LIE ALGEBRAS

github.com/danimalabares/stack

Contents

1. Basic definitions and examples

Definition 1.1. An *algebra* is a vector space over a field $\mathbb{F}$ endowed with a binary operation that is bilinear

$$a(\lambda b + \mu c) = \lambda ab + \mu ac$$
$$(\lambda b + \mu c)a = \lambda ba + \mu ca$$

Definition 1.2. A *Lie algebra* is an algebra $\mathfrak{g}$ with a product $[\cdot,\cdot]$ we call *bracket* that satisfies

(1) $[x,x] = 0 \qquad \forall x \in \mathfrak{g}$,
(2) (Jacobi identity.)

Definition 1.3. A *simple Lie algebra* is a Lie algebra that has no nontrivial proper ideals.

As a rule of thumb I keep in mind the following silly computation:

$$\log\det A = \log\prod_i \lambda_i = \sum \log\lambda_i = \text{“trlog}A''$$

And recall that exponent map goes from $T_eG = \mathfrak{g} \to G$, so that logarithm would go from $G \to \mathfrak{g}$. This is why I remember that the condition on a classical Lie group of *having determinant 1* goes to *having vanishing trace* in the Lie algebra. (Because $\log 1 = 0$.)

Example 1.4. (1) The *special linear Lie algebra*

$$\mathfrak{sl}_n = \{\mathrm{Mat}_n | Tr(A) = 0\}$$

which is just obvious from the slogan above.

(2) The *special orthogonal Lie algebra*

$$\mathfrak{so}_n = \{A \in \mathrm{Mat}_n | A + A^{\mathbf{T}} = 0\}$$

which is obvious from: $\mathrm{SO}(n) =$isometries, so $\langle v, v\rangle = \langle Av, Av\rangle = \langle v, A^{\mathbf{T}}Av\rangle$, so $\mathrm{SO}(n) = \{A \in \mathrm{Mat}_n : A^{-1} = A^{\mathbf{T}}\}$, and then $0 = \log 1 = \log(AA^{\mathbf{T}}) = \log A + \log A^{\mathbf{T}}$.

(3) The *symplectic Lie algebra*

$$\mathfrak{sp}_{2n} = \{A \in \mathrm{Mat}_{2n} : \Omega A + A^{\mathbf{T}}\Omega = 0\}$$

where $\Omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$.

Pending. Why this makes sense?

Example 1.5. If A is an associative Lie algebra, then A with the bracket $[a, b] = ab - ba$ is a Lie algebra, denoted by A_-. (It is an exercise to verify Jacobi's identity.) This gives $\mathfrak{gl}_V := \mathrm{End}(V)_-$.

Definition 1.6. A *Lie subalgebra* is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

Definition 1.7. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an *ideal* if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

Recall that a bilinear form $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is *invariant* if $([a, b], c]) = (a, [b, c])$ for all $a, b, c \in \mathfrak{g}$.

Definition 1.8. For any finite dimensional Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$ we write

$$\mathrm{ad}_a = [a, -]$$

Exercise 1.9. Show that the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $a \mapsto \mathrm{ad}_a$ is a representation, i.e. a morphism of Lie algebras.

Definition 1.10. The *Killing form* is

$$\kappa(x, y) = \mathrm{Tr}_{\mathfrak{g}} \mathrm{ad} x \mathrm{ad} y$$

Exercise 1.11. Prove that an invariant bilinear form on a simple Lie algebra must in fact be symmetric.

Proof. (David.) It's enough to show that $\mathfrak{g}$ is *perfect*, i.e. that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. In this case, let $a, b \in \mathfrak{g}$ and suppose that $b = [x, y]$. Then

$$\begin{aligned}(a, b) &= (a, [x, y]) = (a, -[y, x]) = (-[a, y], x) = ([y, a], x) \\ &= (y, [a, x]) = (y, -[x, a]) = (-[y, x], a) = ([x, y], a) = (b, a)\end{aligned}$$

To confirm that $\mathfrak{g}$ is perfect just observe that $[\mathfrak{g}, \mathfrak{g}]$ is a nontrivial ideal of $\mathfrak{g}$. □

