ALGEBRAIC GEOMETRY

github.com/danimalabares/stack

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1. Sheaves

For a definition of presheaf, see Categories, Definition **??**.

Definition 1.1. Let X be a topological space.

(1) A *sheaf* $\mathcal{F}$ *of sets on* X is a presheaf of sets which satisfies the following additional property: Given any open covering $U = \bigcup_{i\in I} U_i$ and any collection

of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.

(2) A *morphism of sheaves of sets* is simply a morphism of presheaves of sets.
(3) The category of sheaves of sets on X is denoted $Sh(X)$.

Let X be a topological space. Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The *stalk of $\mathcal{F}$ at x* is the set

$$\mathcal{F}_x = \operatorname{colim}_{x \in U} \mathcal{F}(U)$$

where the colimit is over the set of open neighbourhoods U of x in X. The set of open neighbourhoods is partially ordered by (reverse) inclusion: We say $U \geq U' \Leftrightarrow U \subset U'$. The transition maps in the system are given by the restriction maps of $\mathcal{F}$. See Categories, Section **??** for notation and terminology regarding (co)limits over systems. Note that the colimit is a directed colimit. Thus it is easy to describe $\mathcal{F}_x$. Namely,

$$\mathcal{F}_x = \{(U, s) \mid x \in U, s \in \mathcal{F}(U)\}/\sim$$

with equivalence relation given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ with $x \in U''$ and $s|_{U''} = s'|_{U''}$. Given a pair (U, s) we sometimes denote s_x the element of $\mathcal{F}_x$ corresponding to the equivalence class of (U, x). We sometimes use the phrase "image of s in $\mathcal{F}_x$" to denote s_x. For example, given two pairs (U, s) and (U', s') we sometimes say "s is equal to s' in $\mathcal{F}_x$" to indicate that $s_x = s'_x$. Other authors use the terminology "germ of s at x".

2. Abelian sheaves

The following may be used to define the ideal sheaf of a variety:

Lemma 2.1. *Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ be sheaves over X with values on Grp. For every morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$,*

$$\begin{aligned} \operatorname{Ker} f : \mathit{Open}_X^{op} &\longrightarrow \mathit{Sets} \\ U &\longmapsto \operatorname{Ker} f(U) \\ (i : V \to U) &\longmapsto {}^{\operatorname{Ker} f(i) : \operatorname{Ker} f(U) \to \operatorname{Ker} f(V)}_{\quad x \mapsto x} \end{aligned}$$

is a sheaf over X.

Proof. First observe that the correspondence on morphisms is well-defined. Indeed, $\operatorname{Ker} f(U) \subset \mathcal{F}(U) \subset \mathcal{F}(V)$ when $V \subset U$, and we just apply $f(U)$ to notice that $\operatorname{Ker} f(V) \subset \operatorname{Ker} f(U)$.

To see this is a presheaf notice it is obvious that the identity is mapped to the identity by definition of the correspondence of morphisms. It is also obvious that composition is preserved.

To see it is a sheaf consider an open cover U_i of U, and elements $x_i \in \operatorname{Ker} f(U_i)$. Then use the property of $\mathcal{F}$ being a sheaf to reconstruct an element $x \in \mathcal{F}(U)$, whose image under f will be mapped to the identity element of $\mathcal{G}(U)$ because it does so in every point of the cover of U. Thus x is in $\operatorname{Ker} f(U)$ as desired. $\square$

More formally,

Definition 2.2. Let X be a topological space.

(1) A *presheaf of abelian groups on* X or an *abelian presheaf over* X is a presheaf of sets $\mathcal{F}$ such that for each open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an abelian group, and such that all restriction maps ρ^U_V are homomorphisms of abelian groups, see Lemma **??** above.
(2) A *morphism of abelian presheaves over* X $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets which induces a homomorphism of abelian groups $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$.
(3) The category of presheaves of abelian groups on X is denoted $PAb(X)$.

Definition 2.3. Let X be a topological space.
(1) An *abelian sheaf on* X or *sheaf of abelian groups on* X is an abelian presheaf on X such that the underlying presheaf of sets is a sheaf.
(2) The category of sheaves of abelian groups is denoted $Ab(X)$.

Let X be a topological space. In the case of an abelian presheaf $\mathcal{F}$ the sheaf condition with regards to an open covering $U = \bigcup U_i$ is often expressed by saying that the complex of abelian groups

$$0 \to \mathcal{F}(U) \to \prod\nolimits_i \mathcal{F}(U_i) \to \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is exact. The first map is the usual one, whereas the second maps the element $(s_i)_{i\in I}$ to the element

$$(s_{i_0}|_{U_{i_0}\cap U_{i_1}} - s_{i_1}|_{U_{i_0}\cap U_{i_1}})_{(i_0,i_1)} \in \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

In fact, the notion of kernel of a sheaf is not really defined as I did in the beginning of this section, but in the next one, along with several other important things.

3. The abelian category of sheaves of modules

I guess that the reason to introduce coherent sheaves is not the search for an abelian category, after all. Looks like the pathologies avoided by the definition of coherence are not so obvious—something like “wildly infinitely generated”.

Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Definition **??**. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, see Sheaves, Definition **??**. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subset X$ is the sum of the maps induced by φ, ψ. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus $\mathit{Mod}(\mathcal{O}_X)$ is a pre-additive category, see Homology, Definition **??**.

We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subset X$. Clearly this is both a final and an initial object of $\mathit{Mod}(\mathcal{O}_X)$. Given a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each open U, and (d) $\varphi_x = 0$ for all $x \in X$. See Sheaves, Lemma **??**.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition **??**. Thus $Mod(\mathcal{O}_X)$ is an additive category, see Homology, Definition **??**.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define $\mathrm{Ker}(\varphi)$ to be the subsheaf of $\mathcal{F}$ with sections

$$\mathrm{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all open $U \subset X$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}_X$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\mathrm{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$. Moreover, on the level of stalks we have $\mathrm{Ker}(\varphi)_x = \mathrm{Ker}(\varphi_x)$.

On the other hand, we define $\mathrm{Coker}(\varphi)$ as the sheaf of $\mathcal{O}_X$-modules associated to the presheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \mathrm{Coker}(\mathcal{F}(U) \to \mathcal{G}(U)) = \mathcal{G}(U)/\varphi(\mathcal{F}(U)).$$

Since taking stalks commutes with taking sheafification, see Sheaves, Lemma **??** we see that $\mathrm{Coker}(\varphi)_x = \mathrm{Coker}(\varphi_x)$. Thus the map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ is surjective (as a map of sheaves of sets), see Sheaves, Section **??**. To show that this is a cokernel, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}_X$-modules such that $\beta \circ \varphi$ is zero, then you get for every open $U \subset X$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma **??**) we obtain a canonical map $\mathrm{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \mathrm{Coker}(\varphi) \to \mathcal{H}$. The morphism $\mathrm{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

Lemma 3.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is an abelian category. Moreover a complex*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact at $\mathcal{G}_x$.

Proof. By Homology, Definition **??** we have to show that image and coimage agree. By Sheaves, Lemma **??** it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathcal{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. $\square$

Actually the category $Mod(\mathcal{O}_X)$ has many more properties. Here are two constructions we can do.

(1) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the product

$$\prod\nolimits_{i \in I} \mathcal{F}_i$$

which is the sheaf that associates to each open U the product of the modules $\mathcal{F}_i(U)$. This is also the categorical product, as in Categories, Definition **??**.

(2) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the direct sum

$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i$$

which is the *sheafification* of the presheaf that associates to each open U the direct sum of the modules $\mathcal{F}_i(U)$. This is also the categorical coproduct, as in Categories, Definition **??**. To see this you use the universal property of sheafification.

Using these we conclude that all limits and colimits exist in $Mod(\mathcal{O}_X)$.

Lemma 3.2. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. *All limits exist in $Mod(\mathcal{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathcal{O}_X$-modules (i.e., commute with taking sections over opens).*
2. *All colimits exist in $Mod(\mathcal{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.*
3. *Filtered colimits are exact.*
4. *Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathcal{O}_X$-modules.*

Proof. As $Mod(\mathcal{O}_X)$ is abelian (Lemma 3.1) it has all finite limits and colimits (Homology, Lemma **??**). Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description (see discussion above) and Categories, Lemmas **??** and **??**. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks. Part (3) signifies that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of $\mathcal{O}_X$-modules over a directed set I the sequence $0 \to \operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact as well. Since we can check exactness on stalks (Lemma 3.1) this follows from the case of modules which is Algebra, Lemma **??**. We omit the proof of (4). $\square$

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section **??**. See Homology, Lemma **??** for a description of exactness properties in terms of short exact sequences.

Lemma 3.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.*

1. *The functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is left exact. In fact it commutes with all limits.*
2. *The functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is right exact. In fact it commutes with all colimits.*
3. *Pullback $f^{-1} : Ab(Y) \to Ab(X)$ on abelian sheaves is exact.*

Proof. Parts (1) and (2) hold because (f^*, f_*) is an adjoint pair of functors, see Sheaves, Lemma **??** and Categories, Section **??**. Part (3) holds because exactness can be checked on stalks (Lemma 3.1) and the description of stalks of the pullback, see Sheaves, Lemma **??**. $\square$

Lemma 3.4. *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_! : Ab(U) \to Ab(X)$ is exact.*

Proof. Follows from the description of stalks given in Sheaves, Lemma **??**. $\square$

Lemma 3.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathcal{F}_i$ be a sheaf of $\mathcal{O}_X$-modules. For $U \subset X$ quasi-compact open the map*

$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i(U) \longrightarrow \left(\bigoplus\nolimits_{i \in I} \mathcal{F}_i\right)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j \in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i \in I_j} s_{ji}$ with $s_{ji} \in \mathcal{F}_i(U_j)$. Because U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j \in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i \in I'} \mathcal{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed, see Lemma 3.2 above. □

4. Tensor product of sheaves

Here's my unexpected encounter with the definition of tensor product of sheaves. It's not the "fiber is tensor product of fibers" construction, but actually just some notion of "change of ring" sheaf that ends up being adjoint to some "restriction" sheaf. The setting is a mapping of presheaves *of rings* over a space X… (I think the usual definition is this one taking $\mathcal{O}_1$ as the other presheaf we want to tensor).

