SEMINARS

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Contents

1. Neutrinos 1
2. Spheres with minimal equators 2
3. Smoothable compactified Jacobians of nodal curves 4
4. Equivariant spaces of matrices of constant rank 7
5. On wrapped Floer homology barcode entropy and hyperbolic sets 8
6. Revisiting cotangent bundles 11

References 12

1. Neutrinos

Hiroshi Nunokawas, PUC-Rio. Friday Seminar, Seminar Name. June 27, 2025.

Abstract. Hiroshi will come and tell us everything we (not) wanted to know about these mysterious particles, and are not going to be afraid to ask. In particular, about the neutrino oscillation, and the great matrices.

Protons and neutrons have very similar mass of $m_p \approx 940$ MeV, while electrons have mass of $m_e \approx 0.5$ MeV. MeV is 10^6 electronvolts, where one eV is approximately 1.6×10^{-19} J. This is standard in high energy physics, they use electronvolts instead of Joules. Recall that 2J=1N×1m.

Most of the things we see are protons since they are so much larger than electrons. But protons nor neutrons are elementary particles.

Here's the standard model:

$$\begin{array}{cccc}
\text{Quarks} & \begin{bmatrix} u \\ d \end{bmatrix}_L & \begin{bmatrix} c \\ s \end{bmatrix}_L & \begin{bmatrix} t \\ b \end{bmatrix}_L \\
\text{Leptons} & \begin{bmatrix} \nu_e \\ e^- \end{bmatrix}_L & \begin{bmatrix} \nu_\mu \\ \mu^- \end{bmatrix}_L & \begin{bmatrix} \nu_\tau \\ \tau^- \end{bmatrix}_L \\
\text{Generation} & \text{1st} & \text{2nd} & \text{3rd} \\
\text{Bosons} & g,\ \gamma\ ,\ \omega^\pm,\ z & & \\
\text{Higgs Boson} & H & &
\end{array}$$

It is very particular that nature repeats itself three times. The L in those matrix actually means left-handed, and accounts for chirality. Only left-handed fermions have weak interaction. Right-handed have electromagnetic interaction, gravitational interaction, but not weak interaction.

And then there's neutrinos. They have negative helicity (chirality). Being left-handed, mathematically, means to have helicity -1. I think this means that the spin is left-handed. But chirality and helicity are not the same: helicity is observer-dependent, and chirality is not. Almost all neutrinos we can see (% 99.99999...) have negative helicity, but not all of them.

Consider the following:

$$n + \nu_e \leftrightarrow p + e^-$$

But it's not completely correct: we'd better put d instead of n, and u instead of p: the d and u quarks, instead of the neutrons and protons.

Now consider the following reaction: a neutron decays into a proton, an electron and an antineutrino:

$$n \to p + e^+ \overline{\nu}_e$$

Protons is very stable, that's why we are here. But neutron decays in only 15 minutes.

By experimental data, we can conclude that neutrinos' mass is consistent with zero. But if they have mass, it should be much smaller than the electron's $m_v \leq 0.5$ eV. And the electron is already the lightest fermion!

If the mass of the neutrino was zero, i.e. $m_\nu = 0$, then $v_\nu = c$ in vacuum, which would imply that

$$\nu_e \xrightarrow{L} \nu_e \longrightarrow \nu_e$$

$$0:00 \qquad 0:00 \qquad 0:00$$

meaning: time doesn't pass! And this means the state of the particle cannot change.

2. Spheres with minimal equators

Lucas Ambrozio, IMPA. Differential Geometry Seminar, IMPA. June 24, 2025.

Abstract. We will discuss the connection between Riemannian metrics on the sphere with respect to which all equators are minimal hypersurfaces, and algebraic curvature tensors with positive sectional curvatures.

Definition 2.1. An $(n-k)$-*equator* orthogonal to Π is

$$\Sigma_\Pi := \{p \in \mathbb{S}^n : \langle p, x\rangle = 0 \forall x \in \Pi\}$$

for Π a k-dimensional linear subspace of $\mathbb{R}^{n+1}$.

Remark 2.2. Equators are totally geodesic hypersurfaces with the usual sphere metric, which implies they are minimal hypersurfaces.

Problem. Characterize the set $\mathcal{M}_k(U)$ of metrics g on an open set $U \subset \mathbb{S}^n$ such that all k-equators Σ_Π with $\Sigma \cap U \neq \emptyset$ yield are minimal hypersurfaces $\Sigma \cap U$ on (U, g).

