Daniel González Casanova Azuela
github.com/danimalabares/

Complex surfaces

Contents

1 Lecture 1: Kodaira dimension and Hopf manifolds

1.1 Outline

1. Kodaira dimension definition: the function $P(N) = H^0(K^N)$ is polynomial (so probably h rathen than H right?). The degree $\kappa(M)$ is called ***Kodaira dimension*** of M. If P is identically 0, we set $\kappa(M) = -\infty$.

2. Nilmanifolds and solvmanifolds (quotients of (solvable) Lie groups.

3. Kodaira surface definition (it's a nilmanifold).

4. Minimal models. A complex surface is ***minimal*** if it does not contain a smooth rational curve with self-intersection -1. Thorem by Hartshorne: for any complex surface M there exists a minimal surface M_1 and a holomorphic, bimeromorphic map $M \to M_1$. M_1 is called a ***minimal model for*** M**.**

5. Kodaira theorem: a complex surface is projective iaotfh: (i) field o meromorphic functions has trascendental dimension 2, (ii) M admits a holomorphic line bundle L with $c_1(L)^2 > 0$, (iii) the Neron-Severy lattice of M, $\mathsf{NS}(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, contains a class with positive self-intersection.

6. Class VII and VII_0 surfaces definition.

7. Hopf manifolds (Hopf manifolds are VII_0).

1.2 Kodaira-Enriques classification for non-algebraic surfaces: constructions and examples

- ***(Primary) Kodaira surface*** can be defined as $M := G/\Gamma$ with the complex structure defined by the subalgebra $\mathfrak{g}^{1,0} := \langle x + \sqrt{-1}y, z + \sqrt{-1}t \rangle$, which is actually abelian.

1.3 Holomorphic contractions and Hopf manifolds

Hopf manifolds are quotients $\mathbb{C}^{\setminus}\{0\}/\langle\gamma\rangle$ where γ is a ***contraction***, a function that puts any compact set of M inside any neighbourhood of any given points after a finite number of iterations. So for example $\gamma(z) = \frac{1}{2}z$ and then the Hopf manifold consists of the orbits of every point, which are discrete sets within the rays of every point. In fact, every orbit rpeats over and over so that there is one representative in the circle S^1, so that in fact this Hopf manifold is $S^1 \times S^1$. In general, a Hopf manifold H is called ***linear Hopf manifold***if γ is linear, and ***classical Hopf manifold*** if $\gamma = \lambda\,\mathsf{Id}$.

Proposition A Hopf manifold is diffeomorphic to $S^1 \times S^{2n-1}$.

Proof. If H is classical, it's simple; if its linear, approximate by classical; in general approximate by linear. □

A ***Class VII*** surface (also called Kodaira class VII surface) is a complex surface with $\kappa(M) = \infty$ and first betty number $b_1(M) = 1$. Minimal class VII are called ***class*** VII_0 ***surfaces***.

A ***primary Hopf surface*** is a Hopf manifold of dimension 2. A ***secondary Hopf surface is a quotient of a primary Hopf surface*** H ***by a finite group acting freely and holomorphically on*** H.

Claim Hopf surfaces are class VII_0.

2 Lecture 2: Hopf manifolds and algebraic cones

2.1 Algebraic cones

Definition Let P be a projective orbifold (so probably a manifold with mild singularities) and L an ample line bundle on P. An ***open algebraic cone*** $\mathrm{Tot}^0(L)$ is **just the set of nonzero vectors of the bundle**.

In the case of $P \subset \mathbb{C}P^n$ and $L = \mathcal{O}(1)|_P$, the open algebraic cone $\mathrm{Tot}^0(L)$ can be identified with thw set $\pi^{-1}(P)$ **of all** $v \in \mathbb{C}^{n+1} \backslash \{0\}$ **projected to** P **under the standard map** $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}P^n$**.** The ***closed algebraic cone*** is its closure in $\mathbb{C}^{n+1}$. It is an affine subvariety given by the same collection of homogeneous equations as P. Its ***origin*** is zero.

ChatGPT In the case where $P \subset \mathbb{C}P^n$ and $L = \mathcal{O}(1)|_P$, the open algebraic cone $\mathrm{Tot}^0(L)$ can be identified with the set $\pi^{-1}(P)$, where $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}P^n$ is the standard projection. Explicitly, $\pi^{-1}(P)$ consists of all $v \in \mathbb{C}^{n+1} \backslash \{0\}$ that project to points in P.

The **closed algebraic cone** is the Zariski closure of $\pi^{-1}(P)$ in $\mathbb{C}^{n+1}$. It is an affine subvariety defined by the same collection of homogeneous equations as P. Its **origin** is the zero vector in $\mathbb{C}^{n+1}$.

