18.745 Introduction to Lie Algebras September 9, 2010

Lecture 1 — Basic Definitions (I)

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Definition 1.1. An algebra A is a vector space V over a field $\mathbb{F}$, endowed with a binary operation which is bilinear:

$$\begin{aligned} a(\lambda b + \mu c) &= \lambda ab + \mu ac \\ (\lambda b + \mu c)a &= \lambda ba + \mu ca \end{aligned}$$

Example 1.1. The set of $n \times n$ matrices with the matrix multiplication, $\mathrm{Mat}_n(\mathbb{F})$ is an associative algebra: $(ab)c = a(bc)$.

Example 1.2. Given a vector space V, the space of all endomorphisms of V, End V, with the composition of operators, is an associative algebra.

Definition 1.2. A subalgebra B of an algebra A is a subspace closed under multiplication: $\forall a, b \in B, ab \in B$.

Definition 1.3. A Lie algebra is an algebra with product $[a, b]$ (usually called bracket), satisfying the following two axioms:

1. (skew-commutativity) $[a, a] = 0$
2. (Jacobi identity) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

Example 1.3.

1. Take a vector space $\mathfrak{g}$ with bracket $[a, b] = 0$. This is called an abelian Lie algebra;
2. $\mathfrak{g} = \mathbb{R}^3$,$[a, b] = a \times b$ (cross product);
3. Let A be an associative algebra with product ab. Then the space A with the bracket $[a, b] = ab - ba$ is a Lie algebra, denoted by A_-.

Exercise 1.1. Show the Jacobi identity holds in Example 1.3.3 for the following cases.

1. 2-member identity: $a(bc) = (ab)c$
2. 3-member identity: $a(bc) + b(ca) + c(ab) = 0$ and $(ab)c + (bc)a + (ca)b = 0$
3. 4-member identity: $a(bc) - (ab)c - b(ac) + (ba)c = 0$
4. 6-member identity: $[a, bc] + [b, ca] + [c, ab] = 0$

Proof. Expanding the Jacobi identity,

$$\begin{aligned}
& [a,[b,c]]+[b,[c,a]]+[c,[a,b]] \\
= \; & a(bc)-a(cb)-(bc)a+(cb)a+b(ca)-b(ac)-(ca)b+(ac)b+c(ab)-c(ba)-(ab)c+(ba)c \\
= \; & \quad [a(bc)-(ab)c]+[(cb)a-c(ba)] \\
& +[b(ca)-(bc)a]+[(ac)b-a(cb)] \\
& +[c(ab)-(ca)b]+[(ba)c-b(ac)] \\
= \; & \quad [a(bc)+b(ca)+c(ab)] \\
& -[(ab)c+(bc)a+(ca)b] \\
& -[a(cb)+c(ba)+b(ac)] \\
& +[(ac)b+(cb)a+(ba)c] \\
= \; & \quad [a(bc)-(ab)c-b(ac)+(ba)c] \\
& +[b(ca)-(bc)a-c(ba)+(cb)a] \\
& +[c(ab)-(ca)b-a(cb)+(ac)b] \\
= \; & \quad [a(bc)-(bc)a+b(ca)-(ca)b+c(ab)-(ab)c] \\
& -[a(cb)-(cb)a+b(ac)-(ac)b+c(ba)-(ba)c] \\
= \; & 0 \text{ if one of the identities is satisfied.}
\end{aligned}$$

□

Example 1.4. A special case of example 1.3.3: $\mathfrak{gl}_V = (\text{End } V)_{\text{-}}$, where V is a vector space, is a Lie algebra, called the general Lie algebra. In the case $V = \mathbb{F}^n$, we denote $\mathfrak{gl}_V = \mathfrak{gl}_n(\mathbb{F})$, the set of all $n \times n$ matrices with the bracket $[a, b] = ab - ba$.

Remark: Any subalgebra of a Lie algebra is a Lie algebra.

Example 1.5. The two most important classes of subalgebras of $\mathfrak{gl}_V$:

1. $sl_n(\mathbb{F}) = \{a \in \mathfrak{gl}_n(\mathbb{F}) | \text{tr } (a) = 0\}$;
2. Let B be a bilinear form on a vector space V.
   $o_{V,B} = \{a \in \mathfrak{gl}_V | B(a(u), v) = -B(u, a(v)) \forall u, v \in V\}$.

Exercise 1.2. Show that $tr[a, b] = 0 \; \forall a, b \in \text{Mat}_n(\mathbb{F})$. In particular, sl_n is a Lie algebra, called the special Lie algebra.

Proof.

$$\text{tr } [a,b] = \sum_i \sum_j (a_{ji}b_{ij} - b_{ji}a_{ij}) = 0$$

sl_n is trivially a subspace by the linearity of the trace, and we have shown it to be closed under the bracket operation. Hence, sl_n is a subalgebra and is therefore a Lie algebra. □

Exercise 1.3. Show that $o_{V,B}$ is a subalgebra of the Lie algebra $\mathfrak{gl}_V$

Proof. Consider $a, b \in o_{V,B}$. Then

$$B(ab(u), v) = -B(b(u), a(v)) = B(u, ba(v)))$$

from the property of a and b. Similarly,

$$B(ba(u), v) = -B(a(u), b(v)) = B(u, ab(v)))$$

subtracting the second of these from the first,

$$B([a, b](u), v) = B(u, -[a, b](v)) = -B(u, [a, b](v))$$

by the bilinearity of B. Hence, $o_{V,B}$ is closed under the bracket. As $o_{V,B}$ is trivially a subspace of $\mathfrak{gl}_V$, it is also a subalgebra and therefore a Lie algebra. □

Exercise 1.4. Let $V = \mathbb{F}^n$ and let B be the matrix of a bilinear form in the standard basis of $\mathbb{F}^n$. Show that

$$o_{\mathbb{F}^n,B} = \left\{a \in \mathfrak{gl}_n(\mathbb{F}) | a^T B + Ba = 0\right\}$$

where a^T denotes the transpose of matrix a.

Proof. The condition for members of $o_{V,B}$,

$$B(a(u), v) + B(u, a(v)) = 0$$

reads, in terms of the standard basis, employing summation convention (a repeated index is summed over):

$$B(a_{ij}u_j\vec{e_i}, v_k\vec{e_k}) + B(u_j\vec{e_j}, a_{ik}v_k\vec{e_i}) = 0.$$

Hence, from the bilinearity of B,

$$a_{ij}B_{ik}u_jv_k + B_{ji}a_{ik}u_jv_k = 0.$$

This is true $\forall u_j, v_k$. Therefore,

$$(a^T)_{ji}B_{ik} + B_{ji}a_{ik} = 0.$$

□

Remark Special cases of $o_{\mathbb{F}^n,B}$ are the following:

1. $so_{n,B}(\mathbb{F})$ if B is a non-degenerate symmetric matrix; this is called the orthogonal Lie algebra.
2. $sp_{n,B}(\mathbb{F})$ if B is a non-degenerate skew-symmetric matrix; this is called the symplectic Lie algebra.

The three series of Lie algebras $sl_n(\mathbb{F})$, $so_{n,B}(\mathbb{F})$ and $sp_{n,B}(\mathbb{F})$ are the most important for this course's examples.

Convenient notation: If X,Y are subspaces of a Lie algebra $\mathfrak{g}$, then $[X, Y]$ denotes the span of all vectors $[x, y]$, where $x \in X$,$y \in Y$.

Definition 1.4. Let $\mathfrak{g}$ be a Lie algebra. In the above notation, a subspace $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called an ideal if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

Definition 1.5. A derived subalgebra of a Lie algebra $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}]$.

Proposition 1.1. $[\mathfrak{g}, \mathfrak{g}]$ *is an ideal of a Lie algebra* $\mathfrak{g}$

Proof. Let $a \in \mathfrak{g}, b \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. Then $[a, b] \in [\mathfrak{g}, \mathfrak{g}]$. $\square$

We now classify Lie algebras in 1 and 2 dimensions.

Dim 1. $\mathfrak{g} = \mathbb{F}a$, $[a, a] = 0$ so the Abelian Lie algebra is the only one.

Dim 2. Consider $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. Let $\mathfrak{g} = \mathbb{F}x + \mathbb{F}y$, then $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}[x, y]$. Therefore, $\dim[\mathfrak{g}, \mathfrak{g}] \leq 1$.

Case 1. $\dim[\mathfrak{g}, \mathfrak{g}] = 0$, Abelian Lie algebra.

Case 2. $\dim[\mathfrak{g}, \mathfrak{g}] = 1$, $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}b, b \neq 0$. Take $a \in \mathfrak{g} \backslash \mathbb{F}b$. Then $[a, b] \in [\mathfrak{g}, \mathfrak{g}]$, hence $[a, b] = \lambda b$ and $\lambda \neq 0$, otherwise $[\mathfrak{g}, \mathfrak{g}] = 0$. So, replacing a by $\lambda^{-1}a$, we get $[a, b] = b$. Hence, we have found a basis of $g : g = \mathbb{F}a + \mathbb{F}b$ with bracket $[a, b] = b$. So this Lie algebra is isomorphic to the subalgebra $\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \subset \mathfrak{gl}_2(\mathbb{F}) \right\}$, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $[a, b] = b$.

Exercise 1.5. Let $f : \mathrm{Mat}_n(\mathbb{F}) \to \mathbb{F}$ be a linear function such that $f([a, b]) = 0$. Show that $f(a) = \lambda \mathrm{tr}\ (a)$, for some λ independent of $a \in \mathrm{Mat}_n(\mathbb{F})$.

Proof. The condition $f([a, b]) = 0$ means

$$f(a_{ij}b_{jk}e_{ik} - b_{ij}a_{jk}e_{ik}) = 0.$$

By linearity of f,

$$(a_{ij}b_{jk} - b_{ij}a_{jk})f(e_{ik}) = 0 \ \forall a, b \in \mathrm{Mat}_n(\mathbb{F})$$

where summation convention has been used. Let $a = e_{mn}, b = e_{nn}$ for some $m \neq n$. Then $f(e_{mn}) = 0$. Hence

$$f([a, b]) = (a_{ij}b_{ji} - b_{ij}a_{ji})f(e_{ii}) = 0.$$

But $f(e_{ii}) = f(e_{jj}) \ \forall i, j$ as $f(e_{ii}) - f(e_{jj}) = f(e_{ij}e_{ji} - e_{ji}e_{ij}) = 0$. Hence $f(e_{ii}) = \lambda$ for some constant λ, and $f(a) = \mathrm{tr}\ (a)f(e_{ii}) = \lambda \mathrm{tr}\ (a)$ $\square$

18.745 Introduction to Lie Algebras September 14, 2010

Lecture 2 — Some Sources of Lie Algebras

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From Associative Algebras

We saw in the previous lecture that we can form a Lie algebra A_-, from an associative algebra A, with binary operation the commutator bracket $[a, b] = ab - ba$. We also saw that this construction works for algebras satisfying any one of a variety of other conditions.

As Algebras of Derivations

Lie algebras are often constructed as the algebra of derivations of a given algebra. This corresponds to the use of vector fields in geometry.

Definition 2.1. For any algebra A over a field $\mathbb{F}$, a derivation of A is an $\mathbb{F}$-vector space endomorphism D of A satisfying $D(ab) = D(a)b + aD(b)$. Let $\mathrm{Der}(A) \subset \mathfrak{gl}_A$ be denote the space of derivations of A.

For an element a of a Lie algebra $\mathfrak{g}$, define a map $\mathbf{ad}(a) : \mathfrak{g} \to \mathfrak{g}$, by $b \mapsto [a, b]$. This map is referred to as the adjoint operator. Rewriting the Jacobi identity as

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]], \tag{1}$$

we see that $\mathbf{ad}(a)$ is a derivation of $\mathfrak{g}$. Derivations of this form are referred to as inner derivations of $\mathfrak{g}$.

Proposition 2.1.

(a) $\mathrm{Der}(A)$ *is a subalgebra of* $\mathfrak{gl}_A$ *(with the usual commutator bracket).*

(b) The inner derivations of a Lie algebra $\mathfrak{g}$ *form an ideal of* $\mathrm{Der}(\mathfrak{g})$*. More precisely,*

$$[D, \mathbf{ad}(a)] = \mathbf{ad}(D(a)) \textit{ for all } D \in \mathrm{Der}(\mathfrak{g}) \textit{ and } a \in \mathfrak{g}. \tag{2}$$

Proof of (b): We check (2) by applying both sides to $b \in \mathfrak{g}$:

$$[D, \mathbf{ad}(a)]b = D[a, b] - [a, Db] = [Da, b] = \mathbf{ad}(Da)b,$$

where the second equality holds as D is a derivation. □

Exercise 2.1. Prove (a).

Solution: The derivations of A are those maps $D \in \mathfrak{gl}_A$ which satisfy $D(ab) - D(a)b - aD(b) = 0$ for all a and b in A. For fixed a and b, the left hand side of this equation is linear in D, so that the set of endomorphisms satisfying that single equation is a subspace. The set of dervations is the intersection over all a and b in A of these subspaces, which is a subspace.

We are only left to check that the bracket of two derivations is a derivation. For any $a, b \in A$ and $D_1, D_2 \in \mathrm{Der}(A)$ we calculate:

$$\begin{aligned}
[D_1, D_2](ab) &= D_1(D_2(a)b + aD_2(b)) - D_2(D_1(a)b + aD_1(b)) \\
&= D_1D_2(a)b + D_2(a)D_1(b) + D_1(a)D_2(b) + aD_1D_2(b) \\
&\quad - \{D_2D_1(a)b + D_1(a)D_2(b) + D_2(a)D_1(b) + aD_2D_1(b)\} \\
&= D_1D_2(a)b - D_2D_1(a)b + aD_1D_2(b) - aD_2D_1(b) \\
&= [D_1, D_2](a)b + a[D_1, D_2](b).
\end{aligned}$$

Thus the derivations are closed under the bracket, and so form a Lie subalgebra of $\mathfrak{gl}_A$. □

From Poisson Brackets

Exercise 2.2. Let $A = \mathbb{F}[x_1, \ldots, x_n]$, or let A be the ring of C^∞ functions on $x_1, \ldots, x_n$. Define a Poisson bracket on A by:

$$\{f, g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}, \text{ for fixed choices of } \{x_i, x_j\} \in A. \tag{3}$$

Show that this bracket satisfies the axioms of a Lie algebra if and only if $\{x_i, x_i\} = 0$, $\{x_i, x_j\} = -\{x_j, x_i\}$ and any triple x_i, x_j, x_k satisfy the Jacobi identity.

Solution: If the Poisson bracket defines a Lie algebra structure for some choice of values $\{x_i, x_j\}$, then in particular, the axioms of a Lie algebra must be satisfied for brackets of terms x_i. The interesting question is whether the converse holds. We suppose then that the $\{x_i, x_j\}$ are chosen so that $\{x_i, x_j\} = -\{x_j, x_i\}$, and so that the Jacobi identity is satisfied for triples x_i, x_j, x_k.

The bilinearity of the bracket follows from the linearity of differentiation, and the skew-symmetry follows from the assumption of the skew symmetry on the x_i.

At this point we introduce some shorthands to simplify what follows. If f is any function, we write f_i for the derivative of f with respect to x_i. When we are discussing an expression e in terms of three functions f, g, h, we will write $\mathrm{CS}(e)$ for the 'cyclic summation' of e, the expression formed by summing those obtained from e by permuting the f, g, h cyclically. In particular, the Jacobi identity will be $\mathrm{CS}(\{f, \{g, h\}\}) = 0$.

First we calculate the iterated bracket of monomials x_i:

$$\{x_i, \{x_j, x_k\}\} = \sum_l \{x_j, x_k\}_l \{x_i, x_l\} \text{ (an example of the shorthands described).}$$

Now the iterated bracket of any three polynomials (or functions) f, g and h is:

$$\{h, \{f, g\}\} = \sum_{i,j,k,l} \left[f_{il} g_j h_k + g_{jl} f_i h_k\right] \{x_i, x_j\}\{x_k, x_l\} + \sum_{i,j,k,l} f_i g_j h_k \{x_i, x_j\}_l \{x_k, x_l\}.$$

By the assumption that the Jacobi identity holds on the x_i, we have (for any i, j, k):

$$\sum_l \mathrm{CS}(f_i g_j h_k)\{x_i, x_j\}_l \{x_k, x_l\} = 0,$$

for cyclicly permuting the f, g, h corresponds to cyclicly permuting the i, j, k (in the opposite order). Thus we have:

$$\mathrm{CS}\left(\{h, \{f, g\}\}\right) = \sum_{i,j,k,l} \mathrm{CS}(f_{il} g_j h_k + g_{jl} f_i h_k)\{x_i, x_j\}\{x_k, x_l\}.$$

The remaining task can be viewed as finding the $\{x_\alpha, x_\beta\}\{x_\gamma, x_\delta\}$ coefficient in this expression, where we substitute all appearences of $\{x_\beta, x_\alpha\}$ for $-\{x_\alpha, x_\beta\}$, and so on. To do so, we tabulate all the appearances of terms which are multiples of $\{x_\alpha, x_\beta\}\{x_\gamma, x_\delta\}$. We may as well assume here that $\alpha < \beta$ and $\gamma < \delta$.

i	j	k	l	multiple of $\{x_\alpha, x_\beta\}\{x_\gamma, x_\delta\}$	i	j	k	l	multiple of $\{x_\alpha, x_\beta\}\{x_\gamma, x_\delta\}$
α	β	γ	δ	$+ f_{\alpha\delta} g_\beta h_\gamma + g_{\beta\delta} f_\alpha h_\gamma$	γ	δ	α	β	$+ f_{\gamma\beta} g_\delta h_\alpha + g_{\delta\beta} f_\gamma h_\alpha$
β	α	γ	δ	$- f_{\beta\delta} g_\alpha h_\gamma - g_{\alpha\delta} f_\beta h_\gamma$	δ	γ	α	β	$- f_{\delta\beta} g_\gamma h_\alpha - g_{\gamma\beta} f_\delta h_\alpha$
α	β	δ	γ	$- f_{\alpha\gamma} g_\beta h_\delta - g_{\beta\gamma} f_\alpha h_\delta$	γ	δ	β	α	$- f_{\gamma\alpha} g_\delta h_\beta - g_{\delta\alpha} f_\gamma h_\beta$
β	α	δ	γ	$+ f_{\beta\gamma} g_\alpha h_\delta + g_{\alpha\gamma} f_\beta h_\delta$	δ	γ	β	α	$+ f_{\delta\alpha} g_\gamma h_\beta + g_{\gamma\alpha} f_\delta h_\beta$
α	β	γ	δ	$+ g_{\alpha\delta} h_\beta f_\gamma + h_{\beta\delta} g_\alpha f_\gamma$	γ	δ	α	β	$+ g_{\gamma\beta} h_\delta f_\alpha + h_{\delta\beta} g_\gamma f_\alpha$
β	α	γ	δ	$- g_{\beta\delta} h_\alpha f_\gamma - h_{\alpha\delta} g_\beta f_\gamma$	δ	γ	α	β	$- g_{\delta\beta} h_\gamma f_\alpha - h_{\gamma\beta} g_\delta f_\alpha$
α	β	δ	γ	$- g_{\alpha\gamma} h_\beta f_\delta - h_{\beta\gamma} g_\alpha f_\delta$	γ	δ	β	α	$- g_{\gamma\alpha} h_\delta f_\beta - h_{\delta\alpha} g_\gamma f_\beta$
β	α	δ	γ	$+ g_{\beta\gamma} h_\alpha f_\delta + h_{\alpha\gamma} g_\beta f_\delta$	δ	γ	β	α	$+ g_{\delta\alpha} h_\gamma f_\beta + h_{\gamma\alpha} g_\delta f_\beta$
α	β	γ	δ	$+ h_{\alpha\delta} f_\beta g_\gamma + f_{\beta\delta} h_\alpha g_\gamma$	γ	δ	α	β	$+ h_{\gamma\beta} f_\delta g_\alpha + f_{\delta\beta} h_\gamma g_\alpha$
β	α	γ	δ	$- h_{\beta\delta} f_\alpha g_\gamma - f_{\alpha\delta} h_\beta g_\gamma$	δ	γ	α	β	$- h_{\delta\beta} f_\gamma g_\alpha - f_{\gamma\beta} h_\delta g_\alpha$
α	β	δ	γ	$- h_{\alpha\gamma} f_\beta g_\delta - f_{\beta\gamma} h_\alpha g_\delta$	γ	δ	β	α	$- h_{\gamma\alpha} f_\delta g_\beta - f_{\delta\alpha} h_\gamma g_\beta$
β	α	δ	γ	$+ h_{\beta\gamma} f_\alpha g_\delta + f_{\alpha\gamma} h_\beta g_\delta$	δ	γ	β	α	$+ h_{\delta\alpha} f_\gamma g_\beta + f_{\gamma\alpha} h_\delta g_\beta$

Of course, if $\alpha = \gamma$ and $\beta = \delta$, there is repetition, so that the right hand columns of this table must be ignored. Whether or not this is the case, the reader will be able to arrange the entries of this table into cancelling pairs. □

Example 2.1. Let $A = \mathbb{F}[p_1, \ldots, p_n, q_1, \ldots, q_n]$. Let $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\{p_i, q_j\} = -\{q_i, p_j\} = \delta_{i,j}$. Both conditions clearly hold, and explicitly:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}$$

is a Lie algebra bracket, important in classical mechanics.

Via Structure Constants

Given a basis $e_1, \ldots, e_n$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, the bracket is determined by the structure constants $c_{ij}^k \in \mathbb{F}$, defined by:

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

The structure constants must satisfy the obvious skew-symmetry condition ($c_{ii}^k = 0$ and $c_{ij}^k = -c_{ji}^k$), and a more complicated (quadratic) condition corresponding to the Jacobi identity.

Definition 2.2. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$, be two Lie algebras over $\mathbb{F}$ and $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ a linear map. We say that φ is a homomorphism if it preserves the bracket: $\varphi([a,b]) = [\varphi(a), \varphi(b)]$, and an isomorphism if it is bijective. If there exists an isomorphism φ, we say that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic, written $\mathfrak{g}_1 \cong \mathfrak{g}_2$.

Exercise 2.3. Let $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be homomorphism. Then:

(a) $\ker \varphi$ is an ideal of $\mathfrak{g}_1$.

(b) im φ is a subalgebra of $\mathfrak{g}_2$.

(c) im $\varphi \cong g_1/\ker \varphi$.

Solution: Let $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be homomorphism of Lie algebras. Then:

(a) The kernel $\ker \varphi$ is an subspace of $\mathfrak{g}_1$, as in particular φ is $\mathbb{F}$-linear. Furthermore, if $x \in \ker \varphi$ and $y \in \mathfrak{g}_1$, we have $\varphi([x,y]) = [\varphi(x), \varphi(y)] = [0, \varphi(y)] = 0$. Thus $[x,y] \in \ker \varphi$. This shows that $\ker \varphi$ is an ideal of $\mathfrak{g}$.

(b) The image im φ is a subspace, again as φ is $\mathbb{F}$-linear. Now for any $u, v \in$ im φ, we may write $u = \varphi(x)$ and $v = \varphi(y)$ for elements $x, y \in \mathfrak{g}_1$. Then $[u,v] = [\varphi(x), \varphi(y)] = \varphi([x,y]) \in$ im φ. Thus the image is a subalgebra.

(c) Consider the map $\psi : \mathfrak{g}_1/\ker \varphi \to$ im φ given by $x + \ker \varphi \mapsto \varphi(x)$. We must first see that ψ is well defined. If $x + \ker \varphi = x' + \ker \varphi$, then $x' - x \in \ker \varphi$, so that:

$$\varphi(x) = \varphi(x) + \varphi(x' - x) = \varphi(x) + \varphi(x') - \varphi(x) = \varphi(x').$$

Thus our definition of ψ does not depend on choice of representative, and ψ is well defined. It is trivial to see that ψ is a homomorphism. Now suppose that $x + \ker \varphi \in \ker \psi$. Then $\varphi(x) = 0$, so that $x \in \ker \varphi$, and $x + \ker \varphi = 0 + \ker \varphi$. Thus ψ is injective, and that ψ is surjective is obvious. Thus ψ is an isomorphism $\mathfrak{g}_1/\ker \varphi \to$ im φ. □

As the Lie Algebra of an Algebraic (or Lie) Group

Definition 2.3. An algebraic group G over a field $\mathbb{F}$ is a collection $\{P_\alpha\}_{\alpha \in I}$ of polynomials on the space of matrices $\mathrm{Mat}_n(\mathbb{F})$ such that for any unital commutative associative algebra A over $\mathbb{F}$, the set

$$G(A) := \{g \in \mathrm{Mat}_n(A) \mid g \text{ is invertible, and } P_\alpha(g) = 0 \text{ for all } \alpha \in I\}$$

is a group under matrix multiplication.

Example 2.2. The general linear group GL_n. Let $\{P_\alpha\} = \emptyset$, so that $\mathrm{GL}_n(A)$ is the set of invertible matrices with entries in A. This is a group for any A, so that GL_n is an algebraic group.

Example 2.3. The special linear group SL_n. Let $\{P_\alpha\} = \{\det(x_{ij}) - 1\}$, so that $\mathrm{SL}_n(A)$ is the set of invertible matrices with entries in A and determinant 1. This is a group for any A, so that SL_n is an algebraic group.

Exercise 2.4. Given $B \in \mathrm{Mat}_n(\mathbb{F})$, let $O_{n,B}(A) = \{g \in \mathrm{GL}_n(A) : g^{\mathrm{T}} B g = B\}$. Show that this family of groups is given by an algebraic group.

Solution: For any unital commutative associative algebra A over $\mathbb{F}$, the set

$$O_{n,B}(A) = \{g \in \mathrm{GL}_n(A) : g^{\mathrm{T}} B g = B\}$$

is a group under matrix multiplication, as if $g, h \in O_{n,B}(A)$ we have:

$$(gh)^{\mathrm{T}} B(gh) = h^{\mathrm{T}} g^{\mathrm{T}} B g h = h^{\mathrm{T}} B h = B, \text{ and } \left(g^{-1}\right)^{\mathrm{T}} B g^{-1} = \left(g^{\mathrm{T}}\right)^{-1} \left(g^{\mathrm{T}} B g\right) g^{-1} = B.$$

We then only have to show that the condition $g^{\mathrm{T}} B g = B$ can be written as a collection of polynomial equations in the entries g_{ij} of the matrix g, with coefficients in $\mathbb{F}$. This is obvious — we have one polynomial equation for each of the n^2 entries of the matrix, and the coefficients depend only on the entries of B. □

Definition 2.4. Over a given field $\mathbb{F}$, define the algebra of dual numbers D to be

$$D := \mathbb{F}[\epsilon]/(\epsilon^2) = \{a + b\epsilon \mid a, b \in \mathbb{F},\ \epsilon^2 = 0\}.$$

We then define the Lie algebra **Lie** G of an algebraic group G to be

$$\mathbf{Lie}\ G := \{X \in \mathfrak{gl}_n(\mathbb{F}) \mid I_n + \epsilon X \in G(D)\}.$$

Example 2.4. (1) **Lie** $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{F})$, since $(I_n + \epsilon X)^{-1} = I_n - \epsilon X$. ($I_n - \epsilon X$ approximates the inverse to order two, but over dual numbers, order two is ignored).

(2) **Lie** $\mathrm{SL}_n = \mathfrak{sl}_n(\mathbb{F})$.

(3) **Lie** $O_{n,B} = o_{\mathbb{F}^n,B}$.

Exercise 2.5. Prove (2) and (3) from example 2.4.

Solution: For (2), We need only prove the formula $\det(I_n + \epsilon X) = 1 + \epsilon\,\mathrm{tr}\,(X)$. It is trivial when $n = 1$, and we proceed by induction on n. Consider the matrix $I_n + \epsilon X$, and the cofactor expansion of the determinant along the final column. If $i < n$, the (i, n) entry is a multiple of ϵ, and so is every entry of the i^{th} column of the matrix obtained by removing the row and column containing (i, n). Thus the corresponding cofactor has no contribution to the overall determinant. The determinant is therefore $1 + \epsilon X_{nn}$ multiplied by the minor corresponding to (n, n). By induction, the determinant is $(1 + \epsilon X_{nn})(1 + \epsilon(\mathrm{tr}\ (X) - X_{nn}))$. The result follows.

For (3), the following calculation gives the result:

$$(1 + \epsilon X)^{\mathrm{T}} B(1 + \epsilon X) = B + \epsilon X^{\mathrm{T}} B + \epsilon X^{\mathrm{T}} B + \epsilon^2 X^{\mathrm{T}} B X = B + \epsilon(X^{\mathrm{T}} B + X^{\mathrm{T}} B).$$ □

Theorem 2.2. ***Lie*** *G is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$.*

Proof. We first show that **Lie** G is a subspace. Indeed, $X \in$ **Lie** G iff $P_\alpha(I_n + \epsilon X) = 0$ for all α. Using the Taylor expansion:

$$P_\alpha(I_n + \epsilon X) = P_\alpha(I_n) + \sum_{i,j} \frac{\partial P_\alpha}{\partial x_{ij}}(I_n)\, \epsilon\, x_{ij},$$

as $\epsilon^2 = 0$. Now as $P_\alpha(I_n) = 0$ (every group contains the identity), this condition is linear in the x_{ij}, so that **Lie** G is a subspace.

Now suppose that $X, Y \in \textbf{Lie } G$. We wish to prove that $XY - YX \in \textbf{Lie } G$. We have:

$$I_n + \epsilon X \in G(\mathbb{F}[\epsilon]/(\epsilon^2)), \text{ and } I_n + \epsilon' Y \in G(\mathbb{F}[\epsilon']/((\epsilon')^2)).$$

Viewing these as elements of $G(\mathbb{F}[\epsilon, \epsilon']/(\epsilon^2, (\epsilon')^2))$, we have

$$(I_n + \epsilon X)(I_n + \epsilon' Y)(I_n + \epsilon X)^{-1}(I_n + \epsilon' Y)^{-1} = I_n + \epsilon\epsilon'(XY - YX) \in G(\mathbb{F}[\epsilon, \epsilon']/(\epsilon^2, (\epsilon')^2)).$$

Hence $I_n + \epsilon\epsilon'(XY - YX) \in G(\mathbb{F}[\epsilon\epsilon']/((\epsilon\epsilon')^2)) = G(D)$, so that $XY - YX \in \textbf{Lie } G$. $\square$

18.745 Introduction to Lie Algebras	September 16, 2010

Lecture 3 — Engel's Theorem

Prof. Victor Kac — *Scribe: Emily Berger*

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and V a vector space over $\mathbb{F}$. A *representation* of $\mathfrak{g}$ in V is a homomorphism $\pi : \mathfrak{g} \to gl_V$. In other words, it is a linear map $a \to \pi(a)$ from $\mathfrak{g}$ to the space of linear operators on V such that $\pi([a,b]) = \pi(a)\pi(b) - \pi(b)\pi(a)$.

Example 3.1. *Trivial representation* of $\mathfrak{g}$ in V where $\pi(a) = 0$ for all a.

Example 3.2. *Adjoint representation* of $\mathfrak{g}$ in $\mathfrak{g}$: $a \to \mathbf{ad}a$.

Let's check that it is in fact a representation. We must show that

$$\mathbf{ad}[a,b] = (\mathbf{ad}a)(\mathbf{ad}b) - (\mathbf{ad}b)(\mathbf{ad}a).$$

Applying both sides to $c \in \mathfrak{g}$, we check

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]].$$

By skew-symmetry, this is just the Jacobi Identity.

Definition 3.2. The *center* of a Lie algebra $\mathfrak{g}$ is denoted $Z(\mathfrak{g}) = \{a \in \mathfrak{g} | [a,\mathfrak{g}] = 0\}$.

Clearly, $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

Exercise 3.1. Show $Z(gl_n(\mathbb{F})) = \mathbb{F}I_n$, $Z(sl_n(\mathbb{F})) = 0$ if *char* $\mathbb{F} \nmid n$.

Proof. This is clear when $n = 1$ so assume otherwise. Suppose $A = (a_{ij}) \in Z(gl_n(\mathbb{F}))$ and there exists $a_{ij} \neq 0$ with $i \neq j$. Let $B = (b_{ij})$ and consider $AB = C = (c_{ij})$ and $BA = C' = (c'_{ij})$. We wish to show $C \neq C'$ for some B. We have

$$c_{i1} = \sum_{k=1}^{n} a_{ik} b_{k1}$$

and

$$c'_{i1} = \sum_{k=1}^{n} b_{ik} a_{k1}.$$

Define B by $b_{j1} = 1$ and $b_{k1} = 0$ for all $k \neq j$. We then have $c_{i1} = a_{ij} \neq 0$ and that the i^{th} row of B only has the restriction $b_{i1} = 0$. We choose the remaining entries of the row so that $c'_{i1} \neq c_{i1} = a_{ij}$. Therefore $a_{ij} = 0$ for all $i \neq j$. We observe that these restrictions still allow B to have trace zero and determinant non-zero.

Now suppose without loss of generality that $a_{11} \neq a_{22}$. Consider

$$c_{12} = \sum_{k=1}^{n} a_{1k} b_{k2}$$

and let $b_{12} = 1$ and $b_{i2} = 0$ for all $i \neq 1$. These restrictions on B still allow $B \in sl_n(\mathbb{F}) \subset gl_n(\mathbb{F})$. Clearly $\mathbb{F}I_n \subset Z(gl_n(\mathbb{F}))$, so by above $\mathbb{F}I_n = Z(gl_n(\mathbb{F}))$. As well, since we allowed $B \in sl_n(\mathbb{F})$ and $tr(A) = na_{11}$, if *char* $\mathbb{F} \nmid n$, then $Z(sl_n(\mathbb{F})) = 0$. $\square$

Proposition 3.1. *The adjoint representation defines an embedding of $\mathfrak{g}/Z(\mathfrak{g})$ in $gl_{\mathfrak{g}}$.*

Proof. $\mathbf{ad} : \mathfrak{g} \to gl_{\mathfrak{g}}$ is a homomorphism; Ker $\mathbf{ad} = Z(\mathfrak{g})$. Hence $\mathbf{ad}$ induces an embedding $\mathfrak{g}/Z(\mathfrak{g}) \to gl_n$ since $\mathfrak{g}/$ Ker $\varphi \cong$ Im φ. □

Theorem 3.2. *Ado's Theorem*

Any finite dimensional Lie algebra embeds in $gl_n(\mathbb{F})$ for some n. [Presented without proof.]

Remark 1. Proposition 3.1 proves Ado's Theorem in the case $Z(\mathfrak{g}) = 0$.

Exercise 3.2. Let dim $\mathfrak{g} < \infty$. Show dim $Z(\mathfrak{g}) \neq$ dim $\mathfrak{g} - 1$.

Proof. Suppose $\dim Z(\mathfrak{g}) = \dim \mathfrak{g} - 1$ and pick any non-zero $x \in \mathfrak{g} \setminus Z(\mathfrak{g})$. Clearly, x commutes with $Z(\mathfrak{g})$ and with αx which implies $x \in Z(\mathfrak{g})$. This is a contradiction and therefore $\dim Z(\mathfrak{g}) \neq \dim \mathfrak{g} - 1$. □

Definition 3.3. We define $Heis_{2n+1}$ to be the Lie algebra with basis $\{p_i, q_i, c\}$ where $[p_i, q_i] = c = -[q_i, p_i], 1 \leq i \leq n$, and all other bracketed pairs are 0.

Exercise 3.3. Classify all finite dimensional Lie algebras for which dim $Z(\mathfrak{g}) =$ dim $\mathfrak{g} - 2$. Let $\dim \mathfrak{g} = n$ and show either $\mathfrak{g} \cong Ab_{n-3} \oplus$ Heis$_3$ or $\mathfrak{g} \cong Ab_{n-2} \oplus \mathfrak{h}$ where $\mathfrak{h}$ is the two-dimensional non-abelian Lie algebra.

Proof. Suppose $\dim Z(\mathfrak{g}) = \dim \mathfrak{g} - 2$, then $\mathfrak{g}/Z(\mathfrak{g})$ may be generated by two elements, and consider their preimages, say p and q. Let $[p, q] = c \neq 0$ (else $p, q \in Z(\mathfrak{g})$. Suppose $c \in Z(\mathfrak{g})$, then in this case $\mathfrak{g} \cong Ab_{n-2} \oplus Heis_3$. Assume otherwise, that $c = z + a_p p + a_q q$ (without loss of generality, assume $a_p \neq 0$). We have $[c, q] = a_p[p, q] = a_p c$. Let $q' = \frac{q}{a_p}$, then c, q' are linearly independent and $[c, q'] = c$ which shows $\mathfrak{g} \cong Ab_{n-2} \oplus$ two-dimensional non-abelian Lie algebra. □

Constructions of representations from given ones.

Definition 3.4. Representation from direct sum

Given representations π_1, π_2 of $\mathfrak{g}$ in V_i. We have $\pi_1 \oplus \pi_2$ of $\mathfrak{g}$ in $V_1 \oplus V_2 : (\pi_1 \oplus \pi_2)(a) = \pi_1(a) \oplus \pi_2(a)$.

Definition 3.5. Subrepresentation and factor representation

Given a representation of π of $\mathfrak{g}$ in V, if $U \subset V$ is a subspace invariant with respect to all operators $\pi(a), a \in \mathfrak{g}$, we have the subrepresentation π_U of $\mathfrak{g}$ in $U : a \to \pi(a)|_U$.

Moreover, the factor representation $\pi_{V/U}$ of $\mathfrak{g}$ in $V/U : a \to \pi(a)|_{V/U}$.

Definition 3.6. A linear operator A in a vector space V is called nilpotent if $A^N = 0$ for some positive integer N.

Exercise 3.4. Show if dim $V < \infty$, then A is nilpotent if and only if all eigenvalues of A are zero.

Proof. If λ is an eigenvalue of A, then λ^N is an eigenvalue of $A^N = 0$, and therefore $\lambda = 0$. Conversely, suppose all eigenvalues of A are zero, then the characteristic polynomial of A is t^n, and by Cayley-Hamilton $A^n = 0$. □

Lemma 3.3. *Let A be a nilpotent operator in a vector space V, then*

(a) *There exists a non-zero $v \in V$ such that $Av = 0$.*

(b) ***ad*** *A is a nilpotent operator on gl_V.*

Proof. (a) Consider minimal $N > 0$ such that $A^N = 0$, then $A^{N-1} \neq 0$. Choose a non-zero vector $v \in A^{N-1}V \neq 0$. Then $Av = 0$.

Remark 2. ad $A = L_A - R_A$

$L_A(B) = AB$, $R_A(B) = BA$

$L_A R_B = R_B L_A$, due to the associativity of the product of operators. Hence

$$(\mathbf{ad}\ A)^M = \sum_{j=0}^{M} \binom{M}{j} L_A^j R_A^{M-j}.$$

(b) Now we have

$$\mathbf{ad}^{2N} B = \sum_{j=0}^{2N} \binom{2N}{j} A^j B A^{2N-j} = 0$$

since either $j \geq N$ or $2N - j \geq N$.

$\square$

Theorem 3.4. *Engel's Theorem Let V be a non-zero vector space and let $\mathfrak{g} \in gl_V$ be a finite dimensional subalgebra which consists of nilpotent operators. Then there exists a non-zero vector $v \in V$ such that $Av = 0$ for all $A \in \mathfrak{g}$.*

Proof. By induction on $\dim \mathfrak{g}$.

If $\dim \mathfrak{g} = 1$, then $\mathfrak{g} = \mathbb{F}a$ for $a \in gl_V$. By Lemma 3.3(a), Engel's Theorem holds.

We may assume $\dim \mathfrak{g} \geq 2$ and let $\mathfrak{h}$ be a maximal proper subalgebra of $\mathfrak{g}$. Since $\mathbb{F}a$ is always a subalgebra, we have that $\dim \mathfrak{h} \geq 1$. Consider the adjoint representation of $\mathfrak{g}$ (on itself) and consider its restriction to $\mathfrak{h}$, so we have $\mathbf{ad} : \mathfrak{h} \to gl_{\mathfrak{g}}$ is an invariant subspace for the representation (since $\mathfrak{h}$ is a subalgebra). Therefore, we may consider the factor representation in $\mathfrak{g}/\mathfrak{h}$. Then $\pi(\mathfrak{h}) \subset gl_{\mathfrak{g}/\mathfrak{h}}$ and $\dim \pi(\mathfrak{h}) \leq \dim \mathfrak{h} \leq \dim \mathfrak{g}$. Moreover, $\pi(\mathfrak{h})$ consists of nilpotent operators by Lemma 3.3(b).

We may apply the inductive assumption. We have there exists $\bar{a} \in \mathfrak{g}/\mathfrak{h}$, a non-zero vector such that $\pi(h)\bar{a} = 0$ for all $h \in \mathfrak{h}$. If $a \in \mathfrak{g}$ is an arbitrary preimage of $\bar{a}$, we get that $[\mathfrak{h}, a] \subset \mathfrak{h}$ and since $\bar{a} \neq 0$, $a \notin \mathfrak{h}$. Hence, $\mathfrak{h} \oplus \mathbb{F}a$ is a subalgebra of $\mathfrak{g}$. This subalgebra is larger than $\mathfrak{h}$, but $\mathfrak{h}$ was a maximal proper subalgebra of $\mathfrak{g}$, which implies $\mathfrak{h} \oplus \mathbb{F}a = \mathfrak{g}$.

By inductive assumption, there exists non-zero $v \in V$ such that $Av = 0$ for all $A \in \mathfrak{h}$. Let V_0 denote the space of all vectors $v \in V$ satisfying $Av = 0$. We claim that $aV_0 \subset V_0$; indeed: $V_0 = \{v \mid \mathfrak{h}v = 0\}$. So if $v \in V_0$, then we have $h(av) = [h, a] + ah(v) = 0 + 0 = 0$. By Lemma 3.3($a$) there exists a non-zero vector $v \in V_0$ "killed" by a. Therefore v is "killed" by $\mathfrak{h}$ and a, and hence $\mathfrak{g}$. $\square$

Remark 3. If we assume $\dim V < \infty$, then $\mathfrak{g}$ is finite dimensional since $\dim \mathfrak{g} \leq (\dim V)^2 < \infty$. Therefore Engel's Theorem holds if we only assume $\dim V < \infty$.

Corollary 3.5. *Let $\pi : \mathfrak{g} \to gl_V$ be a representation of a Lie algebra $\mathfrak{g}$ in a finite dimensional vector space V such that $\pi(a)$ is a nilpotent operator for all $a \in \mathfrak{g}$. Then there exists a basis of V in which all operators $\pi(a)$, $a \in \mathfrak{g}$ are strictly upper triangular matrices.*

Proof. Induction on $\dim V$.

By Engel's Theorem, there exists a non-zero vector e_1, such that $\pi(e_1) = 0$ for all $a \in \mathfrak{g}$. Since $\mathbb{F}e_1$ is an invariant subspace, we consider the factor representation of $\mathfrak{g}$ in $V/\mathbb{F}e_1$. Apply the inductive assumption to get the basis $\bar{e_2}, ..., \bar{e_n}$ of $V/\mathbb{F}e_1$, in which all matrices of $\pi|_{V/\mathbb{F}e_1}$ are strictly upper triangular. Take $e_2, ..., e_n$ preimages of $\bar{e_2}, ..., \bar{e_n}$. Then in the basis $e_1, ..., e_2$ of V, all matrices $\pi(\mathfrak{g})$ are strictly upper triangular. □

Exercise 3.5. Construct in $sl_3(\mathbb{F})$ a two-dimensional subspace consisting of nilpotent matrices, which do not have a common eigenvector.

Proof. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then the subspace generated by A and B is two dimensional. A has eigenspace generated by $(1, 0, 0)^t$ and B has eigenspace generated by $(0, 0, 1)^t$. Therefore A and B have no common eigenvectors. Any linear combination of A, B say $\alpha A + \beta B$ has characteristic polynomial $-\lambda^3$ and therefore is nilpotent.

