CATEGORIES

github.com/danimalabares/stack

Contents

1. Definitions

We recall the definitions, partly to fix notation.

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Mor}_{\mathcal{C}}(x, y)$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map $\mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $\mathrm{id}_x \in \mathrm{Mor}_{\mathcal{C}}(x, x)$ such that $\mathrm{id}_x \circ \phi = \phi$ and $\psi \circ \mathrm{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is associative, i.e., $(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi)$ whenever these compositions make sense.

Definition 1.2. A *functor* $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
(2) For every $x, y \in \mathrm{Ob}(\mathcal{A})$ a map $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in the following manner: $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for a composable pair (ϕ, ψ) of morphisms of $\mathcal{A}$ and $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

Note that every category $\mathcal{A}$ has an *identity* functor $\mathrm{id}_\mathcal{A}$. In addition, given a functor $G : \mathcal{B} \to \mathcal{C}$ and a functor $F : \mathcal{A} \to \mathcal{B}$ there is a *composition* functor $G \circ F : \mathcal{A} \to \mathcal{C}$ defined in an obvious manner.

Definition 1.3. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) We say F is *faithful* if for any objects $x, y \in \mathrm{Ob}(\mathcal{A})$ the map
$$F : \mathrm{Mor}_\mathcal{A}(x, y) \to \mathrm{Mor}_\mathcal{B}(F(x), F(y))$$
is injective.
(2) If these maps are all bijective then F is called *fully faithful*.
(3) The functor F is called *essentially surjective* if for any object $y \in \mathrm{Ob}(\mathcal{B})$ there exists an object $x \in \mathrm{Ob}(\mathcal{A})$ such that $F(x)$ is isomorphic to y in $\mathcal{B}$.

Definition 1.4. A *subcategory* of a category $\mathcal{B}$ is a category $\mathcal{A}$ whose objects and arrows form subsets of the objects and arrows of $\mathcal{B}$ and such that source, target and composition in $\mathcal{A}$ agree with those of $\mathcal{B}$ and such that the identity morphism of an object of $\mathcal{A}$ matches the one in $\mathcal{B}$. We say $\mathcal{A}$ is a *full subcategory* of $\mathcal{B}$ if $\mathrm{Mor}_\mathcal{A}(x, y) = \mathrm{Mor}_\mathcal{B}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{A})$. We say $\mathcal{A}$ is a *strictly full* subcategory of $\mathcal{B}$ if it is a full subcategory and given $x \in \mathrm{Ob}(\mathcal{A})$ any object of $\mathcal{B}$ which is isomorphic to x is also in $\mathcal{A}$.

If $\mathcal{A} \subset \mathcal{B}$ is a subcategory then the identity map is a functor from $\mathcal{A}$ to $\mathcal{B}$. Furthermore a subcategory $\mathcal{A} \subset \mathcal{B}$ is full if and only if the inclusion functor is fully faithful. Note that given a category $\mathcal{B}$ the set of full subcategories of $\mathcal{B}$ is the same as the set of subsets of $\mathrm{Ob}(\mathcal{B})$.

Remark 1.5. Suppose that $\mathcal{A}$ is a category. A functor F from $\mathcal{A}$ to *Sets* is a mathematical object (i.e., it is a set not a class or a formula of set theory, see Sets, Section **??**) even though the category of sets is "big". Namely, the range of F on objects will be a set $F(\mathrm{Ob}(\mathcal{A}))$ and then we may think of F as a functor between $\mathcal{A}$ and the full subcategory of the category of sets whose objects are elements of $F(\mathrm{Ob}(\mathcal{A}))$.

Example 1.6. A homomorphism $p : G \to H$ of groups gives rise to a functor between the associated groupoids in Example **??**. It is faithful (resp. fully faithful) if and only if p is injective (resp. an isomorphism).

Example 1.7. Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the *category of objects over* X, denoted $\mathcal{C}/X$ as follows. The objects of $\mathcal{C}/X$ are morphisms $Y \to X$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : \mathcal{C}/X \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : \mathcal{C}/X' \to \mathcal{C}/X$ obtained by composition with f, and $p_X \circ F = p_{X'}$.

Example 1.8. Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the *category of objects under* X, denoted $X/\mathcal{C}$ as follows. The objects of $X/\mathcal{C}$ are morphisms $X \to Y$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $X \to Y$ and $X \to Y'$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : X/\mathcal{C} \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : X/\mathcal{C} \to X'/\mathcal{C}$ obtained by composition with f, and $p_{X'} \circ F = p_X$.