Remark 2.3. This problem can be thought of as a problem of finding metrics on $\mathbb{R}^n$ such that k-planes are minimal. To see why project the k-equators to $T_p\mathbb{S}^n$ and pullback those metrics to the sphere.

Let $g \in \mathcal{M}_k(U)$ for $U \subset \mathbb{S}^n$ open and $n \geq 2$.

Theorem 2.4 (Beltrami, Schäfli). *If $k = 1$ then g has constant sectional curvature.*

Theorem 2.5 (Hongan). *If $1 < k < n-1$ then g has constant sectional curvature.*

Then Hongan also managed to produce a classification of these metrics for $k = n-1$.

Remark 2.6. If $T \in \mathrm{GL}(n+1,\mathbb{R})$, then

$$\begin{aligned}\varphi : \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ x &\longmapsto \frac{Tx}{|Tx|}\end{aligned}$$

is a diffeomorphism that maps k-equators into k-equators. Thus if $g \in \mathcal{M}_k(\mathbb{S}^n)$ then so is $\varphi(T)^* g$.

Theorem 2.7. *There exists a $GL(n+1,\mathbb{R})$ equivariant bijection*

$$\mathcal{M}_{n-1}(\mathbb{S}^n) \leftrightarrow \mathit{Curv}_+(\mathbb{R}^{n+1})$$

where the set on the right-hand-side is the set of algebraic curvature tensors (also called curvature-like, i.e. with the same symmetries as the Riemannian curvature tensor) on $\mathbb{R}^{n+1}$ with positive sectional curvature.

The group action is given as follows for $T \in GL(n+1,\mathbb{R})$:

$$(R\cdot T)(x,y,z,w) = \frac{1}{|\det(T)|^{\frac{1}{n+1}}} R(Tx,Ty,Tz,Tw)$$

The point is that $\mathrm{Curv}_+(\mathbb{R}^{n+1})$ is an open cone on a linear space. Here are two simple corollaries:

Lemma 2.8.
(1) *$\mathcal{M}_{n+1}(\mathbb{S}^n)$ is in bijection with an open positive cone of an $\frac{n(n+2)(n+1)^2}{12}$-dimensional real vector space.*
(2) *Every metric on $\mathcal{M}_{n-1}(\mathbb{S}^n)$ is invariant by the antipodal map.*

Algorithm. From any $R \in \mathrm{Curv}_p(\mathbb{R}^{n+1})$ we obtain a symmetric positive definite (positive-definitiness comes from the positiveness of the curvature of R) 2-tensor k_R satisfying

$$(k_R)_p(v,v) = R(pv,pv) > 0$$

Also, k_R has the *Killing property*, i.e. that $\overline{\nabla} k(X,X,X) = 0$ for all $X \in \mathfrak{X}(\mathbb{S}^n)$.

Then we define a positive function on $\mathbb{S}^n$ by

$$D_R := \left(\frac{d\mathrm{Vol}_{k_R}}{dV_g}\right)^{\frac{4}{n-1}} \tag{2.8.1}$$

and finally a Riemannian metric on $\mathbb{S}^n$ in $\mathcal{M}_{n-1}\mathbb{S}^n$ by

$$g_R = \frac{1}{D_R} k_R$$

And to go back, for $g \in \mathcal{M}_{n-1}(\mathbb{S}^n)$ define a positive function on $\mathbb{S}^n$

$$F_g := \left(\frac{dV_g}{dV_{\overline{g}}}\right)^{\frac{4}{n-1}}$$

Then let $k_g := \frac{1}{F_g} g > 0$, which is a positive definite Killing 2-tensor, from which we may define $R_g \in \mathrm{Curv}_+(\mathbb{R}^{n+1})$ with $R_g(pv,pv) = (k_g)_p(v,v)$ for all $p, v \in T\mathbb{S}^n$.

More corollaries:

Lemma 2.9.
(1) *$g \in \mathcal{M}_{n-1}(\mathbb{S}^n)$ is analytic because it is a Killing tensor on $\mathbb{S}^n$, which are well-known.*

(2) *If g is left-invatiant on $\mathbb{S}^3$, seen as unit quaternions, then $g \in \mathcal{M}_2(\mathbb{S}^3$. Moreover, for $a \geq b \geq c > 0$,*

$$aL_i \odot L_i + bL_j \odot L_j + cL_k \odot L_k = k$$

is Killing, $k > 0$, D_k constant and thus $g = \frac{1}{const.}k \in \mathcal{M}_2(\mathbb{S}^3)$.

