A TOUR THROUGH ALGEBRAIC SURFACE

Minicourse by Aline Zanardini (EPFL, Switzerland), IMPA Summer School, 2026.

Notes at github.com/danimalabares/stack

Contents

1. Plan

(1) Basics.
(2) Birational maps and classification.
(3) Elliptic surfaces.
(4) Halphen surfaces.
(5) K3 surfaces and friends.
(6) Research questions.

2. References

(1) A. Beauville. Complex algebraic surfaces.
(2) R. Miranda. Overview of algebraic surfaces.
(3) M. Reid. Chapters on algebraic surfaces.

3. Introduction

Everything will be over the complex numbers. By "surface" we mean "algebraic surface", but we drop the term "algebraic". Informally, a surface is a smooth projective algebraic variety of dimension 2. We may think of surfaces also as compact connect complex manifolds of dimension 2.

$\mathcal{M}(S)$, meromorphic functions on S, has transcendence degree 2 over $\mathbb{C}$. That is, for $p, q \in S$ there exists $f \in \mathcal{M}(S)$ such that $f(p) \neq f(q)$ and for all $p \in S$ there exists $f, g \in \mathcal{M}(S)$ such that (f, g) give local coordinates at p.

Remark 3.1. The remark on transcendence just made implies that S is smooth algebraic of dimension 2. Indeed, compactness implies that there exists meromorphic functions $\varphi_1, \dots, \varphi_n \in \mathcal{M}(S)$ such that $S \xhookrightarrow{alg} \mathbb{P}^n$, $p \mapsto (1 : \varphi_1(p) : \dots : \varphi_n(p))$ and Chow theorem imply that $S = V(f_1, \dots, f_k)$ with $f_i \in \mathbb{C}[x_0, x_1, \dots, x_n]$.

Example 3.2.
1. $S = V(f) \subset \mathbb{P}^3$ with $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ homogeneous of degree d. For example, $f = x_0^d + x_1^d + x_2^d + x_3^d$. Take $d = 1$, then we can assume that $f = x_i$, and then $S \simeq \mathbb{P}^2$. If $d = 2$ we can assume that $f = x_0x_1 - x_2x_3$ and one can verify that $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. If $d = 3$, $S \simeq Bl_6\mathbb{P}^2$. These are all rational surfaces.
   If $d = 4$, S is a K3 surface.
2. (Complete intersections of type $(d_1, \dots, d_{n-1})$ in $\mathbb{P}^n$.) Let $f_1, \dots, f_{n-2} \in \mathbb{C}[x_0, x_1, \dots, x_n]$ homogeneous of degrees $d_1, \dots, d_{n-2}$. Then $S = V(f_1, \dots, f_{n-2})$.
   E.g., intersections of type $(2, 3)$ in $\mathbb{P}^4$ or of type $(2, 2, 2)$ in $\mathbb{P}^5$, which are K3 surfaces.
3. (Ruled surfaces.) Take any curve C and consider $C \times \mathbb{P}^1$ (or anything birational to it). [This is interesting because we can consider the projection to C, and we get a fibration.]

4. Curves on surfaces (divisors)

We will use: $\mathrm{Div}(S)/\sim \simeq \mathrm{Pic}(X) = H^1(S, \mathcal{O}_S^*)$ where $\sim$ is linear equivalence.

$D \subset S$ is a divisor if and only if D is one of the following:

1. (Cartier divisor.) [I can think of D given locally as the zero locus of some function.] $D : f = 0$ for $0 \neq f \in \mathcal{M}(S)$ + gluing condition. This is equivalent to saying that $D|_f = \mathrm{div}(f)$.
2. (Weil divisors.) $D = \sum n_i C_i$ for $n_i \in \mathbb{Z}$, $C_i \subset S$ irreducible curves, where there are only finitely many nonzero coefficients n_i.
3. (Line bundle.) D effective (i.e. $n_i \geq 0$), we associate $\mathcal{O}_S(D)$, a holomorphic line bundle on S along with a nonzero section [whose zero locus defines D. Recall that $U \mapsto \mathcal{O}_S(D)(U)$ is "given" by $g(z, w)$ such that fg is holomorphic where $D|_U = \mathrm{div}(f)$. And we extend by linearity.

[And we say that two divisors are linearly equivalent if their difference is $div(f)$ for some f.]

Example 4.1.
1. $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$ generated by $\mathcal{O}_{\mathbb{P}^2}(1)$, the dual of the tautological bundle, or, equivalently, the class of a hyperplane H, whose sections are polynomials of degree 1 (in the appropriate number of variables).
2. $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n}(1)$. We have $\mathcal{O}_{\mathbb{P}^n}(d) := \mathcal{O}_{\mathbb{P}^n}^{\otimes d}$. For $n = 2$, the sections of $\mathcal{O}(d)$ are plane curves of degree 2.

