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Complex surfaces

Contents

1 Lecture 1: Kodaira dimension and Hopf manifolds

1.1 Outline

1. Kodaira dimension definition: the function $P(N) = H^0(K^N)$ is polynomial (so probably h rathen than H right?). The degree $\kappa(M)$ is called ***Kodaira dimension*** of M. If P is identically 0, we set $\kappa(M) = -\infty$.

2. Nilmanifolds and solvmanifolds (quotients of (solvable) Lie groups.

3. Kodaira surface definition (it's a nilmanifold).

4. Minimal models. A complex surface is ***minimal*** if it does not contain a smooth rational curve with self-intersection -1. Thorem by Hartshorne: for any complex surface M there exists a minimal surface M_1 and a holomorphic, bimeromorphic map $M \to M_1$. M_1 is called a ***minimal model for*** M**.**

5. Kodaira theorem: a complex surface is projective iaotfh: (i) field o meromorphic functions has trascendental dimension 2, (ii) M admits a holomorphic line bundle L with $c_1(L)^2 > 0$, (iii) the Neron-Severy lattice of M, $\mathsf{NS}(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, contains a class with positive self-intersection.

6. Class VII and VII_0 surfaces definition.
7. Hopf manifolds (Hopf manifolds are VII_0).

1.2 Kodaira-Enriques classification for non-algebraic surfaces: constructions and examples

- ***(Primary) Kodaira surface*** can be defined as $M := G/\Gamma$ with the complex structure defined by the subalgebra $\mathfrak{g}^{1,0} := \langle x + \sqrt{-1}y, z + \sqrt{-1}t \rangle$, which is actually abelian.

1.3 Holomorphic contractions and Hopf manifolds

Hopf manifolds are quotients $\mathbb{C}^{\setminus}\{0\}/\langle\gamma\rangle$ where γ is a ***contraction***, a function that puts any compact set of M inside any neighbourhood of any given points after a finite number of iterations. So for example $\gamma(z) = \frac{1}{2}z$ and then the Hopf manifold consists of the orbits of every point, which are discrete sets within the rays of every point. In fact, every orbit rpeats over and over so that there is one representative in the circle S^1, so that in fact this Hopf manifold is $S^1 \times S^1$. In general, a Hopf manifold H is called ***linear Hopf manifold***if γ is linear, and ***classical Hopf manifold*** if $\gamma = \lambda\,\mathsf{Id}$.

Proposition A Hopf manifold is diffeomorphic to $S^1 \times S^{2n-1}$.

Proof. If H is classical, it's simple; if its linear, approximate by classical; in general approximate by linear. □

A ***Class VII*** surface (also called Kodaira class VII surface) is a complex surface with $\kappa(M) = \infty$ and first betty number $b_1(M) = 1$. Minimal class VII are called ***class*** VII_0 ***surfaces***.

A ***primary Hopf surface*** is a Hopf manifold of dimension 2. A ***secondary Hopf surface is a quotient of a primary Hopf surface*** H ***by a finite group acting freely and holomorphically on*** H.

Claim Hopf surfaces are class VII_0.

2 Lecture 2: Hopf manifolds and algebraic cones

2.1 Algebraic cones

Definition Let P be a projective orbifold (so probably a manifold with mild singularities) and L an ample line bundle on P. An ***open algebraic cone*** $\mathsf{Tot}^0(L)$ is **just the set of nonzero vectors of the bundle**.

In the case of $P \subset \mathbb{C}P^n$ and $L = \mathcal{O}(1)|_P$, the open algebraic cone $\mathsf{Tot}^0(L)$ can be identified with thw set $\pi^{-1}(P)$ **of all** $v \in \mathbb{C}^{n+1}\backslash\{0\}$ **projected to** P **under the standard map** $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P^n$**.** The ***closed algebraic cone*** is its closure in $\mathbb{C}^{n+1}$. It is an affine subvariety given by the same collection of homogeneous equations as P. Its ***origin*** is zero.

ChatGPT In the case where $P \subset \mathbb{C}P^n$ and $L = \mathcal{O}(1)|_P$, the open algebraic cone $\mathsf{Tot}^0(L)$ can be identified with the set $\pi^{-1}(P)$, where $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}P^n$ is the standard projection. Explicitly, $\pi^{-1}(P)$ consists of all $v \in \mathbb{C}^{n+1} \backslash \{0\}$ that project to points in P.

The **closed algebraic cone** is the Zariski closure of $\pi^{-1}(P)$ in $\mathbb{C}^{n+1}$. It is an affine subvariety defined by the same collection of homogeneous equations as P. Its **origin** is the zero vector in $\mathbb{C}^{n+1}$.

Hard definition An automorphism $A : P \to P$ is L-***Linearizable*** L admits an A-equivariant structure, in other words, if A can be lifted to an automorphism of the cone $\mathsf{Tot}^0(L)$ which is linear on fibers.

Explanation by ChatGPT The definition essentially asks whether A can be extended to the total space of L in a way that is consistent with the geometric and algebraic structures of L. This "lifting" ensures that the action of A on P interacts harmoniously with the fibers of L.

We need that to define ***Vaisman manifolds***: they are the quotient $\mathsf{Tot}^0(L)/\langle A \rangle$ where $A : \mathsf{Tot}^0(L) \to \mathsf{Tot}^0(L)$ which is linear on fibers and satisfies $|A(v)| = \lambda|v|$ for some number $\lambda > 1$.

