REPRESENTATION THEORY

github.com/danimalabares/stack

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1. Basic definitions

Definition 1.1. A *representation of Lie algebra* $\mathfrak{g}$ is a vector space V and homomorphism

$$\rho : \mathfrak{g} \to \operatorname{End}(V)$$

of Lie algebras, i.e.,

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

Remark 1.2. Another name for a representation of a Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-*module*. This is like the abstract definition of an algebra using a morphism. It's as if the morphism lets us define a sort of product of elements in the codomain by elements in the domain.

It is not possible to classify all representations of a Lie algebra $\mathfrak{g}$. But there is a theorem by Weyl that says that if $\mathfrak{g}$ is finite-dimensional and (semi)simple, then every finite-dimensional representation of $\mathfrak{g}$ (i.e. a representation where V is finite-dimensional) is isomorphic to a direct sum of irreducible representations

2. Borel-Weil-Bott theorem

[These are *very* sketchy notes after a conversation with Lyalya Guseva on Borel-Weil-Bott theorem]

Borel-Weil-Bott theorem is a device for computing cohomologies of sheaves.

Take the Grassmanian $\operatorname{Gr}(n, k)$, which may be obtained as a quotient G of a group G by a parabolic subgroup P. We would like it if P was a semisimple group since those are classified, but unfortunately it is not. So instead we use so-called Levi quotient denoted by L which is semisimple and allows us to understand the cohomologies of the variety G/P.

Fortunately the representation theory of $\operatorname{Gr}(k, n)$ is well known, in fact we get $\mathrm{GL}_k \times \mathrm{GL}_{n-k}$.

There are functors associated to L, which are described by a sequence of numbers $a_1, \ldots, a_n$. Along with other sequences of numbers $k_1, \ldots, k_\ell$, and ρ (the latter is a concept in representation theory but we may ultimately think of it as another sequence of numbers) we may construct an action of the symmetric group S^n acting on these sequences and obtain a result concerning the cohomology $H^{\ell(\sigma)}(L)$, and in particular we find that if two entries in our list of numbers coincide, the cohomology will vanish.

References