COMPLEX ANALYSIS

github.com/danimalabares/stack

Contents

1. Holomorphic functions

Definition 1.1. A function $f : W \subset \mathbb{C} \to \mathbb{C}$ is *holomorphic* at $z_0 \in U$ if

$$\lim_{z\to z_0} \frac{f(z_0 - z) - f(z_0)}{z - z_0}$$

exists. This is equivalent to

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

where h is a complex parameter.

2. Cauchy-Riemann equations

Theorem 2.1. *Let $f : W \subset \mathbb{C} \to \mathbb{C}$ be a function. Let $z = x + iy$ coordinates in $\mathbb{C}$. Define $f := u + iv$ and $z_0 = x_0 + iy_0 \in U$. f is holomorphic at z_0 if and only if*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

3. Integration

The upshot about complex integration (and complex analysis, really) is that differential forms fdz with f holomorphic are exact because that's equivalent to Cauchy-Riemann equations.

The complex integral may first be defined for a complex-valued function f defined on a real integral as

$$\int_a^b f = \int_a^b u + i \int_a^b v$$

And then for curves as

$$\int_\gamma f := \int_a^b f(\gamma(t))\gamma'(t)dt$$

A natural question is: what's the essential difference between complex and real analysis? What is the power behind Cauchy-Riemann equations? The answer is in the proof of Cauchy integral theorem 3.2: Stokes theorem, which in dimension 2 is called Green's theorem, gives an integrand with Cauchy-Riemann equations.

Recall

Theorem 3.1 (Stokes). *The line integral $\int_\gamma pdx + qdy$ defined in Ω depends on the end points of γ if and only if there exists a function $U(x, y)$ in Ω with the partial derivatives $\partial U/\partial x = p$ and $\partial U/\partial y = q$, that is, if $pdx + qdy$ is exact.*

Proof. Fundamental theorem of calculus. □

So, when computing a complex integral $\int_\gamma f$, we are interested in knowing when is the form fdz exact. Which says that there is a function F such that $\partial_x F = f$ and $\partial_y F = if$. In fact, this is the same as asking that F satisfies the Cauchy-Riemann equations (provided f is continuous).

Thus, the integral $\int_\gamma f$ for continuous f depends only on the end points of γ if and only if f is the derivative of an analytic function in Ω.

The theory of complex integration has to do with homology. The following result does not hold for domains that are not simply connected:

Theorem 3.2 (Cauchy). *If $f(z)$ is analytic in an open disk Δ, then*

$$\int_\gamma f(z)dz = 0 \tag{3.2.1}$$

for every closed curve γ in Δ.

Proof. Recall that Green's theorem says that

$$\int_{\partial\Delta} (vdx + udy) = \int_\Delta d(vdx + udy) = \int_\Delta (\partial_x u - \partial_y v)\mathrm{Vol}$$

There's Cauchy-Riemann equations! □

Alternative idea. I prove this using Stokes theorem. The fact that f is holomorphic says that $\overline{\partial} f = 0$. In turn, $d = \partial + \overline{\partial}$, so that $df = \partial f$, which is a $(1, 0)$-form (just as $\overline{\partial}$ maps functions to $(0, 1)$-forms... (must justify...)). This makes $d(fdz) = df \wedge dz$ a $(2, 0)$-form, but there are no such forms on $\mathbb{C}$ (because there are no holomorphic or antiholomorphic 2-forms on a complex dimension 1 space). □

The following limit condition allows for the former result even if the domain has some points removed:

Theorem 3.3. *Let f be analytic in the region Δ' by omitting a finite number of points ζ_j from an open disk Δ. If f satisfies that*

$$\lim_{z\to\zeta_j}(z-\zeta_j)f(z)=0 \tag{3.3.1}$$

for all j, then Eq. 3.2.1 holds for any closed curve γ in Δ'.

Sketch of proof. It suffices to prove that Eq. 3.3.1 implies that we can extend f to a holomorphic function on all of Δ and apply Stokes as in Theorem 3.2. But that won't work: for that we need Cauchy integral formula, and that's why I'm here: to prove Cauchy integral formula.

So I think it might be using logarithmic derivative. The limit above allows to write

$$|f(z)|\leq\frac{\varepsilon}{|z-\zeta_j|}$$

Then integrate. On the left we can bound the integral of f after applying integral triangle inequality, and on the right we have logarithmic derivative of $z-\zeta_j$. This will be a fixed number (the index, see Definition 4.2), so that we have effectively bounded the integral. □

4. Cauchy integral formula

The way to Cauchy's integral formula is basically: holomorphic functions satisfy Cauchy-Riemann equations, which give Cauchy's integral theorem (which is just a consequence Stokes) and then apply this to the function $\frac{f(z)-f(z_0)}{z-z_0}$.

The theorem says that we can compute the value of a holomorphic function at a point as an integral around the point.

To prove a general version we shall use the notion of index of a curve about a point, which tells us how many times the curve winds about the point.

Lemma 4.1. *If the piecewise differentiable closed curve γ does not pass through the point a, then the value of the integral*

$$\int_\gamma\frac{dz}{z-a}$$

is a multple of $2\pi i$.

