SEMINARS

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Contents

1. Friday Seminar, PUC-Rio

1.1. **Neutrinos.** Hiroshi Nunokawas, PUC-Rio. June 27, 2025.

Abstract. Hiroshi will come and tell us everything we (not) wanted to know about these mysterious particles, and are not going to be afraid to ask. In particular, about the neutrino oscillation, and the great matrices.

Protons and neutrons have very similar mass of $m_p \approx 940$ MeV, while electrons have mass of $m_e \approx 0.5$ MeV. MeV is 10^6 electronvolts, where one eV is approximately 1.6×10^{-19} J. This is standard in high energy physics, they use electronvolts instead of Joules. Recall that 2J=1N×1m.

Most of the things we see are protons since they are so much larger than electrons. But protons nor neutrons are elementary particles.

Here's the standard model:

$$\begin{array}{cccc}
\text{Quarks} & \begin{bmatrix} u \\ d \end{bmatrix}_L & \begin{bmatrix} c \\ s \end{bmatrix}_L & \begin{bmatrix} t \\ b \end{bmatrix}_L \\
\text{Leptons} & \begin{bmatrix} \nu_e \\ e^- \end{bmatrix}_L & \begin{bmatrix} \nu_\mu \\ \mu^- \end{bmatrix}_L & \begin{bmatrix} \nu_\tau \\ \tau^- \end{bmatrix}_L \\
\text{Generation} & \text{1st} & \text{2nd} & \text{3rd} \\
\text{Bosons} & g, \gamma\ , \omega^\pm, z & & \\
\text{Higgs Boson} & H & &
\end{array}$$

It is very particular that nature repeats itself three times. The L in those matrix actually means left-handed, and accounts for chirality. Only left-handed fermions have weak interaction. Right-handed have electromagnetic interaction, gravitational interaction, but not weak interaction.

And then there's neutrinos. They have negative helicity (chirality). Being left-handed, mathematically, means to have helicity -1. I think this means that the spin is left-handed. But chirality and helicity are not the same: helicity is observer-dependent, and chirality is not. Almost all neutrinos we can see (% 99.99999...) have negative helicity, but not all of them.

Consider the following:

$$n + \nu_e \leftrightarrow p + e^-$$

But it's not completely correct: we'd better put d instead of n, and u instead of p: the d and u quarks, instead of the neutrons and protons.

Now consider the following reaction: a neutron decays into a proton, an electron and an antineutrino:

$$n \to p + e^+ \overline{\nu}_e$$

Protons is very stable, that's why we are here. But neutron decays in only 15 minutes.

By experimental data, we can conclude that neutrinos' mass is consistent with zero. But if they have mass, it should be much smaller than the electron's $m_v \leq 0.5$ eV. And the electron is already the lightest fermion!

If the mass of the neutrino was zero, i.e. $m_\nu = 0$, then $v_\nu = c$ in vacuum, which would imply that

$$\begin{tikzcd} \nu_e \arrow[r, "L"] & \nu_e \arrow[r] & \nu_e \\ 0:00 & 0:00 & 0:00 \end{tikzcd}$$

meaning: time doesn't pass! And this means the state of the particle cannot change.

2. Differential Geometry Seminar, IMPA

2.1. **Spheres with minimal equators.** Lucas Ambrozio, IMPA. June 24, 2025.

Abstract. We will discuss the connection between Riemannian metrics on the sphere with respect to which all equators are minimal hypersurfaces, and algebraic curvature tensors with positive sectional curvatures.

Definition 2.2. An $(n-k)$-*equator* orthogonal to Π is

$$\Sigma_\Pi := \{p \in \mathbb{S}^n : \langle p, x\rangle = 0 \forall x \in \Pi\}$$

for Π a k-dimensional linear subspace of $\mathbb{R}^{n+1}$.

Remark 2.3. Equators are totally geodesic hypersurfaces with the usual sphere metric, which implies they are minimal hypersurfaces.

Problem. Characterize the set $\mathcal{M}_k(U)$ of metrics g on an open set $U \subset \mathbb{S}^n$ such that all k-equators Σ_Π with $\Sigma \cap U \neq \emptyset$ yield are minimal hypersurfaces $\Sigma \cap U$ on (U, g).