Definition 1.12. A *semisimple Lie algebra* is a direct sum of simple ones

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$$

Theorem 1.13 (Cartan)**.** *The finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.*

2. Nilpotent and solvable Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. Define

$$\mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^2 := [\mathfrak{g}, \mathfrak{g}^1], \quad \ldots, \mathfrak{g}^n := [\mathfrak{g}, \mathfrak{g}^{n-1}]$$

and

$$\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(2)} := [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \ldots, \mathfrak{g}^{(n)} := [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}].$$

Definition 2.1. A Lie algebra $\mathfrak{g}$ is

- *nilpotent* if there exists n such that $\mathfrak{g}^n = 0$,
- *solvable* is there exists n such that $\mathfrak{g}^{(n)} = 0$.

3. Lie's theorem

Lie's theorem says that solvable Lie algebras over closed fields of characterstic not zero have weights.

Definition 3.1. Let $\mathfrak{h}$ be a Lie algebra, $\pi : \mathfrak{h} \to \mathfrak{gl}_V$ a representation of $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$. The *weight space* of $\mathfrak{h}$ attached to λ is

$$V_\lambda^{\mathfrak{h}} := \{v \in V | \pi(h)v = \lambda(h)v, \ \forall h \in \mathfrak{h}\}.$$

If $V_\lambda^{\mathfrak{h}} \neq 0$, we say that λ is a *weight* for π.

Theorem 3.2 (Lie's theorem)**.** *Let $\mathfrak{g}$ be a solvable Lie algebra and π a representation of $\mathfrak{g}$ on a finite dimensional vector space $V \neq 0$, over an algebraically closed field $\mathbb{F}$ of characterstic 0. Then there exists a weight $\lambda \in \mathfrak{g}^*$ for π (that is, $V_\lambda^{\mathfrak{g}} \neq \{0\}$).*

Exercise 3.3. Show the following two corollaries of Lie's theorem:

- for all representations π of a solvable Lie algebra $\mathfrak{g}$ on a finite dimensional vector space $\mathcal{V}$ over an algebraically closed field $\mathbb{F}$, $\mathrm{char}\mathbb{F} = 0$, there exists a basis for V for which the matrices of $\pi(\mathfrak{g})$ are upper triangular;
- a solvable subalgebra $\mathfrak{g} \subset \mathfrak{gl}_V$, (where V is finite-dimensional over an algebraically closed field $\mathbb{F}$, $\mathrm{char}\mathbb{F} = 0$), is contained in the subalgebra of upper triangular matrices over $\mathbb{F}$ for some basis of V.

4. Generalized Eigenspaces and Generalized Weight spaces

Definition 4.1. A *generalized eigenspace of $A \in End(V)$ with eigenvalue* λ is

$$V_\lambda = \{v \in V : (A - \lambda \mathrm{Id})^N = 0 \text{ for some positive integer } N\}.$$

It turns out that any linear operator on an algebraically closed field gives a decomposition into generalized eigenspaces via Jordan canonical form:

Proposition 4.2. *Let A be a linear operator on a finite-dimensional vector space V over an algebraically closed field $\mathbb{F}$, and let $\lambda_1, \ldots, \lambda_s$ be all eigenvalues of A, and $n_1, \ldots, n_s$ their multiplicities. Then one has the generalized eigenspace decomposition:*

$$V = \bigoplus_{i=1}^{s} V_{\lambda_i}, \qquad \dim V_{\lambda_i} = n_i.$$

In particular, for a Lie algebra $\mathfrak{g}$ with a representation (i.e. Lie algebra morphism) $\pi : \mathfrak{g} \to \mathrm{End}(V)$ we have

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda^a, \qquad V_\lambda^a = \{v \in V : (\pi(a)v - \lambda \mathrm{Id})^N v = 0 \text{ for some } N \in \mathbb{N}\}$$

And even more particularly, for the adjoint representation we have

$$V = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_\lambda^a, \qquad \mathfrak{g}_\alpha^a = \{b \in \mathfrak{g} : ([a, \cdot] - \alpha \mathrm{Id})^N b = 0, \text{ for some } N \in \mathbb{N}\}$$