Hard definition An automorphism $A : P \to P$ is L-***Linearizable*** L admits an A-equivariant structure, in other words, if A can be lifted to an automorphism of the cone $\mathrm{Tot}^0(L)$ which is linear on fibers.

Explanation by ChatGPT The definition essentially asks whether A can be extended to the total space of L in a way that is consistent with the geometric and algebraic structures of L. This "lifting" ensures that the action of A on P interacts harmoniously with the fibers of L.

We need that to define ***Vaisman manifolds***: they are the quotient $\mathrm{Tot}^0(L)/\langle A \rangle$ where $A : \mathrm{Tot}^0(L) \to \mathrm{Tot}^0(L)$ which is linear on fibers and satisfies $|A(v)| = \lambda |v|$ for some number $\lambda > 1$.

Right so notice that Vaisman manifolds and Hopa manifolds are similar. Here's a diagram from the board (from Lecture 3):

$$\begin{array}{ccc} \mathrm{Tot}^0(L) & \hookrightarrow & \mathbb{C}^N \backslash \{0\} \\ \downarrow{\scriptstyle /\mathbb{Z}} & & \downarrow{\scriptstyle /\mathbb{Z}} \\ \text{Vaismann} & \hookrightarrow & \text{Hopf} \end{array}$$

Every Vaismann can be embedded to a Hopf-Vaismann (a Hopf that is Veismann). Not any Vaismann is Hopf nor the other way around.

Elliptic non algebraic surfaces are Vaismann

$$\begin{tikzcd}[column sep=small]
& & \text{Non-algebraic surfaces} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{K3} & \text{Class VII} & \text{Hopf non-ellptic} & \text{Elliptic (largest class)} \\
\text{Not Vaisman} & \text{Not Vaisman} & \text{Sometimes Vaismann} & \text{Vaismann}
\end{tikzcd}$$

3 Lecture 3: Locally conformal Kähler manifolds

3.1 Algebraic cones and Vaisman manifolds (reminder)

$$\begin{tikzcd}
M \arrow[r, hook] & \mathbb{C}P^1 \\
C_0(M) \arrow[u, "\mathbb{C}^*\text{-fibered}"] & \mathbb{C}^{n+1}\backslash\{0\} \arrow[u, "\mathbb{C}^0"'] \arrow[d, hook] \\
C(M)\backslash \arrow[u] \arrow[r, hook] & \mathbb{C}
\end{tikzcd}$$

3.2 LCK manifolds in terms of differential forms

So what is Kähler?

(M, I) complex manifolds, g an I-invariant Riemannian metric, "Hermitian metric", $\omega(x, y) := g(Ix, y)$ Hermitian form; $d\omega = 0$ Kähler.

Definition ω Hermitian form, $\omega \in \Lambda^{1,1}_{\mathbb{R}}(M)$, $\omega(x, Ix) > 0$, ω is ***Locally conformally Kähler*** if $d\omega = \omega \wedge \theta$, θ closed 1-form. θ is called the ***Lee form***.

Remark The condition if **conformally invariant**: it is preserved if we replace ω by a conformally equivalent form $f\omega$ for some positive smooth function $f > 0$.Indeed,

$$d(f\omega) = df \wedge \omega + fd\omega = df \wedge \omega + f\theta \wedge \omega = (df + f\theta) \wedge \omega.$$

This makes us notice that a classical Hopf manifold $\frac{\mathbb{C}^n\backslash\{0\}}{\langle \lambda \,\mathrm{Id}\rangle}$ is LCK.

3.3 Chern connection again

There is a connection on a holomorphic bundle compatible with the metric that is called ***Chern connection***.

The point is that the curvature can be written locally as dd^c of some function. And it can be global if you have a non-degenerate holomorphic section taking $\partial\bar{\partial} \log |b|$. But it is dd^c of a function that's the point.

Now there is

Theorem 5.30 ([OV24]) The function that maps $l = \psi : v \mapsto |v|^2$ along with some other stuff like the definition of the function q then there following expression is true:

$$dd^c l = -q(\theta_B) + \omega_\pi.$$

Which leads to

Corollary Let L be a line bundle with negative curvature on a projective manifold. Then the form $\frac{dd^c\psi}{\psi}$ is **homothety invariant** and locally conformally Kähler on $\mathrm{Tot}^0(L)$.

Remark We have just shown that **Vaisman manifolds are LCK**.

3.4 Homotheties, monodromy and objective

We want to give a definition of LCK in terms of a Kähler form on the universal covering. Also might involve local systems. **Under the alternative definition, LCK manifold is a quotient of a Kähler manifold by a free action of cocompact, discrete group acting by homotheties.**

Claim Any conformal map $\varphi : (M, \omega) \to (M_1, \omega_1)$ of Kähler manifolds is a homothety.

3.5 Reminder on connections and curvature

The point is that local systems are flat line bundles.

