

ALGEBRA

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1. MODULES

It looks like most books (at least [?] and [?]) define a module to be an abelian group along with a certain operation with elements of the ring.

Which secretly just says that

Definition 1.1. Let R be a ring. An (*left*) R -module M is an abelian group such that there exists a (left) ring morphism

$$R \rightarrow \text{End}(M).$$

Indeed, the three usual requirements correspond to:

- (1) The endomorphism corresponding to $a \in R$ respect the group structure of the group M :

$$a(x + y) = ax + ay,$$

- (2) The representation map is a map of rings

$$(a + b)x = ax + by, \quad (ab)x = a(bx).$$

2. ALGEBRAS

There appears to be no definition of “algebra” in Stacks Project. But there is one in [?]:

Definition 2.1. If R is a commutative ring, then a *commutative algebra* over R is a commutative ring S together with a ring morphism $R \rightarrow S$.

But actually I was expecting that S would be defined as a ring that is also an R -module. So let us note that a morphism of rings $R \rightarrow S$ gives a representation $R \rightarrow \text{End}(S)$ via left multiplication. But the other way around, given an endomorphism associated to some element in R , how do we assign an element of S so as to produce a map $R \rightarrow S$?

The second item in the following definition is the algebraic counterpart to an affine algebraic set (variety); i.e. the reason why we hear that “affine varieties are in correspondence with finitely presented k -algebras” (wasn’t there a notion of reduced algebra in that phrase...?) And the difference with that and finite type, I think, is that the kernel is finitely generated.

Definition 2.2. Let $R \rightarrow S$ be a ring map.

- (1) We say $R \rightarrow S$ is of *finite type*, or that S is a *finite type R -algebra* if there exist an $n \in \mathbf{N}$ and an surjection of R -algebras $R[x_1, \dots, x_n] \rightarrow S$.
- (2) We say $R \rightarrow S$ is of *finite presentation* if there exist integers $n, m \in \mathbf{N}$ and polynomials $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ and an isomorphism of R -algebras $R[x_1, \dots, x_n]/(f_1, \dots, f_m) \cong S$.

So, probably an “algebra” A over a ring R is when A contains (or, is isomorphic to?) $R[X]$ for some possibly very arbitrary set $X \subset R$.

3. FINITE AND INTEGRAL RING EXTENSIONS

Trivial lemmas concerning finite and integral ring maps. We recall the definition.

Definition 3.1. Let $\varphi : R \rightarrow S$ be a ring map.

- (1) An element $s \in S$ is *integral over R* if there exists a monic polynomial $P(x) \in R[x]$ such that $P^\varphi(s) = 0$, where $P^\varphi(x) \in S[x]$ is the image of P under $\varphi : R[x] \rightarrow S[x]$.
- (2) The ring map φ is *integral* if every $s \in S$ is integral over R .

Lemma 3.2. Let $\varphi : R \rightarrow S$ be a ring map. Let $y \in S$. If there exists a finite R -submodule M of S such that $1 \in M$ and $yM \subset M$, then y is integral over R .

Proof. Consider the map $\varphi : M \rightarrow M$, $x \mapsto y \cdot x$. By Lemma ?? there exists a monic polynomial $P \in R[T]$ with $P(\varphi) = 0$. In the ring S we get $P(y) = P(y) \cdot 1 = P(\varphi)(1) = 0$. \square

Lemma 3.3. A finite ring map is integral.

Proof. Let $R \rightarrow S$ be finite. Let $y \in S$. Apply Lemma 3.2 to $M = S$ to see that y is integral over R . \square

Lemma 3.4. Let $\varphi : R \rightarrow S$ be a ring map. Let s_1, \dots, s_n be a finite set of elements of S . In this case s_i is integral over R for all $i = 1, \dots, n$ if and only if there exists an R -subalgebra $S' \subset S$ finite over R containing all of the s_i .