Immediately after introducing this notion there's the definition of sheaf, then stalks, abelian sheaves, some other notions like an "algebraic structure" and then tensor product will be defined after sheafification—because the following definition is in general not a sheaf.

Furthermore, I add that Vakil leaves it as an exercise to define the tensor product of two $\mathcal{O}_X$ modules (with a hint of defining the presheaf tensor product and sheafifying), which makes me think that after all it *is* just the intuitive definition. Before diving in, also by Vakil (Exercise 26.K): the stalk of the tensor product is the tensor product of the stalks.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of* $\mathcal{F}$. We obtain the restriction functor

$$PMod(\mathcal{O}_2) \longrightarrow PMod(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$PMod(\mathcal{O}_1) \longrightarrow PMod(\mathcal{O}_2)$$

Lemma 4.1. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. □

4.2. **Tipologia dos feixes.**

Definition 4.3. A sheaf of $\mathcal{A}$-modules $\mathcal{F}$ over a sheaf of rings $\mathcal{A}$ (on a topological space X) is called

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5. Locally ringed spaces

6. Morphisms

Induced map on rings, etc.

7. Dominant morphisms

The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

Definition 7.1. A morphism $f : X \to S$ of schemes is called *dominant* if the image of f is a dense subset of S.

8. Morphisms of finite type

Recall that a ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \dots, x_n]$ as an R-algebra, see Algebra, Definition **??**.

Definition 8.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is of *finite type at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
(2) We say that f is *locally of finite type* if it is of finite type at every point of X.
(3) We say that f is of *finite type* if it is locally of finite type and quasi-compact.

9. Flat morphisms

There is a lot of information on Stacks Project about flatness. It looks like the heart of the concept is captured in the commutative-algebraic notion of preserving exact sequences:

Definition 9.1. Let R be a ring.

(1) An R-module M is called *flat* if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.
(2) An R-module M is called *faithfully flat* if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.
(3) A ring map $R \to S$ is called *flat* if S is flat as an R-module.
(4) A ring map $R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module.

Recall that a module M over a ring R is *flat* if the functor $- \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is exact. A ring map $R \to A$ is said to be *flat* if A is flat as an R-module.

10. Invertible modules (line bundles)

Similarly to the case of modules over rings (More on Algebra, Section **??**) we have the following definition.

Definition 10.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An *invertible $\mathcal{O}_X$-module* is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that the functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an equivalence of categories. We say that $\mathcal{L}$ is *trivial* if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$.

Lemma 10.4 below explains the relationship with locally free modules of rank 1.

Lemma 10.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Equivalent are*

1. *$\mathcal{L}$ is invertible, and*
2. *there exists an $\mathcal{O}_X$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \cong \mathcal{O}_X$.*

In this case $\mathcal{L}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module and the module $\mathcal{N}$ in (2) is isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Proof. Assume (1). Then the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ is essentially surjective, hence there exists an $\mathcal{O}_X$-module $\mathcal{N}$ as in (2). If (2) holds, then the functor $- \otimes_{\mathcal{O}_X} \mathcal{N}$ is a quasi-inverse to the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ and we see that (1) holds.

Assume (1) and (2) hold. Denote $\psi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{O}_X$ the given isomorphism. Let $x \in X$. Choose an open neighbourhood U an integer $n \geq 1$ and sections $s_i \in \mathcal{L}(U)$, $t_i \in \mathcal{N}(U)$ such that $\psi(\sum s_i \otimes t_i) = 1$. Consider the isomorphisms

$$\mathcal{L}|_U \to \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{N}|_U \to \mathcal{L}|_U$$

where the first arrow sends s to $\sum s_i \otimes s \otimes t_i$ and the second arrow sends $s \otimes s' \otimes t$ to $\psi(s' \otimes t)s$. We conclude that $s \mapsto \sum \psi(s \otimes t_i)s_i$ is an automorphism of $\mathcal{L}|_U$. This automorphism factors as

$$\mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{L}|_U$$

where the first arrow is given by $s \mapsto (\psi(s \otimes t_1), \ldots, \psi(s \otimes t_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i s_i$. In this way we conclude that $\mathcal{L}|_U$ is a direct summand of a finite free $\mathcal{O}_U$-module.

Assume (1) and (2) hold. Consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

To finish the proof of the lemma we will show this is an isomorphism by checking it induces isomorphisms on stalks. Let $x \in X$. Since we know (by the previous paragraph) that $\mathcal{L}$ is a finitely presented $\mathcal{O}_X$-module we can use Lemma **??** to see that it suffices to show that

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{L}_x, \mathcal{O}_{X,x}) \longrightarrow \mathcal{O}_{X,x}$$

is an isomorphism. Since $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x = (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N})_x = \mathcal{O}_{X,x}$ (Lemma **??**) the desired result follows from More on Algebra, Lemma **??**. $\square$

Lemma 10.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{L}$ of an invertible $\mathcal{O}_Y$-module is invertible.*

Proof. By Lemma 10.2 there exists an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{N} \cong \mathcal{O}_Y$. Pulling back we get $f^*\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N} \cong \mathcal{O}_X$ by Lemma **??**. Thus $f^*\mathcal{L}$ is invertible by Lemma 10.2. ☐

Lemma 10.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any locally free $\mathcal{O}_X$-module of rank 1 is invertible. If all stalks $\mathcal{O}_{X,x}$ are local rings, then the converse holds as well (but in general this is not the case).*

Proof. The parenthetical statement follows by considering a one point space X with sheaf of rings $\mathcal{O}_X$ given by a ring R. Then invertible $\mathcal{O}_X$-modules correspond to invertible R-modules, hence as soon as $\mathrm{Pic}(R)$ is not the trivial group, then we get an example.

Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

Looking over an open covering trivialization $\mathcal{L}$, we see that this map is an isomorphism. Hence $\mathcal{L}$ is invertible by Lemma 10.2.

Assume all stalks $\mathcal{O}_{X,x}$ are local rings and $\mathcal{L}$ invertible. In the proof of Lemma 10.2 we have seen that $\mathcal{L}_x$ is an invertible $\mathcal{O}_{X,x}$-module for all $x \in X$. Since $\mathcal{O}_{X,x}$ is local, we see that $\mathcal{L}_x \cong \mathcal{O}_{X,x}$ (More on Algebra, Section **??**). Since $\mathcal{L}$ is of finite presentation by Lemma 10.2 we conclude that $\mathcal{L}$ is locally free of rank 1 by Lemma **??**. ☐

Now I introduce some of the properties of line bundles, Cartier divisors and so on.

Lemma 10.5. *The ideal sheaf of an effective Cartier divisor (a subscheme locally defined by the vanishing of a single function) is an invertible sheaf.*

Proof. We just need to check that the generator of the ideal sheaf at any affine set is not a zerodivisor. This follows from the ideal sheaf exact sequence, which implies that multplication by the generator is injective:

$$0 \longrightarrow I \cong A \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

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11. Ampleness

First is this lemma that comes from modules.tex. I think these sets X_s are the base points of the bundle. Because look: image of s just means consider the section s of the line bundle as a germ near x. Now a line bundle is a locally free rank-1 $\mathcal{O}_X$-module, so its sections, like s, may be multiplied by germs of functions in the maximal ring $\mathfrak{m}_x$, i.e. the functions that vanish at x. So X_s is the vanishing locus of the section s. If $s(x) \neq 0$, obviously $s \notin \mathfrak{m}_x\mathcal{L}_x$, so $x \in X_s$. Conversely, I would like to show that if $s(x) = 0$ then $s \in \mathfrak{m}_x\mathcal{L}_x$ but I'm not sure how. It's like: a vector field with a zero can be multiplied by a function that vanishes at the point, sure, but what's this function?

Lemma 11.1. *From modules.tex. Let X be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set*

$$X_s = \{x \in X \mid \textit{image } s \notin \mathfrak{m}_x\mathcal{L}_x\}$$

is open in X. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathcal{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

Proof. Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma **??**. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma **??** that s_x is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood U of x such that t_x comes from a section t of $\mathcal{L}^{\otimes -1}$ over U and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that s generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that X_s is open. Moreover, the section t constructed over U above is unique, and hence these glue to give the section s' of the lemma. □

Recall from Modules, Lemma **??** that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where ss' denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

Definition 11.2. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample* if

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

Exercise 11.3. Let L be an ample bundle on a K3 surface M. Prove that $\mathcal{L}^{\otimes 2}$ is globally generated (that is, for each $x \in M$ there exsits a section $h \in H^0(L^{\otimes 2})$ which does not vanish in x).