□

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Lecture 4 — Nilpotent and Solvable Lie Algebras

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4.1 Preliminary Definitions and Examples

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The *lower central series* of $\mathfrak{g}$ is the descending chain of subspaces

$$\mathfrak{g}^1 = \mathfrak{g} \supseteq \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] \supseteq \mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2] \supseteq \ldots \supseteq \mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}] \supseteq \ldots$$

while the *derived series* is

$$\mathfrak{g}^{(0)} = \mathfrak{g} \supseteq \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \supseteq \ldots \supseteq \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \supseteq \ldots$$

We note that

(1) $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$ for $n \geq 1$ by induction

(2) All $\mathfrak{g}^n$ and $\mathfrak{g}^{(n)}$ are ideals in $\mathfrak{g}$

Definition 4.2. A lie algebra $\mathfrak{g}$ is called *nilpotent* (resp. *solvable*) if $\mathfrak{g}^n = 0$ for some $n > 0$ (resp. $\mathfrak{g}^{(n)} = 0$ for some $n > 0$).

If $\mathfrak{g}$ is nilpotent then $\mathfrak{g}$ is solvable. In fact

$$\{\text{abelian}\} \subsetneq \{\text{nilpotent}\} \subsetneq \{\text{solvable}\}$$

Example 4.1. Let $\mathfrak{g} = \mathbb{F}a + \mathbb{F}b$ with $[a, b] = b$, $\mathfrak{g}^{(1)} = \mathfrak{g}^2 = \mathbb{F}b$, $\mathfrak{g}^3 = \mathfrak{g}^4 = \ldots = \mathbb{F}b$ but $\mathfrak{g}^{(2)} = 0$ so $\mathfrak{g}$ is solvable but not nilpotent.

Example 4.2. Let $H_3 = \mathbb{F}p + \mathbb{F}q + \mathbb{F}c$ with $[c, \mathfrak{g}] = 0$ and $[p, q] = c$. Then $H_3^2 = \mathbb{F}c, H_3^3 = 0$.

Example 4.3.

$$\begin{aligned} gl_n(\mathbb{F}) &\supseteq b_n = \{\text{upper triangular matrices}\} \\ &\supseteq \eta_n = \{\text{strictly upper triangular matrices}\} \end{aligned}$$

Exercise 4.1. Show b_n is a solvable (but not nilpotent) Lie algebra and that $[b_n, b_n] = \eta_n (n \geq 2)$. Also show that η_n is a nilpotent Lie algebra.

Proof. Consider $C = AB - BA$ for $A, B \in b_n$. Say $A = (a_{ij}), B = (b_{ij})$, and $C = (c_{ij})$. Then

$$c_{ij} = \sum_{k=1}^{n} (a_{ik} b_{kj} - b_{ik} a_{kj})$$

We notice $a_{ik} = b_{ik} = 0$ if $k < i$ and $b_{kj} = a_{kj} = 0$ if $k > j$. Thus

$$c_{ij} = \sum_{k=i}^{j} (a_{ik} b_{kj} - b_{ik} b_{kj})$$

Then if $i = j$, $c_{ii} = a_{ii} b_{ii} - b_{ii} a_{ii} = 0$, implying $C \in \eta_n$, i.e., $[b_n, b_n] \in \eta_n$. Define the kth diagonal to be the set of c_{ij} where $j - i = k$. We show that the kth diagonal contains no nonzero entries in b_n^{k+1}. We have shown this to be true for $k = 0$. Now assume it is true up to $k = m$ for some $m \geq 0$. Then any $C = [A, B] \in b_n^{m+1}$ is such that $A, B \in b_n^m$ and thus

$$c_{i,i+m} = \sum_{k=1}^{i+m} (a_{ik} b_{k,i+m} - b_{ik} a_{k,i+m})$$

Now $a_{jk} \neq 0$, $b_{jk} \neq 0$ implies $k = i + m$ while $a_{k,i+m} \neq 0$, $b_{k,i+m} \neq 0$ implies $k = i$ but $m \neq 0$ implies $c_{i,i+m} = 0$ which is equivalent to saying all diagonals are zero so b_n is solvable. Next we show $[b_n, b_n] \supseteq \eta_n$ so that we finally know $[b_n, b_n] = \eta_n$. Consider the basis element $e_{ij} (j > i)$ which is defined to have entry (i, j) equal to 1, and all other entires 0. Then $[e_{ij}, e_{jj}] = e_{ij} e_{jj} - e_{jj} e_{ij} = e_{ij} e_{jj} = e_{ij} \Rightarrow e_{ij} \in [b_n, b_n]\ \forall e_{ij}$ such that $j > i \Rightarrow [b_n, b_n] \supseteq \eta_n$ This shows that b_n is not nilpotent. $\square$

4.2 Simple Facts about Nilpotent and Solvable Lie Algebras

First we note

1. Any subalgebra of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable).
2. Any factor algebra of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable)

Exercise 4.2. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be an ideal. Show that if $\mathfrak{h}$ is solvable and $\mathfrak{g}/\mathfrak{h}$ is solvable, then $\mathfrak{g}$ is solvable too.

Proof. First we prove that all the homomorphic images of a solvable algebra are solvable. Let $\mathfrak{g}_1$ be solvable and $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ a surjective homomorphism. We show

$$\phi(g_1^{(i)}) = \mathfrak{g}_2^{(i)}$$

The case $i = 0$ is trivial. Suppose it holds for some $i \geq 0$. Then

$$\begin{aligned} \phi(\mathfrak{g}_1^{(i+1)}) &\supseteq \phi([\mathfrak{g}_1^{(i)}, \mathfrak{g}_1^{(i)}]) \\ &= [\phi(\mathfrak{g}_1^{(i)}), \phi(\mathfrak{g}_1^{(i)})] \\ &= [\mathfrak{g}_2^{(i)}, \mathfrak{g}_2^{(i)}] \\ &= \mathfrak{g}_2^{(i+1)} \end{aligned}$$

Thus if $\mathfrak{g}_1$ is solvable, so is $\mathfrak{g}_2$. Now suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a solvable ideal, say $\mathfrak{h}^{(n)} = 0$, and $(\mathfrak{g}/\mathfrak{h})^{(m)} = 0$. Consider the canonical homomorphism

$\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ From the previous result,

$$\pi(\mathfrak{g}^{(m)}) = (\mathfrak{g}/\mathfrak{h})^{(m)} = 0 \Rightarrow \mathfrak{g}^{(m)} \subseteq I$$

Then $(\mathfrak{g}^{(m)})^{(n)} = \mathfrak{g}^{(m+n)} \subseteq I^{(n)} = 0$ which means that $\mathfrak{g}$ is solvable. $\square$

The last exercise does not hold if we everywhere put "nilpotent" in place of "solvable," as the following example shows.

Example 4.4. Suppose g= Fa+Fb, $[a, b] = b$. $\mathbb{F}b \subset \mathfrak{g}$ is an ideal, $\mathbb{F}b$ and $\mathfrak{g}/\mathbb{F}b$ are 1-dimensional and hence abelian and nilpotent. But $\mathfrak{g}$ is not nilpotent.

Theorem 4.1. *(a) If $\mathfrak{g}$ is a nonzero nilpotent Lie algebra then $Z(\mathfrak{g})$ is nonzero*

(b) If $\mathfrak{g}$ is a finite-dimensional Lie algebra such that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is nilpotent.

Proof. (a) Take $N > 0$ minimal such that $\mathfrak{g}^N = 0$. Since $\mathfrak{g} \neq 0$, $N \geq 2$, but then $\mathfrak{g}^{N-1} \neq 0$ and $[\mathfrak{g}, \mathfrak{g}^{N-1}] = \mathfrak{g}^N = 0$, so $\mathfrak{g}^{N-1} \subset Z(\mathfrak{g})$.

(b) $\bar{\mathfrak{g}} = \mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, i.e., $\bar{\mathfrak{g}}^n = 0$ for some n which implies $\mathfrak{g}^n \subset Z(\mathfrak{g})$, but then $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subset [\mathfrak{g}, Z(\mathfrak{g})] = 0$.

$\square$

4.3 Engel's Characterization of Nilpotent Lie Algebras

Theorem 4.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent iff for each $a \in \mathfrak{g}$, $(ad\ a)^n = 0$ for some $n > 0$. One may always take $n = dim\mathfrak{g}$.*

proof If $\mathfrak{g}$ is nilpotent then $\mathfrak{g}^{n+1} = 0$ for some n. In particular, $(\text{ad a})^n b = 0$ for all $a, b \in \mathfrak{g}$ sinc this is a length $(n+1)$ commutator. For the converse: The adjoint representation gives an injective homomorphism $\mathfrak{g}/Z(\mathfrak{g}) \hookrightarrow gl_{\mathfrak{g}}$ and by assumption the image consists of nilpotent operators. So by Engel's Theorem (from last lecture), $\mathfrak{g}/Z(\mathfrak{g})$ consists of strictly upper triangular matrices in the same basis. Therefore $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent and hence $\mathfrak{g}$ is nilpotent as well.

4.4 How to Classify 2-Step Nilpotent Lie Algebras

Let $\mathfrak{g}$ be n-dimensional and nilpotent with $Z(\mathfrak{g}) \neq 0$ so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent of dimension $n_1 < n$.

Definition 4.3.

- $\mathfrak{g}$ is *1-step nilpotent* if it is abelian
- $\mathfrak{g}$ is *2-step nilpotent* if $\mathfrak{g}/Z(\mathfrak{g})$ is abelian
- $\mathfrak{g}$ is *k-step nilpotent* if $\mathfrak{g}/Z(\mathfrak{g})$ is $(k-1)$-step nilpotent

Let $\mathfrak{g}$ be 2-step nilpotent so $V = \mathfrak{g}/Z(\mathfrak{g})$ is abelian. Consider the bilinear form

$$\begin{aligned} \text{B: V} \times V &\to Z(\mathfrak{g}) \\ \text{(a,b)} &\mapsto [\tilde{a}, \tilde{b}] \end{aligned}$$

where $\tilde{a}$ and $\tilde{b}$ are preimages of a, b under $\mathfrak{g} \to V$ (B is an *alternating form*, i.e., $B(x,x) = 0$ for all x).

Exercise 4.3. Show that 2-step nilpotent Lie algebras are classified by such nondegenerate alternating bilinear forms.

Proof. Suppose $\mathfrak{g}$ is a 2-step Lie algebra so $\mathfrak{g}/Z(\mathfrak{g})$ is abelian. Let $W = Z(\mathfrak{g})$ and $V \cong \mathfrak{g}/Z(\mathfrak{g})$. We check that the form $\phi : (v_1, v_2) \to [v_1, v_2]$ is nondegenerate and alternating. It is clearly alternating by the definition of the bracket, and nondegenerate since if $[v, v] = 0$ then $v \in Z(\mathfrak{g}) \Rightarrow v = 0$. For the other direction, given a triple (v, w, ϕ) such that $\phi : v \times v \to w$, consider $\mathfrak{g} = v \oplus w$. Then 2-step nilpotent Lie algebras with bracket $[v+w, v^{'}+w^{'}] = \phi(v, v^{'})$. This is the case because the bracket is alternating by the definition of ϕ, the bracket satisfies the Jacobi identity (see the next paragraph), and since the bracket is nondegenerate, $V \cong \mathfrak{g}/Z(\mathfrak{g})$. To check that the bracket satisfies the Jacobi identity, check that

$$
\begin{aligned}
[v_1, [v_2, v_3]] + [v_2, [v_3, v_1] + [v_3, [v_1, v_2]] &= \phi(v_1, \phi(v_2, v_3)) + \phi(v_2, \phi(v_3, v_1)) + \phi(v_3, \phi(v_1, v_2)) \\
&= \phi(v_1, 0) + \phi(v_2, 0) + \phi(v_3, 0) = 0
\end{aligned}
$$

We must show these maps are isomorphisms. Let $\alpha : \mathfrak{g} \to (v, w, \phi)$ and $\beta : (v, w, \phi) \to \mathfrak{g}$. We check that $\alpha\beta \cong 1_{(v,w,\phi)}$ and $\beta\alpha \cong 1_{\mathfrak{g}}$. We've seen that β sends a triple to the Lie algebra $v \oplus w$ where w is the center of the form $[v+w, v^{'}+w^{'}] = \phi(v, v^{'}) \in w$ which gets mapped to $(v \oplus w, w, \phi) \cong (v, w, \phi)$ The other direction $\mathfrak{g} \to (\mathfrak{g}/Z(\mathfrak{g}), Z(\mathfrak{g})) \to \mathfrak{g}/(Z(\mathfrak{g}) \oplus Z(\mathfrak{g}))$ giving a bijection. $\square$

You can show that the problem of classifying all nilpotent algebras is equivalent to problems that are known to be impossible. However, you can classify things in some special circumstances.

Exercise 4.4. Show that if $Z(\mathfrak{g}) = \mathbb{F}c$ and $\mathfrak{g}$ is 2-step nilpotent, then $\mathfrak{g}$ is isomorphic to $H_{2n+1} = (\mathbb{F}p_1 + \mathbb{F}p_2 + \ldots + \mathbb{F}p_n) + (\mathbb{F}q_1 + \mathbb{F}q_2 + \ldots + \mathbb{F}q_n) + \mathbb{F}c$ with $[p_i, q_j] = \delta_{ij}$, $[p_i, p_i] = 0$, $[q_i, q_j] = 0$, and $[c, H_{2k+1}] = 0$.

Proof. From the previous exercise there exists a skew-symmetric form B on $V := \mathfrak{g}/\mathbb{F}c$, $B(v_1, v_2) = [v_1, v_2]$ and just as above it is easy to check that it is nondegenerate. But for any non-degenerate skew-symmetric bilinear form B on V over any field $\mathbb{F}$ there exists a basis p_i, q_i such that $B(p_i, q_j) = \delta_{ij}$, $B(p_i, p_j) = 0$, and $B(q_i, p_j) = 0$. Indeed, pick arbitrary $p_1 \in V$ and a q_1 such that $B(p_1, q_1) = 1$, and let $V_1^{\perp}$ be the orthocomplement to $\mathbb{F}p_1 + \mathbb{F}q_1$ in V. Continue by induction on $\dim V$. Then look at the preimages of these p_i and q_i in the space $\mathfrak{g}$, and note that they satisfy the same commutation relations. This implies that $H = (\mathbb{F}p_1 + \mathbb{F}p_2 + \ldots + \mathbb{F}p_n) + (\mathbb{F}q_1 + \mathbb{F}q_2 + \ldots + \mathbb{F}q_n) + \mathbb{F}c$ with the desired commutation relations. $\square$

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Lecture 5 — Lie's Theorem

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1 Lie's Theorem

1.1 Weight spaces

Notation 1.1. Let V be a vector space over a field $\mathbb{F}$. We will denote by V^* the dual vector space.

Definition 1.1. Let $\mathfrak{h}$ be a Lie algebra, $\pi : \mathfrak{h} \to \mathfrak{gl}_V$ a representation of $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$. We define the weight space of $\mathfrak{h}$ attached to λ as

$$V_\lambda^{\mathfrak{h}} = \{v \in V \mid \pi(h)v = \lambda(h)v,\ \forall h \in \mathfrak{h}\}.$$

If $V_\lambda^{\mathfrak{h}} \neq 0$, we will say that λ is a weight for π.

1.2 Lie's Lemma

Lemma 1.1 (Lie's Lemma)**.** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ an ideal both over $\mathbb{F}$ algebrically closed and of characteristic 0. Let π be a representation of $\mathfrak{g}$ in a finite dimensional $\mathbb{F}-$vector space V. Then each weight space $V_\lambda^{\mathfrak{h}}$ for the restricted representation $\pi_{|\mathfrak{h}}$ is invariant under $\mathfrak{g}$.*

Proof. We wish to show that if $v \in V_\lambda^{\mathfrak{h}}$, then $\pi(a)v \in V_\lambda^{\mathfrak{h}}$, $\forall a \in \mathfrak{g}$. But this is verified if and only if

$$\pi(h)\pi(a)v = \lambda(h)\pi(a)v,\ \forall h \in \mathfrak{h},\ \forall a \in \mathfrak{g}. \tag{1}$$

Now,

$$\pi(h)\pi(a)v = [\pi(h), \pi(a)]v + \pi(a)\pi(h)v = \pi([h,a])v + \pi(a)\lambda(h)v. \tag{2}$$

Since $\mathfrak{h}$ is an ideal, $[h,a] \in \mathfrak{h}$, then (2) becomes

$$\pi(h)\pi(a)v = \lambda([h,a])v + \pi(a)\lambda(h)v. \tag{3}$$

Thus, it is sufficient to show that

$$\lambda([h,a]) = 0, \forall h \in \mathfrak{h},$$

whenever $V_\lambda^{\mathfrak{h}} \neq 0$.
Let us fix $a \in \mathfrak{g}$ and let $0 \neq v$ be an element in $V_\lambda^{\mathfrak{h}} \neq \{0\}$. We define

$$W_m = span\ < v, \pi(a)v, \pi^2(a)v, \dots, \pi^m(a)v >,\ \forall m \geq 0,\ W_{-1} = \{0\}.$$

Since V is finite dimensional, then there exists $N \in \mathbb{N}$ s.t. N is the maximal integer for which all the generators of W_N are linearly indipendent. Then we have $W_N = W_{N+1} = \dots$, hence

$\pi(a)W_N \subset W_N$.
We will consider the increasing sequence of subspaces

$$W_{-1} = \{0\} \subset W_0 = \{\mathbb{F}v\} \subset \cdots \subset W_N.$$

Now we claim that $\forall m \geq 0,\ W_m$ is invariant under $\pi(\mathfrak{h})$ and furthermore

$$\forall h \in \mathfrak{h},\ \pi(h)\pi(a)^m v - \lambda(h)\pi(a)^m v \in W_{m-1}. \tag{4}$$

We prove (4) by induction on m. The case $m = 0$ is true since $v \in V_\lambda^{\mathfrak{h}}$.
Suppose that we have proved the assumption for $m-1$: we want to prove it for m.

$$\begin{aligned} & \pi(h)\pi(a)^m v - \lambda(h)\pi(a)^m v = [\pi(h), \pi(a)]\pi(a)^{m-1}v + \pi(a)\pi(h)\pi(a)^{m-1}v - \lambda(h)\pi(a)^m v = \\ = \ & [\pi(h), \pi(a)]\pi(a)^{m-1}v + \pi(a)\pi(h)\pi(a)^{m-1}v - \pi(a)\lambda(h)\pi(a)^{m-1}v. \end{aligned}$$

By induction hypothesis, we have that

$$w = \pi(h)\pi(a)^{m-1}v - \lambda(h)\pi(a)^{m-1}v \in W_{m-2}$$

and $\pi(a)w \in W_{m-1}$, by construction of the W_i's.
Moreover, $\mathfrak{h}$ is an ideal so that $[\pi(h), \pi(a)] \in \pi(\mathfrak{h})$ and by inductive hypothesis

$$[\pi(h), \pi(a)]\pi(a)^{m-1}v \in W_{m-1},$$

thus,

$$\pi(h)\pi(a)^m v - \lambda(h)\pi(a)^m v \in W_{m-1}$$

because it is a sum of elements in W_{m-1}. This concludes the proof of the inductive step.
We know that W_N is invariant both for $\pi(a)$ and for $\pi(h)$, $\forall h \in \mathfrak{h}$. In particular,(4) shows that $\forall h \in \mathfrak{h},\ \pi(h)$ acts on W_N as an upper triangular matrix, with in the basis $\{v, \pi(a)v, \ldots, \pi(a^N)v\}$,

$$\begin{pmatrix} \lambda(h) & * & \ldots & * & * \\ 0 & \ddots & * & \ldots & \vdots \\ 0 & 0 & \lambda(h) & * & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda(h) \end{pmatrix}$$

As a consequence of this, we have that

$$tr_{W_N}([\pi(h), \pi(a)]) = 0 = tr_{W_N}(\pi[h, a]) = N\lambda([h, a])$$

which implies that $\lambda([h, a]) = 0$, since $char\ \mathbb{F} = 0$. This concludes the proof. $\square$

Lie's Theorem. *Let $\mathfrak{g}$ be a solvable Lie algebra and π a representation of $\mathfrak{g}$ on a finite dimensional vector space $V \neq 0$, over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then there exists a weight $\lambda \in \mathfrak{g}^*$ for π, that is $V_\lambda^{\mathfrak{g}} \neq \{0\}$.*

Proof. We can suppose that $\mathfrak{g}$ is finite dimensional, since V is finite dimensional and the representation factors in the following way:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\pi"] \arrow[dr, "\phi"'] & & End(V) \\ & \pi(\mathfrak{g}) \arrow[ur, "i"'] & \end{tikzcd}$$

As $\mathfrak{g}$ is solvable, also $\pi(\mathfrak{g})$ will be solvable. This follows from the fact that $\pi(\mathfrak{g})^{(n)} = \pi(\mathfrak{g}^{(n)})$ - by induction on n.
We will prove Lie's theorem by induction on the dimension of $\mathfrak{g}$, $\dim \mathfrak{g} = m$.
The case $\dim \mathfrak{g} = 0$ is trivial.
Suppose now that we have proved Lie's theorem $\dim \mathfrak{g} = m - 1$, we want to show that the theorem holds also for $\mathfrak{g}$, $\dim \mathfrak{g} = m \geq 1$.
Since $\mathfrak{g}$ is solvable, of positive dimension, $\mathfrak{g}$ properly includes $[\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, any subspace is automatically an ideal.
Take a subspace of codimension one in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, then its inverse image $\mathfrak{h}$ is an ideal of codimension one in $\mathfrak{g}$ (including $[\mathfrak{g}, \mathfrak{g}]$). Thus, we have the following decomposition of $\mathfrak{g}$ as a vector space

$$\mathfrak{g} = \mathfrak{h} + \mathbb{F}a.$$

Now, $\dim \mathfrak{h} = m - 1$ and it is solvable (since an ideal of a solvable Lie algebra is solvable), hence by inductive hypothesis we can find a non-zero weight space $V_\lambda^{\mathfrak{h}} \neq \{0\}$, $\lambda \in \mathfrak{h}^*$. By Lie's lemma, $V_\lambda^{\mathfrak{h}}$ is invariant under the action of $\pi(\mathfrak{g})$. In particular, $\pi(a)V_\lambda^{\mathfrak{h}} \subset V_\lambda^{\mathfrak{h}}$, hence (since F is algebraically closed) there exists $0 \neq v \in V_\lambda^{\mathfrak{h}}$ such that $\pi(a)v = la$, for some $l \in \mathbb{F}$. We define a linear functional $\lambda' \in \mathfrak{g}^*$ on $\mathfrak{g}$ by

$$\lambda'(h + \mu a) = \lambda(h) + \mu l, \ \forall h \in \mathfrak{h}, \ l \in \mathbb{F}.$$

Thus, by construction, we see that v belongs to $V_{\lambda'}^{\mathfrak{g}}$. In particular $V_{\lambda'}^{\mathfrak{g}} \neq \{0\}$. □

Exercise. 5.1 Show that we may relax the assumption on $\mathbb{F}$ in Lie's Lemma. Show Lie's Lemma under the assumptions that $\mathbb{F}$ is algebraically closed and that $\dim V < char\, \mathbb{F}$.

In the proof, we calculated the trace of a certain endomorphism of $W \subseteq V$ to be zero, and also to be Nq, where $N = \dim W$ and q was a quantity that we needed to show was zero. Of course then if the characteristic of $\mathbb{F}$ exceeds the dimension of V, $\dim W$ is non-zero in $\mathbb{F}$ and we may conclude $q = 0$. Given this weaker assumption, the rest goes through unaltered.

Exercise. 5.2 Consider the Heisenberg algebra H_3 and its representation on $\mathbb{F}[x]$ given by

$$\begin{aligned} c &\mapsto Id, \\ p &\mapsto (f(x) \mapsto xf(x)), \ \forall f(x) \in \mathbb{F}[x], \\ q &\mapsto (f(x) \mapsto \frac{d}{dx} f(x)), \ \forall f(x) \in \mathbb{F}[x]. \end{aligned}$$

Show that the ideal generated by x^N, $0 < N = char\, \mathbb{F}$, in $\mathbb{F}[x]$ is invariant for the representation of H_3 and that the induced representation of H_3 on $\mathbb{F}[x]/(x^N)$ has no weight.

(x^N) is a subrepresentation, as it is preserved by the action of p and c, and the action of q annihilates the x^N term when differentiating. Now consider $v = a_0x_0 + a_1x_1 + \ldots + a_{N-1}x^{N-1}$, a representative of an element of the quotient. All members of the quotient will be represented this way. Now if v is a weight vector, its derivative must be kv for some k, so that $ia_i = ka_{i-1}$ for each $i < N$. On the other hand, in order that it's an eigenvector for p, it must have zero constant term, or have $pv = 0$. These two statements show easily that $v = 0$, (as $i \neq 0$ when $i = 1, \ldots, N-1$). Thus there is no weight vector.

Exercise. 5.3 Show the following two corollaries to Lie's Theorem:

- for all representations π of a solvable Lie algebra $\mathfrak{g}$ on a finite dimensional vector space V over an algebraically closed field $\mathbb{F}$, *char* $\mathbb{F} = 0$, there exists a basis for V for which the matrices of $\pi(\mathfrak{g})$ are upper triangular;
- a solvable subalgebra $\mathfrak{g} \subset \mathfrak{gl}_V$ (V is finite dimensional over an algebraically closed field $\mathbb{F}$, *char* $\mathbb{F} = 0$) is contained in the subalgebra of upper triangular matrices over $\mathbb{F}$ for some basis of V.

The second statement is simply an application of the first. We prove the first by induction on the module's dimension. It is trivial in dimension 1. Suppose V is a module. Use Lie's theorem to find a weight v of V. The quotient module $V/\mathbb{F}v$, by induction, can given a basis such that $\mathfrak{g}$ acts by upper triangular matrices. Taking any preimages of that basis in V, and extending it to a basis of V by including v, we obtain a basis of V on which $\mathfrak{g}$ acts by upper triangular matrices.

Exercise. 5.4 Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over the algebrically closed field $\mathbb{F}$, *char* $\mathbb{F} = 0$. Show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

If $[\mathfrak{g}/Z(\mathfrak{g}), \mathfrak{g}/Z(\mathfrak{g})]$ is nilpotent, so is $[\mathfrak{g}, \mathfrak{g}]$, so using the adjoint representation, we may assume $\mathfrak{g}$ is a solvable subalgebra of $\mathfrak{gl}_{\mathfrak{g}}$. Then by the previous, there is a basis in which $\mathfrak{g}$ consists of upper triangular matrices. Then $[\mathfrak{g}, \mathfrak{g}]$ consists of strictly upper triangular matrices, and is a subalgebra of the nilpotent Lie algebras $\mathfrak{u}_n$.

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Lecture 6 — Generalized Eigenspaces & Generalized Weight Spaces	
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Definition 6.1. Let A be a linear operator on a vector space V over field $\mathbb{F}$ and let $\lambda \in \mathbb{F}$, then the subspace

$$V_\lambda = \{v \mid (A - \lambda I)^N v = 0 \text{ for some positive integer N}\}$$

is called a generalized eigenspace of A with eigenvalue λ. Note that the eigenspace of A with eigenvalue λ is a subspace of V_λ.

Example 6.1. A is a nilpotent operator if and only if $V = V_0$.

Proposition 6.1. *Let A be a linear operator on a finite dimensional vector space V over an algebraically closed field $\mathbb{F}$, and let $\lambda_1, ..., \lambda_s$ be all eigenvalues of A, $n_1, n_2, ..., n_s$ be their multiplicities. Then one has the generalized eigenspace decomposition:*

$$V = \bigoplus_{i=1}^{s} V_{\lambda_i} \textit{ where } \dim V_{\lambda_i} = n_i$$

Proof. By the Jordan normal form of A in some basis $e_1, e_2, ...e_n$. Its matrix is of the following form:

$$A = \begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_n} \end{pmatrix},$$

where J_{λ_i} is an $n_i \times n_i$ matrix with λ_i on the diagonal, 0 or 1 in each entry just above the diagonal, and 0 everywhere else.

Let $V_{\lambda_1} = \text{span}\{e_1, e_2, ..., e_{n_1}\}, V_{\lambda_2} = \text{span}\{e_{n_1+1}, ..., e_{n_1+n_2}\}, ...,$ so that J_{λ_i} acts on V_{λ_i}. i.e. V_{λ_i} are A-invariant and $A|_{V_{\lambda_i}} = \lambda_i I_{n_i} + N_i$, N_i nilpotent. □

From the above discussion, we obtain the following decomposition of the operator A, called the classical Jordan decomposition

$$A = A_s + A_n$$

where A_s is the operator which in the basis above is the diagonal part of A, and A_n is the rest ($A_n = A - A_s$). It has the following 3 properties

(i) A_s is a diagonalizable operator (usually called semisimple)

(ii) A_n is a nilpotent operator

(iii) $A_s A_n = A_n A_s$.

(iii) holds since $V = \bigoplus_{i=1}^{s} V_{\lambda_i}$, $AV_{\lambda_i} \in V_{\lambda_i}$, and $A_s|_{V_{\lambda_i}} = \lambda_i I$. Hence $A_s A_n = A_n A_s$.

Definition 6.2. A decomposition of an operator A of the form $A = A_s + A_n$, for which these three properities hold is called a Jordan decomposition of A. We have established its existence, provided that $\dim V < +\infty, \mathbb{F} = \bar{\mathbb{F}}$

Proposition 6.2. *Jordan decomposition is unique under the same assumptions*

Lemma 6.3. *Let A and B be commuting operators on V; i.e., $AB = BA$. Then*

(a) All generalized eigenspaces of A are B-invariant

(b) if $A = A_s + A_n$ is the classical Jordan decomposition, then B commutes with both A_s and A_n.

Proof. (a) is immediate from definition of generalized eigenspace. (b) follows from (a) since each V_{λ_i} is B-invariant, $A_s|_{V_{\lambda_i}} = \lambda_i I_{n_i}$, therefore A and A_s commute on each V_{λ_i}, therefore commute. □

Proof of the proposition. Consider a Jordan decomposition $A = A'_s + A'_n$, and let $A = A_s + A_n$ be the classical Jordan decomposition. Take the difference, we get

$$A_s - A'_s = A_n - A'_n$$

But A'_s commutes with A'_n and itself, hence with A. Hence by taking $B = A'_s$ in lemma (b), we conclude that A'_s commutes with A_s and A_n. Therefore $A'_n = A - A'_s$ also commutes with A_s and A_n. So in (2) we we have difference of commutative operators on both sides. Hence LHS is diagonalizable and RHS is nilpotent (by the binomial formula). But equality of a diagonalizable operator to a nilpotent one is possible only if both are 0.

Question. Is it true in general that Jordan decomposition is unique?

Exercise 6.1. Show that any nonabelian 3-dimensional nilpotent Lie algebra is isomorphic to the Heisenberg algebra H_3.

Proof. If $\mathfrak{g}$ is nonabelian and 3-dimensional, then $Z(\mathfrak{g})$ must have dimension less than 3. By a previous exercise (3.2), $\dim Z(\mathfrak{g}) \neq \dim \mathfrak{g} - 1$, so this dimension cannot be 2. A proposition from lecture 4 states that if $\mathfrak{g}$ is nonzero and nilpotent, $Z(\mathfrak{g})$ is nonzero. Hence $Z(\mathfrak{g})$ is 1-dimensional.

Now by exercise 3.3, the n-dimensional Lie algebras for which $Z(\mathfrak{g})$ has dimension two less than $\mathfrak{g}$ are $Ab_{n-2} \oplus \mathfrak{g}_2$ and $Ab_{n-3} \oplus H_3$, where $\mathfrak{g}_2$ is the 2-dimensional Lie algebra $\mathbb{F}x + \mathbb{F}y$ defined by $[x, y] = y$.

Since $\mathfrak{g}$ is nilpotent, it cannot be $Ab_1 \oplus \mathfrak{g}_2$, because $\mathfrak{g}_2$ is not nilpotent. Then the only remaining possibility is $\mathfrak{g} = Ab_0 \oplus H_3 = H_3$. □

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and π its representation on a finite-dimensional vector space V, over an algebraically closed field $\mathbb{F}$ of characteristic 0. We have the following generalized eigenspace decompositions for a fixed $a \in \mathfrak{g}$.

$$V = \bigoplus_{\lambda \in \mathbb{F}} V^a_\lambda \qquad V^a_\lambda = \left\{ v \in V \mid (\pi(a) - \lambda I)^N v = 0 \text{ for some } N \in \mathbb{N} \right\}$$

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{F}} \mathfrak{g}^a_\alpha \qquad \mathfrak{g}^a_\alpha = \left\{ g \in \mathfrak{g} \mid (\mathbf{ad}\ a - \alpha I)^N g = 0 \text{ for some } N \in \mathbb{N} \right\}$$

We'll prove the following.

Theorem 6.4. $\pi\left(\mathfrak{g}_\alpha^a\right) V_\lambda^a \subseteq V_{\lambda+\alpha}^a$

First, we need a lemma on associative algebras.

Lemma 6.5. *Suppose U is a unital associative algebra over $\mathbb{F}$, and let $a, b \in U$ and $\lambda, \alpha \in \mathbb{F}$. Then*

$$(a-\alpha-\lambda)^N b = \sum_{j=0}^{N} \binom{N}{j} \left((\mathbf{ad}\ a - \alpha I)^j\, b\right)(a-\lambda)^{N-j}.$$

Proof. Write $\mathbf{ad}\ a = L_a - R_a$, where $L_a(x) = ax$ and $R_a(x) = xa$. Then

$$\begin{aligned} L_{a-\alpha-\lambda} &= L_a - \alpha I - \lambda I \\ &= \mathbf{ad}\ a + R_a - \alpha I - \lambda I \\ L_{a-\alpha-\lambda} &= (\mathbf{ad}\ a - \alpha) + R_{a-\lambda} \end{aligned} \tag{1}$$

For any given $a, b \in U$, the operators L_a and R_b commute by associativity of U. Since $\mathbf{ad}\ a$ is just the difference $L_a - R_a$, it commutes with both L_a and R_a. Then since $\alpha I, \lambda I \in \mathbb{F}I \subset Z(U)$, the terms $(\mathbf{ad}\ a - \alpha)$ and $R_{a-\lambda}$ on the right side of (1) commute. Given this, the claimed equality follows from raising both sides of (1) to the Nth power and applying the Binomial Theorem. $\square$

Proof of Theorem 6.4. Applying the lemma to $\pi(\mathfrak{g})$, we have the following for all $g \in \mathfrak{g}$, and thus for all $g \in \mathfrak{g}_\alpha^a$. (Recall that $a \in \mathfrak{g}$ is fixed.)

$$(\pi(a) - \alpha - \lambda)^N \pi(g) = \sum_{j=0}^{N} \binom{N}{j} (\mathbf{ad}\ \pi(a) - \alpha)^j\, \pi(g)\, (\pi(a) - \lambda)^{N-j}$$

Apply both sides of this to $v \in V_\lambda^a$ with $N > \dim V_\lambda^a + \dim \mathfrak{g}_\alpha^a$. By this choice of N, either $j > \dim \mathfrak{g}_\alpha^a$ or $N - j > \dim V_\lambda^a$. If $j > \dim g_\alpha^a$, then $(\mathbf{ad}\ \pi(a) - \alpha)^j\, \pi(g) = 0$ since $g \in g_\lambda^a$. Otherwise, $N - j > \dim V_\lambda^a$, so $(\pi(a) - \lambda)^{N-j}\, v = 0$ since $v \in V_\lambda^a$.

This makes every term in the sum on the right zero, so $(\pi(a) - \alpha - \lambda)^N \pi(g)v = 0$. Then $\pi(g)v$ is a generalized eigenvector of $\pi(a)$ with eigenvalue $\alpha - \lambda$, so $\pi(g)v \in V_{\lambda+\alpha}^a$. Since this holds for all $g \in \mathfrak{g}_\alpha^a$ and $v \in V_\lambda^a$, the claimed inclusion holds. $\square$

By analogy to the definition of a generalized eigenspace, we can define generalized weight spaces of a Lie algebra $\mathfrak{g}$.

Definition 6.3. Let $\mathfrak{g}$ be a Lie algebra with a representation π on a vector space on V, and let $\lambda \in \mathfrak{g}^*$ be a linear functional on $\mathfrak{g}$. The generalized weight space of $\mathfrak{g}$ in V attached to λ is

$$V_\lambda^{\mathfrak{g}} = \left\{ v \in V \mid (\pi(g) - \lambda(g)I)^N\, v = 0 \text{ for some } N \text{ depending on } g, \text{ for all } g \in \mathfrak{g} \right\}.$$

Under the right conditions, a nilpotent subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ permits decomposing V as a direct sum of the generalized weight spaces of $\mathfrak{h}$, each of which is a subrepresentation of $\pi_{\mathfrak{h}}$. The following theorem makes this precise.

Theorem 6.6. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and π its representation on a finite-dimensional vector space V, over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $\mathfrak{h}$ be a nilpotent subalgebra of $\mathfrak{g}$. Then the following equalities hold.*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{\mathfrak{h}} \tag{2}$$

$$\pi\left(\mathfrak{g}_\alpha^{\mathfrak{h}}\right) V_\lambda^{\mathfrak{h}} \subseteq V_{\lambda+\alpha}^{\mathfrak{h}} \tag{3}$$

Remark. In the case of the adjoint representation, we may express these as follows.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha^{\mathfrak{h}} \tag{4}$$

$$\left[\mathfrak{g}_\alpha^{\mathfrak{h}}, \mathfrak{g}_\beta^{\mathfrak{h}}\right] \subseteq \mathfrak{g}_{\alpha+\beta}^{\mathfrak{h}} \tag{5}$$

Proof of Theorem 6.6.

Case 1. For each $a \in \mathfrak{h}$, $\pi(a)$ has only one eigenvalue.

In this case, V is a generalized eigenspace $V_{\lambda(a)}^a$ of every $a \in \mathfrak{h}$, so we just need to check the linearity of λ.

Since $\mathfrak{h}$ is nilpotent, it is solvable. Since we assumed $\mathbb{F}$ to be algebraically closed and with characteristic 0, we can then apply Lie's theorem, which guarantees the existence of a weight λ' with some nonzero weight space $V_{\lambda'}^{\mathfrak{h}}$. Then $\lambda'(a)$ must be the eigenvalue of $\pi(a)$ with which $\pi(a)$ acts on $V_{\lambda'}^{\mathfrak{h}}$, so $\lambda' = \lambda$. Therefore λ is linear, so V is the generalized weight space $V_\lambda^{\mathfrak{h}}$.

Case 2. For some $a_0 \in \mathfrak{h}$, $\pi(a_0)$ has at least two distinct eigenvalues.

Since $\mathfrak{h}$ is nilpotent, **ad** a is a nilpotent operator on $\mathfrak{h}$ for all $a \in \mathfrak{h}$. Thus $\mathfrak{h} \subset \mathfrak{g}_0^a$. Then by Theorem 6.4, $\pi(\mathfrak{h})V_\lambda^a \subseteq V_\lambda^a$ for any $a \in \mathfrak{h}$.

Since $\mathbb{F}$ is algebraically closed, V can be written as a direct sum of the generalized eigenspaces of a_0. Since each $V_\lambda^{a_0}$ is invariant under the action of $\mathfrak{h}$, each $V_\lambda^{a_0}$ is also a representation of $\mathfrak{h}$. Since $\dim V_\lambda^{a_0} < \dim V$, we may apply induction on $\dim V$. This establishes the equality (2).

To finish, we'll prove the inclusion (3). Suppose $\alpha, \lambda \in \mathfrak{h}^*$, and suppose $g \in \mathfrak{g}_\alpha^{\mathfrak{h}}$. Then $g \in \mathfrak{g}_{\alpha(a)}^a$ for all $a \in \mathfrak{h}$. By Theorem 6.4, $\pi(g)V_{\lambda(a)}^a \subset V_{\lambda(a)+\alpha(a)}^a$ for all $a \in \mathfrak{h}$. Then

$$v \in \bigcap_{a \in \mathfrak{h}} V_{\lambda(a)}^a \implies \pi(g)v \in \bigcap_{a \in \mathfrak{h}} V_{\lambda(a)+\alpha(a)}^a.$$

Since $\displaystyle\bigcap_{a \in \mathfrak{h}} V_{\lambda(a)}^a = V_\lambda^{\mathfrak{h}}$ by the definition of a generalized weight space, this establishes (3). □

Exercise 6.2. Suppose $\mathbb{F}$ has characteristic 2, and $V = \mathbb{F}[x]/(x^2)$ is a representation of H_3 where $p \mapsto \frac{\partial}{\partial x}$, $q \mapsto x$, and $c \mapsto I$. Then $V = V_\lambda$, but λ is not a linear function on H_3. Compute λ.

Proof. Suppose p acts as $\frac{\partial}{\partial x}$, q acts as multiplication by x, and c acts as the identity on $\mathbb{F}[x]/(x^2)$. Then:

$$\begin{aligned} p(a+bx) &= b + 0x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ q(a+bx) &= 0 + ax = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ c(a+bx) &= a + bx = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

Then making a basis on $\mathbb{F}[x]/(x^2)$ using 1 and x, we can write the matrix representing some $rp + sq + tc \in H_3$ as the following matrix.

$$\begin{bmatrix} t & r \\ s & t \end{bmatrix}$$

Then finding λ is a matter of solving its characteristic polynomial.

$$\begin{aligned} 0 &= \det \begin{bmatrix} t-\lambda & r \\ s & t-\lambda \end{bmatrix} \\ &= (t-\lambda)^2 - rs \\ \pm\sqrt{rs} &= t - \lambda \\ \lambda &= t \pm \sqrt{rs} \end{aligned}$$

In a field of characteristic 2, we can drop the $\pm$ sign. By passing to the algebraic closure if necessary, we can assume the square root of rs always exists. Thus:

$$\lambda(rp + sq + tc) = t + \sqrt{rs}$$

(To verify that λ is not linear, observe that by this formula, $\lambda(p) = \lambda(q) = 0$, but $\lambda(p+q) = 1$.) $\square$

Exercise 6.3. By the example of the adjoint representation of a nonabelian solvable Lie algebra, show that the generalized weight space decomposition fails if the Lie algebra is solvable but not nilpotent.

Proof. Consider the Lie algebra $\mathfrak{g}_2 = \mathbb{F}x + \mathbb{F}y$, with the bracket operation defined by $[x, y] = y$.

It's apparent by induction that $\mathfrak{g}_2^k = [\mathfrak{g}_2, \mathfrak{g}_2^{k-1}] = \mathbb{F}y$ (for $k \geq 2$), so $\mathfrak{g}_2$ is not nilpotent. However, then $\mathfrak{g}_2^{(1)} = \mathbb{F}y$, which is 1-dimensional, so $\mathfrak{g}_2^{(2)} = 0$, and thus $\mathfrak{g}_2$ is solvable.

Taking x and y as the basis elements of $\mathfrak{g}_2$, the adjoint representation takes x and y to the following matrices.

$$x \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad y \mapsto \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

So we find their eigenvalues by solving their characteristic polynomials.

$$\begin{aligned} \lambda(\lambda - 1) &= 0 \\ \lambda &= 0 \text{ or } 1 \end{aligned} \qquad\qquad \begin{aligned} \lambda^2 &= 0 \\ \lambda &= 0 \end{aligned}$$

The corresponding generalized eigenvectors can be found by lucky guessing. Specifically, **ad** x has x with eigenvalue 0 and y with eigenvalue 1, while **ad** y has all of $\mathfrak{g}_2$ with eigenvalue 0.

So we can get weight spaces $V_0 = \text{span}\{x\}$ and $V_{x^*} = \text{span}\{y\}$, corresponding to the zero linear functional and the linear functional defined by $x \mapsto 1$. The vector space decomposes into the direct sum of these weight spaces, but the *representation* does not! Specifically, V_0 is not closed under the action of y. $\square$

Exercise 6.4. Take $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{h} = \{\text{diagonal matrices}\}$. Find the generalized weight space decomposition in both the tautological and the adjoint representations, and check the inclusions (3) and (5) in Theorem 6.6.

Proof. Suppose $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{h} \subset \mathfrak{g}$ consists of diagonal matrices. Then the generalized eigenvectors of $h \in \mathfrak{h}$ are actual eigenvectors, so every standard basis element of $\mathbb{F}^n$ is an eigenvector. Also, given any linear combination $ae_i + be_j$ of more than one basis element, there is some diagonal matrix that takes e_i to e_i and e_j to zero, so these linear combinations are not generalized eigenvectors of everything in $\mathfrak{h}$. Thus the only candidates for generalized weight spaces are the n axes, each of which is the span of a single standard basis element of $\mathbb{F}^n$.

