

COMPLEX GEOMETRY

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1. COMPLEX MANIFOLDS

Definition 1.1. A *complex manifold* M is a smooth manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all α, β .

Definition 1.2. A function on an open set $U \subset M$ is *holomorphic* if for all α , $f \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n$.

Lemma 1.3. *The sheaf \mathcal{O}_M of holomorphic functions is a sheaf.*

2. HODGE STAR

The construction of the Hodge star in [Voi02, Section 5.1] is essentially: map $\alpha \in \Lambda^{n-k}$ to the operator $\alpha \wedge - \in \text{Hom}(\Lambda^k, \Lambda^n)$ and then use the metric to identify $\text{Hom}(\Lambda^k, \Lambda^n)$ with Λ^k . Notice that the latter means that we take a form α and we map it to the morphism which maps \cdot to the top-form $g(\alpha, \cdot) \text{Vol}$. This says that for all $\alpha \in \Lambda^k$ and $\beta \in \Lambda^{n-k}$

$$(2.0.1) \quad \alpha \wedge * \beta = (\alpha, \beta) \text{Vol}.$$

Lemma 2.1. *The operator $*$ satisfies the identity*

$$(2.1.1) \quad *^2 = (-1)^{k(n-k)} \text{ on } A^k(X).$$

Proof. Observe that $*$ is an isometry, use skew-commutativity of product to obtain the $(-1)^{k(n-k)}$ and finally Equation 2.0.1. \square

3. RIEMANN SURFACES

Definition 3.1. A meromorphic function on a Riemann surface is a holomorphic map to the Riemann sphere which is not identically equal to ∞ .

Definition 3.2. A pole of a meromorphic function on a Riemann surface is any point x such that $F(x) = \infty$.

Definition 3.3. The order of a pole of a meromorphic function on a Riemann surface is the integer k_x such that the coordinate representation of f is $f(x) = z^{k_x}$ near x .

Proposition 3.4. Let $F : X \rightarrow Y$ be a proper, non-constant holomorphic map between connected Riemann surfaces. Then the integer $d(y) = \#f^{-1}(y)$ does not depend on $y \in Y$.

Proof. Using local representation of f as z^k . But see also ??: this number is locally constant because we can find a neighbourhood V of any point y such that any other y' in this neighbourhood has the same number of preimages. Using compactness fix neighbourhoods U_i of all the preimages of y , using that y is a regular value we may suppose the f is a diffeomorphism at the U_i , and then define $V := \cap V_i \setminus f(M \setminus \cup U_i)$. \square

Lemma 3.5. Let X be a compact Riemann surface. If there is a meromorphic function on X having exactly one pole, and that pole has order 1, then X is equivalent to the Riemann sphere.

Proof. A meromorphic function has no critical points! That is, ∞ is a regular value, having one preimage by hypothesis. Then the degree of f is one, so that it must be a bijection (why?). And it has no critical points... (why?) So by Lemma something, its inverse must be holomorphic. \square

Exercise 3.6. Show that the canonical bundle K of a Riemann surface S has no base points.

Proof. Suppose p is a base point of K . We want to construct a meromorphic function with exactly one pole and use Lemma 3.5 to arrive at a contradiction. Any such function is an element of $H^0(\mathcal{O}(p))$. Note that by Serre duality we have:

$$H^0(\mathcal{O}(p)) = H^0(\mathcal{O}(p) \otimes K^* \otimes K) = H^1(K(-p))^*$$

So it's enough to show that $H^1(K(-p))$ is not zero. So consider the sheaf exact sequence twisted by the ideal sheaf $\mathcal{I}_p = \mathcal{O}_S(-p)$ of p :

$$0 \longrightarrow K(-p) \longrightarrow K \longrightarrow K(p) \longrightarrow 0$$

Then we use the exact sequence in cohomology to prove that $H^1(K(-p))$ is not zero. First, $H^1(K(p)) = 0$ because by Serre duality it has the same dimension as the space of holomorphic functions vanishing at p , which is only the zero function since M is compact. Next, $H^1(K)$ is not zero because by Dolbeault theorem it is

$H^{1,1}(M, \mathbb{R})$ which contains real 2-forms. Then $H^1(K(-p))$ cannot be zero because it surjects onto a nontrivial space. \square

4. BELYI FUNCTIONS

To understand Belyi functions we start by considering meromorphic functions $f : X \rightarrow \mathbb{CP}^1$ as ramified coverings:

Proposition 4.1. *A nonconstant meromorphic function $f : X \rightarrow \mathbb{CP}^1$, considered as a mapping of the underlying topological space, is a ramified covering of the sphere $S^2 \cong \mathbb{CP}^1$.*

Be careful: isomorphic complex ramified coverings produce isomorphic Riemann surfaces (by definition), but the converse is certainly false. The same Riemann surface may be obtained by many different and pairwise non-isomorphic coverings. Just consider different meromorphic functions on the same surface.

The following paragraph is an approximate quote:

Proposition 4.1, and the fact that every Riemann surface admits a meromorphic function (which can be seen by Riemann-Roch, cf. [LGZV, Fact 1.8.6]) show that every Riemann surface may be represented by a ramified covering of \mathbb{CP}^1 . The following theorem, which affirms the converse statement, is one of the most fundamental:

Theorem 4.2 (Riemann's existence theorem). *Suppose a base star is fixed in \mathbb{CP}^1 , and the sequence of its terminal vertices is $R = [y_1, \dots, y_k]$. Then for any constellation $[g_1, \dots, g_k]$, $g_i \in S_n$, there exists a compact Riemann surface X and a meromorphic function $f : X \rightarrow \mathbb{CP}^1$ such that y_1, \dots, y_k are the critical values of f (i.e. f' vanishes at these points) and g_1, \dots, g_k are the corresponding monodromy permutations. The ramified covering $f : X \rightarrow \mathbb{CP}^1$ is independent of the choice of the base star in a given homotopy type and is unique up to isomorphism.*

Proof. The heart of this theorem is the correspondence between constellations, which are abstract sets of permutations whose product is the identity and act transitively on the set of n elements (and are also maps on surfaces), and the monodromy group of a covering. Indeed: for a regular point y_0 of a degree- n covering we get an action of the symmetric group of n elements on the fiber $E := \pi^{-1}(y_0)$; taking each generator of the fundamental group to be any of these permutations we obtain a constellation (cf [LGZV, Construction 1.2.13]).

