DIFFERENTIAL GEOMETRY

github.com/danimalabares/stack

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Definition 2.1. Let M_1 and M_2 be Riemannian manifolds. The *product metric* on $M_1 \times M_2$ is $\pi_1^* g_1 + \pi_2^* g_2$.

Definition 2.2. $S^1 \times \ldots \times S^1$ with the product metric is called the *flat torus*.

Exercise 2.3 (Warped product)**.** See [Pet16]. Para (M, g_M) e (N, g_N) e $f : M \to \mathbb{R}_+$, definimos o *warped product* como sendo o produto cartesiano $M \times N$ com a métrica $g_M + f^2 g_N$.

Mostre que se os fatores são completos, o warped product é completo.

Definition 2.4. Let $f : M \to \mathbb{R}$. Define the *gradient* of f as the (unique) vector field ∇f such that

$$\langle \nabla f, X \rangle = Xf, \qquad \forall X \in \mathfrak{X}(M). \tag{2.4.1}$$

Notice that for a frame E_i with dual frame E^i we have

$$\begin{aligned} \langle \nabla f, X \rangle &= g_{ij} E^i \odot E^j (\nabla f, X) = g_{ij} (\nabla f)^i X^j \\ Xf &= df(X) = (df)_i E^i(X) = (df)_i X^i \\ &\implies g^{ij} (df)_i X^j = (\nabla f)^i X^i \end{aligned}$$

so substituting X with a basic vector E_i and $(df)_i$ with $\partial_i f$ we obtain

$$(\nabla f)^i = g^{ij} \partial_i f \iff \nabla f = g^{ij} \partial_i f E_j \tag{2.4.2}$$

Definition 2.5. For any $X \in \mathfrak{X}(M)$,

$$\operatorname{div}(X) := \operatorname{tr}(v \mapsto \nabla_v X)$$

Lemma 2.6. [dC79]*, Chapter III, Exercise 9(b).*

$$\Delta(fg) = f \Delta g + g \Delta f + 2 \langle \nabla f, \nabla g \rangle \tag{2.6.1}$$

Proof. Just compute. □

Lemma 2.7.

$$d i_X \, Vol = div X \, Vol, \qquad \textit{for all } X \in \mathfrak{X}(M). \tag{2.7.1}$$

Proof. Pick a geodesic frame E_i (see Lemma 5.5) and its dual coframe ε^i, i.e. satisfying $\varepsilon^i(E_j) = \delta_{ij}$. Then $\mathrm{Vol} = \varepsilon^1 \wedge \ldots \wedge \varepsilon^n$. Then for any $X = X^i E_i \in \mathfrak{X}(U)$,

$$i_X \, \mathrm{Vol} = \varepsilon^1 \wedge \ldots \wedge \varepsilon^n (X^i E_i, \cdot, \ldots, \cdot) = X^i \varepsilon^1 \wedge \ldots \wedge \varepsilon^n (E_i, \cdot, \ldots, \cdot)$$

How to compute that? Recall that for top-forms we have

$$\varepsilon^1 \wedge \ldots \wedge \varepsilon^n (Z_1, \ldots, Z_n) = \det(\varepsilon^i(Z_j))$$

so for example if $n = 3$

$$\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3(E_1, Z_2, Z_3) = \begin{vmatrix} \varepsilon^1(E_1) & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ \varepsilon^2(E_1) & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ \varepsilon^3(E_1) & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix} = \begin{vmatrix} 1 & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ 0 & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ 0 & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix}$$
$$= \varepsilon^2(Z_2)\varepsilon^3(Z_3) - \varepsilon^2(Z_3)\varepsilon^3(Z_2) = \varepsilon^2 \wedge \varepsilon^3(Z_2, Z_3),$$
$$\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3(E_2, Z_2, Z_3) = \begin{vmatrix} 0 & \varepsilon^1(Z_2) & \varepsilon^1(Z_3) \\ 1 & \varepsilon^2(Z_2) & \varepsilon^2(Z_3) \\ 0 & \varepsilon^3(Z_2) & \varepsilon^3(Z_3) \end{vmatrix} = -\varepsilon^1 \wedge \varepsilon^3(Z_2, Z_3)$$

and so on. When we sum over all i, we get

$$X^i \varepsilon^1 \wedge \ldots \wedge \varepsilon^n(E_i, \cdot, \ldots, \cdot) = \sum_i (-1)^{i+1} X^i \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n.$$

Now take exterior derivative of that, we get

$$\begin{aligned} di_X \operatorname{Vol} &= \sum_i (-1)^{i+1} (dX^i) \wedge \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n \\ &+ \sum_i (-1)^{i+1} X^i \wedge d(\varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n) \end{aligned}$$

And then the first term actually is

$$\sum_i (-1)^{i+1} E_j X^i \varepsilon^j \wedge \varepsilon^1 \wedge \ldots \wedge \widehat{\varepsilon^i} \wedge \ldots \wedge \varepsilon^n = E_i X^i \operatorname{Vol}$$

while the second term vanishes because look,

$$\begin{aligned} d\varepsilon^i(E_j, E_k) &= E_j \varepsilon^i(E_k) - E_k \varepsilon^i(E_j) - \varepsilon^i([E_j, E_k]) \\ &= -\varepsilon^i(\nabla_{E_j} E_k - \nabla_{E_k} E_j) \qquad \text{torsion!} \end{aligned}$$

which vanishes at p because we said that this geodesic frame would have vanishing Christoffel symbols at p. So we conclude:

$$di_X \operatorname{Vol} = E_i X^i \operatorname{Vol}$$

Now you just have to think what is divergence:

$$\begin{aligned} \operatorname{div} X &= \sum_i \left\langle \nabla_{E_i} X^j E_j, E_i \right\rangle = \sum_i \left\langle E_i X^j E_j, E_i \right\rangle + X^j \left\langle \nabla_{E_i} E_j, E_i \right\rangle \\ &= \sum_i \left\langle E_i X^j E_j, E_i \right\rangle = \sum_i E_i X^j \left\langle E_j, E_i \right\rangle = E_i X^i \end{aligned}$$

again using that we are a geodesic frame with vanishing covariant derivative at p. □

3. Connections

Example 3.1. The usual connection in $\mathbb{R}^n$ is given by

$$\nabla_X Y = X(Y^i)\partial_i$$

Proposition 3.2 (Everything I know about connections)**.**

Lemma 3.3. [?], *p.5. If $g : M_1 \to M_2$ and $f : M_2 \to M_3$ are smooth maps of manifolds and E is a vector bundle over M_3, then*

$$g^*(f^*(E)) = (f \circ g)^*(E) \qquad \text{and} \qquad (\nabla^f)^g = \nabla^{f \circ g}$$

The following three lemmas are fundamental properties of isometric immersions and will be used in Section 9. (Cf. Lista 3, Exercise 2.)

Lemma 3.4. *If $f : M \to \tilde{M}$ is an isometric immersion, then $f_*(\nabla_X Y) = (\nabla^f_X f_* Y)^\top$.*

Proof. We have to check that $(\nabla^f_X f_* Y)^\top$ is a torsion-free and compatible connection on TM. □

Lemma 3.5. *Let $f : M \to \tilde{M}$ be an isometric immersion, c is a smooth curve in M and $Y \in \mathfrak{X}_c$. Then*

$$f_* \nabla^\gamma_{\frac{d}{dt}} Y = \left(\nabla^{f\circ\gamma}_{\frac{d}{dt}} f_* Y\right)^\top$$

Proof. This is just a mixture of Lemmas 3.3 and 3.4. □

The following result will be reminded when we define totally geodesic submanifolds, cf. Definition 9.5, addressing the case when also the normal component vanishes:

Lemma 3.6. *Let $f : M \to \tilde{M}$ be an isometric immersion and $\gamma : I \to M$ a smooth curve parametrized by arclength. Show that γ is a geodesic of M if and only if its acceleration on $\tilde{M}$ is perpendicular to M, that is,*

$$\tilde{\nabla}^{f\circ\cong\gamma}_{\frac{d}{dt}} (f \circ \gamma)' \perp f_*(T_{\gamma(t)} M)$$

Proof. Decompose the vector field $\tilde{\nabla}^{f\circ\cong\gamma}_{\frac{d}{dt}} (f\circ\gamma)'$ in tangent and normal part. Then apply Lemma 3.4. □

4. Riemannian submersions and immersions

For any smooth manifold submersion $\pi : \tilde{M} \to M$ there is a bundle isomorphism I love:

$$\pi^* TM \oplus \operatorname{Ker} \pi \cong T\tilde{M} \tag{4.0.1}$$

where $\operatorname{Ker}\pi$ is the *kernel bundle*, whose fiber is the kernel of π_*, and is also called the *vertical bundle*. Proving that $\operatorname{Ker}\pi$ is in fact a bundle and confirming the isomorphism 4.0.1 is a simple exercise.

We call π a *riemannian submersion* only when $\pi^* TM$ is isometric to TM, which makes sense only when $\tilde{M}$ and M are Riemannian manifolds.

Following notation of [**?**], for any subbundle $\xi \subset \eta$ over a Riemannian manifold, the *orthogonal complement bundle* $\xi^\perp$ can be defined fiberwise as the orthogonal complement of the fiber and shown to be a bundle as in the case of the kernel bundle. Then Milnor defines the *normal bundle* when ξ is the tangent bundle of a submanifold. Finally this construction is noted to work just as well for any isometric immersion $f : M \to \tilde{M}$, so that

$$f^* T\tilde{M} \cong TM \oplus T^\perp M \tag{4.0.2}$$

Further, in problem 3-B we are asked to define for $\xi \subset \eta$ the *quotient bundle* and show it's a bundle. And if η has a Riemannian metric, then $\xi^\perp \cong \eta/\xi$. So that's why we sometimes say that we can define the normal bundle using a Riemannian metric but we don't need to.

An interesting thought is: given a subbundle, is there a manifold whose tangent bundle is that subbundle?

5. Geodesics

Example 5.1 (Geodesics of sphere)**.** The geodesics of S^n are

$$\gamma(t) = \cos tp + \sin tv \tag{5.1.1}$$

for $p \in S^n$ and $v \in T_p^1 S^n$.

Derivando como uma simples curva em $\mathbb{R}^{n+1}$, vemos que $\gamma'' = -\gamma$, o que significa que $(\gamma'')^\top = 0$, i.e. γ é uma geodésica de $\mathbb{S}^n$.

Sendo essa uma geodésica partindo de um ponto arbitrário numa direção arbitrária, concluimos por unicidade das geodésicas e *rescaling lemma* que todas as geodésicas de $\mathbb{S}^n$ são como γ.

Example 5.2 (Isometry group of sphere)**.** The isometry group of S^n is

$$\mathrm{O}(n) := \{A \in \mathrm{GL}(n+1) : AA^{\mathbf{T}} = \mathrm{Id}\}$$

It's immediate that $\mathcal{O}(n+1) \subset \mathrm{Isom}(S^n)$ since orthogonal transformations preserve euclidean product:

$$AA^{\mathbf{T}} = \mathrm{Id} \iff \sum_k A_{ik}A_{jk} = \delta_{ij} \iff Ae_i \cdot Ae_j = \delta_{ij}$$
$$\iff Av \cdot Aw = A\left(v^i e_i\right) \cdot A\left(w^j e_j\right) = v^i w^j (Ae_i) \cdot (Ae_j) = v^i \cdot w^j \delta_{ij} = v \cdot w$$

For the other contention let $A : S^n \to S^n$ be an isometry.

Vamos mostrar que A é a restrição de uma função $\tilde{A} \in \mathsf{O}(n+1)$. Defina

$$\begin{aligned} \tilde{A} : \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}^{n+1} \\ (r,\theta) &\longmapsto rA(1,\theta) \\ 0 &\longmapsto 0 \end{aligned}$$

Se mostramos que $\tilde{A}$ é uma isometria linear, é claro que ela é um elemento de $\mathsf{O}(n+1)$ pela conta anterior. De fato, basta mostrar que $\tilde{A}$ é uma isometria, pois toda isometria de espaços de Banach que fixa a origem é linear ([**?**] Teo. 7.11).

Para ver que $\tilde{A}$ é uma isometria de $\mathbb{R}^{n+1}$, afirmo que a distância de p a q está totalmente determinada pelas normas $\|p\|$ e $\|q\|$, e pela distancia esférica entre $\frac{p}{\|p\|}$ e $\frac{q}{\|q\|}$. Note que essa afirmação é na verdade um problema de geometria plana, pois todas essas quantidades podem ser descritas dentro do único plano que contém 0, p e q.

Acabou que essa afirmação é simplesmente a lei dos cosenos, já que a distância esférica entre $\frac{p}{\|p\|}$ e $\frac{q}{\|q\|}$ é exatamente o angulo entre p e q (poque essa distância é um segmento de círculo máximo!):

$$\text{lei dos cosenos:} \qquad d(p,q)^2 = \|p\|^2 + \|q\|^2 - 2\|p\|\|q\| \cos\angle(p,q)$$

Em fim, $\tilde{A}$ é uma isometria porque $d_{\mathbb{R}^{n+1}}(p,q) = d_{\mathbb{R}^{n+1}}(\tilde{A}p, \tilde{A}q)$ pelo argumento anterior.

Proposition 5.3. *If $f : M \to \tilde{M}$ is an isometry,*

$$\begin{tikzcd} B_\varepsilon(0) \subset T_pM \arrow[r, "d_pf"] \arrow[d, "exp_p"'] & B_\varepsilon(0) \subset T_{f(p)}\tilde{M} \arrow[d, "exp_{f(p)}"] \\ B_\varepsilon(p) \subset M \arrow[r, "f"'] & B_\varepsilon(f(p)) \subset \tilde{M} \end{tikzcd}$$

Proof. Isometry maps geodesics to geodesics by Lemma **??**, then uniqueness of solutions of ODEs. $\square$

Proposition 5.4 (Totally Normal Neighbourhoods)**.** *For all $p \in M$ there is a neighbourhood $W \ni p$ and a number $\delta > 0$ such that for every $q \in W$, $B_\delta(q)$ is normal and $W \subset B_\delta(q)$.*

Lemma 5.5. *At every point p in M there is an orthonormal frame $\{E_i\} \subset \mathfrak{X}(U)$ such that*

$$\left(\nabla_{E_i} E_j\right)_p = 0 \tag{5.5.1}$$

(This means the Christoffel symbols of the connection vanish at the point p.)

Proof. This frame can be obtained by taking geodesic coordinates at the point, an orthonormal base $\{e_i\}$ of T_pM, and taking parallel transport of the vectors e_i along radial geodesics emanating from p. This immediately ensures that E_i is orthonormal since parallel transport preserves angles.

To check that Christoffel symbols vanish at p we do as follows. Take a random vector $v \in T_pM$ and its geodesic $\gamma_v(t) = \exp_p(tv)$. I drop the subindex v for the next computations for the next computations. Then

$$0 = \nabla^\gamma_{\frac{d}{dt}} \gamma' = \nabla^\gamma_{\frac{d}{dt}} v^i (E_i \circ \gamma)$$

where the $v = (v^1, \ldots, v^n)$. Indeed: this is very silly but, since the coordinate chart of geodesic coordinates is $\exp_p^{-1}$, the coordinate representation of γ in this chart is as simple as

$$\hat{\gamma}(t) = (\underbrace{\varphi}_{\text{chart}} \circ \gamma)(t) = \exp_p^{-1} \exp_p(tv) = tv.$$

And the composition $E_i \circ \gamma$ just means that we take our local frame *along* γ. Continue:

$$\begin{aligned} &= v^i \nabla^\gamma_{\frac{d}{dt}} E_i \circ \gamma = v^i \nabla_{\gamma_{v,*} \frac{d}{dt}} E_i \\ &= v^i \nabla_{v^j E_j} E_i = v^i v^j \nabla_{E_j} E_i \\ &= v^i v^j \Gamma^k_{ji} E_k \end{aligned}$$

along γ. Now choose $v = e_1$. You get $\Gamma^k_{11} = 0$ for all k along γ_{e_1}. Now choose $v = e_2$, then $\Gamma^k_{22} = 0$ along γ_{e_2}, so at least at p they both vanish. And now choose $v = e_1 + e_2$. You get

$$0 = (v^1)^2 \underbrace{\Gamma^k_{11}}_{=0} + v^1 v^2 \Gamma^k_{12} + v^2 v^1 \Gamma^k_{21} + (v^2)^2 \underbrace{\Gamma^k_{22}}_{=0}$$

So $\Gamma^k_{12} = 0$ since Levi-Civita is torsion-free, i.e. symmetric. And so on. So the all Christoffel symbols vanish at the same time at p. $\square$

Exercise 5.6. [dC79], Chapter III, Exercise 12. If f is *subharmonic*, i.e. $\Delta f \geq 0$, on M compact connected $\implies$ f constant.

Proof. To prove the theorem for subharmonic functions first we show that in fact they are harmonic via Eq. 2.7.1 on $X := \nabla f$ and Stokes:

$$\int_M \Delta f \,\mathrm{Vol} = \int_M \operatorname{div} X \,\mathrm{Vol} = \int_M d(i_X \,\mathrm{Vol}) = \int_{\partial M} i_X \,\mathrm{Vol} = 0$$

meaning that the non-negative function Δf is in fact 0, i.e. f is harmonic. Now we do it again for $X := \nabla(f^2/2)$:

$$\int_M \Delta(f^2/2)\,\mathrm{Vol} = \int_M d(i_X\,\mathrm{Vol}) = \int_{\partial M} i_X\,\mathrm{Vol} = 0$$

And then apply Eq. 2.6.1:

$$0 = \int_M \Delta(f^2/2)\,\mathrm{Vol} = \int_M f\Delta f\,\mathrm{Vol} + \int_M \langle\nabla f, \nabla f\rangle\,\mathrm{Vol}$$

First one vanishes because f is harmonic, so second one is zero which says f is constant! □

6. Curvature

The *curvature tensor* is

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{6.0.1}$$

Proposition 6.1. *The curvature tensor is a tensor and*

1. *It is antisymmetric in the first two and last two entries,*
2. $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle$, *and*
3. *Bianchi identity.*

Given a dimension-2 vector subspace $\sigma \subset T_pM$, the *sectional curvature* of M in σ is

$$K(\sigma) = \frac{\langle R(u,v)v,u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2} \tag{6.1.1}$$

for any base $\{u,v\}$ of σ.

Lemma 6.2. *Sectional curvature does not depend on the choice of base.*

Proposition 6.3. *Let R and R' be two type $(0,4)$ tensors on a vector space $\mathbb{V}$. If their associated sectional curvatures K and K' are equal, then so are R and R'.*

Proof. Polarize and make lots of computations, cf. [dC79]. □

If a manifold has constant sectional curvature c,

$$\langle R(X,Y)Z,W\rangle = c(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Z,Y\rangle) \tag{6.3.1}$$

since the right-hand-side of Eq. 6.3.1 is a so-called curvature tensor, i.e. a tensor satisfying the symmetries of R, and applying Proposition 6.3.

The *Ricci tensor* is

$$\begin{gathered}\mathrm{Ric} : TM \times TM \longrightarrow \mathbb{R} \\ \mathrm{Ric}(X,Y) = \frac{1}{n-1}\mathrm{tr}(Z \mapsto R(Z,X)Y)\end{gathered} \tag{6.3.2}$$

It is symmetric.

The *Ricci curvature* is

$$\begin{gathered}\mathrm{Ric} : T^1M \longrightarrow \mathbb{R} \\ \mathrm{Ric}(X) = \mathrm{Ric}(X,X) = \frac{1}{n-1}\sum_{i=1}^{n-1} K(X,E_i)\end{gathered} \tag{6.3.3}$$

where $\{X, E_1, \dots, E_{n-1}\}$ is an orthonormal basis.

The *scalar curvature* is

$$\operatorname{scal} : M \longrightarrow \mathbb{R}$$

$$\operatorname{scal}(p) = \frac{1}{n}\operatorname{trRic} = \frac{1}{n}\sum_i \operatorname{Ric}(E_i, E_i) \tag{6.3.4}$$

Lemma 6.4.

$$scal(p) = \frac{1}{n}\sum_i Ric(e_i, e_i) = \frac{1}{n(n-1)}\sum_{i,j} K(e_i, e_j) = \frac{1}{Vol(S^{n-1})}\int_{\sigma\subset T_pM} K(\sigma)$$

Exercise 6.5. Lista 5, Exercise 1. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and let $M_1 \times M_2$ be equipped with the product metric $g := g_1 \oplus g_2$. Show that the curvature tensor, the Ricci curvature and the scalar curvature of g are given by

(1) $R = \pi_1^* R_1 + \pi_2^* R^2$,
(2) $\operatorname{Ric} = \pi_1^*\operatorname{Ric}_1 + \pi_2^*\operatorname{Ric}_2$,
(3) $\operatorname{scal} = \pi_1^*\operatorname{scal}_1 + \pi_2^*\operatorname{scal}_2$

where $\pi_i : M_1 \times M_2 \to M_i$ are the projections.

