MUMFORD-TATE GROUPS IN HODGE THEORY

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github.com/danimalabares/stack

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Upshot. The Mumford-Tate group is the stabilizer of the Hodge classes, and is constant on the complement of the Hodge locus (see Definition 21.5).

1. Plan

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2. Introduction

$X \subset \mathbb{C}P^N$ smooth complex subvariety, $\dim_{\mathbb{C}} = n$. First recall we have singular cohomology, $H^k(X, \mathbb{C})$, which is isomorphic to the cohomology of the constant sheaf $\underline{\mathbb{C}}_X$. This cohomology is nonzero for $0 \leq k \leq 2n$.

Recall. $U \subset X$ open, $\Gamma(U, \mathbb{C}) = \{f : U \to \mathbb{C} : f \text{ is locally constant}\} = \prod_{\pi_0(U)} \mathbb{C}$.

Example 2.1. (1) $X = \mathbb{C}P^n$,

$$H^k(\mathbb{C}P^n, \mathbb{C}) = \begin{cases} \mathbb{C} & k = 2m, 0 \leq m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

A way to prove this is using the CW decomposition of $\mathbb{C}P^n$.

(2) $X \subset \mathbb{C}P^2$ hypersurface of degree d; X a Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$. Then $H^0(X, \mathbb{C}) = \mathbb{C}$, $H^1(X, \mathbb{C}) = \mathbb{C}^{2g}$, $H^2(X, \mathbb{C}) = \mathbb{C}$.

We also have the following additional data (a Hodge structure) on $H^k(X, \mathbb{C})$:

- A lattice $H^k(X, \mathbb{Z})/\text{torsion} \subset H^k(X, \mathbb{C})$,
- A (p, q)-decomposition, $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$.

Why is it useful?

- It gives restrictions on possible Betti numbers of algebraic varieties. (So it may tell us that certain complex variety cannot be algebraic, for example.)
- It $f : X \to Y$ is a morphism of algebraic varieties, then $f^* : H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ preserves the Hodge structure.

As an example of the latter statement,

Example 2.2. Let $X \subset \mathbb{C}P^4$ be a "general" hypersurface of degree 5. Then there exists no abelian variety (i.e. a projective variety that is biholomorphic to $\mathbb{C}P^N/\Lambda$ where Λ is a lattice; so, a complex torus that is also a projective variety) A that admits a dominant birational map onto X $f : A \dashrightarrow X$.

3. The p,q decomposition

We use the de Rham complex. Let Ω_X^k be the sheaf of holomorphic k-forms. The de Rham complex is

$$\Omega_{dR}^\bullet = (0 \to \mathcal{O}_X \to \Omega_X^1 \xrightarrow{d} \cdots \to \Omega_X^n \to 0)$$

$\Omega_{dR}^\bullet$ is a resolution of $\mathbb{C}_X$. Therefore $H^k(X, \mathbb{C}) \cong H^k(X, \Omega_{dR}^\bullet$.

Let's define a subcomplex:

$$F^p\Omega_{dR}^\bullet = (0 \to \cdots \to \Omega_X^p \to O_X^{p+1} \to \cdots \to \Omega_X^n \to 0)$$

It's a subcomplex $F^p\Omega_{dr}^\bullet \subset \Omega_{dR}^\bullet$ since the sheaves coincide when $F^p\Omega_{dR}$ are nonzero and they inject otherwise (because it's the zero sheaf).

This gives $H^k(X, F^p\Omega_{dR}^\bullet) \xrightarrow{(*)} H^k(X, \Omega_{dR}^\bullet) = H^k(X, \mathbb{C})$.

Definition 3.1. The *Hodge filtration* is $F^pH^k(X) = \text{Im}(*)$.

Hodge theory tells us the map $(*)$ is in fact injective.

Let $\Lambda_X^{p,q}$ be the sheaf of C^∞-forms on X of type (p,q). They are given locally by $\sum_{\substack{|I|=p\\|J|=q}} \alpha_{IJ} dz_I \wedge d\bar{z}_J$, for $\alpha_{IJ} \in C_X^\infty$.

Then we have an acyclic resolution of Ω_X^p:

$$0 \to \Omega_X^p \hookrightarrow \Lambda_X^{p,0} \xrightarrow{\bar\partial} \Lambda_X^{p,1} \xrightarrow{\bar\partial} \cdots \to \Lambda_X^{p,n} \to 0.$$

$\Lambda_X^{\bullet,\bullet}$ is a fine resolution of $\Omega_X^\bullet$.

Then $H^k(X,\mathbb{C}) \cong H^k(X, \mathrm{Tot}\Lambda_X^{\bullet,\bullet}$, which can be computed by a spectral sequence. The first page of such spectral sequence is given by $E_1^{p,q} = H^q(X,\Omega_X^p)$. This converges to $H^k(X,\mathbb{C})$. This is the Hodge-to-de Rham spectral sequence.

Since our manifolds are projective they admit a Kähler metric ω induced by the inclusion $X \xrightarrow{i} \mathbb{C}P^N$, that is, $\omega = i^*(\text{Fubini-Study metric on } \mathbb{C}P^N)$.

Then $\Lambda_X^k = \bigoplus_{p+q=k} \Lambda_X^{p,q}$, $\Lambda_X^\bullet$ becomes an elliptic complex.

We have

$$\ldots \to \Lambda_X^{k-1} \xrightarrow{d} \Lambda_X^k \to \ldots, \qquad \ldots \to \Lambda_X^k \xrightarrow{d^*} \Lambda_X^{k-1} \to \ldots$$

where d^* is the adjoint of d w.r.t. ω. Then $\Lambda = dd^* + d^*d$ is an elliptic operator. $\mathcal{H}^k = \mathrm{Ker}(\Lambda|_{\Lambda_X^k}) = \mathrm{Ker}(d|_{\Lambda_X^k}) \cap \mathrm{Ker}(d^*|_{\Lambda_X^k})$ are the harmonic forms.

Consider a natural map from the harmonic forms of type (p,q) to the cohomology:

$$\mathcal{H}^{p,q} = \mathcal{H}^k \cap \Lambda_X^{p,q} \xrightarrow{(**)} H^q(X,\Omega_X^p).$$

Fact: since $d\omega = 0$ (X is Kähler), $(**)$ is an isomorphism.

This means that for Kähler manifolds

$$\dim H^k(X,\mathbb{C}) \le \sum_{p+q=k} H^q(X,\Omega_X^p) \underbrace{\le}_{(**)} \dim H^k(X,\mathbb{C})$$

This implies that the Hodge-to-de Rham spectral sequence degenerates at E_1. Therefore,

$$H^k(X,\mathbb{C}) = \mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong H^q(X,\Omega_X^p).$$

The Hodge filtration is

$$F^p H^k(X,\mathbb{C}) = \bigoplus_{\substack{p'+q'=k\\ p'\ge p}} H^{p',q'}(X).$$

4. Symmetries of the p,q decomposition

(1) Since $\overline{\Lambda_X^{p,q}} = \Lambda_X^{q,p}$, we have $\mathcal{H}^{q,p} = \overline{\mathcal{H}^{p,q}}$. This means that if $k \equiv 1 (\mathrm{mod}\ 2)$, $H^k(X,\mathbb{C}) = \mathcal{H}^{k,0} \oplus \ldots \oplus \mathcal{H}^{\frac{k+1}{2},\frac{k-1}{2}} \oplus \mathcal{H}^{0,k} \oplus \ldots \oplus \mathcal{H}^{\frac{k-1}{2},\frac{k+1}{2}}$. This means that the k-th Betti number is even, $b_k(X) \equiv 0 \mathrm{mod}\ 2$.

(2) (Poincare duality.) We have a perfect pairing

$$H^k(X,\mathbb{C}) \otimes H^{2n-k}(X,\mathbb{C}) \longrightarrow \mathbb{C}$$
$$[\alpha]\otimes[\beta] \longmapsto \int_X \alpha \wedge \beta$$

Notice that if $\alpha \in \mathcal{H}^{p,q}$ then $\beta \in \mathcal{H}^{n-p,n-q}$. This induces a perfect pairing

$$\mathcal{H}^{p,q} \otimes \mathcal{H}^{n-p,n-q} \to \mathbb{C}.$$

(3) (Polarization and the Lefschetz operator.) The polarization is the Kähler class of the Kähler form. By X being projective we have that the Kähler class is integral. Moreover, it is the Poincar'e dual of the hyperplane section class. That is, let $h \in H^2(\mathbb{C}P^n, \mathbb{Z})$ be the class of a hyperplane, then $i^*h = [\omega] \in H^2(X, \mathbb{Z})$. We have $\omega \in \mathcal{H}^{1,1}$ and $[\omega] \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. The Lefschetz operator is

$$\begin{aligned} L_\omega : H^{p,q}(X) &\longrightarrow H^{p+1,q+1}(X) \\ [\alpha] &\longmapsto [\alpha \wedge \omega] = [\alpha] \cup [\omega]. \end{aligned}$$

Lefschetz theorem says
(a) $L_\omega^k : H^{n-k}(X, \mathbb{Q}) \xrightarrow{\sim} H^{n+k}(X, \mathbb{Q})$ is an isomorphism for $0 \leq k \leq n$
(b) The dual of L_ω is

$$\begin{aligned} \Lambda_\omega : H^{p,q}(X) &\longrightarrow H^{p-1,q-1}(X) \\ [\alpha] &\longmapsto [i_\alpha \omega]. \end{aligned}$$

$$\begin{aligned} [L_\omega, \Lambda_\omega] &= \Theta \in \mathrm{End}(H^\bullet(X, \mathbb{Q})) \\ \Theta|_{H^k(X,\mathbb{Q})} &= (k-n)\mathrm{Id} \end{aligned}$$

Then L_ω, Λ_ω and Θ span a subalgebra of $\mathrm{End}(H^\bullet(X, \mathbb{Q}))$ isomorphic to $\mathfrak{sl}_2$. This allows us to use what we know about the representation theory of $\mathfrak{sl}_2$. Let $H^k_{\mathrm{prim}}(X, \mathbb{Q}) = (\Lambda|_{H^k(X,\mathbb{Q})})$. Then $H^m(X, \mathbb{Q}) = \bigoplus_{i \geq 0} L^i_\omega H^{m-2i}_{\mathrm{prim}}(X, \mathbb{Q})$ for $0 < m \leq n$. (I think this corresponds to the usual weight space decomposition.)
[Picture of Hodge diamond. Reflection by vertical axis is complex conjugation, 180-degree rotation is Poincar'e duality, $p+q$ =constant is a horizontal line, reflection along horizontal axis is Lefschetz theorem. Warning! This depends on conventions of how we draw the diamond.]

