A TOUR THROUGH ALGEBRAIC SURFACE

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Notes at github.com/danimalabares/stack

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2. References

(1) A. Beauville. Complex algebraic surfaces.
(2) R. Miranda. Overview of algebraic surfaces.
(3) M. Reid. Chapters on algebraic surfaces.

3. Introduction

Everything will be over the complex numbers. By "surface" we mean "algebraic surface", but we drop the term "algebraic". Informally, a surface is a smooth projective algebraic variety of dimension 2. We may think of surfaces also as compact connect complex manifolds of dimension 2.

$\mathcal{M}(S)$, meromorphic functions on S, has transcendence degree 2 over $\mathbb{C}$. That is, for $p, q \in S$ there exists $f \in \mathcal{M}(S)$ such that $f(p) \neq f(q)$ and for all $p \in S$ there exists $f, g \in \mathcal{M}(S)$ such that (f, g) give local coordinates at p.

Remark 3.1. The remark on transcendence just made implies that S is smooth algebraic of dimension 2. Indeed, compactness implies that there exists meromorphic functions $\varphi_1, \ldots, \varphi_n \in \mathcal{M}(S)$ such that $S \xhookrightarrow{alg} \mathbb{P}^n$, $p \mapsto (1 : \varphi_1(p) : \ldots : \varphi_n(p))$ and Chow theorem imply that $S = V(f_1, \ldots, f_k)$ with $f_i \in \mathbb{C}[x_0, x_1, \ldots, x_n]$.

Example 3.2.
(1) $S = V(f) \subset \mathbb{P}^3$ with $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ homogeneous of degree d. For example, $f = x_0^d + x_1^d + x_2^d + x_3^d$. Take $d = 1$, then we can assume that $f = x_i$, and then $S \simeq \mathbb{P}^2$. If $d = 2$ we can assume that $f = x_0x_1 - x_2x_3$ and one can verify that $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. If $d = 3$, $S \simeq Bl_6\mathbb{P}^2$. These are all rational surfaces.
If $d = 4$, S is a K3 surface.
(2) (Complete intersections of type $(d_1, \ldots, d_{n-1})$ in $\mathbb{P}^n$.) Let $f_1, \ldots, f_{n-2} \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ homogeneous of degrees $d_1, \ldots, d_{n-2}$. Then $S = V(f_1, \ldots, f_{n-2})$.
E.g., intersections of type $(2, 3)$ in $\mathbb{P}^4$ or of type $(2, 2, 2)$ in $\mathbb{P}^5$, which are K3 surfaces.
(3) (Ruled surfaces.) Take any curve C and consider $C \times \mathbb{P}^1$ (or anything birational to it). [This is interesting because we can consider the projection to C, and we get a fibration.]

4. Curves on surfaces (divisors)

We will use: $\mathrm{Div}(S)/\sim \simeq \mathrm{Pic}(X) = H^1(S, \mathcal{O}_S^*)$ where $\sim$ is linear equivalence.

$D \subset S$ is a divisor if and only if D is one of the following:

(1) (Cartier divisor.) [I can think of D given locally as the zero locus of some function.] $D : f = 0$ for $0 \neq f \in \mathcal{M}(S)$ + gluing condition. This is equivalent to saying that $D|_f = \mathrm{div}(f)$.
(2) (Weil divisors.) $D = \sum n_i C_i$ for $n_i \in \mathbb{Z}$, $C_i \subset S$ irreducible curves, where there are only finitely many nonzero coefficients n_i.
(3) (Line bundle.) D effective (i.e. $n_i \geq 0$), we associate $\mathcal{O}_S(D)$, a holomorphic line bundle on S along with a nonzero section [whose zero locus defines D. Recall that $U \mapsto \mathcal{O}_S(D)(U)$ is "given" by $g(z, w)$ such that fg is holomorphic where $D|_U = \mathrm{div}(f)$. And we extend by linearity.

[And we say that two divisors are linearly equivalent if their difference is $div(f)$ for some f.]