Suppose that $\pi : S \hookrightarrow \mathbb{P}^n$. The *hyperplane class* is $H|_S := \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. We denote $\mathcal{O}_S(1) = \mathcal{O}_S(H|_S)$. This is always a non-trivial class.

We also have the *canonical class*, which is the class of any divisor K such $\mathcal{O}_S(K) = \Omega_S^2$, the sheaf of holomorphic 2-forms, informally its elements look like $f(z,w)dx \wedge dw$. We denote this as K, K_S or ω_S.

Example 4.2 (Canonical class of projective plane)**.** Take coordinates $Z = X_1/X_0$ and $W = X_2/X_0$ on $U_0 : X_0 \neq 0$. Likewise put $U = X_0/X_1$, $V = X_2/X_1$ at U_1. Then $dZ \wedge dW = -U^{-3}dU \wedge dV$. Thus, the canonical class of $\mathbb{P}^2$ is linearly equivalent to $-3H$. In general, $K_{\mathbb{P}^n} \sim -(n+1)H$.

In practice, we can use this to compute K_S for any $S \hookrightarrow \mathbb{P}^n$ via $\omega_S = \omega_{\mathbb{P}^n} \otimes \Lambda^{n-2} N_S/\mathbb{P}^n$.

5. Numerical invariants

- (Hodge decomposition.)

$$H^k(S,\mathbb{C}) = \bigoplus_{p+q=k} \underbrace{H^q(S,\Omega_S^p)}_{H^{p,q}(S)}.$$

Setting $h^{p,q} = \dim H^{p,q}$ + symmetries, we have the Hodge diamond

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{0,1} & & h^{1,0} & \\ h^{0,2} & & h^{1,1} & & h^{2,0} \\ & h^{1,2} & & h^{2,1} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

where we call q the *irregularity* p_g the *geometric genus* and the *Euler characteristic* is the alternating sum of Betti numbers. Notice that $h^{1,1} \geq 1$.

6. The Néron-Severi group and Lefschetz (1,1) theorem

There is always a map

$$\begin{aligned} c_1 : \mathrm{Pic}(S) &\longrightarrow H^2(S,\mathbb{Z}) \overset{\substack{\text{Poincaré}\\ \text{dual}}}{\cong} H_2(S,\mathbb{Z}) \\ \mathcal{L} &\longmapsto c_1(\mathcal{L}) \end{aligned}$$

We define $\mathrm{NS}(X) := \mathrm{Im} c_1/\text{torsion}$. It is a finitely generated abelian group isomorphic to $\mathbb{Z}^\rho$ where ρ is the Picard rank.

Theorem 6.1.

$$NS(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$$

$NS(S)$, $Div(S)/\sim$ and $Pic(S)$ have more structure! There exists a unique bilinear form

$$\begin{aligned} \cdot : Div(S) \times Div(S) &\longrightarrow \mathbb{Z} \\ (C,D) &\longmapsto C \cdot D \end{aligned}$$

such that

(1) *If C and D are smooth curves which intersect transversally, then* $C.D = \#C \cap D$.
(2) *If* $C \sim C'$, *then* $C \cdot D = C' \sim D$.

Definition 6.2. $C.D = \mathcal{O}_S(C).\mathcal{O}_S(D) = \langle c_1(\mathcal{O}_S(D) \cup c_1(\mathcal{O}_S(D)), [S]_{\text{fund}}\rangle$, and we say that $C.C = \deg \mathcal{N}_{C/S}$.

[An important invariant of a surface is the self-intersection number of the canonical class.] Particular case: $K^2 = c_1^2$ where c_1^2 is the first Chern number.

Example 6.3. (1) $K_{\mathbb{P}^2}^2 = (-3H).(-3H) = 9$.
(2) (Degree of S.) Let $S \hookrightarrow \mathbb{P}^n$. Recall our notation above for $\mathcal{O}_S(1).\mathcal{O}_S(1)$. If $n = 3$, the degree d of S is the degree of $\mathcal{N}_{S/\mathbb{P}^3}$. Thus the adjunction becomes $\omega_S = \mathcal{O}_S(d-4$
(3) C smooth curve, then $f : S \twoheadrightarrow C$, F fiber, then $F^2 = 0$.
(4) $g = \tilde{S} \xrightarrow{d} D$, then $D_1, D_2 \in \mathrm{Div}(S) \implies g^*(D_1) \cdot g^*(D_2) = d(D_1.D_2)$.