5. The Hodge-Riemann relations

For all $[\alpha] \in H^k_{\mathrm{prim}}(X, \mathbb{C}) \cap H^{p,q}(X)$ we have

$$i^{p-q}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \overline{\alpha} \wedge \omega^{n-k} > 0.$$

(Here i is the imaginary unit.)

Define a pairing on $H^k_{\mathrm{prim}}(X, \mathbb{C})$:

$$\psi([\alpha], [\beta]) = (2\pi i)^{-k}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \beta \wedge \omega^{n-k}$$

which is $(-1)^k$-symmetric.

Definition 5.1. The *Weil operator* $C \in \mathrm{End}(H^k(X, \mathbb{C}))$ is given by $C|_{H^{p,q}}(X) = (i)^{p-q}\mathrm{Id}$.

Let $Q([\alpha], [\beta]) = (2\pi i)^k \psi(C[\alpha], [\beta])$.

Then Q is symmetric (exercise) and positive on $H^k(X, \mathbb{R})$. Positive is just the Hodge-Riemann relation.

6. Hodge structures and Mumford-Tate groups

Definition 6.1. A *rational Hodge structure* ($\mathbb{Q}$-HS) of weight $k \in \mathbb{Z}$ is a finite-dimensional $\mathbb{Q}$-vector space V and a decomposition $V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$ such that $V^{q,p} = \overline{V^{p,q}} \forall p, q$.

A $\mathbb{Q}$-HS (without fixed weight) is a $\mathbb{Q}$-vector space V with a decomposition $V = V_{k_1} \oplus \ldots \oplus V_{k_n}$ where V_{k_i} is a $\mathbb{Q}$-HS of weight k_i.

Analogously, define $\mathbb{Z}$-HS, $\mathbb{R}$-HS, etc (i.e. take V to be a finitely generated $\mathbb{Z}$-module, $\mathbb{R}$-vector space, etc.)

Example 6.2. (1) $X \subset \mathbb{C}P^N$ smooth subvariety, then $H^k(X, \mathbb{Z})$ is a $\mathbb{Z}$-HS of weight k.

(2) The *Tate HS* is $\mathbb{Z}(1) := 2\pi i\mathbb{Z} = \mathrm{Ker}(\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$, since $\mathbb{C} = \mathbb{Z}(1) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{Z}(1)^{-1,1}$, so this is a $\mathbb{Q}$-HS of weight -2.

Note that $H^2(\mathbb{C}P^1, \mathbb{Z}) = \mathbb{Z}$ sits inside $H^2(\mathbb{C}P^1, \mathbb{C} = \mathbb{Z} \otimes \mathbb{C} \cong \mathbb{C} = H^{1,1}(\mathbb{C}P^1)$.

Analogously, $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$ and $\mathbb{Q}(1) \otimes \mathbb{C} = \mathbb{Q}(1)^{-1,-1}$ is of weight -2.

7. The Deligne torus

Definition 7.1. $\mathbb{S}$ is the algebraic group such that $\mathbb{S}(\mathbb{R})$ is the $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The group of real points $\mathbb{S}(\mathbb{R})$ is a real Lie group.

Note that

$$\begin{aligned}\mathbb{C}^* &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 0\} \\ &= \{(x, y, y) \in \mathbb{R}^2 : (x^2 + y^2)t = 1\},\end{aligned}$$

which allows to see $\mathbb{C}^*$ as the vanishing locus of the polynomial $(x^2 + y^2)t - 1$, so to see it as an algebraic variety. Now

$$\begin{aligned}\mathbb{S}(\mathbb{C}) &= \{(x, y, t) \in \mathbb{C}^3 : (x^2 + y^2)t = 1\} \\ &= \{(x, y, t) \in \mathbb{C}^3 : \underbrace{(x + iy)}_{z}\underbrace{(x - iy)}_{w} t = 1\} \\ &= \{(z, w, t) \in \mathbb{C}^3 : zwt = 1\} \\ &= \{(z, w) \in \mathbb{C}^2 : z \neq 0, w \neq 0\} \cong \mathbb{C}^* \times \mathbb{C}^*.\end{aligned}$$

Let V be a $\mathbb{Q}$-HS of weight k. Define a representation over $\mathbb{R}$ $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes \mathbb{R})$ as follows: $\forall z \in \mathbb{C}^* \forall v \in V^{p,q}$ $\rho(z) \cdot v = z^p \overline{z}^q v$ if $v \in V \otimes \mathbb{R}$, then $v = \sum_{p+q=k} v^{p,q}$, $v^{q,p} = \overline{v^{p,q}}$, and then $\overline{\rho \cdot v} = \sum_{p+q=k} z^p \overline{z}^q = \sum_{p+q=k} \overline{z}^p z^q v^{q,p} = \rho(z) \cdot v$, so it is in fact a representation. But why? Why is this equality what we need to make sure it is a representation?

Observe that $z \in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$, the eigenvalue is $\rho(z) \cdot v = z^{p+q} \cdot v = z^k \cdot v$. This motivates the following:

Conversely, given a $\mathbb{Q}$-vector space V and a representation $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ of $\mathbb{R}$-groups, such that $\forall r \in \mathbb{R}^*$ $\rho(r) = r^k \cdot \mathrm{Id}$, we have $V \otimes \mathbb{C} = \bigoplus_{p,q} V^{p,q}$ where $v \in V^{p,q}$, $\rho(z, w) = z^p w^q \cdot v$. Since ρ is a representation of $\mathbb{R}$-groups we have $V^{q,p} = \overline{V^{p,q}}$, so by the computation above we have $V^{p,q} = 0$ when $p + q \neq k$, then V becomes a $\mathbb{Q}$ =HS of weight k.

In conclusion, a $\mathbb{Q}$-HS is the same thing as a $\mathbb{Q}$-vector space V and a representation of $\mathbb{R}$-groups $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ such that $\rho|_{\mathbb{R}^*}$ is defined over $\mathbb{Q}$.

(We may also replace $\mathbb{Q}$ with $\mathbb{Z}$, etc. in this construction.)

Back to the Tate group $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$, define for all $z \in \mathbb{C}^*$ and $v \in \mathbb{Q}(1) \otimes \mathbb{C}$ by $\rho(z) \cdot v = |z|^{-2} \cdot v$.

We have operations on HS: $V_1 \oplus V_2, V_1 \otimes V_2, \mathrm{Hom}(V_1, V_2)$. The $\mathbb{Q}$-HS form an abelian category. Let $\mathbb{Q}(m) = \mathbb{Q}(1)^{\otimes m}$ when $m \geq 0$, $\mathbb{Q}(-1) = \mathbb{Q}(1)^*$ and $\mathbb{Q}(-m) = \mathbb{Q}(-1)^{\otimes m}$ for $m \geq 0$.

If V is a $\mathbb{Q}$-HS, then $V(m) = V \otimes \mathbb{Q}(m)$ is the *Tate twist*.

Example 7.2. (1) $X \subset \mathbb{C}P^n$ subvariety. Notice that while $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, such a closure can be obtained by choosing the imaginary unit i or $-i$, so this is not canonical.

However, the first Chern class is canonical:

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0$$

gives the connecting homomorphism $H^1(X, \mathbb{C}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}(1))$ which is in fact a HS of weight 0 (because H^2 has weight 2 and $\mathbb{Z}(1)$ has weight -2).

Analogously, the p-th chern class is $c_p(A \text{ coh. sheaf}) \in H^{2p}(X, \mathbb{Q}(p))$.

(2) $cl : CH^p_{\mathbb{Q}}(X) \to H^{2p}(X, \mathbb{Q})$ the cycle class map, where

$$CH^p_{\mathbb{Q}}(X) = \left\{ \sum_{\substack{Z_i \subset X \\ \text{subvar.}}} \alpha_i [Z_i] : \alpha_i \in \mathbb{Q} \right\} \Big/ \text{rational equiv.}$$

Consider

$$\begin{tikzcd} Z \arrow[r, hook, "i"] & X \\ \tilde{Z} \arrow[u] \arrow[ur, "j"'] & \end{tikzcd}$$

where Z is a subvariety of codimension p and $\tilde{Z}$ is a resolution of singularities. Then we have the pushforward of the fundamental class, $j_*[\tilde{Z}] \in H_{2n-2p}(X, \mathbb{Q})$ where $[\tilde{Z}] \in H_{2n-2p}(\tilde{Z}, \mathbb{Q}) \cong H^{2n-2p}(\tilde{Z}, \mathbb{Q})^*$. Then the Poincare dual of $j_*[\tilde{Z}] := d[Z] \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$. So $H_{2n-2p}(\tilde{Z}, \mathbb{Q})$ is a HS of weight $2(p-n)$ and $[\tilde{Z}] \in H_{p-n,p-n}$.

Definition 7.3. The *space of Hodge classes* is $H^{2p}_{\mathrm{Hdg}}(X) = H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$

Then we have $cl : CH^p_{\mathbb{Q}}(X) \to H^{2p}_{\mathrm{Hdg}}(X) \cong \mathrm{Hom}_{\mathbb{Q}-HS}(\mathbb{Q}(-p), H^{2p}(X, \mathbb{Q})$. The Hodge conjecture is that cl is surjective onto the space of Hodge classes.

8. Polarizations

Let V be a $\mathbb{Q}$-HS of weight k and $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ the corresponding representation of the Deligne torus. The Weil operator in terms of ρ is $C = \rho(i)$.

Definition 8.1. A *polarization* on V is a morphism of $\mathbb{Q}$-HS is a $(-1)^k$-symmetric morphism $\psi : V \otimes V \to \mathbb{Q}(-k)$ such that the bilinear form $Q : V_{\mathbb{R}} \otimes V_{\mathbb{R}} \to \mathbb{R}$ given by $Q(x, y) = (2\pi i)^k \psi(Cx, y)$ is symmetric and positive definite.

Example 8.2. $V = H^k_{\mathrm{prim}}(X, \mathbb{Q})$ with ψ the Hodge-Riemann pairing.

Observe: if V is polarizable (i.e. it admits a polarization; not every HS admits a polarization e.g. HS on nonprojetive varieties) then V is semisimple $V = V_1 \oplus \dots \oplus V_m$, V_i is simple.

Assume $W \subset V$ is a sub-HS, then $W^{\perp_\psi} \subset V$ is a sub-HS. Then $0 \underbrace{=}_{\substack{Q \\ \text{positive} \\ \text{def.}}} W \cap W^{\perp_\psi} \subset V$, Then $V = W \oplus W^{\perp_\psi}$.