Right so notice that Vaisman manifolds and Hopa manifolds are similar. Here's a diagram from the board (from Lecture 3):

$$\begin{tikzcd}
\mathsf{Tot}^0(L) \arrow[r, hook] \arrow[d, "/\mathbb{Z}"] & \mathbb{C}^N \backslash \{0\} \arrow[d, "/\mathbb{Z}"] \\
\text{Vaismann} \arrow[r, hook] & \text{Hopf}
\end{tikzcd}$$

Every Vaismann can be embedded to a Hopf-Vaismann (a Hopf that is Veismann). Not any Vaismann is Hopf nor the other way around.

Elliptic non algebraic surfaces are Vaismann

$$\begin{tikzcd}
 & & \text{Non-algebraic surfaces} \arrow[lld] \arrow[ld] \arrow[d] \arrow[rd] & \\
\text{K3} & \text{Class VII} & \text{Hopf non-ellptic} & \text{Elliptic (largest class)} \\
\text{Not Vaisman} & \text{Not Vaisman} & \text{Sometimes Vaismann} & \text{Vaismann}
\end{tikzcd}$$

3 Lecture 3: Locally conformal Kähler manifolds

3.1 Algebraic cones and Vaisman manifolds (reminder)

$$\begin{tikzcd}
M \arrow[r, hook] & \mathbb{C}P^1 \\
C_0(M) \arrow[u, "\mathbb{C}^*\text{-fibered}"] & \mathbb{C}^{n+1}\backslash\{0\} \arrow[u, "\mathbb{C}^0"] \arrow[d, hook] \\
C(M)\backslash \arrow[u] \arrow[r, hook] & \mathbb{C}
\end{tikzcd}$$

3.2 LCK manifolds in terms of differential forms

So what is Kähler?

(M, I) complex manifolds, g an I-invariant Riemannian metric, "Hermitian metric", $\omega(x, y) := g(Ix, y)$ Hermitian form; $d\omega = 0$ Kähler.

Definition ω Hermitian form, $\omega \in \Lambda^{1,1}_{\mathbb{R}}(M)$, $\omega(x, Ix) > 0$, ω is ***Locally conformally Kähler*** if $d\omega = \omega \wedge \theta$, θ closed 1-form. θ is called the ***Lee form***.

Remark The condition if **conformally invariant**: it is preserved if we replace ω by a conformally equivalent form $f\omega$ for some positive smooth function $f > 0$.Indeed,

$$d(f\omega) = df \wedge \omega + fd\omega = df \wedge \omega + f\theta \wedge \omega = (df + f\theta) \wedge \omega.$$

This makes us notice that a classical Hopf manifold $\frac{\mathbb{C}^n\backslash\{0\}}{\langle\lambda \operatorname{Id}\rangle}$ is LCK.

3.3 Chern connection again

There is a connection on a holomorphic bundle compatible with the metric that is called ***Chern connection***.

The point is that the curvature can be written locally as dd^c of some function. And it can be global if you have a non-degenerate holomorphic section taking $\partial\bar{\partial} \log |b|$. But it is dd^c of a function that's the point.

Now there is

Theorem 5.30 ([?]) The function that maps $l = \psi : v \mapsto |v|^2$ along with some other stuff like the definition of the function q then there following expression is true:

$$dd^c l = -q(\theta_B) + \omega_\pi.$$

Which leads to

Corollary Let L be a line bundle with negative curvature on a projective manifold. Then the form $\frac{dd^c\psi}{\psi}$ is **homothety invariant** and locally conformally Kähler on $\operatorname{Tot}^0(L)$.

Remark We have just shown that **Vaisman manifolds are LCK**.

3.4 Homotheties, monodromy and objective

We want to give a definition of LCK in terms of a Kähler form on the universal covering. Also might involve local systems. **Under the alternative definition, LCK manifold is a quotient of a Kähler manifold by a free action of cocompact, discrete group acting by homotheties.**

Claim Any conformal map $\varphi : (M, \omega) \to (M_1, \omega_1)$ of Kähler manifolds is a homothety.

3.5 Reminder on connections and curvature

The point is that local systems are flat line bundles.

Definition A ***local system*** on a manifold is a locally constant sheaves of vector spaces.

Theorem (Rieman-Surfaces lecture 20) Fix a point $x \in M$. The category of local systems is naturally equivalent to the category of representations of $\pi_1(M, x)$.

Proof.

Step 1 From a locally constant sheaf $\mathbb{V}$ we construct a vector bundle $B := \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$, where $\mathbb{R}_M$ is the constant sheaf on M. Define a connection $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$; where $v_1, \ldots, v_n$ is a basis in $\mathbb{V}(U)$. We have constructed a functor from locally constant sheaves to flat vector bundles.

Step 2 The converse functor takes a flat bundle (B, ∇) on M goes to the sheaf of parallel sections $\nabla b = 0$; this sheaf is locally constant because every vector can be locally extended to a parallel section uniquely (using Frobenius theorem; this is non-trivial).

□

3.6 χ-automorphic forms

The following resembles the way we have define LCK form on a manifold; multipliying by a number something that comes from the universal cover (...?)

