

# COMPLEX GEOMETRY

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## 1. COMPLEX MANIFOLDS

**Definition 1.1.** A *complex manifold*  $M$  is a smooth manifold admitting an open cover  $\{U_\alpha\}$  and coordinate maps  $\varphi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$  such that  $\varphi_\alpha \circ \varphi_\beta^{-1}$  is holomorphic on  $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$  for all  $\alpha, \beta$ .

**Definition 1.2.** A function on an open set  $U \subset M$  is *holomorphic* if for all  $\alpha$ ,  $f \circ \varphi_\alpha^{-1}$  is holomorphic on  $\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n$ .

**Lemma 1.3.** The sheaf  $\mathcal{O}_M$  of holomorphic functions is a sheaf.

## 2. HODGE STAR

The construction of the Hodge star in [Voi02, Section 5.1] is essentially: map  $\alpha \in \Lambda^{n-k}$  to the operator  $\alpha \wedge - \in \text{Hom}(\Lambda^k, \Lambda^n)$  and then use the metric to identify  $\text{Hom}(\Lambda^k, \Lambda^n)$  with  $\Lambda^k$ . Notice that the latter means that we take a form  $\alpha$  and we

map it to the morphism which maps  $\cdot$  to the top-form  $g(\alpha, \cdot)\text{Vol}$ . This says that for all  $\alpha \in \Lambda^k$  and  $\beta \in \Lambda^{n-k}$

$$(2.0.1) \quad \alpha \wedge * \beta = (\alpha, \beta) \text{Vol}.$$

**Lemma 2.1.** *The operator  $*$  satisfies the identity*

$$(2.1.1) \quad *^2 = (-1)^{k(n-k)} \text{ on } \Lambda^k(X).$$

*Proof.* Observe that  $*$  is an isometry, use skew-commutativity of product to obtain the  $(-1)^{k(n-k)}$  and finally Equation 2.0.1.  $\square$

### 3. RIEMANN SURFACES

**Definition 3.1.** A meromorphic function on a Riemann surface is a holomorphic map to the Riemann sphere which is not identically equal to  $\infty$ .

**Definition 3.2.** A pole of a meromorphic function on a Riemann surface is any point  $x$  such that  $F(x) = \infty$ .

**Definition 3.3.** The order of a pole of a meromorphic function on a Riemann surface is the integer  $k_x$  such that the coordinate representation of  $f$  is  $f(x) = z^{k_x}$  near  $x$ .

**Proposition 3.4.** Let  $F : X \rightarrow Y$  be a proper, non-constant holomorphic map between connected Riemann surfaces. Then the integer  $d(y) = \#f^{-1}(y)$  does not depend on  $y \in Y$ .

*Proof.* Using local representation of  $f$  as  $z^k$ . But see also ??: this number is locally constant because we can find a neighbourhood  $V$  of any point  $y$  such that any other  $y'$  in this neighbourhood has the same number of preimages. Using compactness fix neighbourhoods  $U_i$  of all the preimages of  $y$ , using that  $y$  is a regular value we may suppose the  $f$  is a diffeomorphism at the  $U_i$ , and then define  $V := \cap V_i \setminus f(M \setminus \cup U_i)$ .  $\square$

**Lemma 3.5.** Let  $X$  be a compact Riemann surface. If there is a meromorphic function on  $X$  having exactly one pole, and that pole has order 1, then  $X$  is equivalent to the Riemann sphere.

*Proof.* A meromorphic function has no critical points! That is,  $\infty$  is a regular value, having one preimage by hypothesis. Then the degree of  $f$  is one, so that it must be a bijection (why?). And it has no critical points... (why?) So by Lemma something, its inverse must be holomorphic.  $\square$

**Exercise 3.6.** Show that the canonical bundle  $K$  of a Riemann surface  $S$  has no base points.

*Proof.* Suppose  $p$  is a base point of  $K$ . We want to construct a meromorphic function with exactly one pole and use Lemma 3.5 to arrive at a contradiction. Any such function is an element of  $H^0(\mathcal{O}(p))$ . Note that by Serre duality we have:

$$H^0(\mathcal{O}(p)) = H^0(\mathcal{O}(p) \otimes K^* \otimes K) = H^1(K(-p))^*$$

So it's enough to show that  $H^1(K(-p))$  is not zero. So consider the sheaf exact sequence twisted by the ideal sheaf  $\mathcal{I}_p = \mathcal{O}_S(-p)$  of  $p$ :

$$0 \longrightarrow K(-p) \longrightarrow K \longrightarrow K(p) \longrightarrow 0$$

Then we use the exact sequence in cohomology to prove that  $H^1(K(-p))$  is not zero. First,  $H^1(K(p)) = 0$  because by Serre duality it has the same dimension as the space of holomorphic functions vanishing at  $p$ , which is only the zero function since  $M$  is compact. Next,  $H^1(K)$  is not zero because by Dolbeault theorem it is  $H^{1,1}(M, \mathbb{R})$  which contains real 2-forms. Then  $H^1(K(-p))$  cannot be zero because it surjects onto a nontrivial space.  $\square$

#### 4. BELYI FUNCTIONS

To understand Belyi functions we start by considering meromorphic functions  $f : X \rightarrow \mathbb{CP}^1$  as ramified coverings:

**Proposition 4.1.** *A nonconstant meromorphic function  $f : X \rightarrow \mathbb{CP}^1$ , considered as a mapping of the underlying topological space, is a ramified covering of the sphere  $S^2 \cong \mathbb{CP}^1$ .*

Be careful: isomorphic complex ramified coverings produce isomorphic Riemann surfaces (by definition), but the converse is certainly false. The same Riemann surface may be obtained by many different and pairwise non-isomorphic coverings. Just consider different meromorphic functions on the same surface.

The following paragraph is an approximate quote:

Proposition 4.1, and the fact that every Riemann surface admits a meromorphic function (which can be seen by Riemann-Roch, cf. [LGZV, Fact 1.8.6]) show that every Riemann surface may be represented by a ramified covering of  $\mathbb{CP}^1$ . The following theorem, which affirms the converse statement, is one of the most fundamental:

**Theorem 4.2** (Riemann's existence theorem). *Suppose a base star is fixed in  $\mathbb{CP}^1$ , and the sequence of its terminal vertices is  $R = [y_1, \dots, y_k]$ . Then for any constellation  $[g_1, \dots, g_k]$ ,  $g_i \in S_n$ , there exists a compact Riemann surface  $X$  and a meromorphic function  $f : X \rightarrow \mathbb{CP}^1$  such that  $y_1, \dots, y_k$  are the critical values of  $f$  (i.e.  $f'$  vanishes at these points) and  $g_1, \dots, g_k$  are the corresponding monodromy permutations. The ramified covering  $f : X \rightarrow \mathbb{CP}^1$  is independent of the choice of the base star in a given homotopy type and is unique up to isomorphism.*

*Proof.* The heart of this theorem is the correspondence between constellations, which are abstract sets of permutations whose product is the identity and act transitively on the set of  $n$  elements (and are also maps on surfaces), and the monodromy group of a covering. Indeed: for a regular point  $y_0$  of a degree- $n$  covering we get an action of the symmetric group of  $n$  elements on the fiber  $E := \pi^{-1}(y_0)$ ; taking each generator of the fundamental group to be any of these permutations we obtain a constellation (cf [LGZV, Construction 1.2.13]).

