MUMFORD-TATE GROUPS IN HODGE THEORY

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Upshot. The Mumford-Tate group is the stabilizer of the Hodge classes, and remains invariant on the generic fibres of deformations.

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2. Introduction

$X \subset \mathbb{C}P^N$ smooth complex subvariety, $\dim_{\mathbb{C}} = n$. First recall we have singular cohomology, $H^k(X, \mathbb{C})$, which is isomorphic to the cohomology of the constant sheaf $\underline{\mathbb{C}}_X$. This cohomology is nonzero for $0 \leq k \leq 2n$.

Recall. $U \subset X$ open, $\Gamma(U, \mathbb{C}) = \{f : U \to \mathbb{C} : f \text{ is locally constant}\} = \prod_{\pi_0(U)} \mathbb{C}$.

Example 2.1. (1) $X = \mathbb{C}P^n$,

$$H^k(\mathbb{C}P^n, \mathbb{C}) = \begin{cases} \mathbb{C} & k = 2m, 0 \le m \le n \\ 0 & \text{otherwise.} \end{cases}$$

A way to prove this is using the CW decomposition of $\mathbb{C}P^n$.

(2) $X \subset \mathbb{C}P^2$ hypersurface of degree d; X a Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$. Then $H^0(X, \mathbb{C}) = \mathbb{C}$, $H^1(X, \mathbb{C}) = \mathbb{C}^{2g}$, $H^2(X, \mathbb{C}) = \mathbb{C}$.

We also have the following additional data (a Hodge structure) on $H^k(X, \mathbb{C})$:

- A lattice $H^k(X, \mathbb{Z})/\text{torsion} \subset H^k(X, \mathbb{C})$,
- A (p, q)-decomposition, $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$.

Why is it useful?

- It gives restrictions on possible Betti numbers of algebraic varieties. (So it may tell us that certain complex variety cannot be algebraic, for example.)
- It $f : X \to Y$ is a morphism of algebraic varieties, then $f^* : H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ preserves the Hodge structure.

As an example of the latter statement,

Example 2.2. Let $X \subset \mathbb{C}P^4$ be a "general" hypersurface of degree 5. Then there exists no abelian variety (i.e. a projective variety that is biholomorphic to $\mathbb{C}P^N/\Lambda$ where Λ is a lattice; so, a complex torus that is also a projective variety) A that admits a dominant birational map onto X $f : A \dashrightarrow X$.

3. The p,q decomposition

We use the de Rham complex. Let Ω^k_X be the sheaf of holomorphic k-forms. The de Rham complex is

$$\Omega^\bullet_{dR} = (0 \to \mathcal{O}_X \to \Omega^1_X \xrightarrow{d} \cdots \to \Omega^n_X \to 0)$$

$\Omega^\bullet_{dR}$ is a resolution of $\mathbb{C}_X$. Therefore $H^k(X, \mathbb{C}) \cong H^k(X, \Omega^\bullet_{dR}$.

Let's define a subcomplex:

$$F^p\Omega^\bullet_{dR} = (0 \to \cdots \to \Omega^p_X \to O^{p+1}_X \to \cdots \to \Omega^n_X \to 0)$$

It's a subcomplex $F^p\Omega^\bullet_{dr} \subset \Omega^\bullet_{dR}$ since the sheaves coincide when $F^p\Omega_{dR}$ are nonzero and they inject otherwise (because it's the zero sheaf).

This gives $H^k(X, F^p\Omega^\bullet_{dR}) \xrightarrow{(*)} H^k(X, \Omega^\bullet_{dR}) = H^k(X, \mathbb{C})$.

Definition 3.1. The *Hodge filtration* is $F^pH^k(X) = \text{Im}(*)$.

Hodge theory tells us the map $(*)$ is in fact injective.

Let $\Lambda^{p,q}_X$ be the sheaf of C^∞-forms on X of type (p, q). They are given locally by $\sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ} dz_I \wedge d\bar{z}_J$, for $\alpha_{IJ} \in C^\infty_X$.

Then we have an acyclic resolution of Ω^p_X:

$$0 \to \Omega^p_X \hookrightarrow \Lambda^{p,0}_X \xrightarrow{\bar{\partial}} \Lambda^{p,1}_X \xrightarrow{\bar{\partial}} \cdots \to \Lambda^{p,n}_X \to 0.$$

$\Lambda^{\bullet,\bullet}_X$ is a fine resolution of $\Omega^\bullet_X$.

Then $H^k(X,\mathbb{C}) \cong H^k(X, \mathrm{Tot}\Lambda_X^{\bullet;\bullet}$, which can be computed by a spectral sequence. The first page of such spectral sequence is given by $E_1^{p,q} = H^q(X, \Omega_X^p)$. This converges to $H^k(X,\mathbb{C})$. This is the Hodge-to-de Rham spectral sequence.

Since our manifolds are projective they admit a Kähler metric ω induced by the inclusion $X \xrightarrow{i} \mathbb{C}P^N$, that is, $\omega = i^*(\text{Fubini-Study metric on } \mathbb{C}P^N)$.

Then $\Lambda_X^k = \bigoplus_{p+q=k} \Lambda_X^{p,q}$, $\Lambda_X^{\bullet}$ becomes an elliptic complex.

We have

$$\ldots \to \Lambda_X^{k-1} \xrightarrow{d} \Lambda_X^k \to \ldots, \qquad \ldots \to \Lambda_X^k \xrightarrow{d^*} \Lambda_X^{k-1} \to \ldots$$

where d^* is the adjoint of d w.r.t. ω. Then $\Lambda = dd^* + d^*d$ is an elliptic operator. $\mathcal{H}^k = \mathrm{Ker}(\Lambda|_{\Lambda_X^k}) = \mathrm{Ker}(d|_{\Lambda_X^k}) \cap \mathrm{Ker}(d^*|_{\Lambda_X^k})$ are the harmonic forms.

Consider a natural map from the harmonic forms of type (p,q) to the cohomology:

$$\mathcal{H}^{p,q} = \mathcal{H}^k \cap \Lambda_X^{p,q} \xrightarrow{(**)} H^q(X, \Omega_X^p).$$

Fact: since $d\omega = 0$ (X is Kähler), $(**)$ is an isomorphism.

