MUMFORD-TATE GROUPS IN HODGE THEORY

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Notes at github.com/danimalabares/cimpa-floripa

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2. Introduction

$X \subset \mathbb{C}P^N$ smooth complex subvariety, $\dim_{\mathbb{C}} = n$. First recall we have singular cohomology, $H^k(X, \mathbb{C})$, which is isomorphic to the cohomology of the constant sheaf $\underline{\mathbb{C}}_X$. This cohomology is nonzero for $0 \leq k \leq 2n$.

Recall. $U \subset X$ open, $\Gamma(U, \mathbb{C}) = \{f : U \to \mathbb{C} : f \text{ is locally constant}\} = \prod_{\pi_0(U)} \mathbb{C}$.

Example 2.1. (1) $X = \mathbb{C}P^n$,

$$H^k(\mathbb{C}P^n, \mathbb{C}) = \begin{cases} \mathbb{C} & k = 2m, 0 \leq m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

A way to prove this is using the CW decomposition of $\mathbb{C}P^n$.

(2) $X \subset \mathbb{C}P^2$ hypersurface of degree d; X a Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$. Then $H^0(X, \mathbb{C}) = \mathbb{C}$, $H^1(X, \mathbb{C}) = \mathbb{C}^{2g}$, $H^2(X, \mathbb{C}) = \mathbb{C}$.

We also have the following additional data (a Hodge structure) on $H^k(X, \mathbb{C})$:

- A lattice $H^k(X, \mathbb{Z})/\text{torsion} \subset H^k(X, \mathbb{C})$,
- A (p, q)-decomposition, $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$.

Why is it useful?

- It gives restrictions on possible Betti numbers of algebraic varieties. (So it may tell us that certain complex variety cannot be algebraic, for example.)
- It $f : X \to Y$ is a morphism of algebraic varieties, then $f^* : H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ preserves the Hodge structure.

As an example of the latter statement,

Example 2.2. Let $X \subset \mathbb{C}P^4$ be a "general" hypersurface of degree 5. Then there exists no abelian variety (i.e. a projective variety that is biholomorphic to $\mathbb{C}P^N/\Lambda$ where Λ is a lattice; so, a complex torus that is also a projective variety) A that admits a dominant birational map onto X $f : A \dashrightarrow X$.

3. The p,q decomposition

We use the de Rham complex. Let Ω_X^k be the sheaf of holomorphic k-forms. The de Rham complex is

$$\Omega_{dR}^\bullet = (0 \to \mathcal{O}_X \to \Omega_X^1 \xrightarrow{d} \cdots \to \Omega_X^n \to 0)$$

$\Omega_{dR}^\bullet$ is a resolution of $\mathbb{C}_X$. Therefore $H^k(X, \mathbb{C}) \cong H^k(X, \Omega_{dR}^\bullet$.

Let's define a subcomplex:

$$F^p\Omega_{dR}^\bullet = (0 \to \cdots \to \Omega_X^p \to O_X^{p+1} \to \cdots \to \Omega_X^n \to 0)$$

It's a subcomplex $F^p\Omega_{dr}^\bullet \subset \Omega_{dR}^\bullet$ since the sheaves coincide when $F^p\Omega_{dR}$ are nonzero and they inject otherwise (because it's the zero sheaf).

This gives $H^k(X, F^p\Omega_{dR}^\bullet) \xrightarrow{(*)} H^k(X, \Omega_{dR}^\bullet) = H^k(X, \mathbb{C})$.

Definition 3.1. The *Hodge filtration* is $F^pH^k(X) = \mathrm{Im}(*)$.

Hodge theory tells us the map $(*)$ is in fact injective.

Let $\Lambda_X^{p,q}$ be the sheaf of C^∞-forms on X of type (p, q). They are given locally by $\sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ} dz_I \wedge d\bar{z}_J$, for $\alpha_{IJ} \in C_X^\infty$.

