VERTEX ALGEBRAS

github.com/danimalabares/vertex-algebras

Contents

1. Cartan subalgebra, Cartan matrix and Serre relations

Kac-Moody algebras are Lie algebras, whose definition is motivated by the structure of finite-dimensional simple Lie algebras over $\mathbb{C}$.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ has a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ (abelian + ...) (see [Kac10, Definition 8.2]). Fixing $\mathfrak{h} \subset \mathfrak{g}$ gives a *root space decomposition* (see [Kac10, Proposition 8.5])

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta \subset \mathfrak{h}^*$ linear dual, and, by definition

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \; \forall H \in \mathfrak{h}\}$$

Turns out the $\mathfrak{g}_\alpha$ are all 1-dimensional, though this property is lost when we go to Kac-Moody algebras.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

The Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $\kappa(x, y) = \mathrm{Tr}_{\mathfrak{g}} \mathrm{ad}(x)\mathrm{ad}(y)$ is nondegenerate. "This is kind of the definition of semisimple." (Think of $\mathfrak{h}$ as $\mathfrak{g}_0$, btw.)

$\kappa|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\beta} \neq 0$ only when $\beta = -\alpha$. $\kappa|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate. This gives a linear isomorphism $\mathfrak{h} \xrightarrow{\nu} \mathfrak{h}$ via $\nu(H)(H') = \kappa(H, H')$.

So, $\mathfrak{h}^*$ comes with a non-degenerate bilinear form.

The *reflection* $r_\alpha : \mathfrak{h} \to \mathfrak{h}^*$ in $\alpha \in \mathfrak{h}^*$ (usually a root) is $r_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \cdot \alpha$.

"Classify root systems [...] classify semisimple Lie algebras" It is a fact that $r_\alpha(\Delta) = \Delta$ for all $\alpha \in \Delta$, which motivates the definition of *root system* (see [Kac10, Definition 15.1]) and permits classification. (See [Kac10, Lecture 17] for comments on correspondence of root systems and semisimple Lie algebras.)

Example 1.1. $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h} =$ diagonal matrices

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

is a basis of $\mathfrak{h}$. There are 6 roots vectors

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

E_{23}, E_{13}, etc.

Exercise 1.2. $[H_1, E_{12}] = 2E_{12}$, $[H_2, E_{12}] = -E_{12}$, $\alpha_{12} = (2, -1)$.

[Drawing of roots]

Notions of *positive roots* and *simple roots* (set of rank$\mathfrak{g}$ simple roots has ℓ elements, where $\ell = \dim(\mathfrak{h}^*)$. (See [Kac10, Definition 17.1].) This will also fail for Kac-Moody algebras more generally). Next write the *Cartan matrix* (see [Kac10, Definition 17.2])

$$A = (a_{ij}), \qquad a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

for $1 \leq i, j \leq \ell$.

Example 1.3. $\mathfrak{sl}_3$. [Picture, hexagonal pattern]. $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2, (\alpha_1, \alpha_2) = -1$, so

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Example 1.4. $\mathfrak{sl}_5$. [Picture, square pattern]. $|\alpha_2| = 1$, $|\alpha_1| = 2$, $(\alpha_1, \alpha_2) = -2$, so

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

Remark 1.5. Since $\mathfrak{g}_\alpha$ is 1-dimensional, set $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$ and $E_i = E_{\alpha_i}$, $i = 1, 2, \ldots, \ell$ (simple root vectors). It turns out that

$$\mathrm{ad}(E_i)^{1-a_{ij}} E_j = 0.$$

This is called a *Serre relation*.

2. Some infinite dimensional Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, and define the *loop algebra*

$$\begin{aligned} L\mathfrak{g} &= \mathfrak{g}[t, t^{-1}], \text{ (with basis } at^m|^{a \,\in\, \text{a basis of } \mathfrak{g}}_{m \in \mathbb{Z}} \text{)} \\ &= \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \end{aligned}$$

with the Lie bracket

$$[at^m, bt^n] = [a, b]t^{m+n}.$$

"This construction is absurdely general — we don't need $\mathfrak{g}$ to be semisimple [...]"

