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Home assignment 4: Fubini-Study form and its potential

Exercise 4.2

Proof. You can just apply chain rule to obtain the first formula. Then substitute in coordinates. You will find the following form in there:

$$A = \mathsf{Id} - \frac{\langle v, \cdot \rangle}{\|v\|^2} v$$

so for orthogonal vectors to v you get the degeneracy of the form. So Bruno has written the following

$$\partial\bar{\partial} \log \ell = \frac{1}{\ell^2} \sum [\ell \underbrace{\delta_{ij} - z_i - \bar{z}_j]}_{A_{ij}} dz_i \wedge d\bar{z}_j$$

□

Exercise 4.3

(a) Consider the function $|z|^2 = z\bar{z}$ on $\mathbb{C}^*$, and let $\rho = z\frac{d}{dz}$, where z is the complex coordinate on $\mathbb{C}$. Prove that $\mathsf{Lie}_\rho |z|^2 = |z|^2$.

(b) Prove that $\mathsf{Lie}_\rho(\log |z|^2) = \mathsf{const}$.

Solution.

(a)

$$\begin{aligned} \mathsf{Lie}_\rho |z|^2 &= i_\rho d|z|^2 + \cancelto{0}{d i_\rho |z|^2} \\ &= i_\rho d(z\bar{z}) = i_\rho \left(\bar{z} dz + z \cancelto{0}{d\bar{z}} \right) \\ &= i_\rho (\bar{z} dz = \bar{z} dz \left(z \frac{d}{dz} \right) = \bar{z} z. \end{aligned}$$

(b) We have two functions $f : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$. f is $\log$ and $g = z\bar{z} = x^2 + y^2$.

Then by definition

$$\begin{aligned}
d(f\circ g) &= \frac{\partial}{\partial x}(f\circ g)dx + \frac{\partial}{\partial y}(f\circ g)dy \\
&= \frac{df}{dt}\Big|_{g(x,y)}\frac{\partial g}{\partial x}dx + \frac{df}{dt}\Big|_{g(x,y)}\frac{\partial g}{\partial y}dy \\
&= \log'(x^2+y^2)\frac{\partial}{\partial x}(x^2+y^2)dx + \log'(x^2+y^2)\frac{\partial}{\partial y}(x^2+y^2)dy \\
&= \frac{1}{x^2+y^2}2x dx + \frac{1}{x^2+y^2}2y dy \\
&= \frac{2}{x^2+y^2}(x dx + y dy)
\end{aligned}$$

OK now go back to your basic complex geometry knowledge to tell

$$dz = dx + \sqrt{-1}dy, \qquad d\bar{z} = dx - \sqrt{-1}dy$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - s^{-1}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right).$$

which says

$$dx = \frac{1}{2}(dz + d\bar{z}), \qquad dy = \frac{1}{2\sqrt{-1}}(dz - d\bar{z})$$

So that our thing of today is

$$\begin{aligned}
\frac{2}{x^2+y^2}(x dx + y dy) &= \frac{2}{|z|^2}\Big(\operatorname{Re} z\frac{1}{2}(dz+d\bar{z}) + \operatorname{Im} z\frac{1}{2\sqrt{-1}}(dz-d\bar{z})\Big) \\
&= \frac{1}{|z|^2}\Big(\operatorname{Re} z(dz+d\bar{z}) + \frac{1}{\sqrt{-1}}\operatorname{Im} z(dz-d\bar{z})\Big)
\end{aligned}$$

That is $d\log|z|^2$. Now evaluate in $\rho = z\frac{d}{dz}$:

$$d\log|z|^2(\rho) = \frac{1}{|z|^2}\Big(z\operatorname{Re} z + \frac{1}{\sqrt{-1}}z\operatorname{Im} z\Big) = \frac{z}{|z|^2}\bar{z} = 1$$

□

Exercise 4

Solution. Misha: to look for the differential of the action; tangent vector space to the action. $\mathbb{C}^*$ action has a real vector field. *This* is this vector field. □

Remark (Dani) The point is that the vector fields ρ and $\bar{\rho}$ are a basis for the $\mathbb{C}^*$-invariant vector fields, which end up being the vector fields that survive under the quotent. So the vector fields on $\mathbb{C}P^n$ are generated by those two. Now my question is: why is the form non-degenerate? I should find that the vector that gives degeneracy on the first problem is not vertical, i.e. is tangent to the base.