This means that for Kähler manifolds

$$\dim H^k(X,\mathbb{C}) \leq \sum_{p+q=k} H^q(X,\Omega_X^p) \underbrace{\leq}_{(**)} \dim H^k(X,\mathbb{C})$$

This implies that the Hodge-to-de Rham spectral sequence degenerates at E_1. Therefore,

$$H^k(X,\mathbb{C}) = \mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong H^q(X,\Omega_X^p).$$

The Hodge filtration is

$$F^p H^k(X,\mathbb{C}) = \bigoplus_{\substack{p'+q'=k \\ p' \geq p}} H^{p',q'}(X).$$

4. Symmetries of the p,q decomposition

(1) Since $\overline{\Lambda_X^{p,q}} = \Lambda_X^{q,p}$, we have $\mathcal{H}^{q,p} = \overline{\mathcal{H}^{p,q}}$. This means that if $k \equiv 1 (\mathrm{mod}\ 2)$, $H^k(X,\mathbb{C}) = \mathcal{H}^{k,0} \oplus \ldots \oplus \mathcal{H}^{\frac{k+1}{2},\frac{k-1}{2}} \oplus \mathcal{H}^{0,k} \oplus \ldots \oplus \mathcal{H}^{\frac{k-1}{2},\frac{k+1}{2}}$. This means that the k-th Betti number is even, $b_k(X) \equiv 0 \mathrm{mod}\ 2$.

(2) (Poincare duality.) We have a perfect pairing

$$\begin{aligned} H^k(X,\mathbb{C}) \otimes H^{2n-k}(X,\mathbb{C}) &\longrightarrow \mathbb{C} \\ [\alpha] \otimes [\beta] &\longmapsto \int_X \alpha \wedge \beta \end{aligned}$$

Notice that if $\alpha \in \mathcal{H}^{p,q}$ then $\beta \in \mathcal{H}^{n-p,n-q}$. This induces a perfect pairing

$$\mathcal{H}^{p,q} \otimes \mathcal{H}^{n-p,n-q} \to \mathbb{C}.$$

(3) (Polarization and the Lefschetz operator.) The polarization is the Kähler class of the Kähler form. By X being projective we have that the Kähler class is integral. Moreover, it is the Poincar'e dual of the hyperplane section class. That is, let $h \in H^2(\mathbb{C}P^n, \mathbb{Z})$ be the class of a hyperplane, then

$i^*h = [\omega] \in H^2(X,\mathbb{Z})$. We have $\omega \in \mathcal{H}^{1,1}$ and $[\omega] \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. The Lefschetz operator is

$$\begin{aligned} L_\omega : H^{p,q}(X) &\longrightarrow H^{p+1,q+1}(X) \\ [\alpha] &\longmapsto [\alpha \wedge \omega] = [\alpha] \cup [\omega]. \end{aligned}$$

Lefschetz theorem says
(a) $L_\omega^k : H^{n-k}(X,\mathbb{Q}) \xrightarrow{\sim} H^{n+k}(X,\mathbb{Q})$ is an isomorphism for $0 \le k \le n$
(b) The dual of L_ω is

$$\begin{aligned} \Lambda_\omega : H^{p,q}(X) &\longrightarrow H^{p-1,q-1}(X) \\ [\alpha] &\longmapsto [i_\alpha \omega]. \end{aligned}$$

$$[L_\omega, \Lambda_\omega] = \Theta \in \mathrm{End}(H^\bullet(X,\mathbb{Q}))$$

$$\Theta|_{H^k(X,\mathbb{Q})} = (k-n)\mathrm{Id}$$

Then L_ω, Λ_ω and Θ span a subalgebra of $\mathrm{End}(H^\bullet(X,\mathbb{Q}))$ isomorphic to $\mathfrak{sl}_2$. This allows us to use what we know about the representation theory of $\mathfrak{sl}_2$. Let $H^k_{\mathrm{prim}}(X,\mathbb{Q}) = (\Lambda|_{H^k(X,\mathbb{Q})})$. Then $H^m(X,\mathbb{Q}) = \bigoplus_{i\ge 0} L^i_\omega H^{m-2i}_{\mathrm{prim}}(X,\mathbb{Q})$ for $0 < m \le n$. (I think this corresponds to the usual weight space decomposition.)
[Picture of Hodge diamond. Reflection by vertical axis is complex conjugation, 180-degree rotation is Poincar'e duality, $p+q$ =constant is a horizontal line, reflection along horizontal axis is Lefschetz theorem. Warning! This depends on conventions of how we draw the diamond.]

5. The Hodge-Riemann relations

For all $[\alpha] \in H^k_{\mathrm{prim}}(X,\mathbb{C}) \cap H^{p,q}(X)$ we have

$$i^{p-q}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \overline{\alpha} \wedge \omega^{n-k} > 0.$$

(Here i is the imaginary unit.)
Define a pairing on $H^k_{\mathrm{prim}}(X,\mathbb{C})$:

$$\psi([\alpha],[\beta]) = (2\pi i)^{-k}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \beta \wedge \omega^{n-k}$$

which is $(-1)^k$-symmetric.

Definition 5.1. The *Weil operator* $C \in \mathrm{End}(H^k(X,\mathbb{C}))$ is given by $C|_{H^{p,q}}(X) = (i)^{p-q}\mathrm{Id}$.

Let $Q([\alpha],[\beta]) = (2\pi i)^k \psi(C[\alpha],[\beta])$.
Then Q is symmetric (exercise) and positive on $H^k(X,\mathbb{R})$. Positive is just the Hodge-Riemann relation.

6. Hodge structures and Mumford-Tate groups

Definition 6.1. A *rational Hodge structure* ($\mathbb{Q}$-HS) of weight $k \in \mathbb{Z}$ is a finite-dimensional $\mathbb{Q}$-vector space V and a decomposition $V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$ such that $V^{q,p} = \overline{V^{p,q}} \forall p, q$.

A $\mathbb{Q}$-HS (without fixed weight) is a $\mathbb{Q}$-vector space V with a decomposition $V = V_{k_1} \oplus \dots \oplus V_{k_n}$ where V_{k_i} is a $\mathbb{Q}$-HS of weight k_i.

Analogously, define $\mathbb{Z}$-HS, $\mathbb{R}$-HS, etc (i.e. take V to be a finitely generated $\mathbb{Z}$-module, $\mathbb{R}$-vector space, etc.)

Example 6.2.
(1) $X \subset \mathbb{C}P^N$ smooth subvariety, then $H^k(X, \mathbb{Z})$ is a $\mathbb{Z}$-HS of weight k.
(2) The *Tate HS* is $\mathbb{Z}(1) := 2\pi i\mathbb{Z} = \mathrm{Ker}(\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$, since $\mathbb{C} = \mathbb{Z}(1) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{Z}(1)^{-1,1}$, so this is a $\mathbb{Q}$-HS of weight -2.
Note that $H^2(\mathbb{C}P^1, \mathbb{Z}) = \mathbb{Z}$ sits inside $H^2(\mathbb{C}P^1, \mathbb{C} = \mathbb{Z} \otimes \mathbb{C} \cong \mathbb{C} = H^{1,1}(\mathbb{C}P^1)$.
Analogously, $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$ and $\mathbb{Q}(1) \otimes \mathbb{C} = \mathbb{Q}(1)^{-1,-1}$ is of weight -2.

7. The Deligne torus

Definition 7.1. $\mathbb{S}$ is the algebraic group such that $\mathbb{S}(\mathbb{R})$ is the $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The group of real points $\mathbb{S}(\mathbb{R})$ is a real Lie group.