Proof. This just asks that the n in Definition 11.2 is 2 for all $x \in X$. Because, again, $x \in X_s$ means that $s(x) \neq 0$ because if it was, then we could somehow write s as a product of a vanishing function on $\mathfrak{m}_x$ and a local frame of $\Gamma(X, \mathcal{L})$. But I guess for the exercise do this: a line bundle is *ample* if there is n such that the canonical embedding (cf Lemma 11.5) is an embedding, i.e. that $\mathcal{L}^{\otimes n}$ is *very ample*. (Interestingly, the notion very ampleness is defined in morphisms.tex.) □

Now we pass to the part where ampleness gives you an **open immersion** to some projective space. Because, it's only very ampleness that gives an embedding, right? (Actually I think here in stacks project there are no embeddings but closed immersions.)

Definition 11.4. From modules.tex. Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X we define the *associated graded ring* to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

Lemma 11.5. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes*

$$f : X \longrightarrow Y = Proj(S),$$

*to the homogeneous spectrum of S (see Constructions, Section **??**). This morphism has the following properties*

1. *$f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,*
2. *there are $\mathcal{O}_X$-module maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (**??**),*
3. *the composition $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ is the identity map, and*
4. *for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of x such that $f^*\mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.*

Proof. Denote $\psi : S \to \Gamma_*(X, \mathcal{L})$ the identity map. We are going to use the triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ of Constructions, Lemma **??**. By assumption the open subscheme $U(\psi)$ of equals X. Hence $r_{\mathcal{L},\psi} : U(\psi) \to Y$ is defined on all of X. We set $f = r_{\mathcal{L},\psi}$. The maps in part (2) are the components of θ. Part (3) follows from condition (2) in the lemma cited above. Part (1) follows from (3) combined with condition (1) in the lemma cited above. Part (4) follows from the last statement in Constructions, Lemma **??** since the map α mentioned there is an isomorphism. □

Lemma 11.6. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume (a) every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, and (b) X is quasi-compact. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 11.5 above is quasi-compact with dense image.*

Proof. To prove f is quasi-compact it suffices to show that $f^{-1}(D_+(s))$ is quasi-compact for any $s \in S_+$ homogeneous. Write $X = \bigcup_{i=1,\ldots,n} X_i$ as a finite union of affine opens. By Lemma **??** each intersection $X_s \cap X_i$ is affine. Hence $X_s = \bigcup_{i=1,\ldots,n} X_s \cap X_i$ is quasi-compact. Assume that the image of f is not dense to get a contradiction. Then, since the opens $D_+(s)$ with $s \in S_+$ homogeneous form a basis for the topology on $\text{Proj}(S)$, we can find such an s with $D_+(s) \neq \emptyset$ and $f(X) \cap D_+(s) = \emptyset$. By Lemma 11.5 this means $X_s = \emptyset$. By Lemma **??** this means that a power s^n is the zero section of $\mathcal{L}^{\otimes n \deg(s)}$. This in turn means that $D_+(s) = \emptyset$ which is the desired contradiction. □

Lemma 11.7. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume $\mathcal{L}$ is ample. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 11.5 is an open immersion with dense image.*

Proof. By Lemma **??** we see that X is quasi-separated. Choose finitely many $s_1, \ldots, s_n \in S_+$ homogeneous such that X_{s_i} are affine, and $X = \bigcup X_{s_i}$. Say s_i has degree d_i. The inverse image of $D_+(s_i)$ under f is X_{s_i}, see Lemma 11.5. By Lemma **??** the ring map

$$(S^{(d_i)})_{(s_i)} = \Gamma(D_+(s_i), \mathcal{O}_{\text{Proj}(S)}) \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_X)$$

is an isomorphism. Hence f induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus f is an isomorphism of X onto the open subscheme $\bigcup_{i=1,\ldots,n} D_+(s_i)$ of $\text{Proj}(S)$. The image is dense by Lemma 11.6. □

Lemma 11.8. *Let X be a scheme. Let S be a graded ring. Assume X is quasi-compact, and assume there exists an open immersion*

$$j : X \longrightarrow Y = Proj(S).$$

Then $j^*\mathcal{O}_Y(d)$ is an invertible ample sheaf for some $d > 0$.

Proof. This is Constructions, Lemma **??**. □

Proposition 11.9. *Let X be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$. The following are equivalent:*

1. *$\mathcal{L}$ is ample,*
2. *the open sets X_s, with $s \in S_+$ homogeneous, cover X and the associated morphism $X \to Proj(S)$ is an open immersion,*
3. *the open sets X_s, with $s \in S_+$ homogeneous, form a basis for the topology of X,*
4. *the open sets X_s, with $s \in S_+$ homogeneous, which are affine form a basis for the topology of X,*
5. *for every quasi-coherent sheaf $\mathcal{F}$ on X the sum of the images of the canonical maps*
$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$
with $n \geq 1$ equals $\mathcal{F}$,
6. *same property as (5) with $\mathcal{F}$ ranging over all quasi-coherent sheaves of ideals,*
7. *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exists an integer n_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$,*
8. *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exist integers $n > 0$, $k \geq 0$ such that $\mathcal{F}$ is a quotient of a direct sum of k copies of $\mathcal{L}^{\otimes -n}$, and*
9. *same as in (8) with $\mathcal{F}$ ranging over all sheaves of ideals of finite type on X.*

Proof. Lemma 11.7 is (1) $\Rightarrow$ (2). Lemmas **??** and 11.8 provide the implication (1) $\Leftarrow$ (2). The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (3) are clear from Constructions, Section **??**. Lemma **??** is (3) $\Rightarrow$ (1). Thus we see that the first 4 conditions are all equivalent.

Assume the equivalent conditions (1) – (4). Note that in particular X is separated (as an open subscheme of the separated scheme $\text{Proj}(S)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Choose $s \in S_+$ homogeneous such that X_s is affine. We claim that any section $m \in \Gamma(X_s, \mathcal{F})$ is in the image of one of the maps displayed in (5) above. This will imply (5) since these affines X_s cover X. Namely, by Lemma **??** we may write m as the image of $m' \otimes s^{-n}$ for some $n \geq 1$, some $m' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This proves the claim.

Clearly (5) $\Rightarrow$ (6). Let us assume (6) and prove $\mathcal{L}$ is ample. Pick $x \in X$. Let $U \subset X$ be an affine open which contains x. Set $Z = X \setminus U$. We may think of Z as a reduced closed subscheme, see Schemes, Section **??**. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to the closed subscheme Z. By assumption (6), there exists an $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ such that s does not vanish at x (more precisely such that $s \notin \mathfrak{m}_x \mathcal{I}_x \otimes \mathcal{L}_x^{\otimes n}$). We may think of s as a section of $\mathcal{L}^{\otimes n}$. Since it clearly vanishes along Z we see that $X_s \subset U$. Hence X_s is affine, see Lemma **??**. This proves that $\mathcal{L}$ is ample. At this point we have proved that (1) – (6) are equivalent.

Assume the equivalent conditions (1) – (6). In the following we will use the fact that the tensor product of two sheaves of modules which are globally generated is globally generated without further mention (see Modules, Lemma **??**). By (1) we can find elements $s_i \in S_{d_i}$ with $d_i \geq 1$ such that $X = \bigcup_{i=1,\ldots,n} X_{s_i}$. Set $d = d_1 \ldots d_n$. It follows that $\mathcal{L}^{\otimes d}$ is globally generated by

$$s_1^{d/d_1}, \ldots, s_n^{d/d_n}.$$

This means that if $\mathcal{L}^{\otimes j}$ is globally generated then so is $\mathcal{L}^{\otimes j+dn}$ for all $n \geq 0$. Fix a $j \in \{0, \ldots, d-1\}$. For any point $x \in X$ there exists an $n \geq 1$ and a global section s of $\mathcal{L}^{j+dn}$ which does not vanish at x, as follows from (5) applied to $\mathcal{F} = \mathcal{L}^{\otimes j}$ and ample invertible sheaf $\mathcal{L}^{\otimes d}$. Since X is quasi-compact there we may find a finite list of integers n_i and global sections s_i of $\mathcal{L}^{\otimes j+dn_i}$ which do not vanish at any point of X. Since $\mathcal{L}^{\otimes d}$ is globally generated this means that $\mathcal{L}^{\otimes j+dn}$ is globally generated where $n = \max\{n_i\}$. Since we proved this for every congruence class mod d we conclude that there exists an $n_0 = n_0(\mathcal{L})$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. At this point we see that if $\mathcal{F}$ is globally generated then so is $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0$.

We continue to assume the equivalent conditions (1) – (6). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. Denote $\mathcal{F}_n \subset \mathcal{F}$ the image of the canonical map of (5). By construction $\mathcal{F}_n \otimes \mathcal{L}^{\otimes n}$ is globally generated. By (5) we see $\mathcal{F}$ is the sum of the subsheaves $\mathcal{F}_n$, $n \geq 1$. By Modules, Lemma **??** we see that $\mathcal{F} = \sum_{n=1,\ldots,N} \mathcal{F}_n$ for some $N \geq 1$. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated whenever $n \geq N + n_0(\mathcal{L})$ with $n_0(\mathcal{L})$ as above. We conclude that (1) – (6) implies (7).

Assume (7). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. By (7) there exists an integer $n \geq 1$ such that the canonical map

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

is surjective. Let I be the set of finite subsets of $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ partially ordered by inclusion. Then I is a directed partially ordered set. For $i = \{s_1, \ldots, s_{r(i)}\}$ let $\mathcal{F}_i \subset \mathcal{F}$ be the image of the map

$$\bigoplus\nolimits_{j=1,\ldots,r(i)} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

which is multiplication by s_j on the jth factor. The surjectivity above implies that $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$. Hence Modules, Lemma **??** applies and we conclude that $\mathcal{F} = \mathcal{F}_i$ for some i. Hence we have proved (8). In other words, (7) $\Rightarrow$ (8).

