

COMPLEX GEOMETRY

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1. COMPLEX MANIFOLDS

Definition 1.1. A *complex manifold* M is a smooth manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all α, β .

Definition 1.2. A function on an open set $U \subset M$ is *holomorphic* if for all α , $f \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n$.

Lemma 1.3. *The sheaf \mathcal{O}_M of holomorphic functions is a sheaf.*

Complex structures...

2. SHEAF OF HOLOMORPHIC FUNCTIONS

For a definition of sheaf see Algebraic Geometry Definition 1.1.

Lemma 2.1. *Let X be a complex manifold of complex dimension n .*

The functor

$$\begin{aligned}\mathcal{O}_X : \text{Open}_X^{\text{op}} &\longrightarrow \text{Set} \\ U &\longmapsto \mathcal{O}_X(U) := \{f : U \rightarrow \mathbb{C}^n : f \text{ is holomorphic}\} \\ i : V \hookrightarrow U &\longmapsto \mathcal{O}_X(i) : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)\end{aligned}$$

where $\mathcal{O}_X(i)$ is given by restriction, is a sheaf.

Proof. \mathcal{O}_X is a presheaf, i.e. a functor, by definition: restriction maps the identity to the identity and preserves compositions.

To check the condition of being a sheaf let U_i be an open cover of some open set U of X and $f_i : U_i \rightarrow \mathbb{C}^n$ sections of $\mathcal{O}_X(U_i)$, that is, holomorphic functions, which coincide pairwise in intersections. Then it is obvious that the function $f : U \rightarrow \mathbb{C}^n$, $x \mapsto f_i(x)$ for any $U_i \ni x$ is well defined. It is also holomorphic since holomorphicity is a condition defined on open sets. \square

3. ANALYTIC SETS

Definition 3.1. An *analytic set* or *subvariety* Y is a subset of a complex manifold X given locally as the zeros of a finite collection of holomorphic functions.

Lemma 3.2. *Let $Y \subset X$ be an analytic subvariety of a complex manifold X equipped with the subspace topology. Then*

$$\begin{aligned}\mathcal{O}_Y : \text{Open}_Y^{\text{op}} &\longrightarrow \text{Set} \\ U \cap Y &\longmapsto \{f|_{U \cap Y} : f \in \mathcal{O}_X(U)\}\end{aligned}$$

is a sheaf.

Proof. This is a presheaf for the same reasons as the sheaf \mathcal{O}_X is a presheaf. The fact that it is a sheaf follows from the fact that \mathcal{O}_X is a sheaf and by definition of subspace topology; namely for any family of functions defined over an open cover $\{U_i \cap Y\}$ of some open set $U \cap Y$ we can reconstruct a function on U that restricted to $U \cap Y$ is as desired. \square

Recall that any map of sheaves of groups has well-defined kernel, see Lemma 2.1.

Definition 3.3. Let $Y \subset X$ be an analytic subvariety of a complex manifold X . The *ideal sheaf* of Y , denoted \mathcal{I}_Y , is the kernel of the restriction map $\mathcal{O}_X \rightarrow \mathcal{O}_Y$.

This just says that $\mathcal{I}_X(U)$ is the set of functions $f \in \mathcal{O}_X$ that vanish under the restriction map, i.e. that vanish at points of Y .

Since the restriction map is surjective, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

Notice that for any open set $U \subset X$, $\mathcal{I}_Y(U)$ is an ideal of $\mathcal{O}_X(U)$. (Indeed, if a function vanishes at Y then its product with any other function will also vanish.)

Definition 3.4. An analytic variety $V \subset U \subset \mathbb{C}^n$ is *irreducible* if V cannot be written as the union of two distinct analytic varieties $V_1, V_2 \subset U$, both distinct to V .

Lemma 3.5. *If V is an irreducible analytic hypersurface given locally as $V = \{f = 0\}$, then f is irreducible in \mathcal{O}_p .*

Proof. If $f = gh$ and neither of g and h are units and they are distinct, we could express V as the union of two distinct varieties: $V(g)$ and $V(h)$. (If one of them, was a unit then the vanishing set would be all of U .) \square

Definition 3.6. The *germ* of a set in the origin $0 \in \mathbb{C}^n$ is given by a subset $X \subset \mathbb{C}^n$. To subsets X, Y define the same germ if there exists an open neighbourhood $0 \in U \subset \mathbb{C}^n$ with $U \cap X = U \cap Y$. A germ is called *analytic* if there are functions $f_1, \dots, f_k \in \mathcal{O}_n$ such that X and $Z(f_1, \dots, f_k)$ define the same germ.

Now the adventure of proving that the ideal sheaf is a line bundle for hypersurfaces begins.

See Algebraic Geometry Lemma ?? for the proof that the ideal sheaf of an irreducible codimension-1 closed subscheme of a smooth scheme is a line bundle. Now I will try to prove this for the case of an analytic subvariety of a (smooth) complex manifold.

First notice that in the case that Y is an irreducible subvariety, this ideal is prime: if $fg \in \mathcal{I}_Y(U)$, then $V(f) \cup V(g) \supset Y \cap U$ so that $(V(f) \cup V(g)) \cap Y = Y \cap U$, and then it must be true that $V(f) \cap Y \subseteq V(g) \cap Y$ or that $V(f) \cap Y \supseteq V(g) \cap Y \cap U$ since Y is irreducible, and thus, say, $V(f) \supseteq Y \cap U$, that is, $I(V(f)) \subseteq I(Y \cap U)$ so that $f \in I(Y \cap U)$.

If we further ask that Y is a hypersurface, we want to show that \mathcal{I}_Y is locally principal. If $f \in \mathcal{I}_Y(U)$ is the defining function of some hypersurface Y of a variety X on an open set $U \subset X$, we shall have that $\mathcal{I}_Y(U) = (f)$. That is, if g also vanishes at $Y \cap U$, we want to see that $g = fh$ for some $h \in \mathcal{O}_X(U)$.

By Stacks Project 0AFT we need two ingredients: (1) the fact that the ring $\mathcal{O}_X(U)$ is a UFD for all $U \subset X$ and (2) the following lemma.

Lemma 3.7. *The ideal of an irreducible analytic hypersurface is a height-1 prime ideal.*

Proof. The fact that it is prime comes from the fact that Y is irreducible as explained above.

Now suppose that $0 \subset \mathfrak{p} \subseteq I$. Then $V(\mathfrak{p})$ is an analytic variety that contains Y . At regular points of both $V(\mathfrak{p})$ and Y , both are smooth manifolds, but since Y is of codimension 1 and $V(\mathfrak{p})$ does not equal all of Y , we conclude that they coincide. Thus, in a neighbourhood of a regular point we have $\mathfrak{p} = I(V(\mathfrak{p})) = I$ by the holomorphic Nullstellensatz ??. \square

Now we shall use the Gauss Lemma ?? to prove:

Lemma 3.8. *The stalk $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n, 0}$ is a UFD.*

Proof. By induction on n . For $n = 0$ it is trivial. Suppose \mathcal{O}_{n-1} is a UFD. Then by the Gauss Lemma ??, $\mathcal{O}_{n-1}[w]$ is a UFD too. Thus we may express any Weierstrass polynomial g as a product of irreducible elements (uniquely up to multiplication by units).

Let $f \in \mathcal{O}_n$. We want to express f as a product of (unique up to multiplication by units) of irreducible elements. By Weierstrass Preparation Theorem ?? there is a Weierstrass polynomial $g \in \mathcal{O}_n[w]$ and a holomorphic function not vanishing on 0 (i.e. a unit of \mathcal{O}_n) such that $f = gh$. By the previous remark g is factored uniquely up to multiplication by units as $g = g_1 \dots g_m$. This shows existence of the factorization.

