MUMFORD-TATE GROUPS IN HODGE THEORY

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2. Introduction

$X \subset \mathbb{C}P^N$ smooth complex subvariety, $\dim_{\mathbb{C}} = n$. First recall we have singular cohomology, $H^k(X, \mathbb{C})$, which is isomorphic to the cohomology of the constant sheaf $\underline{\mathbb{C}}_X$. This cohomology is nonzero for $0 \leq k \leq 2n$.

Recall. $U \subset X$ open, $\Gamma(U, \mathbb{C}) = \{f : U \to \mathbb{C} : f \text{ is locally constant}\} = \prod_{\pi_0(U)} \mathbb{C}$.

Example 2.1. (1) $X = \mathbb{C}P^n$,

$$H^k(\mathbb{C}P^n, \mathbb{C}) = \begin{cases} \mathbb{C} & k = 2m, 0 \leq m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

A way to prove this is using the CW decomposition of $\mathbb{C}P^n$.

(2) $X \subset \mathbb{C}P^2$ hypersurface of degree d; X a Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$. Then $H^0(X, \mathbb{C}) = \mathbb{C}$, $H^1(X, \mathbb{C}) = \mathbb{C}^{2g}$, $H^2(X, \mathbb{C}) = \mathbb{C}$.

We also have the following additional data (a Hodge structure) on $H^k(X, \mathbb{C})$:

- A lattice $H^k(X, \mathbb{Z})/\text{torsion} \subset H^k(X, \mathbb{C})$,

- A (p,q)-decomposition, $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$.

Why is it useful?

- It gives restrictions on possible Betti numbers of algebraic varieties. (So it may tell us that certain complex variety cannot be algebraic, for example.)
- It $f : X \to Y$ is a morphism of algebraic varieties, then $f^* : H^k(Y,\mathbb{C}) \to H^k(X,\mathbb{C})$ preserves the Hodge structure.

As an example of the latter statement,

Example 2.2. Let $X \subset \mathbb{C}P^4$ be a "general" hypersurface of degree 5. Then there exists no abelian variety (i.e. a projective variety that is biholomorphic to $\mathbb{C}P^N/\Lambda$ where Λ is a lattice; so, a complex torus that is also a projective variety) A that admits a dominant birational map onto X $f : A \dashrightarrow X$.

3. The p,q decomposition

We use the de Rham complex. Let Ω^k_X be the sheaf of holomorphic k-forms. The de Rham complex is

$$\Omega^\bullet_{dR} = (0 \to \mathcal{O}_X \to \Omega^1_X \xrightarrow{d} \cdots \to \Omega^n_X \to 0)$$

$\Omega^\bullet_{dR}$ is a resolution of $\mathbb{C}_X$. Therefore $H^k(X,\mathbb{C}) \cong H^k(X, \Omega^\bullet_{dR}$.

Let's define a subcomplex:

$$F^p\Omega^\bullet_{dR} = (0 \to \cdots \to \Omega^p_X \to O^{p+1}_X \to \cdots \to \Omega^n_X \to 0)$$

It's a subcomplex $F^p\Omega^\bullet_{dr} \subset \Omega^\bullet_{dR}$ since the sheaves coincide when $F^p\Omega_{dR}$ are nonzero and they inject otherwise (because it's the zero sheaf).

This gives $H^k(X, F^p\Omega^\bullet_{dR}) \xrightarrow{(*)} H^k(X, \Omega^\bullet_{dR}) = H^k(X,\mathbb{C})$.

Definition 3.1. The *Hodge filtration* is $F^pH^k(X) = \mathrm{Im}(*)$.

Hodge theory tells us the map $(*)$ is in fact injective.

Let $\Lambda^{p,q}_X$ be the sheaf of C^∞-forms on X of type (p,q). They are given locally by $\sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ} dz_I \wedge d\bar{z}_J$, for $\alpha_{IJ} \in C^\infty_X$.

