ALGEBRAIC GEOMETRY

github.com/danimalabares/stack

Contents

1. Sheaves

For a definition of presheaf, see Categories, Definition **??**.

Definition 1.1. Let X be a topological space.

(1) A *sheaf $\mathcal{F}$ of sets on X* is a presheaf of sets which satisfies the following additional property: Given any open covering $U = \bigcup_{i \in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$
$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$
there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.
(2) A *morphism of sheaves of sets* is simply a morphism of presheaves of sets.
(3) The category of sheaves of sets on X is denoted $Sh(X)$.

Let X be a topological space. Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The *stalk of $\mathcal{F}$ at x* is the set
$$\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$$
where the colimit is over the set of open neighbourhoods U of x in X. The set of open neighbourhoods is partially ordered by (reverse) inclusion: We say $U \geq U' \Leftrightarrow U \subset U'$. The transition maps in the system are given by the restriction maps of $\mathcal{F}$. See Categories, Section **??** for notation and terminology regarding (co)limits over systems. Note that the colimit is a directed colimit. Thus it is easy to describe $\mathcal{F}_x$. Namely,
$$\mathcal{F}_x = \{(U, s) \mid x \in U, s \in \mathcal{F}(U)\}/\sim$$
with equivalence relation given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ with $x \in U''$ and $s|_{U''} = s'|_{U''}$. Given a pair (U, s) we sometimes denote s_x the element of $\mathcal{F}_x$ corresponding to the equivalence class of (U, x). We sometimes use the phrase "image of s in $\mathcal{F}_x$" to denote s_x. For example, given two pairs (U, s) and (U', s') we sometimes say "s is equal to s' in $\mathcal{F}_x$" to indicate that $s_x = s'_x$. Other authors use the terminology "germ of s at x".

2. Abelian sheaves

The following may be used to define the ideal sheaf of a variety:

Lemma 2.1. *Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ be sheaves over X with values on Grp. For every morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$,*
$$\begin{aligned} \mathrm{Ker}\, f : \mathit{Open}_X^{op} &\longrightarrow \mathit{Sets} \\ U &\longmapsto \mathrm{Ker}\, f(U) \\ (i : V \to U) &\longmapsto {\scriptstyle \mathrm{Ker}\, f(i) : \mathrm{Ker}\, \underset{x \mapsto x}{f(U)} \to \mathrm{Ker}\, f(V)} \end{aligned}$$
is a sheaf over X.

Proof. First observe that the correspondence on morphisms is well-defined. Indeed, $\operatorname{Ker} f(U) \subset \mathcal{F}(U) \subset \mathcal{F}(V)$ when $V \subset U$, and we just apply $f(U)$ to notice that $\operatorname{Ker} f(V) \subset \operatorname{Ker} f(U)$.

To see this is a presheaf notice it is obvious that the identity is mapped to the identity by definition of the correspondence of morphisms. It is also obvious that composition is preserved.

To see it is a sheaf consider an open cover U_i of U, and elements $x_i \in \operatorname{Ker} f(U_i)$. Then use the property of $\mathcal{F}$ being a sheaf to reconstruct an element $x \in \mathcal{F}(U)$, whose image under f will be mapped to the identity element of $\mathcal{G}(U)$ because it does so in every point of the cover of U. Thus x is in $\operatorname{Ker} f(U)$ as desired. $\square$

More formally,

Definition 2.2. Let X be a topological space.

(1) A *presheaf of abelian groups on* X or an *abelian presheaf over* X is a presheaf of sets $\mathcal{F}$ such that for each open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an abelian group, and such that all restriction maps ρ^U_V are homomorphisms of abelian groups, see Lemma **??** above.
(2) A *morphism of abelian presheaves over* X $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets which induces a homomorphism of abelian groups $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$.
(3) The category of presheaves of abelian groups on X is denoted $PAb(X)$.

Definition 2.3. Let X be a topological space.

(1) An *abelian sheaf on* X or *sheaf of abelian groups on* X is an abelian presheaf on X such that the underlying presheaf of sets is a sheaf.
(2) The category of sheaves of abelian groups is denoted $Ab(X)$.

Let X be a topological space. In the case of an abelian presheaf $\mathcal{F}$ the sheaf condition with regards to an open covering $U = \bigcup U_i$ is often expressed by saying that the complex of abelian groups

$$0 \to \mathcal{F}(U) \to \prod\nolimits_i \mathcal{F}(U_i) \to \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is exact. The first map is the usual one, whereas the second maps the element $(s_i)_{i\in I}$ to the element

$$(s_{i_0}|_{U_{i_0}\cap U_{i_1}} - s_{i_1}|_{U_{i_0}\cap U_{i_1}})_{(i_0,i_1)} \in \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

In fact, the notion of kernel of a sheaf is not really defined as I did in the beginning of this section, but in the next one, along with several other important things.

3. The abelian category of sheaves of modules

I guess that the reason to introduce coherent sheaves is not the search for an abelian category, after all. Looks like the pathologies avoided by the definition of coherence are not so obvious—something like "wildly infinitely generated".

Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Definition **??**. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, see Sheaves, Definition **??**. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subset X$ is the sum of the maps induced by φ, ψ. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus $Mod(\mathcal{O}_X)$ is a pre-additive category, see Homology, Definition **??**.

We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subset X$. Clearly this is both a final and an initial object of $Mod(\mathcal{O}_X)$. Given a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each open U, and (d) $\varphi_x = 0$ for all $x \in X$. See Sheaves, Lemma **??**.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition **??**. Thus $Mod(\mathcal{O}_X)$ is an additive category, see Homology, Definition **??**.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define $\operatorname{Ker}(\varphi)$ to be the subsheaf of $\mathcal{F}$ with sections

$$\operatorname{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all open $U \subset X$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}_X$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\operatorname{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$. Moreover, on the level of stalks we have $\operatorname{Ker}(\varphi)_x = \operatorname{Ker}(\varphi_x)$.

On the other hand, we define $\operatorname{Coker}(\varphi)$ as the sheaf of $\mathcal{O}_X$-modules associated to the presheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \operatorname{Coker}(\mathcal{F}(U) \to \mathcal{G}(U)) = \mathcal{G}(U)/\varphi(\mathcal{F}(U)).$$

Since taking stalks commutes with taking sheafification, see Sheaves, Lemma **??** we see that $\operatorname{Coker}(\varphi)_x = \operatorname{Coker}(\varphi_x)$. Thus the map $\mathcal{G} \to \operatorname{Coker}(\varphi)$ is surjective (as a map of sheaves of sets), see Sheaves, Section **??**. To show that this is a cokernel, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}_X$-modules such that $\beta \circ \varphi$ is zero, then you get for every open $U \subset X$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma **??**) we obtain a canonical map $\operatorname{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \operatorname{Coker}(\varphi) \to \mathcal{H}$. The morphism $\operatorname{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

Lemma 3.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is an abelian category. Moreover a complex*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact at $\mathcal{G}_x$.

Proof. By Homology, Definition **??** we have to show that image and coimage agree. By Sheaves, Lemma **??** it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathcal{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. $\square$

Actually the category $Mod(\mathcal{O}_X)$ has many more properties. Here are two constructions we can do.

(1) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the product
$$\prod\nolimits_{i \in I} \mathcal{F}_i$$
which is the sheaf that associates to each open U the product of the modules $\mathcal{F}_i(U)$. This is also the categorical product, as in Categories, Definition 6.1.
(2) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the direct sum
$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i$$
which is the *sheafification* of the presheaf that associates to each open U the direct sum of the modules $\mathcal{F}_i(U)$. This is also the categorical coproduct, as in Categories, Definition 11.4. To see this you use the universal property of sheafification.

Using these we conclude that all limits and colimits exist in $Mod(\mathcal{O}_X)$.

Lemma 3.2. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *All limits exist in $Mod(\mathcal{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathcal{O}_X$-modules (i.e., commute with taking sections over opens).*
(2) *All colimits exist in $Mod(\mathcal{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.*
(3) *Filtered colimits are exact.*
(4) *Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathcal{O}_X$-modules.*

Proof. As $Mod(\mathcal{O}_X)$ is abelian (Lemma 3.1) it has all finite limits and colimits (Homology, Lemma **??**). Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description (see discussion above) and Categories, Lemmas **??** and **??**. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks. Part (3) signifies that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of $\mathcal{O}_X$-modules over a directed set I the sequence $0 \to \operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact as well. Since we can check exactness on stalks (Lemma 3.1) this follows from the case of modules which is Algebra, Lemma **??**. We omit the proof of (4). □

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section **??**. See Homology, Lemma **??** for a description of exactness properties in terms of short exact sequences.

Lemma 3.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.*

(1) *The functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is left exact. In fact it commutes with all limits.*
(2) *The functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is right exact. In fact it commutes with all colimits.*
(3) *Pullback $f^{-1} : Ab(Y) \to Ab(X)$ on abelian sheaves is exact.*

Proof. Parts (1) and (2) hold because (f^*, f_*) is an adjoint pair of functors, see Sheaves, Lemma **??** and Categories, Section **??**. Part (3) holds because exactness can be checked on stalks (Lemma 3.1) and the description of stalks of the pullback, see Sheaves, Lemma **??**. □

Lemma 3.4. *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_! : Ab(U) \to Ab(X)$ is exact.*

Proof. Follows from the description of stalks given in Sheaves, Lemma **??**. □

Lemma 3.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathcal{F}_i$ be a sheaf of $\mathcal{O}_X$-modules. For $U \subset X$ quasi-compact open the map*

$$\bigoplus\nolimits_{i\in I} \mathcal{F}_i(U) \longrightarrow \left(\bigoplus\nolimits_{i\in I} \mathcal{F}_i\right)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j\in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i\in I_j} s_{ji}$ with $s_{ji} \in \mathcal{F}_i(U_j)$. Because U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j\in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i\in I'} \mathcal{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed, see Lemma 3.2 above. □

4. Tensor product of sheaves

Here's my unexpected encounter with the definition of tensor product of sheaves. It's not the "fiber is tensor product of fibers" construction, but actually just some notion of "change of ring" sheaf that ends up being adjoint to some "restriction" sheaf. The setting is a mapping of presheaves *of rings* over a space X... (I think the usual definition is this one taking $\mathcal{O}_1$ as the other presheaf we want to tensor).

Immediately after introducing this notion there's the definition of sheaf, then stalks, abelian sheaves, some other notions like an "algebraic structure" and then tensor product will be defined after sheafification—because the following definition is in general not a sheaf.

Furthermore, I add that Vakil leaves it as an exercise to define the tensor product of two $\mathcal{O}_X$ modules (with a hint of defining the presheaf tensor product and sheafifying), which makes me think that after all it *is* just the intuitive definition. Before diving in, also by Vakil (Exercise 26.K): the stalk of the tensor product is the tensor product of the stalks.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of* $\mathcal{F}$. We obtain the restriction functor

$$PMod(\mathcal{O}_2) \longrightarrow PMod(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$\left(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}\right)(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$PMod(\mathcal{O}_1) \longrightarrow PMod(\mathcal{O}_2)$$

Lemma 4.1. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. $\square$

Tipologia dos feixes.

Definition 4.2. A sheaf of $\mathcal{A}$-modules $\mathcal{F}$ over a sheaf of rings $\mathcal{A}$ (on a topological space X) is called

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5. Locally ringed spaces

Recall that we defined ringed spaces in Sheaves, Section **??**. Briefly, a ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$. You can think of $f^\sharp$ as a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$, see Sheaves, Definition **??** and Lemma **??**.

A good geometric example of this to keep in mind is $\mathcal{C}^\infty$-manifolds and morphisms of $\mathcal{C}^\infty$-manifolds. Namely, if M is a $\mathcal{C}^\infty$-manifold, then the sheaf $\mathcal{C}^\infty_M$ of smooth functions is a sheaf of rings on M. And any map $f : M \to N$ of manifolds is smooth if and only if for every local section h of $\mathcal{C}^\infty_N$ the composition $h \circ f$ is a local section of $\mathcal{C}^\infty_M$. Thus a smooth map f gives rise in a natural way to a morphism of ringed spaces

$$f : (M, \mathcal{C}^\infty_M) \longrightarrow (N, \mathcal{C}^\infty_N)$$

see Sheaves, Example **??**. It is instructive to consider what happens to stalks. Namely, let $m \in M$ with image $f(m) = n \in N$. Recall that the stalk $\mathcal{C}^\infty_{M,m}$ is the ring of germs of smooth functions at m, see Sheaves, Example **??**. The algebra of germs of functions on (M, m) is a local ring with maximal ideal the functions which vanish at m. Similarly for $\mathcal{C}^\infty_{N,n}$. The map on stalks $f^\sharp : \mathcal{C}^\infty_{N,n} \to \mathcal{C}^\infty_{M,m}$ maps the maximal ideal into the maximal ideal, simply because $f(m) = n$.

In algebraic geometry we study schemes. On a scheme the sheaf of rings is not determined by an intrinsic property of the space. The spectrum of a ring R (see Algebra, Section **??**) endowed with a sheaf of rings constructed out of R (see below), will be our basic building block. It will turn out that the stalks of $\mathcal{O}$ on $\mathrm{Spec}(R)$ are the local rings of R at its primes. There are two reasons to introduce locally ringed spaces in this setting: (1) There is in general no mechanism that assigns to a continuous map of spectra a map of the corresponding rings. This is why we add as an extra datum the map $f^\sharp$. (2) If we consider morphisms of these spectra in the category of ringed spaces, then the maps on stalks may not be local homomorphisms. Since our geometric intuition says it should we introduce locally ringed spaces as follows.

Definition 5.1. Locally ringed spaces.

(1) A *locally ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.
(2) Given a locally ringed space $(X, \mathcal{O}_X)$ we say that $\mathcal{O}_{X,x}$ is the *local ring of X at x*. We denote $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the *residue field of X at x* is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
(3) A *morphism of locally ringed spaces* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We will usually suppress the sheaf of rings $\mathcal{O}_X$ in the notation when discussing locally ringed spaces. We will simply refer to "the locally ringed space X". We will by abuse of notation think of X also as the underlying topological space. Finally we will denote the corresponding sheaf of rings $\mathcal{O}_X$ as the *structure sheaf of X*. In addition, it is customary to denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$ by $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$. We will say "let $f : X \to Y$ be a morphism of locally ringed spaces" thereby suppressing the structure sheaves even further. In this case, we will by abuse of notation think of $f : X \to Y$ also as the underlying continuous map of topological spaces. The f-map corresponding to f will customarily be denoted $f^\sharp$. The condition that f is a morphism of locally ringed spaces can then be expressed by saying that for every $x \in X$ the map on stalks

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

maps the maximal ideal $\mathfrak{m}_{Y,f(x)}$ into $\mathfrak{m}_{X,x}$.

Let us use these notational conventions to show that the collection of locally ringed spaces and morphisms of locally ringed spaces forms a category. In order to see this we have to show that the composition of morphisms of locally ringed spaces is a morphism of locally ringed spaces. OK, so let $f : X \to Y$ and $g : Y \to Z$ be morphism of locally ringed spaces. The composition of f and g is defined in Sheaves, Definition **??**. Let $x \in X$. By Sheaves, Lemma **??** the composition

$$\mathcal{O}_{Z,g(f(x))} \xrightarrow{g^\sharp} \mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp} \mathcal{O}_{X,x}$$

is the associated map on stalks for the morphism $g \circ f$. The result follows since a composition of local ring homomorphisms is a local ring homomorphism.

A pleasing feature of the definition is the fact that the functor

$$\textit{Locally ringed spaces} \longrightarrow \textit{Ringed spaces}$$

reflects isomorphisms (plus more). Here is a less abstract statement.

Lemma 5.2. *Let X, Y be locally ringed spaces. If $f : X \to Y$ is an isomorphism of ringed spaces, then f is an isomorphism of locally ringed spaces.*

Proof. This follows trivially from the corresponding fact in algebra: Suppose A, B are local rings. Any isomorphism of rings $A \to B$ is a local ring isomorphism. □

6. Open immersions of locally ringed spaces

Definition 6.1. Let $f : X \to Y$ be a morphism of locally ringed spaces. We say that f is an *open immersion* if f is a homeomorphism of X onto an open subset of Y, and the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

The following construction is parallel to Sheaves, Definition **??** (3).

Example 6.2. Let X be a locally ringed space. Let $U \subset X$ be an open subset. Let $\mathcal{O}_U = \mathcal{O}_X|_U$ be the restriction of $\mathcal{O}_X$ to U. For $u \in U$ the stalk $\mathcal{O}_{U,u}$ is equal to the stalk $\mathcal{O}_{X,u}$, and hence is a local ring. Thus $(U, \mathcal{O}_U)$ is a locally ringed space and the morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is an open immersion.

Definition 6.3. Let X be a locally ringed space. Let $U \subset X$ be an open subset. The locally ringed space $(U, \mathcal{O}_U)$ of Example 6.2 above is the *open subspace of X associated to U*.

Lemma 6.4. *Let $f : X \to Y$ be an open immersion of locally ringed spaces. Let $j : V = f(X) \to Y$ be the open subspace of Y associated to the image of f. There is a unique isomorphism $f' : X \cong V$ of locally ringed spaces such that $f = j \circ f'$.*

Proof. Let f' be the homeomorphism between X and V induced by f. Then $f = j \circ f'$ as maps of topological spaces. Since there is an isomorphism of sheaves $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, there is an isomorphism of rings $f^\sharp : \Gamma(U, f^{-1}(\mathcal{O}_Y)) \to \Gamma(U, \mathcal{O}_X)$ for each open subset $U \subset X$. Since $\mathcal{O}_V = j^{-1}\mathcal{O}_Y$ and $f^{-1} = f'^{-1}j^{-1}$ (Sheaves, Lemma **??**) we see that $f^{-1}\mathcal{O}_Y = f'^{-1}\mathcal{O}_V$, hence $\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, f^{-1}(\mathcal{O}_Y))$ is an isomorphism for every $U \subset X$ open. By composing these we get an isomorphism of rings

$$\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, \mathcal{O}_X)$$

for each open subset $U \subset X$, and therefore an isomorphism of sheaves $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$. In other words, we have an isomorphism $f'^\sharp : f'^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$ and therefore an isomorphism of locally ringed spaces $(f', f'^\sharp) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ (use Lemma 5.2). Note that $f = j \circ f'$ as morphisms of locally ringed spaces by construction.

Suppose we have another morphism $f'' : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ such that $f = j \circ f''$. At any point $x \in X$, we have $j(f'(x)) = j(f''(x))$ from which it follows that $f'(x) = f''(x)$ since j is the inclusion map; therefore f' and f'' are the same as morphisms of topological spaces. On structure sheaves, for each open subset $U \subset X$ we have a commutative diagram

$$\begin{tikzcd}
\Gamma(U, f^{-1}(\mathcal{O}_Y)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Gamma(U, \mathcal{O}_X) \\
\Gamma(U, f'^{-1}(\mathcal{O}_V)) \arrow[ur, bend left, "f'^\sharp"] \arrow[ur, bend right, "f''^\sharp"'] &
\end{tikzcd}$$

from which we see that $f'^\sharp$ and $f''^\sharp$ define the same morphism of sheaves. □

From now on we do not distinguish between open subsets and their associated subspaces.

Lemma 6.5. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $U \subset X$, and $V \subset Y$ be open subsets. Suppose that $f(U) \subset V$. There exists a unique morphism of locally ringed spaces $f|_U : U \to V$ such that the following diagram is* *a commutative square of locally ringed spaces*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r] & Y \end{tikzcd}$$

Proof. Omitted. □

In the following we will use without further mention the following fact which follows from the lemma above. Given any morphism $f : Y \to X$ of locally ringed spaces, and any open subset $U \subset X$ such that $f(Y) \subset U$, then there exists a unique morphism of locally ringed spaces $Y \to U$ such that the composition $Y \to U \to X$ is equal to f. In fact, we will even by abuse of notation write $f : Y \to U$ since this rarely gives rise to confusion.

7. Closed immersions of locally ringed spaces

We follow our conventions introduced in Modules, Definition **??**.

Definition 7.1. Let $i : Z \to X$ be a morphism of locally ringed spaces. We say that i is a *closed immersion* if:

(1) The map i is a homeomorphism of Z onto a closed subset of X.
(2) The map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective; let $\mathcal{I}$ denote the kernel.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

Lemma 7.2. *Let $f : Z \to X$ be a morphism of locally ringed spaces. In order for f to be a closed immersion it suffices that there exists an open covering $X = \bigcup U_i$ such that each $f : f^{-1}U_i \to U_i$ is a closed immersion.*

Proof. Omitted. □

Example 7.3. Let X be a locally ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals which is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Let Z be the support of the sheaf of rings $\mathcal{O}_X/\mathcal{I}$. This is a closed subset of X, by Modules, Lemma **??**. Denote $i : Z \to X$ the inclusion map. By Modules, Lemma **??** there is a unique sheaf of rings $\mathcal{O}_Z$ on Z with $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. For any $z \in Z$ the stalk $\mathcal{O}_{Z,z}$ is equal to a quotient $\mathcal{O}_{X,i(z)}/\mathcal{I}_{i(z)}$ of a local ring and nonzero, hence a local ring. Thus $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a closed immersion of locally ringed spaces.

Definition 7.4. Let X be a locally ringed space. Let $\mathcal{I}$ be a sheaf of ideals on X which is locally generated by sections. The locally ringed space $(Z, \mathcal{O}_Z)$ of Example 7.3 above is the *closed subspace of X associated to the sheaf of ideals $\mathcal{I}$.*

Lemma 7.5. *Let $f : X \to Y$ be a closed immersion of locally ringed spaces. Let $\mathcal{I}$ be the kernel of the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $i : Z \to Y$ be the closed subspace of Y associated to $\mathcal{I}$. There is a unique isomorphism $f' : X \cong Z$ of locally ringed spaces such that $f = i \circ f'$.*

Proof. Omitted. □

Lemma 7.6. *Let X, Y be locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals locally generated by sections. Let $i : Z \to X$ be the associated closed subspace. A morphism $f : Y \to X$ factors through Z if and only if the map $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$ is zero. If this is the case the morphism $g : Y \to Z$ such that $f = i \circ g$ is unique.*

Proof. Clearly if f factors as $Y \to Z \to X$ then the map $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Conversely suppose that $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Pick any $y \in Y$, and consider the ring map $f_y^\sharp : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Since the composition $\mathcal{I}_{f(y)} \to \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is zero by assumption and since $f_y^\sharp(1) = 1$ we see that $1 \notin \mathcal{I}_{f(y)}$, i.e., $\mathcal{I}_{f(y)} \neq \mathcal{O}_{X,f(y)}$. We conclude that $f(Y) \subset Z = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$. Hence $f = i \circ g$ where $g : Y \to Z$ is continuous. Consider the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. The assumption $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero implies that the composition $\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$ is zero by adjointness of f_* and f^*. In other words, we obtain a morphism of sheaves of rings $\overline{f^\sharp} : \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y$. Note that $f_*\mathcal{O}_Y = i_*g_*\mathcal{O}_Y$ and that $\mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$. By Sheaves, Lemma **??** we obtain a unique morphism of sheaves of rings $g^\sharp : \mathcal{O}_Z \to g_*\mathcal{O}_Y$ whose pushforward under i is $\overline{f^\sharp}$. We omit the verification that $(g, g^\sharp)$ defines a morphism of locally ringed spaces and that $f = i \circ g$ as a morphism of locally ringed spaces. The uniqueness of $(g, g^\sharp)$ was pointed out above. □

Lemma 7.7. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals which is locally generated by sections. Let $i : Z \to Y$ be the closed subspace associated to the sheaf of ideals $\mathcal{I}$. Let $\mathcal{J}$ be the image of the map $f^*\mathcal{I} \to f^*\mathcal{O}_Y = \mathcal{O}_X$. Then this ideal is locally generated by sections. Moreover, let $i' : Z' \to X$ be the associated closed subspace of X. There exists a unique morphism of locally ringed spaces $f' : Z' \to Z$ such that the following diagram is a commutative square of locally ringed spaces*

$$\begin{tikzcd} Z' \arrow[r, "i'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

Moreover, this diagram is a fibre square in the category of locally ringed spaces.

Proof. The ideal $\mathcal{J}$ is locally generated by sections by Modules, Lemma **??**. The rest of the lemma follows from the characterization, in Lemma 7.6 above, of what it means for a morphism to factor through a closed subspace. □

Let me finish with a proof that haunted my second semester of PhD:

Lemma 7.8. *The ideal sheaf of an irreducible codimension-1 closed subscheme of a smooth scheme is a line bundle.*

Proof. Since Y is codimension-1 irreducible, the ideal sheaf at every affine chart is a codimension-1 prime ideal. This means that there aren't any nontrivial prime ideals of $\mathcal{I}_Y(\mathrm{Spec}\, A) := I$. Let $f \in I$. Since X is smooth, $O_X \,\mathrm{Spec}\, A = A$ is a UFD (why?). Then there exists an irreducible element $g \in I$ such that $gh = f$. The ideal generated by g is prime (again because A is a UFD) and contained in I, so that I is principal. This shows that $\mathcal{I}_Y$ is locally principal, i.e. it is a line bundle. □

8. The spectrum of a ring

We arbitrarily decide that the spectrum of a ring as a topological space is part of the algebra chapter, whereas an affine scheme is part of the chapter on schemes.

Definition 8.1. Let R be a ring.

(1) The *spectrum* of R is the set of prime ideals of R. It is usually denoted $\mathrm{Spec}(R)$.

(2) Given a subset $T \subset R$ we let $V(T) \subset \mathrm{Spec}(R)$ be the set of primes containing T, i.e., $V(T) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \forall f \in T, f \in \mathfrak{p}\}$.
(3) Given an element $f \in R$ we let $D(f) \subset \mathrm{Spec}(R)$ be the set of primes not containing f.

Lemma 8.2. *Let R be a ring.*

(1) *The spectrum of a ring R is empty if and only if R is the zero ring.*
(2) *Every nonzero ring has a maximal ideal.*
(3) *Every nonzero ring has a minimal prime ideal.*
(4) *Given an ideal $I \subset R$ and a prime ideal $I \subset \mathfrak{p}$ there exists a prime $I \subset \mathfrak{q} \subset \mathfrak{p}$ such that $\mathfrak{q}$ is minimal over I.*
(5) *If $T \subset R$, and if (T) is the ideal generated by T in R, then $V((T)) = V(T)$.*
(6) *If I is an ideal and $\sqrt{I}$ is its radical, see basic notion (**??**), then $V(I) = V(\sqrt{I})$.*
(7) *Given an ideal I of R we have $\sqrt{I} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$.*
(8) *If I is an ideal then $V(I) = \emptyset$ if and only if I is the unit ideal.*
(9) *If I, J are ideals of R then $V(I) \cup V(J) = V(I \cap J)$.*
(10) *If $(I_a)_{a \in A}$ is a set of ideals of R then $\bigcap_{a \in A} V(I_a) = V(\bigcup_{a \in A} I_a)$.*
(11) *If $f \in R$, then $D(f) \amalg V(f) = \mathrm{Spec}(R)$.*
(12) *If $f \in R$ then $D(f) = \emptyset$ if and only if f is nilpotent.*
(13) *If $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$.*
(14) *If $I \subset R$ is an ideal, and $\mathfrak{p}$ is a prime of R with $\mathfrak{p} \notin V(I)$, then there exists an $f \in R$ such that $\mathfrak{p} \in D(f)$, and $D(f) \cap V(I) = \emptyset$.*
(15) *If $f, g \in R$, then $D(fg) = D(f) \cap D(g)$.*
(16) *If $f_i \in R$ for $i \in I$, then $\bigcup_{i \in I} D(f_i)$ is the complement of $V(\{f_i\}_{i \in I})$ in $\mathrm{Spec}(R)$.*
(17) *If $f \in R$ and $D(f) = \mathrm{Spec}(R)$, then f is a unit.*

Proof. We address each part in the corresponding item below.

(1) This is a direct consequence of (2) or (3).
(2) Let $\mathfrak{A}$ be the set of all proper ideals of R. This set is ordered by inclusion and is non-empty, since $(0) \in \mathfrak{A}$ is a proper ideal. Let A be a totally ordered subset of $\mathfrak{A}$. Then $\bigcup_{I \in A} I$ is in fact an ideal. Since $1 \notin I$ for all $I \in A$, the union does not contain 1 and thus is proper. Hence $\bigcup_{I \in A} I$ is in $\mathfrak{A}$ and is an upper bound for the set A. Thus by Zorn's lemma $\mathfrak{A}$ has a maximal element, which is the sought-after maximal ideal.
(3) Since R is nonzero, it contains a maximal ideal which is a prime ideal. Thus the set $\mathfrak{A}$ of all prime ideals of R is nonempty. $\mathfrak{A}$ is ordered by reverse-inclusion. Let A be a totally ordered subset of $\mathfrak{A}$. It's pretty clear that $J = \bigcap_{I \in A} I$ is in fact an ideal. Not so clear, however, is that it is prime. Let $xy \in J$. Then $xy \in I$ for all $I \in A$. Now let $B = \{I \in A | y \in I\}$. Let $K = \bigcap_{I \in B} I$. Since A is totally ordered, either $K = J$ (and we're done, since then $y \in J$) or $K \supset J$ and for all $I \in A$ such that I is properly contained in K, we have $y \notin I$. But that means that for all those $I, x \in I$, since they are prime. Hence $x \in J$. In either case, J is prime as desired. Hence by Zorn's lemma we get a maximal element which in this case is a minimal prime ideal.
(4) This is the same exact argument as (3) except you only consider prime ideals contained in $\mathfrak{p}$ and containing I.