And a little more generally we have

Definition 4.3. Let $\mathfrak{h}$ be a Lie algebra with a representation π on a vector space V, and $\lambda \in \mathfrak{h}^*$. A *generalized weight space of* $\mathfrak{h}$ *in* V *attached to* λ is

$$V_\lambda^{\mathfrak{h}} = \left\{ v \in V : (\pi(a) - \lambda(a)\mathrm{Id})^N v = 0, \begin{smallmatrix} \text{for some } N \in \mathbb{N} \\ \text{depending on } a \in \mathfrak{h} \\ \text{for all } a \in \mathfrak{h} \end{smallmatrix} \right\}$$

Under the right conditions, a nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$ permits decomposing V as a direct sum of generalized weight spaces of $\mathfrak{h}$. Namely,

Theorem 4.4. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and π its representation on a finite-dimensional vector space V, over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $\mathfrak{h}$ be a nilpotent subalgebra of $\mathfrak{g}$. Then the following equalities hold:*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{\mathfrak{h}}$$

$$\pi\left(\mathfrak{g}_\alpha^{\mathfrak{h}}\right) V_\lambda^{\mathfrak{h}} \subseteq V_{\lambda+\alpha}^{\mathfrak{h}} \tag{4.4.1}$$

Which in the case of the adjoint representation, looks like:

Definition 4.5. The *generalized root space decomposition* of a Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbb{F}$ of characteristic zero with respect to a nilpotent subalgebra $\mathfrak{h}$ is the generalized weight space decomposition with respect to the adjoint representation. That is,

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha^{\mathfrak{h}}$$

and it has the property that

$$[\mathfrak{g}_\alpha^{\mathfrak{h}}, \mathfrak{g}_\beta^{\mathfrak{h}}] \subseteq \mathfrak{g}_{\alpha+\beta}^{\mathfrak{h}} \tag{4.5.1}$$

It is important to make the distinction between the generalized weight space decomposition and the generalized root space decomposition; "we will see its convenience in later lectures, as we try to better understand the functionals α appearing in the decomposition".

Exercise 4.6. Take $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{h} = \{\text{diagonal matrices}\}$. Find the generalized weight space decomposition in both the tautological and the adjoint representations, and check the inclusions 4.4.1 and 4.5.1.

5. Zarisky Topology and Regular Elements

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra of dimension d. For every element $a \in \mathfrak{g}$ the characeristic polynomial of ad_a must be of the form

$$\det_{\mathfrak{g}}(\mathrm{ad}_a - \lambda \mathrm{Id}) = (-\lambda)^d + c_{d-1}(-\lambda)^{d-1} + \ldots + \det(\mathrm{ad}_a).$$

[to be honest I don't see why the constant term must be de determinant of ad_a, but OK...] since $[a, a] = 0$, $\det(\mathrm{ad}_a) = 0$, i.e., the constant term vanishes.

Exercise 5.1. Show that c_j is a homogeneous polynomial on $\mathfrak{g}$ of degree $d - j$.

Proof. The term "polynomial on $\mathfrak{g}$" is not clear. But we follow [Kac10] to interpret this is only as being a polynomial on the a^i where $a = a^i X_i$ for some basis X_i of $\mathfrak{g}$.

Then we just notice that denoting $[X_i, X_j] := L_{ij}$ we have

$$[a, X_j] = [a^i X_i, X_j] = a^i [X_i, X_j] = a^i L_{ij}$$

which is a linear combination of the a^i. Then the matrix representation of ad_a in terms of the basis X_i is

$$[\mathrm{ad}_a] = \begin{pmatrix} a^i L^1_{i1} & \cdots & a^i L^1_{in} \\ \vdots & & \vdots \\ a^i L^n_{i1} & \cdots & a^i L^n_{in} \end{pmatrix}$$

substracting $\lambda\mathrm{Id}$ and taking determinant we obtain that a term with λ^k would have for coefficient a product of k of the linear combinations $a^i L^j_{i\ell}$ for varying j and ℓ. Such a product is understood to be a homogeneous polynomial of degree k in $\mathfrak{g}$. $\square$

Definition 5.2. I think this goes for any Lie algebra $\mathfrak{g}$:

- The *rank* of $\mathfrak{g}$ is the smallest integer r such that $c_r(a)$ is not the zero polynomial on $\mathfrak{g}$.
- An element $a \in \mathfrak{g}$ is called *regular* if $c_r(a) \neq 0$.
- The *discriminant* of $\mathfrak{g}$ is the nonzero polynomial $c_r(a)$ of degree $d-r$, what?