Note that

$$\begin{aligned}\mathbb{C}^* &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 0\} \\ &= \{(x, y, y) \in \mathbb{R}^2 : (x^2 + y^2)t = 1\},\end{aligned}$$

which allows to see $\mathbb{C}^*$ as the vanishing locus of the polynomial $(x^2 + y^2)t - 1$, so to see it as an algebraic variety. Now

$$\begin{aligned}\mathbb{S}(\mathbb{C}) &= \{(x, y, t) \in \mathbb{C}^3 : (x^2 + y^2)t = 1\} \\ &= \{(x, y, t) \in \mathbb{C}^3 : \underbrace{(x + iy)}_{z}\underbrace{(x - iy)}_{w}t = 1\} \\ &= \{(z, w, t) \in \mathbb{C}^3 : zwt = 1\} \\ &= \{(z, w) \in \mathbb{C}^2 : z \neq 0, w \neq 0\} \cong \mathbb{C}^* \times \mathbb{C}^*.\end{aligned}$$

Let V be a $\mathbb{Q}$-HS of weight k. Define a representation over $\mathbb{R}$ $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes \mathbb{R})$ as follows: $\forall z \in \mathbb{C}^* \forall v \in V^{p,q}$ $\rho(z) \cdot v = z^p \overline{z}^q v$ if $v \in V \otimes \mathbb{R}$, then $v = \sum_{p+q=k} v^{p,q}$, $v^{q,p} = \overline{v^{p,q}}$, and then $\overline{\rho \cdot v} = \sum_{p+q=k} z^p \overline{z}^q = \sum_{p+q=k} \overline{z}^p z^q v^{q,p} = \rho(z) \cdot v$, so it is in fact a representation. But why? Why is this equality what we need to make sure it is a representation?

Observe that $z \in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$, the eigenvalue is $\rho(z) \cdot v = z^{p+q} \cdot v = z^k \cdot v$. This motivates the following:

Conversely, given a $\mathbb{Q}$-vector space V and a representation $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ of $\mathbb{R}$-groups, such that $\forall r \in \mathbb{R}^*$ $\rho(r) = r^k \cdot \mathrm{Id}$, we have $V \otimes \mathbb{C} = \bigoplus_{p,q} V^{p,q}$ where $v \in V^{p,q}$, $\rho(z, w) = z^p w^q \cdot v$. Since ρ is a representation of $\mathbb{R}$-groups we have $V^{q,p} = \overline{V^{p,q}}$, so by the computation above we have $V^{p,q} = 0$ when $p + q \neq k$, then V becomes a $\mathbb{Q}$ =HS of weight k.

In conclusion, a $\mathbb{Q}$-HS is the same thing as a $\mathbb{Q}$-vector space V and a representation of $\mathbb{R}$-groups $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ such that $\rho|_{\mathbb{R}^*}$ is defined over $\mathbb{Q}$.

(We may also replace $\mathbb{Q}$ with $\mathbb{Z}$, etc. in this construction.)

Back to the Tate group $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$, define for all $z \in \mathbb{C}^*$ and $v \in \mathbb{Q}(1) \otimes \mathbb{C}$ by $\rho(z) \cdot v = |z|^{-2} \cdot v$.

We have operations on HS: $V_1 \oplus V_2, V_1 \otimes V_2, \mathrm{Hom}(V_1, V_2)$. The $\mathbb{Q}$-HS form an abelian category. Let $\mathbb{Q}(m) = \mathbb{Q}(1)^{\otimes m}$ when $m \geq 0$, $\mathbb{Q}(-1) = \mathbb{Q}(1)^*$ and $\mathbb{Q}(-m) = \mathbb{Q}(-1)^{\otimes m}$ for $m \geq 0$.

If V is a $\mathbb{Q}$-HS, then $V(m) = V \otimes \mathbb{Q}(m)$ is the *Tate twist*.

Example 7.2. (1) $X \subset \mathbb{C}P^n$ subvariety. Notice that while $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, such a closure can be obtained by choosing the imaginary unit i or $-i$, so this is not canonical.

However, the first Chern class is canonical:

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0$$

gives the connecting homomorphism $H^1(X, \mathbb{C}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}(1))$ which is in fact a HS of weight 0 (because H^2 has weight 2 and $\mathbb{Z}(1)$ has weight -2).

Analogously, the p-th chern class is $c_p(A \text{ coh. sheaf}) \in H^{2p}(X, \mathbb{Q}(p))$.

(2) $cl : CH^p_{\mathbb{Q}}(X) \to H^{2p}(X, \mathbb{Q})$ the cycle class map, where

$$CH^p_{\mathbb{Q}}(X) = \left\{ \sum_{\substack{Z_i \subset X \\ \text{subvar.}}} \alpha_i [Z_i] : \alpha_i \in \mathbb{Q} \right\} \Big/ \text{rational equiv.}$$

Consider

$$\begin{tikzcd} Z \arrow[r, hook, "i"] & X \\ \tilde{Z} \arrow[u] \arrow[ur, "j"'] & \end{tikzcd}$$

where Z is a subvariety of codimension p and $\tilde{Z}$ is a resolution of singularities. Then we have the pushforward of the fundamental class, $j_*[\tilde{Z}] \in H_{2n-2p}(X, \mathbb{Q})$ where $[\tilde{Z}] \in H_{2n-2p}(\tilde{Z}, \mathbb{Q}) \cong H^{2n-2p}(\tilde{Z}, \mathbb{Q})^*$. Then the Poincare dual of $j_*[\tilde{Z}] := d[Z] \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$. So $H_{2n-2p}(\tilde{Z}, \mathbb{Q})$ is a HS of weight $2(p-n)$ and $[\tilde{Z}] \in H_{p-n,p-n}$.

Definition 7.3. The *space of Hodge classes* is $H^{2p}_{\mathrm{Hdg}}(X) = H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$

Then we have $cl : CH^p_{\mathbb{Q}}(X) \to H^{2p}_{\mathrm{Hdg}}(X) \cong \mathrm{Hom}_{\mathbb{Q}-HS}(\mathbb{Q}(-p), H^{2p}(X, \mathbb{Q})$. The Hodge conjecture is that cl is surjective onto the space of Hodge classes.

8. Polarizations

Let V be a $\mathbb{Q}$-HS of weight k and $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R})$ the corresponding representation of the Deligne torus. The Weil operator in terms of ρ is $C = \rho(i)$.

Definition 8.1. A *polarization* on V is a morphism of $\mathbb{Q}$-HS is a $(-1)^k$-symmetric morphism $\psi : V \otimes V \to \mathbb{Q}(-k)$ such that the bilinear form $Q : V_{\mathbb{R}} \otimes V_{\mathbb{R}} \to \mathbb{R}$ given by $Q(x, y) = (2\pi i)^k \psi(Cx, y)$ is symmetric and positive definite.