Example 4.1.
(1) $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$ generated by $\mathcal{O}_{\mathbb{P}^2}(1)$, the dual of the tautological bundle, or, equivalently, the class of a hyperplane H, whose sections are polynomials of degree 1 (in the appropriate number of variables).
(2) $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n}(1)$. We have $\mathcal{O}_{\mathbb{P}^n}(d) := \mathcal{O}_{\mathbb{P}^n}^{\otimes d}$. For $n = 2$, the sections of $\mathcal{O}(d)$ are plane curves of degree 2.

Suppose that $\pi : S \hookrightarrow \mathbb{P}^n$. The *hyperplane class* is $H|_S := \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. We denote $\mathcal{O}_S(1) = \mathcal{O}_S(H|_S)$. This is always a non-trivial class.

We also have the *canonical class*, which is the class of any divisor K such $\mathcal{O}_S(K) = \Omega_S^2$, the sheaf of holomorphic 2-forms, informally its elements look like $f(z, w)dx \wedge dw$. We denote this as K, K_S or ω_S.

Example 4.2 (Canonical class of projective plane)**.** Take coordinates $Z = X_1/X_0$ and $W = X_2/X_0$ on $U_0 : X_0 \neq 0$. Likewise put $U = X_0/X_1$, $V = X_2/X_1$ at U_1. Then $dZ \wedge dW = -U^{-3} dU \wedge dV$. Thus, the canonical class of $\mathbb{P}^2$ is linearly equivalent to $-3H$. In general, $K_{\mathbb{P}^n} \sim -(n+1)H$.

In practice, we can use this to compute K_S for any $S \hookrightarrow \mathbb{P}^n$ via $\omega_S = \omega_{\mathbb{P}^n} \otimes \Lambda^{n-2} N_S/\mathbb{P}^n$.

5. Numerical invariants

- (Hodge decomposition.)

$$H^k(S, \mathbb{C}) = \bigoplus_{p+q=k} \underbrace{H^q(S, \Omega_S^p)}_{H^{p,q}(S)}.$$

Setting $h^{p,q} = \dim H^{p,q}$ + symmetries, we have the Hodge diamond

$$\begin{array}{ccccccccccc} & & h^{0,0} & & & = & & & 1 & & \\ & h^{0,1} & & h^{1,0} & & & & q & & q & \\ h^{0,2} & & h^{1,1} & & h^{2,0} & & p_g & & h^{1,1} & & p_g \\ & h^{1,2} & & h^{2,1} & & & & q & & q & \\ & & h^{2,2} & & & & & & 1 & & \end{array}$$

where we call q the *irregularity* p_g the *geometric genus* and the *Euler characteristic* is the alternating sum of Betti numbers. Notice that $h^{1,1} \geq 1$.

6. The Néron-Severi group and Lefschetz (1,1) theorem

There is always a map

$$\begin{aligned} c_1 : \mathrm{Pic}(S) &\longrightarrow H^2(S, \mathbb{Z}) \overset{\substack{\text{Poincaré}\\ \text{dual}}}{\cong} H_2(S, \mathbb{Z}) \\ \mathcal{L} &\longmapsto c_1(\mathcal{L}) \end{aligned}$$

We define $\mathrm{NS}(X) := \mathrm{Im} c_1/\text{torsion}$. It is a finitely generated abelian group isomorphic to $\mathbb{Z}^\rho$ where ρ is the Picard rank.

Theorem 6.1.
$$NS(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$$
$NS(S)$, $Div(S)/\sim$ and $Pic(S)$ have more structure! There exists a unique bilinear form

$$\begin{aligned} \cdot : Div(S) \times Div(S) &\longrightarrow \mathbb{Z} \\ (C, D) &\longmapsto C \cdot D \end{aligned}$$

such that

(1) *If C and D are smooth curves which intersect transversally, then $C.D = \#C \cap D$.*
(2) *If $C \sim C'$, then $C \cdot D = C' \sim D$.*

Definition 6.2. $C.D = \mathcal{O}_S(C).\mathcal{O}_S(D) = \langle c_1(\mathcal{O}_S(D) \cup c_1(\mathcal{O}_S(D)), [S]_{\text{fund}}\rangle$, and we say that $C.C = \deg \mathcal{N}_{C/S}$.