2. Monomorphisms

Definition 2.1. Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathcal{C}$.

(1) We say that f is a *monomorphism* if for every object W and every pair of morphisms $a, b : W \to X$ such that $f \circ a = f \circ b$ we have $a = b$.
(2) We say that f is an *epimorphism* if for every object W and every pair of morphisms $a, b : Y \to W$ such that $a \circ f = b \circ f$ we have $a = b$.

Definition 2.2. Let $\mathcal{C}$ be a category, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

Lemma 2.3. *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of $PSh(\mathcal{C})$. A map is an isomorphism if and only if it is both injective and surjective.*

3. Presheaves

Definition 3.1. A *presheaf of sets* on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to *Sets*. *Morphisms of presheaves* are natural transformations of functors. The category of presheaves of sets is denoted $PSh(\mathcal{C})$ or $\hat{\mathcal{C}}$.

4. Yoneda lemma

The Yoneda lemma says that the sections over a of a presheaf X can be described completely as natural transformations between the presheaf $\mathrm{Hom}(-, a)$ and X.

Definition 4.1. Let A be a category. The *Yoneda embedding* is the functor

$$h : A \to \hat{A}$$

whose value at an object a of A is the presheaf

$$h_a = \mathrm{Hom}_A(-, a).$$

In other words, the evaluation of the presheaf h_a at an object c of A is the set of maps from c to a.

Theorem 4.2 (Yoneda lemma)**.** *For any presheaf X over A, there is a natural bijection (in Sets I think!!)*

$$\begin{aligned} \mathrm{Hom}_{\widehat{A}}(h_a, X) &\xrightarrow{\simeq} X_a \\ (h_a \xrightarrow{u} X) &\longmapsto u_a(1_a) \end{aligned}$$

5. Internal Hom

Upshot. Internal Hom is when the Hom set of two objects in some category is in also an object of the category. Down-to-earth, that for two sheaves $\mathcal{F}, \mathcal{G}$, $U \mapsto \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U)$ is also a sheaf, called $\mathcal{H}om$.

I start with Stacks Project approach.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section **??** that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is rarely an isomorphism.

Cartesian closed category In the category of sets there is a bijection $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ that depends naturally on X, Y and Z. The notions related to this bijection are "Cartesian closed category", "currying" and "internal Hom".

Definition 5.1. A category $\mathcal{C}$ is *Cartesian closed* if:

1. $\mathcal{C}$ has all finite products (Caveat: some require that $\mathcal{C}$ has all finite limits)
2. For any object Y the functor $-\times Y$ has a right adjoint, which we will denote by $\mathrm{Map}(Y, -)$ or by $-^Y$.

Remark 5.2. By section 3 here, the second property above implies that we get a functor $\mathrm{Map}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, and moreover we get natural isomorphisms $\mathrm{Hom}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Hom}(X \times Y, Z)$ and $\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Map}(X \times Y, Z)$.

6. Product

Definition 6.1. A *product* in a category $\mathcal{C}$ is

$$\begin{tikzcd} P \arrow[r, "a"] \arrow[d, "b"'] & A \\ B & \end{tikzcd}$$

such that for every other

$$\begin{tikzcd} P' \arrow[r, "a'"] \arrow[d, "b'"'] & A \\ B & \end{tikzcd}$$

there exists a unique map $p : P' \to P$ such that $ap = a'$ and $bp = b'$.

For completeness here's Stacks Project formulation:

Definition 6.2. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *product* of x and y is an object $x \times y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times y, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times y, y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(w, y)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times y)$ making the diagram

$$\begin{tikzcd}
w \arrow[rrd, "\beta"] \arrow[rd, dashed, "\gamma"] \arrow[rdd, "\alpha"'] & & \\
& x \times y \arrow[r, "q"'] \arrow[d, "p"'] & y \\
& x &
\end{tikzcd}$$

commute.

And some nice piece of information and a definition also from Stacks Project.

If a product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times y$ to be an object of $\mathcal{C}$ such that

$$h_{x \times y}(w) = h_x(w) \times h_y(w)$$

functorially in w. In other words the product $x \times y$ is an object representing the functor $w \mapsto h_x(w) \times h_y(w)$.

