# FACTORING CREMONA TRANSFORMATIONS

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Notes at github.com/danimalabares/stack

Abstract. O grupo de Cremona em dimensão n é o grupo das transformações birracionais do espaço projetivo de dimensão n. O celebrado teorema de Noether-Castelnuovo (1871-1901) afirma que o grupo de Cremona em dimensão 2 é gerado pelos automorfismos lineares e por uma única transformação quadrática. Em dimensões superiores, não há uma descrição simples do grupo de Cremona em termos de geradores, e a situação é bem mais complicada. Por outro lado, técnicas de geometria birracional, em particular o MMP (Minimal Model Program), fornecem uma maneira de fatorar transformações de Cremona como composições de elos elementares. Essa teoria, conhecida como "Programa de Sarkisov", tem se mostrado extremamente útil no estudo do grupo de Cremona em dimensão superior. Neste minicurso, faremos uma introdução ao MMP e ao Programa de Sarkisov.

Contents



## 1. Cremona group

Let $\mathrm{Cr}_n(k) := \mathrm{Aut}_k k(x_1, \dots, x_n)$.

**Definition 1.1.** $f : X \dashrightarrow Y$ is *rational* if it is defined in an open dense set $U_X \subseteq X$. It is *birational* if it admits an inverse; equivalently, if there is $U_Y \subseteq Y$ open dense such that $f : U_X \to U_Y$ is an isomorphism.

Fix $n = 2$ and consider the map $(x_1, x_2) \mapsto \left(\frac{1}{x_1}, \frac{1}{x_2}\right)$.

**Example 1.2.**

$$\begin{aligned} \sigma : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\ (x_0, x_1, x_2) &\longmapsto (x_1x_2, x_0x_2, x_0x_1). \end{aligned}$$

Then $\sigma = \sigma^{-1}$, with

$$\{x_i = 0\} \mapsto p_i, \qquad \{x_0 = 0\} \mapsto (1:0:0).$$

$\sigma$: standard quadratic transformation.

Question (Enrique, 1895): Is $\mathrm{Cr}_n(k)$ simple? [i.e. does it have any nontrivial normal subgroups?] Do there exists any homomorphisms $\mathrm{Cr}_n(k) \to H$?
$\mathrm{PGL}_{n-1}(k) \cong \mathrm{Aut}(\mathbb{P}^n) \subseteq \mathrm{Cr}_n(k)$.

How to find homomorphism to a group $H$, $\mathrm{Cr}_n(k) \to H$.

(1) Find a set of generators $G$ by $\mathrm{Cr}_n(k)$.

(2) Get a set $R$ of relators. $\mathrm{Cr}_n(k) = \langle G|R \rangle$.
(3) Map the generators to arbitrary elements of $H$ and check that the relators are mapped to $1_H$.

**Theorem 1.3** (Noether-Castelnuovo, 1872). *$Cr_2(\mathbb{C})$ is generated by two automorphisms of $\mathbb{P}^2$ and $\sigma$.*

**Theorem 1.4** (Gizatulin, 1983). *Description of the relators with respect to two generators $\{PGL_3(k), \sigma\}$.*

*Remark* 1.5. Contat-Lamy, 2010. $\mathrm{Cr}_2(k)$ is not simple for $k = \bar{k}$. Lonjoi, 2017: any field. [This is the proof that uses an action on an infinite-dimensional hyperbolic space. The same technique does not work for higher dimensions.]

**Theorem 1.6** (Hudson, 1927 & Pan, 1999). *Any set of [nonlinear] generators for $Cr_n(k)$, $n \geq 3$, is uncountable.*

... and we don't know any relators!

## 2. MMP and Sarkisov theory

[MMP is an algorithm.]

$$X : \substack{\text{smooth}\\ \text{projective}} \longrightarrow \text{MMP} \longrightarrow X_{\min} : \substack{\text{mild singular}\\ \text{projective}}$$

so that

- $X \sim_{bir} X_{\min}$
- $X_{\min}$ is the "simpler" than $X$
  - $K_{X_{\min}}$ is "more positive",
  - $\rho(X_{\min}) \leq \rho(X)$.
- The process is realized in "elementary"steps.

The outputs of MMP are of two types:

(1) minimal models, or
(2) Mori fiber spaces (Mfs)

Sarkisov program. An algorithm for decomposing birational maps among Mfs into "simpler" maps.

[The idea of these maps is to copy the action of the standard quadratic transformation $\sigma$ from Example 1.2: we blow up three points (vertices of a triangle, the $p_i$), then we get a "hexagonal" arangement, and contract three lines (the other three) to get back at a triangle).]

$$\text{triangle} \rightsquigarrow \text{hexagon} \rightsquigarrow \text{triangle}$$

$$\underbrace{\mathbb{P}^2}_{\text{triangle}} \rightsquigarrow \underbrace{\mathbb{F}_1}_{\text{Hirzebruch}} \rightsquigarrow * \rightsquigarrow \mathbb{P}^q \times \mathbb{P}^1 \rightsquigarrow * \rightsquigarrow * \rightsquigarrow \underbrace{\mathbb{P}^2}_{\text{triangle}}$$

**Theorem 2.1** (Corb, 1995 and Hacon-McKeman, 2011). *Any birational map among Mfs can be factores as a composition of Sarkisov links if and only if the Sarkisov links are generators of the groupoid*

$$\underbrace{BirMori}_{\supseteq Cr_n(k)} = \left\{ f : X \dashrightarrow Y \substack{f \ birational\\ X,Y \ Mfs,\\ X,Y \sim_{bir} \mathbb{P}^n} \right\}$$

**Theorem 2.2** (Blanc-Lamy-Zimmerman)**.** *Description of the relators among the Sarkisov links.*

**Theorem 2.3** (BLZ, 2021)**.** *$Cr_n(k)$ is not simple for $n \geq 3$, $\varphi_i$ only appearing in relations of the form $\varphi_i \circ X \circ \varphi_i^{-1} \circ X^{-1} = id$, $\varphi_i^2 = id$.*

$$\begin{aligned} BirMor(\mathbb{P}^n) &\longrightarrow \mathbb{Z}/2\mathbb{Z} \\ \varphi &\longmapsto 1 \\ links \neq \varphi &\longmapsto 0 \end{aligned}$$