Explanation: we compute the polynomial characteristic w.r.t. ad_a for every $a \in \mathfrak{g}$. Express this polynomial as a polynomial with coefficients in $\mathfrak{g}[t]$, (here's the question: this polynomial seems to me to be a polynomial in $\mathbb{F}$.) This polynomial does not have a constant term. But what is the next smallest-degree monomial? 1? 2? That's the rank of the Lie algebra.

6. Cartan subalgebra

Recall that $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if $[a, \mathfrak{h}] \subset \mathfrak{h}$ for all $a \in \mathfrak{g}$. Maybe $\mathfrak{h}$ is not an ideal, but we can consider the largest subalgebra of $\mathfrak{g}$ where $\mathfrak{h}$ is an ideal. This is called the normalizer.

Definition 6.1. Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. The *normalizer* of $\mathfrak{h}$ is $N_{\mathfrak{g}}(\mathfrak{h}) := \{a \in \mathfrak{g} | [a, \mathfrak{h}] \subset \mathfrak{h}\}$

I think behind the Cartan subalgebra is that, roughly, taking product with anything not in $\mathfrak{h}$ leaves $\mathfrak{h}$.

Definition 6.2. A *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$, satisfying the following two conditions:

(1) $\mathfrak{h}$ is a nilpotent Lie algebra,

(2) $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Proposition 6.3. *Let $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{F})$ be a subalgebra containing a diagonal matrix $a = diag(a_1, \dots, a_n)$ with distinct a_i, and let $\mathfrak{h}$ be the subspace of all diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra (of $\mathfrak{g}$ I guess).*

Idea of proof. Consider the following illustrative example:

$$\begin{aligned}
&\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} a\lambda_1 & b\lambda_2 \\ c\lambda_3 & d\lambda_2 \end{pmatrix} - \begin{pmatrix} \lambda_1 a & \lambda_1 b \\ \lambda_2 c & \lambda_2 d \end{pmatrix} \\
&= \begin{pmatrix} 0 & b\lambda_2 - \lambda_1 b \\ \lambda_3 a - \lambda_2 c & 0 \end{pmatrix}
\end{aligned}$$

which says that any non-diagonal matrix would escape $\mathfrak{h}$. □

Theorem 6.4 (Cartan)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$. Let $a \in \mathfrak{g}$ be a regular element (which exists since $\mathbb{F}$ is infinite), and let $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_\lambda^a$ be the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to ad_a. Then $\mathfrak{g}_0^a$ is a Cartan subalgebra.*

Idea of proof. First recall what is $\mathfrak{h} = \mathfrak{g}_0^a$, the set of elements $b \in \mathfrak{g}$ such that $\mathrm{ad}_a^N(b) = 0$. □

Remark 6.5. The rank of a Lie algebra is the dimension of $\mathfrak{g}_0^a = \mathfrak{h}$.

Proposition 6.6. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Then $\mathfrak{g}_0 = \mathfrak{h}$ in the generalized weight space decomposition.*

Idea of proof. Engel's theorem… □

7. Semisimple Lie algebras

Definition 7.1. A *radical* $R(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is a solvable ideal of $\mathfrak{g}$ of maximal possible dimension.

Proposition 7.2. *The radical ideal of $\mathfrak{g}$ contains any solvable ideal of $\mathfrak{g}$ and is unique.*

If $\mathfrak{g}$ is a finite dimensional solvable Lie algebra, then $R(\mathfrak{g}) = \mathfrak{g}$. The opposite case if when $R(\mathfrak{g}) = 0$.

Definition 7.3. A finite-dimensional Lie algebra $\mathfrak{g}$ is called *semisimple* if $R(\mathfrak{g}) = 0$.

