Complex surfaces, home assignment 1: Hopf surfaces and Kodaira surfaces

Rules: This is a class assignment for this week, for discussion in class Wednesday next week.

Exercise 1.1. Let $M \longrightarrow X$ be a holomorphic fibration on a surface M with fiber $\mathbb{C}P^1$. Find the Kodaira dimension of M.

Exercise 1.2. Prove that the primary Kodaira surface (defined, as in lecture 1, as a nilmanifold) has trivial canonical bundle.

Exercise 1.3. Construct a closed, non-degenerate real (1,1)-form on Kodaira surface.

Exercise 1.4. Let H be a classical Hopf surface.

a. Prove that the holomorphic tangent bundle TH globally generated (that is, for each $x \in H$, the projection $H^0(TH) \longrightarrow T_xH$ is surjective).

b. Prove that $H^0(T^*H) = 0$.

c. Prove that $H^0(\mathsf{Sym}^k T^*H) = 0$.

d. (*) Prove that $H^0(T^*H) = 0$ for any Hopf linear surface.

Exercise 1.5. Let H be a linear Hopf surface, $H = \frac{\mathbb{C}^2 \backslash 0}{\langle A \rangle}$ with A a linear contraction. Denote by G the group $\operatorname{Aut}(H)$ of holomorphic automorphisms of H.

a. Prove that G contains a subgroup isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ if A is diagonal.

b. Prove that G contains a subgroup isomorphic to $\mathbb{C}^* \times \mathbb{C}$ if A is a non-diagonal Jordan block.

Exercise 1.6. Let C be a smooth complex curve on a non-projective complex surface M. Assume that $C \subset M$ admits a positive-dimensional family of deformations. Prove that the C is a genus 1 curve.