Then we have an acyclic resolution of Ω_X^p:

$$0 \to \Omega_X^p \hookrightarrow \Lambda_X^{p,0} \xrightarrow{\bar\partial} \Lambda_X^{p,1} \xrightarrow{\bar\partial} \cdots \to \Lambda_X^{p,n} \to 0.$$

$\Lambda_X^{\bullet,\bullet}$ is a fine resolution of $\Omega_X^\bullet$.

Then $H^k(X, \mathbb{C}) \cong H^k(X, \mathrm{Tot}\Lambda_X^{\bullet,\bullet}$, which can be computed by a spectral sequence. The first page of such spectral sequence is given by $E_1^{p,q} = H^q(X, \Omega_X^p)$. This converges to $H^k(X, \mathbb{C})$. This is the Hodge-to-de Rham spectral sequence.

Since our manifolds are projective they admit a Kähler metric ω induced by the inclusion $X \xrightarrow{i} \mathbb{C}P^N$, that is, $\omega = i^*(\text{Fubini-Study metric on } \mathbb{C}P^N)$.

Then $\Lambda_X^k = \bigoplus_{p+q=k} \Lambda_X^{p,q}$, $\Lambda_X^\bullet$ becomes an elliptic complex.

We have

$$\ldots \to \Lambda_X^{k-1} \xrightarrow{d} \Lambda_X^k \to \ldots, \qquad \ldots \to \Lambda_X^k \xrightarrow{d^*} \Lambda_X^{k-1} \to \ldots$$

where d^* is the adjoint of d w.r.t. ω. Then $\Lambda = dd^* + d^*d$ is an elliptic operator. $\mathcal{H}^k = \mathrm{Ker}(\Lambda|_{\Lambda_X^k}) = \mathrm{Ker}(d|_{\Lambda_X^k}) \cap \mathrm{Ker}(d^*|_{\Lambda_X^k})$ are the harmonic forms.

Consider a natural map from the harmonic forms of type (p, q) to the cohomology:

$$\mathcal{H}^{p,q} = \mathcal{H}^k \cap \Lambda_X^{p,q} \xrightarrow{(**)} H^q(X, \Omega_X^p).$$

Fact: since $d\omega = 0$ (X is Kähler), $(**)$ is an isomorphism.

This means that for Kähler manifolds

$$\dim H^k(X,\mathbb{C}) \leq \sum_{p+q=k} H^q(X,\Omega_X^p) \underbrace{\leq}_{(**)} \dim H^k(X,\mathbb{C})$$

This implies that the Hodge-to-de Rham spectral sequence degenerates at E_1. Therefore,

$$H^k(X,\mathbb{C}) = \mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong H^q(X,\Omega_X^p).$$

The Hodge filtration is

$$F^pH^k(X,\mathbb{C}) = \bigoplus_{\substack{p'+q'=k\\ p'\geq p}} H^{p',q'}(X).$$

4. Symmetries of the p,q decomposition

(1) Since $\overline{\Lambda_X^{p,q}} = \Lambda_X^{q,p}$, we have $\mathcal{H}^{q,p} = \overline{\mathcal{H}^{p,q}}$. This means that if $k \equiv 1 (\text{mod } 2)$, $H^k(X,\mathbb{C}) = \mathcal{H}^{k,0} \oplus \ldots \oplus \mathcal{H}^{\frac{k+1}{2},\frac{k-1}{2}} \oplus \mathcal{H}^{0,k} \oplus \ldots \oplus \mathcal{H}^{\frac{k-1}{2},\frac{k+1}{2}}$. This means that the k-th Betti number is even, $b_k(X) \equiv 0 \text{mod } 2$.