The other way around, [LGZV, Proposition 1.2.15], we can define a group homomorphism  $\pi(S^2 \setminus \{p_i\}, y_0) \rightarrow G$  where  $G$  is the group of the constellation; the fact that it is indeed a group homomorphism is due to the fact that both sets of generators satisfy the property that their product is identity. Then for any point  $x$  in the set of  $n$  elements we can consider its stabilizer. This corresponds to a subgroup  $M_x$  of  $\pi(S^2 \setminus \{p_i\}, y_0)$ . Such a subgroup determines a finite-sheeted covering of  $S^2 \setminus \{p_i\}$ . The covering is connected since  $G$  acts transitively on  $E$ . *Habemus superficiem.*  $\square$

The surprising result by Belyi is that for the case  $k = 3$  it will happen that the corresponding Riemann surfaces will be defined over  $\overline{\mathbb{Q}}$ , the field of algebraic numbers. Therefore, the absolute Galois group  $\text{Aut}(\overline{\mathbb{Q}}|\mathbb{Q})$  (that is, the automorphism group of the field  $\overline{\mathbb{Q}}$ ) acts on them, and thus on 3-constellations as well. The mysterious nature of the group and the simplicity of the objects on which it acts, gave rise to the following term which may look a bit strange: *theory of dessins d'enfants*.

**Definition 4.3.** If it is possible to realize a Riemann surface  $X$  by a system of equations with coefficients in a subfield  $K \subseteq \mathbb{C}$ , then we say that  $X$  is *defined over*  $K$ .

**Theorem 4.4** (Belyi). *A Riemann surface  $X$  admits a model over the field  $\overline{\mathbb{Q}}$  of algebraic numbers if and only if there exists a covering  $X \rightarrow \overline{\mathbb{C}}$  unramified outside  $\{0, 1, \infty\}$ . In such a case, the meromorphic function  $f$  can also be chosen in such a way that it will be defined over  $\overline{\mathbb{Q}}$ .*

## 5. SHEAF OF HOLOMORPHIC FUNCTIONS

For a definition of sheaf see Algebraic Geometry Definition 1.1.

**Lemma 5.1.** *Let  $X$  be a complex manifold of complex dimension  $n$ .*

*The functor*

$$\begin{aligned} \mathcal{O}_X : \text{Open}_X^{\text{op}} &\longrightarrow \text{Set} \\ U &\longmapsto \mathcal{O}_X(U) := \{f : U \rightarrow \mathbb{C}^n : f \text{ is holomorphic}\} \\ i : V \hookrightarrow U &\longmapsto \mathcal{O}_X(i) : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V) \end{aligned}$$

*where  $\mathcal{O}_X(i)$  is given by restriction, is a sheaf.*

*Proof.*  $\mathcal{O}_X$  is a presheaf, i.e. a functor, by definition: restriction maps the identity to the identity and preserves compositions.

To check the condition of being a sheaf let  $U_i$  be an open cover of some open set  $U$  of  $X$  and  $f_i : U_i \rightarrow \mathbb{C}^n$  sections of  $\mathcal{O}_X(U_i)$ , that is, holomorphic functions, which coincide pairwise in intersections. Then it is obvious that the function  $f : U \rightarrow \mathbb{C}^n$ ,  $x \mapsto f_i(x)$  for any  $U_i \ni x$  is well defined. It is also holomorphic since holomorphicity is a condition defined on open sets.  $\square$

## 6. ANALYTIC SETS

**Definition 6.1.** An *analytic set* or *subvariety*  $Y$  is a subset of a complex manifold  $X$  given locally as the zeros of a finite collection of holomorphic functions.

**Lemma 6.2.** *Let  $Y \subset X$  be an analytic subvariety of a complex manifold  $X$  equipped with the subspace topology. Then*

$$\begin{aligned} \mathcal{O}_Y : \text{Open}_Y^{\text{op}} &\longrightarrow \text{Set} \\ U \cap Y &\longmapsto \{f|_{U \cap Y} : f \in \mathcal{O}_X(U)\} \end{aligned}$$

*is a sheaf.*

*Proof.* This is a presheaf for the same reasons as the sheaf  $\mathcal{O}_X$  is a presheaf. The fact that it is a sheaf follows from the fact that  $\mathcal{O}_X$  is a sheaf and by definition of subspace topology; namely for any family of functions defined over an open cover

$\{U_i \cap Y\}$  of some open set  $U \cap Y$  we can reconstruct a function on  $U$  that restricted to  $U \cap Y$  is as desired.  $\square$

Recall that any map of sheaves of groups has well-defined kernel, see Lemma 2.1.

**Definition 6.3.** Let  $Y \subset X$  be an analytic subvariety of a complex manifold  $X$ . The *ideal sheaf* of  $Y$ , denoted  $\mathcal{I}_Y$ , is the kernel of the restriction map  $\mathcal{O}_X \rightarrow \mathcal{O}_Y$ .

This just says that  $\mathcal{I}_X(U)$  is the set of functions  $f \in \mathcal{O}_X$  that vanish under the restriction map, i.e. that vanish at points of  $Y$ .

Since the restriction map is surjective, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

Notice that for any open set  $U \subset X$ ,  $\mathcal{I}_Y(U)$  is an ideal of  $\mathcal{O}_X(U)$ . (Indeed, if a function vanishes at  $Y$  then its product with any other function will also vanish.)

**Definition 6.4.** An analytic variety  $V \subset U \subset \mathbb{C}^n$  is *irreducible* if  $V$  cannot be written as the union of two distinct analytic varieties  $V_1, V_2 \subset U$ , both distinct to  $V$ .

**Lemma 6.5.** If  $V$  is an irreducible analytic hypersurface given locally as  $V = \{f = 0\}$ , then  $f$  is irreducible in  $\mathcal{O}_p$ .

*Proof.* If  $f = gh$  and neither of  $g$  and  $h$  are units and they are distinct, we could express  $V$  as the union of two distinct varieties:  $V(g)$  and  $V(h)$ . (If one of them, was a unit then the vanishing set would be all of  $U$ .)  $\square$

**Definition 6.6.** The *germ* of a set in the origin  $0 \in \mathbb{C}^n$  is given by a subset  $X \subset \mathbb{C}^n$ . To subsets  $X, Y$  define the same germ if there exists an open neighbourhood  $0 \in U \subset \mathbb{C}^n$  with  $U \cap X = U \cap Y$ . A germ is called *analytic* if there are functions  $f_1, \dots, f_k \in \mathcal{O}_n$  such that  $X$  and  $Z(f_1, \dots, f_k)$  define the same germ.

Now the adventure of proving that the ideal sheaf is a line bundle for hypersurfaces begins.

See Algebraic Geometry Lemma ?? for the proof that the ideal sheaf of an irreducible codimension-1 closed subscheme of a smooth scheme is a line bundle. Now I will try to prove this for the case of an analytic subvariety of a (smooth) complex manifold.

First notice that in the case that  $Y$  is an irreducible subvariety, this ideal is prime: if  $fg \in \mathcal{I}_Y(U)$ , then  $V(f) \cup V(g) \supset Y \cap U$  so that  $(V(f) \cup V(g)) \cap Y = Y \cap U$ , and then it must be true that  $V(f) \cap Y \subseteq V(g) \cap Y$  or that  $V(f) \cap Y \supseteq V(g) \cap Y \cap U$  since  $Y$  is irreducible, and thus, say,  $V(f) \supseteq Y \cap U$ , that is,  $I(V(f)) \subseteq I(Y \cap U)$  so that  $f \in I(Y \cap U)$ .

If we further ask that  $Y$  is a hypersurface, we want to show that  $\mathcal{I}_Y$  is locally principal. If  $f \in \mathcal{I}_Y(U)$  is the defining function of some hypersurface  $Y$  of a variety  $X$  on an open set  $U \subset X$ , we shall have that  $\mathcal{I}_Y(U) = (f)$ . That is, if  $g$  also vanishes at  $Y \cap U$ , we want to see that  $g = fh$  for some  $h \in \mathcal{O}_X(U)$ .