For the axis V_i spanned by e_i, the linear functional on $\mathfrak{h}$ that takes h to the component $h_{i,i}$ is a weight making V_i a weight space. Thus in the tautological representation, $\mathbb{F}^n$ decomposes as a direct sum of n copies of $\mathbb{F}$.

To do the same with the adjoint representation, suppose the diagonal entries of $h \in \mathfrak{h}$ are h_i. Then for $a \in \mathfrak{g}$, we have:

$$\begin{aligned} ((\mathbf{ad}\ h)a)_{i,j} &= (ha - ah)_{i,j} \\ &= h_i a_{i,j} - a_{i,j} h_j \\ &= (h_i - h_j) a_{i,j} \end{aligned}$$

This shows that **ad** h is diagonalizable, which again implies that its generalized eigenvectors are actual eigenvectors, and so its generalized weight spaces are actually weight spaces.

Possible pairs of eigenvalues are

1. $h_i - h_j$ vs. $h_i - h_k$,
2. $h_i - h_j$ vs. $h_k - h_\ell$,
3. $h_i - h_j$ vs. $h_j - h_i$, and
4. $h_i - h_i$ vs. $h_j - h_j$.

By appropriate choice of h, we can always make distinct eigenvalues in (1) and (2), so the basis elements $e_{i,j}$ of $\mathfrak{g}$ satisfying $i < j$ lie in distinct eigenspaces for some h, and thus they lie in distinct candidate weight spaces. This weight space can be achieved with the linear functional $\lambda_{i,j}$ taking h to $h_i - h_j$.

Since for the theorem we assume the characteristic of $\mathbb{F}$ is not 2, the eigenvalues in (3) will be distinct, so we'll also have $\lambda_{i,j}$ with $i > j$. Finally, both eigenvalues in (4) are always zero, so the zero linear functional has $\mathfrak{h}$ as its weight space.

Combining all of this, the generalized weight space decomposition of $\mathfrak{g}$ in the adjoint representation is $\mathfrak{h}$ plus some 1-dimensional weight spaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \operatorname{span}\{e_{i,j}\}$$

To check the assertion of the theorem from class, first we verify for the tautological representation that:

$$\pi\left(\mathfrak{g}_\alpha^\mathfrak{h}\right) V_\lambda^\mathfrak{h} \subseteq V_{\lambda+\alpha}^\mathfrak{h}$$

Each $V_\lambda^\mathfrak{h}$ is the span of some basis element e_i, with λ corresponding to the map $h \mapsto h_i$, so we really only need to check that, for some appropriate j:

$$\pi\left(\mathfrak{g}_\alpha^\mathfrak{h}\right) e_i \propto e_j$$

In the case $\alpha = 0$ we should get $j = i$, and we do; the space $\mathfrak{g}_0^\mathfrak{h}$ consists of all diagonal matrices, so they act on e_i by scaling.

In the case $\alpha h = h_k - h_\ell$, we then have $\mathfrak{g}_\alpha^\mathfrak{h} = e_{k,\ell}$, and $\alpha + \lambda = h_k - h_\ell + h_i$. We should expect zero if $i \neq \ell$, since we only have nonzero weight spaces for λ of the form $h \mapsto h_{something}$; and indeed this is the case, since if $i \neq \ell$ then $e_{k,\ell} e_i = 0$. Furthermore, if $i = \ell$ then we should get the span of e_k, which is the weight space corresponding to $h \mapsto h_k$. We verify this by observing that $e_{k,i} e_i = e_k$. So in fact we have equality:

$$\pi\left(\mathfrak{g}_\alpha^\mathfrak{h}\right) V_\lambda^\mathfrak{h} = V_{\lambda+\alpha}^\mathfrak{h}$$

The second assertion in (b) of the theorem is essentially the same statement for the adjoint representation. First, if $\alpha = \beta = 0$, then both $\mathfrak{g}_\alpha^\mathfrak{h}$ and $\mathfrak{g}_\beta^\mathfrak{h}$ are equal to $\mathfrak{h}$. Since $\mathfrak{h}$ consists of diagonal matrices, it's commutative, so the bracket is zero and thus is contained in any weight space we like.

If $\alpha = 0$ and $\beta = \{h \mapsto h_i - h_j\}$, then we end up with $[\mathfrak{h}, \operatorname{span}\{e_{i,j}\}]$. Observe that:

$$\begin{aligned} [h, e_{i,j}] &= h e_{i,j} - e_{i,j} h \\ &= (h_i - h_j) e_{i,j} \in \operatorname{span}\{e_{i,j}\} \end{aligned}$$

Finally, if α maps h to $h_i - h_j$ and β maps h to $h_k - h_\ell$, we have essentially $[e_{i,j}, e_{k,\ell}]$. We need either $j = k$ or $i = \ell$ for $\alpha + \beta$ to be a weight, and we indeed see that if neither holds, then $e_{i,j} e_{k,\ell} = 0$. Otherwise, by relabeling α and β, we can assume without loss of generality that $j = k$. This gives us $e_{i,\ell}$ if $i \neq \ell$ and 0 if $i = \ell$, so either way it's in the weight space of $h \mapsto h_i - h_\ell$. $\square$

18.745 Introduction to Lie Algebras	September 30, 2010
Lecture 7 - Zariski Topology and Regular Elements	
Prof. Victor Kac	*Scribe: Daniel Ketover*

Definition 7.1. A topological space is a set X with a collection of subsets F (the closed sets) satisfying the following axioms:

1) $X \in F$ and $\emptyset \in F$
2) The union of a finite collection of closed sets is closed
3) The intersection of arbitrary collection of closed sets is closed
4) (weak separation axiom) For any two points $x, y \in X$ there exists an $S \in F$ such that $x \in S$ but $y \notin S$

A subset $O \subset X$ which is the complement in X of a set in F is called open. The axioms for a topological space can also be phrased in terms of open sets.

Definition 7.2. Let X $= \mathbb{F}^n$ where $\mathbb{F}$ is a field. We define the Zariski topology on X as follows. A set in X is closed if and only if if is the set of common zeroes of a collection (possibly infinite) of polynomials $P_\alpha(x)$ on $\mathbb{F}^n$

Exercise 7.1. Prove that the Zariski topology is indeed a topology.

Proof. We check the axioms. For 1), let $p(x) = 1$, so that $\mathbb{V}(p) = \emptyset$, so $\emptyset \in F$. If $p(x) = 0$, then $\mathbb{V}(p) = \mathbb{F}^n$, so $\mathbb{F}^n \in F$. For 2), first take two closed sets $\mathbb{V}(p_\alpha)$ with $\alpha \in I$ and $\mathbb{V}(q_\beta)$ with $\beta \in J$, and consider the family of polynomials $f_{\alpha\beta} = \{p_\alpha q_\beta \mid \alpha \in I, \beta \in J\}$. Then $\mathbb{V}(f_{\alpha\beta}) = \mathbb{V}(p_\alpha) \cup \mathbb{V}(q_\beta)$. Therefore $\mathbb{V}(p_\alpha) \cup \mathbb{V}(q_\beta)$ is closed. This can be iterated for any n-fold union. For 3) closure under arbitrary intersections is obvious. For 4), suppose $x \neq y \in \mathbb{F}^n$ and $x = (x_1, x_2, ..., x_n)$. Set $X = \mathbb{V}(p_1, p_2, ..., p_n)$ where $p_i(z) = z_i - x_i$. So $x \in X$ but y is not. □

Example 7.1. If $X = \mathbb{F}$ the closed sets in the Zariski topology are precisely the $\emptyset$, $\mathbb{F}$ and finite subsets of $\mathbb{F}$.

Notation 7.1. Given a collection of polynomials S on $\mathbb{F}^n$ we denote by $\mathbb{V}(S) \in \mathbb{F}^n$ the set of common zeroes of all polynomials in S. By definition, $\mathbb{V}(S)$ are all closed subsets in $\mathbb{F}^n$ in the Zariski topology. If S contains precisely one non-constant polynomial, then $\mathbb{V}(S)$ is called a hypersurface.

Proposition 7.1. *Suppose the field $\mathbb{F}$ is infinite and $n \geq 1$ then*

1) *The complement to a hypersurface in $\mathbb{F}^n$ is an infinite set. Consequently, the complement of $\mathbb{V}(S)$, where S contains a nonzero polynomial, is an infinite set.*
2) *Every two non-empty Zariski open subsets of $\mathbb{F}^n$ have a nonempty intersection*
3) *If a polynomial $p(x)$ vanishes on a nonempty Zariski open set then it is identically zero.*

Example 7.2. The condition that $\mathbb{F}$ be infinite is crucial. In 3), for instance, the polynomial $p(x) = x^2 + x$ on $\mathbb{F}_2$ vanishes at 1 and 0 but is not the zero polynomial.

Proof. For 1), We induct on n. When $n = 1$, since any nonzero polynomial has at most deg P roots, the complement of this set is infinite (since $\mathbb{F}$ is). If $n \geq 1$ then any nonzero polynomial $p(x_1, x_2, ..., x_n)$ can be written as a polynomial in one variable with coefficients polynomials in the other variables. So we can write $p(x) = p_d(x_2, x_3, ..., x_n)x_1^d + p_{d-1}(x_2, x_3, ...x_n)x_1^{d-1} + ... + p_0(x_2, x_3, ..., x_n)$ where $p_d(x_2, x_3, ...x_n)$ is a nonzero polynomial. By the inductive hypothesis, we can find $x_2^o, x_3^o, ..., x_n^o \in \mathbb{F}$ such that $p_d(x_2^o, x_3^o, ..., x_n^o) \neq 0$ Now fixing these values, $p(x) = p_d(x_2^o, x_3^o, ..., x_n^o)x_1^d + p_{d-1}(x_2^o, x_3^o, ...x_n^o)x_1^{d-1} + ... + p_0(x_2^o, x_3^o, ..., x_n^o)$ we are back in the $n = 1$ case. We can find an infinite number of values x_1^o such that $p(x) = p_d(x_2^o, x_3^o, ..., x_n^o)(x_1^o)^d + p_{d-1}(x_2^o, x_3^o, ...x_n^o)(x_1^o)^{d-1} + ... + p_0(x_2^o, x_3^o, ..., x_n^o) \neq 0$. Thus we have found an infinite number of points in $\mathbb{F}^n$ where p does not vanish. To prove the second claim of 1), just observe if S a collection of polynomials containing the nonzero polynomial $p(x)$, we have $\mathbb{V}(S)^C \supset \mathbb{V}(p)^C$ so by the first claim, we have that $\mathbb{V}(S)^C$ contains an infinite set.
2) Consider $\mathbb{V}(S_1)$ and $\mathbb{V}(S_2)$ where S_1 and S_2 are nonzero sets of polynomials. It suffices to prove that $\mathbb{V}(p_1)^C \cap \mathbb{V}(p_2)^C$ is nonempty for any fixed $p_1 \in S_1$ and $p_2 \in S_2$ nonzero polynomials. But observe that $\mathbb{V}(p_1p_2) = \mathbb{V}(p_1) \cup \mathbb{V}(p_2)$, so $\mathbb{V}(p_1p_2)^C = \mathbb{V}(p_1)^C \cap \mathbb{V}(p_2)^C$ but by 1) we know that the term on the left is infinite, hence nonempty.
3) If p vanishes on $\mathbb{V}(S)^C$ for S containing a nonzero polynomial q then p vanishes on $\mathbb{V}(q)^C$. If p were nonzero, then by 2), $\mathbb{V}(q)^C \cap \mathbb{V}(p)^C$ is nonempty. This is a contradiction, so p is identically zero.

□

Let $\mathfrak{g}$ be a finite dimensional Lie algebra of dimension d and consider the characteristic polynomial of an endomorphism **ad** a for some $a \in \mathfrak{g}$: $\det_{\mathfrak{g}}(\mathbf{ad}\ a - \lambda) = (-\lambda)^d + c_{d-1}(-\lambda)^{d-1} + ... + \det(\mathbf{ad}\ a)$.

This is a polynomial of degree d. Because $(\mathbf{ad}\ a)(a) = [a, a] = 0$, we know $\det(\mathbf{ad}\ a) = 0$ and the constant term in the polynomial vanishes.

Exercise 7.2. Show that c_j is a homogeneous polynomial on $\mathfrak{g}$ of degree $d - j$.

Proof. By "polynomial in $\mathfrak{g}$" we mean that if we fix a basis $X_1, X_2, ..., X_d$ of $\mathfrak{g}$ and write a general element $a = \sum_i a_i X_i$, then $c_k(a)$ is a polynomial in the a_i. Observe that **ad** a is a d by d matrix whose entries b_{ij} are linear combinations of the a_i. Now consider $\det_{\mathfrak{g}}(\mathbf{ad}\ a - \lambda)$. This is a polynomial in the entries of the matrix $\mathbf{ad}\ a - \lambda$. A general element in the expansion of this determinant is $b_{**}b_{**}...(b_{ii} - \lambda)...(b_{jj} - \lambda)$. In order to get a term with λ^k we have to chose k λ's and $d - k$ of the b_{**} in the expansion of this product. Therefore the coefficient of the polynomial λ^k is homogeneous of degree $d - k$ in the b_{**}, so it is also homogeneous of degree $d - k$ in the a_i since the b_{ij} are linear in the a_i. □

Definition 7.3. The smallest positive integer such that $c_r(a)$ is not the zero polynomial on $\mathfrak{g}$ is called the rank of $\mathfrak{g}$. Note $1 \leq r \leq n$. An element $a \in \mathfrak{g}$ is called regular if $c_r(a) \neq 0$. The nonzero polynomial $c_r(a)$ of degree $d - r$ is called the discriminant of $\mathfrak{g}$.

Proposition 7.2. *Let $\mathfrak{g}$ be a Lie algebra of dimension d, rank r over $\mathbb{F}$*

1) $r = d$ if and only if $\mathfrak{g}$ is nilpotent.

2) If $\mathfrak{g}$ is nilpotent, the set of regular elements is $\mathfrak{g}$.

3) If $\mathfrak{g}$ is not nilpotent, the set of regular elements is infinite if $\mathbb{F}$ is.

Proof. 1) $r = d$ means that $\det(\mathbf{ad}\ a - \lambda) = (-\lambda)^d$ for all a. This is true if and only if all the eigenvalues of $\mathbf{ad}\ a$ are 0, which happens if and only if $\mathbf{ad}\ a$ is nilpotent for all a. By Engel's theorem, this is equivalent to $\mathfrak{g}$ being nilpotent.
2) If $\mathfrak{g}$ is nilpotent, we know $\det(\mathbf{ad}\ a - \lambda) = (-\lambda)^d$ and $c_r = c_d = 1$, which does not vanish anywhere, so all elements are regular.
3) If $\mathfrak{g}$ is not nilpotent, the set of regular elements is by definition $\mathbb{V}(c_r)^C$, which is the complement of a hypersurface, defined by a homogeneous polynomial of degree $d - r > 0$ by Ex 7.2. By part 1 of the previous theorem the complement of a hypersurface is an infinite set if $\mathbb{F}$ is. $\square$

Example 7.3. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$, the rank is n and we will find its discriminant. Also, the set of regular elements consists of all matrices with distinct eigenvalues.

Exercise 7.3. 1) Show that the Jordan decomposition of $\mathbf{ad}\ a$ is given by $\mathbf{ad}\ a = (\mathbf{ad}\ a_s)+(\mathbf{ad}\ a_n)$ in $\mathfrak{gl}_n$
2) If $\lambda_1, ..., \lambda_n$ are eigenvalues of a_s then $\lambda_i - \lambda_j$ are eigenvalues of $\mathbf{ad}\ a_s$
3) $\mathbf{ad}\ a_s$ has the same eigenvalues as $\mathbf{ad}\ a$.

Proof. For 2), If $\lambda_1, ..., \lambda_n$ are eigenvalues of a_s observe that if E_{ij} is the matrix with 1 in the ij slot and zero elsewhere, then $[a_s, E_{ij}] = (a_{sii} - a_{sjj})E_{ij}$ so $\mathbf{ad}\ a_s$ is diagonalizable since the E_{ij} form a basis of $\mathfrak{gl}_n(\mathbb{F})$. The eigenvalues are thus $\lambda_i - \lambda_j$. As for 1), we already showed that $\mathbf{ad}\ a_s$ is diagonalizable, also $\mathbf{ad}\ a_n$ is nilpotent because a_n is. Finally, $\mathbf{ad}\ a_s$ and $\mathbf{ad}\ a_n$ commute: $[\mathbf{ad}\ a_s, \mathbf{ad}\ a_n]v = [a_s, [a_n, v]] - [a_n, [a_s, v]] = -[[a_s, a_n], v] = 0$ (since $[a_s, a_n] = 0$). Thus by uniqueness of Jordan form, $\mathbf{ad}\ a = \mathbf{ad}\ a_s + \mathbf{ad}\ a_n$. As for 3), we know from Jordan canonical form that the semisimple part of an operator and the operator itself have the same eigenvalues – and since $\mathbf{ad}\ a_s$ is the semisimple part of the operator $\mathbf{ad}\ a$, the claim follows. $\square$

Exercise 7.4. Deduce the statement of example 1.3 by showing that the rank of $\mathfrak{gl}_n$ is n and that the discriminant is given by $c_n(a) = \prod_{i \neq j}(\lambda_i - \lambda_j)$. Also compute $c_2(a)$ for $\mathfrak{gl}_2(\mathbb{F})$ in terms of the matrix coefficients of a.

Proof. We must compute $\det(\mathbf{ad}\ a - \lambda)$. This is equal to $\det(\mathbf{ad}\ a_s - \lambda) = \prod_{i,j}(\lambda_i - \lambda_j - \lambda)$. Note when $i = j$, we can pull out the $(-\lambda)$, so we have $\prod_{i,j}(\lambda_i - \lambda_j - \lambda) = (-\lambda)^n \prod_{i \neq j}(\lambda_i - \lambda_j - \lambda)$. Thus we see that the rank of $\mathfrak{gl}_n(\mathbb{F}^n) = n$, and the discriminant is $\prod_{i \neq j}(\lambda_i - \lambda_j)$. An element in $\mathfrak{gl}_n$ is therefore regular if and only if all its eigenvalues are distinct. Finally we compute the discriminant c_2 for $\mathfrak{gl}_2(\mathbb{F})$: $\prod_{i \neq j}(\lambda_i - \lambda_j) = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_1) = -(\lambda_1 + \lambda_2)^2 + 4\lambda_1\lambda_2$. If we write a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the discriminant is $-(a + d)^2 + 4(ad - bc)$. This a clearly a homogeneous polynomial of degree $n^2 - n = 2^2 - 2 = 2$. $\square$

18.745 Introduction to Lie Algebras	October 5, 2010

Lecture 8 — Cartan Subalgebra

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Definition 8.1. Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. Then $N_{\mathfrak{g}}(\mathfrak{h}) = \{a \in \mathfrak{g} | [a, \mathfrak{h}] \subset \mathfrak{h}\}$ is a subalgebra of $\mathfrak{g}$, called the normalizer of $\mathfrak{h}$.

The fact that $N_{\mathfrak{g}}(\mathfrak{h})$ is a subalgebra follows directly from the Jacobi identity. Also note that $\mathfrak{h} \subset N_{\mathfrak{g}}(\mathfrak{h})$ and the normalizer of $\mathfrak{h}$ is the maximal subalgebra containing $\mathfrak{h}$ as an ideal.

Lemma 8.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra such that $\mathfrak{h} \neq \mathfrak{g}$. Then, $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$.*

Proof. Consider the central series: $\mathfrak{g} = \mathfrak{g}^1 \subset \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \subset g^3 \subset ... \subset g^n = 0$
Note that the last equality is true for some $n \in \mathbb{N}$ because $\mathfrak{g}$ is a nilpotent Lie algebra . Take j to be the maximal possible positive integer such that: $\mathfrak{g}^j \not\subseteq \mathfrak{h}$. Clearly we have that $1 < j < n$; but then $[\mathfrak{g}^j, \mathfrak{h}] \subset \mathfrak{g}^{j+1} \subset \mathfrak{h}$ by the choice made on j. Hence $\mathfrak{g}^j \subset N_{\mathfrak{g}}(\mathfrak{h})$, which is not a subspace of $\mathfrak{h}$. Thus, we can conclude that $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$ □

Definition 8.2. A Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$, satisfying the following two conditions:
i) $\mathfrak{h}$ is a nilpotent Lie algebra
ii) $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$

Corollary 8.2. *Any Cartan subalgebra of $\mathfrak{g}$ is a maximal nilpotent subalgebra*

Proof. This follows directly from Lemma 1 and the definition of Cartan subalgebras.

□

Exercise 8.1. Let $\mathfrak{g} = gl_n(\mathbb{F})$ with $char(\mathbb{F}) \neq 2$. Let $\mathfrak{h} = \{\mathfrak{n}_n + \mathbb{F}I_n\}$, where $\mathfrak{n}_n$ is the subalgebra of strictly upper triangular matrices. Then this is a maximal nilpotent subalgebra but not a Cartan subalgebra.

Solution. First, we show that $\mathfrak{n}_n + \mathbb{F}I_n$ is not a Cartan subalgebra of $\mathfrak{g}$. $\mathfrak{h}$ is not Cartan since, as was shown earlier, $[b_n, \mathfrak{n}_n] \subset \mathfrak{n}_n$ and thus $b_n \subset N_{\mathfrak{g}}(\mathfrak{n}_n) \subset N_{\mathfrak{g}}(\mathfrak{n}_n + \mathbb{F}I_n)$.
Now we show that $\mathfrak{n}_n + \mathbb{F}I_n$ is a maximal nilpotent subalgebra. Note that since I_n commutes with everything, then $\mathfrak{h} = \mathfrak{n}_n \bigoplus \mathbb{F}I_n$. Hence, it is nilpotent. Now, it suffices to show that $\mathfrak{h}$ is maximal in gl_n. Suppose there exists some nilpotent subalgebra $\mathfrak{n}_n + \mathbb{F}I_n \subsetneq \mathfrak{h}'$. First note that in fact we have $b_n = N_{gl_n}(\mathfrak{n}_n + \mathbb{F}I_n)$, since if $b = \sum_{i,j} c_{ij}E_{ij} \in N_{\mathfrak{g}}(\mathfrak{n}_n + \mathbb{F}I_n)$ with $c_{i'j'} \neq 0$ for $i' > j'$, then $E_{j'i'} \in \mathfrak{n}_n \subset \mathfrak{n}_n + \mathbb{F}I_n$ and $[b, E_{j'i'}] = \sum_i c_{ij'}E_{ii'} - \sum_j c_{i'j}E_{j'j}$. Note that the $(i', i')^{th}$ and the $(j', j')^{th}$ entries are $c_{i'j'}$ and $-c_{i'j'}$ respectively; both of which are nonzero. Thus, $[b, E_{j'i'}]\mathfrak{n}_n + \mathbb{F}I_n$, unless $char(\mathbb{F}) = 2$. Now, by the Lemma above, we must have that $\mathfrak{n}_n + \mathbb{F}I_n \subsetneq \mathfrak{h} \cap N_{gl_n}(\mathfrak{n}_n + \mathbb{F}I_n)$. Thus, $\mathfrak{h}'$ contains some element of $b_n \backslash \mathfrak{n}_n + \mathbb{F}I_n$; but all elements of $b_n \backslash \mathfrak{n}_n + \mathbb{F}I_n$ have at least two distinct eigenvalues, and thus are not ad-nilpotent. Thus we find a contradiction to Engel's theorem and our assumption must be wrong. We find there is no proper set containing $\mathfrak{n}_n + \mathbb{F}I_n$; which implies that $\mathfrak{h}$ is a maximal nilpotent subalgebra.

Proposition 8.3. *Let $\mathfrak{g} \subset gl_n(\mathbb{F})$ be a subalgebra containing a diagonal matrix $a = diag(a_1, ..., a_n)$ with distinct a_i, and let $\mathfrak{h}$ be the subspace of all diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra.*

Proof. We have to prove that $\mathfrak{h}$ satisfies the two conditions necessary to be a Cartan subalgebra.
i) We know that $\mathfrak{h}$ is abelian; and thus, it is a nilpotent Lie algebra.
ii)Let $b = \sum_{i,j=1}^{n} b_{ij}e_{ij} \in \mathfrak{g}$ such that $[b, \mathfrak{h}] \subset \mathfrak{h}$ (i.e. $b \in N_{\mathfrak{g}}(\mathfrak{h})$). In particular, we have that $[a, b] \in \mathfrak{h}$ for all $a \in \mathfrak{h}$. But $[a, b] = [\sum_k a_k e_{kk}, \sum_{i,j} e_{ij}e_{ij}] = \sum_{ij}(a_i - a_j)b_{ij}e_{ij}$, which will be non-diagonal only if $b_{ij} \neq 0$ for some $i \neq j$. Thus, we can conclude that $b \in \mathfrak{h}$, and therefore $N_g(\mathfrak{h}) = \mathfrak{h}$. □

Remark $\mathfrak{g}$ is a Cartan subalgebra in itself if and only if $\mathfrak{g}$ is a nilpotent Lie algebra.

Theorem 8.4. *(E.Cartan) Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$. Let $a \in \mathfrak{g}$ be a regular element (which exists since $\mathbb{F}$ is infinite), and let $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_\lambda^a$ be the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to ada. Then $\mathfrak{h} = \mathfrak{g}_0^a$ is a Cartan subalgebra.*

Proof. The proof for this theorem uses the fact that Zariski Topology is highly non-Hausdorff, namely any two non-empty Zariski open sets have a non-empty intersection. We will also recall the fact that $[\mathfrak{g}_\lambda^a, \mathfrak{g}_\mu^a] \subset \mathfrak{g}_{\lambda+\mu}^a$ and in particular, if $\lambda = 0$ then $[\mathfrak{h}, \mathfrak{g}_\mu^a] \subset \mathfrak{g}_\mu^a$.
Let $V = \bigoplus_{\lambda \neq 0} \mathfrak{g}_\lambda^a$. Then $\mathfrak{g} = \mathfrak{h} \bigoplus V$ and $[\mathfrak{h}, V] \subset V$.
Consider the following two subsets of $\mathfrak{h}$:
$U = \{h \in \mathfrak{h}$ such that $adh|_{\mathfrak{h}}$ is not a nilpotent operator$\}$
$R = \{h \in \mathfrak{h}$ such that $adh|_V$ is a non-singular operator $\}$
Both U and R are Zariski open subsets of $\mathfrak{h}$. Next we note that $a \in R$ since all zero eigenvalues of ada lie in $\mathfrak{h}$, hence R is non-empty.
Now, we shall prove by contradiction that $\mathfrak{h}$ is a nilpotent subalgebra. Suppose the contrary is true. Then, by Engel's Theorem there exists $h \in \mathfrak{h}$ such that $adh|_{\mathfrak{h}}$ is not nilpotent. But in this case $h \in U$; and hence, $U \neq \emptyset$. Therefore $U \bigcap R \neq \emptyset$. We now take $b \in U \bigcap R$. Then $adh|_{\mathfrak{h}}$ is not nilpotent and $adb|_V$ is invertible. Hence $\mathfrak{g}_0^b \subsetneq \mathfrak{h}$, which contradicts the fact that a is a regular element. Thus, this contradicts the assumption made; and we find that $\mathfrak{h}$ is a nilpotent Lie algebra.
Finally, we need to proof that $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Now, if $b \in N_{\mathfrak{g}}(\mathfrak{h})$, so that $[b, \mathfrak{h}] \subset \mathfrak{h}$, then we have that, in particular, $[b, a] \in \mathfrak{h}$. But since $a \in \mathfrak{h}$ and $\mathfrak{h}$ is a nilpotent Lie algera, then $ada|_{\mathfrak{h}}$ is a nilpotent operator. In particular, $0 = (ada)^N((ada)b) = (ada)^{N+1}(b)$. Hence $b \in \mathfrak{g}_0^a = \mathfrak{h}$, which completes the proof of the theorem. □

Remark The dimension of the Cartan subalgebra constructed in Cartan's Theorem, Theorem 4, equals the rank of $\mathfrak{g}$.

Proposition 8.5. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Consider the generalized weight space decomposition (called root space decomposition) with respect to $\mathfrak{h}$: $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$. Then $\mathfrak{g}_0 = \mathfrak{h}$.*

Proof. Since by Engel's Theorem $adh|_{\mathfrak{h}}$ is nilpotent for all $h \in \mathfrak{h}$, it follows that $\mathfrak{h} \subseteq \mathfrak{g}_0$. But by definition of $\mathfrak{g}$, for all elements $h \in \mathfrak{h}$, $adh|_{\mathfrak{g}_0}$ is a nilpotent operator. Hence $adh|_{\mathfrak{g}_0/\mathfrak{h}}$ is a nilpotent operator for all $h \in \mathfrak{h}$. Therefore, by Engle's Theorem, there exists a non-zero $barb \in \mathfrak{g}_0/\mathfrak{h}$ which is annihilated by all $adh|_{\mathfrak{g}_0/\mathfrak{h}}$. Taking a pre-image $b \in \mathfrak{g}$ of $\bar{b}$, this means that $[b, \mathfrak{h}] \subset \mathfrak{h}$ (i.e. $\mathfrak{h} \neq N_{\mathfrak{g}}(\mathfrak{h})$), which contradicts the fact that $\mathfrak{h}$ is a Cartan subalgebra. □

Remark $\mathfrak{g} = \mathfrak{h} \bigoplus (\bigoplus_{\lambda \in \mathfrak{h}^*, \lambda \neq 0} \mathfrak{g}_\lambda)$ by the latter Proposition.

Next we will apply the last couple of theorems, lemmas and propositions in order to classify all 3-dimensional Lie algebras $\mathfrak{g}$ over an algebraically closed field $\mathbb{F}$ of characteristic zero.
We know that the $rank(\mathfrak{g}) = 3, 2$ or 1; and $rank(\mathfrak{g}) = 3$ if and only if $\mathfrak{g}$ is nilpotent.
$Rank(\mathfrak{g}) = 3$:
We know by exercise 6.1, that any 3-dimensional nilpotent Lie algebra is either abelian or H_3. So any 3-dimensional Lie algebra of $rank(\mathfrak{g}) = 3$ is either abelian or H_3.
$Rank(\mathfrak{g}) = 2$:
In this case $dim(\mathfrak{h}) = 2$. Since $\mathfrak{h}$ must be a nilpotent Lie algebra, we can conclude that $\mathfrak{h}$ is abelian (otherwise $\mathfrak{h}$ would be a 2-dimensional solvable Lie algebra, which is not nilpotent). Hence the root space decomposition is $\mathfrak{g} = \mathfrak{h} \bigoplus \mathbb{F}b$, where $[\mathfrak{h}, b] \subset \mathbb{F}b$. Since $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$ then $[\mathfrak{h}, b] \neq 0$. Hence there exist $a \in \mathfrak{h}$ such that $[a, b] = b$. Also since $\mathfrak{h}$ is 2-dimensional, then there exists $c \in \mathfrak{h}$ such that $[c, b] = 0$. We also know that $[a, c] = 0$. Thus, we can conclude that the only 3-dimensional Lie algebra of rank 2 is a direct sum of a 2-dimensional non-abelian algebra and one dimensional central subalgebra: $\mathfrak{g} = (\mathbb{F}a + \mathbb{F}b) \bigoplus \mathbb{F}c$.
$Rank(\mathfrak{g}) = 1$:

Exercise 8.2. Any 3-dimensional Lie algebra $\mathfrak{g}$ of rank 1 is isomorphic to one of the following Lie algebras with basis h, a, b:
i) $[h, a] = a$, $[h, b] = a + b$, $[a, b] = 0$;
ii) $[h, a] = a$, $[h, b] = \lambda b$, where $\lambda \in \mathbb{F}/\{0\}$ $[a, b] = 0$;
iii) $[h, a] = a$, $[h, b] = -b$, $[a, b] = h$;

Solution. Let $\mathfrak{h} = \mathbb{F}h$,where $h \in \mathfrak{g}$, be a Cartan subalgebra; we have: $\mathfrak{g} = \mathfrak{h} \bigoplus V$, where $[h, V] \subset V$, $dim(V) = 2$, and adh is non-singular on V. We may assume one of the eigenvalues of $adh|_V$ is 1, if we scale h accordingly.
First, suppose $adh|_V$ is not semisimple. Thus, $adh|_V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$in some basis $\{a, b\}$. Thus, $[h, a] = a$ and $[h, b] = a + b$. Also, by using Jacobian Identity, we have $[h, [a, b]] = [[h, a], b] + [a, [h, b]] = [a, b] + [a, a + b] = 2[a, b]$. But we know that the value 2 cannot be an eigenvalue of adh, thus $[a, b] = 0$. Thus, this Lie alebra corresponds to (i).
Now, assume $adh|_V$ is semisimple. Thus, we have that $adh|_V = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$in some basis $\{a, b\}$. We then have that $[h, a] = a$ and $[h, b] = \lambda b$. By following the same procedure as above, we note that $[h, [a, b]] = [[h, a], b] + [a, [h, b]] = [a, b] + [a, \lambda b] = (1 + \lambda)[a, b]$. Thus, we must have that either $[a, b] = 0$, which corresponds to (ii), or $(1 + \lambda)$ is an eigenvalue of $adh|_V$, case (iii). Note that for the case of $[a, b] = 0$, then λ is arbitrary and uniquely defined by $\mathfrak{g}$ up to inverting it by swapping a and b and scaling h accordingly. In the case were $(1 + \lambda)$ is an eigenvalue of $adh|_V$, we must have that $1 + \lambda = 0$ and that $[a, b]$ be a multiple of h. Thus $\lambda = -1$ and, scaling a accordingly, we may assume that $[a, b] = h$. Thus, we get option (iii) with the latter case.

Exercise 8.3. Show that all Lie algebras in exercise 8.2 are non-isomorphic . Those from (i) and (ii) are solvable, and the one from (iii) is isomorphic to $sl_2(\mathbb{F})$, which is not solvable.

Solution. We first show that the Lie algebra from (iii) is isomorphic to $sl_2(\mathbb{F})$. The standard basis for $sl_2(\mathbb{F})$ consist of { h'= $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $a' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$}.Thus, we have

$[h, a'] = 2a', [h, b'] = -2b'$ and $[a', b'] = h$. Now if we scale and set $h = \frac{h'}{2}$, $a = \frac{a'}{\sqrt{2}}$ and $b = \frac{b'}{\sqrt{2}}$; then we get the Lie algebra of case (iii). Thus, Lie algebra (iii) is isomorphic to $sl_2(\mathbb{F})$, which is not solvable. In contrast, it is clear that the algebras (i) and (ii) are solvable by construction.

By conjugacy of Cartan subalgebras, the isomorphism class of $\mathfrak{g}$ needs to be independent of the choice of the Cartan subalgebra. It follows that the algebras of the three types are non-isomorphic to each other. It also follows that the algebras of type (ii), corresponding to parameters λ and λ' are isomorphic if and only if $\lambda\lambda' = 1$

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Lecture 9 — Chevalley's Theorem

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In the last lecture we defined Cartan subalgebras and gave a construction using regular elements. In this lecture we will show that this construction is essentially unique by proving Chevalley's Theorem on conjugacy of Cartan subalgebras.

To state the theorem, we need the notion of the exponential of a nilpotent operator.

Definition 9.1. Let A be a nilpotent operator on a vector space V over a field $\mathbb{F}$ of characteristic 0. Define $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$. (As A is nilpotent, this is a *finite* sum.)

Exercise 9.1. If A and B are commuting nilpotent operators, then we know that $A+B$ is also nilpotent, so e^{A+B}, e^A, and e^B are all defined. Show that $e^{A+B} = e^A e^B$. In particular, deduce that $e^A e^{-A} = I$, so e^A is always an invertible operator.

Solution: As A, B, and $A+B$ are nilpotent, we can find an integer $N \geq 0$ such that $A^{N+1} = B^{N+1} = (A+B)^{(2N+1)} = 0$. Because A and B commute, we can use the binomial theorem to get

$$e^{A+B} = \sum_{n=0}^{2N} \frac{(A+B)^n}{n!} = \sum_{n=0}^{2N} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} = \sum_{\substack{k,\ell \geq 0 \\ k+\ell \leq 2N}} \frac{1}{k!\ell!} A^k B^\ell.$$

Every term above with $k > N$ or $\ell > N$ vanishes, as either $A^k = 0$ or $B^\ell = 0$. Conversely, every term with $k \leq N$ and $\ell \leq N$ is present in the sum. Thus

$$e^{A+B} = \sum_{k,\ell \leq N} \frac{1}{k!\ell!} A^k B^\ell = \left(\sum_{k=0}^{N} \frac{A^k}{k!} \right) \left(\sum_{\ell=0}^{N} \frac{B^\ell}{\ell!} \right) = e^A e^B,$$

as desired. The "deduce" part is immediate, as A and $-A$ commute and $e^0 = I$. □

Exercise 9.2. Let $\mathfrak{g}$ be an arbitrary (not necessarily Lie) algebra over a field $\mathbb{F}$ of characteristic 0, and let D be a nilpotent derivation of $\mathfrak{g}$. Show that e^D is an automorphism of $\mathfrak{g}$.

Solution: We already know from Exercise 9.1 that e^D is a linear isomorphism; the content of this exercise is that it also respects multiplication in $\mathfrak{g}$. Let $\mu : \mathfrak{g} \otimes_F \mathfrak{g} \to \mathfrak{g}$ denote the multiplication map. Define linear transformations D_1 and D_2 on $\mathfrak{g} \otimes_F \mathfrak{g}$ by $D_1(a \otimes b) = D(a) \otimes b$ and $D_2(a \otimes b) = a \otimes D(b)$. It is clear that D_1 and D_2 commute, that they are both nilpotent, and that $e^{D_1}(a \otimes b) = e^D(a) \otimes b$ and $e^{D_2}(a \otimes b) = a \otimes e^D(b)$. Next, because D is a derivation, notice that

$$(\mu \circ (D_1 + D_2))(a \otimes b) = D(a) \cdot b + a \cdot D(b) = D(a \cdot b) = (D \circ \mu)(a \otimes b),$$

so $\mu \circ (D_1 + D_2) = D \circ \mu$. It follows easily that $\mu \circ e^{D_1 + D_2} = e^D \circ \mu$. Applying Exercise 9.1,

$$e^D(a \cdot b) = (e^D \mu)(a \otimes b) = (\mu e^{D_1 + D_2})(a \otimes b) = (\mu e^{D_1} e^{D_2})(a \otimes b) = e^D(a) \cdot e^D(b),$$

as desired. □

We are now ready to state the main result.

Theorem 9.1 (Chevalley)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra of an algebraically closed field $\mathbb{F}$ of characteristic 0. Denote by G the subgroup of the group of automorphisms of $\mathfrak{g}$ which is generated by automorphisms of the form $e^{\operatorname{ad} a}$ for $a \in \mathfrak{g}$ such that $\operatorname{ad} a$ is nilpotent. Then any two Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate by G, i.e. there exists $\sigma \in G$ such that $\sigma(\mathfrak{h}_1) = \mathfrak{h}_2$.*

Remark 9.1. The group G in Chevalley's Theorem is almost (but not quite) the Lie group associated to the Lie algebra $\mathfrak{g}$.

Before proving Chevalley's Theorem, we give a corollary that addresses the question with which we opened the lecture.

Corollary 9.2. *Let $\mathbb{F}$ be an algebraically closed field of characteristic 0 and let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{F}$. Then any Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ is of the form $\mathfrak{g}_0^a$ for some regular element $a \in \mathfrak{g}$, and in particular $\dim \mathfrak{h} = \operatorname{rank} \mathfrak{g}$. Also, all such subalgebras $\mathfrak{g}_0^a$ are isomorphic.*

Proof. Fix a regular element $a \in \mathfrak{g}$. By Chevalley's Theorem, any Cartan subalgebra $\mathfrak{h}$ is conjugate to $\mathfrak{g}_0^a$, say $\mathfrak{h} = \sigma(\mathfrak{g}_0^a)$. Hence $\dim \mathfrak{h} = \dim \mathfrak{g}_0^a = \operatorname{rank} \mathfrak{g}$. But because σ is an algebra automorphism, it is easy to check that $\sigma(\mathfrak{g}_0^a) = \mathfrak{g}_0^{\sigma(a)}$. The dimensionality of this tells us that $\sigma(a) \in \mathfrak{g}$ is a regular element. Finally, the last claim is immediate as conjugate subalgebras are isomorphic. □

To prepare for the proof of Chevalley's Theorem, we first prove two lemmas.

Lemma 9.3. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra and suppose there is a regular element $a \in \mathfrak{g}$ which is in $\mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{g}_0^a$.*

Proof. Being a Cartan subalgebra, $\mathfrak{h}$ is nilpotent. Thus $\operatorname{ad} a|_{\mathfrak{h}}$ is nilpotent, and so $\mathfrak{h} \subseteq \mathfrak{g}_0^a$. Now we know from the last lecture that $\mathfrak{g}_0^a$ is a Cartan subalgebra, so in particular it is nilpotent. We also know from the last lecture that $\mathfrak{h}$, being a Cartan subalgebra, must be a maximal nilpotent subalgebra. Hence $\mathfrak{h} \subseteq \mathfrak{g}_0^a$ implies $\mathfrak{h} = \mathfrak{g}_0^a$, as desired. □

The second lemma is a special case of a result from algebraic geometry.

Lemma 9.4. *Let $f : \mathbb{F}^m \to \mathbb{F}^m$ be a polynomial map, i.e.*

$$f(x_1, \ldots, x_m) = (f_1(x_1, \ldots, x_m), \ldots, f_m(x_1, \ldots, x_m))$$

where the f_i's are polynomials. Suppose that for some $a \in \mathbb{F}^m$ the linear map $df|_{x=a} : \mathbb{F}^m \to \mathbb{F}^m$ is nonsingular. Then the image $f(\mathbb{F}^m)$ contains a nonempty Zariski open subset of $\mathbb{F}^m$.

Exercise 9.3. Prove Lemma 9.4 by the following steps.

1. Note that $\mathrm{d}f|_{x=a}$ is the linear map $\mathbb{F}^m \to \mathbb{F}^m$ given by the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1}^m$.
2. Suppose for contradiction that $F(f_1, \ldots, f_m) = 0$ identically for some nonzero polynomial F. Show that $\det\left(\frac{\partial f_i}{\partial x_j}\right) = 0$. (Thus the polynomials f_i are algebraically independent.)
3. Given algebraically independent elements $y_1, \ldots, y_m \in \mathbb{F}[x_1, \ldots, x_m]$, show that the extension of fields
$$\mathbb{F}(x_1, \ldots, x_m) \supseteq \mathbb{F}(y_1, \ldots, y_m)$$
is finite, i.e. each x_i satisfies a nonzero polynomial equation over $\mathbb{F}(y_1, \ldots, y_m)$.

4. For each $i = 1, 2, \ldots, m$, take a polynomial equation satisfied by x_i over $\mathbb{F}(f_1, \ldots, f_m)$, clear denominators to get a polynomial over $\mathbb{F}[f_1, \ldots, f_m]$, and let $p_i(f_1, \ldots, f_m)$ be the leading coefficient of this polynomial. Show that the set of points $\{y \in \mathbb{F}^m : p_i(y) \neq 0, i = 1, 2, \ldots, m\}$ satisfies Lemma 9.4.

Solution: Step 1 is completely standard. For step 2, suppose for contradiction that the f_i's are algebraically dependent, and let $F(y_1, \ldots, y_m)$ be a nonzero polynomial of minimal degree such that $F(f_1, \ldots, f_m) = 0$ identically. For any $j = 1, 2, \ldots, m$, apply the partial derivative $\partial/\partial x_j$ to the identity $F(f_1, \ldots, f_m) = 0$. By the chain rule, this gives the equation

$$\sum_{i=1}^{m} \left.\frac{\partial F}{\partial y_i}\right|_{(f_1,\ldots,f_m)} \cdot \frac{\partial f_i}{\partial x_j} = 0.$$

Let $J(x)$ be the Jacobian matrix $\left(\left.\frac{\partial f_i}{\partial x_j}\right|_x \right)_{i,j=1}^{m}$ and let $\vec{v}(y)$ be the row vector $\left(\left.\frac{\partial F}{\partial y_i}\right|_{(y_1,\ldots,y_m)} \right)_{i=1}^{m}$. Then the above equations yield $\vec{v}(f(x))J(x) = 0$.