Take $\mathfrak{g} = \mathfrak{sl}_2$. Recall that

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

[Picture with $F, H, E, Ft, Ht, Et, Et^2 \ldots$] E was a root vector, corresponding to the unique root in $\mathfrak{sl}_2$, call it α_1. We seem to have a second simple root α_0, corresponding to Ft.

This looks like it wants to have a Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

We will indeed recover (a variant of) $L\mathfrak{g}$ as a Lie algebra "built from" $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, a Kac-Moody algebra. But note first, $\mathfrak{h} = \mathbb{C}H$ is too small. "Problem with α_0 and α_1 being linearly independent ..."

Exercise 2.1. Consider $L\mathfrak{g} \oplus \mathbb{C}d$, and set $[d, at^m] = mat^m$, $[d, d] = 0$. Check this defines a Lie algebra.

Proof. Skew-commutativity, i.e. for all $x \in L\mathfrak{g} \oplus \mathbb{C}d$,

$$[x, x] = 0, \tag{2.1.1}$$

is immediate from skew commutativity in $L\mathfrak{g}$ and the hypothesis that $[d, d] = 0$.

To confirm Jacobi identity, i.e. that for all $x, y, z \in L\mathfrak{g} \oplus \mathbb{C}d$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \tag{2.1.2}$$

notice that since this is a cyclic sum on x, y, z we only need to consider three elements in $L\mathfrak{g} \oplus \mathbb{C}d$ up to cyclic permutation. The cases in which the three elements are either in $L\mathfrak{g}$ or in $\mathbb{C}d$ are obvious, so that there are only two interesting possibilities:

$$x = d, \qquad y = at^m, \qquad z = bt^n \tag{2.1.3}$$

$$x = d, \qquad y = d, \qquad z = at^n \tag{2.1.4}$$

Case 2.1.3 gives

$$\begin{aligned}
&[d, [at^m, bt^n]] + [at^m, [bt^n, d]] + [bt^n, [d, at^m]] \\
&= [d, [a, b]t^{m+n}] + [at^m, -nbt^n] + [bt^n, mat^m] \\
&= (m+n)[a, b]t^{m+n} - n[a, b]t^{m+n} + m[b, a]t^{m+n} \\
&= (m+n)[a, b]t^{m+n} - n[a, b]t^{m+n} - m[a, b]t^{m+n} \\
&= (m+n)[a, b]t^{m+n} - (m+n)[a, b]t^{m+n} = 0.
\end{aligned}$$

Case 2.1.4 gives

$$\begin{aligned}
&[d, [d, at^m]] + [d, [at^m, d]] + [at^m, [d, d]] \\
&= [d, mat^m] + [d, -mat^m] = 0.
\end{aligned}$$

□

The Kac-Moody algebra turns out to be, not quite this, but slightly larger still.

Recall that an *invariant bilinear form* $(\cdot, \cdot)$ on a Lie algebra $\mathfrak{g}$ is a bilinear form such that

$$([a, b], c) = (a, [b, c]) \qquad \forall a, b, c \in \mathfrak{g}. \tag{2.1.5}$$

Definition 2.2. Given $\mathfrak{g}$ simple, with $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ invariant bilinear form, the *affine Lie algebra* is

$$\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K,$$

with $[K, \hat{\mathfrak{g}}] = 0$, and $[at^m, bt^n] = [a, b]t^{m+n} + m(a, b)\delta_{m,-n}K$.

"For the construction to work it doesn't actually have to be nondegenerate."

Exercise 2.3. Check that the affine Lie algebra $\hat{\mathfrak{g}}$ is a Lie algebra.

Proof. (Skew-commutativity.) Since $[K, \hat{\mathfrak{g}}] = 0$ and $K \in \hat{\mathfrak{g}}$, it is immediate that $[K, K] = 0$. For the case of an element in $L\mathfrak{g}$, we see that $[at^m, at^m] = 0$ by skew-commutativity of the bracket in $\mathfrak{g}$ and the Kronecker delta.