Example 6.6 (Curvature of complex projective space)**.** Defina uma métrica Riemanniana em $\mathbb{C}^{n+1}\setminus\{0\}$ do seguinte modo. Se $Z \in \mathbb{C}^{n+1}\setminus\{0\}$ e $V, W \in T_Z(\mathbb{C}^{n+1}\setminus\{0\})$,

$$\langle V, W\rangle_Z = \frac{\operatorname{Re}(V, W)}{(Z, Z)}$$

onde

$$(Z, W) = z_0\overline{w}_0 + \ldots + z_n\overline{w}_n$$

é o produto hermitiano em $\mathbb{C}^{n+1}$. Observe que a métrica $\langle\cdot,\cdot\rangle$ restrita a $S^{2n+1} \subset \mathbb{C}^{n+1}\setminus\{0\}$ coincide com a métrica induzida por $\mathbb{R}^{2n+2}$.

(1) Mostre que, para todo $0 \le \theta \le 2\pi$, $e^{i\theta} : S^{2n+1} \to S^{2n+1}$ é uma isometria, e que, portanto, é possível definir uma métrica Riemanniana em $\mathbb{P}^n(\mathbb{C})$ de modo que a submersão f seja Riemanniana.
(2) Mostre que, nesta métrica, a curvatura seccional de $\mathbb{P}^n(\mathbb{C})$ é dada por

$$K(\sigma) = 1 + 3\cos^2\varphi$$

onde σ é gerado pelo par ortonormal X, Y, $\cos\varphi = \langle \overline{X}, i\overline{Y}\rangle$, e $\overline{X}$, $\overline{Y}$ são os levantamentos horizontais de X e Y, respetivamente. Em particular, $1 \le K(\sigma) \le 4$.

7. Jacobi Fields

Proposition 7.1 (Everyday Jacobi field)**.** [dC79] *Chapter V, Proposition 2.4. Let $\gamma : [0, a] \to M$ be a normalized geodesic and let J be a Jacobi field through γ with $J(0) = 0$. Let $p := \gamma(0)$ $w := J'(0)$ and $v := \gamma'(0)$. Consider the variation*

$$f(s, t) := exp_p(t(v + sw))$$

Then the associated Jacobi field is J. So in particular the field

$$J(t) = d_{tv}exp_p(tw)$$

is a Jacobi field.

Remark: *the variation can also be taken with respect to any a curve σ on T_vT_pM passing through v at $s = 0$ with velocity w.*

Example 7.2 (Jacobi Fields in constant curvature)**.** If M has constant curvature K and γ is a normalized geodesic, a Jacobi field is the solution of the equation

$$J'' + KJ = 0 \tag{7.2.1}$$

This follows from Eq. 6.3.1 since it implies that for any $X \in \mathfrak{X}_\gamma$, which we may take orthogonal to γ',

$$\langle X, R_{\gamma'}J\rangle = K(\langle X, J\rangle \langle \gamma', \gamma'\rangle - \langle X, \gamma'\rangle \langle J, \gamma'\rangle)$$

so that $R_{\gamma'}J = KJ$. The solutions of Eq. 7.2.1 can be computed taking a parallel vector field W along γ orthogonal to γ' and of unit length. Then

$$J(t) = \begin{cases} \sin(\sqrt{K}t)W(t) & K > 0 \\ tW(t) & K = 0 \\ \sinh(\sqrt{-K}t)W(t) & K < 0 \end{cases} \tag{7.2.2}$$

provided that $J(0) = 0$, $J'(t) \perp \gamma'(t)$ and $J'(0) = W(0)$.

Proposition 7.3. *Differentiating a Jacobi field enough times, we arrive to*

$$\|J(t)\|^2 = \|w\|^2 t^2 - \frac{1}{3}\langle R(w, v)v, w\rangle t^4 + O(t^4). \tag{7.3.1}$$

Sometimes it's useful to work with

Definition 7.4. The *Jacobi tensor* is the map

$$\begin{aligned} \overline{J}_t &: \longrightarrow T_{\gamma(t)}M \\ \overline{J}_t(w) &= d_{tv}\exp_p(tw) \end{aligned}$$

Exercise 7.5. Show that $\bar{J}'(0) = \mathrm{Id}$.

Proof. Just notice that since $\bar{J}_t = d_{tv}\exp_p(tw)$ is linear on tw (because it's a differential) we have

$$\begin{aligned} \overline{J}'(0) &= \frac{d}{dt}\Big|_{t=0}(d_{tv}\exp_p(tw)) \\ &= \frac{d}{dt}\Big|_{t=0} t \cdot (d_{tv}\exp_p(w)) \\ &= d_{tv}\exp_p(w)|_{t=0} + t \cdot (\text{something})|_{t=0} \\ &= \mathrm{Id}_{T_pM} \end{aligned}$$

□

Remark 7.6. Lembre o seguinte fato geral mostrado em aula:

$$\dim\{J \in \mathfrak{X}_\gamma^J : J(0) = 0, J \perp \gamma\} = n - 1 \tag{7.6.1}$$

Isso segue das seguintes observações:

(1) $\dim \mathfrak{X}_\gamma^J = 2n$ porque são soluções de n equações diferenciais ordinárias de segunda ordem, i.e. cada campo de Jacobi está determinado pelas condições iniciais $J(0)$ e $J'(0)$.
(2) $\dim\{J \in \mathfrak{X}_\gamma^J : J(0) = 0\} = n$.

(3) $\dim\{J \in \mathfrak{X}_\gamma^J : J \perp \gamma'\} = 2n - 2$. Para confirmar isso note que pelas simetrias de R e a equação de Jacobi tem-se que $\langle J, \gamma' \rangle'' = 0$, pelo que $\langle J, \gamma' \rangle = a + bt$ para dois números reais a, b. Segue que qualquer $J \in \mathfrak{X}_\gamma^J$ se escreve como $J(t) = a\gamma'(t) + bt\gamma'(t) + \hat{J}(t)$ para algum $\hat{J}$ perpendicular a γ'. (Supondo que $|\gamma'| = 1$.) Então se $J \perp \gamma$, temos que $a = b = 0$, então tiramos dois números do $2n$ que tínhamos.

(4) Segue 7.6.1.

8. Conjugate points

Definition 8.1. A point $p \in M$ is *conjugate* to $q \in M$ along γ if γ is a geodesic joining p and q and there is a Jacobi field $J \in \mathfrak{X}_\gamma^J$ such that $J(0) = 0$ and $J(\ell) = 0$.

Proposition 8.2 (Critical points of exponential)**.** [dC79] *Chapter V, Proposition 3.5. A vector $v \in T_pM$ is a critical point of exp_p if and only if the corresponding $\gamma_v(1)$ is a conjugate point of p along γ_v. Moreover, in such points, the dimension of the kernel of exp_p at this vector is the multiplicity of the conjugate point $\gamma_v(1)$ (i.e. the dimension of the space of Jacobi fields vanishing at the endpoints).*

Proof. ($\Longrightarrow$) Suppose that v is a critical point of $\exp_p$ and let $w \in \ker d_v \exp_p$ then a Jacobi field along γ vanishing at the endpoints is given by

$$J(t) := d_{tv} \exp_p tw$$

($\Longleftarrow$) If you have a critical point along γ, then the differential of $\exp_p$ has kernel.

(Mulitplicity.) The proof every element of the kernel gives a Jacobi field vanishing at the points. Linearly dependent elements will give linearly dependent Jacobi fields. $\square$

9. Isometric immersions

Let $f : M \to \tilde{M}$ be an isometric immersion of Riemannian manifolds. When computing the covariant derivative along two vector fields $X, Y \in \mathfrak{X}(M)$ we get a vetor field along X that may be split into its normal and tangent part thanks to Equation 4.0.2. By Lemma 3.4 we see that the tangent part is in fact the Levi-Civita connection of M since it is a symmetric and torsion free connection on the tangent bundle. Since the Levi-Civita connection of $\tilde{M}$ is torsion free, the normal part is a symmetric tensor we call *second fundamental form*:

$$\nabla^f_X f_* Y = f_* \nabla_X Y + \alpha(X, Y) \tag{9.0.1}$$

Now let's differentiate a normal section $\xi \in T^\perp M$. We call the tangent part *shape operator*. By Lemma 9.1 below, the normal part is a connection we call the *normal connection*.

$$\nabla^f_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{9.0.2}$$

Lemma 9.1. $\nabla^\perp$ *is a connection on $T^\perp M$.*

Notice the adjointness relation

$$\langle \xi, \alpha(X, Y) \rangle = - \langle A_\xi X, Y \rangle \tag{9.1.1}$$

which follows from computing $0 = X \langle Y, \xi \rangle$.

The *Gauss equation*

$$(R_{\nabla^f}(X, Y)Z)^\top = R_\nabla(X, Y)Z - A_{\alpha(Y,Z)}X + A_{\alpha(X,Z)}Y$$

may be derived by substituting Eq. 9.0.1 into Equation 6.0.1. But most likely we will use the following form:
(9.1.2)

$$\left\langle \tilde{R}(X,Y)Z,W\right\rangle = \langle R(X,Y)Z,W\rangle - \langle \alpha(X,Z),\alpha(Y,W)\rangle + \langle \alpha(X,W),\alpha(Y,Z)\rangle$$

Substituting for $X=W$ and $Y=Z$ we obtain the equivalent (by Proposition 6.3) equation

$$K(X,Y) = \tilde{K}(X,Y) + \langle \alpha(X,Y),\alpha(Y,Y)\rangle - \|\alpha(X,Y)\|^2 \tag{9.1.3}$$

Notice that we are seeing the determinant of the second fundamental form, which is the product of its eigenvalues, which may be defined as the *principal curvatures*. In particular, this shows that the Gaussian curvature of a surface in $\mathbb{R}^3$, defined as the determinant of the second fundamental form, coincides with sectional curvature when substituting $\tilde{K}\equiv 0$ for the flat metric on euclidean space. (!!!)

In the case of a hypersurface, the normal vector $\alpha(X,Y)$ must be proportional to the unit normal N, and by Eq. 9.1.1 the proportionally factor is exactly $\langle A_N X, Y\rangle$. Using this and evaluating at an orthonormal pair X,Y we obtain

$$K(X,Y) = \tilde{K}(X,Y) + \langle AX,X\rangle\langle AY,Y\rangle - \langle AX,Y\rangle^2 \tag{9.1.4}$$

In this case we are looking at the determinant of the matrix

$$\begin{pmatrix} \langle AX,X\rangle & \langle AX,Y\rangle \\ \langle AX,Y\rangle & \langle AY,Y\rangle \end{pmatrix} \tag{9.1.5}$$

which is the matrix representation of the shape operator A to the plane spanned by X,Y. (That's just a particular case of the simple fact that the representation of the matrix A_i^j for $Ae_i = A_i^j e_j$ is given by $\langle Ae_i, e_j\rangle = A_i^k\langle e_k,e_j\rangle = A_i^j$.)

Example 9.2 (Sectional curvature of sphere)**.** Using Eq. 9.1.4 we can prove that the sectional curvature of S^n is 1 once we make sure that the shape operator of the sphere is the identity. And this follows from the fact that the connection on $\mathbb{R}^n$ is just directional derivative, see Example 3.1.

More explicitly, the normal component of $\tilde{\nabla}_X N$ will vanish (see discussion in Section 11), so that $\tilde{\nabla}_X N = -A_N X$. But the connection in Euclidean space is just differentiating the vector field, that is, $\tilde{\nabla}_X N = NX$. And finally, who's N? It's $-\mathrm{id}$ if you chose the inner normal. So $\tilde{\nabla}_X N = -X$.

Exercise 9.3. Use Eq. 9.1.4 to show that no hypersurface $M\hookrightarrow\mathbb{R}^{n+1}$ can have constant negative curvature.

Proof. After substituting $\tilde{K}(X,Y)=0$, we see that Eq. 9.1.4 is determinant of the second fundamental form. But... how to argue that this determinant can't be always negative? □

Example 9.4 (Sectional curvature of hyperbolic space)**.** [dC79], Chapter VIII, Exercise 3. The unit normal of hyperbolic space in the hyperboloid model is also the identity. This allows for a similar computation, but Eq. 9.1.4 doesn't hold exactly because the metric is not Riemannian. In fact, the normal vector has norm -1, giving that hyperbolic space has constant curvature -1.

Definition 9.5. A submanifold $M\subset\tilde{M}$ is *totally geodesic* if its second fundamental form vanishes.

Lemma 9.6. *A submanifold is totally geodesic if and only if every geodesic of M is a geodesic of $\tilde{M}$.*

Proof. The direct implication is clear: if $\alpha \equiv 0$ and we take any curve contained in M, it will have zero acceleration in M and $\tilde{M}$ simultaneously. For the converse we check that all the eigenvalues of α are zero. Consider any tangent vector v to M and let γ_v be the geodesic in M with initial velocity v. By hypothesis this geodesic will also be a geodesic of $\tilde{M}$, so that $\alpha(v, v) = 0$. Diagonalizing α we see it must vanish identically. □

Lemma 3.6 says what happens in general: a geodesic of the submanifold doesn't need to be a geodesic of the ambient manifold, but it will always have acceleration normal to the submanifold.

Exercise 9.7. [Pet16], Proposition 5.6.5. Let $f : M \to M$ be an isometry. Show that the connected component of the fixed points of f is a totally geodesic submanifold.

Proof. The trick is to show that the eigenvectors of *df* are in bijection with the fixed point set of f. For this consider an eigenvector, a geodesic realising the vector, map it with f and notice it must coincide with the original geodesic because f is an isometry and the point is a fixed point. This shows that the fixed point set is a submanifold since it is parametrized as a linear subspace under geodesic coordinate charts. It also shows that the fixes point set is totally geodesic. □

Definition 9.8. An immersion $f : M \to \tilde{M}$ is called *minimal* if the trace of its shape operator vanishes at all points of M for any choice of normal vector field.

Definition 9.9. A minimal surface $f : \Sigma \to \mathbb{B}^3 \subset \mathbb{R}^3$ is of *free boundary* if it is orthogonal to the boundary of S^2.

Exercise 9.10 (Mean curvature and Hessian)**.** Lista 4, Exercício 8. Let (M, g) be a Riemannian manifold. Suppose that there is a function $f \in C^\infty(M)$ such that $0 \in \mathbb{R}$ is a regular value of f and let $\Sigma := f^{-1}(0)$.

(1) Let $N = \frac{\nabla f}{|\nabla f| \in \mathfrak{X}^\perp(\Sigma)}$. Mostre que
$$\langle \alpha_\Sigma(X, Y), N \rangle = -\frac{\mathrm{Hess}(f)(X, Y)}{|\nabla f|},$$
para todos $X, Y \in \mathfrak{X}(\Sigma)$.

(2) Show that the mean curvature of Σ is given by
$$H_\Sigma = -\frac{1}{n} \mathsf{div} \left(\frac{\nabla f}{|\nabla f|} \right)$$

Note. The notation in item (1) identifies fields using the differential of the inclusion map, as is customary.

Proof. (1) One one hand,
$$\langle \alpha(X, Y), \nabla f \rangle = - \langle Y, A_{\nabla f} X \rangle$$
by Eq. 9.1.1. On the other hand,
$$\mathrm{Hess}(f)(X, Y) = \left\langle \tilde{\nabla}_X \nabla f, Y \right\rangle = \left\langle -A_{\nabla f} X + \nabla^\perp_X \nabla f, Y \right\rangle,$$
then multiply by $\frac{1}{|\nabla f|}$.

(2) To compute the divergence as the trace of the gradient we need an orthonormal frame of M. So let E_i be the completion of $\frac{\nabla f}{|\nabla f|}$ to an orthonormal frame of M at a point of Σ. Then:

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) &= \sum_{i=1}^{n-1}\left\langle \tilde{\nabla}_{E_i}\frac{\nabla f}{|\nabla f|}, E_i\right\rangle + \underbrace{\left\langle \tilde{\nabla}_{\frac{\nabla f}{|\nabla f|}}\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}\right\rangle}_{=\frac{1}{2}\frac{\nabla f}{|\nabla f|}\left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}\right\rangle = 0} \\
&= \sum_{i=1}^{n-1}\left\langle -A_{\frac{\nabla f}{|\nabla f|}}E_i, E_i\right\rangle \qquad (9.10.1) \\
&= -\operatorname{tr}\left(X \mapsto A_{\frac{\nabla f}{|\nabla f|}}X\right) \\
&= -nH_\Sigma(p)
\end{aligned}$$

□

Notice that since the Laplacian of a function f is defined as divergence of gradient, that is, as the trace of $Z \mapsto \nabla_Z \nabla f$, we have for the ambient manifold Laplacian

$$\Delta_{\tilde{g}} f = \operatorname{div}_{\tilde{g}}(\nabla f) = \operatorname{tr}(Z \mapsto \tilde{\nabla}_Z \nabla f) = \sum_{i=1}^{n-1}\left\langle \tilde{\nabla}_{E_i}\nabla f, E_i\right\rangle + \left\langle \tilde{\nabla}_{\nabla f}\nabla f, \nabla f\right\rangle \tag{9.10.2}$$

where we must add an extra term since a frame for the ambient manifold also needs the normal vector. Then we have

$$\Delta_{\tilde{g}} f - \operatorname{Hess}_{\tilde{g}}(\nabla f, \nabla f) = \sum_{i=1}^{n-1}\left\langle \tilde{\nabla}_{E_i}\nabla f, E_i\right\rangle$$

If we multiply both sides by $\frac{1}{|\nabla f|}$, on the right hand side we arrive at the mean curvature H by Eq. 9.10.1 (after applying Leibniz rule), and on the left hand side. . .

$$H = \frac{1}{|\nabla f|}\left(\Delta_{\tilde{g}} f - \operatorname{Hess}_{\tilde{g}} f\left(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}\right)\right) \tag{9.10.3}$$

Caveat of Eq. 9.10.3. This is Lucas Ambrozio's formula from his talk on Spheres with Minimal Equators (see Seminars, Section **??**). Now it's almost perfect except for putting the $\frac{1}{|\nabla f|}$ inside the Hessian. . .

Example 9.11 (Equatorial spheres)**.** Show that equatorial spheres are minimal hypersurfaces of S^n. (See Seminars, Definition **??**.)

Proof. To use Eq. 9.10.3 we just need to compute the submanifold Laplacian $\Delta_g f$ where $f(x) := \langle x, v\rangle$ for a fixed v which is just the normal vector defining the equator, i.e. the equator is just $\Sigma := f^{-1}(0)$, the normal plane to v intersected with S^n. First we need to compute the gradient of f, which by Eq. 2.4.2 is just

$$\nabla f = g^{ij}\partial_i f E_j = \sum_{i,j} g^{ij} v^i E_j$$

So that's a problem because in practice we don't know the coefficients of the metric, nor do we ever want to compute them. What this really says is: you don't know what's the gradient. The next step would be to compute the Laplacian by differentiating the gradient according to Eq. 9.10.2:

$$\Delta_g f = \sum_{i=1}^{n-1} \langle \nabla_{E_i} \nabla f, E_i \rangle$$

Which just takes me back to Eq. 9.10.1. So, need to compute second fundamental form.

The only easy way I see to do this is use the fact that the geodesics of the sphere are well-known by Example 5.1. We just need to note that the equator is in fact a sphere and that any curve in the equator is a geodesic of the equator if and only if it is a geodesic of the whole S^n (cf. Lemma 9.6). But this is obvious by definition of equator: it's the unitary vectors in a hyperplane passing through the origin, and it is equipped with the metric induced from S^n, which in turn is induced from $\mathbb{R}^{n+1}$, so that geodesics of the equator are of the form Eq. 5.1.1 by the same reasons as why the geodesics of S^n of that form.

The other way to do this, following [dC79], Chapter VI, Exercises 8 (see also Exercise 9.12) is computing the shape operator of the submanifold with respect to $\mathbb{R}^{n+1}$ first, which is easy because Euclidean space is flat, and then computing the shape operator with respect to S^n. To do this we just need to compute inner products of the form $\langle A_{N_k} E_i, E_j \rangle$ since these are exactly the entries of A_{N_k} in the basis E_i (cf. Eq. 9.1.5). But I will do this in the next example (maybe some day do it for S^n too). □

Exercise 9.12 (Clifford torus)**.** [dC79], Chapter VI, Exercises 2 and 8. Show that the map $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$\mathbf{x}(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \qquad (\theta, \varphi) \in \mathbb{R}^2$$

is an immersion of $\mathbb{R}^2$ in the unit sphere $S^3 \subset \mathbb{R}^4$, whose image is a torus T^2 with zero sectional curvature in the induced metric. Finally show that it is a minimal submanifold of S^3. This is called the *Clifford torus*.

Proof. This map is an immersion since the derivatives of sine and cosine cannot vanish simultaneously. Its image is a torus since the map can be seen as a map from $S^1 \times S^1$ to $\mathbb{R}^4$.

It is natural to think that this manifold is the flat torus from Example **??**. And it is, because it is diffeomorphic to a product. We can use the immersion $\mathbf{x}$ on a small domain as a local parametrization of T^2. When differentiating the parametrization we notice that the tangent vectors are decomposed as a pair of vectors tangent to one circle or to the other. Then the metric induced from $\mathbb{R}^4$ and then by S^3 is exactly the product metric from Definition 2.1. By Lemma 6.5, the curvature tensor on $S^1 \times S^1$ vanishes identically since the curvature tensor of any 1-manifold vanishes by antisymmetry in the first or two last entries. This means that this torus is flat. This says that its first fundamental form has determinant -1 when seen as a submanifold of S^3.