Definition 6.3. We say the category $\mathcal{C}$ *has products of pairs of objects* if a product $x \times y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

7. Coproducts of pairs

Definition 7.1. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *coproduct*, or *amalgamated sum* of x and y is an object $x \amalg y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $i \in \mathrm{Mor}_{\mathcal{C}}(x, x \amalg y)$ and $j \in \mathrm{Mor}_{\mathcal{C}}(y, x \amalg y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(x, w)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(y, w)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(x \amalg y, w)$ making the diagram

$$\begin{tikzcd}
& y \arrow[d, "j"] \arrow[rdd, "\beta"] & \\
x \arrow[r, "i"] \arrow[rrd, "\alpha"'] & x \amalg y \arrow[rd, dashed, "\gamma"] & \\
& & w
\end{tikzcd}$$

commute.

If a coproduct exists it is unique up to unique isomorphism. This follows from the Yoneda lemma (applied to the opposite category) as the definition requires $x \amalg y$ to be an object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathcal{C}}(x \amalg y, w) = \mathrm{Mor}_{\mathcal{C}}(x, w) \times \mathrm{Mor}_{\mathcal{C}}(y, w)$$

functorially in w.

Definition 7.2. We say the category $\mathcal{C}$ *has coproducts of pairs of objects* if a coproduct $x \amalg y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

8. Fibre products

Definition 8.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_\mathcal{C}(x, y)$ and $g \in \mathrm{Mor}_\mathcal{C}(z, y)$. A *fibre product* of f and g is an object $x \times_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times_y z, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times_y z, z)$ making the diagram

$$\begin{tikzcd} x \times_y z \arrow[r, "q"'] \arrow[d, "p"'] & z \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

commute, and such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(w, z)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times_y z)$ making the diagram

$$\begin{tikzcd} w \arrow[dr, dashed, "\gamma"'] \arrow[drr, "\beta"] \arrow[ddr, "\alpha"'] & & \\ & x \times_y z \arrow[r, "q"'] \arrow[d, "p"] & z \arrow[d, "g"] \\ & x \arrow[r, "f"] & y \end{tikzcd}$$

commute.

Remark 8.2. I think that a product is a fibre product when we put y to be the terminal object!

If a fibre product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times_y z$ to be an object of $\mathcal{C}$ such that

$$h_{x \times_y z}(w) = h_x(w) \times_{h_y(w)} h_z(w)$$

functorially in w. In other words the fibre product $x \times_y z$ is an object representing the functor $w \mapsto h_x(w) \times_{h_y(w)} h_z(w)$.

Definition 8.3. We say a commutative diagram

$$\begin{tikzcd} w \arrow[r] \arrow[d] & z \arrow[d] \\ x \arrow[r] & y \end{tikzcd}$$

in a category is *cartesian* if w and the morphisms $w \to x$ and $w \to z$ form a fibre product of the morphisms $x \to y$ and $z \to y$.

Definition 8.4. We say the category $\mathcal{C}$ *has fibre products* if the fibre product exists for any $f \in \mathrm{Mor}_\mathcal{C}(x, y)$ and $g \in \mathrm{Mor}_\mathcal{C}(z, y)$.

Definition 8.5. A morphism $f : x \to y$ of a category $\mathcal{C}$ is said to be *representable* if for every morphism $z \to y$ in $\mathcal{C}$ the fibre product $x \times_y z$ exists.

Lemma 8.6. *Let $\mathcal{C}$ be a category. Let $f : x \to y$, and $g : y \to z$ be representable. Then $g \circ f : x \to z$ is representable.*

Proof. Let $t \in \mathrm{Ob}(\mathcal{C})$ and $\varphi \in \mathrm{Mor}_{\mathcal{C}}(t, z)$. As g and f are representable, we obtain commutative diagrams

$$\begin{tikzcd} y \times_z t \arrow[r, "q"'] \arrow[d, "p"'] & t \arrow[d, "\varphi"] \\ y \arrow[r, "g"] & z \end{tikzcd} \qquad \begin{tikzcd} x \times_y (y \times_z t) \arrow[r, "q'"'] \arrow[d, "p'"'] & y \times_z t \arrow[d, "p"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