(Jacobi identity.) As in Exercise 2.1, any choice of x, y, z involving K is immediate by $[K, \hat{\mathfrak{g}}] = 0$. Thus the only interesting case is for Jacobi identity consider the cases

$$\begin{aligned}
&[at^m, [bt^n, ct^\ell]] + [bt^n, [ct^\ell, at^m]] + [ct^\ell, [at^m, bt^n]] \\
&= [at^m, [b, c]t^{n+\ell} + n(b, c)\delta_{n,-\ell}K] \\
&+ [bt^n, [c, a]t^{\ell+m} + \ell(c, a)\delta_{\ell,-m}K] \\
&+ [ct^\ell, [a, b]t^{m+n} + m(a, b)\delta_{m,-n}K] \\
&= [at^m, [b, c]t^{n+\ell}] + [at^m, n(b, c)\delta_{n,-\ell}K] \\
&+ [bt^n, [c, a]t^{\ell+m}] + [bt^n, \ell(c, a)\delta_{\ell,-m}K] \\
&+ [ct^\ell, [a, b]t^{m+n}] + [ct^\ell, m(a, b)\delta_{m,-n}K] \\
&= [a, [b, c]]t^{m+(n+\ell)} + m(a, [b, c])\delta_{m,-(n+\ell)}K \\
&+ [b, [c, a]]t^{n+(\ell+m)} + n(b, [c, a])\delta_{n,-(\ell+m)}K \\
&+ [c, [a, b]]t^{\ell+(m+n)} + \ell(c, [a, b])\delta_{\ell,-(m+n)}K = 0
\end{aligned}$$

It is clear that we obtain a Jacobi equation on $\mathfrak{g}$. To see that the remaining terms vanish, notice that the condition on the Kronecker delta in its three appearances is the same, namely, $m + n + \ell = 0$. In this case, we only need to check that $(a, [b, c]) = (b, [c, a]) = (c, [a, b])$ to conclude. This follows from the invariance of $(\cdot, \cdot)$ (and using that invariance implies it is symmetric). $\square$

We also have

Definition 2.4. The *extended affine Lie algebra* is

$$\tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

with $[d, at^m] = mat^m$ as before, and $[K, d] = 0$.

The extended affine Lie algebra is an example of a Kac-Moody algebra.

Exercise 2.5 (For those who like geometry)**.** Let $R = \mathbb{C}[t, t^{-1}]$. If $D \in \mathrm{Der}(R)$, then $L\mathfrak{g} \oplus \mathbb{C}d$ is a Lie algebra with $[d, a \otimes r] = a \otimes D(r)$. Is $(\mathfrak{g} \otimes R) \oplus \mathrm{Der}(R)$ a Lie algebra? (The Lie alegra $L\mathfrak{g} \oplus \mathbb{C}d$ from Exercise 2.1 is a particular case, for $D = t\frac{d}{dt}$.)

Proof. It is. Checking that $L\mathfrak{g} \oplus \mathbb{C}d$ is a Lie algebra with $[d, a \otimes r] = a \otimes D(r)$ is similar to Exercise 2.1: skew-commutativity is immediate from skew-commutativity in each of the components, while Jacobi identity is verified in two cases. For $x = y = d$ and $z = a \otimes r$ we quickly obtain

$$\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
&= [d, [d, a \otimes r]] + [d, [a \otimes r, d]] + [a \otimes r, [d, d]] \\
&= [d, a \otimes D(r)] + [d, -a \otimes D(r)] \\
&= a \otimes D(D(r)) - a \otimes D(D(r)) = 0
\end{aligned}$$

And for $x = d$, $y = a \otimes r$ and $z = b \otimes s$, we get

$$\begin{aligned}
&[x,[y,z]] + [y,[z,x]] + [z,[x,y]] \\
&= [d,[a\otimes r, b\otimes s]] + [a\otimes r,[b\otimes s, d]] + [b\otimes s,[d,a\otimes r]] \\
&= [d,[a,b]\otimes rs] + [a\otimes r, -b\otimes D(s)] + [b\otimes s, a\otimes D(r)] \\
&= [a,b]\otimes D(rs) - [a,b]\otimes rD(s) + [b,a]\otimes sD(r) = 0.
\end{aligned} \tag{2.5.1}$$