9. The Mumford-Tate group

Let V be a $\mathbb{Q}$-HS and $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes \mathbb{R})$ a representation of $\mathbb{R}$-groups.

Consider $G \subset \mathrm{GL}(V)$ such that $\mathrm{Im}(\rho) \subset G(\mathbb{R})$.

(The idea is that the Mumford-Tate group will recover the rational structure of V, because the representation ρ knows nothing about this rational structure, which must exist since V is a rational vector space.)

Definition 9.1. The *Mumford-Tate group* is

$$MT(V) = \bigcap_{\substack{G \subset \mathrm{GL}(V) \\ \text{subgroup s.t.} \\ \mathrm{Im}(\rho) \subset G(\mathbb{R})}} G = \text{smallest } Q\text{-subgroup of } \mathrm{GL}(V) \text{ containing } \mathrm{Im}\rho.$$

Remark 9.2. We can also consider $U(1) \subset \mathbb{S}(\mathbb{R})$ and $\rho' := \rho|_{U(1)} : U(1) \to \mathrm{GL}(V \otimes \mathbb{R})$. Then the *Hodge group* is $\mathrm{Hdg}(V)$ = smallest Q-subgroup of $\mathrm{GL}(V)$ such that $\mathrm{Im}(\rho') \subset \mathrm{Hdg}(V)(\mathbb{R})$.

$\mathrm{Hdg}(V)$ is always smaller than $MT(V)$.

Example 9.3.

(1) $MT(\mathbb{Q}(1)) = \mathbb{Q}^*$ $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$, $\mathbb{Q}(1) \otimes \mathbb{R} = 2\pi i\mathbb{R} \subset \mathbb{C}$, $\mathrm{GL}(\mathbb{Q}(1) \otimes \mathbb{R}) = \mathbb{R}^*$, since $z \in \mathbb{S}(\mathbb{R})$ acts as $|z|^{-2} \cdot \mathrm{Id}$.

(2) $\mathbb{Q}(0) = \mathbb{Q} \subset \mathbb{C}$, $MT(\mathbb{Q}(0)) = \{1\}$. In general, if V is of weight k, then $\mathbb{Q}^*$ = center of $\mathrm{GL}(V) \subset MT(V)$ and $MT(V)$ is generated by $\mathbb{Q}^*$ and $Hdg(V)$.

10. Tensor construction

Let V be a $\mathbb{Q}$-HS with $MT(V) \subset \mathrm{GL}(V)$ and $\rho : \mathbb{S}(\mathbb{R}) \to MT(V)(\mathbb{R})$. Then

$$T^\bullet(V) = \bigoplus_{e,f \geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$$

is a $MT(V)$-representation.

Proposition 10.1. *A finite-dimensional subspace $W \subset T^\bullet(V)$ is a sub-HS if and only if W is a $MT(V)$-subspace.*

Proof. ($\Longleftarrow$). If W is a $MT(V)$-subrepresentation, then from ρ we can compose with the representation that $MT(V)$ is to obtain $\rho' : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(W \otimes \mathbb{R})$. Note that $\rho|_{\mathbb{R}^*}$ is defined over Q, We get that W is a sub-HS.

($\Longrightarrow$). Assume that $W \subset T^\bullet(V)$ is a sub-HS, $G - \mathrm{Stab}(W) \subset \mathrm{GL}(V)$ is a $\mathbb{Q}$-subgroup. Since W is a sub-HS, the action of $\mathbb{S}(\mathbb{R})$ on $T^\bullet(V)$ preserves W, then $\mathrm{Im}(\rho) \subset G(\mathbb{R})$ and thus $MT(V) \subset G$, which implies that W is a $MT(V)$-subrepresentation. $\square$

As a corollary,

Lemma 10.2. *$x \in T^\bullet(V)$ is $MT(V)$-invariant if and only if X is a $(0,0)$ Hodge element.*

Proposition 10.3. *Assume that V is a $\mathbb{Q}$-HS of weight k and $\psi : V \otimes V \to \mathbb{Q}(-k)$ is a polarization. Then*

$$Hdg(V) \subset \begin{cases} SO(V,\psi) & \text{if } k \equiv 0 (mod\ 2) \\ Sp(V,\psi) & \text{if } k \equiv 1 (mod\ 2) \end{cases}$$

Proof. $\psi : V \otimes V \to \mathbb{Q}(-k)$, where the $\mathbb{Q}(-k)$ is a trivial $Hdg(V)$-module, so that ψ is a morphism of $Hdg(V)$-modules. The action of $Hdg(V)$ preserves the symmetric form ψ if $k \equiv 0(\mathrm{mod}\ 2)$ and the antisymmetric form ψ if $k \equiv 1(\mathrm{mod}\ 2)$. □

Example 10.4. Let $V = H^1(E,\mathbb{Q})$ where E is an elliptic curve and ψ the Hodge-Riemann pairing. Then $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ and $\dim V^{1,0} = 1$. $Hdg(V) \subset Sp(V,\psi) \cong \mathrm{SL}_2(\mathbb{Q})$.

Observe thati, in general, if V is polarizable, then $V = V_1 \oplus \ldots \oplus V_m$, V_i are irreducible $MT(V)$-representations, so $MT(V)$ is reductive.

Back to our elliptic curve example, $Hdg(V) \subset \mathrm{SL}_2(\mathbb{Q})$ is reductive.

There are two possibilities:

(1) $Hdg(V) = \mathrm{SL}^2(\mathbb{Q})$, which implies that $MT(V) = \mathrm{GL}_2(V)$. This happens when E is generic in the moduli space.
(2) $Hdg(V)$ is properly contained in $\mathrm{SL}_2(\mathbb{Q})$. Then $Hdg(V)$ is a 1-dimensional torus. Then $\mathrm{End}_{\mathbb{Q}-HS}(V) \neq \mathbb{Q}$, which implies that E has complex multiplication.

Definition 10.5. A HS V is of *CM-type* if $MT(V)$ is abelian (such HS defines a special point in the moduli space of HS.)

11. Summary so far

Recall:

- A $\mathbb{Q}$-HS is a finite dimensional $\mathbb{Q}$ vector space V with a representation of $\mathbb{R}$-group $\rho : \underbrace{\mathbb{S}(\mathbb{R})}_{\cong \mathbb{C}^*} \to \mathrm{GL}(V \otimes \mathbb{R})$ such that $\rho|_{\mathbb{R}^*}$ is defined over $\mathbb{Q}$.

  This gives a Hodge structure $V \otimes \mathbb{C} = \bigoplus_{p,q} V^{p,q}$, $V^{p,q} = \overline{V^{q,p}}$.

  V is of weight k if $\forall r \in \mathbb{R}^k$, $\rho(r) = r^k \cdot \mathrm{Id}$. $\forall z \in \mathbb{S}(\mathbb{R})$, $\rho(z)|_{V^{p,q}} = z^p \overline{v}^q \mathrm{Id}$
- $MT(V) = \bigcap_{\substack{G \in \mathrm{GL}(V) \\ \mathrm{s.t.\,Im}(\rho) \subset G(\mathbb{R})}} G$ is the Mumford-Tate group.
- $Hdg(V) = \bigcap_{\substack{G \in \mathrm{GL}(V) \\ \mathrm{s.t.\,Im}(\rho|_{\mathrm{U}(1)}) \subset G(\mathbb{R})}} G$ is the Hodge group.
- $T^\bullet V = \bigoplus_{e,f \geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$. A sub-HS of $T^\bullet V$ is the same thing as a sub-$MT(V)$-representation. $T^\bullet V^{MT(V)} = MT(V)$-invariants. $T^\bullet V \cap (T^* V)^{0,0} =$ the space of Hodge $(0,0)$-classes.
- (Polarization.) Assume that V has weight k $\psi : V \otimes V \to \mathbb{Q}(-k)$ a $\mathbb{Q}$-HS morphism such that the following form is positive definite:

$$Q : V_{\mathbb{R}} \otimes V_{\mathbb{R}} \longrightarrow \mathbb{R}$$
$$Q(x,y) = (2\pi i)^k \psi(Cx, y)$$

  where $C = \rho(i) \in Hdg(V)(\mathbb{R})$, then $MT(V), Hdg(V)$ are reductive.

"We are interested in polarizable Hodge structures because that is the case of cohomologies of algebraic varieties."

12. Some facts about reductive groups

Let V be a finite dimensional vector space over k, $\mathrm{char}(k) = 0$ and $G \subset \mathrm{GL}(V)$ an irreducible algebraic k-group.

Remark 12.1. $MT(V)$ and $Hdg(V)$ are connected.

Since by definition G is a subgroup of $\mathrm{GL}(V)$, we can view V as a faithful representation (because the inclusion has no kernel). We say G is *reductive* if V is semisimple, i.e. V is a direct sum of irreducible representations $V = V_1 \oplus \ldots \oplus V_n$. Equivalently, Every G-representation is semisimple. Equivalently, the center $Z^0(G)$ is an algebraic torus and $G/Z(G)$ is semisimple. Equivalently, $G(\mathbb{C})$ admits a compact real form.

$\sigma \in \mathrm{Aut}(G(\mathbb{C}))$ is a *Cartan involution* if $\sigma^2 = \mathrm{id}$ and $G^\sigma = \{g \in G(\mathbb{C}) : \sigma(\bar{g}) = g\}$ is compact.

Example 12.2. Let V be a polarizable $\mathbb{Q}$-HS of weight k. $C^2 = \rho(i^2) = \rho(-1) \in Z(\underbrace{Hdg(V)}_{G})$, $G = Hdg(V)$.

Note that $\sigma = Ad_C : G \to G, g \mapsto CgC^{-1}$, $\mathrm{id} = Ad_C^2, g \mapsto C^2gC^{-2} = g$.

We claim that σ is a Cartan involution on $Hdg(V) = G$. Indeed, Q defines a Hermitian product on $V \otimes \mathbb{C}$. $g \in G^\sigma$,

$$\begin{aligned} Q(gx, \overline{gy}) &= Q(gx, \sigma(g)\bar{y}) \\ &= (2\pi i)^k \psi(\underbrace{Cg}_{\in Hdg}\, x, \underbrace{Cg}_{\in Hdg}\, C^{-1}\bar{y}) \\ &= (2\pi i)^k \psi(x, C^{-1}\bar{y}) \\ &= (2\pi i)^k \psi(Cx, \bar{y}) \\ &= Q(x, \bar{y}). \end{aligned}$$

Thus, $G^\sigma \subset \underbrace{U(V \otimes \mathbb{C}}_{\text{compact}}$ as claimed.

Remark 12.3. A $\mathbb{Q}$-HS is polarizable if and only if $G = Ad_C$ is a Cartan involution.