Definition Let $\tilde{M} \xrightarrow{\pi} M$ be the universal covering of M, and $\xi : \pi_1(M) \to \mathbb{R}^{>0}$ a ***character***, which is just a group homomorphism. Consider the natural action of $\pi_1(M)$ on $\tilde{M}$. An ξ***-automorphic form*** on $\tilde{M}$ is a differential form $\eta \in \Lambda^k(M)$ which satisfies $\gamma^*\eta = \xi(\gamma)\eta$ for any $\gamma \in \pi_1(M)$.

This makes sense because $\pi_1(M)$ acts freely on $\tilde{M}$ (and the quotient is M), so we can pullback η and it gives an other form on $\Lambda^k(M)$.

Proposition 1 (What Lada had said!, this is Claim 3.28 [?]) Let L be a rank 1 local system on M associated to the representation χ (so how is it assocated to χ?). Then the space of χ-automorphic k-forms on $\tilde{M}$ is in natural correspondence with the space of sections of $\Lambda^k(M) \otimes L$. Under this equivalence, the de Rham differential on χ-automorphic forms corrasponds to the operator $d_\nabla : \Lambda^k(M) \otimes L \to \Lambda^{k+1}(M) \otimes L$.

Proof.

Step 1 Pullback the line bundle to the universal cover: $\tilde{L} := \pi^* L$, $\pi : \tilde{M} \to M$. I think $\tilde{L}$ is trivial: "The bundle $\tilde{L}$ is flat and has trivial monodromy, hence it is naturally trivialized by parallel sections".

$\square$

Remark $d_\nabla = d + \theta$

4 Lecture 5: local systems and LCK manifolds

4.1 χ-automorphic forms again

Upshot The point is that L-valued differential forms on M are in correspondence with χ_L-automorphic differential forms *on* $\tilde{M}$.

Proposition 1 (L, ∇) a real flat orientes line bundle. Identify with a local system: associated to χ fix a trivializatino of L. Then sections of $L \otimes \Lambda^1(M)$ are in bijection with χ-automorphic forms on $\tilde{M}$ via

$$\sigma : \Lambda^\bullet(M) \otimes L \longrightarrow \Lambda^\bullet(\tilde{M})$$

$$\sigma(d_\nabla \eta) = d\sigma(\eta).$$

Very incomplete proof.

Step 1 u_1 a nowhere-vanishing section of L, and θ a 1-form such that $\nabla u_1 = u_1 \otimes \theta$.

Extra How to produce the antiderivative of an exact one form: we integrate from x to y.

$\square$

4.2 Lichnerowicz cohomology

Look for **Definition 2.51** for definition of d_∇, the B-***valued de Rham differential*** of the complex $\Lambda^i(M) \otimes B \longrightarrow \Lambda^{i+1}(M) \otimes B$ given by $d_\nabla(\eta \otimes b) := d\eta \otimes b + (-1)^{\tilde{\eta}-1} \eta \wedge \nabla b$ for the (real I think) flat bundle (B, ∇).

Definition Let θ be a closed 1-form on a manifold, and $d_\theta(\alpha) := d\alpha + \theta \wedge \alpha$ be the corresponding differential on $\Lambda^*(M)$. Its cohomology are called ***Morse-Novikov cohomology***, or ***Lichnerowicz cohomology***, denoted $H^*_\theta(M)$.

Theorem Lichnerowitz cohomology of a manifold is equal to the cohomology with coefficientes in a local system defined by (L, ∇).

Proof. Short. $\square$

4.3 Definition of LCK manifolds in terms of an L-calued Kähler form

Definition Let (L, ∇) be an oriented real line bundle with flat connection on a complex manifold M, and $\omega \in L \otimes \Lambda^{1,1}(M)$ a $(1,1)$-form with values in L. We say that ω is an L-***valued Kähler form*** if $\omega(x, Ix) \in L$ is (strictly) positive for any non-zero tangent vector, and $d_\nabla \omega = 0$.

Superremark If we use a trivialization to identify L and $C^\infty M$, ω becomes a $(1,1)$-form and d_∇ becomes d_θ, giving $d_\nabla(\alpha) = d\alpha + \theta \wedge \alpha$. Therefore, **$L$-valued Kähler form on a manifold is the same as an LCK-form**.

4.4 Definition of LCK manifolds in terms of deck transform

Another definition An ***LCK manifold*** is a complex manifold M, $\dim_{\mathbb{C}} M \geqslant 2$ such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

Theorem These two definitions are equivalent.

5 Class 6: Vaisman theorem

5.1 LCK (reminder)

Definition A complex Hermitian manifold of dimension ¿1 (M, I, g, ω) is called ***locally conformally Kähler*** if there is a closed form θ such that $d\omega = \theta \wedge \omega$. θ is the ***Lee form*** and its cohomology class is the ***Lee class***.

The Deal with ω

Fundamental Form:

- The 2-form ω is the **fundamental 2-form** associated with the Hermitian metric g and the complex structure J, defined by:

$$\omega(X, Y) = g(JX, Y).$$

- This ω is not automatically a **symplectic form**, because it may fail to be **closed** ($d\omega \neq 0$).

Kähler Condition:

- When $d\omega = 0$, ω becomes a symplectic form, and the manifold M is Kähler. This means M is simultaneously a symplectic, complex, and Riemannian manifold with a harmonious interaction between these structures.

Locally Conformally Kähler:

- In LCK geometry, ω doesn't satisfy $d\omega = 0$ globally. Instead, it satisfies:

$$d\omega = \theta \wedge \omega,$$

  where θ is the **Lee form** (a closed 1-form). This deviation from $d\omega = 0$ characterizes LCK manifolds.