Non-proof. We are tempted to simply write the integrand as the logarithmic derivative of the function $f(z)=z-a$. But this isn't quite right, we must be careful with the domain of the logarithm. But it is instructive to see the computation:

$$\int_\gamma\frac{dz}{z-a}=\int_\gamma \mathrm{dlog}(z-a)=\int_\gamma \mathrm{dlog}|z-a|+i\int_\gamma \mathrm{darg}(z-a)$$

If γ is closed then $\log|z-a|$ would return to its initial value and $\arg(z-a)$ increases or decreases by a multiple of 2π. The actual proof by Ahlfors is different.

Still, the proof for a circle about a point z_0 is also trivial: this is just the case of the integral $\int_{\partial\Delta}\frac{dz}{z-z_0}$, but the curve here is $\gamma(t)=e^{it}+z_0$ so that when substituting in the parametrization we obtain $\int_0^{2\pi}\frac{ie^{it}}{(e^{it}+z_0)-z_0}=2\pi i$. □

Definition 4.2. The *index* of the point a with respect to the closed curve γ is

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \tag{4.2.1}$$

Theorem 4.3 (Cauchy Integral Formula)**.** *Suppose that $f(z)$ is analytic in an open disk Δ, and let γ be a closed curve in Δ. For any point not on γ,*

$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-a} \tag{4.3.1}$$

where $n(\gamma, a)$ is the index of a with respect to γ.

Proof. This is a simple application of Theorem 3.3 for the function

$$F(z) := \frac{f(z) - f(a)}{z-a}$$

Notice the limit condition holds, so that the integral vanishes! □

Notice that we can differentiate Cauchy integral formula to obtain

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \tag{4.3.2}$$

and more generally

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

Though we can just say that integrals can be differentiating by splitting into real and imaginary part, and using the real derivative, a technical lemma is used by Ahlfors to confirm that indeed we can differentiate under the integral sign.

Lemma 4.4. *[missing]*

The importance of this result is that it shows that a holomorphic function has derivatives of all degrees.

Then we state two interesting results:

Theorem 4.5 (Morera)**.** *If $f(z)$ is defined and continuous in a region Ω, and if $\int_\gamma f dz = 0$ for all closed curves γ in Ω, then $f(z)$ is analytic in Ω.*

This is what Lee calls a *conservative covector field*, i.e. a form whose integral on closed curves vanishes. The proof of Morera's theorem then reduces to the fact that a covector field is conservative if and only if it is exact [Lee12, Theorem 11.42]. The reverse implication is the fundamental theorem of calculus. The forward implication is not very straightforward in [Lee12].

As a final remark, I add that ([Lee12, Propoistion 11.40]) a smooth covector field is conservative if and only if its line integrals are path-independent, in the sense that integrals coincide along two *piecewise* smooth curve segments with the same starting and ending points. That is, form is exact implies integral independent of homotopy class?

We finish with

Theorem 4.6 (Liouville)**.** *A bounded holomorphic function defined on all of $\mathbb{C}$ must be constant.*

Proof. Let γ be a circle of radius r about z, then by the first derivative of Cauchy formula (Eq. 4.3.2),

$$|f'(z)| \leq \frac{M}{2\pi i} \int_\gamma \frac{d\zeta}{|\zeta - z|^2}$$

and compute the integral, of course is $\frac{2\pi i}{r}$. □

A fun application of this is proving the fundamental theorem of algebra. If a complex polynomial had no zeroes, $1/P(z)$ would be analytic in all of $\mathbb{C}$, and since $P(z)$ tends to ∞ as z tends to ∞, $1/P(z)$ is bounded (use Riemann sphere argument).

5. Zeroes of a holomorphic function

Perhaps the best way to remember this section is by the formula

$$f(z) = f_n(z)(z - z_0)^n \tag{5.0.1}$$

which holds for a holomorphic function which vanishes on z_0 and whose derivatives $f^{(\nu)}(z_0)$ vanish for $\nu < n$ (see [Ahl79, p. 126]). Here f_n is a holomorphic function which does not vanish at z_0 (according to [Lee24], but why?). This fact shows that the zero z_0 is isolated, since $f_n(z)$ is continuous and nonzero in z_0, and $(z - z_0)^n$ is also nonzero in a pointed neighbourhood of z_0.

All this is basically due to Cauchy integral formula, as the following theorem shows.

Theorem 5.1 (Removable singularity theorem)**.** *Let $f(z)$ be analytic in a region Ω' obtained by omitting a point z_0 from a region Ω. There exists an analytic function defined on all Ω that coincides with f on Ω' if and only if* $\lim_{z\to z_0}(z - z_0)f(z) = 0$. *Such a function is unique.*

Proof. (What is wrong in this proof? I need to use the limit condition somewhere... see wiki and wiki) The proof is just looking at Cauchy integral formula and noticing that the function defined at the singularity via the integral around it is holomorphic since it is given by an integral. Integrals are holomorphic since partial derivatives, which are limits, go through the integral sign, meaning we can differentiate infinitely many times and obtain a Taylor series expansion. (or perhaps more generally by Technical Lemma 4.4). □

Now consider the function

$$F(z) = \frac{f(z) - f(z_0)}{z - z_0}, \tag{5.1.1}$$

which has $\lim_{z\to z_0}(z - z_0)F(z) = 0$. Then we apply the theorem and find that the holomorphic function extending F, say f_1, is actually f' at z_0 by definition. Substituting F by f_1 in Equation 5.1.1 we obtain $f(z) = f(z_0) + f_1(z)(z - z_0)$. Applying the same argument to f_1 yields $f_1(z) = f_1(z_0) + f_2(z)(z - z_0)$ for some function f_2 that is f'' at z_0.