Proof. If each s_i is integral, then the subalgebra generated by $\varphi(R)$ and the s_i is finite over R . Namely, if s_i satisfies a monic equation of degree d_i over R , then this subalgebra is generated as an R -module by the elements $s_1^{e_1} \dots s_n^{e_n}$ with $0 \leq e_i \leq d_i - 1$. Conversely, suppose given a finite R -subalgebra S' containing all the s_i . Then all of the s_i are integral by Lemma 3.3. \square

Lemma 3.5. Let $R \rightarrow S$ be a ring map. The following are equivalent

- (1) $R \rightarrow S$ is finite,
- (2) $R \rightarrow S$ is integral and of finite type, and
- (3) there exist $x_1, \dots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R .

Proof. Clear from Lemma 3.4. \square

Lemma 3.6. Suppose that $R \rightarrow S$ and $S \rightarrow T$ are integral ring maps. Then $R \rightarrow T$ is integral.

Proof. Let $t \in T$. Let $P(x) \in S[x]$ be a monic polynomial such that $P(t) = 0$. Apply Lemma 3.4 to the finite set of coefficients of P . Hence t is integral over some subalgebra $S' \subset S$ finite over R . Apply Lemma 3.4 again to find a subalgebra $T' \subset T$ finite over S' and containing t . Lemma ?? applied to $R \rightarrow S' \rightarrow T'$ shows that T' is finite over R . The integrality of t over R now follows from Lemma 3.3. \square

Lemma 3.7. *Let $R \rightarrow S$ be a ring homomorphism. The set*

$$S' = \{s \in S \mid s \text{ is integral over } R\}$$

is an R -subalgebra of S .

Proof. This is clear from Lemmas 3.4 and 3.3. \square

Lemma 3.8. *Let $R_i \rightarrow S_i$ be ring maps $i = 1, \dots, n$. Let R and S denote the product of the R_i and S_i respectively. Then an element $s = (s_1, \dots, s_n) \in S$ is integral over R if and only if each s_i is integral over R_i .*

Proof. Omitted. \square

Definition 3.9. Let $R \rightarrow S$ be a ring map. The ring $S' \subset S$ of elements integral over R , see Lemma 3.7, is called the *integral closure* of R in S . If $R \subset S$ we say that R is *integrally closed* in S if $R = S'$.

In particular, we see that $R \rightarrow S$ is integral if and only if the integral closure of R in S is all of S .

Lemma 3.10. *Let $R_i \rightarrow S_i$ be ring maps $i = 1, \dots, n$. Denote the integral closure of R_i in S_i by S'_i . Further let R and S denote the product of the R_i and S_i respectively. Then the integral closure of R in S is the product of the S'_i . In particular $R \rightarrow S$ is integrally closed if and only if each $R_i \rightarrow S_i$ is integrally closed.*

Proof. This follows immediately from Lemma 3.8. \square

Lemma 3.11. *Integral closure commutes with localization: If $A \rightarrow B$ is a ring map, and $S \subset A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subset B$ is the integral closure of A in B .*

Proof. Since localization is exact we see that $S^{-1}B' \subset S^{-1}B$. Suppose $x \in B'$ and $f \in S$. Then $x^d + \sum_{i=1, \dots, d} a_i x^{d-i} = 0$ in B for some $a_i \in A$. Hence also

$$(x/f)^d + \sum_{i=1, \dots, d} a_i/f^i (x/f)^{d-i} = 0$$

in $S^{-1}B$. In this way we see that $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum_{i=1, \dots, d} (a_i/f_i) (x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f' f_1 \dots f_d x)^d + \sum_{i=1, \dots, d} f^i (f')^i f_1^i \dots f_i^{i-1} \dots f_d^i a_i (f' f_1 \dots f_d x)^{d-i} = 0$$

for a suitable $f' \in S$. Hence $f' f_1 \dots f_d x \in B'$ and thus $x/f \in S^{-1}B'$ as desired. \square

Lemma 3.12. *Let $\varphi : R \rightarrow S$ be a ring map. Let $x \in S$. The following are equivalent:*

- (1) x is integral over R , and

(2) for every prime ideal $\mathfrak{p} \subset R$ the element $x \in S_{\mathfrak{p}}$ is integral over $R_{\mathfrak{p}}$.