By Stacks Project 0AFT we need two ingredients: (1) the fact that the ring  $\mathcal{O}_X(U)$  is a UFD for all  $U \subset X$  and (2) the following lemma.

**Lemma 6.7.** The ideal of an irreducible analytic hypersurface is a height-1 prime ideal.

*Proof.* The fact that it is prime comes from the fact that  $Y$  is irreducible as explained above.

Now suppose that  $0 \subset \mathfrak{p} \subseteq I$ . Then  $V(\mathfrak{p})$  is an analytic variety that contains  $Y$ . At regular points of both  $V(\mathfrak{p})$  and  $Y$ , both are smooth manifolds, but since  $Y$  is of codimension 1 and  $V(\mathfrak{p})$  does not equal all of  $Y$ , we conclude that they coincide. Thus, in a neighbourhood of a regular point we have  $\mathfrak{p} = I(V(\mathfrak{p})) = I$  by the holomorphic Nullstellensatz ??.

Now we shall use the Gauss Lemma ?? to prove:

**Lemma 6.8.** *The stalk  $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n, 0}$  is a UFD.*

*Proof.* By induction on  $n$ . For  $n = 0$  it is trivial. Suppose  $\mathcal{O}_{n-1}$  is a UFD. Then by the Gauss Lemma ??,  $\mathcal{O}_{n-1}[w]$  is a UFD too. Thus we may express any Weierstrass polynomial  $g$  as a product of irreducible elements (uniquely up to multiplication by units).

Let  $f \in \mathcal{O}_n$ . We want to express  $f$  as a product of (unique up to multiplication by units) of irreducible elements. By Weierstrass Preparation Theorem ?? there is a Weierstrass polynomial  $g \in \mathcal{O}_n[w]$  and a holomorphic function not vanishing on 0 (i.e. a unit of  $\mathcal{O}_n$ ) such that  $f = gh$ . By the previous remark  $g$  is factored uniquely up to multiplication by units as  $g = g_1 \dots g_m$ . This shows existence of the factorization.

To prove uniqueness suppose that  $f = f_1 \dots f_k$  for some irreducible  $f_1, \dots, f_k \in \mathcal{O}_n$ . Since  $f$  does not vanish in the  $w$  axis, neither can each  $f_i$ , so that we may decompose each of them as  $f_i = g'_i h_i$  by Weierstrass Preparation Theorem. Since  $f_i$  is irreducible, it follows that  $g'_i$  is irreducible. Then we have that

$$f = gh = \prod g'_i \prod h_i$$

so by uniqueness in Weierstrass Preparation Theorem we conclude that  $g = \prod g'_i$ , and by uniqueness from the fact that  $\mathcal{O}_n[w]$  is a UFD we conclude that  $g$  coincides with  $\prod g'_i$  up to multiplication by units.  $\square$

**Lemma 6.9.** *The ideal sheaf of an irreducible codimension-1 analytic subvariety of a smooth complex manifold is a line bundle.*

*Proof.* By definition, our subvariety  $Y$  is locally defined by the zeroes of a single holomorphic function defined on the ambient manifold  $X$ . That is, at every  $p \in Y$  there is an open set  $U \subset X$  such that  $U \cap Y = f^{-1}(0)$  for some  $f \in \mathcal{O}_X(U)$ . Since  $U$  is an open set of a complex manifold, the ring  $\mathcal{O}_X(U)$  is isomorphic to the ring  $\mathcal{O}_n$  of holomorphic functions on  $\mathbb{C}^n$  in a neighbourhood of the origin; that's by Definition 1.1 since a holomorphic function on a manifold is defined as such if its composition with the coordinate chart is a holomorphic function on  $\mathbb{C}^n$ .

By Lemma 6.5, since  $Y$  is irreducible it follows that  $f$  is irreducible and so is the ideal of functions vanishing on  $Y$ .

By Lemma 6.8, we prove identically as in Lemma ?? that the ideal of functions vanishing on  $Y$  is principal.  $\square$

Note the that argument actually works backwards if we assume the ambient ring is Noetherian, see Stacks Project 0AFT.

**Definition 6.10.** Let  $D := S$  be an effective Cartier divisor, that is, an analytic hypersurface of a complex manifold. The ideal sheaf  $\mathcal{I}_S := \mathcal{O}_X(-S)$  of  $S$  is the

dual line bundle of the *line bundle associated to the divisor*  $D$ , which is denoted by  $\mathcal{O}_X(D)$ .

## 7. HILBERT'S NULLSTELLENSATZ

Usually I think of Hilbert's Nullstellensatz as the statement that  $I(V(I)) = \sqrt{I}$ . Following are two simple initial observations.

It is immediate that  $\sqrt{\mathfrak{p}} = \mathfrak{p}$  if  $\mathfrak{p}$  is a prime ideal (of any ring): if  $f^k \in \mathfrak{p}$  then either  $f \in \mathfrak{p}$  or  $f^{k-1} \in \mathfrak{p}$ , etc.

It is also easy to see that for any ideal  $I$  of functions or germs of functions (i.e. either  $I \subset \mathcal{O}_X(U)$  or  $I \subset \mathcal{O}_n$ ) we have  $\sqrt{I} \subseteq I(V(I))$ . Indeed, if  $f^k \in I$ , then  $f^k(x) = 0$  for all  $x \in V(I)$ , and since  $\mathbb{C}$  has no nonzero divisors we see that  $f(x) = 0$ , that is  $f \in I(V(I))$ .

It remains to show that  $I(V(I)) \subseteq \sqrt{I}$ . We shall first suppose that  $I$  is prime and prove that  $I(V(I)) \subseteq I$ . (The complete statement for the radical follows from a purely algebraic process, see [Dem, Theorem 4.22].)

We will need a few preparatory results. First recall that  $S$  is an integral ring over  $R$  if every element of  $S$  is expressed as a root of a monic polynomial with coefficients in  $R$ . (See Stacks Project 00GI.)

It will be handy to keep in mind the following lemma (Stacks Project 00GH):

**Lemma 7.1.** *Let  $R \rightarrow S$  be a ring map. The following are equivalent*

- (1)  $R \rightarrow S$  is finite,
- (2)  $R \rightarrow S$  is integral and of finite type, and
- (3) there exist  $x_1, \dots, x_n \in S$  which generate  $S$  as an algebra over  $R$  such that each  $x_i$  is integral over  $R$ .

This essentially says that an integral extension  $R \rightarrow S$  gives  $S$  the structure of a finitely generated  $R$ -module (I think experts call this a “finite  $R$ -module”) whose generators are integral. This just means that  $S = R[x_1, \dots, x_k]$  and  $x_i \in S$  is the root of a monic polynomial in  $R$ .