To check that $(\mathfrak{g}\otimes R)\oplus \mathrm{Der}(R)$ is a Lie algebra first put the Lie bracket on $\mathrm{Der}(R)$ as $[D,D_1] = DD_1 - D_1D$. It is clear that this bracket is skew-commutative. Jacobi identity reads

$$\begin{aligned}
&[D,[D_1,D_2]] + [D_1,[D_2,D]] + [D_2,[D,D_1]] \\
&= [D, D_1D_2 - D_2D_1] + [D_1, D_2D - DD_2] + [D_2, DD_1 - D_1D] \\
&= D(D_1D_2 - D_2D_1) - (D_1D_2 - D_2D_1)D + D_1(D_2D - DD_2) \\
&- (D_2D - DD_2)D_1 + D_2(DD_1 - D_1D) - (DD_1 - D_1D)D_2 \\
&= DD_1D_2 - DD_2D_1 - D_1D_2D + D_2D_1D + D_1D_2D - D_1DD_2 \\
&- D_2DD_1 + DD_2D_1 + D_2DD_1 - D_2D_1D - DD_1D_2 + D_1DD_2 = 0.
\end{aligned}$$

Now put the bracket on $(\mathfrak{g}\otimes R)\oplus \mathrm{Der}(R)$ as $[D, a\otimes r] = a\otimes D(r)$. Skew-commutativity is immediate. Jacobi identity for $x = D, y = a\otimes r$ and $z = b\otimes s$ is identical to the computation 2.5.1. In the case $x = D$, $y = D_1$ and $z = a\otimes r$, we get

$$\begin{aligned}
&[D,[D_1,a\otimes r]] + [D_1,[a\otimes r, D]] + [a\otimes r,[D,D_1]] \\
&= [D, a\otimes D_1(r)] + [D_1, -a\otimes D(r)] + [a\otimes r,[D,D_1]] \\
&= a\otimes DD_1(r) - a\otimes D_1D(r) + a\otimes [D,D_1](r) = 0.
\end{aligned}$$

$\square$

3. Kac-Moody algebras

Recall the notion of the free Lie algebra on a vector space V of generators (or a set X, think of V as a vector space with basis X):

Definition 3.1. The *free Lie algebra* on V is characterized by the universal property

$$\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathfrak{g} \\
& F(V) \arrow[ur, "\exists!\tilde{f}"'] &
\end{tikzcd}$$

That is, for any linear map $f : V \to \mathfrak{g}$ with $\mathfrak{g}$ Lie algebra, there exists a unique $\tilde{f}$ homomorphism of Lie algebras $F(V)\to\mathfrak{g}$ such that $\tilde{f}\circ i = f$.

$$\mathrm{Hom}_{\mathrm{Lie}}(F(V),\mathfrak{g}) = \mathrm{Hom}_{\mathrm{Vec}}(V,\mathfrak{g})$$

naturally.

That is, F and the forgetful functor $G : \underline{\mathrm{Lie}} \to \underline{\mathrm{Vec}}$ are adjoint:

$$\mathrm{Hom}_{\underline{\mathrm{Lie}}}(F(V),\mathfrak{g}) \xrightarrow{\simeq} \mathrm{Hom}_{\underline{\mathrm{Vec}}}(V, G(\mathfrak{g}))$$

A realisation of $F(V)$. Let
$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$
be the tensor algebra of V.

Then inside $T(V)$ consider $F(V)$ the span of iterated commutators of elements of V.

Proposition 3.2. *This realises the free Lie algebra.*

Proof. In online notes. □

In the finite dimensional simple case, we had
$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$
which we think also as $\alpha_i, \alpha_j \in \mathfrak{h}^*$, and $\alpha_i^\vee = \frac{2}{(\alpha_i,\alpha_i)}\nu^{-1}(\alpha_i) \in \mathfrak{h}$.

Clearly, $\alpha_{ii} = 2$ for all i. a_{ij} misht not equal a_{ji}, but certainly $a_{ij} = 0 \iff a_{ji} = 0$. And $\forall i \neq j$, $a_{ij} \leq 0$.