with the universal property of Definition 8.1. We claim that $x \times_z t = x \times_y (y \times_z t)$ with morphisms $q \circ q' : x \times_z t \to t$ and $p' : x \times_z t \to x$ is a fibre product. First, it follows from the commutativity of the diagrams above that $\varphi \circ q \circ q' = g \circ f \circ p'$. To verify the universal property, let $w \in \mathrm{Ob}(\mathcal{C})$ and suppose $\alpha : w \to x$ and $\beta : w \to y$ are morphisms with $\varphi \circ \beta = g \circ f \circ \alpha$. By definition of the fibre product, there are unique morphisms δ and γ such that

$$\begin{tikzcd} w \arrow[drr, bend left, "\beta"] \arrow[dr, dashed, "\delta"] \arrow[ddr, bend right, "f \circ \alpha"'] & & \\ & y \times_z t \arrow[r, "q"'] \arrow[d, "p"'] & t \arrow[d, "\varphi"] \\ & y \arrow[r, "g"] & z \end{tikzcd}$$

and

$$\begin{tikzcd} w \arrow[drr, bend left, "\delta"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, bend right, "\alpha"'] & & \\ & x \times_y (y \times_z t) \arrow[r, "q'"'] \arrow[d, "p'"'] & y \times_z t \arrow[d, "p"] \\ & x \arrow[r, "f"] & y \end{tikzcd}$$

commute. Then, γ makes the diagram

$$\begin{tikzcd} w \arrow[drr, bend left, "\beta"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, bend right, "\alpha"'] & & \\ & x \times_z t \arrow[r, "q \circ q'"'] \arrow[d, "p'"'] & t \arrow[d, "\varphi"] \\ & x \arrow[r, "g \circ f"] & z \end{tikzcd}$$

commute. To show its uniqueness, let γ' verify $q \circ q' \circ \gamma' = \beta$ and $p' \circ \gamma' = \alpha$. Because γ is unique, we just need to prove that $q' \circ \gamma' = \delta$ and $p' \circ \gamma' = \alpha$ to conclude. We supposed the second equality. For the first one, we also need to use the uniqueness of delta. Notice that δ is the only morphism verifying $q \circ \delta = \beta$ and $p \circ \delta = f \circ \alpha$. We already supposed that $q \circ (q' \circ \gamma') = \beta$. Furthermore, by definition of the fibre product, we know that $f \circ p' = p \circ q'$. Therefore:

$$p \circ (q' \circ \gamma') = (p \circ q') \circ \gamma' = (f \circ p') \circ \gamma' = f \circ (p' \circ \gamma') = f \circ \alpha.$$

Then $q' \circ \gamma' = \delta$, which concludes the proof. $\square$

Lemma 8.7. *Let $\mathcal{C}$ be a category. Let $f : x \to y$ be representable. Let $y' \to y$ be a morphism of $\mathcal{C}$. Then the morphism $x' := x \times_y y' \to y'$ is representable also.*

Proof. Let $z \to y'$ be a morphism. The fibre product $x' \times_{y'} z$ is supposed to represent the functor

$$\begin{array}{lll} w & \mapsto & h_{x'}(w) \times_{h_{y'}(w)} h_z(w) \\ & = & (h_x(w) \times_{h_y(w)} h_{y'}(w)) \times_{h_{y'}(w)} h_z(w) \\ & = & h_x(w) \times_{h_y(w)} h_z(w) \end{array}$$

which is representable by assumption. □

9. Examples of fibre products

In this section we list examples of fibre products and we describe them.

As a really trivial first example we observe that the category of sets has fibre products and hence every morphism is representable. Namely, if $f : X \to Y$ and $g : Z \to Y$ are maps of sets then we define $X \times_Y Z$ as the subset of $X \times Z$ consisting of pairs (x, z) such that $f(x) = g(z)$. The morphisms $p : X \times_Y Z \to X$ and $q : X \times_Y Z \to Z$ are the projection maps $(x, z) \mapsto x$, and $(x, z) \mapsto z$. Finally, if $\alpha : W \to X$ and $\beta : W \to Z$ are morphisms such that $f \circ \alpha = g \circ \beta$ then the map $W \to X \times Z$, $w \mapsto (\alpha(w), \beta(w))$ obviously ends up in $X \times_Y Z$ as desired.

In many categories whose objects are sets endowed with certain types of algebraic structures the fibre product of the underlying sets also provides the fibre product in the category. For example, suppose that X, Y and Z above are groups and that f, g are homomorphisms of groups. Then the set-theoretic fibre product $X \times_Y Z$ inherits the structure of a group, simply by defining the product of two pairs by the formula $(x, z) \cdot (x', z') = (xx', zz')$. Here we list those categories for which a similar reasoning works.