To see it is a minimal submanifold of S^3 it suffices to show it is totally geodesic Lemma 9.6. In other words, we may equivalently show that $\alpha \equiv 0$ or that the geodesics of this torus are geodesics of $\mathbb{R}^n$. However, this time it is not so immediate what are the geodesics of T^2.

The other way to do this, following [dC79], Chapter VI, Exercises 8 (see also Exercise 9.12) is computing the shape operator of the submanifold with respect to $\mathbb{R}^4$ first, which is easy because Euclidean space is flat, and then computing the shape operator with respect to S^3. To do this we just need to compute inner products of the form $\langle A_{N_k} E_i, E_j \rangle$ since these are exactly the entries of A_{N_k} in the basis E_i (cf. Eq. 9.1.5).

Strategy: we need the following ingredients: a point p in the submanifold, a basis E_i of the tangent space of the submanifold and a basis N_k of the normal space. For every basic normal vector N_k we compute its shape operator as a matrix. Then we pass to S^n. We should only consider the normal vectors N_k that are tangent to S^n—this way they are normal to the subsubmanifold Σ as a submanifold of S^n. Then the trace of the shape operator with respect to those normals should be the mean curvature.

Differentiating $\mathbf{x}$ as a local parametrization we see that

$$e_1 = \partial_\theta = (-\sin\theta, \cos\theta, 0, 0), \qquad e_2 = \partial_\varphi = (0, 0, -\sin\varphi, \cos\varphi)$$

are an orthonormal basis of T_pT^2, while

$$n_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \qquad n_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)$$

are an orthonormal basis of $T_p^\perp T^2$. Only n_2 is tangent to S^3 at p since n_1 is in fact the position vector p, which is normal to S^3.

Now observe that by Eqs. 9.1.1 and 9.0.1,

$$\langle A_{n_2} e_i, e_j \rangle = \langle n_k, \alpha(e_i, e_j) \rangle = \left\langle n_k, \nabla^{\mathbb{R}^{n+1}}_{e_i} e_j - \nabla^{T^2}_{e_i} e_j \right\rangle$$

Now

$$\nabla^{\mathbb{R}^{n+1}}_{\partial_\theta} \partial_\theta = -\cos\theta\partial_\theta - \sin\theta\partial_\theta$$

so that

$$\langle n_2, -\cos\theta\partial_\theta - \sin\theta\partial_\theta \rangle = \frac{1}{\sqrt{2}}$$

An analogous computation for ∂_φ shows that

$$A_{n_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

□

Here are some comments after reading [Jos13], Section 5.4. Let $f : M \to N$ be an isometric immersion. This allows to use the vectors

$$e_\alpha := f_* \left(\frac{\partial}{\partial x^\alpha} \right) = \frac{\partial f^i}{\partial x^\alpha} \frac{\partial}{\partial f^i},$$

(where f^i are local coordinates of N near $f(x)$ using that f is an isometry) as a basis of $T_{f(x)}N$. Then, by Eq. 9.0.1, checking that $f(M)$ is then a minimal submanifold, i.e. that the trace of its second fundamental form vanishes, is to say that

$$(\nabla^N_{e_\alpha} e_\alpha)^\perp = 0 \tag{9.12.1}$$

where we are implicitly summing over α.

Choosing normal coordinates ∂_α at M, using Eq. 5.5.1 to obtain that the tangent part to the submanifold vanishes at the point, we get

$$
\begin{aligned}
(\nabla^N_{e_\alpha} e_\alpha)^\perp &= \nabla^N_{e_\alpha} e_\alpha \\
&= \nabla^N_{\frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial f^i}} \frac{\partial f^j}{\partial x^\alpha}\frac{\partial}{\partial f^j} \\
&= \frac{\partial^2 f^j}{(\partial x^\alpha)^2}\frac{\partial}{\partial f^j} + \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\alpha}\Gamma^j_{ik}\frac{\partial}{\partial f^j} \quad \text{for } j = 1, \dots, n.
\end{aligned} \tag{9.12.2}
$$

And it is claimed that in arbitrary coordinates this is just

$$
\Delta_M f^j + \gamma^{\alpha\beta}(x)\Gamma^j_{ik}(f(x))\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} = 0 \quad \text{for } j = 1, \dots, n. \tag{9.12.3}
$$

(It would be a nice exercise to perform the passage from normal to arbitrary coordinates.)

Definition 9.13. A map of manifolds satisfying Eq. 9.12.3 is called *harmonic*.

Notice that this is not asking that the coordinate functions of f must be harmonic. Also notice that this says that images of harmonic isometric immersions are minimal by definition.

Lemma 9.14. *Let $f : M \to N$ be a smooth (isometric, right?) submersion of Riemannian manifolds. f is harmonic if and only if its fibers are minimal submanifolds of M.*

Proof. Here's my idea: do the same trick as in Eqs. 9.12.1 and 9.12.2 but now using f^{-1} instead of f, which we may do since f is a submersion. Since it is also an isometry, we get the same result, again with f^{-1} instead of f. Then the claim becomes: f is harmonic if and only f its local inverse is harmonic. I think this might follow from taking the Laplacian of the composition $ff^{-1} = \mathrm{Id}$, which must vanish because the identity is harmonic. A Leibniz rule should appear (modulo some other metric term? see Eq. 2.6.1). □

Now we prove that the images of minimal isometric immersions are critical points of the volume functional.

Exercise 9.15. Consider an isometric immersion $f_0 : (M, g) \to (\tilde{M}, \tilde{g})$ and a vector field $\xi \in f^*T\tilde{M}$.

Define a variation by:

$$
\begin{aligned}
f : (-\varepsilon, \varepsilon) \times M &\longrightarrow \tilde{M} \\
f(s, p) &= \exp_p(s\xi)
\end{aligned}
$$

Then $f_s := f(s, \cdot) : M \longrightarrow \tilde{M}$ is an immersion (isometric at $s = 0$ by hypothesis). Now we define the volume functional, which simply computes the volume of M_s:

$$
S(s) := \int_M \mathrm{Vol}_{f_s^*\tilde{g}} = \int_M f_s^* \mathrm{Vol}_{\tilde{g}}\,.
$$

Compute $S'(0)$ and show that critical points of S are immersed submanifolds with vanishing mean curvature for every normal vector.

Proof. How to express volume in any coordinate system?

$$\sqrt{\det(f_s^* g)_{ij}}dx^1 \wedge \ldots \wedge dx^n$$

Let's differentiate:

$$\begin{aligned}\frac{d}{ds}S(s) &= \frac{d}{ds}\int_M \mathrm{Vol}_{f_s^*\tilde{g}} = \frac{d}{ds}\int_M \sqrt{\det(f_s^* g)_{ij}}dx^1 \wedge \ldots \wedge dx^n \\ &= \int_M \frac{d}{ds}\sqrt{\det(f_s^* g)_{ij}}dx^1 \wedge \ldots \wedge dx^n\end{aligned}$$

We must differentiate the square root of the determinant of a matrix. By Exercise 34.4,

$$\frac{d}{dt}\det A(t) = \det A(t) \cdot \mathrm{tr}\left(A(t)^{-1} \cdot \frac{d}{dt}A(t)\right)$$

and using that we see that

$$\frac{d}{dt}\sqrt{\det A(t)} = \frac{1}{2}\sqrt{\det A(t)} \cdot \mathrm{tr}\left(A(t)^{-1} \cdot \frac{d}{dt}A(t)\right)$$

So we put $A(s) = (f_s^*\tilde{g})_{ij}$. The good news is that square root of the determinant part is exactly the local coordinate function of $\mathrm{Vol}_{f_0^* \mathrm{Vol}_{\tilde{g}}} = \mathrm{Vol}_g$. Then we only have to integrate that other function, which hopefully is related to the mean curvature.

Let's compute

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}(f_s^*\tilde{g})_{ij} &= \frac{d}{ds}\Big|_{s=0} f_s^*\tilde{g}\,(\partial_i, \partial_j) = \frac{d}{ds}\Big|_{s=0}\tilde{g}(f_*^s\partial_i, f_*^s\partial_j) \\ &= \tilde{g}\left(\nabla^f_{\partial_s} f_*^s\partial_i, f_*^s\partial_j\right) + \tilde{g}\left(f_*^s\partial_i, \nabla^f_{\partial_s} f_*^s\partial_j\right)\Big|_{s=0} \\ &= \tilde{g}\left(\nabla^f_{\partial_s} f_*^0\partial_i, f_*^0\partial_j\right) + \tilde{g}\left(f_*^0\partial_i, \nabla^f_{\partial_s} f_*^0\partial_j\right) \\ &= \tilde{g}\left(\nabla^f_{\partial_i} f_*^0\partial_s, f_*^0\partial_j\right) + \tilde{g}\left(f_*^0\partial_i, \nabla^f_{\partial_j} f_*^0\partial_s\right) \qquad \text{symmetry lemma} \\ &= \tilde{g}(\nabla^f_{\partial_i}\xi, f_*^0\partial_j) + \tilde{g}(f_*^0\partial_i, \nabla^f_{\partial_j}\xi)\end{aligned}$$

Where ∇^f is the pullback connection. The point is that we arrived at the variational field ξ. If we assume (for now) that ξ is normal to M then upon differentiation we get:

$$\nabla^f_{\partial_i}\xi = -A_\xi\partial_i + N$$

where the normal component N vanishes since it is orthogonal to the basic tangent vectors of M. We conclude that

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}(f_s^*\tilde{g})_{ij} &= \tilde{g}\left(-A_\xi\partial_i, \partial_j\right) + \tilde{g}\left(\partial_i, -A_\xi\partial_j\right) \\ &= \tilde{g}(\xi, \alpha(\partial_i, \partial_j)) + \tilde{g}(\alpha(\partial_j, \partial_i), \xi) \\ &= 2\tilde{g}(\xi, \alpha(\partial_i, \partial_j)) \qquad \alpha \text{ is symmetric} \\ &= -2\tilde{g}(A_\xi\partial_i, \partial_j)\end{aligned}$$

Now we have to multiply by the inverse matrix, which is just the inverse of the metric in M because we are at $s = 0$. Then we take trace and obtain the mean curvature, defined as the trace of the shape operator:

$$\mathrm{tr}\left(g^{ij}g(A\partial_j, \partial_k)\right) = g^{ij}g(A\partial_j, \partial_i) = g^{ij}g(A_j^\ell\partial_\ell, \partial_i) = g^{ij}A_j^\ell g_{\ell i} = \delta_j^i A_i^j = A_j^j$$

We conclude that

$$\frac{d}{ds}\Big|_{s=0} S(s) = -\int_M H \operatorname{Vol}_g$$

□

Here are two other important equations in the theory of isometric immersions. The *Codazzi equation* is

$$(R_{\nabla^f}(X,Y)Z)^\perp = (\nabla^\perp_X \alpha)(Y,Z) - (\nabla^\perp_{Y},\alpha)(X,Z) \tag{9.15.1}$$

Codazzi equation may help us prove... something cool about bundles I don't quite remember.

And the *Ricci equation* is

$$\langle R_{\nabla^f}(X,Y)\xi,\eta\rangle = \langle R_{\nabla^\perp}(X,Y)\xi,\eta\rangle - \langle [A_\xi, A_\eta]X, Y\rangle \tag{9.15.2}$$

Ricci equation may help prove that the normal bundle of a high codimension submanifold is trivial.

Theorem 9.16 (Fundamental theorem of submanifolds)**.** *See* [Pet16]*, Exercise 3.4.21. Let M be a simply connected Riemannian manifold and $(E^p, \widetilde{\langle\cdot,\cdot\rangle}, \tilde\nabla)$ be a bundle over M with $\tilde\nabla$ compatible with $\widetilde{\langle\cdot,\cdot\rangle}$. Let $\tilde\alpha : \mathfrak{X}(M)\times\mathfrak{X}(M) \to E$ be a symmetric tensor. Suppose there is a constant c such that $(E,\tilde\nabla,\tilde\alpha)$ satisfies Gauss equation for constant curvature* **??***, Codazzi equation 9.15.1 and Ricci equation 9.15.2. Then there exists a unique isometric immersion $f : M \to \mathbb{Q}^{n+p}_c$ and a bundle isomorphism $\varphi : E \to T^\perp_p M$ such that $\varphi\circ\tilde\alpha = \alpha_f$.*

10. Harmonic maps

Goal. Understand harmonic maps $T^2 \to S^3$. Then gauge theory and algebraic geometry (spectral data).

Let M be a compact Riemann surface, G be a compact Lie group with a bi-invariant metric and $f : M \to G$ a smooth map.

Example 10.1 (To keep in mind)**.** $M = T^2$ and $G = \mathrm{SU}(2) \cong S^3$.

Definition 10.2. A *connection* is a map

$$\nabla^A : \Omega^0(M;G) \to \Omega^1(M;E)$$

where $\Omega^p(M;E) = \Gamma(M, \Lambda^p T^*M \otimes E)$.

Consider the pullback bundle of (TG, ∇^{LC}) over M, which we denote by (E,∇^A).

Locally, we can express

$$\nabla^A_{\mathrm{loc}} = d + A$$

where A is the *connection matrix*, a mtrix of 1-forms.

Extend ∇^A to an operator

$$d_A : \Omega^p(M;E) \to \Omega^{p+1}(M;E)$$

by Leibniz rule

$$d_A(\alpha\wedge\beta = d\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge d_A\beta$$

And we have a *curvature* given by

$$F_A = d_A^2 \in \Omega^2(M,;\mathrm{End}(E))$$

Lemma 10.3. $d_A(df) = 0$

Proof. If $\alpha \in \Omega^1(M;E)$ and $X, Y \in \mathfrak{X}(M)$,

$$d_A(\alpha)(X,Y) = \nabla^A_X(\alpha(Y)) - \nabla^A_Y(\alpha(X)) - \alpha([X,Y])$$

So for $\alpha = df\ldots$ (use that LC is torsion-free!) $\square$

Definition 10.4. The *Hodge star operator* is

$$d_A^* = - * d_A *$$

where

$$\begin{aligned} * : \Omega^p(M;E) &\longrightarrow \Omega^{2-p}(M;E) \\ \alpha \otimes s &\longmapsto *\alpha \otimes s \end{aligned}$$

since our manifold is dimension 2.

Definition 10.5.

$$E(f) = \frac{1}{2}\int_M |df|^2 d\mathrm{Vol}$$

Definition 10.6. A *harmonic map* of smooth manifolds $f : M \to N$ is a critical point of the energy functional.

Lemma 10.7. *The critical points of E satisfy*

$$\tau(f) := tr_g(\nabla df) = 0$$

where ∇df is the induced connection on $T^*M \otimes E$, i.e.

$$\nabla(df)(X,Y) = \nabla^A_X(df(Y)) - df(\nabla^g_X Y)$$

Lemma 10.8. *f is harmonic if and only if $d_A^*(df) = 0$.*

Proof. Choose coordinates such that $\nabla^g_X Y$ vanishes at the point. Then

$$\tau(f) = \nabla^A_{e_1}(df(e_1)) + \nabla^A_{e_1}(df(e_2))$$

Let's compute $d_A^*(df) = - * d_A * (df)$. First we find that $*df = e^2 s_1 - e^1 s_2$ for a dual frame e^i and now no longer writing the tensor product. Then we get that

$$d_A(*df) = de^2 \otimes s_1 - e^2 \nabla^A(s_1) -$$

$\square$

Remark 10.9. In the case of a function $f : \Omega \subset \mathbb{C} \to \mathbb{R}$ we obtain the usual Laplacian (Exercise!).

11. Riccati equation

Take any point and any tangent vector at that point. Take the geodesic and take the geodesic sphere of raduis say r with center in the point. It's a hypersurface. Normal vectors are all multiples of the unit normal vector. But who's that, the unit normal? Ah, it's the speed of the geodesic.

So? The normal derivative of any vector field tangent to the sphere with respect to the unit normal $\gamma'(r) := \nu$ is…

$$0 = X\langle \nu, \nu\rangle = 2\left\langle \nabla^\perp_X \nu, \nu \right\rangle$$

that is, normal derivative of ν with respect to X is perpendicular to ν. But normal derivative is normal so it's actually a multiple of ν, which makes ν have norm zero or else $\nabla^\perp_X \nu = 0$.

Now consider everyday Jacobi variation 7.1 $f(s,t) := \exp_p(t(v+sw))$ with Jacobi field $J(t) = d_{tv}\exp_p(tw)$. Differentiate with using the pullback connection along f:

$$\begin{aligned}\nabla^{\gamma}_{\frac{d}{dt}} J(t) &= \nabla^{f}_{\partial_t} f_s = \nabla^{f}_{\partial_t} \partial_s f \\ &= \nabla^{f}_{\partial_s} \partial_t f = \nabla^{f}_{J} \gamma' = -A_{\gamma'} f\end{aligned}$$

using that by our discussion above the normal derivative component vanishes. We conclude that

$$J' = AJ \tag{11.0.1}$$

Finally we differentiate that to obtain

$$J'' = A'J + AJ' = A'J + A^2 J$$

that is,

$$A' + A^2 + R_{\gamma'} = 0 \tag{11.0.2}$$

since we can let $J(r)$ be any vector whatsoever and make this construction work.

Exercise 11.1. Show that $\lim_{s\to 0} sA(s) = \mathrm{Id}$.

Solution. By Rafael. First we present a naive computation. We will use that $A(t)(J(t)) = J'(t)$, $J(0) = 0$ and $J'(0) \neq 0$. Let $\varepsilon > 0$ and let $s > 0$ be such that

$$\frac{\|J(s) - J(0) - sJ'(0)\|}{s} < \frac{\varepsilon}{2}$$

and

$$\|J'(s) - J'(0)\| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}\left\| A(s)(J(s)) - \frac{J(s)}{s} \right\| &= \left\| J'(s) - \left(\frac{J(s) - J(0)}{s} \right) \right\| \\ &\le \|J'(s) - J'(0)\| + \frac{\|J(s) - J(0) - sJ'(0)\|}{s} \\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

This shows that $A = \frac{1}{s}J$ for small s, but we have differentiated as if we were in $\mathbb{R}^n$. A similar computation but picking s such that

$$\frac{|\langle J(s), e_i(s)\rangle - \langle J(0), e_i(0)\rangle - s\langle J'(0), e_i(0)\rangle|}{s} < \varepsilon/2$$

and

$$|\langle J'(0), e_i(0)\rangle - \langle J'(s), e_i(s)\rangle| < \varepsilon/2$$

for a parallel frame e_i along γ allows to conclude as desired. □

12. Hopf-Rinow Theorem

Theorem 12.1 (Hopf-Rinow)**.** *The following are equivalent for any $p \in M$:*

1. *The domain of the exponential map $\exp_p$ is T_pM.*
2. *Closed and bounded subsets are compact.*
3. *(M, d) is a complete metric space.*
4. *M is geodeiscally complete (i.e. every geodesic is maximally defined in all of $\mathbb{R}$).*
5. *Existence of exhaustions.*

Moreover, each of the former conditions imply that there exists a minimizing geodesic joining any two points.

13. Hadamard's theorem

Theorem 13.1 (Hadamard)**.** *If M is a complete simply connected manifold with $K \leq 0$, then the exponential map is a diffeomorphism.*

Proof. □

14. Manifolds of constant sectional curvature

The following theorem and proof where explained by Dimitri Korshunov (IMPA) at Geometric Structures Seminar on July 3, 2025.

Abstract. Crystallographic group is a discrete co-compact subgroup of the group of isometries of a Euclidean space. A theorem of Bieberbach says that it contains a subgroup of translations of finite index. Or, in other words, any compact flat Riemannian manifold is a finite quotient of a flat torus. We will prove this theorem following a streamlined geometric argument of Vinberg.

Theorem 14.1 (Bieberbach)**.** *In particular, any flat manifold is a quotient of a torus.*

Sketch of proof. First we show that it suffices to find only one pure translation, that is, an inductive argument shows that if we have a pure translation, then we can find others. Perhaps this is the hardest part.

Then we obtain some bounds in operator norm for the pure translations, this part looks straightforward. One of these bounds is that $\|[A,B]-\mathrm{Id}\| \leq 2\|\mathrm{Id}-A\|\|\mathrm{Id}-B\|$. The other of the bounds is roughly as follows. For $g = Ax + a$ and $h = Bx + b$, we have that $[g,h] = [A,B]x + c$ for some c whose norm we shall bound by r later.

We use the first bound to show that if two elements in the group are close to the identity, then their commutator will be even closer. This shows that there is a finite set of small generators $g_1, \ldots, g_k$. Each of them is of the form $g_i = A_i x + a_i$.

To arrive at a contradiction suppose that there are no pure translations to obtain a contradiction.