Definition 6.4. A divisor $D \subset S$ is *nef* if and only if for each $C \subset S$ irreducible, $D \cdot C \geq 0$. We say that D is ample if and only if $D^2 > 0$ and for each irreducible $C \subset S$ we have $D.C \geq 0$.

7. Big theorems

Theorem 7.1. *Let S be a surface.*

(1) *(Noether's formula.)*

$$12\chi = K^2 + e = c_1^2 + c_2$$

where e is the Euler number.
(2) *(Riemann-Roch.) Given $D \in Div(S)$,*

$$\chi(\mathcal{O}_S(D)) = \frac{D.(D-K)}{2} + \chi(\mathcal{O}_S).$$

(3) *(Genus formula.) Given $C \subset S$ smooth,*

$$2g - 2 = C.(C+K).$$

Example 7.2. (1) For $\mathbb{P}^2$, $K^2 = 9$ and $e = 3$. Thus $12\chi = 12 \implies \chi = 1$. The irregularity is $q = H^0(\mathbb{P}^2, \Omega^1) = 0$. The arithmetic genus is $p_g = \dim H^0(\mathbb{P}^2, \Omega^2) = 0$. Thus, the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(2) Any rational surface has a Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & \rho & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The Noether formula gives $12 = K_S^2 + \rho + 2 \implies K_S^2 = 10 - \rho$.

(3) $C \subset \mathbb{P}^2$ smooth of degree d, then $2g - 2 = d^2 - 3d$ and $g = \frac{d^2-3d+2}{2} = \frac{(d-1)(d-2)}{2} = \binom{d-1}{2}$. In particular, if $d = 1$ or 2, then $g = g(C) = 0$ if $d = 3$ then $g(C) = 1$.

8. Birational maps and classification theorem

Question: given two surfaces S and S', when are S and S' birational/biregular or isomorphic?

- Rules surfaces (all birational to $C \times \mathbb{P}^1$ for some C.
- Classify minimal (models of) surfaces by looking at the positive of K (and related invariants q, p_g and κ).

K	κ	p_g	q	Structure
$K^2 > 0$	2			General type
$K^2 = 0$	1			Elliptic*
$K = 0$	0	1	2	abelian
		0	1	hyperelliptic
		1	0	K3
		0	0	Enriques
	$-\infty$	0	≥ 1	ruled surfaces
		0	0	rational

The following proposition is also a definition:

Proposition 8.1. *Let S and S' be two surfaces, then the following statements about S and S' are equivalent:*

(1) *S and S' are birational*
(2) *$\mathcal{M}(S) \simeq \mathcal{M}(S')$ as algebras (over $\mathbb{C}$).*
(3) *There exists $U \subset S$, $V \subset S'$ opens such that $U \simeq V$.*

9. The blow up of the projective plane at a point

Consider $p = (0 : 0 : 1)$ and

$$\begin{aligned} \varphi_p : \mathbb{P}^2 \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ (x_0 : x_1 : x_2) &\longmapsto (x_0 : x_1) \\ q &\longmapsto L_{pq} \end{aligned}$$

Consider also

$$\Gamma := \left\{ ((x_0 : x_1 : x_2), (y_0 : y_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 : \det \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} = 0 \right\}.$$

By construction,

$$\pi : \Gamma \to \mathbb{P}^2$$

is an isomorphism over $U = \mathbb{P}^2 \setminus \{p\}$ and $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1 \simeq \mathbb{P}^1 := E$.

Definition 9.1. $\pi : \Gamma \to \mathbb{P}^2$ as above is called the *blowup* of $\mathbb{P}^2$ at p (also Γ). The curve E is called the *exceptional divisor*.

Note that we have the following local description (chart $x_2 = 1$): $\{(x, \ell) \in \mathbb{C}^2 \times \mathbb{P}^1 : x \in \ell\} \to \mathbb{C}^2$. That is, the exceptional divisor may be identified with the zero section of the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, which means that the self intersection of E is 1, i.e. $E^2 = 1$.

Remark 9.2. (1) The surface $Bl_p\mathbb{P}^2 := \Gamma$ is the Hirzebruch surface $\mathbb{F}_1$ (or Σ_1).
(2) Choosing local coordinates, we can talk about Bl_pS for $p \in S$.
(3) In fact, $\pi : Bl_pS \to S$ is characterized by the following universal property: if $f : \tilde{S} \to S$ is birational such that f^{-1} is not defined at $p \in S$, then f factorizes as $f : \underbrace{\tilde{S} \xrightarrow{g}}_{\text{bir}} \hat{S} := Bl_pS \xrightarrow{\varepsilon} S$ where $\varepsilon^{-1}(S \setminus \{p\}) \simeq S \setminus \{p\}$ and $\varepsilon^{-1}(p) \simeq \mathbb{P}^1$.