(1) The category *Groups* of groups.
(2) The category *G-Sets* of sets endowed with a left G-action for some fixed group G.
(3) The category of rings.
(4) The category of R-modules given a ring R.

10. Pushouts

The dual notion to fibre products is that of pushouts.

Definition 10.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_\mathcal{C}(y, x)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. A *pushout* of f and g is an object $x \amalg_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x, x \amalg_y z)$ and $q \in \mathrm{Mor}_\mathcal{C}(z, x \amalg_y z)$ making the diagram

$$\begin{tikzcd} y \arrow[r, "g"'] \arrow[d, "f"'] & z \arrow[d, "q"] \\ x \arrow[r, "p"] & x \amalg_y z \end{tikzcd}$$

commute, and such that the following universal property holds: For any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(x, w)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(z, w)$ with $\alpha \circ f = \beta \circ g$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(x \amalg_y z, w)$ making the diagram

$$\begin{tikzcd}
y \arrow[r, "g"'] \arrow[d, "f"'] & z \arrow[d, "q"] \arrow[ddr, "\beta"] & \\
x \arrow[r, "p"] \arrow[drr, "\alpha"'] & x \amalg_y z \arrow[dr, dashed, "\gamma"] & \\
 & & w
\end{tikzcd}$$

commute.

It is possible and straightforward to prove the uniqueness of the triple $(x \amalg_y z, p, q)$ up to unique isomorphism (if it exists) by direct arguments. Another possibility is to think of the pushout as the fibre product in the opposite category, thereby getting this uniqueness for free from the discussion in Section 8.

Definition 10.2. We say a commutative diagram

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & z \arrow[d] \\
x \arrow[r] & w
\end{tikzcd}$$

in a category is *cocartesian* if w and the morphisms $x \to w$ and $z \to w$ form a pushout of the morphisms $y \to x$ and $y \to z$.

According to Sergei Burkin, another way to define a pushout is as follows. The diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is a *pushout* if the diagram

$$\begin{tikzcd}
Map(D, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Map(B, X) \arrow[d] \\
Map(C, D) \arrow[r] & Map(D, X)
\end{tikzcd}$$

is a pullback.

In fact, spheres are (homotopy?) pushouts:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & S^n
\end{tikzcd}$$

which is homotopically the same as

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & D^n \arrow[d] \\
D^n \arrow[r] & S^n,
\end{tikzcd}$$

the familiar construction of glueing two disks along their boundary to obtain a sphere.

11. Limits and colimits

The definition of product is actually a particular case of a limit.

Let $\mathcal{C}$ be a category. A *diagram* in $\mathcal{C}$ is simply a functor $M : \mathcal{I} \to \mathcal{C}$. We say that $\mathcal{I}$ is the *index category* or that M is an $\mathcal{I}$-diagram. We will use the notation M_i to denote the image of the object i of $\mathcal{I}$. Hence for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $M(\phi) : M_i \to M_{i'}$.

Definition 11.1. A *limit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\lim_{\mathcal{I}} M$ in $\mathcal{C}$ together with morphisms $p_i : \lim_{\mathcal{I}} M \to M_i$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $p_{i'} = M(\phi) \circ p_i$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to M_i$ (indexed by $i \in \mathrm{Ob}(\mathcal{I})$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $q_{i'} = M(\phi) \circ q_i$ there exists a unique morphism $q : W \to \lim_{\mathcal{I}} M$ such that $q_i = p_i \circ q$ for every object i of $\mathcal{I}$.

Limits $(\lim_{\mathcal{I}} M, (p_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Products of pairs, fibre products, and equalizers are examples of limits. The limit over the empty diagram is a final object of $\mathcal{C}$. In the category of sets all limits exist. The dual notion is that of colimits.

Definition 11.2. A *colimit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\mathrm{colim}_{\mathcal{I}} M$ in $\mathcal{C}$ together with morphisms $s_i : M_i \to \mathrm{colim}_{\mathcal{I}} M$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $s_i = s_{i'} \circ M(\phi)$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $t_i : M_i \to W$ (indexed by $i \in \mathrm{Ob}(\mathcal{I})$) such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $t_i = t_{i'} \circ M(\phi)$ there exists a unique morphism $t : \mathrm{colim}_{\mathcal{I}} M \to W$ such that $t_i = t \circ s_i$ for every object i of $\mathcal{I}$.