Let $r(g)$ be the norm of the translational part of a transformation g. Let r be the maximum of the translational parts of the g_i. Let $S := \{g : r(g) < r\}$. It's a finite set. Let g_0 be minimal in S. After some discussion we show that the linear part of g_0 commutes with the linear part of g_i for all i. In fact, this implies that g_0 itself is in the center of the group.

Finally we obtain a contradiction with compactness… simple argument. □

15. First Variation Formula

Definition 15.1. The *energy function* of a variation $f : (-\varepsilon, \varepsilon) \times [0, a] \to M$ is

$$E(s) := \int_0^a \langle f_t, f_t \rangle_s$$

Recall Schwarz inequality from functional analysis:

$$\left(\int_0^a fg \right)^2 \leq \int_0^a f^2 \int_0^a g^2$$

Setting $f \equiv 1$ and $g = f_t$ we obtain

$$\ell(\gamma) \leq aE(\gamma) \tag{15.1.1}$$

Theorem 15.2 (First Variation Formula)**.** *If $f : (-\varepsilon, \varepsilon) \times [-0, a] \to M$ is a variation of the geodesic $\gamma : [0, a] \to M$, then*

$$\frac{1}{2}E'(0) = -\int_0^a \langle V, \gamma''\rangle + \langle V, \gamma'\rangle |_0^a + \sum_{i=1}^{k-1} \left\langle V, \gamma'(t_i^-) - \gamma'(t_i^+)\right\rangle \tag{15.2.1}$$

Proof. If γ is smooth,

$$E'(s) = \int_0^a \langle f_t, f_t\rangle_s = 2\int_0^a \langle f_{ts,t}\rangle = 2\int_0^a \langle f_s, f_{tt}\rangle - \langle f_s, f_t\rangle |_0^a$$

using that

$$\langle f_s, f_t\rangle_t = \langle f_{st}, f_t\rangle + \langle f_s, f_{tt}\rangle$$

If γ is not smooth, the integral of $\langle f_s, f_t\rangle_t$ must be done part by part, obtaining the sum in Eq. 15.2.1. □

16. Second Variation Formula

Theorem 16.1 (Second Variation Formula)**.** *Let $f : (-\varepsilon, \varepsilon) \times [0, a] \to M$ be a variation of a smooth geodesic γ with perhaps piecewise smooth variation field V. Then*

(16.1.1)

$$E''(0) = \int_0^a |V'| - \int_0^a R_{\gamma'}V + \langle \gamma'(t), \nabla_{\partial_s} V(t)\rangle |_0^a + \sum_i \left\langle V(t_i), V'(t_i^-) - V'(t_i^+)\right\rangle$$

Proof.

$$E''(s) = 2\left(\int_0^a \langle f_s, f_{tt}\rangle - \langle f_s, f_t\rangle |_0^a\right)_s = \int_0^a \langle f_{ss}, f_{tt}\rangle + \int_0^a \langle f_s, f_{tts}\rangle - \langle f_{ss}, f_t\rangle |_0^a - \langle f_s, f_{ts}\rangle |_0^a$$

Now substitute

$$R_{\gamma'}V = R(V, \gamma')\gamma' = f_{tts} - f_{tst}$$

and

$$\langle f_s, f_{ts}\rangle |_0^a = \int_0^a \langle f_s, f_{ts}\rangle_t = \int_0^a \langle f_{st}, f_{ts}\rangle + \int_0^a \langle f_s, f_{tst}\rangle$$

If the variational field is not differentiable we must be careful when applying the fundamental theorem of calculus. □

Definition 16.2. The *index form* is

$$I_a(V, W) := \int_0^a \langle V'W'\rangle - \int_0^a \langle R_{\gamma'}V, W\rangle \tag{16.2.1}$$

for any $V, W \in \mathfrak{X}_\gamma$.

Remark 16.3. Using that the connection is metric,

$$\langle V, W'\rangle' = \langle V', W'\rangle + \langle V, W''\rangle$$
$$\implies \langle V', W'\rangle = \langle V, W'\rangle' - \langle V, W''\rangle$$

and substituting in the definition we obtain that

$$
\begin{aligned}
I_a(V,W) &= \int_0^a \langle V, W'\rangle' - \int_0^a \langle V, W''\rangle - \int_0^a \langle R_{\gamma'}V, W\rangle \\
&= \langle V, W'\rangle\,|_0^a - \int_0^a \langle V, W''\rangle - \int_0^a \langle R_{\gamma'}W, V\rangle \qquad (16.3.1) \\
&= \langle V, W'\rangle\,|_0^a - \int_0^a \langle V, W'' + R_{\gamma'}W\rangle
\end{aligned}
$$

17. Bonnet-Myers Theorem

Theorem 17.1 (Bonnet-Myers)**.** *If M is a complete manifold with Ric $\geq \frac{1}{r^2}$, then diam$M \leq \pi r$. In particular, M is compact.*

Sketch of proof. Let $p, q \in M$ be points with $d(p,q) > \pi r$. We will construct a Jacobi field vanishing at the endpoints, so that these points are conjugate and thus any geodesic γ joining them cannot be minimizing.

Let w_i be vectors in T_pM which along with $\gamma'(0)$ form an orthonormal basis. Define W_i to be the parallel transport of w_i along γ and $J_i := \sin(\pi t/\ell)W_i$.

Computing the second variation formula, which reduces to the index of J_i, which we may write as in Eq. 16.3.1 to obtain

$$\frac{1}{2}E''(0) = -\int_0^\ell \langle J, J'' - R_{\gamma'}J\rangle \approx \int_0^\ell \sin(\pi t)(1 + K(W_i, \gamma'(t))$$

Summing over all indices i we arrive at the Ricci curvature, which must be greater or equal than $1/r^2$ by hypothesis. In comparing with the length of the curve $\ell > \pi r$, we find that the integrand must be negative, which forces one of the indices to be negative. □

Lemma 17.2. [dC79]*, Chapter IX, Corollary 3.2. Let M be a complete Riemannian manifold with Ric $\geq \delta > 0$. Then the universal covering of M is compact and M has finite fundamental group.*

Proof. The universal cover with the pullback metric satisfies the same curvature hypothesis as M, making it compact by Bonnet-Myers Theorem 17.1. To confirm that the deck transformation group is finite notice that if we had an infinite orbit, we could find a limit point whose projection to the base would not admit a regular neighbourhood. □

Exercise 17.3 (Exercício 3)**.** Encontre um exemplo de variedade suave que admite alguma métrica Riemanniana com curvatura escalar positiva, mas não admite uma métrica Riemanniana com Ricci positivo.

Proof. Considere $S^2 \times S^1$. A métrica produto tem curvatura escalar positiva, pois é a soma das curvaturas escalares em cada fator, onde a curvatura escalar de S^2 é constate 1 quanto que a curvatura escalar de S^1 é constante zero. Suponha que $S^2 \times S^1$ admite uma métrica com Ric > 0. Como $S^2 \times S^1$ é compacta, o fibrado unitário dela é compacto. Isso significa que existe uma constante δ tal que Ric $\geq \delta > 0$. O recobrimento universal de $S^2 \times S^1$ com a métrica de recobrimento satisfaz as mesmas hipóteses de curvatura, de modo que podemos aplicar o teorema de Bonnet-Myers para concluir que é compacto e que o grupo de transformações de coberta, ou seja, o grupo fundamental de $S^2 \times S^1$ é finito. □

18. Weinstein's theorem

Theorem 18.1 (Weinstein)**.** *Let M^n be oriented, compact, $K > 0$ and $f \in Iso(M)$. Suppose that if n is even, f preserves orientation, and if n is odd, f reverts orientation. Then f has a fixed point.*

Proof. A fórmula da segunda variação é a fonte de inspiração, olhe:

$$E''(0) = \int |\dot{V}|^2 - \int \langle R_{\dot{\gamma}} V, V \rangle + \langle \dot{\gamma}, \nabla_{\partial_s} f_s \rangle \,|_0^a$$

e pense: o que preciso para tirar algo bacano dessa formulinha?

De alguma maneira nos damos conta de que isso tem a ver com os pontos fixos de uma isometria f. Suponha que f não tem pontos fixos e busquemos uma contradição. Defina

$$g : M \longrightarrow \mathbb{R}$$
$$g(q) = d(q, f(q))$$

então como M é compacta e essa função é contínua, ela tem um ponto mínimo que chamamos de p. Temos que:

$$0 < g(p) = \min g \leq g(q) \qquad \forall q.$$

Vamos usar a segunda fórmula da variação para achar um ponto que avaliado em g fica menor do que $g(p)$. Seja γ a geodésica minimizante entre p e $f(p)$ parametrizada por comprimento de arco (que existe porque M compacta implica completa).

Agora pegue um vetor $v \in T_pM$ unitário e ortogonal a $\dot{\gamma}$ (para que o termo com R na fórmula da seg. var. fique K), transporte ele paralelamente ao longo de γ para obter V, e pegue a variação por geodésicas

$$h(s,t) = \exp_{\gamma(t)}(sV(t)) = \gamma_{V(t)}(s).$$

Então a 2nda fórmula fica beleza: como o campo é paralelo, o termo $|\dot{V}|^2$ morre, o termo da curvatura fica K e o termo com $\nabla_{\partial_s} f_s$ morre também (porque as curvas verticais são geodésicas).

Então como estamos em $K > 0$, acaba que $E''(0) < 0$, então existe uma vizinhança de 0 onde as curvas $f(s,t)$ tem menor energia que $f(0,t)$ (para toda s nessa vizinhança). Agora lembre que a $\ell^2 \leq dE$, e que no caso de geodésicas minimizantes se da a igualdade. Então fica que

$$\frac{1}{d}\ell^2(c^s) \leq E(s) < E(0) = \frac{1}{d}\ell(\gamma)^2 = \frac{1}{d}d(p, f(p))^2 = \frac{1}{d}(\min g)^2$$

lado esquerdo. Defina $\beta(s)$ como a curva "vertical" no tempo 0, ie. $\beta(s) = f(s, 0)$. Queremos mostrar que

$$\frac{1}{d}d(\beta(s), f(\beta(s))^2 \leq \frac{1}{d}\ell^2(c^s)$$

Já que por definição, do lado esquerdo temos $\frac{1}{d}g(\beta(s))^2$, dando uma contradição com a equação anterior. Para confirmar essa desigualdade e concluir a prova só devemos mostrar que a curva c^s liga $\beta(s)$ e $f(\beta(s))$, pois a distância entre esses dois pontos é menor do que o comprimento de qualquer curva que liga esses pontos.

Ou seja, queremos que

$$c^s(0) = \beta(s), \qquad c^s(d) = f(\beta(s)) \qquad \forall s$$

Ou seja, queremos que

$$f \circ \beta = \gamma_{V(d)}$$

Note que $\beta(s) = \gamma_{V(0)}(s)$. Então derive em $s = 0$:

$$f \circ \beta = f \circ \gamma_{V(0)} = \gamma_{V(d)} \iff f_{*,p}V(0) = V(d) = PV(0)$$

onde P é o transporte paralelo ao longo de γ de $p = \gamma(0)$ a $\gamma(d)$. Isso implica que queremos que

$$P^{-1} f_{*,p} V(0) = V(0)$$

Com um desenho muito convincente (ver Manfredo p. 225, esse agumentinho tá tranquilo) fica que como f é uma isometria

$$f_{*,p}\gamma'(0) = \gamma'(d)$$

Decorre daí que $P^{-1} f_{*,p}\gamma'(0) = P^{-1}\gamma'(d) = \gamma'(0)$. Então podemos fixar nossa atenção em

$$A := P^{-1} f_{*,p}|_{\gamma'(p)^{\perp}} \in \mathsf{O}(n-1)$$

Nosso objetivo é analisar o que acontece quando A tem um ponto fixo, $V(0)$, pois essa é uma condição que queremos que seja verdade para que $f \circ \beta = \gamma_{V(d)}$.

Agora segue o argumento dos eigenvalores que diz assim. Defina $A := P^{-1} f_{*,p}$. Suponha que A tem um ponto fixo. Os autovalores vem dados assim:

$$\operatorname{Spec} A = \{\underbrace{\lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}}_{\text{par}}, -1, \dots, -1, 1, \dots, 1\}$$

Ou seja, os eigenvalores que não são 1 ou -1 não afetam o determinante. Daí concluímos que

- Se $n-1$ é ímpar, f preserva orientação. Por que? Deixa os complexos do lado que não importam. Então pensa: se só tenho -1s, então tenho uma quantidade ímpar de -1s porque a dimensão é ímpar. Respira. Nesse caso f inverte orientação. Mas: f deve ter pelo menos um 1! Então a quantidade de -1s é par. Então não: f preserva orientação.
- se $n-1$ é par, f reverte orientação.

E essas são as condições que precisamos para que A tenha um ponto fixo e tudo funcione. Então bota isso aí na hipótese do teorema. □

19. Synge's theorem

Theorem 19.1 (Synge). *Let M^n be compact and $K > 0$. Then*

(1) *If n is even,*

$$\pi_1(M) = \begin{cases} 1 & \text{if } M \text{ is orientable} \\ \mathbb{Z}_2 & \text{if } M \text{ is not orientable} \end{cases}$$

(2) *If n is odd, then M^n is orientable.*

Proof. First suppose n is even and M is orientable. Since M is compact we must have that $K \geq \delta > 0$. Now consider the universal cover $\tilde{M}$. Then it also satisfies the curvature bound (with the pullback metric), making it compact as well by Bonnet-Myers 17.1. Now pick a deck transformation, which must be orientation-preserving if we choose the orientation on $\tilde{M}$ (simply connected is always orientable) making the projection orientation-preserving. Then f must have a fixed point by Weinstein's theorem 18.1, which applies since we have shown that $\tilde{M}$ is compact.

However, deck transformations cannot have fixed points unless they are the identity: this is a consequence of the Unique Lifting Property **??**, which is a slightly more general statement than the uniqueness of curve lifts that we used for Klinenberg's lemma exercise **??**. More exactly: given a covering space and two maps from any space into the base, these coincide if they agree at one point only.

If M is not orientable we can show that the two-sheeted orientable cover of M is its universal covering by the same arguments as above. Then the group of deck transformations is the group of two elements $\mathbb{Z}_2$ since the covering is two-sheeted.

Now suppose n is odd and consider the two-sheeted orientable cover of M. Then by Weinstein's theorem we can say that no deck transformation can be orientation reversing. But since M is non-orientable, the single deck transformation generating the group of deck transformations, which is $\mathbb{Z}_2$, is not orientation preserving: if it was, then the quotient of the two-sheeted orientable covering by the action of the deck transformations would induce an orientation on M. □

Exercise 19.2. Encontre um exemplo de variedade suave que admite uma métrica Riemanniana com Ricci positivo, mas não admite uma métrica Riemanniana com curvatura seccional positiva.

Proof. Considere $S^3 \times S^1$. A métrica produto tem $\mathrm{Ric} > 0$, já que ele é a soma dos tensores de Ricci em cada fator de acordo ao Exercício **??** (a curvatura de Ricci de S^3 é 1, enquanto que a curvatura escalar de S^1 é zero). Agora suponha que existe uma métrica com $K > 0$. Então pelo teorema de Synge 19.1, como $S^3 \times S^1$ é compacta, de dimensão par e orientável, o grupo fundamental dela deveria ser trivial, mas não é o caso. □

20. Index lemma

Lemma 20.1 (Index)**.** [dC79] *Chapter X, Section 2, Lemma 2.2. Let $\gamma : [0, a] \to M$ be a geodesic without conjugate points (this means $\gamma(t)$ is not conjugate to $\gamma(0)$ on $(0, a]$). Let $J \in \mathfrak{X}^J_\gamma$ with $\langle J, \gamma' \rangle = 0$ and let V be a piecewise differentiable vector field along γ with $\langle V, \gamma' \rangle = 0$. Suppose $J(0) = 0 = V(0)$ and that there is $t_0 \in (0, a]$ such that $J(t_0) = V(t_0)$. Then*

$$I_{t_0}(J, J) \leq I_{t_0}(V, V)$$

and equality holds if and only if $V = J$ on $[0, t_0]$.

21. Rauch's comparison theorem

We motivate Rauch's comparison theorem with a result from ordinary differential equations.

Theorem 21.1 (Sturm)**.** *Suppose $f, \tilde{f}, K, \tilde{K} : [0, a] \to \mathbb{R}$ are smooth functions satisfying the following conditions. $\tilde{f} > 0$ in $(0, a]$, $f(0) = \tilde{f}(0) = 0$, $f'(0) = \tilde{f}'(0)$ and*

$$f'' + Kf = 0, \qquad \tilde{f}'' + \tilde{K}\tilde{f} = 0.$$

If $K \leq \tilde{K}$, then $f/\tilde{f}$ is non decreasing. Moreover, if there is any point $t \in [0, a]$ where $f(t) = \tilde{f}(t)$, then $f \equiv \tilde{f}$ and $K \equiv \tilde{K}$.

Proof. Since we want to show $f/\tilde{f}$ is non decreasing we are interested in computing the sign of the derivative of this quotient. We thus calculate $f'\tilde{f} - f\tilde{f}'$. We only have information about second derivatives, so we differentiate to obtain

$$(f'\tilde{f} - f\tilde{f}')' = (\tilde{K} - K)f\tilde{f}$$

We may integrate to find that

$$f'\tilde{f} - f\tilde{f}' = \int_0^t (f'\tilde{f} - f\tilde{f}')' = \int_0^t (\tilde{K} - K)f\tilde{f} \tag{21.1.1}$$

If we show that $f \geq 0$ we get our result. So suppose that t_0 is the smallest number $t_0 \in (0, a]$ where f is zero. Then $f'(t_0) < 0$ since before that f is positive, which in turn is because $f'(0) = \tilde{f}'(0)$ and $\tilde{f}$ is strictly positive. But this contradicts Equation 21.1.1 since the integrand was positive up to t_0.

Now suppose that $f(t_0) = \tilde{f}(t_0)$ for some $t_0 \in (0, a]$. Then Equation 21.1.1 vanishes at t_0, and we conclude that $K \equiv \tilde{K}$ in $[0, t_0]$. Somehow this implies that f also equals $\tilde{f}$...

Here's ChatGPT in action: define $h(t) := f(t) - \tilde{f}(t)$. Notice that on $(0, t_0]$ it satisfies $h'' + Kh = 0$ with $h(0) = h'(0) = 0$. But so does constant zero function, hence $h \equiv 0$. To extend to all of $[0, a]$ use uniqueness of ODEs again. □

Theorem 21.2 (Rauch). *Let γ be a geodesic of M and $\tilde{\gamma}$ a geodesic of $\tilde{M}$. Suppose that $J \in \mathfrak{X}^J_\gamma$ and $\tilde{J} \in \mathfrak{X}^J_{\tilde{\gamma}}$ satisfy the following conditions. $J(0) = 0$, $\tilde{J}(0) = 0$, $|J'(0)| = |\tilde{J}'(0)|$ and $\langle J, \gamma' \rangle = \langle \tilde{J}, \tilde{\gamma}' \rangle$. Also,*

$$J'' + KJ = 0, \qquad \tilde{J}'' + \tilde{K}\tilde{J} = 0$$

If $\tilde{\gamma}$ has no conjugate points and $\tilde{K} \geq K$, then $|J|/|\tilde{J}|$ is nondecreasing. Moreover, if $|J|$ and $|\tilde{J}|$ coincide in one point, then $|J| \equiv |\tilde{J}|$ and $K \equiv \tilde{K}$.

Proof. Let $f := |J|^2$ and $\tilde{f} := |\tilde{J}|^2$. I think the main difference with Sturm is that the condition on Jacobi fields is not a condition on the derivatives of the functions f and $\tilde{f}$ we just defined. Note that

$$f' = 2\langle J', J \rangle, \qquad J'' = 2(\langle J', J' \rangle + \langle J'', J \rangle).$$

While we may substitute $-KJ$ on the second bracket, we still have the first one. Perhaps this is the reason to define $U(t) := J(t)/|J(r)|$ and $\tilde{U}(t) := \tilde{J}(t)/|\tilde{J}(r)|$. But wait, what's that r? There's a trick in this proof.

Notice that U and $\tilde{U}$ are Jacobi fields and $|U(r)| = |\tilde{U}(r)|$. □

Proposition 21.3. *Sejam $p \in M$, $\tilde{p} \in \tilde{M}$ e $i : T_pM \to T_{\tilde{p}}\tilde{M}$ uma isometria. Então $f := exp_p^{-1} \circ i \circ exp_{\tilde{p}}$ é uma contração métrica, i.e. $|f_* w| \leq |w|$ para todo $w \in T_pM$.*

Mas ainda, se exp_p^{-1} está definida numa bola totalmente convexa, f é uma contração métrica.

Proof. Write an abitrary $w \in T_pM$ as a Jacobi variational field of the variation $f(s,t) = \exp_p(t(v + sw))$. The resulting field on $\tilde{M}$ is a Jacobi field with respect to iw using that i is an isometry. The conditions of Rauch theorem are satisfied and the desired inequality is obtained.