Definition A ***local system*** on a manifold is a locally constant sheaves of vector spaces.

Theorem (Rieman-Surfaces lecture 20) Fix a point $x \in M$. The category of local systems is naturally equivalent to the category of representations of $\pi_1(M, x)$.

Proof.

Step 1 From a locally constant sheaf $\mathbb{V}$ we construct a vector bundle $B := \mathbb{V} \otimes_{\mathbb{R}_M} C^\infty M$, where $\mathbb{R}_M$ is the constant sheaf on M. Define a connection $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$; where $v_1, \ldots, v_n$ is a basis in $\mathbb{V}(U)$. We have constructed a functor from locally constant sheaves to flat vector bundles.

Step 2 The converse functor takes a flat bundle (B, ∇) on M goes to the sheaf of parallel sections $\nabla b = 0$; this sheaf is locally constant because every vector can be locally extended to a parallel section uniquely (using Frobenius theorem; this is non-trivial).

□

3.6 χ-automorphic forms

The following resembles the way we have define LCK form on a manifold; multiplying by a number something that comes from the universal cover (...?)

Definition Let $\tilde{M} \xrightarrow{\pi} M$ be the universal covering of M, and $\xi : \pi_1(M) \to \mathbb{R}^{>0}$ a ***character***, which is just a group homomorphism. Consider the natural action of $\pi_1(M)$ on $\tilde{M}$. An ***ξ-automorphic form*** on $\tilde{M}$ is a differential form $\eta \in \Lambda^k(M)$ which satisfies $\gamma^*\eta = \xi(\gamma)\eta$ for any $\gamma \in \pi_1(M)$.

This makes sense because $\pi_1(M)$ acts freely on $\tilde{M}$ (and the quotient is M), so we can pullback η and it gives an other form on $\Lambda^k(M)$.

Proposition 1 (What Lada had said!, this is Claim 3.28 [OV24]) Let L be a rank 1 local system on M associated to the representation χ (so how is it assocated to χ?). Then the space of χ-automorphic k-forms on $\tilde{M}$ is in natural correspondence with the space of sections of $\Lambda^k(M) \otimes L$. Under this equivalence, the de Rham differential on χ-automorphic forms corrasponds to the operator $d_\nabla : \Lambda^k(M) \otimes L \to \Lambda^{k+1}(M) \otimes L$.

Proof.

Step 1 Pullback the line bundle to the universal cover: $\tilde{L} := \pi^*L$, $\pi : \tilde{M} \to M$. I think $\tilde{L}$ is trivial: "The bundle $\tilde{L}$ is flat and has trivial monodromy, hence it is naturally trivialized by parallel sections".

□

Remark $d_\nabla = d + \theta$

4 Lecture 5: local systems and LCK manifolds

4.1 χ-automorphic forms again

Upshot The point is that L-valued differential forms on M are in correspondence with χ_L-automorphic differential forms *on* $\tilde{M}$.

Proposition 1 (L, ∇) a real flat orientes line bundle. Identify with a local system: associated to χ fix a trivializatino of L. Then sections of $L \otimes \Lambda^1(M)$ are in bijection with χ-automorphic forms on $\tilde{M}$ via

$$\sigma : \Lambda^\bullet(M) \otimes L \longrightarrow \Lambda^\bullet(\tilde{M})$$

$$\sigma(d_\nabla \eta) = d\sigma(\eta).$$

Very incomplete proof.

Step 1 u_1 a nowhere-vanishing section of L, and θ a 1-form such that $\nabla u_1 = u_1 \otimes \theta$.

Extra How to produce the antiderivative of an exact one form: we integrate from x to y.

□

4.2 Lichnerowicz cohomology

Look for **Definition 2.51** for definition of d_∇, the B-***valued de Rham differential*** of the complex $\Lambda^i(M) \otimes B \longrightarrow \Lambda^{i+1}(M) \otimes B$ given by $d_\nabla(\eta \otimes b) := d\eta \otimes b + (-1)^{\tilde{\eta}-1}\eta \wedge \nabla b$ for the (real I think) flat bundle (B, ∇).

Definition Let θ be a closed 1-form on a manifold, and $d_\theta(\alpha) := d\alpha + \theta \wedge \alpha$ be the corresponding differential on $\Lambda^*(M)$. Its cohomology are called ***Morse-Novikov cohomology***, or ***Lichnerowicz cohomology***, denoted $H^*_\theta(M)$.

Theorem Lichnerowitz cohomology of a manifold is equal to the cohomology with coefficientes in a local system defined by (L, ∇).