To prove uniqueness suppose that $f = f_1 \dots f_k$ for some irreducible $f_1, \dots, f_k \in \mathcal{O}_n$. Since f does not vanish in the w axis, neither can each f_i , so that we may decompose each of them as $f_i = g'_i h_i$ by Weierstrass Preparation Theorem. Since f_i is irreducible, it follows that g'_i is irreducible. Then we have that

$$f = gh = \prod g'_i \prod h_i$$

so by uniqueness in Weierstrass Preparation Theorem we conclude that $g = \prod g'_i$, and by uniqueness from the fact that $\mathcal{O}_n[w]$ is a UFD we conclude that g coincides with $\prod g'_i$ up to multiplication by units. \square

Lemma 3.9. *The ideal sheaf of an irreducible codimension-1 analytic subvariety of a smooth complex manifold is a line bundle.*

Proof. By definition, our subvariety Y is locally defined by the zeroes of a single holomorphic function defined on the ambient manifold X . That is, at every $p \in Y$ there is an open set $U \subset X$ such that $U \cap Y = f^{-1}(0)$ for some $f \in \mathcal{O}_X(U)$. Since U is an open set of a complex manifold, the ring $\mathcal{O}_X(U)$ is isomorphic to the ring \mathcal{O}_n of holomorphic functions on \mathbb{C}^n in a neighbourhood of the origin; that's by Definition 1.1 since a holomorphic function on a manifold is defined as such if its composition with the coordinate chart is a holomorphic function on \mathbb{C}^n .

By Lemma 3.5, since Y is irreducible it follows that f is irreducible and so is the ideal of functions vanishing on Y .

By Lemma 3.8, we prove identically as in Lemma ?? that the ideal of functions vanishing on Y is principal. \square

Note the that argument actually works backwards if we assume the ambient ring is Noetherian, see Stacks Project 0AFT.

Definition 3.10. Let $D := S$ be an effective Cartier divisor, that is, an analytic hypersurface of a complex manifold. The ideal sheaf $\mathcal{I}_S := \mathcal{O}_X(-S)$ of S is the dual line bundle of the *line bundle associated to the divisor* D , which is denoted by $\mathcal{O}_X(D)$.

4. HILBERT'S NULLSTELLENSATZ

Usually I think of Hilbert's Nullstellensatz as the statement that $I(V(I)) = \sqrt{I}$. Following are two simple initial observations.

It is immediate that $\sqrt{\mathfrak{p}} = \mathfrak{p}$ if \mathfrak{p} is a prime ideal (of any ring): if $f^k \in \mathfrak{p}$ then either $f \in \mathfrak{p}$ or $f^{k-1} \in \mathfrak{p}$, etc.

It is also easy to see that for any ideal I of functions or germs of functions (i.e. either $I \subset \mathcal{O}_X(U)$ or $I \subset \mathcal{O}_n$) we have $\sqrt{I} \subseteq I(V(I))$. Indeed, if $f^k \in I$, then $f^k(x) = 0$ for all $x \in V(I)$, and since \mathbb{C} has no nonzero divisors we see that $f(x) = 0$, that is $f \in I(V(I))$.

It remains to show that $I(V(I)) \subseteq \sqrt{I}$. We shall first suppose that I is prime and prove that $I(V(I)) \subseteq I$. (The complete statement for the radical follows from a purely algebraic process, see [Dem, Theorem 4.22].)

We will need a few preparatory results. First recall that S is an integral ring over R if every element of S is expressed as a root of a monic polynomial with coefficients in R . (See Stacks Project 00GI.)

It will be handy to keep in mind the following lemma (Stacks Project 00GH):

Lemma 4.1. *Let $R \rightarrow S$ be a ring map. The following are equivalent*

- (1) $R \rightarrow S$ is finite,
- (2) $R \rightarrow S$ is integral and of finite type, and
- (3) there exist $x_1, \dots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R .

This essentially says that an integral extension $R \rightarrow S$ gives S the structure of a finitely generated R -module (I think experts call this a “finite R -module”) whose generators are integral. This just means that $S = R[x_1, \dots, x_k]$ and $x_i \in S$ is the root of a monic polynomial in R .

Proposition 4.2. $\mathcal{O}_n/\mathcal{I}$ is a finite integral extension of \mathcal{O}_d .

Proof. For this it’s enough to check that the map $\mathcal{O}_d \rightarrow \mathcal{O}_n/\mathcal{I}$ is finite. That is, that every element of $\mathcal{O}_n/\mathcal{I}$ can be expressed as a tuple of elements of \mathcal{O}_d . (We can think of this tuple as a linear combination, or cycle.) Step 1: produce some Weierstrass polynomials. . . \square

Lemma 4.3. *If $\mathcal{I} \subset \mathcal{O}_n$ is prime and $A = V(\mathcal{I})$, then $\mathcal{I}_{A,0} = \mathcal{I}$.*

Proof. Proposition 4.2 says: take a polynomial $f \in \mathcal{O}_n$ and consider its equivalence class $\tilde{f} \in \mathcal{O}_n/\mathcal{I}$. Then \tilde{f} is the root of some monic polynomial $t^r + b_{r-1}t^{r-1} + \dots + b_1t + b_0$ with $b_i \in \mathcal{O}_d$. That is, putting f instead of t we get an equation

$$f^r + b_1f^{r-1} + \dots + b_r = 0 \quad \text{mod } \mathcal{I}.$$

In other words, the holomorphic function $f^r + b_1f^{r-1} + \dots + b_r$ is in \mathcal{I} . Since f vanishes on $(A,0)$, this says that b_r vanishes in $(A,0)$ too. Now suppose that this implied that the class $\tilde{b}_r = 0$. We are supposed to use that Proposition 4.2 also tells us is that the above equation is irreducible, which I think means that $F = \sum b_it^i \in \mathcal{O}_d[t]$ is irreducible. In turn this gives that $r = 1$. \square

Let’s just store what was learned today:

- (1) **Noether normalization lemma.** See this video. Let k be a (infinite) field, $A \neq 0$ a finitely generated k -algebra. Then there exists elements $y_1, \dots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \dots, y_r]$. That is,

$$\begin{array}{c} A \\ \left| \begin{array}{c} \text{integral} \\ \text{ext.} \end{array} \right. \\ k[y_1, \dots, y_r] \\ \left| \begin{array}{c} \text{transc.} \\ \text{ext.} \end{array} \right. \\ k. \end{array}$$

- (2) This can be used to prove the weak Nullstellensatz, i.e. let k be an algebraically closed field, and $I \subseteq k[x_1, \dots, x_n]$ an ideal such that $V(I) = \emptyset$. Then $I = k[x_1, \dots, x_n]$.

To prove it using Noether's normalization lemma, suppose that the $I \neq k[x_1, \dots, x_n]$. Let \mathfrak{m} be a maximal ideal such that $I \subseteq \mathfrak{m}$. Then $A = k[x_1, \dots, x_n]/\mathfrak{m} = k[y_1, \dots, y_n]$, where the first string is algebraically independent while the last string is integral. But because the whole extension is algebraic, then y_1, \dots, y_n are integral over k . Then we can map $y_i \mapsto \alpha_i$, a root of its defining polynomial over k . This means we do have a point where all polynomials in \mathfrak{m} vanish.

Note. I'm not sure what we mean with "the first string" and the "second string".

- (3) Also reviewed Misha's proof of the Nullstellensatz. Looks accessible, though it is not holomorphic!
- (4) Perhaps start in Demailly just after Prop. 4.13.