13. Reductive groups and their representations

Let V be a finite-dimensional vector space over k, $\mathrm{char} k = 0$. $T^\bullet V = \bigoplus_{e,f \geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$.

Theorem 13.1. *Let $G \subset GL(V)$ be a reductive group. Then*

1. *Any finite-dimensional representation of G is a subgroup of $(T^\bullet V)^{\oplus N}$ for some $N > 0$.*
2. *Assume that $H \subset G$ is an algebraic subgroup. There exists a G-representation W and a 1-dimensional subspace $L \subset W$ such that $H = Stab_W(L)$.*
3. *If H is reductive, then there exists a G-representation W and $x \in W$ such that $H = Stab_W(x)$.*

We need some preparations for the proof:

(1) $G \subset \mathrm{GL}(V)$ Let $k[G]$ be the algebra of regular functions on G.

$$\begin{tikzcd}
\mathrm{GL}(V) \arrow[rrr] \arrow[drrr] & & & \mathrm{End}(V) \oplus \mathrm{End}(V^*) \\
g \arrow[dr, mapsto] & & & \\
& (g,(g^t)^{-1}) & & \{(A,B) : A \cdot B^t = \mathrm{Id}\} \arrow[uu, hook]
\end{tikzcd}$$

So we have

$$\mathrm{Sym}(V \otimes V^*)^{\otimes 2} \twoheadrightarrow K[\mathrm{GL}(V)] \twoheadrightarrow K[G].$$

(2) $K[G]$ is a Hopf algebra. Indeed, we have

$$\begin{aligned}
G \times G &\xrightarrow{m} G & \qquad K[G] &\xrightarrow{\Delta} K[G] \otimes K[G] \\
\{1\} &\xhookrightarrow{e} G & K[G] &\xrightarrow{\varepsilon} K \\
G &\xrightarrow{inv} G & K[G] &\xrightarrow{coinv} K[G].
\end{aligned}$$

and

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
K[C] \arrow[r, "\Delta"] \arrow[d, "\Delta"] & K[G]^{\otimes 2} \arrow[d, "\mathrm{id} \otimes \Delta"] \\
K[G]^{\otimes 2} \arrow[r, "\Delta \otimes \mathrm{id}"] & K[G]
\end{tikzcd}$$

Then V is a G-representation, i.e. a *comodule*. We have

$$\begin{gathered}
G \times V^* \to V^* \\
\mathrm{Sym}(V) \to \mathrm{Sym}(V) \otimes K[G] \\
V \to V \otimes K[G] \\
\mu : V \to V \otimes K[G]
\end{gathered}$$

and the following diagram commutes

$$\begin{tikzcd}
V \arrow[r, "\mu"] \arrow[d, "\mu"] & V \otimes K[G] \arrow[d, "\mu \otimes \mathrm{id}"] \\
V \otimes K[G] \arrow[r, "\mathrm{id} \otimes \Delta"'] & V \otimes K[G] \otimes K[G].
\end{tikzcd} \tag{13.1.1}$$

Equation 13.1.1 defines a $K[G]$-comodule structure on $K[G]$.

(3) Any G-representation is a union of finite-dimensional subrepresentations. (In particular this applies to $K[G]$.)

Indeed, let $\{a_i\}$ be a basis of $K[G]$. Fix $x \in V$ and denote

$$\mu(x) = \sum_{i,\text{finite}} v_i \otimes e_i \qquad \text{and} \qquad \Delta e_i = \sum_{j,k} a_{ijk} e_j \otimes e_k, \qquad a_{ijk} \in k.$$

Then

$$\sum_i \mu(v_i) \otimes e_i = \sum_{i,j,k} a_{ijk} v_i \otimes e_j \otimes e_k$$
$$= \sum_i \left(\sum_{j,k} a_{ijk} v_k \otimes v_k \otimes e_j \right) \otimes e_i.$$

$\implies \mu(v_i) = \sum_{j,k} a_{kji} v_k \otimes e_j$. $W = \mathrm{span}(x, v_i) \implies \mu : W \to W \otimes K[G]$, $\dim W < \infty$.

Now we can prove Theorem 13.1.

Proof. (1) Let W be a finite-dimensional G-representation. It is enough to show that W is a subrepresentation of $K[G]^{\oplus N}$ for some $N > 0$.

(13.1.2)
$$\begin{tikzcd} W \arrow[r, "\mu"] \arrow[d, "\mu"'] & W \otimes K[G] \arrow[d, "\mu\otimes\mathrm{id}"] \\ W \otimes K[G] \arrow[r, "\mathrm{id}\otimes\Delta"'] & W \otimes K[G] \otimes K[G] \end{tikzcd}$$

Equation 13.1.2 means that μ defines a morphism of $K[G]$-comodules

$$W \hookrightarrow \underbrace{W \otimes K[G]}_{K[G]^{\oplus \dim W}}.$$

(2) Let $H \subset G$ be an algebraic k-subgroup. [We want to show] There exists a G-representation W and $L \subset W$ with $\dim L = 1$ such that $H = Stab_W(L)$.

$k[H] = k[G]/I$ for an ideal I. For $f \in K[G], h, g \in G$ we have $gf(h) = f(g^{-1}h)$.

Suppose that $H = V(I)$ for an ideal I. Note that $g \in H$ if and only if $g \cdot I = I$ since $g \cdot H = V(gI)$ since $0 = f(x) = f(g^{-1}gx) = gf(gx) = 0$. That is, $H = Stab_{K[G]}(I)$.

Let $f_1, \dots, f_m$ be generators of I. Then there exists $\tilde{W} \subset k[G]$, a finite-dimensional G-subgroup with $f_1, \dots, f_m \in W$, $H = Stab_{\tilde{W}}(\tilde{W} \cap I)$. Consider $W = \Lambda^d \tilde{W}$ and $L = \Lambda^d(W \cap I)$, which satisfy the conditions.

(3) If H is reductive, $H = Stab_W(L)$ $W = L \oplus W'$, $W \otimes W^* = (\underbrace{L \otimes L^*}_{\text{trivial}} \oplus (\dots)$.

Let $0 \neq x \in L \otimes L^*$. Then $H = Stab_{W \otimes W^*} Stab(x)$.

∎

As a corollary,

Lemma 13.2. *If V is a polarizable $\mathbb{Q}$-HS, then [the Mumford-Tate group is reductive] $MT(V)$ = Stabilizer of all $(0,0)$-Hodge classes in $T^\bullet V$.*

Proof. Let $G = MT(V)$ and

$$G' = \bigcap_{\substack{x \in (0,0)\text{-Hodge} \\ \text{classes}}} Stab(x) \subset \mathrm{GL}(V).$$

$G \subset G'$.

By part 3 of Theorem **??**, $G = Stab_W(x)$ where W is a representation of $\mathrm{GL}(V)$. By part 1 of Theorem **??**, $W \subset (T^\bullet V)^{\oplus N}$. x is fixed by $G = MT(V)$, then x is a $(0,0)$-Hodge class. Thus $G' \subset G$. □

[Approximate comment] We have shown that the representations of $MT(V)$ live in that tensor algebra. We would like to find a universal object such that any Hodge structure is realised as a representation of such group. This is analogous to Galois theory, where we take an inverse limit over Glois groups of field extensions, which provides a universal object and an equivalence of categories.

14. Short summary

V a $\mathbb{Q}$-Hs, polarizable. $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes \mathbb{R})$, $MT(V) \subset \mathrm{GL}(V)$, the smallest subgroup over $\mathbb{Q}$ such that $MT(V)(\mathbb{R})$ contains $\mathrm{Im}(\rho)$. Polarizable $\implies MT(V)$ is reductive. $MT(V) =$ stabilizer of all $(0,0)$ Hodge classes in $T^\bullet V = \bigoplus_{e,f\geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$.

15. Absolute Mumford-Tate group

Let $V' \subset V$ be a sub-HS. $MT(V) \subset Stab_V(V') = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$.

We also have a map $MT(V) \twoheadrightarrow MT(V')$.

The absolute MT group should be something like the inverse limit $\lim \longleftarrow MT(V)$.

Let $(\mathbb{Q} - HS)$ be the category of $\mathbb{Q}$-HS.

(1) $\mathbb{Q}$-linear abelian category.
(2) Has $\otimes$, which is commutative with unit $1 = \mathbb{Q}(0)$, which a Hodge structure such that $\mathrm{End}(\mathbb{Q}(0)) = \mathbb{Q}$. We also have duality, $\forall V \exists V^*$ and $\mathbb{Q}(0) = q \to V \otimes V^*$ which is the "diagonal embedding", and $V \otimes V^* \to 1 = \mathbb{Q}(0)$, which is the trace.
(3) There exists a functor $\omega : (\mathbb{Q} - HS) \to (\mathbb{Q} - v.s.)$, forgetful functor. ω is called *fibre functor*. A $\otimes$-functor that is exact and faithful (injective on Hom's).

The pair ($\mathbb{Q}$-HS, ω) is called a *neutral Tannakian category.*

Theorem 15.1. *A neutral Tannakian category $(\mathcal{C}, \omega)$ (where $\mathcal{C}$ is K-linear, char$K = 0$) is equivalent to the category of finite dimensional representations of an affine group scheme over K $SpcK[G]$, $K[G]$ Hopf algebra. G is a pro-alg. group called the Tannakian fundamental group.*

Apply this to $(\mathbb{Q} - HS), \omega, \implies G = MT_{\mathbb{Q}-HS}$ is the *absolute MT group.*

Let $(\mathbb{Q} - HS)^{pol}$ be the full $\otimes$-subcategory of polarizable HS. $MT_{(\mathbb{Q}-HS)^{\mathrm{pol}}}$ is pro-reductive.

16. How to construct the Tannakian fundamental group

We have $\omega : \mathcal{C} \to (K - v.s.)$. Let $\mathrm{Aut}(\omega)^{\otimes}$ be the automorphisms ot ω as a $\otimes$-functor. That is, if $g \in \mathrm{Aut}(\omega)^{\otimes}$, we have $\forall X \in \mathcal{C}$, $g_x \in \mathrm{GL}(\omega(X))$.

We also have:

- Compatibility with morphisms, i.e. for every $f : X \to Y$ we have

$$\begin{tikzcd} \omega(X) \arrow[r, "\omega(f)"] \arrow[d, "g_X"'] & \omega(Y) \arrow[d, "g_Y"] \\ \omega(X) \arrow[r, "\omega(f)"'] & \omega(Y) \end{tikzcd}$$

commutes, i.e. $g_Y \circ \omega(f) = \omega(f) \circ g_X$.
- Compatibility with $\otimes$:

$$g_{X\otimes Y} = g_X \otimes g_Y$$
$$g_1 = 1 \in K^* = \mathrm{GL}(K), g_{X^*} = (g_X^*)^{-1}.$$

$\mathrm{Aut}(\omega)^{\otimes}$ is the group of K-points of the Tannakian fundamental group.

