VERTEX ALGEBRAS

github.com/danimalabares/vertex-algebras

Contents

1. Cartan subalgebra, Cartan matrix and Serre relations

Kac-Moody algebras are Lie algebras, whose definition is motivated by the structure of finite-dimensional simple Lie algebras over $\mathbb{C}$.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ has a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ (abelian $+ \dots$) (see [Kac10, Definition 8.2]). Fixing $\mathfrak{h} \subset \mathfrak{g}$ gives a *root space decomposition* (see [Kac10, Proposition 8.5])

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta \subset \mathfrak{h}^*$ linear dual, and, by definition

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$$

Turns out the $\mathfrak{g}_\alpha$ are all 1-dimensional, though this property is lost when we go to Kac-Moody algebras.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

The Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $\kappa(x, y) = \mathrm{Tr}_{\mathfrak{g}} \mathrm{ad}(x)\mathrm{ad}(y)$ is nondegenerate. "This is kind of the definition of semisimple." (Think of $\mathfrak{h}$ as $\mathfrak{g}_0$, btw.)

$\kappa|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\beta} \neq 0$ only when $\beta = -\alpha$. $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate. This gives a linear isomorphism $\mathfrak{h} \xrightarrow{\nu} \mathfrak{h}$ via $\nu(H)(H') = \kappa(H, H')$.

So, $\mathfrak{h}^*$ comes with a non-degenerate bilinear form.

The *reflection* $r_\alpha : \mathfrak{h} \to \mathfrak{h}^*$ in $\alpha \in \mathfrak{h}^*$ (usually a root) is $r_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \cdot \alpha$.

"Classify root systems [...] classify semisimple Lie algebras" It is a fact that $r_\alpha(\Delta) = \Delta$ for all $\alpha \in \Delta$, which motivates the definition of *root system* (see [Kac10, Definition 15.1]) and permits classification. (See [Kac10, Lecture 17] for comments on correspondence of root systems and semisimple Lie algebras.)

Example 1.1. $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{h} =$ diagonal matrices

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

is a basis of $\mathfrak{h}$. There are 6 roots vectors

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

E_{23}, E_{13}, etc.

Exercise 1.2. $[H_1, E_{12}] = 2E_{12}$, $[H_2, E_{12}] = -E_{12}$, $\alpha_{12} = (2, -1)$.

[Drawing of roots]

Notions of *positive roots* and *simple roots* (set of rank$\mathfrak{g}$ simple roots has ℓ elements, where $\ell = \dim(\mathfrak{h}^*)$. (See [Kac10, Definition 17.1].) This will also fail for Kac-Moody algebras more generally). Next write the *Cartan matrix* (see [Kac10, Definition 17.2])

$$A = (a_{ij}), \qquad a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

for $1 \leq i, j \leq \ell$.

Example 1.3. $\mathfrak{sl}_3$. [Picture, hexagonal pattern]. $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2, (\alpha_1, \alpha_2) = -1$, so

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Example 1.4. $\mathfrak{sl}_5$. [Picture, square pattern]. $|\alpha_2| = 1$, $|\alpha_1| = 2$, $(\alpha_1, \alpha_2) = -2$, so

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

Remark 1.5. Since $\mathfrak{g}_\alpha$ is 1-dimensional, set $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$ and $E_i = E_{\alpha_i}$, $i = 1, 2, \ldots, \ell$ (simple root vectors). It turns out that

$$\mathrm{ad}(E_i)^{1-a_{ij}} E_j = 0.$$

This is called a *Serre relation.*

2. Some infinite dimensional Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, and define the *loop algebra*

$$\begin{aligned} L\mathfrak{g} &= \mathfrak{g}[t, t^{-1}], \text{ (with basis } at^m|^{a \in \text{ a basis of } \mathfrak{g}}_{m \in \mathbb{Z}} \text{)} \\ &= \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \end{aligned}$$

with the Lie bracket

$$[at^m, bt^n] = [a, b]t^{m+n}.$$

"This construction is absurdely general — we don't need $\mathfrak{g}$ to be semisimple [...]"

Take $\mathfrak{g} = \mathfrak{sl}_2$. Recall that

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

[Picture with $F, H, E, Ft, Ht, Et, Et^2 \ldots$] E was a root vector, corresponding to the unique root in $\mathfrak{sl}_2$, call it α_1. We seem to have a second simple root α_0, corresponding to Ft.

This looks like it wants to have a Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

We will indeed recover (a variant of) $L\mathfrak{g}$ as a Lie algebra "built from" $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, a Kac-Moody algebra. But note first, $\mathfrak{h} = \mathbb{C}H$ is too small. "Problem with α_0 and α_1 being linearly independent ..."

Exercise 2.1. Consider $L\mathfrak{g} \oplus \mathbb{C}d$, and set $[d, at^m] = mat^m$, $[d, d] = 0$. Check this defines a Lie algebra.

Proof. Skew-commutativity, i.e. for all $x \in L\mathfrak{g} \oplus \mathbb{C}d$,

$$[x, x] = 0, \tag{2.1.1}$$

is immediate from skew commutativity in $L\mathfrak{g}$ and the hypothesis that $[d, d] = 0$.