5. P,Q DECOMPOSITION

The upshot is the following:

$$T^2(V \oplus W) = T^2(V) \otimes T^0(W) \oplus T^1(V) \otimes T^1(W) \oplus T^0(V) \otimes T^2(W).$$

Explicitly, the decomposition of a general element on the LHS is

$$\begin{aligned} (v_1, w_1) \otimes (v_2, w_2) &= (v_1, 0) \otimes (v_2, w_2) + (0, w_1) \otimes (v_2, w_2) \\ &= (v_1, 0) \otimes (v_2, 0) + (v_1, 0) \otimes (0, w_2) \\ &\quad + (0, w_1) \otimes (v_2, 0) + (0, w_1) \otimes (0, w_2) \end{aligned}$$

which is valid because the tensor product of a vector space T^2X is just the pairs $x \otimes y$ subject to relations that $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$ and addition for y and scalar product. So we just identify elements

$$\begin{aligned} (v_1, 0) \otimes (v_2, 0) &\rightsquigarrow v_1 \otimes v_2 \in T^2(V) \otimes T^0(W) := T^{2,0}(V \oplus W) \\ (v_1, 0) \otimes (0, w_2) &\rightsquigarrow v_1 \otimes w_2 \in T^1(V) \otimes T^1(W) := T^{1,1}(V \oplus W) \\ (0, w_1) \otimes (v_2, 0) &\rightsquigarrow w_1 \otimes v_2 \in T^1(W) \otimes T^1(V) \cong T^1(V) \otimes T^1(W) := T^{1,1}(V \oplus W) \\ (0, w_1) \otimes (0, w_2) &\rightsquigarrow w_1 \otimes w_2 \in T^0(V) \otimes T^2(W) := T^{0,2}(V \oplus W). \end{aligned}$$

We have proved that any element in $T^2(V \oplus W)$ can be written as an element in the vector space $T^2(V) \otimes T^0(W) + T^1(V) \otimes T^1(W) + T^0(V) \otimes T^2(W)$. Since the intersection of the factors is disjoint, the sum can be taken direct.

Notice that we have two copies of $T^{1,1}(V \oplus W)$. By the binomial theorem and isomorphisms of the kind $A \otimes B \cong B \otimes A$, we readily see that

$$T^k(V \oplus W) \cong \bigoplus_{p+q=k} \binom{k}{p} T^p(V) \otimes T^q(W).$$

Of course the binomial coefficient matters little when the underlying field is \mathbb{C} .

The next step is to take the quotient on the LHS by the relation $v \otimes v = 0$, i.e., the ideal generated by $v \otimes v$. When we compute $(v_1, w_1) \otimes (v_1, w_1)$ as above, this relation becomes $v_1 \otimes v_1 = 0$ and $w_1 \otimes w_2 = 0$ on the first and last components. If

we want the middle component to vanish as well we must ask that the isomorphism $T^1(W) \otimes T^1(V) \cong T^1(V) \otimes T^1(W)$ exchanges sign. Then we obtain that

$$\Lambda^2(V \oplus W) = \Lambda^{2,0}(V \oplus W) \oplus \Lambda^{1,1}(V \oplus W) \oplus \Lambda^{0,2}(V \oplus W)$$

by defining $\Lambda^{p,q}(V \oplus W) := \Lambda^p(V) \otimes \Lambda^q(W)$.

To make sure that this generalizes to all p, q write, e.g. $p + q = 3$,

$$\begin{aligned} (v_1, w_1) \otimes (v_1, w_1) \otimes (v_1, w_1) &= (v_1, 0) \otimes (v_1, 0) \otimes (v_1, 0) \\ &\quad + (v_1, 0) \otimes (v_1, 0) \otimes (0, w_1) \\ &\quad + \dots \end{aligned}$$

so that imposing the relation $v \otimes v$ on $T^3(V \oplus W)$ implies imposing it on every factor. [Further arguments needed here. . . but looks like just boring computations. . .]

The main result of Hodge theory is that when we take $T^{1,0}(M) \oplus T^{0,1}(M)$, the eigenspaces of the complex structure and pass to complex singular cohomology we get

$$(5.0.1) \quad H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$$

where $H^{p,q}(M)$ is the set of differential forms represented by a closed form in $\Lambda^{p,q}(M)$.

Missing: what does it even mean that $\overline{\Lambda^{p,q}(V)} = \Lambda^{q,p}(V)$? Recall that \bar{V} is just V but with the scalar product $\lambda v := \bar{\lambda}v$.

Exercise 5.1. Show that $(1, 1)$ -forms are invariant under the action of the complex structure I . And conversely, a 2-form is of type $(1, 1)$ only if it is I -invariant.

Proof. Let $\alpha, \beta \in \Lambda^{1,1}(M)$. Recall that $\Lambda^{1,1}(M) = \Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M)$. For any vectors v, w tangent to some point $p \in M$,

$$\begin{aligned} I(\alpha \wedge \beta)(v, w) &= (\alpha \wedge \beta)(Iv, Iw) \\ &= \alpha(Iv)\beta(Iw) - \beta(Iv)\alpha(Iw) \\ &= -\sqrt{-1}\sqrt{-1}(\alpha(v)\beta(w) - \beta(v)\alpha(w)) \\ &= \alpha(v)\beta(w) - \beta(v)\alpha(w) \\ &= (\alpha \wedge \beta)(v, w). \end{aligned}$$

The converse follows easily by the direct sum splitting of $\Lambda^2(M)$ and noting that a $(2, 0)$ or $(0, 2)$ -form γ satisfies $I \cdot \gamma(v, w) = -\gamma(v, w)$ in an analogous computation. \square

In particular this implies that ω , the fundamental form of the Riemannian metric g and the complex structure I is a $(1, 1)$ -form. So this shows that Kähler classes are in $H^{1,1}(M)$.

6. KÄHLER METRICS

Definition 6.1. A *hermitian structure* on a complex manifold X is a Riemannian metric g on X which is an isometry for the complex structure.

Definition 6.2. A *Kähler structure* is an hermitian structure g for which the fundamental form is closed.

Examples of Kähler forms

Proposition 6.3 (Kähler identities).

7. HODGE THEORY

Harmonic forms and Hodge theorem. Definition of $NS(X)$ and Lefschetz theorem on $(1,1)$ -classes (i.e. Pic surjects on integral classes). Hard Lefschetz theorem (?). Hodge-Riemann bilinear relation (?).

Theorem 7.1 (Hodge index). *The intersection form on $H^{1,1}(X, \mathbb{R})$ has index $(1, h^{1,1} - 1)$.*

Hodge conjecture.

8. CONES

Recall that $NS(X)$ is the quotient group $Pic(X)/H^1(\mathcal{O}_X) \subset H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$ coming from the exponential exact sequence. This has an intersection form. In the case of a K3 we readily see that $h^1 = 0$ so $NS = Pic$. In fact, the intersection form coincides with the usual cup product on 2-cocycles of singular cohomology.

For a Kähler manifold we have

- The *Kähler cone*. Classes in $H^{1,1}(X, \mathbb{R})$ that are represented by a Kähler form.
- The *positive cone*. Classes $NS(X)_{\mathbb{R}} := NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ with $\alpha^2 > 0$.
- The *ample cone*. Classes $NS(X)_{\mathbb{R}}$ with $\alpha^2 > 0$ and $\alpha.C > 0$ for all algebraic curves C . (See 28.1.) This is true, but hard to prove, see [?, p. 150]. Another description is that the ample cone is the set of classes in $NS(X)_{\mathbb{R}}$ that are finite sums $\sum a_i L_i$ for $L_i \in NS(X)$ ample and $a_i \in \mathbb{R}_{>0}$.
- The *nef cone*. Classes (in $NS(X)_{\mathbb{R}}$) with $\alpha.C = \int_C \alpha \geq 0$ for any algebraic curve. There is also a notion of nef for analytic cycles which basically says that the form is approximated by Kähler forms. In [DP01, Theorem 4.7(ii)] we have that a $(1,1)$ -class is nef (in the analytic sense) if and only if $\int_Y \alpha \wedge \omega^{p-1} \geq 0$ for every irreducible analytic set $Y \subset X$, $p = \dim Y$ and ω Kähler class.