**Proposition 7.2.**  $\mathcal{O}_n/\mathcal{I}$  is a finite integral extension of  $\mathcal{O}_d$ .

*Proof.* For this it's enough to check that the map  $\mathcal{O}_d \rightarrow \mathcal{O}_n/\mathcal{I}$  is finite. That is, that every element of  $\mathcal{O}_n/\mathcal{I}$  can be expressed as a tuple of elements of  $\mathcal{O}_d$ . (We can think of this tuple as a linear combination, or cycle.) Step 1: produce some Weierstrass polynomials. . .  $\square$

**Lemma 7.3.** *If  $\mathcal{I} \subset \mathcal{O}_n$  is prime and  $A = V(\mathcal{I})$ , then  $\mathcal{I}_{A,0} = \mathcal{I}$ .*

*Proof.* Proposition 7.2 says: take a polynomial  $f \in \mathcal{O}_n$  and consider its equivalence class  $\tilde{f} \in \mathcal{O}_n/\mathcal{I}$ . Then  $\tilde{f}$  is the root of some monic polynomial  $t^r + b_{r-1}t^{r-1} + \dots + b_1t + b_0$  with  $b_i \in \mathcal{O}_d$ . That is, putting  $f$  instead of  $t$  we get an equation

$$f^r + b_1f^{r-1} + \dots + b_r = 0 \quad \text{mod } \mathcal{I}.$$

In other words, the holomorphic function  $f^r + b_1f^{r-1} + \dots + b_r$  is in  $\mathcal{I}$ . Since  $f$  vanishes on  $(A,0)$ , this says that  $b_r$  vanishes in  $(A,0)$  too. Now suppose that this implied that the class  $\tilde{b}_r = 0$ . We are supposed to use that Proposition 7.2 also tells us that the above equation is irreducible, which I think means that  $F = \sum b_i t^i \in \mathcal{O}_d[t]$  is irreducible. In turn this gives that  $r = 1$ .  $\square$

Let's just store what was learned today:

- (1) **Noether normalization lemma.** See this video. Let  $k$  be a (infinite) field,  $A \neq 0$  a finitely generated  $k$ -algebra. Then there exists elements  $y_1, \dots, y_r \in A$  which are algebraically independent over  $k$  and such that  $A$  is integral over  $k[y_1, \dots, y_r]$ . That is,

$$\begin{array}{c} A \\ \left| \begin{array}{l} \text{integral} \\ \text{ext.} \end{array} \right. \\ k[y_1, \dots, y_r] \\ \left| \begin{array}{l} \text{transc.} \\ \text{ext.} \end{array} \right. \\ k. \end{array}$$

- (2) This can be used to prove the weak Nullstellensatz, i.e. let  $k$  be an algebraically closed field, and  $I \subseteq k[x_1, \dots, x_n]$  an ideal such that  $V(I) = \emptyset$ . Then  $I = k[x_1, \dots, x_n]$ .

To prove it using Noether's normalization lemma, suppose that the  $I \neq k[x_1, \dots, x_n]$ . Let  $\mathfrak{m}$  be a maximal ideal such that  $I \subseteq \mathfrak{m}$ . Then  $A = k[x_1, \dots, x_n]/\mathfrak{m} = k[y_1, \dots, y_n]$ , where the first string is algebraically independent while the last string is integral. But because the whole extension is algebraic, then  $y_1, \dots, y_n$  are integral over  $k$ . Then we can map  $y_i \mapsto \alpha_i$ , a root of its defining polynomial over  $k$ . This means we do have a point where all polynomials in  $\mathfrak{m}$  vanish.

**Note.** I'm not sure what we mean with "the first string" and the "second string".

- (3) Also reviewed Misha's proof of the Nullstellensatz. Looks accessible, though it is not holomorphic!
- (4) Perhaps start in Demailly just after Prop. 4.13.

## 8. LINE BUNDLES

I have followed mostly [GH78] for this, and also some [Lee24].

For a smooth hypersurface  $X \subset Y$  locally defined as the vanishing of  $f_i \in \mathcal{O}_Y(U_i)$  for an open cover  $U_i$  of  $Y$ , we define the line bundle  $\mathcal{O}_Y(X)$  by the cocycle  $f_{ij} := f_i/f_j$ .

We may also see  $\mathcal{O}_Y(X)$  as a locally free sheaf by assigning to  $U_i$  the group of holomorphic sections of  $\mathcal{O}_Y(X)$ . As explained in [GH78, p. 138], tensoring by a section with divisor  $X$  gives an identification between this group of sections and the meromorphic sections on  $M$  with poles along  $X$ . Ultimately this means that the sheaf corresponding to  $\mathcal{O}_Y(X)$  is locally generated by  $1/f_i$ .

In turn this means that its dual, denoted  $\mathcal{O}_Y(-X)$ , is a line bundle defined by the cocycle  $(f_i/f_j)^{-1}$  and a locally principal sheaf generated by  $f_i$ , that is, the ideal sheaf of  $X$ .

## 9. SINGULARITIES AND DIMENSION

The key statement is that the set of smooth points of a complex variety is dense.



## 10. RIEMANN-ROCH FORMULAS

**Theorem 10.1** (Riemann-Roch for curves). *Let  $D$  be a divisor on a compact Riemann surface  $X$ , that is,  $D$  is a collection of  $d$  points on  $X$ .*

$$(10.1.1) \quad h^0(D) - h^0(K - D) = d - g + 1$$

**Exercise 10.2.** Prove that if  $X$  is a complex smooth curve with  $g \geq 2$  then  $h^0(T_X) = 0$ .

*Proof.* First you notice that the dimension of the holomorphic 1-forms is the genus. This is just because the genus  $g$  is defined as  $h^1(\mathcal{O}_X)$ , which is just  $h^0(K_X)$  by Serre duality. Then you say well if you have a holomorphic vector field, pair it with the nonzero holomorphic 1-form. This gives a nonzero function, but it must be constant because  $X$  is compact. Then it is actually nowhere vanishing. This says the vector field cannot vanish anywhere, which means the tangent bundle of the curve is trivial. Apparently elliptic curves, i.e.  $g = 1$  are the  $\square$

**Theorem 10.3** (Riemann-Roch for line bundles on surfaces). *Let  $L$  be a line bundle on a complex surface  $X$ , and  $K_X = \Omega^2 X$  the canonical bundle of  $X$ . Then*

$$(10.3.1) \quad \chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}(L, L - K_X)$$

## 11. ADJUNCTION FORMULA

Perhaps the most accessible statement that could be interpreted as “adjunction formula” is that the line bundle associated to a smooth hypersurface is the normal bundle.

**Theorem 11.1** (Adjunction formula I). *If  $X$  is a smooth hypersurface of a smooth complex manifold  $Y$ , then*

$$(11.1.1) \quad \mathcal{O}_Y(X)|_X \simeq \mathcal{N}_{X/Y},$$

where  $\mathcal{N}_{X/Y} := T'_Y/T'_X$ .

*Proof.* If  $X$  is locally defined by the vanishing of  $f_i$  on an open cover  $U_i$  of  $Y$ , then  $\mathcal{O}_Y(X)$  is given by the cocycle  $f_{ij} := f_i f_j$ .

To prove Equation 11.1.1 it's enough to show that  $\mathcal{O}_Y(X)|_X \otimes \mathcal{N}_{X/Y}^\vee$  is trivial, which can be done by constructing a nonvanishing section. Such a section is given by

$$df_i = d(f_{ij} f_j) = df_{ij} \cdot f_j + f_{ij} df_j,$$

since the first term vanishes along  $X$  (because  $f_j$  does) and the remaining terms define precisely a section of  $T'^\vee_Y/T'^\vee_X$ , the conormal bundle — indeed, sections of such a bundle are holomorphic differentials vanishing along  $X$ .  $\square$

**Theorem 11.2** (Adjunction formula II). *Let  $X$  be a complex submanifold of a complex manifold  $Y$ . Then*

$$(11.2.1) \quad (K_Y \otimes \mathcal{O}_Y(X))|_X \cong K_X.$$

*Proof.* From the holomorphic tangent bundle exact sequence we get

$$0 \longrightarrow T'_X \longrightarrow T'_Y|_X \longrightarrow \mathcal{N}_{X/Y} := T'_Y/T'_X \longrightarrow 0$$

which in fact has been claimed not to split. But one can prove by hand that

$$T'_Y|_X \cong T'_X \oplus \mathcal{N}_{X/Y}$$

by using the local trivializations of  $T'_X$  and  $N_{X/Y}$ , and then conclude using the local trivializations of both  $T'_X \oplus N_{X/Y}$  and  $T'_Y|_X$ , and the fact that their dimensions coincide, to construct a bundle isomorphism among them.