The other way around, [LGZV, Proposition 1.2.15], we can define a group homomorphism $\pi(S^2 \setminus \{p_i\}, y_0) \rightarrow G$ where G is the group of the constellation; the fact that it is indeed a group homomorphism is due to the fact that both sets of generators satisfy the property that their product is identity. Then for any point x in the set of n elements we can consider its stabilizer. This corresponds to a subgroup M_x of $\pi(S^2 \setminus \{p_i\}, y_0)$. Such a subgroup determines a finite-sheeted covering of $S^2 \setminus \{p_i\}$. The covering is connected since G acts transitively on E . *Habemus superficiem.* \square

The surprising result by Belyi is that for the case $k = 3$ it will happen that the corresponding Riemann surfaces will be defined over $\overline{\mathbb{Q}}$, the field of algebraic numbers. Therefore, the absolute Galois group $\text{Aut}(\overline{\mathbb{Q}}|\mathbb{Q})$ (that is, the automorphism group of the field $\overline{\mathbb{Q}}$) acts on them, and thus on 3-constellations as well. The

mysterious nature of the group and the simplicity of the objects on which it acts, gave rise to the following term which may look a bit strange: *theory of dessins d'enfants*.

Definition 4.3. If it is possible to realize a Riemann surface X by a system of equations with coefficients in a subfield $K \subseteq \mathbb{C}$, then we say that X is *defined over* K .

Theorem 4.4 (Belyi). *A Riemann surface X admits a model over the field $\overline{\mathbb{Q}}$ of algebraic numbers if and only if there exists a covering $X \rightarrow \overline{\mathbb{C}}$ unramified outside $\{0, 1, \infty\}$. In such a case, the meromorphic function f can also be chosen in such a way that it will be defined over \mathbb{Q} .*

5. SHEAF OF HOLOMORPHIC FUNCTIONS

For a definition of sheaf see Algebraic Geometry Definition 1.1.

Lemma 5.1. *Let X be a complex manifold of complex dimension n . The functor*

$$\begin{aligned} \mathcal{O}_X : \text{Open}_X^{\text{op}} &\longrightarrow \text{Set} \\ U &\longmapsto \mathcal{O}_X(U) := \{f : U \rightarrow \mathbb{C}^n : f \text{ is holomorphic}\} \\ i : V \hookrightarrow U &\longmapsto \mathcal{O}_X(i) : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V) \end{aligned}$$

where $\mathcal{O}_X(i)$ is given by restriction, is a sheaf.

Proof. \mathcal{O}_X is a presheaf, i.e. a functor, by definition: restriction maps the identity to the identity and preserves compositions.

To check the condition of being a sheaf let U_i be an open cover of some open set U of X and $f_i : U_i \rightarrow \mathbb{C}^n$ sections of $\mathcal{O}_X(U_i)$, that is, holomorphic functions, which coincide pairwise in intersections. Then it is obvious that the function $f : U \rightarrow \mathbb{C}^n$, $x \mapsto f_i(x)$ for any $U_i \ni x$ is well defined. It is also holomorphic since holomorphicity is a condition defined on open sets. \square

6. ANALYTIC SETS

Definition 6.1. An *analytic set* or *subvariety* Y is a subset of a complex manifold X given locally as the zeros of a finite collection of holomorphic functions.

Lemma 6.2. *Let $Y \subset X$ be an analytic subvariety of a complex manifold X equipped with the subspace topology. Then*

$$\begin{aligned} \mathcal{O}_Y : \text{Open}_Y^{\text{op}} &\longrightarrow \text{Set} \\ U \cap Y &\longmapsto \{f|_{U \cap Y} : f \in \mathcal{O}_X(U)\} \end{aligned}$$

is a sheaf.

Proof. This is a presheaf for the same reasons as the sheaf \mathcal{O}_X is a presheaf. The fact that it is a sheaf follows from the fact that \mathcal{O}_X is a sheaf and by definition of subspace topology; namely for any family of functions defined over an open cover $\{U_i \cap Y\}$ of some open set $U \cap Y$ we can reconstruct a function on U that restricted to $U \cap Y$ is as desired. \square

Recall that any map of sheaves of groups has well-defined kernel, see Lemma 2.1.

Definition 6.3. Let $Y \subset X$ be an analytic subvariety of a complex manifold X . The *ideal sheaf* of Y , denoted \mathcal{I}_Y , is the kernel of the restriction map $\mathcal{O}_X \rightarrow \mathcal{O}_Y$.

This just says that $\mathcal{I}_X(U)$ is the set of functions $f \in \mathcal{O}_X$ that vanish under the restriction map, i.e. that vanish at points of Y .

Since the restriction map is surjective, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

Notice that for any open set $U \subset X$, $\mathcal{I}_Y(U)$ is an ideal of $\mathcal{O}_X(U)$. (Indeed, if a function vanishes at Y then its product with any other function will also vanish.)

Definition 6.4. An analytic variety $V \subset U \subset \mathbb{C}^n$ is *irreducible* if V cannot be written as the union of two distinct analytic varieties $V_1, V_2 \subset U$, both distinct to V .