Why Locally Conformal?

- Locally, there exists a function f such that rescaling the metric by e^{-f} makes ω closed:

$$e^{-f}\omega \text{ is Kähler.}$$

- This means LCK manifolds are "almost" Kähler but need a conformal adjustment locally.

5.1.1 Some more notes on complex geometry

That the Kähler form is the differential of a plurisubharmonic function ψ. that is $\omega = dd^c\psi = \sqrt{-1}\partial\partial\psi$.

And it is a $(1,1)$-form.

Any positive $(1,1)$ form looks like this: $\sum \alpha I x_i \wedge x_i$ for some positive functions $\alpha_i \geqslant 0$.

How to prove adjunction formula, that the canonical bundle of a submanifold is the normal bundle of the submanifold tensor product the canonical bundle of the ambient manifold restricted to the submanifold, $K_M = N_M \otimes K_X|_M$: contract a form of the ambient space with a normal section!

5.2 Vaisman theorem

Suppose you have a LCK manifold. Suppose you want to replace θ by $\theta' = \theta + df$.

$$\begin{aligned}
d(e^f\omega) &= e^f(d\omega + df \wedge \omega) \\
&= e^f((\theta + df) \wedge \omega), \qquad \omega \text{ is LCK} \\
&= e^f(\theta' \wedge \omega, \qquad \theta \text{ and } \theta' \text{ cohomologous} \\
\implies d\omega' &= (\theta + df) \wedge \omega' \\
\implies d\omega' &= \theta' \wedge \omega'.
\end{aligned}$$

And the converse is also true:

> conformally equivalent LCK metric give rise to homologous Lee forms, and any closed 1 form cohomologous to the Lee form is a Lee form conformally equivalent LCK metric.

Theorem (Vaisman) M LCK, if $[\theta] = 0$ (=θ is not extact) then M is not of Kähler type.

Remark (M, I) Kähler compact: for $\alpha \in \Lambda^1(M, \mathbb{C})$ there exists unique representatives $\alpha = \alpha^{1,0} + \alpha^{0,1}$ that are closed, i.e. $d\alpha^{1,0} = d\alpha^{0,1} = 0$.

Proof of Vaisman theorem. Take a form θ and multiply it by its complex conjugate $I\theta$. I think here we took local coordinates: $\theta = x_1$ and $I\theta = y_1$. So $\omega = \sum x_i \wedge y_i$. And then here's what I don't understand: $x_1 \wedge y_1 \left(\sum_{i=1}^n x_i \wedge y_i\right)^{n-1} = (n-1)! x_1 \wedge y_1 \wedge \prod_{i=2}^4 x_i \wedge y_i$. And that's positive!

And that gives the contradiction that

$$0 = \int dd^c(\omega^{n-1}) > 0$$

because any exact form has intergral zero by Stokes. □

Definition ***Vaismann manifold*** is (M, I, ω) complex hermitian, $d\omega = \theta \wedge \omega$, $\nabla\theta = 0$, ∇ Levi-Civita connection of $g = (\omega(x, Iy)$.

(I think here we may be considering the musical dual of θ to take the covariant derivative.)

Equivalent definition G complex Lie group acting on an LCK manifold conformally and holomorphically, then (M, I) is Vaismann.

Remark The theorem that both definitions are equivalent is hard to prove.

5.3 Vaisman examples

Theorem Diagonal Hopf is Vaisman.

Diagonal Hopf is then the A in $\mathbb{C}^n / \langle A \rangle$ is diagonalizable.

I think

Theorem Z Hopf. Z is Vaisman iff is Z is diagonal.

5.4 The fundamental foliation

Definition M Vaisman manifold, $\theta^\sharp$ its Lee fild,, and Σ a 2-dimensional real foliation generated by $\theta^\sharp, I\theta^\sharp$. It is called ***the fundamental foliation*** of M.

Question So at

Theorem M Vaisman, Σ its canonical foliation.

1. Σ is independent from the metric (it's canonical).
2. There exists a 2-form $\omega \in \Lambda^{1,1}(M)$ which is semipositive on every transversal to Σ, $\omega_0|_\Sigma = 0$, exact.

It's a contact structure, right? On slides: 2. There exists a positive, exact $(1, 1)$-form ω_0 with $\sum = \ker \omega_0$

Remark This 2-form is easy to see in these examples: the pullback of Fubini-Study metric: its pullback is exact! Remember that the pullback of any bundle to the $\mathrm{Tot}^0(M)$ is trivial.

3. $Z \subset M$ tangent to Σ,
4. $Z \subset M$ is Vaisman.

6 Lecture 7: elliptic operators of order 2

1. "The ring of symbols"

   Theorem
   $$\bigoplus_k \frac{\mathrm{Dif}^k(M)}{\mathrm{Dif}^{k-1}(M)} = \bigoplus_k \mathrm{Sym}^{k-m}(TM)$$

   So on the lefthandside we have the graded algebra induced by the filtration $\mathrm{Dif}^0 \subset \mathrm{Dif}^1 \subset \ldots$ of differential operators of different orders on M.