Repeating this process yields:

Lemma 5.2 (Finite Taylor expansion)**.** *If f is analytic in a connected open set $W \subset \mathbb{C}$ and $a \in W$, then we can write*

(5.2.1)

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n$$

for some function f_n analytic on W.

Theorem 5.3. *If $f : W \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic function defined on an open set W and $f(a) = 0$ for some $a \in W$, and f is not identically zero, then there is a disk $D_r(a) \subseteq W$ such that $f(z) \neq 0$ for $z \in D_r(a) \setminus \{a\}$ and a positive integer m called the order or multiplicity of the zero a such that $f(z) = (z-a)^m h(z)$ for some holomorphic function h that does not vanish at a. The order of a zero is equal to the smallest integer m such that $f^{(m)}(a) \neq 0$.*

Proof. If f is not identically zero, there must exist a first derivative $f^{(h)}(a)$ that is not zero, since otherwise f would vanish identically in W. This follows from ...

Then by Lemma 5.2 we obtain that $f(z) = f_h(z-a)^n$. □

6. Taylor expansion

Actually this part is quite later in Ahlfors book because it requires the theory of convergence of series. To pass from the finite Taylor expansion to the infinite version we notice that the term f_{n+1} in Equation 5.2.1 is of the form

$$f_{n+1}(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)}$$

where γ is a curve of radius ρ. Then "we obtain at once"

$$|f_{n+1}(z)(z-z_0)^{n+1}| \leq \frac{M|z-z_0|^{n+1}}{\rho^n(\rho - |z-z_0|}.$$

Then we must observe that the bound converges uniformly to zero in every disk of radius smaller than ρ, and by Weierstrass theorem there should be a limit function identical to zero.

7. Argument Principle

A simple version of the Argument Principle does not need Residue theorem:

Exercise 7.1. Let f be a holomorphic function on a disk, non-zero on $\partial\Delta$, and let $S_k(f) := \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f'}{f} z^k dz$. Prove the $S_k(f) = \sum d_i \alpha_i^k$ where α_i are all zeroes of f and d_i their multiplicities.

Proof. We use Lemma 5.2 to write $f(z) = (z - z_1)h_1(z)$ where z_1 is a zero of f. Applying that to h_1 for another zero z_2 of f, and continuing in this way, we obtain

$$f(z) = (z-z_1)(z-z_2)\dots(z-z_n)h(z). \tag{7.1.1}$$

Computing the quotient we obtain

$$\frac{f'(z)}{f(z)} = \frac{\text{derivative of } (z-z_1)\dots(z-z_n)}{(z-z_1)\dots(z-z_n)} + \frac{h'(z)}{h(z)}$$

The right hand term will vanish upon integration since it is a holomorphic function (with no poles because $h(z) \neq 0$). The left-hand term will give a sum of $\frac{1}{z-z_i}$ upon differentiation. As it is, the result of integration is the sum of the orders of each zero i.e. the number of zeroes counted without multiplicity (because the integrals are all 1, and there's one for each zero without multiplicity).

Multiplying by z^k yields the desired result by Cauchy formula 4.3.1. More exactly, multiplying z^k in the formula gives instead of a sum of integrals $\int \frac{1}{z-z_i}$, a sum of integrals $\int \frac{z^k}{z-z_i}$ which by Cauchy formula give z_i^k (disregarding the factor $2\pi\sqrt{-1}$). ☐

Theorem 7.2 (Argument Principle)**.** *If f is meromorphic in Ω with zeros a_j and poles b_k, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta = \sum_i n(\gamma, a_i) - \sum_k n(\gamma, b_k) \tag{7.2.1}$$

Lemma 7.3 (Rouché theorem)**.** *Let γ be homologous to zero in Ω and such that $n(\gamma, z)$ is either 0 or 1 for any point z not on γ. Suppose that f and g are analytic in Ω and satisfy that $|f-g| < |f|$ on γ. Then f and g have the same number of zeros enclosed by γ.*

Compare with Misha's version

Theorem 7.4 (Rouché theorem)**.** *Let f_t be a family of holomorphic functions on a disk Δ, continuously depending on a parameter $t \in \mathbb{R}$ and non-zero everywhere on its boundary $\partial\Delta$. Prove that the number of zeros of f_t in Δ is constant.*

Proof. Consider the map $t \mapsto f_t \mapsto S(f_t)$, which is continuous by hypothesis and because S, being an integral, is continuous. Then we obtain a continuous map $\mathbb{R} \to \mathbb{R}$ with integer values, meaning it must be constant. ☐

A similar argument may be used to prove that a holomorphic function F defined on $\Delta \times \Delta$ gives a holomorphic map

$$y_0 \mapsto \int_{\partial\Delta} \frac{F'(x, y_0)}{F(x, y_0)}\phi(x)dx = \sum_i d_i\phi(\alpha_i) \tag{7.4.1}$$

for any holomorphic function $\phi : \Delta \to \mathbb{C}$.