Lemma 6.5. If V is an irreducible analytic hypersurface given locally as $V = \{f = 0\}$, then f is irreducible in \mathcal{O}_p .

Proof. If $f = gh$ and neither of g and h are units and they are distinct, we could express V as the union of two distinct varieties: $V(g)$ and $V(h)$. (If one of them, was a unit then the vanishing set would be all of U .) \square

Definition 6.6. The *germ* of a set in the origin $0 \in \mathbb{C}^n$ is given by a subset $X \subset \mathbb{C}^n$. To subsets X, Y define the same germ if there exists an open neighbourhood $0 \in U \subset \mathbb{C}^n$ with $U \cap X = U \cap Y$. A germ is called *analytic* if there are functions $f_1, \dots, f_k \in \mathcal{O}_n$ such that X and $Z(f_1, \dots, f_k)$ define the same germ.

Now the adventure of proving that the ideal sheaf is a line bundle for hypersurfaces begins.

See Algebraic Geometry Lemma ?? for the proof that the ideal sheaf of an irreducible codimension-1 closed subscheme of a smooth scheme is a line bundle. Now I will try to prove this for the case of an analytic subvariety of a (smooth) complex manifold.

First notice that in the case that Y is an irreducible subvariety, this ideal is prime: if $fg \in \mathcal{I}_Y(U)$, then $V(f) \cup V(g) \supset Y \cap U$ so that $(V(f) \cup V(g)) \cap Y = Y \cap U$, and then it must be true that $V(f) \cap Y \subseteq V(g) \cap Y$ or that $V(f) \cap Y \supseteq V(g) \cap Y \cap U$ since Y is irreducible, and thus, say, $V(f) \supseteq Y \cap U$, that is, $I(V(f)) \subseteq I(Y \cap U)$ so that $f \in I(Y \cap U)$.

If we further ask that Y is a hypersurface, we want to show that \mathcal{I}_Y is locally principal. If $f \in \mathcal{I}_Y(U)$ is the defining function of some hypersurface Y of a variety X on an open set $U \subset X$, we shall have that $\mathcal{I}_Y(U) = (f)$. That is, if g also vanishes at $Y \cap U$, we want to see that $g = fh$ for some $h \in \mathcal{O}_X(U)$.

By Stacks Project 0AFT we need two ingredients: (1) the fact that the ring $\mathcal{O}_X(U)$ is a UFD for all $U \subset X$ and (2) the following lemma.

Lemma 6.7. The ideal of an irreducible analytic hypersurface is a height-1 prime ideal.

Proof. The fact that it is prime comes from the fact that Y is irreducible as explained above.

Now suppose that $0 \subset \mathfrak{p} \subseteq I$. Then $V(\mathfrak{p})$ is an analytic variety that contains Y . At regular points of both $V(\mathfrak{p})$ and Y , both are smooth manifolds, but since

Y is of codimension 1 and $V(\mathfrak{p})$ does not equal all of Y , we conclude that they coincide. Thus, in a neighbourhood of a regular point we have $\mathfrak{p} = I(V(\mathfrak{p})) = I$ by the holomorphic Nullstellensatz ??.

Now we shall use the Gauss Lemma ?? to prove:

Lemma 6.8. *The stalk $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.*

Proof. By induction on n . For $n = 0$ it is trivial. Suppose \mathcal{O}_{n-1} is a UFD. Then by the Gauss Lemma ??, $\mathcal{O}_{n-1}[w]$ is a UFD too. Thus we may express any Weierstrass polynomial g as a product of irreducible elements (uniquely up to multiplication by units).

Let $f \in \mathcal{O}_n$. We want to express f as a product of (unique up to multiplication by units) of irreducible elements. By Weierstrass Preparation Theorem ?? there is a Weierstrass polynomial $g \in \mathcal{O}_n[w]$ and a holomorphic function not vanishing on 0 (i.e. a unit of \mathcal{O}_n) such that $f = gh$. By the previous remark g is factored uniquely up to multiplication by units as $g = g_1 \dots g_m$. This shows existence of the factorization.

To prove uniqueness suppose that $f = f_1 \dots f_k$ for some irreducible $f_1, \dots, f_k \in \mathcal{O}_n$. Since f does not vanish in the w axis, neither can each f_i , so that we may decompose each of them as $f_i = g'_i h_i$ by Weierstrass Preparation Theorem. Since f_i is irreducible, it follows that g'_i is irreducible. Then we have that

$$f = gh = \prod g'_i \prod h_i$$

so by uniqueness in Weierstrass Preparation Theorem we conclude that $g = \prod g'_i$, and by uniqueness from the fact that $\mathcal{O}_n[w]$ is a UFD we conclude that g coincides with $\prod g'_i$ up to multiplication by units. \square

Lemma 6.9. *The ideal sheaf of an irreducible codimension-1 analytic subvariety of a smooth complex manifold is a line bundle.*

Proof. By definition, our subvariety Y is locally defined by the zeroes of a single holomorphic function defined on the ambient manifold X . That is, at every $p \in Y$ there is an open set $U \subset X$ such that $U \cap Y = f^{-1}(0)$ for some $f \in \mathcal{O}_X(U)$. Since U is an open set of a complex manifold, the ring $\mathcal{O}_X(U)$ is isomorphic to the ring \mathcal{O}_n of holomorphic functions on \mathbb{C}^n in a neighbourhood of the origin; that's by Definition 1.1 since a holomorphic function on a manifold is defined as such if its composition with the coordinate chart is a holomorphic function on \mathbb{C}^n .

By Lemma 6.5, since Y is irreducible it follows that f is irreducible and so is the ideal of functions vanishing on Y .