Proof. Short. □

4.3 Definition of LCK manifolds in terms of an L-calued Kähler form

Definition Let (L, ∇) be an oriented real line bundle with flat connection on a complex manifold M, and $\omega \in L \otimes \Lambda^{1,1}(M)$ a $(1, 1)$-form with values in L. We say that ω is an L-***valued Kähler form*** if $\omega(x, Ix) \in L$ is (strictly) positive for any non-zero tangent vector, and $d_\nabla\omega = 0$.

Superremark If we use a trivialization to identify L and $C^\infty M$, ω becomes a $(1, 1)$-form and d_∇ becomes d_θ, giving $d_\nabla(\alpha) = d\alpha + \theta \wedge \alpha$. Therefore, **$L$-valued Kähler form on a manifold is the same as an LCK-form**.

4.4 Definition of LCK manifolds in terms of deck transform

Another definition An ***LCK manifold*** is a complex manifold M, $\dim_{\mathbb{C}} M \geqslant 2$ such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

Theorem These two definitions are equivalent.

5 Class 6: Vaisman theorem

5.1 LCK (reminder)

Definition A complex Hermitian manifold of dimension ¿1 (M, I, g, ω) is called ***locally conformally Kähler*** if there is a closed form θ such that $d\omega = \theta \wedge \omega$. θ is the ***Lee form*** and its cohomology class is the ***Lee class***.

5.2 Vaisman theorem

Suppose you have a LCK manifold. Suppose you want to replace θ by $\theta' = \theta + df$.

$$\begin{aligned} d(e^f\omega) &= e^f(d\omega + df \wedge \omega) \\ &= e^f((\theta + df) \wedge \omega) \\ \implies d\omega' &= (\theta + df) \wedge \omega' \\ d\omega' &= \theta' \wedge \omega'. \end{aligned}$$

And the converse is also true:

> conformally equivalent LCK metric give rise to homologous Lee forms, and any closed 1 form cohomologous to the Lee form is a Lee form conformally equivalent LCK metric.

Theorem (Vaisman) M LCK, if $[\theta] = 0$ (=θ is not extact) then M is not of Kähler type.

Remark (M, I) Kähler compact: for $\alpha \in \Lambda^1(M, \mathbb{C})$ there exists unique representatives $\alpha = \alpha^{1,0} + \alpha^{0,1}$ that are closed, i.e. $d\alpha^{1,0} = d\alpha^{0,1} = 0$.

Proof of Vaisman theorem. Take a form θ and multiply it by its complex conjugate $I\theta$. I think here we took local coordinates: $\theta = x_1$ and $I\theta = y_1$. So $\omega = \sum x_i \wedge y_i$. And then here's what I don't understand: $x_1 \wedge y_1 \left(\sum_{i=1}^n x_i \wedge y_i\right)^{n-1} = (n-1)! x_1 \wedge y_1 \wedge \prod_{i=2}^4 x_i \wedge y_i$. And that's positive!

And that gives the contradiction that

$$0 = \int dd^c(\omega^{n-1}) > 0$$

because any exact form has intergral zero by Stokes. $\square$

Definition ***Vaismann manifold*** is (M, I, ω) complex hermitian, $d\omega = \theta \wedge \omega$, $\nabla\theta = 0$, ∇ Levi-Civita connection of $g = (\omega(x, Iy)$.

(I think here we may be considering the musical dual of θ to take the covariant derivative.)

Equivalent definition G complex Lie group acting on an LCK manifold conformally and holomorphically, then (M, I) is Vaismann.

Remark The theorem that both definitions are equivalent is hard to prove.

5.3 Vaisman examples

Theorem Diagonal Hopf is Vaisman.

Diagonal Hopf is then the A in $\mathbb{C}^n / \langle A \rangle$ is diagonalizable.

I think

Theorem Z Hopf. Z is Vaisman iff is Z is diagonal.

5.4 The fundamental foliation

Definition M Vaisman manifold, $\theta^\sharp$ its Lee fild,, and Σ a 2-dimensional real foliation generated by $\theta^\sharp, I\theta^\sharp$. It is called ***the fundamental foliation*** of M.

Question So at

Theorem M Vaisman, Σ its canonical foliation.

1. Σ is independent from the metric (it's canonical).
2. There exists a 2-form $\omega \in \Lambda^{1,1}(M)$ which is semipositive on every transversal to Σ, $\omega_0|_\Sigma = 0$, exact.

   It's a contact structure, right? On slides: 2. There exists a positive, exact $(1,1)$-form ω_0 with $\sum = \ker \omega_0$

   Remark This 2-form is easy to see in these examples: the pullback of Fubini-Study metric: its pullback is exact! Remember that the pullback of any bundle to the $\mathrm{Tot}^0(M)$ is trivial.

3. $Z \subset M$ tangent to Σ,
4. $Z \subset M$ is Vaisman.

References

[OV24] Liviu Ornea and Misha Verbitsky. Principles of locally conformally kahler geometry, 2024.