Example 8.2. $V = H^k_{\mathrm{prim}}(X, \mathbb{Q})$ with ψ the Hodge-Riemann pairing.

Observe: if V is polarizable (i.e. it admits a polarization; not every HS admits a polarization e.g. HS on nonprojetive varieties) then V is semisimple $V = V_1 \oplus \ldots \oplus V_m$, V_i is simple.

Assume $W \subset V$ is a sub-HS, then $W^{\perp_\psi} \subset V$ is a sub-HS. Then $0 \underbrace{=}_{\substack{Q \\ \text{positive} \\ \text{def.}}} W \cap W^{\perp_\psi} \subset V$, Then $V = W \oplus W^{\perp_\psi}$.

9. The Mumford-Tate group

Let V be a $\mathbb{Q}$-HS and $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes \mathbb{R})$ a representation of $\mathbb{R}$-groups.

Consider $G \subset \mathrm{GL}(V)$ such that $\mathrm{Im}(\rho) \subset G(\mathbb{R})$.

(The idea is that the Mumford-Tate group will recover the rational structure of V, because the representation ρ knows nothing about this rational structure, which must exist since V is a rational vector space.)

Definition 9.1. The *Mumford-Tate group* is

$$MT(V) = \bigcap_{\substack{G \subset \mathrm{GL}(V) \\ \text{subgroup s.t.} \\ \mathrm{Im}(\rho) \subset G(\mathbb{R})}} G = \text{smallest } Q\text{-subgroup of } \mathrm{GL}(V) \text{ containing } \mathrm{Im}\rho.$$

Remark 9.2. We can also consider $U(1) \subset \mathbb{S}(\mathbb{R})$ and $\rho' := \rho|_{U(1)} : U(1) \to \mathrm{GL}(V \otimes \mathbb{R})$. Then the *Hodge group* is $\mathrm{Hdg}(V)$ = smallest Q-subgroup of $\mathrm{GL}(V)$ such that $\mathrm{Im}(\rho') \subset \mathrm{Hdg}(V)(\mathbb{R})$.

$\mathrm{Hdg}(V)$ is always smaller than $MT(V)$.

Example 9.3.
(1) $MT(\mathbb{Q}(1)) = \mathbb{Q}^*$ $\mathbb{Q}(1) = 2\pi i\mathbb{Q} \subset \mathbb{C}$, $\mathbb{Q}(1) \otimes \mathbb{R} = 2\pi i \mathbb{R} \subset \mathbb{C}$, $\mathrm{GL}(\mathbb{Q}(1) \otimes \mathbb{R}) = \mathbb{R}^*$, since $z \in \mathbb{S}(\mathbb{R})$ acts as $|z|^{-2} \cdot \mathrm{Id}$.
(2) $\mathbb{Q}(0) = \mathbb{Q} \subset \mathbb{C}$, $MT(\mathbb{Q}(0)) = \{1\}$. In general, if V is of weight k, then $\mathbb{Q}^*$ = center of $\mathrm{GL}(V) \subset MT(V)$ and $MT(V)$ is generated by $\mathbb{Q}^*$ and $Hdg(V)$.

10. Tensor construction

Let V be a $\mathbb{Q}$-HS with $MT(V) \subset \mathrm{GL}(V)$ and $\rho : \mathbb{S}(\mathbb{R}) \to MT(V)(\mathbb{R})$. Then

$$T^\bullet(V) = \bigoplus_{e,f \geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$$

is a $MT(V)$-representation.

Proposition 10.1. *A finite-dimensional subspace $W \subset T^\bullet(V)$ is a sub-HS if and only if W is a $MT(V)$-subspace.*

Proof. ($\Longleftarrow$). If W is a $MT(V)$-subrepresentation, then from ρ we can compose with the representation that $MT(V)$ is to obtain $\rho' : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(W \otimes \mathbb{R})$. Note that $\rho|_{\mathbb{R}^*}$ is defined over Q, We get that W is a sub-HS.

($\Longrightarrow$). Assume that $W \subset T^\bullet(V)$ is a sub-HS, $G - \mathrm{Stab}(W) \subset \mathrm{GL}(V)$ is a $\mathbb{Q}$-subgroup. Since W is a sub-HS, the action of $\mathbb{S}(\mathbb{R})$ on $T^\bullet(V)$ preserves W, then $\mathrm{Im}(\rho) \subset G(\mathbb{R})$ and thus $MT(V) \subset G$, which implies that W is a $MT(V)$-subrepresentation. $\square$

As a corollary,

Lemma 10.2. *$x \in T^\bullet(V)$ is $MT(V)$-invariant if and only if X is a $(0,0)$ Hodge element.*

Proposition 10.3. *Assume that V is a $\mathbb{Q}$-HS of weight k and $\psi : V \otimes V \to \mathbb{Q}(-k)$ is a polarization. Then*

$$Hdg(V) \subset \begin{cases} SO(V,\psi) & \text{if } k \equiv 0(mod\ 2) \\ Sp(V,\psi) & \text{if } k \equiv 1(mod\ 2) \end{cases}$$

Proof. $\psi : V \otimes V \to \mathbb{Q}(-k)$, where the $\mathbb{Q}(-k)$ is a trivial $Hdg(V)$-module, so that ψ is a morphism of $Hdg(V)$-modules. The action of $Hdg(V)$ preserves the symmetric form ψ if $k \equiv 0(\text{mod } 2)$ and the antisymmetric form ψ if $k \equiv 1(\text{mod } 2)$. □

Example 10.4. Let $V = H^1(E,\mathbb{Q})$ where E is an elliptic curve and ψ the Hodge-Riemann pairing. Then $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ and $\dim V^{1,0} = 1$. $Hdg(V) \subset Sp(V,\psi) \cong \mathrm{SL}_2(\mathbb{Q})$.

Observe thati, in general, if V is polarizable, then $V = V_1 \oplus \ldots \oplus V_m$, V_i are irreducible $MT(V)$-representations, so $MT(V)$ is reductive.

Back to our elliptic curve example, $Hdg(V) \subset \mathrm{SL}_2(\mathbb{Q})$ is reductive.

There are two possibilities:

(1) $Hdg(V) = \mathrm{SL}^2(\mathbb{Q})$, which implies that $MT(V) = \mathrm{GL}_2(V)$. This happens when E is generic in the moduli space.
(2) $Hdg(V)$ is properly contained in $\mathrm{SL}_2(\mathbb{Q})$. Then $Hdg(V)$ is a 1-dimensional torus. Then $\mathrm{End}_{\mathbb{Q}-HS}(V) \neq \mathbb{Q}$, which implies that E has complex multiplication.

Definition 10.5. A HS V is of *CM-type* if $MT(V)$ is abelian (such HS defines a special point in the moduli space of HS.)