For the metric result we just integrate along a curve realising the distance and compute. □

22. Moore's theorem

Theorem 22.1 (Moore)**.** *Let $M^n \subset \tilde{M}^{n+p}$ be a compact submanifold of a Hadamard manifold $\tilde{M}$. If $K \leq \tilde{K}$, then $p \geq n$.*

23. Focal points

Definition 23.1. Let M be a submanifold of $\tilde{M}$ and p a point not in M. We say p is a *focal point* of M if there is a geodesic γ orthogonal to M and a Jacobi field $J \in \mathfrak{X}_\gamma^J$ such that

$$J'(0) \in T_{\gamma(0)}M, \qquad J'(0) + A_{\gamma'(0)}J(0) \in T^\perp_{\gamma(0)}M$$

Proposition 23.2. *The focal points of M are the singularities of $exp|_{T^\perp M}$.*

24. Morse index theorem

Definition 24.1 (Index of index form)**.**

$$i(I_a) := \sup\{\dim L \subset \mathfrak{X}_\gamma : I_a|_{L\times L} < 0\}$$

For the proof of Morse Index Theorem **??** we will use the following objects

$$\begin{aligned} \mathcal{V}_a &= \{V \in \mathfrak{X}_\gamma : \text{d.p.p.}, V(0) = 0, V(a) = 0\} \\ \mathcal{V}_t^+ &:= \{V \in \mathcal{V}_t : V(t_i) = 0\} \\ \mathcal{V}_t^- &:= \{V \in \mathcal{V}_t : V|_{[t_j,t_{j+1}]} \in \mathfrak{X}^J_{\gamma|_{[t_j,t_{j+1}]}}\} \end{aligned} \tag{24.1.1}$$

Proposition 24.2. [dC79], *Chapter X, Proposition 2.3. The kernel of the index form i*

$$\operatorname{Ker} I_t := \{V \in \mathcal{V}_t : I_t(V,W) = 0 \forall W \in \mathcal{V}_t\} = V_t \cap \mathfrak{X}_\gamma^J$$

that is, it is composed of smooth Jacobi fields along γ.

Sketch of proof. The point here is to show that if $V \in \operatorname{Ker}$ then it is (1) a Jacobi field in each interval where V is smooth, and (2) smooth in the whole interval.

We use the fact that V is in the kernel to show that (1) the norm of $V - R_{\gamma'}V$ is zero, and (2) the left and right derivatives of V coincide. The first part involves multiplying by a positive function f that vanishes at the singular points; this will make the "singular part" of the index form, i.e. the sum, vanish, so that we get only with the integral part, which is the norm of the Jacobi equation, which will vanish since we are in the kernel. The second part I'm not sure—why do we need to show that V is differentiable, or C^2 (since it's the solution of an ODE). □

Lemma 24.3. [dC79], *Chapter X, Corollary 2.4. The index form is degenerate if and only if the initial and final points of γ are conjugate. In this case, the nullity, i.e. dimension of* $\operatorname{Ker} I_a$ *coincides with the nullity of $\gamma(a)$ as a conjugate point of $\gamma(p)$, that is, the dimension of the space of Jacobi fields vanishing at the endpoints.*

Proof. Just by Proposition 24.2 and by the fact that $\mathcal{V}_a$ is defined as the smooth vector fields vanishing in the endpoints. □

Proposition 24.4. [dC79], *Chapter XI, Proposition 2.5.* $\mathcal{V}_a = \mathcal{V}^+ \oplus \mathcal{V}^-$.

Proof. Assume that the partition $0 < t_1 < \ldots < t_k = a$ of $[0,a]$ is such that every segment $\gamma|_{[t_{i-1},t_i]}$ is contained in a totally geodesic neighbourhood, so that no such segment can have conjugate points.

Let $V \in \mathcal{V}_a$ and pick a field $J \in \mathcal{V}^-$ defined as the piecewise Jacobi field with pairwise boundary conditions $J(t_j) = V(t_j)$. The assumtion that $\gamma|_{[t_{i-1},t_i]}$ has no conjugate points ensures the uniqueness of such J. Then $V - J \in \mathcal{V}^+$ and we obtain the result. $\square$

Theorem 24.5 (Morse Index)**.** *O índice $i(I_t)$ é finito e igual ao número de pontos conjugados em $[0,t)$ ao longo de γ.*

Proof. (1) Define

$$\mathcal{V}_a = \{V \in \mathfrak{X}_\gamma : \text{d.p.p. } V(0) = 0, V(a) = 0\}$$
$$\mathcal{V}_t^+ := \{V \in \mathcal{V}_t : V(t_i) = 0\}$$
$$\mathcal{V}_t^- := \{V \in \mathcal{V}_t : V|_{[t_j,t_{j+1}]} \in \mathfrak{X}^J_{\gamma|_{[t_j,t_{j+1}]}}\}$$

(2) Note que
$$\mathcal{V}_t = \mathcal{V}_t^+ \oplus \mathcal{V}_t^-$$
(3) Note que
$$\mathcal{V}_t^- = \bigoplus_{j=1}^{i-1} T_{\gamma(t_j)}M$$
porque para qualquer escolha de vetores $v_1 \in T_{\gamma(t_1)}M, \ldots, v_{i-1} \in T_{\gamma(t_{i-1})}M$ existe um único campo $J \in \mathcal{V}^-$ tal que $J(t_j) = v_j$, por ser a solução da EDO com condições de contorno.
(4) Usamos Proposition 24.2.
(5) ...

$\square$

Theorem 24.6 (Jacobi's theorem)**.** *Se γ é minimizante, então não pode ter pontos conjugados. More generally, if $\gamma : I \to M$ is a geodesic and $\gamma(a)$ is conjugate to $\gamma(0)$ through γ then for every $\delta > 0$, $I_{a+\delta} \not\geq 0$.*

Proof. Se γ é minimizante, a segunda fórmula da variação é positiva para todo campo em $\mathcal{V} = \{V \in \mathfrak{X}_\gamma : V \text{ d.p.p. }, V(a) = 0, V(b) = 0\}$. Ou seja, a forma do índice é uma forma bilinear positiva, e portanto tem signatura zero, e pelo Teorema do Índice de Morse **??** o número de pontos conjugados é zero. $\square$

It is possible to prove this theorem without using Morse Index Theorem **??** by constructing a variation where the index form is negative, see Exercise **??**.

25. Cut locus

Definition 25.1. O *cut point* de p ao longo de γ é $\gamma(\rho(\gamma))$ onde

$$\rho(\gamma) = \sup\{t : \gamma|_{[0,t]} \text{ é minimizante}\}$$

Proposition 25.2. [dC79] *Chapter XIII, Proposition 2.2. Let γ be a minimizing geodesic joining p and q. Then q is the cut point of p if and only if either of the following conditions hold:*

(1) *q is the first conjugate point of p along γ.*
(2) *There is another, distinct, minimizing geodesic σ joining p and q.*

Sketch of proof. ($\Longrightarrow$) Since $q := \gamma(r)$ is the cut point of p, we know that for every point $\gamma(r+\varepsilon)$ along γ after q there must be some other geodesic γ_ε which realizes the distance between p and $\gamma(r+\varepsilon)$

Taking the limit as $\varepsilon \to 0$ we obtain another geodesic (because we are taking limit inside $S^n \subset T_pM$, so we get convergence; and because "things will behave well under the limit") which puts us in the second condition of the theorem, unless the limit geodesic is the same as γ setwise.

Now there's a critical observation: the two geodesics γ and γ_ε meet at the point $\gamma(r+\varepsilon)$, so you can write $\gamma_\varepsilon(r+\delta(\varepsilon)) = \gamma(r+\varepsilon)$ for some function $\delta(\varepsilon)$ (that could be negative, doesn't matter). But since γ_ε is the minimizing geodesic from $\gamma(0)$ to this point, then $|\delta(\varepsilon)| < \varepsilon$.

Right, the point is that defining

$$u_\varepsilon := (r+\varepsilon)v, \qquad u'_\varepsilon := r+\delta(\varepsilon))v_\varepsilon$$

we see that $\exp_p$ is not injective, since these two vectors hit that point.

This being true for every $\varepsilon > 0$, there is no neighbourhood of vr where $\exp_p$ is injective—much less a diffeomorphism. So it cannot be a local diffeomorphism, and we know that critical points of $\exp_p$ are conjugate points of p. (Remember: if $\exp_p$ has a critical point, its differential has kernel, and the map $d_{tv}\exp_p w$, for w in the kernel, gives a Jacobi field. See Proposition 8.2) ☐

Exercise 25.3. Calcule o diâmetro de S^2, $\mathbb{T}^2$, $\mathbb{R}P^2$.

Solution. O diâmetro de S^2 pode ser calculado via o teorema de Bonnet Myers **??**: nenhuma geodésica é minimizante depois de atingir comprimento πr, e temos uma geodésica que atinge esse comprimento: qualquer uma!

O diâmetro de $\mathbb{T}^2$ é 1. Isso é por simples geometria euclidiana: é o diâmetro do cubo! Por definição, a métrica de $\mathbb{T}^2$ é a induzida pela projeção quociente.

O diâmetro de $\mathbb{R}P^2$ é $\pi/2$. Qualquer geodésica que liga dois pontos a distância $\pi/2$ é minimizante, pois estamos na métrica esférica e podemos pegar cartas esféricas onde dois pontos a essa distância estão contidos. Por outro lado, se tivéssemos dois pontos a distância maior, a geodésica esférica que percorre o ponto antípoda ao inicial, chega no ponto final mais rapidamente que inicial; portanto geodésicas de comprimento maior que $\pi/2$ não são minimizantes. ☐

Lemma 25.4. *If q is the cut point of p through γ, then p is the cut point of p through γ.*

Lemma 25.5.

Proposition 25.6. *The function*

$$\begin{aligned} \rho : T_p^1 &\longrightarrow [0, +\infty] \\ v &\longmapsto \textit{cut point along } \gamma_v \end{aligned}$$

is continuous.

Lemma 25.7. *Because it's the image of the unit sphere in T_pM, which is closed.*

Lemma 25.8.

Lemma 25.9. [dC79] *Chapter XIII, Corollary 2.8. If $q \in M \setminus C_m(p)$, there exists a unique minimizing geodesic joining p and q.*

Proof. By Hopf-Rinow 12.1, we know that there exists a minimizing geodesic joining any two points. If there was more than one such geodesic, by Proposition 25.2 we get that q is the cut point of q along either of these geodesics, a contradiction. □

The former Lemma 25.9 shows that the exponential map $\exp_p$ is injective outside the cut locus of p. This motivates the following definition.

Definition 25.10. The *injectivity radius* of a manifold M is

$$i(M) = \inf\{d(p, C_m(p)) : p \in M\}$$

26. Bishop-Gromov Theorem

Theorem 26.1 (Bishop-Gromov)**.** *Let $p \in M$ and $0 \leq t \leq i(p) = d(p, C_m(p)$. Denote $B_t(p)$ the ball of raduis t centered in p and $B_{t,k}$ the ball of radius t in a space of constant curvature k.*

If $Ric \geq k$, then $Vol(B_t(p))/Vol(B_{t,k})$ is a non increasing function.

Moreover, if there are numbers $0 < s < r \leq i(p)$ such that

$$\frac{Vol(B_s(p))}{Vol(B_{s,k}} = \frac{Vol(B_r(p))}{Vol(B_{r,k})}$$

then $B_r(p) = B_{r,k}$ isometrically.

Proof. Use the exponential map a parametrization of the geodesic sphere with radius r and centre in $p \in M$. (This parametrization works at least locally, but why have we said in lecture that "it's a global chart"?) Then

$$\mathrm{Vol}(S_r^n(p)) = \int_{S_r^n(p)} \mathrm{Vol}_{S_r^n(p)} = \int_{S_r^n(0)} \exp_p^* \mathrm{Vol}_{S_r^n} = \int_{S_r^n(0)} |\det d_{rv} \exp_p| \mathrm{Vol}_{S_r^n(0)}$$

This determinant can be given by a basis of tangent vectors to $S_r(0)$, each of which yields a Jacobi field via Proposition 7.1. These Jacobi fields are the columns of the matrix $\mathbb{J}$, which is the Jacobian matrix of the exponential we are interested in computing.

Now we want to differentiate this with respect to r because we aim to apply Sturm's theorem 21.1. By Exercise 34.4, we know that the derivative of the determinant of a one-parameter family of invertible matrices $\mathbb{J}(r)$ is given by $\det \mathbb{J}(r) \mathrm{tr}(\mathbb{J}^{-1}(r)\mathbb{J}'(r))$.

By the discussion in Section 11, we know that $AJ = J'$ where A is the shape operator with respect to the unit normal $\gamma'(r)$ and J is a Jacobi field defined by 7.1. This gives $A = J'J^{-1}$.

Thediagonal entries of the matrix $\mathbb{J}$ are vectors of this kind, so that its trace, which depends only on the diagonal by Exercise 34.1, is precisely the trace of the shape operator A, which we have defined as $H/(n-1)$ where H is called the mean curvature.

This data translates to the following equation

$$\det \mathbb{J}' = \det \mathbb{J} H \tag{26.1.1}$$

Now recall that any self-adjoint operator can be expressed uniquely as a multiple of the identity plus a symmetric traceless matrix (ref?). This allows to write

$$A = H\mathrm{Id} + A_0 \tag{26.1.2}$$

with A_0 symmetric and traceless. Differentiating Eq. 26.1.2 and using that the shape operator satisfies Riccati equation 11.0.2 we shall obtain that

$$H' + H^2 + \mathcal{R} = 0$$

where $\mathcal{R} := \mathrm{Ric}(\gamma') + |A_0|^2$. Then

$$\det \mathbb{J}'' + \mathcal{R} \det \mathbb{J} = 0, \det \mathbb{J}(0) =, \det \mathbb{J}'(0) = 1 \tag{26.1.3}$$

Notice that in the case of constant curvature k, Eq. 26.1.1 becomes

$$\det \overline{\mathbb{J}}' = \det \overline{\mathbb{J}} k \tag{26.1.4}$$

which in turn yields

$$\det \overline{\mathbb{J}}'' + k \det \overline{\mathbb{J}} = 0 \tag{26.1.5}$$

Now we want to compare the volume of the sphere in M with the volume of a sphere of the same raduis in the space of constant curvature $\mathbb{Q}_k^{n+1}$:

$$\frac{\mathrm{Vol}(S_r^n(p))}{\mathrm{Vol}(S_{r,k}^n)} = \frac{\int_{S^n(0)} \det \mathbb{J}}{\int_{S^n(0)} \det \overline{\mathbb{J}}} = \frac{1}{\mathrm{Vol}(S^n(0))} \int_{S^n(0)} \frac{\det \mathbb{J}}{\det \overline{\mathbb{J}}}$$

By Sturm Theorem 21.1, which we may apply since both $\det \mathbb{J}$ and $\det \overline{\mathbb{J}}$ satisfy Eqs. 26.1.3 and 26.1.5, and moreover they both vanish at $r = 0$ and their derivatives are 1 at $r = 0$ by Exercise 7.5, and of course, since we are supposing that $\mathrm{Ric} M \geq k$, we conclude that the integrand is a non **decreasing** function.

Remark 26.2 (Pregunta). Según Sturm, el cociente de las funciones en cuestión, en este caso, $\frac{\det \mathbb{J}}{\det \overline{\mathbb{J}}}$ debería ser una función no decreciente, pero el resultado que buscamos es que sea no **creciente**.

Now we turn to computing the ratios of the volumes of balls of radius r.

$$\begin{aligned}\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_{r,k})} &= \frac{\int_{S^n(0)} \int_0^r \det \mathbb{J}(t) dt \mathrm{Vol} S^n(0)}{\mathrm{Vol}(S^n(0)) \int_0^r \det \overline{\mathbb{J}}(t) dt} \\ &= \frac{1}{\mathrm{Vol}(S^n(0))} \frac{\int_0^r}{}\end{aligned}$$

Remark 26.3 (Preguntas). I have two questions:

(1) We have said in lecture that we may apply Fubini's theorem by Gauss' lemma. Why? By Gauss' Lemma the normal vectors are orthogonal to the spheres, the constructions related to Riccati equation are valid (cf. Section 11). By why does this allow to use Fubini?
(2) We can pull out the volume in the denominator because the volumes of spheres in the space of constant curvature somehow does not depend on the radius r. But how exactly? I think $\det \overline{\mathbb{J}}(r)$ does depend on r (in the space of constant curvature), see Eq. 26.1.4.

Then the equation continues to

$$
\begin{aligned}
&= \frac{1}{\mathrm{Vol}(S^n(0))}\int_{S^n(0)} \frac{\int_0^r \det \mathbb{J}(t) dt \mathrm{Vol} S^n(0)}{\int_0^r \det \overline{J} dt} \\
&= \frac{1}{\mathrm{Vol}(S^n(0))}\int_{S^n(0)} \frac{\int_0^r \frac{\det \mathbb{J}(t)}{\det \overline{\mathbb{J}}(t)} \det \overline{\mathbb{J}}(t) dt \mathrm{Vol} S^n(0)}{\int_0^r \det \overline{\mathbb{J}} dt} \\
&= \frac{1}{\mathrm{Vol}(S^n(0))}\int_{S^n(0)} \frac{\int_0^r \frac{\det \mathbb{J}(t)}{\det \overline{\mathbb{J}}(t)} d\mu}{\mu[0,r]} \mathrm{Vol} S^n(0)
\end{aligned}
$$

where we have introduced a measure $\mu := \det \overline{\mathbb{J}} dt$.

By our discussion before, the integrand is a non decreasing function, almost as required.

Finally, notice that by the rigidity part on Sturm's Theorem 21.1, the condition

$$\frac{\mathrm{Vol}(B_s(p))}{\mathrm{Vol}(B_{s,k}} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_{r,k})}$$

implies that $\det \mathbb{J} = \det \overline{\mathbb{J}}$ and $k = \mathcal{R}$. This makes $A_0 = 0$ from Eq. 26.1.2 so that A is proportional to the identity. By some mixture of Eqs. 7.2.1 and 11.0.1, this means that the Jacobi fields along γ are of the form Eq. 7.2.2. In turn, this makes the map of Proposition 21.3 an isometry. **Who is i then?** □

27. Cheng's Theorem

Cheng's theorem says what happens in case of equality in the conditions of Bonnet-Myers Theorem 17.1.

Theorem 27.1 (Cheng)**.** *Let M be a complete Riemannian manifold with $Ric \geq 1$. If $diam M = \pi$ then $M = S^n$ isometrically.*

Proof. By Bonnet-Myers Theorem 17.1, M is a compact manifold with diameter π. This means that M is the closure of any ball of radius π. Then, by Bishop-Gromov Theorem 26.1, it's enough to show the rigidity condition holds for radius π. That is, we must show that there is $s < \pi$ such that

$$\frac{\mathrm{Vol}(B_s(p))}{\mathrm{Vol}(B_{s,1}} = \frac{\mathrm{Vol}(B_\pi(p)}{\mathrm{Vol} B_{\pi,1}}$$

Consider two points p_1 and p_2 at distance π. Then the balls of radius $\pi/2$ with center at these points cannot intersect. This says that

$$\mathrm{Vol} M \geq B_{\pi/2}(p_1) + B_{\pi/2}(p_2)$$

But since the volume of M is exactly the volume of $B_\pi(p)$ for any p, we see that

$$\frac{\mathrm{Vol}(B_\pi(p))}{2} \leq B_{\pi/2}(p_i), \qquad i = 1, 2$$

□

28. Toponogov's theorem

Definition 28.1. [dC79], p. 285. A *geodesic triangle* is the set formed by three minimizing geodesics γ_i so that $\gamma_{i+1}(0) = \gamma_i(\ell(\gamma_i))$ cyclically. (Just to note that for some authors geodesic triangles are minimizing by definition. Looks like for [**?**] it's not the case.)

First notice that the following result can be proved using Rauch's theorem 21.2:

Theorem 28.2 (Toponogov, local version)**.** *Let $o, p_1, p_2 \in B$ where B is a totally convex ball of M. Let γ_1 and γ_2 be the normalized geodesics joining o to p_1 and to p_2.*

Suppose $\tilde{o}, \tilde{p}_1, \tilde{p}_2 \in \tilde{B}$ is a triple on another manifold $\tilde{M}$ such that the distances from $\tilde{o}$ to $\tilde{p}_1$ and from $\tilde{o}$ to $\tilde{p}_2$ coincide with those in M.

Then for any $t \in [0, \ell(\gamma_1)]$ and $s \in [0, \ell(\sigma)]$,

$$\tilde{d}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(s)) \leq d(\gamma_1(t), \gamma_2(s))$$

Proof. This is an application of Proposition 21.3 for i as given by the condition of mapping $\gamma_1'(0)$ to $\gamma_2'(0)$ and the angle between them to the corresponding vectors and angle on $\tilde{M}$. $\square$

The global version of that Theorem 28.2 is the following. Notice it coincides with our intuition from comparing plane triangles with spherical ones; spherical are fatter.

Theorem 28.3 (Toponogov, hinge version)**.** *M complete, $K \geq k$. γ_1, γ_2 normalized geodesics with $\gamma_1(0) = \gamma_2(0)$. Suppose γ_1 is minimizing.*

If $k > 0$ suppose additionally that $\ell(\gamma_2) \leq \pi/\sqrt{k}$ (because that's where conjugate points appear).