Then we have an acyclic resolution of Ω^p_X:

$$0 \to \Omega^p_X \hookrightarrow \Lambda^{p,0}_X \xrightarrow{\bar\partial} \Lambda^{p,1}_X \xrightarrow{\bar\partial} \cdots \to \Lambda^{p,n}_X \to 0.$$

$\Lambda^{\bullet;\bullet}_X$ is a fine resolution of $\Omega^\bullet_X$.

Then $H^k(X,\mathbb{C}) \cong H^k(X, \mathrm{Tot}\Lambda^{\bullet;\bullet}_X$, which can be computed by a spectral sequence. The first page of such spectral sequence is given by $E_1^{p,q} = H^q(X, \Omega^p_X)$. This converges to $H^k(X,\mathbb{C})$. This is the Hodge-to-de Rham spectral sequence.

Since our manifolds are projective they admit a Kähler metric ω induced by the inclusion $X \xrightarrow{i} \mathbb{C}P^N$, that is, $\omega = i^*(\text{Fubini-Study metric on } \mathbb{C}P^N)$.

Then $\Lambda^k_X = \bigoplus_{p+q=k} \Lambda^{p,q}_X$, $\Lambda^\bullet_X$ becomes an elliptic complex.

We have

$$\ldots \to \Lambda^{k-1}_X \xrightarrow{d} \Lambda^k_X \to \ldots, \qquad \ldots \to \Lambda^k_X \xrightarrow{d^*} \Lambda^{k-1}_X \to \ldots$$

where d^* is the adjoint of d w.r.t. ω. Then $\Lambda = dd^* + d^*d$ is an elliptic operator. $\mathcal{H}^k = \mathrm{Ker}(\Lambda|_{\Lambda^k_X}) = \mathrm{Ker}(d|_{\Lambda^k_X}) \cap \mathrm{Ker}(d^*|_{\Lambda^k_X})$ are the harmonic forms.

Consider a natural map from the harmonic forms of type (p,q) to the cohomology:

$$\mathcal{H}^{p,q} = \mathcal{H}^k \cap \Lambda_X^{p,q} \xrightarrow{(**)} H^q(X, \Omega_X^p).$$

Fact: since $d\omega = 0$ (X is Kähler), $(**)$ is an isomorphism.

This means that for Kähler manifolds

$$\dim H^k(X,\mathbb{C}) \leq \sum_{p+q=k} H^q(X,\Omega_X^p) \underbrace{\leq}_{(**)} \dim H^k(X,\mathbb{C})$$

This implies that the Hodge-to-de Rham spectral sequence degenerates at E_1. Therefore,

$$H^k(X,\mathbb{C}) = \mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong H^q(X,\Omega_X^p).$$

The Hodge filtration is

$$F^pH^k(X,\mathbb{C}) = \bigoplus_{\substack{p'+q'=k \\ p'\geq p}} H^{p',q'}(X).$$

4. Symmetries of the p,q decomposition

(1) Since $\overline{\Lambda_X^{p,q}} = \Lambda_X^{q,p}$, we have $\mathcal{H}^{q,p} = \overline{\mathcal{H}^{p,q}}$. This means that if $k \equiv 1 (\text{mod } 2)$, $H^k(X,\mathbb{C}) = \mathcal{H}^{k,0} \oplus \ldots \oplus \mathcal{H}^{\frac{k+1}{2},\frac{k-1}{2}} \oplus \mathcal{H}^{0,k} \oplus \ldots \oplus \mathcal{H}^{\frac{k-1}{2},\frac{k+1}{2}}$. This means that the k-th Betti number is even, $b_k(X) \equiv 0 \text{mod } 2$.