The implication (8) $\Rightarrow$ (9) is trivial.

Finally, assume (9). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. By Lemma **??** (this is where we use the condition that X be quasi-separated) we see that $\mathcal{I} = \operatorname{colim}_\alpha I_\alpha$ with each I_α quasi-coherent of finite type. Since by assumption each of the I_α is a quotient of negative tensor powers of $\mathcal{L}$ we conclude the same for $\mathcal{I}$ (but of course without the finiteness or boundedness of the powers). Hence we conclude that (9) implies (6). This ends the proof of the proposition. □

The following proofs were used for Exercise 13.7.

Lemma 11.10. *Let D be a divisor on a complete, nonsingular curve X. If $h^0(D) \neq 0$ then $degD \geq 0$.*

Proof. If D has sections, we can take the zero locus of any of its sections to produce an effective divisor linearly equivalent to D. Since degree depends only on linear equivalence and effective divisors have non negative degree. □

Proposition 11.11. *Let D be a divisor on a complete, nonsingular curve X. Then the complete linear system has no base points if and only if for every point $P \in X$,*

$$\dim |D - P| = \dim |D| - 1$$

Proof. To show that D has no base points amounts to showing that not every section of D. That is, that the injective map $0 \to H^0(D - p) \to H^0(D)$ is not surjective. □

Lemma 11.12. *Let D be a divisor on a curve X of genus g. If $degD \geq 2g$, then $|D|$ has no base points.*

Proof. First we prove that $\deg D \geq 2g$ implies that D and $D - P$ are *nonspecial*, i.e. that $h^0(K - D) = 0 = h^0(K - (D - P))$. Then we apply Riemann-Roch to both D and $D - P$ and the fact that $\deg(D - P) = \deg(D) - 1$ to find that $\dim |D - P| = \dim |D| - 1$ and apply Proposition 11.11.

To prove that D is nonspecial first apply Riemann-Roch to K to obtain that $\deg K = 2g - 2$. Indeed, $h^0(0) = 1$ and $h^0(K) = p_g$ by Serre duality on $H^1(\mathcal{O}_X)$ recalling definition of genus as $h^1(\mathcal{O}_X)$. Then apply Riemann-Roch to both D and $K - D$ to prove that $\deg D > 2g - 2$ implies $\deg(K - D) < 0$; start with

$$h^0(K - D) - h^0(K - (K - D)) = -(h^0(D) - h^0(K - D))$$

An analogous result will be valid for $D - P$ since its degree is also greater than $2g - 2$.

Then apply Lemma 11.10. □

12. Closed immersions of locally ringed spaces

We follow our conventions introduced in Modules, Definition **??**.

Definition 12.1. Let $i : Z \to X$ be a morphism of locally ringed spaces. We say that i is a *closed immersion* if:

(1) The map i is a homeomorphism of Z onto a closed subset of X.
(2) The map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective; let $\mathcal{I}$ denote the kernel.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

Lemma 12.2. *Let $f : Z \to X$ be a morphism of locally ringed spaces. In order for f to be a closed immersion it suffices that there exists an open covering $X = \bigcup U_i$ such that each $f : f^{-1}U_i \to U_i$ is a closed immersion.*

Proof. Omitted. □

Example 12.3. Let X be a locally ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals which is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Let Z be the support of the sheaf of rings $\mathcal{O}_X/\mathcal{I}$. This is a closed subset of X, by Modules, Lemma **??**. Denote $i : Z \to X$ the inclusion map. By Modules, Lemma **??** there is a unique sheaf of rings $\mathcal{O}_Z$ on Z with $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. For any $z \in Z$ the stalk

$\mathcal{O}_{Z,z}$ is equal to a quotient $\mathcal{O}_{X,i(z)}/\mathcal{I}_{i(z)}$ of a local ring and nonzero, hence a local ring. Thus $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a closed immersion of locally ringed spaces.

Definition 12.4. Let X be a locally ringed space. Let $\mathcal{I}$ be a sheaf of ideals on X which is locally generated by sections. The locally ringed space $(Z, \mathcal{O}_Z)$ of Example 12.3 above is the *closed subspace of X associated to the sheaf of ideals $\mathcal{I}$.*

Lemma 12.5. *Let $f : X \to Y$ be a closed immersion of locally ringed spaces. Let $\mathcal{I}$ be the kernel of the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $i : Z \to Y$ be the closed subspace of Y associated to $\mathcal{I}$. There is a unique isomorphism $f' : X \cong Z$ of locally ringed spaces such that $f = i \circ f'$.*

Proof. Omitted. □

Lemma 12.6. *Let X, Y be locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals locally generated by sections. Let $i : Z \to X$ be the associated closed subspace. A morphism $f : Y \to X$ factors through Z if and only if the map $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$ is zero. If this is the case the morphism $g : Y \to Z$ such that $f = i \circ g$ is unique.*

Proof. Clearly if f factors as $Y \to Z \to X$ then the map $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Conversely suppose that $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Pick any $y \in Y$, and consider the ring map $f^\sharp_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Since the composition $\mathcal{I}_{f(y)} \to \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is zero by assumption and since $f^\sharp_y(1) = 1$ we see that $1 \notin \mathcal{I}_{f(y)}$, i.e., $\mathcal{I}_{f(y)} \neq \mathcal{O}_{X,f(y)}$. We conclude that $f(Y) \subset Z = \text{Supp}(\mathcal{O}_X/\mathcal{I})$. Hence $f = i \circ g$ where $g : Y \to Z$ is continuous. Consider the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. The assumption $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero implies that the composition $\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$ is zero by adjointness of f_* and f^*. In other words, we obtain a morphism of sheaves of rings $\overline{f^\sharp} : \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y$. Note that $f_*\mathcal{O}_Y = i_*g_*\mathcal{O}_Y$ and that $\mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$. By Sheaves, Lemma **??** we obtain a unique morphism of sheaves of rings $g^\sharp : \mathcal{O}_Z \to g_*\mathcal{O}_Y$ whose pushforward under i is $\overline{f^\sharp}$. We omit the verification that $(g, g^\sharp)$ defines a morphism of locally ringed spaces and that $f = i \circ g$ as a morphism of locally ringed spaces. The uniqueness of $(g, g^\sharp)$ was pointed out above. □

Lemma 12.7. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals which is locally generated by sections. Let $i : Z \to Y$ be the closed subspace associated to the sheaf of ideals $\mathcal{I}$. Let $\mathcal{J}$ be the image of the map $f^*\mathcal{I} \to f^*\mathcal{O}_Y = \mathcal{O}_X$. Then this ideal is locally generated by sections. Moreover, let $i' : Z' \to X$ be the associated closed subspace of X. There exists a unique morphism of locally ringed spaces $f' : Z' \to Z$ such that the following diagram is a commutative square of locally ringed spaces*

$$\begin{tikzcd} Z' \arrow[r, "i'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

Moreover, this diagram is a fibre square in the category of locally ringed spaces.

Proof. The ideal $\mathcal{J}$ is locally generated by sections by Modules, Lemma **??**. The rest of the lemma follows from the characterization, in Lemma 12.6 above, of what it means for a morphism to factor through a closed subspace. □

13. Adjunction formulas

There are several statements called adjunction formula in different texts. All of them concern "subvarieties", that is, closed embedded subschemes.

Exercise 13.1 (Genus formula for a curve on a surface)**.** Let $C \to X$ be a closed embedded subscheme of dimension 1 (as a topological space, i.e. pure dimension) inside a smooth surface X. Then $2p_a - 2 = (\mathcal{O}_X(C), \mathcal{O}_X(C))$.

Proof. Consider the ideal sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

This sequence splits since there is an obvious inverse morphism to the inclusion $\mathcal{O}_X(-C) \to \mathcal{O}_X$, namely mapping a function f to Then $\mathcal{O}_X \cong \mathcal{O}_X(-C) \oplus \mathcal{O}_C$. $\chi(\mathcal{O}_X) = \chi$ □

14. Normalization

Definition 14.1. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the scheme[1]

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over X. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

The factorization is the composition of the canonical morphism $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ (see Constructions, Lemma **??**) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \to f_*\mathcal{O}_Y$. We can characterize the normalization as follows.

Lemma 14.2. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

(1) *the morphism ν is integral, and*
(2) *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f'"'] & Z \arrow[d, "\pi"] \\ X' \arrow[r, "\nu"] \arrow[ur, "h"] & X \end{tikzcd}$$

for some unique morphism $h : X' \to Z$.

Moreover, the morphism $f' : Y \to X'$ is dominant and in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

[1]The scheme X' need not be normal, for example if $Y = X$ and $f = \mathrm{id}_X$, then $X' = X$.

15. Reflexive sheaves

Definition 15.1. Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The *reflexive hull* of $\mathcal{F}$ is the $\mathcal{O}_X$-module

$$\mathcal{F}^{**} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

We say $\mathcal{F}$ is *reflexive* if the natural map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is an isomorphism.

Lemma 15.2. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *If $\mathcal{F}$ is reflexive, then $\mathcal{F}$ is torsion free.*
(2) *The map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is injective if and only if $\mathcal{F}$ is torsion free.*

Remark 15.3 (Talk at IMPA, 11 June). Torsion could also be defined so that the sheaf can inject onto its dual. In this talk we discussed the moduli space of reflexive/torsion-free sheaves, which turned out to be parametrized by c_1, c_2 and c_3. This was denoted by $R(c_1, c_2, c_3)$. Actually I think it may have been Manolache that proved the existence of this moduli space.