Taking determinant of both bundles we obtain the result since exterior power of direct sum goes to tensor product.  $\square$

**Lemma 11.3.** *Let  $S$  be any curve on a surface  $X$ . Then*

$$(11.3.1) \quad 2p_a(S) - 2 = (S.S - K_X)$$

*Proof.* If  $S$  is nonsingular we may use the normal bundle short exact sequence. If  $S$  is singular we need to use the ideal sheaf short exact sequence, which means we think of  $S$  either as a closed embedded subscheme of  $X$  or as an analytic subvariety. Taking Euler class we obtain

$$\chi(\mathcal{O}(-S)) + \chi(\mathcal{O}_S) = \chi(\mathcal{O}_X)$$

Since  $p_a(S) := 1 - \chi(\mathcal{O}_S)$ , we obtain that  $p_a(S) = 1 - \chi(\mathcal{O}_X) - \chi(\mathcal{O}(-S))$ . To compute  $\chi(\mathcal{O}(-S))$  we use Riemann-Roch formula ?? for curves on surfaces, which gives

$$\chi(\mathcal{O}(-S)) = \chi(\mathcal{O}_X) + \frac{(\mathcal{O}(-S). \mathcal{O}(-S) + \omega_X^*)}{2}$$

Taking duals on both entries of the intersection form and substituting in our previous expression for  $p_a(S)$  we obtain the result.  $\square$

**Exercise 11.4.** Let  $S$  be a singular, irreducible complex curve on a K3 surface. Prove that  $(S.S) \geq 0$ .

*Proof.* This is just an application of Eq. 11.3.1. Since  $K_X = \mathcal{O}_X$  it suffices to show that  $p_a(S)$  is not zero. This follows fact that  $S$  is singular, since its arithmetic genus is defined as the genus of its normalization, which must be strictly positive since there exists at least one singularity.  $\square$

## 12. CHOW'S THEOREM

**Proposition 12.1.** *Every codimension- $p$  submanifold of a complex  $n$ -manifold has slice charts, i.e. for any  $p \in M$  there's a coordinate chart  $U \ni p$  such that points of  $M$  have coordinates  $(z_1, \dots, z_n, 0, \dots, 0)$ .*

*Proof.* This may be taken as the definition of smooth (real or complex) submanifold, or as a proposition using inverse function theorem (real or complex).  $\square$

**Proposition 12.2.** *Stalk of holomorphic functions is a PID.*

*Proof.* This can be proved via Weierstrass Preparation theorem ?? which implies Lemma ??, and this in turn implies Lemma ??.

Also, it can be shown directly for smooth hypersurfaces as in [Lee24] Lemma 3.38.  $\square$

**Theorem 12.3** (Chow for hypersurfaces). *Every complex codimension-1 submanifold of  $\mathbb{CP}^n$  is algebraic.*

### 13. AMPLENESS

**Definition 13.1.** Let  $L$  be a holomorphic bundle on a complex manifold  $X$ . A point  $x \in X$  is a *base point* of  $L$  if  $s(x) = 0$  for all  $s \in H^0(X, L)$ .

**Definition 13.2.** Let  $L$  be a holomorphic bundle on a complex manifold  $X$ . The *base locus*  $Bs(L)$  is the set of all base points of  $L$ .

**Proposition 13.3.** Let  $L$  be a holomorphic line bundle on a complex manifold  $X$  and suppose that  $s_0, \dots, s_N \in H^0(X, L)$  is a basis. Then

$$\begin{aligned} \varphi_L : X \setminus Bs(L) &\longrightarrow \mathbb{P}^N \\ x &\longmapsto (s_0(x) : \dots : s_N(x)) \end{aligned}$$

defines a holomorphic map such that  $\varphi_L^* \mathcal{O}_{\mathbb{P}^N}(1) \cong L|_{X \setminus Bs(L)}$ .

**Definition 13.4.** The map  $\varphi_L$  in Proposition 13.3 is said to be associated to the *complete linear system*  $H^0(X, L)$  (i.e. the sections of  $L$  are what we call a complete linear system) whereas a subspace of  $H^0(X, L)$  is called a *linear system* of  $L$ .

**Definition 13.5.**  $L$  is *globally generated* by the sections  $s_0, \dots, s_N$  if  $Bs(L, s_0, \dots, s_N) = \emptyset$ .

**Definition 13.6.** A line bundle  $L$  on a complex manifold is called *ample* if for some  $k > 0$  and some linear system in  $H^0(X, L^k)$  the associated map  $\varphi$  is an embedding.

**Exercise 13.7.** Let  $L$  be an ample bundle on a K3 surface  $M$ . Prove that  $L^{\otimes 2}$  is globally generated (that is, for each  $x \in M$  there exists a section  $h \in H^0(L^{\otimes 2})$  which does not vanish in  $x$ ).

*Proof.* As an outline for exposition:

- (1) Show that  $L$  has sections (by ampleness+Kodaira Vanishing+Riemann-Roch), pick a section  $C$  and note that if  $L^{\otimes 2}$  had base points they would be in  $C$ .
- (2) Show that restriction map is surjective (Kodaira Vanishing).
- (3) Show that inclusion map is not surjective. For this it's enough to show that  $\dim |D| = \dim |D - p| + 1$ . This follows by writing the formula of Riemann-Roch for both of these Weil divisors (have to pass to Weil divisors). But you will need to make sure that  $h^0(K - D)$  and  $h^0(K - (D - p))$  are zero.

First we need to show that  $L$  has sections. To see this I will show that since  $L$  is ample,  $L^2 > 0$ . Indeed, if  $mL$  is very ample we apply by Riemann-Roch, Kodaira Vanishing and the fact that  $\chi(\mathcal{O}_X) = 2$  on a K3 to obtain  $h^0(mL) = 2 + \frac{1}{2}m^2L^2$  which must be positive since  $mL$  has sections by being very ample. Notice  $L^2$  cannot be zero as  $h^0(mL) = 2$  would imply we have an embedding of a K3 surface into  $\mathbb{P}^1$ .

Applying again Riemann-Roch and Kodaira vanishing we see that  $h^0(L) > 0$ . Let  $C$  be a section of  $L$ .

Notice that if  $L^{\otimes 2}$  had any base points, they would have to be on  $C$ . Indeed, since  $C$  is the vanishing set of a section  $s \in H^0(L)$ , we must have  $s \otimes s \in H^0(L^{\otimes 2})$  vanishing in any base point of  $L^{\otimes 2}$ .

Then we consider the ideal exact sequence for  $C$  and tensor by  $L^{\otimes 2}$  to obtain

$$0 \longrightarrow L^{\otimes 2}(-C) \longrightarrow L^{\otimes 2} \longrightarrow L^{\otimes 2} \otimes \mathcal{O}_C \longrightarrow 0$$

Notice that the term on the left is actually  $L$  since the ideal sheaf  $\mathcal{O}_M(-C) = \mathcal{I}_C$  is dual to  $L$  because  $C$  is defined as the vanishing set of a section of  $L$ . Passing to cohomology we obtain

$$H^0(L^{\otimes 2}) \longrightarrow H^0(L^{\otimes 2} \otimes \mathcal{O}_C) \longrightarrow H^1(L)$$

where the latter term vanishes by Kodaira Vanishing because  $L$  is ample. This says that every section of  $L^{\otimes 2}|_C$  is the restriction to  $C$  of a section of  $L^{\otimes 2}$ . In turn this shows that it's enough to show that the bundle  $L^{\otimes 2}$  has no base points along  $C$ .