In [Kac10, Lecture 11] we have the tools needed for a characterization of semisimple Lie algebras in terms of the Killing form:

Theorem 7.4 (Cartan)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0. Then the Killing form on $\mathfrak{g}$ is non-degenerate is and only if $\mathfrak{g}$ is semisimple. Moreover, if $\mathfrak{g}$ is semisimple and $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then the restriction of the Killing form to $\mathfrak{a}$, $K|_{\mathfrak{a}\times\mathfrak{a}}$, is also nondegenerate and coincides with the Killing form of $\mathfrak{a}$.*

I think we need semisimplicity, i.e. nondegeneracy of the Killing form for the following definition:

Definition 7.5. Given a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$, we define the *coroot*

$$\alpha_i^\vee = \frac{2}{(\alpha_i, \alpha_i)}\nu^{-1}(\alpha_i) \tag{7.5.1}$$

where $\nu : \mathfrak{h} \to \mathfrak{h}^*$ is the pairing induced by the Killing form.

8. $\mathfrak{g}$-modules

Definition 8.1. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-*module* (also called a *representation of* $\mathfrak{g}$ is a vector space V and a homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

of Lie algebras, i.e.,

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

A vector space $W \subset V$ such that $vW \subset W$ for all $x \in \mathfrak{g}$ is called a $\mathfrak{g}$-*submodule.*

Definition 8.2. An *irreducible* $\mathfrak{g}$-*module* is one that has no $\mathfrak{g}$-submodules other than the trivial ones (namely, 0 and V itself).

It is possible to classify and describe the irreducible finite-dimensional $\mathfrak{g}$-modules. This is in contrast with all the modules of a Lie algebra $\mathfrak{g}$, which is "impossible in general".

9. Verma module

Let M be a $\mathfrak{g}$-module. Recall Definition 3.1: let $\mathfrak{h}$ be a Lie algebra, $\pi : \mathfrak{h} \to \mathfrak{gl}_V$ a representation of $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$. The *weight space* of $\mathfrak{h}$ attached to λ is

$$V_\lambda^{\mathfrak{h}} := \{v \in V | \pi(h)v = \lambda(h)v, \ \forall h \in \mathfrak{h}\}.$$

We have that a weight space in M is a nonzero subspace $M_\lambda \subset M$ such that

$$hv = \lambda(h)v \qquad \forall h \in \mathfrak{h} \subset \mathfrak{g} \text{ and } v \in M,$$

and in this case we call λ a weight. If M is a direct sum of weight spaces, we call it a *weight module.*

The *support* of a weight module M is

$$\mathrm{supp}(M) := \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\} \tag{9.0.1}$$

We call elements of M_λ *vectors of weight* λ.

Definition 9.1. $\Lambda \in \mathfrak{h}^*$ is a *highest weight* of M if $\Lambda \in \mathrm{supp}(M)$ and for all positive roots $\alpha \in \Delta_+$ we have that $\Lambda + \alpha \notin \mathrm{supp}(M)$.

Exercise 9.2. The highest weight of an irreducible $\mathfrak{g}$-module is unique.

Proof. □

Definition 9.3. Let $\Lambda \in \mathfrak{h}^*$ (a highest weight, I suppose). The *Verma module* $M(\Lambda)$ of highest weight Λ is

$$M(\Lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+} \mathbb{C}_\Lambda,$$

where $\mathbb{C}_\Lambda = \mathbb{C}v_\Lambda$ is the 1-dimensional $(\mathfrak{h} \oplus \mathfrak{n}_+)$-module defined by $\mathfrak{n}_+ v_\Lambda = 0$ and $hv_\Lambda = \langle \Lambda, h\rangle v_\Lambda$ for all $h \in \mathfrak{h}$.

Long story short, by the PBW Theorem we have an isomorphism $U(\mathfrak{n}_-) \to M(\Lambda)$ of $U(\mathfrak{n}_-)$ modules.