8. Holomorphic functions in several variables

Lemma 8.1. [Lee24]*, Theorem 1.21. Let $U \subseteq \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$. The following are equivalent:*

(1) *f is holomorphic (i.e. it is continuous and has a complex partial derivative with respect to each variable at each point of U)*
(2) *f is smooth and satisfies the following Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x^j} = \frac{\partial v}{\partial y^j}, \qquad \frac{\partial u}{\partial y^j} = -\frac{\partial v}{\partial x^j} \tag{8.1.1}$$

where $z^j = x^j + \sqrt{-1}y^j$ and $f(s) = u(z) + \sqrt{-1}v(x)$.
(3) *For each $p = (p^1, \dots, p^n) \in U$ there exists a neighbourhood of p in U on which f is equal to the sum of an absolutely convergent power series of the form*

$$f(z) = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n}(z^1 - p^1)\dots(z^n - p^n) \tag{8.1.2}$$

Proof. I will prove that if f is holomorphic then it has a Taylor series for $n = 2$. First apply Cauchy integral formula on each variable to obtain

$$f(z^1, z^2) = \frac{1}{(2\pi\sqrt{-1})^2}\int_{\substack{|z^1-w^1|=r\\ |z^2-w^2|=r}} \frac{f(w^1, w^2)}{(w^1-z^1)(w^2-z^2)}dw^1dw^2$$

Now observe:

$$\frac{1}{w^1 - z^1} = \frac{1}{w^1 - p^1 + p^1 - z^1} = \frac{1}{w^1 - p^1} \frac{1}{1 - \frac{p^1 - z^1}{w^1 - p^1}} \tag{8.1.3}$$

And on the right-hand-side we have a geometric series so that we may write

$$\frac{1}{w^1 - z^1} = \frac{1}{w^1 - p^1} \sum_{k=0}^{\infty} \left(\frac{p^1 - z^1}{w^1 - p^1} \right)^k$$

finally substituting this into (8.1.3) we may take the products $(p^1 - z^1)^{k_1}(p^2 - z^2)^{k_2}$ out of the integral and define the remaining term as $a_{k_1 k_2}$. □

9. Taylor series in several variables

10. Identity theorem in several variables

Theorem 10.1 (Identity theorem)**.** *If two holomorphic functions $f, g : W \subset \mathbb{C}^n \to \mathbb{C}$ coincide in an open subset of the connected open set W, then they coincide in all of W.*

Proof. Let U be the set where the function $h := f - g$ and all its partial derivatives vanish. Then U is open since for every point in U there is a neighbourhood where h is expressed as a Taylor series, whose coefficients must be given by the partial derivatives of h. By definition of U, we see that h must be zero in such a neighbourhood. U is also closed by continuity of partial derivatives of all orders. □

11. Germs of holomorphic functions

Definition 11.1. Let $U, U' \subset \mathbb{C}^n$ be neighbourhoods of 0 and $f \in \mathcal{O}_U$, $f' \in \mathcal{O}_{U'}$ holomorphic functions. We say that f and f' have the same germ, $f \sim f'$, if $f|_{U \cap U'} = f'|_{U \cap U'}$. Clearly (?), $\sim$ gives an equivalence relation on the set of pairs $(U \ni 0, f \in \mathcal{O}_U)$. An equivalence class is called **germ of a holomorphic function**. The space of germs in 0 of holomorphic functions on $\mathbb{C}^n$ is denoted $\mathcal{O}_n$.

Exercise 11.2. Prova that the ring $\mathcal{O}_n$ of germs of holomorphic functions is not finitely generated over $\mathbb{C}$ for any $n > 0$.

Proof. I think the existence of e^z as a holomorphic function satisfying the differential equation $f' = f$ is enough to show that the coefficients of its Taylor polynomial are all nonzero. This argument works for several variables as well. □

Definition 11.3. A *formal power series* in the variables $t_1, \dots, t_n$ is a sum $\sum_{i=0}^{\infty} P_i(t_1, \dots, t_n)$ where P_i are homogeneous polynomials of degree i. Addition of power series is defined componentwise, and multiplication is defined via

$$\left(\sum_{i=0}^{\infty} P_i(t_1, \dots, t_n) \right) \left(\sum_{i=0}^{\infty} Q_i(t_1, \dots, t_n) \right) = \sum_{i=0}^{\infty} R_i(t_1, \dots, t_n)$$

where $R_d(t_1, \dots, t_n) = \sum_{i+j=d} P_i(t_1, \dots, t_n) Q_j(t_1, \dots, t_n)$.

We can think of germs of functions in $\mathcal{O}_n$ as elements in the ring of power series $\mathbb{C}[\![t_1, \dots, t_n]\!]$. I think there is no problem to prove this statement, nor the fact that power series is actually a ring with units the nonzero constants and zero the zero constant.

Exercise 11.4. Prove that $\mathcal{O}_n$ has no zero divisors.

Proof. Suppose that $PQ = 0$ but neither of P nor Q are zero. Then $P_i \neq 0$ for some i and $Q_j \neq 0$ for some j. Then we can write

$$P_iQ_j = -\sum_{\substack{p+q=i+j \\ p\neq i, q\neq j}} P_pQ_q$$

And then what. Other way is by induction. For $n = 0$ we are the complex numbers so no problem. Suppose that $\mathcal{O}_n$ has no zero divisors. Then it looks like we can deal with degrees smaller than $n+1$, but the d-th term is not a simple product of P_iQ_j with $i + j = d$, but a sum. So not sure too. □

Definition 11.5. A ring R is called *local* if it contains an ideal $I \subset R$ such that all elements $r \notin I$ are invertible.