(5) (T) is the smallest ideal containing T. Hence if $T \subset I$, some ideal, then $(T) \subset I$ as well. Hence if $I \in V(T)$, then $I \in V((T))$ as well. The other inclusion is obvious.
(6) Since $I \subset \sqrt{I}, V(\sqrt{I}) \subset V(I)$. Now let $\mathfrak{p} \in V(I)$. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some n. Hence $x^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, a boring induction argument gets you that $x \in \mathfrak{p}$. Hence $\sqrt{I} \subset \mathfrak{p}$ and $\mathfrak{p} \in V(\sqrt{I})$.
(7) Let $f \in R \setminus \sqrt{I}$. Then $f^n \notin I$ for all n. Hence $S = \{1, f, f^2, \ldots\}$ is a multiplicative subset, not containing 0. Take a prime ideal $\bar{\mathfrak{p}} \subset S^{-1}R$ containing $S^{-1}I$. Then the pull-back $\mathfrak{p}$ in R of $\bar{\mathfrak{p}}$ is a prime ideal containing I that does not intersect S. This shows that $\bigcap_{I\subset\mathfrak{p}} \mathfrak{p} \subset \sqrt{I}$. Now if $a \in \sqrt{I}$, then $a^n \in I$ for some n. Hence if $I \subset \mathfrak{p}$, then $a^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$. Thus the equality is shown.
(8) I is not the unit ideal if and only if I is contained in some maximal ideal (to see this, apply (2) to the ring R/I) which is therefore prime.
(9) If $\mathfrak{p} \in V(I) \cup V(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ which means that $I \cap J \subset \mathfrak{p}$. Now if $I \cap J \subset \mathfrak{p}$, then $IJ \subset \mathfrak{p}$ and hence either I of J is in $\mathfrak{p}$, since $\mathfrak{p}$ is prime.
(10) $\mathfrak{p} \in \bigcap_{a\in A} V(I_a) \Leftrightarrow I_a \subset \mathfrak{p}, \forall a \in A \Leftrightarrow \mathfrak{p} \in V(\bigcup_{a\in A} I_a)$
(11) If $\mathfrak{p}$ is a prime ideal and $f \in R$, then either $f \in \mathfrak{p}$ or $f \notin \mathfrak{p}$ (strictly) which is what the disjoint union says.
(12) If $a \in R$ is nilpotent, then $a^n = 0$ for some n. Hence $a^n \in \mathfrak{p}$ for any prime ideal. Thus $a \in \mathfrak{p}$ as can be shown by induction and $D(a) = \emptyset$. Now, as shown in (7), if $a \in R$ is not nilpotent, then there is a prime ideal that does not contain it.
(13) $f \in \mathfrak{p} \Leftrightarrow uf \in \mathfrak{p}$, since u is invertible.
(14) If $\mathfrak{p} \notin V(I)$, then $\exists f \in I \setminus \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so $\mathfrak{p} \in D(f)$. Also if $\mathfrak{q} \in D(f)$, then $f \notin \mathfrak{q}$ and thus I is not contained in $\mathfrak{q}$. Thus $D(f) \cap V(I) = \emptyset$.
(15) If $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Hence if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, if $fg \notin \mathfrak{p}$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.
(16) $\mathfrak{p} \in \bigcup_{i\in I} D(f_i) \Leftrightarrow \exists i \in I, f_i \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in \mathrm{Spec}(R) \setminus V(\{f_i\}_{i\in I})$
(17) If $D(f) = \mathrm{Spec}(R)$, then $V(f) = \emptyset$ and hence $fR = R$, so f is a unit.

□

The lemma implies that the subsets $V(T)$ from Definition 8.1 form the closed subsets of a topology on $\mathrm{Spec}(R)$. And it also shows that the sets $D(f)$ are open and form a basis for this topology.

Definition 8.3. Let R be a ring. The topology on $\mathrm{Spec}(R)$ whose closed sets are the sets $V(T)$ is called the *Zariski* topology. The open subsets $D(f)$ are called the *standard opens* of $\mathrm{Spec}(R)$.

It should be clear from context whether we consider $\mathrm{Spec}(R)$ just as a set or as a topological space.

Lemma 8.4. *Suppose that $\varphi : R \to R'$ is a ring homomorphism. The induced map*

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}(R') \longrightarrow \mathrm{Spec}(R), \quad \mathfrak{p}' \longmapsto \varphi^{-1}(\mathfrak{p}')$$

is continuous for the Zariski topologies. In fact, for any element $f \in R$ we have $\mathrm{Spec}(\varphi)^{-1}(D(f)) = D(\varphi(f))$.

Proof. It is basic notion (**??**) that $\mathfrak{p} := \varphi^{-1}(\mathfrak{p}')$ is indeed a prime ideal of R. The last assertion of the lemma follows directly from the definitions, and implies the first. □

If $\varphi' : R' \to R''$ is a second ring homomorphism then the composition

$$\mathrm{Spec}(R'') \longrightarrow \mathrm{Spec}(R') \longrightarrow \mathrm{Spec}(R)$$

equals $\mathrm{Spec}(\varphi' \circ \varphi)$. In other words, Spec is a contravariant functor from the category of rings to the category of topological spaces.

Lemma 8.5. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The map $R \to S^{-1}R$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(S^{-1}R) \longrightarrow \{\mathfrak{p} \in \mathrm{Spec}(R) \mid S \cap \mathfrak{p} = \emptyset\}$$

where the topology on the right hand side is that induced from the Zariski topology on $\mathrm{Spec}(R)$*. The inverse map is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p} = \mathfrak{p}(S^{-1}R)$.*

Proof. Denote the right hand side of the arrow of the lemma by D. Choose a prime $\mathfrak{p}' \subset S^{-1}R$ and let $\mathfrak{p}$ the inverse image of $\mathfrak{p}'$ in R. Since $\mathfrak{p}'$ does not contain 1 we see that $\mathfrak{p}$ does not contain any element of S. Hence $\mathfrak{p} \in D$ and we see that the image is contained in D. Let $\mathfrak{p} \in D$. By assumption the image $\overline{S}$ does not contain 0. By basic notion (**??**) $\overline{S}^{-1}(R/\mathfrak{p})$ is not the zero ring. By basic notion (**??**) we see $S^{-1}R/S^{-1}\mathfrak{p} = \overline{S}^{-1}(R/\mathfrak{p})$ is a domain, and hence $S^{-1}\mathfrak{p}$ is a prime. The equality of rings also shows that the inverse image of $S^{-1}\mathfrak{p}$ in R is equal to $\mathfrak{p}$, because $R/\mathfrak{p} \to \overline{S}^{-1}(R/\mathfrak{p})$ is injective by basic notion (**??**). This proves that the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ is bijective onto D with inverse as given. It is continuous by Lemma 8.4. Finally, let $D(g) \subset \mathrm{Spec}(S^{-1}R)$ be a standard open. Write $g = h/s$ for some $h \in R$ and $s \in S$. Since g and $h/1$ differ by a unit we have $D(g) = D(h/1)$ in $\mathrm{Spec}(S^{-1}R)$. Hence by Lemma 8.4 and the bijectivity above the image of $D(g) = D(h/1)$ is $D \cap D(h)$. This proves the map is open as well. ☐

Lemma 8.6. *Let R be a ring. Let $f \in R$. The map $R \to R_f$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(R_f) \longrightarrow D(f) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p} \cdot R_f$.

Proof. This is a special case of Lemma 8.5. ☐

It is not the case that every "affine open" of a spectrum is a standard open. See Example **??**.

Lemma 8.7. *Let R be a ring. Let $I \subset R$ be an ideal. The map $R \to R/I$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(R/I) \longrightarrow V(I) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p}/I$.

Proof. It is immediate that the image is contained in $V(I)$. On the other hand, if $\mathfrak{p} \in V(I)$ then $\mathfrak{p} \supset I$ and we may consider the ideal $\mathfrak{p}/I \subset R/I$. Using basic notion (**??**) we see that $(R/I)/(\mathfrak{p}/I) = R/\mathfrak{p}$ is a domain and hence $\mathfrak{p}/I$ is a prime ideal. From this it is immediately clear that the image of $D(f + I)$ is $D(f) \cap V(I)$, and hence the map is a homeomorphism. ☐

Lemma 8.8. *Let R be a ring. The space* $\mathrm{Spec}(R)$ *is quasi-compact.*

Proof. It suffices to prove that any covering of $\mathrm{Spec}(R)$ by standard opens can be refined by a finite covering. Thus suppose that $\mathrm{Spec}(R) = \cup D(f_i)$ for a set of elements $\{f_i\}_{i\in I}$ of R. This means that $\cap V(f_i) = \emptyset$. According to Lemma 8.2 this means that $V(\{f_i\}) = \emptyset$. According to the same lemma this means that the ideal generated by the f_i is the unit ideal of R. This means that we can write 1 as a *finite* sum: $1 = \sum_{i\in J} r_i f_i$ with $J \subset I$ finite. And then it follows that $\mathrm{Spec}(R) = \cup_{i\in J} D(f_i)$. □

Lemma 8.9. *Let R be a ring. The topology on $X = \mathrm{Spec}(R)$ has the following properties:*

1. *X is quasi-compact,*
2. *X has a basis for the topology consisting of quasi-compact opens, and*
3. *the intersection of any two quasi-compact opens is quasi-compact.*

Proof. The spectrum of a ring is quasi-compact, see Lemma 8.8. It has a basis for the topology consisting of the standard opens $D(f) = \mathrm{Spec}(R_f)$ (Lemma 16.1) which are quasi-compact by the first remark. The intersection of two standard opens is quasi-compact as $D(f) \cap D(g) = D(fg)$. Given any two quasi-compact opens $U, V \subset X$ we may write $U = D(f_1) \cup \ldots \cup D(f_n)$ and $V = D(g_1) \cup \ldots \cup D(g_m)$. Then $U \cap V = \bigcup D(f_i g_j)$ which is quasi-compact. □

9. Affine schemes

Let R be a ring. Consider the topological space $\mathrm{Spec}(R)$ associated to R, see Algebra, Section **??**. We will endow this space with a sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ and the resulting pair $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ will be an affine scheme.

Recall that $\mathrm{Spec}(R)$ has a basis of open sets $D(f)$, $f \in R$ which we call standard opens, see Algebra, Definition **??**. In addition, the intersection of two standard opens is another: $D(f) \cap D(g) = D(fg)$, $f, g \in R$.

Lemma 9.1. *Let R be a ring. Let $f \in R$.*

1. *If $g \in R$ and $D(g) \subset D(f)$, then*
   (a) *f is invertible in R_g,*
   (b) *$g^e = af$ for some $e \geq 1$ and $a \in R$,*
   (c) *there is a canonical ring map $R_f \to R_g$, and*
   (d) *there is a canonical R_f-module map $M_f \to M_g$ for any R-module M.*
2. *Any open covering of $D(f)$ can be refined to a finite open covering of the form $D(f) = \bigcup_{i=1}^n D(g_i)$.*
3. *If $g_1, \ldots, g_n \in R$, then $D(f) \subset \bigcup D(g_i)$ if and only if $g_1, \ldots, g_n$ generate the unit ideal in R_f.*

Proof. Recall that $D(g) = \mathrm{Spec}(R_g)$ (see Algebra, Lemma **??**). Thus (a) holds because f maps to an element of R_g which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma **??**. Write the inverse of f in R_g as a/g^d. This means $g^d - af$ is annihilated by a power of g, whence (b). For (c), the map $R_f \to R_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. The equality $M_f = M \otimes_R R_f$ can be used to obtain the map on modules, or we can define $M_f \to M_g$ by mapping x/f^n to $a^n x/g^{ne}$.

Recall that $D(f)$ is quasi-compact, see Algebra, Lemma **??**. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma **??**. □

In Sheaves, Section **??** we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas **??** and **??**. Moreover, we showed in Sheaves, Lemma **??** that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

Definition 9.2. Let R be a ring.

(1) A *standard open covering* of $\mathrm{Spec}(R)$ is a covering $\mathrm{Spec}(R) = \bigcup_{i=1}^n D(f_i)$, where $f_1, \ldots, f_n \in R$.
(2) Suppose that $D(f) \subset \mathrm{Spec}(R)$ is a standard open. A *standard open covering* of $D(f)$ is a covering $D(f) = \bigcup_{i=1}^n D(g_i)$, where $g_1, \ldots, g_n \in R$.

Let R be a ring. Let M be an R-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \mathrm{Spec}(R)$ is a standard open. If $f, g \in R$ are such that $D(f) = D(g)$, then by Lemma 16.1 above there are canonical maps $M_f \to M_g$ and $M_g \to M_f$ which are mutually inverse. Hence we may choose any f such that $U = D(f)$ and define

$$\widetilde{M}(U) = M_f.$$

Note that if $D(g) \subset D(f)$, then by Lemma 16.1 above we have a canonical map

$$\widetilde{M}(D(f)) = M_f \longrightarrow M_g = \widetilde{M}(D(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = R$, then $\widetilde{R}$ is a presheaf of rings on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \mathrm{Spec}(R)$. Suppose that x corresponds to the prime $\mathfrak{p} \subset R$. By definition of the stalk we see that

$$\widetilde{M}_x = \mathrm{colim}_{f \in R, f \notin \mathfrak{p}} M_f$$

Here the set $\{f \in R, f \notin \mathfrak{p}\}$ is preordered by the rule $f \geq f' \Leftrightarrow D(f) \subset D(f')$. If $f_1, f_2 \in R \setminus \mathfrak{p}$, then we have $f_1 f_2 \geq f_1$ in this ordering. Hence by Algebra, Lemma **??** we conclude that

$$\widetilde{M}_x = M_{\mathfrak{p}}.$$

Next, we check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_f \to \bigoplus M_{g_i} \to \bigoplus M_{g_i g_j}.$$

Note that $D(g_i) = D(f g_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_f \to \bigoplus M_{f g_i} \to \bigoplus M_{f g_i g_j}.$$

In addition, by Lemma 16.1 above we see that $g_1, \ldots, g_n$ generate the unit ideal in R_f. Thus we may apply Algebra, Lemma **??** to the module M_f over R_f and the elements $g_1, \ldots, g_n$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section **??** that there exists a unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which agrees with $\widetilde{R}$ on the standard opens. Note that by our computation of stalks above, the stalks of this sheaf of rings are all local rings.

Similarly, for any R-module M there exists a unique sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma **??**.

Definition 9.3. Let R be a ring.

(1) The *structure sheaf* $\mathcal{O}_{\mathrm{Spec}(R)}$ *of the spectrum of* R is the unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which agrees with $\widetilde{R}$ on the basis of standard opens.
(2) The locally ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is called the *spectrum* of R and denoted $\mathrm{Spec}(R)$.
(3) The sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules extending $\widetilde{M}$ to all opens of $\mathrm{Spec}(R)$ is called the sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

Lemma 9.4. *Let R be a ring. Let M be an R-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules associated to M.*

(1) *We have* $\Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) = R$.
(2) *We have* $\Gamma(\mathrm{Spec}(R), \widetilde{M}) = M$ *as an R-module.*
(3) *For every $f \in R$ we have* $\Gamma(D(f), \mathcal{O}_{\mathrm{Spec}(R)}) = R_f$.
(4) *For every $f \in R$ we have* $\Gamma(D(f), \widetilde{M}) = M_f$ *as an R_f-module.*
(5) *Whenever $D(g) \subset D(f)$ the restriction mappings on $\mathcal{O}_{\mathrm{Spec}(R)}$ and $\widetilde{M}$ are the maps $R_f \to R_g$ and $M_f \to M_g$ from Lemma 16.1.*
(6) *Let $\mathfrak{p}$ be a prime of R, and let $x \in \mathrm{Spec}(R)$ be the corresponding point. We have $\mathcal{O}_{\mathrm{Spec}(R),x} = R_{\mathfrak{p}}$.*
(7) *Let $\mathfrak{p}$ be a prime of R, and let $x \in \mathrm{Spec}(R)$ be the corresponding point. We have $\widetilde{M}_x = M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module.*

Moreover, all these identifications are functorial in the R module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of R-modules to the category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

Proof. Assertions (1) - (7) are clear from the discussion above. The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{\mathfrak{p}}$ is exact and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma **??**. □

Definition 9.5. An *affine scheme* is a locally ringed space isomorphic as a locally ringed space to $\mathrm{Spec}(R)$ for some ring R. A *morphism of affine schemes* is a morphism in the category of locally ringed spaces.

It turns out that affine schemes play a special role among all locally ringed spaces, which is what the next section is about.

10. The category of affine schemes

This might be a cornerstone of AG. The point is that once we take affine charts, we re in the affine category. And this category turns out to be equivalent to the category of rings. /o/ See below for some important properties of the correspondence; namely fibred products and tensor product counterparts.

Note that if Y is an affine scheme, then its points are in canonical $1-1$ bijection with prime ideals in $\Gamma(Y, \mathcal{O}_Y)$.

Lemma 10.1. *Let X be a locally ringed space. Let Y be an affine scheme. Let $f \in \mathrm{Mor}(X, Y)$ be a morphism of locally ringed spaces. Given a point $x \in X$ consider the ring maps*

$$\Gamma(Y, \mathcal{O}_Y) \xrightarrow{f^\sharp} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$$

Let $\mathfrak{p} \subset \Gamma(Y, \mathcal{O}_Y)$ denote the inverse image of $\mathfrak{m}_x$. Let $y \in Y$ be the corresponding point. Then $f(x) = y$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r] & \mathcal{O}_{X,x} \\ \Gamma(Y, \mathcal{O}_Y) \arrow[r] \arrow[u] & \mathcal{O}_{Y,f(x)} \arrow[u] \end{tikzcd}$$

(see the discussion of f-maps below Sheaves, Definition **??**). Since the right vertical arrow is local we see that $\mathfrak{m}_{f(x)}$ is the inverse image of $\mathfrak{m}_x$. The result follows. □

Lemma 10.2. *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The set*

$$D(f) = \{x \in X \mid \textit{image } f \notin \mathfrak{m}_x\}$$

is open. Moreover $f|_{D(f)}$ has an inverse.

Proof. This is a special case of Modules, Lemma **??**, but we also give a direct proof. Suppose that $U \subset X$ and $V \subset X$ are two open subsets such that $f|_U$ has an inverse g and $f|_V$ has an inverse h. Then clearly $g|_{U\cap V} = h|_{U \cap V}$. Thus it suffices to show that f is invertible in an open neighbourhood of any $x \in D(f)$. This is clear because $f \notin \mathfrak{m}_x$ implies that $f \in \mathcal{O}_{X,x}$ has an inverse $g \in \mathcal{O}_{X,x}$ which means there is some open neighbourhood $x \in U \subset X$ so that $g \in \mathcal{O}_X(U)$ and $g \cdot f|_U = 1$. □

Lemma 10.3. *In Lemma 10.2 above, if X is an affine scheme, then the open $D(f)$ agrees with the standard open $D(f)$ defined previously (in Algebra, Definition* **??***).*

Proof. Omitted. □

Lemma 10.4. *Let X be a locally ringed space. Let Y be an affine scheme. The map*

$$\mathrm{Mor}(X, Y) \longrightarrow \mathrm{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

Proof. Since Y is affine we have $(Y, \mathcal{O}_Y) \cong (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some ring R. During the proof we will use facts about Y and its structure sheaf which are direct consequences of things we know about the spectrum of a ring, see e.g. Lemma 9.4.

Motivated by the lemmas above we construct the inverse map. Let $\psi_Y : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. First, we define the corresponding map of spaces

$$\Psi : X \longrightarrow Y$$

by the rule of Lemma 10.1. In other words, given $x \in X$ we define $\Psi(x)$ to be the point of Y corresponding to the prime in $\Gamma(Y, \mathcal{O}_Y)$ which is the inverse image of $\mathfrak{m}_x$ under the composition $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\psi_Y} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$.

We claim that the map $\Psi : X \to Y$ is continuous. The standard opens $D(g)$, for $g \in \Gamma(Y, \mathcal{O}_Y)$ are a basis for the topology of Y. Thus it suffices to prove that $\Psi^{-1}(D(g))$ is open. By construction of Ψ the inverse image $\Psi^{-1}(D(g))$ is exactly the set $D(\psi_Y(g)) \subset X$ which is open by Lemma 10.2. Hence Ψ is continuous.

Next we construct a Ψ-map of sheaves from $\mathcal{O}_Y$ to $\mathcal{O}_X$. By Sheaves, Lemma **??** it suffices to define ring maps $\psi_{D(g)} : \Gamma(D(g), \mathcal{O}_Y) \to \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X)$ compatible with restriction maps. We have a canonical isomorphism $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, because Y is an affine scheme. Because $\psi_Y(g)$ is invertible on $D(\psi_Y(g))$ we see that there is a canonical map

$$\Gamma(Y, \mathcal{O}_Y)_g \longrightarrow \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X) = \Gamma(D(\psi_Y(g)), \mathcal{O}_X)$$

extending the map ψ_Y by the universal property of localization. Note that there is no choice but to take the canonical map here! And we take this, combined with the canonical identification $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, to be $\psi_{D(g)}$. This is compatible with localization since the restriction mapping on the affine schemes are defined in terms of the universal properties of localization also, see Lemmas 9.4 and 16.1.

Thus we have defined a morphism of ringed spaces $(\Psi, \psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ recovering ψ_Y on global sections. To see that it is a morphism of locally ringed spaces we have to show that the induced maps on local rings

$$\psi_x : \mathcal{O}_{Y, \Psi(x)} \longrightarrow \mathcal{O}_{X,x}$$

are local. This follows immediately from the commutative diagram of the proof of Lemma 10.1 and the definition of Ψ.

Finally, we have to show that the constructions $(\Psi, \psi) \mapsto \psi_Y$ and the construction $\psi_Y \mapsto (\Psi, \psi)$ are inverse to each other. Clearly, $\psi_Y \mapsto (\Psi, \psi) \mapsto \psi_Y$. Hence the only thing to prove is that given ψ_Y there is at most one pair (Ψ, ψ) giving rise to it. The uniqueness of Ψ was shown in Lemma 10.1 and given the uniqueness of Ψ the uniqueness of the map ψ was pointed out during the course of the proof above. ☐

Lemma 10.5. *The category of affine schemes is equivalent to the opposite of the category of rings. The equivalence is given by the functor that associates to an affine scheme the global sections of its structure sheaf.*

Proof. This is now clear from Definition 9.5 and Lemma 10.4. ☐

Lemma 10.6. *Let Y be an affine scheme. Let $f \in \Gamma(Y, \mathcal{O}_Y)$. The open subspace $D(f)$ is an affine scheme.*

Proof. We may assume that $Y = \mathrm{Spec}(R)$ and $f \in R$. Consider the morphism of affine schemes $\phi : U = \mathrm{Spec}(R_f) \to \mathrm{Spec}(R) = Y$ induced by the ring map $R \to R_f$. By Algebra, Lemma **??** we know that it is a homeomorphism onto $D(f)$. On the other hand, the map $\phi^{-1}\mathcal{O}_Y \to \mathcal{O}_U$ is an isomorphism on stalks, hence an isomorphism. Thus we see that ϕ is an open immersion. We conclude that $D(f)$ is isomorphic to U by Lemma 6.4. ☐

Lemma 10.7. *The category of affine schemes has finite products, and fibre products. In other words, it has finite limits. Moreover, the products and fibre products* *in the category of affine schemes are the same as in the category of locally ringed spaces. In a formula, we have (in the category of locally ringed spaces)*

$$\mathrm{Spec}(R) \times \mathrm{Spec}(S) = \mathrm{Spec}(R \otimes_{\mathbf{Z}} S)$$

and given ring maps $R \to A$, $R \to B$ we have

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_R B).$$

Proof. This is just an application of Lemma 10.4. First of all, by that lemma, the affine scheme $\mathrm{Spec}(\mathbf{Z})$ is the final object in the category of locally ringed spaces. Thus the first displayed formula follows from the second. To prove the second note that for any locally ringed space X we have

$$\begin{aligned}
\mathrm{Mor}(X, \mathrm{Spec}(A \otimes_R B)) &= \mathrm{Hom}(A \otimes_R B, \mathcal{O}_X(X)) \\
&= \mathrm{Hom}(A, \mathcal{O}_X(X)) \times_{\mathrm{Hom}(R, \mathcal{O}_X(X))} \mathrm{Hom}(B, \mathcal{O}_X(X)) \\
&= \mathrm{Mor}(X, \mathrm{Spec}(A)) \times_{\mathrm{Mor}(X, \mathrm{Spec}(R))} \mathrm{Mor}(X, \mathrm{Spec}(B))
\end{aligned}$$

which proves the formula. See Categories, Section 8 for the relevant definitions. □

Lemma 10.8. *Let X be a locally ringed space. Assume $X = U \amalg V$ with U and V open and such that U, V are affine schemes. Then X is an affine scheme.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ by the sheaf property. By Lemma 10.4 there is a canonical morphism of locally ringed spaces $X \to \mathrm{Spec}(R)$. By Algebra, Lemma **??** we see that as a topological space $\mathrm{Spec}(\mathcal{O}_X(U)) \amalg \mathrm{Spec}(\mathcal{O}_X(V)) = \mathrm{Spec}(R)$ with the maps coming from the ring homomorphisms $R \to \mathcal{O}_X(U)$ and $R \to \mathcal{O}_X(V)$. This of course means that $\mathrm{Spec}(R)$ is the coproduct in the category of locally ringed spaces as well. By assumption the morphism $X \to \mathrm{Spec}(R)$ induces an isomorphism of $\mathrm{Spec}(\mathcal{O}_X(U))$ with U and similarly for V. Hence $X \to \mathrm{Spec}(R)$ is an isomorphism. □

11. Quasi-coherent sheaves on affines

Recall that we have defined the abstract notion of a quasi-coherent sheaf in Modules, Definition **??**. In this section we show that any quasi-coherent sheaf on an affine scheme $\mathrm{Spec}(R)$ corresponds to the sheaf $\widetilde{M}$ associated to an R-module M.

Lemma 11.1. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let M be an R-module. There exists a canonical isomorphism between the sheaf $\widetilde{M}$ associated to the R-module M (Definition 16.3) and the sheaf $\mathcal{F}_M$ associated to the R-module M (Modules, Definition **??**). This isomorphism is functorial in M. In particular, the sheaves $\widetilde{M}$ are quasi-coherent. Moreover, they are characterized by the following mapping property*

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{F}))$$

for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Here a map $\alpha : \widetilde{M} \to \mathcal{F}$ corresponds to its effect on global sections.

Proof. By Modules, Lemma **??** we have a morphism $\mathcal{F}_M \to \widetilde{M}$ corresponding to the map $M \to \Gamma(X, \widetilde{M}) = M$. Let $x \in X$ correspond to the prime $\mathfrak{p} \subset R$. The induced map on stalks are the maps $\mathcal{O}_{X,x} \otimes_R M \to M_{\mathfrak{p}}$ which are isomorphisms because $R_{\mathfrak{p}} \otimes_R M = M_{\mathfrak{p}}$. Hence the map $\mathcal{F}_M \to \widetilde{M}$ is an isomorphism. The mapping property follows from the mapping property of the sheaves $\mathcal{F}_M$. □

Lemma 11.2. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. There are canonical isomorphisms*

(1) $\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$*, see Modules, Section* **??**.
(2) $\widetilde{T^n(M)} \cong T^n(\widetilde{M})$, $\widetilde{Sym^n(M)} \cong Sym^n(\widetilde{M})$*, and* $\widetilde{\wedge^n(M)} \cong \wedge^n(\widetilde{M})$*, see Modules, Section* **??**.
(3) *if M is a finitely presented R-module, then* $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_R(M, N)}$*, see Modules, Section* **??**.

First proof. Using Lemma 11.1 and Modules, Lemma **??** we see that the functor $M \mapsto \widetilde{M}$ can be viewed as π^* for a morphism π of ringed spaces. And pulling back modules commutes with tensor constructions by Modules, Lemmas **??** and **??**. The morphism $\pi : (X, \mathcal{O}_X) \to (\{*\}, R)$ is flat for example because the stalks of $\mathcal{O}_X$ are localizations of R (Lemma 9.4) and hence flat over R. Thus pullback by π commutes with internal hom if the first module is finitely presented by Modules, Lemma **??**. $\square$

Second proof. Proof of (1). By Lemma 11.1 to give a map $\widetilde{M \otimes_R N}$ into $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ we have to give a map on global sections $M \otimes_R N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})$ which exists by definition of the tensor product of sheaves of modules. To see that this map is an isomorphism it suffices to check that it is an isomorphism on stalks. And this follows from the description of the stalks of $\widetilde{M}$ (either in Lemma 9.4 or in Modules, Lemma **??**), the fact that tensor product commutes with localization (Algebra, Lemma **??**) and Modules, Lemma **??**.

Proof of (2). This is similar to the proof of (1), using Algebra, Lemma **??** and Modules, Lemma **??**.

Proof of (3). Since the construction $M \mapsto \widetilde{M}$ is functorial there is an R-linear map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. The target of this map is the global sections of $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. Hence by Lemma 11.1 we obtain a map of $\mathcal{O}_X$-modules $\widetilde{\mathrm{Hom}_R(M, N)} \to \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. We check that this is an isomorphism by comparing stalks. If M is finitely presented as an R-module then $\widetilde{M}$ has a global finite presentation as an $\mathcal{O}_X$-module. Hence we conclude using Algebra, Lemma **??** and Modules, Lemma **??**. $\square$

Third proof of part (1). For any $\mathcal{O}_X$-module $\mathcal{F}$ we have the following isomorphisms functorial in M, N, and $\mathcal{F}$

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}, \mathcal{F}) &= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F}))) \\
&= \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{N}, \mathcal{F})) \\
&= \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, \Gamma(X, \mathcal{F}))) \\
&= \mathrm{Hom}_R(M \otimes_R N, \Gamma(X, \mathcal{F})) \\
&= \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M \otimes_R N}, \mathcal{F})
\end{aligned}$$

The first equality is Modules, Lemma **??**. The second equality is the universal property of $\widetilde{M}$, see Lemma 11.1. The third equality holds by definition of $\mathcal{H}om$. The fourth equality is the universal property of $\widetilde{N}$. Then fifth equality is Algebra, Lemma **??**. The final equality is the universal property of $\widetilde{M \otimes_R N}$. By the Yoneda lemma (Categories, Lemma **??**) we obtain (1). □

Lemma 11.3. *Let* $(X, \mathcal{O}_X) = (\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)})$, $(Y, \mathcal{O}_Y) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ *be affine schemes. Let* $\psi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *be a morphism of affine schemes, corresponding to the ring map* $\psi^\sharp : R \to S$ *(see Lemma 10.5).*

1. *We have* $\psi^*\widetilde{M} = \widetilde{S \otimes_R M}$ *functorially in the* R*-module* M.
2. *We have* $\psi_*\widetilde{N} = \widetilde{N_R}$ *functorially in the* S*-module* N.

Proof. The first assertion follows from the identification in Lemma 11.1 and the result of Modules, Lemma **??**. The second assertion follows from the fact that $\psi^{-1}(D(f)) = D(\psi^\sharp(f))$ and hence

$$\psi_*\widetilde{N}(D(f)) = \widetilde{N}(D(\psi^\sharp(f))) = N_{\psi^\sharp(f)} = (N_R)_f = \widetilde{N_R}(D(f))$$

as desired. □

Lemma 11.3 above says in particular that if you restrict the sheaf $\widetilde{M}$ to a standard affine open subspace $D(f)$, then you get $\widetilde{M_f}$. We will use this from now on without further mention.