(2) (Poincare duality.) We have a perfect pairing

$$\begin{aligned} H^k(X,\mathbb{C}) \otimes H^{2n-k}(X,\mathbb{C}) &\longrightarrow \mathbb{C} \\ [\alpha]\otimes[\beta] &\longmapsto \int_X \alpha\wedge\beta \end{aligned}$$

Notice that if $\alpha \in \mathcal{H}^{p,q}$ then $\beta \in \mathcal{H}^{n-p,n-q}$. This induces a perfect pairing

$$\mathcal{H}^{p,q} \otimes \mathcal{H}^{n-p,n-q} \to \mathbb{C}.$$

(3) (Polarization and the Lefschetz operator.) The polarization is the Kähler class of the Kähler form. By X being projective we have that the Kähler class is integral. Moreover, it is the Poincar'e dual of the hyperplane section class. That is, let $h \in H^2(\mathbb{C}P^n,\mathbb{Z})$ be the class of a hyperplane, then $i^*h = [\omega] \in H^2(X,\mathbb{Z})$. We have $\omega \in \mathcal{H}^{1,1}$ and $[\omega] \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. The Lefschetz operator is

$$\begin{aligned} L_\omega : H^{p,q}(X) &\longrightarrow H^{p+1,q+1}(X) \\ [\alpha] &\longmapsto [\alpha\wedge\omega] = [\alpha]\cup[\omega]. \end{aligned}$$

Lefschetz theorem says

(a) $L_\omega^k : H^{n-k}(X,\mathbb{Q}) \xrightarrow{\sim} H^{n+k}(X,\mathbb{Q})$ is an isomorphism for $0 \leq k \leq n$

(b) The dual of L_ω is

$$\begin{aligned} \Lambda_\omega : H^{p,q}(X) &\longrightarrow H^{p-1,q-1}(X) \\ [\alpha] &\longmapsto [i_\alpha\omega]. \end{aligned}$$

$$\begin{aligned} [L_\omega,\Lambda_\omega] &= \Theta \in \text{End}(H^\bullet(X,\mathbb{Q})) \\ \Theta|_{H^k(X,\mathbb{Q})} &= (k-n)\text{Id} \end{aligned}$$

Then L_ω, Λ_ω and Θ span a subalgebra of $\mathrm{End}(H^\bullet(X,\mathbb{Q}))$ isomorphic to $\mathfrak{sl}_2$. This allows us to use what we know about the representation theory of $\mathfrak{sl}_2$. Let $H^k_{\mathrm{prim}}(X,\mathbb{Q}) = (\Lambda|_{H^k(X,\mathbb{Q})})$. Then $H^m(X,\mathbb{Q}) = \bigoplus_{i\geq 0} L^i_\omega H^{m-2i}_{\mathrm{prim}}(X,\mathbb{Q})$ for $0 < m \leq n$. (I think this corresponds to the usual weight space decomposition.)
[Picture of Hodge diamond. Reflection by vertical axis is complex conjugation, 180-degree rotation is Poincar'e duality, $p+q$ =constant is a horizontal line, reflection along horizontal axis is Lefschetz theorem. Warning! This depends on conventions of how we draw the diamond.]

5. The Hodge-Riemann relations

For all $[\alpha] \in H^k_{\mathrm{prim}}(X,\mathbb{C}) \cap H^{p,q}(X)$ we have

$$i^{p-q}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \overline{\alpha} \wedge \omega^{n-k} > 0.$$

(Here i is the imaginary unit.)

Define a pairing on $H^k_{\mathrm{prim}}(X,\mathbb{C})$:

$$\psi([\alpha],[\beta]) = (2\pi i)^{-k}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \beta \wedge \omega^{n-k}$$

which is $(-1)^k$-symmetric.

Definition 5.1. The *Weil operator* $C \in \mathrm{End}(H^k(X,\mathbb{C}))$ is given by $C|_{H^{p,q}}(X) = (i)^{p-q}\mathrm{Id}$.

Let $Q([\alpha],[\beta]) = (2\pi i)^k \psi(C[\alpha],[\beta])$.

Then Q is symmetric (exercise) and positive on $H^k(X,\mathbb{R})$. Positive is just the Hodge-Riemann relation.