11. Summary so far

Recall:

- A $\mathbb{Q}$-HS is a finite dimensional $\mathbb{Q}$ vector space V with a representation of $\mathbb{R}$-group $\rho : \underbrace{\mathbb{S}(\mathbb{R})}_{\cong \mathbb{C}^*} \to \mathrm{GL}(V \otimes \mathbb{R})$ such that $\rho|_{\mathbb{R}^*}$ is defined over $\mathbb{Q}$.

  This gives a Hodge structure $V \otimes \mathbb{C} = \bigoplus_{p,q} V^{p,q}$, $V^{p,q} = \overline{V^{q,p}}$.

  V is of weight k if $\forall r \in \mathbb{R}^k$, $\rho(r) = r^k \cdot \mathrm{Id}$. $\forall z \in \mathbb{S}(\mathbb{R})$, $\rho(z)|_{V^{p,q}} = z^p \overline{v}^q \mathrm{Id}$
- $MT(V) = \bigcap_{\substack{G \in \mathrm{GL}(V) \\ \text{s.t.}\,\mathrm{Im}(\rho) \subset G(\mathbb{R})}} G$ is the Mumford-Tate group.
- $Hdg(V) = \bigcap_{\substack{G \in \mathrm{GL}(V) \\ \text{s.t.}\,\mathrm{Im}(\rho|_{\mathrm{U}(1)}) \subset G(\mathbb{R})}} G$ is the Hodge group.
- $T^\bullet V = \bigoplus_{e,f \geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$. A sub-HS of $T^\bullet V$ is the same thing as a sub-$MT(V)$-representation. $T^\bullet V^{MT(V)} = MT(V)$-invariants. $T^\bullet V \cap (T^* V)^{0,0} =$ the space of Hodge $(0,0)$-classes.
- (Polarization.) Assume that V has weight k $\psi : V \otimes V \to \mathbb{Q}(-k)$ a $\mathbb{Q}$-HS morphism such that the following form is positive definite:

$$Q : V_{\mathbb{R}} \otimes V_{\mathbb{R}} \longrightarrow \mathbb{R}$$
$$Q(x,y) = (2\pi i)^k \psi(Cx,y)$$

  where $C = \rho(i) \in Hdg(V)(\mathbb{R})$, then $MT(V), Hdg(V)$ are reductive.

"We are interested in polarizable Hodge structures because that is the case of cohomologies of algebraic varieties."

12. Some facts about reductive groups

Let V be a finite dimensional vector space over k, $\mathrm{char}(k) = 0$ and $G \subset \mathrm{GL}(V)$ an irreducible algebraic k-group.

Remark 12.1. $MT(V)$ and $Hdg(V)$ are connected.

Since by definition G is a subgroup of $\mathrm{GL}(V)$, we can view V as a faithful representation (because the inclusion has no kernel). We say G is *reductive* if V is semisimple, i.e. V is a direct sum of irreducible representations $V = V_1 \oplus \ldots \oplus V_n$. Equivalently, Every G-representation is semisimple. Equivalently, the center $Z^0(G)$ is an algebraic torus and $G/Z(G)$ is semisimple. Equivalently, $G(\mathbb{C})$ admits a compact real form.

$\sigma \in \mathrm{Aut}(G(\mathbb{C}))$ is a *Cartan involution* if $\sigma^2 = \mathrm{id}$ and $G^\sigma = \{g \in G(\mathbb{C}) : \sigma(\bar{g}) = g\}$ is compact.

Example 12.2. Let V be a polarizable $\mathbb{Q}$-HS of weight k. $C^2 = \rho(i^2) = \rho(-1) \in Z(\underbrace{Hdg(V)}_{G})$, $G = Hdg(V)$.

Note that $\sigma = Ad_C : G \to G, g \mapsto CgC^{-1}$, $\mathrm{id} = Ad_C^2, g \mapsto C^2gC^{-2} = g$.

We claim that σ is a Cartan involution on $Hdg(V) = G$. Indeed, Q defines a Hermitian product on $V \otimes \mathbb{C}$. $g \in G^\sigma$,

$$\begin{aligned} Q(gx, \overline{gy}) &= Q(gx, \sigma(g)\bar{y}) \\ &= (2\pi i)^k \psi(\underbrace{Cg}_{\in Hdg} x, \underbrace{Cg}_{\in Hdg} C^{-1}\bar{y}) \\ &= (2\pi i)^k \psi(x, C^{-1}\bar{y}) \\ &= (2\pi i)^k \psi(Cx, \bar{y}) \\ &= Q(x, \bar{y}). \end{aligned}$$

Thus, $G^\sigma \subset \underbrace{U(V \otimes \mathbb{C}}_{\text{compact}}$ as claimed.

Remark 12.3. A $\mathbb{Q}$-HS is polarizable if and only if $G = Ad_C$ is a Cartan involution.

13. Reductive groups and their representations

Let V be a finite-dimensional vector space over k, $\mathrm{char} k = 0$. $T^\bullet V = \bigoplus_{e,f \geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$.

Theorem 13.1. *Let $G \subset GL(V)$ be a reductive group. Then*

(1) *Any finite-dimensional representation of G is a subgroup of $(T^\bullet V)^{\oplus N}$ for some $N > 0$.*
(2) *Assume that $H \subset G$ is an algebraic subgroup. There exists a G-representation W and a 1-dimensional subspace $L \subset W$ such that $H = Stab_W(L)$.*
(3) *If H is reductive, then there exists a G-representation W and $x \in W$ such that $H = Stab_W(x)$.*

We need some preparations for the proof:

(1) $G \subset \mathrm{GL}(V)$ Let $k[G]$ be the algebra of regular functions on G.

$$\begin{tikzcd}
\mathrm{GL}(V) \arrow[rr] \arrow[drr] & & \mathrm{End}(V) \oplus \mathrm{End}(V^*) \\
g \arrow[r, mapsto] & (g,(g^t)^{-1}) & \{(A,B) : A \cdot B^t = \mathrm{Id}\} \arrow[u, hook]
\end{tikzcd}$$

So we have

$$\mathrm{Sym}(V \otimes V^*)^{\otimes 2} \twoheadrightarrow K[\mathrm{GL}(V)] \twoheadrightarrow K[G].$$

(2) $K[G]$ is a Hopf algebra. Indeed, we have

$$\begin{array}{cc}
G \times G \xrightarrow{m} G & K[G] \xrightarrow{\Delta} K[G] \otimes K[G] \\
\{1\} \xhookrightarrow{e} G & K[G] \xrightarrow{\varepsilon} K \\
G \xrightarrow{inv} G & K[G] \xrightarrow{coinv} K[G].
\end{array}$$

and

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
K[C] \arrow[r, "\Delta"] \arrow[d, "\Delta"] & K[G]^{\otimes 2} \arrow[d, "\mathrm{id}\otimes\Delta"] \\
K[G]^{\otimes 2} \arrow[r, "\Delta\otimes\mathrm{id}"] & K[G]
\end{tikzcd}$$

Then V is a G-representation, i.e. a *comodule*. We have

$$\begin{aligned}
G \times V^* &\to V^* \\
\mathrm{Sym}(V) &\to \mathrm{Sym}(V) \otimes K[G] \\
V &\to V \otimes K[G] \\
\mu : V &\to V \otimes K[G]
\end{aligned}$$

and the following diagram commutes

(13.1.1)
$$\begin{tikzcd}
V \arrow[r, "\mu"] \arrow[d, "\mu"] & V \otimes K[G] \arrow[d, "\mu\otimes\mathrm{id}"] \\
V \otimes K[G] \arrow[r, "\mathrm{id}\otimes\Delta"'] & V \otimes K[G] \otimes K[G].
\end{tikzcd}$$

Equation 13.1.1 defines a $K[G]$-comodule structure on $K[G]$.