One further property. Set
$$\varepsilon_i = \frac{2}{(\alpha_i, \alpha_i)}, \quad \text{and} \quad D = \substack{\text{diagonal matrix} \\ \text{with entries } \varepsilon_i}$$
Then $A = DB$, where $B = ((\alpha_i, \alpha_j))$ is symmetric. If a matrix A is equal to (diag)(symm), we call it *symmetrizable*.

Definition 3.3. A *generalized Cartan matrix* is an integer matrix $A = (a_{ij})$ which is

- symmetrizable,
- $a_{ii} = 2$ for all i,
- $a_{ij} = 0 \iff a_{ji} = 0$,
- $a_{ij} \leq 0$ for $i \neq j$.

Definition 3.4. A *realisation* of a generalized Cartan matrix is a complex vector space $\mathfrak{h}$, and two sets
$$\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\}, \quad \text{and,}$$
$$\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$
such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$, $1 \leq i, j \leq n$.

Exercise 3.5. $\dim(\mathfrak{h}) \geq 2n - \operatorname{rank}(A)$.

Proof. For $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, a realisation is given by
$$\Pi^\vee = \{H_1, H_0\}, \qquad \Pi = \{\alpha_0, \alpha_1\}$$
$$\mathfrak{h} = \mathbb{C}H, \mathbb{C}d, \mathbb{C}K,$$
$$\mathfrak{h}^* = \mathbb{C}\alpha_1 + \mathbb{C}\delta + \mathbb{C}\Lambda_0$$
(Canonical dual, $\langle \alpha_1, H \rangle = 2$, $\langle \delta, d \rangle = 1 = \langle \Lambda_0, K \rangle$, every other pairing 0.)

Then
$$\begin{cases} \alpha_1 = \alpha_1 \\ \alpha_0 = \delta - \alpha_1 \end{cases} \qquad \begin{cases} \alpha_1^\vee = H \\ \alpha_0^\vee = K - H \end{cases}$$

So we obtain

$$\begin{aligned}
\langle \alpha_0^\vee, \alpha_1\rangle &= \langle K - H, \alpha_1\rangle = 2\\
\langle \alpha_1^\vee, \alpha_0\rangle &= \langle H, \delta - \alpha_1\rangle = -2\\
\langle \alpha_0^\vee, \alpha_0\rangle &= \langle K - H, \delta - \alpha_1\rangle = +2
\end{aligned}$$

□

Finally let's define Kac-Moody algebras.

Let A be a generalized Cartan matrix. Let

$$\tilde{\mathfrak{n}}_+ = F(e_1, \ldots, e_n),$$

the free Lie algebra on n generators, and similarly

$$\tilde{\mathfrak{n}}_- = F(f_1, \ldots, f_n).$$

Let $\mathfrak{h}$ be a realisation of A. Set $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.

Make $\tilde{\mathfrak{g}}(A)$ a Lie algebra by defining

- $[\mathfrak{h}, \mathfrak{h}] = 0$,
- $\forall H \in \mathfrak{h}$, $[H, e_i] = \langle \alpha_i, H\rangle\, e_i = \alpha_i(H) e_i$. And similarly, $[H, f_i] = -\alpha_i(H) f_i$.
- $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$.

Then $\tilde{\mathfrak{g}}(A)$ is a Lie algebra (though not yet the Kac-Moody algebra). See Kac, [Kac90, Thorem 1.2].

Remark 3.6. In $\mathfrak{h}$ we have a lattice

$$\begin{aligned}
Q^\vee &= \mathbb{Z}\alpha_1^\vee + \ldots + \mathbb{Z}\alpha_n^\vee, \quad \text{and}\\
Q &= \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_n \text{ in } \mathfrak{h}^*
\end{aligned}$$

(root and coroot lattices). $\tilde{\mathfrak{g}}(A)$ is naturally Q-graded, with

$$\tilde{\mathfrak{g}}(A)_\beta = \text{span}\{\text{commutators of } e_i \text{ with } \sum \alpha_i = \beta\}.$$

$\tilde{g}(A) = \mathfrak{h}$.