Suppose that $\tilde{\gamma}_1, \tilde{\gamma}_2$ is a corresponding hinge in $\mathbb{Q}^2_k$, that is, $\ell(\gamma_i) = \ell(\tilde{\gamma}_i)$ and $\angle(\gamma_1'(0), \gamma_2'(0) = \angle\tilde{\gamma}_1'(0), \tilde{\gamma}_2'(0)$. Then

$$d(\gamma_1(\ell_1), \gamma_2(\ell_2)) \leq \tilde{d}(\tilde{\gamma}_1(\ell_1), \tilde{\gamma}_2(\ell_2)) \tag{28.3.1}$$

This is not intuitive for me: the distance between the remaining side should be fatter in the manifold that is more curved.

The following version is equivalent to the hinge version and is intuitive for me:

Theorem 28.4 (Toponogov)**.** *Let M be complete with $K \geq k$. If γ_i is a minimizing geodesic triangle in M, then there is a unique minimizing geodesic triangle $\tilde{\gamma}_i$ in $\mathbb{Q}^2_k$ with $\ell(\tilde{\gamma}_i) = \ell(\gamma_i)$ for $i = 0, 1, 2$, and such that*

$$\tilde{d}(\tilde{\gamma}_1(t), \tilde{\gamma}(s)) \leq d(\gamma_1(t), \gamma_2(s) \tag{28.4.1}$$

for all $t \in [0, \ell(\gamma_1)]$ and $s \in [0, \ell(\gamma_2)]$.

Another equivalent version:

Theorem 28.5 (Toponogov)**.** *Let M be complete with $K \geq k$. If γ_i is a minimizing geodesic triangle in M, then there is a unique minimizing geodesic triangle $\tilde{\gamma}_i$ in $\mathbb{Q}^2_k$ with $\ell(\tilde{\gamma}_i) = \ell(\gamma_i)$ for $i = 0, 1, 2$ and such that*

$$\tilde{d}(\tilde{o}, \tilde{\gamma}(t)) \leq d(o, \gamma_0(t)) \qquad \forall t \in [0, \ell(\gamma_0)] \tag{28.5.1}$$

All these versions are equivalent and we will only prove the following one:

Theorem 28.6 (Toponogov, metric version)**.** *Let M be a complete manifold and $K \geq k$. Let $o, p_1, p_2 \in M$, distinct points. Let γ_1 and γ_2 be normalized geodesics joining o to p_1 and to p_2. Suppose γ_1 is minimizing (in the prove we show that γ_2 is minimizing too). Let γ_0 be a nonconstant geodesic joining p_1 to p_2 and suppose*

$$\ell(\gamma_0) \leq \ell(\gamma_1) + \ell(\gamma_2)$$

(which is a weaker condition than asking that γ_0 is minimizing as well).

If $k > 0$ suppose additionally that $\ell(\gamma_0) \leq \pi/\sqrt{k}$.

Then there exists a geodesic triangle $\{\tilde{\gamma}_i : i = 1, 2, 0\}$ in $\mathbb{Q}_k^2$ such that $\ell(\tilde{\gamma}_i) = \ell(\gamma_i)$ and

$$\tilde{d}(\tilde{o}, \tilde{\gamma}_0(t)) \leq d(o, \gamma_0(t)) \qquad \forall t \in [0, \ell(\gamma_0)] \tag{28.6.1}$$

Outline of proof. This is the one we proved in class. Define distance functions $\rho := d(\sigma, \cdot)$ and $\tilde{\rho} := \tilde{d}(\tilde{\sigma}, \cdot)$. □

Lemma 28.7. *If equality holds in Eq. 28.3.1, then γ_1 and γ_2 are the boundary of a totally geodesic surface $\Delta \subset M$ with $K_\Delta \equiv k$.*

Proof. Using the map from Cartan's lemma. Use Rauch's theorem 21.2 to compare M with $\mathbb{Q}_k$, and again to compare Δ with M. As you might expect, use Gauss' equation 9.1.2 to show that the second fundamental form of Δ vanishes. The essence of the proof is that $\alpha(\gamma', \gamma') = 0$ for the geodesics emanating from p, which is more or less straightforward, and then showing that also $\alpha(\sigma', \sigma') = 0$ for geodesics emanating from another vertex of the triangle, which requires to "look at things from another point of view". □

Theorem 28.8 (Toponogov, angle version)**.** [CEA08] *Theorem 2.2.(A). Let M be a complete manifold with $K_M \geq H$.*

Let $(\gamma_1, \gamma_2, \gamma_3)$ determine a geodesic triangle in M. Suppose γ_1, γ_3 are minimal and if $H > 0$, suppose $\ell(\gamma_2) \leq \frac{\pi}{\sqrt{H}}$. Then M^H, the simply connected 2-dimensional space of constant curvature H, there exists a geodesic triangle $(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$ such that $\ell(\gamma_i) = \ell(\tilde{\gamma}_i)$ and $\tilde{\alpha}_1 \leq \alpha_1$, $\tilde{\alpha}_3 \leq \tilde{\alpha}_3$.

Except in case $H > 0$ and $\ell(\gamma_i) = \frac{\pi}{\sqrt{H}}$ for some i, the triangle in M^H is uniquely determined.

29. Gromov's theorem

Theorem 29.1 (Gromov)**.** *M^n complete, $K \geq 0$, then π admits a set of 3^n generators. In particular, it is finitely generated.*

Proof. Consider the universal cover $\tilde{M}$ of M with the pullback metric. Fix a point $x \in \tilde{M}$ and consider the set

$$\{f \in \Gamma : d(x, f(x)) < r\}, \qquad r > 0$$

By **??** every point must have a disjoint neighbourhood … □

30. Cartan's Theorem

Definition 30.1. The *free homotopy set* is the set of homotopy classes of loops based on any point of M.

Definition 30.2. A *geodesic loop* is a geodesic whose starting and ending points are the same. It may not be differentiable at such point.

Definition 30.3. A *closed geodesic* a smooth geodesic loop.

Theorem 30.4 (Cartan)**.** *Let M^n be compact, and $\omega \in \hat{\pi}_1 M$ nontrivial. Then there is a closed geodesic γ such that $[\gamma] = \omega$.*

Proof. Consider a sequence of loops whose lengths converge to the number

$$\ell := \inf\{\ell(c) : c \in \omega\} > 0$$

Consider a sequence c_n of curves in ω such that their lengths converge to ℓ. Then apply Arzelà-Ascoli Theorem **??** to ensure they converge to some curve c and show this is a geodesic in ω.

First suppose that each of the c_n is a piecewise geodesic parametrized by arclength (why can we do this?). We may then pick the supremum L of all the lengths since we are assuming that their lengths decrease. Then

$$d(c_n(t), c_n(s)) \leq \ell\left(c_n|_{[t,s]}\right) \leq N|t-s|$$

This says that $\{c_n\}$ is equicontinuous (the same N works for all n) as a family of maps from □

31. Preissman's Theorem

Definition 31.1. An isometry $f \in \mathrm{Iso}(M)$ is called a *translation* if there exists a geodesic of M such that $f(\gamma) = \gamma$. That is, $f \circ \gamma$ is a reparametrization of γ.

Theorem 31.2 (Preissman)**.** *Let M^n be compact with $K < 0$. If $1 \neq H \subset \pi_1(M)$ is abelian, then $H = \mathbb{Z}$.*

Theorem 31.3 (Preissman)**.** *If M is compact and $K < 0$, then $\pi_1(M)$ is not abelian.*

32. Byers' Theorem

Lemma 32.1. *Let M^n be a complete manifold with $K < 0$. Suppose that every element of Deck(π), where π is the universal covering, fixes the same geodesic $\tilde{\gamma}$. Then M is not compact.*

Theorem 32.2 (Byers)**.** *Let M be compact, $K < 0$ and H soluble subgroup of $\pi_1(M)$, $H \neq \{e\}$. Then H is cyclic infinite. Moreover, $\pi_1(M)$ does not have any cyclic subgroup of finite index.*

33. Cheeger-Gromoll Siplitting Theorem

Theorem 33.1 (The Strong Maximum Principle)**.** [Pet16] *Theorem 7.1.7. If $f : (M, g) \to \mathbb{R}$ is continuous and subharmonic, then f is constant in a neighbourhood of every local maximum. In particular, if f has a global maximum, then f is constant.*

Proof. If $\Delta f > 0$ then we can find a support function. Notice that Prof. Florit shows the theorem for functions satisfying $\Delta f \leq 0$. □

The following lemma follows from the lemma above and gives a bound on the Laplacian of the distance function.

Lemma 33.2 (Calabi)**.** *If Ric ≥ 0, for $\rho := d(p, \cdot)$, it holds that $\Delta\rho \leq (n-1)/\rho$ on $M \setminus C_m(p) \cup \{p\}$.*

Theorem 33.3 (Cheeger-Gromoll). *Let M be complete with $Ric \geq 0$. If M has a line, then M is isometric to $N \times \mathbb{R}$ for some Riemannian manifold N.*

34. Exercises: basic constructions

Exercise 34.1. Show that the trace is independent of coordinates.

Solution. Let A be an endomorphism of a vector space. By definition

$$\mathrm{tr} A = A_i^i$$

Now if P is any invertible matrix,

$$\mathrm{tr} PAP^{-1} = P_i^j A_j^k \overline{P}_k^i = P_i^j \overline{P}_k^i A_j^k = \delta_k^j A_j^k = A_j^j$$

where $\overline{P}$ denotes the indices of the inverse of P. □

The following two lemmas are preparatory results for Exercise 34.4.

Lemma 34.2. *Let $\gamma_1, ..., \gamma_m : (a,b) \to \mathbb{R}^n$ be smooth curves and define $f : (a,b) \to \mathbb{R}$ by $f(t) = \det(\gamma_1(t), \ldots, \gamma_n(t))$. Then*

$$f'(t_0) = \sum_{k=1}^{m} \det(\gamma_1(t_0), \ldots, \gamma_k'(t_0), \ldots, \gamma_m(t_0)).$$

Proof. This follows from multilinearity of the determinant using the definition of derivative of a map $g : \mathbb{R}^n \to \mathbb{R}^m$ as given by

$$g(x+h) - g(x) = L(x)h + \varepsilon(x;h)$$

where ε is $o(h)$, meaning that $\lim_{h\to 0} \frac{\varepsilon(x;h)}{h} = 0$.

More exactly, since γ_k are smooth we may write

$$\gamma(t_0 + h) = \gamma_k(t_0) + h\gamma_k'(t_0) + \varepsilon_k(h)$$

We substitute into the determinant... □

Lemma 34.3. *Let $M \in Mat_{n\times n}(\mathbb{R})$ be any matrix. Let $f : \mathbb{T} \to \mathbb{R}$ be defined by $f(t) := \det(Id + tM)$. Then $f'(0) = tr(M)$.*

Proof. This follows easily from Lemma 34.2. □

Exercise 34.4. Let $A(t)$ be a one parameter family of matrices. Compute $\det A(t)'$.

Naive solution. The determinant of a matrix is given by the product of its eigenvalues λ_i. Taking logarithms we see that

$$\log \det A(t) = \log \prod_i \lambda_i = \sum_i \log \lambda_i$$

and differentiating we obtain

$$\frac{\det A(t)'}{\det A(t)} = \sum_i \frac{\lambda_i'}{\lambda_i} = \mathrm{tr} A'(t) A(t)^{-1}.$$

with the caveat that we must confirm whether the functions λ_i are differentiable. (Still pending...) □

Solution. Consider the determinant as a function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$. Is total derivative at $A \in \mathrm{GL}(n, \mathbb{R})$ is a linear map $\mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ which when evaluated at a matrix $H \in \mathrm{GL}(n\mathbb{R})$ is defined by

$$\det(A)'H = \lim_{t\to 0} \frac{\det(A + tH) - \det(A)}{t}$$

Now define $f(t) := \det(A + tH)$ so that the latter equation reads $f'(0)$. Then observe that

$$f(t) = \det(A + tH) = \det(A(\mathrm{Id} + tA^{-1}H) = \det(A)\det(\mathrm{Id} + tA^{-1}H).$$

By Lemma 34.3 we conclude that $f(t) = \det(A)\mathrm{tr}(A^{-1}H)$. Letting $H = A(t)'$ we obtain the result since by the chain rule $\det(A(t))' = \det(A(t))'A(t)'$. □

Exercise 34.5. Let P and Q be Riemannian manidols. Sow that the Levi-Civita connection of the product metric is given by $\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^P_{Y_1}X_1 + \nabla^Q_{Y_2}X_2$.

Exercise 34.6. Let P and Q be Riemannian manifolds. Show that for any $p \in P$ and $q \in Q$, $P \times \{q\}$ and $\{q\} \times P$ are totally geodesic submanifolds of $P \times Q$.

Solution. Let $v \in T_{(p,q)}(P \times \{q_0\})$ and consider the geodesic $\exp_p^{P\times Q}(tv)$. Then its acceleration vanishes since the Levi-Civita connection of the product metric is given by Exercise 34.5, and the component along Q vanishes. □

35. Exercises: global differential geometry

Exercise 35.1. [dC79], Chapter X, Exercise 1. Seja M uma variedade Riemanniana completa com curvatura seccional $K \leq K_0$ onde K_0 é uma constante positiva. Sejam $p, q \in M$ e seja γ_0 e γ_1 duas geodésicas distinas unindo p a q com $\ell(\gamma_0) \leq \ell(\gamma_1)$. Admita que γ_0 é homotópica a γ_1, isto é, existe uma família contínua de curvas α_t, $t \in [0, 1]$ tal que $\alpha_0 = \gamma_0$ e $\alpha_1 = \gamma_1$. Prove que existe $t_0 \in [0, 1]$ tal que

$$\ell(\gamma_0) + \ell(\alpha_{t_0}) \geq \frac{2\pi}{\sqrt{K_0}}$$

Exercise 35.2. Use Klingenberg lemma from Exercise 35.1 to prove Hadamard's theorem.

Proof. Suponha que M é uma variedade completa, simplesmente conexa e com curvatura positiva. Por completitude, sabemos que para qualquer $p \in M$ o domínio de $\exp_p$ é todo T_pM. Como $K \leq 0$, sabemos que M não pode ter pontos conjugados, de modo que $\exp_p$ é um difeomorfismo local. Também por completitude sabemos que $\exp_p$ é sobrejetiva: qualquer ponto q está ligado a p mediante uma geodésica, e essa geodésica coincide com uma geodésica partindo de p dada como a imagem de uma reta em T_pM sob $\exp_p$.

Portanto, o desafio é provar que $\exp_p$ também é injetiva. Suponha que não é o caso, i.e. considere dois pontos v_1 e v_2 em T_pM tais que $\exp_p v_1 = \exp_p v_2 := q$. As imagens das retas geradas por v_1 e v_2 sob $\exp_p$ são duas geodésicas γ_1 e γ_2 em M ligando p e q.

Como M é simplesmente conexa, existe uma homotopia entre γ_1 e γ_2. Podemos aplicar o lema de Klingenberg para obter uma curva α_{K_0} tal que

$$\ell(\gamma_1) + \ell(\alpha_{K_0}) \geq 2\pi/\sqrt{K_0}$$

para qualquer $K_0 > 0$. Isso mostra que o comprimento das curvas na homotopia não está limitado, o que não é possível já que a homotopia é uma função contínua definida em um compacto, pelo qual a sua imagem debe ser compacto e portanto limitado. **Errado! Por exemplo, um círculo percorrido miles de vezes tem comprimento arbitrariamente grande e está contido num compacto.** □

Exercise 35.3. γ geodésica sem pontos conjugados em $[0, a]$. Então γ é localmente minimizante.

Proof. Temos dois casos:

(1) Se $E''(0) < 0$ usamos o teorema do índice. Isto é, não é possível que $E''(0) < 0$ porque isso implica que o índice da forma do índice é diferente de zero, ou seja, existem pontos conjugados.
(2) Se $E''(0) = 0$,
    (0) Como γ não tem pontos conjugados, então I não tem nulidade em todo $\mathcal{V}$.
    (a) $I_{\mathcal{V}^-}$ não tem índice nem nulidade. Segue do ponto anterior + algum passo na prova.
    (b) $I_{\mathcal{V}^-}$ não tem índice.
    (c) 2+3 dão que $I_{\mathcal{V}^-} > 0$.
    (d) Logo $I > 0$.

□

Exercise 35.4. [dC79] Chapter IX, Exercise 5. Sejam N_1 e N_2 duas subvariedades fechadas e disjuntas de uma variedade Riemanniana compacta.

(1) Mostre que a distância entre N_1 e N_2 é realizada por uma geodésica γ perpendicular a ambas N_1 e N_2.
(2) Mostre que, para qualquer variação ortogonal $h(t, s)$ de γ, com $h(0, s) \in N_1$ e $h(\ell, s) \in N_2$, tem-se para a fórmula da segunda variação a seguinte expressão

$$\frac{1}{2}E''(0) = I_\ell(V, V) + \left\langle V(\ell), A^{(2)}_{\gamma'(\ell)}V(\ell)\right\rangle - \left\langle V(0), A^{(1)}_{\gamma'(0)}(V(0))\right\rangle$$

Proof.
(1) Since both submanifolds are compact, there exists a minimizing geodesic γ joining them. Then we can apply the first variation formula. Consider a variation that fixes γ everywhere but in a small neighbourhood of one of the contact points. We realise by Eq. 15.2.1 that it must be orthogonal. A similar procedure proves the other endpoint also arrives orthogonally.
(2) By the previous item, we know that γ' is orthogonal to the curves $h(0, s)$ and $h(\ell, s)$. Then we can differentiate using the formula for normal sections at the endpoints of γ:

$$\tilde{\nabla}_V\gamma' = A^{(i)}_{\gamma'}V + \nabla^{\perp,i}_V\gamma'$$

When computing $V\langle\gamma', \gamma'\rangle$ we obtain $\left\langle\nabla^{\perp,i}_V\gamma', \gamma'\right\rangle = 0$ since the shape operator is tangent to the submanifolds. If M_1 and M_2 are hypersurfaces we conclude that the normal vector $\nabla^{\perp,i}_V\gamma'$ vanishes since it must be a multiple of γ'. Otherwise I'm not sure why this term would vanish.

Now, other than the Index form, in the Second Variation Formula 16.1.1 we have $\langle \nabla_{\partial_s} V, \gamma' \rangle$. Notice that

$$\langle f_s, f_t \rangle_s = \langle f_{ss}, f_t \rangle + \langle f_s, f_{st} \rangle$$

The left-hand-side will vanish because the variation is orthogonal, giving the equality we want (modulo a sign) since f_{st} is precisely $\tilde{\nabla}_V \gamma'$ by the Symmetry Lemma.

□

Exercise 35.5. Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_M > 0$. Suppose M_1, M_2 are minimal (compact?) complete hypersurfaces. Show that $M_1 \cap M_2 \neq \emptyset$.

Proof. Let γ be a minimizing geodesic joining $p \in M_1$ to $q \in M_2$ and minimizing the distance between M_1 and M_2. Let $e_1, \ldots, e_{n-1} \in T_pM$ be such that along with $\gamma'(0)$ form an orthonormal basis of T_pM. Let E_i be the parallel transport of e_i along γ. Notice that $E_i(\ell)$ is tangent to M_2 since it is orthogonal to $\gamma'(\ell)$ and M_2 is a hypersurface, but it is not immediate that the variational curve will be contained in M_2 if we define the variation using the exponential map. This is essential for the second variational formula to vanish since γ is minimizing among curves joining M_1 and M_2.

We need to produce the variation not from the vector field but from the variational curves. Take rectifying coordinates for M_1 at p. Define the curve c^i as the i-th coordinate curve on the whole neighbourhood of p along γ.

Now take the velocity of c^i for some $t > 0$. Parallel transport this vector along γ until it reaches a rectifying neighbourhood of q. Along the path of γ we can use the exponential to define a variation. Once in the rectifying neighbourhood of q, we must define by hand the variational curves to ensure that the curve at q is contained in M_2.

Suppose that $\gamma(t_0)$ is in the rectifying neighbourhood of q. Consider a smooth function $\varphi : [t_0, \ell] \to \mathbb{R}$ that is zero at t_0 and 1 at ℓ. We can smoothly deform the exponential curves to a curve contained in M_2 at $t = \ell$ by defining

$$c_i^s(t) := (1 - \varphi)\exp_{\gamma(t)} sE_i(t) + \varphi s E_i(t)$$

Notice that the variational vector is orthogonal to M_2 at q, so that we may apply Exercise 35.4. Thus we can express the second variation formula in such a way that when summing over all indices we obtain the mean curvatures of each submanifold, which vanish, thus obtaining a contradiction with the hypothesis that Ric > 0. □

Exercise 35.6. Prove que $M^{2n}, K_M > 0$ compacta, então ela é simplesmente conexa.

Exercise 35.7. M^{2n+1}, $K_M > 0$ compacta $\implies$ orientável.