   Proof. We give a pairing
   $$\frac{\mathrm{Dif}^k}{\mathrm{Dif}^{k-1}} \otimes \frac{\mathfrak{m}^k}{\mathfrak{m}^{k-1}}$$
   and we recall from Hartshorne that $\frac{\mathfrak{m}^k}{\mathfrak{m}^{k+1}} = \mathrm{Sym}^k(T^*M)$. □

2. Definition: ***symbol*** of the differential operator $D \in \mathrm{Dif}^k(M)$ is its image in $\mathrm{Sym}^k(TM)$
3. Remark. A differential operator gives us a polynomial function on the cotangent bundle because $\mathrm{Sym}^k(T^*M) =$ order k homogeneous polynomial functions on T^*M.
4. Definition: $D \in \mathrm{Dif}^k$ is ***elliptic*** if $\sigma(D)$ is positive or negative everywhere on $T^*M \backslash \{0\}$.
5. Remark. The symbol of an elliptic operator of second order is positive definite or negative definite. We assume is positive definite.
6.

   Strong maximum principle (version with boundary) (Hopf) M manifold with boundary. D elliptic of second order. $f \in C^\infty(M)$ $D(f) \geqslant 0$. Then all local maxima of f are on ∂M or f is constant.

   Proof assuming $D(f) > 0$. Because the Hessian is negative semidefinite □

7. Weak maximum principle. I think here /cha

7 Lecture 8: adjoint operators in Hodge theory

7.1 Adjoint connection (reminder)

First recall that a connection on a vector bundle B induces a connection on the dual bundle: if $\nabla : B \to B \otimes \Lambda^1(M)$, exists a unique connection $\nabla^* : B^* \to B^* \otimes \Lambda^1(M)$ satisfying $d\langle b, \beta\rangle = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$.

The connection ∇^* is called ***adjoint connection*** to ∇. The connection $\nabla = \nabla^*$ happens precisely when ∇ preserves the metric tension, consider a section of $B^* \otimes B^*$ and in this case ∇ is called an ***orthogonal connection***.

7.2 Adjoint connection and L^2-product

Upshot A scalar product on the space of sections of a vector bundle B. Because we want to define adjoint differential operators on the infinite-dimensional space of sections of the bundle. You multiply sections pointwise, you obtain a function, you integrate that function, you get a number.

M Riemannian manifold, b, b' sections of B. $(,)$ scalar product on B, $(b, b,)_{L^2} = \int (b, b')\,\mathsf{Vol}$.

Lemma (Integration by parts)

Proof. The key observation is that

$$\int \mathsf{Lie}_X(\langle b, b'\rangle)\,\mathsf{Vol} = 0$$

because $\mathsf{Lie}_X(\eta) = d i_X \eta + \cancelto{0}{i_X d\eta}$ so $\mathsf{Lie}_X(\eta)$ is exact. □

7.3 Adjoint operators

Definition $A : F \to G$ linear map on vector spaces with scalar products. $A : G \to G$ is ***dual*** if $(Ax, y) = (x, A^* y)$. $A^* y$ is a vector such that $\langle A^* y, x\rangle = \langle Ax, y\rangle$.

Existence is obvious and uniqueness $(\dots)$

Claim A diferential operator on vector bundles with scalar products $D : B_1 \to B_2$, then its adjoint $D^* : B_2 \to B_1$ is also a differential operator of the same order.

Remark Most of you know that $d^* = \pm * d *$ which is a composition of linear operators, it's the Hodge star.

Proof of claim. $C^\infty M$-linear. Then we can take the dual point by point (dual exists because its finite dimensional), and it works because it's linear and it's a vector bundle. Then we claim that all first order differential operators are combinations of a fixed connection ∇_X (connections are differential operators). Then somehow we have shown that actually ∇_X coincides by ∇_X^* (integrating by parts). □

7.4 Laplacian on differential forms

Start with a Riemannian manifold, say, compact. You can do it with non-compact: if you take care about having things with compact support, and it will work, but we don't want to do it because it takes some extra effort and we won't need it.

Then the sections of $\Lambda^*(M)$ we define scalar product ver naturally:

$$(\eta, \eta')_{L^2} = \int (\eta, \eta') \operatorname{Vol}_g$$

now

$$d^* = \text{dual to } d$$

Also, but this is not related to our course since we did not define Hodge star operator, $d^* = \pm * d*$.

Definition Laplacian is $\Delta = dd^* + d^*d$

Remark It's self dual (self-adjoint) because $*$ is self dual . And it's positive-definite

8 Lecture 9: Atiyah-Singer index theorem

Very famous theorem. It's topology. Application of analysis for topology.

8.1 Fredholm operators

Definition A continuous operator $F : H_1 \to H_2$ of Hilbert spaces is called ***Fredholm*** if *its image is closed* (that's new I think) and its kernel and cokernel are finite dimensional.

Next is a condition of invertibility of Fredholm maps. Uses Banach-Schouder theorem. We take the kernel of the map and quotient its domain (we get injectivity); and the restrict the codomain to the image. But we do have to use that BS theorem.

Claim $F : H_1 \to H_2$ is Fredholm iff there is a map $G : H_2 \to H_1$ such tat $\mathrm{Id} - FG$ and $\mathrm{Id} - GF$ have finite rank.

Definition ***Index*** of F is $\dim\ker F - \dim\operatorname{coker} F$.

This one is also in [**?**] (though not proved there):

Proposition The class of Fredholm operators is an open subset of $\mathcal{L}(E, F)$ (with the norm topology, so the norm of an operator is the supremum of its values on the unit sphere).