The upshot about Demailly-Păun theorem is that Kähler classes are numerically positive (recall that “numerically” means w.r.t. intersection form) on *analytic* cycles. This distinguishes complex analytic geometry from algebraic geometry. This is why Demailly-Păun say that their theorem is a generalization of Nakai-Moishezon: because the latter is for *algebraic* cycles, indeed, found in Hartshorne’s book.

The proof goes roughly as follows. It’s clear that $\mathcal{K} \subset \mathcal{P}$ (\mathcal{P} is the analytically numerically positive cone) because, since the Kähler form comes from the Riemannian metric and the curves are complex, which means that they are invariant under the complex structure, we have integrals $\int_Y \omega$ for curves $Y \subset X$ but $\omega(v, Jv) > 0$ for a basis v, Jv of $T_p Y$. That is to say, there are no complex Lagrangian curves.

But that’s elementary. The point of the theorem is proving that a form in $\bar{\mathcal{K}} \cap \mathcal{P}$, i.e. in the closure of the Kähler cone and which is also numerically analytically positive, is in fact Kähler.

So, the theorem says that the boundary of the Kähler cone does not intersect the numerically analytically positive cone.

Theorem 8.1 (Demailly-Paun). *Let X be a compact Kähler manifold. The Kähler cone is a connected component of the cone of 1,1-classes numerically positive on analytic cycles.*

Exercise 8.2. Let M be a K3 surface such that $\text{rkNS}(M) < 20$. Using Demailly-Paun theorem, prove that there exists a non-zero vector $\eta \in H^{1,1}(M)$ which satisfies $\eta^2 = 0$ and belongs to the boundary of the Kähler cone.

Proof. Let $\alpha \in \text{NS}(X)^\perp \setminus \{0\}$, which is nonempty since $\dim \text{NS}(X) < 20 = h^{1,1}$. By the Hodge index theorem 7.1 the intersection form on $H^{1,1}(X, \mathbb{R})$ has index $(1, 19)$. If we assume that X is projective then we know that the restriction of the Fubini-Study form ω is integral. Since ω is positive (by being a Kähler form), we see by the index of the intersection form that α must be negative.

Define $\eta_t := t\alpha + (1-t)\omega$ for $t \in [0, 1]$. Then $\eta_t^2 = t^2\alpha^2 + (1-t)^2\omega^2$. Since at $t = 0$ we have a positive number and at $t = 1$ a negative number, we know that there exists t_0 such that $\eta_{t_0}^2 = 0$ by continuity of the intersection form.

Now we need to show that η_{t_0} is in the boundary of the Kähler cone. For this it's enough to show that for every $\varepsilon > 0$ $\eta_{t_0+\varepsilon}$ is in the Kähler, which by Demailly-Paun means that we have to show that it is positive on analytic cycles.

By our choice of ω and α we see that

$$\int_Y \eta_{t_0+\varepsilon} = \int_Y (t_0 + \varepsilon)\alpha + (1 - (t_0 + \varepsilon))\omega$$

and the integral of α vanishes since α is in the orthogonal complement of $\text{NS}(X)$, which contains all analytic curves by X being projective (Chow theorem). \square

9. HODGE STAR

The construction of the Hodge star in [Voi02, Section 5.1] is essentially: map $\alpha \in \Lambda^{n-k}$ to the operator $\alpha \wedge - \in \text{Hom}(\Lambda^k, \Lambda^n)$ and then use the metric to identify $\text{Hom}(\Lambda^k, \Lambda^n)$ with Λ^k . Notice that the latter means that we take a form α and we map it to the morphism which maps \cdot to the top-form $g(\alpha, \cdot)\text{Vol}$. This says that for all $\alpha \in \Lambda^k$ and $\beta \in \Lambda^{n-k}$

$$(9.0.1) \quad \alpha \wedge * \beta = (\alpha, \beta) \text{Vol}.$$

Lemma 9.1. *The operator $*$ satisfies the identity*

$$(9.1.1) \quad *^2 = (-1)^{k(n-k)} \text{ on } \Lambda^k(X).$$

Proof. Observe that $*$ is an isometry, use skew-commutativity of product to obtain the $(-1)^{k(n-k)}$ and finally Equation 9.0.1. \square

10. LINE BUNDLES

I have followed mostly [GH78] for this, and also some [Lee24].

For a smooth hypersurface $X \subset Y$ locally defined as the vanishing of $f_i \in \mathcal{O}_Y(U_i)$ for an open cover U_i of Y , we define the line bundle $\mathcal{O}_Y(X)$ by the cocycle $f_{ij} := f_i/f_j$.

We may also see $\mathcal{O}_Y(X)$ as a locally free sheaf by assigning to U_i the group of holomorphic sections of $\mathcal{O}_Y(X)$. As explained in [GH78, p. 138], tensoring by a section with divisor X gives an identification between this group of sections and the meromorphic sections on M with poles along X . Ultimately this means that the sheaf corresponding to $\mathcal{O}_Y(X)$ is locally generated by $1/f_i$.

In turn this means that its dual, denoted $O_Y(-X)$, is a line bundle defined by the cocycle $(f_i/f_j)^{-1}$ and a locally principal sheaf generated by f_i , that is, the ideal sheaf of X .

11. EXAMPLES OF LINE BUNDLES

Example 11.1 (Determinant bundle). Let $\pi : E \rightarrow M$ be a vector bundle of rank r . The *determinant bundle* of E is given by the following data. A space $\tilde{E} := \coprod_{p \in M} \Lambda^r E_p$ with the obvious projection map and topology induced from such map. Since we have a cover U_i of M along with local trivializations $U_i \xrightarrow{\varphi_i} U_i \times \mathbb{C}^r$ we may just postcompose φ_i with the functor Λ^r on the second entry, giving trivializations

$$U_i \xrightarrow[\simeq]{\varphi_i} U_i \times \mathbb{C}^r \xrightarrow{\text{Id} \times \Lambda^r} U_i \times \Lambda^r \mathbb{C}^r = U_i \times \mathbb{C}$$

We can also apply Λ^r to the transition functions g_{ij} on the intersections $U_{ij} := U_i \cap U_j$

$$\begin{array}{ccc} U_{ij} & \xrightarrow{\varphi_i} & U_{ij} \times \mathbb{C}^r \\ & & \downarrow g_{ij} \in \text{GL}_r(\mathbb{C}) \\ U_{ij} & \xrightarrow{\varphi_j} & U_{ij} \times \mathbb{C}^r \end{array}$$

giving maps in $\text{GL}_1(\mathbb{C})$. Recall that Λ^r is functorial on linear maps $g_{ij} \in \text{GL}_n(\mathbb{C})$ because it is given by the determinant of g , and the determinant is multiplicative. We thus obtain the cocycle condition on the maps g_{ij} after applying Λ^r .

12. RIEMANN-ROCH FORMULAS

Theorem 12.1 (Riemann-Roch for curves). *Let D be a divisor on a compact Riemann surface X , that is, D is a collection of d points on X .*

$$(12.1.1) \quad h^0(D) - h^0(K - D) = d - g + 1$$

Exercise 12.2. Prove that if X is a complex smooth curve with $g \geq 2$ then $h^0(T_X) = 0$.

