BIRATIONAL MAPS FROM THE COMMUTATIVE ALGEBRA VIEWPOINT

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Notes at github.com/danimalabares/stack

O estudo da equivalência de variedades algébricas a partir da biracionalidade é um dos temas centrais da Geometria Algébrica. Nessa direção, a classificação de variedades pode ser vista como parte da chamada geometria birracional. De modo análogo, podemos concentrar-nos na estrutura dos mapas racionais que definem biracionalidades entre variedades — este será o ponto de partida do minicurso. Apresentarei aplicações de métodos de Álgebra Comutativa, como o uso de syzygies no estudo dos ideais de base de mapas racionais, e discutirei critérios algébricos que fornecem ferramentas computacionais para verificar a biracionalidade de tais mapas. Introduziremos também alguns métodos no Macaulay2 para testar biracionalidade. Além disso, veremos como diferentes teorias de multiplicidades — como mixed multiplicity e j-multiplicity — desempenham um papel importante na abordagem algébrica dos mapas racionais.

1. Plan

- A. Simis, A. Doria, - 2012 (Adv. Math) Criterio de Biracionalidade
- Simis, - 2012: Plane Cremona.
- Simis, -, 2017. Bounds for degrees of Birational maps with CM graph.
- Simis, Chardin 2021: degree of rational map versus syzygy.
- Mostafazadeh, 2024: loci de $\mathrm{Bir}(X)$.
- `RationalMaps.m2`

2. Introdução

Let R be a ring. We are interested in Mod_R and R-algebras categories. The first one is abelian, while the second one isn't. There are other diferences.

We are interested in discussing flat R-algebras and flat R-modules.

Theorem 2.1 (Grothendieck Generic freeness theorem)**.** *Let R be noetherian domain, $\varphi : R \to S$, S a finitely generated R-algebra. Then there exists $f \in R$ such that $\varphi : R_f \to S_{\varphi(f)}$, $S_{\varphi(f)}$ is a free R_f-module. $S_{\varphi(f)}$ is a flat R_f-module.*

Theorem 2.2 (Dimension of fibers theorem (Matsumura 15.1))**.** *$\varphi : R \to S$, R Noetherian, S Noetherian, $Q \in \mathrm{Spec}(S)$, $p = \varphi^{-1}(Q) \in \mathrm{Spec}(R)$,*

(1) *$htQ \leq htp + \dim \frac{S_Q}{pS_Q}$ [Shafarevich section 3].*

(2) *If φ is flat then $htQ = htp + \dim \frac{S_Q}{pS_Q}$.*

If $Q \not\ni \varphi(f) \implies S_Q$ is a free R_p-module (by Grothendieck's theorem). $\dim S_Q = \dim R_p + \dim \frac{S_Q}{pS_Q}$.

Topologia de Zariski.

$$\mathrm{Spec}(S) = \{Q : Q \text{ primo}\}$$

$$\text{fechado } = V(I) = \{Q : Q \supseteq I\}.$$

S is a domain, $\mathrm{Spec}(S)$ is an irreducible space, $\implies$ open nonempty sets are dense.

k field. An algebraic set is $V(I) = Z(I)$ for an ideal $I \subset k[x_1, \ldots, x_n]$. I is prime if and only if $X := V(I)$ is irreducible. Recall that a polynomial map is given by polynomial functions in the entries. Any such map yields in the category of algebras a map

$$k[Y] = A[Y] = \mathcal{O}_Y = \frac{k[y_1, \ldots, y_m]}{J} \xrightarrow{\varphi^*} \frac{k[x_1, \ldots, x_n]}{I}$$
$$y_i \mapsto \overline{f_i(x)}$$

where $Y := V(J)$. $\varphi^*(g) = 0\ \forall g \in J$, $g \in J \iff g(f_1, \ldots, f_m) \in I$ where f_i are the coordinate polynomial functions of φ.

φ is an isomorphism if it has a right and left inverse morphism. $\varphi : X \to X$ is an automorphism if it is an isomorphism.

Question: determine the automorphism group $\mathrm{Aut}(X)$.

If $X = \mathbb{A}^1$, a morphism

$$\varphi : k[x] \longrightarrow k[x]$$
$$x \longmapsto f(x)$$

has an inverse if and only if there exists $f(x)$ such that $f(g(x)) = x$. [This forces f to have degree 1 since composition $f \circ g$ must have degree $\deg f \cdot \deg g = \deg x = 1$.] So $f(x) = ax + b$, $a \neq 0$. Thus, we can parametrize the automorphism by two numbers a, b with $a \neq 0$. This is a quasi-affine variety.

For $X = \mathbb{A}^2$ the situation is a little more involved. de Jonequeres, Triangular group. Hierzch Jung 1942, chre. Wout Van der Kulk, 1953, chr arbitrary.

$$\mathrm{Aut}(\mathbb{A}^2) = \langle \text{linear, triangular} \rangle$$

For $\mathrm{Aut}(\mathbb{A}^3)$, the question if whether $\mathrm{Aut}(\mathbb{A}^3) = \langle \text{linear, triangular} \rangle$. This is false in general

Definition 2.3. An auotomorphism $\sigma \in \mathrm{Aut}(\mathbb{A}^3)$ is called *tame* if $\sigma \in \langle \text{linear, triangular} \rangle$ and *wild* otherwise.

Nagata's conjecture is that a $\sigma : k[x, y, z] \to k[x, y, z]$, $(x, y, z) \mapsto (x + (x^2 - yz)z, y + 2(x^2yz)x + (x^2 - yz)^2 z, z)$ is wild.

Shiguero Kuroda 2014. Introduce a monomial order $\leq$ in $k[x, y, z]$. [The following is not a criterion; this just says "if this happens, then the morphism is wild", but not an iff.] Let

$$\sigma : k[x, y, z] \longrightarrow k[x, y, z]$$
$$\longmapsto (f_1, f_2, f_3)$$

σ is wild if

(1) $\mathrm{int}(f_1), \mathrm{int}(f_2), \mathrm{int}(f_3)$ are linearly dependent over $\mathbb{Z}$, pairwise linear over $\mathbb{Z}$.
(2) $\mathrm{int}(f_{i_1}) \neq p\mathrm{int}(f_{i_2}) + q(\mathrm{int}(f_{i_3})$ for $p, q \in \mathbb{Z}_{\geq 0}$.

Jacobian conjecture: a polynomial map $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ [with coordinate functions] $(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ is an automorphism if and only if $\det \mathrm{Jacob}(\varphi) = \text{constant} \neq 0$.

The forward implication is clear: if $\varphi\psi = \mathrm{id}$ then $\mathrm{Jac}(\varphi)(\psi)\mathrm{Jac}(\psi) = \mathrm{Jac}(\mathrm{id}) = 1$.

[Bass-Conell-Wright, 1982] Let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ a polynomial map with $\det \mathrm{Jac}\varphi =$ constant. Then φ is invertible (its inverse is a polynomial map) [note that by the holomorphic inverse function theorem we already know that there exists a local inverse that is a series, but here we see it's actually polynomial] if and only if φ is injective if and only if φ is proper ($\varphi^{-1}(\text{compact}) = \text{compact}$).

Theorem 2.4. *If the Jacobian conjecture is valid for all n and for any polynomial of degree ≤ 3, then it's valid.*

Theorem 2.5. *If the Jacobian conjecture is valid over $\mathbb{C}$ then it is valid over any domain.*