So that makes the domain of the index function a reasonable space. Then: [**?**]: the index map $A \mapsto \operatorname{ind} A$is continuous. But not only that:

Theorem The index function is locally constant.

So, apparently obviously, the open set of domain of Fredholm operators is not connected, so for example, I think the shift function So, apparently obviously, the open set of domain of Fredholm operators is not connected, so for example, I think the shift function.

8.2 Sobolev norm

Definition B vector bundle with metric over a Riemannian manifold. Define L^2_p metric for asection $b \in B$

$$|b|_p^2 = |b|_{L_p^2} = \sum_{i=0}^{p} |\nabla^i b|$$

Of course in the non-compact case we must take things with compact support.

Remark This norm is not complete, must take closure.

Definition L^2_p-***topology*** on $C^\infty M$ is topology defined by L^2_p-norm.

Remark (should be simple) If D is a differential operator (on a compact mfld for simplicity) of order k. Then $D : C^\infty M, L^2_p \to C^\infty M, L^2_{p-k}$ is (continuous?)

If you have a bound on k derivatives and you take more derivatives...

8.3 Elliptic operators

Definition $D : B \to B$ order k differential operator, then $\mathsf{symb}(D) \in \mathsf{Sym}^k(T) \otimes \mathsf{End}(B)$ k (this is agrued using a matrix of symbols) =degree k polynomial functions on T^*M with values in $\mathsf{End}(B)$.

D is ***elliptic*** if (after interpreting its symbol as a polynomial function) $\mathsf{symb}(D)(\nu)$ is invertible fora ll $0 \neq \nu \in T^*M$.

Theorem (Elliptic operator is Fredholm (I didn't find it in [?])) B_1, B_2 vector bundles, $D : B_1 \to B_2$ elliptic of order p. Then $D(B_1, L^2_{k+p} \to (B_2, L^2_k)$ is Fredholm.

Super hard to prove but super important and basic.

Definition $D : B_1 \to BV_2$ elliptic. Its ***index*** is the index of the map $D : (B_1, L^2_p \to (B_2, L^2)$

Corollary Let D_t be a continuous family of elliptic operators. The map $t \mapsto \mathsf{ind}(D_t)$ is constant. so that if D_t are elliptic they remain elliptic.

Proof. Because we have seen that index is locally constant. □

8.4 Index theorem for elliptic operators on $C^\infty M$

Theorem All elliptic operators on $C^\infty M$ have index 0.

Proof.

Step 1 $\mathsf{symb}\, D$ is a homogeneous function. (by properties of the symbol) we conclude that $\mathsf{deg}\, D$ is even.

$\square$

9 Lecture 10: Gauduchon metrics

9.1 Positive $(1,1)$ and $(n-1, n-1)$-forms

Cornerstone result of linear algebra If g_0 is a positive definite scalar product on V and $g_1 \in \mathsf{Sym}^2(V^*)$ then there is a basis such that g_0 is the identity matrix and g_1 is a diagonal matrix.

Everyone should know the proof, but we won't do it here. Better off, take a hermitian vector space (V, I). We will modify the previous result so that there exists a basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ so that $I(x_i) = y_i$ and $I(y_i) = x_i$. The basis is orthonormal with respect to some hermitian metric g_0. Then $\omega_0 = g(I\cdot, \cdot) = \sum_i x_i \wedge y_i$. That's the identity guy. There's also $\omega_1 = g_1(I\cdot, \cdot) = \sum_i \alpha_i x_i \wedge y_i$ where α_i are the eigenvalues of $g_1 \circ g_0^{-1}$. So that's the diagonal guy.

Moving on. $\Lambda^{1,1}(V)$ is the space of invariant 2-forms. We say $\omega \in \Lambda^{1,1}_{\mathbb{R}}(V)$ is positive if $\omega(x, Ix) \geqslant 0 \forall x$. It is ***strictly positive*** if $>$. Now, using the previous result we see that positivity is equivalent to $\omega = \sum \alpha_i x_i \wedge y_i \geqslant 0$.

There is a pairing

$$\Lambda^{1,1}(V) \otimes \Lambda^{n-1,n-1}(V) \to \mathsf{Vol}(V)$$

And looks like also

$$\Lambda^{1,1}(V) \times \mathsf{Vol} \xrightarrow{\sim} \Lambda^{n-1,n-1}(V).$$

Claim (Equivalences for Positivity for $n-1, n-1$ forms) Let $P \in \Lambda^{n-1,n-1}(V)$. TFAE:

(i) $\exists! z$such that $i_z \mathsf{Vol} = P$. z is positive as a $(1,1)$-form.

(ii) This one looks like all we did today, an orthonormal basis, $Ix_i = y_i$.

(iii) There's a third one.

Definition An $n-1, n-1$ form is called ***positive*** if either of the conditions from the claim hold, and ***strictly positive*** "if in the interior".

Next is an exercise given to everyone and never solved.

Claim $(n-1)$th power of positive form is positive, and moreover, the map $\alpha \mapsto \alpha^{n-1}$ defines a homeomorphism (bijective continuous invertible) between strictly positive $(1,1)$ and $(n-1, n-1)$ forms.

Proof.

$\square$

Remark (What this is good for) We have just proved that the map $\omega \mapsto \omega^{n-1}$ defines a homeomorphism from the cone (because we can multiply by nonzero positive numbers) of positive $(1,1)$-forms and the cone of strictly positive $(n-1, n-1)$-forms.