Proof. First you notice that the dimension of the holomorphic 1-forms is the genus. This is just because the genus p_a is defined as $h^1(\mathcal{O}_X)$, which is just $h^0(K_X)$ by Serre duality. Then you say well if you have a holomorphic vector field, pair it with the nonzero holomorphic 1-form. This gives a nonzero function, but it must be constant because X is compact. Then it is actually nowhere vanishing. This says the vector field cannot vanish anywhere, which means the tangent bundle of the curve is trivial. Apparently elliptic curves, i.e. $g = 1$ are the \square

Theorem 12.3 (Riemann-Roch for line bundles on surfaces). *Let L be a line bundle on a complex surface X , and $K_X = \Omega^2 X$ the canonical bundle of X . Then*

$$(12.3.1) \quad \chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}(L, L - K_X)$$

13. ADJUNCTION FORMULAS

Perhaps the most accessible statement that could be interpreted as “adjunction formula” is that the line bundle associated to a smooth hypersurface is the normal bundle.

Theorem 13.1 (Adjunction formula I). *If X is a smooth hypersurface of a smooth complex manifold Y , then*

$$(13.1.1) \quad \mathcal{O}_Y(X)|_X \simeq \mathcal{N}_{X/Y},$$

where $\mathcal{N}_{X/Y} := T'_Y/T'_X$.

Proof. If X is locally defined by the vanishing of f_i on an open cover U_i of Y , then $\mathcal{O}_Y(X)$ is given by the cocycle $f_{ij} := f_i f_j$.

To prove Equation 13.1.1 it's enough to show that $\mathcal{O}_Y(X)|_X \otimes \mathcal{N}_{X/Y}^\vee$ is trivial, which can be done by constructing a nonvanishing section. Such a section is given by

$$df_i = d(f_{ij} f_j) = df_{ij} \cdot f_j + f_{ij} df_j,$$

since the first term vanishes along X (because f_j does) and the remaining terms define precisely a section of T_Y^\vee/T_X^\vee , the conormal bundle — indeed, sections of such a bundle are holomorphic differentials vanishing along X . \square

The next adjunction formula is essentially a consequence of the following linear algebra fact:

Lemma 13.2. *If*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is a short exact sequence of vector bundles (of possibly different ranks), then

$$\det(E) \cong \det(E') \otimes \det(E'').$$

Proof. First notice that each of the three determinant bundles gives fiberwise the top-exterior power of the corresponding vector spaces, thus varying the rank of the exterior power taken. Then just define a map

$$x_1 \wedge \dots \wedge x_{\text{rk} E'} \otimes x_{\text{rk} E' + 1} \wedge \dots \wedge x_{\text{rk} E' + \text{rk} E''} \mapsto x_1 \wedge \dots \wedge x_{\text{rk} E'} \wedge x'_{\text{rk} E' + 1} \wedge \dots \wedge x'_{\text{rk} E' + \text{rk} E''}$$

where $x'_{\text{rk} E' + i}$ maps to $x_{\text{rk} E' + i}$ under the surjection given by the exact sequence.

It's clear that such a map is injective: if any monomial in the image is zero, then one of the factors must be zero, and then the element we began with also vanishes. Being two vector spaces of dimension 1, the map must also be surjective. This gives an isomorphism in every fiber.

To prove that this extends to an isomorphism of bundles all we have to do is show that this is a biholomorphic map on the total spaces of the bundles. But holomorphicity is a local property, and locally both bundles are isomorphic because the exact sequence splits, so the map above is trivial.

In local coordinates of the total spaces $\coprod_{p \in M} \Lambda^{\text{rk} E'} E'_p$ and $\coprod_{p \in M} \Lambda^{\text{rk} E''} E''_p$, which have local frames given by a single local section, the map above takes the coordinates to their product, which is clearly holomorphic. Thus this is a holomorphic bundle isomorphism. \square

Theorem 13.3 (Adjunction formula II). *Let X be a complex submanifold of a complex manifold Y . Then*

$$(13.3.1) \quad (K_Y \otimes \mathcal{O}_Y(X))|_X \cong K_X.$$

Proof. From the holomorphic tangent bundle exact sequence we get

$$0 \longrightarrow T'_X \longrightarrow T'_Y|_X \longrightarrow N_{X/Y} := T'_Y/T'_X \longrightarrow 0$$

Taking determinant of this exact sequence we obtain the result. \square

Lemma 13.4. *Let S be any curve on a surface X . Then*

$$(13.4.1) \quad 2p_a(S) - 2 = (S.S - K_X)$$

Proof. If S is nonsingular we may use the normal bundle short exact sequence. If S is singular we need to use the ideal sheaf short exact sequence, which means we think of S either as a closed embedded subscheme of X or as an analytic subvariety. Taking Euler class we obtain

$$\chi(\mathcal{O}(-S)) + \chi(\mathcal{O}_S) = \chi(\mathcal{O}_X)$$

Since $p_a(S) := 1 - \chi(\mathcal{O}_S)$, we obtain that $p_a(S) = 1 - \chi(\mathcal{O}_X) - \chi(\mathcal{O}(-S))$. To compute $\chi(\mathcal{O}(-S))$ we use Riemann-Roch formula ?? for curves on surfaces, which gives

$$\chi(\mathcal{O}(-S)) = \chi(\mathcal{O}_X) + \frac{(\mathcal{O}(-S). \mathcal{O}(-S) + \omega_X^*)}{2}$$

Taking duals on both entries of the intersection form and substituting in our previous expression for $p_a(S)$ we obtain the result. \square

Exercise 13.5. Let S be a singular, irreducible complex curve on a K3 surface. Prove that $(S.S) \geq 0$.

Proof. This is just an application of Eq. 13.4.1. Since $K_X = \mathcal{O}_X$ it suffices to show that $p_a(S)$ is not zero. This follows fact that S is singular, since its arithmetic genus is defined as the genus of its normalization, which must be strictly positive since there exists at least one singularity. \square

14. CHOW'S THEOREM

Proposition 14.1. *Every codimension- p submanifold of a complex n -manifold has slice charts, i.e. for any $p \in M$ there's a coordinate chart $U \ni p$ such that points of M have coordinates $(z_1, \dots, z_n, 0, \dots, 0)$.*

Proof. This may be taken as the definition of smooth (real or complex) submanifold, or as a proposition using inverse function theorem (real or complex). \square

Proposition 14.2. *Stalk of holomorphic functions is a PID.*

Proof. This can be proved via Weierstrass Preparation theorem ?? which implies Lemma ??, and this in turn implies Lemma ??.

Also, it can be shown directly for smooth hypersurfaces as in [Lee24] Lemma 3.38. \square

Theorem 14.3 (Chow for hypersurfaces). *Every complex codimension-1 submanifold of \mathbb{CP}^n is algebraic.*

15. AMPLENESS

Definition 15.1. Let L be a holomorphic bundle on a complex manifold X . A point $x \in X$ is a *base point* of L if $s(x) = 0$ for all $s \in H^0(X, L)$.

Definition 15.2. Let L be a holomorphic bundle on a complex manifold X . The *base locus* $Bs(L)$ is the set of all base points of L .

Proposition 15.3. Let L be a holomorphic line bundle on a complex manifold X and suppose that $s_0, \dots, s_N \in H^0(X, L)$ is a basis. Then

$$\begin{aligned} \varphi_L : X \setminus Bs(L) &\longrightarrow \mathbb{P}^N \\ x &\longmapsto (s_0(x) : \dots : s_N(x)) \end{aligned}$$

defines a holomorphic map such that $\varphi_L^* \mathcal{O}_{\mathbb{P}^N}(1) \cong L|_{X \setminus Bs(L)}$.

Definition 15.4. The map φ_L in Proposition 15.3 is said to be associated to the *complete linear system* $H^0(X, L)$ (i.e. the sections of L are what we call a complete linear system) whereas a subspace of $H^0(X, L)$ is called a *linear system* of L .

Definition 15.5. L is *globally generated* by the sections s_0, \dots, s_N if $Bs(L, s_0, \dots, s_N) = \emptyset$.