6. Hodge structures and Mumford-Tate groups

Definition 6.1. A *rational Hodge structure* ($\mathbb{Q}$-HS) of weight $k \in \mathbb{Z}$ is a finite-dimensional $\mathbb{Q}$-vector space V and a decomposition $V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$ such that $V^{q,p} = \overline{V^{p,q}} \forall p,q$.

A $\mathbb{Q}$-HS (without fixed weight) is a $\mathbb{Q}$-vector space V with a decomposition $V = V_{k_1} \oplus \ldots \oplus V_{k_n}$ where V_{k_i} is a $\mathbb{Q}$-HS of weight k_i.

Analogously, define $\mathbb{Z}$-HS, $\mathbb{R}$-HS, etc (i.e. take V to be a finitely generated $\mathbb{Z}$-module, $\mathbb{R}$-vector space, etc.)

Example 6.2.
(1) $X \subset \mathbb{C}P^N$ smooth subvariety, then $H^k(X,\mathbb{Z})$ is a $\mathbb{Z}$-HS of weight k.
(2) The *Tate HS* is $\mathbb{Z}(1) := 2\pi i\mathbb{Z} = \mathrm{Ker}(\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$, since $\mathbb{C} = \mathbb{Z}(1) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{Z}(1)^{-1,1}$, so this is a $\mathbb{Q}$-HS of weight -2.

Note that $H^2(\mathbb{C}P^1,\mathbb{Z}) = \mathbb{Z}$ sits inside $H^2(\mathbb{C}P^1, \mathbb{C} = \mathbb{Z} \otimes \mathbb{C} \cong \mathbb{C} = H^{1,1}(\mathbb{C}P^1)$.

Analogously, $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$ and $\mathbb{Q}(1) \otimes \mathbb{C} = \mathbb{Q}(1)^{-1,-1}$ is of weight -2.

7. The Deligne torus

Definition 7.1. $\mathbb{S}$ is the algebraic group such that $\mathbb{S}(\mathbb{R})$ is the $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The group of real points $\mathbb{S}(\mathbb{R})$ is a real Lie group.

Note that

$$\begin{aligned}\mathbb{C}^* &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\}\\ &= \{(x,y,y) \in \mathbb{R}^2 : (x^2+y^2)t = 1\},\end{aligned}$$

which allows to see $\mathbb{C}^*$ as the vanishing locus of the polynomial $(x^2+y^2)t-1$, so to see it as an algebraic variety. Now

$$\begin{aligned}\mathbb{S}(\mathbb{C}) &= \{(x,y,t) \in \mathbb{C}^3 : (x^2+y^2)t = 1\}\\ &= \{(x,y,t) \in \mathbb{C}^3 : \underbrace{(x+iy)}_{z}\underbrace{(x-iy)}_{w}t = 1\}\\ &= \{(z,w,t) \in \mathbb{C}^3 : zwt = 1\}\\ &= \{(z,w) \in \mathbb{C}^2 : z \neq 0, w \neq 0\} \cong \mathbb{C}^* \times \mathbb{C}^*.\end{aligned}$$

Let V be a $\mathbb{Q}$-HS of weight k. Define a representation over $\mathbb{R}$ $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes \mathbb{R})$ as follows: $\forall z \in \mathbb{C}^* \forall v \in V^{p,q}$ $\rho(z)\cdot v = z^p \overline{z}^q v$ if $v \in V \otimes \mathbb{R}$, then $v = \sum_{p+q=k} v^{p,q}$, $v^{q,p} = \overline{v^{p,q}}$, and then $\overline{\rho \cdot v} = \sum_{p+q=k} z^p \overline{z}^q = \sum_{p+q=k} \overline{z}^p z^q v^{q,p} = \rho(z)\cdot v$, so it is in fact a representation. But why? Why is this equality what we need to make sure it is a representation?