(3) Any G-representation is a union of finite-dimensional subrepresentations. (In particular this applies to $K[G]$.)

Indeed, let $\{a_i\}$ be a basis of $K[G]$. Fix $x \in V$ and denote

$$\mu(x) = \sum_{i,\text{finite}} v_i \otimes e_i \qquad \text{and} \qquad \Delta e_i = \sum_{j,k} a_{ijk} e_j \otimes e_k, \qquad a_{ijk} \in k.$$

Then

$$\begin{aligned}\sum_i \mu(v_i)\otimes e_i &= \sum_{i,j,k} a_{ijk} v_i \otimes e_j \otimes e_k \\ &= \sum_i \left(\sum_{j,k} a_{ijk} v_k \otimes v_k \otimes e_j\right)\otimes e_i.\end{aligned}$$

$\implies \mu(v_i) = \sum_{j,k} a_{kji} v_k \otimes e_j$. $W = \mathrm{span}(x, v_i) \implies \mu : W \to W \otimes K[G]$, $\dim W < \infty$.

Now we can prove Theorem 13.1.

Proof. (1) Let W be a finite-dimensional G-representation. It is enough to show that W is a subrepresentation of $K[G]^{\oplus N}$ for some $N > 0$.

(13.1.2)
$$\begin{tikzcd} W \arrow[r, "\mu"] \arrow[d, "\mu"'] & W \otimes K[G] \arrow[d, "\mu\otimes\mathrm{id}"] \\ W \otimes K[G] \arrow[r, "\mathrm{id}\otimes\Delta"'] & W \otimes K[G] \otimes K[G] \end{tikzcd}$$

Equation 13.1.2 means that μ defines a morphism of $K[G]$-comodules

$$W \hookrightarrow \underbrace{W \otimes K[G]}_{K[G]^{\oplus \dim W}}.$$

(2) Let $H \subset G$ be an algebraic k-subgroup. [We want to show] There exists a G-representation W and $L \subset W$ with $\dim L = 1$ such that $H = Stab_W(L)$.

$k[H] = k[G]/I$ for an ideal I. For $f \in K[G], h, g \in G$ we have $gf(h) = f(g^{-1}h)$.

Suppose that $H = V(I)$ for an ideal I. Note that $g \in H$ if and only if $g \cdot I = I$ since $g \cdot H = V(gI)$ since $0 = f(x) = f(g^{-1}gx) = gf(gx) = 0$. That is, $H = Stab_{K[G]}(I)$.

Let $f_1, \ldots, f_m$ be generators of I. Then there exists $\tilde{W} \subset k[G]$, a finite-dimensional G-subgroup with $f_1, \ldots, f_m \in W$, $H = Stab_{\tilde{W}}(\tilde{W} \cap I)$. Consider $W = \Lambda^d \tilde{W}$ and $L = \Lambda^d (W \cap I)$, which satisfy the conditions.

(3) If H is reductive, $H = Stab_W(L)$ $W = L \oplus W'$, $W \otimes W^* = (\underbrace{L \otimes L^*}_{\text{trivial}} \oplus (\ldots)$.

Let $0 \neq x \in L \otimes L^*$. Then $H = Stab_{W\otimes W^*} Stab(x)$. ∎

As a corollary,

Lemma 13.2. *If V is a polarizable $\mathbb{Q}$-HS, then [the Mumford-Tate group is reductive] $MT(V) =$ Stabilizer of all $(0,0)$-Hodge classes in $T^\bullet V$.*

Proof. Let $G = MT(V)$ and

$$G' = \bigcap_{\substack{x\in(0,0)\text{-Hodge}\\ \text{classes}}} Stab(x) \subset \mathrm{GL}(V).$$

$G \subset G'$.

By part 3 of Theorem **??**, $G = Stab_W(x)$ where W is a representation of $\mathrm{GL}(V)$. By part 1 of Theorem **??**, $W \subset (T^\bullet V)^{\oplus N}$. x is fixed by $G = MT(V)$, then x is a $(0,0)$-Hodge class. Thus $G' \subset G$. □

[Approximate comment] We have shown that the representations of $MT(V)$ live in that tensor algebra. We would like to find a universal object such that any Hodge structure is realised as a representation of such group. This is analogous to Galois theory, where we take an inverse limit over Glois groups of field extensions, which provides a universal object and an equivalence of categories.

14. Short summary

V a $\mathbb{Q}$-Hs, polarizable. $\rho : \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V \otimes \mathbb{R})$, $MT(V) \subset \mathrm{GL}(V)$, the smallest subgroup over $\mathbb{Q}$ such that $MT(V)(\mathbb{R})$ contains $\mathrm{Im}(\rho)$. Polarizable $\implies$ $MT(V)$ is reductive. $MT(V) =$ stabilizer of all $(0,0)$ Hodge classes in $T^\bullet V = \bigoplus_{e,f\geq 0} V^{\otimes e} \otimes (V^*)^{\otimes f}$.

15. Absolute Mumford-Tate group

Let $V' \subset V$ be a sub-HS. $MT(V) \subset Stab_V(V') = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$.

We also have a map $MT(V) \twoheadrightarrow MT(V')$.

The absolute MT group should be something like the inverse limit $\lim \longleftarrow MT(V)$.

Let $(\mathbb{Q} - HS)$ be the category of $\mathbb{Q}$-HS.

(1) $\mathbb{Q}$-linear abelian category.
(2) Has $\otimes$, which is commutative with unit $1 = \mathbb{Q}(0)$, which a Hodge structure such that $\mathrm{End}(\mathbb{Q}(0)) = \mathbb{Q}$. We also have duality, $\forall V \exists V^*$ and $\mathbb{Q}(0) = q \to V \otimes V^*$ which is the "diagonal embedding", and $V \otimes V^* \to 1 = \mathbb{Q}(0)$, which is the trace.
(3) There exists a functor $\omega : (\mathbb{Q} - HS) \to (\mathbb{Q} - v.s.)$, forgetful functor. ω is called *fibre functor*. A $\otimes$-functor that is exact and faithful (injective on Hom's).