Definition 15.6. A line bundle L on a complex manifold is called *ample* if for some $k > 0$ and some linear system in $H^0(X, L^k)$ the associated map φ is an embedding.

Exercise 15.7. Let L be an ample bundle on a K3 surface M . Prove that $L^{\otimes 2}$ is globally generated (that is, for each $x \in M$ there exists a section $h \in H^0(L^{\otimes 2})$ which does not vanish in x).

Proof. As an outline for exposition:

- (1) Show that L has sections (by ampleness+Kodaira Vanishing+Riemann-Roch), pick a section C and note that if $L^{\otimes 2}$ had base points they would be in C .
- (2) Show that restriction map is surjective (Kodaira Vanishing).
- (3) Show that inclusion map is not surjective. For this it's enough to show that $\dim |D| = \dim |D - p| + 1$. This follows by writing the formula of Riemann-Roch for both of these Weil divisors (have to pass to Weil divisors). But you will need to make sure that $h^0(K - D)$ and $h^0(K - (D - p))$ are zero.

First we need to show that L has sections. To see this I will show that since L is ample, $L^2 > 0$. Indeed, if mL is very ample we apply by Riemann-Roch, Kodaira Vanishing and the fact that $\chi(\mathcal{O}_X) = 2$ on a K3 to obtain $h^0(mL) = 2 + \frac{1}{2}m^2L^2$ which must be positive since mL has sections by being very ample. Notice L^2 cannot be zero as $h^0(mL) = 2$ would imply we have an embedding of a K3 surface into \mathbb{P}^1 .

Applying again Riemann-Roch and Kodaira vanishing we see that $h^0(L) > 0$. Let C be a section of L .

Notice that if $L^{\otimes 2}$ had any base points, they would have to be on C . Indeed, since C is the vanishing set of a section $s \in H^0(L)$, we must have $s \otimes s \in H^0(L^{\otimes 2})$ vanishing in any base point of $L^{\otimes 2}$.

Then we consider the ideal exact sequence for C and tensor by $L^{\otimes 2}$ to obtain

$$0 \longrightarrow L^{\otimes 2}(-C) \longrightarrow L^{\otimes 2} \longrightarrow L^{\otimes 2} \otimes \mathcal{O}_C \longrightarrow 0$$

Notice that the term on the left is actually L since the ideal sheaf $\mathcal{O}_M(-C) = \mathcal{I}_C$ is dual to L because C is defined as the vanishing set of a section of L . Passing to cohomology we obtain

$$H^0(L^{\otimes 2}) \longrightarrow H^0(L^{\otimes 2} \otimes \mathcal{O}_C) \longrightarrow H^1(L)$$

where the latter term vanishes by Kodaira Vanishing because L is ample. This says that every section of $L^{\otimes 2}|_C$ is the restriction to C of a section of $L^{\otimes 2}$. In turn this shows that it's enough to show that the bundle $L^{\otimes 2}$ has no base points along C .

By adjunction formula for (possibly singular) curves on smooth surfaces and by the fact that M is smooth, we see that $2p_a(C) - 2 = (L.L)$ so that

$$4p_a(C) - 4 = (2L.L) = \deg_C(L^{\otimes 2})$$

Notice that since $L^2 > 0$ we exclude the cases that $p_a(C) = 0, 1$.

To conclude pick a point $p \in C$. Saying that p is not a base point of $L^{\otimes 2}|_C$ is the same as saying that not every section of $L^{\otimes 2}|_C$ vanishes at p , that is, that $h^0(L^{\otimes 2}|_C(-p)) < h^0(L^{\otimes 2}|_C)$.

Now I will prove that since the degree of $L^{\otimes 2}|_C$ and $L^{\otimes 2}|_C(p)$ is greater than $2p_a(C)$ we have that $h^0(\omega_C \otimes (L^{\otimes 2}|_C)^\vee) = 0 = h^0(\omega_C \otimes (L^{\otimes 2}|_C(-p))^\vee)$. For this we need to see these line bundles as Weil divisors by simply taking sections, whose vanishing sets are finite sets of points with multiplicities. Then their degree is the sum of the multiplicities. This definition makes degree additive with respect to tensor product (we may take local sections and sum degrees). By Riemann-Roch on the curve C , we know that $\deg \omega_C = 2p_a - 2$. Then

$$\deg_C(\omega_C \otimes L^{\otimes 2}) = \deg_C(\omega_C) + \deg_C(L^{\otimes 2}) = 2p_a - 2 - (4p_a(C) - 4) < 0$$

and a similar computation works for $L^{\otimes 2}(-p)$ as it has the same degree of $L^{\otimes 2}$ minus 1. This implies that neither of these bundles can have sections since any section would provide a linearly equivalent effective divisor. This would be a contradiction since both $L^{\otimes 2}$ and $L^{\otimes 2}(-p)$ have sections. Therefore $h^0(\omega_C \otimes (L^{\otimes 2}|_C)^\vee) = 0 = h^0(\omega_C \otimes (L^{\otimes 2}|_C(-p))^\vee)$ as claimed.

Finally we apply Riemann-Roch to $L^{\otimes 2}$ and $L^{\otimes 2}(-p)$ to obtain that $h^0(L^{\otimes 2}) > h^0(L^{\otimes 2}(-p))$. \square

16. BERTINI'S THEOREM

17. SERRE DUALITY

Theorem 17.1 (Serre duality). [Voi02] II.5.32. *The pairing*

$$H^q(X, \mathcal{E}) \otimes H^{n-q}(X, \mathcal{E}^* \otimes K_X) \rightarrow H^n(X, K_X) \cong \mathbb{C}$$

is perfect.

So when you put the dual \vee on one of these you get isomorphism.

Our course version says:

$$H^k(X, \mathcal{L})^\vee = H^{n-k}(X, \omega_X \otimes \mathcal{L}^*)$$

18. KODAIRA VANISHING THEOREM

Theorem 18.1 (Kodaira Vanishing). *Suppose k is a field of characteristic 0, and X is a smooth projective k -variety. Then for any ample invertible sheaf L , $H^i(X, K_X \otimes L) = 0$ for $i > 0$.*

Proof. No proof in [Vak25]. \square

19. KODAIRA EMBEDDING THEOREM

The forward implication is easy and sometimes not considered as part of the theorem.

Compare this theorem with Nakai-Moishezon Criterion 28.1.

Theorem 19.1 (Kodaira Embedding). *Suppose M is a compact complex manifold. A holomorphic line bundle $L \rightarrow M$ is ample if and only if it is positive. Thus M is projective if and only if it admits a holomorphic line bundle.*

Proof. The easy implication as follows. If L is ample then there is N such that $L^{\otimes N}$ is very ample, meaning by Misha's definition that the canonical map is an embedding such that $L^{\otimes N} = \varphi^* \mathcal{O}(1)$, which has degree 1 by definition as in [Vak25, 15.4.14]. \square

20. NÉRON-SEVERI GROUP

Instead of considering the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{M}_X^*) \longrightarrow H^0(X, \mathcal{M}_X^* / \mathcal{O}_X^*) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow 0$$

which says that $\text{Pic}(X)$ is isomorphic to Cartier divisors modulo linear equivalence, we consider

$$0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}(X) / \text{Pic}^0(X) := \text{NS}(X) \longrightarrow 0$$

where $\text{Pic}^0(X)$ is the connected component of the identity in the Picard scheme, which is a scheme representing some functor. The idea is that $\text{Pic}^0(X)$ -equivalence is like linear equivalence, but different. The Néron-Severi group describes line bundles modulo this other equivalence.

21. HYPERCOMPLEX MANIFOLDS

Exercise 21.1. Let M be a compact hypercomplex manifold of real dimension 4, equipped with a quaternionic Hermitian structure, and V the space of closed $\text{SU}(2)$ -invariant 2-forms. Prove that V is finite-dimensional.