(3) *R curvature tensor of $(\mathbb{C}P^2, g_{FS})$. We may not remember what's the curvature tensor, but we know the sectional curvature is $1 \leq sec(R) \leq 4$,*

$$(k_R)_p(v,w) = \overline{g}(v,w) + 3\overline{g}(Jp,v)\overline{g}(Jp,w)$$

and $D_R = 4^{\frac{4}{3-1}} = 4$, so that by 2.8.1 we obtain $g_R = \frac{1}{4}k_R$, which is a Berger metric on $\mathbb{S}^3$ with scalar curvature 0.

Now define

$$\Sigma_V = \{p \in \mathbb{S}^n : \langle p, v\rangle = 0\} = V^{-1}(0)$$

where $V(x) := \langle x, v\rangle$ for all $x \in \mathbb{S}^n$. Then the normal vector field is $\nabla V/|\nabla V|_g$, and the second fundamental form is given by

$$A = \frac{1}{|\nabla V|_g}\mathrm{Hess}_g V$$

and its mean curvature by

$$H = \frac{1}{|\nabla V|_g}\left(\Delta_g V - \mathrm{Hess}_g V\left(\frac{\nabla V}{|\nabla V|}, \frac{\nabla V}{|\nabla V|}\right)\right) \tag{2.9.1}$$

For every $v \in \mathbb{S}^n$ and $p \in \Sigma_V$, we see that $H_{\Sigma_V} = 0$ iff

$$|\nabla V|_g^2(p)\Delta_g V(p) - \mathrm{Hess}_g V(\nabla V(p), \nabla V(p)) = 0$$

And for $\overline{g}$,

$$\mathrm{Hess}_{\overline{g}} V + V\overline{g} = 0 \implies \mathrm{Hess}_{\overline{g}} V(X,X) = 0$$

for all $X \in T_p\mathbb{S}^n$ and $p \in \Sigma_v$. Then

$$J_g(X,Y,Z) = g(\nabla_X Y - \overline{\nabla}_X Y, Z)$$

$$J_g(X,Y,\nabla V) = \mathrm{Hess}_{\overline{g}} - \mathrm{Hess}$$

Problems.

(1) Similar story for $\mathbb{C}P^n, \mathbb{H}P^n$?
(2) Complete metrics on $\mathbb{R}^n$ with minimal hyperplanes.
(3) Find geometric invariants of metrics on $\mathcal{M}_{n-1}(\mathbb{S}^n)$ (may be useful to study (M^n, g), $n \geq 4$, sec > 0.

3. Smoothable compactified Jacobians of nodal curves

Nicola Pagani, University of Liverpool and Bologna. Seminar of Algebraic Geometry UFF. August 20, 2025.

Abstract. Building from examples, we introduce an abstract notion of a 'compactified Jacobian' of a nodal curve. We then define a compactified Jacobian to be 'smoothable' whenever it arises as the limit of Jacobians of smooth curves. We give a complete combinatorial characterization of smoothable compactified Jacobians in terms of some 'vine stability conditions', which we will also introduce. This is a joint work with Fava and Viviani.

Let C be a smooth curve and $d \in \mathbb{Z}$. Define

$$J_C^d = \{L : L \text{ is a line bundle of degree } d\}/\sim$$

which is a smooth projective variety of dimension $g(C)$.

If C is nodal we still can consider J_C^d.

(1) One connected component. Then the Jacobian is $\mathbb{P}^1$ minus two points. This is not universally closed, so it is not proper.
(2) Two components intersecting at one point. The pullback of the normalization splits the degree in intintely many ways, givieng that J_C^{-1} is an infinite set of points. This is not of finite type, so it is not proper.
(3) The curve has two components intersecting at two points. This gives J_C^{-2}, which is a mixture of the two former items. (Probably not proper too.)

Now consider

$$\mathrm{TF}_C^d = \{\mathcal{F} : \text{coherent on } C, \text{ torsion-free, rank-1 on } C\}/\sim$$

This satisfies the existence oart of the valu point of properness.

Now we consider the moduli. Now we consider the ideal sheaf of the (singular?) point(s?):

(1) One component. The stack is proper!
(2) Two components intersecting once. Now we get stacky points, $x = [\bullet/\mathbb{G}_m]$. These points have generic stabilizer. The resulting stack is not separated because a morphism of a curve, say $\mathbb{P}^1$ minus a point ... there are infinitely many ways to extend a morphism from this thing to a line bundle. So you cannot include any of these stacky points. Recall that a sheaf is *simple* if its automorphism group is $\mathbb{G}_m$.
(3) The ideal sheaf of both nodes $\mathcal{I}(N_1, N_2)$ has a positive dimensional automorphism group. The stack is not proper.