The pair ($\mathbb{Q}$-HS, ω) is called a *neutral Tannakian category.*

Theorem 15.1. *A neutral Tannakian category $(\mathcal{C}, \omega)$ (where $\mathcal{C}$ is K-linear, $charK = 0$) is equivalent to the category of finite dimensional representations of an affine group scheme over K $SpcK[G]$, $K[G]$ Hopf algebra. G is a pro-alg. group called the Tannakian fundamental group.*

Apply this to $(\mathbb{Q} - HS), \omega, \implies G = MT_{\mathbb{Q}-HS}$ is the *absolute MT group.*

Let $(\mathbb{Q} - HS)^{pol}$ be the full $\otimes$-subcategory of polarizable HS. $MT_{(\mathbb{Q}-HS)^{\mathrm{pol}}}$ is pro-reductive.

16. How to construct the Tannakian fundamental group

We have $\omega : \mathcal{C} \to (K - v.s.)$. Let $\mathrm{Aut}(\omega)^\otimes$ be the automorphisms ot ω as a $\otimes$-functor. That is, if $g \in \mathrm{Aut}(\omega)^\otimes$, we have $\forall X \in \mathcal{C}$, $g_x \in \mathrm{GL}(\omega(X))$.

We also have:

- Compatibility with morphisms, i.e. for every $f : X \to Y$ we have

$$\begin{tikzcd} \omega(X) \arrow[r, "\omega(f)"] \arrow[d, "g_X"'] & \omega(Y) \arrow[d, "g_Y"] \\ \omega(X) \arrow[r, "\omega(f)"'] & \omega(Y) \end{tikzcd}$$

  commutes, i.e. $g_Y \circ \omega(f) = \omega(f) \circ g_X$.
- Compatibility with $\otimes$:

$$g_{X\otimes Y} = g_X \otimes g_Y$$
$$g_1 = 1 \in K^* = \mathrm{GL}(K), g_{X^*} = (g_X^*)^{-1}.$$

$\mathrm{Aut}(\omega)^{\otimes}$ is the group of K-points of the Tannakian fundamental group.

[Our category $(\mathbb{Q} - HS)^{\mathrm{pol}}$ is equivalent to the category of representations of this group. But the problem is that this group is huge. So we shall restrict to a smaller subcategory.]

Let $V \in (\mathbb{Q} - HS)^{\mathrm{pol}}$ and $\langle V\rangle$ the full $\otimes$-subcategory generated by V. Objects are sub-HS of $T^\bullet V$ and their direct sums.

What is the Tannakian fundamental group of $\langle V\rangle$?

Consider $\omega : \langle V\rangle \to (\mathbb{Q} - v.s.)$. For $g \in \mathrm{Aut}(\omega)^{\otimes}$ we have that $g_V \in \mathrm{GL}(V)$ is uniquely determined by g.

Let $x \in T^\bullet V \cap (T^\bullet V)^{0,0}$. We have an embedding

$$1 = \mathbb{Q}(0) \hookrightarrow T^\bullet V$$
$$1 \mapsto x.$$

We have $g_{\mathbb{Q}(0)} = 1 \implies g_V \in \mathrm{Stab}_{T^\bullet V}(x)$. Thus $g_V \in MT(V)$. We also have the converse, so we see that the Tannakian fundamental group of $\langle V\rangle$ is $MT(V)$. This could be taken as an equivalent definition of $MT(V)$

17. Hodge structures of abelian varieties

Let $(\mathbb{Q} - HS)^{\mathrm{ab}}$ be the full $\otimes$-subcategory of $(\mathbb{Q} - HS)^{\mathrm{pol}}$ genertated by $H^1(A, \mathbb{Q})$ where A is an abelian varieties.

Recall that if A is a projective variety it is biholomorphic to $\mathbb{C}^n/\Lambda$ and $H^1(A, \mathbb{C}) = H^{1,0}(A) \oplus H^{0,1}(A)$.

[Suppose that you have some Hodge structure, say, on the cohomology of your favourite variety. Does it belong to this category or not? It is not clear — there might be some complicated Hodge structures in there]

Let $V = H^1(A, \mathbb{Q})$, $MT(V) \subset \mathrm{GL}(V)$. Let $\mathfrak{mt}(V) = \mathrm{Lie}(MT(V)) \subset \mathfrak{gl}(V) = V \otimes V^*$, a sub-HS (of $\mathfrak{gl}(V)$, which is always a Hodge structure). That is, the Lie algebra of $MT(V)$ always carries a Hodge structure. This HS may only have types $(-1, 1), (0, 0), (1, -1)$ because $((1, 0) + (0, 1)) \otimes ((-1, 0), (0, -1))$.

Proposition 17.1. *Let $V \in (\mathbb{Q} - HS)^{ab}$. Then the HS on $\mathfrak{mt}(V)$ may have only Hodge types $(-1, 1), (0, 0), (1, -1)$.*

Proof. We may assume that $V \subset T^\bullet W$ where $W = H^1(A, \mathbb{Q})$ by the tensor construction. Then we must have $MT(W) \twoheadrightarrow MT(V)$, which on the level of Lie algebras becomes $\mathfrak{mt}(W) \twoheadrightarrow \mathfrak{mt}(V)$, which is a surjection of HS. $\square$

The next example shows how we may use this result.

Example 17.2. (1) Let $X =$ a K3 surface, $X \subset \mathbb{P}^3$, a smooth quartic. Then $K_X = \mathcal{O}_X$, $h^{2,0} = h^{0,2} = 1$, $H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$. The HS is $V = H^2(X, \mathbb{Q})$ and the intersection product is a map $V \otimes V \to \mathbb{Q}(-2)$. Recall that the Hodge group is the image of $U(n)$ under the representation from the Deligne torus. In this case we have $MT(V) \supset Hdg(V) \subset \mathrm{SO}(V, \psi)$. Also

$$\mathfrak{mt}(V) = \underbrace{\mathfrak{hdg}(V)}_{\mathrm{Lie}(Hdg(V))} \oplus \underbrace{\mathbb{Q}(0)}_{\text{center of } \mathfrak{gl}(V)} .$$

And also $(1,-1), (0,0), (-1,1)$. Now since $H^{2,0}$ is 1-dimensional we can compute

$$\Lambda^2(V \otimes \mathbb{C}) = \underbrace{(H^{2,0} \otimes H^{1,1})}_{3,1} \oplus \underbrace{(\Lambda^2 H^{1,1} \oplus H^{2,0} \otimes H^{0,2})}_{2,2} \oplus \underbrace{(H^{1,1} \otimes H^{0,2})}_{1,3}$$

(2) $X \subset \mathbb{C}P^3$ hypersurface of degree ≥ 5. Then $h^{2,0} > 1$, $V = H^2(X, \mathbb{Q})$, $\mathfrak{hdg}(V) \subset \mathfrak{so}(V, \psi)$. Hodge types: $(2,2), (1,-1), (0,0), (-1,1), (-2,-2)$.

