VERTEX ALGEBRAS

github.com/danimalabares/vertex-algebras

Contents

1. Cartan subalgebra, Cartan matrix and Serre relations

Kac-Moody algebras are Lie algebras, whose definition is motivated by the structure of finite-dimensional simple Lie algebras over $\mathbb{C}$.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ has a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ (abelian + ...) (see [Kac10, Definition 8.2]). Fixing $\mathfrak{h} \subset \mathfrak{g}$ gives a *root space decomposition* (see [Kac10, Proposition 8.5])

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta \subset \mathfrak{h}^*$ linear dual, and, by definition

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$$

Turns out the $\mathfrak{g}_\alpha$ are all 1-dimensional, though this property is lost when we go to Kac-Moody algebras.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

The Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $\kappa(x, y) = \mathrm{Tr}_{\mathfrak{g}} \mathrm{ad}(x)\mathrm{ad}(y)$ is nondegenerate. “This is kind of the definition of semisimple.” (Think of $\mathfrak{h}$ as $\mathfrak{g}_0$, btw.)

$\kappa|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\beta} \neq 0$ only when $\beta = -\alpha$. $\kappa|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate. This gives a linear isomorphism $\mathfrak{h} \xrightarrow{\nu} \mathfrak{h}$ via $\nu(H)(H') = \kappa(H, H')$.

So, $\mathfrak{h}^*$ comes with a non-degenerate bilinear form.

The *reflection* $r_\alpha : \mathfrak{h} \to \mathfrak{h}^*$ in $\alpha \in \mathfrak{h}^*$ (usually a root) is $r_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \cdot \alpha$.

“Classify root systems [...] classify semisimple Lie algebras” It is a fact that $r_\alpha(\Delta) = \Delta$ for all $\alpha \in \Delta$, which motivates the definition of *root system* (see [Kac10, Definition 15.1]) and permits classification. (See [Kac10, Lecture 17] for comments on correspondence of root systems and semisimple Lie algebras.)

Example 1.1. $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h} =$ diagonal matrices

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

is a basis of $\mathfrak{h}$. There are 6 roots vectors

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

E_{23}, E_{13}, etc.

Exercise 1.2. $[H_1, E_{12}] = 2E_{12}$, $[H_2, E_{12}] = -E_{12}$, $\alpha_{12} = (2, -1)$.

[Drawing of roots]

Notions of *positive roots* and *simple roots* (set of rank$\mathfrak{g}$ simple roots has ℓ elements, where $\ell = \dim(\mathfrak{h}^*)$. (See [Kac10, Definition 17.1].) This will also fail for Kac-Moody algebras more generally). Next write the *Cartan matrix* (see [Kac10, Definition 17.2])

$$A = (a_{ij}), \qquad a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

for $1 \le i, j \le \ell$.

Example 1.3. $\mathfrak{sl}_3$. [Picture, hexagonal pattern]. $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2, (\alpha_1, \alpha_2) = -1$, so

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Example 1.4. $\mathfrak{sl}_5$. [Picture, square pattern]. $|\alpha_2| = 1$, $|\alpha_1| = 2$, $(\alpha_1, \alpha_2) = -2$, so

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

Remark 1.5. Since $\mathfrak{g}_\alpha$ is 1-dimensional, set $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$ and $E_i = E_{\alpha_i}$, $i = 1, 2, \ldots, \ell$ (simple root vectors). It turns out that

$$\mathrm{ad}(E_i)^{1-a_{ij}} E_j = 0.$$

This is called a *Serre relation*.

2. Some infinite dimensional Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, and define the *loop algebra*

$$\begin{aligned} L\mathfrak{g} &= \mathfrak{g}[t, t^{-1}], \text{ (with basis } at^m|^{a \,\in\, \text{a basis of } \mathfrak{g}}_{m \in \mathbb{Z}} \text{)} \\ &= \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \end{aligned}$$

with the Lie bracket

$$[at^m, bt^n] = [a, b]t^{m+n}.$$

"This construction is absurdely general — we don't need $\mathfrak{g}$ to be semisimple [...]"