Remark 2.4. This problem can be thought of as a problem of finding metrics on $\mathbb{R}^n$ such that k-planes are minimal. To see why project the k-equators to $T_p\mathbb{S}^n$ and pullback those metrics to the sphere.

Let $g \in \mathcal{M}_k(U)$ for $U \subset \mathbb{S}^n$ open and $n \geq 2$.

Theorem 2.5 (Beltrami, Schäfli)**.** *If* $k = 1$ *then* g *has constant sectional curvature.*

Theorem 2.6 (Hongan)**.** *If* $1 < k < n-1$ *then* g *has constant sectional curvature.*

Then Hongan also managed to produce a classification of these metrics for $k = n-1$.

Remark 2.7. If $T \in \mathrm{GL}(n+1,\mathbb{R})$, then

$$\begin{aligned}\varphi : \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ x &\longmapsto \frac{Tx}{|Tx|}\end{aligned}$$

is a diffeomorphism that maps k-equators into k-equators. Thus if $g \in \mathcal{M}_k(\mathbb{S}^n)$ then so is $\varphi(T)^*g$.

Theorem 2.8. *There exists a $GL(n+1,\mathbb{R})$ equivariant bijection*

$$\mathcal{M}_{n-1}(\mathbb{S}^n) \leftrightarrow Curv_+(\mathbb{R}^{n+1})$$

where the set on the right-hand-side is the set of algebraic curvature tensors (also called curvature-like, i.e. with the same symmetries as the Riemannian curvature tensor) on $\mathbb{R}^{n+1}$ with positive sectional curvature.

The group action is given as follows for $T \in GL(n+1,\mathbb{R})$:

$$(R\cdot T)(x,y,z,w) = \frac{1}{|\det(T)|^{\frac{1}{n+1}}} R(Tx,Ty,Tz,Tw)$$

The point is that $\mathrm{Curv}_+(\mathbb{R}^{n+1})$ is an open cone on a linear space. Here are two simple corollaries:

Lemma 2.9. (1) *$\mathcal{M}_{n+1}(\mathbb{S}^n)$ is in bijection with an open positive cone of an $\frac{n(n+2)(n+1)^2}{12}$-dimensional real vector space.*
(2) *Every metric on $\mathcal{M}_{n-1}(\mathbb{S}^n)$ is invariant by the antipodal map.*

Algorithm. From any $R \in \mathrm{Curv}_p(\mathbb{R}^{n+1})$ we obtain a symmetric positive definite (positive-definitiness comes from the positiveness of the curvature of R) 2-tensor k_R satisfying

$$(k_R)_p(v,v) = R(pv,pv) > 0$$

Also, k_R has the *Killing property*, i.e. that $\overline{\nabla} k(X,X,X) = 0$ for all $X \in \mathfrak{X}(\mathbb{S}^n)$.

Then we define a positive function on $\mathbb{S}^n$ by

$$D_R := \left(\frac{d\mathrm{Vol}_{k_R}}{dV_g}\right)^{\frac{4}{n-1}} \tag{2.9.1}$$

and finally a Riemannian metric on $\mathbb{S}^n$ in $\mathcal{M}_{n-1}\mathbb{S}^n$ by

$$g_R = \frac{1}{D_R} k_R$$

And to go back, for $g \in \mathcal{M}_{n-1}(\mathbb{S}^n)$ define a positive function on $\mathbb{S}^n$

$$F_g := \left(\frac{dV_g}{dV_{\overline{g}}}\right)^{\frac{4}{n-1}}$$

Then let $k_g := \frac{1}{F_g} g > 0$, which is a positive definite Killing 2-tensor, from which we may define $R_g \in \mathrm{Curv}_+(\mathbb{R}^{n+1})$ with $R_g(pv,pv) = (k_g)_p(v,v)$ for all $p, v \in T\mathbb{S}^n$.

More corollaries:

Lemma 2.10. (1) *$g \in \mathcal{M}_{n-1}(\mathbb{S}^n)$ is analytic because it is a Killing tensor on $\mathbb{S}^n$, which are well-known.*

(2) *If g is left-invatiant on $\mathbb{S}^3$, seen as unit quaternions, then $g \in \mathcal{M}_2(\mathbb{S}^3$. Moreover, for $a \geq b \geq c > 0$,*

$$aL_i \odot L_i + bL_j \odot L_j + cL_k \odot L_k = k$$

is Killing, $k > 0$, D_k constant and thus $g = \frac{1}{const.}k \in \mathcal{M}_2(\mathbb{S}^3)$.