To confirm Jacobi identity, i.e. that for all $x, y, z \in L\mathfrak{g} \oplus \mathbb{C}d$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \tag{2.1.2}$$

notice that since this is a cyclic sum on x, y, z we only need to consider three elements in $L\mathfrak{g} \oplus \mathbb{C}d$ up to cyclic permutation. The cases in which the three elements are either in $L\mathfrak{g}$ or in $\mathbb{C}d$ are obvious, so that there are only two interesting possibilities:

$$x = d, \qquad y = at^m, \qquad z = bt^n \tag{2.1.3}$$

$$x = d, \qquad y = d, \qquad z = at^n \tag{2.1.4}$$

Case 2.1.3 gives

$$\begin{aligned}
&[d, [at^m, bt^n]] + [at^m, [bt^n, d]] + [bt^n, [d, at^m]] \\
&= [d, [a, b]t^{m+n}] + [at^m, -nbt^n] + [bt^n, mat^m] \\
&= (m + n)[a, b]t^{m+n} - n[a, b]t^{m+n} + m[b, a]t^{m+n} \\
&= (m + n)[a, b]t^{m+n} - n[a, b]t^{m+n} - m[a, b]t^{m+n} \\
&= (m + n)[a, b]t^{m+n} - (m + n)[a, b]t^{m+n} = 0.
\end{aligned}$$

Case 2.1.4 gives

$$\begin{aligned}
&[d, [d, at^m]] + [d, [at^m, d]] + [at^m, [d, d]] \\
&= [d, mat^m] + [d, -mat^m] = 0.
\end{aligned}$$

□

The Kac-Moody algebra turns out to be, not quite this, but slightly larger still.

Recall that an *invariant bilinear form* $(\cdot, \cdot)$ on a Lie algebra $\mathfrak{g}$ is a bilinear form such that

$$([a, b], c) = (a, [b, c]) \qquad \forall a, b, c \in \mathfrak{g}. \tag{2.1.5}$$

Exercise 2.2. Prove that an invariant bilinear form on a simple Lie algebra must in fact be symmetric.

Proof. It's enough to show that $\mathfrak{g}$ is *perfect*, i.e. that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. In this case, let $a, b \in \mathfrak{g}$ and suppose that $b = [x, y]$. Then

$$\begin{aligned}
(a, b) &= (a, [x, y]) = (a, -[y, x]) = (-[a, y], x) = ([y, a], x) \\
&= (y, [a, x]) = (y, -[x, a]) = (-[y, x], a) = ([x, y], a) = (b, a)
\end{aligned}$$

To confirm that $\mathfrak{g}$ is perfect just observe that $[\mathfrak{g}, \mathfrak{g}]$ is a nontrivial ideal of $\mathfrak{g}$. □

Definition 2.3. Given $\mathfrak{g}$ simple, with $(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ invariant bilinear form, the *affine Lie algebra* is

$$\hat{\mathfrak{g}} = L\mathfrak{g}\oplus\mathbb{C}K,$$

with $[K,\hat{\mathfrak{g}}]=0$, and $[at^m,bt^n]=[a,b]t^{m+n}+m(a,b)\delta_{m,-n}K$.

"For the construction to work it doesn't actually have to be nondegenerate."

Exercise 2.4. Check that the affine Lie algebra $\hat{\mathfrak{g}}$ is a Lie algebra.

Proof. (Skew-commutativity.) Since $[K,\hat{\mathfrak{g}}]=0$ and $K\in\hat{\mathfrak{g}}$, it is immediate that $[K,K]=0$. For the case of an element in $L\mathfrak{g}$, we see that $[at^m,at^m]=0$ by skew-commutativity of the bracket in $\mathfrak{g}$ and the Kronecker delta.

(Jacobi identity.) As in Exercise 2.1, any choice of x,y,z involving K is immediate by $[K,\hat{\mathfrak{g}}]=0$. Thus the only interesting case is for Jacobi identity consider the cases

$$\begin{aligned}
&[at^m,[bt^n,ct^\ell]]+[bt^n,[ct^\ell,at^m]]+[ct^\ell,[at^m,bt^n]]\\
&=[at^m,[b,c]t^{n+\ell}+n(b,c)\delta_{n,-\ell}K]\\
&+[bt^n,[c,a]t^{\ell+m}+\ell(c,a)\delta_{\ell,-m}K]\\
&+[ct^\ell,[a,b]t^{m+n}+m(a,b)\delta_{m,-n}K]\\
&=[at^m,[b,c]t^{n+\ell}]+[at^m,n(b,c)\delta_{n,-\ell}K]\\
&+[bt^n,[c,a]t^{\ell+m}]+[bt^n,\ell(c,a)\delta_{\ell,-m}K]\\
&+[ct^\ell,[a,b]t^{m+n}]+[ct^\ell,m(a,b)\delta_{m,-n}K]\\
&=[a,[b,c]]t^{m+(n+\ell)}+m(a,[b,c])\delta_{m,-(n+\ell)}K\\
&+[b,[c,a]]t^{n+(\ell+m)}+n(b,[c,a])\delta_{n,-(\ell+m)}K\\
&+[c,[a,b]]t^{\ell+(m+n)}+\ell(c,[a,b])\delta_{\ell,-(m+n)}K=0
\end{aligned}$$

It is clear that we obtain a Jacobi equation on $\mathfrak{g}$. To see that the remaining terms vanish, notice that the condition on the Kronecker delta in its three appearances is the same, namely, $m+n+\ell=0$. In this case, we only need to check that $(a,[b,c])=(b,[c,a])=(c,[a,b])$ to conclude. This follows from the invariance of $(\cdot,\cdot)$ and the fact that $\mathfrak{g}$ simple using Exercise 2.2. □

We also have

Definition 2.5. The *extended affine Lie algebra* is

$$\tilde{\mathfrak{g}} = L\mathfrak{g}\oplus\mathbb{C}K\oplus\mathbb{C}d,$$

with $[d,at^m]=mat^m$ as before, and $[K,d]=0$.

The extended affine Lie algebra is an example of a Kac-Moody algebra.

