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Home assignment 1: Hopf surfaces and Kodaira surfaces

Exercise 1.2 Prove that a the primary Kodaira surface (defines, as in lecture 1, as a nilmanifold) has trivial canonical bundle.

Exercise 1.3 Construct a closed, non-degenerate (1,1)-form.

Solution. So the idea is to use, somehow, Lie algebra: $\mathfrak{g}^*$, the dual of $\mathsf{Lie}(S^1 \times Heis)$, which is defined as generated by x, y, z, t and $[x, y] = z$. □

Definition A ***classical Hopf surface*** is $\frac{\mathbb{C}^n}{\langle\gamma\rangle}$ with $\gamma = \lambda\,\mathsf{Id}$.

Exercise 1.4 Let H be a classical Hopf surface.

(a) Prove that the holomorphic tangent bundle TH (is?) globally generated (that is, for each $x \in H$, the projection $H^0(TH) \longrightarrow T_xH$ is surjective).

(b) Prove that $H^0(T^*H) = 0$.

(c) Prove that $H^0(\mathsf{Sym}^k\, T^*H) = 0$.

(d) (*) Prove that $H^0(T^*H) = 0$ for any Hopf linear surface.

Solution. I guess I just don't know the definition of $H^0(X)$. Looks like there's some group action involved.

(a) Answer by ChatGPT: the vector fields $X_k = x_k \frac{\partial}{\partial x_k}$ descend to H: $X_k(\lambda p) = \lambda X_k(p)$. These vector fields globally generate $H^0(TH)$ since they do so in the cover $\mathbb{C}^n \backslash \{0\}/\langle\gamma\rangle$.

A holomorphic tangent bundle equipped with a transitive group action is globally generated. Indeed: any basis for the tangent space at any point is made into a section via the group action.

To find a transitive group action on H we need a group of transformations that preserves de action of γ, that is $Ax = A\lambda x$.

(b) Recall: we are trying to show that an invariant 1-form vanishes. Then you would pair this form with a vector field that

(c) If a vector bundle is globally generated, then more or less so is its symmetric power. Then we do the same game of taking dual.

(d) A linear contraction is when has eigenvalues of norm smaller than 1. (Unsolved in class.)

□

Exercise 1.6 Let C be a smooth complex curve on a non-projective complex surface M. Assume that $C \subset M$ admits a positive-dominsional family of deformations. Prove that the C is a genus 1 curve.

Solution by Yulia. It's an application of Adjunction Formula.

Remark Kodaira theorem (lecture 1). For any non-algebraic surface

Theorem (Kodaira) For any non-algebraic surface X, if $a, b \in \mathsf{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$, with $a^2 = 0$, then $ab = 0$.

Since it deforms then $C^2 = 0$ bid (can't be negative) so $g(C) = g$. So adjunction says

$$2g - 2 = K_X \cdot C + C^2 = 0.$$

Proof of Adjunction formula by Yulia. I think it's just something like tensor product of graded algebras like Künneth. "Determinant takes direct sum to product" $C \subset X$.

$$\det T_{X|_C} = \det T_C \otimes \det(N_C)$$

but since C is a curve it's al 1 dimensional so we could have not put the det.

This gives $\Omega^2(\Omega_{X|_C}) = \Lambda^1(C) \otimes N_C^*$ "cotangent times conormal budndles". Right but how to get from there to adjunction formula?

Remember that adjunction formula is about relating ambient manifold to submanifold.

(***degree*** of a bundle has to do with taking a section...)

So... the previous equation translates to

$$C^2 + K_X \cdot C = K_C.$$

$$00 \longrightarrow T_C \longrightarrow T_{X|_C}$$

$$arrow[r] \qquad N_C \longrightarrow 0$$

□

and dualize:

$$0 \longrightarrow N_C^* \longrightarrow \Omega^1(X|_C) \longrightarrow T_C^* \longrightarrow 0$$

□

References