Alan Muniz

Nesta palestra discutiremos a classificação de feixes reflexivos de posto dois e seus espaços de módulos. Apresentaremos algumas ferramentas básicas usadas na construção e determinação de tais feixes. Aplicaremos estas técnicas para o caso de feixes com segunda classe de Chern igual a quatro, obtido recentemente em colaboração com Marcos Jardim.

15.4. **Distributions on manifolds.** Here's the abstract from a talk by Marcos Jardim at Geometric Structures:

“I will revise the work done over the past 10 years with various collaborators on distributions and foliations on 3-folds, especially on the projective space, with a focus on properties of the tangent sheaf and singular scheme."

Here are two key ideas: if the distribution is codimension 1 we can write:

$$0 \longrightarrow F \longrightarrow TX \xrightarrow{\omega} I_Z \otimes L \longrightarrow 0$$

where L is a line bundle and $\omega \in H^0(\Omega_X \otimes L)$, and $Z = \{p : \omega(p) = 0\}$.

When codimension is 2 then $\mathcal{D}$ is given by a holomorphic vector field ν: $T_p = \langle \nu(p) \rangle$.

It can be encoded as an exact sequence

$$0 \longrightarrow L \xrightarrow{\nu} TX \longrightarrow N \longrightarrow 0$$

where L is a line bundle and $\nu \in H^0(TX \otimes L^\vee)$; $Z = \{p | \nu(p) = 0\}$.

Remark 15.5. Saturation means that $Z \subset X$ is a union of curves and points.

And again, distributions are parametrized by Chern classes.

Two interesting open questions:

(1) **Conjeture.** if $\mathcal{D}$ is a codimension 1 foliation of degree d on $\mathbb{P}^3$, then $c_2(F) \leq d^2 - d + 1$ and bound is attained by rational foliations of type $(1, d+1)$. (True for $d \leq 2$.)
(2) **Conjecture (with Pepe Seade).** $\mathcal{D}$ is a codimension 1 foliation on a smooth projective 3-fold, then Sing$\mathcal{D}$ is connected.

Theorem 15.6 (Jardim-Muniz)**.** *Conditions on Chern classes used to understand moduli space $R(c_1, c_2, c_3)$. $c_2 = 4$ gives (?). For $c_3 \leq 6$, possible "spectrum" exists...*

16. Stability

Question. What is stability?

(1) Stable objects in an abelian category are the "building blocks": we can reconstruct the whole category from them.
(2) An abelian subcategory (hart) $\subset$ a triangulated
(3) stability defined via stability function on $\mathcal{A}$.
(4) Q. Can we reconstruct $\mathcal{T}$ from the semistable elements of $\mathcal{A}$
(5) **Example.** $\mathcal{A} = \mathrm{Coh}X$ is heart of $D^b(X)$ w/ funny function.
(6) Stability condition is hart + stability function.
(7) Bridgeland Stabl:= the stability conditions are a complex manifold of complex codimension $\mathrm{rk}\Lambda$:

$$\mathcal{Z} : \mathrm{Stab}(\mathcal{T}) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C})$$

(8) There's a chamber structure; moduli space changes across chambers.
(9) I think we typically think of vectors in $\mathrm{Hom}(\Lambda, \mathbb{C})$ as Chern classes, to characterize the moduli spaces.
(10) Existence: given a projective variety X, are there stability conditions on $D^{\mathrm{b}}(X)$? Yes for fano 3fold pic rk 1.
(11) Moduli spaces: is $M_\sigma(v)$ a projective scheme? Cannot use usual git techniques to study. A stack!
(12) Picture: blue + black are walls. Q. What are β and α?
(13) thm: bridgeland stable = gieseker stable ?
(14) Q. slope stability = bridgeland stability? A. Not always.
(15) DT/PT correspondance: only one wall between PT and G chambers
(16) Polynomial stability function. This is an asymptotic version of BS.
(17) There are some ρ's. Arrangements of ρ_i are polynomial stability conditions on a threefold.
(18) Pata-Thomas introduced stability for rank 1 objects. Bayer compares the— wall. Q. Same for Bridge S—only one wall?
(19) Recall Gieseker stability.
(20) Def. A *stable triple*: when $\gcd(\mathrm{ch}_0, \mathrm{ch}_2, \mathrm{ch}_3) = 1$, every PT stable object comes from three conditions (missing).
(21) What happens when you cross the blue wall? Both $\mathcal{G}$ and $\mathcal{T}$ are projective. What happens at the blue wall?
(22) For X smooth threefold with $\mathrm{rkPic} = 1$, $\mathcal{G}(v)$, $v = (r, 0, 0, -n)$, $\mathcal{G}(v)$ is a known sheaf object and $\mathcal{T} = \emptyset$.
(23) X sm 3 rkpic1, Fake wall; $\mathcal{G} = \mathcal{T}$.
(24) (Extra.) red circle is a wall for a weaker form of stability.

Definition 16.1 (Talk at impa)**.** A rank-2 sheaf $\mathcal{F}$ is *semistable (stable)*if $H^0(\mathcal{F}(t)) = 0$ for $-t \geq (>) \frac{c_1 F}{2}$

Compare with

Definition 16.2 (moduli-curves.tex)**.** Let $f : X \to S$ be a family of curves. We say f is a *semistable family of curves* if

(1) $X \to S$ is a prestable family of curves, and
(2) X_s has genus ≥ 1 and does not have a rational tail for all $s \in S$.

- The twistor diagram.

$$
\begin{tikzcd}
 & \ell_\infty \arrow[dl] \arrow[dr, dotted] & \\
\mathbb{C}P^2 \arrow[rr, hook] & & \mathbb{C}P^3 \arrow[dd, "\substack{\text{twistor}\\ \text{map}}"'] \\
 & & \\
\mathbb{C}^2\mathbb{R}^4 \arrow[u, hook] \arrow[rr] & & S^4
\end{tikzcd}
$$

- Instantons are solutions to some Yang-Mills solution.
- Expository paper of Donaldson arXiv:2205.08639
- ADHM construction, 1978. The first appearance of Algebraic Geometry in Mathematical Physics. See Hitchin-Kobayashi correspondence.
- Donaldson: "unashamedly computational".
- Expository paper by Simon.
- Take bundle (E, ∇) with an anti-self dual (ADS) connection on S^4 and pullback to $\mathbb{C}P^3$ via the twistor map

  $$\tau[x : y : z : w] = [x + jy : z + jw] \qquad \text{note: } \tau^{-1}(p) = \mathbb{C}P^1$$

  Then:
  - Restriction to fibres are trivial.
  - Invariant under anti holomorphic involution (check this formula!) $[x : y : z : w] \mapsto [-y : x : -w : z]$.
  - $\mathcal{E}$ is also an instanton bundle.
  - Penrose Transform: $H^1(\mathcal{E}(-2)) \cong \operatorname{Ker} \Delta = 0$ where Δ is a Laplacian.
  - Definition of instanton sheaf on $\mathbb{P}^3$ via $c_1(E) = 0$ and some vanishing of cohomologies.
  - Passage from [differential equations? algebraic geometry?] to linear algebra: via *monads*. The point is that instanton sheaf is equivalent to "E being the cohomology of a linear monad"; theorem by Horps in the 60's, and is the main tool used by ADHM. Indeed, ADHM equations come from the cohomology sequences of the so-called monads.
  - Mathamatical inst bund:= locally free instanton sheaves.

Properties.

- The only instanton of rank 1 on $\mathbb{P}^3$ is the structure sheaf.
- non-trivial rank 2 locally free instanton sheaf ois $\mu_0 stable$
- double dual is locally free and also instanton
- non-trivial rank 2 instanton sheaaf is Fieseker stable therefore it makes sance fo define moduls space of instanton sheaves as an open subset of $\mathcal{M}(c) = \mathcal{G}(k, 0, 2, 0)$.

Then studied the irreducibility (Tikhomirov) and smoothness (Jardim-Verbitsky, 2014. Uses "3rd hyperkähler quotient") of $\mathcal{I}(c)$, the moduli space of rank 2 locally gree instanton sheaves of charge c. But nobody likes this results; want new proofs.

In contrast, $\mathcal{M}(c)$ of rank 2-instanton sheaves of charge c is not irreducible in general! $\mathcal{M}(1)$ and $\mathcal{M}(2)$ are irreducible, $\mathcal{M}(3)$ has exactly 2 irreducible components of dimension 21; $\mathcal{M}(4)$ has 4 irredicuble components: the locally free is irreducible, and the other 3 that intersect the closure of the locally free, $\overline{\mathcal{I}(c)}$. 3 components of dimension 29 and one of dimension 32

Remark 16.3. In general the μ moduli space is not projective, but the Gieseker is.

Is $\mathcal{M}(c)$ connected? Use $\mathbb{C}^*$ action. True for $c \leq 4$; every component intersects $\overline{\mathcal{I}(c)}$ in this range!

Definition 16.4 (Elementary transformation)**.** F of rank 2 locally free instanton, Q of rank 0 instanton with 1 dimensional sheaf $h^p Q(-2)$ for $p = 0, 1$. So in this conditions if we have an epimorphism

$$F \overset{\varphi,\text{ surj}}{\to}\} Q$$

we get that $\operatorname{Ker} \varphi$ is an instanton.

So we might be interested in

$$E \hookrightarrow E^{**} \xrightarrow{\text{surj.}} Q_E$$

Let's have a look at $\mathcal{M}(3) = \overline{\mathcal{I}(3)} \cup \overline{C(0,1,3}$. Consider a generic line bundle of degree 0 on a planar cubic (cwhich is encoded in that we have intersection of something of cimension 1 and something of simension 3), $Linn\mathrm{Pic}^0(C)$ so we have an epimorphism

$$0 \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \longrightarrow (i_* L) \longrightarrow 0$$

yielding a bundle E as previously outlined.