Exercise 2.6 (For those who like geometry)**.** Let $R=\mathbb{C}[t,t^{-1}]$. If $D\in\mathrm{Der}(R)$, then $L\mathfrak{g}\oplus\mathbb{C}d$ is a Lie algebra with $[d,a\otimes r]=a\otimes D(r)$. Is $(\mathfrak{g}\otimes R)\oplus\mathrm{Der}(R)$ a Lie algebra? (The Lie alegra $L\mathfrak{g}\oplus\mathbb{C}d$ from Exercise 2.1 is a particular case, for $D=t\frac{d}{dt}$.)

Proof. Checking that $L\mathfrak{g}\oplus\mathbb{C}d$ is a Lie algebra with $[d,a\otimes r]=a\otimes D(r)$ is similar to Exercise 2.1: skew-commutativity is immediate from skew-commutativity in each of the components, while Jacobi identity is verified in two cases. For $x = y = d$ and $z = a \otimes r$ we quickly obtain

$$\begin{aligned}&[x,[y,z]] + [y,[z,x]] + [z,[x,y]]\\ &= [d,[d,a\otimes r]] + [d,[a\otimes r,d]] + [a\otimes r,[d,d]]\\ &= [d, a\otimes D(r)] + [d, -a\otimes D(r)] = 0.\end{aligned}$$

And for $x = d$, $y = a \otimes r$ and $z = b \otimes s$, we get

$$\begin{aligned}&[x,[y,z]] + [y,[z,x]] + [z,[x,y]]\\ &= [d,[a\otimes r, b\otimes s]] + [a\otimes r,[b\otimes s,d]] + [b\otimes s,[d,a\otimes r]]\\ &= [d,[a,b]\otimes rs] + [a\otimes r, -b\otimes D(s)] + [b\otimes s, a\otimes D(r)]\\ &= [a,b]\otimes D(rs) - [a,b]\otimes rD(s) + [b,a]\otimes sD(r) = 0.\end{aligned} \tag{2.6.1}$$

To check whether $(\mathfrak{g}\otimes R)\oplus \mathrm{Der}(R)$ is a Lie algebra first put the Lie bracket on $\mathrm{Der}(R)$ as $[D, D_1] = DD_1 - D_1D$. It is clear that this bracket is skew-commutative. Jacobi identity reads

$$\begin{aligned}&[D,[D_1,D_2]] + [D_1,[D_2,D]] + [D_2,[D,D_1]]\\ &= [D, D_1D_2 - D_2D_1] + [D_1, D_2D - DD_2] + [D_2, DD_1 - D_1D]\\ &= D(D_1D_2 - D_2D_1) - (D_1D_2 - D_2D_1)D + D_1(D_2D - DD_2)\\ &- (D_2D - DD_2)D_1 + D_2(DD_1 - D_1D) - (DD_1 - D_1D)D_2\\ &= DD_1D_2 - DD_2D_1 - D_1D_2D + D_2D_1D + D_1D_2D - D_1DD_2\\ &- D_2DD_1 + DD_2D_1 + D_2DD_1 - D_2D_1D - DD_1D_2 + D_1DD_2 = 0.\end{aligned}$$

Now put the bracket on $(\mathfrak{g}\otimes R)\oplus \mathrm{Der}(R)$ as $[D, a\otimes r] = a\otimes D(r)$. Skew-commutativity is immediate. Jacobi identity for $x = D, y = a\otimes r$ and $z = b\otimes s$ is identical to the computation 2.6.1. In the case $x = D$, $y = D_1$ and $z = a\otimes r$, we get

$$\begin{aligned}&[D,[D_1,a\otimes r]] + [D_1,[a\otimes r,D]] + [a\otimes r,[D,D_1]]\\ &= [D, a\otimes D_1(r)] + [D_1, -a\otimes D(r)] + [a\otimes r,[D,D_1]]\\ &= a\otimes DD_1(r) - a\otimes D_1D(r) - a\otimes [D,D_1](r) = 0\end{aligned}$$

□

3. Kac-Moody algebras

Recall the notion of the free Lie algebra on a vector space V of generators (or a set X, think of V as a vector space with basis X):

Definition 3.1. The *free Lie algebra* on V is characterized by the universal property

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathfrak{g} \\ & F(V) \arrow[ur, "\exists ! \tilde{f}"'] & \end{tikzcd}$$

That is, for any linear map $f : V \to \mathfrak{g}$ with $\mathfrak{g}$ Lie algebra, there exists a unique $\tilde{f}$ homomorphism of Lie algebras $F(V) \to \mathfrak{g}$ such that $\tilde{f}\circ i = f$.

$$\mathrm{Hom}_{\mathrm{Lie}}(F(V),\mathfrak{g}) = \mathrm{Hom}_{\mathrm{Vec}}(V,\mathfrak{g})$$

naturally.

That is, F and the forgetful functor $G : \underline{\mathrm{Lie}} \to \underline{\mathrm{Vec}}$ are adjoint:

$$\mathrm{Hom}_{\underline{\mathrm{Lie}}}(F(V), \mathfrak{g}) \xrightarrow{\simeq} \mathrm{Hom}_{\underline{\mathrm{Vec}}}(V, G(\mathfrak{g}))$$

A realisation of $F(V)$. Let

$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

be the tensor algebra of V.

Then inside $T(V)$ consider $F(V)$ the span of iterated commutators of elements of V.

Proposition 3.2. *This realises the free Lie algebra.*

Proof. In online notes. □

In the finite dimensional simple case, we had

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$

which we think also as $\alpha_i, \alpha_j \in \mathfrak{h}^*$, and $\alpha_i^\vee = \frac{2}{(\alpha_i,\alpha_i)}\nu^{-1}(\alpha_i) \in \mathfrak{h}$.

Clearly, $\alpha_{ii} = 2$ for all i. a_{ij} might not equal a_{ji}, but certainly $a_{ij} = 0 \iff a_{ji} = 0$. And $\forall i \neq j$, $a_{ij} \leq 0$.