Theorem 3.7 (Gabber-Kac). *Denote by $I \subset \tilde{\mathfrak{g}}(A)$ the maximal Q-graded ideal, such that $I \cap \mathfrak{h} = \{0\}$. Then I is generated by the Serre relations*

$$ad(e_i)^{1-a_{ij}} e_j \qquad \textit{and} \qquad ad(f_i)^{1-a_{ij}} f_j,\ i \neq j.$$

Proof. [Kac90, Theorem 9.11]. □

(The existence of the ideal I does not need the theorem; the importance of the theorem is providing an expression for the generators.)

Definition 3.8. The *Kac-Moody algebra* $\mathfrak{g}(A)$ is $\tilde{\mathfrak{g}}(A)/I$.

4. Affine Kac-Moody algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, with $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ invariant bilinear form,

$$([x, y], z) = (z, [y, z]) \qquad \forall x, y, z \in \mathfrak{g}$$

(Eg. the Killing form $\kappa(x, y) = \text{Tr}_{\mathfrak{g}}\text{ad}(x)\text{ad}(y)$ is invariant.)

Typically we normalise $(\cdot,\cdot)$ so that $(\alpha, \alpha) = 2$ for the long roots of $\mathfrak{g}$.

Then $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ (affine Lie algebra),

$$[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a, b)K, \qquad [K, \hat{\mathfrak{g}}] = 0$$

and $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d$, $[d, K] = 0$, $[d, at^m] = mat^m$, (affine Kac-Moody algebra or "extended affine Lie algebra")

Theorem 4.1. *$\tilde{\mathfrak{g}}$ is a Kac-Moody algebra.*

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, ($\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$.)

The simple roots and coroots. $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. We identify $\tilde{\mathfrak{h}}^*$ with $\mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ where

$$\begin{aligned} \Lambda_0(\mathfrak{h}) &= \delta(\mathfrak{h}) = 0 \\ \Lambda_0(d) &= \delta(K) = 0 \\ \Lambda_0(K) &= \delta(d) = 1 \end{aligned}$$

The *real coroots* are

$$\hat{\Delta}^{V,re} = \{E_\alpha t^m | \alpha \in \Delta, m \in \mathbb{Z}\}$$

and there are also imaginary roots and coroots

$$\hat{\Delta}^{V,im} = \{Ht^m | H \in \mathfrak{h}, m \in \mathbb{Z} \setminus \{0\}\}$$

Roots:

$$\begin{aligned} \hat{\Delta}^{re} &= \{\alpha + m\delta | \alpha \in \Delta, m \in \mathbb{Z}\} \\ \hat{\Delta}^{im} &= \{m\delta | m \neq 0\} \end{aligned}$$

Xt^m:

$$\begin{aligned} [H, Xt^m] &= [H, x]t^m, \qquad H \in \mathfrak{h} \\ [K, xt^m] &= 0 \\ [d, xt^m] &= mxt^m \end{aligned}$$

so it $x \in \mathfrak{g}_\alpha$, $xt^m \in \tilde{\mathfrak{g}}_{\alpha+m\delta}$.

The invariant bilinear form $(\cdot,\cdot)$ from $\mathfrak{g}\times\mathfrak{g}$ extends uniquely to $(\cdot,\cdot) : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \mathbb{C}$.
$(d, d) = (K, K) = 0$, $(d, K) = 1$ and $(d, \mathfrak{h}) = (K, \mathfrak{h}) = 0$.

So, in $\tilde{\mathfrak{h}}^*$:

$$\begin{aligned} (\Lambda_0, \Lambda_0) &= (\delta, \delta) = 0 \\ (\Lambda_0, \mathfrak{h}^*) &= (\delta, \mathfrak{h}^*) = 0 \\ (\Lambda_0, \delta) &= 1. \end{aligned}$$

Hence, $|\alpha + m\delta|^2 = |\alpha|^2$, $|m\delta|^2 = 0$.

Example 4.2. $\widetilde{\mathfrak{sl}_2}, \tilde{\mathfrak{h}}^* = \mathrm{span}\{\alpha, \Lambda_0, \delta\}$ with Gram matrix ...

We can make a choice of positive roots,

$$\hat{\Delta}_+ = \{\alpha + m\delta | \alpha \in \Delta, m > 0\} \cup \{m\delta | m > 0\} \cup \Delta_+$$

Obviously, if $\alpha \in \Delta_+$ is simple, $\alpha \in \hat{\Delta}_+$ is simple.