So we are studying the components via pushforwards of sheaves on complete intersection curves inside $\mathbb{P}^3$

"when we do alementary transformation of rational (not rationl?) we get something on the boundary of locally free."

Perverse instanton sheaves. Like an instnaton sheaf in monad description but with some restrictions on the cohomologies. This leads to definition of 0-*rimensional instanton*, a perverse instanton such that $\mathcal{H}^0 = 0$.

Instantons and quivers.

Framed instantons. Fix a line $j : L \to \mathbb{P}^3$ and E a perverse sheaf; a *framing* at L is an isomorphis [missing].

Apply GIT to the ADHM data to construct a moduli space

$$\mathcal{P}(r, c) = \mathcal{V}(r, c)^{\text{st}} /\!/ \mathrm{GL}(V_c)$$

and it will follow that $\mathbb{P}(?, ?)$ is connected.

Considerations on quaternionic spaces lead to generalization of what has been discussed so far to higher dimensions. I.e. an *instanton sheaf* on $\mathbb{P}^n$ is... [other cohomological conditions] sheaf

Why are instantons interesting? They are the simplest; may provide examples for Bridgeland stability.

In Kuznetsov (2012) and Faenzi (2014) introduced *rank 2 instanton bundles on Fano 3-folds*. An *instanton bundle* on X is a μ-stable ... and some Chern class is called the *charge*. An *instanton sheaf* (introduced by Marcos-Gaia) is ....

"Since we are imposing μ-stability on the defintion we can consider the moduli $\mathcal{I}_X(c) \subset \mathcal{G}_X(2, -r_X, c, 0)$".

There's also monad representations as an ingredient.

Here are two questions that invite us to join the instanton fever:

Task 1. Construct rank 2-instanton sheaves that do not deform into locally free ones, and obtain the new irreducible components of $\mathcal{G}(2, -r_X, c, 0)$.

Task 2. Nonlocally instanton sheaves that can be deformed into non-locally free ones: the *instanton boundary* $\overline{\mathcal{I}(c)}/\mathcal{I}(X)$[formula right?]

Recipe to construct your own instanton.

(1) (Make a bunch of instantons.) Find an appropriate curve to use Serre correspondence to find some rank 2 instanton sheaf:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \underbrace{E}_{\exists} \longrightarrow \mathcal{I}_{\sqcup \text{lines}} \longrightarrow 0$$

where E is a locally free instanton of charge = the number of lines -1.

(2) (Is your family of instantons generic?) You look at the Exts. "Therefore, the family of instantons only defines a locally closed subset within a generically smooth irreducible component of $\mathcal{G}$".

(3) "Find suitable rank 0 intranton sheaves to perform an elementary transformations on the examples obtained in Step 1." But they are non-locally free.

(4) Now I have my instantons, I know they are locally free: but how to prove that the elementary transformations deform to locally free ones? Looks like the challenge is to prove that the deformation is locally free.

In the papers by the group there are several particular cases when the deformations are locally free. But they don't have a general result that would work for 3-folds.

What you need to call a thing an instanton.

- Minimal cohomology possible; try to kill as much cohomology as you can.
- Fixing c_1 (which may determine other Chern classes).
- Some stability condition like μ-stability or quiver stability. Here is an example that is not μ-semistable: $T\mathbb{P}^3(-1) \oplus \Omega_{\mathbb{P}^3}(1)$
- Whenever possible, look for a monadic representation. (The monadic representation comes from ADHM — the beginnings of this theory. And it's still here!)

17. Coherent sheaves

Lemma 17.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply Cohomology, Lemma **??**. As our basis $\mathcal{B}$ for the topology of X we are going to use the affine opens of X. As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X.

Next we check that conditions (1), (2) and (3) of Cohomology, Lemma **??** hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings of any element of $\mathcal{B}$, see Schemes, Lemma **??**. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma **??**. $\square$

18. Hilbert polynomial

The following lemma will be made obsolete by the more general Lemma **??**.

Lemma 18.1. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function*

$$d \longmapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$$

is a polynomial.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \text{Spec}(k)$ and $\mathcal{F}(d) = \mathcal{F}$. Hence in this case the function is constant, i.e., a polynomial of degree 0. Assume $n > 0$. By Lemma **??** we may assume k is infinite. Apply Lemma **??**. Applying Lemma **??** to the twisted sequences $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*\mathcal{G}(d) \to 0$ we obtain

$$\chi(\mathbf{P}^n_k, \mathcal{F}(d)) - \chi(\mathbf{P}^n_k, \mathcal{F}(d-1)) = \chi(H, \mathcal{G}(d))$$

See Remark **??**. Since $H \cong \mathbf{P}^{n-1}_k$ by induction the right hand side is a polynomial. The lemma is finished by noting that any function $f : \mathbf{Z} \to \mathbf{Z}$ with the property that the map $d \mapsto f(d) - f(d-1)$ is a polynomial, is itself a polynomial. We omit the proof of this fact (hint: compare with Algebra, Lemma **??**). $\square$

Definition 18.2. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function $d \mapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$ is called the *Hilbert polynomial* of $\mathcal{F}$.

The Hilbert polynomial has coefficients in $\mathbf{Q}$ and not in general in $\mathbf{Z}$. For example the Hilbert polynomial of $\mathcal{O}_{\mathbf{P}^n_k}$ is

$$d \longmapsto \binom{d+n}{n} = \frac{d^n}{n!} + \ldots$$

This follows from the following lemma and the fact that

$$H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(d)) = k[T_0, \ldots, T_n]_d$$

(degree d part) whose dimension over k is $\binom{d+n}{n}$.

Lemma 18.3. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$ with Hilbert polynomial $P \in \mathbf{Q}[t]$. Then*

$$P(d) = \dim_k H^0(\mathbf{P}^n_k, \mathcal{F}(d))$$

for all $d \gg 0$.

Proof. This follows from the vanishing of cohomology of high enough twists of $\mathcal{F}$. See Cohomology of Schemes, Lemma **??**. $\square$

For completeness I include earlier notes from [Har77] on the matter.

The fact that M is finitely generated is what makes the following two definitions make sense.

Definition 18.4. The *Hilbert function* of a finitely generated graded $S = k[x_0, \dots, x_r]$ -module M is

$$H_M(d) = \dim_k M_d$$

Definition 18.5. Define F_0 to be the free S-module on the generators of M. Elements in the kernel M_1 of the inclusion are called *sysygies.* By Hilbert's basis theorem, M_1 is also finitely generated, so we may choose a set of generators and repeat this process.

Theorem 18.6 (Hilbert Syzygy Theorem)**.** *Any finitely generated S-module M has a finite graded free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} \quad \longrightarrow F_{m-q} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

Moreover, we may take $m \leq r+1$, the number of variables in S.

Lemma 18.7. *Suppose that $S = k[x_0, \dots, x_r]$ is a polynomial ring. If the graded S-module M has finite free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \cdots [r] \qquad F_1 \xrightarrow{\varphi_1} \quad \longrightarrow F_0$$

with each F_i a finitely generated free module, $F_i = \bigoplus_j S(-a_{i,j})$, then

(18.7.1) $$H_M(d) = \sum_{i=0} (-1)^i \sum_j \binom{r+d-a_{i,j}}{r}$$

Lemma 18.8. *There is a polynomial $P_M(d)$ called the Hilbert polynomial such that, if M has free resolution as above, then $P_M(d) = H_M(d)$ for $d \geq max_{i,j}\{a_{i,j} - r\}$.*

Proof. When d satisfies this bound then the binomial coefficients in Eq. 18.7.1 are polynomials of degree r in d. $\square$

Theorem 18.9 (Hilbert-Serre)**.** *Let M be a finitely generated graded $S = k[x_0, \dots, x_n]$. Then there exists a unique polynomial p_M such that $p_M(\ell) = \dim S_\ell$ for large enough ℓ.*

Definition 18.10. The polynomial P_M of Hilbert-Serre Theorem [**?**] is the *Hilbert polynomial* of the finitely generated $k[x_0, \dots, x_n]$-module M.

Definition 18.11. If $Y \subset \mathbb{P}^n$ is an algebraic set of dimension r, we define the *degree of Y* to be $r!$ times the leading coefficient of the Hilbert polynomial of the homogeneous coordinate ring $S(Y)$.

Exercise 18.12. Let H be a very ample divisor on the surface X, corresponding to a projective embedding $X \subseteq \mathbb{P}^N$. If we write the Hilbert polynomial of X as $P(z) = \frac{1}{2}az^2 + bz + c$, show that $a = H^2$, $b = \frac{1}{2}H^2 + 1 - \pi$, where π is the genus of a nonsingular curve representing H, and $c = 1 + p_a$.

19. Nakai-Moishezon Criterion

Theorem 19.1 (Nakai-Moishezon Criterion)**.** *A divisor D on the surface X is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C in X.*

Proof. The direct implication is easy: since D is ample, mD is very ample for some m, so that m^2D^2 is the self-intersection number of mD. By exercise 18.12, D^2 is the leading coefficient of the Hilbert polynomial of X as a subscheme of $\mathbb{P}^n$. This means that D^2 is twice the leading coefficient of the Hilbert polynomial of a projective variety for large enough m, so that it must be a positive number (it's the dimension of one of the graded components of the coordinate ring of the surface). □

20. Hilbert scheme

Upshot [**?**, p. 6]**.** We wish to parametrize subschemes of a projective space (or perhaps a more general scheme?). Since there are too many such subschemes we restrict ourselves to schemes with a given Hilbert polynomial, since the latter "encodes the most important numerical invariants of schemes". The Hilbert scheme is introduced via a theorem by Grothendieck as the object that represents the functor $\mathbf{Hilb}_{P,r}$ that maps a reduced scheme B to the set of proper flat families

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}^r \times B \arrow[d, "\pi_B"] \\ & B \end{tikzcd}$$

with $\mathcal{X}$ having Hilbert polynomial P.