Proof. It is clear that (1) implies (2). Assume (2). Consider the R -algebra $S' \subset S$ generated by $\varphi(R)$ and x . Let \mathfrak{p} be a prime ideal of R . Then we know that $x^d + \sum_{i=1, \dots, d} \varphi(a_i)x^{d-i} = 0$ in $S_{\mathfrak{p}}$ for some $a_i \in R_{\mathfrak{p}}$. Hence we see, by looking at which denominators occur, that for some $f \in R$, $f \notin \mathfrak{p}$ we have $a_i \in R_f$ and $x^d + \sum_{i=1, \dots, d} \varphi(a_i)x^{d-i} = 0$ in S_f . This implies that S'_f is finite over R_f . Since \mathfrak{p} was arbitrary and $\text{Spec}(R)$ is quasi-compact (Lemma ??) we can find finitely many elements $f_1, \dots, f_n \in R$ which generate the unit ideal of R such that S'_{f_i} is finite over R_{f_i} . Hence we conclude from Lemma ?? that S' is finite over R . Hence x is integral over R by Lemma 3.4. \square

Lemma 3.13. Let $R \rightarrow S$ and $R \rightarrow R'$ be ring maps. Set $S' = R' \otimes_R S$.

- (1) If $R \rightarrow S$ is integral so is $R' \rightarrow S'$.
- (2) If $R \rightarrow S$ is finite so is $R' \rightarrow S'$.

Proof. We prove (1). Let $s_i \in S$ be generators for S over R . Each of these satisfies a monic polynomial equation P_i over R . Hence the elements $1 \otimes s_i \in S'$ generate S' over R' and satisfy the corresponding polynomial P'_i over R' . Since these elements generate S' over R' we see that S' is integral over R' . Proof of (2) omitted. \square

Lemma 3.14. Let $R \rightarrow S$ be a ring map. Let $f_1, \dots, f_n \in R$ generate the unit ideal.

- (1) If each $R_{f_i} \rightarrow S_{f_i}$ is integral, so is $R \rightarrow S$.
- (2) If each $R_{f_i} \rightarrow S_{f_i}$ is finite, so is $R \rightarrow S$.

Proof. Proof of (1). Let $s \in S$. Consider the ideal $I \subset R[x]$ of polynomials P such that $P(s) = 0$. Let $J \subset R$ denote the ideal (!) of leading coefficients of elements of I . By assumption and clearing denominators we see that $f_i^{n_i} \in J$ for all i and certain $n_i \geq 0$. Hence J contains 1 and we see s is integral over R . Proof of (2) omitted. \square

Lemma 3.15. Let $A \rightarrow B \rightarrow C$ be ring maps.

- (1) If $A \rightarrow C$ is integral so is $B \rightarrow C$.
- (2) If $A \rightarrow C$ is finite so is $B \rightarrow C$.

Proof. Omitted. \square

Lemma 3.16. Let $A \rightarrow B \rightarrow C$ be ring maps. Let B' be the integral closure of A in B , let C' be the integral closure of B' in C . Then C' is the integral closure of A in C .

Proof. Omitted. \square

Lemma 3.17. Suppose that $R \rightarrow S$ is an integral ring extension with $R \subset S$. Then $\varphi : \text{Spec}(S) \rightarrow \text{Spec}(R)$ is surjective.