[Our category $(\mathbb{Q} - HS)^{\mathrm{pol}}$ is equivalent to the category of representations of this group. But the problem is that this group is huge. So we shall restrict to a smaller subcategory.]

Let $V \in (\mathbb{Q} - HS)^{\mathrm{pol}}$ and $\langle V \rangle$ the full $\otimes$-subcategory generated by V. Objects are sub-HS of $T^\bullet V$ and their direct sums.

What is the Tannakian fundamental group of $\langle V \rangle$?

Consider $\omega : \langle V \rangle \to (\mathbb{Q} - v.s.)$. For $g \in \mathrm{Aut}(\omega)^{\otimes}$ we have that $g_V \in \mathrm{GL}(V)$ is uniquely determined by g.

Let $x \in T^\bullet V \cap (T^\bullet V)^{0,0}$. We have an embedding

$$1 = \mathbb{Q}(0) \hookrightarrow T^\bullet V$$
$$1 \mapsto x.$$

We have $g_{\mathbb{Q}(0)} = 1 \implies g_V \in \mathrm{Stab}_{T^\bullet V}(x)$. Thus $g_V \in MT(V)$. We also have the converse, so we see that the Tannakian fundamental group of $\langle V \rangle$ is $MT(V)$. This could be taken as an equivalent definition of $MT(V)$

17. Hodge structures of abelian varieties

Let $(\mathbb{Q} - HS)^{\mathrm{ab}}$ be the full $\otimes$-subcategory of $(\mathbb{Q} - HS)^{\mathrm{pol}}$ genertated by $H^1(A, \mathbb{Q})$ where A is an abelian varieties.

Recall that if A is a projective variety it is biholomorphic to $\mathbb{C}^n/\Lambda$ and $H^1(A, \mathbb{C}) = H^{1,0}(A) \oplus H^{0,1}(A)$.

[Suppose that you have some Hodge structure, say, on the cohomology of your favourite variety. Does it belong to this category or not? It is not clear — there might be some complicated Hodge structures in there]

Let $V = H^1(A, \mathbb{Q})$, $MT(V) \subset \mathrm{GL}(V)$. Let $\mathfrak{mt}(V) = \mathrm{Lie}(MT(V)) \subset \mathfrak{gl}(V) = V \otimes V^*$, a sub-HS (of $\mathfrak{gl}(V)$, which is always a Hodge structure). That is, the Lie algebra of $MT(V)$ always carries a Hodge structure. This HS may only have types $(-1, 1), (0, 0), (1, -1)$ because $((1, 0) + (0, 1)) \otimes ((-1, 0), (0, -1))$.

Proposition 17.1. *Let $V \in (\mathbb{Q} - HS)^{ab}$. Then the HS on $\mathfrak{mt}(V)$ may have only Hodge types* $(-1, 1), (0, 0), (1, -1)$.

Proof. We may assume that $V \subset T^\bullet W$ where $W = H^1(A, \mathbb{Q})$ by the tensor construction. Then we must have $MT(W) \twoheadrightarrow MT(V)$, which on the level of Lie algebras becomes $\mathfrak{mt}(W) \twoheadrightarrow \mathfrak{mt}(V)$, which is a surjection of HS. $\square$

The next example shows how we may use this result.

Example 17.2. (1) Let X = a K3 surface, $X \subset \mathbb{P}^3$, a smooth quartic. Then $K_X = \mathcal{O}_X$, $h^{2,0} = h^{0,2} = 1$, $H^2(X,\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$. The HS is $V = H^2(X,\mathbb{Q})$ and the intersection product is a map $V \otimes V \to \mathbb{Q}(-2)$. Recall that the Hodge group is the image of $U(n)$ under the representation from the Deligne torus. In this case we have $MT(V) \supset Hdg(V) \subset \mathrm{SO}(V,\psi)$. Also

$$\mathfrak{mt}(V) = \underbrace{\mathfrak{hdg}(V)}_{\mathrm{Lie}(Hdg(V))} \oplus \underbrace{\mathbb{Q}(0)}_{\text{center of } \mathfrak{gl}(V)} .$$

And also $(1,-1),(0,0),(-1,1)$. Now since $H^{2,0}$ is 1-dimensional we can compute

$$\Lambda^2(V\otimes\mathbb{C}) = \underbrace{(H^{2,0}\otimes H^{1,1})}_{3,1} \oplus \underbrace{(\Lambda^2 H^{1,1} \oplus H^{2,0}\otimes H^{0,2})}_{2,2} \oplus \underbrace{(H^{1,1}\otimes H^{0,2})}_{1,3}$$

(2) $X \subset \mathbb{C}P^3$ hypersurface of degree ≥ 5. Then $h^{2,0} > 1$, $V = H^2(X,\mathbb{Q})$, $\mathfrak{hdg}(V) \subset \mathfrak{so}(V,\psi)$. Hodge types: $(2,2),(1,-1),(0,0),(-1,1),(-2,-2)$.

18. Variations of Hodge structures

A *family* is $\pi : \mathcal{X} \to B$ with $\mathcal{X}, B$ complex manifolds, π is a submersion, B is connected and $\mathcal{X} \hookrightarrow \mathbb{P}^N \times B$.

Then $R^k\pi_*\mathbb{Z}/\text{tors.}$ is a local system over B. In fact, $(R^k\pi_*\mathbb{Z})_t = H^k(\mathcal{X}_t,\mathbb{Z})$ where $\mathcal{X}_t = \pi^{-1}(t)$. That is, $R^k\pi_*\mathbb{Z}$ is a sheaf with the information of the cohomologies of the fibres.

Consider

$$0 \longrightarrow \pi^*\Omega^1_B \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/B} \longrightarrow 0$$

Assume $\dim B = 1$ (This is used to prove the property of the Gauss-Manin connection, though it can be also be done by restricting to curves.) Then

$$0 \longrightarrow \pi^*\Omega^1_B \otimes \Omega^{k-1}_{\mathcal{X}} \longrightarrow \Omega^k_{\mathcal{X}} \longrightarrow \Omega^k_{\mathcal{X}/B} \longrightarrow 0$$

$$0 \longrightarrow \pi^*\Omega^1_B \otimes \Omega^\bullet_{\mathcal{X}/B}[-1] \longrightarrow \Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}/B} \simeq \pi^{-1}\mathcal{O}_B \longrightarrow 0 \tag{18.0.1}$$

We can construct the vector bundle

$$\mathcal{V}_k = (R^k\pi_*\mathbb{C}) \otimes \mathcal{O}_B = R^k\pi_*(\pi^{-1}\mathcal{O}) \simeq R^k\pi_*(\Omega^\bullet_{\mathcal{X}/B}).$$

On $\mathcal{V}_k$ we have a flat connection from Equation 18.0.1. We get

$$R^k\pi_*(\Omega^\bullet_{\mathcal{X}/B} \longrightarrow R^{k+1}\pi_*(\pi^*\Omega^1_B \otimes \Omega^\bullet_{\mathcal{X}/B}[-1] \longrightarrow \mathcal{V}_k \longrightarrow \Omega^{1\,\nabla}_B \otimes R^k\pi_*\Omega^\bullet_{\mathcal{X}/B} = \Omega^1_B \otimes \mathcal{V}^k.$$

The map ∇ is called the *Gauss-Manin* connection.

Define $F^pV_k = R^k\pi_*(\Omega^{\geq p}_{\mathcal{X}/B})$. Then $F^p\mathcal{V}_k \subset \mathcal{V}_k$ is a holomorphic subbundle.

The *Hodge filtration* on $\mathcal{V}_k$.

(1) $\nabla : \mathcal{V}_k \to \Omega^1_B \otimes \mathcal{V}_k$ a flat connection.
(2) The Hodge filtration

$$0 \subset \ldots \subset F^p\mathcal{V}_k \subset F^{p-1}\mathcal{V}_k \subset \ldots \subset V_k.$$

(3) (Griffiths transversality.) When we restrict this construction to F^p the relation we obtain is *Griffiths transversality.* $\nabla(F^k\mathcal{V}_k) \subset \Omega^1_B \otimes F^{p-1}\mathcal{V}_k$.

Definition 18.1. A *polarization* on $\mathcal{V}_k$ is defined by a section of $R^2\pi_*\mathbb{Z}(1)$.

By the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 1$$

we obtain maps

$$R^1\pi_*\mathbb{C}^* \to R^2\pi_*\mathbb{Z}(1),$$

and

$$\mathcal{O}_{\mathbb{P}^N}(1) \in H^1(\mathcal{X}, \mathbb{C}^*) \to H^0(B, R^1\pi_*\mathbb{C}^*) \to H^0(B, R^2\pi_*\mathbb{Z}(1)).$$

From there we get

$$\psi : V_B \otimes V_B \to \mathbb{Z}_B(-k),$$

where $V_B = R^k\pi_*\mathbb{Z}/\text{tors.}$, a morphism of local systems that defines polarizations on the fibres of π.

Definition 18.2. A *$\mathbb{Z}$-variation of Hodge-Structure over B* is a $\mathbb{Z}$-local system V_B with a fibre $\mathbb{Z}^r$ for some $r \geq 0$ and a finite filtration of holomorphic subbundles on $\mathcal{V} = V_B \otimes \mathcal{O}_B$

$$0 \subset \ldots \subset F^p\mathcal{V} \subset F^{p-1}\mathcal{V} \subset \ldots \subset \mathcal{V}$$

[which satisfies Griffiths transversality] such that

(1) for all $t \in B$ $(V_{B,t}, F^\bullet\mathcal{V}_t)$ is a $\mathbb{Z}$-HS.
(2) $\nabla(F^p\mathcal{V}) \subset \Omega^1_B \otimes F^{p-1}\mathcal{V}$ where ∇ is the flat connection on $\mathcal{V}$.

19. The Hodge locus

[Next we shall try to study the MT group on families. Since MT essentially is the satbilizer of the Hodge classes, we look for those. There are fibres that may have more Hodge classes than others. This leads to the following concept.]

Let B be a complex manifold and V_B a $\mathbb{Z}$-local system with a VHS. $F^p\mathcal{V}$ of weight $2k$. We have

$$\begin{tikzcd} \mathrm{Tot}(\mathcal{V}) \arrow[r, "p"] & B \\ \mathrm{Tot}(V_B). \arrow[u, hook] \arrow[ur, "\text{covering}"'] & \end{tikzcd}$$

Note that $\mathrm{Tot}(V_B) = \amalg W_i$ where W_i are the connected components of $\mathrm{Tot}(V_B)$.