Take $\mathfrak{g} = \mathfrak{sl}_2$. Recall that

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

[Picture with $F, H, E, Ft, Ht, Et, Et^2 \ldots$] E was a root vector, corresponding to the unique root in $\mathfrak{sl}_2$, call it α_1. We seem to have a second simple root α_0, corresponding to Ft.

This looks like it wants to have a Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

We will indeed recover (a variant of) $L\mathfrak{g}$ as a Lie algebra "built from" $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, a Kac-Moody algebra. But note first, $\mathfrak{h} = \mathbb{C}H$ is too small. "Problem with α_0 and α_1 being linearly independent ..."

We can consider $L\mathfrak{g} \oplus \mathbb{C}d$, and set $[d, at^m] = mat^n$

Exercise 2.1. Check this defines a Lie algebra.

The Kac-Moody algebra turns out to be, not quite this, but slightly larger still.

Definition 2.2. Given $\mathfrak{g}$ simple, with $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ invariant bilinear form, there is a Lie algebra

$$\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K,$$

with $[K, \hat{\mathfrak{g}}] = 0$, and $[at^m, bt^n] = [a, b]t^{m+n} + m(a, b)\delta_{m_1} - nK$. $[K, \hat{g}] = 0$, K: central.

"For the construction to work it doesn't actually have to be nondegenerate."

This is called an *affine Lie algebra*. We also have $\tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$, *extended affine Lie algebra*, $[d, at^m] = mat^m$ as before, and $[K, d] = 0$.

The extended affine Lie algebra is an example of a Kac-Moody algebra.

3. Kac-Moody algebras

Recall the notion of the free Lie algebra on a vector space V of generators (or a set X, think of V as a vector space with basis X):

Definition 3.1. The *free Lie algebra* on V is characterized by the universal property

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathfrak{g} \\ & F(V) \arrow[ur, "\exists! \tilde{f}"'] & \end{tikzcd}$$

That is, for any linear map $f : V \to \mathfrak{g}$ with $\mathfrak{g}$ Lie algebra, there exists a unique $\tilde{f}$ homomorphism of Lie algebras $F(V) \to \mathfrak{g}$ such that $\tilde{f} \circ i = f$.

$$\mathrm{Hom}_{\mathrm{Lie}}(F(V), \mathfrak{g}) = \mathrm{Hom}_{\mathrm{Vec}}(V, \mathfrak{g})$$

naturally.

That is, F and the forgetful functor $G : \underline{\mathrm{Lie}} \to \underline{\mathrm{Vec}}$ are adjoint:

$$\mathrm{Hom}_{\underline{\mathrm{Lie}}}(F(V), \mathfrak{g}) \xrightarrow{\simeq} \mathrm{Hom}_{\underline{\mathrm{Vec}}}(V, G(\mathfrak{g}))$$

A realisation of $F(V)$. Let

$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

be the tensor algebra of V.

Then inside $T(V)$ consider $F(V)$ the span of iterated commutators of elements of V.

Proposition 3.2. *This realises the free Lie algebra.*

Proof. In online notes. □

In the finite dimensional simple case, we had

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$

which we think also as $\alpha_i, \alpha_j \in \mathfrak{h}^*$, and $\alpha_i^\vee = \frac{2}{(\alpha_i,\alpha_i)}\nu^{-1}(\alpha_i) \in \mathfrak{h}$.

Clearly, $\alpha_{ii} = 2$ for all i. a_{ij} misht not equal a_{ji}, but certainly $a_{ij} = 0 \iff a_{ji} = 0$. And $\forall i \neq j$, $a_{ij} \leq 0$.

One further property. Set

$$\varepsilon_i = \frac{2}{(\alpha_i, \alpha_i)}, \quad \text{and} \quad D = \substack{\text{diagonal matrix} \\ \text{with entries } \varepsilon_i}$$

Then $A = DB$, where $B = ((\alpha_i, \alpha_j))$ is symmetric. If a matrix A is equal to (diag)(symm), we call it *symmetrizable*.

Definition 3.3. A *generalized Cartan matrix* is an integer matrix $A = (a_{ij})$ which is

- symmetrizable,
- $a_{ii} = 2$ for all i,
- $a_{ij} = 0 \iff a_{ji} = 0$,
- $a_{ij} \leq 0$ for $i \neq j$.