Proposition 9.3. *Consider $\pi : \tilde{S} = Bl_pS \to S$ as above, with exceptional divisor E,*

(1) $E^2 = -1$,
(2) $E \cdot \pi^*D = 0$ *for all* $D \in Div(S)$,
(3) *If $C \subset S$ has mutliplicity m at p, then $\tilde{C} := \overline{\pi^{-1}(C) \setminus E} \subset \tilde{S}$ satisfies $\tilde{C} \cdot E = m$ and $\tilde{C}^2 = C^2 - m^2$*
(4) $K_{\tilde{S}} = \pi^*K_S + E \implies K_{\tilde{S}}^2 = K_S^2 - 1$
(5) *q, p_g and χ are the same for $\tilde{S}$ as for S*
(6) *e, $h^{1,1}$ and b_2 all go up by 1*
(7) $Pic(\tilde{S}) = \pi^*(Pic(S)) \oplus \mathbb{Z}E$.

10. Castelnuovo's criterion

[It turns out that if you find a surface with a -1 curve, then that surface is the blow up of another!]

Theorem 10.1. *If on a surface $\tilde{S}$ one finds a curve $E \simeq \mathbb{P}^1$ with $E^2 = -1$, then $\tilde{S} = Bl_pS$ for some $p \in S$.*

Definition 10.2. A surface S is *minimal* if it has no (-1)-curves (i.e. a curve $\simeq \mathbb{P}^1$ with self-intersection -1).

(If and only if $f : S \dashrightarrow \tilde{S}$ and S minimal, then f is an isomorphism.)

11. Mori's point of view

If S is a surface such that K_S is not nef [so, by definition of nef there must exist an irreducible curve that intersects the canonical divisor negatively, and this curve in fact is a (-1) curve:] then there exists a (-1)-curve on S.

12. Kodaira dimension

Let S be a surface.

Definition 12.1. For each $n \geq 0$, we define the *pluri-genera*

$$P_n := \dim H^0(S, \mathcal{O}_S(nK))$$

and the *Kodaira dimension*

$$\kappa := \begin{cases} -\infty & \text{if } P_n = 0 \forall n \\ \text{smallest } k \text{ such that } P_n = O(n^k) & \text{otherwise} \end{cases}$$

Remark 12.2. (By Daniel.) In one of Misha's course we have that the function $\dim H^0(nK)$ is known to be polynomial for all projective varieties (Birkar, Cascini, Hacon, McKernan). Conjecturally it is always polynomial. Thus, the definition of Kodaira dimension is nothing more than the degree of this polynomial.

Further, in [**?**, Definition 2.2.26] we see that this should be equivalent to defining it as the transcendence degree over $\mathbb{C}$ of the fraction field of the ring $\bigoplus_{m\geq 0} H^0(X, K_X^{\oplus m})$, which is endowed with a natural product which, in general for vector bundles E and F, maps $H^0(X,E) \otimes H^0(X,F) \to H^0(X, E \otimes F)$.

13. Results of Mori's theory

Theorem 13.1. *Given a surface S, there exists a chain*

$$S = S^0 \xrightarrow{\sigma_1} S^1 \to \cdots \xrightarrow{\sigma_N} S_N = S'$$

where each σ_i is the contraction of a (-1) curve E_i and S' satisfies
(1) *$K_{S'}$ is nef or*
(2) *S' is a $\mathbb{P}^1$-bundle over a curve or*
(3) *$S' \simeq \mathbb{P}^2$.*

Theorem 13.2. *If a surface S is such that K_S is not nef then there exists $\varphi : S \to X$, with $\dim X = 0, 1$ or 2, contracting at least one curve to a point, such that $-K_S \cdot C > 0$ for every curve C contained in a fiber of φ.*

Note:

$$\dim X = \begin{cases} 0 \rightsquigarrow & -K \text{ is ample} \\ 1 \rightsquigarrow & S \text{ is a conic bundle} \\ 2 \rightsquigarrow & S = Bl_{p_1,...,p_n}. \end{cases}$$

14. Examples and the cubic surface revisited

(1) Slogan: "blowups resolve singularities". Consider the plane cubic

$$C = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : \underbrace{x_1^2 x_2 - x_0^2(x_0 + x_2) =}_{:= f_C} 0\},$$

the point $p = (0 : 0 : 1)$ and $\pi : Bl_p\mathbb{P}^2 \to \mathbb{P}^2$, then

$$\pi^{-1}(C) = \begin{cases} f_C = 0 \\ x_0 y_0 = y_0 x_1 \end{cases}$$

[This curve has a simple node. In the blowup, the curve will intersect the exceptional divisor in two points — we have "separated" the singularity into two points]

(2) $Bl_1\mathbb{P}^1 \times \mathbb{P}^1 \simeq Bl_1\mathbb{F}_1 \simeq Bl_2\mathbb{P}^2$. [Consider two curves L_1 and L_2 on $\mathbb{P}^1 \times \mathbb{P}^1$. Their strict transforms along with the exceptional divisor become (-1) curves. Then contract $\tilde{L}_1$, this gives $\mathbb{F}_1$, the Hirzebruch surface, which is just the blowup at a point of $\mathbb{P}^2$. $\mu(\tilde{L}_2) = -1$, $\mu(E \setminus \tilde{L}_1) = 0$.]