One further property. Set

$$\varepsilon_i = \frac{2}{(\alpha_i, \alpha_i)}, \quad \text{and} \quad D = \substack{\text{diagonal matrix} \\ \text{with entries } \varepsilon_i}$$

Then $A = DB$, where $B = ((\alpha_i, \alpha_j))$ is symmetric. If a matrix A is equal to (diag)(symm), we call it *symmetrizable.*

Definition 3.3. A *generalized Cartan matrix* is an integer matrix $A = (a_{ij})$ which is

- symmetrizable,
- $a_{ii} = 2$ for all i,
- $a_{ij} = 0 \iff a_{ji} = 0$,
- $a_{ij} \leq 0$ for $i \neq j$.

Definition 3.4. A *realisation* of a generalized Cartan matrix is a complex vector space $\mathfrak{h}$, and two sets

$$\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\}, \quad \text{and,}$$
$$\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$, $1 \leq i, j \leq n$.

Exercise 3.5. $\dim(\mathfrak{h}) \geq 2n - \mathrm{rank}(A)$.

Proof. For $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, a realisation is given by

$$\Pi^\vee = \{H_1, H_0\}, \qquad \Pi = \{\alpha_0, \alpha_1\}$$

$$\mathfrak{h} = \mathbb{C}H, \mathbb{C}d, \mathbb{C}K,$$
$$\mathfrak{h}^* = \mathbb{C}\alpha_1 + \mathbb{C}\delta + \mathbb{C}\Lambda_0$$

(Canonical dual, $\langle \alpha_1, H\rangle = 2$, $\langle \delta, d\rangle = 1 = \langle \Lambda_0, K\rangle$, every other pairing 0.)

Then

$$\begin{cases} \alpha_1 = \alpha_1 \\ \alpha_0 = \delta - \alpha_1 \end{cases} \qquad \begin{cases} \alpha_1^\vee = H \\ \alpha_0^\vee = K - H \end{cases}$$

So we obtain

$$\begin{aligned} \langle \alpha_0^\vee, \alpha_1\rangle &= \langle K - H, \alpha_1\rangle = 2 \\ \langle \alpha_1^\vee, \alpha_0\rangle &= \langle H, \delta - \alpha_1\rangle = -2 \\ \langle \alpha_0^\vee, \alpha_0\rangle &= \langle K - H, \delta - \alpha_1\rangle = +2 \end{aligned}$$

□

Finally let's define Kac-Moody algebras.

Let A be a generalized Cartan matrix. Let

$$\tilde{\mathfrak{n}}_+ = F(e_1, \ldots, e_n),$$

the free Lie algebra on n generators, and similarly

$$\tilde{\mathfrak{n}}_- = F(f_1, \ldots, f_n).$$

Let $\mathfrak{h}$ be a realisation of A. Set $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.

Make $\tilde{\mathfrak{g}}(A)$ a Lie algebra by defining

- $[\mathfrak{h}, \mathfrak{h}] = 0$,
- $\forall H \in \mathfrak{h}$, $[H, e_i] = \langle \alpha_i, H\rangle\, e_i = \alpha_i(H) e_i$. And similarly, $[H, f_i] = -\alpha_i(H) f_i$.
- $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$.

Then $\tilde{\mathfrak{g}}(A)$ is a Lie algebra (though not yet the Kac-Moody algebra). See Kac, [Kac90, Thorem 1.2].

Remark 3.6. In $\mathfrak{h}$ we have a lattice

$$\begin{aligned} Q^\vee &= \mathbb{Z}\alpha_1^\vee + \ldots + \mathbb{Z}\alpha_n^\vee, \quad \text{and} \\ Q &= \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_n \text{ in } \mathfrak{h}^* \end{aligned}$$

(root and coroot lattices). $\tilde{\mathfrak{g}}(A)$ is naturally Q-graded, with

$$\tilde{\mathfrak{g}}(A)_\beta = \operatorname{span}\{\text{commutators of } e_i \text{ with } \sum \alpha_i = \beta\}.$$

$\tilde{g}(A) = \mathfrak{h}$.

Theorem 3.7 (Gabber-Kac). *Denote by $I \subset \tilde{\mathfrak{g}}(A)$ the maximal Q-graded ideal, such that $I \cap \mathfrak{h} = \{0\}$. Then I is generated by the Serre relations*

$$ad(e_i)^{1-a_{ij}} e_j \qquad \textit{and} \qquad ad(f_i)^{1-a_{ij}} f_j,\ i \neq j.$$

Proof. [Kac90, Theorem 9.11]. □

(The existence of the ideal I does not need the theorem; the importance of the theorem is providing an expression for the generators.)

Definition 3.8. The *Kac-Moody algebra* $\mathfrak{g}(A)$ is $\tilde{\mathfrak{g}}(A)/I$.

4. Affine Kac-Moody algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, with $(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ invariant bilinear form,

$$([x,y],z)=(z,[y,z])\qquad \forall x,y,z\in\mathfrak{g}$$

(Eg. the Killing form $\kappa(x,y)=\mathrm{Tr}_{\mathfrak{g}}\mathrm{ad}(x)\mathrm{ad}(y)$ is invariant.)

Typically we normalise $(\cdot,\cdot)$ so that $(\alpha,\alpha)=2$ for the long roots of $\mathfrak{g}$.