[An important invariant of a surface is the self-intersection number of the canonical class.] Particular case: $K^2 = c_1^2$ where c_1^2 is the first Chern number.

Example 6.3. (1) $K_{\mathbb{P}^2}^2 = (-3H).(-3H) = 9$.
(2) (Degree of S.) Let $S \hookrightarrow \mathbb{P}^n$. Recall our notation above for $\mathcal{O}_S(1).\mathcal{O}_S(1)$. If $n = 3$, the degree d of S is the degree of $\mathcal{N}_{S/\mathbb{P}^3}$. Thus the adjunction becomes $\omega_S = \mathcal{O}_S(d - 4$
(3) C smooth curve, then $f : S \twoheadrightarrow C$, F fiber, then $F^2 = 0$.
(4) $g = \tilde{S} \xrightarrow{d} D$, then $D_1, D_2 \in \mathrm{Div}(S) \implies g^*(D_1) \cdot g^*(D_2) = d(D_1.D_2)$.

Definition 6.4. A divisor $D \subset S$ is *nef* if and only if for each $C \subset S$ irreducible, $D \cdot C \geq 0$. We say that D is ample if and only if $D^2 > 0$ and for each irreducible $C \subset S$ we have $D.C \geq 0$.

7. Big theorems

Theorem 7.1. *Let S be a surface.*
(1) *(Noether's formula.)*
$$12\chi = K^2 + e = c_1^2 + c_2$$
where e is the Euler number.
(2) *(Riemann-Roch.) Given $D \in Div(S)$,*
$$\chi(\mathcal{O}_S(D)) = \frac{D.(D-K)}{2} + \chi(\mathcal{O}_S).$$
(3) *(Genus formula.) Given $C \subset S$ smooth,*
$$2g - 2 = C.(C + K).$$

Example 7.2. (1) For $\mathbb{P}^2$, $K^2 = 9$ and $e = 3$. Thus $12\chi = 12 \implies \chi = 1$. The irregularity is $q = H^0(\mathbb{P}^2, \Omega^1) = 0$. The arithmetic genus is $p_g = \dim H^0(\mathbb{P}^2, \Omega^2) = 0$. Thus, the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(2) Any rational surface has a Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & \rho & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The Noether formula gives $12 = K_S^2 + \rho + 2 \implies K_S^2 = 10 - \rho$.

(3) $C \subset \mathbb{P}^2$ smooth of degree d, then $2g - 2 = d^2 - 3d$ and $g = \frac{d^2-3d+2}{2} = \frac{(d-1)(d-2)}{2} = \binom{d-1}{2}$. In particular, if $d = 1$ or 2, then $g = g(C) = 0$ if $d = 3$ then $g(C) = 1$.

8. Birational maps and classification theorem

Question: given two surfaces S and S', when are S and S' birational/biregular or isomorphic?

- Rules surfaces (all birational to $C \times \mathbb{P}^1$ for some C.
- Classify minimal (models of) surfaces by looking at the positive of K (and related invariants q, p_g and κ).

K	κ	p_g	q	Structure
$K^2 > 0$	2			General type
$K^2 = 0$	1			Elliptic*
		1	2	abelian
		0	1	hyperelliptic
$K = 0$	0	1	0	K3
		0	0	Enriques
	$-\infty$	0	≥ 1	ruled surfaces
		0	0	rational

The following proposition is also a definition:

Proposition 8.1. *Let S and S' be two surfaces, then the following statements about S and S' are equivalent:*

(1) *S and S' are birational*
(2) *$\mathcal{M}(S) \simeq \mathcal{M}(S')$ as algebras (over $\mathbb{C}$).*
(3) *There exists $U \subset S$, $V \subset S'$ opens such that $U \simeq V$.*

9. The blow up of the projective plane at a point

Consider $p = (0 : 0 : 1)$ and

$$\begin{aligned} \varphi_p : \mathbb{P}^2 \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\longmapsto (x_0 : x_1) \\ q &\longmapsto L_{pq} \end{aligned}$$

Consider also

$$\Gamma := \left\{ ((x_0 : x_1 : x_2), (y_0 : y_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 : \det \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} = 0 \right\}.$$

By construction,

$$\pi : \Gamma \to \mathbb{P}^2$$

is an isomorphism over $U = \mathbb{P}^2 \setminus \{p\}$ and $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1 \simeq \mathbb{P}^1 := E$.

