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Home assignment 4: Fubini-Study form and its potential

Exercise 4.3

(a) Consider the function $|z|^2 = z\bar{z}$ on $\mathbb{C}^*$, and let $\rho = z\frac{d}{dz}$, where z is the complex coordinate on $\mathbb{C}$. Prove that $\mathsf{Lie}_\rho |z|^2 = |z|^2$.

(b) Prove that $\mathsf{Lie}_\rho(\log |z|^2) = \mathsf{const}$.

Solution.

(a)

$$\begin{aligned}
\mathsf{Lie}_\rho |z|^2 &= i_\rho d|z|^2 + \cancelto{0}{d i_\rho |z|^2} \\
&= i_\rho d(z\bar{z}) = i_\rho\left(\bar{z}dz + \cancelto{0}{z d\bar{z}}\right) \\
&= i_\rho(\bar{z}dz = \bar{z}dz\left(z\frac{d}{dz}\right) = \bar{z}z.
\end{aligned}$$

(b) We have two functions $f : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$. f is $\log$ and $g = z\bar{z} = x^2 + y^2$. Then by definition

$$\begin{aligned}
d(f \circ g) &= \frac{\partial}{\partial x}(f \circ g)dx + \frac{\partial}{\partial y}(f \circ g)dy \\
&= \frac{df}{dt}\Big|_{g(x,y)}\frac{\partial g}{\partial x}dx + \frac{df}{dt}\Big|_{g(x,y)}\frac{\partial g}{\partial y}dy \\
&= \log'(x^2 + y^2)\frac{\partial}{\partial x}(x^2 + y^2)dx + \log'(x^2 + y^2)\frac{\partial}{\partial y}(x^2 + y^2)dy \\
&= \frac{1}{x^2 + y^2}2x dx + \frac{1}{x^2 + y^2}2y dy \\
&= \frac{2}{x^2 + y^2}(x dx + y dy)
\end{aligned}$$

OK now go back to your basic complex geometry knowledge to tell

$$dz = dx + \sqrt{-1}dy, \qquad d\bar{z} = dx - \sqrt{-1}dy$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - s^{-1}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right).$$

which says

$$dx = \frac{1}{2}\left(dz + d\bar{z}\right), \qquad dy = \frac{1}{2\sqrt{-1}}(dz - d\bar{z})$$

So that our thing of today is

$$\begin{aligned}
\frac{2}{x^2+y^2}(x\,dx + y\,dy) &= \frac{2}{|z|^2}\Big(\operatorname{Re} z \frac{1}{2}(dz + d\bar{z}) + \operatorname{Im} z \frac{1}{2\sqrt{-1}}(dz - d\bar{z})\Big) \\
&= \frac{1}{|z|^2}\Big(\operatorname{Re} z(dz + d\bar{z}) + \frac{1}{\sqrt{-1}} \operatorname{Im} z(dz - d\bar{z})\Big)
\end{aligned}$$

That is $d \log |z|^2$. Now evaluate in $\rho = z\frac{d}{dz}$:

$$d \log |z|^2(\rho) = \frac{1}{|z|^2}\Big(z \operatorname{Re} z + \frac{1}{\sqrt{-1}} z \operatorname{Im} z\Big) = \frac{z}{|z|^2}\bar{z} = 1$$

□