Definition 11.3. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *product* $\prod_{i \in I} M_i$ is by definition $\lim_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is a final object of the category. The morphisms $p_i : \prod M_i \to M_i$ are called the *projection morphisms*.

Definition 11.4. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *coproduct* $\coprod_{i \in I} M_i$ is by definition $\mathrm{colim}_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the coproduct is an initial object of the category. Note that the coproduct comes equipped with morphisms $M_i \to \coprod M_i$. These are sometimes called the *coprojections*.

12. Simplicial sets

Definition 12.1. The *simplex category* is the category of ordinals, i.e. non-empty finite ordered sets

$$[n] = \{0 < 1 < \ldots < n\}, \quad n = 0, 1, \ldots$$

with order preserving maps of sets.

Objects of the simplicial category Δ are not called simplices. Instead, simplices are type of simplicial set:

Definition 12.2. A *simplicial set* is a presheaf on Δ, i.e. an element of $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sets})$.

Definition 12.3. The *n-simplex* is the representable simplicial set $\mathrm{Hom}(-, [n]) := \Delta^n$.

So far the upshot for me is that simplicial sets are like a generalization of a triangulated topological space. The following construction shows how to associate to $S \in \mathsf{sSet}$ a topological space $|S| \in \mathsf{Top}$. In fact, the weak equivalences in sSet can be defined using weak equivalences in Top just like in Hatcher.

Here's some copy-paste from [Hau25, Chapter 1] transcribed by ChatGPT:

The category Δ is generated by

- the *face maps* $d_i : [n-1] \hookrightarrow [n]$ that skip $i \in [n]$,
- the *degeneracy maps* $s_i : [n+1] \xrightarrow{\text{surj.}} [n]$ that repeat $i \in [n]$,

subject to certain relations.

Definition 12.4. The *topological n-simplex* $|\Delta^n|$ is the topological space

$$|\Delta^n| := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1,\ 0 \leq x_i \leq 1\}$$

(with the subspace topology from $\mathbb{R}^{n+1}$). For $\varphi : [n] \to [m]$ we can define a continuous map $\varphi_* : |\Delta^n| \to |\Delta^m|$ by

$$\varphi_*(x_0, \ldots, x_n)_i = \sum_{j : \varphi(j) = i} x_j.$$

This gives a functor $|\Delta^\bullet| : \Delta \to \mathsf{Top}$.

We can then define the *singular simplicial set functor*

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{Set}_\Delta$$

as

$$\mathrm{Sing}(X) = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^\bullet|, X).$$

This has a left adjoint $|-| : \mathbf{Set}_\Delta \to \mathbf{Top}$, called the *geometric realization functor*, which is the unique colimit-preserving functor that extends $|\Delta^\bullet|$ via the Yoneda embedding. More concretely, we can define $|S|$ for a simplicial set S as the quotient of $\coprod_n S_n \times |\Delta^n|$ where we identify $(\sigma, \varphi_* p)$ with $(\varphi^* \sigma, p)$ for $\varphi : [n] \to [m]$, $\sigma \in S_n$ and $p \in |\Delta^m|$. Informally, we build the topological space $|S|$ out of simplices according to the "blueprint" S.

If we say that a morphism $S \to T$ in $\mathbf{Set}_\Delta$ is a weak equivalence if $|S| \to |T|$ is a weak homotopy equivalence, then the relative category consisting of $\mathbf{Set}_\Delta$ with these weak equivalences describes the same homotopy theory as that of topological spaces; for example, the counit map $|\mathrm{Sing}\, X| \to X$ for a topological space X is always a weak homotopy equivalence. We can also describe the weak equivalences of simplicial sets as homotopy equivalences (or describe them via homotopy groups) if we restrict to a class of nice objects, which we will introduce next.

Exercise 12.5. It should be possible to show that the geometric realization of Δ^n is in fact the topological n-simplex in $\mathbb{R}^n$, right?

13. ∞-groupoids

Apparently the philosophy is that we will not formally construct ∞-categories (nor ∞-groupoids) but barely start using them. So we admit "facts" such as "there are objects called ∞-*groupoids*". We shall admit that although there are points (and paths), we cannot distinguish between points if there is a path joining two points. Thus we don't really have points but a set $\pi_0 X$ of path components.

