contract M and blow up q, $\rightsquigarrow$ we obtain a RES with section.

(5) (Halpne.) Given $p_1, \ldots, p_8 \in \mathbb{P}^2$, the locus of the 9th base point of a Halpen pencil of index m is a curve of degree $3\psi(m)$ having ordinary multiple points at $p_1, \ldots, p_8$ of multiplicity $\psi(m)$, where

$$\psi(m) = m^2 \left(1 - \frac{1}{q_2^2}\right)\left(1 - \frac{1}{q_2^2}\right) \cdots$$

where $m = q_1^{r_1} q_2^{r_2} \cdots$ prime factorization. $m = 2 \implies \psi(m) = 3$.

21. K3 surfaces

Definition 21.1. A surface S is a *K3* if $\Omega^2 = \mathcal{O}_S$ and $h^1(S, \mathcal{O}_S) = 0$.

So, $\omega_S = \mathcal{O}_S$. Also $\Omega^2 = \mathcal{O}_S$ means that there exists a global nowhere zero holomorphic 2-form σ.

Example 21.2.

(1) $S \subset \mathbb{P}^3$ smooth quartic.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

The long exact sequence in cohomology gives $h^1(S, \mathcal{O}_S) = 0$. Adjunction gives $\omega_S = \omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4)|_S = \mathcal{O}_S$.

(2) $\pi : S \xrightarrow{2:1} \mathbb{P}^2 \supset B$ smooth sextic. $\pi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$ gives $h^1(S, \mathcal{O}_S) = 0$. (If $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(-3)$, B is a section of $\mathcal{L}^{-2}$.) Since $\omega_S = \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{L}^{-1}$. This implies that $\omega_S \simeq \mathcal{O}_S$.

Remark 21.3. We can allow ADE singularities on B and take a minimal resolution

$$\begin{tikzcd}
\tilde{S} \arrow[d, "2:1"'] & S \text{ K3 surface} \arrow[l, "\substack{\text{min}\\ \text{res}}"'] \arrow[d, "2:1"] \\
\mathbb{P}^2 \arrow[d, dashed, "\substack{B\\ \text{sextic}\\ \text{w/ 9 noded}}"] & \text{Halphen index 2} \arrow[d, "|-2K_Y| \ni F"] \\
\mathbb{P}^1 \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

(3) S a K3 surface with a fixed point free involution i then S/i is an *Enriques surface* [a friend of K3], $p_g = q = 0, 2K = 0$.

(4) (Kummer surfaces.) A abelian surface (e.g. $E \times E^1$) $a \mapsto -a$ has 16 fixed points. $Kum(A) := \widetilde{A/(-1)}$ is a K3 surface.

The Hodge diamond of a K3:

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

[The H^2 lattice is]

$$H^2(S, \mathbb{Z}) = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3},$$

an even unimodular lattice of signature $(\underbrace{3}_{>0}, \underbrace{19}_{<0})$.

22. Curves on a K3 surface

Let S be a K3 surface and $C \subset S$ a curve. By adjunction formula, $C^2 = 2p_a(C) - 2$, so $C^2 \geq -2$.

In particular, if C is smooth and $C^2 = -2$, $C = \mathbb{P}^1$.

Assume that C is smooth and irreducible, and set $g = g(C)$, $\mathcal{L} = \mathcal{O}_S(C)$.

Then $\mathcal{L}^2 = 2g - 2$, $h^0(S, \mathcal{L}) = g + 1$. Moreover, if $g \geq 1$, then $|C|$ is base point free and the induced morphism $S \to \mathbb{P}^9$ restricts to the canonical map on C.

Definition 22.1. A *polarized K3 surface* of degree $2d$ is a pair $(S, \mathcal{L})$ with S a K3 and $\mathcal{L}$ an ample (ind in Pic) line bundle of degree $\mathcal{L}^2 = 2d$

Proposition 22.2. *For any $g \geq 3$ there exists a polarized K3 of degree $2g-2$ in $\mathbb{P}^9$.*

Example 22.3.
- $2d = 4$. Smooth quartic in $\mathbb{P}^3$ or a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched aloong a $(4,4)$ curve.
- $2d = 2$. $S \xrightarrow{2:1} \mathbb{P}^2 \supset$ sextic.