Definition 9.1. $\pi : \Gamma \to \mathbb{P}^2$ as above is called the *blowup* of $\mathbb{P}^2$ at p (also Γ). The curve E is called the *exceptional divisor*.

Note that we have the following local description (chart $x_2 = 1$): $\{(x, \ell) \in \mathbb{C}^2 \times \mathbb{P}^1 : x \in \ell\} \to \mathbb{C}^2$. That is, the exceptional divisor may be identified with the zero section of the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, which means that the self intersection of E is 1, i.e. $E^2 = 1$.

Remark 9.2.
(1) The surface $Bl_p\mathbb{P}^2 := \Gamma$ is the Hirzebruch surface $\mathbb{F}_1$ (or Σ_1).
(2) Choosing local coordinates, we can talk about Bl_pS for $p \in S$.
(3) In fact, $\pi : Bl_pS \to S$ is characterized by the following universal property: if $f : \tilde{S} \to S$ is birational such that f^{-1} is not defined at $p \in S$, then f factorizes as $f : \underbrace{\tilde{S} \xrightarrow{g} \hat{S}}_{\text{bir}} := Bl_pS \xrightarrow{\varepsilon} S$ where $\varepsilon^{-1}(S \setminus \{p\}) \simeq S \setminus \{p\}$ and $\varepsilon^{-1}(p) \simeq \mathbb{P}^1$.

Proposition 9.3. *Consider $\pi : \tilde{S} = Bl_pS \to S$ as above, with exceptional divisor E,*
(1) $E^2 = -1$,
(2) $E \cdot \pi^*D = 0$ *for all* $D \in Div(S)$,
(3) *If $C \subset S$ has mutliplicity m at p, then $\tilde{C} := \overline{\pi^{-1}(C) \setminus E} \subset \tilde{S}$ satisfies $\tilde{C} \cdot E = m$ and $\tilde{C}^2 = C^2 - m^2$*
(4) $K_{\tilde{S}} = \pi^*K_S + E \implies K_{\tilde{S}}^2 = K_S^2 - 1$
(5) *q, p_g and χ are the same for $\tilde{S}$ as for S*
(6) *e, $h^{1,1}$ and b_2 all go up by 1*
(7) $Pic(\tilde{S}) = \pi^*(Pic(S)) \oplus \mathbb{Z}E$.

10. Castelnuovo's criterion

[It turns out that if you find a surface with a -1 curve, then that surface is the blow up of another!]

Theorem 10.1. *If on a surface $\tilde{S}$ one finds a curve $E \simeq \mathbb{P}^1$ with $E^2 = -1$, then $\tilde{S} = Bl_pS$ for some $p \in S$.*

Definition 10.2. A surface S is *minimal* if it has no (-1)-curves (i.e. a curve $\simeq \mathbb{P}^1$ with self-intersection -1).

(If and only if $f : S \dashrightarrow \tilde{S}$ and S minimal, then f is an isomorphism.)

11. Mori's point of view

If S is a surface such that K_S is not nef [so, by definition of nef there must exist an irreducible curve that intersects the canonical divisor negatively, and this curve in fact is a (-1) curve:] then there exists a (-1)-curve on S.

12. Kodaira dimension

Let S be a surface.