By adjunction formula for (possibly singular) curves on smooth surfaces and by the fact that  $M$  is smooth, we see that  $2p_a(C) - 2 = (L.L)$  so that

$$4p_a(C) - 4 = (2L.L) = \deg_C(L^{\otimes 2})$$

Notice that since  $L^2 > 0$  we exclude the cases that  $p_a(C) = 0, 1$ .

To conclude pick a point  $p \in C$ . Saying that  $p$  is not a base point of  $L^{\otimes 2}|_C$  is the same as saying that not every section of  $L^{\otimes 2}|_C$  vanishes at  $p$ , that is, that  $h^0(L^{\otimes 2}|_C(-p)) < h^0(L^{\otimes 2}|_C)$ .

Now I will prove that since the degree of  $L^{\otimes 2}|_C$  and  $L^{\otimes 2}|_C(p)$  is greater than  $2p_a(C)$  we have that  $h^0(\omega_C \otimes (L^{\otimes 2}|_C)^\vee) = 0 = h^0(\omega_C \otimes (L^{\otimes 2}|_C(-p))^\vee)$ . For this we need to see these line bundles as Weil divisors by simply taking sections, whose vanishing sets are finite sets of points with multiplicities. Then their degree is the sum of the multiplicities. This definition makes degree additive with respect to tensor product (we may take local sections and sum degrees). By Riemann-Roch on the curve  $C$ , we know that  $\deg \omega_C = 2p_a - 2$ . Then

$$\deg_C(\omega_C \otimes L^{\otimes 2}) = \deg_C(\omega_C) + \deg_C(L^{\otimes 2}) = 2p_a - 2 - (4p_a(C) - 4) < 0$$

and a similar computation works for  $L^{\otimes 2}(-p)$  as it has the same degree of  $L^{\otimes 2}$  minus 1. This implies that neither of these bundles can have sections since any section would provide a linearly equivalent effective divisor. This would be a contradiction since both  $L^{\otimes 2}$  and  $L^{\otimes 2}(-p)$  have sections. Therefore  $h^0(\omega_C \otimes (L^{\otimes 2}|_C)^\vee) = 0 = h^0(\omega_C \otimes (L^{\otimes 2}|_C(-p))^\vee)$  as claimed.

Finally we apply Riemann-Roch to  $L^{\otimes 2}$  and  $L^{\otimes 2}(-p)$  to obtain that  $h^0(L^{\otimes 2}) > h^0(L^{\otimes 2}(-p))$ .  $\square$

#### 14. BERTINI'S THEOREM

#### 15. SERRE DUALITY

**Theorem 15.1** (Serre duality). [Voi02] II.5.32. *The pairing*

$$H^q(X, \mathcal{E}) \otimes H^{n-q}(X, \mathcal{E}^* \otimes K_X) \rightarrow H^n(X, K_X) \cong \mathbb{C}$$

*is perfect.*

So when you put the dual  $\vee$  on one of these you get isomorphism.

Our course version says:

$$H^k(X, \mathcal{L})^\vee = H^{n-k}(X, \omega_X \otimes \mathcal{L}^*)$$

## 16. KODAIRA VANISHING THEOREM

**Theorem 16.1** (Kodaira Vanishing). *Suppose  $k$  is a field of characteristic 0, and  $X$  is a smooth projective  $k$ -variety. Then for any ample invertible sheaf  $L$ ,  $H^i(X, K_X \otimes L) = 0$  for  $i > 0$ .*

*Proof.* No proof in [Vak25].  $\square$

## 17. KODAIRA EMBEDDING THEOREM

The forward implication is easy and sometimes not considered as part of the theorem.

Compare this theorem with Nakai-Moishezon Criterion 25.1.

**Theorem 17.1** (Kodaira Embedding). *Suppose  $M$  is a compact complex manifold. A holomorphic line bundle  $L \rightarrow M$  is ample if and only if it is positive. Thus  $M$  is projective if and only if it admits a holomorphic line bundle.*

*Proof.* The easy implication as follows. If  $L$  is ample then there is  $N$  such that  $L^{\otimes N}$  is very ample, meaning by Misha's definition that the canonical map is an embedding such that  $L^{\otimes N} = \varphi^* \mathcal{O}(1)$ , which has degree 1 by definition as in [Vak25, 15.4.14].  $\square$

## 18. NÉRON-SEVERI GROUP

Instead of considering the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{M}_X^*) \longrightarrow H^0(X, \mathcal{M}_X^* / \mathcal{O}_X^*) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow 0$$

which says that  $\text{Pic}(X)$  is isomorphic to Cartier divisors modulo linear equivalence, we consider

$$0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}(X) / \text{Pic}^0(X) := \text{NS}(X) \longrightarrow 0$$

where  $\text{Pic}^0(X)$  is the connected component of the identity in the Picard scheme, which is a scheme representing some functor. The idea is that  $\text{Pic}^0(X)$ -equivalence is like linear equivalence, but different. The Néron-Severi group describes line bundles modulo this other equivalence.

## 19. HYPERCOMPLEX MANIFOLDS

**Exercise 19.1.** Let  $M$  be a compact hypercomplex manifold of real dimension 4, equipped with a quaternionic Hermitian structure, and  $V$  the space of closed  $\text{SU}(2)$ -invariant 2-forms. Prove that  $V$  is finite-dimensional.

*Proof by Arpan.* Consider the Hodge star operator with respect to the Riemannian metric on  $M$ .

- (1) A closed anti-self-dual form is harmonic. Indeed, by some characterization a form is harmonic  $\iff$  it is  $d$ -closed and  $d^*$ -closed. So if  $\alpha$  is self-dual and  $d$ -closed we get  $d^* \alpha = (-1)^{-\bullet} * d * \alpha = 0$  since  $d\alpha = 0$ .
- (2) The space of harmonic forms is finite-dimensional (analysis, cf. Fredholm theory).
- (3) I should be able to prove that an  $\text{SU}(2)$ -invariant closed form is anti self-dual.

- (a) Define  $\Lambda^+ := \text{self-dual 2-forms}$  and  $\Lambda^- := \text{anti-self dual 2-forms}$ . Notice that  $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ , this is by 2.1, since we can see  $*$  as an involutive endomorphism of  $\Lambda^2$ , so that it has eigenvalues  $\pm 1$ , given the desired decomposition as its eigenspace decomposition.
- (b) **Claim.**  $\Lambda^+ = \text{span}(\omega_I, \omega_J, \omega_K)$ .
- (c) **Claim.**  $\omega_I, \omega_J$  and  $\omega_K$  are  $\text{SU}(2)$ -invariant.  
The last two items imply that  $\text{SU}(2)\Lambda^+ = \Lambda^+$ .
- (d) **Claim.**  $\text{SU}(2) \curvearrowright \Lambda^+$  has no fixed points.  
We conclude that if  $\alpha$  is  $\text{SU}(2)$ -invariant (i.e. a fixed point of  $\text{SU}(2) \curvearrowright \Lambda^2$ ) its positive part will vanish, so that  $\alpha$  is anti-self-dual.

□

*Proof.* Idea: show that Hodge star  $*\omega$  is in the  $\text{SU}(2)$ -orbit of  $\omega$  and conclude that  $\int \omega \wedge *\omega = 0$ , implying that  $\|\omega\| = 0$ . □

### Exercise 19.2.

*Proof.* Idea: find a counterexample. The easiest should be a Kummer surface. It looks possible to find an almost hypercomplex structure on  $\mathbb{C}^2$  passing to the torus and then to the  $\mathbb{Z}_2$  quotient, but not clear what will happen after blowing up. □

## 20. DEFORMATION THEORY (INTRODUCTION)

Here I discuss deformations in the category of complex spaces.

Let's just do a summary of [Voi02, Chapter 9].