Then $\hat{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K$ (affine Lie algebra),

$$[at^m,bt^n]=[a,b]t^{m+n}+m\delta_{m,-n}(a,b)K,\qquad [K,\hat{\mathfrak{g}}]=0$$

and $\tilde{\mathfrak{g}}=\hat{\mathfrak{g}}\oplus\mathbb{C}d$, $[d,K]=0$, $[d,at^m]=mat^m$, (affine Kac-Moody algebra or "extended affine Lie algebra")

Theorem 4.1. *$\tilde{\mathfrak{g}}$ is a Kac-Moody algebra.*

Let $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$, ($\mathfrak{g}_\alpha=\mathbb{C}E_\alpha$.)

The simple roots and coroots. $\tilde{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}K\oplus\mathbb{C}d$. We identify $\tilde{\mathfrak{h}}^*$ with $\mathfrak{h}^*\oplus\mathbb{C}\Lambda_0\oplus\mathbb{C}\delta$ where

$$\Lambda_0(\mathfrak{h})=\delta(\mathfrak{h})=0$$
$$\Lambda_0(d)=\delta(K)=0$$
$$\Lambda_0(K)=\delta(d)=1$$

The *real coroots* are

$$\hat{\Delta}^{V,re}=\{E_\alpha t^m|\alpha\in\Delta,m\in\mathbb{Z}\}$$

and there are also imaginary roots and coroots

$$\hat{\Delta}^{V,im}=\{Ht^m|H\in\mathfrak{h},m\in\mathbb{Z}\setminus\{0\}\}$$

Roots:

$$\hat{\Delta}^{re}=\{\alpha+m\delta|\alpha\in\Delta,m\in\mathbb{Z}\}$$
$$\hat{\Delta}^{im}=\{m\delta|m\neq 0\}$$

Xt^m:

$$[H,Xt^m]=[H,x]t^m,\qquad H\in\mathfrak{h}$$
$$[K,xt^m]=0$$
$$[d,xt^m]=mxt^m$$

so it $x\in\mathfrak{g}_\alpha$, $xt^m\in\tilde{\mathfrak{g}}_{\alpha+m\delta}$.

The invariant bilinear form $(\cdot,\cdot)$ from $\mathfrak{g}\times\mathfrak{g}$ extends uniquely to $(\cdot,\cdot):\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}\to\mathbb{C}$.

$(d,d)=(K,K)=0$, $(d,K)=1$ and $(d,\mathfrak{h})=(K,\mathfrak{h})=0$.

So, in $\tilde{\mathfrak{h}}^*$:

$$(\Lambda_0,\Lambda_0)=(\delta,\delta)=0$$
$$(\Lambda_0,\mathfrak{h}^*)=(\delta,\mathfrak{h}^*)=0$$
$$(\Lambda_0,\delta)=1.$$

Hence, $|\alpha+m\delta|^2=|\alpha|^2$, $|m\delta|^2=0$.

Example 4.2. $\widetilde{\mathfrak{sl}_2}$, $\tilde{\mathfrak{h}}^*=\mathrm{span}\{\alpha,\Lambda_0,\delta\}$ with Gram matrix ...

We can make a choice of positive roots,

$$\hat{\Delta}_+ = \{\alpha + m\delta | \alpha \in \Delta, m > 0\} \cup \{m\delta | m > 0\} \cup \Delta_+$$

Obviously, if $\alpha \in \Delta_+$ is simple, $\alpha \in \hat{\Delta}_+$ is simple.

Notation. Let $\theta \in \Delta_+$ be a the highest root. ($\nexists \alpha \in \Delta_+$ such that $\alpha - \theta \in \mathbb{Z}_+\Delta_+$.) and $\alpha = \delta - \theta$.

Then $\alpha_0 \in \hat{\Delta}_+$ is simple and the set of simple roots is $\hat{\Pi} = \{\alpha_0, \underbrace{\alpha_1, \dots, \alpha_\ell}_{\text{the finite simple roots}}\}$.

where $\ell = \operatorname{rank}(\mathfrak{g})$.

The coroot corresponding to α_0 is

$$\alpha_0^\vee = K - \theta^\vee, \qquad \theta^\vee = \frac{2}{(\theta,\theta)}\nu^{-1}(\theta) \in \mathfrak{h}$$

$$\text{and} \qquad E_{\alpha_0} = E_{-\theta} t.$$

Now, in any Kac-Moody algebra, we have

$$\begin{aligned} &\text{roots} \qquad \Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^* \\ &\text{coroots} \qquad \Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}, \end{aligned}$$

and *reflections* $r_i \in \mathrm{GL}(\mathfrak{h}^*)$, defined by

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i.$$

One can check that

$$(r_i\lambda, r_i\mu) = (\lambda, \mu) \qquad \forall \lambda, \mu \in \mathfrak{h}^*$$

The *Weyl group* W is $\langle r_i | i = 1, \dots, \ell \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$.

Example 4.3. For $\widetilde{\mathfrak{sl}_2}$, r_1 is easy,

$$\begin{aligned} r_1(\alpha) &= -\alpha \qquad (\text{as in } \mathfrak{sl}_2) \\ r_1(\delta) &= \delta, \qquad r_1(\Lambda_0) = \Lambda_0. \end{aligned}$$

To compute r_0 take an arbitrary element $m\alpha_1 + k\Lambda_0 + f\delta$ and do:

$$r_0(m\alpha_1 + k\Lambda_0 + f\delta) = m\alpha_1 + k\Lambda_0 + f\delta - \langle \alpha_0^\vee, m\alpha_1 + k\Lambda_0 + f\delta \rangle \alpha_0$$

$$\alpha_0 = \delta - \alpha_1, \qquad \alpha_0^\vee = K - \alpha^\vee$$

so we obtain

$$\begin{aligned} &= m\alpha_1 + k\Lambda_0 + f\delta - (k - 2m)(\delta - \alpha_1) \\ &(k - m)\alpha_1 + k\Lambda_0 + (f - k + 2m)\delta. \end{aligned}$$

Relative to basis $\{\alpha_1, \Lambda_0, \delta\}$.

$$r_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} m \\ k \\ f \end{pmatrix}$$

$$t = r_1 r_0 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

Notice that δ is fixed by all r_i. Also $m\alpha + k\Lambda_0 + f\delta$, the *coefficient* of Λ_0 is fixed by all r_i.