Definition 3.4. A *realisation* of a generalized Cartan matrix is a complex vector space $\mathfrak{h}$, and two sets

$$\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\}, \quad \text{and,}$$
$$\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$, $1 \leq i, j \leq n$.

Exercise 3.5. $\dim(\mathfrak{h}) \geq 2n - \operatorname{rank}(A)$.

Proof. For $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, a realisation is given by

$$\Pi^\vee = \{H_1, H_0\}, \qquad \Pi = \{\alpha_0, \alpha_1\}$$

$$\mathfrak{h} = \mathbb{C}H, \mathbb{C}d, \mathbb{C}K,$$
$$\mathfrak{h}^* = \mathbb{C}\alpha_1 + \mathbb{C}\delta + \mathbb{C}\Lambda_0$$

(Canonical dual, $\langle \alpha_1, H \rangle = 2$, $\langle \delta, d \rangle = 1 = \langle \Lambda_0, K \rangle$, every other pairing 0.)

Then

$$\begin{cases} \alpha_1 = \alpha_1 \\ \alpha_0 = \delta - \alpha_1 \end{cases} \qquad \begin{cases} \alpha_1^\vee = H \\ \alpha_0^\vee = K - H \end{cases}$$

So we obtain

$$\langle \alpha_0^\vee, \alpha_1 \rangle = \langle K - H, \alpha_1 \rangle = 2$$
$$\langle \alpha_1^\vee, \alpha_0 \rangle = \langle H, \delta - \alpha_1 \rangle = -2$$
$$\langle \alpha_0^\vee, \alpha_0 \rangle = \langle K - H, \delta - \alpha_1 \rangle = +2$$

$\square$

Finally let's define Kac-Moody algebras.

Let A be a generalized Cartan matrix. Let

$$\tilde{\mathfrak{n}}_+ = F(e_1, \ldots, e_n),$$

the free Lie algebra on n generators, and similarly

$$\tilde{\mathfrak{n}}_- = F(f_1, \ldots, f_n).$$

Let $\mathfrak{h}$ be a realisation of A. Set $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.

Make $\tilde{\mathfrak{g}}(A)$ a Lie algebra by defining

- $[\mathfrak{h}, \mathfrak{h}] = 0$,
- $\forall H \in \mathfrak{h}$, $[H, e_i] = \langle \alpha_i, H \rangle \, e_i = \alpha_i(H) e_i$. And similarly, $[H, f_i] = -\alpha_i(H) f_i$.
- $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$.

Then $\tilde{\mathfrak{g}}(A)$ is a Lie algebra (though not yet the Kac-Moody algebra). See Kac, [Kac90, Thorem 1.2].

Remark 3.6. In $\mathfrak{h}$ we have a lattice

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \ldots + \mathbb{Z}\alpha_n^\vee, \quad \text{and}$$
$$Q = \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_n \text{ in } \mathfrak{h}^*$$

(root and coroot lattices). $\tilde{\mathfrak{g}}(A)$ is naturally Q-graded, with

$$\tilde{\mathfrak{g}}(A)_\beta = \text{span}\{\text{commutators of } e_i \text{ with } \sum \alpha_i = \beta\}.$$

$\tilde{g}(A) = \mathfrak{h}$.

Theorem 3.7 (Gabber-Kac). *Denote by $I \subset \tilde{\mathfrak{g}}(A)$ the maximal Q-graded ideal, such that $I \cap \mathfrak{h} = \{0\}$. Then I is generated by the Serre relations*

$$ad(e_i)^{1-a_{ij}} e_j \qquad \textit{and} \qquad ad(f_i)^{1-a_{ij}} f_j, \; i \neq j.$$

Proof. [Kac90, Theorem 9.11]. □

(The existence of the ideal I does not need the theorem; it's importance is providing an expression for the generators.)

Definition 3.8. The *Kac-Moody algebra* $\mathfrak{g}(A)$ is $\tilde{\mathfrak{g}}(A)/I$.

References

[Kac90] V.G. Kac, *Infinite-dimensional lie algebras*, Progress in mathematics, Cambridge University Press, 1990.

[Kac10] Victor Kac, *Lecture notes of 18.745 – introduction to lie algebras (fall 2010)*, `https://math.mit.edu/classes/18.745/classnotes.html`, 2010, Lecture notes, MIT.