Lemma 11.4. *Let* $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ *be an affine scheme. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. Then* $\mathcal{F}$ *is isomorphic to the sheaf associated to the* R*-module* $\Gamma(X, \mathcal{F})$.

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Since every standard open $D(f)$ is quasi-compact we see that X is locally quasi-compact, i.e., every point has a fundamental system of quasi-compact neighbourhoods, see Topology, Definition **??**. Hence by Modules, Lemma **??** for every prime $\mathfrak{p} \subset R$ corresponding to $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the quasi-coherent sheaf associated to some $\mathcal{O}_X(U)$-module M. In other words, we get an open covering by U's with this property. By Lemma 16.1 for example we can refine this covering to a standard open covering. Thus we get a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ and R_{f_i}-modules M_i and isomorphisms $\varphi_i : \mathcal{F}|_{D(f_i)} \to \mathcal{F}_{M_i}$ for some R_{f_i}-module M_i. On the overlaps we get isomorphisms

$$\mathcal{F}_{M_i}|_{D(f_if_j)} \xrightarrow{\varphi_i^{-1}|_{D(f_if_j)}} \mathcal{F}|_{D(f_if_j)} \xrightarrow{\varphi_j|_{D(f_if_j)}} \mathcal{F}_{M_j}|_{D(f_if_j)}.$$

Let us denote these ψ_{ij}. It is clear that we have the cocycle condition

$$\psi_{jk}|_{D(f_if_jf_k)} \circ \psi_{ij}|_{D(f_if_jf_k)} = \psi_{ik}|_{D(f_if_jf_k)}$$

on triple overlaps.

Recall that each of the open subspaces $D(f_i)$, $D(f_if_j)$, $D(f_if_jf_k)$ is an affine scheme. Hence the sheaves $\mathcal{F}_{M_i}$ are isomorphic to the sheaves $\widetilde{M_i}$ by Lemma 11.1 above. In particular we see that $\mathcal{F}_{M_i}(D(f_if_j)) = (M_i)_{f_j}$, etc. Also by Lemma 11.1 above we see that ψ_{ij} corresponds to a unique $R_{f_if_j}$-module isomorphism

$$\psi_{ij} : (M_i)_{f_j} \longrightarrow (M_j)_{f_i}$$

namely, the effect of ψ_{ij} on sections over $D(f_if_j)$. Moreover these then satisfy the cocycle condition that

$$\begin{tikzcd}
(M_i)_{f_jf_k} \arrow[rr, "\psi_{ik}"] \arrow[dr, "\psi_{ij}"'] & & (M_k)_{f_if_j} \\
& (M_j)_{f_if_k} \arrow[ur, "\psi_{jk}"'] &
\end{tikzcd}$$

commutes (for any triple i, j, k).

Now Algebra, Lemma **??** shows that there exist an R-module M such that $M_i = M_{f_i}$ compatible with the morphisms ψ_{ij}. Consider $\mathcal{F}_M = \widetilde{M}$. At this point it is a formality to show that $\widetilde{M}$ is isomorphic to the quasi-coherent sheaf $\mathcal{F}$ we started out with. Namely, the sheaves $\mathcal{F}$ and $\widetilde{M}$ give rise to isomorphic sets of glueing data of sheaves of $\mathcal{O}_X$-modules with respect to the covering $X = \bigcup D(f_i)$, see Sheaves, Section **??** and in particular Lemma **??**. Explicitly, in the current situation, this boils down to the following argument: Let us construct an R-module map

$$M \longrightarrow \Gamma(X, \mathcal{F}).$$

Namely, given $m \in M$ we get $m_i = m/1 \in M_{f_i} = M_i$ by construction of M. By construction of M_i this corresponds to a section $s_i \in \mathcal{F}(U_i)$. (Namely, $\varphi_i^{-1}(m_i)$.) We claim that $s_i|_{D(f_if_j)} = s_j|_{D(f_if_j)}$. This is true because, by construction of M, we have $\psi_{ij}(m_i) = m_j$, and by the construction of the ψ_{ij}. By the sheaf condition of $\mathcal{F}$ this collection of sections gives rise to a unique section s of $\mathcal{F}$ over X. We leave it to the reader to show that $m \mapsto s$ is a R-module map. By Lemma 11.1 we obtain an associated $\mathcal{O}_X$-module map

$$\widetilde{M} \longrightarrow \mathcal{F}.$$

By construction this map reduces to the isomorphisms φ_i^{-1} on each $D(f_i)$ and hence is an isomorphism. □

Lemma 11.5. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. The functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ define quasi-inverse equivalences of categories*

$$\mathit{QCoh}(\mathcal{O}_X) \rightleftarrows \mathit{Mod}_R$$

between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of R-modules.

Proof. See Lemmas 11.1 and 11.4 above. □

From now on we will not distinguish between quasi-coherent sheaves on affine schemes and sheaves of the form $\widetilde{M}$.

Lemma 11.6. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Kernels and cokernels of maps of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. This follows from the exactness of the functor $\widetilde{\ }$ since by Lemma 11.1 we know that any map $\psi : \widetilde{M} \to \widetilde{N}$ comes from an R-module map $\varphi : M \to N$. (So we have $\mathrm{Ker}(\psi) = \widetilde{\mathrm{Ker}(\varphi)}$ and $\mathrm{Coker}(\psi) = \widetilde{\mathrm{Coker}(\varphi)}$.) □

Lemma 11.7. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. The direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. The same holds for colimits.*

Proof. Suppose $\mathcal{F}_i$, $i \in I$ is a collection of quasi-coherent sheaves on X. By Lemma 11.5 above we can write $\mathcal{F}_i = \widetilde{M_i}$ for some R-module M_i. Set $M = \bigoplus M_i$. Consider the sheaf $\widetilde{M}$. For each standard open $D(f)$ we have

$$\widetilde{M}(D(f)) = M_f = \Big(\bigoplus M_i\Big)_f = \bigoplus M_{i,f}.$$

Hence we see that the quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ is the direct sum of the sheaves $\mathcal{F}_i$. A similar argument works for general colimits. □

Lemma 11.8. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. This is clear in case both $\mathcal{F}_1$ and $\mathcal{F}_2$ are quasi-coherent because the functor $M \mapsto \widetilde{M}$ is exact, see Lemma 9.4. Similarly in case both $\mathcal{F}_2$ and $\mathcal{F}_3$ are quasi-coherent. Now, suppose that $\mathcal{F}_1 = \widetilde{M_1}$ and $\mathcal{F}_3 = \widetilde{M_3}$ are quasi-coherent. Set $M_2 = \Gamma(X, \mathcal{F}_2)$. We claim it suffices to show that the sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact. Namely, if this is the case, then (by using the mapping property of Lemma 11.1) we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M_1} \arrow[r] \arrow[d] & \widetilde{M_2} \arrow[r] \arrow[d] & \widetilde{M_3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0
\end{tikzcd}$$

and we win by the snake lemma.

The "correct" argument here would be to show first that $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$. This is actually not all that hard, but it is perhaps better to postpone this till later. Instead we use a small trick.

Pick $m \in M_3 = \Gamma(X, \mathcal{F}_3)$. Consider the following set

$$I = \{f \in R \mid \text{the element } fm \text{ comes from } M_2\}.$$

Clearly this is an ideal. It suffices to show $1 \in I$. Hence it suffices to show that for any prime $\mathfrak{p}$ there exists an $f \in I$, $f \notin \mathfrak{p}$. Let $x \in X$ be the point corresponding to $\mathfrak{p}$. Because surjectivity can be checked on stalks there exists an open neighbourhood U of x such that $m|_U$ comes from a local section $s \in \mathcal{F}_2(U)$. In fact we may assume that $U = D(f)$ is a standard open, i.e., $f \in R$, $f \notin \mathfrak{p}$. We will show that for some $N \gg 0$ we have $f^N \in I$, which will finish the proof.

Take any point $z \in V(f)$, say corresponding to the prime $\mathfrak{q} \subset R$. We can also find a $g \in R$, $g \notin \mathfrak{q}$ such that $m|_{D(g)}$ lifts to some $s' \in \mathcal{F}_2(D(g))$. Consider the difference $s|_{D(fg)} - s'|_{D(fg)}$. This is an element m' of $\mathcal{F}_1(D(fg)) = (M_1)_{fg}$. For some integer $n = n(z)$ the element $f^n m'$ comes from some $m'_1 \in (M_1)_g$. We see that $f^n s$ extends to a section σ of $\mathcal{F}_2$ on $D(f) \cup D(g)$ because it agrees with the restriction of $f^n s' + m'_1$ on $D(f) \cap D(g) = D(fg)$. Moreover, σ maps to the restriction of $f^n m$ to $D(f) \cup D(g)$.

Since $V(f)$ is quasi-compact, there exists a finite list of elements $g_1, \ldots, g_m \in R$ such that $V(f) \subset \bigcup D(g_j)$, an integer $n > 0$ and sections $\sigma_j \in \mathcal{F}_2(D(f) \cup D(g_j))$ such that $\sigma_j|_{D(f)} = f^n s$ and σ_j maps to the section $f^n m|_{D(f)\cup D(g_j)}$ of $\mathcal{F}_3$. Consider the differences

$$\sigma_j|_{D(f)\cup D(g_j g_k)} - \sigma_k|_{D(f)\cup D(g_j g_k)}.$$

These correspond to sections of $\mathcal{F}_1$ over $D(f) \cup D(g_j g_k)$ which are zero on $D(f)$. In particular their images in $\mathcal{F}_1(D(g_j g_k)) = (M_1)_{g_j g_k}$ are zero in $(M_1)_{g_j g_k f}$. Thus some high power of f kills each and every one of these. In other words, the elements $f^N \sigma_j$, for some $N \gg 0$ satisfy the glueing condition of the sheaf property and give rise to a section σ of $\mathcal{F}_2$ over $\bigcup (D(f) \cup D(g_j)) = X$ as desired. $\square$

12. Closed subspaces of affine schemes

Example 12.1. Let R be a ring. Let $I \subset R$ be an ideal. Consider the morphism of affine schemes $i : Z = \mathrm{Spec}(R/I) \to \mathrm{Spec}(R) = X$. By Algebra, Lemma **??** this is a homeomorphism of Z onto a closed subset of X. Moreover, if $I \subset \mathfrak{p} \subset R$ is a prime corresponding to a point $x = i(z)$, $x \in X$, $z \in Z$, then on stalks we get the map

$$\mathcal{O}_{X,x} = R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/IR_{\mathfrak{p}} = \mathcal{O}_{Z,z}$$

Thus we see that i is a closed immersion of locally ringed spaces, see Definition 7.1. Clearly, this is (isomorphic) to the closed subspace associated to the quasi-coherent sheaf of ideals $\widetilde{I}$, as in Example 7.3.

Lemma 12.2. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let $i : Z \to X$ be any closed immersion of locally ringed spaces. Then there exists a unique ideal $I \subset R$ such that the morphism $i : Z \to X$ can be identified with the closed immersion $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ constructed in Example 12.1 above.*

Proof. This is kind of silly! Namely, by Lemma 7.5 we can identify $Z \to X$ with the closed subspace associated to a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ as in Definition 7.4 and Example 7.3. By our conventions this sheaf of ideals is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Hence the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ is locally on X the cokernel of a map $\bigoplus_{j\in J} \mathcal{O}_U \to \mathcal{O}_U$. Thus by definition, $\mathcal{O}_X/\mathcal{I}$ is quasi-coherent. By our results in Section 11 it is of the form $\widetilde{S}$ for some R-module S. Moreover, since $\mathcal{O}_X = \widetilde{R} \to \widetilde{S}$ is surjective we see by Lemma 11.8 that also $\mathcal{I}$ is quasi-coherent, say $\mathcal{I} = \widetilde{I}$. Of course $I \subset R$ and $S = R/I$ and everything is clear. $\square$

13. Schemes

Definition 13.1. A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted *Sch*.

Let X be a scheme. We will use the following (very slight) abuse of language. We will say $U \subset X$ is an *affine open*, or an *open affine* if the open subspace U is an affine scheme. We will often write $U = \mathrm{Spec}(R)$ to indicate that U is isomorphic to $\mathrm{Spec}(R)$ and moreover that we will identify (temporarily) U and $\mathrm{Spec}(R)$.

Lemma 13.2. *Let X be a scheme. Let $j : U \to X$ be an open immersion of locally ringed spaces. Then U is a scheme. In particular, any open subspace of X is a scheme.*

Proof. Let $U \subset X$. Let $u \in U$. Pick an affine open neighbourhood $u \in V \subset X$. Because standard opens of V form a basis of the topology on V we see that there exists a $f \in \mathcal{O}_V(V)$ such that $u \in D(f) \subset U$. And $D(f)$ is an affine scheme by Lemma 10.6. This proves that every point of U has an open neighbourhood which is affine. $\square$

Clearly the lemma (or its proof) shows that any scheme X has a basis (see Topology, Section **??**) for the topology consisting of affine opens.

Example 13.3. Let k be a field. An example of a scheme which is not affine is given by the open subspace $U = \mathrm{Spec}(k[x,y]) \setminus \{(x,y)\}$ of the affine scheme $X = \mathrm{Spec}(k[x,y])$. It is covered by two affines, namely $D(x) = \mathrm{Spec}(k[x,y,1/x])$ and $D(y) = \mathrm{Spec}(k[x,y,1/y])$ whose intersection is $D(xy) = \mathrm{Spec}(k[x,y,1/xy])$. By the sheaf property for $\mathcal{O}_U$ there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_U) \to k[x,y,1/x] \times k[x,y,1/y] \to k[x,y,1/xy]$$

We conclude that the map $k[x,y] \to \Gamma(U, \mathcal{O}_U)$ (coming from the morphism $U \to X$) is an isomorphism. Therefore U cannot be affine since if it was then by Lemma 10.5 we would have $U \cong X$.

14. Immersions of schemes

In Lemma 13.2 we saw that any open subspace of a scheme is a scheme. Below we will prove that the same holds for a closed subspace of a scheme.

Note that the notion of a quasi-coherent sheaf of $\mathcal{O}_X$-modules is defined for any ringed space X in particular when X is a scheme. By our efforts in Section 11 we know that such a sheaf is on any affine open $U \subset X$ of the form $\widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

Lemma 14.1. *Let X be a scheme. Let $i : Z \to X$ be a closed immersion of locally ringed spaces.*

1. *The locally ringed space Z is a scheme,*
2. *the kernel $\mathcal{I}$ of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of ideals,*
3. *for any affine open $U = \mathrm{Spec}(R)$ of X the morphism $i^{-1}(U) \to U$ can be identified with $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ for some ideal $I \subset R$, and*
4. *we have $\mathcal{I}|_U = \widetilde{I}$.*

In particular, any sheaf of ideals locally generated by sections is a quasi-coherent sheaf of ideals (and vice versa), and any closed subspace of X is a scheme.

Proof. Let $i : Z \to X$ be a closed immersion. Let $z \in Z$ be a point. Choose any affine open neighbourhood $i(z) \in U \subset X$. Say $U = \mathrm{Spec}(R)$. By Lemma 12.2 we know that $i^{-1}(U) \to U$ can be identified with the morphism of affine schemes $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. First of all this implies that $z \in i^{-1}(U) \subset Z$ is an affine neighbourhood of z. Thus Z is a scheme. Second this implies that $\mathcal{I}|_U$ is $\widetilde{I}$. In other words for every point $x \in i(Z)$ there exists an open neighbourhood such that $\mathcal{I}$ is quasi-coherent in that neighbourhood. Note that $\mathcal{I}|_{X \setminus i(Z)} \cong \mathcal{O}_{X \setminus i(Z)}$. Thus the restriction of the sheaf of ideals is quasi-coherent on $X \setminus i(Z)$ also. We conclude that $\mathcal{I}$ is quasi-coherent. $\square$

Definition 14.2. Let X be a scheme.

(1) A morphism of schemes is called an *open immersion* if it is an open immersion of locally ringed spaces (see Definition 6.1).
(2) An *open subscheme* of X is an open subspace of X in the sense of Definition 6.3; an open subscheme of X is a scheme by Lemma 13.2.
(3) A morphism of schemes is called a *closed immersion* if it is a closed immersion of locally ringed spaces (see Definition 7.1).
(4) A *closed subscheme* of X is a closed subspace of X in the sense of Definition 7.4; a closed subscheme is a scheme by Lemma 14.1.
(5) A morphism of schemes $f : X \to Y$ is called an *immersion*, or a *locally closed immersion* if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.

It follows from the lemmas in Sections 6 and 7 that any open (resp. closed) immersion of schemes is isomorphic to the inclusion of an open (resp. closed) subscheme of the target.

Our definition of a closed immersion is halfway between Hartshorne and EGA. Hartshorne defines a closed immersion as a morphism $f : X \to Y$ of schemes which induces a homeomorphism of X onto a closed subset of Y such that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, see [**?**, Page 85]. We will show this is equivalent to our notion in Lemma **??**. In [**?**], Grothendieck and Dieudonné first define closed subschemes via the construction of Example 7.3 using quasi-coherent sheaves of ideals and then define a closed immersion as a morphism $f : X \to Y$ which induces an isomorphism with a closed subscheme. It follows from Lemma 14.1 that this agrees with our notion.

Pedagogically speaking the definition above is a disaster/nightmare. In teaching this material to students, we have found it often convenient to define a closed immersion as an affine morphism $f : X \to Y$ of schemes such that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. Namely, it turns out that the notion of an affine morphism (Morphisms, Section **??**) is quite natural and easy to understand.

For more information on closed immersions we suggest the reader visit Morphisms, Sections **??** and **??**.

We will discuss locally closed subschemes and immersions at the end of this section.

Remark 14.3. If $f : X \to Y$ is an immersion of schemes, then it is in general not possible to factor f as an open immersion followed by a closed immersion. See Morphisms, Example **??**.

Lemma 14.4. *Let $f : Y \to X$ be an immersion of schemes. Then f is a closed immersion if and only if $f(Y) \subset X$ is a closed subset.*

Proof. If f is a closed immersion then $f(Y)$ is closed by definition. Conversely, suppose that $f(Y)$ is closed. By definition there exists an open subscheme $U \subset X$ such that f is the composition of a closed immersion $i : Y \to U$ and the open immersion $j : U \to X$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals associated to the closed immersion i. Note that $\mathcal{I}|_{U \setminus i(Y)} = \mathcal{O}_{U \setminus i(Y)} = \mathcal{O}_{X \setminus i(Y)}|_{U \setminus i(Y)}$. Thus we may glue (see Sheaves, Section **??**) $\mathcal{I}$ and $\mathcal{O}_{X \setminus i(Y)}$ to a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$. Since every point of X has a neighbourhood where $\mathcal{J}$ is quasi-coherent, we see that $\mathcal{J}$ is quasi-coherent (in particular locally generated by sections). By construction

$\mathcal{O}_X/\mathcal{J}$ is supported on U and, restricted there, equal to $\mathcal{O}_U/\mathcal{I}$. Thus we see that the closed subspaces associated to $\mathcal{I}$ and $\mathcal{J}$ are canonically isomorphic, see Example 7.3. In particular the closed subspace of U associated to $\mathcal{I}$ is isomorphic to a closed subspace of X. Since $Y \to U$ is identified with the closed subspace associated to $\mathcal{I}$, see Lemma 7.5, we conclude that $Y \to U \to X$ is a closed immersion. $\square$

Let $f : Y \to X$ be an immersion. Let $Z = \overline{f(Y)} \setminus f(Y)$ which is a closed subset of X. Let $U = X \setminus Z$. The lemma implies that U is the biggest open subspace of X such that $f : Y \to X$ factors through a closed immersion into U. We define a *locally closed subscheme of* X as a pair (Z, U) consisting of a closed subscheme Z of an open subscheme U of X such that in addition $\overline{Z} \cup U = X$. We usually just say "let Z be a locally closed subscheme of X" since we may recover U from the morphism $Z \to X$. The above then shows that any immersion $f : Y \to X$ factors uniquely as $Y \to Z \to X$ where Z is a locally closed subspace of X and $Y \to Z$ is an isomorphism.

The interest of this is that the collection of locally closed subschemes of X forms a set. We may define a partial ordering on this set, which we call inclusion for obvious reasons. To be explicit, if $Z \to X$ and $Z' \to X$ are two locally closed subschemes of X, then we say that Z *is contained in* Z' simply if the morphism $Z \to X$ factors through Z'. If it does, then of course Z is identified with a unique locally closed subscheme of Z', and so on.

For more information on immersions, we refer the reader to Morphisms, Section **??**.

15. Base change in algebraic geometry

One motivation for the introduction of the language of schemes is that it gives a very precise notion of what it means to define a variety over a particular field. For example a variety X over $\mathbf{Q}$ is synonymous (Varieties, Definition **??**) with $X \to \mathrm{Spec}(\mathbf{Q})$ which is of finite type, separated, irreducible and reduced[1]. In any case, the idea is more generally to work with schemes over a given *base scheme*, often denoted S. We use the language: "let X be a scheme over S" to mean simply that X comes equipped with a morphism $X \to S$. In diagrams we will try to picture the *structure morphism* $X \to S$ as a downward arrow from X to S. We are often more interested in the properties of X relative to S rather than the internal geometry of X. For example, we would like to know things about the fibres of $X \to S$, what happens to X after base change, and so on.

We introduce some of the language that is customarily used. Of course this language is just a special case of thinking about the category of objects over a given object in a category, see Categories, Example 1.7.

Definition 15.1. Let S be a scheme.

(1) We say X is a *scheme over* S to mean that X comes equipped with a morphism of schemes $X \to S$. The morphism $X \to S$ is sometimes called the *structure morphism*.
(2) If R is a ring we say X is a *scheme over* R instead of X is a scheme over $\mathrm{Spec}(R)$.

[1] Of course algebraic geometers still quibble over whether one should require X to be geometrically irreducible over $\mathbf{Q}$.

(3) A *morphism $f : X \to Y$ of schemes over S* is a morphism of schemes such that the composition $X \to Y \to S$ of f with the structure morphism of Y is equal to the structure morphism of X.
(4) We denote $\mathrm{Mor}_S(X, Y)$ the set of all morphisms from X to Y over S.
(5) Let X be a scheme over S. Let $S' \to S$ be a morphism of schemes. The *base change* of X is the scheme $X_{S'} = S' \times_S X$ over S'.
(6) Let $f : X \to Y$ be a morphism of schemes over S. Let $S' \to S$ be a morphism of schemes. The *base change* of f is the induced morphism $f' : X_{S'} \to Y_{S'}$ (namely the morphism $\mathrm{id}_{S'} \times_{\mathrm{id}_S} f$).
(7) Let R be a ring. Let X be a scheme over R. Let $R \to R'$ be a ring map. The *base change* $X_{R'}$ is the scheme $\mathrm{Spec}(R') \times_{\mathrm{Spec}(R)} X$ over R'.

Here is a typical result.

Lemma 15.2. *Let S be a scheme. Let $f : X \to Y$ be an immersion (resp. closed immersion, resp. open immersion) of schemes over S. Then any base change of f is an immersion (resp. closed immersion, resp. open immersion).*

Proof. We can think of the base change of f via the morphism $S' \to S$ as the top left vertical arrow in the following commutative diagram:

$$\begin{tikzcd} X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \arrow[dd, bend left] \\ Y_{S'} \arrow[r] \arrow[d] & Y \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

The diagram implies $X_{S'} \cong Y_{S'} \times_Y X$, and the lemma follows from Lemma **??**. $\square$

In fact this type of result is so typical that there is a piece of language to express it. Here it is.

Definition 15.3. Properties and base change.

(1) Let $\mathcal{P}$ be a property of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that $\mathcal{P}$ is *preserved under base change* if whenever X/S has $\mathcal{P}$, any base change $X_{S'}/S'$ has $\mathcal{P}$.
(2) Let $\mathcal{P}$ be a property of morphisms of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that *preserved under base change* if whenever $f : X \to Y$ over S has $\mathcal{P}$, any base change $f' : X_{S'} \to Y_{S'}$ over S' has $\mathcal{P}$.

At this point we can say that "being a closed immersion" is preserved under arbitrary base change.

Definition 15.4. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$ be a point. The *scheme theoretic fibre X_s of f over s*, or simply the *fibre of f over s*, is the scheme fitting in the following fibre product diagram

$$\begin{tikzcd} X_s = \mathrm{Spec}(\kappa(s)) \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd}$$

We think of the fibre X_s always as a scheme over $\kappa(s)$.

Lemma 15.5. *Let $f : X \to S$ be a morphism of schemes. Consider the diagrams*

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}(\mathcal{O}_{S,s}) \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_{S,s}) \arrow[r] & S \end{tikzcd}$$

In both cases the top horizontal arrow is a homeomorphism onto its image.

Proof. Choose an open affine $U \subset S$ that contains s. The bottom horizontal morphisms factor through U, see Lemma **??** for example. Thus we may assume that S is affine. If X is also affine, then the result follows from Algebra, Remark **??**. In the general case the result follows by covering X by open affines. $\square$

16. Proj of a graded ring

This section may be taken as the preamble to the question: what is projective space anyway? Yes, projective space is Proj of a graded ring S. And yes, projective space is a scheme. So in particular it's locally affine.
In this section we construct Proj of a graded ring following [**?**, II, Section 2].

Let S be a graded ring. Consider the topological space $\mathrm{Proj}(S)$ associated to S, see Algebra, Section **??**. We will endow this space with a sheaf of rings $\mathcal{O}_{\mathrm{Proj}(S)}$ such that the resulting pair $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ will be a scheme.

Recall that $\mathrm{Proj}(S)$ has a basis of open sets $D_+(f)$, $f \in S_d$, $d \geq 1$ which we call *standard opens*, see Algebra, Section **??**. This terminology will always imply that f is homogeneous of positive degree even if we forget to mention it. In addition, the intersection of two standard opens is another: $D_+(f) \cap D_+(g) = D_+(fg)$, for $f, g \in S$ homogeneous of positive degree.

Lemma 16.1. *Let S be a graded ring. Let $f \in S$ homogeneous of positive degree.*

(1) *If $g \in S$ homogeneous of positive degree and $D_+(g) \subset D_+(f)$, then*
   (a) *f is invertible in S_g, and $f^{\deg(g)}/g^{\deg(f)}$ is invertible in $S_{(g)}$,*
   (b) *$g^e = af$ for some $e \geq 1$ and $a \in S$ homogeneous,*
   (c) *there is a canonical S-algebra map $S_f \to S_g$,*
   (d) *there is a canonical S_0-algebra map $S_{(f)} \to S_{(g)}$ compatible with the map $S_f \to S_g$,*
   (e) *the map $S_{(f)} \to S_{(g)}$ induces an isomorphism*
   $$(S_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong S_{(g)},$$
   (f) *these maps induce a commutative diagram of topological spaces*
   $$\begin{tikzcd} D_+(g) \arrow[d] & \{\mathbf{Z}\text{-graded primes of } S_g\} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Spec}(S_{(g)}) \arrow[d] \\ D_+(f) & \{\mathbf{Z}\text{-graded primes of } S_f\} \arrow[l] \arrow[r] & \mathrm{Spec}(S_{(f)}) \end{tikzcd}$$
   where the horizontal maps are homeomorphisms and the vertical maps are open immersions,

(g) *there are compatible canonical S_f and $S_{(f)}$-module maps $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ for any graded S-module M, and*
(h) *the map $M_{(f)} \to M_{(g)}$ induces an isomorphism*
$$(M_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong M_{(g)}.$$

(2) *Any open covering of $D_+(f)$ can be refined to a finite open covering of the form $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$.*
(3) *Let $g_1, \ldots, g_n \in S$ be homogeneous of positive degree. Then $D_+(f) \subset \bigcup D_+(g_i)$ if and only if $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$.*

Proof. Recall that $D_+(g) = \text{Spec}(S_{(g)})$ with identification given by the ring maps $S \to S_g \leftarrow S_{(g)}$, see Algebra, Lemma **??**. Thus $f^{\deg(g)}/g^{\deg(f)}$ is an element of $S_{(g)}$ which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma **??**. We conclude that (a) holds. Write the inverse of f in S_g as a/g^d. We may replace a by its homogeneous part of degree $d\deg(g) - \deg(f)$. This means $g^d - af$ is annihilated by a power of g, whence $g^e = af$ for some $a \in S$ homogeneous of degree $e\deg(g) - \deg(f)$. This proves (b). For (c), the map $S_f \to S_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. This clearly induces a map of the subrings $S_{(f)} \to S_{(g)}$ of degree zero elements as well. We can similarly define $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ by mapping x/f^n to $a^n x/g^{ne}$. The statements writing $S_{(g)}$ resp. $M_{(g)}$ as principal localizations of $S_{(f)}$ resp. $M_{(f)}$ are clear from the formulas above. The maps in the commutative diagram of topological spaces correspond to the ring maps given above. The horizontal arrows are homeomorphisms by Algebra, Lemma **??**. The vertical arrows are open immersions since the left one is the inclusion of an open subset.

The open $D_+(f)$ is quasi-compact because it is homeomorphic to $\text{Spec}(S_{(f)})$, see Algebra, Lemma **??**. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma **??**. $\square$

In Sheaves, Section **??** we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas **??** and **??**. Moreover, we showed in Sheaves, Lemma **??** that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

Definition 16.2. Let S be a graded ring. Suppose that $D_+(f) \subset \text{Proj}(S)$ is a standard open. A *standard open covering* of $D_+(f)$ is a covering $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, where $g_1, \ldots, g_n \in S$ are homogeneous of positive degree.