(3) Consider two cubics C_1 and C_2. They intersect in 9 points. Do

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ p &\longmapsto (C_1(p) : C_2(p)) \end{aligned}$$

Let $S = Bl_{p_1,...,p_9}\mathbb{P}^2$. Then

(a) $K_S^2 = 0$ [Because K^2 of $\mathbb{P}^2$ is 9, and for every blowup we substract 1!]

(b) $\mathrm{Pic}(S)$ "=" $\pi^*\mathrm{Pic}(\mathbb{P}^2) \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_q$.

(c) $-K_S = -\pi^*(K_{\mathbb{P}^2}) - E_1 - \ldots - E_q$ is nef. (In fact, this is the class of a fiber of $S \to \mathbb{P}^1$.)

(4) (Continuation of previous item.) $-K_S$ nef + genus formula $\implies C^2 \geq -2$ for every $C \subset S$ irreducible and rational. This is an elliptic surface!

[We have taken a 1-dimensional linear system. But why not take a higher-dimensional linear system?]

(5) Fix $p_1, \ldots, p_6$ in $\mathbb{P}^2$ in general position (not three in a line, not six in a conic). Consider $f_1, f_2, f_3, f_4 \in \mathbb{C}[x_0, x_1, x_2]$ homogeneous of degree 3 vanishing at p_i and consider

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\ p &\longmapsto (f_1(p) : f_2(p) : f_3(p) : f_4(p)). \end{aligned}$$

The resolved morphism $Bl_6\mathbb{P}^2 \to \mathbb{P}^3$ is an embedding with image of degree 3 [a cubic surface on $\mathbb{P}^3$].

15. Elliptic surfaces

Definition 15.1. An *elliptic surface* is a surface S together with $f : S \to C$ holomorphic (with connected fibers) from S to a smooth curve C such that the generic fiber is a smooth curve of genus 1.

Moreover, we assume that there do not exist (-1)-curves on fibers of f.

Warning: I don't assume the existence of a section [a distinguished point on the fibers], and I'm not saying that all fibers are smooth.

Singular fibers have been classified by Kodaira: $I_0, I_1, (m)I_n, II, IV, IV^* = \tilde{E}_6, III^* = \tilde{E}_7, II^* = \tilde{E}_8$ and I_n.

Example 15.2.

$$\left\{\begin{matrix}\text{pencils}\\ \text{of cubics}\\ \text{w/ smooth}\\ \text{member}\end{matrix}\right\} \leftsquigarrow\rightsquigarrow \left\{\begin{matrix}\text{RES}\\ \text{w/ section}\end{matrix}\right\}.$$

$$\mathcal{P} \mapsto S_{\mathcal{P}}.$$

Question: given $\mathcal{P}$ and $S_{\mathcal{P}}$ how many singular fibers does $S_{\mathcal{P}}$ have?

Lemma 15.3. *The discriminant locus in $\mathbb{P}^9$ [this $\mathbb{P}^9$ parametrizes cubics in $\mathbb{P}^2$] has degree 12.*

Note that the Hodge diamond $S_{\mathcal{P}}$ is

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

(Notice that adding up the center column gives 12.)

Example 15.4. [We take a conic C and a triple line $3L$ (it has to be triple so that it is a pencil of conics) which is tangent to C at an inflection point. Then we blow up the point of intersection 9 times, each time picking a point in the intersection of the strict transform of the original cubic and the triple line.]

Blowing up 9 times give a surface $S_{\mathcal{P}}$ with a II^* fiber.

Question: How is Kodaira's classification obtained?

(1) If $F = \sum n_i C_i$ is singular fiber, then the intersection form restricts to $\operatorname{span}(\{C_i\})$. This restriction is negative semi-definite with a one-dimensional kernel spanned by F.
(2) If F is irreducible and $p_a(F) = 1$, then F =smooth,nodal or cuspidal. If F is reducible, then each C_i is a (-2)-curve.

This two items combined imply that the dual graph [vertices are curves,

two vertices joined if they intersect] is a Dynkin diagram (ADE type) [by a result of graph theory].