Proof by Arpan. Consider the Hodge star operator with respect to the Riemannian metric on M .

- (1) A closed anti-self-dual form is harmonic. Indeed, by some characterization a form is harmonic \iff it is d -closed and d^* -closed. So if α is self-dual and d -closed we get $d^* \alpha = (-1)^{-\bullet} * d * \alpha = 0$ since $d\alpha = 0$.
- (2) The space of harmonic forms is finite-dimensional (analysis, cf. Fredholm theory).
- (3) I should be able to prove that an $\text{SU}(2)$ -invariant closed form is anti self-dual.

- (a) Define $\Lambda^+ := \text{self-dual 2-forms}$ and $\Lambda^- := \text{anti-self dual 2-forms}$. Notice that $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, this is by 9.1, since we can see $*$ as an involutive endomorphism of Λ^2 , so that it has eigenvalues ± 1 , given the desired decomposition as its eigenspace decomposition.
- (b) **Claim.** $\Lambda^+ = \text{span}(\omega_I, \omega_J, \omega_K)$.
- (c) **Claim.** ω_I, ω_J and ω_K are $\text{SU}(2)$ -invariant.
The last two items imply that $\text{SU}(2)\Lambda^+ = \Lambda^+$.
- (d) **Claim.** $\text{SU}(2) \curvearrowright \Lambda^+$ has no fixed points.
We conclude that if α is $\text{SU}(2)$ -invariant (i.e. a fixed point of $\text{SU}(2) \curvearrowright \Lambda^2$) its positive part will vanish, so that α is anti-self-dual.

□

Proof. Idea: show that Hodge star $*\omega$ is in the $\text{SU}(2)$ -orbit of ω and conclude that $\int \omega \wedge *\omega = 0$, implying that $\|\omega\| = 0$. □

Exercise 21.2.

Proof. Idea: find a counterexample. The easiest should be a Kummer surface. It looks possible to find an almost hypercomplex structure on \mathbb{C}^2 passing to the torus and then to the \mathbb{Z}_2 quotient, but not clear what will happen after blowing up. □

22. DEFORMATION THEORY (INTRODUCTION)

Here I discuss deformations in the category of complex spaces.

Let's just do a summary of [Voi02, Chapter 9].

Let \mathcal{X} be a complex manifold, B a complex manifold and $\phi : \mathcal{X} \rightarrow B$ a holomorphic map. We call ϕ a *family* if it is a proper holomorphic submersion.

A theorem by Ehresmann says that any “deformation” of smooth real manifolds is trivial:

Theorem 22.1 (Ehresmann). *If $\phi : \mathcal{X} \rightarrow B$ is a proper submersion of differentiable manifolds (no complex structure), with B contractible and $0 \in B$ some distinguished point, then the total space \mathcal{X} is trivial as a manifold over B , i.e. there exists a diffeomorphism*

$$T : \mathcal{X} \xrightarrow{\cong} X_0 \times B$$

of manifolds over B , i.e. commuting with projection.

Proof. The main tool is the theorem of tubular neighbourhoods, which says that there is a diffeomorphism of a neighbourhood W of X_0 in \mathcal{X} and a neighbourhood of X_0 in its normal bundle. (We have a normal bundle because X_0 is a submanifold of \mathcal{X} .)

In particular this gives a differentiable retraction $T_0 : W \rightarrow X_0$, which allows us to consider the map

$$(T_0, \phi) : W \rightarrow X_0 \times B.$$

Since T_0 is a retraction (it's the identity when restricted to X_0) and ϕ is a submersion, the map (T_0, ϕ) has nonsingular differential along X_0 . Then we can invert it locally along X_0 , and since X_0 is compact we obtain a whole neighbourhood $W' \subset W \subset \mathcal{X}$ where the differential is nonzero and thus $(T_0, \phi)|_{W'}$ is an embedding (using the inverse function theorem).

Finally we restrict the image of $(T_0, \phi)|_{W'}$ to a neighbourhood $U \subset B$ of 0, using properness of ϕ to ensure that $\phi^{-1}(U) \subset W'$ — though I'm not exactly sure how.

We have constructed a diffeomorphism $\phi^{-1}(U) \rightarrow X_0 \times U$ commuting with the projection.

This proof was only local; the global version is in [Voi02] though it's not used in the rest of the text. \square

In the complex situation, if $\phi : \mathcal{X} \rightarrow B$ is a family of complex manifolds and $0 \in B$, then “up to replacing B by a neighbourhood of 0”, there exists a smooth trivialization $T = (T_0, \phi) : \mathcal{X} \rightarrow X_0 \times B$ such that the fibres of T_0 are complex submanifolds of \mathcal{X} .

Now let $\phi : \mathcal{X} \rightarrow B$ be a family of complex manifolds. Recall the pullback bundle ϕ^*B and the fact that the differential ϕ_* can be thought of as a gadget that maps tangent vectors to \mathcal{X} not to vectors tangent to B but to vectors in the pullback bundle ϕ^*B . Restricting everything to the fibre $X = \phi^{-1}(0)$ we get

$$(22.1.1) \quad 0 \longrightarrow T_X \longrightarrow T_{\mathcal{X}|X} \longrightarrow \phi^*T_{B|X} \longrightarrow 0.$$

There's what seems to be a really nice result saying that $H^1(\mathcal{F})$ parametrizes isomorphism classes of extensions of \mathcal{F} by the trivial bundle, but for now let me just define it roughly as follows. The category of \mathcal{O}_X is most surely abelian with enough injectives. This means that objects have associated complexes, and we can take cohomologies of these complexes. This says, roughly, that short exact sequences of vector bundles have associated cohomology long exact sequences. Then the Kodaira-Spencer map is the map

$$\rho : T_{B,0} = H^0(X, \phi^*T_{B|X}) \rightarrow H^1(X, T_X)$$

induced by the long exact sequence associated to 22.1.1. Here the equality $T_{B,0} = H^1(X, \phi^*T_{B|X})$ is just because the pullback of the tangent bundle of B to any fiber is trivial.

Now I will explain first order deformations. Consider a family $\mathcal{X} \rightarrow B$. We introduce a complex analytic subscheme B_ε of B giving an ideal sheaf \mathcal{M}_0^2 where \mathcal{M}_0 is the maximal ideal of $\mathcal{O}_{B,0}$ consisting of the holomorphic functions vanishing at 0. This means that a function in \mathcal{M}_0^2 is (a finite sum of terms) of the form $f = gh$ for $g, h \in \mathcal{M}_0$, which means that $df = d(gh) = h dg + g dh = 0$ at 0 since both g and h vanish at 0. That is, this sheaf contains functions that vanish at 0 and whose first derivative vanishes at 0.

This means that B_ε is a point: it's the vanishing locus of the set of functions that vanish at a single point. However, it is endowed with an unusual structure sheaf. Let v be a tangent vector to B at 0 and f a function in the strange sheaf $\mathcal{M}_0/\mathcal{M}_0^2$. Then by the computation above, since v is a differential operator, v annihilates f . This means that this sheaf detects not only the point 0 but also any tangent vector at 0. Let's denote $B_\varepsilon = \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)$ just because its structure sheaf is $\mathbb{C}[\varepsilon]/(\varepsilon^2)$.

In fact, this is so throughout deformation theory: $\text{Spec } k[\varepsilon]/(\varepsilon^2)$ is topologically a point! But it's a *fat point*: the support of its structure sheaf is a point, but as a scheme, it's not a point. And the same happens with the family \mathcal{X} over $\text{Spec } k[\varepsilon]/(\varepsilon^2)$, topologically it's just the fibre X ! But its structure sheaf is different: “it contains the directions in \mathcal{X} which are normal to X ”.