Let S be a graded ring. Let M be a graded S-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \text{Proj}(S)$ is a standard open. If $f, g \in S$ are homogeneous of positive degree such that $D_+(f) = D_+(g)$, then by Lemma 16.1 above there are canonical maps $M_{(f)} \to M_{(g)}$ and $M_{(g)} \to M_{(f)}$ which are mutually inverse. Hence we may choose any f such that $U = D_+(f)$ and define
$$\widetilde{M}(U) = M_{(f)}.$$

Note that if $D_+(g) \subset D_+(f)$, then by Lemma 16.1 above we have a canonical map

$$\widetilde{M}(D_+(f)) = M_{(f)} \longrightarrow M_{(g)} = \widetilde{M}(D_+(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = S$, then $\widetilde{S}$ is a presheaf of rings on the basis of standard opens. And for general M we see that $\widetilde{M}$ is a presheaf of $\widetilde{S}$-modules on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \text{Proj}(S)$. Suppose that x corresponds to the homogeneous prime ideal $\mathfrak{p} \subset S$. By definition of the stalk we see that

$$\widetilde{M}_x = \text{colim}_{f \in S_d, d>0, f \not\in \mathfrak{p}} M_{(f)}$$

Here the set $\{f \in S_d, d > 0, f \not\in \mathfrak{p}\}$ is preordered by the rule $f \geq f' \Leftrightarrow D_+(f) \subset D_+(f')$. If $f_1, f_2 \in S \setminus \mathfrak{p}$ are homogeneous of positive degree, then we have $f_1 f_2 \geq f_1$ in this ordering. In Algebra, Section **??** we defined $M_{(\mathfrak{p})}$ as the module whose elements are fractions x/f with x, f homogeneous, $\deg(x) = \deg(f)$, $f \not\in \mathfrak{p}$. Since $\mathfrak{p} \in \text{Proj}(S)$ there exists at least one $f_0 \in S$ homogeneous of positive degree with $f_0 \not\in \mathfrak{p}$. Hence $x/f = f_0 x/f f_0$ and we see that we may always assume the denominator of an element in $M_{(\mathfrak{p})}$ has positive degree. From these remarks it follows easily that

$$\widetilde{M}_x = M_{(\mathfrak{p})}.$$

Next, we check the sheaf condition for the standard open coverings. If $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(g_i)} \to \bigoplus M_{(g_i g_j)}.$$

Note that $D_+(g_i) = D_+(f g_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(f g_i)} \to \bigoplus M_{(f g_i g_j)}.$$

By Lemma 16.1 we see that $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$, and that the modules $M_{(f g_i)}$, $M_{(f g_i g_j)}$ are the principal localizations of the $S_{(f)}$-module $M_{(f)}$ at these elements and their products. Thus we may apply Algebra, Lemma **??** to the module $M_{(f)}$ over $S_{(f)}$ and the elements $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section **??** that there exists a unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the standard opens. Note that by our computation of stalks above and Algebra, Lemma **??** the stalks of this sheaf of rings are all local rings.

Similarly, for any graded S-module M there exists a unique sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma **??**.

Definition 16.3. Let S be a graded ring.

(1) The *structure sheaf $\mathcal{O}_{Proj(S)}$ of the homogeneous spectrum of S* is the unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the basis of standard opens.
(2) The locally ringed space $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ is called the *homogeneous spectrum* of S and denoted $\text{Proj}(S)$.

(3) The sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules extending $\widetilde{M}$ to all opens of $\text{Proj}(S)$ is called the sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

Lemma 16.4. *Let S be a graded ring. Let M be a graded S-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{Proj(S)}$-modules associated to M.*

(1) *For every $f \in S$ homogeneous of positive degree we have*
$$\Gamma(D_+(f), \mathcal{O}_{Proj(S)}) = S_{(f)}.$$
(2) *For every $f \in S$ homogeneous of positive degree we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ as an $S_{(f)}$-module.*
(3) *Whenever $D_+(g) \subset D_+(f)$ the restriction mappings on $\mathcal{O}_{Proj(S)}$ and $\widetilde{M}$ are the maps $S_{(f)} \to S_{(g)}$ and $M_{(f)} \to M_{(g)}$ from Lemma 16.1.*
(4) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{O}_{Proj(S),x} = S_{(\mathfrak{p})}$.*
(5) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{F}_x = M_{(\mathfrak{p})}$ as an $S_{(\mathfrak{p})}$-module.*
(6) *There is a canonical ring map $S_0 \longrightarrow \Gamma(Proj(S), \widetilde{S})$ and a canonical S_0-module map $M_0 \longrightarrow \Gamma(Proj(S), \widetilde{M})$ compatible with the descriptions of sections over standard opens and stalks above.*

Moreover, all these identifications are functorial in the graded S-module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of graded S-modules to the category of $\mathcal{O}_{Proj(S)}$-modules.

Proof. Assertions (1) - (5) are clear from the discussion above. We see (6) since there are canonical maps $M_0 \to M_{(f)}$, $x \mapsto x/1$ compatible with the restriction maps described in (3). The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{(\mathfrak{p})}$ is exact (see Algebra, Lemma **??**) and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma **??**. □

Remark 16.5. The map from M_0 to the global sections of $\widetilde{M}$ is generally far from being an isomorphism. A trivial example is to take $S = k[x, y, z]$ with $1 = \deg(x) = \deg(y) = \deg(z)$ (or any number of variables) and to take $M = S/(x^{100}, y^{100}, z^{100})$. It is easy to see that $\widetilde{M} = 0$, but $M_0 = k$.

Lemma 16.6. *Let S be a graded ring. Let $f \in S$ be homogeneous of positive degree. Suppose that $D(g) \subset \text{Spec}(S_{(f)})$ is a standard open. Then there exists an $h \in S$ homogeneous of positive degree such that $D(g)$ corresponds to $D_+(h) \subset D_+(f)$ via the homeomorphism of Algebra, Lemma **??**. In fact we can take h such that $g = h/f^n$ for some n.*

Proof. Write $g = h/f^n$ for some h homogeneous of positive degree and some $n \geq 1$. If $D_+(h)$ is not contained in $D_+(f)$ then we replace h by hf and n by $n+1$. Then h has the required shape and $D_+(h) \subset D_+(f)$ corresponds to $D(g) \subset \text{Spec}(S_{(f)})$. □

Lemma 16.7. *Let S be a graded ring. The locally ringed space $Proj(S)$ is a scheme. The standard opens $D_+(f)$ are affine opens. For any graded S-module M the sheaf $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{Proj(S)}$-modules.*

Proof. Consider a standard open $D_+(f) \subset \text{Proj}(S)$. By Lemmas 16.1 and 16.4 we have $\Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}) = S_{(f)}$, and we have a homeomorphism $\varphi : D_+(f) \to \text{Spec}(S_{(f)})$. For any standard open $D(g) \subset \text{Spec}(S_{(f)})$ we may pick an $h \in S_+$ as in Lemma 16.6. Then $\varphi^{-1}(D(g)) = D_+(h)$, and by Lemmas 16.4 and 16.1 we see

$$\Gamma(D_+(h), \mathcal{O}_{\text{Proj}(S)}) = S_{(h)} = (S_{(f)})_{h^{\deg(f)}/f^{\deg(h)}} = (S_{(f)})_g = \Gamma(D(g), \mathcal{O}_{\text{Spec}(S_{(f)})}).$$

Thus the restriction of $\mathcal{O}_{\text{Proj}(S)}$ to $D_+(f)$ corresponds via the homeomorphism φ exactly to the sheaf $\mathcal{O}_{\text{Spec}(S_{(f)})}$ as defined in Schemes, Section **??**. We conclude that $D_+(f)$ is an affine scheme isomorphic to $\text{Spec}(S_{(f)})$ via φ and hence that $\text{Proj}(S)$ is a scheme.

In exactly the same way we show that $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules. Namely, the argument above will show that

$$\widetilde{M}|_{D_+(f)} \cong \varphi^*\left(\widetilde{M_{(f)}}\right)$$

which shows that $\widetilde{M}$ is quasi-coherent. $\square$

Lemma 16.8. *Let S be a graded ring. The scheme Proj(S) is separated.*

Proof. We have to show that the canonical morphism $\text{Proj}(S) \to \text{Spec}(\mathbf{Z})$ is separated. We will use Schemes, Lemma **??**. Thus it suffices to show given any pair of standard opens $D_+(f)$ and $D_+(g)$ that $D_+(f) \cap D_+(g) = D_+(fg)$ is affine (clear) and that the ring map

$$S_{(f)} \otimes_{\mathbf{Z}} S_{(g)} \longrightarrow S_{(fg)}$$

is surjective. Any element s in $S_{(fg)}$ is of the form $s = h/(f^n g^m)$ with $h \in S$ homogeneous of degree $n\deg(f) + m\deg(g)$. We may multiply h by a suitable monomial $f^i g^j$ and assume that $n = n'\deg(g)$, and $m = m'\deg(f)$. Then we can rewrite s as $s = h/f^{(n'+m')\deg(g)} \cdot f^{m'\deg(g)}/g^{m'\deg(f)}$. So s is indeed in the image of the displayed arrow. $\square$

Lemma 16.9. *Let S be a graded ring. The scheme Proj(S) is quasi-compact if and only if there exist finitely many homogeneous elements $f_1, \ldots, f_n \in S_+$ such that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$. In this case Proj$(S) = D_+(f_1) \cup \ldots \cup D_+(f_n)$.*

Proof. Given such a collection of elements the standard affine opens $D_+(f_i)$ cover $\text{Proj}(S)$ by Algebra, Lemma **??**. Conversely, if $\text{Proj}(S)$ is quasi-compact, then we may cover it by finitely many standard opens $D_+(f_i)$, $i = 1, \ldots, n$ and we see that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$ by the lemma referenced above. $\square$

Lemma 16.10. *Let S be a graded ring. The scheme Proj(S) has a canonical morphism towards the affine scheme* $\text{Spec}(S_0)$*, agreeing with the map on topological spaces coming from Algebra, Definition* **??**.

Proof. We saw above that our construction of $\widetilde{S}$, resp. $\widetilde{M}$ gives a sheaf of S_0-algebras, resp. S_0-modules. Hence we get a morphism by Schemes, Lemma **??**. This morphism, when restricted to $D_+(f)$ comes from the canonical ring map $S_0 \to S_{(f)}$. The maps $S \to S_f$, $S_{(f)} \to S_f$ are S_0-algebra maps, see Lemma 16.1. Hence if the homogeneous prime $\mathfrak{p} \subset S$ corresponds to the $\mathbf{Z}$-graded prime $\mathfrak{p}' \subset S_f$ and the (usual) prime $\mathfrak{p}'' \subset S_{(f)}$, then each of these has the same inverse image in S_0. $\square$

Lemma 16.11. *Let S be a graded ring. If S is finitely generated as an algebra over S_0, then the morphism Proj$(S) \to \mathrm{Spec}(S_0)$ satisfies the existence and uniqueness parts of the valuative criterion, see Schemes, Definition* **??**.

Proof. The uniqueness part follows from the fact that $\mathrm{Proj}(S)$ is separated (Lemma 16.8 and Schemes, Lemma **??**). Choose $x_i \in S_+$ homogeneous, $i = 1, \ldots, n$ which generate S over S_0. Let $d_i = \deg(x_i)$ and set $d = \mathrm{lcm}\{d_i\}$. Suppose we are given a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Proj}(S) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(S_0) \end{tikzcd}$$

as in Schemes, Definition **??**. Denote $v : K^* \to \Gamma$ the valuation of A, see Algebra, Definition **??**. We may choose an $f \in S_+$ homogeneous such that $\mathrm{Spec}(K)$ maps into $D_+(f)$. Then we get a commutative diagram of ring maps

$$\begin{tikzcd} K & S_{(f)} \arrow[l, "\varphi"'] \\ A \arrow[u] & S_0 \arrow[l] \arrow[u] \end{tikzcd}$$

After renumbering we may assume that $\varphi(x_i^{\deg(f)}/f^{d_i})$ is nonzero for $i = 1, \ldots, r$ and zero for $i = r+1, \ldots, n$. Since the open sets $D_+(x_i)$ cover $\mathrm{Proj}(S)$ we see that $r \geq 1$. Let $i_0 \in \{1, \ldots, r\}$ be an index minimizing $\gamma_i = (d/d_i)v(\varphi(x_i^{\deg(f)}/f^{d_i}))$ in Γ. For convenience set $x_0 = x_{i_0}$ and $d_0 = d_{i_0}$. The ring map φ factors though a map $\varphi' : S_{(fx_0)} \to K$ which gives a ring map $S_{(x_0)} \to S_{(fx_0)} \to K$. The algebra $S_{(x_0)}$ is generated over S_0 by the elements $x_1^{e_1} \ldots x_n^{e_n}/x_0^{e_0}$, where $\sum e_i d_i = e_0 d_0$. If $e_i > 0$ for some $i > r$, then $\varphi'(x_1^{e_1} \ldots x_n^{e_n}/x_0^{e_0}) = 0$. If $e_i = 0$ for $i > r$, then we have

$$\begin{aligned} d\deg(f)v(\varphi'(x_1^{e_1} \ldots x_r^{e_r}/x_0^{e_0})) &= dv(\varphi'(x_1^{e_1\deg(f)} \ldots x_r^{e_r\deg(f)}/x_0^{e_0\deg(f)})) \\ &= d\sum e_i v(\varphi'(x_i^{\deg(f)}/f^{d_i})) - e_0 v(\varphi'(x_0^{\deg(f)}/f^{d_0})) \\ &= \sum e_i d_i \gamma_i - e_0 d_0 \gamma_0 \\ &\geq \sum e_i d_i \gamma_0 - e_0 d_0 \gamma_0 = 0 \end{aligned}$$

because γ_0 is minimal among the γ_i. This implies that $S_{(x_0)}$ maps into A via φ'. The corresponding morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(S_{(x_0)}) = D_+(x_0) \subset \mathrm{Proj}(S)$ provides the morphism fitting into the first commutative diagram of this proof. $\square$

We saw in the proof of Lemma 16.11 that, under the hypotheses of that lemma, the morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is quasi-compact as well. Hence (by Schemes, Proposition **??**) we see that $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is universally closed in the situation of the lemma. We give several examples showing these results do not hold without some assumption on the graded ring S.

Example 16.12. Let $\mathbf{C}[X_1, X_2, X_3, \ldots]$ be the graded $\mathbf{C}$-algebra with each X_i in degree 1. Consider the ring map

$$\mathbf{C}[X_1, X_2, X_3, \ldots] \longrightarrow \mathbf{C}[t^\alpha; \alpha \in \mathbf{Q}_{\geq 0}]$$

which maps X_i to $t^{1/i}$. The right hand side becomes a valuation ring A upon localization at the ideal $\mathfrak{m} = (t^\alpha; \alpha > 0)$. Let K be the fraction field of A. The above gives a morphism $\mathrm{Spec}(K) \to \mathrm{Proj}(\mathbf{C}[X_1, X_2, X_3, \ldots])$ which does not extend to a morphism defined on all of $\mathrm{Spec}(A)$. The reason is that the image of $\mathrm{Spec}(A)$ would be contained in one of the $D_+(X_i)$ but then X_{i+1}/X_i would map to an element of A which it doesn't since it maps to $t^{1/(i+1)-1/i}$.

Example 16.13. Let $R = \mathbf{C}[t]$ and

$$S = R[X_1, X_2, X_3, \ldots]/(X_i^2 - tX_{i+1}).$$

The grading is such that $R = S_0$ and $\deg(X_i) = 2^{i-1}$. Note that if $\mathfrak{p} \in \mathrm{Proj}(S)$ then $t \notin \mathfrak{p}$ (otherwise $\mathfrak{p}$ has to contain all of the X_i which is not allowed for an element of the homogeneous spectrum). Thus we see that $D_+(X_i) = D_+(X_{i+1})$ for all i. Hence $\mathrm{Proj}(S)$ is quasi-compact; in fact it is affine since it is equal to $D_+(X_1)$. It is easy to see that the image of $\mathrm{Proj}(S) \to \mathrm{Spec}(R)$ is $D(t)$. Hence the morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(R)$ is not closed. Thus the valuative criterion cannot apply because it would imply that the morphism is closed (see Schemes, Proposition **??**).

Example 16.14. Let A be a ring. Let $S = A[T]$ as a graded A algebra with T in degree 1. Then the canonical morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$ (see Lemma 16.10) is an isomorphism.

Example 16.15. Let $X = \mathrm{Spec}(A)$ be an affine scheme, and let $U \subset X$ be an open subscheme. Grade $A[T]$ by setting $\deg T = 1$. Define S to be the subring of $A[T]$ generated by A and all fT^i, where $i \geq 0$ and where $f \in A$ is such that $D(f) \subset U$. We claim that S is a graded ring with $S_0 = A$ such that $\mathrm{Proj}(S) \cong U$, and this isomorphism identifies the canonical morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$ of Lemma 16.10 with the inclusion $U \subset X$.

Suppose $\mathfrak{p} \in \mathrm{Proj}(S)$ is such that every $fT \in S_1$ is in $\mathfrak{p}$. Then every generator fT^i with $i \geq 1$ is in $\mathfrak{p}$ because $(fT^i)^2 = (fT)(fT^{2i-1}) \in \mathfrak{p}$ and $\mathfrak{p}$ is radical. But then $\mathfrak{p} \supset S_+$, which is impossible. Consequently $\mathrm{Proj}(S)$ is covered by the standard open affine subsets $\{D_+(fT)\}_{fT \in S_1}$.

Observe that, if $fT \in S_1$, then the inclusion $S \subset A[T]$ induces a graded isomorphism of $S[(fT)^{-1}]$ with $A[T, T^{-1}, f^{-1}]$. Hence the standard open subset $D_+(fT) \cong \mathrm{Spec}(S_{(fT)})$ is isomorphic to $\mathrm{Spec}(A[T, T^{-1}, f^{-1}]_0) = \mathrm{Spec}(A[f^{-1}])$. It is clear that this isomorphism is a restriction of the canonical morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$. If in addition $gT \in S_1$, then $S[(fT)^{-1}, (gT)^{-1}] \cong A[T, T^{-1}, f^{-1}, g^{-1}]$ as graded rings, so $D_+(fT) \cap D_+(gT) \cong \mathrm{Spec}(A[f^{-1}, g^{-1}])$. Therefore $\mathrm{Proj}(S)$ is the union of open subschemes $D_+(fT)$ which are isomorphic to the open subschemes $D(f) \subset X$ under the canonical morphism, and these open subschemes intersect in $\mathrm{Proj}(S)$ in the same way they do in X. We conclude that the canonical morphism is an isomorphism of $\mathrm{Proj}(S)$ with the union of all $D(f) \subset U$, which is U.

17. Projective space

I have skipped lots of sections between the former and this one.
Projective space is one of the fundamental objects studied in algebraic geometry. In this section we just give its construction as Proj of a polynomial ring. Later we will discover many of its beautiful properties.

Lemma 17.1. *Let $S = \mathbf{Z}[T_0, \ldots, T_n]$ with $\deg(T_i) = 1$. The scheme*

$$\mathbf{P}^n_{\mathbf{Z}} = \mathit{Proj}(S)$$

represents the functor which associates to a scheme Y the pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ where

1. *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
2. *$s_0, \ldots, s_n$ are global sections of $\mathcal{L}$ which generate $\mathcal{L}$*

up to the following equivalence: $(\mathcal{L}, (s_0, \ldots, s_n)) \sim (\mathcal{N}, (t_0, \ldots, t_n)) \Leftrightarrow$ there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{N}$ with $\beta(s_i) = t_i$ for $i = 0, \ldots, n$.

Proof. This is a special case of Lemma **??** above. Namely, for any graded ring A we have

$$\begin{array}{rcl} \mathrm{Mor}_{gradedrings}(\mathbf{Z}[T_0, \ldots, T_n], A) & = & A_1 \times \ldots \times A_1 \\ \psi & \mapsto & (\psi(T_0), \ldots, \psi(T_n)) \end{array}$$

and the degree 1 part of $\Gamma_*(Y, \mathcal{L})$ is just $\Gamma(Y, \mathcal{L})$. □

Definition 17.2. The scheme $\mathbf{P}^n_{\mathbf{Z}} = \mathrm{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ is called *projective n-space over $\mathbf{Z}$*. Its base change $\mathbf{P}^n_S$ to a scheme S is called *projective n-space over S*. If R is a ring the base change to $\mathrm{Spec}(R)$ is denoted $\mathbf{P}^n_R$ and called *projective n-space over R*.

Given a scheme Y over S and a pair $(\mathcal{L}, (s_0, \ldots, s_n))$ as in Lemma 17.1 the induced morphism to $\mathbf{P}^n_S$ is denoted

$$\varphi_{(\mathcal{L}, (s_0, \ldots, s_n))} : Y \longrightarrow \mathbf{P}^n_S$$

This makes sense since the pair defines a morphism into $\mathbf{P}^n_{\mathbf{Z}}$ and we already have the structure morphism into S so combined we get a morphism into $\mathbf{P}^n_S = \mathbf{P}^n_{\mathbf{Z}} \times S$. Note that this is the S-morphism characterized by

$$\mathcal{L} = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} \mathcal{O}_{\mathbf{P}^n_R}(1) \quad \text{and} \quad s_i = \varphi^*_{(\mathcal{L}, (s_0, \ldots, s_n))} T_i$$

where we think of T_i as a global section of $\mathcal{O}_{\mathbf{P}^n_S}(1)$ via (19.1.3).

Lemma 17.3. *Projective n-space over $\mathbf{Z}$ is covered by $n + 1$ standard opens*

$$\mathbf{P}^n_{\mathbf{Z}} = \bigcup\nolimits_{i=0,\ldots,n} D_+(T_i)$$

where each $D_+(T_i)$ is isomorphic to $\mathbf{A}^n_{\mathbf{Z}}$ affine n-space over $\mathbf{Z}$.

Proof. This is true because $\mathbf{Z}[T_0, \ldots, T_n]_+ = (T_0, \ldots, T_n)$ and since

$$\mathrm{Spec}\left(\mathbf{Z}\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\right) \cong \mathbf{A}^n_{\mathbf{Z}}$$

in an obvious way. □

Lemma 17.4. *Let S be a scheme. The structure morphism $\mathbf{P}^n_S \to S$ is*

1. *separated,*
2. *quasi-compact,*
3. *satisfies the existence and uniqueness parts of the valuative criterion, and*
4. *universally closed.*

Proof. All these properties are stable under base change (this is clear for the last two and for the other two see Schemes, Lemmas **??** and **??**). Hence it suffices to prove them for the morphism $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$. Separatedness is Lemma 16.8. Quasi-compactness follows from Lemma 17.3. Existence and uniqueness of the valuative criterion follow from Lemma 16.11. Universally closed follows from the above and Schemes, Proposition **??**. □

Remark 17.5. What's missing in the list of properties above? Well to be sure the property of being of finite type. The reason we do not list this here is that we have not yet defined the notion of finite type at this point. (Another property which is missing is "smoothness". And I'm sure there are many more you can think of.)

Lemma 17.6 (Segre embedding)**.** *Let S be a scheme. There exists a closed immersion*

$$\mathbf{P}^n_S \times_S \mathbf{P}^m_S \longrightarrow \mathbf{P}^{nm+n+m}_S$$

called the Segre embedding.

Proof. It suffices to prove this when $S = \mathrm{Spec}(\mathbf{Z})$. Hence we will drop the index S and work in the absolute setting. Write $\mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[X_0, \ldots, X_n])$, $\mathbf{P}^m = \mathrm{Proj}(\mathbf{Z}[Y_0, \ldots, Y_m])$, and $\mathbf{P}^{nm+n+m} = \mathrm{Proj}(\mathbf{Z}[Z_0, \ldots, Z_{nm+n+m}])$. In order to map into $\mathbf{P}^{nm+n+m}$ we have to write down an invertible sheaf $\mathcal{L}$ on the left hand side and $(n+1)(m+1)$ sections s_i which generate it. See Lemma 17.1. The invertible sheaf we take is

$$\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}^m}(1)$$

The sections we take are

$$s_0 = X_0Y_0,\ s_1 = X_1Y_0, \ldots,\ s_n = X_nY_0,\ s_{n+1} = X_0Y_1, \ldots,\ s_{nm+n+m} = X_nY_m.$$

These generate $\mathcal{L}$ since the sections X_i generate $\mathcal{O}_{\mathbf{P}^n}(1)$ and the sections Y_j generate $\mathcal{O}_{\mathbf{P}^m}(1)$. The induced morphism φ has the property that

$$\varphi^{-1}(D_+(Z_{i+(n+1)j})) = D_+(X_i) \times D_+(Y_j).$$

Hence it is an affine morphism. The corresponding ring map in case $(i,j) = (0,0)$ is the map

$$\mathbf{Z}[Z_1/Z_0, \ldots, Z_{nm+n+m}/Z_0] \longrightarrow \mathbf{Z}[X_1/X_0, \ldots, X_n/X_0, Y_1/Y_0, \ldots, Y_n/Y_0]$$

which maps Z_i/Z_0 to the element X_i/X_0 for $i \leq n$ and the element $Z_{(n+1)j}/Z_0$ to the element Y_j/Y_0. Hence it is surjective. A similar argument works for the other affine open subsets. Hence the morphism φ is a closed immersion (see Schemes, Lemma **??** and Example **??**.) □

The following two lemmas are special cases of more general results later, but perhaps it makes sense to prove these directly here now.

Lemma 17.7. *Let R be a ring. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme. Let*

$$I_d = \mathrm{Ker}\left(R[T_0, \ldots, T_n]_d \longrightarrow \Gamma(Z, \mathcal{O}_{\mathbf{P}^n_R}(d)|_Z)\right)$$

Then $I = \bigoplus I_d \subset R[T_0, \ldots, T_n]$ is a graded ideal and $Z = Proj(R[T_0, \ldots, T_n]/I)$.

Proof. It is clear that I is a graded ideal. Set $Z' = \mathrm{Proj}(R[T_0, \ldots, T_n]/I)$. By Lemma 20.5 we see that Z' is a closed subscheme of $\mathbf{P}^n_R$. To see the equality $Z = Z'$ it suffices to check on an standard affine open $D_+(T_i)$. By renumbering the homogeneous coordinates we may assume $i = 0$. Say $Z \cap D_+(T_0)$, resp. $Z' \cap D_+(T_0)$

is cut out by the ideal J, resp. J' of $R[T_1/T_0, \ldots, T_n/T_0]$. Then J' is the ideal generated by the elements $F/T_0^{\deg(F)}$ where $F \in I$ is homogeneous. Suppose the degree of $F \in I$ is d. Since F vanishes as a section of $\mathcal{O}_{\mathbf{P}^n_R}(d)$ restricted to Z we see that F/T_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $f \in J$. If f has total degree d in $T_1/T_0, \ldots, T_n/T_0$, then we can write $f = F/T_0^d$ for some $F \in R[T_0, \ldots, T_n]_d$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(T_i)$ is cut out by some ideal $J_i \subset R[T_0/T_i, \ldots, T_n/T_i]$. Moreover,

$$J \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] = J_i \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]$$

The left hand side is the localization of J with respect to the element T_i/T_0 and the right hand side is the localization of J_i with respect to the element T_0/T_i. It follows that $T_0^{d_i} F/T_i^{d+d_i}$ is an element of J_i for some d_i sufficiently large. This proves that $T_0^{\max(d_i)} F$ is an element of I, because its restriction to each standard affine open $D_+(T_i)$ vanishes on the closed subscheme $Z \cap D_+(T_i)$. Hence $f \in J'$ and we conclude $J \subset J'$ as desired. $\square$

The following lemma is a special case of the more general Properties, Lemmas **??** or **??**.

Lemma 17.8. *Let R be a ring. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbf{P}^n_R$. For $d \geq 0$ set*

$$M_d = \Gamma(\mathbf{P}^n_R, \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(d)) = \Gamma(\mathbf{P}^n_R, \mathcal{F}(d))$$

Then $M = \bigoplus_{d \geq 0} M_d$ is a graded $R[T_0, \ldots, R_n]$-module and there is a canonical isomorphism $\mathcal{F} = \widetilde{M}$.

Proof. The multiplication maps

$$R[T_0, \ldots, R_n]_e \times M_d \longrightarrow M_{d+e}$$

come from the natural isomorphisms

$$\mathcal{O}_{\mathbf{P}^n_R}(e) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{F}(d) \longrightarrow \mathcal{F}(e+d)$$

see Equation (19.1.4). Let us construct the map $c : \widetilde{M} \to \mathcal{F}$. On each of the standard affines $U_i = D_+(T_i)$ we see that $\Gamma(U_i, \widetilde{M}) = (M[1/T_i])_0$ where the subscript $_0$ means degree 0 part. An element of this can be written as m/T_i^d with $m \in M_d$. Since T_i is a generator of $\mathcal{O}(1)$ over U_i we can always write $m|_{U_i} = m_i \otimes T_i^d$ where $m_i \in \Gamma(U_i, \mathcal{F})$ is a unique section. Thus a natural guess is $c(m/T_i^d) = m_i$. A small argument, which is omitted here, shows that this gives a well defined map $c : \widetilde{M} \to \mathcal{F}$ if we can show that

$$(T_i/T_j)^d m_i|_{U_i \cap U_j} = m_j|_{U_i \cap U_j}$$

in $M[1/T_iT_j]$. But this is clear since on the overlap the generators T_i and T_j of $\mathcal{O}(1)$ differ by the invertible function T_i/T_j.

Injectivity of c. We may check for injectivity over the affine opens U_i. Let $i \in \{0, \ldots, n\}$ and let s be an element $s = m/T_i^d \in \Gamma(U_i, \widetilde{M})$ such that $c(m/T_i^d) = 0$. By the description of c above this means that $m_i = 0$, hence $m|_{U_i} = 0$. Hence $T_i^e m = 0$ in M for some e. Hence $s = m/T_i^d = T_i^e/T_i^{e+d} = 0$ as desired.

Surjectivity of c. We may check for surjectivity over the affine opens U_i. By renumbering it suffices to check it over U_0. Let $s \in \mathcal{F}(U_0)$. Let us write $\mathcal{F}|_{U_i} = \widetilde{N_i}$ for some $R[T_0/T_i, \ldots, T_0/T_i]$-module N_i, which is possible because $\mathcal{F}$ is quasi-coherent. So s corresponds to an element $x \in N_0$. Then we have that

$$(N_i)_{T_j/T_i} \cong (N_j)_{T_i/T_j}$$

(where the subscripts mean "principal localization at") as modules over the ring

$$R\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}, \frac{T_0}{T_j}, \ldots, \frac{T_n}{T_j}\right].$$

This means that for some large integer d there exist elements $s_i \in N_i$, $i = 1, \ldots, n$ such that

$$s = (T_i/T_0)^d s_i$$

on $U_0 \cap U_i$. Next, we look at the difference

$$t_{ij} = s_i - (T_j/T_i)^d s_j$$

on $U_i \cap U_j$, $0 < i < j$. By our choice of s_i we know that $t_{ij}|_{U_0 \cap U_i \cap U_j} = 0$. Hence there exists a large integer e such that $(T_0/T_i)^e t_{ij} = 0$. Set $s_i' = (T_0/T_i)^e s_i$, and $s_0' = s$. Then we will have

$$s_a' = (T_b/T_a)^{e+d} s_b'$$

on $U_a \cap U_b$ for all a, b. This is exactly the condition that the elements s_a' glue to a global section $m \in \Gamma(\mathbf{P}^n_R, \mathcal{F}(e+d))$. And moreover $c(m/T_0^{e+d}) = s$ by construction. Hence c is surjective and we win. □

Lemma 17.9. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf and let $s_0, \ldots, s_n$ be global sections of $\mathcal{L}$ which generate it. Let $\mathcal{F}$ be the kernel of the induced map $\mathcal{O}_X^{\oplus n+1} \to \mathcal{L}$. Then $\mathcal{F} \otimes \mathcal{L}$ is globally generated.*

Proof. In fact the result is true if X is any locally ringed space. The sheaf $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module of rank n. The elements

$$s_{ij} = (0, \ldots, 0, s_j, 0, \ldots, 0, -s_i, 0, \ldots, 0) \in \Gamma(X, \mathcal{L}^{\oplus n+1})$$

with s_j in the ith spot and $-s_i$ in the jth spot map to zero in $\mathcal{L}^{\otimes 2}$. Hence $s_{ij} \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L})$. A local computation shows that these sections generate $\mathcal{F} \otimes \mathcal{L}$.