Proof. I want to show that M^{2n+1} is diffeomorphic to its double orientable cover $\tilde{M}$. Since M is odd-dimensional, so is $\tilde{M}$. □

O exercício anterior usa Exercício 6 da lista 6. Ou, equivalentemente,

Exercise 35.8. Se M^{2n+1} tem $K_M > 0$ e $\gamma : S^1 \to M$ inverta orientação, então é possível encurtar γ na sua classe de homotopia.

Exercise 35.9. Let N, M be compact. Then $N \times M$ does not admit a metric with $K < 0$.

Proof. If both N and M are simply connected then they cannot be compact by Hadamard's theorem. Assume first that both are not simply connected. Then each has a nontrivial homotopy class, and such a pair generates an abelian subgroup of $\pi_1(M \times N)$ that is not isomorphic to $\mathbb{Z}$, a contradiction with Preissman Theorem 31.2.

Now suppose that N is simply connected and M is not. The universal cover of $M \times N$, if it admits a metric with negative sectional curvature, must be a Hadamard manifold, so that $\tilde{M} \times N \cong \mathbb{R}^n$. Then the induced map on cohomology is injective map (**why?**) $\pi^* : H^k(M) \to H^k(\tilde{M} \times N) = H^k(\mathbb{R}^n) = 0$. This means by Poincaré duality that $0 = H^n(M) \cong H_0(M) \neq 0$, a contradiction. □

Exercise 35.10. Show that $S^n \times \mathbb{R}$ does not admite a complete metric with $\mathrm{Ric} > 0$.

Proof. If it did, then by Preissman theorem 31.2 then $S^n \times \mathbb{R} \cong N \times \mathbb{R}$, but the upshot is that that the latter isomorphism is an isometry, and the $\mathbb{R}$ factor is flat, so we can't have strictly positive Ricci. □

Exercise 35.11. Let (M^n, g) be complete and such that there exists a compact set $K \subset M$ such that $M \setminus K \cong \mathbb{R}^n \setminus \tilde{K}$ for some compact set $\tilde{K} \subset \mathbb{R}^n$. If $\mathrm{Ric} M \geq 0$, then $M^n \cong \mathbb{R}^n$.

Proof. Since $M \setminus K$ is isometric to $\mathbb{R}^n \setminus \tilde{K}$, there must be a line (oops! this is not obvious). But suppose for now that there *is* a line on M. Then by the Splitting theorem 33.3 there is an isomorphism $M \cong N \times \mathbb{R}$. (*My original idea:* If we manage to show that N also has a line we would have that $M \cong N_2 \times \mathbb{R}^2$. if we can further do this process and find that all N_i have lines, then we end up with $M \cong N_k \times \mathbb{R}^k$ for some manifold N_k without lines.)

Let's fix some notation. Let $f : M \setminus K \to \mathbb{R}^n \setminus \tilde{K}$ be the given isometry and $\varphi : N^{n-1} \times \mathbb{R} \to M$ the isometry given by the Splitting theorem. Consider the restriction $f \circ \varphi|_{N^{n-1} \times \{a\}}$ for some $a \in \mathbb{R}$. Observe que $N^{n-1} \times \{a\}$ é totalmente geodésica (cf. Exercise 34.6) , e portanto também a sua imagem sob $f \circ \varphi$. Toda subvariedade totalmente geodésica de $\mathbb{R}^n$ é um plano, e acabamos.

However, notice that it is not immediate from the isomorphism f that M has a line. Fix a line L in $\mathbb{R}^n$. Then its inverse image under f is minimizing in $M \setminus K$, but there could be a curve that minimizes distance between some two points of $f^{-1}(L)$ that is not contained in $f^{-1}(L)$.

Suppose that $f^{-1}(L)$ is not minimizing in M. Then there are two points j and $-j$ in $f^{-1}(L)$ and a minimizing geodesic σ joining them and intersecting K. Then the path that joins any point before j and any point after j on σ, before σ intersects K, is a minimizing segment contained in $M \setminus K$ and should thus be contained in $f^{-1}(M)$. This shows that $f^{-1}(K)$ accumulates near K, which is not possible. □

Exercise 35.12. Mostre que $S^1 \times S^1 \times \mathbb{R}$ (**não?**) admite uma métrica de curvatura $K \equiv -1$. *Dica.* Teorema de Preissman 31.2.

Proof. Porque $\mathbb{Z} \times \mathbb{Z}$ é abeliano mas não é isomorfo a $\mathbb{Z}$. □

Exercise 35.13. $\mathbb{R}P^n \times \mathbb{R}P^m$ admite uma métrica com curvatura positiva?

Proof. Se m, n são os dois ímpares, são orientáveis, então o produto é orientável e de dimensão par, e pelo Teorema de Synge 19.1 deve ser simplesmente conexa, absurdo.

Se as duas tem dimensão par, considere o recobrimento duplo orientável, que deve ser compacto (pode usar Bonnet-Myers 17.1), com curvatura positiva e dimensão par, e portanto é simplesmente conexo. Ou seja, trata-se do recobrimento universal. Porém, ele é um recobrimento duplo e portanto a fibra tem dois elementos, e desse jeito o Deck $\cong \pi_1(\mathbb{R}P^n \times \mathbb{R}P^m)$ deve ser $\mathbb{Z}_2$, mas não é.

E se uma é de dimensão par e a outra ímpar... é igualzinho que no caso anterior. □

Exercise 35.14. Considere $S^n, \mathbb{T}^n$ e $\mathbb{C}P^n$. Existem métricas com $K > 0$, $K \leq 0$, $K < 0$, $K \geq 0$, $\mathrm{Ric} > 0$, $\mathrm{Ric} \geq 0$? Faça uma tabela.

Exercise 35.15. Let M^n be a Hadamard manifold and $S_r^{n-1} \subset M$. Mostre que a curvatura média satisfaz $H_S \geq \frac{1}{r}$. *Dica.* A curvatura média da esfera em $\mathbb{R}^n$ é $\frac{1}{r}$.

Proof. Considere uma geodésica que liga p, o centro da esfera, e o bordo dela. Considere um referencial de campos de Jacobi J_i. Lembre que cada um dele satisfaz $AJ = J'$. Então

$$H = \sum_i \left\langle A \frac{J_i}{|J_i|}, J_i \right\rangle =$$

□

Exercise 35.16. M^n compact, $\mathrm{Ric} \geq 0$, then $\tilde{M} = N \times \mathbb{R}^n$ for some compact manifold N, where $\tilde{M}$ is the universal cover of M.

Proof. If $\tilde{M}$ is compact we are done. So suppose it is not. Then there exists a ray. We want to construct a line to use Splitting Theorem 33.3. Consider the fundamental domain of the cover $\tilde{M}$, which exists since M is compact: we may take a finite covering of normal neighbourhoods of M, so that the preimage under π is a compact set containing at least one representative of each class.

Let p_i be a sequence along the ray γ so that p_i goes to infinity. For each of them there exists a deck transformation F_i that puts p_i inside K. Consider the ray $F_i \circ \gamma$. □

Exercise 35.17. Lista 8, Exercício 12. Let M be a complete Riemannian manifold not compact with $K \geq 0$. Let σ be a ray of M. Define for $t > 0$ the *semispace*

$$H_t^\sigma := M \setminus \bigcup_{s>0} \Big(\sigma(t+s), s).$$

Show that H_t^σ is totally convex, that is, is a geodesic segment $\alpha : [0,1] \to M$ has its endpoints in H_t^σ, i.e. $\alpha(0), \alpha(1) \in H_t^\sigma$, then that segment is completely contained in $H_t^\sigma : \alpha([0,1]) \subset H_t^\sigma$. *Hint.* Suppose that it's false and use Toponogov.

Proof. Suppose that there is a geodesic $\gamma : [0,1] \to M$ with $\gamma(0), \gamma(1) \in M$, but it intersects H_t^σ. Consider the triangle joining p, $\sigma(t+s)$ and some other point.

Choose the other point so that you obtain a contradiction using the structure of H_t^σ of embedded balls—if a point is in any ball, it must be in all the next ones. □

36. Lista 3

Exercise 36.1 (Curvas minimizantes)**.** (1) Seja γ uma curva suave por partes parametrizada por comprimento de arco, conectando p a q, pontos de uma variedade Riemanniana (M, g). Mostre que se $d(p, q) = \ell(\gamma)$, então γ é uma geodésica. Dizemos que γ realiza a distância entre p e q.

(2) Suponha que $\gamma, \sigma : [0, 2] \to M$ são geodésicas distintas e satisfazem: $\gamma(0) = \sigma(0) := p$, $\gamma(1) = \sigma(1) := q$, γ e σ realizam a distância entre p e q. Mostre que γ não realiza a distância entre p e $\gamma(1 + s)$ para nenhum $s > 0$.

Proof. (1) Primeiro suponha que γ é suave. Podemos ver que trata-se de uma geodésica usando seu campo de velocidades como campo variacional na primeira fórmula da variação:

$$E'(0) = \int_0^a |\gamma'| = 0 \implies \gamma' \equiv 0$$

Uma geodésica quebrada não pode ser minimizante porque em uma vizinhança totalmente normal (cf. 5.4) do ponto singular podemos achar uma geodésica radial (suave) que acurta a distância entre qualquer ponto antes do ponto singular e qualquer ponto depois do ponto singular.

(2) Se as duas geodésicas se intersectam não tangencialmente, podemos construir um caminho mais curto entre p e $\gamma(1 + s)$ ao longo de σ, a menos de suavizar a quina perto de q. Se as curvas se intersectam tangencialmente devem ser a mesma.

□

37. Lista 5

Exercise 37.1. [dC79] Capítulo XII, Exercício 6. Uma geodésica $\gamma : [0, \infty) \to M$ em uma variedade Riemannana M é um *raio partindo de* $\gamma(0)$ se ela é minimizante entre $\gamma(0)$ e $\gamma(s)$ para todo $s \in (0, \infty)$. Admita que M é completa, não compacta, e seja $p \in M$. Mostre que M contém um raio partindo de p.

Proof. Como M é compacta e completa, ela não pode ser limitada. Então existe uma sequência de pontos p_i tal que $\lim_{i\to\infty} d(p_i, p) = \infty$. Para cada ponto podemos pegar uma geodésica minimizante γ_n ligando p e p_n. Podemos associar a cada geodésica um vetor unitário $v_i \in S^n \subset T_pM$ tal que $\gamma_i(t) = \exp_p(tv_i)$.

Como $S^n \subset T_pM$ é compacta existe um vetor v limite da sequência v_i. A geodésica $\gamma(t) := \exp_p(tv)$ é um raio, pois por continuidade de $\exp_p$ e da distância Riemanniana,

$$d(\gamma(0), \gamma(t)) = d\left(p, \exp_p(tv)\right) = d\left(p, \exp_p\left(t \lim_{i\to\infty} v_i\right)\right) = \lim_{i\to\infty} d(p, \gamma_i(t))$$

Pegando i suficientemente grande, teremos que γ_i é minimizante entre p e $\gamma_i(t)$. Isso significa que $d(p, \gamma_i(t))$ está dada como o comprimento de γ_i até esse ponto, e pegando o limite concluímos que o comprimento de γ é a distância entre $\gamma(0)$ e $\gamma(t)$. □

Definition 37.2. Let (M, g) be a Riemannian manifold. A geodesic $\gamma : \mathbb{R} \to M$ is called a *line* if $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in \mathbb{R}$.

Exercise 37.3. Mostre que toda métrica completa em $S^n \times \mathbb{R}$ admite uma linha.

Proof. Considere o conjunto $S^n \times \mathbb{R} \backslash (S^n \times \{0\})$. Trata-se de um aberto disconexo, e portanto não fechado, nem compacto nem limitado. Dentro desse conjunto podemos pegar duas sequências de pontos p_i e q_i, onde a segunda coordenada de p_i tem signo positivo, e a segunda coordenada de q_i tem signo negativo para toda i. Considere a geodésica γ_i que liga p_i com q_i, que deve passar pela esfera $S^n \times \{0\}$. Então obtemos uma sequência v_i de vetores no fibrado tangente unitário de $S^n \times \{0\}$ dadas como as velocidades das geodésicas γ_i. Como tal fibrado é compacto, achamos um limite v dessa sequência que por um argumento análogo ao exercício anterior, converge a um vetor cuja geodésica associada é uma linha. (γ é minimizante, e como está parametrizada por comprimento de arco obtemos a condição dada na Definição 37.2. □

38. Lista 6

Exercise 38.1. Encontre um exemplo de variedade suave que admite alguma métrica Riemanniana com curvatura escalar positiva, mas não admite uma métrica Riemanniana com Ricci positivo.

Proof. Se o problema for para $\mathrm{Ric} \geq \delta > 0$, poderíamos usar o teorema de Bonnet-Myers 17.1 como segue. Um produto de uma esfera com uma reta tem curvatura escalar positiva, mas não admite uma métrica com $\mathrm{Ric} \geq \delta > 0$ porque não é compacto.

Ideia errada: Considere $S^1 \times S^1$. É claro que na métrica produto temos curvatura escalar positiva (erro! As variedades de dimensão 1 tem tensor de curvatura nulo pela antisimetria nas primeiras (ou últimas) duas entradas!) pelo Exercício **??**. Como é uma superfície, a curvatura de Ricci coincide com a curvatura seccional. (Ricci é o promédio das curvaturas seccionais.) A curvatura seccional é a curvatura Gaussiana. Se $S^1 \times S^1$ tivesse uma métrica com curvatura Gaussiana positiva, pelo teorema de Gauss-Bonnet teria característica de Euler positiva, mas a sua característica de Euler é zero. No nosso curso, aplicamos o teorema de Synge 19.1 para uma variedade orientável de dimensão par com curvatura escalar positiva, que deve ter grupo fundamental trivial, uma contradição com o fato de que $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.

Nem tudo está perdido: considere $\mathbb{R}P^2$. Na métrica induzida pela projeção quociente desde S^2 temos curvatura escalar constante 1, e é uma superfície, de modo que a curvatura seccional coincide com a curvatura de Ricci, aplicamos o teorema de Synge 19.1, mas agora $\mathbb{R}P^2$ não é orientável, então seu grupo deveria ser $\mathbb{Z}_2 \ldots$ pois é.

Depois reparei que esse argumento está totalmente errado: no caso das superfícies nem só a curvatura escalar coincide com a Ricci... também com a seccional!

$S^3 \times S^1$. Então sim: pelo teorema de Synge 19.1, uma variedade orientável de dimensão par com curvatura *escalar* positiva deveria ter grupo fundamental trivial, que não é o caso. Mas já não é verdade que a curvatura de Ricci coincide com a curvatura seccional; poderíamos ter um plano com curvatura seccional negativa e ainda ter Ricci positivo. Temos que descartar a possibilidade de ter um plano com curvatura seccional não positiva. □

Exercise 38.2. Encontre um exemplo de variedade suave que admite uma métrica Riemanniana com Ricci positivo, mas não admite uma métrica Riemanniana com curvatura seccional positiva.

Proof. Considere $S^3 \times S^1$. A métrica produto tem Ric > 0, já que ele é a soma dos tensores de Ricci em cada fator de acordo ao Exercício **??** (a curvatura de Ricci de S^3 é 1, enquanto que a curvatura escalar de S^1 é zero). Agora suponha que existe uma métrica com $K > 0$. Então pelo teorema de Synge 19.1, como $S^3 \times S^1$ é compacta, de dimensão par e orientável, o grupo fundamental dela deveria ser trivial, mas não é o caso. $\square$

Exercise 38.3. Let (M, g) be a complete, connected Riemannian manifold with positive curvature. Let M_1 and M_2 be two totally geodesic submanifolds such that $\dim M_1 + \dim M_2 \geq n$. Show that $M_1 \cap M_2 \neq \emptyset$.

Solução. Suponha que a interseção não é vazia. Então existe uma geodésica minimizante não trivial que minimiza a distância entre A e B. Sabemos que essa geodésica intersecta tanto A quanto B ortogonalmente. Queremos ver que $E''(0) < 0$, ou seja que γ não pode ser minimizante, uma contradição.

Suponha que conseguimos construir uma variação de γ tal que todos os pontos iniciais $f(s, 0)$ estão em A, e o mesmo acontece com os pontos finais, i.e. $f(s, \ell) \in B$. Então as derivadas das curvas $f(s, 0)$ e $f(s, \ell)$ em $s = 0$ são perpendiculares a $\gamma'(0)$ e $\gamma'(\ell)$. Ou seja, temos que $\langle \gamma'(0), \nabla_{\partial_s} V \rangle |_0^\ell = 0$.

Suponha ainda que essa variação tem campo variacional paralelo. Então acabou porque a segunda fórmula da variação diz que

$$E''(0) = -\int_0^\ell \langle R_{\gamma'} V, V \rangle < 0$$

Vamos construir essa variação. O passo inicial é fácil: pegamos qualquer vetor $v \in T_a A$ e transportamos paralelamente ao longo de γ para obter o campo paralelo $V \in \mathfrak{X}_\gamma$. Como A é totalmente geodésica, a variação $f(s, t) := \exp_{\gamma(0)}(sV(0))$ fica dentro de A.

Para concluir basta ver que $V(\ell) \in T_b B$, já que B também é totalmente geodésica. Isso segue da última hipótese. Podemos realizar esse processo para cada vetor básico de $T_a A$, obtendo $\dim A$ vetores linearmente independentes em $T_b B$ (já que o transporte paralelo é uma isometria). Se nenhum deles ficasse em $T_b B$, poderíamos construir um espaço de dimensão $\dim A$ estritamente contido em $T_b M \setminus T_b B$, absurdo pois isso implicaria que $\dim A < \operatorname{codim} B$, porém, $\dim A + \dim B \geq n$.

Edit. Depois de reler a prova do exercício seguinte, parece que não era necessário usar que γ intersecta ortogonalmente A e B, pois a variação que usei foi por geodésicas, então $\nabla_{\partial_s} V = 0$. $\square$

Exercise 38.4. Seja M^{2n} uma variedade Riemanniana de dimensão par, completa, orientável e com curvatura seccional $K > 0$. Seja γ uma geodésica fechada em M de comprimento $\ell(\gamma)$. Mostre que existem curvas livremente homotópicas a γ em M, arbitrariamente próximas de γ, que possuem comprimento menor que $\ell(\gamma)$.

Proof. Considere qualquer vector unitário ortogonal a $\dot{\gamma}$, defina $V \in \mathfrak{X}''_\gamma$ como sendo o transporte paralelo de v ao longo de γ. Note que $\dot{V} = 0$. Defina a variação $f(s, t) = \exp_{\gamma(t)} sV(t)$. Note que $\nabla_{\partial_s} f_s = 0$.

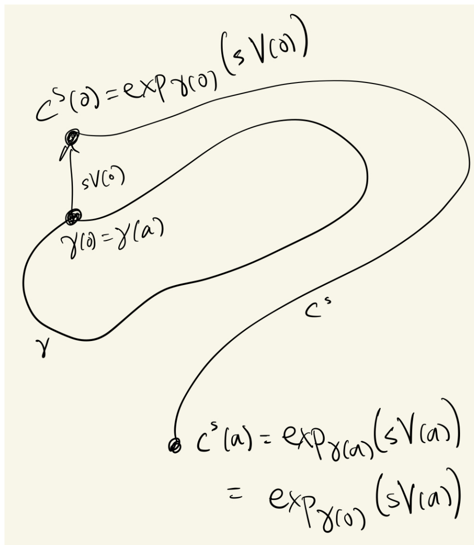
A segunda fórmula da variação nos diz que

$$E''(0) = -\int_0^a \langle R_{\dot{\gamma}} V, V \rangle = -\int_0^a K < 0$$

já que $K > 0$. Como γ é uma geodésica, temos que para s pequeno

$$\frac{1}{a}\ell^2(c^s) \leq E(s) < E(0) = \frac{1}{a}\ell^2(\gamma)$$

Note que essa variação pode não preservar o fechamento das curvas. Para resolver isso olhemos ao seguinte desenho:



Fica claro que se $V(0) = V(a)$ as curvas c^s são fechadas para toda s. Então a condição que precisamos é que

$$P^{\gamma}_{0,a}V(0) = V(a)$$

onde P é o transporte paralelo. Dito de outra forma, basta ver que P tem um ponto fixo—além de $\gamma'(0) = \gamma'(a)$, que já é um ponto fixo. Esse argumento é uma imitação da prova do teorema de Weinstein: como P preserva orientação, o determinante dele deve ser positivo, ou seja, o produto dos seus autovalores deve ser positivo. Restringindo-nos ao subespaço $\gamma'(0)^{\perp}$, que tem dimensão ímpar, concluímos que para que o produto dos autovalores de P seja positivo, deve ter pelo menos um que seja 1. □

39. Lista 7

Exercise 39.1. Seja $\gamma : [0, a] \to M$ uma geodésica em uma variedade Riemanniana M.

Prove que se γ é minimizante, então γ não possui pontos conjugados em $(0, a)$. Encontre um exemplo de geodésica $\gamma : [0, a] \to M$ sem pontos conjugados que não é minimizante.