Observe that $z \in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$, the eigenvalue is $\rho(z)\cdot v = z^{p+q}\cdot v = z^k \cdot v$. This motivates the following:

Conversely, given a $\mathbb{Q}$-vector space V and a representation $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ of $\mathbb{R}$-groups, such that $\forall r \in \mathbb{R}^*$ $\rho(r) = r^k \cdot \mathrm{Id}$, we have $V \otimes \mathbb{C} = \bigoplus_{p,q} V^{p,q}$ where $v \in V^{p,q}$, $\rho(z,w) = z^p w^q \cdot v$. Since ρ is a representation of $\mathbb{R}$-groups we have $V^{q,p} = \overline{V^{p,q}}$, so by the computation above we have $V^{p,q} = 0$ when $p+q \neq k$, then V becomes a $\mathbb{Q}$ =HS of weight k.

In conclusion, a $\mathbb{Q}$-HS is the same thing as a $\mathbb{Q}$-vector space V and a representation of $\mathbb{R}$-groups $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ such that $\rho|_{\mathbb{R}^*}$ is defined over $\mathbb{Q}$.

(We may also replace $\mathbb{Q}$ with $\mathbb{Z}$, etc. in this construction.)

Back to the Tate group $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$, define for all $z \in \mathbb{C}^*$ and $v \in \mathbb{Q}(1) \otimes \mathbb{C}$ by $\rho(z)\cdot v = |z|^{-2}\cdot v$.

We have operations on HS: $V_1 \oplus V_2, V_1 \otimes V_2, \mathrm{Hom}(V_1, V_2)$. The $\mathbb{Q}$-HS form an abelian category. Let $\mathbb{Q}(m) = \mathbb{Q}(1)^{\otimes m}$ when $m \geq 0$, $\mathbb{Q}(-1) = \mathbb{Q}(1)^*$ and $\mathbb{Q}(-m) = \mathbb{Q}(-1)^{\otimes m}$ for $m \geq 0$.

If V is a $\mathbb{Q}$-HS, then $V(m) = V \otimes \mathbb{Q}(m)$ is the *Tate twist*.

Example 7.2. (1) $X \subset \mathbb{C}P^n$ subvariety. Notice that while $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, such a closure can be obtained by choosing the imaginary unit i or $-i$, so this is not canonical.

However, the first Chern class is canonical:

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0$$

gives the connecting homomorphism $H^1(X,\mathbb{C}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}(1))$ which is in fact a HS of weight 0 (because H^2 has weight 2 and $\mathbb{Z}(1)$ has weight -2).

Analogously, the p-th chern class is $c_p(A \text{ coh. sheaf}) \in H^{2p}(X,\mathbb{Q}(p))$.

(2) $cl : CH^p_{\mathbb{Q}}(X) \to H^{2p}(X, \mathbb{Q})$ the cycle class map, where

$$CH^p_{\mathbb{Q}}(X) = \left\{ \sum_{\substack{Z_i \subset X \\ \text{subvar.}}} \alpha_i [Z_i] : \alpha_i \in \mathbb{Q} \right\} \Big/ \text{rational equiv.}$$

Consider

$$\begin{tikzcd} Z \arrow[r, hook, "i"] & X \\ \tilde{Z} \arrow[u] \arrow[ur, "j"'] & \end{tikzcd}$$

where Z is a subvariety of codimension p and $\tilde{Z}$ is a resolution of singularities. Then we have the pushforward of the fundamental class, $j_*[\tilde{Z}] \in H_{2n-2p}(X, \mathbb{Q})$ where $[\tilde{Z}] \in H_{2n-2p}(\tilde{Z}, \mathbb{Q}) \cong H^{2n-2p}(\tilde{Z}, \mathbb{Q})^*$. Then the Poincare dual of $j_*[\tilde{Z}] := d[Z] \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$. So $H_{2n-2p}(\tilde{Z}, \mathbb{Q})$ is a HS of weight $2(p-n)$ and $[\tilde{Z}] \in H_{p-n,p-n}$.