23. Elliptic K3s

(1)
$$\begin{tikzcd} & Kum(E \times E^1) \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 \simeq E/(-1) & & \mathbb{P}^1 \simeq E^1/(-1) \end{tikzcd}$$

(2)
$$S \xrightarrow[\text{res}]{\text{min}} \tilde{S} \xrightarrow{2:1} \mathbb{P}^1 \supset {}_{\text{w/ node}}^{\text{sextic}}.$$

The elliptic pencil comes from the pencil of lines in $\mathbb{P}^1$ through the node.

(3)
$$\begin{tikzcd} & S \text{ elliptic} \arrow[d, "2:1"] \\ \underbrace{F_1 \cup F_2}_{\text{smooth}} & X \text{ a RES w/ section} \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$

Proposition 23.1. *Let S be a K3 surface.*

1. *S is elliptic $\iff \exists L \in Pic(S)$ with $L^2 = 0$.*
2. *S is elliptic with a section $\iff U \xhookrightarrow{primitive} NS(S)$ [where U is the hyperbolic lattice].*
3. *(If the Picard rank is big enough, the surface will be elliptic.) If $\rho(S) \geq 5$, then S is elliptic $\rho(S) \geq 12 \implies \exists$ a section.*

To visualize the first point of the proposition, consdier the following example.

Example 23.2. A smooth quartic in $\mathbb{P}^3$ is elliptic iff it contains a line.

Remark 23.3.
- A K3 may admit more than one elliptic fibration.
- Since the base is always $\mathbb{P}^1$ [as you remember from the examples], there will always be a singular fiber. If $\underbrace{S}_{K3} \to C$ with $g(C) > 0$ then $h^0(\Omega_C) > 0$ but $H^0(\Omega_C) \hookrightarrow H^0(\Omega_S)$.
- Any $\underbrace{S}_{K3} \overset{\text{flat}}{\twoheadrightarrow} \mathbb{P}^1$ is an elliptic fibration.

24. Weierstrass model for K3

For an elliptic K3 $S \to \mathbb{P}^1$ with section, $R^1\pi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^1}(-2)$. So, a Weierstrass model is given by

$$y^1 = x^3 + A(t)x + B(t), \qquad t \in \mathbb{P}^1$$

with $\deg A = 8, \deg B = 12$.

The volume form can be written locally as

$$\frac{dx \wedge dt}{2y}.$$

25. Automorphisms

[Automorphisms will act in differential forms, cohomology, etc. A fundamental question is if an automorphism will preserve the volume form or not.]

symplectic $\iff$ σ is preserved

2 types

non- symplectic otherwise

(Purely non-symplectic if it acts as multiplication by a primitive $n-$-th root of 1.) We restrict to finite order.

Proposition 25.1. *Let S be a K3 surface with volume form σ.*

(1) *If $f \in Aut(S)$ acts as $f^*\sigma = \underbrace{\zeta_s}_{primitive} \sigma$, then $\varphi(n) \leq 20$ and all $n \neq 20$ satisfying this appear.*
   If $n = 2$, $Fix(f)$ contains only curves (no isolated points) or $\emptyset$.
   If $n \geq 3$, $Fix(f) = C_g \coprod R_1 \coprod \ldots \coprod R_k \coprod \{p_1, \ldots, p_m\}$, $g \geq 0$. (Nikulin, Sarti+Artebani, Hoffmann Brandhorst.)

(2) *If $f^*\sigma = \sigma$, then $Fix(f) = \{p_1, \ldots, p_k\}$ with $1 \leq k \leq 8$ and $ord(f) \leq 8$.*

n	2	3	4	5	6	7	8
Fix(f)	8	6	4	4	2	3	2
$\rho(S)$	9	13	15	17	17	19	19

(See Salgado, Garbagnati. Comparin-Prieto-Troncoso-Montero (on involutions).)

Example 25.2. Consider the elliptic K3 given by [the Weierstrass model ...]

$$y^2 = x^3 + t^2x + t^{10}$$

[we have written in this in a simpler way, check degrees] and the automorphism

$$f : (x, y, t) \mapsto (\zeta_7 x, \zeta_7^5 y, -\zeta_7 t)$$

(order 14).

Note:

$$\begin{aligned}A(t,s) &= t^2 s^6\\ B(t,s) &= t^{10}s^2\\ \Delta(t,s) &= 4A^3+27B^2 = \text{const. } s^4t^6(s^{14}-t^{14}).\end{aligned}$$

- At $t=0$, $a=2, b=10, \delta=6$. I_0^*.
- At $s=0$, $a=6, b=2, \delta=4$. IV.
- At each root of $s^{14}-t^{14}$, $a=b=0, \delta=1$, I_1.