Let  $\mathcal{X}$  be a complex manifold,  $B$  a complex manifold and  $\phi : \mathcal{X} \rightarrow B$  a holomorphic map. We call  $\phi$  a *family* if it is a proper holomorphic submersion.

A theorem by Ehresmann says that any “deformation” of smooth real manifolds is trivial:

**Theorem 20.1** (Ehresmann). *If  $\phi : \mathcal{X} \rightarrow B$  is a proper submersion of differentiable manifolds (no complex structure), with  $B$  contractible and  $0 \in B$  some distinguished point, then the total space  $\mathcal{X}$  is trivial as a manifold over  $B$ , i.e. there exists a diffeomorphism*

$$T : \mathcal{X} \xrightarrow{\cong} X_0 \times B$$

*of manifolds over  $B$ , i.e. commuting with projection.*

*Proof.* The main tool is the theorem of tubular neighbourhoods, which says that there is a diffeomorphism of a neighbourhood  $W$  of  $X_0$  in  $\mathcal{X}$  and a neighbourhood of  $X_0$  in its normal bundle. (We have a normal bundle because  $X_0$  is a submanifold of  $\mathcal{X}$ .)

In particular this gives a differentiable retraction  $T_0 : W \rightarrow X_0$ , which allows us to consider the map

$$(T_0, \phi) : W \rightarrow X_0 \times B.$$

Since  $T_0$  is a retraction (it's the identity when restricted to  $X_0$ ) and  $\phi$  is a submersion, the map  $(T_0, \phi)$  has nonsingular differential along  $X_0$ . Then we can invert it locally along  $X_0$ , and since  $X_0$  is compact we obtain a whole neighbourhood  $W' \subset W \subset \mathcal{X}$  where the differential is nonzero and thus  $(T_0, \phi)|_{W'}$  is an embedding (using the inverse function theorem).

Finally we restrict the image of  $(T_0, \phi)|_{W'}$  to a neighbourhood  $U \subset B$  of 0, using properness of  $\phi$  to ensure that  $\phi^{-1}(U) \subset W'$  — though I'm not exactly sure how.

We have constructed a diffeomorphism  $\phi^{-1}(U) \rightarrow X_0 \times U$  commuting with the projection.

This proof was only local; the global version is in [Voi02] though it's not used in the rest of the text.  $\square$

In the complex situation, if  $\phi : \mathcal{X} \rightarrow B$  is a family of complex manifolds and  $0 \in B$ , then “up to replacing  $B$  by a neighbourhood of 0”, there exists a smooth trivialization  $T = (T_0, \phi) : \mathcal{X} \rightarrow X_0 \times B$  such that the fibres of  $T_0$  are complex submanifolds of  $\mathcal{X}$ .

Now let  $\phi : \mathcal{X} \rightarrow B$  be a family of complex manifolds. Recall the pullback bundle  $\phi^*B$  and the fact that the differential  $\phi_*$  can be thought of as a gadget that maps tangent vectors to  $\mathcal{X}$  not to vectors tangent to  $B$  but to vectors in the pullback bundle  $\phi^*B$ . Restricting everything to the fibre  $X = \phi^{-1}(0)$  we get

$$(20.1.1) \quad 0 \longrightarrow T_X \longrightarrow T_{\mathcal{X}|X} \longrightarrow \phi^*T_{B|X} \longrightarrow 0.$$

There's what seems to be a really nice result saying that  $H^1(\mathcal{F})$  parametrizes isomorphism classes of extensions of  $\mathcal{F}$  by the trivial bundle, but for now let me just define it roughly as follows. The category of  $\mathcal{O}_X$  is most surely abelian with enough injectives. This means that objects have associated complexes, and we can take cohomologies of these complexes. This says, roughly, that short exact sequences of vector bundles have associated cohomology long exact sequences. Then the Kodaira-Spencer map is the map

$$\rho : T_{B,0} = H^0(X, \phi^*T_{B|X}) \rightarrow H^1(X, T_X)$$

induced by the long exact sequence associated to 20.1.1. Here the equality  $T_{B,0} = H^1(X, \phi^*T_{B|X})$  is just because the pullback of the tangent bundle of  $B$  to any fiber is trivial.

Now I will explain first order deformations. Consider a family  $\mathcal{X} \rightarrow B$ . We introduce a complex analytic subscheme  $B_\varepsilon$  of  $B$  giving an ideal sheaf  $\mathcal{M}_0^2$  where  $\mathcal{M}_0$  is the maximal ideal of  $\mathcal{O}_{B,0}$  consisting of the holomorphic functions vanishing at 0. This means that a function in  $\mathcal{M}_0^2$  is (a finite sum of terms) of the form  $f = gh$  for  $g, h \in \mathcal{M}_0$ , which means that  $df = d(gh) = h dg + g dh = 0$  at 0 since both  $g$  and  $h$  vanish at 0. That is, this sheaf contains functions that vanish at 0 and whose first derivative vanishes at 0.

This means that  $B_\varepsilon$  is a point: it's the vanishing locus of the set of functions that vanish at a single point. However, it is endowed with an unusual structure sheaf. Let  $v$  be a tangent vector to  $B$  at 0 and  $f$  a function in the strange sheaf  $\mathcal{M}_0/\mathcal{M}_0^2$ . Then by the computation above, since  $v$  is a differential operator,  $v$  annihilates  $f$ . This means that this sheaf detects not only the point 0 but also any tangent vector at 0. Let's denote  $B_\varepsilon = \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)$  just because its structure sheaf is  $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ .

In fact, this is so throughout deformation theory:  $\text{Spec } k[\varepsilon]/(\varepsilon^2)$  is topologically a point! But it's a *fat point*: the support of its structure sheaf is a point, but as a scheme, it's not a point. And the same happens with the family  $\mathcal{X}$  over  $\text{Spec } k[\varepsilon]/(\varepsilon^2)$ , topologically it's just the fibre  $X$ ! But its structure sheaf is different: “it contains the directions in  $\mathcal{X}$  which are normal to  $X$ ”.

Perhaps the most important part of this construction comes when considering a fat point  $B_\varepsilon \subset B$ . This gives a first-order deformation  $\phi^{-1}(B_\varepsilon) \subset \mathcal{X}$

$$\begin{array}{ccc} X & \longrightarrow & \phi^{-1}(B_\varepsilon) \subset \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & B_\varepsilon \subset B \end{array}$$

Using the constructions above, one shows that to any such deformation we can associate a cocycle in  $H^1(X, T_X)$ . Thus, first order deformations are “exactly parametrised” by the group  $H^1(X, T_X)$ . This is the main idea behind Kodaira-Spencer map, which is well-explained in Wikipedia.

I will present a similar construction to Kodaira-Spencer map that works for embedded deformations instead of just deformations. (That’s with the intention of understanding the forthcoming exercise Exercise 21.3.)

Since we are in complex space category I follow [Voi02, p. 224] for the main setup, but this is also a mixture of [?, p. 147, 150].

The result is in fact quite reasonable geometrically: the dimension of directions in which we can deform a submanifold should be non other than the dimension of the normal bundle.

## 21. DEFORMATION THEORY

**Definition 21.1.** Let  $X$  be a complex manifold. A *deformation* of  $X$  is a holomorphic submersion  $\pi : \mathcal{X} \rightarrow B$  of complex manifolds such that  $\pi^{-1}(0) \cong X$  for some  $0 \in B$ .

If  $\mathcal{X}$  and  $\mathcal{Y}$  are two deformations of  $X$  over the same base, a *morphism of deformations* is a map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  such that the diagram

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \searrow & & \swarrow \\ & B & \end{array}$$

commutes. The *space of deformations* of  $X$  is the set of deformations modulo the equivalence relation of two deformations being isomorphic. A priori we don’t know if this set has any structure.