Alternative proof. Consider the morphism $\varphi : X \to \mathbf{P}^n_{\mathbf{Z}}$ associated to the pair $(\mathcal{L}, (s_0, \ldots, s_n))$. Since the pullback of $\mathcal{O}(1)$ is $\mathcal{L}$ and since the pullback of T_i is s_i, it suffices to prove the lemma in the case of $\mathbf{P}^n_{\mathbf{Z}}$. In this case the sheaf $\mathcal{F}$ corresponds to the graded $S = \mathbf{Z}[T_0, \ldots, T_n]$ module M which fits into the short exact sequence

$$0 \to M \to S^{\oplus n+1} \to S(1) \to 0$$

where the second map is given by $T_0, \ldots, T_n$. In this case the statement above translates into the statement that the elements

$$T_{ij} = (0, \ldots, 0, T_j, 0, \ldots, 0, -T_i, 0, \ldots, 0) \in M(1)_0$$

generate the graded module $M(1)$ over S. We omit the details. □

18. Quasi-coherent sheaves on Proj

Let S be a graded ring. Let M be a graded S-module. We saw in Lemma 16.4 how to construct a quasi-coherent sheaf of modules $\widetilde{M}$ on $\mathrm{Proj}(S)$ and a map

$$M_0 \longrightarrow \Gamma(\mathrm{Proj}(S), \widetilde{M}) \tag{18.0.1}$$

of the degree 0 part of M to the global sections of $\widetilde{M}$. The degree 0 part of the nth twist $M(n)$ of the graded module M (see Algebra, Section **??**) is equal to M_n. Hence we can get maps

$$M_n \longrightarrow \Gamma(\mathrm{Proj}(S), \widetilde{M(n)}). \tag{18.0.2}$$

We would like to be able to perform this operation for any quasi-coherent sheaf $\mathcal{F}$ on $\mathrm{Proj}(S)$. We will do this by tensoring with the nth twist of the structure sheaf, see Definition 19.1. In order to relate the two notions we will use the following lemma.

Lemma 18.1. *Let S be a graded ring. Let $(X, \mathcal{O}_X) = (Proj(S), \mathcal{O}_{Proj(S)})$ be the scheme of Lemma 16.7. Let $f \in S_+$ be homogeneous. Let $x \in X$ be a point corresponding to the homogeneous prime $\mathfrak{p} \subset S$. Let M, N be graded S-modules. There is a canonical map of $\mathcal{O}_{Proj(S)}$-modules*

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

which induces the canonical map $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$ on sections over $D_+(f)$ and the canonical map $M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \to (M \otimes_S N)_{(\mathfrak{p})}$ on stalks at x. Moreover, the following diagram

$$\begin{tikzcd} M_0 \otimes_{S_0} N_0 \arrow[r] \arrow[d] & (M \otimes_S N)_0 \arrow[d] \\ \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \arrow[r] & \Gamma(X, \widetilde{M \otimes_S N}) \end{tikzcd}$$

is commutative where the vertical maps are given by (18.0.1).

Proof. To construct a morphism as displayed is the same as constructing a $\mathcal{O}_X$-bilinear map

$$\widetilde{M} \times \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

see Modules, Section **??**. It suffices to define this on sections over the opens $D_+(f)$ compatible with restriction mappings. On $D_+(f)$ we use the $S_{(f)}$-bilinear map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$, $(x/f^n, y/f^m) \mapsto (x \otimes y)/f^{n+m}$. Details omitted. $\square$

Remark 18.2. In general the map constructed in Lemma 18.1 above is not an isomorphism. Here is an example. Let k be a field. Let $S = k[x, y, z]$ with k in degree 0 and $\deg(x) = 1$, $\deg(y) = 2$, $\deg(z) = 3$. Let $M = S(1)$ and $N = S(2)$, see Algebra, Section **??** for notation. Then $M \otimes_S N = S(3)$. Note that

$$\begin{aligned} S_z &= k[x, y, z, 1/z] \\ S_{(z)} &= k[x^3/z, xy/z, y^3/z^2] \cong k[u, v, w]/(uw - v^3) \\ M_{(z)} &= S_{(z)} \cdot x + S_{(z)} \cdot y^2/z \subset S_z \\ N_{(z)} &= S_{(z)} \cdot y + S_{(z)} \cdot x^2 \subset S_z \\ S(3)_{(z)} &= S_{(z)} \cdot z \subset S_z \end{aligned}$$

Consider the maximal ideal $\mathfrak{m} = (u, v, w) \subset S_{(z)}$. It is not hard to see that both $M_{(z)}/\mathfrak{m}M_{(z)}$ and $N_{(z)}/\mathfrak{m}N_{(z)}$ have dimension 2 over $\kappa(\mathfrak{m})$. But $S(3)_{(z)}/\mathfrak{m}S(3)_{(z)}$ has dimension 1. Thus the map $M_{(z)} \otimes N_{(z)} \to S(3)_{(z)}$ is not an isomorphism.

19. Invertible sheaves on Proj

Here's the construction of the twisted sheaves $\mathcal{O}_X(n)$. The point is that there is a good way to pass from an S-module M to an $\mathcal{O}_X$-module, called $\widetilde{M}$. So basically you just define the twists by $M(d)_n = M_{n+d}$ and apply that construction.

Pick an element of degree d. Then think of the module generated by this element. The least degree piece of such a module is the degree d piece of the original module, that is we have $M(-d)$.

Recall from Algebra, Section **??** the construction of the twisted module $M(n)$ associated to a graded module over a graded ring.

Definition 19.1. Let S be a graded ring. Let $X = \text{Proj}(S)$.

(1) We define $\mathcal{O}_X(n) = \widetilde{S(n)}$. This is called the nth *twist of the structure sheaf of Proj*(S).
(2) For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

We are going to use Lemma 18.1 to construct some canonical maps. Since $S(n) \otimes_S S(m) = S(n+m)$ we see that there are canonical maps

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m). \tag{19.1.1}$$

These maps are not isomorphisms in general, see the example in Remark 18.2. The same example shows that $\mathcal{O}_X(n)$ is *not* an invertible sheaf on X in general. Tensoring with an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ we get maps

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n+m). \tag{19.1.2}$$

The maps (18.0.2) on global sections give a map of graded rings

$$S \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)). \tag{19.1.3}$$

And for an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ the maps (19.1.2) give a graded module structure

$$\bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)) \times \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \longrightarrow \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \tag{19.1.4}$$

and via (19.1.3) also a S-module structure. More generally, given any graded S-module M we have $M(n) = M \otimes_S S(n)$. Hence we get maps

$$\widetilde{M}(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \longrightarrow \widetilde{M(n)}. \tag{19.1.5}$$

On global sections (18.0.2) defines a map of graded S-modules

$$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M(n)}). \tag{19.1.6}$$

Here is an important fact which follows basically immediately from the definitions.

Lemma 19.2. *Let S be a graded ring. Set $X = Proj(S)$. Let $f \in S$ be homogeneous of degree $d > 0$. The sheaves $\mathcal{O}_X(nd)|_{D_+(f)}$ are invertible, and in fact trivial for all $n \in \mathbf{Z}$ (see Modules, Definition **??**). The maps (19.1.1) restricted to $D_+(f)$*

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(nd+m)|_{D_+(f)},$$

the maps (19.1.2) restricted to $D_+(f)$

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{F}(m)|_{D_+(f)} \longrightarrow \mathcal{F}(nd+m)|_{D_+(f)},$$

and the maps (19.1.5) restricted to $D_+(f)$

$$\widetilde{M}(nd)|_{D_+(f)} = \widetilde{M}|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(nd)|_{D_+(f)} \longrightarrow \widetilde{M(nd)}|_{D_+(f)}$$

are isomorphisms for all $n, m \in \mathbf{Z}$.

Proof. The (not graded) S-module maps $S \to S(nd)$, and $M \to M(nd)$, given by $x \mapsto f^n x$ become isomorphisms after inverting f. The first shows that $S_{(f)} \cong S(nd)_{(f)}$ which gives an isomorphism $\mathcal{O}_{D_+(f)} \cong \mathcal{O}_X(nd)|_{D_+(f)}$. The second shows that the map $S(nd)_{(f)} \otimes_{S_{(f)}} M_{(f)} \to M(nd)_{(f)}$ is an isomorphism. The case of the map (19.1.2) is a consequence of the case of the map (19.1.1). □

Lemma 19.3. *Let* S *be a graded ring. Let* M *be a graded* S*-module. Set* $X = Proj(S)$. *Assume* X *is covered by the standard opens* $D_+(f)$ *with* $f \in S_1$, *e.g., if* S *is generated by* S_1 *over* S_0. *Then the sheaves* $\mathcal{O}_X(n)$ *are invertible and the maps (19.1.1), (19.1.2), and (19.1.5) are isomorphisms. In particular, these maps induce isomorphisms*

$$\mathcal{O}_X(1)^{\otimes n} \cong \mathcal{O}_X(n) \quad \text{and} \quad \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{M}(n) \cong \widetilde{M(n)}$$

Thus (18.0.2) becomes a map

$$M_n \longrightarrow \Gamma(X, \widetilde{M}(n)) \tag{19.3.1}$$

and (19.1.6) becomes a map

$$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M}(n)). \tag{19.3.2}$$

Proof. Under the assumptions of the lemma X is covered by the open subsets $D_+(f)$ with $f \in S_1$ and the lemma is a consequence of Lemma 19.2 above. □

Lemma 19.4. *Let* S *be a graded ring. Set* $X = Proj(S)$. *Fix* $d \geq 1$ *an integer. The following open subsets of* X *are equal:*

1. *The largest open subset* $W = W_d \subset X$ *such that each* $\mathcal{O}_X(dn)|_W$ *is invertible and all the multiplication maps* $\mathcal{O}_X(nd)|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(md)|_W \to \mathcal{O}_X(nd+md)|_W$ *(see 19.1.1) are isomorphisms.*
2. *The union of the open subsets* $D_+(fg)$ *with* $f, g \in S$ *homogeneous and* $\deg(f) = \deg(g) + d$.

Moreover, all the maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ *(see 19.1.5) are isomorphisms.*

Proof. If $x \in D_+(fg)$ with $\deg(f) = \deg(g)+d$ then on $D_+(fg)$ the sheaves $\mathcal{O}_X(dn)$ are generated by the element $(f/g)^n = f^{2n}/(fg)^n$. This implies x is in the open subset W defined in (1) by arguing as in the proof of Lemma 19.2.

Conversely, suppose that $\mathcal{O}_X(d)$ is free of rank 1 in an open neighbourhood V of $x \in X$ and all the multiplication maps $\mathcal{O}_X(nd)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_X(md)|_V \to \mathcal{O}_X(nd+md)|_V$ are isomorphisms. We may choose $h \in S_+$ homogeneous such that $x \in D_+(h) \subset V$. By the definition of the twists of the structure sheaf we conclude there exists an element s of $(S_h)_d$ such that s^n is a basis of $(S_h)_{nd}$ as a module over $S_{(h)}$ for all $n \in \mathbf{Z}$. We may write $s = f/h^m$ for some $m \geq 1$ and $f \in S_{d+m\deg(h)}$. Set $g = h^m$

so $s = f/g$. Note that $x \in D_+(g)$ by construction. Note that $g^d \in (S_h)_{d\deg(g)}$. By assumption we can write this as a multiple of $s^{\deg(g)} = f^{\deg(g)}/g^{\deg(g)}$, say $g^d = a/g^e \cdot f^{\deg(g)}/g^{\deg(g)}$. Then we conclude that $g^{d+e+\deg(g)} = af^{\deg(g)}$ and hence also $x \in D_+(f)$. So x is an element of the set defined in (2).

The existence of the generating section $s = f/g$ over the affine open $D_+(fg)$ whose powers freely generate the sheaves of modules $\mathcal{O}_X(nd)$ easily implies that the multiplication maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 19.1.5) are isomorphisms. Compare with the proof of Lemma 19.2. $\square$

Recall from Modules, Lemma **??** that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x\mathcal{L}_x\}$ is open.

Lemma 19.5. *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. Let $W = W_d \subset X$ be the open subscheme defined in Lemma 19.4. Let $n \geq 1$ and $f \in S_{nd}$. Denote $s \in \Gamma(W, \mathcal{O}_W(nd))$ the section which is the image of f via (19.1.3) restricted to W. Then*

$$W_s = D_+(f) \cap W.$$

Proof. Let $D_+(ab) \subset W$ be a standard affine open with $a, b \in S$ homogeneous and $\deg(a) = \deg(b) + d$. Note that $D_+(ab) \cap D_+(f) = D_+(abf)$. On the other hand the restriction of s to $D_+(ab)$ corresponds to the element $f/1 = b^n f/a^n (a/b)^n \in (S_{ab})_{nd}$. We have seen in the proof of Lemma 19.4 that $(a/b)^n$ is a generator for $\mathcal{O}_W(nd)$ over $D_+(ab)$. We conclude that $W_s \cap D_+(ab)$ is the principal open associated to $b^n f/a^n \in \mathcal{O}_X(D_+(ab))$. Thus the result of the lemma is clear. $\square$

The following lemma states the properties that we will later use to characterize schemes with an ample invertible sheaf.

Lemma 19.6. *Let S be a graded ring. Let $X = Proj(S)$. Let $Y \subset X$ be a quasi-compact open subscheme. Denote $\mathcal{O}_Y(n)$ the restriction of $\mathcal{O}_X(n)$ to Y. There exists an integer $d \geq 1$ such that*

1. *the subscheme Y is contained in the open W_d defined in Lemma 19.4,*
2. *the sheaf $\mathcal{O}_Y(dn)$ is invertible for all $n \in \mathbf{Z}$,*
3. *all the maps $\mathcal{O}_Y(nd) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \longrightarrow \mathcal{O}_Y(nd+m)$ of Equation (19.1.1) are isomorphisms,*
4. *all the maps $\widetilde{M}(nd)|_Y = \widetilde{M}|_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X(nd)|_Y \to \widetilde{M(nd)}|_Y$ (see 19.1.5) are isomorphisms,*
5. *given $f \in S_{nd}$ denote $s \in \Gamma(Y, \mathcal{O}_Y(nd))$ the image of f via (19.1.3) restricted to Y, then $D_+(f) \cap Y = Y_s$,*
6. *a basis for the topology on Y is given by the collection of opens Y_s, where $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$, and*
7. *a basis for the topology of Y is given by those opens $Y_s \subset Y$, for $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$ which are affine.*

Proof. Since Y is quasi-compact there exist finitely many homogeneous $f_i \in S_+$, $i = 1, \ldots, n$ such that the standard opens $D_+(f_i)$ give an open covering of Y. Let $d_i = \deg(f_i)$ and set $d = d_1 \ldots d_n$. Note that $D_+(f_i) = D_+(f_i^{d/d_i})$ and hence we see immediately that $Y \subset W_d$, by characterization (2) in Lemma 19.4 or by (1) using Lemma 19.2. Note that (1) implies (2), (3) and (4) by Lemma 19.4. (Note that

(3) is a special case of (4).) Assertion (5) follows from Lemma 19.5. Assertions (6) and (7) follow because the open subsets $D_+(f)$ form a basis for the topology of X and are affine. □

Lemma 19.7. *Let S be a graded ring. Set $X = Proj(S)$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $M = \bigoplus_{n\in\mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ as a graded S-module, using (19.1.4) and (19.1.3). Then there is a canonical $\mathcal{O}_X$-module map*

$$\widetilde{M} \longrightarrow \mathcal{F}$$

functorial in $\mathcal{F}$ such that the induced map $M_0 \to \Gamma(X, \mathcal{F})$ is the identity.

Proof. Let $f \in S$ be homogeneous of degree $d > 0$. Recall that $\widetilde{M}|_{D_+(f)}$ corresponds to the $S_{(f)}$-module $M_{(f)}$ by Lemma 16.4. Thus we can define a canonical map

$$M_{(f)} \longrightarrow \Gamma(D_+(f), \mathcal{F}), \quad m/f^n \longmapsto m|_{D_+(f)} \otimes f|_{D_+(f)}^{-n}$$

which makes sense because $f|_{D_+(f)}$ is a trivializing section of the invertible sheaf $\mathcal{O}_X(d)|_{D_+(f)}$, see Lemma 19.2 and its proof. Since $\widetilde{M}$ is quasi-coherent, this leads to a canonical map

$$\widetilde{M}|_{D_+(f)} \longrightarrow \mathcal{F}|_{D_+(f)}$$

via Schemes, Lemma **??**. We obtain a global map if we prove that the displayed maps glue on overlaps. Proof of this is omitted. We also omit the proof of the final statement. □

20. Functoriality of Proj

This is reminiscent of the olden days. The point is that Proj is covariant, i.e. for rings $S' \subset S$ we have Proj$S \subset$ Proj$S$ (under the right conditions, which is the point of this section), meaning that, indeed, **Projs are projective schemes**.

A graded ring map $\psi : A \to B$ does not always give rise to a morphism of associated projective homogeneous spectra. The reason is that the inverse image $\psi^{-1}(\mathfrak{q})$ of a homogeneous prime $\mathfrak{q} \subset B$ may contain the irrelevant prime A_+ even if $\mathfrak{q}$ does not contain B_+. The correct result is stated as follows.

Lemma 20.1. *Let A, B be two graded rings. Set $X = Proj(A)$ and $Y = Proj(B)$. Let $\psi : A \to B$ be a graded ring map. Set*

$$U(\psi) = \bigcup\nolimits_{f\in A_+ \ homogeneous} D_+(\psi(f)) \subset Y.$$

Then there is a canonical morphism of schemes

$$r_\psi : U(\psi) \longrightarrow X$$

and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^*\left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the following properties:

(1) *For every $d \geq 0$ the diagram*

$$\begin{tikzcd}
A_d \arrow[rr, "\psi"] \arrow[d] & & B_d \arrow[d] \\
\Gamma(X, \mathcal{O}_X(d)) \arrow[r, "\theta"] & \Gamma(U(\psi), \mathcal{O}_Y(d)) & \Gamma(Y, \mathcal{O}_Y(d)) \arrow[l]
\end{tikzcd}$$

is commutative.

(2) *For any $f \in A_+$ homogeneous we have $r_\psi^{-1}(D_+(f)) = D_+(\psi(f))$ and the restriction of r_ψ to $D_+(\psi(f))$ corresponds to the ring map $A_{(f)} \to B_{(\psi(f))}$ induced by ψ.*

Proof. Clearly condition (2) uniquely determines the morphism of schemes and the open subset $U(\psi)$. Pick $f \in A_d$ with $d \geq 1$. Note that $\mathcal{O}_X(n)|_{D_+(f)}$ corresponds to the $A_{(f)}$-module $(A_f)_n$ and that $\mathcal{O}_Y(n)|_{D_+(\psi(f))}$ corresponds to the $B_{(\psi(f))}$-module $(B_{\psi(f)})_n$. In other words θ when restricted to $D_+(\psi(f))$ corresponds to a map of $\mathbf{Z}$-graded $B_{(\psi(f))}$-algebras

$$A_f \otimes_{A_{(f)}} B_{(\psi(f))} \longrightarrow B_{\psi(f)}$$

Condition (1) determines the images of all elements of A. Since f is an invertible element which is mapped to $\psi(f)$ we see that $1/f^m$ is mapped to $1/\psi(f)^m$. It easily follows from this that θ is uniquely determined, namely it is given by the rule

$$a/f^m \otimes b/\psi(f)^e \longmapsto \psi(a)b/\psi(f)^{m+e}.$$

To show existence we remark that the proof of uniqueness above gave a well defined prescription for the morphism r and the map θ when restricted to every standard open of the form $D_+(\psi(f)) \subset U(\psi)$ into $D_+(f)$. Call these r_f and θ_f. Hence we only need to verify that if $D_+(f) \subset D_+(g)$ for some $f, g \in A_+$ homogeneous, then the restriction of r_g to $D_+(\psi(f))$ matches r_f. This is clear from the formulas given for r and θ above. □

Lemma 20.2. *Let A, B, and C be graded rings. Set $X = Proj(A)$, $Y = Proj(B)$ and $Z = Proj(C)$. Let $\varphi : A \to B$, $\psi : B \to C$ be graded ring maps. Then we have*

$$U(\psi \circ \varphi) = r_\psi^{-1}(U(\varphi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. □

Lemma 20.3. *With hypotheses and notation as in Lemma 20.1 above. Assume $A_d \to B_d$ is surjective for all $d \gg 0$. Then*

(1) *$U(\psi) = Y$,*
(2) *$r_\psi : Y \to X$ is a closed immersion, and*
(3) *the maps $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are surjective but not isomorphisms in general (even if $A \to B$ is surjective).*

Proof. Part (1) follows from the definition of $U(\psi)$ and the fact that $D_+(f) = D_+(f^n)$ for any $n > 0$. For $f \in A_+$ homogeneous we see that $A_{(f)} \to B_{(\psi(f))}$ is surjective because any element of $B_{(\psi(f))}$ can be represented by a fraction $b/\psi(f)^n$ with n arbitrarily large (which forces the degree of $b \in B$ to be large). This proves (2). The same argument shows the map

$$A_f \to B_{\psi(f)}$$

is surjective which proves the surjectivity of θ. For an example where this map is not an isomorphism consider the graded ring $A = k[x, y]$ where k is a field and $\deg(x) = 1$, $\deg(y) = 2$. Set $I = (x)$, so that $B = k[y]$. Note that $\mathcal{O}_Y(1) = 0$

in this case. But it is easy to see that $r_\psi^*\mathcal{O}_X(1)$ is not zero. (There are less silly examples.) $\square$

Lemma 20.4. *With hypotheses and notation as in Lemma 20.1 above. Assume $A_d \to B_d$ is an isomorphism for all $d \gg 0$. Then*

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ *is an isomorphism, and*
(3) *the maps* $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are isomorphisms.*

Proof. We have (1) by Lemma 20.3. Let $f \in A_+$ be homogeneous. The assumption on ψ implies that $A_f \to B_f$ is an isomorphism (details omitted). Thus it is clear that r_ψ and θ restrict to isomorphisms over $D_+(f)$. The lemma follows. $\square$

Lemma 20.5. *With hypotheses and notation as in Lemma 20.1 above. Assume $A_d \to B_d$ is surjective for $d \gg 0$ and that A is generated by A_1 over A_0. Then*

(1) $U(\psi) = Y$,
(2) $r_\psi : Y \to X$ *is a closed immersion, and*
(3) *the maps* $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are isomorphisms.*

Proof. By Lemmas 20.4 and 20.2 we may replace B by the image of $A \to B$ without changing X or the sheaves $\mathcal{O}_X(n)$. Thus we may assume that $A \to B$ is surjective. By Lemma 20.3 we get (1) and (2) and surjectivity in (3). By Lemma 19.3 we see that both $\mathcal{O}_X(n)$ and $\mathcal{O}_Y(n)$ are invertible. Hence θ is an isomorphism. $\square$

Lemma 20.6. *With hypotheses and notation as in Lemma 20.1 above. Assume there exists a ring map $R \to A_0$ and a ring map $R \to R'$ such that $B = R' \otimes_R A$. Then*

(1) $U(\psi) = Y$,
(2) *the diagram*

$$\begin{tikzcd} Y = \mathit{Proj}(B) \arrow[r, "r_\psi"'] \arrow[d] & \mathit{Proj}(A) = X \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

is a fibre product square, and

(3) *the maps* $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are isomorphisms.*

Proof. This follows immediately by looking at what happens over the standard opens $D_+(f)$ for $f \in A_+$. $\square$

Lemma 20.7. *With hypotheses and notation as in Lemma 20.1 above. Assume there exists a $g \in A_0$ such that ψ induces an isomorphism $A_g \to B$. Then $U(\psi) = Y$, $r_\psi : Y \to X$ is an open immersion which induces an isomorphism of Y with the inverse image of $D(g) \subset \mathrm{Spec}(A_0)$. Moreover the map θ is an isomorphism.*

Proof. This is a special case of Lemma 20.6 above. $\square$

Lemma 20.8. *Let S be a graded ring. Let $d \geq 1$. Set $S' = S^{(d)}$ with notation as in Algebra, Section* **??**. *Set $X = \mathit{Proj}(S)$ and $X' = \mathit{Proj}(S')$. There is a canonical isomorphism $i : X \to X'$ of schemes such that*

(1) *for any graded S-module M setting $M' = M^{(d)}$, we have a canonical isomorphism $\widetilde{M} \to i^*\widetilde{M'}$,*

(2) *we have canonical isomorphisms* $\mathcal{O}_X(nd) \to i^*\mathcal{O}_{X'}(n)$

and these isomorphisms are compatible with the multiplication maps of Lemma 18.1 and hence with the maps (19.1.1), (19.1.2), (19.1.3), (19.1.4), (19.1.5), and (19.1.6) (see proof for precise statements.

Proof. The injective ring map $S' \to S$ (which is not a homomorphism of graded rings due to our conventions), induces a map $j : \mathrm{Spec}(S) \to \mathrm{Spec}(S')$. Given a graded prime ideal $\mathfrak{p} \subset S$ we see that $\mathfrak{p}' = j(\mathfrak{p}) = S' \cap \mathfrak{p}$ is a graded prime ideal of S'. Moreover, if $f \in S_+$ is homogeneous and $f \notin \mathfrak{p}$, then $f^d \in S'_+$ and $f^d \notin \mathfrak{p}'$. Conversely, if $\mathfrak{p}' \subset S'$ is a graded prime ideal not containing some homogeneous element $f \in S'_+$, then $\mathfrak{p} = \{g \in S \mid g^d \in \mathfrak{p}'\}$ is a graded prime ideal of S not containing f whose image under j is $\mathfrak{p}'$. To see that $\mathfrak{p}$ is an ideal, note that if $g, h \in \mathfrak{p}$, then $(g+h)^{2d} \in \mathfrak{p}'$ by the binomial formula and hence $g + h \in \mathfrak{p}'$ as $\mathfrak{p}'$ is a prime. In this way we see that j induces a homeomorphism $i : X \to X'$. Moreover, given $f \in S_+$ homogeneous, then we have $S_{(f)} \cong S'_{(f^d)}$. Since these isomorphisms are compatible with the restrictions mappings of Lemma 16.1, we see that there exists an isomorphism $i^\sharp : i^{-1}\mathcal{O}_{X'} \to \mathcal{O}_X$ of structure sheaves on X and X', hence i is an isomorphism of schemes.

Let M be a graded S-module. Given $f \in S_+$ homogeneous, we have $M_{(f)} \cong M'_{(f^d)}$, hence in exactly the same manner as above we obtain the isomorphism in (1). The isomorphisms in (2) are a special case of (1) for $M = S(nd)$ which gives $M' = S'(n)$. Let M and N be graded S-modules. Then we have

$$M' \otimes_{S'} N' = (M \otimes_S N)^{(d)} = (M \otimes_S N)'$$

as can be verified directly from the definitions. Having said this the compatibility with the multiplication maps of Lemma 18.1 is the commutativity of the diagram

$$\begin{tikzcd} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \arrow[r] \arrow[d, "(1)\otimes(1)"'] & \widetilde{M \otimes_S N} \arrow[d, "(1)"] \\ i^*(\widetilde{M'} \otimes_{\mathcal{O}_{X'}} \widetilde{N'}) \arrow[r] & i^*(\widetilde{M' \otimes_{S'} N'}) \end{tikzcd}$$

This can be seen by looking at the construction of the maps over the open $D_+(f) = D_+(f^d)$ where the top horizontal arrow is given by the map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$ and the lower horizontal arrow by the map $M'_{(f^d)} \times N'_{(f^d)} \to (M' \otimes_{S'} N')_{(f^d)}$. Since these maps agree via the identifications $M_{(f)} = M'_{(f^d)}$, etc, we get the desired compatibility. We omit the proof of the other compatibilities. □

21. Reduced schemes

So far a reduced scheme, for me, is just a scheme where there are no "repetitions": (x^2) is not reduced because it gives the same information as (x). Algebraically this means that the coordinate ring is reduced, which is defined by asking that there are no nilpotent elements, i.e. elements such that some power vanishes. (There's no definition of reduced ring in Stacks Project since it's taken as part of the basic algebraic knowledge).

The definition is on stalks but the first lemma shows it's equivalent to define it on any open set. This is in virtue of some "injectivity" lemma from the ring at U

to the product of all stalks parametrized in U; essentially that if a section vanishes when projected to all the stalks then it's zero.

Definition 21.1. Let X be a scheme. We say X is *reduced* if every local ring $\mathcal{O}_{X,x}$ is reduced.