By Lemma 6.8, we prove identically as in Lemma ?? that the ideal of functions vanishing on Y is principal. \square

Note that that argument actually works backwards if we assume the ambient ring is Noetherian, see Stacks Project 0AFT.

Definition 6.10. Let $D := S$ be an effective Cartier divisor, that is, an analytic hypersurface of a complex manifold. The ideal sheaf $\mathcal{I}_S := \mathcal{O}_X(-S)$ of S is the dual line bundle of the line bundle associated to the divisor D , which is denoted by $\mathcal{O}_X(D)$.

7. HILBERT'S NULLSTELLENSATZ

Usually I think of Hilbert's Nullstellensatz as the statement that $I(V(I)) = \sqrt{I}$. Following are two simple initial observations.

It is immediate that $\sqrt{\mathfrak{p}} = \mathfrak{p}$ if \mathfrak{p} is a prime ideal (of any ring): if $f^k \in \mathfrak{p}$ then either $f \in \mathfrak{p}$ or $f^{k-1} \in \mathfrak{p}$, etc.

It is also easy to see that for any ideal I of functions or germs of functions (i.e. either $I \subset \mathcal{O}_X(U)$ or $I \subset \mathcal{O}_n$) we have $\sqrt{I} \subseteq I(V(I))$. Indeed, if $f^k \in I$, then $f^k(x) = 0$ for all $x \in V(I)$, and since \mathbb{C} has no nonzero divisors we see that $f(x) = 0$, that is $f \in I(V(I))$.

It remains to show that $I(V(I)) \subseteq \sqrt{I}$. We shall first suppose that I is prime and prove that $I(V(I)) \subseteq I$. (The complete statement for the radical follows from a purely algebraic process, see [Dem, Theorem 4.22].)

We will need a few preparatory results. First recall that S is an integral ring over R if every element of S is expressed as a root of a monic polynomial with coefficients in R . (See Stacks Project 00GI.)

It will be handy to keep in mind the following lemma (Stacks Project 00GH):

Lemma 7.1. *Let $R \rightarrow S$ be a ring map. The following are equivalent*

- (1) $R \rightarrow S$ is finite,
- (2) $R \rightarrow S$ is integral and of finite type, and
- (3) there exist $x_1, \dots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R .

This essentially says that an integral extension $R \rightarrow S$ gives S the structure of a finitely generated R -module (I think experts call this a “finite R -module”) whose generators are integral. This just means that $S = R[x_1, \dots, x_k]$ and $x_i \in S$ is the root of a monic polynomial in R .

Proposition 7.2. $\mathcal{O}_n/\mathcal{I}$ is a finite integral extension of \mathcal{O}_d .

Proof. For this it's enough to check that the map $\mathcal{O}_d \rightarrow \mathcal{O}_n/\mathcal{I}$ is finite. That is, that every element of $\mathcal{O}_n/\mathcal{I}$ can be expressed as a tuple of elements of \mathcal{O}_d . (We can think of this tuple as a linear combination, or cycle.) Step 1: produce some Weierstrass polynomials. . . \square

Lemma 7.3. *If $\mathcal{I} \subset \mathcal{O}_n$ is prime and $A = V(\mathcal{I})$, then $\mathcal{I}_{A,0} = \mathcal{I}$.*

Proof. Proposition 7.2 says: take a polynomial $f \in \mathcal{O}_n$ and consider its equivalence class $\tilde{f} \in \mathcal{O}_n/\mathcal{I}$. Then \tilde{f} is the root of some monic polynomial $t^r + b_{r-1}t^{r-1} + \dots + b_1t + b_0$ with $b_i \in \mathcal{O}_d$. That is, putting f instead of t we get an equation

$$f^r + b_1f^{r-1} + \dots + b_r = 0 \quad \text{mod } \mathcal{I}.$$

In other words, the holomorphic function $f^r + b_1f^{r-1} + \dots + b_r$ is in \mathcal{I} . Since f vanishes on $(A, 0)$, this says that b_r vanishes in $(A, 0)$ too. Now suppose that this implied that the class $\tilde{b}_r = 0$. We are supposed to use that Proposition 7.2 also tells us is that the above equation is irreducible, which I think means that $F = \sum b_it^i \in \mathcal{O}_d[t]$ is irreducible. In turn this gives that $r = 1$. \square

Let's just store what was learned today:

- (1) **Noether normalization lemma.** See this video. Let k be a (infinite) field, $A \neq 0$ a finitely generated k -algebra. Then there exists elements $y_1, \dots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \dots, y_r]$. That is,

$$\begin{array}{c} A \\ \left| \begin{array}{c} \text{integral} \\ \text{ext.} \end{array} \right. \\ k[y_1, \dots, y_r] \\ \left| \begin{array}{c} \text{transc.} \\ \text{ext.} \end{array} \right. \\ k. \end{array}$$

- (2) This can be used to prove the weak Nullstellensatz, i.e. let k be an algebraically closed field, and $I \subseteq k[x_1, \dots, x_n]$ an ideal such that $V(I) = \emptyset$. Then $I = k[x_1, \dots, x_n]$.

To prove it using Noether's normalization lemma, suppose that the $I \neq k[x_1, \dots, x_n]$. Let \mathfrak{m} be a maximal ideal such that $I \subseteq \mathfrak{m}$. Then $A = k[x_1, \dots, x_n]/\mathfrak{m} = k[y_1, \dots, y_n]$, where the first string is algebraically independent while the last string is integral. But because the whole extension is algebraic, then y_1, \dots, y_n are integral over k . Then we can map $y_i \mapsto \alpha_i$, a root of its defining polynomial over k . This means we do have a point where all polynomials in \mathfrak{m} vanish.