Then

$$t(m\alpha_1+k\Lambda_0 + f\delta) = (m-k)\alpha_1 + k\Lambda_0 + (f-k+2m)\delta.$$

Think of t as a translation.

The number k in

$$\mathfrak{h}^* \ni \hat{\lambda} = \lambda + k\Lambda_0 + f\delta$$

is called the *level* of $\hat{\lambda}$.

$\hat{\mathfrak{h}}$ = union of (hyper)planes of constant level which are stable under W. The roots α are all of level 0.

[Picture] "r_1 changes the sign of the finite path". And $t = r_1 r_0$ is a sort of translation. Indeed, in general we can consider $t_{\alpha_i} = r_{\alpha_i} \circ r_0 \in W$,

$$t_\alpha(\beta + m\delta) = \beta + (m + (\beta, \alpha_i))\delta$$

One can describe the action of t_α on $\hat{\lambda}$ in general (e.g. see [Kac90, Chapter 6])

Proposition 4.4. *For the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ with $\hat{W} = \langle r_0, r_1, \ldots, r_\ell\rangle$ its Weyl group (and $W = \langle r_1, \ldots, r_\ell\rangle \subset \hat{W}$ the Weyl group of $\mathfrak{g}$), then $\hat{W} \simeq W \times t_{Q^\vee}$ (where it should be semidirect product instead of $\times \ldots$) where $Q^\vee$ is the coroot lattice of $\mathfrak{g}$.*

Remark 4.5. For general Kac-Moody algebras, the Weyl groups are much larger, hyperbolic reflection groups.

In the affine case, $\hat{W}$ fixes level k, and $|\hat{\lambda}|$. One gets, in the intersection, paraboloids [Picture of section of hyperboloid that is a parabola].

5. Weyl character formula

Highest weight representations of Kac-Moody algebras. Let $\lambda \in \mathfrak{h}^*$, where $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a Kac-Moody algebra. We define a *Verma module*

$$M(\Lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n}_+} \mathbb{C}v_\Lambda$$

where $\mathfrak{h} + \mathfrak{n}_+$ acts on V_Λ by:

$$Xv_\Lambda = 0, \qquad \forall x \in \mathfrak{n}_+, Hv_\Lambda \qquad = \Lambda(H)v_\Lambda, \qquad \forall H \in \mathfrak{h}$$

So $\mathbb{C}v_\Lambda$ is a $U(\mathfrak{h}+\mathfrak{n}_+)$-module,

$$\begin{array}{c} U(\mathfrak{h}+\mathfrak{n}_+) \\ \downarrow \\ U(\mathfrak{g}) \end{array}$$

By the PBW theorem, $M(\Lambda)$ has a linear $\mathbb{C}$-basis.

Let $\{F_{\alpha,i} : i = 1, \ldots, \dim \mathfrak{g}_\alpha\}$ be a basis of $\mathfrak{g}_{-\alpha}$, $\forall \alpha \in \Delta_+$. Also choose a total order on Δ_+. (Some sort of lexicographical order that takes longer to write than to say.)

$$F_{\alpha_1,i_1}, F_{\alpha_2,i_2}, \ldots, F_{\alpha_s,i_s}, v_\Lambda$$
$$\alpha_1 \le \alpha_2 \le \ldots \le \alpha_2 \text{ and if } \alpha_p = \alpha_{p+1}, i_p \le i_{p+1}$$

We have $M(\Lambda)_\lambda = \{m | Hm = \lambda(H)m\}$ weight spaces.

$$M(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} M(\Lambda)_\lambda$$

The vector v_Λ is in $M(\Lambda)_\Lambda$ by definition,

$$F_{\alpha,i}V_\Lambda \in M(\Lambda)_{\Lambda-\alpha}$$
$$H(Fv_\Lambda) = \underbrace{[H,F]v_\Lambda}_{=-\alpha(H)Fv_\Lambda} + \underbrace{FHv_\Lambda}_{=\Lambda(H)FV_\Lambda}$$

So $\chi_{M(\Lambda)} = \sum_{\lambda\in\mathfrak{h}^*} \dim M(\Lambda)_\lambda e^\lambda$ is computed by counting monomials y with fixed $\sum_i \alpha_i$.

$$\chi_{M(\Lambda)} = e^\Lambda \prod_{\alpha\in\Delta_+} \frac{1}{(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \tag{5.0.1}$$

The product on Eq. 5.0.1 is called *Weyl denominator*.

Exercise 5.1. Convince yourself of this.

Example 5.2. $\mathfrak{g} = \mathfrak{sl}_2$, [Picture]

$$\begin{aligned}\chi_{M(\Lambda)} &= e^\Lambda + e^{\Lambda-\alpha} + e^{\Lambda-2\alpha} + \dots \\ &= e^\Lambda(1 + e^{-\alpha} + e^{-2\alpha} + \dots \\ &= e^\Lambda \frac{1}{1-e^{-\alpha}}.\end{aligned}$$

For certain Λ, $M(\Lambda)$ is *reducible* (i.e. there exists a submodule $0 \neq N \subseteq M(\Lambda)$ (with proper contention).

Lemma 5.3. *For any submodule N,*

$$N = \bigoplus_{\mu\in\mathfrak{h}^*} N \cap M(\Lambda)_\mu.$$

Corollary. The sum of all <u>proper</u> submodules of $M(\Lambda)$ is proper, in particular there is a maximal proper submodule.