(3) *R curvature tensor of $(\mathbb{C}P^2, g_{FS})$. We may not remember what's the curvature tensor, but we know the sectional curvature is $1 \leq sec(R) \leq 4$,*

$$(k_R)_p(v,w) = \overline{g}(v,w) + 3\overline{g}(Jp,v)\overline{g}(Jp,w)$$

and $D_R = 4^{\frac{4}{3-1}} = 4$, so that by 2.9.1 we obtain $g_R = \frac{1}{4}k_R$, which is a Berger metric on $\mathbb{S}^3$ with scalar curvature 0.

Now define

$$\Sigma_V = \{p \in \mathbb{S}^n : \langle p, v\rangle = 0\} = V^{-1}(0)$$

where $V(x) := \langle x, v\rangle$ for all $x \in \mathbb{S}^n$. Then the normal vector field is $\nabla V/|\nabla V|_g$, and the second fundamental form is given by

$$A = \frac{1}{|\nabla V|_g}\mathrm{Hess}_g V$$

and its mean curvature by

$$H = \frac{1}{|\nabla V|_g}\left(\Delta_g V - \mathrm{Hess}_g V\left(\frac{\nabla V}{|\nabla V|}, \frac{\nabla V}{|\nabla V|}\right)\right) \tag{2.10.1}$$

For every $v \in \mathbb{S}^n$ and $p \in \Sigma_V$, we see that $H_{\Sigma_V} = 0$ iff

$$|\nabla V|_g^2(p)\Delta_g V(p) - \mathrm{Hess}_g V(\nabla V(p), \nabla V(p)) = 0$$

And for $\overline{g}$,

$$\mathrm{Hess}_{\overline{g}} V + V\overline{g} = 0 \implies \mathrm{Hess}_{\overline{g}} V(X,X) = 0$$

for all $X \in T_p\mathbb{S}^n$ and $p \in \Sigma_v$. Then

$$J_g(X,Y,Z) = g(\nabla_X Y - \overline{\nabla}_X Y, Z)$$

$$J_g(X,Y,\nabla V) = \mathrm{Hess}_{\overline{g}} - \mathrm{Hess}$$

Problems.

(1) Similar story for $\mathbb{C}P^n, \mathbb{H}P^n$?
(2) Complete metrics on $\mathbb{R}^n$ with minimal hyperplanes.
(3) Find geometric invariants of metrics on $\mathcal{M}_{n-1}(\mathbb{S}^n)$ (may be useful to study (M^n, g), $n \geq 4$, sec > 0.

3. UFF Algebraic Geometry Seminar

3.1. **Smoothable compactified Jacobians of nodal curves.** Nicola Pagani,

University of Liverpool and Bologna. August 20, 2025.

Abstract. Building from examples, we introduce an abstract notion of a 'compactified Jacobian' of a nodal curve. We then define a compactified Jacobian to be 'smoothable' whenever it arises as the limit of Jacobians of smooth curves. We give a complete combinatorial characterization of smoothable compactified Jacobians in terms of some 'vine stability conditions', which we will also introduce. This is a joint work with Fava and Viviani.

Let C be a smooth curve and $d \in \mathbb{Z}$. Define

$$J_C^d = \{L : L \text{ is a line bundle of degree } d\}/\sim$$

which is a smooth projective variety of dimension $g(C)$.

If C is nodal we still can consider J_C^d.

(1) One connected component. Then the Jacobian is $\mathbb{P}^1$ minus two points. This is not universally closed, so it is not proper.
(2) Two components intersecting at one point. The pullback of the normalization splits the degree in intintely many ways, givieng that J_C^{-1} is an infinite set of points. This is not of finite type, so it is not proper.
(3) The curve has two components intersecting at two points. This gives J_C^{-2}, which is a mixture of the two former items. (Probably not proper too.)

Now consider

$$\mathrm{TF}_C^d = \{\mathcal{F} : \text{coherent on } C, \text{ torsion-free, rank-1 on } C\}/\sim$$

This satisfies the existence oart of the valu point of properness.