16. The Weistrass model

Let $S \to C$ be an elliptic surface with a section. A choice of a section defines a divisor of degree 1 on the generic fiber S_η, i.e. a point p.

Looking at $H^0(S_\eta, np)$ for every n, we get:

$$H^0(p) = \langle 1 \rangle, \qquad H^0(2p) = \langle 1, x \rangle, \qquad H^0(3p) = \langle 1, x, y \rangle,$$
$$\ldots, \qquad H^0(6p) = \left\langle 1, x, y, x^2, xy, x^3, y^2 \right\rangle,$$

so there must exist a relation. Up to further tricks, we get to $y^2 = x^3 + ax + b$.

Making the construction global gives the Weierstrass model.

- Any $S \to C$ is birational to a Weierstrass fibration $W \to C$ (where W can be singular, while C is smooth) where all fibers are irreducible and have $p_a = 1$.

Geometrically, W is obtained from S by contracting all fiber components that do not meet the chosen section

[Once we show that we can do this, it will suffice to study Weierstrass fibrations.]

[From now on W is a Weierstrass fibration.] Given $\pi_i : W \to C$ (with a section σ) consider the fundamental line bundle $\mathcal{L} = (\pi_* N_{\sigma/K})^\vee = (R^1\pi_*\mathcal{O}_W)^\vee$. Then for all $n \geq 2$ we have a splitting $\pi_*\mathcal{O}_W(n\sigma) \simeq \mathcal{O}_C \oplus \mathcal{L}^{-2} \oplus \ldots \oplus \mathcal{L}^{-n}$. In particular, we have a section

$$\pi^*\pi_*\mathcal{O}_W(3\sigma) \twoheadrightarrow \mathcal{O}_W(3\sigma)$$

gives a map $f : W \to \mathbb{P}(\pi_*\mathcal{O}_W(3\sigma))$ that exhibits W as a relative cubic in a $\mathbb{P}^2$-bundle over C.

A global equation is given by

$$Y^2 Z = X^3 + AXZ^2 + BZ^3$$

where (Z, X, Y, A, B) are interpreted as sections of $(\mathcal{O}_C, \mathcal{L}^2, \mathcal{L}^3, \mathcal{L}^4, \mathcal{L}^6)$. σ is given by $Z = X = 0$.

Remark 16.1. The singular fibers of a minimal resolution of W are determined by the order of vanishing of A, B and $\Delta = 4A^3 + 27B^2$.

Consequences:

(1) $\omega_W = \pi^*(\omega_C \otimes \mathcal{L})$ implies that $K_W^2 = 0$. $e = R \cdot \deg\mathcal{L} =$ number of singular fibers.
(2) $\kappa(W) \leq 1$.
(3) $\deg\mathcal{L} = \chi(W)$, $e = R \cdot \deg\mathcal{L} =$ number of singular fibers.
(4) If $C = \mathbb{P}^1$, then $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(d)$ for some d.

$$\begin{cases} d = 1 \leftrightsquigarrow & \text{(general) RES w/section} \\ d = 2 \leftrightsquigarrow & \text{K3s, } K_W = 0 \\ 3 \geq 3 \leftrightsquigarrow & \kappa = 1 \text{ and } K_W = \alpha F \text{for some } \alpha > 0. \end{cases}$$

Invariants. $\pi : W \to C$. Set $g = g(C)$.

	q	p_g
W not a product	g	$g + \deg\mathcal{L} - 1$
product	$g + 1$	$g + \deg\mathcal{L}$.

We can also compute the pluri-genera P_n and deduce:

- $g = 0$: done.
- $g = 1$:

$$\begin{cases} \mathcal{L} = \mathcal{O}_C & E \times E \\ \mathcal{L} \text{ is torsion of order 2,3,4 or 6 hyp} & \\ \deg(\mathcal{L}) \geq 1 & \kappa = 1 \end{cases}$$

- $g \geq 2$: $\kappa = 1$.

Proposition 16.2. *Let S be a minimal surface with $\kappa = 1$. Then*

(1) $K_S^2 = 0$ *and*
(2) *there exists $f : S \twoheadrightarrow C$ elliptic fibration.*

Proof.

- S contains a curve $E = \sum n_i C_i$ ($n_i \in \{0, 1\}, \operatorname{supp}(E)$ is connected) with $E \overset{num}{\not\sim} 0$ and $E^2 = K_S E = 0$.
  In fact $K_S \overset{num}{\sim} rE$ for some $r \in \mathbb{Q}$.
- Some multiple of E moves in an elliptic pencil. Some multiple of E is the movable part of $\tilde{r}K_S$ for some $\tilde{r} \in \mathbb{Q}$.

