Complex surfaces, home assignment 2: Cohomology of local systems

Rules: This is a class assignment for this week, for discussion in class Wednesday next week.

Remark 2.1. Throughout this assignment, all manifolds are assumed to be connected.

Exercise 2.1. Let $\lambda > 0$ be a real number. Define **weight λ homogeneous forms** on $\mathbb{R}^n \backslash 0$ as differential forms η which satisfy $\rho_t^*\eta = \lambda^t\eta$, where ρ_t is a homothety map $z \longrightarrow tz$, $t > 0$. Prove that a closed weight λ form is a differential of an exact weight λ form any $\lambda \neq 0$.

Hint. Use Cartan's formula $\operatorname{Lie}_v(\alpha) = di_v\alpha + i_v d\alpha$.

Exercise 2.2. Let $M = \mathbb{R}^n \backslash 0/(x \sim 2x)$ be a Hopf manifold, θ a closed, non-exact 1-form, $d_\theta = d + \theta$, and $H_\theta^*(M)$ cohomology of the complex $\Lambda^*(M), d_\theta$ ("Morse-Novikov cohomology"). Prove that $H_\theta^i(M) = 0$ for all i.

Hint. Use the previous exercise.

Exercise 2.3. Let θ be an exact 1-form on a manifold M. Construct an isomorphism between the complexes $(\Lambda^i(M), d_\theta)$ and $(\Lambda^i(M), d)$.

Exercise 2.4. Let θ be a closed 1-form on a connected manifold. Prove that $H_\theta^0(M) \neq 0$ if and only if θ is exact.

Definition 2.1. **A complex of vector spaces** is a collection of $\{A_i, i \in \mathbb{Z}\}$ of vector spaces equipped with **the differential** $d: A_i \longrightarrow A_{i+1}$ such that $d^2 = 0$; its **cohomology groups** are $\frac{\ker d}{\operatorname{im} d}$. **Morphism of complexes** $\phi: (A_*, d_A) \longrightarrow (B_*, d_B)$ is a collection of homomorphisms $\phi_i: A_i \longrightarrow B_i$ commuting with the differentials. **The cone** of such a morphism is a complex $C_i := B_i \oplus A_{i+1}$ with the differential $d_B + (-1)^i d_A + \phi: B_i \oplus A_{i+1} \longrightarrow B_{i+1} \oplus A_{i+2}$.

Exercise 2.5. Let $A_* \xrightarrow{\phi} B_*$ be a morphism of complexes of vector spaces, and C_* its cone. Construct a long exact sequence

$$... \longrightarrow H^{i-1}(C_*) \longrightarrow H^i(A_*) \longrightarrow H^i(B_*) \longrightarrow H^i(C_*) \longrightarrow H^{i+1}(A_*) \longrightarrow ...$$

Exercise 2.6. Let X be a manifold, $M := X \times S^1$, and $\pi: M \longrightarrow S^1$ the standard projection. Denote by $T_X^1(M)$ the bundle of vectors tangent to the fibers of π, $\Lambda_X^1(M)$ its dual, and $\Lambda_X^*(M)$ the corresponding Grassmann algebra. Consider the de Rham differential d_X acting on $\Lambda_X^*(M)$ fiberwise along X.

a. Prove that the cohomology of $(\Lambda^*_X(M), d_X)$ is isomorphic to $H^*(X)\otimes C^\infty S^1$.

b. Let $\frac{d}{dt}$ be the derivative along the circle. Prove that $\frac{d}{dt}$ commutes with d_X.

c. Let $C^\infty_0 S^1$ denote the functions which average to zero. Prove that $\frac{d}{dt}$ is invertible on $C^\infty_0 S^1$.

d. Prove that the kernel and the cokernel of $H^*(X)\otimes C^\infty S^1 \xrightarrow{d/dt} H^*(X)\otimes C^\infty S^1$ are naturally isomorphic to $H^*(X)$.

e. Prove that the map $\frac{d}{dt} + \mathsf{Id}:\ C^\infty S^1 \longrightarrow C^\infty S^1$ is an isomorphism.

Exercise 2.7. We work in assumptions of the previous exercise.

a. Prove that $\Lambda^*(M)$ is (as a graded vector bundle) naturally isomorphic to $\Lambda^*_X(M)\oplus\Lambda^*_X(M)\wedge dt$, where dt is the constant coordinate 1-form on S^1 lifted to M.

b. Let $\Phi:\ \Lambda^*_X(M)\longrightarrow\Lambda^*_X(M)\wedge dt$ denote the operator $\alpha\mapsto\frac{d}{dt}\alpha\wedge dt$. Prove that

$$\Phi:\ (\Lambda^*_X(M), d_X)\longrightarrow(\Lambda^*_X(M)\wedge dt, d_X)$$

is a morphism of complexes, and the de Rham algebra of M is naturally isomorphic to its cone.

c. Let $\theta := dt$, and let $d_\theta:\ \Lambda^*(M)\longrightarrow\Lambda^*(M)$ be the corresponding Morse-Novikov differential. Prove that $(\Lambda^*(M), d_\theta)$ is isomorphic to the cone of $\Phi+\mathbb{I}$, where $\mathbb{I}(\alpha)=\alpha\wedge dt$.

d. Using the previous exercise, prove that Φ acts on the cohomology $H^*(X)\otimes C^\infty S^1$ of $(\Lambda^*_X(M), d_X)$ as $\frac{d}{dt}$, and $\Phi+\mathbb{I}$ acts as an isomorphism.

Exercise 2.8. Let X be a compact manifold, and $M = X\times S^1$, $\pi: M\longrightarrow S^1$ the standard projection, $\theta:=\pi^* dt$. Prove that $H^i_\theta(M)=0$ for all i.

Hint. Use the previous exercise.