18. Variations of Hodge structures

A *family* is $\pi : \mathcal{X} \to B$ with $\mathcal{X}, B$ complex manifolds, π is a submersion, B is connected and $\mathcal{X} \hookrightarrow \mathbb{P}^N \times B$.

Then $R^k\pi_*\mathbb{Z}/\mathrm{tors.}$ is a local system over B. In fact, $(R^k\pi_*\mathbb{Z})_t = H^k(\mathcal{X}_t, \mathbb{Z})$ where $\mathcal{X}_t = \pi^{-1}(t)$. That is, $R^k\pi_*\mathbb{Z}$ is a sheaf with the information of the cohomologies of the fibres.

Consider

$$0 \longrightarrow \pi^*\Omega^1_B \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/B} \longrightarrow 0$$

Assume $\dim B = 1$ (This is used to prove the property of the Gauss-Manin connection, though it can be also be done by restricting to curves.) Then

$$0 \longrightarrow \pi^*\Omega^1_B \otimes \Omega^{k-1}_{\mathcal{X}} \longrightarrow \Omega^k_{\mathcal{X}} \longrightarrow \Omega^k_{\mathcal{X}/B} \longrightarrow 0$$

$$0 \longrightarrow \pi^*\Omega^1_B \otimes \Omega^\bullet_{\mathcal{X}/B}[-1] \longrightarrow \Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}/B} \simeq \pi^{-1}\mathcal{O}_B \longrightarrow 0 \tag{18.0.1}$$

We can construct the vector bundle

$$\mathcal{V}_k = (R^k\pi_*\mathbb{C}) \otimes \mathcal{O}_B = R^k\pi_*(\pi^{-1}\mathcal{O}) \simeq R^k\pi_*(\Omega^\bullet_{\mathcal{X}/B}).$$

On $\mathcal{V}_k$ we have a flat connection from Equation 18.0.1. We get

$$R^k\pi_*(\Omega^\bullet_{\mathcal{X}/B} \longrightarrow R^{k+1}\pi_*(\pi^*\Omega^1_B \otimes \Omega^\bullet_{\mathcal{X}/B}[-1] \longrightarrow \mathcal{V}_k \longrightarrow \Omega^{1\nabla}_B \otimes R^k\pi_*\Omega^\bullet_{\mathcal{X}/B} = \Omega^1_B \otimes \mathcal{V}^k.$$

The map ∇ is called the *Gauss-Manin* connection.

Define $F^pV_k = R^k\pi_*(\Omega^{\geq p}_{\mathcal{X}/B})$. Then $F^p\mathcal{V}_k \subset \mathcal{V}_k$ is a holomorphic subbundle.

The *Hodge filtration* on $\mathcal{V}_k$.

(1) $\nabla : \mathcal{V}_k \to \Omega^1_B \otimes \mathcal{V}_k$ a flat connection.
(2) The Hodge filtration

$$0 \subset \ldots \subset F^p\mathcal{V}_k \subset F^{p-1}\mathcal{V}_k \subset \ldots \subset V_k.$$

(3) (Griffiths transversality.) When we restrict this construction to F^p the relation we obtain is *Griffiths transversality.* $\nabla(F^k\mathcal{V}_k) \subset \Omega^1_B \otimes F^{p-1}\mathcal{V}_k$.

Definition 18.1. A *polarization* on $\mathcal{V}_k$ is defined by a section of $R^2\pi_*\mathbb{Z}(1)$.

By the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 1$$

we obtain maps

$$R^1\pi_*\mathbb{C}^* \to R^2\pi_*\mathbb{Z}(1),$$

and

$$\mathcal{O}_{\mathbb{P}^N}(1) \in H^1(\mathcal{X}, \mathbb{C}^*) \to H^0(B, R^1\pi_*\mathbb{C}^*) \to H^0(B, R^2\pi_*\mathbb{Z}(1)).$$

From there we get

$$\psi : V_B \otimes V_B \to \mathbb{Z}_B(-k),$$

where $V_B = R^k\pi_*\mathbb{Z}/\text{tors.}$, a morphism of local systems that defines polarizations on the fibres of π.

Definition 18.2. A $\mathbb{Z}$*-variation of Hodge-Structure over* B is a $\mathbb{Z}$-local system V_B with a fibre $\mathbb{Z}^r$ for some $r \geq 0$ and a finite filtration of holomorphic subbundles on $\mathcal{V} = V_B \otimes \mathcal{O}_B$

$$0 \subset \ldots \subset F^p\mathcal{V} \subset F^{p-1}\mathcal{V} \subset \ldots \subset \mathcal{V}$$

[which satisfies Griffiths transversality] such that

(1) for all $t \in B$ $(V_{B,t}, F^\bullet\mathcal{V}_t)$ is a $\mathbb{Z}$-HS.
(2) $\nabla(F^p\mathcal{V}) \subset \Omega^1_B \otimes F^{p-1}\mathcal{V}$ where ∇ is the flat connection on $\mathcal{V}$.

19. The Hodge locus

[Next we shall try to study the MT group on families. Since MT essentially is the satbilizer of the Hodge classes, we look for those. There are fibres that may have more Hodge classes than others. This leads to the following concept.]

Let B be a complex manifold and V_B a $\mathbb{Z}$-local system with a VHS. $F^p\mathcal{V}$ of weight $2k$. We have

$$\begin{tikzcd} \mathrm{Tot}(\mathcal{V}) \arrow[r, "p"] & B \\ \mathrm{Tot}(V_B). \arrow[u, hook, "\text{covering}"] \arrow[ur] & \end{tikzcd}$$

Note that $\mathrm{Tot}(V_B) = \amalg W_i$ where W_i are the connected components of $\mathrm{Tot}(V_B)$.

In this setting, a Hodge class is a point x in one of W_i such that $x \in F^k\mathcal{V} \cap \overline{F^k\mathcal{V}}$ $\iff$ $x \in F^k\mathcal{V} \cap W_i$.

Let $Z_i = \mathrm{Tot}(F^k\mathcal{V}) \cap W_i$. Note that Z_i is a closed subvariety (maybe singular) of $\mathrm{Tot}(\mathcal{V})$.

If the projections of these Z_i, $p(Z_i)$, is not equal to B, then $p(Z_i)$ is in the Hodge locus. That is, the *Hodge locus* is

$$B^{\mathrm{Hdg}}(V_B) = \bigcup_{i:p(Z_i)\neq B} p(Z_i)$$

$$B^{\mathrm{Hdg}} = \bigcup_{\substack{e,f \\ \text{countable} \\ \text{union}}} B^{\mathrm{Hdg}}(\underbrace{T^{e,f}V_B}_{V_B^{\otimes e}\otimes(V_B^*)^{\otimes f}}).$$

[Our goal is to consider the complement of the Hodge locus and study there the MT group.]