9.2 Harnack inequality

Theorem (Harnack) L elliptic operator, $\Omega \Subset \Omega_1$, $L : C^\infty \Omega_1 \to C^\infty \Omega_1$ "Any elliptic eq. has infinitely many solutions". Then $\exists C > 1$ depending on L, Ω and Ω_1, such that each solution $Lu = 0$ for u nonnegative, $\sup_\Omega u \leqslant C \inf_\Omega U$

But we only need the corollary:

Corollary if $u \geqslant 0$ is a solution of $Lu = 0$ then $u > 0$.

9.3 Gauduchon metrics

Definition a hermitian form ω on an n-manifold M is ***Gauduchon*** if $dd^c \omega^{n-1} = 0$. So that's a top-form. Well the zero top-form.

Theorem (Gauduchon) ω hermitian, then there exists a unique up to a constant function $\psi > 0$ such that $\psi\omega$ is Gauduchon.

Proof. So the idea is that dd^c is always laplacian. It's the trace of... □

10 Lecture 11: Bott-Chern cohomology and defect of a complex surface

10.1 Bott-Chern cohomology

Definition ***Bott-Chern cohomology*** of a complex manifold M is $H^{p,q}_{BC}(M)$ is the cohomology of dd^c (I think).

Remark There is no multiplicative structure on BC cohomology.

Theorem M compact complex manifold. $H^{p,q}_{BC}(M)$ is finite-dimensional.

Proof. Later today. □

10.2 Elliptic complexes

Definition ***Elliptic complex*** of vector bundles is when the symbols of the differentials give an exact sequence (I think).

Definition (Fredholm complex)

Corollary Cohomology of any elliptic complex is finite-dimensional.

Theorem BC cohomology of a compact complex manifold is finite-dimensional.

Proof. (This proof also proved finite-dimensionality of Dolbeaut cohomology.) $\square$

10.3 dd^c lemma

Theorem (dd^c lemma) M compact Kähler. $\eta \in \Lambda^{p,q}(M)$ d-exact, then $\eta \in \operatorname{img} dd^c$.

Lemma (dd^c-Lemma) *(This note was written by ChatGPT.)* Let X be a compact complex manifold. The following conditions are equivalent for a differential form α:

1. α is d-closed and d^c-exact: $d\alpha = 0$ and $\alpha = d^c\beta$ for some β.
2. α is d^c-closed and d-exact: $d^c\alpha = 0$ and $\alpha = d\gamma$ for some γ.
3. α is dd^c-exact: there exists η such that $\alpha = dd^c\eta$.

Remark dd^c is equivalent to the natural map $H^*_{BC}(M) \to H^*(M)$ being injective.

10.4 An inequality

Definition (M, ω) an hermitian surface. $\eta \in \Lambda^{1,1}(M)$ is called ***primitive*** if $\eta \perp \omega$ everywhere.

Theorem If $\alpha \in \Lambda^{1,1}(M)$ is primitive, then

$$\frac{\alpha \wedge \alpha}{\mathrm{Vol}} = -\|\alpha\|^2 \iff \alpha \wedge \alpha = -\|\alpha\|^2 \,\mathrm{Vol}$$

Proof. We can express $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$ using the 1 and -1 eigenspaces of Hodge star operator. (This is also in `k3.pdf`; search for "eigenspeces of the Hodge star operator".) $\square$

10.5 Top important statement

Theorem M compact complex surface. $p : H^{1,1}_{BC}(M) \to H^2(M)$ standard map. Then $\dim \ker p \leqslant 1$. So that kernel is forms that are exact but not Bott-Chern exact.

Proof. Consider the operator $D(f) = dd^c f \wedge \omega$ mapping functions to 4-forms. Because D is elliptic, we get that $\operatorname{rk} \operatorname{coker} D = 1$. Using Gauduchon form; an integral will vanish! $\square$

10.6 The most important invariant of a surface: the defect

Corollary Let $x \in \ker P$, $x \neq 0$, then $\int x \wedge \omega > 0$ or < 0 for any ω Gauduchon. Then a linear combination of ω_1 and ω_2 gives a zero integral. But by the previous proof (which I didn't type), the form is exact and it cannot give 0.

Proof. Suppose there is ω_1 giving a positive integral and ω_2 giving a negative integral. $\square$

Definition The ***defect*** of a surface is the number $\dim\ker P$. It's denoted $\delta(M)$. By the previous theorem it can only be 0 or 1. We will show that the surface is Kähler iff $\delta(M) = 1$.

11 Lecture 12: Cohomology of a complex surface

11.1 Lemma 1 (reminder)

Looks like the key observation is that the orthogonal complement of ω is 3-dimensional. What for? To show that there is an explicit form for $\Lambda^+(M)$ and $\Lambda^-(M)$, namely

$$\Lambda^+(M) = \langle \omega_I, \omega_J, \omega_K \rangle \qquad \Lambda^-(M) = \langle \omega_I, \omega_J, \omega_K \rangle^\perp$$

Quote (Misha) That $1,1$ forms orthogonal forms to ω are $\Lambda^-(M)$.

That is,

Lemma 1

$$\Lambda^-(V) = \{\alpha : I\alpha = \alpha, \alpha \perp \omega\}$$

where (V, I) is given by $V = \mathbb{R}^n$, $I^2 = -\mathsf{Id}$, g an I-invariant scalar product and $\Lambda^2(V) = \Lambda^+(V) \oplus \Lambda^-(V)$, and of course $\omega(x, y) = g(Ix, y)$.