Let $\mathcal{L}$ be the locus of points x such that $\det(J(x)) \neq 0$ and let $\mathcal{L}'$ be the locus of points x such that $\vec{v}(f(x)) \neq 0$. At any point of $\mathcal{L}'$, the matrix J has a nonzero null vector, so it is singular: thus $\mathcal{L} \cap \mathcal{L}' = \emptyset$. Both $\mathcal{L}$ and $\mathcal{L}'$ are Zariski open subsets of $\mathbb{F}^m$, and $\mathcal{L}$ is nonempty by assumption. As the base field $\mathbb{F}$ is infinite, the only way for the nonempty open set $\mathcal{L}$ not to intersect the open set $\mathcal{L}'$ is to have $\mathcal{L}' = \emptyset$. But this means that the polynomials in $\vec{v}$, i.e. the partial derivatives $\partial F/\partial y_i$, are polynomials in $y_1, \cdots, y_n$ which vanish identically upon substitution of the f_i's. As $\deg(\partial F/\partial y_i) < \deg F$, our minimality assumption on $\deg F$ then implies $\partial F/\partial y_i = 0$. Over a field of characteristic zero, a polynomial whose partial derivatives all vanish identically must be constant. But F is not constant; contradiction.

For step 3, note that both $\mathbb{F}(x_1, \ldots, x_m)$ and $\mathbb{F}(y_1, \ldots, y_m)$ have transcendence degree m over $\mathbb{F}$, and an extension of fields with the same transcendence degree must be algebraic. Hence each x_i is algebraic over $\mathbb{F}(y_1, \ldots, y_m)$, and in particular they generate a finite extension.

For step 4, note that the stated set is certainly a nonempty Zariski open set, as it is the nonvanishing locus of the nonzero polynomial $p_1 \cdots p_m$. We need to show that it is in the image of f, so choose any y such that $p_1(y), \ldots, p_m(y) \neq 0$. Because the elements f_i are algebraically independent, we can define an evaluation homomorphism

$$e_y : \mathbb{F}[f_1, \ldots, f_m] \to \mathbb{F}$$

mapping each f_i to y_i. By our assumption on y, e_y maps each p_i to a nonzero field element. Thus e_y extends uniquely to a map on the localization, $\hat{e}_y : \mathbb{F}[f_1, \ldots, f_m, p_1^{-1}, \ldots, p_m^{-1}] \to \mathbb{F}$. For each i, it follows from the definition of p_i that x_i is integral over the ring $\mathbb{F}[f_1, \ldots, f_m, p_1^{-1}, \ldots, p_m^{-1}]$. We now use the following fact from commutative algebra:

Given an integral extension of rings $R \subseteq R'$ and a ring homomorphism from R to an algebraically closed field, there exists an extension of that ring homomorphism to R'.

Applying this, we get an extension of $\hat{e}_y$ to a homomorphism $\hat{\hat{e}}_y : \mathbb{F}[x_1, \ldots, x_m] \to \mathbb{F}$. Unwrapping definitions, the fact that $\hat{\hat{e}}_y$ extends e_y tells us that $f(\hat{\hat{e}}_y(x_1), \ldots, \hat{\hat{e}}_y(x_m)) = e_y(f_1, \ldots, f_m) = y$, so that indeed y is in the image of f. $\square$

Example 9.1. Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. The image is $f(\mathbb{R}) = \mathbb{R}_{\geq 0}$, which does not contain a nonempty Zariski open set (because nonempty Zariski open sets in one dimension are cofinite). Thus the algebraic closure assumption in Lemma 9.4 is necessary. (The characteristic assumption is not necessary, though.)

Example 9.2. It is also important in Lemma 9.4 that the map f be algebraic. There is a well-known example of a smooth (but not algebraic) map from $\mathbb{R}$ to the torus which does not include any nonempty open set in its image.

We conclude with the proof of the main result.

Proof of Chevalley's Theorem: Let $\mathfrak{h}$ be any Cartan subalgebra of $\mathfrak{g}$. As $\mathfrak{h}$ is nilpotent, we can define a corresponding generalized root space decomposition $\mathfrak{g} = \oplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha^{\mathfrak{h}}$. For later use, we observe that $\mathfrak{g}_0^{\mathfrak{h}} = \mathfrak{h}$ by a proposition from last lecture.

We claim that for any $\alpha \neq 0$ and any $x \in \mathfrak{g}_\alpha^{\mathfrak{h}}$, $\operatorname{ad} x$ is nilpotent. To prove this, recall that the root space decomposition satisfies $[\mathfrak{g}_\beta^{\mathfrak{h}}, \mathfrak{g}_\gamma^{\mathfrak{h}}] \subseteq \mathfrak{g}_{\beta+\gamma}^{\mathfrak{h}}$ for all β, γ. Thus for any β and any $N \geq 1$, $(\operatorname{ad} x)^N \mathfrak{g}_\beta^{\mathfrak{h}} \subseteq \mathfrak{g}_{\beta+N\alpha}^{\mathfrak{h}}$. As $\alpha \neq 0$ and $\operatorname{char} F = 0$, $\{\beta + N\alpha : N \geq 1\}$ is an infinite set of distinct functionals. There are only finitely many nonzero root spaces $\mathfrak{g}_\gamma^{\mathfrak{h}}$, so for some N we must have $(\operatorname{ad} x)^N \mathfrak{g}_\beta^{\mathfrak{h}} = 0$. This holds for all root spaces $\mathfrak{g}_\beta$. Replacing N by its maximum over the finite number of nonzero root spaces, we have $(\operatorname{ad} x)^N \mathfrak{g} = 0$, which proves the claim.

Our next goal is to show that there is a Zariski open subset of $\mathfrak{g}$ consisting of images of elements of $\mathfrak{h}$ under the action of the group G. Let $\{\alpha_i\}$ be the set of nonzero functionals α such that $\mathfrak{g}_\alpha^{\mathfrak{h}} \neq 0$, and let $\{b_j\}_{j=1}^m$ be a basis for $\oplus_i \mathfrak{g}_{\alpha_i}^{\mathfrak{h}}$ consisting of a union of bases for each $g_{\alpha_i}^{\mathfrak{h}}$. Then any element of $\mathfrak{g}$ has a unique expansion of the form $h + \sum_{j=1}^m x_j b_j$ for some $h \in \mathfrak{h}$ and some scalars $x_j \in \mathbb{F}$. Define a map $f : \mathfrak{g} \to \mathfrak{g}$ by

$$f\left(h + \sum_{j=1}^m x_j b_j\right) = e^{x_1 \operatorname{ad} b_1} e^{x_2 \operatorname{ad} b_2} \cdots e^{x_m \operatorname{ad} b_m}(h).$$

This is well-defined because each $\operatorname{ad} b_j$ is nilpotent by the result of the last paragraph. Moreover, it is a polynomial map. Looking to apply Lemma 9.4, we compute the differential $\mathrm{d}f$ at an arbitrary point $a \in \mathfrak{h}$, evaluated at a point $b + h$ (where $h \in \mathfrak{h}$ and $b = \sum x_j b_j$). This is

$$\mathrm{d}f|_{x=a}(b+h) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} [f(a + t(b+h))] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[e^{tx_1 \operatorname{ad} b_1} e^{tx_2 \operatorname{ad} b_2} \cdots e^{tx_m \operatorname{ad} b_m}(a + th)\right].$$

To compute this derivative, it suffices to expand the function we are differentiating to first order in t. Doing so, we find

$$\mathrm{d}f|_{x=a}(b+h) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[\left(\prod_{j=1}^m (I + tx_j \operatorname{ad} b_j)\right)(a + th)\right] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[(a+th) + \sum_{j=1}^m tx_j (\operatorname{ad} b_j)(a)\right],$$

which equals $h + \sum_{j=1}^m x_j [b_j, a] = h + [b, a]$. Thus the linear operator $\mathrm{d}f|_{x=a}$ restricts to the identity on $\mathfrak{h}$ and to $-\operatorname{ad} a$ on $\oplus_i \mathfrak{g}_{\alpha_i}^{\mathfrak{h}}$. On each space $\mathfrak{g}_{\alpha_i}^{\mathfrak{h}}$, the only eigenvalue of $-\operatorname{ad} a$ is $-\alpha_i(a)$. Thus if we can find $a \in \mathfrak{h}$ such that $\alpha_i(a) \neq 0$ for each i, then $\mathrm{d}f|_{x=a}$ will act invertibly on each generalized root space and so it will be nonsingular as an operator on $\mathfrak{g}$.

It is easy to prove that this is possible. Indeed, for each i we can find $a_i \in \mathfrak{h}$ such that $\alpha_i(a_i) \neq 0$, just because the functional $\alpha_i \in \mathfrak{h}^*$ is nonzero. We identify the span of the a_i's with affine space via the association $f = (f_i) \mapsto a = \sum_i f_i a_i$. For each i, the condition $\alpha_i(a) \neq 0$ defines a nonempty Zariski open subset. The (finite) intersection of these over all i is then also nonempty, as desired.

We have now proven that the polynomial map f has the property that $\mathrm{d}f|_{x=a}$ is nonsingular at some point a. Then by Lemma 9.4, the image of f contains a nonempty Zariski open subset of $\mathfrak{g}$. Let $\Omega_{\mathfrak{h}}$ be such a subset. Notice that, by definition, the image of f (and thus the subset $\Omega_{\mathfrak{h}}$) consists of points of the form $\sigma(h)$ for some $\sigma \in G$ and $h \in \mathfrak{h}$.

The arguments so far hold for any Cartan subalgebra of $\mathfrak{g}$. Thus, letting $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two arbitrary Cartan subalgebras, we obtain corresponding nonempty Zariski open subsets $\Omega_{\mathfrak{h}_1}$ and $\Omega_{\mathfrak{h}_2}$ of $\mathfrak{g}$. Let Ω_r be the subset of all regular elements of $\mathfrak{g}$; this is also a nonempty Zariski open subset. The intersection $\Omega_{\mathfrak{h}_1} \cap \Omega \mathfrak{h}_2 \cap \Omega_r$ is then nonempty. Rephrasing this, there exists a regular element $x \in \mathfrak{g}$, elements $h_1 \in \mathfrak{h}_1$ and $h_2 \in \mathfrak{h}_2$, and automorphisms $\sigma_1, \sigma_2 \in G$ such that $\sigma_1(h_1) = x = \sigma_2(h_2)$. As x is regular and σ_1 is an algebra automorphism of $\mathfrak{g}$, $h_1 = \sigma_1^{-1}(x)$ is also regular. Thus by Lemma 9.3, $\mathfrak{h}_1 = \mathfrak{g}_0^{h_1}$. Similarly, $\mathfrak{h}_2 = \mathfrak{g}_0^{h_2}$. The automorphism $\sigma = \sigma_2^{-1}\sigma_1 \in G$ maps h_1 to h_2, and so (as in the proof of Corollary 9.2)

$$\sigma(\mathfrak{h}_1) = \sigma(\mathfrak{g}_0^{h_1}) = \mathfrak{g}_0^{\sigma(h_1)} = \mathfrak{g}_0^{h_2} = \mathfrak{h}_2.$$

This finishes the proof of Chevalley's Theorem. $\square$

18.745 Introduction to Lie Algebras	October 12, 2010

Lecture 10 — Trace Form & Cartan's criterion

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Definition 10.1. Let $\mathfrak{g}$ be a Lie algebra and π a representation of $\mathfrak{g}$ on a finite dimensional vector space V. The associated trace form is a bilinear form on $\mathfrak{g}$, given by the following formula:

$$(a, b)_V = \text{tr } (\pi(a)\pi(b))$$

Proposition 10.1. *(i) The trace form is symmetric, i.e.* $(a, b)_V = (b, a)_V$.

(ii) The trace form is invariant, i.e. $([a, b], c)_V = (a, [b, c])_V$.

Proof. (i) follows from the fact that tr $(AB) =$ tr (BA). For (ii), note the following:

$$\begin{aligned}([a, b], c)_V &= \text{tr } (\pi([a, b])\pi(c)) = \text{tr } (\pi(a)\pi(b)\pi(c) - \pi(b)\pi(a)\pi(c)) \\ &= \text{tr } (\pi(a)\pi(b)\pi(c) - \pi(a)\pi(c)\pi(b)) = \text{tr } (\pi(a)\pi([b, c])) \\ &= (a, [b, c])_V\end{aligned}$$

□

Definition 10.2. If $\dim \mathfrak{g} < \infty$, then the trace form of the adjoint representation is called the Killing form:

$$\kappa(a, b) = \text{tr } ((\textbf{ad } a)(\textbf{ad } b))$$

Exercise 10.1. Let $\mathbb{F}$ be a field of characteristic 0. Suppose $(\cdot, \cdot)$ be an invariant bilinear form on $\mathfrak{g}$. Show that if $v \in \mathfrak{g}$ is such that **ad** a is nilpotent, then $(e^{\text{ad } v}a, e^{\text{ad } v}b) = (a, b)$. In other words, the bilinear form is invariant with respect to the group G generated by $e^{\text{ad } v}$, **ad** v nilpotent.

Proof.

$$\begin{aligned}(e^{\text{ad } v}a, e^{\text{ad } v}b) &= (a + (\textbf{ad } v)a + \frac{1}{2!}(\textbf{ad } v)^2 + \cdots, b + (\textbf{ad } v)b + \frac{1}{2!}(\textbf{ad } v)^2 b + \cdots) \\ &= (a, b) + \sum_{i \geq 1} \sum_{j+k=i} \frac{1}{j!k!}((\textbf{ad } v)^j a, (\textbf{ad } v)^k b)\end{aligned}$$

Thus it suffices to prove that for $i \geq 1$, $\sum_{j+k=i} \frac{1}{j!k!}((\textbf{ad } v)^j a, (\textbf{ad } v)^k b) = 0$. Note the following:

$$((\textbf{ad } v)^j a, (\textbf{ad } v)^k b) = ((\textbf{ad } v)(\textbf{ad } v)^{j-1} a, (\textbf{ad } v)^k b) = -((\textbf{ad } v)^{j-1} a, (\textbf{ad } v)^{k+1} b)$$

Here we have used the invariance of $(\cdot, \cdot)$. So this means $((\textbf{ad } v)^i a, b) = -((\textbf{ad } v)^{i-1}, (\textbf{ad } v)b) = ((\textbf{ad } v)^{i-2}, (\textbf{ad } v)^2 b) = \cdots$. So:

$$\sum_{j+k=i} \frac{1}{j!k!}((\textbf{ad } v)^j a, (\textbf{ad } v)^k b) = \sum_{j+k=i} \frac{1}{j!k!}(-1)^k((\textbf{ad } v)^i a, b) = \frac{1}{i!}((\textbf{ad } v)^i a, b) \sum (-1)^k \frac{i!}{j!k!} = 0$$

Above we have used the fact that $\sum_{k=0}^{i} (-1)^k \binom{i}{k} = 0$. □

Exercise 10.2. Show that the trace form of $\mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F})$ associated to the standard representation is non-degenerate and the Killing form on $\mathfrak{sl}_n(\mathbb{F})$ is also non-degenerate, provided char $\mathbb{F} \nmid 2n$. Find the kernel of the Killing form on $\mathfrak{gl}_n(\mathbb{F})$.

Proof. To show the trace form on $\mathfrak{gl}_n(\mathbb{F})$ for the standard representation is non-degenerate, if $\sum_{i,j} a_{ij}e_{ij}$ lies in the kernel where for some r, s, $a_{rs} \neq 0$, then $\text{tr}((\sum_{i,j} a_{ij}e_{ij})e_{sr}) = \text{tr}(a_{rs}e_{rs}e_{sr}) = a_{rs} \neq 0$ since $\text{tr}(e_{i,j}e_{s,r}) = 0$ unless $j = s, i = r$, which is a contradiction. To show the trace form on $\mathfrak{sl}_n(\mathbb{F})$ for the standard representation is non-degenerate, if $x = \sum_{i,j} a_{ij}e_{ij}$ lies in the kernel and some $a_{rs} \neq 0$, then by the same argument we have a contradiction. So say $a_{rs} = 0$ for $r \neq s$, so $x = \sum_{a_{ii}e_{ii}}$. If $a_{jj} \neq a_{kk}$ for some j, k, then $\text{tr}(\sum a_{ii}e_{ii}, e_{jj} - e_{kk}) = a_{jj} - a_k \neq 0$, which is a contradiction. So $a_{jj} = a_{kk} \forall j, k$, so $\text{tr}x = na_{11} = 0$, so $a_{11} = 0$ (since char $\mathbb{F} \nmid n$, and $x = 0$).

For the killing form on $\mathfrak{gl}_n(\mathbb{F})$, consider the basis of $\mathfrak{gl}_n(\mathbb{F})$, $\{e_{ij}\}$. Then:

$$\textbf{ad } e_{ij}\textbf{ad } e_{kl}(e_{gh}) = [e_{ij}, \delta_{lg}e_{kl} - \delta_{lk}e_{gl}] = \delta_{jk}\delta_{lg}e_{kh} - \delta_{ki}\delta_{lg}e_{kj} - \delta_{jg}\delta_{hk}e_{il} + \delta_{li}\delta_{hk}e_{gj}$$

The coefficient of e_{gh} in this expansion is $a_{gh} = \delta_{gi}\delta_{jk}\delta_{lg} - \delta_{jk}\delta_{hg}\delta_{hi}\delta_{lg} - \delta_{gi}\delta_{lh}\delta_{jg}\delta_{hk} + \delta_{hj}\delta_{li}\delta_{hk}$. So:

$$\begin{aligned}\kappa_{\mathfrak{gl}_n}(e_{ij}e_{kl}) &= \sum_{g,h} a_{gh} = \sum_{g,h}(\delta_{gi}\delta_{jk}\delta_{lg} - \delta_{gk}\delta_{hg}\delta_{hi}\delta_{lg} - \delta_{gl}\delta_{kl}\delta_{jg}\delta_{hk} + \delta_{hj}\delta_{li}\delta_{hk}) \\ &= n\delta_{il}\delta_{jk} - \delta_{kl}\delta_{ij} - \delta_{ij}\delta_{kl} + n\delta_{jk}\delta_{il} = 2n\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl} = 2n\text{tr}(e_{ij}e_{kl}) - 2\text{tr}(e_{ij})\text{tr}(e_{kl})\end{aligned}$$

It follows that $\kappa_{\mathfrak{gl}_n}(x, y) = 2n\text{tr}(xy) - 2\text{tr}(x)\text{tr}(y)$ by bilinearity. To calculate the radical of $\kappa_{\mathfrak{gl}_n}$, note if $x = \sum_{i,j} x_{i,j}e_{i,j}$, $\kappa_{\mathfrak{gl}_n}(x, e_{kl}) = 2nx_{lk} - 2(\sum x_{ii})\delta_{kl}$. If this is always 0, $x_{lk} = 0$ when $k \neq l$, and $nx_{kk} = \sum_i x_{ii}$, so $x = \lambda I$ for some λ (since char$\mathbb{F} \nmid 2n$). So the radical of $\kappa_{\mathfrak{gl}_n}$ is $\mathbb{F}I$. By a theorem from Lecture 11, since $\mathfrak{sl}_n(\mathbb{F})$ is an ideal of $\mathfrak{gl}_n(\mathbb{F})$, $\kappa_{\mathfrak{sl}_n}$ is the restriction of $\kappa_{\mathfrak{gl}_n}$ to $\mathfrak{sl}_n(\mathbb{F})$; hence $\kappa_{\mathfrak{sl}_n}(x, y) = 2n\text{tr}(x)\text{tr}(y)$. Since it is a scalar multiple of the trace form of the standard representation, which is non-degenerate, it follows that the radical of $\kappa_{\mathfrak{sl}_n}$ is trivial. $\square$

Lemma 10.2 (Cartan)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{F} = \overline{\mathbb{F}}$, a field of characteristic* 0 *(so that $\mathbb{Q} \subset \mathbb{F}$). Let π be a representation of $\mathfrak{g}$ in a finite dimensional vector space V. Let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}$, and consider the generalized weight space decomposition of V and the generalized root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$:*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \pi\left(\mathfrak{g}_\alpha\right) V_\lambda \subseteq V_{\lambda+\alpha}, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

Pick $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$, so that $h = [e, f] \in \mathfrak{g}_0 = \mathfrak{h}$. Suppose that $V_\lambda \neq 0$. Then $\lambda(h) = r\alpha(h)$, where $r \in \mathbb{Q}$ depends only on λ and α but not on h.

Proof. Let $U = \bigoplus_{n \in \mathbb{Z}} V_{\lambda+n\alpha} \subset V$. Then dim$U < \infty$, and U is $\pi(e), \pi(f)$ and $\pi(h)$ invariant. But $[\pi(e), \pi(f)] = \pi(h)$, hence $\text{tr}_U(\pi(h)) = 0$. Thus we have the following:

$$0 = \text{tr}_U(\pi(h)) = \sum_n \text{tr}_{V_{\lambda+n\alpha}}(\pi(h)) = \sum_n (\lambda + n\alpha)(h)\text{dim } V_{\lambda+n\alpha}$$

In the last line we have used the fact that the matrix of $\pi(h)|_{V_{\lambda+n\alpha}}$ takes the following form:

$$A = \begin{pmatrix} (\lambda+n\alpha)(h) & * & & \\ & (\lambda+n\alpha)(h) & * & \\ & & \ddots & \\ & & & (\lambda+n\alpha)(h) \end{pmatrix}$$

$$\implies \lambda(h)(\sum_n \dim V_{\lambda+n\alpha}) = -\alpha(h)\sum_n n \dim V_{\lambda+n\alpha}$$
$$\implies \lambda(h) = r\alpha(h), r = -\frac{\sum_n n \dim V_{\lambda+n\alpha}}{\sum_n \dim V_{\lambda+n\alpha}}$$

Note in the above that $V_\lambda \neq 0$, so $\sum_n \dim V_{\lambda+n\alpha} \neq 0$. □

Theorem 10.3 (Cartan's criterion)**.** *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$, where V is a finite dimensional vector space over $\mathbb{F} = \overline{\mathbb{F}}$, a field of characteristic 0. Then the following are equivalent:*

1. *$(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$, i.e. $(a, b)_V = 0$ for $a \in \mathfrak{g}, b \in [\mathfrak{g}, \mathfrak{g}]$.*
2. *$(a, a)_V = 0$ for all $a \in [\mathfrak{g}, \mathfrak{g}]$.*
3. *$\mathfrak{g}$ is solvable.*

Proof. (i) $\implies$ (ii): Obvious.

(iii) $\implies$ (i): By Lie's theorem, in some basis of V, all matrices of $\mathfrak{g}$ are upper triangular, and thus $[\mathfrak{g}, \mathfrak{g}]$ is strictly upper triangular. Thus $\pi(ab)$ is strictly upper triangular and $(a, b)_V = 0$ if $a \in \mathfrak{g}, b \in [\mathfrak{g}, \mathfrak{g}]$.

(ii) $\implies$ (iii): Suppose not. Then the derived series of $\mathfrak{g}$ stabilizes, so suppose $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^(k)$ for some k with $\mathfrak{g}^{(k)} \neq 0$. Then $(a, a)_V = 0$ for $a \in \mathfrak{g}^{(k)}$ and $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^{(k)}$. We reach the desired contradiction using the following Lemma. □

Lemma 10.4. *If $\mathfrak{g} \subset \mathfrak{gl}_V$, such that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then $(a, a)_V \neq 0$ for some $a \in \mathfrak{g}$.*

Proof. Proof by contradiction. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and let:

$$V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda, \mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha$$

Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, we obtain that $\mathfrak{h} = \sum_{\alpha\in\mathfrak{h}^*}[g_\alpha, g_{-\alpha}]$. Hence by Cartan's Lemma $\lambda(h) = r_{\lambda,\alpha}\alpha(h)$ for $r_{\lambda,\alpha} \in \mathbb{Q}, \neq 0$ if $V_\lambda \neq 0$. The assumption that $(a, a)_V = 0$ means that, for all $h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$:

$$0 = (h, h)_V = \mathrm{tr}_V(\pi(h)^2) = \sum_{\lambda\in\mathfrak{h}*} \mathrm{tr}_{V_\lambda}(\pi(h)^2) = \sum_{\lambda\in\mathfrak{h}*} \lambda(h)^2 \dim V_\lambda = \alpha(h)^2 \sum_\lambda r^2_{\lambda,\alpha} \dim V_\lambda$$

In the above, we have used the fact that $\mathrm{tr}_{V_\lambda}(\pi(h)^2) = \lambda(h)^2 \dim V_\lambda$, which follows from the Jordan form theorem, since $\pi(h)|_{V_\lambda}$ can be expressed as an upper triangular matrix with $\lambda(h)$-s on the diagonal. It follows from this calculation that $\alpha(h) = 0$, and hence $\lambda(h) = 0$ for all $h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Since $\mathfrak{h} = \sum_{\alpha\in\mathfrak{h}^*}[g_\alpha, g_{-\alpha}]$, it follows that $\lambda(h) = 0$ for all $h \in \mathfrak{h}$. Since $\lambda \in \mathfrak{h}*$ was arbitrary, this means $V = V_0$.

Since $\pi(\mathfrak{g}_\alpha)V = \pi(\mathfrak{g}_\alpha)V_0 \subset V_\alpha = 0$ for $\alpha \neq 0$, it follows that $\mathfrak{g}_\alpha = 0$ for $\alpha \neq 0$. Hence $\mathfrak{g} = \mathfrak{g}_0$, and $\mathfrak{g}$ is nilpotent. This contradicts the fact that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. ☐

Corollary 10.5. *A finite dimensional Lie algebra* $\mathfrak{g}$ *over* $\mathbb{F} = \overline{\mathbb{F}}$ *is solvable iff* $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

Proof. Consider the adjoint representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Its kernel is $Z(\mathfrak{g})$. So $\mathfrak{g}$ is solvable iff $\mathrm{ad}\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ is a solvable. But by Cartan's criterion, $\mathrm{ad}\mathfrak{g}$ is solvable iff $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$. ☐

Exercise 10.3. Consider the following 4-dim solvable Lie algebra $D = \mathrm{Heis}_3 + \mathbb{F}d$, where $\mathrm{Heis}_3 = \mathbb{F}p + \mathbb{F}q + \mathbb{F}c$, with the relations $[d, p] = p, [d, q] = -q, [d, c] = 0$. Define on D the bilinear form $(p, q) = (c, d) = 1, \text{rest } = 0$. Show that this is a non-degenerate symmetric invariant bilinear form, but $(D, [D, D]) \neq 0$, so Cartan's criterion fails for this bilinear form.

Proof. It is symmetric by construction. It is nondegenerate since if $a_1p+a_2q+a_3c+a_4d$ lies in the kernel, taking the bilinear form with q, p, d, c respectively gives $a_1 = a_2 = a_3 = a_4 = 0$. Cartan's criterion fails since $(p, q) \neq 0$ and $p \in D, q \in [D, D]$. To check that it is invariant:

$$\begin{aligned}
B([a_1p + a_2q + a_3c + a_4d,&b_1p + b_2q + b_3c + b_4d], c_1p + c_2q + c_3c + c_4d)\\
&= B((a_1b_2 - a_2b_1)c + (a_4b_1 - a_1b_4)p + (a_2b_4 - a_4b_2)q, c_1p + c_2q + c_3c + c_4d)\\
&= -(a_1b_2 - a_2b_1)c_4 + (a_4b_1 - a_1b_4)c_1 + (a_2b_4 - a_4b_2)c_1\\
B(a_1p + a_2q + a_3c + a_4d,&[b_1p + b_2q + b_3c + b_4d, c_1p + c_2q + c_3c + c_4d])\\
&= B(a_1p + a_2q + a_3c + a_4d, (b_1c_2 - b_2c_1)c + (b_4c_1 - b_1c_4)p + (b_2c_4 - b_4c_2)q)\\
&= a_1(b_2c_4 - b_4c_2) + a_2(b_4c_1 - b_1c_4) + a_4(b_1c_2 - b_2c_1)
\end{aligned}$$

By comparision, it is clear $B([r, s], t) = B(r, [s, t])$ for $r = a_1p + a_2q + a_3c + a_4d, s = b_1p + b_2q + b_3c + b_4d, t = c_1p + c_2q + c_3c + c_4d$, so B is invariant, as required. ☐

Remark. Very often one can remove the condition $\mathbb{F} = \overline{\mathbb{F}}$ by the following trick. Let $\mathbb{F} \subset \overline{\mathbb{F}}$ be the algebraic closure. Given a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, let $\overline{\mathfrak{g}} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{g}$ be a Lie algebra over $\overline{\mathbb{F}}$.

Exercise 10.4.

1. $\mathfrak{g}$ is solvable (resp. nilpotent) iff $\overline{\mathfrak{g}}$ is.
2. Derive Cartan's criterion and Corollary for char $\mathbb{F} = 0$ but not $\mathbb{F} = \overline{\mathbb{F}}$.
3. Show that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ is nilpotent iff $\mathfrak{g}$ is solvable when char $\mathbb{F} = 0$
4. $\mathfrak{g}_0^a$ is a Cartan sub-algebra for every regular element of $a \in \mathfrak{g}$ for any field $\mathbb{F}$.

Proof. By construction of $\overline{\mathfrak{g}}$, if we pick a basis $a_1, ..., a_n$ of $\mathfrak{g}$, so that $\mathfrak{g} = \mathbb{F}a_1 + \cdots + \mathbb{F}a_n$, then $\overline{\mathfrak{g}} = \overline{\mathbb{F}}a_1 + \cdots + \overline{\mathbb{F}}a_n$ with the same bracket relations holding.

1. Note $\overline{[\mathfrak{g},\mathfrak{g}]} = [\overline{\mathfrak{g}},\overline{\mathfrak{g}}]$. To see this, $[\mathfrak{g},\mathfrak{g}]$ is the $\mathbb{F}$-span of $[a_i,a_j]$ with certain linear relations holding between them, so both $\overline{[\mathfrak{g},\mathfrak{g}]}$ and $[\overline{\mathfrak{g}},\overline{\mathfrak{g}}]$ are the $\overline{\mathbb{F}}$-span of $[a_i,a_j]$ with certain linear relations holding between them; and the Lie algebra structure is the same. Iterating this, we have $\overline{\mathfrak{g}^{(i)}} = \overline{\mathfrak{g}}^{(i)}$. So $\mathfrak{g}$ is solvable $\leftrightarrow \exists i, \mathfrak{g}^{(i)} = 0 \leftrightarrow \exists i, \overline{\mathfrak{g}}^{(i)} = 0 \leftrightarrow \overline{\mathfrak{g}}$ is solvable. A similar argument shows that $\overline{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]} = [\overline{\mathfrak{g}},[\overline{\mathfrak{g}},\overline{\mathfrak{g}}]]$, and more generally, $\overline{\mathfrak{g}^i} = \overline{g}^i$. So $\mathfrak{g}$ is nilpotent $\leftrightarrow \exists i, \mathfrak{g}^i = 0 \leftrightarrow \exists i, \overline{g}^i = 0 \leftrightarrow \overline{g}$ is nilpotent.

2. In Cartan's Criterion, note that the second condition $(a,a)_V = 0 \forall a \in [\mathfrak{g},\mathfrak{g}]$ is equivalent to the condition $(a,b)_V = 0 \forall a,b \in [\mathfrak{g},\mathfrak{g}]$, since $(\cdot,\cdot)_V$ is symmetric (to see this, expand $(a+b,a+b)_V = 0$ and note it is characteristic 0). Thus by Cartan's criterion, we have $(\mathfrak{g},[\mathfrak{g},\mathfrak{g}])_V = 0 \leftrightarrow (\overline{\mathfrak{g}},[\overline{\mathfrak{g}},\overline{\mathfrak{g}}])_V = 0 \leftrightarrow (a,b)_V = 0 \forall a,b \in [\overline{\mathfrak{g}},\overline{\mathfrak{g}}] \leftrightarrow (a,b)_V = 0 \forall a,b \in [\mathfrak{g},\mathfrak{g}]$. So the first two conditions of Cartan's criterion are equivalent for $\mathrm{char}\mathbb{F} = 0$. By Cartan's criterion, $(a,b)_V = 0 \forall a,b \in [\mathfrak{g},\mathfrak{g}] \leftrightarrow (a,b)_V = 0 \forall a,b \in [\overline{\mathfrak{g}},\overline{\mathfrak{g}}] \leftrightarrow \overline{\mathfrak{g}}$ is solvable $\leftrightarrow \mathfrak{g}$ is solvable, so the last two conditions of Cartan's criterion are equivalent for $\mathrm{char}\mathbb{F} = 0$. For the Corollary, $\mathfrak{g}$ is solvable $\leftrightarrow \overline{\mathfrak{g}}$ is solvable $\leftrightarrow \kappa(\overline{g},[\overline{g},\overline{g}]) = 0 \leftrightarrow \kappa(\mathfrak{g},[\mathfrak{g},\mathfrak{g}]) = 0$, so the Corollary is true for all $\mathrm{char}\mathbb{F} = 0$.

3. This was proven in a previous exercise when $\mathbb{F} = \overline{\mathbb{F}}$. So $\mathfrak{g}$ is solvable $\to \overline{\mathfrak{g}}$ is solvable $\to [\overline{\mathfrak{g}},\overline{\mathfrak{g}}]$ is nilpotent $\to [\mathfrak{g},\mathfrak{g}]$ is nilpotent.

4. a is a regular element of $\mathfrak{g} \implies a$ is a regular element of $\overline{\mathfrak{g}}$ (since the discriminant of a is same in both $\mathfrak{g}$ and $\overline{\mathfrak{g}}$). Hence $\overline{\mathfrak{g}}_0^a$ is a Cartan sub-algebra of $\overline{\mathfrak{g}}$, and hence $\mathfrak{g}_0^a$ is a Cartan sub-algebra of $\mathfrak{g}$. To see the last step, a sub-algebra is Cartan iff it is nilpotent and self-normalizing, and $\overline{\mathfrak{g}_0^a} = \overline{g}_0^a$, so since $\overline{\mathfrak{g}}_0^a$ is nilpotent, $\mathfrak{g}_0^a$ is nilpotent, and $N_{\overline{\mathfrak{g}}}(\overline{g}_0^a) = \overline{\mathfrak{g}}_0^a \to N_{\mathfrak{g}}(\mathfrak{g}_0^a) = \mathfrak{g}_0^a$.

$\square$

18.745 Introduction to Lie Algebras	October 14, 2010

Lecture 11 — The Radical and Semisimple Lie Algebras

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Exercise 11.1. *Let $\mathfrak{g}$ be a Lie algebra. Then*

1. *if $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are ideals, then $\mathfrak{a}+\mathfrak{b}$ and $\mathfrak{a}\cap\mathfrak{b}$ are ideals, and if $\mathfrak{a}$ and $\mathfrak{b}$ are solvable then $\mathfrak{a}+\mathfrak{b}$ and $\mathfrak{a}\cap\mathfrak{b}$ are solvable.*

2. *If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal and $b \subset \mathfrak{g}$ is a subalgebra, then $\mathfrak{a}+\mathfrak{b}$ is a subalgebra*

Solution:

Let $x \in \mathfrak{g}, a \in \mathfrak{a}, b \in \mathfrak{b}$. Then $[x, a+b] = [x,a]+[x,b] \in \mathfrak{a}+\mathfrak{b}$ since $\mathfrak{a}, \mathfrak{b}$ are ideals. Similarly, if $y \in \mathfrak{a}\cap\mathfrak{b}$ then $[x,y] \in \mathfrak{a}$, $[x,y] \in \mathfrak{b}$, so $\mathfrak{a}\cap\mathfrak{b}$ is an ideal too.

By an easy induction, $(\mathfrak{a}\cap\mathfrak{b})^{(i)} \subset \mathfrak{a}^{(i)}$, which is eventually zero if $\mathfrak{a}$ is solvable, so $\mathfrak{a}\cap\mathfrak{b}$ is solvable.

We showed earlier (in lecture 4) that if $\mathfrak{h} \subset \mathfrak{g}$ and both $\mathfrak{h}, \mathfrak{g}/\mathfrak{h}$ are solvable, then $\mathfrak{g}$ is solvable. Consider $(\mathfrak{a}+\mathfrak{b})/\mathfrak{b}$. By hypothesis, $\mathfrak{b}$ is solvable. Noether's Third Isomorphism Theorem shows $(\mathfrak{a}+\mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a}\cap\mathfrak{b})$ (more explicitly, look at the homomorphisms $\mathfrak{a} \to \mathfrak{a}+\mathfrak{b}$ and $\mathfrak{a}+\mathfrak{b} \to (\mathfrak{a}+\mathfrak{b})/\mathfrak{b}$, notice that the kernel of their composition is exactly $\mathfrak{a}\cap\mathfrak{b}$). Since $\mathfrak{a}$ is solvable, $\mathfrak{a}/(\mathfrak{a}\cap\mathfrak{b})$ is solvable. Hence $\mathfrak{a}+\mathfrak{b}$ is solvable.

For part 2, given any $a_1, a_2 \in \mathfrak{a}$ and $b_1, b_2 \in \mathfrak{b}$, $[a_1+b_1, a_2+b_2] = [a_1,a_2]+[b_1,a_2]+[a_1,b_2]+[b_1,b_2]$. Notice the sum of the first three terms is in $\mathfrak{a}$ since it is an ideal, and the last term is in $\mathfrak{b}$ since it is a subalgebra, thus the entire sum is in $\mathfrak{a}+\mathfrak{b}$. The closure of $\mathfrak{a}+\mathfrak{b}$ under addition and scalar multiplication is obvious. Thus the statement is proven.

Definition 11.1. *A radical $R(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is a solvable ideal of $\mathfrak{g}$ of maximal possible dimension.*

Proposition 11.1. *The radical of $\mathfrak{g}$ contains any solvable ideal of $\mathfrak{g}$ and is unique.*

Proof. If $\mathfrak{a}$ is a solvable ideal of $\mathfrak{g}$, then $\mathfrak{a}+R(\mathfrak{g})$ is again a solvable ideal. Since $R(\mathfrak{g})$ is of maximal dimension, $\mathfrak{a}+R(\mathfrak{g}) = R(\mathfrak{g})$ and $\mathfrak{a} \subset R(\mathfrak{g})$. For uniqueness, if there are two distinct maximal dimensional solvable ideals of $\mathfrak{g}$, then by above explanation, the sum is actually equal to both ideals. Thus we have a contradiction. □

If $\mathfrak{g}$ is a finite dimensional solvable Lie algebra, then $R(\mathfrak{g}) = \mathfrak{g}$. The opposite case is when $R(\mathfrak{g}) = 0$.

Definition 11.2. *A finite dimensional Lie algebra $\mathfrak{g}$ is called semisimple if $R(\mathfrak{g}) = 0$.*

Proposition 11.2. *A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if either of the following two conditions holds:*

1. *Any solvable ideal of $\mathfrak{g}$ is 0.*
2. *Any abelian ideal of $\mathfrak{g}$ is 0.*

Proof. The first condition is obviously equivalent to semisimplicity.

Suppose that $\mathfrak{g}$ contains a non-zero solvable ideal $\mathfrak{r}$. For some k, we have

$$\mathfrak{r} \supset \mathfrak{r}^{(1)} \supset \mathfrak{r}^{(2)} \supset \cdots \supsetneq \mathfrak{r}^{(k)} = 0,$$

hence $\mathfrak{r}^{(k-1)}$ is a nonzero abelian ideal, since all the $\mathfrak{r}^{(i)}$ are ideals of $\mathfrak{g}$. Abelian ideals are solvable, so the other direction is obvious. $\square$

Remark 11.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $R(\mathfrak{g})$ its radical. Then $\mathfrak{g}/R(\mathfrak{g})$ is a semisimple Lie algebra. Indeed, if $\mathfrak{g}/R(\mathfrak{g})$ contains a non-zero solvable ideal $\mathfrak{f}$, then its preimage $\mathfrak{r}$ contains $R(\mathfrak{g})$ properly, so that $\mathfrak{r}/R(\mathfrak{g}) \cong \mathfrak{f}$, which is solvable, hence $\mathfrak{r}$ is a larger solvable ideal than $R(\mathfrak{g})$, a contradiction.*

So an arbitary finite dimensional Lie algebra "reduces" to a solvable Lie algebra $R(\mathfrak{g})$ and a semisimple Lie algebra $\mathfrak{g}/R(\mathfrak{g})$.

In the case char $\mathbb{F} = 0$ a stronger result holds:

Theorem 11.1. *(Levi decomposition) If $\mathfrak{g}$ is a finite dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0, then there exists a semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$, complementary to the radical $R(\mathfrak{g})$, such that $\mathfrak{g} = \mathfrak{s} \oplus R(\mathfrak{g})$ as vector spaces.*

Definition 11.3. *A decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ (direct sum of vector spaces), where $\mathfrak{h}$ is a subalgebra and $\mathfrak{r}$ is an ideal is called a semi-direct sum of $\mathfrak{h}$ and $\mathfrak{r}$ and is denoted by $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{r}$. The special case when $\mathfrak{h}$ is an ideal as well corresponds to the direct sum: $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r} = \mathfrak{h} \oplus \mathfrak{r}$.*

Remark 11.2. *The open end of $\ltimes$ goes on the side of the ideal. When both are ideals, we use $\times$ or $\oplus$, and the sum is direct.*

Exercise 11.2. *Let $\mathfrak{h}$ and $\mathfrak{r}$ be Lie algebras and let $\gamma : \mathfrak{h} \to \mathrm{Der}(\mathfrak{r})$ be a Lie algebra homomorphism. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ be the direct sum of vector spaces and extend the bracket on $\mathfrak{h}$ and on $\mathfrak{r}$ to the whole of $\mathfrak{g}$ by letting*

$$[h, r] = -[r, h] = \gamma(h)(r)$$

for $h \in \mathfrak{h}$ and $r \in \mathfrak{r}$. Show that this provides $\mathfrak{g}$ with a Lie algebra structure, $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{r}$, and that any semidirect sum of $\mathfrak{h}$ and $\mathfrak{r}$ is obtained in this way. Finally, show that $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r}$ if and only if $\gamma = 0$.

Solution:

The bracket so defined is clearly skew-symmetric. Restricted to $\mathfrak{h}$ or $\mathfrak{r}$ it satisfies the Jacobi identity. Take $h_1, h_2 \in \mathfrak{h}$ and $r_1, r_2 \in \mathfrak{r}$. Then we have $[h_1, [r_1, r_2]] + [r_1, [r_2, h_1]] + [r_2, [h_1, r_1]] = \gamma(h_1)([r_1, r_2]) - [r_1, \gamma(h_1)(r_2)] + [r_2, \gamma(h_1)(r_1)] = [\gamma(h_1)(r_1), r_2] + [r_1, \gamma(h_1)(r_2)] - [r_1, \gamma(h_1)(r_2)] + [r_2, \gamma(h_1)(r_1)] = 0$. We used the fact that $\gamma(h_1)$ is a derivation.

Furthermore, $[h_1, [h_2, r_1]] + [h_2, [r_1, h_1]] + [r_1, [h_1, h_2]] = [h_1, \gamma(h_2)(r_1)] - [h_2, \gamma(h_1)(r_1)] - \gamma([h_1, h_2])(r_1) = \gamma(h_1)\gamma(h_2)(r_1) - \gamma(h_2)\gamma(h_1)(r_1) - (\gamma(h_1)\gamma(h_2)(r_1) - \gamma(h_2)\gamma(h_1)(r_1)) = 0$. We used the fact that $\gamma : \mathfrak{h} \to \mathrm{Der}(\mathfrak{r})$ is a Lie algebra homomorphism. It follows that $[,]$ is a Lie bracket. Clearly $\mathfrak{r}$ is an ideal under the bracket, so $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{r}$.

Conversely, by the Jacobi identity, any bracket on $\mathfrak{h} \ltimes \mathfrak{r}$ is a homomorphism from $\mathfrak{h}$ to $\mathrm{Der}(\mathfrak{r})$ (essentially we just reverse the two calculations above). Finally, if $\gamma = 0$ then $[h, r] = 0$ for all $h \in \mathfrak{h}, r \in \mathfrak{r}$, and if $[h, r] = 0$ for all h, r then plainly $\gamma(h) = 0$ for all h, so $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r}$ if and only if $\gamma = 0$.