Note. I'm not sure what we mean with "the first string" and the "second string".

- (3) Also reviewed Misha's proof of the Nullstellensatz. Looks accessible, though it is not holomorphic!
- (4) Perhaps start in Demailly just after Prop. 4.13.

8. SINGULARITIES AND DIMENSION

The key statement is that the set of smooth points of a complex variety is dense.

9. RIEMANN-ROCH FORMULAS

Theorem 9.1 (Riemann-Roch for curves). *Let D be a divisor on a compact Riemann surface X , that is, D is a collection of d points on X .*

$$(9.1.1) \quad h^0(D) - h^0(K - D) = d - g + 1$$

Exercise 9.2. Prove that if X is a complex smooth curve with $g \geq 2$ then $h^0(T_X) = 0$.

Proof. First you notice that the dimension of the holomorphic 1-forms is the genus. This is just because the genus p_a is defined as $h^1(\mathcal{O}_X)$, which is just $h^0(K_X)$ by Serre duality. Then you say well if you have a holomorphic vector field, pair it with the nonzero holomorphic 1-form. This gives a nonzero function, but it must be constant because X is compact. Then it is actually nowhere vanishing. This says the vector field cannot vanish anywhere, which means the tangent bundle of the curve is trivial. Apparently elliptic curves, i.e. $g = 1$ are the □

Theorem 9.3 (Riemann-Roch for line bundles on surfaces). *Let L be a line bundle on a complex surface X , and $K_X = \Omega^2 X$ the canonical bundle of X . Then*

$$(9.3.1) \quad \chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}(L, L - K_X)$$

10. ADJUNCTION FORMULA

Perhaps the most accessible statement that could be interpreted as “adjunction formula” is that the line bundle associated to a smooth hypersurface is the normal bundle.

Theorem 10.1 (Adjunction formula I). *If Y is a smooth hypersurface of a smooth complex manifold X , then $\mathcal{O}_X(Y) \simeq \mathcal{N}_{Y/X}$.*

Lemma 10.2. *Let S be any curve on a surface X . Then*

$$(10.2.1) \quad 2p_a(S) - 2 = (S, S - K_X)$$

Proof. If S is nonsingular we may use the normal bundle short exact sequence. If S is singular we need to use the ideal sheaf short exact sequence, which means we think of S either as a closed embedded subscheme of X or as an analytic subvariety. Taking Euler class we obtain

$$\chi(\mathcal{O}(-S)) + \chi(\mathcal{O}_S) = \chi(\mathcal{O}_X)$$

Since $p_a(S) := 1 - \chi(\mathcal{O}_S)$, we obtain that $p_a(S) = 1 - \chi(\mathcal{O}_X) - \chi(\mathcal{O}(-S))$. To compute $\chi(\mathcal{O}(-S))$ we use Riemann-Roch formula ?? for curves on surfaces, which gives

$$\chi(\mathcal{O}(-S)) = \chi(\mathcal{O}_X) + \frac{(\mathcal{O}(-S), \mathcal{O}(-S) + \omega_X^*)}{2}$$

Taking duals on both entries of the intersection form and substituting in our previous expression for $p_a(S)$ we obtain the result. \square

Exercise 10.3. Let S be a singular, irreducible complex curve on a K3 surface. Prove that $(S, S) \geq 0$.

Proof. This is just an application of Eq. 10.2.1. Since $K_X = \mathcal{O}_X$ it suffices to show that $p_a(S)$ is not zero. This follows fact that S is singular, since its arithmetic genus is defined as the genus of its normalization, which must be strictly positive since there exists at least one singularity. \square

11. CHOW’S THEOREM

Proposition 11.1. *Every codimension- p submanifold of a complex n -manifold has slice charts, i.e. for any $p \in M$ there’s a coordinate chart $U \ni p$ such that points of M have coordinates $(z_1, \dots, z_n, 0, \dots, 0)$.*

Proof. This may be taken as the definition of smooth (real or complex) submanifold, or as a proposition using inverse function theorem (real or complex). \square

Proposition 11.2. *Stalk of holomorphic functions is a PID.*

Proof. This can be proved via Weierstrass Preparation theorem ?? which implies Lemma ??, and this in turn implies Lemma ??.

Also, it can be shown directly for smooth hypersurfaces as in [Lee24] Lemma 3.38. \square

Theorem 11.3 (Chow for hypersurfaces). *Every complex codimension-1 submanifold of \mathbb{CP}^n is algebraic.*

12. AMPLENESS

Definition 12.1. Let L be a holomorphic bundle on a complex manifold X . A point $x \in X$ is a *base point* of L if $s(x) = 0$ for all $s \in H^0(X, L)$.

Definition 12.2. Let L be a holomorphic bundle on a complex manifold X . The *base locus* $\text{Bs}(L)$ is the set of all base points of L .

Proposition 12.3. *Let L be a holomorphic line bundle on a complex manifold X and suppose that $s_0, \dots, s_N \in H^0(X, L)$ is a basis. Then*

$$\begin{aligned} \varphi_L : X \setminus \text{Bs}(L) &\longrightarrow \mathbb{P}^N \\ x &\longmapsto (s_0(x) : \dots : s_N(x)) \end{aligned}$$

defines a holomorphic map such that $\varphi_L^ \mathcal{O}_{\mathbb{P}^N}(1) \cong L|_{X \setminus \text{Bs}(L)}$.*

Definition 12.4. The map φ_L in Proposition 12.3 is said to be associated to the *complete linear system* $H^0(X, L)$ (i.e. the sections of L are what we call a complete linear system) whereas a subspace of $H^0(X, L)$ is called a *linear system* of L .

Definition 12.5. L is *globally generated* by the sections s_0, \dots, s_N if $\text{Bs}(L, s_0, \dots, s_N) = \emptyset$.