Lemma 21.2. *A scheme X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for all $U \subset X$ open.*

Proof. Assume that X is reduced. Let $f \in \mathcal{O}_X(U)$ be a section such that $f^n = 0$. Then the image of f in $\mathcal{O}_{U,u}$ is zero for all $u \in U$. Hence f is zero, see Sheaves, Lemma **??**. Conversely, assume that $\mathcal{O}_X(U)$ is reduced for all opens U. Pick any nonzero element $f \in \mathcal{O}_{X,x}$. Any representative $(U, f \in \mathcal{O}(U))$ of f is nonzero and hence not nilpotent. Hence f is not nilpotent in $\mathcal{O}_{X,x}$. ☐

Lemma 21.3. *An affine scheme* $\mathrm{Spec}(R)$ *is reduced if and only if R is reduced.*

Proof. The direct implication follows immediately from Lemma 21.2 above. In the other direction it follows since any localization of a reduced ring is reduced, and in particular the local rings of a reduced ring are reduced. ☐

Lemma 21.4. *Let X be a scheme. Let $T \subset X$ be a closed subset. There exists a unique closed subscheme $Z \subset X$ with the following properties: (a) the underlying topological space of Z is equal to T, and (b) Z is reduced.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sub presheaf defined by the rule

$$\mathcal{I}(U) = \{f \in \mathcal{O}_X(U) \mid f(t) = 0 \text{ for all } t \in T \cap U\}$$

Here we use $f(t)$ to indicate the image of f in the residue field $\kappa(t)$ of X at t. Because of the local nature of the condition it is clear that $\mathcal{I}$ is a sheaf of ideals. Moreover, let $U = \mathrm{Spec}(R)$ be an affine open. We may write $T \cap U = V(I)$ for a unique radical ideal $I \subset R$. Given a prime $\mathfrak{p} \in V(I)$ corresponding to $t \in T \cap U$ and an element $f \in R$ we have $f(t) = 0 \Leftrightarrow f \in \mathfrak{p}$. Hence $\mathcal{I}(U) = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = I$ by Algebra, Lemma **??**. Moreover, for any standard open $D(g) \subset \mathrm{Spec}(R) = U$ we have $\mathcal{I}(D(g)) = I_g$ by the same reasoning. Thus $\widetilde{I}$ and $\mathcal{I}|_U$ agree (as ideals) on a basis of opens and hence are equal. Therefore $\mathcal{I}$ is a quasi-coherent sheaf of ideals.

At this point we may define Z as the closed subspace associated to the sheaf of ideals $\mathcal{I}$. For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z \cap U = \mathrm{Spec}(R/I)$ where I is a radical ideal and hence Z is reduced (by Lemma 21.3 above). By construction the underlying closed subset of Z is T. Hence we have found a closed subscheme with properties (a) and (b).

Let $Z' \subset X$ be a second closed subscheme with properties (a) and (b). For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z' \cap U = \mathrm{Spec}(R/I')$ for some ideal $I' \subset R$. By Lemma 21.3 the ring R/I' is reduced and hence I' is radical. Since $V(I') = T \cap U = V(I)$ we deduced that $I = I'$ by Algebra, Lemma **??**. Hence Z' and Z are defined by the same sheaf of ideals and hence are equal. ☐

Definition 21.5. Let X be a scheme. Let $Z \subset X$ be a closed subset. A *scheme structure on Z* is given by a closed subscheme Z' of X whose underlying set is equal to Z. We often say "let $(Z, \mathcal{O}_Z)$ be a scheme structure on Z" to indicate this. The *reduced induced scheme structure* on Z is the one constructed in Lemma 21.4. The *reduction X_{red} of X* is the reduced induced scheme structure on X itself.

Often when we say "let $Z \subset X$ be an irreducible component of X" we think of Z as a reduced closed subscheme of X using the reduced induced scheme structure.

Remark 21.6. Let X be a scheme. Let $T \subset X$ be a locally closed subset. In this situation we sometimes also use the phrase "reduced induced scheme structure on T". It refers to the reduced induced scheme structure from Definition 21.5 when we view T as a closed subset of the open subscheme $X \setminus \partial T$ of X. Here $\partial T = \overline{T} \setminus T$ is the "boundary" of T in the topological space of X.

Lemma 21.7. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let Y be a reduced scheme. A morphism $f : Y \to X$ factors through Z if and only if $f(Y) \subset Z$ (set theoretically). In particular, any morphism $Y \to X$ factors as $Y \to X_{red} \to X$.*

Proof. Assume $f(Y) \subset Z$ (set theoretically). Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. For any affine opens $U \subset X$, $\mathrm{Spec}(B) = V \subset Y$ with $f(V) \subset U$ and any $g \in \mathcal{I}(U)$ the pullback $b = f^\sharp(g) \in \Gamma(V, \mathcal{O}_Y) = B$ maps to zero in the residue field of any $y \in V$. In other words $b \in \bigcap_{\mathfrak{p} \subset B} \mathfrak{p}$. This implies $b = 0$ as B is reduced (Lemma 21.2, and Algebra, Lemma **??**). Hence f factors through Z by Lemma 7.6. $\square$

22. Dominant morphisms

The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

Definition 22.1. A morphism $f : X \to S$ of schemes is called *dominant* if the image of f is a dense subset of S.

23. Morphisms of finite type

Recall that a ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, see Algebra, Definition 2.2.

Definition 23.1. Let $f : X \to S$ be a morphism of schemes.

1. We say that f is of *finite type at $x \in X$* if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
2. We say that f is *locally of finite type* if it is of finite type at every point of X.
3. We say that f is of *finite type* if it is locally of finite type and quasi-compact.

24. Flat morphisms

The essential technical property for for defining flatness is the preservation of exact sequences. Right-exactness is true in general; it follows from currying in category of commutative rings. But the functor $- \otimes_R N$ does not preserve injectivity of maps— that's the point of flatness.

Lemma 24.1 (Internal Hom for R-modules)**.** *For any three R-modules M, N, P,*

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

Proof. An R-linear map $\hat{f} \in \mathrm{Hom}_R(M \otimes_R N, P)$ corresponds to an R-bilinear map $f : M \times N \to P$. For each $x \in M$ the mapping $y \mapsto f(x, y)$ is R-linear by the universal property. Thus f corresponds to a map $\phi_f : M \to \mathrm{Hom}_R(N, P)$. This map is R-linear since

$$\phi_f(ax + y)(z) = f(ax + y, z) = af(x, z) + f(y, z) = (a\phi_f(x) + \phi_f(y))(z),$$

for all $a \in R$, $x \in M$, $y \in M$ and $z \in N$. Conversely, any $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ defines an R-bilinear map $M \times N \to P$, namely $(x, y) \mapsto f(x)(y)$. So this is a natural one-to-one correspondence between the two modules $\mathrm{Hom}_R(M \otimes_R N, P)$ and $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. $\square$

Lemma 24.2 (Tensor product is right exact)**.** *Let*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules and homomorphisms, and let N be any R-module. Then the sequence

$$M_1 \otimes N \xrightarrow{f \otimes 1} M_2 \otimes N \xrightarrow{g \otimes 1} M_3 \otimes N \to 0 \tag{24.2.1}$$

is exact. In other words, the functor $- \otimes_R N$ is right exact, in the sense that tensoring each term in the original right exact sequence preserves the exactness.

Proof. For every R-module P we apply the functor $\mathrm{Hom}(-, \mathrm{Hom}(N, P))$ to the first exact sequence. We obtain

$$0 \to \mathrm{Hom}(M_3, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_2, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_1, \mathrm{Hom}(N, P))$$

which is exact by Lemma **??** (1). By Lemma 24.1 this becomes the sequence

$$0 \to \mathrm{Hom}(M_3 \otimes N, P) \to \mathrm{Hom}(M_2 \otimes N, P) \to \mathrm{Hom}(M_1 \otimes N, P)$$

which is therefore also exact. Then using Lemma **??** (1) again, we arrive at the desired exact sequence. $\square$

Remark 24.3. However, tensor product does NOT preserve exact sequences in general. In other words, if $M_1 \to M_2 \to M_3$ is exact, then it is not necessarily true that $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for arbitrary R-module N.

Example 24.4. Consider the injective map $2 : \mathbf{Z} \to \mathbf{Z}$ viewed as a map of $\mathbf{Z}$-modules. Let $N = \mathbf{Z}/2$. Then the induced map $\mathbf{Z} \otimes \mathbf{Z}/2 \to \mathbf{Z} \otimes \mathbf{Z}/2$ is NOT injective. This is because for $x \otimes y \in \mathbf{Z} \otimes \mathbf{Z}/2$,

$$(2 \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

Therefore the induced map is the zero map while $\mathbf{Z} \otimes N \neq 0$.

Definition 24.5. For R-modules N, if the functor $- \otimes_R N$ is exact, i.e. tensoring with N preserves all exact sequences, then N is said to be *flat* R-module. We will discuss this later in Section **??**.

Epiphany: in the category of commutative rings pushout is tensor product. So think of Spec as a functor from $\mathit{CRing}^{\mathrm{op}}$ to *Sch*, then pushout goes to pullback, and what's an example of a pullback? Fibre! So, the coordinate ring of a fiber is essentially given by the residue field at the point that parametrizes it! (tensored with the coordinate ring of the deformation space).

There is a lot of information on Stacks Project about flatness. It looks like the heart of the concept is captured in the commutative-algebraic notion of preserving exact sequences:

Definition 24.6. Let R be a ring.

(1) An R-module M is called *flat* if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.
(2) An R-module M is called *faithfully flat* if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.
(3) A ring map $R \to S$ is called *flat* if S is flat as an R-module.
(4) A ring map $R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module.

Recall that a module M over a ring R is *flat* if the functor $- \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is exact. A ring map $R \to A$ is said to be *flat* if A is flat as an R-module.

25. Tor groups and flatness

A characterization of flatness is that an R-module M is flat if and only if its torsion functors vanish. That's the last proposition in this section.

I just wanted to highlight Lemma 25.2, which says what happens in general with right exact sequences and tensor product: as discussed in my introduction to Section 24, the functor $- \otimes_R M$ is not left exact: instead there are some Tor modules that appear. So yes, it makes sense that the vanishing of those modules is related to flatness.

In this section we use some of the homological algebra developed in the previous section to explain what Tor groups are. Namely, suppose that R is a ring and that M, N are two R-modules. Choose a resolution $F_\bullet$ of M by free R-modules. See Lemma **??**. Consider the homological complex

$$F_\bullet \otimes_R N : \ldots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N$$

We define $\mathrm{Tor}_i^R(M, N)$ to be the ith homology group of this complex. The following lemma explains in what sense this is well defined.

Lemma 25.1. *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1 and that $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = \mathrm{Coker}(d_{F,1}) \to M_2 = \mathrm{Coker}(d_{G,1})$, see Lemma* **??**. *Then the induced maps*

$$H_i(\alpha) : H_i(F_\bullet \otimes_R N) \longrightarrow H_i(G_\bullet \otimes_R N)$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H_i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. The proof of this lemma is identical to the proof of Lemma **??**. $\square$

Not only does this lemma imply that the Tor modules are well defined, but it also provides for the functoriality of the constructions $(M, N) \mapsto \mathrm{Tor}_i^R(M, N)$ in the first variable. Of course the functoriality in the second variable is evident. We leave it to the reader to see that each of the Tor_i^R is in fact a functor

$$\text{Mod}_R \times \text{Mod}_R \to \text{Mod}_R.$$

Here Mod_R denotes the category of R-modules, and for the definition of the product category see Categories, Definition **??**. Namely, given morphisms of R-modules $M_1 \to M_2$ and $N_1 \to N_2$ we get a commutative diagram

$$\begin{tikzcd}
\text{Tor}_i^R(M_1, N_1) \arrow[r] \arrow[d] & \text{Tor}_i^R(M_1, N_2) \arrow[d] \\
\text{Tor}_i^R(M_2, N_1) \arrow[r] & \text{Tor}_i^R(M_2, N_2)
\end{tikzcd}$$

Lemma 25.2. *Let R be a ring and let M be an R-module. Suppose that $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of R-modules. There exists a long exact sequence*

$$Tor_1^R(M, N') \to Tor_1^R(M, N) \to Tor_1^R(M, N'') \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

Proof. The proof of this is the same as the proof of Lemma **??**. □

Consider a homological double complex of R-modules

$$\begin{tikzcd}
\ldots \arrow[r, "d"] & A_{2,0} \arrow[r, "d"] & A_{1,0} \arrow[r, "d"] & A_{0,0} \\
\ldots \arrow[r, "d"] & A_{2,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,1} \arrow[u, "\delta"] \\
\ldots \arrow[r, "d"] & A_{2,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,2} \arrow[u, "\delta"] \\
& \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"]
\end{tikzcd}$$

This means that $d_{i,j} : A_{i,j} \to A_{i-1,j}$ and $\delta_{i,j} : A_{i,j} \to A_{i,j-1}$ have the following properties

(1) Any composition of two $d_{i,j}$ is zero. In other words the rows of the double complex are complexes.
(2) Any composition of two $\delta_{i,j}$ is zero. In other words the columns of the double complex are complexes.
(3) For any pair (i, j) we have $\delta_{i-1,j} \circ d_{i,j} = d_{i,j-1} \circ \delta_{i,j}$. In other words, all the squares commute.

The correct thing to do is to associate a spectral sequence to any such double complex. However, for the moment we can get away with doing something slightly easier.

Namely, for the purposes of this section only, given a double complex $(A_{\bullet,\bullet}, d, \delta)$ set $R(A)_j = \text{Coker}(A_{1,j} \to A_{0,j})$ and $U(A)_i = \text{Coker}(A_{i,1} \to A_{i,0})$. (The letters R and U are meant to suggest Right and Up.) We endow $R(A)_\bullet$ with the structure of a complex using the maps δ. Similarly we endow $U(A)_\bullet$ with the structure of a complex using the maps d. In other words we obtain the following huge commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r, "d"] & U(A)_2 \arrow[r, "d"] & U(A)_1 \arrow[r, "d"] & U(A)_0 & \\
\ldots \arrow[r, "d"] & A_{2,0} \arrow[u] \arrow[r, "d"] & A_{1,0} \arrow[u] \arrow[r, "d"] & A_{0,0} \arrow[u] \arrow[r] & R(A)_0 \\
\ldots \arrow[r, "d"] & A_{2,1} \arrow[u, "\delta"] \arrow[r, "d"] & A_{1,1} \arrow[u, "\delta"] \arrow[r, "d"] & A_{0,1} \arrow[u, "\delta"] \arrow[r] & R(A)_1 \arrow[u, "\delta"] \\
\ldots \arrow[r, "d"] & A_{2,2} \arrow[u, "\delta"] \arrow[r, "d"] & A_{1,2} \arrow[u, "\delta"] \arrow[r, "d"] & A_{0,2} \arrow[u, "\delta"] \arrow[r] & R(A)_2 \arrow[u, "\delta"] \\
 & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"]
\end{tikzcd}$$

(This is no longer a double complex of course.) It is clear what a morphism $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ of double complexes is, and it is clear that this induces morphisms of complexes $R(\Phi) : R(A)_\bullet \to R(B)_\bullet$ and $U(\Phi) : U(A)_\bullet \to U(B)_\bullet$.

Lemma 25.3. *Let $(A_{\bullet,\bullet}, d, \delta)$ be a double complex such that*

(1) *Each row $A_{\bullet,j}$ is a resolution of $R(A)_j$.*
(2) *Each column $A_{i,\bullet}$ is a resolution of $U(A)_i$.*

Then there are canonical isomorphisms

$$H_i(R(A)_\bullet) \cong H_i(U(A)_\bullet).$$

The isomorphisms are functorial with respect to morphisms of double complexes with the properties above.

Proof. We will show that $H_i(R(A)_\bullet))$ and $H_i(U(A)_\bullet)$ are canonically isomorphic to a third group. Namely

$$\mathbf{H}_i(A) := \frac{\{(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mid d(a_{i,0}) = \delta(a_{i-1,1}), \ldots, d(a_{1,i-1}) = \delta(a_{0,i})\}}{\{d(a_{i+1,0}) + \delta(a_{i,1}), d(a_{i,1}) + \delta(a_{i-1,2}), \ldots, d(a_{1,i}) + \delta(a_{0,i+1})\}}$$

Here we use the notational convention that $a_{i,j}$ denotes an element of $A_{i,j}$. In other words, an element of $\mathbf{H}_i$ is represented by a zig-zag, represented as follows for $i = 2$

$$\begin{tikzcd}
a_{2,0} \arrow[r, mapsto] & d(a_{2,0}) = \delta(a_{1,1}) & \\
 & a_{1,1} \arrow[u, mapsto] \arrow[r, mapsto] & d(a_{1,1}) = \delta(a_{0,2}) \\
 & & a_{0,2} \arrow[u, mapsto]
\end{tikzcd}$$

Naturally, we divide out by "trivial" zig-zags, namely the submodule generated by elements of the form $(0, \ldots, 0, -\delta(a_{t+1,t-i}), d(a_{t+1,t-i}), 0, \ldots, 0)$. Note that there are canonical homomorphisms

$$\mathbf{H}_i(A) \to H_i(R(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{0,i}$$

and

$$\mathbf{H}_i(A) \to H_i(U(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{i,0}$$

First we show that these maps are surjective. Suppose that $\overline{r} \in H_i(R(A)_\bullet)$. Let $r \in R(A)_i$ be a cocycle representing the class of $\overline{r}$. Let $a_{0,i} \in A_{0,i}$ be an element which maps to r. Because $\delta(r) = 0$, we see that $\delta(a_{0,i})$ is in the image of d. Hence there exists an element $a_{1,i-1} \in A_{1,i-1}$ such that $d(a_{1,i-1}) = \delta(a_{0,i})$. This in turn implies that $\delta(a_{1,i-1})$ is in the kernel of d (because $d(\delta(a_{1,i-1})) = \delta(d(a_{1,i-1})) = \delta(\delta(a_{0,i})) = 0$. By exactness of the rows we find an element $a_{2,i-2}$ such that $d(a_{2,i-2}) = \delta(a_{1,i-1})$. And so on until a full zig-zag is found. Of course surjectivity of $\mathbf{H}_i \to H_i(U(A))$ is shown similarly.

To prove injectivity we argue in exactly the same way. Namely, suppose we are given a zig-zag $(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i})$ which maps to zero in $H_i(R(A)_\bullet)$. This means that $a_{0,i}$ maps to an element of $\mathrm{Coker}(A_{i,1} \to A_{i,0})$ which is in the image of $\delta : \mathrm{Coker}(A_{i+1,1} \to A_{i+1,0}) \to \mathrm{Coker}(A_{i,1} \to A_{i,0})$. In other words, $a_{0,i}$ is in the image of $\delta \oplus d : A_{0,i+1} \oplus A_{1,i} \to A_{0,i}$. From the definition of trivial zig-zags we see that we may modify our zig-zag by a trivial one and assume that $a_{0,i} = 0$. This immediately implies that $d(a_{1,i-1}) = 0$. As the rows are exact this implies that $a_{1,i-1}$ is in the image of $d : A_{2,i-1} \to A_{1,i-1}$. Thus we may modify our zig-zag once again by a trivial zig-zag and assume that our zig-zag looks like $(a_{i,0}, a_{i-1,1}, \ldots, a_{2,i-2}, 0, 0)$. Continuing like this we obtain the desired injectivity.

If $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ is a morphism of double complexes both of which satisfy the conditions of the lemma, then we clearly obtain a commutative diagram

$$\begin{tikzcd}
H_i(U(A)_\bullet) \arrow[d] & \mathbf{H}_i(A) \arrow[l] \arrow[r] \arrow[d] & H_i(R(A)_\bullet) \arrow[d] \\
H_i(U(B)_\bullet) & \mathbf{H}_i(B) \arrow[l] \arrow[r] & H_i(R(B)_\bullet)
\end{tikzcd}$$

This proves the functoriality. □

Remark 25.4. The isomorphism constructed above is the "correct" one only up to signs. A good part of homological algebra is concerned with choosing signs for various maps and showing commutativity of diagrams with intervention of suitable signs. For the moment we will simply use the isomorphism as given in the proof above, and worry about signs later.

Lemma 25.5. *Let R be a ring. For any $i \geq 0$ the functors $Mod_R \times Mod_R \to Mod_R$, $(M, N) \mapsto Tor_i^R(M, N)$ and $(M, N) \mapsto Tor_i^R(N, M)$ are canonically isomorphic.*

Proof. Let $F_\bullet$ be a free resolution of the module M and let $G_\bullet$ be a free resolution of the module N. Consider the double complex $(A_{i,j}, d, \delta)$ defined as follows:

(1) set $A_{i,j} = F_i \otimes_R G_j$,
(2) set $d_{i,j} : F_i \otimes_R G_j \to F_{i-1} \otimes G_j$ equal to $d_{F,i} \otimes \mathrm{id}$, and
(3) set $\delta_{i,j} : F_i \otimes_R G_j \to F_i \otimes G_{j-1}$ equal to $\mathrm{id} \otimes d_{G,j}$.

This double complex is usually simply denoted $F_\bullet \otimes_R G_\bullet$.

Since each G_j is free, and hence flat we see that each row of the double complex is exact except in homological degree 0. Since each F_i is free and hence flat we see that each column of the double complex is exact except in homological degree 0. Hence the double complex satisfies the conditions of Lemma 25.3.

To see what the lemma says we compute $R(A)_\bullet$ and $U(A)_\bullet$. Namely,

$$\begin{aligned} R(A)_i &= \mathrm{Coker}(A_{1,i} \to A_{0,i}) \\ &= \mathrm{Coker}(F_1 \otimes_R G_i \to F_0 \otimes_R G_i) \\ &= \mathrm{Coker}(F_1 \to F_0) \otimes_R G_i \\ &= M \otimes_R G_i \end{aligned}$$

In fact these isomorphisms are compatible with the differentials δ and we see that $R(A)_\bullet = M \otimes_R G_\bullet$ as homological complexes. In exactly the same way we see that $U(A)_\bullet = F_\bullet \otimes_R N$. We get

$$\begin{aligned} \mathrm{Tor}_i^R(M, N) &= H_i(F_\bullet \otimes_R N) \\ &= H_i(U(A)_\bullet) \\ &= H_i(R(A)_\bullet) \\ &= H_i(M \otimes_R G_\bullet) \\ &= H_i(G_\bullet \otimes_R M) \\ &= \mathrm{Tor}_i^R(N, M) \end{aligned}$$

Here the third equality is Lemma 25.3, and the fifth equality uses the isomorphism $V \otimes W = W \otimes V$ of the tensor product.

Functoriality. Suppose that we have R-modules M_ν, N_ν, $\nu = 1, 2$. Let $\varphi : M_1 \to M_2$ and $\psi : N_1 \to N_2$ be morphisms of R-modules. Suppose that we have free resolutions $F_{\nu,\bullet}$ for M_ν and free resolutions $G_{\nu,\bullet}$ for N_ν. By Lemma **??** we may choose maps of complexes $\alpha : F_{1,\bullet} \to F_{2,\bullet}$ and $\beta : G_{1,\bullet} \to G_{2,\bullet}$ compatible with φ and ψ. We claim that the pair (α, β) induces a morphism of double complexes

$$\alpha \otimes \beta : F_{1,\bullet} \otimes_R G_{1,\bullet} \longrightarrow F_{2,\bullet} \otimes_R G_{2,\bullet}$$

This is really a very straightforward check using the rule that $F_{1,i} \otimes_R G_{1,j} \to F_{2,i} \otimes_R G_{2,j}$ is given by $\alpha_i \otimes \beta_j$ where α_i, resp. β_j is the degree i, resp. j component of α, resp. β. The reader also readily verifies that the induced maps $R(F_{1,\bullet} \otimes_R G_{1,\bullet})_\bullet \to R(F_{2,\bullet} \otimes_R G_{2,\bullet})_\bullet$ agrees with the map $M_1 \otimes_R G_{1,\bullet} \to M_2 \otimes_R G_{2,\bullet}$ induced by $\varphi \otimes \beta$. Similarly for the map induced on the $U(-)_\bullet$ complexes. Thus the statement on functoriality follows from the statement on functoriality in Lemma 25.3. $\square$

Remark 25.6. An interesting case occurs when $M = N$ in the above. In this case we get a canonical map $\mathrm{Tor}_i^R(M, M) \to \mathrm{Tor}_i^R(M, M)$. Note that this map is not the identity, because even when $i = 0$ this map is not the identity! For example, if V is a vector space of dimension n over a field, then the switch map $V \otimes_k V \to V \otimes_k V$ has $(n^2 + n)/2$ eigenvalues $+1$ and $(n^2 - n)/2$ eigenvalues -1. In characteristic 2 it is not even diagonalizable. Note that even changing the sign of the map will not get rid of this.

Lemma 25.7. *Let R be a Noetherian ring. Let M, N be finite R-modules. Then $Tor_p^R(M, N)$ is a finite R-module for all p.*

Proof. This holds because $\mathrm{Tor}_p^R(M, N)$ is computed as the cohomology groups of a complex $F_\bullet \otimes_R N$ with each F_n a finite free R-module, see Lemma **??**. $\square$

Lemma 25.8. *Let R be a ring. Let M be an R-module. The following are equivalent:*

1. *The module M is flat over R.*
2. *For all $i > 0$ the functor $Tor_i^R(M, -)$ is zero.*
3. *The functor $Tor_1^R(M, -)$ is zero.*
4. *For all ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*
5. *For all finitely generated ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*

Proof. Suppose M is flat. Let N be an R-module. Let $F_\bullet$ be a free resolution of N. Then $F_\bullet \otimes_R M$ is a resolution of $N \otimes_R M$, by flatness of M. Hence all higher Tor groups vanish.

It now suffices to show that the last condition implies that M is flat. Let $I \subset R$ be an ideal. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Apply Lemma 25.2. We get an exact sequence

$$\mathrm{Tor}_1^R(M, R/I) \to M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0$$

Since obviously $M \otimes_R R = M$ we conclude that the last hypothesis implies that $M \otimes_R I \to M$ is injective for every finitely generated ideal I. Thus M is flat by Lemma **??**. □

Remark 25.9. The proof of Lemma 25.8 actually shows that

$$\mathrm{Tor}_1^R(M, R/I) = \mathrm{Ker}(I \otimes_R M \to M).$$

26. Singularities

As in [Vak25], a Noetherian local ring A is *regular* if $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. The stalk $\mathcal{O}_{X,p}$ is always local so we say that p is *regular* if its stalk is regular.

27. Birational morphisms

So far I get: they are isomorphisms on a dense open set, but more formally, here in Stacks, they give **isomorphisms on the stalks of the generic points**.

You may be used to the notion of a birational map of varieties having the property that it is an isomorphism over an open subset of the target. However, in general a birational morphism may not be an isomorphism over any nonempty open, see Example **??**. Here is the formal definition.

Definition 27.1. Let X, Y be schemes. Assume X and Y have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if

1. f induces a bijection between the set of generic points of irreducible components of X and the set of generic points of the irreducible components of Y, and
2. for every generic point $\eta \in X$ of an irreducible component of X the local ring map $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

We will see below that the fibres of a birational morphism over generic points are singletons. Moreover, we will see that in most cases one encounters in practice the existence of a birational morphism between irreducible schemes X and Y implies X and Y are birational schemes.

Lemma 27.2. *Let $f : X \to Y$ be a morphism of schemes having finitely many irreducible components. If f is birational then f is dominant.*

Proof. Follows from Lemma **??** and the definition. ☐

Lemma 27.3. *Let $f : X \to Y$ be a birational morphism of schemes having finitely many irreducible components. If $y \in Y$ is the generic point of an irreducible component, then the base change $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ is an isomorphism.*

Proof. We may assume $Y = \mathrm{Spec}(B)$ is affine and irreducible. Then X is irreducible too. If we prove the result for any nonempty affine open $U \subset X$, then the result holds for X (small argument omitted). Hence we may assume X is affine too, say $X = \mathrm{Spec}(A)$. Let $y \in Y$ correspond to the minimal prime $\mathfrak{q} \subset B$. By assumption A has a unique minimal prime $\mathfrak{p}$ lying over $\mathfrak{q}$ and $B_\mathfrak{q} \to A_\mathfrak{p}$ is an isomorphism. It follows that $A_\mathfrak{q} \to \kappa(\mathfrak{p})$ is surjective, hence $\mathfrak{p}A_\mathfrak{q}$ is a maximal ideal. On the other hand $\mathfrak{p}A_\mathfrak{q}$ is the unique minimal prime of $A_\mathfrak{q}$. We conclude that $\mathfrak{p}A_\mathfrak{q}$ is the unique prime of $A_\mathfrak{q}$ and that $A_\mathfrak{q} = A_\mathfrak{p}$. Since $A_\mathfrak{q} = A \otimes_B B_\mathfrak{q}$ the lemma follows. ☐

Example 27.4. Here are two examples of birational morphisms which are not isomorphisms over any open of the target.

First example. Let k be an infinite field. Let $A = k[x]$. Let $B = k[x, \{y_\alpha\}_{\alpha \in k}]/((x-\alpha)y_\alpha, y_\alpha y_\beta)$. There is an inclusion $A \subset B$ and a retraction $B \to A$ setting all y_α equal to zero. Both the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are birational but not an isomorphism over any open.

Second example. Let A be a domain. Let $S \subset A$ be a multiplicative subset not containing 0. With $B = S^{-1}A$ the morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is birational. If there exists an open U of $\mathrm{Spec}(A)$ such that $f^{-1}(U) \to U$ is an isomorphism, then there exists an $a \in A$ such that each every element of S becomes invertible in the principal localization A_a. Taking $A = \mathbf{Z}$ and S the set of odd integers give a counter example.

Lemma 27.5. *Let $f : X \to Y$ be a birational morphism of schemes having finitely many irreducible components over a base scheme S. Assume one of the following conditions is satisfied*

1. *f is locally of finite type and Y reduced,*
2. *f is locally of finite presentation.*

Then there exist dense opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is an isomorphism. In particular if X and Y are irreducible, then X and Y are S-birational.

Proof. There is an immediate reduction to the case where X and Y are irreducible which we omit. Moreover, after shrinking further and we may assume X and Y are affine, say $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. By assumption A, resp. B has a unique minimal prime $\mathfrak{p}$, resp. $\mathfrak{q}$, the prime $\mathfrak{p}$ lies over $\mathfrak{q}$, and $B_\mathfrak{q} = A_\mathfrak{p}$. By Lemma **??** we have $B_\mathfrak{q} = A_\mathfrak{q} = A_\mathfrak{p}$.

Suppose $B \to A$ is of finite type, say $A = B[x_1, \ldots, x_n]$. There exist a $b_i \in B$ and $g_i \in B \setminus \mathfrak{q}$ such that b_i/g_i maps to the image of x_i in $A_\mathfrak{q}$. Hence $b_i - g_i x_i$ maps to zero in $A_{g_i'}$ for some $g_i' \in B \setminus \mathfrak{q}$. Setting $g = \prod g_i g_i'$ we see that $B_g \to A_g$ is surjective. If moreover Y is reduced, then the map $B_g \to B_\mathfrak{q}$ is injective and hence $B_g \to A_g$ is injective as well. This proves case (1).