Let  $Y$  be another complex manifold. A *deformation of  $X$  in  $Y$*  is a deformation of  $X$  such that  $\mathcal{X} \subset Y \times B$ .

The *fat point* is a ringed space  $(B_\varepsilon, \mathbb{C}[\varepsilon]/(\varepsilon^2))$  whose underlying topological space  $B_\varepsilon$  is a point and structure sheaf is the ring of dual numbers  $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ .

A *first order deformation* of  $X$  is a deformation such that the base  $B$  is a fat point.

The following proposition is a mixture of ideas from [Voi02, p. 224] and [Ser06, Examples 3.2.4(ii)].



**Proposition 21.2.** *Let  $X$  be a smooth hypersurface in a complex manifold  $Y$ . For every deformation of  $X$  in  $Y$  with base of dimension 1 we can construct a section of the bundle  $N_{X/Y}$ . Different deformations give different sections up to rescaling.*

*Proof.* Let

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \subset Y \times B \\ \downarrow & & \downarrow \pi \\ 0 & \longrightarrow & B \end{array}$$

be a deformation of  $X$  in  $Y$  and suppose that  $\dim B = 1$ . This means that  $\mathcal{X}$  is a complex submanifold of the product  $Y \times B$  and  $\pi : \mathcal{X} \rightarrow B$  is a holomorphic submersion.

Since  $X$  is a hypersurface, we can find an open cover  $V_i$  of  $M$  such that  $V_i \cap X := U_i$  is given locally by the vanishing of a function  $f_i \in \mathcal{O}_Y(V_i)$ . In terms of coordinates we find that  $V_i \cap X$  is given by  $(z_1, \dots, z_{n-1}, 0)$ .

Since  $\pi$  is a submersion, by the holomorphic inverse function theorem we obtain for every  $V_i \subset \mathcal{X}$  (such that  $V_i \cap X \neq \emptyset$ ) a biholomorphism from some  $B_i \subset B$  containing 0 to  $V_i$ . This gives coordinates  $(0, \dots, b)$  to  $V_i$ . We conclude that  $V_i \cong U_i \times B_i$  along  $X$ .

Now we “restrict” to a first order deformation. Consider the fat point subscheme  $B_\varepsilon$  of  $B$  (I think this is the reason why  $B$  must have dimension 1, in Voisin notation, that a first order neighbourhood of 0 in  $B$  is indeed a fat point) and let  $X_\varepsilon = \pi^{-1}(B_\varepsilon)$ , which is a ringed space whose topological space is the same as  $X$ , but its structure sheaf is induced from  $B_\varepsilon$ .

(I think that to prove this part I have to look carefully at the induced sheaf  $\pi^*(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ .) Since we are on a local trivialization, the structure sheaf of the product  $U_i \times B_i$  is the tensor product of the structure sheaves (reference?), so that the structure sheaf of  $X_\varepsilon$  at  $U_i \times B_\varepsilon$  is given by

$$\mathcal{O}_{X_\varepsilon}(U_i) = \mathcal{O}_Y(U_i) \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2).$$

Functions here look like finite sums of functions of the form  $f + \varepsilon g$  for  $f, g \in \Gamma(U_i, \mathcal{O}_{U_i})$ .

The next key step is to argue that  $X_\varepsilon$  is locally cut by functions of the kind  $f_i + \varepsilon g_i$  for some  $g_i \in \mathcal{O}_M(U_i)$ . Here the  $f_i$  are the same as those which define  $X$ . I think a proof is that from the ideal sheaf exact sequence of  $X$  in  $Y$

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0$$

we obtain after tensoring by  $\mathbb{C}[\varepsilon]/(\varepsilon^2)$

$$0 \longrightarrow \mathcal{I}_X \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathcal{O}_Y \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathcal{O}_X \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow 0.$$

(But is the functor  $- \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)$  exact? Why? So far we didn’t impose a flatness condition on  $\pi$ .) The sheaf in the right coincides with the structure sheaf of  $X_\varepsilon$  when evaluated on a trivial open set of the form  $U_i \otimes B_\varepsilon$ . This shows that the ideal sheaf of  $X_\varepsilon$  as a ringed space is  $\mathcal{I}_X \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)$ . Since  $\mathcal{I}_X$  is locally principal generated by  $f_i$ , we conclude that indeed  $X_\varepsilon$  must be cut locally by functions of the form  $f_i + \varepsilon g_i$  for some  $g_i \in \mathcal{O}_Y(U_i)$

Recall that the line bundle associated to  $X$  is  $\mathcal{O}_Y(X)$ , given by the cocycle  $f_i/f_j := f_{ij}$ . A version of adjunction formula (more precisely, [GH78, Adjunction formula I, p. 146]) establishes that this line bundle, when restricted to  $C$ , is in fact

the normal bundle of  $X$  in  $Y$ , defined as the quotient bundle  $T'_Y/T'_X := N_{X/Y}$ . In symbols,  $\mathcal{O}_Y(X)|_X \cong N_{X/Y}$ .

Similarly, the line bundle associated to  $\mathcal{X}$  is given by the cocycle

$$F_{ij} = f_{ij} + \varepsilon g_{ij}$$

for  $g_{ij} = \frac{g_i - f_{ij}g_j}{f_j}$ . Indeed, note that  $\frac{1}{f_j + \varepsilon g_j} = \frac{1}{f_j} - \varepsilon \frac{g_j}{f_j^2}$  and compute  $F_{ij} = (f_i + \varepsilon g_j)/(f_j + \varepsilon g_j)$ .

Then

$$f_i + \varepsilon g_i = (f_{ij} + \varepsilon g_{ij})(f_j + \varepsilon g_j)$$

gives

$$g_i = f_{ij}g_j + g_{ij}f_j$$

and since  $f_j$  vanishes along  $X$ , we are left with  $g_i = f_{ij}g_j$  which is how a section of the sheaf  $N_{X/Y}$  is obtained.  $\square$

**Exercise 21.3.** Let  $C$  be a smooth genus  $g$  curve which can be embedded in a K3 surface  $M$ , and  $\mathcal{X}$  “the family of all deformations” of  $C$  in  $M$ . Prove that  $\dim \mathcal{X} \leq g$ .

*Proof.* By Adjunction formula II (Theorem 11.2) and  $M$  being a K3 surface we get that  $h^0(C, N_{C/M}) = g$ .

If the deformation space  $X$  exists and is a complex manifold, the map constructed in Proposition 21.2 would be a holomorphic (why holomorphic!?) injective map  $X \rightarrow H^0(C, N_{C/Y})$ . This obliges  $\dim X \leq h^0(C, N_{C/Y}) = g$ . Indeed, if not, the differential must have a kernel, and integrating along a direction in the kernel we obtain the germ of a complex curve along which the map is constant, which is impossible since it is injective.  $\square$

## 22. KÄHLER CONE

**Exercise 22.1.** Let  $M$  be a K3 surface such that  $\text{rkNS}(M) < 20$ . Using Demailly-Paun theorem, prove that there exists a non-zero vector  $\eta \in H^{1,1}(M)$  which satisfies  $\eta^2 = 0$  and belongs to the boundary of the Kähler cone.

Here’s an idea: By Demailly-Paun for K3 (on page 272 of slides.pdf), we have that the Kähler cone is contained in the positive cone. Then perhaps use [?, Theorem 7.1] to conclude to show find a class  $\eta$  in the closure of the Kähler cone. If we can show that  $\eta$  also has  $\eta^2 = 0$ , we conclude that it must be in the *boundary* of the Kähler cone, since by Demailly-Paun the Kähler cone is contained in the positive cone.

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