> It is a useful construction because it allows us to compute because it is an expression of this thing in terms of the complex structure. That's all.

11.2 Reminder on Bott-Chern cohomology

$$H^{p,q}_{\mathsf{BC}} = \frac{\ker d \big/ \Lambda^{p,q}(M)}{\mathsf{img}\, dd^c}$$

Question What's up with the quotient on the numerator?

11.3 New stuff: intersection form on $H^{1,1}_{\mathsf{BC}}(M)$

Proposition M surface with $\delta(M) > 0$.

The intersection form $\alpha \mapsto \int \alpha \wedge \alpha$ is negative definite on the image of $H^{1,1}_{\mathsf{BC}}(M, \mathbb{R})$ in $H^2(M, \mathbb{R})$.

Proof. because every cohomology can be represented by something that has degree zero and degree zzro is negative definite. $\square$

And remember that this is a step towards showing that positive defect imples Kähler. Or was it nonKähler?

11.4 Holomorphic 1-forms on a surface

Introduction. Surfaces are typically fibrations. Like elliptic fibrations; fibers are tori. So some interesting forms on the surface are pullbacks. And some of this will be holomorphic differentials

Lemma All holomorphic 1-forms on a compact complex surface are closed. That is, $\alpha \in \Lambda^{1,0}(M)$ holomorphic 1-form, then $d\alpha = 0$.

Proof. $d\alpha = \partial\alpha \in \Lambda^{2,0}(M)$. Then

$$0 < \int d\alpha \wedge d\bar{\alpha} = \int \partial\alpha = \bar{\partial}\bar{\alpha}$$

. ☐

Claim 1 $\mathcal{H}^{1,1}(M) \oplus \overline{\mathcal{H}^{1,0}(M)} \hookrightarrow H^1(M,\mathbb{C})$

Proof. Also very simple. ☐

It will turn out that it is actually isomorphic when defect is zero, and codimension 1 when defect is 1.

Claim 2 $R : \overline{\mathcal{H}^{1,0}(M)} \to H^{0,1}_{\bar{\partial}}(M)$ is injective.

Claim 3 If $\delta(M) = 0$, that same map is surjective.

Proof. If not, we could construct a generator of $\ker P$, which is impossible. ☐

Now we can put this all together in one exact sequence; i.e. the following proposition is the three claims put in one. Yes: if $\ker P = 0$ we get some implications right?

Proposition 4

$$0 \longrightarrow \overline{\mathcal{H}^{1,1}(M)} \xrightarrow{R} H^{0,1}_{\bar{\partial}}(M) \xrightarrow{\partial} H^{1,1}_{BC}(M,\mathbb{R}) \xrightarrow{P} H^2(M,\mathbb{R})$$

is exact.

This is how we have understood Dolbeault cohomology with respect to defect. Now let's go to deRham.

11.5 de Rham cohomology for a complex surface with $\delta(M) = 0$

Proposition The map

$$\tau : H^1(M,\mathbb{R}) \to H^{0,1}_{\bar{\partial}}(M)$$

taking a close form η to $[\eta^{0,1}]$ is injective.

Proof. I was remembering Frölicher spectral sequence during this proof. ☐

Claim $\tau : H^1(M,\mathbb{R}) \to H^{0,1}_{\bar\partial}(M)$ is sujective **when defect is zero**, i.e. $\delta(M) = 0$.

Dani's toughts After going back to all that Frölicher sequence document I wrote once upon a time, and listening to the ideas that are around these lectures, I see that what lies below everything is that old idea of finding harmonic representatives of cohomology classes. Probably that doesn't make sense.

So it looks like these constructions will allow us for distinguishing when the surface is Kähler. But of course we are in surface case! In the Frölicher notes I said: If M is compact Kähler, there is a harmonic representative of every cohomology class, which says that the Frölicher sequence first page vanishes.

And do recall that Frölicher sequence (so, the convergence of it I suppose) is a statement *similar* to Hodge theorem, which is $H^k(X, C) = \bigoplus_{p+q=k} H^{p,q}(X)$ according to Voisin.

Proposition 5 M surface $\delta M = 1$ then $\ker P$ is generated by $d^c[\theta]$ where $[\theta] \in H^1(M,\mathbb{R})$, θ closed, $H^1(M,\mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)} \oplus \langle\theta\rangle$.

Corollary $b_1(M)$ is odd $\iff$ $\delta(M) = 1$

Corollary

11.6 Frölicher spectral sequence

Definition M complex compact. We say that ***Frölicher-HdR degenerates in*** $E_1^{p,p}$ degenerates isf any $\alpha \in H^{p,q}_{\bar\partial}(M)$ can be represented by $\bar\partial$-closed α such that $\partial\alpha \in \operatorname{img}\bar\partial$.

Remark In Kähler manifolds, you have Hodge theory, so you have degeneration (Dani: I think he means that you have the harmonic representative).

So it looks like the result is that that happens for complex surfaces anyways?

Corollary Hodge de Rham F spectral sequence degenerates on complex surface in $E_0^{1,0}$ and $E_1^{0,1}$.

So I'd say:

> It's not that we have Hodge theory, but we have something similar. (But that's just me.)

Quote (Misha during the proof of the corollary) . . . so we have that result, that image of ∂ is image of $\bar\partial$