Proof of (2). By the argument given in the previous paragraph we may assume that $B \to A$ is surjective. As f is locally of finite presentation the kernel $J \subset B$ is a finitely generated ideal. Say $J = (b_1, \ldots, b_r)$. Since $B_\mathfrak{q} = A_\mathfrak{q}$ there exist $g_i \in B \setminus \mathfrak{q}$ such that $g_i b_i = 0$. Setting $g = \prod g_i$ we see that $B_g \to A_g$ is an isomorphism. □

Lemma 27.6. *Let S be a scheme. Let X and Y be irreducible schemes locally of finite presentation over S. Let $x \in X$ and $y \in Y$ be the generic points. The following are equivalent*

1. *X and Y are S-birational,*
2. *there exist nonempty opens of X and Y which are S-isomorphic, and*
3. *x and y map to the same point s of S and $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as $\mathcal{O}_{S,s}$-algebras.*

Proof. We have seen the equivalence of (1) and (2) in Lemma **??**. It is immediate that (2) implies (3). To finish we assume (3) holds and we prove (1). By Lemma **??** there is a rational map $f : U \to Y$ which sends $x \in U$ to y and induces the given isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. Thus f is a birational morphism and hence induces an isomorphism on nonempty opens by Lemma **??**. This finishes the proof. □

Lemma 27.7. *Let S be a scheme. Let X and Y be integral schemes locally of finite type over S. Let $x \in X$ and $y \in Y$ be the generic points. The following are equivalent*

1. *X and Y are S-birational,*
2. *there exist nonempty opens of X and Y which are S-isomorphic, and*
3. *x and y map to the same point $s \in S$ and $\kappa(x) \cong \kappa(y)$ as $\kappa(s)$-extensions.*

Proof. We have seen the equivalence of (1) and (2) in Lemma **??**. It is immediate that (2) implies (3). To finish we assume (3) holds and we prove (1). Observe that $\mathcal{O}_{X,x} = \kappa(x)$ and $\mathcal{O}_{Y,y} = \kappa(y)$ by Algebra, Lemma **??**. By Lemma **??** there is a rational map $f : U \to Y$ which sends $x \in U$ to y and induces the given isomorphism $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. Thus f is a birational morphism and hence induces an isomorphism on nonempty opens by Lemma **??**. This finishes the proof. □

28. Invertible modules (line bundles)

Similarly to the case of modules over rings (More on Algebra, Section **??**) we have the following definition.

Definition 28.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An *invertible $\mathcal{O}_X$-module* is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that the functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an equivalence of categories. We say that $\mathcal{L}$ is *trivial* if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$.

Lemma 27.4 below explains the relationship with locally free modules of rank 1.

Lemma 28.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Equivalent are*

1. *$\mathcal{L}$ is invertible, and*
2. *there exists an $\mathcal{O}_X$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \cong \mathcal{O}_X$.*

In this case $\mathcal{L}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module and the module $\mathcal{N}$ in (2) is isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Proof. Assume (1). Then the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ is essentially surjective, hence there exists an $\mathcal{O}_X$-module $\mathcal{N}$ as in (2). If (2) holds, then the functor $- \otimes_{\mathcal{O}_X} \mathcal{N}$ is a quasi-inverse to the functor $- \otimes_{\mathcal{O}_X} \mathcal{L}$ and we see that (1) holds.

Assume (1) and (2) hold. Denote $\psi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{O}_X$ the given isomorphism. Let $x \in X$. Choose an open neighbourhood U an integer $n \geq 1$ and sections $s_i \in \mathcal{L}(U)$, $t_i \in \mathcal{N}(U)$ such that $\psi(\sum s_i \otimes t_i) = 1$. Consider the isomorphisms

$$\mathcal{L}|_U \to \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathcal{N}|_U \to \mathcal{L}|_U$$

where the first arrow sends s to $\sum s_i \otimes s \otimes t_i$ and the second arrow sends $s \otimes s' \otimes t$ to $\psi(s' \otimes t)s$. We conclude that $s \mapsto \sum \psi(s \otimes t_i)s_i$ is an automorphism of $\mathcal{L}|_U$. This automorphism factors as

$$\mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{L}|_U$$

where the first arrow is given by $s \mapsto (\psi(s \otimes t_1), \ldots, \psi(s \otimes t_n))$ and the second arrow by $(a_1, \ldots, a_n) \mapsto \sum a_i s_i$. In this way we conclude that $\mathcal{L}|_U$ is a direct summand of a finite free $\mathcal{O}_U$-module.

Assume (1) and (2) hold. Consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

To finish the proof of the lemma we will show this is an isomorphism by checking it induces isomorphisms on stalks. Let $x \in X$. Since we know (by the previous paragraph) that $\mathcal{L}$ is a finitely presented $\mathcal{O}_X$-module we can use Lemma **??** to see that it suffices to show that

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{L}_x, \mathcal{O}_{X,x}) \longrightarrow \mathcal{O}_{X,x}$$

is an isomorphism. Since $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x = (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N})_x = \mathcal{O}_{X,x}$ (Lemma **??**) the desired result follows from More on Algebra, Lemma **??**. $\square$

Lemma 28.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{L}$ of an invertible $\mathcal{O}_Y$-module is invertible.*

Proof. By Lemma 27.2 there exists an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{N} \cong \mathcal{O}_Y$. Pulling back we get $f^*\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N} \cong \mathcal{O}_X$ by Lemma **??**. Thus $f^*\mathcal{L}$ is invertible by Lemma 27.2. $\square$

Lemma 28.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any locally free $\mathcal{O}_X$-module of rank 1 is invertible. If all stalks $\mathcal{O}_{X,x}$ are local rings, then the converse holds as well (but in general this is not the case).*

Proof. The parenthetical statement follows by considering a one point space X with sheaf of rings $\mathcal{O}_X$ given by a ring R. Then invertible $\mathcal{O}_X$-modules correspond to invertible R-modules, hence as soon as $\mathrm{Pic}(R)$ is not the trivial group, then we get an example.

Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

Looking over an open covering trivialization $\mathcal{L}$, we see that this map is an isomorphism. Hence $\mathcal{L}$ is invertible by Lemma 27.2.

Assume all stalks $\mathcal{O}_{X,x}$ are local rings and $\mathcal{L}$ invertible. In the proof of Lemma 27.2 we have seen that $\mathcal{L}_x$ is an invertible $\mathcal{O}_{X,x}$-module for all $x \in X$. Since $\mathcal{O}_{X,x}$ is local, we see that $\mathcal{L}_x \cong \mathcal{O}_{X,x}$ (More on Algebra, Section **??**). Since $\mathcal{L}$ is of finite presentation by Lemma 27.2 we conclude that $\mathcal{L}$ is locally free of rank 1 by Lemma **??**. $\square$

Now I introduce some of the properties of line bundles, Cartier divisors and so on.

Lemma 28.5. *The ideal sheaf of an effective Cartier divisor (a subscheme locally defined by the vanishing of a single function) is an invertible sheaf.*

Proof. We just need to check that the generator of the ideal sheaf at any affine set is not a zerodivisor. This follows from the ideal sheaf exact sequence, which implies that multplication by the generator is injective:

$$0 \longrightarrow I \cong A \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

$\square$

29. Ampleness

First is this lemma that comes from modules.tex. I think these sets X_s are the base points of the bundle. Because look: image of s just means consider the section s of the line bundle as a germ near x. Now a line bundle is a locally free rank-1 $\mathcal{O}_X$-module, so its sections, like s, may be multiplied by germs of functions in the maximal ring $\mathfrak{m}_x$, i.e. the functions that vanish at x. So X_s is the vanishing locus of the section s. If $s(x) \neq 0$, obviously $s \notin \mathfrak{m}_x \mathcal{L}_x$, so $x \in X_s$. Conversely, I would like to show that if $s(x) = 0$ then $s \in \mathfrak{m}_x \mathcal{L}_x$ but I'm not sure how. It's like: a vector field with a zero can be multiplied by a function that vanishes at the point, sure, but what's this function?

Lemma 29.1. *From modules.tex. Let X be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set*

$$X_s = \{x \in X \mid \textit{image } s \notin \mathfrak{m}_x \mathcal{L}_x\}$$

is open in X. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathcal{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

Proof. Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma **??**. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma **??** that s_x is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood U of x such that t_x comes from a section t of $\mathcal{L}^{\otimes -1}$ over U and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that s generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that X_s is open. Moreover, the section t constructed over U above is unique, and hence these glue to give the section s' of the lemma. $\square$

Recall from Modules, Lemma **??** that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where ss' denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

Definition 29.2. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample* if

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

Exercise 29.3. Let L be an ample bundle on a K3 surface M. Prove that $\mathcal{L}^{\otimes 2}$ is globally generated (that is, for each $x \in M$ there exsits a section $h \in H^0(L^{\otimes 2})$ which does not vanish in x).

Proof. This just asks that the n in Definition 28.2 is 2 for all $x \in X$. Because, again, $x \in X_s$ means that $s(x) \neq 0$ because if it was, then we could somehow write s as a product of a vanishing function on $\mathfrak{m}_x$ and a local frame of $\Gamma(X, \mathcal{L})$. But I guess for the exercise do this: a line bundle is *ample* if there is n such that the canonical embedding (cf Lemma 28.5) is an embedding, i.e. that $\mathcal{L}^{\otimes n}$ is *very ample*. (Interestingly, the notion very ampleness is defined in morphisms.tex.) $\square$

Now we pass to the part where ampleness gives you an **open immersion** to some projective space. Because, it's only very ampleness that gives an embedding, right? (Actually I think here in stacks project there are no embeddings but closed immersions.)

Definition 29.4. From modules.tex. Let $(X, \mathcal{O}_X)$ be a ringed space. Given an invertible sheaf $\mathcal{L}$ on X we define the *associated graded ring* to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

Lemma 29.5. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes*

$$f : X \longrightarrow Y = \mathit{Proj}(S),$$

*to the homogeneous spectrum of S (see Constructions, Section **??**). This morphism has the following properties*

(1) *$f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,*
(2) *there are $\mathcal{O}_X$-module maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (**??**),*
(3) *the composition $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ is the identity map, and*
(4) *for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of x such that $f^*\mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.*

Proof. Denote $\psi : S \to \Gamma_*(X, \mathcal{L})$ the identity map. We are going to use the triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ of Constructions, Lemma **??**. By assumption the open subscheme $U(\psi)$ of equals X. Hence $r_{\mathcal{L},\psi} : U(\psi) \to Y$ is defined on all of X. We set $f = r_{\mathcal{L},\psi}$. The maps in part (2) are the components of θ. Part (3) follows from condition (2) in the lemma cited above. Part (1) follows from (3) combined with condition (1) in the lemma cited above. Part (4) follows from the last statement in Constructions, Lemma **??** since the map α mentioned there is an isomorphism. $\square$

Lemma 29.6. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume (a) every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, and (b) X is quasi-compact. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 28.5 above is quasi-compact with dense image.*

Proof. To prove f is quasi-compact it suffices to show that $f^{-1}(D_+(s))$ is quasi-compact for any $s \in S_+$ homogeneous. Write $X = \bigcup_{i=1,\ldots,n} X_i$ as a finite union of affine opens. By Lemma **??** each intersection $X_s \cap X_i$ is affine. Hence $X_s = \bigcup_{i=1,\ldots,n} X_s \cap X_i$ is quasi-compact. Assume that the image of f is not dense to get a contradiction. Then, since the opens $D_+(s)$ with $s \in S_+$ homogeneous form a basis for the topology on $\text{Proj}(S)$, we can find such an s with $D_+(s) \neq \emptyset$ and $f(X) \cap D_+(s) = \emptyset$. By Lemma 28.5 this means $X_s = \emptyset$. By Lemma **??** this means that a power s^n is the zero section of $\mathcal{L}^{\otimes n \deg(s)}$. This in turn means that $D_+(s) = \emptyset$ which is the desired contradiction. ☐

Lemma 29.7. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume $\mathcal{L}$ is ample. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 28.5 is an open immersion with dense image.*

Proof. By Lemma **??** we see that X is quasi-separated. Choose finitely many $s_1, \ldots, s_n \in S_+$ homogeneous such that X_{s_i} are affine, and $X = \bigcup X_{s_i}$. Say s_i has degree d_i. The inverse image of $D_+(s_i)$ under f is X_{s_i}, see Lemma 28.5. By Lemma **??** the ring map

$$(S^{(d_i)})_{(s_i)} = \Gamma(D_+(s_i), \mathcal{O}_{\text{Proj}(S)}) \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_X)$$

is an isomorphism. Hence f induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus f is an isomorphism of X onto the open subscheme $\bigcup_{i=1,\ldots,n} D_+(s_i)$ of $\text{Proj}(S)$. The image is dense by Lemma 28.6. ☐

Lemma 29.8. *Let X be a scheme. Let S be a graded ring. Assume X is quasi-compact, and assume there exists an open immersion*

$$j : X \longrightarrow Y = Proj(S).$$

Then $j^*\mathcal{O}_Y(d)$ is an invertible ample sheaf for some $d > 0$.

Proof. This is Constructions, Lemma **??**. ☐

Proposition 29.9. *Let X be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$. The following are equivalent:*

1. *$\mathcal{L}$ is ample,*
2. *the open sets X_s, with $s \in S_+$ homogeneous, cover X and the associated morphism $X \to Proj(S)$ is an open immersion,*
3. *the open sets X_s, with $s \in S_+$ homogeneous, form a basis for the topology of X,*
4. *the open sets X_s, with $s \in S_+$ homogeneous, which are affine form a basis for the topology of X,*
5. *for every quasi-coherent sheaf $\mathcal{F}$ on X the sum of the images of the canonical maps*

   $$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

   with $n \geq 1$ equals $\mathcal{F}$,

(6) *same property as (5) with $\mathcal{F}$ ranging over all quasi-coherent sheaves of ideals,*
(7) *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exists an integer n_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$,*
(8) *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exist integers $n > 0$, $k \geq 0$ such that $\mathcal{F}$ is a quotient of a direct sum of k copies of $\mathcal{L}^{\otimes -n}$, and*
(9) *same as in (8) with $\mathcal{F}$ ranging over all sheaves of ideals of finite type on X.*

Proof. Lemma 28.7 is (1) $\Rightarrow$ (2). Lemmas **??** and 28.8 provide the implication (1) $\Leftarrow$ (2). The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (3) are clear from Constructions, Section **??**. Lemma **??** is (3) $\Rightarrow$ (1). Thus we see that the first 4 conditions are all equivalent.

Assume the equivalent conditions (1) – (4). Note that in particular X is separated (as an open subscheme of the separated scheme $\text{Proj}(S)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Choose $s \in S_+$ homogeneous such that X_s is affine. We claim that any section $m \in \Gamma(X_s, \mathcal{F})$ is in the image of one of the maps displayed in (5) above. This will imply (5) since these affines X_s cover X. Namely, by Lemma **??** we may write m as the image of $m' \otimes s^{-n}$ for some $n \geq 1$, some $m' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This proves the claim.

Clearly (5) $\Rightarrow$ (6). Let us assume (6) and prove $\mathcal{L}$ is ample. Pick $x \in X$. Let $U \subset X$ be an affine open which contains x. Set $Z = X \setminus U$. We may think of Z as a reduced closed subscheme, see Schemes, Section **??**. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to the closed subscheme Z. By assumption (6), there exists an $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ such that s does not vanish at x (more precisely such that $s \notin \mathfrak{m}_x \mathcal{I}_x \otimes \mathcal{L}_x^{\otimes n}$). We may think of s as a section of $\mathcal{L}^{\otimes n}$. Since it clearly vanishes along Z we see that $X_s \subset U$. Hence X_s is affine, see Lemma **??**. This proves that $\mathcal{L}$ is ample. At this point we have proved that (1) – (6) are equivalent.

Assume the equivalent conditions (1) – (6). In the following we will use the fact that the tensor product of two sheaves of modules which are globally generated is globally generated without further mention (see Modules, Lemma **??**). By (1) we can find elements $s_i \in S_{d_i}$ with $d_i \geq 1$ such that $X = \bigcup_{i=1,\ldots,n} X_{s_i}$. Set $d = d_1 \ldots d_n$. It follows that $\mathcal{L}^{\otimes d}$ is globally generated by

$$s_1^{d/d_1}, \ldots, s_n^{d/d_n}.$$

This means that if $\mathcal{L}^{\otimes j}$ is globally generated then so is $\mathcal{L}^{\otimes j+dn}$ for all $n \geq 0$. Fix a $j \in \{0, \ldots, d-1\}$. For any point $x \in X$ there exists an $n \geq 1$ and a global section s of $\mathcal{L}^{j+dn}$ which does not vanish at x, as follows from (5) applied to $\mathcal{F} = \mathcal{L}^{\otimes j}$ and ample invertible sheaf $\mathcal{L}^{\otimes d}$. Since X is quasi-compact there we may find a finite list of integers n_i and global sections s_i of $\mathcal{L}^{\otimes j+dn_i}$ which do not vanish at any point of X. Since $\mathcal{L}^{\otimes d}$ is globally generated this means that $\mathcal{L}^{\otimes j+dn}$ is globally generated where $n = \max\{n_i\}$. Since we proved this for every congruence class mod d we conclude that there exists an $n_0 = n_0(\mathcal{L})$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. At this point we see that if $\mathcal{F}$ is globally generated then so is $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0$.

We continue to assume the equivalent conditions (1) – (6). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. Denote $\mathcal{F}_n \subset \mathcal{F}$ the image of the canonical map of (5). By construction $\mathcal{F}_n \otimes \mathcal{L}^{\otimes n}$ is globally generated. By (5) we see $\mathcal{F}$ is the sum of the subsheaves $\mathcal{F}_n$, $n \geq 1$. By Modules, Lemma **??** we see that $\mathcal{F} = \sum_{n=1,\ldots,N} \mathcal{F}_n$ for some $N \geq 1$. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated whenever $n \geq N + n_0(\mathcal{L})$ with $n_0(\mathcal{L})$ as above. We conclude that (1) – (6) implies (7).

Assume (7). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. By (7) there exists an integer $n \geq 1$ such that the canonical map

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

is surjective. Let I be the set of finite subsets of $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ partially ordered by inclusion. Then I is a directed partially ordered set. For $i = \{s_1, \ldots, s_{r(i)}\}$ let $\mathcal{F}_i \subset \mathcal{F}$ be the image of the map

$$\bigoplus\nolimits_{j=1,\ldots,r(i)} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

which is multiplication by s_j on the jth factor. The surjectivity above implies that $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$. Hence Modules, Lemma **??** applies and we conclude that $\mathcal{F} = \mathcal{F}_i$ for some i. Hence we have proved (8). In other words, (7) $\Rightarrow$ (8).

The implication (8) $\Rightarrow$ (9) is trivial.

Finally, assume (9). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. By Lemma **??** (this is where we use the condition that X be quasi-separated) we see that $\mathcal{I} = \operatorname{colim}_\alpha I_\alpha$ with each I_α quasi-coherent of finite type. Since by assumption each of the I_α is a quotient of negative tensor powers of $\mathcal{L}$ we conclude the same for $\mathcal{I}$ (but of course without the finiteness or boundedness of the powers). Hence we conclude that (9) implies (6). This ends the proof of the proposition. □

The following proofs were used for Exercise 13.7.

Lemma 29.10. *Let D be a divisor on a complete, nonsingular curve X. If $h^0(D) \neq 0$ then $degD \geq 0$.*

Proof. If D has sections, we can take the zero locus of any of its sections to produce an effective divisor linearly equivalent to D. Since degree depends only on linear equivalence and effective divisors have non negative degree. □

Proposition 29.11. *Let D be a divisor on a complete, nonsingular curve X. Then the complete linear system has no base points if and only if for every point $P \in X$,*

$$\dim |D - P| = \dim |D| - 1$$

Proof. To show that D has no base points amounts to showing that not every section of D. That is, that the injective map $0 \to H^0(D - p) \to H^0(D)$ is not surjective. □

Lemma 29.12. *Let D be a divisor on a curve X of genus g. If $degD \geq 2g$, then $|D|$ has no base points.*

Proof. First we prove that $\deg D \geq 2g$ implies that D and $D - P$ are *nonspecial*, i.e. that $h^0(K - D) = 0 = h^0(K - (D - P))$. Then we apply Riemann-Roch to both D and $D - P$ and the fact that $\deg(D - P) = \deg(D) - 1$ to find that $\dim |D - P| = \dim |D| - 1$ and apply Proposition 28.11.

To prove that D is nonspecial first apply Riemann-Roch to K to obtain that $\deg K = 2g-2$. Indeed, $h^0(0) = 1$ and $h^0(K) = p_g$ by Serre duality on $H^1(\mathcal{O}_X)$ recalling definition of genus as $h^1(\mathcal{O}_X)$. Then apply Riemann-Roch to both D and $K - D$ to prove that $\deg D > 2g-2$ implies $\deg(K-D) < 0$; start with

$$h^0(K-D) - h^0(K-(K-D)) = -(h^0(D) - h^0(K-D))$$

An analogous result will be valid for $D - P$ since its degree is also greater than $2g-2$.

Then apply Lemma 28.10. □

30. Adjunction formulas

There are several statements called adjunction formula in different texts. All of them concern "subvarieties", that is, closed embedded subschemes.

Exercise 30.1 (Genus formula for a curve on a surface)**.** Let $C \to X$ be a closed embedded subscheme of dimension 1 (as a topological space, i.e. pure dimension) inside a smooth surface X. Then $2p_a - 2 = (\mathcal{O}_X(C), \mathcal{O}_X(C))$.

Proof. Consider the ideal sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

This sequence splits since there is an obvious inverse morphism to the inclusion $\mathcal{O}_X(-C) \to \mathcal{O}_X$, namely mapping a function f to Then $\mathcal{O}_X \cong \mathcal{O}_X(-C) \oplus \mathcal{O}_C$. $\chi(\mathcal{O}_X) = \chi$ □

31. Normalization

Definition 31.1. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the scheme[2]

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over X. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

The factorization is the composition of the canonical morphism $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ (see Constructions, Lemma **??**) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \to f_*\mathcal{O}_Y$. We can characterize the normalization as follows.

Lemma 31.2. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

(1) *the morphism ν is integral, and*

[2]The scheme X' need not be normal, for example if $Y = X$ and $f = \mathrm{id}_X$, then $X' = X$.

(2) *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f'"'] & Z \arrow[d, "\pi"] \\ X' \arrow[r, "\nu"] \arrow[ur, "h"] & X \end{tikzcd}$$

for some unique morphism $h : X' \to Z$.

Moreover, the morphism $f' : Y \to X'$ is dominant and in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

32. Reflexive sheaves

Definition 32.1. Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The *reflexive hull* of $\mathcal{F}$ is the $\mathcal{O}_X$-module

$$\mathcal{F}^{**} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

We say $\mathcal{F}$ is *reflexive* if the natural map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is an isomorphism.

Lemma 32.2. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(1) *If $\mathcal{F}$ is reflexive, then $\mathcal{F}$ is torsion free.*
(2) *The map $j : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ is injective if and only if $\mathcal{F}$ is torsion free.*

Remark 32.3 (Talk at IMPA, 11 June). Torsion could also be defined so that the sheaf can inject onto its dual. In this talk we discussed the moduli space of reflexive/torsion-free sheaves, which turned out to be parametrized by c_1, c_2 and c_3. This was denoted by $R(c_1, c_2, c_3)$. Actually I think it may have been Manolache that proved the existence of this moduli space.

Alan Muniz

Nesta palestra discutiremos a classificação de feixes reflexivos de posto dois e seus espaços de módulos. Apresentaremos algumas ferramentas básicas usadas na construção e determinação de tais feixes. Aplicaremos estas técnicas para o caso de feixes com segunda classe de Chern igual a quatro, obtido recentemente em colaboração com Marcos Jardim.

32.4. **Distributions on manifolds.** Here's the abstract from a talk by Marcos Jardim at Geometric Structures:

"I will revise the work done over the past 10 years with various collaborators on distributions and foliations on 3-folds, especially on the projective space, with a focus on properties of the tangent sheaf and singular scheme."

Here are two key ideas: if the distribution is codimension 1 we can write:

$$0 \longrightarrow F \longrightarrow TX \xrightarrow{\omega} I_Z \otimes L \longrightarrow 0$$

where L is a line bundle and $\omega \in H^0(\Omega_X \otimes L)$, and $Z = \{p : \omega(p) = 0\}$.

When codimension is 2 then $\mathcal{D}$ is given by a holomorphic vector field ν: $T_p = \langle \nu(p) \rangle$.

It can be encoded as an exact sequence

$$0 \longrightarrow L \xrightarrow{\nu} TX \longrightarrow N \longrightarrow 0$$

where L is a line bundle and $\nu \in H^0(TX \otimes L^\vee)$; $Z = \{p | \nu(p) = 0\}$.

Remark 32.5. Saturation means that $Z \subset X$ is a union of curves and points.

And again, distributions are parametrized by Chern classes.
Two interesting open questions:

(1) **Conjeture.** if $\mathcal{D}$ is a codimension 1 foliation of degree d on $\mathbb{P}^3$, then $c_2(F) \leq d^2 - d + 1$ and bound is attained by rational foliations of type $(1, d+1)$. (True for $d \leq 2$.)
(2) **Conjecture (with Pepe Seade).** $\mathcal{D}$ is a codimension 1 foliation on a smooth projective 3-fold, then Sing$\mathcal{D}$ is connected.

Theorem 32.6 (Jardim-Muniz)**.** *Conditions on Chern classes used to understand moduli space $R(c_1, c_2, c_3)$. $c_2 = 4$ gives (?). For $c_3 \leq 6$, possible "spectrum" exists...*

33. Stability

Question. What is stability?

(1) Stable objects in an abelian category are the "building blocks": we can reconstruct the whole category from them.
(2) An abelian subcategory (hart) $\subset$ a triangulated
(3) stability defined via stability function on $\mathcal{A}$.
(4) Q. Can we reconstruct $\mathcal{T}$ from the semistable elements of $\mathcal{A}$
(5) **Example.** $\mathcal{A} = \mathrm{Coh}X$ is heart of $D^b(X)$ w/ funny function.
(6) Stability condition is hart + stability function.
(7) Bridgeland Stabl:= the stability conditions are a complex manifold of complex codimension rkΛ:
$$\mathcal{Z} : \mathrm{Stab}(\mathcal{T}) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C})$$
(8) There's a chamber structure; moduli space changes across chambers.
(9) I think we typically think of vectors in $\mathrm{Hom}(\Lambda, \mathbb{C})$ as Chern classes, to characterize the moduli spaces.
(10) Existence: given a projective variety X, are there stability conditions on $D^{\mathrm{b}}(X)$? Yes for fano 3fold pic rk 1.
(11) Moduli spaces: is $M_\sigma(v)$ a projective scheme? Cannot use usual git techniques to study. A stack!
(12) Picture: blue + black are walls. Q. What are β and α?
(13) thm: bridgeland stable = gieseker stable ?
(14) Q. slope stability = bridgeland stability? A. Not always.
(15) DT/PT correspondance: only one wall between PT and G chambers
(16) Polynomial stability function. This is an asymptotic version of BS.
(17) There are some ρ's. Arrangements of ρ_i are polynomial stability conditions on a threefold.
(18) Pata-Thomas introduced stability for rank 1 objects. Bayer compares the— wall. Q. Same for Bridge S—only one wall?
(19) Recall Gieseker stability.
(20) Def. A *stable triple*: when $\gcd(\mathrm{ch}_0, \mathrm{ch}_2, \mathrm{ch}_3) = 1$, every PT stable object comes from three conditions (missing).
(21) What happens when you cross the blue wall? Both $\mathcal{G}$ and $\mathcal{T}$ are projective. What happens at the blue wall?

(22) For X smooth threefold with rkPic $= 1$, $\mathcal{G}(v)$, $v = (r, 0, 0, -n)$, $\mathcal{G}(v)$ is a known sheaf object and $\mathcal{T} = \emptyset$.
(23) X sm 3 rkpic1, Fake wall; $\mathcal{G} = \mathcal{T}$.
(24) (Extra.) red circle is a wall for a weaker form of stability.

Definition 33.1 (Talk at impa)**.** A rank-2 sheaf $\mathcal{F}$ is *semistable (stable)*if $H^0(\mathcal{F}(t)) = 0$ for $-t \geq (>) \frac{c_1 F}{2}$

Compare with

Definition 33.2 (moduli-curves.tex)**.** Let $f : X \to S$ be a family of curves. We say f is a *semistable family of curves* if

(1) $X \to S$ is a prestable family of curves, and
(2) X_s has genus ≥ 1 and does not have a rational tail for all $s \in S$.

- The twistor diagram.

$$\begin{tikzcd}
 & \ell_\infty \arrow[dl] \arrow[dr, dotted] & \\
\mathbb{C}P^2 \arrow[rr, hook] & & \mathbb{C}P^3 \arrow[dd, "\text{twistor map}"'] \\
 & & \\
\mathbb{C}^2\mathbb{R}^4 \arrow[u, hook] \arrow[rr] & & S^4
\end{tikzcd}$$

- Instantons are solutions to some Yang-Mills solution.
- Expository paper of Donaldson arXiv:2205.08639
- ADHM construction, 1978. The first appearance of Algebraic Geometry in Mathematical Physics. See Hitchin-Kobayashi correspondence.
- Donaldson: "unashamedly computational".
- Expository paper by Simon.
- Take bundle (E, ∇) with an anti-self dual (ADS) connection on S^4 and pullback to $\mathbb{C}P^3$ via the twistor map

  $$\tau[x : y : z : w] = [x + jy : z + jw] \qquad \text{note: } \tau^{-1}(p) = \mathbb{C}P^1$$

  Then:
  - Restriction to fibres are trivial.
  - Invariant under anti holomorphic involution (check this formula!) $[x : y : z : w] \mapsto [-y : x : -w : z]$.
  - $\mathcal{E}$ is also an instanton bundle.
  - Penrose Transform: $H^1(\mathcal{E}(-2)) \cong \operatorname{Ker} \Delta = 0$ where Δ is a Laplacian.
  - Definition of instanton sheaf on $\mathbb{P}^3$ via $c_1(E) = 0$ and some vanishing of cohomologies.
  - Passage from [differential equations? algebraic geometry?] to linear algebra: via *monads*. The point is that instanton sheaf is equivalent to "E being the cohomology of a linear monad"; theorem by Horps in the 60's, and is the main tool used by ADHM. Indeed, ADHM equations come from the cohomology sequences of the so-called monads.
  - Mathamatical inst bund:= locally free instanton sheaves.