Definition 3.1. A *fined compactified Jacobian* of C is an open connected substack of $\mathrm{TF}^d(C)$ that is also proper.

Remark 3.2. This thing is automatically an algebraic space.

Definition 3.3. A *compactified Jacobian* is an open connected of $\mathrm{TF}^d(C)$ that admits a proper, good moduli space.

Consider the Artin stack $\mathfrak{X} \xrightarrow{\Gamma} X$ [...] is a *good moduli space* if

(1) Every moduli factors

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[dr] & \mathcal{I} \text{ (ACC. space)} \\ & X \end{tikzcd}$$

(2) $\pi_* \mathcal{O}_{\mathfrak{X}} = \mathcal{O}_X$.

We expect to find a notion of stability condition to produce these things [...] $[\bullet/\mathbb{G}_m]$ would be the polystable representative.

Definition 3.4. A compactified Jacobian $\overline{J_C}$ is *smoothable* if all smoothings $\mathcal{C} \to \Delta = \{0, \eta\}$ (with $\mathcal{C}_0 = C$),

$$J^d_{\mathcal{C}_\eta} \cup C \to \overline{J_C}$$

is proper.

Definition 3.5. Let X be a curve.

$$\mathrm{BCON}(X) = \{Y \subseteq X \text{s.t. } Y, Y^c \text{ are connected}\}$$

Definition 3.6. A v-curve is a generalization of items (2) and (3) in the lists above [it looks like two long snakes $\sim$ that intersect several times, and t is the number of nodes]. A *v-condition* is a pair $n = (n_1, n_2)$ such that

$$n_1 + n_2 = \begin{cases} d + 1 - t & \text{we say the s.c. is nondegenerate} \\ d - t & \text{degenerate} \end{cases}$$

$\mathcal{F}$ on X is *n-(semi)stable* if $\deg \mathcal{F}_{X_i} > n_i$ ($\deg \mathcal{F}_{X_i} \geq n_i$) for $i = 1, 2$.

$\mathcal{F}_{X_i} = \mathcal{F}|_{X_i}$ torsion.

$$\deg(\mathcal{F}_{X_i}) + \deg(\mathcal{F}_{X_2}) = d - |\mathrm{sing}(F)|.$$

Then

$$\overline{J_C}(n) = \{\mathcal{F} \text{ is semistable}\},$$

a smooth compact Jacobian.

Definition 3.7. A *degeneratation* of v-stab. on X is $n : \mathrm{BCON}(X) \to \mathbb{Z}$ such that

(1)

$$n_Y + n_{Y^c} + |Y \cap Y^c| = \begin{cases} d + 1 & \text{we say } Y \text{ is } n\text{-nondegenerate} \\ d & Y \text{ is } n\text{-degenerate} \end{cases}$$

(2) Y_i no pa. common component $n_{Y_1} + n_{Y_2} + \dots$

$$\begin{tikzcd} & Y_1 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ Y_2 \arrow[rr, leftrightarrow] & & Y_3 \end{tikzcd}$$

Theorem 3.8 (-, et al)**.** *(bijection between stability conditions and nodal curves) The map*

$$\left\{ \begin{smallmatrix} sm.\ comp. \\ Jac\ of\ X \end{smallmatrix} \right\} \to \left\{ \begin{smallmatrix} v\text{-}stab. \\ cond.\ of\ X \end{smallmatrix} \right\}$$

$$n \mapsto \overline{J_X}(n) = \{n\text{-}semistable\ sheaves\}$$

is a bijection. (The arrow should be from right to left!)

F. Viviani had proved it for fine compact Jacobians.

4. Equivariant spaces of matrices of constant rank

Ada Boralevi, France. Algebraic Geometry Seminar, IMPA. August 27, 2025.

Abstract. A space of matrices of constant rank is a vector subspace V, say of dimension n+1, ofthe set of matrices of size axb over a field k, such that any nonzero element of V has fixed rank r. It is a classical problem to look for different ways to construct such spaces of matrices. In this talk I will give an introduction up to the state of the art of the topic, and report on my latest joint project with D. Faenzi and D. Fratila, where we give a classification of all spaces of matrices of constant corank one associated to irreducible representation of a reductive group.

We are interested in vector spaces $U \subset \mathrm{Mat}_{m,n}(\mathbb{C})$, with $m \leq n$, of *constant rank, i.e. such that for all* $f \in U$, $r := rankf$ *is the same.*

Let $\ell(r,m,n) := max \dim U : U$ *is of rank* r.