It is an easy exercise to show that this definition is equivalent to having a unique maximal ideal.

Exercise 11.6. Prove that the ring $\mathbb{C}[\![t_1, \dots, t_n]\!]$ is not finitely generated over $\mathcal{O}_n \subset \mathbb{C}[\![t_1, \dots, t_n]\!]$.

Proof. I thought that $\mathcal{O}_n$ would be the same as $\mathbb{C}[\![t_1, .., t_n]\!]$... the natural map is surely injective but why not surjective? There are power series that are not holomorphic functions? Maybe because of radius of convergence? No, because germs of holomorphic functions can be defined very near the origin... Every power series has a nonzero radius of convergence, right? □

Now I will discuss zeroes of holomorphic functions of several variables.

Definition 11.7. Let $f \in \mathcal{O}_n$ be a germ of holomorphic function on $\mathbb{C}^n$. Write its Taylor series $f(z) = \sum_{i=0}^{\infty} P_i(t_1, \dots, t_n)$, where P_i are homogeneous polynomials of degree i. We say that f has a *zero of order (or multiplicity) k in* 0 if $P_0 = \dots = P_{k-1} = 0$. In this situation *principal part* of the function f is the homogeneous polynomial P_k.

The following exercise shows that a holomorphic change of coordinates will preserve the order of a zero, and the principal part of the new function will be determined by the differential of the change of coordinate map. We will need to change coordinates when we do Weierstrass Preparation theorem **??**.

Exercise 11.8. Let $\Phi(t_1, \dots, t_n) = (F_1(t_1, \dots, t_n), \dots, F_n(t_1, \dots, t_n))$ be the holomorphic coordinate change around zero (?), with $F_i(0, \dots, 0) = 0$ and $A := \left(\frac{\partial F_i}{\partial t_j}\right)_0$ its differential (I suppose $\det A \neq 0$. Prove that

(1) For any germ $f \in \mathcal{O}_n$ which has multiplicity k, the function $\Phi^*(f)$ has zero of the same multiplicity.
(2) The principal part of $\Phi^*(f)$ is obtained from the principal part of f by action of A.

Sketch of proof. (1) The condition that $\det A \neq 0$ implies that all F_i must have linear term—otherwise their partial derivatives would vanish when evaluated at zero. When substituting $f(z_1, \dots, z_n) = \sum_{|\alpha|\geq k} \alpha z^\alpha$ with Φ we find that there must be a term of order k.

(2) The case for $n = 1$ is clear. The general case follows, I think, from the observation that the derivative A recovers the linear terms, which, as shown in the previous item, correspond to the principal part of $\Phi^* f$.

□

Exercise 11.9. Let Q be a non-zero homogeneous polynomial on $t_0, \dots, t_n$, and $V(Q)$ its zero set, which we consider as a subset in $\mathbb{C}P^n$.

(1) Prove that $\mathbb{C}P^n \setminus V(Q)$ is non-empty.
(2) Prove that $V(Q) \subset \mathbb{C}P^n$ is a set of measure 0.

Proof. (1) This follows from Identity Theorem 10.1. Indeed, if $V(Q)$ was all of $\mathbb{C}^n$, it would vanish on an open set, implying that Q is identically zero.

(2) $V(Q)$ may be decomposed in the sets of regular and singular points. Regular points have submanifold charts, while singular points have measure zero by Sard's theorem.

□

Exercise 11.10. Let $f_1, f_2, \dots \in \mathcal{O}_n$ be a countable collection of germs, which vanish with multiplicity $k_1, k_2, \dots$. Prove that there exists a coordinate system $z_1, \dots, z_n$ such that $\lim_{z_n \to 0} \frac{f_i(0, z_n)}{z_n^{k_i}} \neq 0$ for all i.

Sketch of proof. I don't understand this exercise: evaluating any germ f_i on $(0, z_n)$ will make all terms that have any variable other than z_n vanish. Thus the first term will be a homogeneous polynomial in z_n, which is the principal part of f_i evaluated in $(0, z_n)$. But of course taking quotient by z_n all terms with powers of z_n higher than 1 will vanish, barely leaving the principal part of f_i evaluated at $(0, 1)$. But this works regardless of the coordinate system. □

12. Elementary symmetric polynomials and Newton formula

The main result in this section is to show that the elementary symmetric polynomials can be given in terms of the Newton polynomials. More exactly, as elements of the polynomial ring with rational coefficients and Newton polynomials as indeterminates. In turn, this says that the elementary symmetric polynomials are Weierstrass polynomials as long as the Newton polynomials are holomorphic (which is easy to prove using the Argument Principle, Exercise 7.1). The elementary symmetric polynomials, in the form of a product, are exactly what we get when we factor the zeroes of a holomorphic function as in Equation 7.1.1. So essentially this paragraph says the proof of Weierstrass Preparation Theorem 13.5.