Notation. Let $\theta \in \Delta_+$ be a the highest root. ($\nexists \alpha \in \Delta_+$ such that $\alpha - \theta \in \mathbb{Z}_+\Delta_+$.) and $\alpha = \delta - \theta$.

Then $\alpha_0 \in \hat{\Delta}_+$ is simple and the set of simple roots is $\hat{\Pi} = \{\alpha_0, \underbrace{\alpha_1, \dots, \alpha_\ell}_{\text{the finite simple roots}}\}$.

where $\ell = \mathrm{rank}(\mathfrak{g})$.

The coroot corresponding to α_0 is

$$\alpha_0^\vee = K - \theta^\vee, \qquad \theta^\vee = \frac{2}{(\theta,\theta)}\nu^{-1}(\theta) \in \mathfrak{h}$$
$$\text{and} \qquad E_{\alpha_0} = E_{-\theta} t.$$

Now, in any Kac-Moody algebra, we have

$$\begin{aligned} \text{roots} \qquad & \Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^* \\ \text{coroots} \qquad & \Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}, \end{aligned}$$

and *reflections* $r_i \in \mathrm{GL}(\mathfrak{h}^*)$, defined by

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \, \alpha_i.$$

One can check that

$$(r_i\lambda, r_i\mu) = (\lambda, \mu) \qquad \forall \lambda, \mu \in \mathfrak{h}^*$$

The *Weyl group* W is $\langle r_i | i = 1, \dots, \ell \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$.

Example 4.3. For $\widetilde{\mathfrak{sl}_2}$, r_1 is easy,

$$\begin{aligned} r_1(\alpha) &= -\alpha \qquad (\text{as in } \mathfrak{sl}_2) \\ r_1(\delta) &= \delta, \qquad r_1(\Lambda_0) = \Lambda_0. \end{aligned}$$

To compute r_0 take an arbitrary element $m\alpha_1 + k\Lambda_0 + f\delta$ and do:

$$r_0(m\alpha_1 + k\Lambda_0 + f\delta) = m\alpha_1 + k\Lambda_0 + f\delta - \langle \alpha_0^\vee, m\alpha_1 + k\Lambda_0 + f\delta \rangle \, \alpha_0$$
$$\alpha_0 = \delta - \alpha_1, \qquad \alpha_0^\vee = K - \alpha^\vee$$

so we obtain

$$\begin{aligned} &= m\alpha_1 + k\Lambda_0 + f\delta - (k - 2m)(\delta - \alpha_1) \\ &(k - m)\alpha_1 + k\Lambda_0 + (f - k + 2m)\delta. \end{aligned}$$

Relative to basis $\{\alpha_1, \Lambda_0, \delta\}$.

$$r_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} m \\ k \\ f \end{pmatrix}$$

$$t = r_1 r_0 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

Notice that δ is fixed by all r_i. Also $m\alpha + k\Lambda_0 + f\delta$, the **coefficient** of Λ_0 is fixed by all r_i.

Then

$$\begin{aligned} t(m\alpha_1 + &k\Lambda_0 + f\delta) = \\ &= (m - k)\alpha_1 + k\Lambda_0 + (f - k + 2m)\delta \end{aligned}$$

Think of t as a translation.

The number k in

$$\mathfrak{h}^* \ni \hat{\lambda} = \lambda + k\Lambda_0 + f\delta$$

is called the *level* of $\hat{\lambda}$.

$\hat{\mathfrak{h}}$ = union of (hyper)planes of constant level which are stable under W. The roots α are all of level 0.

[Picture] "r_1 changes the sign of the finite path". And $t = r_1 r_0$ is a sort of translation. Indeed, in general we can consider $t_{\alpha_i} = r_{\alpha_i} \circ r_0 \in W$,

$$t_\alpha(\beta + m\delta) = \beta + (m + (\beta, \alpha_i))\delta$$

One can describe the action of t_α on $\hat{\lambda}$ in general (e.g. see [Kac90, Chapter 6])

Proposition 4.4. *For the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ with $\hat{W} = \langle r_0, r_1, \dots, r_\ell \rangle$ its Weyl group (and $W = \langle r_1, \dots, r_\ell \rangle \subset \hat{W}$ the Weyl group of $\mathfrak{g}$), then $\hat{W} \simeq W \times t_{Q^\vee}$ (where it should be semidirect product instead of $\times \dots$) where $Q^\vee$ is the coroot lattice of $\mathfrak{g}$.*

Remark 4.5. For general Kac-Moody algebras, the Weyl groups are much larger, hyperbolic reflection groups.