Questions.

(1) $\ell(r,m,n) = ?$ *In general not known.*
(2) *Find relations among* ℓ, r, m *and* n.
(3) *Construction of examples and classification.*

Example 4.1. (1) $\ell(1,m,n) = n$,

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

(2) rank = 2? There are two cases ([Atkinson '83], [Eisenbud-Haus '88])
- Compression spaces,

$$\begin{pmatrix} * & * & \cdots & * \\ * & 0 & \cdots & 0 \\ \vdots & & & \\ * & 0 & \cdots & 0 \end{pmatrix}$$

- Skew-symmetric matrices of 3×3.

(3) $\ell(r,m,n) \geq n - r + 1$. Because you can put a matrix of $m \times n$ (with the first r rows that can have nonzero entries):

$$\begin{pmatrix} x_1 & x_2 & \cdots & & x_{n-r+1} \\ & x_1 & x_2 & \cdots & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

Theorem 4.2 (Westwick '86)**.** (1) $n - r + 1 \leq \ell(r,m,n) \leq 2n - 2r + 1$.
(2) *If* $n - r + 1 \not| \frac{(m-1)!}{(r-1)!} \implies \ell$

We can see these spaces as (subvarieties?) of determinantal varieties $M_n = \{f \in Mat_{m,n}(\mathbb{C}) : rank(f) \leq r\}$. *[Interpretation via secant varieties inside Segre embedding.]*

Consider a map $\varphi : U \to \mathrm{Hom}(V,W)$. *Then* $\varphi \in U^* \otimes V^* \otimes W$. *We get that* φ *is of constant rank if and only if some kernel, image and cokernel are vector bundles.*

Focus of today. *What happens when U, V, W are irreducible representations of a complex reductive group G?*

Question. *What is the natural equivariant morphism*

$$U \to \mathrm{Hom}(V, W) = V^* \otimes W$$

of constant rank?

Consider the case of $G = SL_2$. All the irreducible representations (which are self-dual) are given by $V(m-1) \cong \mathbb{C}[x, y]_{deg=m-1}$.

Recall the Clebsh-Gordon decomposition ($m \leq n$)

$$V(m-1) \otimes V(n-1) = \bigoplus_{i=0}^{m-1} V(n-m+2i)$$

Theorem 4.3 (B-Faenzi-Lella '22)**.**

$$V(n+m-2) \hookrightarrow \mathrm{Hom}(V(m-1), V(n-1))$$

is of constant rank (corank 1) if and only if

$$n - m + 2i | m - 1$$

Theorem 4.4 (B. Faenzi, Fratila '25)**.** *Let $V(\nu)$, $V(\mu)$ and $V(\lambda)$ be irreducible representations of a comple reductive group G, with*

$$\dim(V(\mu)) \leq \dim(V(\lambda))_{-}$$

If there exists a morphism of representations

$$\varphi : V(\nu) to \, \mathrm{Hom}(V(\lambda), V(\mu))$$

then φ is of constant corank 1 if and only if there exists a simple root α_i such that

(1) $\lambda = \mu + \nu - \alpha_i$,
(2) ν *is a multiple of* ν,
(3) ν *is a multple of* ω_i

5. On wrapped Floer homology barcode entropy and hyperbolic sets

Rafael Fernandes, UC Santa Cruz. Differential Geometry Seminar, IMPA. September 4, 2025.

Abstract. *In this talk, we will discuss the interplay between the wrapped Floer homology barcode and topological entropy. The concept of barcode entropy was introduced by Çineli, Ginzburg, and Gürel and has been shown to be related to the topological entropy of the underlying dynamical system in various settings. Specifically, we will explore how, in the presence of a topologically transitive, locally maximal hyperbolic set for the Reeb flow on the boundary of a Liouville domain, barcode entropy is bounded below by the topological entropy restricted to the hyperbolic set.*

Let M^n be a manifold. $\omega \in \Omega^2(M)$ is symplectic if $d\omega = 0$ and it is nondegenerate.

Example 5.1. $\mathbb{R}^n$ is symplectic with canonical Darboux form.

Recall the definition of Hamiltonian vector field associated to a function $H \in C^\infty(M)$.

Definition 5.2. A diffeomorphism $\varphi : M \to M$ is called *non-degenerate* if $\Phi(\varphi) \cap \Delta \subset M \times M$ (pitchfork, i.e. transversal intersection!).