Theorem 20.1 (Grothendieck, '66)**.** *The functor* $\mathbf{Hilb}_{P,r}$ *is representable by a projective scheme* $\mathcal{H}_{P,r}$*.*

SEE Hilbert schemes of subschemes.

21. Deformation theory

The following definition is from [lucas-defos], which in turn comes from [Ser06]

Definition 21.1. Let X be an algebraic $\mathbb{C}$-scheme.

(1) A *deformation* of X is a Cartesian diagram ξ

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}\mathbb{C} \arrow[r, "s"] & S \end{tikzcd}$$

where π is a flat surjective morphism of algebraic $\mathbb{C}$-schemes and S is connected. (Recall that flatness accounts for "continuity".)

(2) A *local deformation* of X is a deformation ξ where $S = \mathrm{Spec}A$ for A a noetherian local $\mathbb{C}$ algebra with residue field $\mathbb{C}$.

(3) An *infinitesimal deformation of* X is a local deformation with A an artinian local $\mathbb{C}$-algebra with residue field $\mathbb{C}$. X is called *rigid* if all infinitesimal deformations are trivial.

(4) An *inifinitesimal deformation of order* n is an infinitesimal deformation when $S = \mathrm{Spec}(\mathbb{C}[t]/(t^{n+1})$.

Upshot. An interpretation of the so-called *dual numbers* $k[t]/(t^2)$ (see [Har09]) as the tangent space of something is thinking of Taylor polynomials: after quotienting by t^2 we loose the tails of the polynomials and are left with the first derivative information only.

So what is the deformation space? Is it a moduli space of curves, that is, points are curves obtained by deforming the curve? Or is it the fibration whose fibers are curves and there is a central fiber that is the original curve?

There is the following interpretation in [continued-fractions] p. 39: the space of first order deformation classes of X is $D(\mathbb{C}[t]/(t^2)$. This is said to "“represent” the tangent space $\mathbb{T}^1_X$ of the hypothetical deformation space of X”. (I put double quotations because the word “represent” is quoted in the original text.) Further, if X is nonsingular and compact, then $\mathbb{T}^1_X = H^1(X, T_X)$.

Which basically I interpret as: the dimension of the deformation space of a smooth compact variety is $H^1(X, T_X)$.

Example 21.2. Fix $g \geq 2$. The dimension of the deformation space of a nonsingular projective curve X is $3g-3$. This is “the dimension of the moduli space of curves of genus g”.

We can compute this number by Riemann-Roch formula on the bundle $-K_X$. Indeed, since X is a curve, $\Omega^1_X = K_X$ and thus $-K_X = T_X$. We get

$$\begin{aligned} h^0(-K) - h^0(K-(-K)) &= \deg(-K) - g + 1 \\ &= -2g + 2 - g + 1 \qquad \text{degree additive and } \deg K = 2g-2 \\ &= -3g + 3 \end{aligned}$$

Now by Serre duality $h^0(K-(-K)) = h^1(-K) = h^1(T_X)$, and it turns out that a Riemann surface of genus $g \geq 2$ has no holomorphic vector fields, so that $h^0(-K) = h^0(T_X) = 0$.

Exercise 21.3. Let C be a smooth genus g curve which can be embedded in a K3 surface M, and X the family of all deformations of C in M.

(1) Prove that $\dim X \leq g$.
(2) Let $\mathcal{X}_g$ be the space of all curves of genus g (smooth?) which can be possibly embedded to a K3 surface. Prove that each irreducible component Z of $\mathcal{X}_g$ satisfies $\dim_{\mathbb{C}} \leq g + 19$. Deduce that there exists a compact complex curve which cannot be embedded in a K3 surface.

Exercise 21.4. Let C be a smooth curve embedded in a K3 surface X. Show that the dimension of the deformation space of C is $\leq g$.

Proof. The deformation space of a variety is the space of isomorphism classes of deformations as explained above. It turns out that there is a way to associate 1-cocyles of the tangent sheaf to deformations, so that in fact the deformation space Def_1 is isomorphic to $H^1(X, \mathcal{T}_X)$ for any variety X.

For our curve C we thus know that the dimension of the space of deformations (deformations not necessarily contained in X) is $h^1(\mathcal{T}_C)$. The family of deformations of C that are contained in X is the Hilbert space of curves with fixed Hilbert polynomial $P(t)$ after quotienting by $\mathbb{C}s$, where s is the section whose vanishing locus is C. This says that the number we are looking for is $h^0(X, \mathcal{O}(C)) - 1$.

Now I will show that $h^0(X, \mathcal{O}(C)) = 1 + g$ (see [**?**, Lemma 1.2.1, Remark 1.2.2]).

Consider the ideal sheaf exact sequence twisted by $\mathcal{O}(C)$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_C = \mathcal{O}(C)|_C \longrightarrow 0$$

By X being a K3 we know that $H^1(\mathcal{O}_X) = 0$, so that we have the short exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}(C)) \longrightarrow H^0(\mathcal{O}(C)|_C) \longrightarrow 0$$

so that $h^0(\mathcal{O}(C)) = 1 + h^1(\mathcal{O}(C)|_C)$. A version of adjunction formula says $\omega_C \cong (K_X \otimes \mathcal{O}(C)|_C$, and using that $K_X = \mathcal{O}_X$ we obtain $h^0(\mathcal{O}(C)|_C) = h^0(\omega_C) = g$.

To

For the record I put other thoughts I went through in solving this exercise.

Recall from [**?**, p. 146] that the normal bundle $\mathcal{N}$ of a hypersurface of a smooth variety X satisfies $\mathcal{N}^\vee \cong \mathcal{O}_X(-C)|_C$. Taking duals we get that $\mathcal{N} \cong \mathcal{O}_X(C)|_C$.

By adjunction formula $2g-2 = (\mathcal{O}(C), \mathcal{O}(C))$. Applying Riemann-Roch to $\mathcal{O}(C)$ (which is by definition the dual of the ideal sheaf of C, which is a line bundle on X), we obtain that $\chi(\mathcal{O}(C)) = 2 + \frac{1}{2}(\mathcal{O}(C), \mathcal{O}(C))$. Then $\chi(\mathcal{O}(C)) = g + 1$.

Now recall that $\chi(\mathcal{O}(C)) = h^0(\mathcal{O}(C) - h^1(\mathcal{O}(C)) + h^2(\mathcal{O}(C))$. By Serre duality and X being a K3 surface we see that $h^2(\mathcal{O}(C)) \cong h^0(\mathcal{O}(-C))$, which is the ideal sheaf of C. Any section of such a sheaf would vanish along C, and since X is compact we conclude there cannot be any such section.

Now we show that also $h^1(\mathcal{O}(C)) = 0$ to conclude that $h^0(\mathcal{O}(C)) = g + 1$.

We thus conclude that $h^0(\mathcal{O}(C))$

Since C is smooth we can use the normal exact sequence

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_X|_C \longrightarrow \mathcal{N} \longrightarrow 0$$

Taking Euler characteristic we see that $\chi(\mathcal{T}_C) + \chi(\mathcal{N}) = \chi(\mathcal{T}_X|_C)$.

This means that we should be done once we compute $\chi(\mathcal{T}_X|_C)$. For this we can use Riemman-Roch formula for coherent sheaves on a curve, which tells us that

$$\deg(\mathcal{T}_X|_C) = \chi(\mathcal{T}_X|_C) - \mathrm{rk}(\mathcal{T}_X|_C) \cdot \chi(\mathcal{O}_C)$$

But of course we know that since C is a curve, by Serre duality we get that $\chi(\mathcal{O}_C) = 1 - g$. (Indeed: $h^1(\mathcal{O}_C) = h^0(\Omega^1_C) := p_a(C)$.) And now the question is what is the degree. Apparently this is just the first Chern class c_1 of the restricted tangent bundle. So what is it? And what is $h^0(\mathcal{T}_C)$ if the genus is 0 or 1, and that's it.

I know that $h^0(\mathcal{T}_X) = 0$ and $h^1(\mathcal{T}_X) = 20$ by X being a K3 surface, but I'm not sure what happens when we restrict to C.