[Pictures of the fibers and how the automorphisms act on them - fixed components etc.]

26. Summary

Slogan. We can parametrize the smooth pencils of cubics as long as we remove the RESs with bad singular fibers.

$$\mathrm{PGL}_3 \curvearrowright \left\{\begin{smallmatrix}\text{smooth}\\ \text{pencils of}\\ \text{plane cubics}\end{smallmatrix}\right\} \longleftrightarrow \left\{\begin{smallmatrix}\text{RES}\\ \text{w/ a section}\end{smallmatrix}\right\}$$
$$\mathcal{P} \mapsto S_{\mathcal{P}}.$$

Question: can we parametrize orbits? Does the geometry of the RESs play any role?
Answer: R. Miranda gives an answer in the '80s using GIT.

$$\left\{\begin{smallmatrix}\text{RES}\\ \text{w/ section}\end{smallmatrix}\right\} \longleftrightarrow \left\{\begin{smallmatrix}\text{Weierstrass eq.}\\ y^2=x^3+A_4(t)+B_6(t)\end{smallmatrix}\right\}$$

Question. Can we parametrize such Weierstrass equations?
Answer. R. Miranda also gives an answer via GIT.

27. Quick intro to GIT

Consider a projective embedding $G \curvearrowright X \xhookrightarrow{\mathcal{L}} \mathbb{P}^N$ using a line bundle $\mathcal{L}$, where X and $\mathcal{L}$ equipped with an action of a group G. [The idea is to use this embedding and line bundle to construct the algebraic quotient.]

Definition 27.1.

$$X//G := \operatorname{Proj}\left(\bigoplus_{d\geq 0} H^0(X,\mathcal{L}^{\otimes d})^G\right).$$

We have a map

$$X \xrightarrow{\pi} X//G.$$

[Notice that π is not well-defined at some places (this is what we meant before with the RESs with bad singular fibers that we must remove), namely where all the invariant sections vanish. These points have a name:]

- $x \in X$ is *unstable* if and only if $\pi(x)$ is not defined.
- $X \supset X^{ss} \supset X^s$.

$$X^{ss}\text{“}=\text{”}\text{ points in } X \text{ where } \pi \text{ is defined}$$

$$X^{s}\text{“}=\text{”}{}^{\text{semistable points w/ closed}}_{\text{orbit and finite stabilizer}}.$$

[The first step in constructing the GIT quotient is understanding what are the unstable and semistable points. This is what the naive questions above are supposed to motivate: we intend to find the unstable and semistable points of the corresponding GIT problem in terms of the fibers of the surfaces.]

Problem. $X =$ Grassmanian of lines in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. $G = \mathrm{SL}(3) \curvearrowright X$, $\mathcal{L} \leftrightsquigarrow$ Plücker embedding.

Miranda proved:

(1) A pencil $\mathcal{P}$ of plane cubics is stable if and only if $\mathcal{P}$ is smooth and $S_{\mathcal{P}}$ contains only reduced fibers.
(2) Moreover, if $\mathcal{P}$ is smooth, then $\mathcal{P}$ is semistable if and only if $S_{\mathcal{P}}$ has no fibers of type II^*, III^* or IV^*.

28. Natural generalizations

(1) What about Halphen pencils/surfaces?
(2) [Recall the construction above on $X =$ Grasmannian of lines in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$: why only in $\mathbb{P}^2$? Why only degree 3?] What about higher dimensional linear systems in higher dimension?
(3) What about a generalization of the Weierstrass models?
(1') What about higher genus fibrations?

Now we give answers to those questions:

(1) For $m = 2$ [recall there is an m associated to Halphen surfaces], part of my PhD thesis. For any m (joint w/ Masafumi Hattori),

$$\mathcal{P} \mapsto S_{\mathcal{P}} \mapsto \mathbb{P}^1.$$

Theorem 28.1. *If $lct(S_{\mathcal{P}}, F) > \frac{1}{2m}$ (resp. $\geq$) for any fiber F, then $\mathcal{P}$ is stable (resp. semistable). Type I_n, II, III, IV, II^**

type	I_n^*	I_n	II	III	IV	II^*	III^*	IV^*	mI_n
lct	1/2	1	5/6	3/4	2/3	1/6	1/4	1/3	$1/m$