Exercise 11.3. *Let $\mathfrak{g} \subset gl_n(\mathbb{F})$ be a subspace consisting of matrices with arbitrary first m rows and 0 for the rest of the rows. Find $R(\mathfrak{g})$ and a Levi decomposition of $\mathfrak{g}$.*

Solution:

Write $x \in \mathfrak{g}$ as (A, B), where A is the upper-left $m \times m$ matrix block and B is the upper-right $m \times (n-m)$ block of x. Take $y = (A', B')$. Then it is clear that $[x, y] = ([A, A'], AB' - A'B)$. Hence, if $\mathfrak{h} \subset gl_m \hookrightarrow \mathfrak{g}$ is an ideal of $\mathfrak{gl}_m$, then $(\mathfrak{h}, 0) + R$ is an ideal of $\mathfrak{g}$, where R denotes the set of all (A, B) with $A = 0$. Furthermore, $\mathfrak{h}$ is solvable if and only if $\mathfrak{h} + R$ is solvable, because of $[R, R] = 0$ and the above identity. Notice that the radical of $\mathfrak{g}$ obviously contains R. Hence, the radical of $\mathfrak{g}$ corresponds to the radical of $\mathfrak{gl}_m$, i.e. $R(\mathfrak{g}) = (R(\mathfrak{gl}_m), 0) + R$. But, by the notes and problem 4 below, the radical of gl_m is $\mathbb{F}I$, the scalar matrices. Hence $R(\mathfrak{g}) = (\mathbb{F}I, 0) + R$ (sum of ideals). The complement of this can obviously be the subalgebra $(\mathfrak{sl}_m, 0)$.

Theorem 11.2. *Let V be a finite-dimensional vector space over an algebraically closed field of characteristic 0 and let $\mathfrak{g} \subset \mathfrak{gl}_V$ be a subalgebra, which is irreducible i.e. any subspace $U \subset V$, which is $\mathfrak{g}-$invariant, is either 0 or V. Then one of two possibilities hold:*

1. *$\mathfrak{g}$ is semisimple*
2. *$\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{sl}_V) \oplus \mathbb{F}I$ and $\mathfrak{g} \cap \mathfrak{sl}_V$ is semisimple.*

Proof. If $\mathfrak{g}$ is not semisimple, then $R(\mathfrak{g})$ is a non-zero solvable ideal in $\mathfrak{g}$. By Lie's theorem, there exists $\lambda \in R(\mathfrak{g})^*$ such that $V_\lambda = \{v \in V | av = \lambda(a)v, a \in R(\mathfrak{g})\}$ is nonzero. By Lie's lemma, V_λ is invariant. Hence, by irreducibility $V_\lambda = V$. Hence $a = \lambda(a)I_V$ for all $a \in R(\mathfrak{g})$, so $R(\mathfrak{g}) = \mathbb{F}I$. Hence $(\mathfrak{g} \cap \mathfrak{sl}_V) \cap R(\mathfrak{g}) = 0$, which proves that we have case 2, as $\mathfrak{g} \cap \mathfrak{sl}_V$ is semisimple since it is the complement of the radical.

□

Exercise 11.4. *Let V be finite-dimensional over a field $\mathbb{F}$ which is algebraically closed and characteristic 0. Show that $\mathfrak{gl}_V$ and $\mathfrak{sl}_V$ are irreducible subalgebras of $\mathfrak{gl}_V$. Deduce that $\mathfrak{sl}_V$ is semisimple.*

Solution: Suppose $W \subset V, W \neq 0$ is fixed by $\mathfrak{sl}_n$. Take nonzero vector $w = \sum c_i v_i \in W$, where $\{v_i\}$ is a basis of V. Suppose $c_k \neq 0$, and pick $\ell \neq k$. Then $e_{\ell k} \in \mathfrak{sl}_n$, and $e_{\ell k} w = c_k e_\ell \in W$. For every $m \neq \ell, e_{m\ell} \in \mathfrak{sl}_n$, so $e_{m\ell} e_\ell = e_m \in W$. Hence $W = V$. Since $\mathfrak{sl}_n \subset \mathfrak{gl}_n$, $\mathfrak{gl}_n$ is also irreducible. It follows from the theorem that $\mathfrak{sl}_n \cap \mathfrak{sl}_n = \mathfrak{sl}_n$ is semisimple.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Recall the Killing form on $\mathfrak{g}$: $K(a,b) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\ a)(\mathrm{ad}\ b)$.

Theorem 11.3. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0. Then the Killing form on $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g}$ is semisimple. Moreover, if $\mathfrak{g}$ is semisimple and $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then the restriction of the Killing form to $\mathfrak{a}, K|_{\mathfrak{a}\times\mathfrak{a}}$, is also non-degenerate and coincides with the Killing form of $\mathfrak{a}$.*

Exercise 11.5. *Let V be a finite-dimensional vector space with a symmetric bilinear form $(,)$. Let U be a subspace such that the restriction $(,)|_{U\times U}$ is non-degenerate. Denote $U^\perp = \{v \in V | (v, U) = 0\}$. Then $V = U \oplus U^\perp$.*

Solution: We pick an arbitrary basis $u_1, ..., u_m$ of U, and then extend it to a basis of V: $u_1, ..., u_m, ..., u_n$. Let the matrix associated with the given bilinear form relative to this basis be $Q = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$, where A is an $m \times m$ invertible matrix. We want to change the basis (more specifically, the part $u_{m+1}, ..., u_n$ to $u'_{m+1}, ..., u'_n$) to make part B vanish. Suppose that the base change matrix is $P = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$, where the sizes of the blocks match that of Q. Then the new matrix associated to the bilinear form is $P^T Q P = \begin{pmatrix} 1 & 0 \\ X^T & 1 \end{pmatrix} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & AX + B \\ X^T A + B^T & * \end{pmatrix}$. Thus we can just take $X = -A^{-1}B$ and the new matrix will have zero upper right block. It is obvious now that we have the desired decomposition by noticing that $U^\perp$ equals span of $u_{m+1}, ..., u_n$.

Lemma 11.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $(,)$ be a symmetric invariant bilinear form on $\mathfrak{g}$. Then*

1. *If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{a}^\perp$ is also an ideal.*
2. *If $(,)|_{\mathfrak{a}\times\mathfrak{a}}$ is non-degenerate, then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, a direct sum of Lie algebras.*

Proof. 1. $v \in \mathfrak{a}^\perp$ means $(v, \mathfrak{a}) = 0$. If $b \in \mathfrak{g}$, then $([v,b], \mathfrak{a}) = (v, [b, \mathfrak{a}]) = 0$, since the form is invariant and $\mathfrak{a}$ is an ideal. Hence $\mathfrak{a}^\perp$ is an ideal.

2. Follows from the preceeding exercise and part 1.

□

Proof of the theorem. Suppose K is non-degenerate on $\mathfrak{g}$, but $\mathfrak{g}$ is not semisimple. Hence there exists an abelian ideal $\mathfrak{a} \subset \mathfrak{g}$. But then $K(\mathfrak{a}, \mathfrak{g}) = 0$, contradicting non-degeneracy of K. Indeed, if $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$, then $(\text{ad } x)(\text{ad } y)z = [x, [y, z]] \in \mathfrak{a}$ for all $z \in \mathfrak{g}$ (and 0 for all $z \in \mathfrak{a}$. If follows that in the basis $e_1, \ldots, e_k$ of $\mathfrak{a}$, $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n$ basis of $\mathfrak{g}$, the matrix of $(\text{ad } x)(\text{ad } y)$ is of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. But trace of this matrix is 0, so $K(x, y) = 0$.

Conversely, let $\mathfrak{g}$ be semisimple. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. If $K_{\mathfrak{a}\times\mathfrak{a}}$ is degenerate, so that $\mathfrak{a} \cap \mathfrak{a}^\perp \neq 0$, hence $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$ such that $K(\mathfrak{b}, \mathfrak{b}) = 0$. By considering the adjoing representation of $\mathfrak{b}$ in $\mathfrak{g}$ and applying the Cartan criterion we conclude that $\mathfrak{b}$ is solvable. Since $\mathfrak{g}$ is semisimple, we deduce that $\mathfrak{b} = 0$. Thus if $\mathfrak{g}$ is semisimple, the Killing form is non-degenerate, by taking $\mathfrak{a} = \mathfrak{g}$.

As for the second part, we already proved that $K|_{\mathfrak{a}\times\mathfrak{a}}$ is non-degenerate. Hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. By lemma it is a direct sum of ideals, so $[\mathfrak{a}, \mathfrak{a}^\perp] = 0$. Hence K for $\mathfrak{a}$ equals K of $\mathfrak{g}$ restricted to $\mathfrak{a}$. □

Definition 11.4. *A Lie algebra $\mathfrak{g}$ is called simple if its only ideals are 0 and $\mathfrak{g}$ and $\mathfrak{g}$ is not abelian.*

Corollary 11.1. *Any semisimple, finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0 is a direct sum of simple Lie algebras.*

Proof. If $\mathfrak{g}$ is semisimple, but not simple, and if $\mathfrak{a}$ is an ideal, then by the theorem, the Killing form restricted to $\mathfrak{a}$ is non-degenerate, hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, where $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are also semisimple. After finitely many steps it can be decomposed into simple algebras. □

18.745 Introduction to Lie Algebras	October 19, 2010

Lecture 12 — Structure Theory of Semisimple Lie Algebras (I)

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In lecture 10, we saw that Cartan's criterion holds without requiring the base field to be algebraically closed. Similarly, we can deduce the semisimplicity criterion proved in the last lecture assuming only that the base field has characteristic zero.

Exercise 12.1. Let $\mathfrak{g}$ denote a Lie algebra over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. Show that $\mathfrak{g}$ is semisimple if and only if the Killing form on $\mathfrak{g}$ is nondegenerate.

Solution: When we proved the theorem in lecture 11, we used the algebraic closure of the base field $\mathbb{F}$ only to apply Cartan's criterion. We know from exercise 10.4, however, that Cartan's criterion is valid assuming merely that $\mathbb{F}$ has characteristic 0. So the same proof applies in this case. □

We should not be led to think, however, that the assumption of algebraic closure is never necessary. Merely requiring that the base field has characteristic zero is not enough for instance to derive Chevalley's theorem on the conjugacy of Cartan subalgebras.

Exercise 12.2. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. Show that there are two distinct conjugacy classes of Cartan subalgebras given by

$$\mathfrak{h}_1 = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathfrak{h}_2 = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Solution: Let $\{e, f, h\}$ be the standard basis of $\mathfrak{g}$ where $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. We want to show that there exist two distinct conjugacy classes of Cartan subalgebras in $\mathfrak{g}$ given by $\mathbb{R}h$ and $\mathbb{R}(e - f)$.

If $w = ae + bf + ch$, then the characteristic polynomial of $\mathbf{ad}\ w$ is

$$t^3 - 4(c^2 + ab) . \tag{1}$$

This fact follows from an immediate computation. So an element $w = ae + bf + ch$ is regular iff $c^2 + ab \neq 0$, and nilpotent otherwise. In particular h and $e - f$ are regular with respective eigenvalues $\{0, \pm 2\}$ and $\{0, \pm 2i\}$.

Suppose we have a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\overline{\mathfrak{g}} = \mathfrak{sl}(\mathbb{C})$ and $\overline{h} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{h}$. By exercise 10.2, we know that $\mathfrak{g}$ is not nilpotent, so $\mathfrak{h}$ has dimension 1 or 2. If $\mathfrak{h}$ has dimension 2, then by extending a basis of $\mathfrak{g}$ to a basis of $\overline{\mathfrak{g}}$, we see that $\overline{\mathfrak{h}}$ is nilpotent and self-normalizing. Yet the Cartan subalgebras of $\overline{\mathfrak{g}}$ are 1-dimensional. Therefore $\mathfrak{h}$ is 1-dimensional.

By exercise 10.4, if w is a regular element, then $\mathfrak{g}_0^w \supset \mathbb{R}w$ is a Cartan subalgebra. So $\mathfrak{g}_0^w = \mathbb{R}w$, implying $\mathbb{R}w$ is Cartan. Conversely, if $\mathbb{R}w$ is Cartan, then w is regular. Otherwise w would be nilpotent contradicting the fact that $\mathbb{R}w$ is self-normalizing. Therefore all Cartan subalgebras are of the form $\mathbb{R}w$ for a regular element w.

By (1), we see that w has eigenvalues $\left\{0, \pm(c^2 + ab)^{1/2}\right\}$. Since scaling determines a Lie algebra isomorphism, and $c^2 + ab \neq 0$, it follows that w is conjugate to an element of $\mathbb{R}h$ or $\mathbb{R}(e - f)$. But h is not conjugate to an element of $\mathbb{R}(e - f)$, since conjugate elements have the same eigenvalues.

□

In this lecture and the few that follow, we will study the structure of finite dimensional semisimple Lie algebras with the aim of classifying them. This will amount to a detailed knowledge of root space decompostions. The first step is to use the Killing form to understand Cartan subalgebras and their actions under the adjoint representation.

Unless otherwise stated, we will assume throughout that our base field $\mathbb{F}$ is algebraically closed of characteristic zero.

We begin by generalizing our notion of Jordan decomposition to an arbitrary Lie algebra.

Definition 12.1. An *abstract Jordan decomposition* of an element of a Lie algebra $\mathfrak{g}$ is a decomposition of the form $a = a_s + a_n$, where

(a) $\mathbf{ad}\ a_s$ is a diagonalizable (equivalently semisimple) endomorphism of $\mathfrak{g}$.

(b) $\mathbf{ad}\ a_n$ is a nilpotent endomorphism.

(c) $[a_s, a_n] = 0$.

Exercise 12.3. Abstract Jordan decomposition in a Lie algebra $\mathfrak{g}$ is unique when it exists if and only if $Z(\mathfrak{g}) = 0$.

Solution: $(\Rightarrow)$ Suppose $c \in Z(\mathfrak{g})$ is not 0, and that $a \in \mathfrak{g}$ has abstract Jordan decomposition $a = a_s + a_n$. Let $a'_s = a_s - c$ and $a'_n = a_n + c$. We note the following facts: $a = (a_s - c) + (a_n + c) = a'_s + a'_n$; $[a'_s, a'_n] = [a_s, a_n] = 0$; $\mathbf{ad}\ a'_s = \mathbf{ad}\ a_s$ and $\mathbf{ad}\ a'_n = \mathbf{ad}\ a_n$ as $Z(\mathfrak{g}) = \ker \mathbf{ad}$. Since $\mathbf{ad}\ a_n$ is nilpotent, and $\mathbf{ad}\ a_s$ is semisimple, we conclude that $a = a'_s + a'_n$ is an abstract Jordan decomposition. But $a'_s \neq a_s$ as $c \neq 0$. So the decomposition for a is not unique.

$(\Leftarrow)$ Suppose that $a = a_s + a_n = a'_s + a'_n$ are abstract Jordan decompositions for some $a \in \mathfrak{g}$. Since $\mathbf{ad}\ (a_s + a_n)$ and $\mathbf{ad}\ (a'_s + a'_n)$ are commuting operators, and $\mathbf{ad}\ a = \mathbf{ad}\ a_s + \mathbf{ad}\ a_n$ and $\mathbf{ad}\ a = \mathbf{ad}\ a'_s + \mathbf{ad}\ a'_n$ are Jordan decompositions, we know from a lemma of lecture 6, that

$$[\mathbf{ad}\ a'_s, \mathbf{ad}\ a_s + \mathbf{ad}\ a_n] = 0, \quad [\mathbf{ad}\ a'_n, \mathbf{ad}\ a_s + \mathbf{ad}\ a_n] = 0$$

and consequently

$$[\mathbf{ad}\ a'_s, \mathbf{ad}\ a_s] = 0 \quad [\mathbf{ad}\ a'_n, \mathbf{ad}\ a_n] = 0\ .$$

Since the sum of commuting semisimple (resp. nilpotent) operators is semisimple (resp. nilpotent),

$$\mathbf{ad}\ a'_s - \mathbf{ad}\ a_s = \mathbf{ad}\ a_n - \mathbf{ad}\ a'_n \ \Rightarrow\ \mathbf{ad}\ a'_s - \mathbf{ad}\ a_s = 0,\ \mathbf{ad}\ a_n - \mathbf{ad}\ a'_n = 0\ .$$

As $\ker \mathbf{ad} = Z(\mathfrak{g}) = 0$ we conclude that $a_n = a'_n$ and $a_s = a'_s$.

□

Our first result in this lecture will be to show the existence of abstract Jordan decompositions under the assumption of semisimplicity.

Theorem 12.1. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{F}$.*

(a) $Z(\mathfrak{g}) = 0$.

(b) All derivations of $\mathfrak{g}$ *are inner.*

(c) Any $a \in \mathfrak{g}$ *admits a unique Jordan decomposition.*

Proof. (a): One of our equivalent definitions of semisimplicity is that $\mathfrak{g}$ has no nontrivial abelian ideals. Hence $Z(\mathfrak{g}) = 0$.

(b): By construction $\mathfrak{g} \xrightarrow{\mathbf{ad}}$ Der $\mathfrak{g}$ with kernel $Z(\mathfrak{g})$. Since $Z(\mathfrak{g}) = 0$, this implies $\mathfrak{g} \cong \mathbf{ad}\ \mathfrak{g}$.

Consider the trace form on Der $\mathfrak{g}$ under the tautological representation. The restriction to $\mathbf{ad}\ \mathfrak{g}$ is the Killing form on $\mathfrak{g}$, which is nondegenerate by semsimplicity. From a lemma of lecture 11, we have that Der $\mathfrak{g} = \mathbf{ad}\ \mathfrak{g} \oplus \mathbf{ad}\ \mathfrak{g}^{\perp}$ as a direct sum of Lie algebras.

We need to show that $\mathbf{ad}\ \mathfrak{g}^{\perp} = 0$. In the contrary case, there exists a nonzero element $D \in \mathbf{ad}\ \mathfrak{g}^{\perp}$. For $a \in \mathfrak{g}$, we have $[D, \mathbf{ad}\ a] = 0$. We recall from exercise 2.1 that $[D, \mathbf{ad}\ a] = \mathbf{ad}\ D(a)$. Hence $D(a) \in Z(\mathfrak{g})$ for any $a \in \mathfrak{g}$. Since $Z(\mathfrak{g}) = 0$, we conclude that $D = 0$ in contradiciton to our assumption.

(c): Fix $a \in \mathfrak{g}$. Consider the classical Jordan decompostion of $\mathbf{ad}\ a = A_s + A_n$, where A_s is diagonalizable, A_n is nilpotent, and $[A_s, A_n] = 0$.

Consider the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to $\mathbf{ad}\ a$

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_{\lambda}^{a} .$$

Recall that $\mathfrak{g}_{\lambda}^{a} = \left\{(\mathbf{ad}\ a - \lambda I)^N = 0\right\}$, and $[\mathfrak{g}_{\lambda}, \mathfrak{g}_{\mu}] \subset \mathfrak{g}_{\lambda+\mu}$ with $A_s \mid_{\mathfrak{g}_{\lambda}} = \lambda I$.

We want first to show that A_s is a derivation. By linearity, it is enough to check this for $x \in \mathfrak{g}_{\lambda}^{a}$ and $y \in \mathfrak{g}_{\mu}^{a}$. We have

$$A_s([x, y]) = (\lambda + \mu) \cdot [x, y] = [\lambda x, y] + [x, \mu y] = [A_s x, y] + [x, A_s y] .$$

This means that $A_s \in$ Der $\mathfrak{g}$.

By part (b), all derivations are inner. So there exists $a_s \in \mathfrak{g}$ such that $\mathbf{ad}\ a_s = A_s$. Letting $a_n = a - a_s$, we have that $\mathbf{ad}\ a_n = A_n$. It remains to check that $[a_s, a_n] = 0$. Since

$$\mathbf{ad}\ ([a_s, a_n]) = [\mathbf{ad}\ a_s, \mathbf{ad}\ a_n] = [A_s, A_n] = 0$$

part (a) gives the result.

□

With the previous result under our belts, let us get a firmer understanding of the Cartan subalgebras of $\mathfrak{g}$ from Theorem 12.1.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and consider the generalized root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_{\alpha} \quad \text{with} \quad \mathfrak{g}_0 = \mathfrak{h}$$

where $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subset \mathfrak{g}_{\alpha+\beta}$. Recall that in the case of the adjoint representation, we say generalized root space decomposition instead of generalized weight space decomposition. It is important to make this distinction; we will see its convenience in later lectures, as we try to better understand the functionals α appearing in the decomposition.

Theorem 12.2. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be as defined, and K denote the Killing form on $\mathfrak{g}$.*

(a) $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$.

(b) $K\mid_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is nondegenerate. Consequently $K\mid_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate.

(c) $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$.

*(d) $\mathfrak{h}$ consists of semisimple elements; namely, each **ad** h for $h \in \mathfrak{h}$ is diagonalizable. Consequently $\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a \ \ \forall h \in \mathfrak{h}\}$.*

Proof. (a): Let $a \in \mathfrak{g}_\alpha$ and $b \in \mathfrak{g}_\beta$ with $\alpha + \beta \neq 0$. Note that

$$((\mathbf{ad}\ a)(\mathbf{ad}\ b))^N \mathfrak{g}_\gamma \subset \mathfrak{g}_{\gamma + N\alpha + N\beta} = 0$$

for $N \gg 0$ as $\mathfrak{g}$ is finite dimensional. So we can choose N sufficiently large so that the operator $((\mathbf{ad}\ a)(\mathbf{ad}\ b))^N$ is zero on each summand of the decomposition. Therefore $(\mathbf{ad}\ a)(\mathbf{ad}\ b)$ is nilpotent. By exercise 3.4, the eigenvalues of $(\mathbf{ad}\ a)(\mathbf{ad}\ b)$ are zero implying that

$$K(a, b) = \mathrm{tr}_{\ \mathfrak{g}}((\mathbf{ad}\ a)(\mathbf{ad}\ b)) = 0\ .$$

(b): By semisimplicity, we know that K is nondegenerate on $\mathfrak{g}$. By part (a), we have $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha \neq -\beta$. So necessarily $K\mid_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is nondegenerate.

(c): Consider $\mathbf{ad}\ \mathfrak{h}$, and note that $K\mid_{\mathfrak{h}\times\mathfrak{h}}$ is the trace form on $\mathbf{ad}\ \mathfrak{h}$ under the tautological representation. As $\mathfrak{h}$ is solvable, indeed nilpotent, we know that $\mathbf{ad}\ \mathfrak{h}$ is solvable. So by Cartan's criterion, we find that

$$0 = \mathrm{tr}_{\ \mathfrak{g}}(\mathbf{ad}\ \mathfrak{h}, [\mathbf{ad}\ \mathfrak{h}, \mathbf{ad}\ \mathfrak{h}]) = K(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}])\ .$$

By part (b), however, $K\mid_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate. Therefore we conclude that the derived subalgebra $[\mathfrak{h}, \mathfrak{h}]$ is zero.

(d): Consider $h \in \mathfrak{h}$. By Theorem 12.1 (c), h has an abstract Jordan decomposition of the form $h = h_s + h_n$, where $h_s, h_n \in \mathfrak{g}$ are such that $\mathbf{ad}\ h_s$ is diagonalizable, $\mathbf{ad}\ h_n$ is nilpotent, and $[h_s, h_n] = 0$.

By part (c), $[h, \mathfrak{h}] = 0$. Hence for $h' \in \mathfrak{h}$, we have $0 = \mathbf{ad}\ ([h', h]) = [\mathbf{ad}\ h', \mathbf{ad}\ h]$. So by a lemma from lecture 6, we know that $[\mathbf{ad}\ h', (\mathbf{ad}\ h)_s] = 0$, yielding

$$0 = [\mathbf{ad}\ h', (\mathbf{ad}\ h)_s] = [\mathbf{ad}\ h', \mathbf{ad}\ h_s] = \mathbf{ad}\ ([h', h_s])\ .$$

By Theorem 12.1 (a), this implies $[h', h_s] = 0$ for $h' \in \mathfrak{h}$. But since $\mathfrak{h}$ is a maximal nilpotent subalgebra, we conclude that $h_s \in \mathfrak{h}$.

It remains to be shown that $h_n = 0$. Note that $h_n = h - h_s \in \mathfrak{h}$.

Since $\mathbf{ad}\ \mathfrak{h}$ is solvable, Lie's theorem implies that there exists a basis of $\mathfrak{gl}_{\mathfrak{g}}$ such that the elements of $\mathbf{ad}\ \mathfrak{h}$ are upper triangular. As $\mathbf{ad}\ h_n$ is nilpotent, it is strictly upper triangular. Therefore $\mathrm{tr}\ ((\mathbf{ad}\ h')(\mathbf{ad}\ h_n)) = 0$ for $h' \in \mathfrak{h}$; namely $K(\mathfrak{h}, h_n) = 0$. By part (b), $K\mid_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate. So $h_n = 0$, implying that $h = h_s$ is diagonalizable. $\square$

For arbitrary Lie algebras, Cartan subalgebras can have an unwieldy structure, and an unpredictable action under the adjoint representation. We see from the previous result, however, that with the assumption of semisimplicity the situation is much more transparent.

We can rewrite the root space decomposition as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where Δ is the set of all $\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq 0$. Recall that $\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a \ \ \forall h \in \mathfrak{h}\}$.

Definition 12.2. We call $\alpha \in \Delta$ a *root* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ the corresponding *root space*.

The rest of the classification will be concerned more or less with gathering information about roots of $\mathfrak{g}$. Having put the Killing form to good use in determining the structure of Cartan subalgebras, we would like to extend it to Δ. We have a canonical linear map

$$\nu : \mathfrak{h} \ni h \mapsto K(h, \bullet) \in \mathfrak{h}^* \ .$$

Since $K\mid_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate by Theorem 12.2 (b), this implies ν is injective. As $\mathfrak{h}$ and $\mathfrak{h}^*$ have the same dimension, ν determines a vector space isomorphism.

Definition 12.3. Abusing notation, we define a bilinear form $K : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{F}$ by the rule $K(\gamma, \mu) = K(\nu^{-1}(\gamma), \nu^{-1}(\mu))$. Note that $K(\nu(h), \nu(h')) = \nu(h)(h') = \nu(h')(h)$.

Theorem 12.3. *(a) If $\alpha \in \Delta$, $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, then $[e, f] = K(e, f)\nu^{-1}(\alpha)$.*

(b) If $\alpha \in \Delta$, then $K(\alpha, \alpha) \neq 0$.

Proof. (a): We know that $[e, f] \in \mathfrak{h}$. By Theorem 12.2 (b), $K\mid_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate. So it is enough to show that $K([e, f] - K(e, f)\nu^{-1}(\alpha), h') = 0$ for any $h' \in \mathfrak{h}$. We have

$$\begin{aligned} K([e, f] - K(e, f)\nu^{-1}(\alpha), h') &= K([e, f], h') - K(e, f)\nu(\nu^{-1}(\alpha))(h') \\ &= -K(e, [h', f]) - K(e, f)\alpha(h') \\ &= \alpha(h')K(e, f) - K(e, f)\alpha(h') = 0 \ . \end{aligned}$$

Note that we have used the invariance of K, and the fact that $h' \in \mathfrak{h}$, $f \in \mathfrak{g}_{-\alpha}$. This gives the result.

(b): By Theorem 12.2 (b), $K\mid_{\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}}$ is nondegenerate. So we can find $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ such that $K(e, f) = 1$. By part (a), we know that $[e, f] = \nu^{-1}(\alpha)$. Hence

$$[\nu^{-1}(\alpha), e] = \alpha(\nu^{-1}(\alpha))e = K(\alpha, \alpha)e$$

and similarly

$$[\nu^{-1}(\alpha), f] = \alpha(\nu^{-1}(\alpha))f = K(\alpha, \alpha)f \ .$$

Suppose to the contrary that $K(\alpha, \alpha) = 0$. By the above relations, we obtain a Lie algebra

$$\mathfrak{a} = \mathbb{F}e + \mathbb{F}f + \mathbb{F}\nu^{-1}(\alpha)$$

isomorphic to Heis_3. Recall that Heis_3 is solvable with center $[\mathrm{Heis}_3, \mathrm{Heis}_3]$.

So applying Lie's theorem to the restriction of the adjoint representation to $\mathfrak{a}$, we can find a basis of $\mathfrak{gl}_{\mathfrak{g}}$ such that $\nu^{-1}(\alpha) = [\mathfrak{a}, \mathfrak{a}]$ is strictly upper triangular. But by Theorem 12.2 (d), we know that $\nu^{-1}(\alpha) \in \mathfrak{h}$ is diagonalizable. So we conclude that $\nu^{-1}(\alpha) = 0$, and $\alpha = 0$. This contradicts definition 12.2, and the fact that $\alpha \in \Delta$.

□

Exercise 12.4. (a) Show that all derivations of the 2-dimensional nonabelian Lie algebra are inner. (b) Find Der Heis_3. Note that not all derivations are inner.

Solution (a) Let $\mathfrak{g}$ be the 2-dimensional nonabelian Lie algebra over $\mathbb{F}$ with ordered basis $\{e, f\}$ where $[e, f] = f$. Choose $D \in \mathrm{Der}\ (\mathfrak{g})$ with

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where } a, b, c, d \in \mathbb{F}.$$

We have

$$\begin{aligned} D([e, f]) = [De, f] + [e, Df] &\iff \\ D(f) = [ae + cf, f] + [e, be + df] &\iff \\ be + df = (a + d)f &\iff a = b = 0\ . \end{aligned}$$

Thus $D = \mathbf{ad}\ de - cf$ implying D is inner.

(b) Let $\mathfrak{g} = \mathrm{Heis}_3$ over $\mathbb{F}$ with ordered basis $\{p, q, c\}$ where $[p, q] = c$. Choose $D \in \mathrm{Der}\ (\mathrm{Heis}_3)$ with $D = \sum_{1 \le i,j \le 3} a_{ij} \cdot e_{ij}$. Note that

$$[D, \mathbf{ad}\ p] = \begin{pmatrix} 0 & a_{13} & 0 \\ 0 & a_{23} & 0 \\ -a_{21} & a_{33} - a_{22} & -a_{23} \end{pmatrix}, \quad [D, \mathbf{ad}\ q] = \begin{pmatrix} -a_{13} & 0 & 0 \\ -a_{23} & 0 & 0 \\ -a_{33} + a_{11} & a_{12} & a_{13} \end{pmatrix}.$$

Recall from exercise 2.1 that $[D, \mathbf{ad}\ p] = \mathbf{ad}\ D(p)$ and $[D, \mathbf{ad}\ q] = \mathbf{ad}\ D(q)$. Since $\mathbf{ad}\ p = e_{32}$ and $\mathbf{ad}\ q = -e_{31}$, it follows that $a_{13} = a_{23} = 0$ and $a_{33} = a_{11} + a_{22}$; namely

$$D = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} + a_{22} \end{pmatrix}.$$

By checking on basis elements, we see that every matrix of this form is indeed a derivation.

Hence the derivations of $\mathfrak{g}$ form a 6-dimensional subspace, while the inner derivations form a 2-dimensional subspace.

□

Exercise 12.5. Show that Theorem 12.1 parts (b) and (c), and Theorem 12.2 part (c) hold for $\mathrm{char}(\mathbb{F}) = 0$.

Solution (12.1 (b)): Note that Theorem 12.1 (a) makes no assumptions on $\mathbb{F}$. We check that the lemma from lecture 11, does not require that the $\mathbb{F}$ be algebraically closed. Hence we only use algebraic closure to deduce that the Killing form on $\mathfrak{g}$ is nondegenerate. This follows from exercise 12.1 merely assuming that $\mathbb{F}$ has characteristic zero.

(12.1(c)): Let $\overline{\mathfrak{g}} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{g}$. Choose $a \in \mathfrak{g}$. We know that there exist $a_s, a_n \in \overline{\mathfrak{g}}$ such that $a = a_s + a_n$ is an abstract Jordan decomposition. Choose a basis $\{g_1, \ldots, g_n\}$ of $\mathfrak{g}$, and note that $\{\overline{g}_1, \ldots, \overline{g}_n\} = \{1 \otimes g_1, \ldots, 1 \otimes g_n\}$ is a basis of $\overline{\mathfrak{g}}$.

Claim. $a_s \in \mathfrak{g}$:

Since $\mathrm{char}(\mathbb{F}) = 0$, this implies that $\overline{\mathbb{F}}/\mathbb{F}$ is a Galois extension. Recall that $f \in \mathbb{F}$ iff $\sigma(f) = f$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.

Let $a_s = \sum_{1 \le i \le n} \alpha_i \overline{g}_i$. Suppose to the contrary that $\alpha_j \in \overline{\mathbb{F}} - \mathbb{F}$. There exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ such that $\sigma(\alpha_j) \neq \alpha_j$. Define a map

$$\Phi : \overline{\mathfrak{g}} \ni \sum_{1 \le i \le n} \beta_i \overline{g}_i \mapsto \sum_{1 \le i \le n} \sigma(\beta_i) \overline{g}_i \in \overline{\mathfrak{g}} \ .$$

Note that Φ is an additive bijection, and $\Phi \mid_{\mathfrak{g}}$ is the identity. Moreover

$$\Phi([\overline{g}_i, \overline{g}_j]) = [\overline{g}_i, \overline{g}_j] = [\Phi \overline{g}_i, \Phi \overline{g}_j] \Longrightarrow \Phi([g, g']) = [\Phi g, \Phi g'] \ \ \forall g, g' \in \overline{\mathfrak{g}}$$

since $[\overline{g}_i, \overline{g}_j] = [g_i, g_j] \in \mathfrak{g}$.

Hence we check that $a = \Phi a_s + \Phi a_n$ is an abstract Jordan decomposition of a. By exercise 12.3, we know that $\Phi a_s = a_s$. This is a contradiction.

□

Consequently $a_n \in \mathfrak{g}$. So $a = a_s + a_n$ is an abstract Jordan decomposition of a in $\mathfrak{g}$.

(12.2(c)): Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Choose $a \in \mathfrak{h}$ nonzero. By the previous result, we know that a has an abstract Jordan decompostion $a = a_s + a_n$. Hence we have a generalized eigenspace decomposition $\mathfrak{g} = \oplus_\alpha \mathfrak{g}^a_\alpha$, where $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. Noting that the Killing form on $\mathfrak{g}$ is nondegenerate by exercise 12.1, the proofs of Theorem 12.2 (a) and (b) are valid replacing root spaces by generalized eigenspaces. So $K \mid_{\mathfrak{g}^a_0 \times \mathfrak{g}^a_0}$ is nondegenerate.

Note that $\mathfrak{h} \subset \mathfrak{g}^a_0$ since $\mathfrak{h}$ is nilpotent. Extending an orthogonal basis of $\mathfrak{h}$ to an orthogonal basis of $\mathfrak{g}^a_0$,we see that

$$K \mid_{\mathfrak{h} \times \mathfrak{h}} \text{ degenerate } \Longrightarrow K \mid_{\mathfrak{g}^a_0 \times \mathfrak{g}^a_0} \text{ degenerate } .$$

Therefore $K \mid_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate.

By exercise 10.4, we know that Cartan's criterion applies to $\mathfrak{g}$. Therefore the argument in Theorem 12.2 (c) is valid in this case.

□

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Lecture 13 — Structure Theory of Semisimple Lie Algebras II

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Throughout this lecture, let $\mathfrak{g}$ be a finite dimensional semisimple Lie Algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0.

So far, we have proved:

1. The Killing form K of $\mathfrak{g}$ is non-degenerate.
2. The algebra $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{h}$. Furthermore, $\mathfrak{h}$ is abelian and diagonalizable on $\mathfrak{g}$, and we have:
$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right),$$
where,
$$\begin{aligned} \mathfrak{g}_\alpha &= \{a \in \mathfrak{g} | [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{h}\}, \\ \Delta &= \{\alpha \in \mathfrak{h}^* | \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}, \\ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] &\subseteq \mathfrak{g}_{\alpha+\beta}, \text{ which is 0 if } \alpha + \beta \notin \Delta \cup \{0\}, \text{ where } \mathfrak{g}_0 = \mathfrak{h}. \end{aligned}$$
3. The restriction $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$, *i.e.* $K(a, b)$ with $a \in \mathfrak{g}_\alpha$ and $b \in \mathfrak{g}_{-\alpha}$, is non-degenerate, so it induces a pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. In particular, we have $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$.
4. The restriction $K|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate, hence we have an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ given by $\nu(h)(h') = K(h, h')$ for all $h, h' \in \mathfrak{h}$. The map ν induces a bilinear form on $\mathfrak{h}^*$ by $K(\alpha, \beta) = \beta\left(\nu^{-1}(\alpha)\right) = \alpha\left(\nu^{-1}(\beta)\right)$ for all $\alpha, \beta \in \mathfrak{h}^*$. We proved that $K(\alpha, \alpha) \neq 0$ if $\alpha \in \Delta$.
5. For all $\alpha \in \Delta$, $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$, we have:
$$[e, f] = K(e, f)\nu^{-1}(\alpha).$$

Now, given $\alpha \in \Delta$, pick non-zero $E \in \mathfrak{g}_\alpha$ and $F \in \mathfrak{g}_{-\alpha}$ such that $K(E, F) = \frac{2}{K(\alpha,\alpha)}$. Let $H = \frac{2\nu^{-1}(\alpha)}{K(\alpha,\alpha)}$. Then, we can check that:

$$\begin{aligned} [H, E] &= 2E, \\ [H, F] &= -2F, \\ [E, F] &= H. \end{aligned}$$

The choice of E and F is possible by 3 and the last claim of 4. We only verify the first equality, the second being analogous and the third coming from 5. We have:

$$[\frac{2\nu^{-1}(\alpha)}{K(\alpha, \alpha)}, E] = \frac{2\alpha\left(\nu^{-1}(\alpha)\right)}{K(\alpha, \alpha)}E = \frac{2K(\alpha, \alpha)}{K(\alpha, \alpha)}E = 2E,$$

where the first equality comes from $\nu^{-1}(\alpha) \in \mathfrak{h}$ and $E \in \mathfrak{g}_\alpha$.

If we now let $\mathfrak{a}_\alpha = \mathbb{F}E + \mathbb{F}F + \mathbb{F}H$, then $\mathfrak{a}_\alpha$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ via:

$$E \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Lemma 1.1 (Key Lemma for $\mathfrak{sl}_2(\mathbb{F})$)**.** *Let $\mathbb{F}$ be a field of characteristic 0. Let π be a representation of $\mathfrak{sl}_2(\mathbb{F})$ in a vector space V over $\mathbb{F}$ and let $v \in V$ be a non-zero vector such that $\pi(E)v = 0$ and $\pi(H)v = \lambda v$ for some $\lambda \in \mathbb{F}$ (such vector is called a singular vector of weight λ).*

Then:

a) *$\pi(H)\pi(F)^n v = (\lambda - 2n)\pi(F)^n v$ for any $n \in \mathbb{Z}_{\geq 0}$.*

b) *$\pi(E)\pi(F)^n = n(\lambda - n + 1)\pi(F)^{n-1}v$ for any $n \in \mathbb{Z}_{\geq 1}$.*

c) *If $\dim V < \infty$, then $\lambda \in \mathbb{Z}_{\geq 0}$, the vectors $\pi(F)^j v$ for $0 \leq j \leq \lambda$ are linearly independent, and $\pi(F)^{\lambda+1}v = 0$.*

Proof. **a)** We prove this by induction on n. For $n = 0$, the result is given to us. Suppose it holds for $n = k - 1$ for some $k > 0$. Then:

$$\begin{aligned}
\pi(H)\pi(F)^k v &= \pi(F)\pi(H)\pi(F)^{k-1}v + [\pi(H), \pi(F)]\pi(F)^{k-1}v \\
&= \pi(F)\left(\lambda - 2(k-1)\right)\pi(F)^{k-1}v + \pi\left([H, F]\right)\pi(F)^{k-1}v \\
&= \left(\lambda - 2(k-1)\right)\pi(F)^k v + \pi(-2F)\pi(F)^{k-1}v \\
&= \left(\lambda - 2(k-1)\right)\pi(F)^k v - 2\pi(F)^k v \\
&= \left(\lambda - 2k\right)\pi(F)^k v.
\end{aligned}$$

This completes the induction.

Exercise 13.1. b) Again, we use induction on n. For the case $n = 1$, we have $\pi(E)\pi(F)v = \pi(F)\pi(E)v + [\pi(E), \pi(F)]v = 0 + \pi\left([E, F]\right)v = \pi(H)v = \lambda v$. Suppose the result holds for $n = k - 1$ for some $k > 1$. Then:

$$\begin{aligned}
\pi(E)\pi(F)^k v &= \pi(F)\pi(E)\pi(F)^{k-1}v + [\pi(E), \pi(F)]\pi(F)^{k-1}v \\
&= \pi(F)(k-1)\left(\lambda - (k-1) + 1\right)\pi(F)^{k-2}v + \pi\left([E, F]\right)\pi(F)^{k-1}v \\
&= (k-1)\left(\lambda - (k-1) + 1\right)\pi(F)^{k-1}v + \pi(H)\pi(F)^{k-1}v \\
&= (k-1)\left(\lambda - (k-1) + 1\right)\pi(F)^{k-1}v + \left(\lambda - 2(k-1)\right)\pi(F)^{k-1}v \\
&= k(\lambda - k + 1)\pi(F)^{k-1}v.
\end{aligned}$$

This completes the proof.

c) Suppose $\lambda \notin \mathbb{Z}_{\geq 0}$. Then, the term $n(\lambda - n + 1)$ is non-zero for all $n \geq 1$. Hence, by induction, we get $\pi(F)^n v \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$. But by **a)**, this implies that all vectors $\pi(F)^n v$ are eigenvectors of $\pi(H)$ with distinct eigenvalues. This allows us to conclude $\dim V = \infty$, therefore proving the first claim. If now, $\lambda \in \mathbb{Z}_{\geq 0}$, by the same argument we see that the vectors $\pi(F)^n v$ are linearly independent for $0 \leq n \leq \lambda$, and moreover, if $\pi(F)^{\lambda+1} v \neq 0$, then by induction we see that $\pi(F)^n v \neq 0$ for all $n > \lambda + 1$, and so there are infinitely many linearly independent vectors. Hence, if $\dim V < \infty$, then $\pi(F)^{\lambda+1} v = 0$, proving the last two claims.

$\square$

Exercise 13.2. Using the notation of the Key Lemma for $\mathfrak{sl}_2(\mathbb{F})$, if v instead satisfies that $\pi(F)v = 0$ and $\pi(H)v = \lambda v$, then we have:

a) $\pi(H)\pi(E)^n v = (\lambda + 2n)\pi(E)^n v$ for any $n \in \mathbb{Z}_{\geq 0}$.

b) $\pi(F)\pi(E)^n = -n(\lambda + n - 1)\pi(E)^{n-1} v$ for any $n \in \mathbb{Z}_{\geq 1}$.

c) If $\dim V < \infty$, then $-\lambda \in \mathbb{Z}_{\geq 0}$, the vectors $\pi(E)^j v$ for $0 \leq j \leq -\lambda$ are linearly independent, and $\pi(F)^{-\lambda+1} v = 0$.

Proof. It is enough to check that the function $\psi(E) = F$, $\psi(F) = E$, $\psi(H) = -H$, is an automorphism of $\mathfrak{a}_\alpha$. Indeed:

$$[\psi(H), \psi(E)] = 2\psi(E),$$
$$[\psi(H), \psi(F)] = -2\psi(F),$$
$$[\psi(E), \psi(F)] = \psi(H).$$

Thus, the Key Lemma shows that if $\pi(\psi(E))v = 0$ and $\pi(\psi(H))v = \lambda' v$ with $\lambda' \in \mathbb{F}$, then:

a) $\pi(\psi(H))\pi(\psi(F))^n v = (\lambda' - 2n)\pi(\psi(F))^n v$ for any $n \in \mathbb{Z}_{\geq 0}$.

b) $\pi(\psi(E))\pi(\psi(F))^n = n(\lambda' - n + 1)\pi(\psi(F))^{n-1} v$ for any $n \in \mathbb{Z}_{\geq 1}$.

c) If $\dim V < \infty$, then $\lambda' \in \mathbb{Z}_{\geq 0}$, the vectors $\pi(\psi(F))^j v$ for $0 \leq j \leq \lambda'$ are linearly independent, and $\pi(\psi(F))^{\lambda'+1} v = 0$.