Definition 12.6. A line bundle L on a complex manifold is called *ample* if for some $k > 0$ and some linear system in $H^0(X, L^k)$ the associated map φ is an embedding.

Exercise 12.7. Let L be an ample bundle on a K3 surface M . Prove that $L^{\otimes 2}$ is globally generated (that is, for each $x \in M$ there exists a section $h \in H^0(L^{\otimes 2})$ which does not vanish in x).

Proof. As an outline for exposition:

- (1) Show that L has sections (by ampleness+Kodaira Vanishing+Riemann-Roch), pick a section C and note that if $L^{\otimes 2}$ had base points they would be in C .
- (2) Show that restriction map is surjective (Kodaira Vanishing).
- (3) Show that inclusion map is not surjective. For this it's enough to show that $\dim |D| = \dim |D - p| + 1$. This follows by writing the formula of Riemann-Roch for both of these Weil divisors (have to pass to Weil divisors). But you will need to make sure that $h^0(K - D)$ and $h^0(K - (D - p))$ are zero.

First we need to show that L has sections. To see this I will show that since L is ample, $L^2 > 0$. Indeed, if mL is very ample we apply by Riemann-Roch, Kodaira Vanishing and the fact that $\chi(\mathcal{O}_X) = 2$ on a K3 to obtain $h^0(mL) = 2 + \frac{1}{2}m^2L^2$ which must be positive since mL has sections by being very ample. Notice L^2 cannot be zero as $h^0(mL) = 2$ would imply we have an embedding of a K3 surface into \mathbb{P}^1 .

Applying again Riemann-Roch and Kodaira vanishing we see that $h^0(L) > 0$. Let C be a section of L .

Notice that if $L^{\otimes 2}$ had any base points, they would have to be on C . Indeed, since C is the vanishing set of a section $s \in H^0(L)$, we must have $s \otimes s \in H^0(L^{\otimes 2})$ vanishing in any base point of $L^{\otimes 2}$.

Then we consider the ideal exact sequence for C and tensor by $L^{\otimes 2}$ to obtain

$$0 \longrightarrow L^{\otimes 2}(-C) \longrightarrow L^{\otimes 2} \longrightarrow L^{\otimes 2} \otimes \mathcal{O}_C \longrightarrow 0$$

Notice that the term on the left is actually L since the ideal sheaf $\mathcal{O}_M(-C) = \mathcal{I}_C$ is dual to L because C is defined as the vanishing set of a section of L . Passing to cohomology we obtain

$$H^0(L^{\otimes 2}) \longrightarrow H^0(L^{\otimes 2} \otimes \mathcal{O}_C) \longrightarrow H^1(L)$$

where the latter term vanishes by Kodaira Vanishing because L is ample. This says that every section of $L^{\otimes 2}|_C$ is the restriction to C of a section of $L^{\otimes 2}$. In turn this shows that it's enough to show that the bundle $L^{\otimes 2}$ has no base points along C .

By adjunction formula for (possibly singular) curves on smooth surfaces and by the fact that M is smooth, we see that $2p_a(C) - 2 = (L.L)$ so that

$$4p_a(C) - 4 = (2L.L) = \deg_C(L^{\otimes 2})$$

Notice that since $L^2 > 0$ we exclude the cases that $p_a(C) = 0, 1$.

To conclude pick a point $p \in C$. Saying that p is not a base point of $L^{\otimes 2}|_C$ is the same as saying that not every section of $L^{\otimes 2}|_C$ vanishes at p , that is, that $h^0(L^{\otimes 2}|_C(-p)) < h^0(L^{\otimes 2}|_C)$.

Now I will prove that since the degree of $L^{\otimes 2}|_C$ and $L^{\otimes 2}|_C(p)$ is greater than $2p_a(C)$ we have that $h^0(\omega_C \otimes (L^{\otimes 2}|_C)^\vee) = 0 = h^0(\omega_C \otimes (L^{\otimes 2}|_C(-p))^\vee)$. For this we need to see these line bundles as Weil divisors by simply taking sections, whose vanishing sets are finite sets of points with multiplicities. Then their degree is the sum of the multiplicities. This definition makes degree additive with respect to tensor product (we may take local sections and sum degrees). By Riemann-Roch on the curve C , we know that $\deg \omega_C = 2p_a - 2$. Then

$$\deg_C(\omega_C \otimes L^{\otimes 2}) = \deg_C(\omega_C) + \deg_C(L^{\otimes 2}) = 2p_a - 2 - (4p_a(C) - 4) < 0$$

and a similar computation works for $L^{\otimes 2}(-p)$ as it has the same degree of $L^{\otimes 2}$ minus 1. This implies that neither of these bundles can have sections since any section would provide a linearly equivalent effective divisor. This would be a contradiction since both $L^{\otimes 2}$ and $L^{\otimes 2}(-p)$ have sections. Therefore $h^0(\omega_C \otimes (L^{\otimes 2}|_C)^\vee) = 0 = h^0(\omega_C \otimes (L^{\otimes 2}|_C(-p))^\vee)$ as claimed.

Finally we apply Riemann-Roch to $L^{\otimes 2}$ and $L^{\otimes 2}(-p)$ to obtain that $h^0(L^{\otimes 2}) > h^0(L^{\otimes 2}(-p))$. \square

13. BERTINI'S THEOREM

14. SERRE DUALITY

Theorem 14.1 (Serre duality). [Voi02] II.5.32. *The pairing*

$$H^q(X, \mathcal{E}) \otimes H^{n-q}(X, \mathcal{E}^* \otimes K_X) \rightarrow H^n(X, K_X) \cong \mathbb{C}$$

is perfect.