For $t \in [0,1]$, we compute

$$K_{\tilde{X}} + t\tilde{D} - \mu^*(K_{\mathbb{C}^2} + tD = (1-2t)E_1 + (2-3t)E_2 + (4-6t)E_3$$

(2) Consider

$$\underbrace{\mathrm{Gr}(K, H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d)}_{X_{K,d,n}} \curvearrowleft \mathrm{SL}(n+1)$$

(HZ) $\mathcal{L} \in X_{K,d,n}$ is GIT unstable/not stable if and only if there exists a choice of generators for $\mathcal{L}$, $H_1, \ldots, H_{k+1}$ such that $H_1 + \ldots + H_{k+1}$ is GIT unstable/not stable. [The action, which is compatible also in symmetric power, exterior algebra, etc., at the level of degree ¿? hypersurfaces will help determine the stable points.] This implies (by Hacking, Kim-Lee) that the pair

$$\left(\mathbb{P}^n, \frac{n+1}{d(k+1)}(H^1 + \ldots + H_k)\right)$$

is not lc (not klt).

Work in progress (HPZ). The converse where the converse of the last implication holds if $n = 2, d = 3$ and for any K.

Now we consider

$$\left\{\begin{smallmatrix}\text{nets of cubics}\\ \text{through } p_1,\dots,p_7.\end{smallmatrix}\right\}$$

$$\begin{tikzcd}
Bl_{p_1,\dots,p_7}\mathbb{P}^2 \arrow[rr, "\substack{\text{res}\\ \text{sing.}}"] \arrow[drr, dashed, "2:1"'] & & Y \arrow[d, "2:1"] \\
& & \mathbb{P}^2 \supset \underbrace{B}_{\text{quartic}}
\end{tikzcd}$$

M is GIT (semi)stable if and only if so is B.

Now we introduce a new question that is not in the above list:

(4) Let

$$\begin{tikzcd}
X \text{ RES w/ section} \arrow[d, "\text{Blow up } f_1"] \\
Y \arrow[d, "f"', "2:1"] \\
\mathbb{P}^2 \supset B + p.
\end{tikzcd}$$

Question. Given X, classify all pairs (B, p).

This morning (CGMVZ). We did this for sufaces with 6 double fibers. ([For completeness:] 6II, $6I_2$, $2I_2 + 4II$, $2II + 4I_2$, $3I_2 + 3II$.)

Now let's describe the higher genus scenario:

(1') Two constructions:

(a)

$$\begin{tikzcd}
& \tilde{S} \arrow[dl] \arrow[d, "\text{Blow up } f^{-1}(p)"] \\
& S \text{ a K3} \arrow[d] \\
\mathbb{P}^1 & \mathbb{P}^2 \cup B \text{ sextic } + p
\end{tikzcd}$$

(b) (With Livia Campo.) $T_{a,b}$ toric 3-fold with weight matrix [we didn't introduce this gadget yet]

$$\begin{pmatrix} u & v & x & y & z & \\ 1 & 1 & 1 & 0 & -a & -3a+b \\ 0 & 0 & 1 & 1 & 3 & \end{pmatrix}$$

This gives

$$\begin{tikzcd} T_{a,b} \arrow[r] & \mathbb{P}^1_{(u,v)} \\ X \arrow[u, hook] \arrow[ur, "\text{of bidigree}\ \binom{-6a+2b}{a}"'] & \end{tikzcd}$$

$$\leftrightsquigarrow$$

$$\begin{tikzcd} X_{a,b} \arrow[d, "2:1"] \\ \mathbb{F}_a \end{tikzcd}$$

When $a = 0$ we can then parametrize certain divisors in $\mathbb{P}^1 \times \mathbb{P}^1$. The case $(a, b) = (0, 1)$ is the Horikawa model of the rational surface $Bl_{13}\mathbb{P}^2$ where the 13 points are the base locus of a pencil of quartics with a common node.

[Finally we address the problem of generalizing the Weierstrass model:]

(3) For Halpen surfaces, at least 2 approaches.

$$\begin{tikzcd} S \arrow[d, "|-mK_S|"] \\ \mathbb{P}^1. \end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd} S \arrow[d, "\substack{\text{gen. of} \\ \text{deg. } m}"] \\ \mathbb{P}^1 \times \mathbb{P}^1. \end{tikzcd}$$

We have an equation $y^m = \dots$.

The second approach is (j/ with S. Rollenske). [Diagram of blowing up.] For $m = 3$, we obtain S as a relative cubic inside a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$.