□

17. Examples

- (The Fermat quartic.) Consider $S \subset \mathbb{P}^3$ given by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ and for each $(\lambda : \mu) \in \mathbb{P}^1$, let

$$C_{(\lambda:\mu)} = \begin{cases} \lambda(x_0^2 + \zeta^2 x_1^2 - \mu(x_2^2 - \zeta^2 x_3^2 = 0 \\ \mu(x_0^2 - \zeta^2 x_1) + \lambda(x_2^2 + \zeta^2 x_3^2) = 0 \end{cases}$$

where $\zeta \in \mathcal{M}_8$ (an eight root of unity), primitive. These curves live in S and

$$\pi : S \longrightarrow \mathbb{P}^1$$
$$(x_0 : x_1 : x_2 : x_3) \longmapsto (x_0^2 - \zeta^2 x_1^2 : -x_2^2 - \zeta x_3^2)$$

is an elliptic surface.

- Consider a smooth plane quartic Q and choose a point $p \notin Q$.

$$\begin{tikzcd} S \arrow[d, "Bl_{q_1,q_2}Y \text{ where } \pi^{-1}(p)=\{q_1,q_2\}"] \\ Y \simeq Bl_7\mathbb{P}^2 \arrow[d, "\pi"', "2:1"] \\ \mathbb{P}^2 \supset Q. \end{tikzcd}$$

where $\pi^{-1}(\ell)$ has $g = 1$.

[This produces a rational elliptic surface with a section]

Notice that there exist elliptic surfaces with multiple fibers (hence without a section).

Example 17.1. Start with three lines in the plane intersecting pairwise, and a conic intersecting tangentially once every line [Fano plane]. We blow up these three points and obtain a type $2I_3$ fiber.

In general, we take pencils of sextics with nodes at 9 points. These are in correspondence with Halpen surfaces.

18. Halphen surfaces

[These are a class of rational surfaces.]

Proposition 18.1. *Let $\pi : S \to C$ be an elliptic surface with S rational. Then*

(1) $C = \mathbb{P}^1$, $b_2(S) = 10$.
(2) *π has at most one multiple fibre mF (necessarily type mI_n for some n and some m.*
(3) *either π has no multiple fiber and $-K_S \sim F$ (any fiber), or π has one muliple fiber of multiplicity m and $-mK_S \sim S_p$*
(4) *if π has no multiple fibers, there exists a section.*

Proof of 2. and 3. Canonical bundle formula:

$$\omega_S = \pi^*(\omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_S\left(\sum_p (m(p)-1)S_p)\right). \tag{18.1.1}$$

Then, given any smooth fiber F, for any $n \in \mathbb{N}$, we have $nK_S \sim n\left(\sum_p \frac{m(p)-1}{m(p)} - 1\right)^{\alpha} F$.

Note that $\alpha < 0$ because no multiple of K_S can be effective. Thus, $m(p) = 1$ except for at most 1 at p.

Why is 18.1.1 true? In general, for any relatively minimal elliptic surface $\pi : S \to C$ with multiple fibers $m_1F_1, \ldots, m_kF_k$:

$$\omega_S = \pi^*(\omega_C \otimes \underbrace{R^1\pi_*\mathcal{O}_S}_{\deg\chi})^\vee) \otimes \mathcal{O}_S\left(\sum m(p) - 1S_p\right).$$

Why? Because $\pi_*\omega_S$ is locally free, thus $\omega_S = \pi^*(\omega_C \otimes \mathcal{L}) \otimes \mathcal{O}_S(D)$ where D is the zero divisor of $\pi^*\pi_*\omega_S \to \omega_S$. Argue that $D = \sum n_iF_i$ with $n_i < m_i$ using adjunction:

$$\mathcal{O}_{F_i} \simeq \omega_{F_i} = \omega_S \otimes \underbrace{\mathcal{O}_{F_i}(F_i)}_{\text{torsion}} = \mathcal{O}_{F_i}(F_i)^{\otimes(n_i+1)}$$

thus $n_i + 1$ has to divide m_i. □

Remark 18.2. Let $\pi : S \to \mathbb{P}^1$ be a rational elliptic surface (RES) and note that $K_S^2 = 0$. So, S is not minimal and there exists a (-1)-curve. By Proposition 18.1, there are two possibilities:

(1) There are no multiple fibers, $M.S_n = 1$.
(2) There exists a multiple fiber mF and $MS_n = m$,

Definition 18.3. A RES as in Item 2 is called a *Halpen surface* of index m. Alternatively, a rational surface S is a *Halphen surface* if there exists m such that $|-mK_S|$ is one dimensionsional and has no fixed component and no base points. The smallest such m is the *index*. (We include $m = 1 \iff$ there exists a section.)