Notation. $L(\Lambda) = M(\Lambda) \Big/ \left(\begin{smallmatrix}\text{max. proper}\\ \text{submodule}\end{smallmatrix}\right)$

Example 5.4. $\mathfrak{sl}_2$. $\Lambda = 3\omega$ (ω: fundamental weight, $\alpha = 2\omega$.) $L(3\omega) = \mathbb{C}\left\langle e^{3\omega}, e^{-\omega}, e^{-3\omega}\right\rangle$.
[Picture]

Definition 5.5. A $\mathfrak{g}$-module is *integrable* if

- $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$ (weight module).
- For all simple roots α_i; e_i and f_i are locally nilpotent on V (i.e. for all $v \in V$ there exists N such that $e_i^N v = f_i^N v = 0$.)

Remark 5.6.
- Vermas are not integrable.
- $\dim V < \infty \implies V$ integrable.
- $\mathfrak{g}$ itself (Kac-Moody) is integrable.

Dominant integrable weights. Let $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ be the simple coroots.

Definition 5.7. The *dominant integral weights* are the weights that pair with the coroots to give integers:

$$P_+ = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0,1}, i = 1, \dots, \ell\}.$$

For $L(\Lambda)$ to be integrable, it is necessary that $\Lambda \in P_+$.

Indeed, suppose $L(\Lambda)$ is integrable. Then $f_i^N v_\Lambda = 0$ in $L(\Lambda)$, or rather

$$\underbrace{e_i f_i^{N+1} v_\Lambda}_{K f_i^N = 0} \in M(\Lambda),$$

and K can only be zero if $\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$. Applying for all i, we find $\Lambda \in P_+$ is necessary.

Proposition 5.8. $L(\Lambda)$ *is integrable if and only if* $\Lambda \in P_+$.

Proof. For the converse, use induction and Serre relations (we know the result for the highest weight, and want to prove for others). ☐

Example 5.9. $\mathfrak{sl}_3$. [Picture]

Example 5.10. $\widehat{\mathfrak{sl}_2}$. [Picture, P_+ looks like diagonal lines.]

$$\alpha_0^\vee = K - H, \qquad \alpha_1^\vee = H \in \mathfrak{sl}_2, \qquad \langle \delta, \alpha_i^\vee \rangle = 0, \quad i = 0, 1$$

Remark 5.11. For affine Kac-Moody algebras, almost nothing about the structure of $M(\Lambda)$ depends on the coefficient of δ in Λ. So it's common to consider

$$M(\Lambda) = M_k(\lambda) = M(k\Lambda_0 + \lambda), \qquad \lambda \in \mathfrak{h}^*$$

where k, the level of Λ, is super important.

Then

$$\underbrace{\hat{P}_+}_{\substack{\delta\text{-coef.}\\ =0}} = \bigcup_{k \in \mathbb{Z}_{\geq 0}} \{k\Lambda_0 + \lambda | \lambda \in P_+^k\}\{k\Lambda_0 + \lambda | \lambda \in P_+^k\}.$$

$$P_+^k = \{\lambda \in P_+ | \langle \lambda, \theta \rangle \leq k\} \subset P_+ \text{ for } \mathfrak{g}.$$

Consider $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ integrable, and $V_\lambda \neq 0$. Let $i \in \{1, \dots, \ell\}$. Consider $U = \bigoplus_{n \in \mathbb{Z}} V_{\lambda + n\alpha_i} \subset V$, and the action of e_i, f_i and $h_i = [e_i, f_i]$.

$\mathfrak{sl}_2 \curvearrowright U$, locally integrable. By structure of $\mathfrak{sl}_2$-represenations, U must be finite-dimensional with "symmetrical" weight space multiplicities, i.e.,

$$\{\lambda + n\alpha_i | n \in \mathbb{Z}\} \cap \{\text{weights of } V\} = \{\lambda + n\alpha_i | -p \leq n \leq q\}.$$

and

$$\langle \lambda - p\alpha_i, h_i \rangle = - \langle \lambda + q\alpha_i, h_i \rangle .$$

Consequently, the reflection $r_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ has the same multiplicity as λ.

[Picture]

Now consider $M(\lambda)$ and $L(\Lambda)$ for $\Lambda \in P_+$. Actually, for general Λ, $M(\Lambda)$, while not necessarilly irreducible, has an (Ω-)composition series* by irreducibles $L(\lambda)$.

$$M(\Lambda) = V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_n = 0,$$

such that V_i/V_{i+1} is $\simeq L(\lambda_i)$ (i.e. an irreducible highest weight module) for some $\lambda_i = \Lambda - \beta_i$, $\beta_i \in Q$. For Kac-Moody algebras we also consider the case that $(V_i/V_{i+1}) = 0$ for all $\mu \in \Omega + Q_+$, (which is the Ω-composition series).

[Picture.]

For an Ω-composition series, as above, we have

$$\mathrm{ch}_{M(\Lambda)} - \sum_{\lambda \geq \Omega} \underbrace{[M(\Lambda) : L(\lambda)]}_{\substack{\#\text{ of times } L(\lambda)\\ \text{appears in the compos. series}}} \mathrm{ch}_{L(\lambda)} \in \langle e^\mu : \mu \ngeq \Omega \rangle$$

Sending "$\Omega \to -\infty$", the identity

$$\mathrm{ch}_{M(\Lambda)} = \sum_{\lambda \leq \Lambda} [M(\Lambda) : L(\lambda)] \mathrm{ch}_{L(\lambda)}$$

makes sense.

Notation. $b_{\Lambda,\lambda} = [M(\Lambda) : L(\lambda)]$.