Now, let $\lambda' = -\lambda$ and evaluate ψ to obtain the result. $\square$

Theorem 1.2. *The root space decomposition of $\mathfrak{g}$ with respect to a Cartan Subalgebra $\mathfrak{h}$ and the set of roots Δ satisfy the following properties:*

a) $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Delta$.

b) *If* $\alpha, \beta \in \Delta$, *then* $\{\beta + n\alpha\}_{n \in \mathbb{Z}} \cap (\Delta \cup \{0\})$ *is a finite connected string* $\{\beta - p\alpha, \beta - (p - 1)\alpha, \ldots, \beta, \ldots, \beta + (q-1)\alpha, \beta + q\alpha\}$, *where* $p, q \in \mathbb{Z}_{\geq 0}$ *and* $p - q = \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$.

c) *If* $\alpha, \beta, \alpha + \beta \in \Delta$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

d) *If* $\alpha \in \Delta$, *then* $n\alpha \in \Delta$ *if and only if* $n = 1$ *or* $n = -1$.

Proof. **a)** Suppose that $\dim \mathfrak{g}_\alpha > 1$ for some $\alpha \in \Delta$, then $\dim \mathfrak{g}_{-\alpha} > 1$ by non-degeneracy of the restriction $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$, property 3 above. Consider the adjoint representation of the subalgebra $\mathfrak{a}_\alpha = \mathbb{F}E + \mathbb{F}F + \mathbb{F}H$ on $\mathfrak{g}$. In particular, recall $H = \frac{2\nu^{-1}(\alpha)}{K(\alpha,\alpha)}$. Since $\dim \mathfrak{g}_{-\alpha} > 1$, there exists a non-zero vector $v \in \mathfrak{g}_{-\alpha}$ such that $K(E, v) = 0$. Hence, $(\mathbf{ad}\ E)v = [E, v] = K(E, v)\nu^{-1}(\alpha)$=0. But $(\mathbf{ad}\ H)v = [H, v] = \frac{2[\nu^{-1}(\alpha),v]}{K(\alpha,\alpha)} = \frac{-2\alpha(\nu^{-1}(\alpha))v}{K(\alpha,\alpha)} = \frac{-2K(\alpha,\alpha)v}{K(\alpha,\alpha)} = -2v$. Hence, $\dim \mathfrak{g} = \infty$ by the Key Lemma, which yields a contradiction.

b) Let q be the largest integer such that $\beta + q\alpha \in \Delta \cup \{0\}$. Notice $q \geq 0$. Pick a non-zero vector $v \in \mathfrak{g}_{\beta+q\alpha}$. Then, $(\mathbf{ad}\ E)v = 0$ since it lies in $\mathfrak{g}_{\beta+(q+1)\alpha}$. Also, $(\mathbf{ad}\ H)v = (\beta + q\alpha)(H)v = \left(\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} + 2q\right)v$. Hence, by the Key Lemma: $\lambda := \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} + 2q \in \mathbb{Z}_{\geq 0}$ and $(\mathbf{ad}\ F)^j v$ are non-zero vectors for $0 \leq j \leq \lambda$. But $(\mathbf{ad}\ F)^j v \in \mathfrak{g}_{\beta+(q-j)\alpha}$, so $\beta+q\alpha, \beta+(q-1)\alpha, \ldots, \beta+(q-\lambda)\alpha \in \Delta \cup \{0\}$. Define $p := -(q - \lambda) = q + \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$. Let p' be the largest integer for which $\beta - p'\alpha \in \Delta \cup \{0\}$. Again, notice $p' \geq 0$. Pick a non-zero vector $v' \in \mathfrak{g}_{\beta-p'\alpha}$. Then, $(\mathbf{ad}\ F)v' = 0$ and $(\mathbf{ad}\ H)v' = \left(\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} - 2p'\right)v'$. By the corollary of the Key Lemma, we conclude that $-\lambda' := 2p' - \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} \in \mathbb{Z}_{\geq 0}$ and that $\beta - p'\alpha, \beta - (p' - 1)\alpha, \ldots, \beta - (p' + \lambda')\alpha \in \Delta \cup \{0\}$. Define $q' := -(p' + \lambda') = p' - \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$. Since q and p' are the largest integers for which $\beta + q\alpha \in \Delta \cup \{0\}$ (resp. $\beta - p'\alpha \in \Delta \cup \{0\}$), we conclude that $q \geq q'$ and $p' \geq p$. Hence, $\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} = p - q \leq p' - q' = \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}$, showing that $p = p'$, $q = q'$, $p, q \in \mathbb{Z}_{\geq 0}$.

c) Pick the largest integers p and q such that $\beta - p\alpha, \beta + q\alpha \in \Delta \cup \{0\}$. Pick a non-zero vector $v \in \mathfrak{g}_{\beta-p\alpha}$. Then, $(\mathbf{ad}\ F)v = 0$ and $(\mathbf{ad}\ H)v = \left(\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} - 2p\right)v$. By the corollary of the Key Lemma, $(\mathbf{ad}\ E)^j v \neq 0$ for $0 \leq j \leq 2p - \frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} = p + q$. But $q \geq 1$ since $\alpha + \beta \in \Delta$, so $(\mathbf{ad}\ E)^{p+1}v$ is a non-zero vector. Its corresponding root is $\alpha + \beta$, and $(\mathbf{ad}\ E)^p v \in \mathfrak{g}_\beta$, so $[E, \mathfrak{g}_\beta] \neq 0$. Hence, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ since $\dim \mathfrak{g}_{\alpha+\beta} = 1$.

d) Let $\beta = n\alpha$, $n \neq 0$. Then, $\frac{2K(\alpha,\beta)}{K(\beta,\beta)} = \frac{2n}{n^2} = \frac{2}{n} \in \mathbb{Z}$. Hence, either $n = 2, 1, -1$ or -2. However, $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0$ by **a)** (resp. $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\alpha}] = 0$), so 2α (resp. -2α) is not a root because otherwise, **c)** would imply that $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$ (resp. $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\alpha}] = \mathfrak{g}_{-2\alpha}$).

□

Exercise 13.3. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$. We know K is non-degenerate, so $\mathfrak{g}$ is semisimple. We will find all possibilities for p and q in the proof above. Suppose $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with associated root system Δ. Under an inner automorphism σ of $\mathfrak{g}$, the Cartan subalgebra $\mathfrak{h}$ is sent to a conjugate Cartan Subalgebra $\mathfrak{h}' := \sigma(\mathfrak{h})$, and Δ is sent to the root system Δ' consisting of all linear functionals on $\mathfrak{h}'$ of the form $\alpha\sigma^{-1}$ with $\alpha \in \Delta$. Hence, we have the root space decomposition $\mathfrak{g} = \mathfrak{h}' \oplus \left(\bigoplus_{\alpha' \in \Delta'} \mathfrak{g}_{\alpha'}\right)$, where $\mathfrak{g}_{\alpha'} = \{a \in \mathfrak{g} | [h, a] = \alpha'(h)a \text{ for all } h \in \mathfrak{h}'\}$. However, inner automorphisms preserve the trace and we can check that $K(\alpha, \beta) = K(\alpha\sigma^{-1}, \beta\sigma^{-1})$, so the values of p and q are independent of the choice of Cartan subalgebra.

To construct $\mathfrak{h}$, take a diagonal matrix $a = \text{diag}\ (a_1, a_2, \ldots, a_n) \in \mathfrak{g}$ all of whose diagonal entries are distinct. By the extension of Exercise 3 in Lecture 7 to $\mathfrak{sl}_n(\mathbb{F})$, a is regular in $\mathfrak{g}$. Hence, $\mathfrak{g}_0^a$ is a Cartan subalgebra of $\mathfrak{g}$, so let $\mathfrak{h} = \mathfrak{g}_0^a$. As $(\mathbf{ad}\ a)^N e_{i,j} = (a_i - a_j)^N e_{i,j}$ for all $N \geq 0$, we see that $\mathfrak{h}$ is precisely the set of diagonal matrices of $\mathfrak{g}$. A basis for $\mathfrak{h}^*$ is given by $\{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \dots, \varepsilon_{n-1}-\varepsilon_n\}$, where $\varepsilon_i(b) = b_i$ for any $b = \text{diag}\,(b_1, b_2, \dots, b_n) \in \mathfrak{h}$ and $i \in \{1, \dots, n\}$. We can check that:

$$\mathfrak{g}_{\varepsilon_i-\varepsilon_j} = \mathbb{F}e_{i,j} \text{ for all } i, j \in \{1, \dots, n\},\ i \neq n.$$

Hence, the set $\Delta := \{\varepsilon_i - \varepsilon_j | i, j \in \{1, \dots, n\},\ i \neq j\}$ is a root system for $\mathfrak{g}$. For no pair of roots $\alpha, \beta \in \Delta$ it is true that $\beta + 3\alpha \in \Delta \cup \{0\}$. Thus, the only possibilities for (q, p) are $(2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)$.

When $n = 2$, we can only have $(q, p) = (2, 0), (0, 2)$. Let $\alpha := \varepsilon_1 - \varepsilon_2$. Then $\Delta = \{\pm\alpha\}$, so $\alpha = \alpha + (0)\alpha$, $0 = \alpha - (1)\alpha$, $-\alpha = \alpha - (2)\alpha$ and we have $\alpha, 0, -\alpha \in \Delta \cup \{0\}$, giving all possible values for p and q.

If $n = 3$, we can only have $(q, p) = (2, 0), (0, 2), (1, 0), (0, 1)$. We have the pairs $(2, 0)$ and $(0, 2)$ by the previous case. Now, letting $\beta := \varepsilon_1 - \varepsilon_3$ and $\gamma := \varepsilon_2 - \varepsilon_3$ so that $\Delta : \{\pm\alpha, \pm\beta, \pm\gamma\}$, we see that $\alpha - \beta, \alpha \in \Delta \cup \{0\}$ but $\alpha - 2\beta, \alpha + \beta \notin \Delta \cup \{0\}$, and similar relations hold among α, γ and β, γ by symmetry.

If $n \geq 4$, we can only have $(q, p) = (2, 0), (0, 2), (1, 0), (0, 1), (0, 0)$. The first four pairs come from the previous two cases. The fifth pair $(0, 0)$ occurs if we let $\delta = \varepsilon_3 - \varepsilon_4$ and then notice that $\alpha - \delta, \alpha + \delta \notin \Delta \cup \{0\}$.

In general, let $\alpha_{ij} := \varepsilon_i - \varepsilon_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. Then, for all multisets $\{i, j, k, l\} \subseteq \{1, \dots, n\}$:

if $\{i, j\} \cap \{k, l\} = 2$, then α_{ij} and α_{kl} are related by pairs $(2, 0), (0, 2)$;

if $\{i, j\} \cap \{k, l\} = 1$, then α_{ij} and α_{kl} are related by pairs $(1, 0), (0, 1)$;

if $\{i, j\} \cap \{k, l\} = 0$, then α_{ij} and α_{kl} are related by the pair $(0, 0)$.

18.745 Introduction to Lie Algebras	October 22, 2010

Lecture 14 — The Structure of Semisimple Lie Algebras III

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In this lecture we will prove more consequences of semisimplicity. Then we will prove a theorem that is used to prove that a Lie algebra is semisimple. Lastly, we will begin to examine examples of semisimple Lie algebras.

Recall some of the facts that we already know about semisimple Lie algebras. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{F}$, an algebraically closed field of characteristic 0. Choose a Cartan subalgebra $\mathfrak{h}$ and consider the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right), \qquad [\mathfrak{h}, \mathfrak{h}] = 0, \qquad dim \mathfrak{g}_\alpha = 1.$$

We use this decomposition to rewrite the Killing form K as a sum over root spaces. For $h_1, h_2 \in \mathfrak{h}$ we have:

$$K(h_1, h_2) = tr_{\mathfrak{g}}(\mathbf{ad}\ h_1)(\mathbf{ad}\ h_2) = \sum_{\alpha \in \Delta} \alpha(h_1)\alpha(h_2). \tag{1}$$

We proved in lecture 12 that $K|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate and creates an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ by $\nu(h)(h') = K(h, h')$. From this isomorphism we define a bilinear form K on $\mathfrak{h}^*$:

$$K(\lambda_1, \lambda_2) = \sum_{\alpha \in \Delta} \alpha(\nu^{-1}(\lambda_1))\alpha(\nu^{-1}(\lambda_2)) = \sum_{\alpha \in \Delta} K(\lambda_1, \alpha) K(\lambda_2, \alpha). \tag{2}$$

Definition 14.1. $\mathfrak{h}^*_{\mathbb{Q}} \subset \mathfrak{h}^*$ is the $\mathbb{Q}$-*span of* Δ. (Note that $\mathbb{Q} \subset \mathbb{F}$ since $charF = 0$.)

Now, a few consequences of semisimplicity:

Theorem 14.1. *For* $\mathfrak{g}$ *as above, the following are true:*

1. Δ *spans* $\mathfrak{h}^*$ *over* $\mathbb{F}$.
2. $K(\alpha, \beta) \in \mathbb{Q}$ *for all* $\alpha, \beta \in \Delta$.
3. $K|_{h^*_{\mathbb{Q}} \times \mathfrak{h}^*_{\mathbb{Q}}}$ *is a positive definite symmetric bilinear form with values in* $\mathbb{Q}$.

Proof. 1. Suppose Δ does not span $\mathfrak{h}^*$, then there exists a non-zero $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta$. This implies that $[h, \mathfrak{g}_\alpha] = 0$ for all α from the definition of root space. It is also true that $[h, \mathfrak{h}] = 0$, as from lecture 12 the Cartan subalgebra $\mathfrak{h}$ is abelian. Therefore, h is in $Z(\mathfrak{g})$. But $\mathfrak{g}$ is semisimple, so $Z(\mathfrak{g}) = 0$, which is a contradiction.

2. From Equation (2) we have:

$$K(\lambda,\lambda) = \sum_{\alpha\in\Delta} K(\lambda,\alpha)^2 \quad \forall\lambda\in\mathfrak{h}^*. \tag{3}$$

For $\lambda\in\Delta$, $K(\lambda,\lambda)\neq 0$, so:

$$\frac{4}{K(\lambda,\lambda)} = \sum_{\alpha\in\Delta}\left(\frac{2K(\lambda,\alpha)}{K(\lambda,\lambda)}\right)^2, \quad \lambda\in\Delta. \tag{4}$$

It follows that $\frac{4}{K(\lambda,\lambda)}\in\mathbb{Z}$ for $\lambda\in\Delta$ because by the string condition $\frac{2K(\lambda,\alpha)}{K(\lambda,\lambda)} = p-q\in\mathbb{Z}$. This implies that $K(\lambda,\lambda)\in\mathbb{Q}$ for $\lambda\in\Delta$.

But since $\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)}\in\mathbb{Z}$ for $\alpha,\beta\in\Delta$ and $K(\alpha,\alpha)\in\mathbb{Q}$ we conclude that $K(\alpha,\beta)\in\mathbb{Q}$.

3. From part 2, $K(\alpha,\beta)\in\mathbb{Q}$, hence $K(\lambda,\alpha)\in\mathbb{Q}$ for $\lambda\in\mathfrak{h}^*_{\mathbb{Q}}$ $\alpha\in\Delta$. Since by part 1, Δ spans $\mathfrak{h}^*$ and the Killing form is non-degenerate on $\mathfrak{h}$, its restriction to $\mathfrak{h}^*_{\mathbb{Q}}$ is non-degenerate as well.

Hence by equation (3), $K(\lambda,\lambda)\geq 0$ for $\lambda\in\mathfrak{h}^*_{\mathbb{Q}}$ since it is the sum of rational squares. This proves that K is positive and semi-definite.

It is a theorem of linear algebra that any non-degenerate positive semi-definite symmetric bilinear form is positive definite. This proves part 3. □

Every semisimple algebra is the direct sum of simple algebras. The following exercise shows that this decomposition is unique up to permutation.

Exercise 14.1. Recall that a semisimple Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^N s_j$ where s_j are simple Lie algebras.

Prove that this decomposition is unique up to permutation of the summands and prove that any ideal of $\mathfrak{g}$ is a subsum of this sum.

Proof. We first prove the second part, take any ideal $\mathfrak{h}$. $\mathfrak{h}\cap s_j$ is an ideal of the subalgebra s_j. s_j is simple so this is either 0 or s_j. So $\mathfrak{h} = \bigoplus_{j=1}^N(\mathfrak{h}\cap s_j) = \bigoplus_{j\in A} s_j$, which proves the second part.

For the first part, consider a second decomposition $\mathfrak{g} = \bigoplus_{i=1}^M(t_i)$. Each t_i is an ideal and therefore is the direct sum of a subset of the s_j. t_i is simple, so it must be the direct sum of exactly one s_j. Similarly, each s_j is the direct sum of exactly one t_i. This proves the first part. □

We will now examine how the decomposition of a Lie algebra corresponds to a decomposition of its root space.

Let $\mathfrak{g}$ be the direct sum of two ideals, $\mathfrak{g} = \mathfrak{g}_1\oplus\mathfrak{g}_2$, where both $\mathfrak{g}_i$ are semisimple. Consider for each of them the root space decomposition. To do this we choose a Cartan subalgebra $\mathfrak{h}_i$ in each $\mathfrak{g}_i$.

$$\mathfrak{g}_i = \mathfrak{h}_i\oplus\left(\bigoplus_{\alpha\in\Delta_i}\mathfrak{g}_\alpha\right) \qquad \mathfrak{g} = \mathfrak{h}\oplus\left(\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha\right)$$

Where $\mathfrak{h}$ is $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $\Delta = \Delta_1 \sqcup \Delta_2$.

If $\alpha \in \Delta_1$ and $\beta \in \Delta_2$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, then we have that $\alpha + \beta \notin \Delta$ and $\alpha + \beta \neq 0$. In other words,

$$\Delta = \Delta_1 \sqcup \Delta_2, \qquad \alpha + \beta \notin \Delta \cup 0 \;\; if \;\; \alpha \in \Delta_1, \beta \in \Delta_2. \tag{5}$$

Definition 14.2. The set Δ in a vector space V is called *indecomposable* if it cannot be decomposed into a disjoint union of non-empty subsets Δ_1 and Δ_2 such that equation 5 holds.

With this notion, we can give a simple way to check if a semisimple Lie algebra is simple.

Theorem 14.2. *Let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} g_\alpha)$ be a decomposition of a finite dimensional Lie algebra $\mathfrak{g}$ into a direct sum of subspaces such that the following properties hold:*

1. *$\mathfrak{h}$ is an abelian subalgebra and $dim g_\alpha = 1$ for all $\alpha \in \Delta, where \mathfrak{g}_\alpha = \{a \in \mathfrak{g} | [h, a] = \alpha(h) a \forall h \in \mathfrak{h}\}$,*
2. *$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F} h_\alpha$, where $h_\alpha \in \mathfrak{h}$ is such that $\alpha(h) \neq 0$,*
3. *$\mathfrak{h}^*$ is spanned by Δ.*

Then $\mathfrak{g}$ is a semisimple Lie algebra. Moreover if the set Δ is indecomposable, then $\mathfrak{g}$ is simple.

Lemma 14.3. *Let $\mathfrak{h}$ be an abelian Lie algebra and π its representation in a vector space V such that V has a weight space decomposition: $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V | \pi(h) v = \lambda(h) v, \forall h \in \mathfrak{h}\}$.*

If $U \subset V$ is a $\pi(\mathfrak{h})$-invariant subspace, then $U = \bigoplus_{\lambda \in \mathfrak{h}^*} (U \cap V_\lambda)$.

Proof. For $u \in U$, let:

$$u = \sum_{i=1}^{n} v_{\lambda_i}, \qquad v_{\lambda_i} \in V_{\lambda_i}, \qquad \lambda_i \neq \lambda_j. \tag{6}$$

We will prove that all v_{λ_i} are in U by induction on n. For the case $n = 1$, $v_{\lambda_1} = u \in U$.

For the case, $n > 1$ we apply $\pi(h)$ to both sides:

$$\pi(h) u = \sum_{i=1}^{n} \lambda_i(h) v_{\lambda_i} \tag{7}$$

where $h \in \mathfrak{h}$ is chosen such that $\lambda_i(h) \neq \lambda_j(h)$ for $i \neq j$.

From equations 6 and 7, $\pi(h) u - \lambda_1(h) u = \sum_{i=2}^{n} (\lambda_i(h) - \lambda_1(h)) v_{\lambda_i}$, where each term is not 0 by assumption.

By the inductive assumption, $v_{\lambda_i} \in U$ for $i \geq 2$, hence also $v_{\lambda_1} \in U$. $\square$

Now for the proof of the theorem.

Proof. We want to prove that if $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, then $\mathfrak{a} = 0$. Note that since $\mathfrak{a}$ is an ideal, it is invariant with respect to **ad** h on $\mathfrak{g}$. Hence by the lemma, either $\mathfrak{g}_\alpha$ is in $\mathfrak{a}$ for some α or $\mathfrak{h} \cap \mathfrak{a}$ is non-zero. This uses the fact that $dim \mathfrak{g}_\alpha = 1$. In the first case, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}h_\alpha \subset \mathfrak{a}$, but $[h_\alpha, \mathfrak{g}_\alpha] \neq 0$ since $\alpha(h_\alpha) \neq 0$. So $\mathfrak{a}$ contains the non-abelian subalgebra $\mathbb{F}h_\alpha \oplus \mathfrak{g}_\alpha$, which is impossible since $\mathfrak{a}$ is abelian.

In the second case, where $\mathfrak{h} \cap \mathfrak{a}$ is non-zero, let $h \in \mathfrak{a}$, $h \in \mathfrak{h}$, $h \neq 0$. By condition 3, $\mathfrak{h}^*$ is spanned by Δ, so $\alpha(h) \neq 0$ for some $\alpha \in \Delta$. Hence $[h, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha \in \mathfrak{a}$, which again contradicts that $\mathfrak{a}$ is abelian. So a=0.

This proves that $\mathfrak{g}$ is semisimple.

Now the proof of simplicity, given that Δ is indecomposable.

Since $\mathfrak{g}$ is semisimple, we have that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where the $\mathfrak{g}_i$ are non-zero ideals. In the contrary case, by our discussion before Definition 14.2, this implies that Δ is decomposable. This is a contradiction, so $\mathfrak{g}$ is simple. $\square$

The following is an easy method to determine if a set Δ is indecomposable.

Exercise 14.2. Prove that a finite subset: $\Delta \subset V \setminus \{0\}$ is an indecomposable set if and only if for any $\alpha, \beta \in \Delta$, there exists a sequence $\gamma_1, \gamma_2 \dots \gamma_s$ such that $\alpha = \gamma_1$, $\beta = \gamma_s$ and $\gamma_i + \gamma_{i+1} \in \Delta$ for $i = 1 \dots s-1$.

Also for any $\Delta \subset V \setminus \{0\}$, construct its canonical decomposition into a disjoint union of indecomposable sets.

Proof. Consider the contrapositive. Suppose Δ is decomposable. Therefore, $\Delta = \Delta_1 \cup \Delta_2$. Take some $\alpha \in \Delta_1, \beta \in \Delta_2$. No sequence can exist with $\gamma_1 = \alpha$ and $\gamma_s = \beta$. For at some step in any such sequence $\gamma_i \in \Delta_1$ and $\gamma_{i+1} \in \Delta_2$. But $\gamma_1 + \gamma_2 \in \Delta$ contradicts the definition of a decomposition.

Suppose for some $\alpha, \beta \in \Delta$ no sequence exists. Let Δ_1 be the set of roots for which a sequence with α exists. Let Δ_2 be the set of roots for which a sequence with α does not exist. This is clearly a disjoint partition of Δ. Further Δ_1 and Δ_2 are non-empty as $\alpha \in \Delta_1$ $\beta \in \Delta_2$. For $\alpha' \in \Delta_1$, $\beta' \in \Delta_2$ $\alpha' + \beta' \notin \Delta$ as otherwise one can concatenate the sequence from $\alpha = \gamma_1$ to $\alpha' = \gamma_s$ and from $\alpha' = \gamma_s$ to $\beta' = \gamma_{s+1}$. Thus, $\Delta_1 \sqcup \Delta_2$, where Δ_1 is indecomposable. Now apply the same argument to Δ_2, etc. Since Δ is a finite set, we obtain the decomposition of Δ in a finite number of steps. $\square$

14.1 Examples of Semisimple Lie Algebras

Example 14.1. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field with characteristic 0. Let the Cartan subalgebra $\mathfrak{h}$ be the set of all traceless diagonal matrices. $\mathfrak{h}$ lies in D_n, the space of all diagonal matrices. Denote by ϵ_i the following linear function on D: $\epsilon_i(A) = a_i$, the ith coordinate of the diagonal of matrix A. The set $\{\epsilon_i \mid i = 1 \dots n\}$ is clearly a basis of D^*. If $char\mathbb{F} \nmid n$, another basis is $\{\epsilon_i - \epsilon_{i+1}, \epsilon_1 + \dots + \epsilon_n \mid i = 1 \dots n-1\}$, since it generates the first basis and has the same size. $\mathfrak{h} = \{a \in D \mid (\epsilon_1 + \dots + \epsilon_n)a = 0\}$. Therefore, $\mathfrak{h}^* = D^*/(\epsilon_1 + \dots + \epsilon_n)$, and $\{\epsilon_i - \epsilon_{i+1} \mid i \neq j\}$ is a basis for h^*.

The root space decomposition of $\mathfrak{g} = \mathfrak{sl}_n(F)$ is $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\Delta = \{\epsilon_i - \epsilon_j | i \neq j\}$ and $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{F} e_{ij}$.

To prove $\mathfrak{sl}_n(\mathbb{F})$ is semisimple, we check the conditions of theorem 14.2. 1. is clearly true. 2. $[e_{ij}, e_{ji}] = e_{ii} - e_{jj}, (\epsilon_i - \epsilon_j)(e_{ii} - e_{jj}) = 2 \neq 0$. This is why $\mathbb{F}$ cannot be characteristic 2. 3. When $char \nmid n$, the $\epsilon_i - \epsilon_j$ span $\mathfrak{h}$, so Δ spans $\mathfrak{h}$.

This implies $\mathfrak{sl}_n(\mathbb{F})$ is semisimple for any $n \neq 2$ and $char\mathbb{F} \nmid n$. To show it is simple, we prove that Δ is indecomposable.

Let $\alpha = \epsilon_i - \epsilon_j$, $\beta = \epsilon_s - \epsilon_t$. Using exercise 14.2, let $\gamma_1 = \alpha$, $\gamma_2 = \epsilon_j - \epsilon_s$, $\gamma_3 = \beta$. This is a string from α to β as $\gamma_1 - \gamma_2 = \epsilon_i - \epsilon_s \in \Delta$ and $\gamma_2 - \gamma_3 = \epsilon_j - \epsilon_t \in \Delta$. This proves Δ is indecomposable and that $\mathfrak{sl}_n(\mathbb{F})$ is simple.

Exercise 14.3. The above argument fails if $char\mathbb{F} \mid n$. As $\mathfrak{sl}_n(\mathbb{F})$ contains a non-trivial abelian ideal, $Z(\mathfrak{sl}_n(\mathbb{F}))$ since $I_n \in Z(\mathfrak{sl}_n(\mathbb{F}))$. How does the argument fail?

Proof. The argument fails because Δ does not span $\mathfrak{h}^*$. When $char\mathbb{F} \mid n$, $I_n \in \mathfrak{h}$ and Δ is still $\{\epsilon_i - \epsilon_j \mid i \neq j\}$. Since $(\epsilon_i - \epsilon_j)(I_n) = 0$, Δ can't span $\mathfrak{h}^*$. □

18.745 Introduction to Lie Algebras	October 28, 2010

Lecture 15 — Classical (Semi) Simple Lie Algebras and Root Systems

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Recall

$$O_{V,B}(\mathbb{F}) = \{a \in gl_V(\mathbb{F}) \mid B(au, v) + B(u, av) = 0, \text{ for all } u, v \in V\} \subset gl_V(\mathbb{F})$$

where V is a vector space over $\mathbb{F}$, B is a bilinear form : $V\text{x}V \to \mathbb{F}$. Choosing a basis of V and denoting by B the matrix of the bilinear form in this basis, we proved we get the subalgebra

$$o_{n,B}(\mathbb{F}) = \{a \in gl_n(\mathbb{F}) \mid a^T B + Ba = 0\} \subset gl_n(\mathbb{F}).$$

For different choices of basis, we get isomorphic Lie algebras $o_{n,B}(\mathbb{F})$.

Now, consider the case where B is a symmetric non-degenerate bilinear form. If $\mathbb{F}$ is algebraically closed and char $\mathbb{F} \neq 2$, one can choose a basis in which the matrix of B is any symmetric non-degenerate matrix.

Example 15.1. I_N where $N = \dim V$.

We will choose a basis such that

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 1 \\ 0 & 0 & & & \iddots & \\ 0 & & \ddots & 1 & & \vdots \\ \vdots & & 1 & \ddots & & 0 \\ & \iddots & & & 0 & 0 \\ 1 & & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and denote by $so_N(\mathbb{F})$ the corresponding Lie algebra $o_{N,B}(\mathbb{F})$.

Exercise 15.1. Show $so_N(\mathbb{F}) = \{a \in gl_N(\mathbb{F}) | a + a' = 0\}$ where a' is the transposition of a with respect to the anti-diagonal.

Proof. $so_N(\mathbb{F}) = \{a \in gl_N\mathbb{F}) \mid a^T B + Ba = 0\}$ where B is the matrix consisting of ones along the anti-diagonal.

As $B = B^T$, we have $a^T B = a^T B^T = (Ba)^T$. Viewing B as a permutation matrix, we get Ba permutes the rows by $\text{row}_i \to \text{row}_{n-i}$. Transposing and reapplying B, we get $B(Ba)^T = a'$ and $BBa = a$. Hence we obtain the following sequence of implications

$$a^T B + Ba = 0$$

$$(Ba)^T + Ba = 0$$

$$B(Ba)^T + a = 0$$

and finally

$$a' + a = 0.$$

Therefore, $so_N(\mathbb{F}) = \{a \in gl_N(\mathbb{F}) | a + a' = 0\}$.

$\square$

Example 15.2. $so_2(\mathbb{F}) = \{\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \alpha \in \mathbb{F}\}$, which is one-dimensional abelian, hence not semisimple.

Proposition 15.1. *Assume $N \geq 3$, then $so_N(\mathbb{F})$ is semisimple.*

Proof. We show this by the study of the root space decomposition.

Case 1: $N = 2n + 1$ (odd). Let

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & & & \\ & & a_n & & & & \\ & & & 0 & & & \\ & & & & -a_n & & \\ & & & & & \ddots & \\ & & & & & & -a_1 \end{pmatrix}. \subset so_{2n+1}(\mathbb{F})$$

This is a Cartan subalgebra since it contains a diagonal matrix with distinct entries.

Case 2: $N = 2n$ (even). Let

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & -a_n & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix}.$$

This is a Cartan subalgebra for the same reason.

In both cases, $\dim \mathfrak{h} = n$ and $\epsilon_1, ..., \epsilon_n$ form a basis $\mathfrak{h}^*$. Note that $\epsilon_{N+1-j}|_{\mathfrak{h}} = -\epsilon_j|_{\mathfrak{h}}$ and $\epsilon_{\frac{N+1}{2}}|_{\mathfrak{h}} = 0$ if N is odd.

Next, all eigenvectors for **ad** $\mathfrak{h}$ are elements $e_{i,j} - e_{N+1-j,N+1-i}, i, j \in \{1, 2, ..., N\}$ and the root is $\epsilon_i - \epsilon_j|_{\mathfrak{h}}$.

Hence the set of roots is:

$N = 2n + 1 : \Delta_{so_N(\mathbb{F})} = \{\epsilon_i - \epsilon_j, \epsilon_i, -\epsilon_i, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i, j \in \{1, ..., n\}, i \neq j\}$

$N = 2n : \Delta_{so_N(\mathbb{F})} = \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i, j \in \{1, ..., n\}, i \neq j\}$ $\square$

Exercise 15.2. a) Using the root space decomposition, prove that $so_N(\mathbb{F})$ is semisimple if $N \geq 3$.

b) Show $so_N(\mathbb{F})$ is simple if $N = 3$ or $N \geq 5$ by showing that Δ is indecomposable.

Thus we have another two series of simple Lie algebras: $so_{2n+1}(\mathbb{F})$ for $n \geq 1$ (type B) and $so_{2n}(\mathbb{F})$ for $n \geq 3$ (type D).

Proof. a) We must check (1), (2), and (3) of the semisimplicity criterion.

(1) is clear for B and D and (3) is clear for B. For (3) in case D, we have roots $\epsilon_i - \epsilon_j$ and $\epsilon_i + \epsilon_j$, adding and dividing by 2 (as char $\mathbb{F} \neq 2$) gives us ϵ_i, hence (3) holds.

(2) We compute $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = [e_{ij} - e_{N+1-j,N+1-i}, e_{ji} - e_{N+1-i,N+1-j}] = (e_{ii} - e_{N+1-i,N+1-i}) + (e_{N+1-j,N+1-j} - e_{jj}) = h_\alpha \in \mathfrak{h}$. As $\alpha(h_\alpha) \neq 0$, so_N is semisimple in $N \geq 3$.

b) To show simple for $N = 3$ and $N \geq 5$, we show that Δ is indecomposable. This is clear for $N = 3$. We list pairs and corresponding paths for $n \geq 3$. $N = 5$ is done separately. For ease of notation, we write ϵ_i as i and remark that any root is connected to its negative by the path of length one.

- $(i + j) \to (j + k)$ via $(i + j, -k - i, j + k)$
- $(i + j) \to (i - j)$ via $(i + j, k - i, j - k, i - j)$
- $(i + j) \to (i)$ via $(i + j, -j, i)$

  Therefore when $N > 5$, we may concatenate and find paths through $(i + j)$. When $N = 5$, we had the issue of connecting $(i + j)$ to $(i - j)$. As N is odd in this case, we may use the path $(i + j, -i, i - j)$. Therefore, $so_N(\mathbb{F})$ is simple for $N = 3$ and $N \geq 5$.

□

Exercise 15.3. Show $\Delta_{so_4(\mathbb{F})} = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1\} \sqcup \{\epsilon_1 + \epsilon_2, -\epsilon_1 - \epsilon_2\}$ is the decomposition into decomposables. Deduce that $so_4(\mathbb{F})$ is isomorphic to $sl_2(\mathbb{F}) \oplus sl_2(\mathbb{F})$.

Proof. We have the decomposition

$$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1\} \sqcup \{\epsilon_1 + \epsilon_2, -\epsilon_1 - \epsilon_2\}.$$

This is in fact a decomposition since the collection of roots distance one from $\epsilon_1 - \epsilon_2$ is $\epsilon_2 - \epsilon_1$ as $\epsilon_1 - \epsilon_2 + \epsilon_1 + \epsilon_2 = 2\epsilon_1 \notin \Delta$ and $\epsilon_1 - \epsilon_2 + -\epsilon_1 - \epsilon_2 = -2\epsilon_2 \notin \Delta$. Hence, we have a decomposition.

To show isomorphic to $sl_2 \oplus sl_2$. Consider the basis of $\mathfrak{h}$, $x = e_{11} - e_{44}$ and $y = e_{22} - e_{33}$. If we change bases to from x, y to $x + y, x - y$. Let

$$e = e_{12} - e_{34}, f = e_{13} - e_{24}, g = e_{31} - e_{42}, h = e_{21} - e_{43}.$$

Then

$$[x + y, e] = 0, [x + y, f] = 2f, [x + y, g] = -2g, [x + y, h] = 0, [g + f] = x + y$$

and

$$[x - y, e] = 2e, [x - y, f] = 0, [x - y, g] = 0, [x - y, h] = -2h, [e, h] = x - y$$

and finally

$$[x+y, x-y] = [e,f] = [e,g] = [h,f] = [h,g] = 0.$$

Therefore, we have two copies of sl_2 formed by $\{f, g, x+y\}$ and $\{e, h, x-y\}$, and therefore an isomorphism. $\square$

Next consider the case where B is a skew-symmetric non-degenerate bilinear form. If $\mathbb{F}$ is of characteristic $\neq 2$, one can choose a basis in which the matrix of B is any skew-symmetric non-degenerate matrix where $N = \dim V = 2n$ (even). We get

$$sp_{n,B} = \{a \in gl_n(\mathbb{F}) \mid a^T B + B a^T = 0\} \subset gl_n(\mathbb{F}).$$

The best choice of B is

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 1 \\ 0 & 0 & & & \iddots & \\ 0 & & \ddots & 1 & & \vdots \\ \vdots & & -1 & \ddots & & 0 \\ & \iddots & & & 0 & 0 \\ -1 & & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 15.4. Repeat the discussion we've done for $so_N(\mathbb{F})$ in the case $sp_{2n}(\mathbb{F})$. First:

$$sp_{2n}(\mathbb{F}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ all } a,b,c,d \text{ are } n \text{ x } n \text{ such that } b = b', c = c', d = -a' \}$$

Next, let $\mathfrak{h}$ be the set of all diagonal matrices in $sp_{2n}(\mathbb{F})$

$$\mathfrak{h} = \{ \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_r & & & \\ & & & -a_r & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix}, a_i \in \mathbb{F}\}$$

Find all eigenvectors for **ad** $\mathfrak{h}$. Show that the set of roots is

$$\Delta_{sp_{2n}(\mathbb{F})} = \{\epsilon_i - \epsilon_j, 2\epsilon_i, -2\epsilon_i, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i,j \in \{1,...,n\}, i \neq j\}$$

Show always indecomposable and deduce that $sp_{2n}(\mathbb{F})$ simple for all $n \geq 1$.

These Lie algebras are called type C simple Lie algebras.

Proof. We must check (1),(2), and (3) of the semisimplicity criterion. (1) is clear.

For (2), split a matrix M into its four quadrants. Label the upper left quadrant A. The upper left half of the upper right quadrant B, with that portion of the anti-diagonal X. Finally, the upper left half of the lower left quadrant C, with that portion of the anti-diagonal Y. We then have the following eigenvectors.

- $e_{ij} - e_{N+1-j,N+1-i}$ if $e_{ij} \in A$
- $e_{ij} + e_{N+1-j,N+1-i}$ if $e_{ij} \in B \cup C$
- e_{ij} if $e_{ij} \in X \cup Y$

with eigenvalues $\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j, 2\epsilon_i, -2\epsilon_i$ for $e_{ij} \in A, B, C, X, Y$ respectively.

To compute $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, we have

$$[e_{ij} - e_{N+1-j,N+1-i}, e_{ji} - e_{N+1-i,N+1-j}] = (e_{ii} - e_{N+1-i,N+1-i}) + (e_{N+1-j,N+1-j} - e_{jj}) = h_\alpha \in \mathfrak{h}$$

$$[e_{ij} + e_{N+1-j,N+1-i}, e_{ji} + e_{N+1-i,N+1-j}] = (e_{ii} - e_{N+1-i,N+1-i}) - (e_{N+1-j,N+1-j} - e_{jj}) = h_\alpha \in \mathfrak{h}$$

$$[e_{ij}, e_{jj}] = e_{ii} - e_{jj} = h_\alpha \in \mathfrak{h}.$$

In each case $\alpha(h_\alpha) \neq 0$, and therefore (2) holds.

Finally, (3) is clear, so we only must show in-decomposablility to get simplicity.

From Exercise 15.2, when $n \geq 3$, have that all pairs of the form $\pm\epsilon_i \pm \epsilon_j$ are connected to $\epsilon_i - \epsilon_j$. We may connect $2\epsilon_j$ to $\epsilon_i - \epsilon_j$ as $2\epsilon_j + \epsilon_i - \epsilon_j = \epsilon_i + \epsilon_j \in \Delta$ and therefore through concatenation we are done.

When $n = 2$ we may connect $\epsilon_i - \epsilon_j$ to $\epsilon_i + \epsilon_j$ as $\epsilon_i - \epsilon_j + \epsilon_i + \epsilon_j = 2\epsilon_i \in \Delta$, so we have in-decomposability.

When $n = 1$, indecomposability is clear.

□

Remark 1. Thus we get four series of simple Lie algebras $A_n = sl_n(\mathbb{F})(n \geq 1), B_n = so_{2n+1}(\mathbb{F})(n \geq 1), C_n = sp_{2n}(\mathbb{F})(n \geq 1), D_n = so_{2n}(\mathbb{F}), (n \geq 4)$ called the classical simple Lie algebras.

Proposition 15.2. *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. Then*

a) Any symmetric invariant bilinear form is either non-degenerate or identically zero.

b) Any two non-degenerate such bilinear forms are proportional: $(a,b)_1 = \lambda(a,b)_2$.

Proof. a) If $(\cdot,\cdot)$ is an invariant bilinear form and I is its kernel, then I is an ideal, hence $\mathfrak{g}$ simple implies that either $I = 0$ or $I = \mathfrak{g}$.

b) Choose a basis of $\mathfrak{g}$ and let B_i be the matrix of $(\cdot,\cdot)_i$ in the basis. $\mathrm{Det}(B_i) \neq 0$. Consider $\det(B_1 - \lambda B_2) = \det(B_2)\det(B_1B_2^{-1} - \lambda I) = 0$ if λ is an eigenvalue of $B_1B_2^{-1}$. Hence the form $(a,b)_1 - \lambda(a,b)_2$ is a degenerate, invariant, bilinear form as det is 0. Hence the form $(a,b)_1 - \lambda(a,b)_2$ is identically zero by (a), which implies $(a,b)_1 = \lambda(a,b)_2$. □

Corollary 15.3. *If $\mathfrak{g} \subset gl_N(\mathbb{F})$ is a simple Lie algebra, then the Killing from on $\mathfrak{g}$ is proportional to the trace form $(a,b) = tr\ ab$ on $\mathfrak{g}$.*

Example 15.3. On $gl_N(\mathbb{F})$: (1) tr $e_{ii}e_{ij} = \delta_{ij}$ (with e_{ii} basis of D), hence the induced bilinear form on $D^* = (\epsilon_i, \epsilon_j) = \delta_{ij}$ (2). Hence for all classical simple Lie algebras A,B,C,D, the Killing form is a positive constant multiple of (1) and on $\mathfrak{h}^*$ is a positive constant multiple of (2).

Definition 15.1. Let V be a finite dimensional real Euclidean space, i.e. V finite dimensional vector space over $\mathbb{R}$ with symmetric positive definite bilinear form $(\cdot,\cdot)$.

Let $\Delta \subset V$ be a subset of V. Then the pair (V,Δ) is called a root system if:

i) Δ finite, $0 \notin \Delta$, Δ spans V over $\mathbb{R}$;

ii) (String Condition) For any $\alpha, \beta \in \Delta$, the set $\{\beta + j\alpha \mid j \in \mathbb{Z}\} \bigcap (\Delta \cup \{0\})$ is a string $\beta + p\alpha, \beta + (p-1)\alpha, ..., \beta - q\alpha$ where $p, q \in \mathbb{Z}$, and $p - q = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$;

iii) For all $\alpha \in \Delta$, we have $k\alpha \in \Delta$ if and only if $k = 1$ or $k = -1$.

Example 15.4. The basic example: Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, $\mathfrak{h}$ a Cartan subalgebra, $\Delta \subset \mathfrak{h}^*_{\mathbb{Q}}$ the set of roots, $(\cdot,\cdot)$ the Killing form on $\mathfrak{h}^*_{\mathbb{Q}}$ which is $\mathbb{Q}$-valued and positive definite.

Let $V = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Q}}$, i.e. linear combinations of roots with real coefficients and extend the Killing form by bilinearity. Then the pair (V,Δ) is a root system, called the $\mathfrak{g}$ root system.

Remark 2. This construction is independent of the choice of the Cartan subalgebra $\mathfrak{h}$ due to Chevalley's Theorem.

Exercise 15.5. Let (V,Δ) be a root system. Then Δ is indecomposable if and only if there does not exist non-trivial decomposition $(V,\Delta) = (V_1,\Delta_1) \oplus (V_2,\Delta_2)$ where $V = V_1 \oplus V_2$, $V_1 \perp V_2$, $\Delta_i \subset V_i$, and $\Delta = \Delta_1 \cup \Delta_2$. (Hint: Use String Condition)

Moreover, the decomposition of $\Delta = \bigsqcup \Delta_i$ into indecomposable sets corresponds to decomposition of the root system in the orthogonal direct sum of indecomposable root systems.