Proposition 18.4. *If $\pi : S \to \mathbb{P}^1$ is a Halphen surface of index m, there exists $f : S \to \mathbb{P}^2$ birational such tat $\pi \circ f^{-1}$ is a Halpen pencil of index m (i.e. a pencil of plane curves with degree $3M$ with 9 base points, each of multiplicity m). Conversely, the resolution of any such pencil gives a Halpen surface.*

Example 18.5. $Q : y^2 + xz = 0$, $L : y = 0$, $C : (y^2 + xz)(\alpha y + z) + \beta yx^2$, $\alpha, \beta \neq 0$.

The pencil $\lambda(4Q + L) + \mu(3C) = 0$ is a Halphen pencil of index 3.

[The intersection of C, Q and L consists of two points, $p_1 = (0 : 0 : 1)$ and $p_2 = (1 : 0 : 0)$.] Blow up at p_1 seven times and at p_2 two times. [We obtain one the exceptional Dynkin diagram fibers.]

Example 18.6. Consider any smooth cubic C. Take $p_1, \dots, p_9 \in C$ such that $p_1 \oplus \dots \oplus p_9$ is m-torsion. Then there exists a curve D of degree $3m$ passing though $p_1, \dots, p_9$ with multiplicity m, thus $\lambda(D) + \mu(mC) = 0$ is a Halpen pencil.

19. Singular fibers

Recall that

$$\left\{\begin{matrix}\text{Halphen pencils}\\ \lambda C_{3m} + \mu(mC) = 0\end{matrix}\right\} \leftrightsquigarrow \left\{\begin{matrix}\text{Halphen surfaces}\\ \pi : S \xrightarrow{|-mK_S|} \mathbb{P}^1\end{matrix}\right\}$$

$$C_{3m} \leftrightsquigarrow F \in |-mK_S|.$$

Proposition 19.1 (Z)**.** *If F is of type II^*, III^* or IV^*, then C_{3m} is non-reduced.*

Proof. $R \subset S$ rational smooth $\implies R^2 \geq -2$, $\implies$ exceptional curves in S can only appear in disjoint chains. Now, if F is as in the statement, then at least one curve in [picture of fiber] is not exceptional. Here are the fibers: □

Proposition 19.2 (Z)**.** *Let $S \to \mathbb{P}^1$ be a Halphen surface of index 2 and assume that π contains a fiber F of type II^*. Then there exists $\varphi : S \to \mathbb{P}^2$ such that the sextic $\varphi(F)$ is one of the following: [Picture of conic of multiplicity 3, line of multiplicity 3 intersecting cubic, two lines of multiplicity 3, quadric and line of multiplicity 4, a line and a line with multiplicity 5].*

Example 19.3. $C : y^2 z = x(x-2)(x - \alpha z)$, $\alpha \neq 0, 1$. $L : z = 0$, $\tilde{L} : x = 0$, $\mathcal{P} : \lambda(L + 5\tilde{L}) + \mu(2C) = 0$.

[A point of intersection is $p_1 = (0 : 1 : 0)$, where the line is tangent at the inflection point of C. Another point of intersection is $p_2 = (0 : 0 : 1)$.] Blow up p_1 4 times and p_2 5 times.

20. Constructions

(1)

$$\begin{tikzcd} X \arrow[d, "m:1"] \\ S \arrow[d, "|-mK_S|\ni F \text{ smooth}"] \\ \mathbb{P}^1 \end{tikzcd}$$

For $m = 2$, X is a K3. For $m \geq 3$, $\kappa(X) = 1$ and contains a $(-m)$ P^1 section.

(2) Type II degenerations of K3 surfaces and degenerations of Enriques surfaces (see Alexeev and Engel).

(3) It $\pi : S \to \mathbb{P}^1$ is a general Halphen surface of index 2 and we blow up we get a Coble surface.

(4) The Halpen transform

$$\begin{tikzcd} S \arrow[d, "|-mK_S|"] \\ \mathbb{P}^1 \end{tikzcd}$$

$|F + M|$ curves of $g = 1$ through q. Contract M and blow up q, $\rightsquigarrow$ we obtain a RES with section.

(5) (Halpne.) Given $p_1, \dots, p_8 \in \mathbb{P}^2$, the locus of the 9th base point of a Halpen pencil of index m is a curve of degree $3\psi(m)$ having ordinary multiple points at $p_1, \dots, p_8$ of multiplicity $\psi(m)$, where

$$\psi(m) = m^2 \left(1 - \frac{1}{q_2^2}\right)\left(1 - \frac{1}{q_2^2}\right) \dots$$

where $m = q_1^{r_1} q_2^{r_2} \dots$ prime factorization. $m = 2 \implies \psi(m) = 3$.