In the affine case, $\hat{W}$ fixes level k, and $|\hat{\lambda}|$. One gets, in the intersection, paraboloids [Picture of section of hyperboloid that is a parabola].

5. Weyl character formula

Highest weight representations of Kac-Moody algebras. Let $\lambda \in \mathfrak{h}^*$, wheere $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a Kac-Moody algebra. We define a *Verma module*

$$M(\Lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n}_+} \mathbb{C}v_\Lambda$$

where $\mathfrak{h} + \mathfrak{n}_+$ acts on V_Λ by:

$$Xv_\Lambda = 0, \qquad \forall x \in \mathfrak{n}_+, Hv_\Lambda \qquad = \Lambda(H)v_\Lambda, \qquad \forall H \in \mathfrak{h}$$

So $\mathbb{C}v_\Lambda$ is a $U(\mathfrak{h} + \mathfrak{n}_+)$-module,

$$\begin{tikzcd} U(\mathfrak{h}+\mathfrak{n}_+) \arrow[d] \\ U(\mathfrak{g}) \end{tikzcd}$$

By the PBW theorem, $M(\Lambda)$ has a linear $\mathbb{C}$-basis.

Let $\{F_{\alpha,i} : i = 1, \dots, \dim \mathfrak{g}_\alpha\}$ be a basis of $\mathfrak{g}_{-\alpha}$, $\forall \alpha \in \Delta_+$. Also choose a total order on Δ_+. (Some sort of lexicographical order that takes longer to write than to say.)

$$F_{\alpha_1,i_1}, F_{\alpha_2,i_2}, \dots, F_{\alpha_s,i_s}, v_\Lambda$$
$$\alpha_1 \le \alpha_2 \le \dots \le \alpha_2 \text{ and if } \alpha_p = \alpha_{p+1}, i_p \le i_{p+1}$$

We have $M(\Lambda)_\lambda = \{m | Hm = \lambda(H)m\}$ weight spaces.

$$M(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} M(\Lambda)_\lambda$$

The vector v_Λ is in $M(\Lambda)_\Lambda$ by definition,

$$F_{\alpha,i}V_\Lambda \in M(\Lambda)_{\Lambda-\alpha}$$
$$H(Fv_\Lambda) = \underbrace{[H,F]v_\Lambda +}_{=-\alpha(H)Fv_\Lambda} \underbrace{FHv_\Lambda}_{=\Lambda(H)FV_\Lambda}$$

So $\chi_{M(\Lambda)} = \sum_{\lambda\in\mathfrak{h}^*} \dim M(\Lambda)_\lambda e^\lambda$ is computed by counting monomials y with fixed $\sum_i \alpha_i$.

$$\chi_{M(\Lambda)} = e^{\Lambda} \prod_{\alpha \in \Delta_+} \frac{1}{(1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}.$$

Exercise 5.1. Convince yourself of this.

For example, $\mathfrak{g} = \mathfrak{sl}_2$,

$$\begin{aligned}\chi_{M(\Lambda)} &= e^{\Lambda} + e^{\Lambda - \alpha} + e^{\Lambda - 2\alpha} + \dots \\ &= e^{\Lambda}(1 + e^{-\alpha} + e^{-2\alpha} + \dots \\ &= e^{\Lambda} \frac{1}{1 - e^{-\alpha}}.\end{aligned}$$

Next week: the Weyl character formula.

References

[Kac90] V.G. Kac, *Infinite-dimensional lie algebras*, Progress in mathematics, Cambridge University Press, 1990.

[Kac10] Victor Kac, *Lecture notes of 18.745 – introduction to lie algebras (fall 2010)*, `https://math.mit.edu/classes/18.745/classnotes.html`, 2010, Lecture notes, MIT.