Proof. For the first direction, suppose we have a decomposition $\Delta = \Delta_1 \sqcup \Delta_2, \Delta_i \subset V_i, \alpha \in \Delta_i, \beta \in \Delta_2$. Therefore, $\alpha + \beta \notin \Delta \cup \{0\}$, hence $q = 0$. As well, clearly $-\alpha \in \Delta_2$, so $p = 0$. Therefore, $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 0$, hence $(\alpha,\beta) = 0$. Let $V_i = span(\Delta_i)$, then by above $V_1 \perp V_2$, and thus $V = V_1 \oplus V_2$.

On the other hand, suppose $V = V_1 \oplus V_2, V_1 \perp V_2$. Choose $\Delta_i = \Delta \cap V_i$. We show $\Delta_1 \sqcup \Delta_2$ is a decomposition. In the contrary case, choose $\alpha \in \Delta_i, \beta \in \Delta_2$ and suppose $\alpha + \beta \in \Delta \cup \{0\}$. Then, $\alpha + \beta \neq 0$ since $(\alpha, -\alpha) \neq 0$, so $\alpha + \beta \in \Delta$. Without loss of generality, $\alpha + \beta \in \Delta_1 \subset V_1$, as $\beta \in \Delta_2 \subset V_2$ and $V_1 \perp V_2$, we have $0 = (\alpha+\beta, \beta) = (\alpha,\beta) + (\beta,\beta) = (\beta,\beta)$. This is a contradiction, hence $\alpha + \beta \notin \Delta$, so we have a decomposition.

By the above argument, it is clear that the decomposition into indecomposables corresponds to the orthogonal decomposition with respect to $(\cdot,\cdot)$.

$\square$

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Lecture 16 — Root Systems and Root Lattices

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Recall that a root system is a pair (V, Δ), where V is a finite dimensional Euclidean space over $\mathbb{R}$ with a positive definite bilinear form $(\cdot, \cdot)$ and Δ is a finite subset, such that:

1. $0 \notin \Delta; \mathbb{R}\Delta = V$;
2. If $\alpha \in \Delta$, then $n\alpha \in \Delta$ if and only if $n = \pm 1$;
3. (String property) if $\alpha, \beta \in \Delta$, then $\{\beta + j\alpha | j \in \mathbb{Z}\} \cap (\Delta \cup 0) = \{\beta - p\alpha, \dots, \beta, \dots, \beta + q\alpha\}$where $p - q = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$.

(V, Δ) is called indecomposable if it cannot be decomposed into a non-trivial orthogonal direct sum. $r = dimV$ is called the rank of (V, Δ) and elements of Δ are called roots.

Definition 16.1. An isomorphism of an indecomposable root system (V, Δ) and (V_1, Δ_1) is a vector space isomorphism $\varphi : V \to V_1$, such that $\varphi(\Delta) = \Delta_1$, and $(\varphi(\alpha), \varphi(\beta))_1 = c(\alpha, \beta)$ for all $\alpha, \beta \in \Delta$, where c is a positive constant, independent of α and β. In particular, replacing $(\cdot, \cdot)$ by $c(\cdot, \cdot)$, where $c > 0$, we get, by definition, an isomorphic root system.

Example 16.1. Root systems of rank 1: $(\mathbb{R}, \Delta = \{\alpha, -\alpha\})$, $\alpha \neq 0$, $(\alpha, \beta) = \alpha\beta$. This root system is isomorphic to that of $sl_2(\mathbb{F})$, $so_3(\mathbb{F})$, and $sp_2(\mathbb{F})$.

Proposition 16.1. *Let (V, Δ) be an indecomposable root system with the bilinear form $(\cdot, \cdot)$. Then*

1. *Any other bilinear form $(\cdot, \cdot)_1$ for which the string property holds is proportional to $(\cdot, \cdot)$, i.e. $(\alpha, \beta)_1 = c(\alpha, \beta)$ for some positive $c \in \mathbb{R}$, independent of α and β.*
2. *If $(\alpha, \alpha) \in \mathbb{Q}$ for some $\alpha \in \Delta$, then $(\beta, \gamma) \in \mathbb{Q}$ for all $\beta, \gamma \in \Delta$.*

Proof. Fix $\alpha \in \Delta$. Since (V, Δ) is indecomposable, for any $\beta \in \Delta$ there exists a sequence $\gamma_0, \gamma_1, \dots, \gamma_k$ such that $\alpha = \gamma_0$, $\beta = \gamma_k$, $(\gamma_i, \gamma_{i+1}) \neq 0$ for all $i = 0, \dots, k - 1$. Define c by $(\alpha, \alpha)_1 = c(\alpha, \alpha)$. By the string property $p - q = 2\frac{(\alpha,\gamma_1)}{(\alpha,\alpha)} = 2\frac{(\alpha,\gamma_1)_1}{(\alpha,\alpha)_1}$. Hence $(\alpha, \gamma_1)_1 = c(\alpha, \gamma_1)$. Likewise, by the string property, $\frac{2(\alpha,\gamma_1)}{(\gamma_1,\gamma_1)} = \frac{2(\alpha,\gamma_1)_1}{(\gamma_1,\gamma_1)_1}$. Hence $(\gamma_1, \gamma_1) = c(\gamma_1, \gamma_1)$. Continuing this way, we show that $(\gamma_2, \gamma_2)_1 = c(\gamma_2, \gamma_2), \dots (\beta, \beta)_1 = c(\beta, \beta)$. Since $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\alpha,\beta)_1}{(\alpha,\alpha)_1}$, we conclude that $(\alpha, \beta)_1 = c(\alpha, \beta)$ for all $\alpha, \beta \in \Delta$. Since Δ spans V, we conclude that (1) holds. The same argument proves (2).

□

Definition 16.2. A lattice in an Euclidean space V is a discrete subgroup $(Q, +)$ of V, which spans V over $\mathbb{R}$, i.e. $\mathbb{R}Q = V$. For example, $\mathbb{Z}^n \subset \mathbb{R}^n$.

Proposition 16.2. *If Δ is a finite set in an Euclidean space V, spanning V over $\mathbb{R}$, such that $(\alpha, \beta) \in \mathbb{Q}$ for all $\alpha, \beta \in \Delta$, then $\mathbb{Z}\Delta$ is a lattice in V.*

Proof. The only thing to prove is that $\mathbb{Z}\Delta$ is a discrete set. Choose a basis $\beta_1, \dots, \beta_r$ of V among the vectors of Δ. Then for any $\alpha \in \Delta$, we have $\alpha = \sum_{i=1}^r c_i\beta_i$, $c_i \in \mathbb{R}$. Hence, $(\alpha, \beta_j) = \sum_{i=1}^r c_i(\beta_i, \beta_j)$. But $((\beta_i, \beta_j))_{i,j=1}^r$ is a Gramm matrix of a basis, hence it is non-singular. Hence the c_i's can be computed by Cramer's rule, so all $c_i \in \mathbb{Q}$. So $\mathbb{Z}\Delta \subset \mathbb{Q}\{\beta_1, \dots, \beta_r\}$. But since Δ is finite, we conclude that $\mathbb{Z}\Delta \subset \frac{1}{N}\mathbb{Z}\{\beta_1, \dots, \beta_r\}$ where N is a positive integer. But $\frac{1}{N}\mathbb{Z}\{\beta_1, \dots, \beta_r\}$ is discrete, hence $\mathbb{Z}\Delta$ is discrete. $\square$

Example 16.2. $\{1, \sqrt{2}\} \subset \mathbb{R}$, then $\mathbb{Z}\{1, \sqrt{2}\}$is not a discrete set.

Corollary 16.3. *If (V, Δ) is a root system, then $Q := \mathbb{Z}\Delta$ is a lattice, called the root lattice.*

Proof. By the two propositions, the corollary holds if (V, Δ) is indecomposable, hence holds for any root system (V, Δ). $\square$

We will list four series of root systems of rank r known to us. In all cases $(\epsilon_i, \epsilon_j) = \delta_{ij}$.

Type	$\mathfrak{g}$	V	Δ	Q
A	$sl_{r+1}(\mathbb{F})$	$\{\sum_{i=1}^{r+1} a_i\epsilon_i \mid a_i \in \mathbb{R}, \sum_{i=1}^{r+1} a_i = 0\}$	$\{\epsilon_i - \epsilon_j \mid 1 \le i, j \le r+1\}$	$\{\sum_{i=1}^{r+1} a_i\epsilon_i \mid a_i \in \mathbb{Z}, \sum_{i=1}^{r+1} a_i = 0\}$
B	$so_{2r+1}(\mathbb{F})$	$\{\sum_{i=1}^{r} a_i\epsilon_i \mid a_i \in \mathbb{R}\}$	$\{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i \mid 1 \le i, j \le r, i \ne j\}$	$\{\sum_{i=1}^{r} a_i\epsilon_i \mid a_i \in \mathbb{Z}\}$
C	$sp_{2r}(\mathbb{F})$	$\{\sum_{i=1}^{r} a_i\epsilon_i \mid a_i \in \mathbb{R}\}$	$\{\pm\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i \mid 1 \le i, j \le r, i \ne j\}$	$\{\sum_{i=1}^{r} a_i\epsilon_i \mid a_i \in \mathbb{Z}, \sum_{i=1}^{r} a_i \in 2\mathbb{Z}\}$
D	$so_{2r}(\mathbb{F}), r \ge 3$	$\{\sum_{i=1}^{r} a_i\epsilon_i \mid a_i \in \mathbb{R}\}$	$\{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i, j \le r, i \ne j\}$	$\{\sum_{i=1}^{r} a_i\epsilon_i \mid a_i \in \mathbb{Z}, \sum_{i=1}^{r} a_i \in 2\mathbb{Z}\}$

Remark Explanation: in case A, V is a factor space of $\widetilde{V} = \sum_{i=1}^{r+1} a_i\epsilon_i$ by a 1-dimensional subspace $\mathbb{R}(\epsilon_1 + \dots + \epsilon_{r+1})$. Notice that $(\epsilon_1 + \dots + \epsilon_{r+1})^\perp = V$ in the table, so $\widetilde{V} = V \oplus \mathbb{R}(\epsilon_1 + \dots + \epsilon_{r+1})$ with direct sum. Secondly, why is $\Delta_A = \{\sum_{i=1}^{r+1} a_i\epsilon_i \mid a_i \in \mathbb{Z}, \sum_{i=1}^{r+1} a_i = 0\}$. Clearly, $\mathbb{Z}\{\epsilon_i - \epsilon_j \mid i \ne j\}$ is included in this set. To show the reverse inclusion, write

$$Q \ni \sum_{i=1}^{r+1} a_i\epsilon_i = a_1(\epsilon_1 - \epsilon_2) + (a_1 + a_2)(\epsilon_2 - \epsilon_3) + \dots + (a_1 + a_2 + \dots + a_r)(\epsilon_r - \epsilon_{r+1}) + (a_1 + \dots + a_{r+1})(\epsilon_{r+1}),$$

where the coefficient of the last term is zero. So the reverse inclusion is also true. For case B, the form of the root lattice is clearly correct.

Exercise 16.1. Explain the root lattices in cases C and D.

Proof. C. $\mathbb{Z}\Delta_{C_r} \subset Q_{C_r}$ as each member of Δ_{C_r} is of the form $a_1\epsilon_i + a_2\epsilon_j$ with $a_1 = \pm 1$, $a_2 = \pm 1$. So each element of $\mathbb{Z}\Delta_{C_r}$is of the form $\sum_{i=1}^r a_i\epsilon_i$ where $a_i \in \mathbb{Z}$, $\sum_{i=1}^r a_i \in 2\mathbb{Z}$. $Q_{C_r} \subset \mathbb{Z}\Delta_{C_r}$ as any $\sum_{i=1}^r a_i\epsilon_i$ with $a_i \in \mathbb{Z}$, $\sum_{i=1}^r a_i \in 2\mathbb{Z}$ can be written in the form

$$\sum_{i=1}^{r} a_i\epsilon_i = a_1(\epsilon_1 - \epsilon_2) + (a_2 + a_1)(\epsilon_2 - \epsilon_3) + \dots + (a_r + a_{r-1} + \dots + a_1)(\epsilon_r - \epsilon_1) + (a_r + \dots + a_1)\epsilon_1.$$

All these coefficients belong to $\mathbb{Z}$ and the final coefficient of $\epsilon_1$belongs to $2\mathbb{Z}$ so can be written as $\frac{a_r + \dots + a_1}{2} 2\epsilon_1$, hence $\sum_{i=1}^r a_i\epsilon_i \in \mathbb{Z}\Delta_{C_r}$.

D. The proof for Q_{D_r} is identical except $2\epsilon_i \in \Delta_{D_r}$, so write

$$\sum_{i=1}^{r} a_i\epsilon_i = a_1(\epsilon_1 - \epsilon_2) + (a_2 + a_1)(\epsilon_2 - \epsilon_3) + \\ \ldots + \frac{a_r + \ldots + a_1}{2}(\epsilon_r - \epsilon_1) + \frac{a_r + \ldots + a_1}{2}(\epsilon_r - \epsilon_2) + \frac{a_r + \ldots + a_1}{2}(\epsilon_1 + \epsilon_2)$$

as $r \geq 3$, and this belongs to $\mathbb{Z}\Delta_{D_r}$.

□

Definition 16.3. A lattice Q is called integral (respectively even) if $(\alpha, \beta) \in \mathbb{Z}$ (respectively $(\alpha, \alpha) \in 2\mathbb{Z}$) for all $\alpha, \beta \in Q$. Note that an even lattice is always integral: if $\alpha, \beta \in Q$, Q even, then $(\alpha+\beta, \alpha+\beta) = (\alpha, \alpha) + (\beta, \beta) + 2(\alpha, \beta)$, $(\alpha, \alpha), (\beta, \beta) \in 2\mathbb{Z}$. Hence $2(\alpha, \beta) \in 2\mathbb{Z}$, so $(\alpha, \beta) \in \mathbb{Z}$.

Example 16.3. For a positive integer r let

$$E_r = \{\sum_{i=1}^{r} a_i\epsilon_i |\text{ either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \frac{1}{2}, \text{ and } \sum_{i=1}^{r} a_i \in 2\mathbb{Z}\},$$

with $E_r \subset \mathbb{R}^r$,$(\epsilon_i, \epsilon_j) = \delta_{ij}$.

Proposition 16.4. *E_r is an even lattice if and only if r is divisible by 8.*

Proof. We use that $a^2 \pm a \in 2\mathbb{Z}$ if $a \in \mathbb{Z}$. Let $\alpha = \sum_{i=1}^{r} a_i\epsilon_i \in E_r$. Case 1: all $a_i \in \mathbb{Z}$, then $(\alpha, \alpha) = \sum_{i=1}^{r} a_i^2 = \sum_{i=1}^{r} a_i \bmod 2 \equiv 0 \bmod 2$ by the condition of E_r, so $(\alpha, \alpha) \in 2\mathbb{Z}$. Case 2: write $\alpha = \rho + \beta$ where $\rho = (\frac{1}{2}, \ldots, \frac{1}{2})$ and β has integer coefficients b_i. Then $(\alpha, \alpha) = (\rho, \rho) + 2(\rho, \beta) + (\beta, \beta) = (\rho, \rho) + \sum_{i=1}^{r}(b_i^2 + b_i) = \frac{r}{4} + n$, $n \in 2\mathbb{Z}$. So (α, α) is even if and only if r is a multiple of 8. □

Theorem 16.5. *Let Q be an even lattice in an Euclidean space V, and assume the subset $\Delta = \{\alpha \in Q | (\alpha, \alpha) = 2\}$ spans V over $\mathbb{R}$. Then $(V, \Delta)$is a root system.*

Proof. Axiom (1) of a root system is clear, as is axiom (2): if $(\alpha, \alpha) = 2$, then $(n\alpha, n\alpha) = 2$ iff. $n = \pm 1$. It remains to show the string property. Reversing the sign of α if necessary, we may assume $(\alpha, \beta) \geq 0$. Note that for $\alpha, \beta \in \Delta$ we have:

$$0 \leq (\alpha - \beta, \alpha - \beta) = (\alpha, \alpha) - 2(\alpha, \beta) + (\beta, \beta) = 4 - 2(\alpha, \beta),$$

where $(\alpha, \beta) \in \mathbb{Z}$, $(\alpha, \beta) \geq 0$. So the only possibilities are $(\alpha, \beta) = 0, 1$ or 2. In the last case, hence $q = 0$, $p = 2$, so $p - q = (\alpha, \beta) = 2$ and the string property is satisfied. □

Exercise 16.2. Complete the proof, for $(\alpha, \beta) = 0$ or 1.

Proof. For $(\alpha, \beta) = 1$, $\alpha - \beta \in \Delta$, $\alpha + \beta \notin \Delta$, so $p = 1$, $q = 0$, and $p - q = (\alpha, \beta)$. For $(\alpha, \beta) = 0$, $\alpha + \beta \notin \Delta$, $\alpha - \beta \notin \Delta$, $p = 0$, $q = 0$, and $p - q = (\alpha, \beta)$. So the string property holds generally, and the theorem holds. □

The most remarkable lattice is E_8 (which is even by proposition.)

Exercise 16.3. Show that

$\Delta_{E_8} := \{\alpha \in E_8 | (\alpha, \alpha) = 2\} = \{\pm\epsilon_i \pm \epsilon_j | i \neq j\} \cup \{\frac{1}{2}(\pm\epsilon_1 \pm \ldots \pm \epsilon_8)|$ even number of minus signs$\}$,

that $|\Delta_{E_8}|= 240$, and that $\mathbb{R}\Delta_{E_8}$=V.

Proof.

$$E_8 = \{\sum_{i=1}^{8} a_i\epsilon_i | \text{ all } a_i \in \mathbb{Z}, \sum_{i=1}^{8} a_i \in 2\mathbb{Z}\} \cup \{\sum_{i=1}^{8} a_i\epsilon_i | \text{ all } a_i \in \mathbb{Z} + \frac{1}{2}, \sum_{i=1}^{8} a_i \in 2\mathbb{Z}\}.$$

The elements from the first set satisfying $(\alpha, \alpha) = 2$ are clearly $\{\pm\epsilon_i \pm \epsilon_j | i \neq j\}$ as $(\epsilon_i, \epsilon_j) = \delta_{ij}$, and the second set must have all $a_i = \pm\frac{1}{2}$ else $(\alpha, \alpha) > 2$, and as $\sum_{i=1}^{8} a_i \in 2\mathbb{Z}$, there must be an even number of minus signs.

$$\begin{aligned}|\Delta_{E_8}| &= |\{\epsilon_i + \epsilon_j | i \neq j\}| + |\{-\epsilon_i - \epsilon_j | i \neq j\}| + |\{\epsilon_i - \epsilon_j | i \neq j\}| + \left|\{\frac{1}{2}(\pm\epsilon_1 \pm \ldots \pm \epsilon_8)| \text{ even number of minus signs}\}\right| \\ &= \frac{8.7}{2} + \frac{8.7}{2} + 8.7 + 2^7 = 240.\end{aligned}$$

Clearly $\epsilon_i = \frac{\epsilon_i+\epsilon_j}{2} + \frac{\epsilon_i-\epsilon_j}{2} \in \mathbb{R}\Delta_{E_8}$, and $\{\epsilon_i\}$ form a basis of V, hence $\mathbb{R}\Delta = V$.

□

So $(\mathbb{R}^8, \Delta_{E_8})$ is a root system by the theorem, which is called the root system of type E_8.

Exercise 16.4. Consider the following subsystem of the root system of type E_8 : take $\rho = (\frac{1}{2}, \ldots, \frac{1}{2})$ and let $\Delta_{E_7} = \{\alpha \in \Delta_{E_8} | (\alpha, \rho) = 0\}$, $Q_{E_7} = \{\alpha \in Q_{E_8} | (\alpha, \rho) = 0\}$, $V_{E_7} = \{v \in V_{E_8} | (v, \rho) = 0\}$. Show that (V_{E_7}, Δ_{E_7}) is a root system of rank 7, and that $|\Delta_{E_7}| = 126$.

Proof. Clearly $\Delta_{E_7} = \{\alpha \in Q_{E_7} | (\alpha, \alpha) = 2\}$. Q_{E_7} is an even lattice in V_{E_7} as it is a subgroup of Q_{E_8} and clearly $\mathbb{R}Q_{E_7} = V_{E_7}$, so $(V_{E_7}, \Delta_{E_7})$is a root system of rank 7 as $V_{E_8} = V_{E_7} \oplus \mathbb{R}(\frac{1}{2}, \ldots, \frac{1}{2})$, $V_{E_7} = (\frac{1}{2}, \ldots, \frac{1}{2})^{\perp}$, and $dim V_{E_8} = 8$.

$$\Delta_{E_7} = \{\epsilon_i - \epsilon_j | i \neq j\} \cup \{\frac{\pm\epsilon_1 \pm \ldots \pm \epsilon_8}{2} | \text{ 4 minus signs.}\}$$

Hence,

$$|\Delta_{E_7}| = 56 + \binom{8}{4} = 126$$

□

Exercise 16.5. Let $\Delta_{E_6} = \{\alpha \in \Delta_{E_7} | (\alpha, \epsilon_7 + \epsilon_8) = 0\}$, $Q_{E_6} = \{\alpha \in Q_{E_7} | (\alpha, \epsilon_7 + \epsilon_8) = 0\}$, $V_{E_6} = \{v \in V_{E_7} | (v, \epsilon_7 + \epsilon_8) = 0\}$. Show that (V_{E_6}, Δ_{E_6}) is a root system of rank 6, and that $|\Delta_{E_6}| = 72$.

Proof. Clearly $\Delta_{E_6} = \{\alpha \in Q_{E_6} | (\alpha, \alpha) = 2\}$. Q_{E_6} is an even lattice in V_{E_6} as it is a subgroup of Q_{E_7} and clearly $\mathbb{R}Q_{E_6} = V_{E_6}$, so (V_{E_6}, Δ_{E_6}) is a root system of rank 6 as $V_{E_7} = V_{E_6} \oplus \mathbb{R}(\epsilon_7 + \epsilon_8)$, $V_{E_6} = (\epsilon_7 + \epsilon_8)^{\perp}$, and $dim V_{E_7} = 7$.

$\Delta_{E_6} = \{\epsilon_i - \epsilon_j | \text{ if } i = 7, j = 8, \text{ if } i = 8, j = 7\} \cup \{\frac{\pm\epsilon_1 \pm \ldots \pm \epsilon_8}{2} | \text{ 4 minus signs and } \epsilon_{7,}\epsilon_8 \text{ have opposite signs}\}$

Hence,

$$|\Delta_{E_6}| = 6.5 + 2 + 2\binom{6}{3} = 72.$$

$\square$

18.745 Introduction to Lie Algebras	November 04, 2010

Lecture 17 — Cartan Matrices and Dynkin Diagrams

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Previously, given a semisimple Lie algebra $\mathfrak{g}$ we constructed its associated root system (V, Δ). (The construction depends on choosing a Cartan subalgebra, but by Chevalley's theorem, the root systems constructed from the same $\mathfrak{g}$ are isomorphic.) Next, given a root system we'll construct a *Cartan matrix* A, and from this we'll eventually see how to reconstruct $\mathfrak{g}$.

We'll see that to every root system there corresponds a semisimple Lie algebra, so it's important to know all the root systems. Last time we saw the four series A_r, B_r, C_r, and D_r, and the three exceptions E_6, E_7, and E_8. The remaining two exceptions are F_4 and G_2, which we will describe in the following exercises.

Exercise 17.1. Define:

$$\begin{aligned} V &= \bigoplus_{i=1}^{4} \mathbb{R}\epsilon_i; \qquad (\epsilon_i, \epsilon_j) = \delta_{ij}; \\ Q_{F_4} &= \left\{ \sum_{i=1}^{4} a_i\epsilon_i \mid \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \right\}; \text{ and} \\ \Delta_{F_4} &= \{\alpha \in Q_{F_4} \mid (\alpha, \alpha) = 1 \text{ or } 2\}. \end{aligned}$$

Show that (V, Δ_{F_4}) is an indecomposable root system of rank 4 with 48 roots.

Solution. A simple verification shows $\Delta = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j : i \neq j\} \cup \{1/2(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$, which has 48 elements.

All that is difficult is to check that the string property holds. For this, we tabulate enough α, β with the corresponding data $p, q, (\alpha, \beta), (\alpha, \alpha)$. I will use a shorthand for writing vectors — a string of four numbers $\eta_1\eta_2\eta_3\eta_4$ represents the vector $\eta_1\varepsilon_1 + \eta_2\varepsilon_2 + \eta_3\varepsilon_3 + \eta_4\varepsilon_4$. Depending on how many numbers are presumed zero in such a string, we take the other numbers η to be ± 1 or $\pm 1/2$ as appropriate.

α	β	p	q	$p-q$	(α,β)	(α,α)
1000	$\eta_1\eta_2 00$	$1+\eta_1$	$1-\eta_1$	$2\eta_1$	η_1	1
	$0\eta_2\eta_3 0$	0	0	0	0	1
	$\eta_1\eta_2\eta_3\eta_4$	$1/2+\eta_1$	$1/2-\eta_1$	$2\eta_1$	η_1	1
1100	$\eta_1 000$	$1/2(1+\eta_1)$	$1/2(1-\varepsilon_1)$	η_1	η_1	2
	$00\eta_3 0$	0	0	0	0	2
	$\eta_1 0\eta_3 0$	$1/2(1+\eta_1)$	$1/2(1-\eta_1)$	η_1	η_1	2
	$\eta_1\eta_2\eta_3\eta_4$	$\delta_{\eta_1=\eta_2=1/2}$	$\delta_{\eta_1=\eta_2=-1/2}$	$\eta_1+\eta_2$	$\eta_1+\eta_2$	2
$1/2(1111)$	$\eta_1 000$	$1/2(1+\eta_1)$	$1/2(1-\eta_1)$	η_1	$1/2\,\eta_1$	1
	$\eta_1\eta_2 00$	$\delta_{\eta_1=\eta_2=1}$	$\delta_{\eta_1=\eta_2=-1}$	$1/2(\eta_1+\eta_2)$	$1/2(\eta_1+\eta_2)$	1
	$\eta_1\eta_2\eta_3\eta_4$	$2\delta_{4+}+\delta_{3+}+\delta_{2+}$	$2\delta_{0+}+\delta_{1+}+\delta_{2+}$	$\sum\eta_i$	$\sum\eta_i$	1

In the last line here we have written δ_{k+} to mean 1 when there are exactly k positive η_i and zero otherwise. While considering each calculation of p and q, that the roots actually appear in strings without gaps is easily checked. Thus we have a root system. To see that it is indecomposable, we need to note that all the roots are equivalent under the the equivalence relation generated by $\alpha \sim \alpha'$ when $(\alpha, \alpha') \neq 0$. Note that $\pm\varepsilon_i \sim {}^1\!/\!{}_2(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$, so that $\{\varepsilon_i, {}^1\!/\!{}_2(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$ is contained in an equivalence class. Next, $\pm\varepsilon_i \pm \varepsilon_j \sim \varepsilon_i$, showing that all roots are equivalent. Thus the root system is indecomposable.

Exercise 17.2. Show that the following describes an indecomposable root system with 12 roots:

$$\begin{aligned} &V_{G_2} = V_{A_2}; \\ &Q_{G_2} = Q_{A_2}; \text{ and} \\ &\Delta_{G_2} = \{\alpha \in Q_{A_2} \mid (\alpha, \alpha) = 2 \text{ or } 6\}. \end{aligned}$$

Solution. $\Delta = \{(a_1, a_2, a_3) \mid a_1 + a_2 + a_3 = 0,\ a_i \in \mathbb{Z},\ \sum a_i^2 \in \{2, 6\}\}$. Now in order to have square sum 2, exactly two of the a_i must equal ± 1, so we have $\pm(1, -1, 0), \pm(1, 0, -1), \pm(0, 1, -1)$. In order to have square sum 6, exactly one must be ± 2, and the other two must both be ∓ 1: $\pm(2, -1, -1), \pm(-1, 2, -1), \pm(-1, -1, 2)$.

We tabulate enough of the relevant quantities below to verify that G_2 is a root system, in the style of the previous exercise. (Here ε stands for 1 or -1.)

α	β	p	q	$p-q$	(α,β)	(α,α)
$(1,-1,0)$	$(\varepsilon,-\varepsilon,0)$	$1+\varepsilon$	$1-\varepsilon$	2ε	2ε	2
	$(\varepsilon,0,-\varepsilon)$	${}^1\!/\!{}_2(3+\varepsilon)$	${}^1\!/\!{}_2(3-\varepsilon)$	ε	ε	2
	$(2\varepsilon,-\varepsilon,-\varepsilon)$	${}^3\!/\!{}_2(1+\varepsilon)$	${}^3\!/\!{}_2(1+\varepsilon)$	3ε	3ε	2
	$(-\varepsilon,-\varepsilon,2\varepsilon)$	0	0	0	0	2
$(2,-1,-1)$	$(\varepsilon,-\varepsilon,0)$	${}^1\!/\!{}_2(1+\varepsilon)$	${}^1\!/\!{}_2(1-\varepsilon)$	ε	3ε	6
	$(0,\varepsilon,-\varepsilon)$	0	0	0	0	6
	$(2\varepsilon,-\varepsilon,-\varepsilon)$	$1+\varepsilon$	$1-\varepsilon$	2ε	6ε	6
	$(-\varepsilon,2\varepsilon,-\varepsilon)$	${}^1\!/\!{}_2(1-\varepsilon)$	${}^1\!/\!{}_2(1+\varepsilon)$	$-\varepsilon$	-3ε	6

Finally, to see that G_2 is irreducible, note that no two of the shorter roots are perpendicular, and each of the longer roots is not perpendicular to one of the shorter roots.

Definition 17.1. Suppose (V, Δ) is a root system and $f : V \to \mathbb{R}$ is a linear map such that $f(\alpha) \neq 0$ for all $\alpha \in \Delta$. Then:

(i) $\alpha \in \Delta$ is *positive* if $f(\alpha) > 0$ and *negative* if $f(\alpha) < 0$.

(ii) A positive root is *simple* if it cannot be written as the sum of two positive roots.

(iii) A *highest root* $\theta \in \Delta$ is a root where f is maximal; that is, $f(\theta) \geq f(\alpha)$ for all $\alpha \in \Delta$.

Notation 17.1.

- Δ_+ is the set of positive roots, and Δ_- is the set of negative roots.
- $\Pi \subset \Delta_+$ is the set of simple roots.
- Π is *indecomposable* if it can't be written as a disjoint union of orthogonal sets $\Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$.

Theorem 17.1 (Dynkin). *(a) If $\alpha, \beta \in \Pi$ and $\alpha \neq \beta$, then $\alpha - \beta \notin \Delta$ and $(\alpha, \beta) \leq 0$.*

(b) *Every positive root is a nonnegative integer linear combination of simple roots; i.e.* $\Delta_+ \subseteq \mathbb{Z}_{\geq 0}\Pi$.

(c) *If* $\alpha \in \Delta_+ \setminus \Pi$ *then* $\alpha - \gamma \in \Delta$ *for some* $\gamma \in \Delta$*; moreover, then* $\alpha - \gamma \in \Delta_+$.

(d) Π *is a basis of* V *over* $\mathbb{R}$ *and of the lattice* Q *over* $\mathbb{Z}$*. Hence the integer linear combinations from part (b) are unique.*

(e) Δ *is indecomposable if and only if* Π *is indecomposable.*

Proof. (a) This is a proof by contradiction. If $\alpha - \beta = \gamma \in \Delta$, then either

- $\gamma \in \Delta_+$, so $\alpha = \beta + \gamma$, which contradicts $\alpha \in \Pi$; or
- $\gamma \in \Delta_-$, so $\beta = \alpha + (-\gamma)$, which contradicts $\beta \in \Pi$.

(b) If α is simple then we're done. Otherwise, $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Delta_+$. Then $f(\alpha) = f(\beta) + f(\gamma)$, so both $f(\beta)$ and $f(\gamma)$ are strictly less than $f(\alpha)$. Repeat this process with β and γ until all summands are simple (which must happen in finitely many steps since Δ is finite), thus yielding α as a sum of simple roots.

(c) Suppose $\alpha \in \Delta_+ \setminus \Pi$. If $\alpha - \gamma \notin \Delta$ for all $\gamma \in \Pi$, then the string condition would imply $\frac{2(\gamma,\alpha)}{(\gamma,\gamma)} \leq 0$ for all $\gamma \in \Pi$. Then by (b),

$$(\alpha, \alpha) = \left(\alpha, \sum_{\gamma \in \Pi} k_\gamma \gamma\right) \leq 0,$$

which would imply $\alpha = 0$ and thus $\alpha \notin \Delta$. Hence $\alpha - \gamma \in \Delta$ for some $\gamma \in \Pi \subset \Delta$.

Now if $\alpha - \gamma = \beta \in \Delta_-$, then $\gamma = \alpha + (-\beta)$, which would contradict γ being simple. Therefore $\alpha - \gamma \in \Delta_+$.

(d) From (b) we have that Π spans Δ_+ over $\mathbb{Z}$. Then since $\Delta = \Delta_+ \cup \Delta_- = \Delta_+ \cup -(\Delta_+)$ and $Q = \mathbb{Z}\Delta$, Π spans Q over $\mathbb{Z}$ and thus spans V over $\mathbb{R}$.

To prove linear independence of Π, suppose the contrary—that there existed a nontrivial linear combination of simple roots $\sum_i k_i \alpha_i = 0$. Split this into positive and negative parts, moving the negative parts to the other side to obtain

$$\gamma := \sum_i a_i \alpha_i = \sum_i b_i \alpha_i,$$

where all a_i and b_i are nonnegative and $a_i b_i = 0$ for all i. Since all α_i are positive, $f(\gamma) > 0$, so $\gamma \neq 0$ and $(\gamma, \gamma) > 0$. However, by (a) we also have

$$(\gamma, \gamma) = \left(\sum a_i \alpha_i, \sum b_j \alpha_j\right) \leq 0,$$

thus giving us a contradiction.

(e) If (V, Δ) is decomposable, then by (d), so is Π. Conversely, if Π decomposes as $\Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$, then we will show $\Delta = (\mathbb{Z}\Pi_1 \cap \Delta) \cup (\mathbb{Z}\Pi_2 \cap \Delta)$.

Suppose the contrary—then $\alpha = \gamma_1 + \gamma_2$ for some $\alpha \in \Delta$ and $\gamma_i \in \mathbb{Z}_{\geq 0}\Pi_i$. By flipping the sign of α if necessary, we can assume $\alpha \in \Delta_+$. By (b), we can subtract simple roots until γ_1 is simple. Then $\Pi_1 \perp \Pi_2$ implies

$$\frac{2(\alpha, \gamma_1)}{(\gamma_1, \gamma_1)} = \frac{2(\gamma_1, \gamma_1)}{(\gamma_1, \gamma_1)} = 2,$$

so by the string property $\beta := \alpha - 2\gamma_1 = \gamma_2 - \gamma_1$ is a root (it can't be zero since $\Pi_1 \perp \Pi_2$).

Flipping the sign of β if necessary, we can assume $\beta \in \Delta_+$. However, the decomposition $\beta = \gamma_2 - \gamma_1$ (or $\gamma_1 - \gamma_2$ if we flipped the sign) can be made into an integer linear combination of simple roots with mixed signs (by expanding γ_2 in terms of simple roots). By (d), this linear combination is unique, so the nonnegative linear combination guaranteed by (b) cannot exist. □

Exercise 17.3. Prove that if (V, Δ) is an indecomposable root system and $f : V \to \mathbb{R}$ is a linear map such that $f(\alpha) \neq 0$ for all $\alpha \in \Delta$, then there exists a unique highest root $\theta \in \Delta$.

Solution. Suppose first that $\theta = \sum \lambda_i \alpha_i$ is a highest root. We wish first to show that $\lambda_i > 0$ for all i (it is a basic property of root systems that $\lambda_i \geq 0$ for all i). Suppose on the contrary that $\lambda_j = 0$. Expanding $(\alpha_j, \theta) = \sum \lambda_i(\alpha_j, \alpha_i)$, as all the simple roots are at obtuse or right angles, we have $(\alpha_j, \theta) \leq 0$, with equality iff $\alpha_j \perp \alpha_i$ whenever $\lambda_i \neq 0$. Now as $\theta + \alpha_j$ cannot be a root, the string condition implies that $(\alpha_j, \theta) = 0$. Thus $\alpha_j \perp \alpha_i$ whenever $\lambda_i \neq 0$ and $\lambda_j = 0$. Thus, if $\lambda_j = 0$ for some j, the simple roots decompose into disjoint perpendicular subsets $\{\alpha_j : \lambda_j = 0\}$ and $\{\alpha_j : \lambda_j \neq 0\}$. This is impossible, as the root system is indecomposable.

Now suppose we have two distinct highest roots θ and $\tilde{\theta}$. Then $(\theta, \tilde{\theta}) \leq 0$, as otherwise $\theta - \tilde{\theta}$ is a root (yet $f(\theta - \tilde{\theta}) = 0$ which is impossible). Thus $0 \geq (\theta, \tilde{\theta}) = \sum \lambda_i(\alpha_i, \tilde{\theta})$, showing that $(\alpha_i, \tilde{\theta}) = 0$ (as the string condition implies that $(\alpha_i, \tilde{\theta}) \geq 0$ for each i). Yet then $\tilde{\theta}$ is perpendicular to a basis of V, so $\theta = 0$, a contradiction.

Definition 17.2. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots of Δ (corresponding to f). The matrix $A = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right)_{ij=1}^{r}$ is called the *Cartan matrix*. We will show later that it is independent of choice of f.

Proposition 17.2. *The Cartan matrix has all entries integers, and the following properties:*

(a) $A_{ii} = 2$ *for all* i.

(b) If $i \neq j$ *then* $A_{ij} \leq 0$, *and* $A_{ij} = 0 \iff A_{ji} = 0$.

(c) All principle values of A *are positive. In particular,* $\det A > 0$.

Proof. (a) is immediate, and (b) follows by Theorem 17.1(a). For (c), we note that we may factorise A as $\mathrm{diag}\left(\frac{2}{(\alpha_i, \alpha_i)}\right)_{i=1}^{r} \cdot ((\alpha_i, \alpha_j))_{ij=1}^{r}$. The first term may be ignored, while the second is the Gram matrix for the inner product with respect to Π. The result follows by Sylvester's criterion. □

Definition 17.3. If $V, \Delta)$ is indecomposable, we have a unique highest root θ (by the above exercise). Let $\alpha_0 = -\theta$, and $\Pi_0 = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$. The matrix $\widetilde{A} = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right)_{ij=0}^{r}$ is called the *extended Cartan matrix*.

Exercise 17.4. $\widetilde{A}$ satisfies all of the properties of Proposition 17.2, except $\det \widetilde{A} = 0$.

Solution. The proof of properties (a) and (b) are exactly the same as for the standard Cartan matrix A. However, we need to see tht the principle minors are still all positive, except for the determinant (which is zero).

Again, $\tilde{A}$ factors as $\tilde{A} = \operatorname{diag}(2/(\alpha_i, \alpha_i))_{i=0}^n \cdot ((\alpha_i, \alpha_j))_{ij=0}^n$, and the first matrix has all diagonal entries positive, so we only need to investigate the principle minors of $Q = ((\alpha_i, \alpha_i))_{i=0}^n$.

Given any proper subset I of $\{0, \ldots, n\}$, I is a subset of some subset I' of $\{0, \ldots, n\}$ with n elements. Now the α_i with $i \in I'$ are a basis of V, as the highest root is $\sum \lambda_i \alpha_i$ with the λ_i nonzero. Thus the submatrix of Q corresponding to I' is a Gram matrix for the inner product, and thus has all principle minors zero. In particular, the principle minor of Q corresponding to I is nonzero (by Sylvester's criterion), and thus so is the corresponding principle minor of $\tilde{A}$. Of course, the determinant of Q (and thus A) is zero as the α_i ($i = 0, \ldots, n$) are not linearly independent.

Definition 17.4. A $r \times r$ matrix satisfying all of the properties of Proposition 17.2 is called an *abstract Cartan matrix.*

Let's classify the abstract Cartan matrices. The only 1×1 such matrix is (2), the Cartan matrix of type A_1. There are more possibilities for 2×2 abstract Cartan matrices A. We know that $A = \left(\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right)$ for nonnegative integers a, b. Moreover, $4 - ab > 0$, so that $ab = 3$. There are four possibilities for A (up to taking the transpose):

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

The Dynkin diagram $D(A)$ depicts the Cartan matrix A by a graph with r vertices in bijection with the simple roots. For any two distinct simple roots α_i and α_j, the corresponding 2×2 Cartan matrix will be one of the above four, or a transpose thereof. The vertices corresponding to a chosen pair of roots are joined as follows in each case:

$$\begin{array}{cccccc} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} & \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \\ \circ \quad \circ & \circ\!-\!\!\!-\!\circ & \circ\!\Rightarrow\!\circ & \circ\!\Leftarrow\!\circ & \circ\!\Rrightarrow\!\circ & \circ\!\Lleftarrow\!\circ \end{array}$$

In each diagram, the left node corresponds to the first row and column of the matrix. Note that when the two roots in question have different lengths, the arrow points to the shorter root. The diagram formed in this way is the *Dynkin diagram* $D(A)$.

Remark. Any subdiagram of a Dynkin diagram is again a Dynkin diagram. Π is indecomposable if and only if the Dynkin diagram is connected.

We can now calculate Cartan matrices and Dynkin diagrams for various root systems. In each of the following root systems, we have $\{\varepsilon_i\}$ an orthonormal set of vectors (which is enough to calculate inner products).

For A_r, $\Delta = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq r + 1, i \neq j\}$. Letting $f(\varepsilon_i) = r + 1 - i$, the simple roots may be indentified easily, as they all take value 1 under f. We find:

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq r + 1\} \supset \Pi = \{e_i - e_{i+1} : 1 \leq i \leq r\}, \text{ and } \theta = \varepsilon_1 - \varepsilon_{r+1}.$$

For the rest of the root systems treated in this lecture, we use the same function f (although there is no basis vector ε_{r+1}).

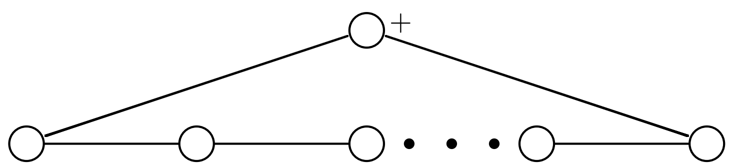
For B_r, $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i : 1 \leq i, j \leq r, i \neq j\}$. We find:

$$\Pi = \{e_i - e_{i+1} : 1 \leq i \leq r-1\} \cup \{\varepsilon_r\}, \text{ and } \theta = \varepsilon_1 + \varepsilon_2.$$

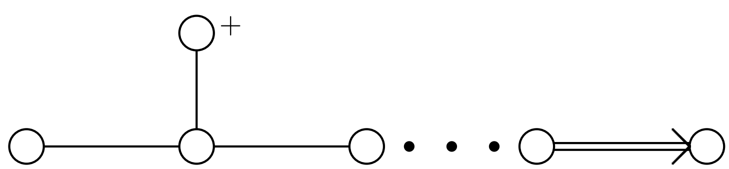
The Cartan matrices are shown below when $r = 6$ — the pattern can be read off easily enough. The augmented Cartan matrices are the whole matrix, where the standard matrix is the bottom right 6×6 block.

$$A_6 : \left(\begin{array}{c|cccccc} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ \hline -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right) \qquad B_6 : \left(\begin{array}{c|cccccc} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{array}\right)$$

A_r :



B_r :



These diagrams are the extended Dynkin diagrams, while the Dynkin diagrams are obtained by removing the node marked with +. (This comment applies to all four Dynking diagrams shown.)

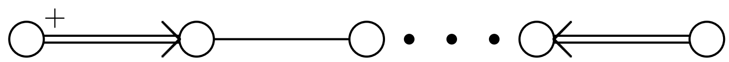
Exercise 17.5. Perform the same analysis for the root systems C_r ($r \geq 2$) and D_r ($r \geq 3$).

Solution. For C_r, $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid i \neq j\}$, $\Delta_+ = \{\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j \mid i \neq j\} \cup \{2\varepsilon_i\}$ and $\Pi = \{\alpha_i\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i < r$) and $\alpha_r = 2\varepsilon_r$. Furthermore, $\alpha_0 = -2\varepsilon_1$.