40. Lista 8

Exercise 40.1. Prop. 2.12 do capítulo XIII, [dC79]. Seja $p \in M$. Suponha exista um ponto $q \in C_m(p)$ que realiza a distância de p a $C_m(p)$. Então:

(1) ou existe uma geodésica minimizante γ de p a q ao longo da qual q é conjugado a p,

(2) ou existem exatamente duas geodésicas minimizantes γ e σ de p a q; além disto, $\gamma'(\ell) = -\sigma'(\ell)$, $\ell = d(p,q)$.

Proof. No final da secção 1 do Capítulo XIII está especificado que as variedades que se consideram no capítulo são completas. Portanto podemos supor que existe uma geodésica minimizante γ ligando p e q.

Então podemos aplicar a Proposição 25.2: um ponto q é o cut point de p ao longo de uma geodésica minimizante γ se e somente se alguma das seguintes condições é verdadeira: (a) q é o primeiro ponto conjugado a p ao longo de γ, ou (b) existem duas geodésicas minimizantes ligando p e q.

Se (a) é verdadeira, terminamos. Se (b) é verdadeira, temos que existe outra geodésica minimizante σ ligando p e q.

Intento pessoal (não funcionou): considere a variação por geodésicas

$$f(s,t) := \exp_{\gamma(t)}(s\exp^{-1}_{\gamma(t)}\sigma(t))$$

cujo campo de Jacobi

$$J(t) = d_{t\exp^{-1}_{\gamma(t)}\sigma(t)}\exp_p(\exp^{-1}_{\gamma(t)}\sigma(t))$$

se anula em $t = 0, 1$. Pensei que esse campo de Jacobi estava bem definido pelo Corolário 2.8, Cap. XIII (Lema 25.9), que assegura que $\exp_p$ é injetiva fora do cut locus de p. Porém, precisaria que a exponencial ao longo de γ, $\exp_{\gamma(t)}$, for injetiva fora do cut locus de p. Essa condição seria garantida se $d(p,q) = i(M)$, ou seja, se a exponencial $\exp_{p'}$ for injetiva na bola de raio $d(p,q)$ para qualquer $p' \in M$. Em conclusão: minha variação não está bem definida. (Se estivesse, com isso terminaria o exercício, pois obtemos que o único caso em que p não é conjugado a q é se $\gamma'(\ell) = -\sigma'(\ell)$, i.e. se o campo de Jacobi associado à variação é nulo.)

A variação certa é produzida no [dC79] supondo que $\gamma'(\ell) \neq -\sigma'(\ell)$; isso implica que existe um vetor $V \in T_qM$ tal que $\langle \gamma'(\ell), V\rangle < 0$ e $\langle \sigma'(\ell), V\rangle < 0$.

Uma maneira simples de conferir a existência desse vetor V é olhando para o plano gerado por $\gamma'(\ell)$ e $\sigma'(\ell)$ (usando que são linearmente independentes). O conjunto de vetores V nesse plano satisfazendo $\langle \gamma'(\ell), V\rangle < 0$ é um semiespaço, e o mesmo acontece com os vetores satisfazendo $\langle \sigma'(\ell), V\rangle < 0$. Esses dois semiespaços devem ter interseção não vazia precisamente porque $\gamma'(\ell) \neq -\sigma'(\ell)$.

Agora usamos o fato de que p e q não são conjugados para achar uma vizinhança do vetor $\ell\gamma'(0)$ onde $\exp_p$ é um difeomorfismo e assim levantar alguma curva r que realize o vetor V ao espaço tangente, digamos $v : (-\varepsilon, \varepsilon) \to T_qM$.

A variação então está dada por

$$f(s,t) := \exp_p\left(\frac{t}{\ell}v(s)\right)$$

Podemos aplicar a primeira fórmula da variação, na qual: o termo com a integral é zero porque γ é uma geodésica; o termo da sumatoria é zero porque trata-se de uma variação suave; e o primeiro termo dos extremos é zero porque a variação fixa o ponto de partida. Obtemos que

$$E'(0) = \langle V, \gamma'(0)\rangle < 0$$

Note que Manfredo escreve a equação anterior usando o funcional de distância. Isso também é válido: de fato a primeira fórmula da variação pode ser deduzida de maneira análoga (feito em sala para variações próprias) para o funcional de distância no caso de geodésicas parametrizadas por comprimento de arco (cf. Teo. 6.3 [Lee19]).

O fato da derivada do funcional de comprimento ser negativa nos diz que para valores perto de $s = 0$ podemos achar curvas com menor comprimento. Mais precisamente, existe um $s > 0$ tal que $\ell(f(s, \cdot)) < \ell(\gamma)$.

É claro que o mesmo procedimento genera uma variação $\tilde{f}$ de σ, e podemos supor que a mesma $s > 0$ faz $\ell(\tilde{f}(s, \cdot)) < \ell(\sigma)$.

Concluímos seguindo o argumento do Professor Manfredo: se $\ell(\gamma_s) = \ell(\sigma_s)$, então temos duas geodésicas com o mesmo comprimento ligando p e $\gamma_s(\ell) = \sigma_s(\ell)$; isso significa que existe um ponto $\tilde{t} \in (0, \ell]$ tal que $\gamma_s(\tilde{t})$ é o cut point de p. Porém, isso contradiz o fato de que q realiza a distância entre p e o seu cut locus.

Finalmente, se $\ell(\gamma_s) < \ell(\sigma_s)$, segue que o cut point de p ao longo de σ_s está a distância menor do que q, que de novo contradiz a nossa hipótese. □

Exercise 40.2. Proposição 2.13, Cap. XIII [dC79]. Se a curvatura seccional K de uma variedade Riemanniana completa M satisfaz

$$0 < K_{\min} \leq K \leq K_{\max},$$

Então

(1) $i(M) \geq \pi/\sqrt{K_{\max}}$, ou
(2) existe uma geodésica fechada γ em M, cujo comprimento é menor do que o de qualquer outra geodésica fechada em M, tal que

$$i(M) = \frac{1}{2}\ell(\gamma)$$

Proof. Suponha que (1) não é verdadeiro.

Note que pelo Teorema de Bonnet-Myers 17.1, M é compacta e portanto $C_m(p)$ é compacto para todo p. Isso nos permite achar dois pontos p e q cuja distância é o raio de injetividade $i(M)$.

Então podemos usar o exercício anterior para p e q. Primeiro vamos ver o que acontece no caso da segunda possibilidade daquele exercício, i.e. que existam exatamente duas geodésicas ligando p e q, digamos γ é σ, tais que $\gamma'(\ell) = -\sigma'(\ell)$ onde $\ell = d(p, q)$. Considere $\gamma * \overline{\sigma}$, onde $*$ é a concatenação de curvas e $\overline{\sigma}$ é a curva σ percorrida em sentido contrário. É claro que $\gamma * \overline{\sigma}$ é uma geodésica fechada de comprimento $2d(p, q)$. Note que por definição $d(p, q) = \ell = i(M)$, como queríamos.

Essa geodésica fechada tem a distância mínima entre as geodésicas fechadas, já que se tivéssemos alguma outra com distância menor, podemos pegar um ponto qualquer nela e o seu cut point ficaria a distância exatamente a metade do comprimento do laço (pois existem duas geodésicas que chegam nele: cada metade do laço partindo em direções opostas do ponto inicial escolhido), contradizendo o fato de que $i(M) = d(p, q)$.

Portanto, basta descartar o primeiro caso do exercício anterior. Para chegar a uma contradição, suponha que p é conjugado a q, i.e. que existe um campo de Jacobi J que se anula em p e em q. Podemos usar o teorema de Rauch 21.2 para comparar esse campo com $\tilde{J}$, um campo em $S^n_{K_{\max}}$, a esfera de curvatura constante $K_{\max}$. Como $K \leq K_{\max}$, concluímos que $\tilde{J}$ deve se anular em $\ell = d(p, q) = i(M)$. Absurdo, pois estamos supondo que (1) não é verdadeiro, i.e. que $i(M) < \pi/\sqrt{K_{\max}}$. □

Exercise 40.3. [dC79], Capítulo XIII, Proposição 3.4. Se a curvatura seccional K de uma variedade Riemanniana M^n, compacta, orientável e de dimensão par, satisfaz $0 < K \leq 1$, então $i(M) \geq \pi$.

Proof. Para obter uma contradição, suponha que existe um ponto $p \in M$ tal que $d(p, C_m(p)) < \pi$. Como M é compacta, sabemos que $C_m(p)$ é compacto e portanto existe uma geodésica γ ligando p com $q \in C_m(p)$ e realizando a distância $d(p, C_m(p))$.

Pelo teorema de Rauch, sabemos que qualquer geodésica não pode ter pontos conjugados antes de alcançar comprimento π. Explicitamente, se J é um campo de Jacobi ao longo de alguma geodésica γ tal que $J(0) = 0$ e $J(\ell(\gamma)) = 0$, comparando com um campo $\tilde{J}$ em S^n tal que $\tilde{J}(0) = 0$, $|\tilde{J}'(0)| = |J'(0)|$ e $\langle J, \gamma' \rangle = \left\langle \tilde{J}, \tilde{\gamma} \right\rangle$, concluímos que $|\tilde{J}| \leq |J|$ ao longo de γ. Como as geodésicas de S^n não tem pontos conjugados antes de atingir comprimento π, concluímos que J não pode se anular antes desse ponto.

Isso mostra que, como estamos supondo que $d(p, C_m(p)) < \pi$, devemos estar no caso (2) do Exercício 40.1, i.e. existem exatamente duas geodésicas γ e σ ligando p e q, e $\gamma'(\ell) = -\sigma'(\ell)$. Repetindo o procedimento do Exercício anterior 40.3, podemos usar essas duas geodésicas para achar uma geodésica fechada de comprimento $2i(M)$, cujo comprimento é menor do que o comprimento de qualquer outra geodésica fechada.

Remark 40.4. Na prova do teorema de Synge 19.1, o Professor Manfredo afirma que "Como M é compacta e tem curvatura positiva, $K \geq \delta > 0$"; que não me parece imediato, pois a curvatura seccional não é uma função definida em M. Porém, não preciso mostrar isso para garantir a existência do loop geodésico (via o exercício anterior); apenas é necessário que M seja compacta para garantir a existência do ponto q.

Para concluir seguimos a prova de [dC79]. Vamos a usar o exercício 6 da lista 6, (cf. 38.4), onde mostrei que, nestas condições, i.e. M de dimensão par, completa, orientável e com curvatura seccional positiva, existem curvas livremente homotópicas a qualquer geodésica fechada γ, que possuem comprimento menor que $\ell(\gamma)$. Lembre que essas curvas foram construídas mediante a função exponencial aplicada a um campo paralelo ao longo de γ, de forma que são curvas suaves; vou usar esse fato depois.

A ideia é chegar numa contradição mostrando que existe uma terceira geodésica ligando p e q. Essa geodésica se obtém como o limite de uma sequência de geodésicas associada à variação dada pelo Exercício 6 da Lista 6.

Seja c_s uma das curvas da variação com comprimento estritamente menor do que $\ell(\gamma)$. Pegue o ponto inicial $c_s(0) := \tilde{p}_s$ e o ponto $\tilde{q}_s$ em c_s tal que $d(\tilde{p}_s, \tilde{q}_s)$ é máxima ao longo de c_s. Existe uma geodésica $\tilde{\gamma}_s$ que liga $\tilde{p}_s$ e $\tilde{q}_s$.

A seguir mostrarei que tomando limite quando $s \to 0$, obteremos uma terceira geodésica minimizante $\tilde{\gamma}_0$ que liga p e q, que não é possível de novo pelo Exercício 40.1.

De fato, podemos dar explicitamente $\tilde{\gamma}_0$ como sendo $exp_q(tw)$ onde w é o vetor limite das velocidades iniciais de cada $\tilde{\gamma}_s$, supondo que elas são de tamanho 1 para assegurar a existência do limite por compacidade do fibrado tangente unitário. Note que a geodésica obtida minimiza a distância entre p e q: a geodésica liga p com q porque, por um lado, o limite dos pontos iniciais é p por definição, e, por outro lado, porque os pontos finais são os pontos que maximizam a distância dentro de cada geodésica. A geodésica limite é minimizante por continuidade da função distância.

Como cada curva c_s é diferenciável, temos um vetor tangente a ela em cada ponto $\tilde{q}_s$. Como $\tilde{\gamma}$ é minimizante, quando aplicamos a fórmula da primeira variação,vemos que $\tilde{\gamma}_s'$ é ortogonal a $c_s'(\tilde{q}_s)$. Como o a métrica é contínua, concluímos que $\tilde{\gamma}_0'(q) \perp \gamma'(q)$. □

Exercise 40.5. Seja M uma variedade Riemanniana completa, de curvatura seccional não negativa. Sejam $\gamma, \sigma : [0, \infty) \to M$ geodésicas tais que $\gamma(0) = \sigma(0)$. Se γ é um raio e $\angle(\gamma'(0), \sigma'(0)) < \frac{1}{2}\pi$, então $\lim_{t\to\infty} \rho(\sigma(0), \sigma(t)) = \infty$ (i.e. σ vai para o infinito).

Meu intento de prova. Seja $t \in \mathbb{R}$. Considere uma geodésica minimizante τ_t ligando $\gamma(t)$ com $\sigma(t)$. Defina o angulo $\beta_t := \angle(-\gamma'(t), \tau_t'(0))$ e o número s_t como sendo o tempo em que τ_t chega em σ, i.e. $\tau_t(s_t) = \sigma(t)$. Ver Figura 1.

Como $0 \leq K$, podemos comparar a "hinge", i.e. a informação do angulo e vetores incidentes no ponto $\gamma(t)$, $(-\gamma'(t), \tau_t'(0), \beta)$ com uma hinge no espaço euclidiano. Obtemos que

$$\rho_{\mathbb{R}^n}(\tilde{\gamma}(0), \tilde{\tau}_t(s_t)) \leq \rho(\gamma(0), \tau_t(s_t)) = \rho(\sigma(0), \sigma(t))$$

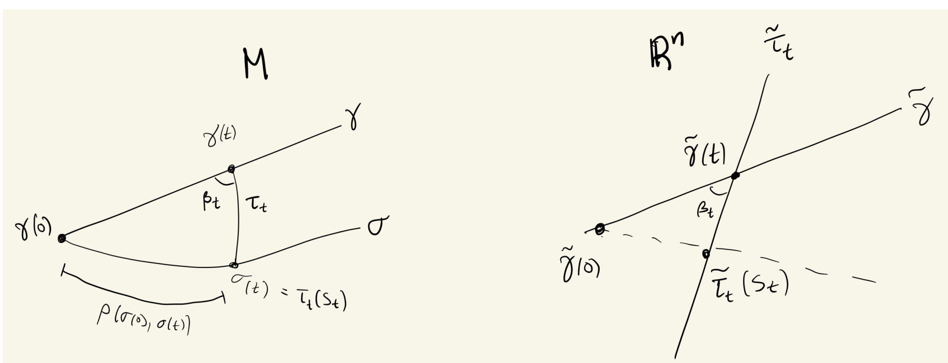
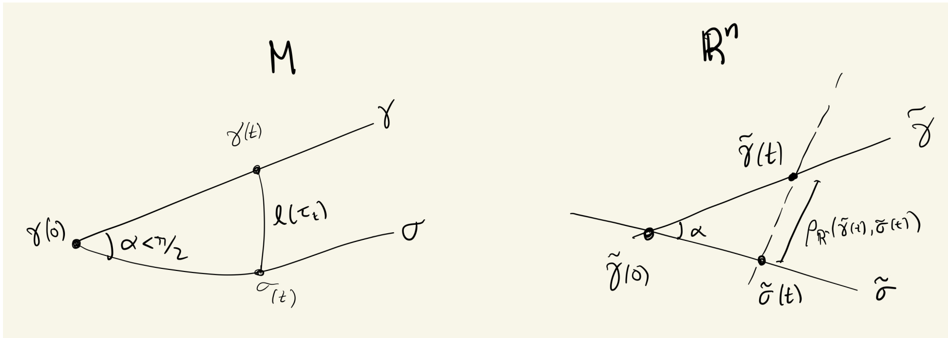


FIGURE 1.

Podemos calcular a distância $\rho_{\mathbb{R}^n}(\tilde{\gamma}(0), \tilde{\tau}_t(s_t))$ usando a lei de cosenos euclidiana desde que conheçamos β_t e $\ell(\tau_t)$:

$$\rho_{\mathbb{R}^n}(\tilde{\gamma}(0), \tilde{\tau}_t(s_t))^2 = t^2 + \ell(\tau_t)^2 - 2t\ell(\tau_t)\cos\beta_t$$

Para calcular $\ell(\tau_t)$ podemos usar o Teorema de Toponogov de novo, dessa vez comparando a hinge $(\gamma'(0), \sigma'(0), \alpha)$ com uma correspondente no espaço euclidiano:



Obtemos que

$$\rho_{\mathbb{R}^n}(\tilde{\gamma}(t), \tilde{\sigma}(t)) \leq \ell(\tau_t)$$

Como α é constante conforme t muda, podemos fixar $\tilde{\gamma}$ e $\tilde{\sigma}$ para fazer a nossa comparação. Conforme t avança, nos movemos a velocidade constante ao longo de $\tilde{\gamma}$ en direção ao infinito, e embora não sabemos se a distância $\rho(\sigma(0), \sigma(t))$ cresse ou diminue, a distância do nosso interesse $\rho_{\mathbb{R}^n}(\tilde{\gamma}(t), \tilde{\sigma}(t))$ vai sempre crescendo, e concluimos que diverge ao infinito.

Portanto o problema acaba se conseguimos achar uma cota inferior positiva para β_t. Parece que isso é equivalente a provar que a distância entre γ e σ está acotada inferiormente por uma constante positiva...

A lei de cosenos do lado Euclideano nos diz que

$$\rho_{\mathbb{R}^n}(\tilde{\gamma}(t), \tilde{\sigma}(t))^2 = t^2 + |\tilde{\sigma}(t)|^2 - 2t|\tilde{\sigma}(t)| \cos \alpha \leq t^2 + |\tilde{\sigma}(t)|^2 - 2t|\tilde{\sigma}(t)|$$

Supondo que $\tilde{\gamma}(0) = \tilde{\sigma}(0) = 0 \in \mathbb{R}^n$. Parece que para medir essa distância deveríamos conhecer $|\tilde{\sigma}(t)| = \rho(\sigma(0), \sigma(t))$... o que parece levar o problema de volta ao problema inicial. □

Prova de [CEA08]. Por desigualdade triangular, para quaisquer $t, s \in \mathbb{R}$,

$$\rho(\sigma(t), \sigma(0)) \geq \rho(\gamma(s), \sigma(t)) - \rho(\gamma(s), \gamma(0)).$$

Como o lado esquerdo não depende de s, vemos que basta tomar o limite quando $s \to \infty$ e mostrar que esse limite está inferiormente limitado por um número que depende linearmente de t, i.e. $t \cos \alpha$.

Considere uma geodésica minimizante $\tau_{s,t}$ ligando $\gamma(s)$ e $\sigma(t)$. Podemos usar o teorema de Toponogov 28.8 na versão de ângulos (que está enunciada em [CEA08]) já que duas das geodésicas no triângulo $\gamma(0), \gamma(s), \sigma(t)$ são minimizantes por hipótese. Como a cota inferior da curvatura é 0 não temos problemas com o comprimento de σ.

Obtemos que existe um triângulo Euclidiano com lados do mesmo comprimento tal que os ângulos correspondentes são menores o iguais que os ângulos com que começamos.

Denote $A := \rho(\gamma(0), \gamma(s)) = s$, $B := \ell(\sigma(t)) = t$ e $C := \rho(\gamma(s), \sigma(t)) = \ell(\tau_{s,t})$. Nosso objetivo é limitar inferiormente $C - A$ conforme $s \to \infty$. Aplicamos a lei de cosenos no lado Euclidiano para obter

$$\begin{aligned}
C^2 &= A^2 + B^2 - 2AB\cos\tilde{\alpha} \\
\implies C^2 - A^2 &= B^2 - 2AB\cos\tilde{\alpha} \\
\implies C - A &= \frac{B^2 - 2AB\cos\tilde{\alpha}}{C + A}
\end{aligned}$$

Onde só $A = s$ e C dependem de s. O comportamento assintótico desse quociente quando $s \to \infty$ é claro geometricamente: como A e C são dois lados de um triângulo, eles crescem proporcionadamente. Formalmente, como $C \leq A+B$, obtemos $C+A \leq 2A + B$. Daí

$$\begin{aligned}
\frac{1}{C+A} &\geq \frac{1}{2A+B} \\
\implies \frac{B^2 - 2AB\cos\tilde{\alpha}}{C+A} &\geq \frac{B^2 - 2AB\cos\tilde{\alpha}}{2A+B} = \frac{A}{A}\frac{B^2/A - 2B\cos\tilde{\alpha}}{2 + B/A}
\end{aligned}$$

que (a menos de analizar o que acontece se o numerador é negativo. . .) converge a $B\cos\tilde{\alpha}$ conforme $s \to \infty$. Concluímos que $C - A$ deve convergir a $-t\cos\tilde{\alpha} \leq -t\cos\alpha$. Tirando o signo obtemos a desigualdade desejada. □

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