So when you put the dual \vee on one of these you get isomorphism.

Our course version says:

$$H^k(X, \mathcal{L})^\vee = H^{n-k}(X, \omega_X \otimes \mathcal{L}^*)$$

15. KODAIRA VANISHING THEOREM

Theorem 15.1 (Kodaira Vanishing). *Suppose k is a field of characteristic 0, and X is a smooth projective k -variety. Then for any ample invertible sheaf L , $H^i(X, K_X \otimes L) = 0$ for $i > 0$.*

Proof. No proof in [Vak25]. \square

16. KODAIRA EMBEDDING THEOREM

The forward implication is easy and sometimes not considered as part of the theorem.

Compare this theorem with Nakai-Moishezon Criterion 23.1.

Theorem 16.1 (Kodaira Embedding). *Suppose M is a compact complex manifold. A holomorphic line bundle $L \rightarrow M$ is ample if and only if it is positive. Thus M is projective if and only if it admits a holomorphic line bundle.*

Proof. The easy implication as follows. If L is ample then there is N such that $L^{\otimes N}$ is very ample, meaning by Misha's definition that the canonical map is an embedding such that $L^{\otimes N} = \varphi^* \mathcal{O}(1)$, which has degree 1 by definition as in [Vak25, 15.4.14]. \square

17. HYPERCOMPLEX MANIFOLDS

Exercise 17.1. Let M be a compact hypercomplex manifold of real dimension 4, equipped with a quaternionic Hermitian structure, and V the space of closed $SU(2)$ -invariant 2-forms. Prove that V is finite-dimensional.

Proof by Arpan. Consider the Hodge star operator with respect to the Riemannian metric on M .

- (1) A closed anti-self-dual form is harmonic. Indeed, by some characterization a form is harmonic \iff it is d -closed and d^* -closed. So if α is self-dual and d -closed we get $d^* \alpha = (-1)^{-\bullet} * d * \alpha = 0$ since $d\alpha = 0$.
- (2) The space of harmonic forms is finite-dimensional (analysis, cf. Fredholm theory).
- (3) I should be able to prove that an $SU(2)$ -invariant closed form is anti self-dual.
 - (a) Define $\Lambda^+ :=$ self-dual 2-forms and $\Lambda^- :=$ anti-self dual 2-forms. Notice that $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, this is by 2.1, since we can see $*$ as an involutive endomorphism of Λ^2 , so that it has eigenvalues ± 1 , given the desired decomposition as its eigenspace decomposition.
 - (b) **Claim.** $\Lambda^+ = \text{span}(\omega_I, \omega_J, \omega_K)$.
 - (c) **Claim.** ω_I, ω_J and ω_K are $SU(2)$ -invariant.
The last two items imply that $SU(2)\Lambda^+ = \Lambda^+$.
 - (d) **Claim.** $SU(2) \curvearrowright \Lambda^+$ has no fixed points.
We conclude that if α is $SU(2)$ -invariant (i.e. a fixed point of $SU(2) \curvearrowright \Lambda^2$) its positive part will vanish, so that α is anti-self-dual.

\square

Proof. Idea: show that Hodge star $*\omega$ is in the $SU(2)$ -orbit of ω and conclude that $\int \omega \wedge *\omega = 0$, implying that $\|\omega\| = 0$. \square

Exercise 17.2.

Proof. Idea: find a counterexample. The easiest should be a Kummer surface. It looks possible to find an almost hypercomplex structure on \mathbb{C}^2 passing to the torus and then to the \mathbb{Z}_2 quotient, but not clear what will happen after blowing up. \square

18. DEFORMATION THEORY

Here I discuss deformations in the category of complex spaces.

Let's just do a summary of [Voi02, Chapter 9].

Let \mathcal{X} be a complex manifold, B a complex manifold and $\phi : \mathcal{X} \rightarrow B$ a holomorphic map. We call ϕ a *family* if it is a proper holomorphic submersion.

A theorem by Ehresmann says that any “deformation” of smooth real manifolds is trivial:

Theorem 18.1 (Ehresmann). *If $\phi : \mathcal{X} \rightarrow B$ is a proper submersion of differentiable manifolds (no complex structure), with B contractible and $0 \in B$ some distinguished point, then the total space \mathcal{X} is trivial as a manifold over B , i.e. there exists a diffeomorphism*

$$T : \mathcal{X} \xrightarrow{\cong} X_0 \times B$$

of manifolds over B , i.e. commuting with projection.

Proof. The main tool is the theorem of tubular neighbourhoods, which says that there is a diffeomorphism of a neighbourhood W of X_0 in \mathcal{X} and a neighbourhood of X_0 in its normal bundle. (We have a normal bundle because X_0 is a submanifold of \mathcal{X} .)

In particular this gives a differentiable retraction $T_0 : W \rightarrow X_0$, which allows us to consider the map

$$(T_0, \phi) : W \rightarrow X_0 \times B.$$

Since T_0 is a retraction (it's the identity when restricted to X_0) and ϕ is a submersion, the map (T_0, ϕ) has nonsingular differential along X_0 . Then we can invert it locally along X_0 , and since X_0 is compact we obtain a whole neighbourhood $W' \subset W \subset \mathcal{X}$ where the differential is nonzero and thus $(T_0, \phi)|_{W'}$ is an embedding (using the inverse function theorem).

Finally we restrict the image of $(T_0, \phi)|_{W'}$ to a neighbourhood $U \subset B$ of 0, using properness of ϕ to ensure that $\phi^{-1}(U) \subset W'$ — though I'm not exactly sure how. We have constructed a diffeomorphism $\phi^{-1}(U) \rightarrow X_0 \times U$ commuting with the projection.