Definition 7.3. The *space of Hodge classes* is $H^{2p}_{\mathrm{Hdg}}(X) = H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$

Then we have $cl : CH^p_{\mathbb{Q}}(X) \to H^{2p}_{\mathrm{Hdg}}(X) \cong \mathrm{Hom}_{\mathbb{Q}-HS}(\mathbb{Q}(-p), H^{2p}(X, \mathbb{Q}))$. The Hodge conjecture is that cl is surjective onto the space of Hodge classes.

8. Polarizations

Let V be a $\mathbb{Q}$-HS of weight k and $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ the corresponding representation of the Deligne torus. The Weil operator in terms of ρ is $C = \rho(i)$.

Definition 8.1. A *polarization* on V is a morphism of $\mathbb{Q}$-HS is a $(-1)^k$-symmetric morphism $\psi : V \otimes V \to \mathbb{Q}(-k)$ such that the bilinear form $Q : V_{\mathbb{R}} \otimes V_{\mathbb{R}} \to \mathbb{R}$ given by $Q(x, y) = (2\pi i)^k \psi(Cx, y)$ is symmetric and positive definite.

Example 8.2. $V = H^k_{\mathrm{prim}}(X, \mathbb{Q})$ with ψ the Hodge-Riemann pairing.

Observe: if V is polarizable (i.e. it admits a polarization; not every HS admits a polarization e.g. HS on nonprojetive varieties) then V is semisimple $V = V_1 \oplus \ldots \oplus V_m$, V_i is simple.

Assume $W \subset V$ is a sub-HS, then $W^{\perp_\psi} \subset V$ is a sub-HS. Then $0 \underbrace{=}_{\substack{Q \\ \text{positive} \\ \text{def.}}} W \cap W^{\perp_\psi} \subset V$, Then $V = W \oplus W^{\perp_\psi}$.

9. The Mumford-Tate group

Let V be a $\mathbb{Q}$-HS and $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes \mathbb{R})$ a representation of $\mathbb{R}$-groups.

Consider $G \subset \mathrm{GL}(V)$ such that $\mathrm{Im}(\rho) \subset G(\mathbb{R})$.

(The idea is that the Mumford-Tate group will recover the rational structure of V, because the representation ρ knows nothing about this rational structure, which must exist since V is a rational vector space.)

Definition 9.1. The *Mumford-Tate group* is

$$MT(V) = \bigcap_{\substack{G \subset \mathrm{GL}(V) \\ \text{subgroup s.t.} \\ \mathrm{Im}(\rho) \subset G(\mathbb{R})}} G = \text{smallest } Q\text{-subgroup of } \mathrm{GL}(V) \text{ containing } \mathrm{Im}\rho.$$

Remark 9.2. We can also consider $U(1) \subset \mathbb{S}(\mathbb{R})$ and $\rho' := \rho|_{U(1)} : U(1) \to \mathrm{GL}(V \otimes \mathbb{R})$. Then the *Hodge group* is $\mathrm{Hdg}(V)$ = smallest Q-subgroup of $\mathrm{GL}(V)$ such that $\mathrm{Im}(\rho') \subset \mathrm{Hdg}(V)(\mathbb{R})$.

$\mathrm{Hdg}(V)$ is always smaller than $MT(V)$.

Example 9.3.
(1) $MT(\mathbb{Q}(1)) = \mathbb{Q}^*$ $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$, $\mathbb{Q}(1) \otimes \mathbb{R} = 2\pi i\mathbb{R} \subset \mathbb{C}$, $\mathrm{GL}(\mathbb{Q}(1) \otimes \mathbb{R}) = \mathbb{R}^*$, since $z \in \mathbb{S}(\mathbb{R})$ acts as $|z|^{-2} \cdot \mathrm{Id}$.
(2) $\mathbb{Q}(0) = \mathbb{Q} \subset \mathbb{C}$, $MT(\mathbb{Q}(0)) = \{1\}$. In general, if V is of weight k, then $\mathbb{Q}^*$ = center of $\mathrm{GL}(V) \subset MT(V)$ and $MT(V)$ is generated by $\mathbb{Q}^*$ and $Hdg(V)$.