Let M^{2n} be a closed symplectic manifold. Arnold's conjecture says

(1) *If $\varphi = \varphi_H$ (Hamiltonian flow) is nondegenerate, then*
$$\#\mathit{Fix}(\varphi_H) \geq \sum_{i=0}^{2n} \dim H_i(M,k) = \dim H_*(M,k)$$
(2) *If $\varphi = \varphi_H$ is degenerate, then*
$$\#\mathit{Fix}(\varphi) \geq \mathit{Cl}(M) + 1$$
where $\mathit{Cl}(M)$ is the maximum number of homology classes we can add before getting to zero.

Why do we care? Because
$$\#\mathit{Fix}(\varphi_H) \leftrightarrow \{\text{1-periodic orbits of } X_H\}$$
Idea by Floer. Construct an invariant that would say something about periodic orbits.

Question. *Can Floer theory capture other "dynamical information"? (Other than the periodic orbits.)*

A persistence module is a pair (V, Π), where $V = \{V_t\}_{t\in\mathbb{R}}$ is a family of $\mathbb{F}$-vector spaces and $\Pi = \{\Pi_{st}\}_{s\leq t}$ is a family of maps such that

(1) $\Pi_{ss} =, \Pi_{ts} \circ \Pi_{rt} = \Pi_{rs}$.
(2) *$\exists s \subset \mathbb{R}$ such that Π_{st} is an isomorphism for s, t in the same connected component of $R \setminus S$.*
(3) *Π_{st} have finite rank.*
(4) *$\exists s_0$ $V_s = \{0\}$, $s \leq s_0$.*
(5) *$V_t = \lim_{s\to t} V_s$ (lower limit!!)*

Theorem 5.3. *Any persistence module is a sum of integral persistence modules,*
$$(V, \Pi) \cong \bigoplus_{I\in B(V)} F(I).$$

Example 5.4. Heart and sphere. There is a noise in the persistence module of the heart due to an unnecessary critical point.

(M^{2n}, ω) a Liouville domain is a compact symplectic manifold and $X \in \mathfrak{X}(M)$ with $X \cap \partial M$ (pitchfork, i.e. transversal intersection!) pointing outwards and preserved by the symplectic form, i.e. $\mathcal{L}_X\omega = \omega$ ($\omega = d\alpha$).

When we restrict ω to the boundary, we obtain a contact form and get some interesting dynamics.

A Lagrangian $(L, \partial L) \subset (M, \partial M)$ is asympotically conical if

(1) *$\partial L \subset \partial M$ is Legendrian.*
(2) *$L \cap [1-\varepsilon, 1] \times \partial M = [1-\varepsilon, 1] \times \partial L$.*

Remark 5.5. Take a Hamiltonian $H : \hat{M} \to \mathbb{R}$ such that
$$\begin{cases} H(r,x) = h(r) & r = 1 \\ H(r,x) = rT - B \end{cases}$$
then $X_H = h'(r)R_\alpha$.

For $L_0, L, A \subset$ Lagrangians, H linear at infinite, then $A_H^{L_0\to L_1}$,

$$A_H^{L_0\to L_1} : P_{L_0\to L_1} \longrightarrow \mathbb{R}$$
$$\gamma \longmapsto \int_0^1 \gamma^*\alpha - \int_0^1 H(x(t))dt$$

where $P_{L_0\to L} = \{\gamma : [0,1] \to \hat{M} : \gamma(0) \in L_0, \gamma(1) \in L_1\}$ is the set of chords.

Remark 5.6. $\mathrm{crit}(A_H^{L_0\to L_1}) = \{$1-chords of X_H from L_0 to $L_1\}$.

Putting a metric on $P_{L_0\to L_1}$ we can consider $\varphi : \mathbb{R}\times[0,1] \to \hat{M}$, solutions of some PDE which is some kind of generalization of a gradient, $-\nabla A_H^{L_0\to L_1}$. These solutions can be put in a moduli space

$$\tilde{\mathcal{M}}(x_-, x_+, H, J) = \{\varphi \textit{ solutions s.t. } \dots\}$$

Then we define a boundary operator ∂.

Theorem 5.7. $\partial^2 = 0$

So that we have a homology, called wrapped Floer homology $HW^t(H, L_0, L_1, J)$

Remark 5.8. We have $H \le K \rightsquigarrow HW^t(H, L_0, L_1, J) \to HW^t(K, L_0, L_1, K)$.

Definition 5.9. For $t \ge 0$

$$HW^t(M, L_0, L_1) = \underline{\lim}_H HW^t(H, L_0, L_1, J)$$

(Where we have taken direct limit.)