In this setting, a Hodge class is a point x in one of W_i such that $x \in F^k\mathcal{V} \cap \overline{F^k\mathcal{V}}$ $\iff$ $x \in F^k\mathcal{V} \cap W_i$.

Let $Z_i = \mathrm{Tot}(F^k\mathcal{V}) \cap W_i$. Note that Z_i is a closed subvariety (maybe singular) of $\mathrm{Tot}(\mathcal{V})$.

If the projections of these Z_i, $p(Z_i)$, is not equal to B, then $p(Z_i)$ is in the Hodge locus. That is, the *Hodge locus* is

$$B^{\mathrm{Hdg}}(V_B) = \bigcup_{i: p(Z_i) \neq B} p(Z_i)$$

$$B^{\mathrm{Hdg}} = \bigcup_{\substack{e,f \\ \text{countable} \\ \text{union}}} B^{\mathrm{Hdg}}(\underbrace{T^{e,f}V_B}_{V_B^{\otimes e} \otimes (V_B^*)^{\otimes f}}).$$

[Our goal is to consider the complement of the Hodge locus and study there the MT group.]

20. Third summary

Recall: $\mathbb{Z}$-VHS of weight k

- B complex manifold,
- V_B a local system over B with fibre $\mathbb{Z}^2$, $\mathcal{V} = V_B \otimes \mathcal{O}_B$,
- The Hodge filtration $F^p\mathcal{V} \subset \mathcal{V}$ (filtration by holomorphic subbundles)
$$0 = F^{k+1}\mathcal{V} \subset F^k\mathcal{V} \subset \ldots \subset F^0\mathcal{V} = \mathcal{V}.$$
- $\nabla : \mathcal{V} \to \Omega^1_B \otimes \mathcal{V}$ the flat connection induced by V_B. Griffiths transversality condition $\nabla(F^p\mathcal{V}) \subset \Omega^1_B \otimes F^{p-1}\mathcal{V}$.
- Polarization given by a morphism of local systems $\psi : V_B \otimes V_B \to \mathbb{Z}_B$ such that for every $t \in B$, ψ_t defines a polarization on $(V_{B,t}, F^p\mathcal{V}_t)$. ψ is $(-1)^k$-symmetric. We have a representation $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V_{B,t} \otimes \mathbb{R})$, $C = \rho(1)$, $Q(x,y) = (2\pi i)^k \psi(Cx, y)$, $Q > 0$ on $V_{B,t} \otimes \mathbb{R}$.
- Hodge locus. Assume weight $= 2k$. $x \in V_{B,t}$ is a Hodge class if and only if $x \in F^k\mathcal{V} \cap V_{B,t}$.

$$\begin{tikzcd}
\mathrm{Tot}(V_B) \arrow[r, hook] \arrow[d, equal] & \mathrm{Tot}(\mathcal{V}) \arrow[d, "p"] \\
\amalg W_i \arrow[r] & B
\end{tikzcd}$$

where W_i are the connected components and $p|_{W_i} \to B$ is a covering.

Let $Z_i = W_i \cap \mathrm{Tot}(F^k\mathcal{V})$, which is a closed complex subvariety of $\mathrm{Tot}(\mathcal{V})$. There are two possibilities:

(1) Either $Z_i = W_i$ and $p(Z_i) = B$,
(2) or $\dim Z_i = \dim W_i = \dim B \implies p(Z_i)$ is nowhere dense. [This one corresponds to the Hodge locus.]
$$B^{Hdg}(V_B) = \bigcup_{p(Z_i) \neq B} p(Z_i)$$

Then the Hodge locus is
$$B^{Hdg} = \bigcup_{e,f \geq 0} B^{Hdg}(T^{e,f}V_B$$
where $T^{e,f}V_B = V_B^{\otimes e} \otimes (V_B^*)^{\otimes f}$.

Note that $B \neq B^{Hdg}$ because B^{Hdg} is a countable union of nowhere dense subsets.

21. Hodge locus (continued)

Proposition 21.1. *$\forall i$, $p|_{Z_i} : Z_i \to B$ is a proper map. Thus $p(Z_i) \subset B$ is a closed subvariety.*

Proof. Let $x \in Z_i$. Define
$$A = (2\pi i)\psi(x,x) \underbrace{=}_{\substack{x \text{ is of} \\ \text{type } (k,k)}} Q(x,x) \geq 0$$
A does not depend on the choice of $x \in Z_i$ because ψ is flat, i.e. $\nabla\psi = 0$.

Let $S_A = \{z \in \mathrm{Tot}(\mathcal{V}^{k,k}) : Q(z, \bar{z}) = A\}$. Then $p|_{S_A} : S_A \to B$ is a sphere bundle $\implies$ proper map. Since Z_i is a closed subset of S_A, we conclude that $p|_{Z_i}$ is also a proper map. $\square$

Remark 21.2. When B is quasi-projective $p(Z_i)$ is algebraic (Cattani-Deligne-Kaplan).

If $W_i = Z_i \implies W_i$ spans a sub-local system on V_B, that is in $\mathcal{V}^{k,k}$.

Definition 21.3. V_B^{HT} is the biggest sub-local system in V_B that is Hodge-Tate, i.e. in $\mathcal{V}^{k,k}$.

Let $t_0 \in B \setminus B^{Hdg}$. Denote MT_{t_0} the MT group of the corresponding Hodge structure, i.e. $MT_{t_0} = MT(V_{B,t}, F^\bullet \mathcal{V}_{t_0})$. We do parallel transport along the loops in B based in t_0, and obtain an action of the fundamental group: the monodromy representation is $\mu : \pi_1(B, t_0) \to \mathrm{GL}(V_{B,t_0})$.

Proposition 21.4. *MT_{t_0} contains a finite index subgroup of Im(μ).*

Proof. Recall that MT_{t_0} is the stabilizer of all $(0,0)$ Hodge classes in all tensor powers, i.e. in $(T^\bullet V_B)_{t_0}$.

There exists Hodge classes $x_j \in (T^{e_j,f_j} V_B)_{t_0} \cap T^{e_j,f_j}\mathcal{V})^{0,0}_{t_0}$ where $j = 1, \ldots, m$ such that $MT_{t_0} = \bigcap_{j=1}^m Stab(x_j)$.

Note that $t_0 \notin B^{Hdg} \implies x_j \in (T^{e_j,f_j} V_B)^{HT}_{t_0}$, which is a sublocal system. Now consider the monodromy action: $\pi_1(B, t_0)$ acts on $(T^{e_j,f_j} V_B)^{HT}_{t_0}$. By Proposition 21.1, all x_j's have finite $\pi_1(B, t_0)$ orbits. Thus there exists a finite index subgroup $\pi_1(B, t_0)$ that stabilizes all x_j's. Thus, the image of this subgroup under the monodromy representation lies in MT_{t_0} (because it stabilizes all x_j's). $\square$

Definition 21.5. For any $t \in B$, define
$$G_t = \bigcap_{x \in (T^\bullet V_B)^{HT}_t} Stab(x).$$
Then G_t define a local system of algebraic groups with fibre G called the *generic MT group.*

For any $t \in B \setminus B^{Hdg}$, $G_t = MT_t$ and for any $t \in B^{Hdg}$, $MT_t \subset G_t$

Example 21.6. Let B be the space of smooth hypersurfaces $X \subset \mathbb{C}P^n$ of degree d. Then B is a Zariski open subset of $\mathbb{C}P^{\binom{n+d}{n}-1}$, i.e. it is a quasi-projective variety.

Let $\pi : \mathcal{X} \to B$ be the universal family.

Assume that $n = 2r$. $V_B = R^{2r-1}\pi_*\mathbb{Z}$ is a $\mathbb{Z}$-VHS$/B$.

$\psi : V_B \otimes V_B \to \mathbb{Z}_B$ is skew-symmetric, $\mu : \pi_1(B, t_0) \to Sp(V_{B,t_0}, \psi)$.

Theorem 21.7 (Kazhdan-Margulis)**.** *Im(μ) is Zariski dense in $Sp(V_{B,t_0} \otimes \mathbb{Q}, \psi)$.*

And a corollary is that

Lemma 21.8. *The generic Hodge group of the family $\mathcal{X}$ is $Sp(V_{B,t_0} \otimes \mathbb{Q}, \psi)$.*

Consider $n = 4$, $d = 5$, i.e. $X \subset \mathbb{C}P^5$ a quintic 3-fold. This is a Calabi-Yau, i.e. $K_X = \mathcal{O}_X$. Thus $h^{3,0} = 1, h^{2,1} = 101, h^{1,2} = 101$, $h^{0,3} = 1$.

Assume $X = \mathcal{X}_t$ for $t \in B \setminus B^{Hdg}$, $Hdg(X) = Sp(H^3, \psi)$, Lie$(Hdg(X)) \cong S^2H^3(3)$ where S^2 is the symmetric square, and it is twisted by 3.

In the Hodge structure $S^2H^3(3)$ we have Hodge classes of types $(3,-3), (2,-2), (1,-1), (0,0), \ldots, (-3,3)$. Thus $H^3(X, \mathbb{Q}) \notin (\mathbb{Q} - HS)^{ab}$.

22. Period domains and period maps

[A variation of Hodge structures is just a bundle. The classifying space of these bundles is called period.]

Let V_B be a local system over B. Pass to the universal covering $u : \tilde{B} \to B$. Consider $u^* V_B = V \otimes \mathbb{Z}_{\tilde{B}}$. Let $V = \mathbb{Z}^2$, $\mathcal{V} = V \otimes \mathcal{O}_{\tilde{B}}$. $F^p\mathcal{V} \subset \mathcal{V}$, $t_0 \in \tilde{B} \setminus B^{Hdg}$, $Hdg(V_{B,u(t_0)}) = G$, a reductive group. We also have $\rho_0 : U(1) \to G(\mathbb{R})$.

Definition 22.1. Let $\mathcal{D}$ be the connected component of $\mathrm{Hom}(U(1), G(\mathbb{R}))$ containing ρ_0.

Lemma 22.2. *$\mathcal{D} \cong G(\mathbb{R})^0/Stab(\rho_0)$, where $G(\mathbb{R})$ acts on $\mathrm{Hom}(U(1), G(\mathbb{R}))$ by conjugation on the image, i.e. for $g \in G(\mathbb{R})$ we put $(g \cdot \rho_0)(z) = g\rho_0(z)g^{-1}$.*

Proof. $\mathrm{Im}(\rho_0)$ is a compact abelian subgroup of $G(\mathbb{R})$, thus $\mathrm{Im}(\rho_0)$ is contained in a maximal compact torus in $G(\mathbb{R})$.