There are also homotopies between paths, and homotopies between homotopies, and so on.

Also, there are *maps* or *morphisms* between groupoids, homotopies between morphisms, homotopies between homotopies, and so on. In fact, all those things form an ∞-groupoid we denote by $\mathsf{Map}(X, Y)$.

We can *compose* maps, and there is an *identity morphism* for every ∞-groupoid X. Composition is unital and associative in the only way that makes sense (?), which is up to homotopy.

Definition 13.1. An *equivalence* of groupoids is a pair of maps that may be composed not be the identities, but homotopical to the identities.

Dani: it looks like the main idea is to care about anything only up to homotopy.

Sets are groupoids and for any set

$$\mathrm{Hom}_{\mathsf{Set}}(\pi_0 X, S) \xrightarrow{\simeq} \mathsf{Map}(X, S)$$

is an equivalence.

Then the 1-point set ends up being the terminal ∞-groupoid. The empty set is also an ∞-groupoid, and it is the initial one.

Given morphisms $X \to Z$ and $Y \to Z$, there exists a pullback square

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

where "square" means not that it is commutative, but that there exists an homotopy between the compositions (commutative up to homotopy). Looks like it basically a fibre product (see Definition **??**) up to homotopy.

The *fibre* $f^{-1}(b)$ at b of a map $f : E \to B$ is defined as the pullback

$$\begin{tikzcd} f^{-1}(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ \{b\} \arrow[r] & B. \end{tikzcd}$$

The *product* $X \times Y$ of two ∞-groupoids X and Y is defined as the pullback

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y \arrow[r] & *. \end{tikzcd}$$

Composition of squares is another square, and the composition of two pullback squares is another pullback square.

Definition 13.2. For points $x, y \in X$, the *path space* $X(x,y)$ is the pullback

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \{x\} \arrow[d] \\ \{y\} \arrow[r] & X. \end{tikzcd}$$

Definition 13.3. The *n-th homotopy group* of a groupoid is $\pi_0 \Omega^n_x$.

In fact $\pi_1(X,x)$ is a group and $\pi_n(X,x)$ is an abelian group for $n > 1$.

Fact. Homotopy groups detect equivalences: a map of groupoids is an qequivalence if and only if the induced maps on all homotopy groups are isomorphisms.

Monomorphisms of ∞-groupoids are...

Lemma 13.4. *A monomorphism of ∞-groupoids $f : X \to Y$ is a monomorphism if and only if the fibres are all either empty of contractible.*

14. ∞-CATEGORIES

The *Kan condition* is

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "i"] & X \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

for all i, and is called strict/weak whether the existence is unique or not. The *inner Kan condition* is if we require for $i \neq 0$ and $i \neq n$.

Why is this so important?

Definition 14.1. The *nerve* functor $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is defined by

$$\mathcal{C} \mapsto \mathsf{Hom}_{\mathsf{Cat}}([\bullet], \mathcal{C}),$$

so that $N\mathcal{C}_n$ is the set of all composable sequences of n morphisms.

Notice that $N\mathcal{C}_0$ is the set of all objects of $\mathcal{C}$ and $N\mathcal{C}_1$ is the set of all morphisms. This is crucial for the following exercise:

Exercise 14.2. $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is fully faithful.

Proof. To a functor $F \in \mathsf{Hom}(\mathcal{C}, \mathcal{D})$ we assign a simplicial set $N(F)_n : N(\mathcal{C})_n \to N(\mathcal{D})_n$ defined in the obvious way: we map a sequence

$$\bullet \xrightarrow{f_1} \bullet \to \cdots \bullet \xrightarrow{f_n} \bullet$$

to

$$\bullet \xrightarrow{F(f_1)} \bullet \to \cdots \bullet \xrightarrow{F(f_n)} \bullet$$

This functor is well defined as a functor of simplicial sets by functoriality of F. To check fully faithfullness we use the fact that we can reconstruct a category from $N(F)_0$ and $N(F)_1$. $\square$

Thus, the Kan condition allows to see categories as simplicial sets. Here are possible generalizations of this:

	strict	weak
all i	groupoids	∞-groupoids
inner	categories	∞-categories.

References

[Hau25] Rune Haugseng, *Yet another introduction to infty-categories*, 2025.