REPRESENTATION THEORY

github.com/danimalabares/stack

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1. Basic definitions

Definition 1.1. A *representation of Lie algebra* $\mathfrak{g}$ is a vector space V and homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

of Lie algebras, i.e.,

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

Remark 1.2. Another name for a representation of a Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-*module*. This is like the abstract definition of an algebra using a morphism. It's as if the morphism lets us define a sort of product of elements in the codomain by elements in the domain.

It is not possible to classify all representations of a Lie algebra $\mathfrak{g}$. But there is a theorem by Weyl that says that if $\mathfrak{g}$ is finite-dimensional and (semi)simple, then every finite-dimensional representation of $\mathfrak{g}$ (i.e. a representation where V is finite-dimensional) is isomorphic to a direct sum of irreducible representations

2. Representations of SL(2,C)

Klein classified nontrivial finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ up to conjugacy:

- Cyclic groups $\mathcal{C}_m$, $m \geq 2$. $|\mathcal{C}_m| = m$,

$$\mathcal{C}_m = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^m = 1 \right\}.$$

- Dihedral groups $\mathcal{D}_m$, $m \geq 2$,

$$\mathcal{D}_m = \left\langle \mathcal{C}_{2m}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

For example, $\mathcal{D}_2$ is the *quaternion group* $\left\langle \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{i}, \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{j} \right\rangle$.

- Binary tetrahedral group $\mathcal{T}$, with $|\mathcal{T}| = 24$.
- Binary octahedral group, $\mathcal{O}$, with $|\mathcal{O}| = 48$.

- Binary icosahedral group $\mathcal{I}$ with $|I| = 120$.

  These three are related to the rotation groups of the Platonic solids by the 2-to-1 covering $\mathrm{SU}_2 \to \mathrm{SO}_3(\mathbb{R})$. To classify the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$: first show that any such subgroup is conjugate to a subgroup of SU_2 so that its image is a finite subgroup of $\mathrm{SO}_3(\mathbb{R})$. We can classify those by considering their action on the sphere and showing that they have to be the rotation group of one of the Platonic solids or a dihedral group or a cyclic group (related to planar polygons).

At some point of the XX century it was realised that this classification is an instance of an ADE classification.

Now we explain the (historically second explanation of this) McKay correspondence. If Γ is a nontrivial finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, define the *McKay graph* $\tilde{\Delta}_\Gamma$ as follows: the vertices are $0, 1, 2, \ldots, \ell$ which index the irreducible representations $R_0, R_1, \ldots, R_\ell$ of Γ.

Let Q be the 2-dimensional Γ coming from $\Gamma < \mathrm{SL}_2(\mathbb{C})$. (Note that Q is one of the R_i's unless $\Gamma = \mathcal{C}_m$, in which case Q is reducible.) The number of edges between i and j in $\tilde{\Delta}_\Gamma$ is the multiplicity of R_j in $Q \otimes R_i$. So: take the representation R_i, tensor it with the 2-dimensional representation Q, express it as a direct sum of irreducible representations and perhaps R_j will appear with nonzero multiplicity, in which case you put R_j with that number of edges. So in general it's a multigraph, though in most cases its a simple graph. Also note that this multiplicity coincides with the multiplicity of R_i in $Q \otimes R_j$ since $Q^* \cong Q$; that is the graph is not directed.

The surprising thing is that when we do this we obtain the affine Dynkin diagrams. McKay's famous observation says:

(1) $\tilde{\Delta}_\Gamma$ is a simply-laced affine Dynkin diagram.
(2) This is a bijection:

$$\left\{ \substack{\text{nontrvial finite} \\ \Gamma \subset \mathrm{SL}_2(\mathbb{C})} \right\} \Big/ \text{conj.} \quad \leftrightsquigarrow \quad \left\{ \substack{\text{simply-laced} \\ \text{affine Dynkin diagrams}} \right\}.$$

The correspondance reads

$$\begin{aligned} \mathcal{C}_m &\longleftrightarrow \widetilde{A}_{m-1} \\ \mathcal{D}_m &\longleftrightarrow \widetilde{D}_{m+2} \\ \mathcal{T} &\longleftrightarrow \tilde{E}_6 \\ \mathcal{O} &\longleftrightarrow \tilde{E}_7 \\ \mathcal{I} &\longleftrightarrow \tilde{E}_8. \end{aligned}$$

Now we explain the geometric approach. Given a finite $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$, consider the quotient $Y_\Gamma = \Gamma \backslash Q$ as an algebraic variety, that is, (since Q is just a representation of Γ, i.e. a 2-dimensional affine space with an action of Γ) we can take its quotient in the sense described before: Y_Γ is the affine variety with coordinate algebra $\mathbb{C}[x,y]^\Gamma$.

Lemma 2.1. *The action of* Γ *is free on* $Q \setminus \{0\}$ *is free.*

3. Borel-Weil-Bott theorem

[These are *very* sketchy notes after a conversation with Lyalya Guseva on Borel-Weil-Bott theorem]

Borel-Weil-Bott theorem is a device for computing cohomologies of sheaves.

Take the Grassmanian $\mathrm{Gr}(n,k)$, which may be obtained as a quotient G of a group G by a parabolic subgroup P. We would like it if P was a semisimple group since those are classified, but unfortunately it is not. So instead we use so-called Levi quotient denoted by L which is semisimple and allows us to understand the cohomologies of the variety G/P.

Fortunately the representation theory of $\mathrm{Gr}(k,n)$ is well known, in fact we get $\mathrm{GL}_k \times \mathrm{GL}_{n-k}$.

There are functors associated to L, which are described by a sequence of numbers $a_1, \ldots, a_n$. Along with other sequences of numbers $k_1, \ldots, k_\ell$, and ρ (the latter is a concept in representation theory but we may ultimately think of it as another sequence of numbers) we may construct an action of the symmetric group S^n acting on these sequences and obtain a result concerning the cohomology $H^{\ell(\sigma)}(L)$, and in particular we find that if two entries in our list of numbers coincide, the cohomology will vanish.

References