Fact: G-reductive implies that all maximal compact tori are conjugate by the action of $G(\mathbb{R})^0$.

$\rho : U(1) \to G(\mathbb{R})$ such that $[\rho] \in \mathcal{D}$, then there exists $g \in G(\mathbb{R})$ such that $g \cdot \rho \in T_0$, $[g, \rho] \in \mathcal{D}$, $\mathrm{Hom}(U(1), T_0)$ is a discrete topological space, thus $g\rho = \rho_0 \iff \rho = g^{-1}\rho_0$. □

Let $K = Stab(\rho_0) \subset G(\mathbb{R})^0$.

$C = \rho_0(i)$, $\sigma = Ad_C$ is a Cartan involution on $G(\mathbb{R})$. $G^\sigma = \{g \in G(\mathbb{C}) : \bar{g} = (g)\}$ is compact. We also have that $\sigma|_K = \mathrm{id} \implies K \subset G^\sigma \implies K$ is compact.

Consider the Lie algebra of G, i.e. $\mathfrak{g} = \mathrm{Lie}(G)$. Consider the Hodge decomposition induced by ρ_0, namely $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^{p,-p}$. We see that the action is trivial on the component $(0,0)$:

$$\begin{aligned} z \in U(1) &\implies \bar{z} = z^{-1}, \\ \rho_0(z)|_{\mathfrak{g}^{p,-p}} &= z^{2p}\mathrm{Id} \\ &\implies \mathfrak{g}^{0,0} \cap \mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}K. \end{aligned}$$

Note that $F^p\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r \geq p} \mathfrak{g}^{r,-2}$, $F^0\mathfrak{g}_{\mathbb{C}}$ = Lie subalgebra that preserves the Hodge filtration on V_{B,t_0}. $P \subset G(\mathbb{C})$ be such that $\mathrm{Lie}(P) = F^0\mathfrak{g}_{\mathbb{C}}$.

Definition 22.3. $\hat{\mathcal{D}} = G(\mathbb{C})/P$ where P is the stabilizer of the Hodge filtration on $V_{B,t_0} \otimes \mathbb{C}$.

$\hat{\mathcal{D}}$ is a flag variety.

[We want to see how to embed $\mathcal{D}$ in $\hat{\mathcal{D}}$.] Consider $\mathrm{Lie}(P) \cap \mathrm{Lie}(G(\mathbb{R})) = F^0\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(K) \implies$ Let $x_0 = [1] \in \hat{\mathcal{D}}$ be the $G(\mathbb{R})^\bullet$-orbit of x_0 is $\mathcal{D} = G(\mathbb{R})^\bullet/K \implies \mathcal{D} \subset \hat{\mathcal{D}}$.

Note that

$$\begin{aligned}\dim_{\mathbb{R}}(\mathcal{D}) &= \dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}} - \dim_{\mathbb{R}}(\mathfrak{g}^{0,0} \cap \mathfrak{g}_{\mathbb{R}}) \\ &= \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - \dim \mathbb{C}\mathfrak{g}^{0,0} \\ &= 2 \cdot \dim_{\mathbb{C}} \left(\bigoplus_{r<0} \mathfrak{g}^{r,-r} \right) \\ &= 2 \cdot \dim_{\mathbb{C}} \hat{\mathcal{D}} \\ &= \dim_{\mathbb{R}} \hat{\mathcal{D}}.\end{aligned}$$

Thus we have an open embedding $\mathcal{D} \subset \hat{\mathcal{D}}$.

Over $\tilde{B}$ we have a flat bundle $V \otimes \mathcal{O}_{\tilde{B}}$ and a filtration $F^p\mathcal{V} \subset \mathcal{V}$ by holomorphic subbundles. Thus we get a period map $p : \tilde{B} \to \mathcal{D} \subset \hat{\mathcal{D}}$.

Griffiths transversality is [means] the following condition

$$dp : T\tilde{B} \to p^*\mathcal{H}$$

where $\mathcal{H}$ is given as follows. $F^{-1}\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}}$ defines a homogeneous subbundle of $T\hat{\mathcal{D}}$ which we call the *horizontal subbundle* $\mathcal{H}$.

If

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1} \tag{22.3.1}$$

then $F^{-1}\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$ which implies that $\mathcal{H} = T\hat{\mathcal{D}}$. Thus Griffiths transversality holds automatically for any holomorphic map $\tilde{B} \to \mathcal{D}$.

Proposition 22.4. *If Equation 22.3.1 holds, then $\mathcal{D}$ is a Hermitian symmetric space.*

Proof. The Ad_C [Weil operator] acts on $\mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}$ as $-\mathrm{Id}$. Thus Ad_C gives a holomorphic isometry fixing $x_0 \in \mathcal{D}$ and acting as $-\mathrm{Id}$ on $T_{x_0}\mathcal{D}$. □

23. The Kuga-Satake construction

Let X be a compact hyperkähler manifold (aka IHS manifold). Recall that this means that X is a compact smooth manifold, simply connected equipped with a Riemannian metric g such that $\mathrm{Hol}(\nabla^g) = \mathrm{Sp}(n)$ where ∇^g is the Levi-Civita connection and $\mathrm{Sp}(n)$ is the compact form of $\mathrm{Sp}(2n, \mathbb{C})$, i.e. $\mathrm{Aut}(\mathbb{H}^n, \langle \cdot, \cdot \rangle)$, the quaternions equipped with the canonical quaternionic Hermitian product $\langle v, w \rangle := \sum_{i=1}^n v_i \overline{w_i}$.

The holonomy principle implies that there exist complex structures I, J and K such that $IJ = JI = K$.

Let X_I be the complex manifold X equipped with I. Again by the holonomy princple, there is a holomorphically symplectic form $\sigma_I \in H^0(X, \Omega^2_{X_I})$. In fact, this is unique up to scaling and $H^0(X, \Omega^2_{X_I}) \cong \mathbb{C}\sigma_I$. Also, σ_I^n is a holomorphic volume form, so that $K_{X_I} \cong \mathcal{O}_{X_I}$, so that these are Calabi-Yau manifolds. $\dim_{\mathbb{C}} X_I = 2n$.

Example 23.1 (IHS manifolds)**.**
1. S complex K3 surfaces. E.g. quartics in $\mathbb{C}P^3$ [by CY theorem there is a hyperkähler metric; it's difficult to construct such a metric explicitly].
2. Let S be a complex K3 surface. Then $\mathrm{Hilb}^n(S)$, the Hilbert scheme of length-n 0-dimensional subschemes of S.

(3) Let $A = \mathbb{C}^2/\Lambda$. Consider the Albanese map $\mathrm{Hilb}^{n+1}(A) \to A$. Then the fiber is called a *Generalized Kummer variety* $K^n A := Alb^{-1}(0)$.
(4) O'Grady: exceptional examples of dimension 6, 10.

24. Cohomology of an IHS manifold

$V := H^2(X, \mathbb{Q})$ is a K3-type Hodge structure. Since $H^0(X, \Omega^2_{X_T}) = \mathbb{C}\sigma$ where σ is the holomorphic symplectic form, we must have $h^{2,0} = h^{0,2} = 1$.

Theorem 24.1 (Beauville-Bogomolov-Fujiki)**.** *There exists a constant $c_X \in \mathbb{Q}$ and a quadratic form $q \in S^2V^*$ of signature $(3, b_2(X) - 3)$ such that $\forall a \in V$*

$$\int_X a^{2n} = c_X q(a)^n \qquad \textit{(Fujiki relation)}$$

where $\int_X$ is the pairing with the fundamental class defined by I.

[We have the integral of a polynomial on V, and it happens to be the n-th power of a quadratic form q. In fact, the integral does not depend on the choice of complex structure.]

We call q the *BBF form*. The HS on H^2 is given by

$$\rho : U(1) \to \mathrm{SO}(V \otimes \mathbb{R}, q).$$

If ω is a Kähler form on X, then $q([\omega]) > 0$. If the class of ω is rational, i.e. $[\omega] \in H^2(X, \mathbb{Q})$, then $H^2(X, \mathbb{Q})_{\mathrm{prim}} = [\omega]^\perp$ is polarized by q.

Since ω is Kähler, we have the Lefschetz triple

$$L_\omega = [\omega] \cup -, \qquad \Lambda_\omega = \text{dual of } L_\omega, \qquad [L_\omega, \Lambda_\omega] = \Theta,$$

[which give us a representation of $\mathfrak{sl}_2$].

Consider L_ω, Λ_ω for all Kähler forms on all deformations of (X, I). The smallest Lie subalgebra of $\mathrm{End}(H^\bullet(X, \mathbb{R}))$ that contains all L_ω, Λ_ω?

25. Mukai extension

Let (V, q), $\tilde{V} = V \oplus U$, where U is the hyperbolic plane $q_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $\tilde{q} = q \oplus q_n$. We have a grading

$$\tilde{V} = \underbrace{\mathbb{Q}e_0}_{0} \oplus \underbrace{V}_{2} \oplus \underbrace{\mathbb{Q}e_4}_{4}.$$

Then

$$\begin{aligned} \mathfrak{so}(\tilde{V}, \tilde{q}) &\simeq \Lambda^2 \tilde{V} \\ &= \underbrace{e_0 \wedge V}_{\substack{-2 \\ \Lambda_\omega \in}} \oplus \underbrace{(\underbrace{\Lambda^2 V}_{\mathfrak{so}(V,q)} \oplus e_0 \wedge e_n)}_{0} \oplus \underbrace{e_n \wedge V}_{\substack{2 \\ L_\omega \in}} \end{aligned}$$

is in fact the minimal Lie subalgebra containing the Lefschetz operators:

Theorem 25.1 (Verbitsky, Looijenga-Lunts)**.** *The smallest subalegbra of End$(H^\bullet(X, \mathbb{R}))$ containing L_ω, Λ_ω is isomorphic to $\mathfrak{so}(\tilde{V}, \tilde{q})$.*

This implies that we have an action of $\mathfrak{so}(V,q)$ on $H^k(X,\mathbb{Q})$ for all k. Thus, $H^k(X,\mathbb{Q})$ is a representation of $\mathrm{Spin}(X,q)$ for $k \equiv 0(\mathrm{mod})$ of $\mathrm{SO}(V,q)$.

The HS of $H^k(X,\mathbb{Q})$ are induced by $\rho : U(1) \to \mathrm{Spin}(V \otimes \mathbb{R}, q)$.

[The idea is to understand the cohomology as a representation of $\mathfrak{so}(\tilde{V},\tilde{q})$.]