The α_i will eventually play the roles of the zeroes of the holomorphic function, but in this section they are just indeterminates in the ring $\mathbb{Z}[\alpha_1, \dots, \alpha_n]$. We start by defining three types of polynomials in this ring.

Definition 12.1. Let $e_i \in \mathbb{Z}[\alpha_1, \dots, \alpha_n]$ be the coefficients of the polynomial

$$t^n + e_1 t^{n-1} + \ldots + e_{n-1} t + e_n := \prod_{i=1}^{n} (1 + \alpha_i)$$

Then e_i are called *elementary symmetric polynomials* on α_i.

Notice that $e_1 = \sum_i \alpha_i = p_1$ for the Newton polynomial p_1 defined as follows. Also $e_n = \prod_i \alpha_i$, and in general $e_k = \sum_{1 \le i_1 < \ldots < i_k \le n} \alpha_{i_1} \ldots \alpha_{i_k}$.

Definition 12.2. A *Newton polynomial* is $p_j := \sum_{i=1}^n \alpha_i^j$.

Definition 12.3. A *complete homogeneous symmetric polynomial* of degree k is h_k obtained as a sum of all homogeneous monomials of degree k, that is, $\alpha_1^k + \ldots + \alpha_n^k + \alpha_1^{k-1}\alpha_2 + \ldots$.

Corresponding to the above definitions we have the generating functions $E(t) := \sum_{i=0}^n e_i t^i$, $P(t) := \sum_{i=1}^\infty p_i t^i$ and $H(t) := \sum_{i=0}^\infty h_i t^i$ which are formal series in $\mathbb{Z}[\alpha_1, \ldots, \alpha_n][\![t]\!]$.

Exercise 12.4. Prove that $H(t) = \prod_{i=1}^n \frac{1}{1-t\alpha_i}$.

Proof. Let us write (possibly as a formal definition)

$$\prod_{i=1}^n \frac{1}{1-t\alpha_i} = \prod_{i=1}^n \sum_{k=0}^\infty \alpha_i^k t^k.$$

We also write $f_i \overset{\text{ops}}{\longleftrightarrow} \{\alpha_i^k\}_{k=1}^\infty$ to mean that f_i is the power series associated to the sequence $\{\alpha_i^k\}_{k=1}^\infty$. Then the equation above is the product of the f_i. Then all we have to prove is that

$$\prod_{i=1}^n f_i \overset{\text{ops}}{\longleftrightarrow} \left\{ \sum_{i_1+\ldots+i_n=k} \alpha_1^{i_1} \ldots \alpha_n^{i_n} \right\}_{k=1}^\infty = \{h_k\}_{k=0}^\infty$$

This is just a generalization of the formula for product of power series for a product of n power series. □

Exercise 12.5. Prove that $E(t) = \prod_{i=1}^n (1 + t\alpha_i)$.

Proof. This is just writing $\prod_{i=1}^n (1+t\alpha_i) = \prod_{i=1}^n t(\frac{1}{t} + \alpha_i)$ and continue until we get to $E(t)$. □

It follows from the two previous exercises that $H(t)E(-t) = 1$. Using Exercise 12.5 and applying logarithm we obtain that $\frac{E'(-t)}{E(-t)} = -\sum_{i=1}^n \frac{\alpha_i}{1-t\alpha_i}$. Expanding this formula as geometric power series we obtain that

$$P(t) = -\frac{E'(-t)}{E(-t)} \tag{12.5.1}$$

Exercise 12.6. Prove that p_i can be expressed as polynomials of e_i (with integer coefficients).

Proof. Using Eq. 12.5.1, and $H(t)E(-t) = 1$, we have

$$P(t) = E'(-t)H(t).$$

Expanding the power series we obtain that the k-th term is

$$p_k = \sum_{i=0}^k (-1)^{i+1} i e_i h_{k-i}$$

□

Exercise 12.7. Prove that h_i can be expressed as polynomials of e_i with integer coefficients. Prove that e_i can be expressed as polynomials of h_i with integer coefficients.

Sketch of proof. By $H(t)E(-t) = 1$ we see that $\frac{E'(-t)}{E(-t)} = \frac{H(t)}{H'(t)}$. Both denominators are expressed as power series in α_i. Multiplying by $E(-t)$ as in Exercise 12.5 would let $E'(-t)$ be expressed as a power series in h_i and α_i, while multiplying by $H(t)$ as in Exercise 12.4 would let $H'(t)$ expressed in terms of e_i and α_i. ☐

Exercise 12.8 (Newton formula)**.** Prove that $ke_k = \sum_{i=1}^{k-1}(-1)^i e_{k-i}p_i$.

Proof. Note that

$$P(t)E(-t) = \left(\sum_{k=0}^{\infty} p_i t^i\right)\left(\sum_{i=0}^{n}(-1)^i e_i t^i\right) = \sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^{k-i}e_{k-i}p_i t^k$$

using the product formula of power series, where we define $E(-t)$ as a power series by letting $e_i = 0$ for $i > n$. By Eq. 12.5.1, this equals $E'(-t)$, which we may also see as a power series. Comparing the k-th term yields the result modulo a minus sign. ☐

Finally,

Exercise 12.9. Prove that e_i are expressed as polynomials on p_i with rational coefficients.