Properties.

- The only instanton of rank 1 on $\mathbb{P}^3$ is the structure sheaf.
- non-trivial rank 2 locally free instanton sheaf ois $\mu_0 stable$
- double dual is locally free and also instanton
- non-trivial rank 2 instanton sheaaf is Fieseker stable therefore it makes sance fo define moduls space of instanton sheaves as an open subset of $\mathcal{M}(c) = \mathcal{G}(k, 0, 2, 0)$.

Then studied the irreducibility (Tikhomirov) and smoothness (Jardim-Verbitsky, 2014. Uses "3rd hyperkähler quotient") of $\mathcal{I}(c)$, the moduli space of rank 2 locally gree instanton sheaves of charge c. But nobody likes this results; want new proofs.

In contrast, $\mathcal{M}(c)$ of rank 2-instanton sheaves of charge c is not irreducible in general! $\mathcal{M}(1)$ and $\mathcal{M}(2)$ are irreducible, $\mathcal{M}(3)$ has exactly 2 irreducible components of dimension 21; $\mathcal{M}(4)$ has 4 irredicuble components: the locally free is irreducible, and the other 3 that intersect the closure of the locally free, $\overline{\mathcal{I}(c)}$. 3 components of dimension 29 and one of dimension 32

Remark 33.3. In general the μ moduli space is not projective, but the Gieseker is.

Is $\mathcal{M}(c)$ connected? Use $\mathbb{C}^*$ action. True for $c \leq 4$; every component intersects $\overline{\mathcal{I}(c)}$ in this range!

Definition 33.4 (Elementary transformation)**.** F of rank 2 locally free instanton, Q of rank 0 instanton with 1 dimensional sheaf $h^p Q(-2)$ for $p = 0, 1$. So in this conditions if we have an epimorphism

$$F \overset{\varphi,\ \mathrm{surj}}{\to}\} Q$$

we get that $\operatorname{Ker}\varphi$ is an instanton.

So we might be interested in

$$E \hookrightarrow E^{**} \xrightarrow{\text{surj.}} Q_E$$

Let's have a look at $\mathcal{M}(3) = \overline{\mathcal{I}(3)} \cup \overline{C(0,1,3}$. Consider a generic line bundle of degree 0 on a planar cubic (cwhich is encoded in that we have intersection of something of cimension 1 and something of simension 3), $Linn \mathrm{Pic}^0(C)$ so we have an epimorphism

$$0 \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \longrightarrow (i_* L) \longrightarrow 0$$

yielding a bundle E as previously outlined.

So we are studying the components via pushforwards of sheaves on complete intersection curves inside $\mathbb{P}^3$

"when we do alementary transformation of rational (not rationl?) we get something on the boundary of locally free."

Perverse instanton sheaves. Like an instnaton sheaf in monad description but with some restrictions on the cohomologies. This leads to definition of 0-*rimensional instanton*, a perverse instanton such that $\mathcal{H}^0 = 0$.

Instantons and quivers.

Framed instantons. Fix a line $j : L \to \mathbb{P}^3$ and E a perverse sheaf; a *framing* at L is an isomorphis [missing].

Apply GIT to the ADHM data to construct a moduli space

$$\mathcal{P}(r,c) = \mathcal{V}(r,c)^{\mathrm{st}}//\mathrm{GL}(V_c)$$

and it will follow that $\mathbb{P}(?,?)$ is connected.

Considerations on quaternionic spaces lead to generalization of what has been discussed so far to higher dimensions. I.e. an *instanton sheaf* on $\mathbb{P}^n$ is... [other cohomological conditions] sheaf

Why are instantons interesting? They are the simplest; may provide examples for Bridgeland stability.

In Kuznetsov (2012) and Faenzi (2014) introduced *rank 2 instanton bundles on Fano 3-folds*. An *instanton bundle* on X is a μ-stable ... and some Chern class is called the *charge*. An *instanton sheaf* (introduced by Marcos-Gaia) is ....

"Since we are imposing μ-stability on the defintion we can consider the moduli $\mathcal{I}_X(c) \subset \mathcal{G}_X(2, -r_X, c, 0)$".

There's also monad representations as an ingredient.

Here are two questions that invite us to join the instanton fever:

Task 1. Construct rank 2-instanton sheaves that do not deform into locally free ones, and obtain the new irreducible components of $\mathcal{G}(2, -r_X, c, 0)$.

Task 2. Nonlocally instanton sheaves that can be deformed into non-locally free ones: the *instanton boundary* $\overline{\mathcal{I}(c)}/\mathcal{I}(X)$[formula right?]

Recipe to construct your own instanton.

(1) (Make a bunch of instantons.) Find an appropriate curve to use Serre correspondence to find some rank 2 instanton sheaf:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \underbrace{E}_{\exists} \longrightarrow \mathcal{I}_{\sqcup \mathrm{lines}} \longrightarrow 0$$

where E is a locally free instanton of charge = the number of lines -1.

(2) (Is your family of instantons generic?) You look at the Exts. "Therefore, the family of instantons only defines a locally closed subset within a generically smooth irreducible component of $\mathcal{G}$".

(3) "Find suitable rank 0 intranton sheaves to perform an elementary transformations on the examples obtained in Step 1." But they are non-locally free.

(4) Now I have my instantons, I know they are locally free: but how to prove that the elementary transformations deform to locally free ones? Looks like the challenge is to prove that the deformation is locally free.

In the papers by the group there are several particular cases when the deformations are locally free. But they don't have a general result that would work for 3-folds.

What you need to call a thing an instanton.

- Minimal cohomology possible; try to kill as much cohomology as you can.
- Fixing c_1 (which may determine other Chern classes).
- Some stability condition like μ-stability or quiver stability. Here is an example that is not μ-semistable: $T\mathbb{P}^3(-1) \oplus \Omega_{\mathbb{P}^3}(1)$

- Whenever possible, look for a monadic representation. (The monadic representation comes from ADHM — the beginnings of this theory. And it's still here!)

34. Coherent sheaves

Lemma 34.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply Cohomology, Lemma **??**. As our basis $\mathcal{B}$ for the topology of X we are going to use the affine opens of X. As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X. Next we check that conditions (1), (2) and (3) of Cohomology, Lemma **??** hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings of any element of $\mathcal{B}$, see Schemes, Lemma **??**. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma **??**. □

35. Hilbert polynomial

The following lemma will be made obsolete by the more general Lemma **??**.

Lemma 35.1. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function*

$$d \longmapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$$

is a polynomial.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and $\mathcal{F}(d) = \mathcal{F}$. Hence in this case the function is constant, i.e., a polynomial of degree 0. Assume $n > 0$. By Lemma **??** we may assume k is infinite. Apply Lemma **??**. Applying Lemma **??** to the twisted sequences $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*\mathcal{G}(d) \to 0$ we obtain

$$\chi(\mathbf{P}^n_k, \mathcal{F}(d)) - \chi(\mathbf{P}^n_k, \mathcal{F}(d-1)) = \chi(H, \mathcal{G}(d))$$

See Remark **??**. Since $H \cong \mathbf{P}^{n-1}_k$ by induction the right hand side is a polynomial. The lemma is finished by noting that any function $f : \mathbf{Z} \to \mathbf{Z}$ with the property that the map $d \mapsto f(d) - f(d-1)$ is a polynomial, is itself a polynomial. We omit the proof of this fact (hint: compare with Algebra, Lemma **??**). □

Definition 35.2. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function $d \mapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$ is called the *Hilbert polynomial* of $\mathcal{F}$.

The Hilbert polynomial has coefficients in $\mathbf{Q}$ and not in general in $\mathbf{Z}$. For example the Hilbert polynomial of $\mathcal{O}_{\mathbf{P}^n_k}$ is

$$d \longmapsto \binom{d+n}{n} = \frac{d^n}{n!} + \ldots$$

This follows from the following lemma and the fact that

$$H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(d)) = k[T_0, \ldots, T_n]_d$$

(degree d part) whose dimension over k is $\binom{d+n}{n}$.

Lemma 35.3. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$ with Hilbert polynomial $P \in \mathbf{Q}[t]$. Then*

$$P(d) = \dim_k H^0(\mathbf{P}^n_k, \mathcal{F}(d))$$

for all $d \gg 0$.

Proof. This follows from the vanishing of cohomology of high enough twists of $\mathcal{F}$. See Cohomology of Schemes, Lemma **??**. ☐

For completeness I include earlier notes from [Har77] on the matter.

The fact that M is finitely generated is what makes the following two definitions make sense.

Definition 35.4. The *Hilbert function* of a finitely generated graded $S = k[x_0, \ldots, x_r]$ -module M is

$$H_M(d) = \dim_k M_d$$

Definition 35.5. Define F_0 to be the free S-module on the generators of M. Elements in the kernel M_1 of the inclusion are called *sysygies*. By Hilbert's basis theorem, M_1 is also finitely generated, so we may choose a set of generators and repeat this process.

Theorem 35.6 (Hilbert Syzygy Theorem)**.** *Any finitely generated S-module M has a finite graded free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} \quad \longrightarrow F_{m-q} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

Moreover, we may take $m \leq r+1$, the number of variables in S.

Lemma 35.7. *Suppose that $S = k[x_0, \ldots, x_r]$ is a polynomial ring. If the graded S-module M has finite free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \cdots [r] \qquad F_1 \xrightarrow{\varphi_1} \quad \longrightarrow F_0$$

with each F_i a finitely generated free module, $F_i = \bigoplus_j S(-a_{i,j})$, then

$$H_M(d) = \sum_{i=0} (-1)^i \sum_j \binom{r+d-a_{i,j}}{r} \tag{35.7.1}$$

Lemma 35.8. *There is a polynomial $P_M(d)$ called the Hilbert polynomial such that, if M has free resolution as above, then $P_M(d) = H_M(d)$ for $d \geq max_{i,j}\{a_{i,j} - r\}$.*

Proof. When d satisfies this bound then the binomial coefficients in Eq. 34.7.1 are polynomials of degree r in d. ☐

Theorem 35.9 (Hilbert-Serre)**.** *Let M be a finitely generated graded $S = k[x_0, \ldots, x_n]$. Then there exists a unique polynomial p_M such that $p_M(\ell) = \dim S_\ell$ for large enough ℓ.*

Definition 35.10. The polynomial P_M of Hilbert-Serre Theorem [**?**] is the *Hilbert polynomial* of the finitely generated $k[x_0, \ldots, x_n]$-module M.

Definition 35.11. If $Y \subset \mathbb{P}^n$ is an algebraic set of dimension r, we define the *degree of* Y to be $r!$ times the leading coefficient of the Hilbert polynomial of the homogeneous coordinate ring $S(Y)$.

Exercise 35.12. Let H be a very ample divisor on the surface X, corresponding to a projective embedding $X \subseteq \mathbb{P}^N$. If we write the Hilbert polynomial of X as $P(z) = \frac{1}{2}az^2 + bz + c$, show that $a = H^2$, $b = \frac{1}{2}H^2 + 1 - \pi$, where π is the genus of a nonsingular curve representing H, and $c = 1 + p_a$.

36. Nakai-Moishezon Criterion

Theorem 36.1 (Nakai-Moishezon Criterion)**.** *A divisor D on the surface X is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C in X.*

Proof. The direct implication is easy: since D is ample, mD is very ample for some m, so that m^2D^2 is the self-intersection number of mD. By exercise 34.12, D^2 is the leading coefficient of the Hilbert polynomial of X as a subscheme of $\mathbb{P}^n$. This means that D^2 is twice the leading coefficient of the Hilbert polynomial of a projective variety for large enough m, so that it must be a positive number (it's the dimension of one of the graded components of the coordinate ring of the surface). $\square$

37. Hilbert scheme

Upshot [**?**, p. 6]**.** We wish to parametrize subschemes of a projective space (or perhaps a more general scheme?). Since there are too many such subschemes we restrict ourselves to schemes with a given Hilbert polynomial, since the latter "encodes the most important numerical invariants of schemes". The Hilbert scheme is introduced via a theorem by Grothendieck as the object that represents the functor $\mathbf{Hilb}_{P,r}$ that maps a reduced scheme B to the set of proper flat families

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}^r \times B \arrow[d, "\pi_B"] \\ & B \end{tikzcd}$$

with $\mathcal{X}$ having Hilbert polynomial P.

Theorem 37.1 (Grothendieck, '66)**.** *The functor* $\mathbf{Hilb}_{P,r}$ *is representable by a projective scheme* $\mathcal{H}_{P,r}$.

SEE Hilbert schemes of subschemes.

38. Deformation theory

I start by reading Stacks Project.

The first notion is *thickening of ringed spaces*, which I ultimately think of as a closed subscheme $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ with a nilpotent ideal sheaf, which in an imprecise way means that $\mathcal{I}_X^n = 0$ for some n, and in a precise way its given on sections.

The following definition is from [lucas-defos], which in turn comes from [Ser06]

Definition 38.1. Let X be an algebraic $\mathbb{C}$-scheme.

(1) A *deformation* of X is a Cartesian diagram ξ

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \mathbb{C} \arrow[r, "s"] & S \end{tikzcd}$$

where π is a flat surjective morphism of algebraic $\mathbb{C}$-schemes and S is connected. (Recall that flatness accounts for "continuity".)

(2) A *local deformation* of X is a deformation ξ where $S = \operatorname{Spec} A$ for A a noetherian local $\mathbb{C}$ algebra with residue field $\mathbb{C}$.
(3) An *infinitesimal deformation of* X is a local deformation with A an artinian local $\mathbb{C}$-algebra with residue field $\mathbb{C}$. X is called *rigid* if all infinitesimal deformations are trivial.
(4) An *inifinitesimal deformation of order* n is an infinitesimal deformation when $S = \operatorname{Spec}(\mathbb{C}[t]/(t^{n+1})$.

Upshot. An interpretation of the so-called *dual numbers* $k[t]/(t^2)$ (see [Har09]) as the tangent space of something is thinking of Taylor polynomials: after quotienting by t^2 we loose the tails of the polynomials and are left with the first derivative information only.

So what is the deformation space? Is it a moduli space of curves, that is, points are curves obtained by deforming the curve? Or is it the fibration whose fibers are curves and there is a central fiber that is the original curve?

There is the following interpretation in [continued-fractions] p. 39: the space of first order deformation classes of X is $D(\mathbb{C}[t]/(t^2)$. This is said to ""represent" the tangent space $\mathbb{T}^1_X$ of the hypothetical deformation space of X". (I put double quotations because the word "represent" is quoted in the original text.) Further, if X is nonsingular and compact, then $\mathbb{T}^1_X = H^1(X, T_X)$.

Which basically I interpret as: the dimension of the deformation space of a smooth compact variety is $H^1(X, T_X)$.

Example 38.2. Fix $g \geq 2$. The dimension of the deformation space of a nonsingular projective curve X is $3g-3$. This is "the dimension of the moduli space of curves of genus g".

We can compute this number by Riemann-Roch formula on the bundle $-K_X$. Indeed, since X is a curve, $\Omega^1_X = K_X$ and thus $-K_X = T_X$. We get

$$\begin{aligned} h^0(-K) - h^0(K-(-K)) &= \deg(-K) - g + 1 \\ &= -2g+2-g+1 \qquad \text{degree additive and } \deg K = 2g-2 \\ &= -3g+3 \end{aligned}$$

Now by Serre duality $h^0(K-(-K)) = h^1(-K) = h^1(T_X)$, and it turns out that a Riemann surface of genus $g \geq 2$ has no holomorphic vector fields, so that $h^0(-K) = h^0(T_X) = 0$.

Exercise 38.3. Let C be a smooth genus g curve which can be embedded in a K3 surface M, and X the family of all deformations of C in M.

(1) Prove that $\dim X \leq g$.
(2) Let $\mathcal{X}_g$ be the space of all curves of genus g (smooth?) which can be possibly embedded to a K3 surface. Prove that each irreducible component Z of $\mathcal{X}_g$ satisfies $\dim_{\mathbb{C}} \leq g + 19$. Deduce that there exists a compact complex curve which cannot be embedded in a K3 surface.

Exercise 38.4. Let C be a smooth curve embedded in a K3 surface X. Show that the dimension of the deformation space of C is $\leq g$.

Proof. The deformation space of a variety is the space of isomorphism classes of deformations as explained above. It turns out that there is a way to associate 1-cocyles of the tangent sheaf to deformations, so that in fact the deformation space Def_1 is isomorphic to $H^1(X, \mathcal{T}_X)$ for any variety X.

For our curve C we thus know that the dimension of the space of deformations (deformations not necessarily contained in X) is $h^1(\mathcal{T}_C)$. The family of deformations of C that are contained in X is the Hilbert space of curves with fixed Hilbert polynomial $P(t)$ after quotienting by $\mathbb{C}s$, where s is the section whose vanishing locus is C. This says that the number we are looking for is $h^0(X, \mathcal{O}(C)) - 1$.

Now I will show that $h^0(X, \mathcal{O}(C)) = 1 + g$ (see [**?**, Lemma 1.2.1, Remark 1.2.2]).

Consider the ideal sheaf exact sequence twisted by $\mathcal{O}(C)$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_C = \mathcal{O}(C)|_C \longrightarrow 0$$

By X being a K3 we know that $H^1(\mathcal{O}_X) = 0$, so that we have the short exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}(C)) \longrightarrow H^0(\mathcal{O}(C)|_C) \longrightarrow 0$$

so that $h^0(\mathcal{O}(C)) = 1 + h^1(\mathcal{O}(C)|_C)$. A version of adjunction formula says $\omega_C \cong (K_X \otimes \mathcal{O}(C)|_C$, and using that $K_X = \mathcal{O}_X$ we obtain $h^0(\mathcal{O}(C)|_C) = h^0(\omega_C) = g$.

To

For the record I put other thoughts I went through in solving this exercise.

Recall from [**?**, p. 146] that the normal bundle $\mathcal{N}$ of a hypersurface of a smooth variety X satisfies $\mathcal{N}^\vee \cong \mathcal{O}_X(-C)|_C$. Taking duals we get that $\mathcal{N} \cong \mathcal{O}_X(C)|_C$.

By adjunction formula $2g-2 = (\mathcal{O}(C), \mathcal{O}(C))$. Applying Riemann-Roch to $\mathcal{O}(C)$ (which is by definition the dual of the ideal sheaf of C, which is a line bundle on X), we obtain that $\chi(\mathcal{O}(C)) = 2 + \frac{1}{2}(\mathcal{O}(C), \mathcal{O}(C))$. Then $\chi(\mathcal{O}(C)) = g + 1$.

Now recall that $\chi(\mathcal{O}(C)) = h^0(\mathcal{O}(C) - h^1(\mathcal{O}(C)) + h^2(\mathcal{O}(C))$. By Serre duality and X being a K3 surface we see that $h^2(\mathcal{O}(C)) \cong h^0(\mathcal{O}(-C))$, which is the ideal sheaf of C. Any section of such a sheaf would vanish along C, and since X is compact we conclude there cannot be any such section.

Now we show that also $h^1(\mathcal{O}(C)) = 0$ to conclude that $h^0(\mathcal{O}(C)) = g + 1$.

We thus conclude that $h^0(\mathcal{O}(C))$

Since C is smooth we can use the normal exact sequence

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_X|_C \longrightarrow \mathcal{N} \longrightarrow 0$$

Taking Euler characteristic we see that $\chi(\mathcal{T}_C) + \chi(\mathcal{N}) = \chi(\mathcal{T}_X|_C)$.

This means that we should be done once we compute $\chi(\mathcal{T}_X|_C)$. For this we can use Riemman-Roch formula for coherent sheaves on a curve, which tells us that

$$\deg(\mathcal{T}_X|_C) = \chi(\mathcal{T}_X|_C) - \mathrm{rk}(\mathcal{T}_X|_C) \cdot \chi(\mathcal{O}_C)$$

But of course we know that since C is a curve, by Serre duality we get that $\chi(\mathcal{O}_C) = 1 - g$. (Indeed: $h^1(\mathcal{O}_C) = h^0(\Omega^1_C) := p_a(C)$.) And now the question is what is the degree. Apparently this is just the first Chern class c_1 of the restricted tangent bundle. So what is it? And what is $h^0(\mathcal{T}_C)$ if the genus is 0 or 1, and that's it.

I know that $h^0(\mathcal{T}_X) = 0$ and $h^1(\mathcal{T}_X) = 20$ by X being a K3 surface, but I'm not sure what happens when we restrict to C.

If $g \geq 2$ we know that $h^0(\mathcal{T}_C) = 0$, so that $\boxed{\chi(\mathcal{T}_C) = h^1(\mathcal{T}_C)}$. To compute the latter Euler characteristic, which is given by definition by $\chi(\mathcal{T}_X|_C) = h^0(\mathcal{T}_X|_C) - h^1(\mathcal{T}_X|_C)$, we first note that $h^0(\mathcal{T}_X|_C) = 0$, because this is the dimension of the global holomorphic vector fields on X restricted to C, which is constant along C since X is smooth. And then this number is actually zero by Hodge numbers of a K3 surface.

and taking cohomology long exact sequence we obtain

$$\cdots \longrightarrow H^0(T_X) \longrightarrow H^0(\mathcal{N}) \longrightarrow H^1(T_C) \longrightarrow$$

$$H^1(T_X) \longrightarrow H^1(\mathcal{N}) \longrightarrow \cdots$$

Or is it $h^1(\mathcal{N})$? See [?]. If this was the case, then we can use adjunction formula as above to get that $2g-2 = (\mathcal{N},\mathcal{N}) = \deg_C\mathcal{N}$. Then we may find $h^0(\mathcal{N})$ via Riemann-Roch:

$$h^0(\mathcal{N}) - h^0(K_C - \mathcal{N}) = \deg\mathcal{N} - g + 1$$

Note that $h^0(N_C - \mathcal{N}) = h^1(-K_C + N + K_C) = h^1(\mathcal{N})$ via Serre duality, and by Riemann-Roch on a surface as above we see that

$$g+1 = \chi(\mathcal{N}) = h^0(\mathcal{N}) - h^1(\mathcal{N}) \implies h^1(\mathcal{N}) = -g - 1 + h^0(\mathcal{N})$$

so that

$$h^0(\mathcal{N}) - (-g-1+h^0(\mathcal{N})) = \deg\mathcal{N} - g + 1$$
$$\implies \text{oops! I lost } h^0(\mathcal{N}) \text{ in this operation}\ldots$$

Maybe if I just use normal exact sequence and realise that

$$h^0(\mathcal{N}) = h^0(\mathcal{T}_X|_C) - h^0(\mathcal{T}_C)$$

I know that $h^0(\mathcal{T}_C) = 0$ for $g > 1$, so the question is how to compute the restricted holomorphic vector fields. □

39. Continued fractions

Definition of HJ continued fraction. For $i > 2$ they are in bijection with $\mathbb{Q}_{>1}$.

The basic diagram of this course starts with a surface S (eg. Hirzebruch surface $S = \mathbb{F}_m$). Blowing up leads to X, and contracting Wahl chains on X leads to W, a normal projective surface that has only Wahl singularities. Then we construct $\mathbb{Q}$-Gorenstein smoothings W_t. (These $\mathbb{Q}$-Gorenstein smoothings have Milnor number =0.)

Continued fractions have minimal models:

- $[1,1]$ means a 0 curve, $\mathbb{P}^1$.
- $[1]$ means a -1 curve, $\mathbb{P}^1$.
- For $\frac{m}{q} \in \mathbb{Q}_{>1}$, the continued fraction $[e_1,\ldots,e_r]$ means a chain, which is a sequence of lines that intersect transversally with $-e_1,\ldots,-e_r$. This is mapped to $\frac{1}{m}(1,q)$.

Third lecture.

Here's some slogans/recap:

(1) The most important cyclic quotient singularities (c.q.s.) are Wahl $\frac{1}{n^2}(na-1)$. There is a model to deal with this kind o singularities using continued fractions. This is very silly but what I picked up is that "you add a 2 in the end and add +1 to the first number", so for example $[4] \rightsquigarrow [5,2] \rightsquigarrow [6,2,2]$. But on the second step the $[5,2]$ also goes to $[2,5,3]$ in a way I don't understand. This is called the Wahl algorithm.

(2) (See [KSB88]) There is a notion of M-resolution, which is a drawing of several curves Γ_i intersecting at points P_i that may be Wahl singularities or smooth points with the key property that $\Gamma_i \cdot K \geq 0$. We have "toric boundary for P_i". These M-resolutions are in 1-1 correspondence with smoothings of $\frac{1}{m}(1,q)$, and in turn in 1-1 correspondence with continued fractions $K\left(\frac{m}{m-q}\right) = \{k_1, \dots, k_s] : 1 \leq k_i \leq b_i \ \forall i\}$ where $\frac{m}{m-q} = [b_1, \dots, b_s]$.

Today we consider the fibers to be $W_t = \mathbb{P}^2$ and try to find W. Set $m_1, m_2, m_3 \in \mathbb{Z}_{>0}$. Define

$$\mathbb{P}(m_1,m_2,m_3) := \mathbb{P}^2/(\mathbb{Z}/m_1 \oplus \mathbb{Z}/m_2 \oplus \mathbb{Z}/m_3) = \mathbb{C}^3 \backslash \{0\}/(\lambda \in \mathbb{C}^* \lambda(x,y,z) = (\lambda^{m_1}x, \lambda^{m_2}y, \lambda^{m_3}z))$$

For $\mathrm{gdc}(d, m_i) = 1$ we have $\mathbb{P}(dm_1, dm_2, dm_3) = \mathbb{P}(m_1, m_2, m_3)$.

For a triangle $xyz = 0$ given by three lines Γ_i we have cqs singularities of the kind $\frac{1}{m_1}(m_2, m_3)$. In this case $K_W = -(m_1+m_2+m_3)\xi = -\Gamma_1 - \Gamma_2 - \Gamma_3$ for $\xi^2 = \frac{1}{m_1 m_2 m_3}$, and $\mathrm{Cl}(W) = \mathbb{Z}\langle \xi \rangle$. Since these are Wahl singularities, we must have that the m_i are squares, i.e. $m_i = n_i^2$ for some n_i. We must have:

$$\begin{aligned} K_W^2 = (m_1+m_2+m_3)^2 \frac{1}{m_1 m_2 m_3} = 9 = K_{\mathbb{P}^2}^2 \\ \implies (n_1^2+n_2^2+n_3^2) - 9n_1^2n_2^2n_3^2 = 0 \\ \implies (n_1^2+n_2^2+n_3^2 - 2n_1n_2n_3) \cdot (\text{positive factor}) = 0 \\ \implies n_1^2+n_2^2+n_3^2 = 3n_1n_2n_3 \end{aligned}$$

The last equation is known as *Markov equation.*

Example 39.1. For $\mathbb{P}(1,1,4) = W$, a triangle with a Wahl singularity $\frac{1}{4}(1,1)$ in one vertex. Blowing up gives the Hirzebruch surface $\mathbb{F}_4$, so that a minimal resolution is the triangle. Compare with [Hacking-Prokhorov-2010]. This example satisfies the Markov equation for $n_1 = 1, n_2 = 1, n_3 = 2$.

Theorem 39.2 ([HP2010)**.** *] If $\mathbb{P}^2 \rightsquigarrow W$ with only log terminal singularities then W is a partial $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}(a^2,b^2,c^2)$ where $a^2+b^2+c^2 = 3abc$.*

By the Markov equation condition all the singularities must be Wahl. The triple (a,b,c) is called *Markov triple.* Any permutation of a Markov triple is another Markov triple. Is (a,b,c) is Markov then so is $(a,b,3ab-c)$. This allows to construct a *Markov tree.* There is so-called Markov conjecture (due to Frobenius) still unsolved.

40. Stanley Reisner

Antes de introduzir matroides, Os conjuntos f_s, que são os conjuntos de tamanho s, eles tem um significado geométrico?

41. Fano varieties

Definition 41.1. A *Fano variety* is a projective variety with $-K_X$ ample.

Definition 41.2.

$$r(X) := \min\{r : \frac{c_1(X)}{r} \in H^2(X, \mathbb{Z})\}$$

Exercise 41.3. By Kodaira vanishing theorem **??**, you can show that the cohomology $H^i(X, L)$ for a Fano variety X vanishes. You just have to put $L = \mathcal{O}(k)$ with $k \geq -r$, where r is the Fano index.

Exercise 41.4. Show that $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ holds for Fano varieties.

Remark 41.5. If $H^3(X, \mathbb{Z}) = 0$ of a Fano 3-fold, then its derived category is generated by 4 elements.

42. Quivers

Definition 42.1. A *quiver* is a set of vertices Q_0, a set of arrows Q_1 equipped with the maps of source s and target t that to each arrow they assign the point that is source or target of the arrow.

Definition 42.2. A *representation* of a quiver is a set of finite dimensional vector spaces equipped with maps between them realising a given quiver (incomplete…).

There is a notion of projective representation, which I missed to write. But it is analogous to the injective representation:

Definition 42.3. Given a quiver Q, the *injective representation* of Q_0 is given by, for $i \in Q_0$,

$$I(i)_j = \begin{cases} k & i = j \\ k^{d'} & j \neq i \end{cases}$$

where d' is the number of paths from j to i.

43. Stacks

My first definition of stack can be extracted from

Definition 43.1. A *superstack* is a stack over the étale site SSch of superschemes, i.e. it is a category fibered in groupoids over the category of superschemes, the latter equipped with the étale topology, satisfying the descent condition.

Here are some other definitions:

Definition 43.2. Let $\mathfrak{X}$ be a stack over $\mathrm{Sch}_{\text{ét}}$. An *algebraic space* is such that there exists morphism $\mathcal{U} \to \mathfrak{X}$ where $\mathcal{U}$ is a scheme, that is schematic, étale and injective (check this one).

$\mathfrak{X} \to y$ is *representable* if there exists a scheme $\mathcal{U}$ and a map $\mathcal{U} \to y$ such that the fibered product

$$\begin{tikzcd} \mathcal{U} \times_y \mathfrak{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{X} \arrow[d] \\ \mathcal{U} \arrow[r] & y \end{tikzcd}$$

is an algebraic space.

Finally, a stack is *algebraic* (resp. *Deligne-Mumford*) is there exists a representable surjective morphism $\mathcal{U} \to \mathfrak{X}$ that is smooth (resp. étale).

A *stable map* over a projective variety X is an element of the first Chow group $\beta \in A_1$, where (C, g) is an algebraic curve and $f : C \to X$ with $[f(C)] = \beta$.

The curves that are points under this map (contractible) are **stable**.

References

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