This proof was only local; the global version is in [Voi02] though it's not used in the rest of the text. \square

In the complex situation, if $\phi : \mathcal{X} \rightarrow B$ is a family of complex manifolds and $0 \in B$, then “up to replacing B by a neighbourhood of 0”, there exists a smooth trivialization $T = (T_0, \phi) : \mathcal{X} \rightarrow X_0 \times B$ such that the fibres of T_0 are complex submanifolds of \mathcal{X} .

Now let $\phi : \mathcal{X} \rightarrow B$ be a family of complex manifolds. Recall the pullback bundle ϕ^*B and the fact that the differential ϕ_* can be thought of as a gadget that maps tangent vectors to \mathcal{X} not to vectors tangent to B but to vectors in the pullback bundle ϕ^*B . Restricting everything to the fibre $X = \phi^{-1}(0)$ we get

$$(18.1.1) \quad 0 \longrightarrow T_X \longrightarrow T_{\mathcal{X}|X} \longrightarrow \phi^*T_{B|X} \longrightarrow 0.$$

There's what seems to be a really nice result saying that $H^1(\mathcal{F})$ parametrizes isomorphism classes of extensions of \mathcal{F} by the trivial bundle, but for now let me just define it roughly as follows. The category of \mathcal{O}_X is most surely abelian with enough injectives. This means that objects have associated complexes, and we can take cohomologies of these complexes. This says, roughly, that short exact sequences of vector bundles have associated cohomology long exact sequences. Then the Kodaira-Spencer map is the map

$$\rho : T_{B,0} = H^0(X, \phi^* T_{B|X}) \rightarrow H^1(X, T_X)$$

induced by the long exact sequence associated to 18.1.1. Here the equality $T_{B,0} = H^1(X, \phi^* T_{B|X})$ is just because the pullback of the tangent bundle of B to any fiber is trivial.

Now I will explain first order deformations. Consider a family $\mathcal{X} \rightarrow B$. We introduce a complex analytic subscheme B_ε of B giving an ideal sheaf \mathcal{M}_0^2 where \mathcal{M}_0 is the maximal ideal of $\mathcal{O}_{B,0}$ consisting of the holomorphic functions vanishing at 0. This means that a function in \mathcal{M}_0^2 is (a finite sum of terms) of the form $f = gh$ for $g, h \in \mathcal{M}_0$, which means that $df = d(gh) = h dg + g dh = 0$ at 0 since both g and h vanish at 0. That is, this sheaf contains functions that vanish at 0 and whose first derivative vanishes at 0.

This means that B_ε is a point: it's the vanishing locus of the set of functions that vanish at a single point. However, it is endowed with an unusual structure sheaf. Let v be a tangent vector to B at 0 and f a function in the strange sheaf $\mathcal{M}_0/\mathcal{M}_0^2$. Then by the computation above, since v is a differential operator, v annihilates f . This means that this sheaf detects not only the point 0 but also any tangent vector at 0. Let's denote $B_\varepsilon = \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)$ just because its structure sheaf is $\mathbb{C}[\varepsilon]/(\varepsilon^2)$.

In fact, this is so throughout deformation theory: $\text{Spec } k[\varepsilon]/(\varepsilon^2)$ is topologically a point! But it's a *fat point*: the support of its structure sheaf is a point, but as a scheme, it's not a point. And the same happens with the family \mathcal{X} over $\text{Spec } k[\varepsilon]/(\varepsilon^2)$, topologically it's just the fibre X ! But its structure sheaf is different: "it contains the directions in \mathcal{X} which are normal to X ".

Then a cocycle construction shows that for B of complex dimension 1, the first order deformations $\phi : X_\varepsilon \rightarrow B_\varepsilon$ associated to the family $\mathcal{X} \rightarrow B$ are "exactly parametrised" by the group $H^1(X, T_X)$.

Exercise 18.2. Let C be a smooth genus g curve which can be embedded in a K3 surface M , and \mathcal{X} "the family of all deformations" of C in M . Prove that $\dim \mathcal{X} \leq g$.

Proof. For this exercise I assume that $\mathcal{X} \rightarrow B$ is a family such that any fiber is contained in the K3 surface M . By the result explained above all we need to do is compute $h^1(X, T_X)$. Indeed, the dimension of \mathcal{X} is described well enough locally via the first order deformation $X_\varepsilon \rightarrow B_\varepsilon = \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)$. In other words, the dimension of the space of first order deformations encodes the local structure of the (hypothetical) deformation space and thus the dimensions of both coincide. Notice that this argument works just as well if we don't assume that all the fibres of \mathcal{X} are contained in the K3 surface M . For now I would just claim that this is indeed the case, i.e. that any family of deformations of C has the same dimension.

As for how to compute $h^1(T_C)$, the only thing that makes sense is using the normal bundle exact sequence. This gives a long exact sequence in cohomology,

$$0 \rightarrow H^0(T_C) \rightarrow H^0(T_X|_C) \rightarrow H^0(N) \rightarrow H^1(T_C) \rightarrow H^1(T_X|_C) \rightarrow H^1(N) \rightarrow 0$$

and then I just need to compute neighbouring terms of $H^1(T_X)$, namely $H^0(N)$ and $H^1(T_X|_C)$. In fact, an old computation shows that $h^0(N) = g + 1$. About $H^1(T_X|_C)$ I'm not sure... Notice that if we want $h^1(T_C) \leq g$ we need the map $H^0(N) \rightarrow H^1(T_C)$ to have kernel, so $H^0(T_X|_C)$ cannot be zero!

Other notes. Isn't the restricted tangent bundle just the pullback bundle by the inclusion? At every point we put the holomorphic tangent space of the K3 surface? Also see wiki on Teichmüller space for some remarks about the complex structure approach. \square

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