Taking direct limit of the homology, we make sure the homology theory is independent of the choice of objects (I think, complex structure and Hamiltonian) we used to construct it.

Proposition 5.10. *$t \to HW^t(M, L_0, L_1)$ is a peristence module $B(M, L_0, L_1)$.*

Finally we can define barcode entropy. Fix $\varepsilon > 0$, $t \ge 0$,

$$b_\varepsilon(M, L_0, L_1, t) = \#\{ \textit{ of bars in } B(M, L_0, L_1) \textit{with length} \ge \varepsilon \textit{ and start before } t\}$$

Then

$$\bar{h}^{HW}(M_0, L_0, L_1) = \lim_{\varepsilon\to 0} \textit{lim sup}_{t\to\infty} \frac{\textit{log}^+(b_\varepsilon(M, L_0, L_1, t)}{t}$$

Consider a contact manifold $(\Sigma, \lambda, L_0, L_1)$ and A the Lagrangians. (Example raising question of filling.)

Theorem 5.11 (M '24)**.** *$\bar{h}^{HW}$ is independent of the filling.*

Theorem 5.12 (M '24)**.** *$\bar{h}^{HW}(M, L_0, L_1) \le h_{top}(\alpha)$.*

Theorem 5.13 (M '25)**.** *Consider (M, L_0, L_1). Let K be a compact topologically transitive hyperbolic set for the Reeb flow α. Assume $W^s_\delta(q) \subset \partial L_0$, $W^s_\delta(p) \subset \partial L_1$. Then*

$$\bar{h}^{HW}(M, L_0, L_1) \ge h_{top}(\alpha|_K) > 0.$$

Which says it captures dynamics beyond unconditional phenomena. In lower dimensions these tend to coincide, but in higher dimension we don't know. This is related to "the sup over hyperbolic sets [...]"

Here's a conjecture:

$$\textit{sup}_{L_0,L_1} \bar{h}^{HW}(M, L_0, L_1) = h_{top}(\alpha).$$

Extra comments. *One of the aims is to describe topological entropy h_{top} using Floer theory. Theorems by Çineli-Ginzburg-Gürel show bounds of topological entropy and barcode entropy (one of which is for hyperbolic sets).*

There is a notion of admissible Hamiltonian and Reeb vector fields which is related to some asymptotical behaviour "linear at infinity". I understand that admissible vector fields give the interesting chords for the Floer homology construction.

6. Revisiting cotangent bundles

Mieugel Cueca, KU Leuven. Symplectic Geometry Joint Seminar, IMPA. September 5, 2025.

Abstract. *Cotangent bundles provide key examples of symplectic manifolds. On the other hand, one can think of Lie groupoids as generalizations of manifolds. In this context, Alan Weinstein constructed their cotangent bundles and proved that they are so-called symplectic groupoids. In this talk, I will recall this construction and explain what happens when one replaces a Lie groupoid with a Lie 2 (or n)-groupoid. If time permits, I will exhibit some of their main applications. This is joint work with Stefano Ronchi.*

Recall the basic properties of the cotangent bundle T^*M for a symplectic manifold:

(1) *It's a vector bundle.*
(2) $\langle \cdot, \cdot \rangle : TM \otimes T^*M \to \mathbb{R}_M$ *is the dual pairing.*
(3) $\omega_{can} \in \Omega^2(T^*M)$ *is symplectic.*
(4) $\mathcal{L}_\varepsilon \omega_{can} = \omega_{can}$, ε *Euler vector field.*

Main goal. *Reproduce the above for Lie n-groupoids.*

For $n = 0$ we get the above situation. For $n = 1$ [Duzud-Weinstein], [Prezun], for $n \geq 2$? and we care about $n = 2$.

Definition 6.1. A *Lie n-groupoid* $\mathcal{G} : \Delta^{\text{op}} \to \mathsf{Man}$ such that

$$P_{e,j} : \mathcal{G}_\ell \to \Lambda^\ell_j \mathcal{G}$$

are surjective submersions $\forall \ell, j$ are differomorphisms for $\ell > n$.

Remark 6.2. This sort of manifolds-valued presheaf category is generated by

$$d^\ell_i : \mathcal{G}_\ell \to \mathcal{G}_{\ell-1} \text{face maps} 0 \leq i, j \leq \ell$$
$$s^\ell_j : \mathcal{G}_\ell \to \mathcal{G}_{\ell+1} \qquad \text{degeneracies}$$

The tangent is a functor, it satisfies

$$T_\bullet(\mathcal{G}) = T_k(\mathcal{G}) = T\mathcal{G}_k$$

(It looks like T preserves diagrams.)