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. We have to show $\mathfrak{p}S_{\mathfrak{p}} \neq S_{\mathfrak{p}}$, see Lemma ??. The localization $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{p}}$ is injective (as localization is exact) and integral by Lemma 3.11 or 3.13. Hence we may replace R, S by $R_{\mathfrak{p}}, S_{\mathfrak{p}}$ and we may assume R is local with maximal ideal \mathfrak{m} and it suffices to show that $\mathfrak{m}S \neq S$. Suppose $1 = \sum f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$ in order to get a contradiction. Let $R \subset S' \subset S$ be such that $R \rightarrow S'$ is finite and $s_i \in S'$, see Lemma 3.4. The equation $1 = \sum f_i s_i$ implies

that the finite R -module S' satisfies $S' = \mathfrak{m}S'$. Hence by Nakayama's Lemma ?? we see $S' = 0$. Contradiction. \square

Lemma 3.18. *Let R be a ring. Let K be a field. If $R \subset K$ and K is integral over R , then R is a field and K is an algebraic extension. If $R \subset K$ and K is finite over R , then R is a field and K is a finite algebraic extension.*

Proof. Assume that $R \subset K$ is integral. By Lemma 3.17 we see that $\text{Spec}(R)$ has 1 point. Since clearly R is a domain we see that $R = R_{(0)}$ is a field (Lemma ??). The other assertions are immediate from this. \square

Lemma 3.19. *Let k be a field. Let S be a k -algebra over k .*

- (1) *If S is a domain and finite dimensional over k , then S is a field.*
- (2) *If S is integral over k and a domain, then S is a field.*
- (3) *If S is integral over k then every prime of S is a maximal ideal (see Lemma ?? for more consequences).*

Proof. The statement on primes follows from the statement “integral + domain \Rightarrow field”. Let S integral over k and assume S is a domain. Take $s \in S$. By Lemma 3.4 we may find a finite dimensional k -subalgebra $k \subset S' \subset S$ containing s . Hence S is a field if we can prove the first statement. Assume S finite dimensional over k and a domain. Pick $s \in S$. Since S is a domain the multiplication map $s : S \rightarrow S$ is surjective by dimension reasons. Hence there exists an element $s_1 \in S$ such that $ss_1 = 1$. So S is a field. \square

Lemma 3.20. *Suppose $R \rightarrow S$ is integral. Let $\mathfrak{q}, \mathfrak{q}' \in \text{Spec}(S)$ be distinct primes having the same image in $\text{Spec}(R)$. Then neither $\mathfrak{q} \subset \mathfrak{q}'$ nor $\mathfrak{q}' \subset \mathfrak{q}$.*

Proof. Let $\mathfrak{p} \subset R$ be the image. By Remark ?? the primes $\mathfrak{q}, \mathfrak{q}'$ correspond to ideals in $S \otimes_R \kappa(\mathfrak{p})$. Thus the lemma follows from Lemma 3.19. \square

Lemma 3.21. *Suppose $R \rightarrow S$ is finite. Then the fibres of $\text{Spec}(S) \rightarrow \text{Spec}(R)$ are finite.*

Proof. By the discussion in Remark ?? the fibres are the spectra of the rings $S \otimes_R \kappa(\mathfrak{p})$. As $R \rightarrow S$ is finite, these fibre rings are finite over $\kappa(\mathfrak{p})$ hence Noetherian by Lemma ???. By Lemma 3.20 every prime of $S \otimes_R \kappa(\mathfrak{p})$ is a minimal prime. Hence by Lemma ?? there are at most finitely many. \square

Lemma 3.22. *Let $R \rightarrow S$ be a ring map such that S is integral over R . Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let \mathfrak{q} be a prime of S mapping to \mathfrak{p} . Then there exists a prime \mathfrak{q}' with $\mathfrak{q} \subset \mathfrak{q}'$ mapping to \mathfrak{p}' .*

Proof. We may replace R by R/\mathfrak{p} and S by S/\mathfrak{q} . This reduces us to the situation of having an integral extension of domains $R \subset S$ and a prime $\mathfrak{p}' \subset R$. By Lemma 3.17 we win. \square

The property expressed in the lemma above is called the “going up property” for the ring map $R \rightarrow S$, see Definition ??.