Remark 5.12. Recall the partial order on weights that $\lambda \leq \Lambda$ if $\Lambda - \lambda \in Q = \sum \mathbb{Z}_+ \alpha_i$. $b_{\Lambda,\lambda} = 1$ if $\lambda = \Lambda$ and $b_{\Lambda,\lambda} = 0$ if not $(\lambda \leq \Lambda)$.

If we choose a total order on $\mathfrak{h}^*$, compatible with $\leq$. Then $\{b_{\Lambda,\lambda}\}$ is a lower triangular matrix with 1 on the diagonal. We an define $\{m_{\Lambda,\lambda}\}$ the *inverse matrix*. It's again lower triangular, 1 on the diagonal, and all $m_{\Lambda,\lambda}$ are <u>integers</u> (maybe negative now). And we have

$$\mathrm{ch}_{L(\Lambda)} = \sum_{\lambda \leq \Lambda} m_{\Lambda,\lambda} \mathrm{ch}_{M(\lambda)}. \tag{5.12.1}$$

Example 5.13. $\mathfrak{sl}_2$. $M(3) \underset{L(3)}{\supset} M(-5) \underset{L(-5)}{\supset} 0$. Since $M(-5)$ is already irreducible. [Missing...]

We want to discover $m_{\Lambda,\lambda}$. In general massively difficult. For $\Lambda \in P_+$, $m_{\Lambda,\lambda}$ easy. Multiply Eq. 5.12.1 by R

$$R\mathrm{ch}_{L(\Lambda)} = \sum_{\lambda \geq \Lambda} m_{\Lambda,\lambda} e^{\lambda} . e^{\rho} R \mathrm{ch}_{L(\Lambda)} = \sum_{\lambda \leq \Lambda} m_{\Lambda,\lambda} e^{\lambda+\rho}.$$

What's ρ? It's $\rho \in \mathfrak{h}^*$ chosen so that $\langle \rho, \alpha_i^\vee \rangle = 1$, and it's called the *Weyl vector*.

Remark 5.14. For $\mathfrak{g}$ finite-dimensional, $\rho = \sum_{i=1}^{\ell} \omega_i$ necessarily. (And equals $\frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$.

For $\mathfrak{g}$ finite dimensional and the affine $\hat{\mathfrak{g}}$, $\hat{\rho} = h^\vee \Lambda_0 + \rho$ works.

For $w \in W = \langle r_i | i = 1, \ldots, \ell \rangle$, define $w(\mathrm{ch}_V) = \sum \dim V_\mu e^{W(\mu)}$. We saw $w(\mathrm{ch}_V) = \mathrm{ch}_V$ if V integrable. In particular $w(\mathrm{ch}_{L(\Lambda)} = \mathrm{ch}_{L(\Lambda)}$, $\Lambda \in P_+$.

Lemma 5.15. $m_{\Lambda,\lambda} = 0$ *unless* $\lambda + \rho = w(\Lambda + \rho)$ *for some* $w \in W$

Claim. $r_i(e^\rho R) = -e^\rho R$. So $w(e^\rho R) = \det(w) e^\rho R$ for all $w \in W$.

Proof.

$$R = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}.$$

Note that

(1) $\mathrm{mult}(r_i(\alpha)) = \mathrm{mult}(\alpha)$ for all $\alpha \in \Delta$. (Since $\mathfrak{g}$ is integrable!)
(2) $r_i(\Delta_+) = \{-\alpha_i\} \cup (\Delta_+ \setminus \{\alpha_i\}$.

Any $\alpha \in \Delta_+$ is of the form $\alpha = \sum_{i=1}^{\ell} k_i\alpha_i$. If $\alpha \neq \alpha_i$, some $k_{j_0} \neq 0$, $j_0 \neq i$ and

$$\begin{aligned} r_i(\alpha) &= \sum_j k_j\alpha_j \langle \alpha, \alpha_i^\vee \rangle \alpha_i \\ &= \sum k_j'\alpha_j \\ &= e^\rho(e^{-\alpha_i} - 1)\left(\prod_{\alpha\in\Delta_+\setminus\alpha_i}\right) \\ &= -e^\rho R. \end{aligned}$$

for $k_{j_0}' = k_{j_0} > 0$. Can't have a mixture of signs, so $r_i(\alpha) \in \Delta_\perp$.

So

$$r_i(e^\rho R) = \prod_{\alpha\in\Delta_+\setminus\alpha_i} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)} \cdot (1 - e^{t\alpha_i} \cdot e^{\rho-\alpha_i}$$

$$r_i = \rho - \langle \alpha_i^\vee, \rho \rangle \alpha_i = \rho - \alpha_i$$

Hence

$$\sum_{\lambda\leq\Lambda} m_{\Lambda,\lambda} e^{\lambda+\rho}$$

is W-skew-invariant. (Lemma 5.15.)

Hence

$$\sum_{\lambda\leq\Lambda} m_{\Lambda,\lambda} e^{\lambda+\rho} = \sum_{w\in W} \det(w) \cdot e^{w(\Lambda+\rho)}$$

In conclusion, the *Weyl character formula*

$$\mathrm{ch}_{L(\Lambda)} = \sum_{w\in W} \det(w) \cdot \frac{e^{w(\Lambda+\rho)-\rho}}{R}$$

□

Corollary (Weyl denominator formula).

$$e^\rho R = \sum_{w\in W} \det(w) e^{w(\rho)}.$$

Next time: Affine case, θ-functions. Modular forms (Poisson summation.)

References

[Kac90] V.G. Kac, *Infinite-dimensional lie algebras*, Progress in mathematics, Cambridge University Press, 1990.

[Kac10] Victor Kac, *Lecture notes of 18.745 – introduction to lie algebras (fall 2010)*, `https://math.mit.edu/classes/18.745/classnotes.html`, 2010, Lecture notes, MIT.