Proof. ☐

13. Weierstrass preparation theorem

The upshot about Weierstrass preparation theorem is that we would like it if $\mathcal{O}_n$ was an Euclidean domain, i.e. that there was a division algorithm, as in Number Theory Definition **??**. I think that what fails has to do with the fact that holomorphic functions are infinite series, so that we cannot put an Euclidean function like the degree. But instead, we have Weierstrass preparation theorem (which gives a Weierstrass *division* theorem. . . ?)

Before starting let's say the proof again. Start with a germ of holomorphic function. Factor its zeroes as in Equation 7.1.1. This gives a product of the zeroes times a holomorphic function

We denote by $B_r(z_1, \dots, z_{n-1})$ the ball of radius r in $\mathbb{C}^{n-1} \subset \mathbb{C}^n$.

Exercise 13.1. Let F be a holomorphic function on a neighbourhood of 0 in $\mathbb{C}^n$, such that $\lim_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} \neq 0, \infty$. Consider the projection map $\Pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1})$.

1. Prove that for an appropriate pair r, r', the resitriction of F to the polydisk $\Delta(n-1,1) := B_r(z_1, \dots, z_{n-1}) \times \Delta_{r'}(z_n)$ nowhere vanishes on the set $\Pi^{-1}(\partial\Delta_{r'}(z_n))$.
2. Prove that in this case the resitriction of F to this polydisk has precisely k zeroes $\alpha_1, \dots, \alpha_k$ on each fiber of Π.
3. Prove that $\sum_{i=1}^{k} \alpha_i^d$ is a holomorphic function on $B_r(z_1, \dots, z_{n-1})$.
4. Prove that any elementary symmetric polynomial on α_i gives a holomorphic function on $B_r(z_1, \dots, z_{n-1})$.

Proof. (1) Suppose that for every (r, r') there are points $z_1 \in B_r(z_1, \dots, z_{n-1})$ and $z_n \in \partial\Delta_{r'}(z_n)$ where F vanishes. Then we obtain a convergent sequence where F vanishes, and by Identity Principle 10.1 we conclude that F mus be identically zero.

(2) This is just applying Argument principle, i.e. the integral $\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{F'(y_0,z)}{F(y_0,z)}dz$ equals the number of zeroes. Since it is a continuous (holomorphic?) function on Δ_y and integer valued, it must be constant.

The condition that $\lim_{z_n\to 0}\frac{F(0,z_n)}{z_n^k}\neq 0,\infty$ means that the fiber at 0 has a zero of order k:

$$\frac{F(0,z_n)}{z_n^k} = \frac{F(0)}{z_n^k} + F'(0)\frac{z_n}{z_n^k} + \ldots + \frac{F^{(k)}(0)}{k!}\frac{z_n^k}{z_n^k} + \ldots$$

Thus, on other fibers we can have more zeroes, but without multiplicities they are always k.

(3)

□

Definition 13.2. A *Weierstrass polynomial* is a function $f \in \mathcal{O}_{n-1}[z_n]$, that is, a function which is polynomial in the last variables with coefficients that are analytic functions on the first $n-1$ variables.

Exercise 13.3. Let F be an analytic function in a neighbourhood of 0 in $\mathbb{C}^n$, such that $\lim_{z_n\to 0}\neq 0,\infty$. Consider the projection map $\Pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$, $(z_1,\ldots,z_n)\mapsto(z_1,\ldots,z_{n-1})$, and let $P(z_n)\in\mathcal{O}_{n-1}[z_n]$ be the Weierstrass polynomial given by $P(z_n) = \sum_{i=0}^k e_i z_n^i$, where e_i are the elementary symmetric polynomials on the zeros $\alpha_1,\ldots,\alpha_k$ defined in the previous exercise. Prove that $F = P(z_n)u$, where u is a germ of an invertible holomorphic function.

Proof. For every y_0 we can write

$$F(y_0,z_n) = (z_n-\alpha_1)\ldots(z_n-\alpha_n)h(z_n)$$

where implicitly all α_i and h depend on y_0. The product of $(z_n-\alpha_i)$ is the definition of P. Varying y_0 we obtain the result. □

This argument works for every function that has a zero of order k in $0\in\mathbb{C}^n$.

As I recall the following is [GH78] formulation of the theorem:

Theorem 13.4 (Weierstrass preparation theorem)**.** *If $f : U\subset\mathbb{C}^n\to\mathbb{C}$ is holomorphic and f is not identically zero in the coordinate axis $z_n := w$, there is a unique germ of a monic Weierstrass polynomial g whose coefficients are holomorphic functions on the first $n-1$ variables and a germ of a holomorphic function h with $h(0)\neq 0$ (i.e. h is a unit of $\mathcal{O}_n$) such that $f = gh$.*

But I will stick to the formulation in [Dem]:

Theorem 13.5 (Weierstrass preparation theorem)**.** *Let g be holomorphic on a neighourhood of 0 in $\mathbb{C}^n$, such that $g(0,z_n)/z_n^s$ has a nonzero finite limit at $z_n = 0$. With the "above" choice of r' and r_n, one can write*

$$g(z) = u(z)P(z',z_n)$$

where u is an invertible holomorphic function in a neighbourhood of the polydisk $\overline{\Delta}(r',r_n)$ and P is a Weierstrass polynomial with coefficient functions defined for $|z'|\leq r'$ in $\mathbb{C}^{n-1}$.