10. Weyl character formula

There are at least two Weyl character formulas. This one is [**?**, Theorem 10.14]:

Theorem 10.1 (Weyl Character Formula)**.** *If (π, V_μ) is an irreducible representation of $\mathfrak{g}$ with highest weight μ, then*

$$\chi_\pi(H) = \frac{\sum_{w\in W} \det(w) e^{\langle w\cdot(\mu+\delta), H\rangle}}{\sum_{w\in W} \det(w) e^{\langle w\cdot\delta, H\rangle}} \tag{10.1.1}$$

for all $H \in \mathfrak{h}$ for which the denominator is nonzero.

And this one is [Kac10, Theorem 25.3]:

Theorem 10.2 (Weyl Character Formula)**.** *Let $R = \prod_{\alpha\in\Delta_+}(1 - e^{-\alpha}$. If $\Lambda \in P_+$, then*

$$e^\rho R ch L(\Lambda) = \sum_{w\in W} \det w e^{w(\Lambda+\rho)} \tag{10.2.1}$$

Exercise 10.3. Let $\mathfrak{g} = \mathfrak{sl}_3$. Compute $\mathrm{ch}_{L(\lambda)}$ for some λ,

(1) $\lambda = 0$,
(2) $\lambda = \omega_1$,
(3) $\lambda = \omega_1 + \omega_2$,

etc.

Proof. No matter what formula I use, I must compute the determinants of the Weyl reflections and the ρ (which I think is δ for Hall), which is just the sum of all the fundamental weights, which in turn are the dual of the coroot vectors. Finally applying the reflections to ρ and to $\lambda + \rho$ would yield the result.

(1) (Find roots of $\mathfrak{sl}_3$.) We look for $\alpha \in \mathfrak{h}^* = \{\text{diagonal matrices}\}$ such that

$$[H, E_{ij}] = \alpha(H)E_{ij} \qquad \forall H \in \mathfrak{h}$$

where E_{ij} is the matrix that has zero in every entry but in the (i, j)-th where it has a 1. One obtains that $[H, E_{ij}] = (h_i - h_j)E_{ij}$, so that the roots are $\alpha_{ij}(h) = h_i - h_j$.

(2) (Compute the Killing form.) Recall that by definition $\kappa(H, H') = \mathrm{Tr}(\mathrm{ad}_H \mathrm{ad}_{H'})$. But in this case, we have $\kappa(H, H') = \mathrm{Tr}(HH')$ because $\mathrm{ad}_H \mathrm{ad}_{H'} = [H, [H', A]]$

The first thing Jethro did was to consider a basis

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and then compute the Killing form via

$$H_1H_1 = 2, \qquad H_1H_2 = -1, \qquad H_2H_1 = -1, \qquad H_2H_2 = 2.$$

(3) (Find the dual basis vectors.) Now we use the musical isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$, $H \mapsto (-, H)$ to find

$$\nu(H_1) =$$

(4) (Find fundamental weights.) It looks like $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = 2\omega_2 - \omega_1$. This says $\omega_1 = 2\omega_2 - \alpha_2 = 2(2\omega_1 - \alpha_1) - \alpha_2$ so $3\omega_1 = 2\alpha_1 + \alpha_2$. And then $\omega_2 = 2(\frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2) - \alpha_1 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$.

(5) (Find ρ.) Then $\rho = \omega_1 + \omega_2 = \alpha_1 + \alpha_2$.

(6) (Find Weyl reflections.) To find Weyl reflections I need to compute first the numbers (α_i, α_j). I think this is still not very clear — have to use Killing form? But OK, I know we must have $(\alpha_i, \alpha_i) = 2$, and the both of the crossed ones give -1. This should give by bilinearity the values on ω_i also.
(7) (Find determinants of Weyl reflections.) I guess for this I have to evaluate the reflections in each of the w
(8) (Evaluate ρ and $\lambda + \rho$ on Weyl reflections.)
(9) (Find the Weyl character.)

What's funny in this whole story is that in the end we can just find the Weyl character geometrically. For every $\lambda = m\omega_1 + n\omega_2$ I want to compute its orbit under the Weyl group W. This means

I would start by finding the determinant of the Weyl reflections... □

References

[Kac10] Victor Kac, *Lecture notes of 18.745 – introduction to lie algebras (fall 2010)*, `https://math.mit.edu/classes/18.745/classnotes.html`, 2010, Lecture notes, MIT.