Lemma 3.23. *Let $R \rightarrow S$ be a finite and finitely presented ring map. Let M be an S -module. Then M is finitely presented as an R -module if and only if M is finitely presented as an S -module.*

Proof. One of the implications follows from Lemma ???. To see the other assume that M is finitely presented as an S -module. Pick a presentation

$$S^{\oplus m} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

As S is finite as an R -module, the kernel of $S^{\oplus n} \rightarrow M$ is a finite R -module. Thus from Lemma ??? we see that it suffices to prove that S is finitely presented as an R -module.

Pick $y_1, \dots, y_n \in S$ such that y_1, \dots, y_n generate S as an R -module. By Lemma 3.2 each y_i is integral over R . Choose monic polynomials $P_i(x) \in R[x]$ with $P_i(y_i) = 0$. Consider the ring

$$S' = R[x_1, \dots, x_n]/(P_1(x_1), \dots, P_n(x_n))$$

Then we see that S is of finite presentation as an S' -algebra by Lemma ???. Since $S' \rightarrow S$ is surjective, the kernel $J = \text{Ker}(S' \rightarrow S)$ is finitely generated as an ideal by Lemma ???. Hence J is a finite S' -module (immediate from the definitions). Thus $S = \text{Coker}(J \rightarrow S')$ is of finite presentation as an S' -module by Lemma ???. Hence, arguing as in the first paragraph, it suffices to show that S' is of finite presentation as an R -module. Actually, S' is free as an R -module with basis the monomials $x_1^{e_1} \dots x_n^{e_n}$ for $0 \leq e_i < \deg(P_i)$. Namely, write $R \rightarrow S'$ as the composition

$$R \rightarrow R[x_1]/(P_1(x_1)) \rightarrow R[x_1, x_2]/(P_1(x_1), P_2(x_2)) \rightarrow \dots \rightarrow S'$$

This shows that the i th ring in this sequence is free as a module over the $(i-1)$ st one with basis $1, x_i, \dots, x_i^{\deg(P_i)-1}$. The result follows easily from this by induction. Some details omitted. \square

Lemma 3.24. *Let R be a ring. Let $x, y \in R$ be nonzerodivisors. Let $R[x/y] \subset R_{xy}$ be the R -subalgebra generated by x/y , and similarly for the subalgebras $R[y/x]$ and $R[x/y, y/x]$. If R is integrally closed in R_x or R_y , then the sequence*

$$0 \rightarrow R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \rightarrow 0$$

is a short exact sequence of R -modules.

Proof. Since $x/y \cdot y/x = 1$ it is clear that the map $R[x/y] \oplus R[y/x] \rightarrow R[x/y, y/x]$ is surjective. Let $\alpha \in R[x/y] \cap R[y/x]$. To show exactness in the middle we have to prove that $\alpha \in R$. By assumption we may write

$$\alpha = a_0 + a_1 x/y + \dots + a_n (x/y)^n = b_0 + b_1 y/x + \dots + b_m (y/x)^m$$

for some $n, m \geq 0$ and $a_i, b_j \in R$. Pick some $N > \max(n, m)$. Consider the finite R -submodule M of R_{xy} generated by the elements

$$(x/y)^N, (x/y)^{N-1}, \dots, x/y, 1, y/x, \dots, (y/x)^{N-1}, (y/x)^N$$

We claim that $\alpha M \subset M$. Namely, it is clear that $(x/y)^i (b_0 + b_1 y/x + \dots + b_m (y/x)^m) \in M$ for $0 \leq i \leq N$ and that $(y/x)^i (a_0 + a_1 x/y + \dots + a_n (x/y)^n) \in M$ for $0 \leq i \leq N$. Hence α is integral over R by Lemma 3.2. Note that $\alpha \in R_x$, so if R is integrally closed in R_x then $\alpha \in R$ as desired. \square

REFERENCES