If $g \geq 2$ we know that $h^0(\mathcal{T}_C) = 0$, so that $\boxed{\chi(\mathcal{T}_C) = h^1(\mathcal{T}_C)}$. To compute the latter Euler characteristic, which is given by definition by $\chi(\mathcal{T}_X|_C) = h^0(\mathcal{T}_X|_C) - h^1(\mathcal{T}_X|_C)$, we first note that $h^0(\mathcal{T}_X|_C) = 0$, because this is the dimension of the global holomorphic vector fields on X restricted to C, which is constant along C since X is smooth. And then this number is actually zero by Hodge numbers of a K3 surface.

and taking cohomology long exact sequence we obtain

$$\cdots \longrightarrow H^0(T_X) \longrightarrow H^0(\mathcal{N}) \longrightarrow H^1(T_C) \longrightarrow$$

$$H^1(T_X) \longrightarrow H^1(\mathcal{N}) \longrightarrow \cdots$$

Or is it $h^1(\mathcal{N})$? See [**?**]. If this was the case, then we can use adjunction formula as above to get that $2g-2=(\mathcal{N},\mathcal{N})=\deg_C\mathcal{N}$. Then we may find $h^0(\mathcal{N})$ via Riemann-Roch:

$$h^0(\mathcal{N})-h^0(K_C-\mathcal{N})=\deg\mathcal{N}-g+1$$

Note that $h^0(N_C-\mathcal{N})=h^1(-K_C+N+K_C)=h^1(\mathcal{N})$ via Serre duality, and by Riemann-Roch on a surface as above we see that

$$g+1=\chi(\mathcal{N})=h^0(\mathcal{N})-h^1(\mathcal{N}) \implies h^1(\mathcal{N})=-g-1+h^0(\mathcal{N})$$

so that

$$h^0(\mathcal{N})-(-g-1+h^0(\mathcal{N}))=\deg\mathcal{N}-g+1$$
$$\implies \text{oops! I lost } h^0(\mathcal{N}) \text{ in this operation...}$$

Maybe if I just use normal exact sequence and realise that

$$h^0(\mathcal{N})=h^0(\mathcal{T}_X|_C)-h^0(\mathcal{T}_C)$$

I know that $h^0(\mathcal{T}_C)=0$ for $g>1$, so the question is how to compute the restricted holomorphic vector fields. □

22. Continued fractions

Definition of HJ continued fraction. For $i>2$ they are in bijection with $\mathbb{Q}_{>1}$.

The basic diagram of this course starts with a surface S (eg. Hirzebruch surface $S=\mathbb{F}_m$). Blowing up leads to X, and contracting Wahl chains on X leads to W, a normal projective surface that has only Wahl singularities. Then we construct $\mathbb{Q}$-Gorenstein smoothings W_t. (These $\mathbb{Q}$-Gorenstein smoothings have Milnor number =0.)

Continued fractions have minimal models:

- $[1,1]$ means a 0 curve, $\mathbb{P}^1$.
- $[1]$ means a -1 curve, $\mathbb{P}^1$.
- For $\frac{m}{q}\in\mathbb{Q}_{>1}$, the continued fraction $[e_1,\dots,e_r]$ means a chain, which is a sequence of lines that intersect transversally with $-e_1,\dots,-e_r$. This is mapped to $\frac{1}{m}(1,q)$.

Third lecture.
Here's some slogans/recap:

(1) The most important cyclic quotient singularities (c.q.s.) are Wahl $\frac{1}{n^2}(na-1)$. There is a model to deal with this kind o singularities using continued fractions. This is very silly but what I picked up is that "you add a 2 in the end and add +1 to the first number", so for example $[4]\rightsquigarrow[5,2]\rightsquigarrow[6,2,2]$. But on the second step the $[5,2]$ also goes to $[2,5,3]$ in a way I don't understand. This is called the Wahl algorithm.

(2) (See [KSB88]) There is a notion of M-resolution, which is a drawing of several curves Γ_i intersecting at points P_i that may be Wahl singularities or smooth points with the key property that $\Gamma_i\cdot K\geq 0$. We have "toric boundary for P_i". These M-resolutions are in 1-1 correspondence with smoothings of $\frac{1}{m}(1,q)$, and in turn in 1-1 correspondence with continued fractions $K\left(\frac{m}{m-q}\right)=\{k_1,\dots,k_s]:1\leq k_i\leq b_i\ \forall i\}$ where $\frac{m}{m-q}=[b_1,\dots,b_s]$.

Today we consider the fibers to be $W_t = \mathbb{P}^2$ and try to find W. Set $m_1, m_2, m_3 \in \mathbb{Z}_{>0}$. Define

$$\mathbb{P}(m_1, m_2, m_3) := \mathbb{P}^2/(\mathbb{Z}/m_1 \oplus \mathbb{Z}/m_2 \oplus \mathbb{Z}/m_3) = \mathbb{C}^3 \backslash \{0\}/(\lambda \in \mathbb{C}^* \lambda(x,y,z) = (\lambda^{m_1}x, \lambda^{m_2}y, \lambda^{m_3}z))$$

For $\mathrm{gdc}(d, m_i) = 1$ we have $\mathbb{P}(dm_1, dm_2, dm_3) = \mathbb{P}(m_1, m_2, m_3)$.

For a triangle $xyz = 0$ given by three lines Γ_i we have cqs singularities of the kind $\frac{1}{m_1}(m_2, m_3)$. In this case $K_W = -(m_1 + m_2 + m_3)\xi = -\Gamma_1 - \Gamma_2 - \Gamma_3$ for $\xi^2 = \frac{1}{m_1 m_2 m_3}$, and $\mathrm{Cl}(W) = \mathbb{Z}\langle \xi \rangle$. Since these are Wahl singularities, we must have that the m_i are squares, i.e. $m_i = n_i^2$ for some n_i. We must have:

$$\begin{aligned} K_W^2 = (m_1 + m_2 + m_3)^2 \frac{1}{m_1 m_2 m_3} &= 9 = K_{\mathbb{P}^2}^2 \\ \implies (n_1^2 + n_2^2 + n_3^2) - 9n_1^2 n_2^2 n_3^2 &= 0 \\ \implies (n_1^2 + n_2^2 + n_3^2 - 2n_1 n_2 n_3) \cdot (\text{positive factor}) &= 0 \\ \implies n_1^2 + n_2^2 + n_3^2 &= 3n_1 n_2 n_3 \end{aligned}$$

The last equation is known as *Markov equation.*

Example 22.1. For $\mathbb{P}(1,1,4) = W$, a triangle with a Wahl singularity $\frac{1}{4}(1,1)$ in one vertex. Blowing up gives the Hirzebruch surface $\mathbb{F}_4$, so that a minimal resolution is the triangle. Compare with [Hacking-Prokhorov-2010]. This example satisfies the Markov equation for $n_1 = 1, n_2 = 1, n_3 = 2$.

Theorem 22.2 ([HP2010)**.** *] If $\mathbb{P}^2 \rightsquigarrow W$ with only log terminal singularities then W is a partial $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}(a^2, b^2, c^2)$ where $a^2 + b^2 + c^2 = 3abc$.*

By the Markov equation condition all the singularities must be Wahl. The triple (a, b, c) is called *Markov triple.* Any permutation of a Markov triple is another Markov triple. Is (a, b, c) is Markov then so is $(a, b, 3ab - c)$. This allows to construct a *Markov tree.* There is so-called Markov conjecture (due to Frobenius) still unsolved.

23. Stanley Reisner

Antes de introduzir matroides, Os conjuntos f_s, que são os conjuntos de tamanho s, eles tem um significado geométrico?

24. Fano varieties

Definition 24.1. A *Fano variety* is a projective variety with $-K_X$ ample.

Definition 24.2.

$$r(X) := \min\{r : \frac{c_1(X)}{r} \in H^2(X, \mathbb{Z})\}$$

Exercise 24.3. By Kodaira vanishing theorem **??**, you can show that the cohomology $H^i(X, L)$ for a Fano variety X vanishes. You just have to put $L = \mathcal{O}(k)$ with $k \geq -r$, where r is the Fano index.

Exercise 24.4. Show that $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ holds for Fano varieties.

Remark 24.5. If $H^3(X, \mathbb{Z}) = 0$ of a Fano 3-fold, then its derived category is generated by 4 elements.

25. Quivers

Definition 25.1. A *quiver* is a set of vertices Q_0, a set of arrows Q_1 equipped with the maps of source s and target t that to each arrow they assign the point that is source or target of the arrow.

Definition 25.2. A *representation* of a quiver is a set of finite dimensional vector spaces equipped with maps between them realising a given quiver (incomplete. . .).

There is a notion of projective representation, which I missed to write. But it is analogous to the injective representation:

Definition 25.3. Given a quiver Q, the *injective representation* of Q_0 is given by, for $i \in Q_0$,

$$I(i)_j = \begin{cases} k & i = j \\ k^{d'} & j \neq i \end{cases}$$

where d' is the number of paths from j to i.

26. Stacks

My first definition of stack can be extracted from

Definition 26.1. A *superstack* is a stack over the étale site SSch of superschemes, i.e. it is a category fibered in groupoids over the category of superschemes, the latter equipped with the étale topology, satisfying the descent condition.

Here are some other definitions:

Definition 26.2. Let $\mathfrak{X}$ be a stack over $\mathrm{Sch}_{\text{ét}}$. An *algebraic space* is such that there exists morphism $\mathcal{U} \to \mathfrak{X}$ where $\mathcal{U}$ is a scheme, that is schematic, étale and injective (check this one).

$\mathfrak{X} \to y$ is *representable* if there exists a scheme $\mathcal{U}$ and a map $\mathcal{U} \to y$ such that the fibered product

$$\begin{tikzcd} \mathcal{U} \times_y \mathfrak{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{X} \arrow[d] \\ \mathcal{U} \arrow[r] & y \end{tikzcd}$$

is an algebraic space.

Finally, a stack is *algebraic* (resp. *Deligne-Mumford*) is there exists a representable surjective morphism $\mathcal{U} \to \mathfrak{X}$ that is smooth (resp. étale).

A *stable map* over a projective variety X is an element of the first Chow group $\beta \in A_1$, where (C, g) is an algebraic curve and $f : C \to X$ with $[f(C)] = \beta$.

The curves that are points under this map (contractible) are **stable**.

References

[Har77] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977.
[Har09] R. Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, Springer New York, 2009.
[Ser06] Edoardo Sernesi, *Deformations of algebraic schemes*, 2006.