 Daniel González Casanova Azuela
github.com/danimalabares/sur

# Home assignment 2: Cohomology of local systems

**Remark 2.1** Throughout this assignment, all manifolds are assumed to be connected.

**Exercise 2.1** (W$\lambda$ closed forms are W$\lambda$ exact) Let $\lambda > 0$ be a real number. Define ***weight $\lambda$ homogeneous forms*** on $\mathbb{R}^n \backslash 0$ as differential forms $\eta$ which satisfy $\rho_t^* = t^\lambda \eta$, where $\rho_t$ is a homothety map $z \mapsto tz$, $t \in \mathbb{R}$. Prove that a closed weight $\lambda$ form is a differential of a weight $\lambda$ form, for any $\lambda \neq 1$.

*Solution by Bruno.* (To be added.) □

**Exercise 2.2** (MN cohomology of Hopf manifold vanishes) Let $M = \mathbb{R}^n \backslash \{0\} \big/ (x \sim 2x$ be a Hopf manifold, $\theta$ a closed, non-exact 1-form, $d_\theta = d + \theta$, and $H_\theta^*(M)$ cohomology of the complex $(\Lambda^*(M), d_\theta)$ (***Morse-Novikov cohomology***). Prove that $H_\theta^i(M) = 0$ for all $i$.

**Hint.** Use the previous exercise.

*Solution.* In view of Exercise 2.1 we are inclined to look for a "weight $\lambda$" form in the class of any given closed differential form $\alpha \in \Lambda^*(M)$. Then the *usual* cohomology class of $\alpha$ vanishes.

Misha: the standard way to do this is to "average" the forms using a compact Lie group. So maybe ask GPT to what this means exactly. □

**Exercise 2.3** ($\theta$ exact $\implies$ MN cohomology=dr cohomology) Let $\theta$ be an exact 1-form on a manifold $M$. Construct an isomorphism between the complexes $(\Lambda^i(M), d_\theta)$ and $(\Lambda^i(M), d)$.

*Solution by Bruno.* Using exponential map. Do you rememeber? □

**Exercise 2.4** ($\theta$ closed $\implies$ MN vanishes iff $\theta$ exact) Let $\theta$ be a closed 1-form on a connected manifold. Prove that $H_\theta^0(M) \neq 0$ if and only if $\theta$ is exact.

*Solution by Bruno.* I don't remember. There is an easy way using sheaves? □

**Definition 2.1** A ***complex of vector spaces*** is a collection of $\{A_i, i \in \mathbb{Z}\}$ of vector spaces equipped with the ***differential*** $d : A_i \to A_{i+1}$ such that $d^2 = 0$;its ***cohomology groups*** are $\frac{\ker d}{\operatorname{img} d}$. ***Morphism of complexes*** $\phi : (A_*, d_A) \to (B_*, d_B)$ is a collection of homomorphisms

$\phi_i : A_i \to B_i$ commuting with the differentials. ***The cone*** of such a morphism is a complex $C_i := B_i \oplus A_{i+1}$ with the differential $d_B + (-1)^i d_A + \phi : B_i \oplus A_{i+1} \to B_{i+1 \oplus A_{i+2}}$.

$$\begin{aligned} d_B + (-1)^i d_A + \phi : B_i \oplus A_{i+1} &\longrightarrow B_{i+1} \oplus A_{i+2} \\ (b^i, a^{i+1}) &\longmapsto (d_B b^i, (-1)^i d_A a^{i+1} \underbrace{\cancel{,\phi}}_{?}) \end{aligned}$$

**Exercise 2.5** Let $A_* \xrightarrow{\phi} B_*$ be a morphism of complexes of vector spaces, and $C_*$ its cone. Construct a long exact sequence

$$\cdots \longrightarrow H^{i-1}(C_*) \longrightarrow H^i(A_*) \longrightarrow H^i(B_*) \longrightarrow H^i(C_*) \longrightarrow H^{i+1}(A_*) \longrightarrow \cdots$$

**Note:** the indices are different in the answer.

*Solution by Lada.* I realise that

$$0 \longrightarrow B_i \longrightarrow B_i \oplus A_{i+1} \longrightarrow A_{i+1} \longrightarrow 0$$

which by the fundamental theorem of homological algebra gives a long exact sequence

$$\cdots \longrightarrow H^k(B) \longrightarrow H^k(B_i \oplus A_{i+1}) = H^k(C_i) \longrightarrow H^k(A_{i+1}) \longrightarrow H^{k+1}(B_{i+1}) \longrightarrow \cdots$$

□

**Exercise 2.6** (The trivial circle bundle) Let $X$ be a manifold, $M := X \times S^1$, and $\pi : M \to S^1$ the standard projection. Denote by $T^1_X(M)$ the bundle of vectors tangent to the fibers of $\pi$, $\Lambda^1_X(M)$ its dual, and $\Lambda^*_X(M)$ the corresponding Grassmann algebra. Consider the de Rham differential $d_X$ acting on $\Lambda^*_X(M)$ fiberwise along $X$.