Proof. We do the proof in steps

Step 0. First notice that since I will use this theorem to prove that $\mathcal{O}_n$ is a UFD, I will start with an arbitrary function $g\in\mathcal{O}_n$. (Right now it's not clear why we need that $g(0) = 0$.) But then it's straightforward to change coordinates so that g is not identically zero on the z_n-axis: just complete the point where g is nonzero to a basis $\{b_i\}_{i=1}^{n-1}$ and precompose with a linear map sending to $(0, \dots, 0, 1)$ to that point, and the rest of the canonical basis to the corresponding basis vectors b_i. The composition of this linear map with g remains holomorphic and is nonvanishing in the z_n-axis.

(See Exercise 11.10 for a proof that we can suppose that the coordinate change preserves the order of the zero, but again it's not clear why I need that.)

Step 1. Now we define the correct domain; skip this for now until it comes into play.

Then $g(0, z_n)$ is a holomorphic function with a zero on 0. By the Taylor expansion we know that this zero must be isolated, so that there is a number $r_n > 0$ such that $f(0, z_n) \neq 0$ for $0 < |z_n| \leq r_n$.

Since the circle $|z_n| = r_n$ is compact and $f(0, z_n) \neq 0$ on this circle, we can fix $r' > 0$ such that $f(z', z_n) \neq 0$ for $z' < r'$ and $|z_n| < r_n + \varepsilon$.

Step 2. Now we apply Argument principle. By Exercise 7.1, we know that $S_0 = \frac{1}{2\pi\sqrt{-1}} \int \frac{f'(z',z)}{f(z',z)} dz$ counts the number of zeroes of $f(z', -)$ on $\Delta(r_n)$ for fixed $z' \in \Delta(r')$.

Notice that the map S_0 is holomorphic as a function of z' just because the complex derivative, defined as a limit, can go into the integral. Moreover, since $S_0(z')$ is an integer, it's constant (as a function of z') by continuity. Thus we have exactly s functions $w_1(z'), \dots, w_s(z')$ which give the s zeroes of $f(z', -)$ at every $z' \in \Delta(r')$. Now we apply argument principle with multiplication by z^k to obtain $S_k(z') = \sum w_i(z')^k$ (without multiplicities).

Step 3. Now we use elementary symmetric polynomials. We can think of $S_k(z') = \sum_{i=1}^{s} w_i^k = p_k$ as the Newton polynomials in the indeterminates w_i. In the last step we showed they are holomorphic functions of z'. In Exercise 12.9 we showed that the elementary symmetric polynomials e_i can be expressed as polynomials of p_i with rational coefficients, that is, $e_i \in \mathbb{Q}[p_1, \dots, p_k]$. We conclude that the e_i are also holomorphic functions of z'. This shows that

$$\begin{aligned} P(z', z_n) :&= \prod_{i=1}^{k} (z_n - w_i(z')) \\ &= t^n + e_1 t^{n-1} + \ldots + e_{n-1} t + e_n, \end{aligned}$$

where the e_i are the elementary symmetric polynomials, defined exactly by the above relation (see Section 12), is a Weiestrass polynomial (because its coefficients are holomorphic functions).

Step 4. Just apply Cauchy formula to figure out the final equation. Factor the zeroes of $f(z', -)$ as in Equation 7.1.1 to obtain $f(z', z_n) = (\prod_i z_n - w_i(z'))\, h_{z'}(z_n) := Ph$.

The classical way is to notice that the function f/P is holomorphic in z_n because f and P have the same zeroes, and we write the integral formula to complete to a holomorphic function (but I think it should be enough with my argument). □

Recall from Number Theory Lemma **??** that if R is a UFD, then $R[x]$ is a UFD.

Lemma 13.6. *The stalk $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.*

Proof. By induction on n. For $n = 0$ it is trivial. Suppose $\mathcal{O}_{n-1}$ is a UFD. Then by Gauss' Lemma **??**, $\mathcal{O}_{n-1}[w]$ is a UFD too. Thus we may express any Weierstrass polynomial g as a product of irreducible elements (uniquely up to multiplication by units).

Let $f \in \mathcal{O}_n$. We want to express f as a product of (unique up to multiplication by units) of irreducible elements. By Weierstrass Preparation Theorem **??** there is a Weierstrass polynomial $g \in \mathcal{O}_n[w]$ and a holomorphic function not vanishing on 0 (i.e. a unit of $\mathcal{O}_n$) such that $f = gh$. By the previous remark g is factored uniquely up to multiplication by units as $g = g_1 \dots g_m$. This shows existence of the factorization.

To prove uniqueness suppose that $f = f_1 \dots f_k$ for some irreducible $f_1, \dots, f_k \in \mathcal{O}_n$. Since f does not vanish in the w axis, neither can each f_i, so that we may decompose each of them as $f_i = g_i' h_i$ by Weierstrass Preparation Theorem. Since f_i is irreducible, it follows that g_i' is irreducible. Then we have that

$$f = gh = \prod g_i' \prod h_i$$

so by uniqueness in Weierstrass Preparation Theorem we conclude that $g = \prod g_i'$, and by uniqueness from the fact that $\mathcal{O}_n[w]$ is a UFD we conclude that g coincides with $\prod g_i'$ up to multiplication by units. $\square$

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