Dold-Kan. *The category* SVect *of simplicial vector spaces has objects*

$$\mathbb{V}_\bullet \qquad \mathbb{V}_n \longrightarrow \cdots \longrightarrow \mathbb{V}_2 \xrightarrow{3\ arrows} \mathbb{V}_1 \xrightarrow{2\ arrows} \mathbb{V}_0$$

where the $\mathbb{V}_i$ are vector spaces.

There is a functor

$$\mathsf{SVect} \xrightarrow{N} \{\textit{chain complexes} \geq 0\}$$
$$\mathbb{V}_\bullet \to N\mathbb{V} = (N_\ell \mathbb{V}\, \mathrm{Ker}\, P_{\ell,\ell}, \partial = d_\ell)$$

Theorem 6.3 (Dold-Kan)**.** *That's an equivalence of categories. [Confirm this!]*

Those categories are monodial:

$$(\mathsf{SVect}, \otimes), \qquad (\mathbb{V}_\bullet \otimes \mathbb{W})_\ell = \mathbb{V}_\ell \otimes \mathbb{W}_\ell$$
$$(\mathsf{ch}_{\geq 0}, \otimes), \qquad (V \otimes W)_i = \bigoplus_{\ell+k=1} V_\ell \otimes W_k$$

And N is Lax monoidal with Lax structure given by the Eilenberg-Zilber map, though we won't explain the details of this.

There are duals given by internal Hom:

$$\mathbb{V}^{n*} = \underline{\mathrm{Hom}}(\mathbb{V}, B^n\mathbb{R})$$

Where the internal Hom is given by

$$\underline{\mathrm{Hom}}(\mathbb{V}, B^n\mathbb{R})_\ell = \mathrm{Hom}_{\mathsf{SVect}}(\mathbb{V} \otimes \Delta_n[\ell], B^n\mathbb{R})$$

for an object $\Delta[\ell] = \mathbb{R}[\Delta[\ell]]$.

Properties.

(1) $\mathbb{V}^{n*}$ *isa simplicial vector bundle.*
(2) $N(V^{n*})$ *and* $N(\mathbb{V})^*[n]$ *is a quasi isomorphism.*
(3) $\langle \cdot, \cdot \rangle : \mathbb{V} \otimes \mathbb{V}^{n*} \to B^n\mathbb{R}$ *is non-degenerate on homology.*
(4) $\mathbb{V} \hookrightarrow (\mathbb{V}^{n*})^{n*}$ *Mont. a eq.*

The vector bundle case.

$$\mathit{Maps}(\Delta[i], \mathcal{G})_k = \mathrm{Hom}_{\mathsf{SSet}}(\Delta[i] \times \Delta[k], \mathcal{G})$$

Proposition 6.4. *Let $\mathcal{G}$ be a Lie n-groupoid.*

(1) $\mathit{Maps}(\Delta[i], \mathcal{G})$ *Lie n-groupoids ME $\mathcal{G}$.*
(2) $\mathit{Maps}(\Delta[i], \mathcal{G})_0 = \mathcal{G}_i$.
(3) $ev : \Delta[i] \times \mathit{Maps}(\Delta[i], \mathcal{G}) \to \mathcal{G}$.

[Staircase looking diagram.]

Definition 6.5. $\mathcal{G}_\bullet$.

$$T_i^{n*}\mathcal{G} = \mathrm{Hom}_{\mathsf{SVect}}(1^*\Pi_{\Delta[i]}(T\mathcal{G}), B^n\mathbb{R}_{\mathcal{G}_i})$$
$$(d_j, F)_{K|d_j\mathcal{G}}(x^a) = (F_k)|_{\mathcal{G}}(x^{\delta_j a}).$$

Proposition 6.6. *$\mathcal{G}$ Lie n-groupoid, then $T^{n*}\mathcal{G}$ satisfy*

(1) *is a vector bundle n-groupoid*
(2) *dual to $T\mathcal{G}$*

$$\langle \cdot, \cdot \rangle : T\mathcal{G} \otimes T^{n*}\mathcal{G} \to B^n\mathbb{R}_{\mathcal{G}}$$

non-degenerate on homology.

(3) *n-shifted symplectic*

$$T_n^{n*}\mathcal{G} \xrightarrow{p} T^*\mathcal{G}_n$$

and $p^*\omega_{can}$.

[More computations I missed]

References