Perhaps the most important part of this construction comes when considering a fat point $B_\varepsilon \subset B$. This gives a first-order deformation $\phi^{-1}(B_\varepsilon) \subset \mathcal{X}$

$$\begin{array}{ccc} X & \longrightarrow & \phi^{-1}(B_\varepsilon) \subset \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & B_\varepsilon \subset B \end{array}$$

Using the constructions above, one shows that to any such deformation we can associate a cocycle in $H^1(X, T_X)$. Thus, first order deformations are “exactly parametrised” by the group $H^1(X, T_X)$. This is the main idea behind Kodaira-Spencer map, which is well-explained in Wikipedia.

I will present a similar construction to Kodaira-Spencer map that works for embedded deformations instead of just deformations. (That’s with the intention of understanding the forthcoming exercise Exercise 23.3.)

Since we are in complex space category I follow [Voi02, p. 224] for the main setup, but this is also a mixture of [?, p. 147, 150].

The result is in fact quite reasonable geometrically: the dimension of directions in which we can deform a submanifold should be non other than the dimension of the normal bundle.

23. DEFORMATION THEORY

Definition 23.1. Let X be a complex manifold. A *deformation* of X is a holomorphic submersion $\pi : \mathcal{X} \rightarrow B$ of complex manifolds such that $\pi^{-1}(0) \cong X$ for some $0 \in B$.

If \mathcal{X} and \mathcal{Y} are two deformations of X over the same base, a *morphism of deformations* is a map $f : \mathcal{X} \rightarrow \mathcal{Y}$ such that the diagram

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \searrow & & \swarrow \\ & B & \end{array}$$

commutes. The *space of deformations* of X is the set of deformations modulo the equivalence relation of two deformations being isomorphic. A priori we don’t know if this set has any structure.

Let Y be another complex manifold. A *deformation of X in Y* is a deformation of X such that $\mathcal{X} \subset Y \times B$.

The *fat point* is a ringed space $(B_\varepsilon, \mathbb{C}[\varepsilon]/(\varepsilon^2))$ whose underlying topological space B_ε is a point and structure sheaf is the ring of dual numbers $\mathbb{C}[\varepsilon]/(\varepsilon^2)$.

A *first order deformation* of X is a deformation such that the base B is a fat point.

The following proposition is a mixture of ideas from [Voi02, p. 224] and [Ser06, Examples 3.2.4(ii)].

Proposition 23.2. *Let X be a smooth hypersurface in a complex manifold Y . For every deformation of X in Y with base of dimension 1 we can construct a section of the bundle $N_{X/Y}$. Different deformations give different sections up to rescaling.*

Proof. Let

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \subset Y \times B \\ \downarrow & & \downarrow \pi \\ 0 & \longrightarrow & B \end{array}$$

be a deformation of X in Y and suppose that $\dim B = 1$. This means that \mathcal{X} is a complex submanifold of the product $Y \times B$ and $\pi : \mathcal{X} \rightarrow B$ is a holomorphic submersion.

Since X is a hypersurface, we can find an open cover V_i of M such that $V_i \cap X := U_i$ is given locally by the vanishing of a function $f_i \in \mathcal{O}_Y(V_i)$. In terms of coordinates we find that $V_i \cap X$ is given by $(z_1, \dots, z_{n-1}, 0)$.

Since π is a submersion, by the holomorphic inverse function theorem we obtain for every $V_i \subset \mathcal{X}$ (such that $V_i \cap X \neq \emptyset$) a biholomorphism from some $B_i \subset B$ containing 0 to V_i . This gives coordinates $(0, \dots, b)$ to V_i . We conclude that $V_i \cong U_i \times B_i$ along X .

Now we “restrict” to a first order deformation. Consider the fat point subscheme B_ε of B (I think this is the reason why B must have dimension 1, in Voisin notation, that a first order neighbourhood of 0 in B is indeed a fat point) and let $X_\varepsilon = \pi^{-1}(B_\varepsilon)$, which is a ringed space whose topological space is the same as X , but its structure sheaf is induced from B_ε .

(I think that to prove this part I have to look carefully at the induced sheaf $\pi^*(\mathbb{C}[\varepsilon]/(\varepsilon^2))$.) Since we are on a local trivialization, the structure sheaf of the product $U_i \times B_i$ is the tensor product of the structure sheaves (reference?), so that the structure sheaf of X_ε at $U_i \times B_\varepsilon$ is given by

$$\mathcal{O}_{X_\varepsilon}(U_i) = \mathcal{O}_Y(U_i) \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2).$$

Functions here look like finite sums of functions of the form $f + \varepsilon g$ for $f, g \in \Gamma(U_i, \mathcal{O}_{U_i})$.

The next key step is to argue that X_ε is locally cut by functions of the kind $f_i + \varepsilon g_i$ for some $g_i \in \mathcal{O}_M(U_i)$. Here the f_i are the same as those which define X . I think a proof is that from the ideal sheaf exact sequence of X in Y

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0$$

we obtain after tensoring by $\mathbb{C}[\varepsilon]/(\varepsilon^2)$

$$0 \longrightarrow \mathcal{I}_X \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathcal{O}_Y \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathcal{O}_X \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow 0.$$

(But is the functor $- \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)$ exact? Why? So far we didn’t impose a flatness condition on π .) The sheaf in the right coincides with the structure sheaf of X_ε when evaluated on a trivial open set of the form $U_i \otimes B_\varepsilon$. This shows that the ideal sheaf of X_ε as a ringed space is $\mathcal{I}_X \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Since \mathcal{I}_X is locally principal generated by f_i , we conclude that indeed X_ε must be cut locally by functions of the form $f_i + \varepsilon g_i$ for some $g_i \in \mathcal{O}_Y(U_i)$

Recall that the line bundle associated to X is $\mathcal{O}_Y(X)$, given by the cocycle $f_i/f_j := f_{ij}$. A version of adjunction formula (more precisely, [GH78, Adjunction formula I, p. 146]) establishes that this line bundle, when restricted to C , is in fact

the normal bundle of X in Y , defined as the quotient bundle $T'_Y/T'_X := N_{X/Y}$. In symbols, $\mathcal{O}_Y(X)|_X \cong N_{X/Y}$.

Similarly, the line bundle associated to \mathcal{X} is given by the cocycle

$$F_{ij} = f_{ij} + \varepsilon g_{ij}$$

for $g_{ij} = \frac{g_i - f_{ij}g_j}{f_j}$. Indeed, note that $\frac{1}{f_j + \varepsilon g_j} = \frac{1}{f_j} - \varepsilon \frac{g_j}{f_j^2}$ and compute $F_{ij} = (f_i + \varepsilon g_i)/(f_j + \varepsilon g_j)$.

Then

$$f_i + \varepsilon g_i = (f_{ij} + \varepsilon g_{ij})(f_j + \varepsilon g_j)$$

gives

$$g_i = f_{ij}g_j + g_{ij}f_j$$

and since f_j vanishes along X , we are left with $g_i = f_{ij}g_j$ which is how a section of the sheaf $N_{X/Y}$ is obtained. \square

Exercise 23.3. Let C be a smooth genus g curve which can be embedded in a K3 surface M , and \mathcal{X} “the family of all deformations” of C in M . Prove that $\dim \mathcal{X} \leq g$.

Proof. By Adjunction formula II (Theorem 13.3) and M being a K3 surface we get that $h^0(C, N_{C/M}) = g$.

If the deformation space X exists and is a complex manifold, the map constructed in Proposition 23.2 would be a holomorphic (why holomorphic?) injective map $X \rightarrow H^0(C, N_{C/Y})$. This obliges $\dim X \leq h^0(C, N_{C/Y}) = g$. Indeed, if not, the differential must have a kernel, and integrating along a direction in the kernel we obtain the germ of a complex curve along which the map is constant, which is impossible since it is injective. \square

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