(2) (Poincare duality.) We have a perfect pairing

$$\begin{aligned} H^k(X,\mathbb{C}) \otimes H^{2n-k}(X,\mathbb{C}) &\longrightarrow \mathbb{C} \\ [\alpha]\otimes[\beta] &\longmapsto \int_X \alpha\wedge\beta \end{aligned}$$

Notice that if $\alpha \in \mathcal{H}^{p,q}$ then $\beta \in \mathcal{H}^{n-p,n-q}$. This induces a perfect pairing

$$\mathcal{H}^{p,q} \otimes \mathcal{H}^{n-p,n-q} \to \mathbb{C}.$$

(3) (Polarization and the Lefschetz operator.) The polarization is the Kähler class of the Kähler form. By X being projective we have that the Kähler class is integral. Moreover, it is the Poincar'e dual of the hyperplane section class. That is, let $h \in H^2(\mathbb{C}P^n,\mathbb{Z})$ be the class of a hyperplane, then $i^*h = [\omega] \in H^2(X,\mathbb{Z})$. We have $\omega \in \mathcal{H}^{1,1}$ and $[\omega] \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. The Lefschetz operator is

$$L_\omega : H^{p,q}(X) \to H^{p+1,q+1}(X).$$

$$\begin{aligned} L_\omega : H^{p,q}(X) &\longrightarrow H^{p+1,q+1}(X) \\ [\alpha] &\longmapsto [\alpha\wedge\omega] = [\alpha]\cup[\omega]. \end{aligned}$$

Lefschetz theorem says

(a) $L_\omega^k : H^{n-k}(X,\mathbb{Q}) \xrightarrow{\sim} H^{n+k}(X,\mathbb{Q})$ is an isomorphism for $0 \leq k \leq n$

(b) The dual of L_ω is

$$\begin{aligned} \Lambda_\omega : H^{p,q}(X) &\longrightarrow H^{p-1,q-1}(X) \\ [\alpha] &\longmapsto [i_\alpha\omega]. \end{aligned}$$

$$\begin{aligned} [L_\omega,\Lambda_\omega] &= \Theta \in \text{End}(H^\bullet(X,\mathbb{Q})) \\ \Theta|_{H^k(X,\mathbb{Q})} &= (k-n)\text{Id} \end{aligned}$$

Then L_ω, Λ_ω and Θ span a subalgebra of $\text{End}(H^\bullet(X,\mathbb{Q}))$ isomorphic to $\mathfrak{sl}_2$. This allows us to use what we know about the representation theory of $\mathfrak{sl}_2$. Let $H^k_{\text{prim}(X,\mathbb{Q})}(\Lambda|_{H^k(X,\mathbb{Q})}$. Then $H^m(X,\mathbb{Q}) =$

$\bigoplus_{i\geq 0} L_\omega^i H_{\mathrm{prim}}^{m-2i}(X,\mathbb{Q})$ for $0 < m \leq n$. (I think this corresponds to the usual weight space decomposition.)
[Picture of Hodge diamond. Reflection by vertical axis is complex conjugation, 180-degree rotation is Poincar'e duality, $p+q$ =constant is a horizontal line, reflection along horizontal axis is Lefschetz theorem. Warning! This depends on conventions of how we draw the diamond.]

5. The Hodge-Riemann relations

For all $[\alpha] \in H^k_{\mathrm{prim}}(X,\mathbb{C}) \cap H^{p,q}(X)$ we have

$$i^{p-q}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \overline{\alpha} \wedge \omega^{n-k} > 0.$$

(Here i is the imaginary unit.)

Define a pairing on $H^k_{\mathrm{prim}}(X,\mathbb{C})$:

$$\psi([\alpha],[\beta]) = (2\pi i)^{-k}(-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \beta \wedge \omega^{n-k}$$

which is $(-1)^k$-symmetric.

Definition 5.1. The *Weil operator* $C \in \mathrm{End}(H^k(X,\mathbb{C}))$ is given by $C|_{H^{p,q}}(X) = (i)^{p-q}\mathrm{Id}$.

Let $Q([\alpha],[\beta]) = (2\pi i)^k \psi(C[\alpha],[\beta])$.

Then Q is symmetric (exercise) and positive on $H^k(X,\mathbb{R})$. Positive is just the Hodge-Riemann relation.