Let $(\mathbb{Q}-HS)^{\mathrm{tori}}$ be the full $\otimes$-subcategory of $(\mathbb{Q}-HS)$ generated by $H^1(T,\mathbb{Q})$ for tori $T = \mathbb{C}/\Lambda$. We have

$$(\mathbb{Q}-HS)^{\mathrm{tori}} \supset (\mathbb{Q}-HS)^{ab} \subset (\mathbb{Q}-HS)^{pol}.$$

Theorem 25.2 (Kurnosov-Soldatenkov-Verbitsky)**.** *Let X be an ISH manifold. Then for all k, $H^k(X,\mathbb{Q}) \in (Q-HS)^{tori}$. If X is projective then*

$$H^k(X,\mathbb{Q}) \in (\mathbb{Q}-HS)^{ab}.$$

26. Idea of the proof

[We need to understand cohomology groups as representations of $Spin(V,q)$.]

(1) (Clifford algebras.) Let (V,q) be a quadratic vector space over $\mathbb{Q}$. The Clifford algebra $Cl(V,q) = \bigoplus_{i\geq 0} V^{\otimes i}/J$ where J is the ideal generated by $v\otimes v - q(v)\cdot 1$ for all $v \in V$.

In $Cl(V,q)$, $v^2 = q(v)\cdot 1$, $uv - vu = 2q(u,v)\cdot 1$ Also,

$$\alpha(v_1 \otimes \ldots \otimes v_k) = (-1)^k v_1 \otimes \ldots \otimes v_k, \qquad \beta(v_1 \otimes \ldots \otimes v_k) = v_k \otimes v_{k-1} \otimes \ldots \otimes v_1$$

descend to $Cl(V,q)$.

Foro all $a \in cl(V,q)$ write $\bar{a} = \alpha\beta(a)$, $Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q)$ is a $\mathbb{Z}/2$-grading.

Denote $Cl(V,q) = \mathcal{C}$ and $\mathcal{C}^\times$ the invertible elements of $\mathcal{C}$. For all $x \in V$ such that $q(x) \neq 0$, $x \in \mathcal{C}^\times$ since $x^2 = q(x)\cdot 1 \implies x^{-1} = \frac{x}{q(x)}$.

We have a natural embedding $V \hookrightarrow \mathcal{C}$. Let $\{x \in \mathcal{C}^\times : \alpha(x)Vx^{-1} \subset V\} := G$.

If $x \in V$, $q(x) \neq 0$, $y \in V$,

$$\begin{aligned}\alpha(x)yx^{-1} &= -xy\frac{x}{q(x)} \\ &= (yx - 2q(x,y)\frac{x}{q(x)} \\ &= y - 2\frac{q(x,y)}{q(x)}x,\end{aligned}$$

i.e. a reflection.

$x \in G$, $y \in V$,

$$\begin{aligned}q(\alpha(x)yx^{-1}) &= \underbrace{\alpha(x)yx^{-1}}_{V}\alpha(x)yx^{-1} \\ &= -\alpha(\alpha(x)yx^{-1})\alpha(x)yx^{-1} \\ &= xy\alpha(x)^{-1}\alpha(x)yx^{-1} \\ &= q(y).\end{aligned}$$

We get a representation $\mu : G \to O(V,q)$. μ is surjective (because $\mathrm{Im}(\mu)$ contains reflections [and there's a theorem that $O(V,q)$ is generated by reflections]).

$\mathrm{Ker}(\mu) = \mathbb{Q}^* \cdot 1$. [The kernel is quite big so let's make it a bit smaller.] Define a norm $N : G \to \mathbb{Q}^*$ by $N(a) = a \cdot \bar{a}$. Define $\mathrm{Ker}(N) := Pin(V, q)$. Then $Pin(V, q) \cap \mathcal{C}^0 = \mathrm{Spin}(V, q)$. Then $\mu : \mathrm{Spin}(V, q) \to \mathrm{SO}(V, q)$. Thus, $\mathrm{Ker}\,\mu = \{\pm 1\}$, [the double cover we were looking for].

(2) (Key observation.) The morphism $\rho : U(1) \to \mathrm{SO}(V \otimes \mathbb{R}, q)$ defining the HS on $H^2(X, \mathbb{Q})$ lifts to $\mathrm{Spin}(V \otimes \mathbb{R}, q)$.

To see why, write

$$V \otimes \mathbb{R} = \underbrace{(V^{2,0} \oplus V^{0,2})}_{V_{\mathbb{R}}^{+}} \oplus V_{\mathbb{R}}^{1,1}.$$

Then $\dim V_{\mathbb{R}}^{+} = 2$.

We have a double cover

$$\begin{tikzcd} \rho : U(1) \arrow[r] \arrow[dr, "2:1"'] & \mathrm{SO}(V_{\mathbb{R}}^{+}) \arrow[d, equal] \\ & U(1). \end{tikzcd}$$

We have

$$Cl(V \otimes \mathbb{R}, q) = \underbrace{Cl(V_{\mathbb{R}}^{+})}_{4-\dim} \otimes Cl(V_{\mathbb{R}}^{1,1}).$$

Then $V_{\mathbb{R}}^{+} = \langle e, f \rangle$ — an orthonormal basis, $Cl^0(V_{\mathbb{R}}^{+}) = \langle 1, ef \rangle \simeq \mathbb{R}^2$.

$$\begin{tikzcd} \mathrm{Spin}(V_{\mathbb{R}}^{+}) \cong U(1) \arrow[d, "2:1"] \\ \mathrm{SO}(V_{\mathbb{R}}^{+}). \end{tikzcd}$$

(3) (Definition of the Hodge structure.) [What's the use of lifting?] We have a lift $\tilde{\rho} : U(1) \to \mathrm{Spin}(V \otimes \mathbb{R}, q)$. Note that $Cl(V_{\mathbb{R}}^{+})$ is two copies of the standard representation of $U(1)$.

$$Cl(V \otimes \mathbb{R}, q) \simeq \bigoplus \text{standard reps of } U(1) \simeq \mathrm{Spin}(V_{\mathbb{R}}^{+}).$$

Denote $H := Cl(V, q)$, $\implies H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$. Then $H \in (\mathbb{Q} - HS)^{\mathrm{tori}}$.

(4) (Constructing the embedding.) We want to construct $V \hookrightarrow H^*$. [We should use some bilinear form on the Clifford algebra.] In fact, there exists a non-degenerate pairing $\tau : \mathcal{C} \times \mathcal{C} \to \mathbb{Q}$, $\tau(x, y) = \mathrm{Tr}(x\bar{y})$, where we consider $x\bar{y}$ as an endomorphism of $\mathcal{C}$ acting by left multiplication and Tr is the trace of this endomorphism. In fact, $\tau(x, y) = \tau(y, x)$, $\tau(ax, y) = \tau(x, \bar{a}y)$ and τ is Pin-invariant, i.e. $\forall \in Pin(V, q)$, $\tau(ax, ay) = \tau(x, y)$.

Analogously, on $H = \mathcal{C}^{\oplus m}$ we have $\tau : H \otimes H \to \mathbb{Q}$.

To construct the embedding, for any $x \in V$ let $\omega_x(u, v) = \tau(xu, v)$. Then $\omega_x(u, v) = -\omega_x(v, u)$, $\omega_x \in \Lambda^2 H^*$. This defines $\varphi : V \to \Lambda^2 H^*$, $x \mapsto \omega_x$, the embedding we wanted.

φ is a morphism of $\mathrm{Spin}(V, q)$-modules [representations]. Thus, φ is a morphism of Hodge structures.

[This finishes the work for K3 surfaces. For general HK manifolds we need to work a bit more, how to embed all cohomology groups? In fact, this is where we will use the m in $H = \mathcal{C}^{\oplus m}$.]

(5) (Extra work needed for general HK manifolds.) Let $\dim H = 4N$ and $\mathrm{Im}(\varphi) = W$. For $p \in S^{2N}H^*$, $p(\omega) = \omega^{2N} \in \Lambda^{2N}(H^*) \simeq \mathbb{Q}$.

[p is a polynomial, when we restrict it:] $p|_W \in (S^{2N}W^*)^{\mathrm{Spin}(V,q)}$. The classical invariant theory implies [that this polynomial is a power of the quadratic form, i.e.] $p = A \cdot q^N, A \in \mathbb{Q}$. So $p(\omega) = \omega^{2N} = Aq(\omega)^N$, a "Fujiki relation".

$$\mathfrak{so}(\tilde{V}, \tilde{q}) = \underbrace{V}_{\substack{2 \\ \Lambda_x \in}} \oplus \underbrace{\mathfrak{so}(\tilde{V}, \tilde{q})}_{0} \oplus \underbrace{V}_{\substack{2 \\ L_x \in}}$$

Let $\eta \in \Lambda^\bullet H^*$, $x \in V$,

$$L_x\eta = \omega_x \wedge \eta, \qquad \Lambda_x\eta = i_\eta\omega_x \tag{26.0.1}$$

[Where i denotes interior multiplication].

$$\begin{aligned} \varphi : V &\hookrightarrow \Lambda^2 H^* \\ x &\longmapsto \omega_x. \end{aligned}$$

Proposition 26.1. *Equation 26.0.1 define a $\mathfrak{so}(\tilde{V}, \tilde{q})$-representation on $\Lambda^\bullet H^*$. The corresponding Spin$(\tilde{V}, \tilde{q})$-representation is faithful.*

In conclusion, for all $\mathfrak{so}(\tilde{V}, \tilde{q})$ representation U exists $m > 0$ such that for $H = \mathcal{C}^{\oplus m}$, $U \subset \Lambda^\bullet H^*$ as a $\mathfrak{so}(\tilde{V}, \tilde{q})$ subrepresentation. U is contained in a direct sum of $(\Lambda^\bullet \mathcal{C}^*)^{\otimes i} \simeq \Lambda^\bullet(\mathcal{C}^{*\oplus i})$.

Take $U = H^2(X, \mathbb{Q})$, qed an embedding of $\mathrm{Spin}(\tilde{V}, \tilde{q})$ representations, $H^\bullet(X, \mathbb{Q}) \hookrightarrow \Lambda^\bullet H^*$ is an embedding of HS.

[Thus the HS on the cohomology of X are abelian.]

27. References

(1) Deligne, Milne, Ogus, Shih, "Hodge cycles, motives and Shimura varieties". We have seen mostly Chapter 1 and a bit of Chapter 2.
(2) Deligne:
- "La conj. de Weil pour les surfaces K3"
- "Théorie de Hodge II"
- "Variétes de Shimura: ..."

(3) Kingler: "Hodge theory, between algebraicity and transcendence + reference therein".