(a) (cohomology of the vectors tangent to fibers is cohomology of base $\otimes$ smooth functions on circle) Prove that the cohomology of $(\Lambda^*_X(M), d_X)$is isomorphic to $H^*(X) \otimes C^\infty S^1$.

(b) Let $\frac{d}{dt}$ be the derivative along the circle. Prove that $\frac{d}{dt}$ commutes with $d_X$.

(c) Let $C^\infty_0 S^1$ denote the functions wich average to zero. Prove that $\frac{d}{dt}$ is invertible on $C^\infty$.

(d) Prove that the kernel and the cokernel of $H^*(X) \otimes C^\infty S^1 \xrightarrow{d/dt} H^*(X) \otimes C^\infty S^1$ are naturally isomorphic to $H^*(X)$.

*Solution.*

1. Now I remember: it's Künneth formula! As I recall,

$$H^*(X \times S^1) = H^*(X) \otimes H^*(S^1).$$

Now I realise: it's the cohomology of the vectors tangent to the fibers of the circle bundle! I think it's the kernel of the map induced by the projection $\pi : X \times S^1 \to S^1$, namely

$$T_X(M) = \ker \pi_* = \{v \in TM : \pi_*(v) = 0 \in TS^1\}$$

So this will inject in the total bundle:

$$0 \longrightarrow T_X(M) \longrightarrow T(X \times S^1) \longrightarrow TS^1 \longrightarrow 0$$

Take Grassman algebra functor to get

$$0 \longrightarrow \Lambda^*_X(M) \longrightarrow \Lambda^*(M) \longrightarrow \Lambda^*(S^1) \longrightarrow 0$$

Giving the sequence

$$\cdots \to 0 \to H^0_X(M) \to H^0(M) \to H^0(S^1) \to$$

$$\to H^1_X(M) \to H^1(M) \to H^1(S^1) \to H^2_X(M) \to \cdots$$

To compute the cohomology of $M$ we use Künneth formula:

| $p+q$ | $H^{p+q}(M)$ |
|---|---|
| 0 | $H^0(X) \otimes H^0(S^1) = H^0(X) \otimes \mathbb{R}$ |
| 1 | $H^1(X) \otimes H^0(S^1) \oplus H^0(X) \otimes H^1(S^1)$ |
| | $= H^1(X) \otimes \mathbb{R} \oplus H^0(X) \otimes \mathbb{R}$ |
| 2 | $H^0(X) \otimes H^2(S^1) \oplus H^1(X) \otimes H^1(S^1) \oplus H^2(X) \otimes H^0(S^1)$ |
| | $= H^1(X) \otimes \mathbb{R} \oplus H^2(X) \otimes \mathbb{R}$ |
| 3 | $H^3(X) \otimes \mathbb{R} \oplus H^2(X) \otimes \mathbb{R}$ |
| 4 | $H^4(X) \otimes \mathbb{R} \oplus H^3(X) \otimes \mathbb{R}$ |
| 5 | $H^5(X) \oplus \mathbb{R} \oplus H^4(X) \otimes \mathbb{R}$ |

Giving

$$\cdots \to 0 \to H^0_X(M) \to H^0(X) \otimes \mathbb{R} \to H^0(S^1) \to$$

$$\to H^1_X(M) \to H^1(X) \otimes \mathbb{R} \oplus H^0(X) \otimes \mathbb{R} \to H^1(S^1) \to \cdots$$

$$\to H^2_X(M) \to H^1(X) \otimes \mathbb{R} \oplus H^2(X) \otimes \mathbb{R} \to 0 \to \cdots$$

$$\to H^3_X(M) \to H^3(X) \otimes \mathbb{R} \oplus H^2(X) \otimes \mathbb{R} \to 0 \to \cdots$$

So I wonder how to finish. Perhaps using some spectral sequence?

2. Here we must think that $\frac{d}{dt}$ is a derivation $f \mapsto \left.\frac{df}{dt}\right|_t$, where the latter is actually a smooth function depending on $t \in S^1$.

Then I'd like to see $d_X$ as a derivation, but I don't know how. The following probably doesn't make sense: $f \mapsto d_X f$, which is a form, so to get a number I should pair it with a vector field. So actually there *is* a canonical vector field on $S^1$, say, unit tangent vector field in a fixed direction. This would give me a number, what number?

The number is: $d_X f\left(\frac{d}{dt}\right) = \frac{df}{dt} dt\left(\frac{d}{dt}\right) = \frac{df}{dt}\Big|_t$

3. (ChatGPT.) First notice that for any $f \in C^\infty S^1$ the integral of its derivative vanishes by the fundamental theorem of calculus. To construct an inverse define for $g \in C^\infty S^1$ the primitive $f(t) = \int_0^t g(y)\,dy$. Then $\int_{S^1} f(t)\,dt$ vanishes because…

$\square$

**Exercise 2.7** (The trivial circle bundle continued)

**Exercise 2.8** (MN of trivial circle bundle vanishes for $X$ compact and $\theta = \pi^* dt$)