10. Tensor construction

Let V be a $\mathbb{Q}$-HS with $MT(V) \subset \mathrm{GL}(V)$ and $\rho : \mathbb{S}(\mathbb{R}) \to MT(V)(\mathbb{R})$. Then

$$T^\bullet(V) = \bigoplus_{e,f\geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$$

is a $MT(V)$-representation.

Proposition 10.1. *A finite-dimensional subspace $W \subset T^\bullet(V)$ is a sub-HS if and only if W is a $MT(V)$-subspace.*

Proof. ($\Longleftarrow$). If W is a $MT(V)$-subrepresentation, then from ρ we can compose with the representation that $MT(V)$ is to obtain $\rho' : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(W \otimes \mathbb{R})$. Note that $\rho|_{\mathbb{R}^*}$ is defined over Q, We get that W is a sub-HS.

($\Longrightarrow$). Assume that $W \subset T^\bullet(V)$ is a sub-HS, $G - \mathrm{Stab}(W) \subset \mathrm{GL}(V)$ is a $\mathbb{Q}$-subgroup. Since W is a sub-HS, the action of $\mathbb{S}(\mathbb{R})$ on $T^\bullet(V)$ preserves W, then $\mathrm{Im}(\rho) \subset G(\mathbb{R})$ and thus $MT(V) \subset G$, which implies that W is a $MT(V)$-subrepresentation. $\square$

As a corollary,

Lemma 10.2. *$x \in T^\bullet(V)$ is $MT(V)$-invariant if and only if X is a $(0,0)$ Hodge element.*

Proposition 10.3. *Assume that V is a $\mathbb{Q}$-HS of weight k and $\psi : V \otimes V \to \mathbb{Q}(-k)$ is a polarization. Then*

$$Hdg(V) \subset \begin{cases} SO(V,\psi) & \text{if } k \equiv 0 (mod\ 2) \\ Sp(V,\psi) & \text{if } k \equiv 1 (mod\ 2) \end{cases}$$

Proof. $\psi : V \otimes V \to \mathbb{Q}(-k)$, where the $\mathbb{Q}(-k)$ is a trivial $Hdg(V)$-module, so that ψ is a morphism of $Hdg(V)$-modules. The action of $Hdg(V)$ preserves the symmetric form ψ if $k \equiv 0(\mathrm{mod}\ 2)$ and the antisymmetric form ψ if $k \equiv 1(\mathrm{mod}\ 2)$. $\square$

Example 10.4. Let $V = H^1(E, \mathbb{Q})$ where E is an elliptic curve and ψ the Hodge-Riemann pairing. Then $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ and $\dim V^{1,0} = 1$. $Hdg(V) \subset Sp(V,\psi) \cong \mathrm{SL}_2(\mathbb{Q})$.

Observe thati, in general, if V is polarizable, then $V = V_1 \oplus \ldots \oplus V_m$, V_i are irreducible $MT(V)$-representations, so $MT(V)$ is reductive.

Back to our elliptic curve example, $Hdg(V) \subset \mathrm{SL}_2(\mathbb{Q})$ is reductive.

There are two possibilities:

(1) $Hdg(V) = \mathrm{SL}^2(\mathbb{Q})$, which implies that $MT(V) = \mathrm{GL}_2(V)$. This happens when E is generic in the moduli space.

(2) $Hdg(V)$ is properly contained in $\mathrm{SL}_2(\mathbb{Q})$. Then $Hdg(V)$ is a 1-dimensional torus. Then $\mathrm{End}_{\mathbb{Q}-HS}(V) \neq \mathbb{Q}$, which implies that E has complex multiplication.

Definition 10.5. A HS V is of *CM-type* if $MT(V)$ is abelian (such HS defines a special point in the moduli space of HS.)