Definition 12.1. For each $n \geq 0$, we define

$$P_n := \dim H^0(S, \mathcal{O}_S(nK))$$

and the *Kodaira dimension*

$$\kappa := \begin{cases} -\infty & \text{if } P_n = 0 \forall n \\ \substack{\text{smallest } k \text{ such that} \\ P_n = O(n^k)} & . \end{cases}$$

13. Results of Mori's theory

Theorem 13.1. *Given a surface S, there exists a chain*

$$S = S^0 \xrightarrow{\sigma_1} S^1 \to \cdots \xrightarrow{\sigma_N} S_N = S'$$

where each σ_i is the contraction of a (-1) curve E_i and S' satisfies

1. *$K_{S'}$ is nef or*
2. *S' is a $\mathbb{P}^1$-bundle over a curve or*
3. *$S' \simeq \mathbb{P}^2$.*

Theorem 13.2. *If a surface S is such that K_S is not nef then there exists $\varphi : S \to X$, with $\dim X = 0, 1$ or 2, contracting at least one curve to a point, such that $-K_S \cdot C > 0$ for every curve C contained in a fiber of φ.*

Note:

$$\dim X = \begin{cases} 0 \rightsquigarrow & -K \text{ is ample} \\ 1 \rightsquigarrow & S \text{ is a conic bundle} \\ 2 \rightsquigarrow & S = Bl_{p_1, \ldots, p_n}. \end{cases}$$

14. Examples and the cubic surface revisited

(1) Slogan: "blowups resolve singularities". Consider the plane cubic

$$C = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : \underbrace{x_1^2 x_2 - x_0^2(x_0 + x_2) =}_{:= f_C} 0\},$$

the point $p = (0 : 0 : 1)$ and $\pi : Bl_p\mathbb{P}^2 \to \mathbb{P}^2$, then

$$\pi^{-1}(C) = \begin{cases} f_C = 0 \\ x_0 y_0 = y_0 x_1 \end{cases}$$

[This curve has a simple node. In the blowup, the curve will intersect the exceptional divisor in two points — we have "separated" the singularity into two points]

(2) $Bl_1\mathbb{P}^1 \times \mathbb{P}^1 \simeq Bl_1\mathbb{F}_1 \simeq Bl_2\mathbb{P}^2$. [Consider two curves L_1 and L_2 on $\mathbb{P}^1 \times P^1$. Their strict transforms along with the exceptional divisor become (-1) curves. Then contract $\tilde{L}_1$, this gives $\mathbb{F}_1$, the Hirzebruch surface, which is just the blowup at a point of $\mathbb{P}^2$. $\mu(\tilde{L}_2) = -1$, $\mu(E \setminus \tilde{L}_1) = 0$.]

(3) Consider two cubics C_1 and C_2. They intersect in 9 points. Do

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$
$$p \longmapsto (C_1(p) : C_2(p))$$

Let $S = Bl_{p_1,...,p_9}\mathbb{P}^2$. Then
  (a) $K_S^2 = 0$ [Because K^2 of $\mathbb{P}^2$ is 9, and for every blowup we substract 1!]
  (b) $\mathrm{Pic}(S)$ "=" $\pi^*\mathrm{Pic}(\mathbb{P}^2) \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_q$.
  (c) $-K_S = -\pi^*(K_{\mathbb{P}^2}) - E_1 - \ldots - E_q$ is nef. (In fact, this is the class of a fiber of $S \to \mathbb{P}^1$.)

(4) (Continuation of previous item.) $-K_S$ nef + genus formula $\implies C^2 \geq -2$ for every $C \subset S$ irreducible and rational. This is an elliptic surface!

[We have taken a 1-dimensional linear system. But why not take a higher-dimensional linear system?]

(5) Fix $p_1, \ldots, p_6$ in $\mathbb{P}^2$ in general position (not three in a line, not six in a conic). Consider $f_1, f_2, f_3, f_4 \in \mathbb{C}[x_0, x_1, x_2]$ homogeneous of degree 3 vanishing at p_i and consider

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$$
$$p \longmapsto (f_1(p) : f_2(p) : f_3(p) : f_4(p)).$$

The resolved morphism $Bl_6\mathbb{P}^2 \to \mathbb{P}^3$ is an embedding with image of degree 3 [a cubic surface on $\mathbb{P}^3$].