Now we consider the moduli. Now we consider the ideal sheaf of the (singular?) point(s?):

(1) One component. The stack is proper!
(2) Two components intersecting once. Now we get stacky points, $x = [\bullet/\mathbb{G}_m]$. These points have generic stabilizer. The resulting stack is not separated because a morphism of a curve, say $\mathbb{P}^1$ minus a point ... there are infinitely many ways to extend a morphism from this thing to a line bundle. So you cannot include any of these stacky points. Recall that a sheaf is *simple* if its automorphism group is $\mathbb{G}_m$.
(3) The ideal sheaf of both nodes $\mathcal{I}(N_1, N_2)$ has a positive dimensional automorphism group. The stack is not proper.

Definition 3.2. A *fined compactified Jacobian* of C is an open connected substack of $\mathrm{TF}^d(C)$ that is also proper.

Remark 3.3. This thing is automatically an algebraic space.

Definition 3.4. A *compactified Jacobian* is an open connected of $\mathrm{TF}^d(C)$ that admits a proper, good moduli space.

Consider the Artin stack $\mathfrak{X} \xrightarrow{\Gamma} X$ [...] is a *good moduli space* if

(1) Every moduli factors

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[drr] & \mathcal{I} \text{ (ACC. space)} & \\ & & X \end{tikzcd}$$

(2) $\pi_* \mathcal{O}_{\mathfrak{X}} = \mathcal{O}_X$.

We expect to find a notion of stability condition to produce these things [...] $[\bullet/\mathbb{G}_m]$ would be the polystable representative.

Definition 3.5. A compactified Jacobian $\overline{J_C}$ is *smoothable* if all smoothings $\mathcal{C} \to \Delta = \{0, \eta\}$ (with $\mathcal{C}_0 = C$),

$$J^d_{\mathcal{C}_\eta} \cup C \to \overline{J_C}$$

is proper.

Definition 3.6. Let X be a curve.

$$\mathrm{BCON}(X) = \{Y \subseteq X \text{s.t. } Y, Y^c \text{ are connected}\}$$

Definition 3.7. A v-curve is a generalization of items (2) and (3) in the lists above [it looks like two long snakes $\sim$ that intersect several times, and t is the number of nodes]. A *v-condition* is a pair $n = (n_1, n_2)$ such that

$$n_1 + n_2 = \begin{cases} d + 1 - t & \text{we say the s.c. is nondegenerate} \\ d - t & \text{degenerate} \end{cases}$$

$\mathcal{F}$ on X is *n-(semi)stable* if $\deg \mathcal{F}_{X_i} > n_i$ ($\deg \mathcal{F}_{X_i} \geq n_i$) for $i = 1, 2$.

$\mathcal{F}_{X_i} = \mathcal{F}|_{X_i}$ torsion.

$$\deg(\mathcal{F}_{X_i}) + \deg(\mathcal{F}_{X_2}) = d - |\mathrm{sing}(F)|.$$

Then

$$\overline{J_C}(n) = \{\mathcal{F} \text{ is semistable}\},$$

a smooth compact Jacobian.

Definition 3.8. A *degeneratation* of v-stab. on X is $n : \mathrm{BCON}(X) \to \mathbb{Z}$ such that

(1)

$$n_Y + n_{Y^c} + |Y \cap Y^c| = \begin{cases} d + 1 & \text{we say } Y \text{ is } n\text{-nondegenerate} \\ d & Y \text{ is } n\text{-degenerate} \end{cases}$$

(2) Y_i no pa. common component $n_{Y_1} + n_{Y_2} + \dots$

$$\begin{tikzcd} & Y_1 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ Y_2 \arrow[rr, leftrightarrow] & & Y_3 \end{tikzcd}$$

Theorem 3.9 (-, et al)**.** *(bijection between stability conditions and nodal curves) The map*

$$\left\{\begin{smallmatrix} \textit{sm. comp.} \\ \textit{Jac of } X \end{smallmatrix}\right\} \to \left\{\begin{smallmatrix} v\textit{-stab.} \\ \textit{cond. of } X \end{smallmatrix}\right\}$$

$$n \mapsto \overline{J_X}(n) = \{n\text{-}\textit{semistable sheaves}\}$$

is a bijection. (The arrow should be from right to left!)

F. Viviani had proved it for fine compact Jacobians.

References