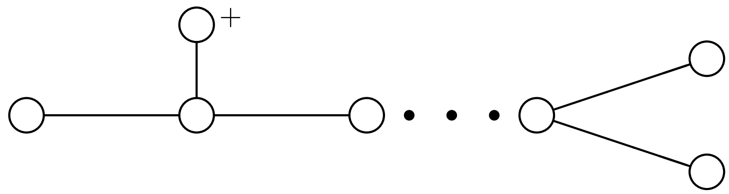
For D_r, $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}$, $\Delta_+ = \{\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j \mid i \neq j\}$ and $\Pi = \{\alpha_i\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i < r$) and $\alpha_r = \varepsilon_{r-1} + \varepsilon_r$. Furthermore, $\alpha_0 = -\varepsilon_1 - \varepsilon_2$.

$$C_6 : \left(\begin{array}{c|cccccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right) \qquad D_6 : \left(\begin{array}{c|cccccc} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{array}\right)$$

C_r :



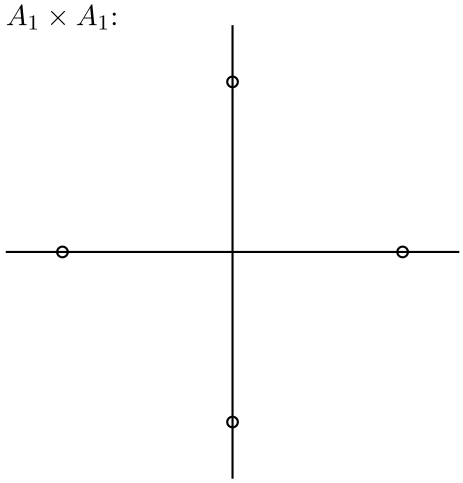
D_r :



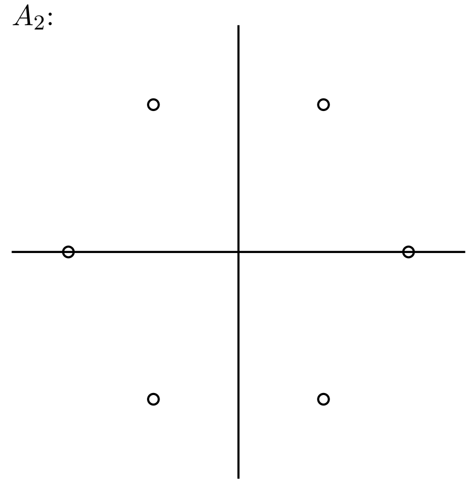
Exercise 17.6. Draw on the plane the root systems $A_1 \times A_1$, A_2, B_2 and G_2.

Solution.

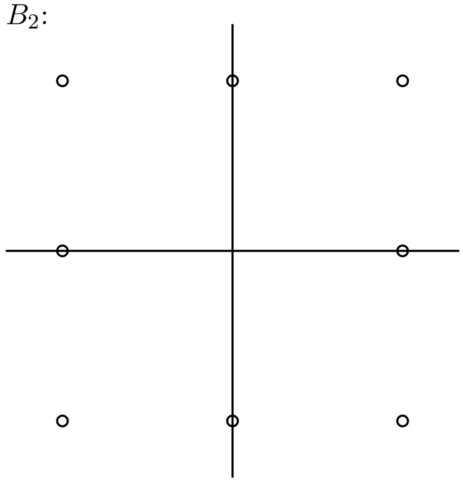
$A_1 \times A_1$:



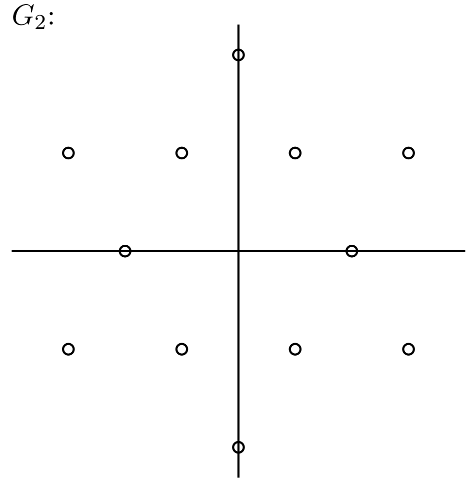
A_2:



B_2:



G_2:



18.745 Introduction to Lie Algebras	October 21, 2010
Lecture 18 — Classification of Dynkin Diagrams	
Prof. Victor Kac	*Scribe: Benjamin Iriarte*

1 Examples of Dynkin diagrams.

In the examples that follow, we will compute the Cartan matrices for the indecomposable root systems that we have encountered earlier. We record these as Dynkin diagrams, summarized in Figure 1. Later in the lecture, we will prove that these are actually the Dynkin diagrams of all possible indecomposable root systems. We also compute extended Dynkin diagrams specifically for the purposes of this proof.

In the following examples, the rank of the root system is always denoted by r, and we get simple roots $\alpha_1, \ldots, \alpha_r$. The largest root, used in the extended Dynkin diagrams, is denoted by θ.

Example 1.1. $A_r (r \geq 2)$: This corresponds to $\mathfrak{sl}_{r+1}(\mathbb{F})$. We have $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ and $\Delta_{\mathfrak{sl}_{r+1}(\mathbb{F})} = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \ldots, r+1\} \text{ and } i \neq j\} \subset V$, where V is the subspace of $\bigoplus_{i=1}^{r+1} \mathbb{R}\varepsilon_i$ on which the sum of coordinates (in the basis $\{\varepsilon_i \mid i \in \{1, \ldots, r+1\}\}$) is zero.

Take $f \in V^*$ given by $f(\varepsilon_1) = r+1, f(\varepsilon_2) = r, \ldots, f(\varepsilon_{r+1}) = 1$.

Then, $\Delta_+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$, $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid i \in \{1, \ldots, r\}\}$ and $\theta = \varepsilon_1 - \varepsilon_{r+1}$.

For the Cartan matrix, recall that:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Hence, we obtain the Dynkin diagrams in Figure 1a.

Example 1.2. $B_r (r \geq 3)$: This corresponds to $\mathfrak{so}_{2r+1}(\mathbb{F})$. We have $\Delta_{\mathfrak{so}_{2r+1}(\mathbb{F})} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid i, j \in \{1, \ldots, r\} \text{ and } i \neq j\} \subset V = \bigoplus_{i=1}^{r} \mathbb{R}\varepsilon_i$.

Take f given by $f(\varepsilon_1) = r, \ldots, f(\varepsilon_r) = 1$.

Then, $\Delta_+ = \{\pm\varepsilon_i \pm \varepsilon_j, \varepsilon_i \mid i, j \in \{1, \ldots, r\} \text{ and } i \neq j\}$, $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid i \in \{1, \ldots, r-1\}\} \sqcup \{\varepsilon_r\}$ and $\theta = \varepsilon_1 + \varepsilon_2$.

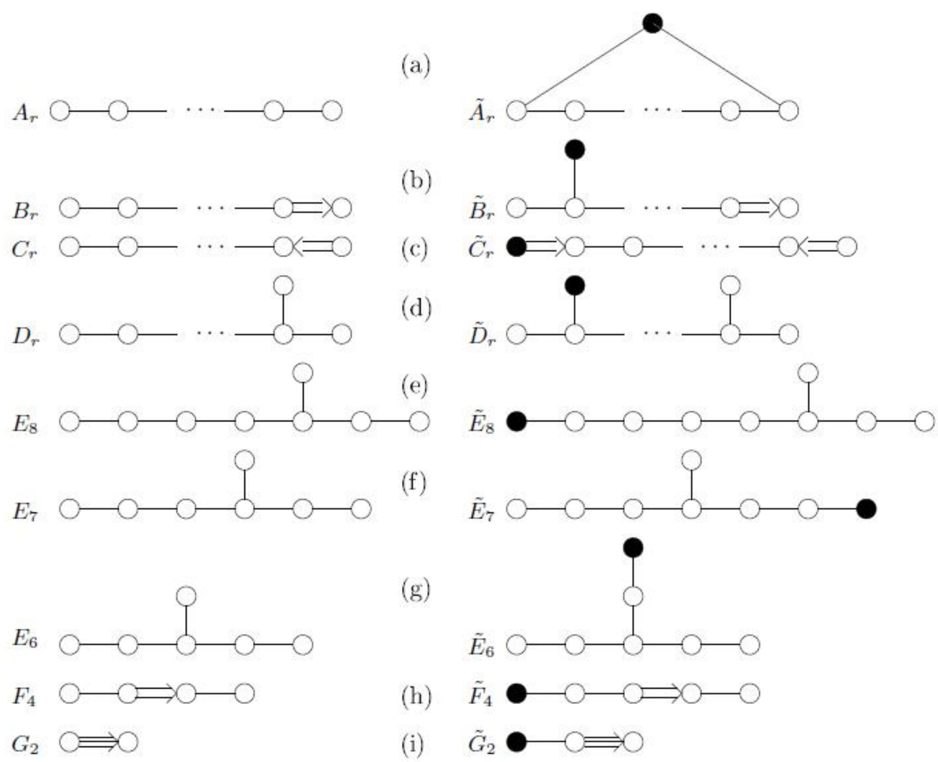


Figure 1: Dynkin diagrams and extended Dynkin diagrams of all the indecomposable root systems, from top to bottom, these correspond to: $A_r, B_r, C_r, D_r, E_8, E_7, E_6, F_4, G_2$.

The Cartan matrix is then:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

Hence, we get the Dynkin diagrams in Figure 1b.

Example 1.3. $C_r (r \geq 1)$: This corresponds to $\mathfrak{sp}_{2r}(\mathbb{F})$. We have $\Delta_{\mathfrak{sp}_{2r}(\mathbb{F})} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \ldots, r\}\} \subset V = \bigoplus_{i=1}^r \mathbb{R}\varepsilon_i$. We take the same f as in the previous case. Then, $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid i \in \{1, \ldots, r-1\}\} \sqcup \{2\varepsilon_r\}$ and $\theta = 2\varepsilon_1$. This gives the Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The corresponding Dynkin diagrams are shown in Figure 1c.

Example 1.4. $D_r (r \geq 4)$: This corresponds to $\mathfrak{so}_{2r}(\mathbb{F})$. We have $\Delta_{\mathfrak{so}_{2r}(\mathbb{F})} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \ldots, r\}$ and $i \neq j\}$. Define $f \in V^*$ by $f(\varepsilon_1) = r-1, \ldots, f(\varepsilon_r) = 0$.

Then $\Delta = \{\varepsilon_i \pm \varepsilon_j \mid i < j\}$, $\Pi = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_{r-1} + \varepsilon_r\}$ and $\theta = \varepsilon_1 + \varepsilon_2$.

The Cartan matrix is:

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & 2 & -1 & -1 \\ \vdots & & 0 & -1 & 2 & 0 \\ 0 & \ldots & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Hence, we have the Dynkin diagrams in Figure 1d.

Example 1.5. E_8: We have $\Delta_{E_8} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \ldots, 8\}$ and $i \neq j\} \sqcup \{\underbrace{\tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_8)}_{\text{even number of + signs}}\}$ with $V = \bigoplus_{i=1}^8 \mathbb{R}\varepsilon_i$.

Let $f(\varepsilon_1) = 32, f(\varepsilon_2) = 6, f(\varepsilon_3) = 5, \ldots, f(\varepsilon_7) = 1, f(\varepsilon_8) = 0$.

Then, $\Delta_+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j\} \sqcup \{\underbrace{\tfrac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_8)}_{\text{even number of + signs}}\}$. Also, $\Pi = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \ldots \varepsilon_7 - \varepsilon_8, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_7 + \varepsilon_8)\}$ and $\theta = \varepsilon_1 + \varepsilon_2$. We get the diagrams E_8 and $\tilde{E}_8$ in Figure 1e.

Exercise 1.1. E_6: We have:

$$\begin{aligned}\Delta_{E_6} =& \{\alpha \in \Delta_{E_8} \mid \alpha_1 + \cdots + \alpha_6 = 0 \text{ and } \alpha_7 + \alpha_8 = 0\} \\ =& \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \ldots, 6\} \text{ and } i \neq j\} \sqcup \{\pm(\varepsilon_7 - \varepsilon_8)\} \\ & \sqcup \{\underbrace{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \cdots \pm \varepsilon_6) \pm \frac{1}{2}(\varepsilon_7 - \varepsilon_8)}_{3\,+\,\text{signs in the first 6 terms}}\}.\end{aligned}$$

Pick $f(\varepsilon_1) = 32, f(\varepsilon_2) = 7, f(\varepsilon_3) = 6, \ldots, f(\varepsilon_7) = 2, f(\varepsilon_8) = 1$. We can then check that $\Pi = \{\varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_5 - \varepsilon_6, \varepsilon_7 - \varepsilon_8, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8)\}$, in particular, $\varepsilon_1 - \varepsilon_2 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 - \varepsilon_8)$. Also, $\theta = \varepsilon_1 - \varepsilon_6$ for obvious reasons. We obtain the Dynkin diagrams in Figure 1g.

E_7 : Also, $\Delta_{E_7} = \{\alpha \in \Delta_{E_8} \mid \alpha_1 + \alpha_2 + \cdots + \alpha_8 = 0\}$, so:

$$\begin{aligned}\Delta_{E_7} =& \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \ldots, 8\} \text{ and } i \neq j\} \\ & \sqcup \{\underbrace{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \cdots \pm \varepsilon_8)}_{4\,+\,\text{signs}}\}.\end{aligned}$$

Again, pick $f(\varepsilon_1) = 32, f(\varepsilon_2) = 7, f(\varepsilon_3) = 6, \ldots, f(\varepsilon_7) = 2, f(\varepsilon_8) = 1$. Then, $\Pi = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \ldots, \varepsilon_7 - \varepsilon_8, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)\}$ and $\theta = \varepsilon_1 - \varepsilon_8$ for obvious reasons. We obtain the Dynkin diagrams in Figure 1f.

Exercise 1.2. F_4: We have $\Delta_{F_4} = \{\pm\varepsilon_i, \pm(\varepsilon_i - \varepsilon_j), \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \mid i, j \in \{1, \ldots, 4\} \text{ and } i \neq j\}$. Pick $f(\varepsilon_1) = 30,\ f(\varepsilon_2) = 3,\ f(\varepsilon_3) = 2,\ f(\varepsilon_4) = 1$. Then, $\Pi = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$ and so $\theta = \varepsilon_1 + \varepsilon_2$. This produces the Dynkin diagrams of Figure 1h.

G_2: Finally, $\Delta_{G_2} = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\}$. Pick $f(\varepsilon_1) = 2, f(\varepsilon_2) = 1, f(\varepsilon_3) = 4$. Then, $\Pi = \{\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$, and so $\theta = 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2$. We obtain the Dynkin diagrams of Figure 1i.

2 Classification of Dynkin Diagrams.

Theorem 2.1. *A complete non-redundant list of connected Dynkin diagrams is the following:* $D(A_r)(r \geq 2)$, $D(B_r)(r \geq 3)$, $D(C_r)(r \geq 1)$, $D(D_r)(r \geq 4)$, $D(E_6)$, $D(E_7)$, $D(E_8)$, $D(F_4)$ *and* $D(G_2)$. *(See Figure 1 for these Dynkin diagrams and their extended versions.)*

Remark 2.2. *Note that* $D(A_1) = D(B_1) = D(C_1)$, $D(B_2) = D(C_2)$ *and* $D(D_3) = D(A_3)$.

Proof. We prove that there are no other Dynkin diagrams. To do this, we find all connected graphs with connections of the 4 types depicted in Figure 2, such that the matrix of any subgraph has a positive determinant, in particular any subgraph must be a Dynkin diagram. Equivalently, we require that the determinant of all principal minors of the corresponding Cartan matrix are positive. In particular, our graphs contain no extended Dynkin diagrams as induced subgraphs, since these have determinant 0.

Figure 2: The four Dynkin diagram connection types, corresponding to the four types of 2×2 Cartan matrix minors.

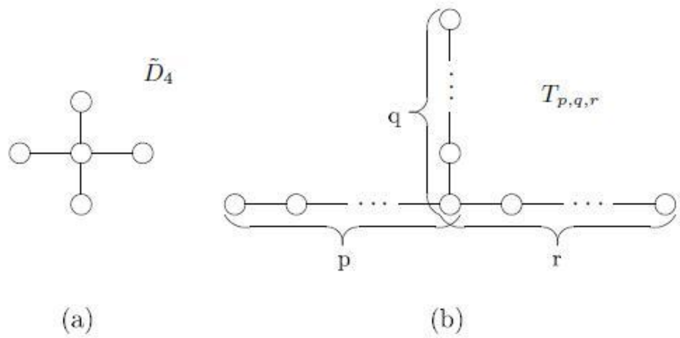


Figure 3: Some diagrams for the simply laced case.

Part 1. We first classify all simply-laced Dynkin diagrams $D(A)$, i.e. diagrams using only $\bigcirc\quad\bigcirc$ or $\bigcirc - \bigcirc$ connections (which correspond to a symmetric Cartan matrix A). Such a diagram contains no cycles, since otherwise it contains $D(\tilde{A}_r)$. It has simple edges and it may or may not contain branching vertices. If there are no branching vertices, we get $D(A_r)$. If there are two branching vertices, the diagram contains $D(\tilde{D}_r)$, which is not possible. If there is precisely one branching vertex, it has at most 3 branches, since $D(\tilde{D}_4)$ is the 4-star in Figure 3a.

Therefore, it remains to consider the case when our graph $D(A)$ is of the form $T_{p,q,r}$, $p \geq q \geq r \geq 2$, depicted in Figure 3b.

If $r = 3$, then $T_{p,q,r}$ contains $D(\tilde{E}_6) = T_{3,3,3}$, which is impossible, so $r = 2$.

If $q = 2$, we have type D.

Now, assume $q \geq 3$. If $q > 3$, then $D(A)$ contains $D(\tilde{E}_7) = T_{4,4,2}$, which is impossible, so we are left with the case $T_{p,3,2}$ with $p \geq 3$.

The case $p = 3$ is E_6, the case $p = 4$ is E_7 and the case $p = 5$ is E_8. The case $p = 6$ is $\tilde{E}_8$, so $T_{p,3,2}$ with $p > 5$ is impossible.

Part 2. We now classify all non-simply-laced diagrams $D(A)$, i.e. those containing double- or triple-edge connections (corresponding to a non-symmetric Cartan matrix A). This can be done by computing many large determinants, but we would rather argue using the following result:

Exercise 2.1. Let A be an $r \times r$ matrix, and let B (resp. C) be the submatrices of A obtained by removing the first row and column (resp. the first two rows and two columns). We have:

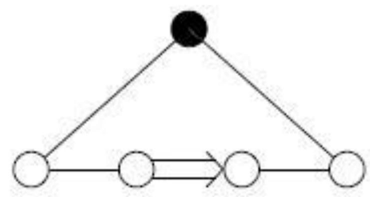


Figure 4: A loop with one double-connection.

a) If

$$A = \left(\begin{array}{c|cccc} 2 & -a & 0 & \cdots & 0 \\ \hline -b & & & & \\ 0 & & & & \\ \vdots & & & B & \\ 0 & & & & \end{array}\right),$$

then, $\det(A) = 2\det(B) - ab\det(C)$.

b) If

$$A = \begin{pmatrix} c_1 & -a_1 & 0 & \cdots & 0 & -b_r \\ -b_1 & c_2 & -a_2 & 0 & & 0 \\ 0 & -b_2 & c_3 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -a_{r-2} & 0 \\ 0 & & \ddots & -b_{r-2} & c_{r-1} & -a_{r-1} \\ -a_r & 0 & \cdots & 0 & -b_{r-1} & c_r \end{pmatrix},$$

then, $\det(A - \varepsilon e_{12}) = \det(A) - \varepsilon\left(b_1 \det(C) + \prod_{i=2}^r a_i\right)$. In particular, if $b_1 > 0$, $a_i > 0$ for $i = 2, \ldots, r$, $\det(C) > 0$ and $\varepsilon > 0$, then $\det(A - \varepsilon e_{12}) < \det(A)$.

Proof. We check these items separately.

a) This is clear, just do cofactor expansion along the first row of A to obtain directly $\det(A) = 2\det(B) - ab\det(C)$.

b) This is also clear. Consider the first row of $A - \varepsilon e_{12}$, write $(c_1, -a_1 - \varepsilon, 0, 0, \ldots, 0, -b_r) = (c_1, -a_1, 0, \ldots, 0, -b_r) + (0, -\varepsilon, 0, \ldots, 0)$, use the multilinearity of det to write $\det(A - \varepsilon e_{12}) = \det(A) + \varepsilon \det D$, where D is obtained from A removing column 2 and row 1, and then do cofactor expansion along the first column of D to get $\det(D) = -b_1 \det(C) + (-1)^r(-1)^{r-1} \prod_{i=2} a_i = -b_1 - \prod_{i=2} a_i$, where the second term comes directly from a lower triangular matrix.

$\square$

Exercise 2.1 implies that in the case of a non-simply laced diagram $D(A)$, there are no cycles since then $\det(A) < \det(\tilde{A}_r) = 0$, a contradiction. For example, the diagram in Figure 4 has $A = \tilde{A}_4 - E_{12}$, so $\det A < 0$.

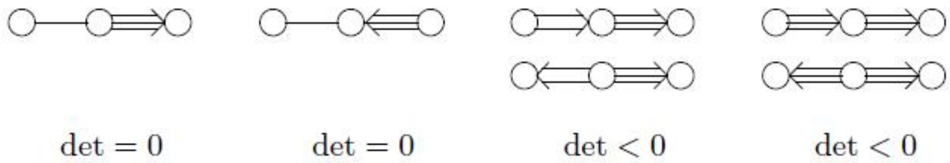


Figure 5: Possible neighbors of G_2 in a diagram.

Next, looking at the extended Dynkin diagrams, if $D(A)$ contains G_2 (Figure 1i), then it must be exactly G_2 by Exercise 2.1. Indeed, otherwise, using Figure 5 we see that there is a principal submatrix of A of the form:

$$M = \begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \text{ with } a, b > 0. \text{ Let } M' = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \text{ and } M'' = \begin{pmatrix} 2 \end{pmatrix}.$$

Hence, by Exercise 2.1, $\det(M) = 2\det(M') - ab\det(M'') = 2(1 - ab) \leq 0$, a contradiction. (The matrix of the second graph in the figure is actually A^T, but all the determinants are the same.)

It remains to consider the case when $D(A)$ has only simple or double connections. Looking at the extended Dynkin diagrams, $D(\tilde{C}_r)$ (in Figure 1c) cannot be a subdiagram. By Exercise 2.1, the variants with flipped arrow directions also do not work. Indeed, using the notation of the exercise, they are obtained from the Cartan matrix A of $D(\tilde{C}_r)$ by replacing some of A, B or C by their transposes, which does not change any of the determinants in the calculation. Thus, Exercise 2.1a shows that each of these variants have also determinant 0.

Therefore, $D(A)$ may contain only one double connection. But then, we cannot have branching points, since $D(\tilde{B}_r)$ contains a double edge and a branching point. So, the only remaining case is a line with a left-right double edge, having p single edges to the left, and q single edges to the right. If $p = 0$, we get $D(C_r)$. If $q = 0$, we get $D(B_r)$. If $p = q = 1$, we get $D(F_4)$. The diagram $D(\tilde{F}_4)$ has $p = 2, q = 1$; its transpose has $p = 1, q = 2$. Hence, if $p > 1$ or $q > 1$, $D(A)$ contains $D(\tilde{F}_4)$, which is impossible.

□

We have now shown that any finite-dimensional simple Lie algebra yields one of a very restricted set of Dynkin diagrams (and hence Cartan matrices). The next step in the classification of semisimple Lie algebras will be to give a construction associating an abstract Cartan matrix to a Lie algebra, and hence to prove that these four classes plus five exceptional algebras are in fact the only finite-dimensional simple Lie algebras.

We depict the plan of our actions:

$$\begin{tikzcd}
\text{s.s. Lie alg. } \mathfrak{g} \arrow[r, "\text{Cartan } \mathfrak{h}"] & \underset{\text{root system}}{(V = \mathfrak{h}^*_{\mathbb{R}}, \Delta)} \arrow[r, "\text{choose } f \in V^*"] & \Pi \arrow[r] & \underset{\text{Cartan matrix}}{A} \arrow[r, phantom, "\subseteq"] & \text{abstract Cartan matrix } A \arrow[lllll] \\
\mathfrak{g}(A) \arrow[u]
\end{tikzcd}$$

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Lecture: 19 Classification of simple finite dimensional Lie algebras over $\mathbb{F}$

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Throughout this lecture $\mathbb{F}$ will be an algebraically closed field of characteristic 0.

Theorem 19.1. *(a) Semisimple Lie algebras over $\mathbb{F}$ are isomorphic if and only if they have the same Dynkin diagram.*
(b) A complete non-redundant list of simple finite dimensional Lie algebras over $\mathbb{F}$ is the following: $\mathfrak{sl}_n(\mathbb{F})$ $(n \geq 2)$, $\mathfrak{so}_n(\mathbb{F})$ $(n \geq 7)$, $\mathfrak{sp}_{2n}(\mathbb{F})$ $(n \geq 2)$ and five exceptions (E_6,E_7,E_8,F_4 and C_2).

Exercise 19.1. Deduce from (a), that the following are isomorphisms: $\mathfrak{sl}_2 \cong \mathfrak{so}_3 \cong \mathfrak{sp}_2$; $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \bigoplus \mathfrak{sl}_2$; $\mathfrak{so}_5 \cong \mathfrak{sp}_4$; $\mathfrak{so}_6 \cong \mathfrak{sl}_4$.
Solution. We know by part (a) of Theorem 19.1 that a semisimple Lie algebras over $\mathbb{F}$ are isomorphic if and only if they have the same Dynkin diagram. Thus we will show that the following are isomorphic to each other by showing their Dynkin Diagrams to be the same. i)$sl_2 \cong so_3 \cong sp_2$.
We know that the Dynkin diagram for $\mathfrak{sl}_2$ is: $\bigcirc$. Similarly, the Dynkin diagram for $\mathfrak{so}_3$ is: $\bigcirc$.
Finally, the Dynkin diagram for $\mathfrak{sp}_2$ is: $\bigcirc$
ii) $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \bigoplus \mathfrak{sl}_2$ Both Dynkian diagrams consist of two disconnected nodes.
iii) $\mathfrak{so}_5 \cong \mathfrak{sp}_4$ Now, the Dynkin diagram for $\mathfrak{so}_5$ is given by: $\bigcirc\!\Longrightarrow\!\bigcirc$.
The Dynkin diagram for $\mathfrak{sp}_4$ is: $\bigcirc\!\Longrightarrow\!\bigcirc$.
iv)$\mathfrak{so}_6 \cong \mathfrak{sl}_4$ We know that the Dynkin diagram for $\mathfrak{so}_6$ is: $\bigcirc\!-\!\bigcirc\!-\!\bigcirc$.
In the same way, the Dynkin diagram for $\mathfrak{sl}_4$ is: $\bigcirc\!-\!\bigcirc\!-\!\bigcirc$.
Thus, $\mathfrak{so}_6 \cong \mathfrak{sl}_4$ by Thm. 19.1.

Proof of Theorem 19.1a) Let $\mathfrak{g}$ be a semisimple finite dimensional Lie algebra over $\mathbb{F}$. Choose a Cartan subalgebra $\mathfrak{h}$ and consider the root space decomposition: $\mathfrak{g} = \mathfrak{h} \bigoplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$, where $\Delta \subset \mathfrak{h}^*$ is the set of roots. Choose a linear function f on $\mathfrak{h}^*_\alpha$ which does not vanish on Δ, and let $\Delta_+ = \{\alpha \in \Delta | f(\alpha) > 0\}$ and $\Delta_- = \{\alpha \in \Delta | f(\alpha) < 0\}$.
Then $\Delta = \Delta_+ \coprod \Delta_-$, where $\Delta_- = -\Delta_+$. Let $\Pi \subseteq \Delta_+$ be the set of simple roots. We will prove that nothing depends on f.
Let $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$, which are obviously subalgebras of $\mathfrak{g}$. Then we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \bigoplus \mathfrak{h} \bigoplus \mathfrak{n}_+$$

as vector spaces.

Exercise 19.2. Show that if $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$ or $\mathfrak{sp}_n$, choosing $\mathfrak{h}$ to be all diagonal matrices in $\mathfrak{g}$; then $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) consists of all strictly upper (resp. lower) triangular matrices in $\mathfrak{g}$.

Solution. For $\mathfrak{sl}_n$, we know that the root space corresponding to $\epsilon_i - \epsilon_j$ is E_{ij} where E_{ij} is 1 in the (i,j) slot and zero elsewhere. In the standard ordering of the roots, $\epsilon_i - \epsilon_j$ is positive if and only if $i < j$. Thus the positive roots spaces correspond to strictly upper triangular matrices and negative root spaces correspond to strictly lower triangular matrices.
For $\mathfrak{so}_n$, we have:

$$\mathfrak{so}_n(\mathbb{F}) = \mathfrak{h} \bigoplus (\bigoplus_{i,j} \mathbb{F} F_{ij}), \tag{2}$$

where $F_{ij} = E_{ij} - E_{n+1-j,n+1-i}$ and $\mathfrak{h} = (a_1, , ...a_r, -a_r, ... - a_1)$. Then the positive roots are: $\epsilon_i - \epsilon_j (i < j)$ and $\epsilon_i + \epsilon_j (i \neq j)$, if $n = 2r$. Thinking of the four r by r blocks in $\mathfrak{so}_n$, the $\epsilon_i + \epsilon_j (i \neq j)$ root spaces fill in the upper right hand block, and the $\epsilon_i - \epsilon_j (i < j)$ fill in the piece above the diagonal for the upper left half. Thus the positive root spaces span the set of upper triangular matrices in $\mathfrak{so}_n$ (which are by definition antisymmetric with respect to the antidiagonal). The odd case is similar.

For $\mathfrak{sp}_n(\mathbb{F})$, we have again the Cartan subalgebra $\mathfrak{h} = (a_1, , ...a_r, -a_r, ... - a_1)$. Here the positive roots are $\epsilon_i + \epsilon_j$ (where i is permitted to equal j) and $\epsilon_i - \epsilon_j$ where $i < j$. We know that $\mathfrak{sp}_n(\mathbb{F})$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $d = -a'$, $b = b'$, $c = c'$ (where $'$ denotes transposition with respect to the antidiagonal). The positive root spaces corresponding to $\epsilon_i - \epsilon_j$ are $F_{ij} = E_{ij} - E_{n+1-j,n+1-i}$ for $1 \leq i, j \leq r$ $(i \leq j)$ - these fill in the part of a above the main diagonal in $\mathfrak{sp}_n(\mathbb{F})$ and also the part of d above this main diagonal by $d = -a'$. The root spaces corresponding to $\epsilon_i + \epsilon_j$ are $F_{ij} = E_{ij} + E_{n+1-j,n+1-i}$ for $1 \leq i \leq r$, $r + 1 \leq j \leq n$, and these spaces fill in the block b. Thus the positive root spaces span the strictly upper triangular matrices.

Continuation of the proof of Theorem 19.1. Recall that for $\alpha \in \Delta$ we can choose $E_\alpha \in \mathfrak{g}_\alpha$ and $F_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $K(E_\alpha, F_\alpha) = 2/K(\alpha, \alpha)$. Then $H_\alpha = 2\nu^{-1}(\alpha)/K(\alpha, \alpha)$ so that $\mathbb{F}E_\alpha + \mathbb{F}F_{-\alpha} + \mathbb{F}H_\alpha$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. Now given $\Pi = (\alpha_1, ..., \alpha_r)$, set $E_i = E_{\alpha_i}$,$F_i = F_{\alpha_i}$,$H_i = H_{\alpha_i}$ for $1 \leq i \leq r$. Then we have:

1) $[H_i, H_j] = 0$ (since the H_i are in a Cartan subalgebra)
2) $[H_i, E_j] = 2(K(\alpha_i, \alpha_j)/K(\alpha_i, \alpha_i))E_j = a_{ij}E_j$, where $a_{ij} = \alpha_j(H_i)$.
3) $[H_i, F_j] = -a_{ij}F_j$
4) $[E_i, F_j] = \delta_{ij}H_j$ (since $\alpha_i - \alpha_j$ is not a root when $i \neq j$ because the α_i are simple)

Definition 19.1. These relations on the E_i, F_i, and H_i are called the Chevalley relations. The E_i, F_i and H_i are called the Chevallay generators.

Lemma 19.1. The E_i (respectively F_i) generate $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$). Consequently the E_i, F_i and H_i generate $\mathfrak{g}$.

Proof. Given $\alpha \in \Delta$, writing $\alpha = \sum_i^r n_i\alpha_i$. We call $\sum_i^r n_i$ the height of the root α. We prove the lemma by induction on the height of a given root. When the height is 1, the root is simple so the E_i certainly generate it. For the inductive step, observe that if $\alpha \in \Delta_+$ but not simple, then for some simple root α_i we have $\mathfrak{g}_\alpha = [\mathfrak{g}_{\alpha-\alpha_i}, \mathfrak{g}_{\alpha_i}]$. Thus we have exhibited $\mathfrak{g}_\alpha$ as a bracket of a term with lower height and one of the E_i, so we are done. □

Definition 19.2. Denote by $\tilde{\mathfrak{g}}(A)$ the Lie algebra with generators E_i, F_i and H_i $(1 \leq i \leq r)$ subject to the Chevalley relations. This Lie algebra is infinite dimensional if $r \geq 1$ but is closely related to $\mathfrak{g}$: we have the surjective homomorphism $\phi : \tilde{\mathfrak{g}}(A) \to \mathfrak{g}$ sending E_i to E_i, F_i to F_i and H_i to H_i.

Lemma 19.2. a) Let $\tilde{\mathfrak{n}}_+$ (resp $\tilde{\mathfrak{n}}_-$) be the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $E_1, ...E_r$ (resp. $F_1, ..., F_r$) and $\mathfrak{h}$ the span of $H_1, ..., H_r$. Then $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{n}}_- \oplus\ \mathfrak{h}$ (direct sum of vector spaces)

b) $\tilde{\mathfrak{n}}_+ = \bigoplus_{\alpha\in\mathbb{Q}_+} \tilde{\mathfrak{g}}_\alpha$ and $\tilde{\mathfrak{n}}_- = \bigoplus_{\alpha\in\mathbb{Q}_+} \tilde{\mathfrak{g}}_{-\alpha}$ where $\mathbb{Q}_+ = \mathbb{Z}_+\Pi/0$

c) If I is an ideal in $\tilde{\mathfrak{g}}(A)$ then $I = (\mathfrak{h}\cap I) \oplus (\bigoplus_{\alpha\in\mathbb{Q}_+\cup-\mathbb{Q}_+} \tilde{\mathfrak{g}}_\alpha \cap I)$.

d) $\tilde{\mathfrak{g}}(A)$ contains a unique proper maximal ideal $I(A)$ provided $D(A)$ is connected.

Proof. For a), we will first show that 1) $[H_i, \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}] \subset \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$, 2) $[E_i, \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}] \subset \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$ and 3) $[F_i, \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}] \subset \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$. Since a general element of $\tilde{\mathfrak{g}}(A)$ is an iterated bracket of the E_i, F_i and H_i, by the Jacobi identity, 1), 2) and 3) together imply that $\tilde{\mathfrak{g}}(A) \subseteq \tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$. Therefore $\tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$ is a subalgebra of $\tilde{\mathfrak{g}}(A)$ and must coincide with $\tilde{\mathfrak{g}}(A)$ since it contains all generators. To prove 1), observe that $\tilde{\mathfrak{n}}_+$ is the span of commutators of the form $[E_{i_1}, ..., E_{i_n}]$ so that $[H_i, [E_{i_1}, ...E_{i_n}]] = [[H_i, E_{i_1}], [E_{i_2}, ...E_{i_n}] + [E_{i_1}, [H_i, E_{i_2}], ...E_{i_n}] \in \tilde{\mathfrak{n}}_+$ since $[H_i, E_{i_k}] \in \tilde{\mathfrak{n}}_+$ by the Chevalley relations. The same is true for $\tilde{\mathfrak{n}}_-$. This proves 1). For 2), first observe $[E_{i_1}, ...E_{i_n}] \in \tilde{\mathfrak{n}}_+$. Also we can write $[F_i, [E_{i_1}, ...E_{i_n}]] = [[F_i, E_{i_1}], [E_{i_2}, ...E_{i_n}] + [E_{i_1}, [F_i, E_{i_2}], ...E_{i_n}] + ...$ By the Chevalley relations, $[F_i, E_{i_k}] \in \mathfrak{h}$, so each summand in this decomposition, appealing to the Jacobi identity again, is in $\tilde{\mathfrak{n}}_+$, so $[F_i, [E_{i_1}, ...E_{i_n}]] \in \tilde{\mathfrak{n}}_+$. 3) is established similarly. Thus we have shown $\tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \ \mathfrak{h} = \tilde{\mathfrak{g}}(A)$. To show that the sum is direct, let $h \in \mathfrak{h}$ satisfy $\alpha_i(h) = 1$ for all i (writing $h = \sum x_i H_i$, finding such an h is equivalent to solving the linear system $\sum a_{ij}x_i = 1$, which is possible since $\det A \neq 0$). Then $\mathbf{ad}\ h$ acts by positive eigenvalues on $\tilde{\mathfrak{n}}_+$, negative eigenvalues on $\tilde{\mathfrak{n}}_-$, and by 0 on $\mathfrak{h}$. This proves the sum is direct.
To prove d), we'll need a weaker statement than c), namely, that $I = (I\cap\tilde{\mathfrak{n}}_-)\oplus(I\cap\tilde{\mathfrak{n}}_+)\oplus(I\cap\mathfrak{h})$. To prove this we will invoke an earlier lemma which said that for any $\mathfrak{h}$-module $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ and any $\mathfrak{h}$-invariant subspace U, we have $U = \oplus_\lambda U \cap V_\lambda$. Apply this to $\mathfrak{h} = \mathbb{F}h$ and $\pi = \mathbf{ad}$ and $V = \tilde{\mathfrak{g}}(A)$. Since I is an invariant subspace under $\mathfrak{h}$, this gives $I = \bigoplus_\lambda I \cap V_\lambda$ where V_λ is the eigenspace of $\tilde{\mathfrak{g}}(A)$ with height λ. Appealing to the lemma again, the sum of the positive eigenspaces in this decomposition is $\tilde{\mathfrak{n}}_+ \cap I$, the sum of the negative eigenspaces is $\tilde{\mathfrak{n}}_- \cap I$, and zero eigenspace is $\mathfrak{h}\cap I$ establishing $I = (I\cap\tilde{\mathfrak{n}}_-) \oplus (I\cap\tilde{\mathfrak{n}}_+) \oplus (I\cap\mathfrak{h})$.

Let $D(A)$ be connected and I be a proper ideal of $\tilde{\mathfrak{g}}(A)$. Then $I\cap\mathfrak{h} = 0$. Indeed, if $a \in I\cap\mathfrak{h}$ is nonzero, then $\alpha_i(a) \neq 0$ for some i, so that $[a, E_i] = \alpha_i(a)E_i \neq 0$, so $E_i \in I$. Hence $H_i \in I$ by the Chevalley relations. Also by the Chevalley relations, E_j and F_j are contained in I for all j such that $a_{ij} \neq 0$. Since $D(A)$ is connected, it follows that all E_j and F_j are contained in I. By Chevalley relations, this implies the H_i are all contained in I, which would mean $I = \tilde{\mathfrak{g}}(A)$, contradicting the properness of I. Thus we have the decomposition $I = (\tilde{\mathfrak{n}}_+ \cap I) \oplus (\tilde{\mathfrak{n}}_- \cap I)$ for any proper ideal, hence for the sum of all proper ideals, $I(A)$. Hence $I(A)$ is the unique proper, maximal ideal. $\square$

Exercise 19.3. Prove statements b) and c) in the theorem.

Proof. b) Since $\tilde{\mathfrak{n}}_+$ is by definition the span of commutators of the form $[E_{i_1}, ..., E_{i_n}]$, then it is spanned by the subspaces $\tilde{\mathfrak{g}}_\alpha$ where α runs through $\mathbb{Q}_+$ (since each $\tilde{\mathfrak{g}}_\alpha$ consist precisely of terms $[E_{i_1}, ..., E_{i_n}]$ where, if $\alpha = \sum n_i\alpha_i$, each E_i appears n_i times). To see that the sum is direct, first observe that since $\det A \neq 0$, for each j we can find $h_j \in \mathfrak{h}$ such that $\alpha_i(h_j) = \delta_{ij}$. By the Chevalley relations, we have $[h_j, E_i] = \delta_{ij}E_i$. Thus given an iterated bracket in $\tilde{\mathfrak{n}}_+$, $[E_{i_1}, ..., E_{i_n}]$, we have that $[h_j, [E_{i_1}, ..., E_{i_n}]] = c[E_{i_1}, ..., E_{i_n}]$ where c is the number of times j appears in the index set $i_1, ..., i_n$. On each $\tilde{\mathfrak{g}}_\alpha$, $\alpha = \sum n_i\alpha_i$, $\mathbf{ad}\ h_j$ acts by the constant n_j. This means each $\tilde{\mathfrak{g}}_\alpha$ is a joint eigenspace for the $\mathbf{ad}\ h_i$, so that sum is direct. The same is true for $\tilde{\mathfrak{n}}_-$.
c) Given the decomposition we already proved, $I = (I\cap\tilde{\mathfrak{n}}_-)\oplus(I\cap\tilde{\mathfrak{n}}_+)\oplus(I\cap\mathfrak{h})$, it is enough to show $I\cap\tilde{\mathfrak{n}}_+ = \bigoplus I\cap\tilde{\mathfrak{g}}_\alpha$. But $\mathfrak{n}_+$ is an $\mathfrak{h}$-module under $\mathbf{ad}$ and $I\cap\tilde{\mathfrak{n}}_+$ is an invariant subspace since I

is an ideal. Therefore by the lemma we had earlier, $I \cap \tilde{\mathfrak{n}}_+ = \bigoplus I \cap \tilde{\mathfrak{g}}_\alpha$ since by b), $\mathfrak{n}_+ = \bigoplus \tilde{\mathfrak{g}}_\alpha$ is a decomposition of $\tilde{\mathfrak{n}}_+$ by weights of $\mathbf{ad}\ \mathfrak{h}$ acting on $\tilde{\mathfrak{n}}_+$. $\square$

Continuation of the proof of Theorem 19.1. By part d) of the lemma $I(A)$ is the unique proper maximal ideal in $\tilde{\mathfrak{g}}(A)$. Now set $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/I(A)$. Since $\mathfrak{g}$ is simple, ker $(\phi : \tilde{\mathfrak{g}}(A) \to \mathfrak{g})$ is a maximal proper ideal. By Lemma 2d, ker $\phi = I(A)$. Hence ϕ induces an isomorphism between $\mathfrak{g}(A)$ and $\mathfrak{g}$. This proves the "if" part of the Theorem 19.1a. The "only if" part will follow once we show the independence of A from the choice of f.

So far, we have shown that if $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_m, \mathfrak{sp}_n$ then $\mathfrak{g} \cong \mathfrak{g}(A)$ where A is the Cartan matrix of $\mathfrak{g}$. The only remaining simple, finite dimensional Lie algebras can be $\mathfrak{g}(A)$ where $A = E_6, E_7, E_8, F_4, G_2$. Hence to complete part b) of the theorem, we need to prove that the dimension of $\mathfrak{g}(A)$ is finite in these five cases. We will prove this by exhibiting explicit constructions of these five.

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Lecture 20 — Explicitly constructing Exceptional Lie Algebras

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First consider the simply-laced case: a symmetric Cartan matrix, root system Δ, root lattice $Q = \mathbb{Z}\Delta$, satisfying $\Delta = \{\alpha \in Q : (\alpha, \alpha) = 2\}$. We will construct $\mathfrak{g}$, a semisimple Lie algebra, satisfying $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathbb{F}E_\alpha)$. We will think of $\mathfrak{h}$ as $\mathbb{F}\otimes_{\mathbb{Z}} Q$. The brackets should be the following:

1. $[h, h'] = 0 \ \forall \ h, h' \in \mathfrak{h}$,
2. $[h, E_\alpha] = -[E_\alpha, h] = (\alpha, h)E_\alpha$,
3. $[E_\alpha, E_{-\alpha}] = -\alpha$ for $\alpha \in \Delta$,
4. $[E_\alpha, E_\beta] = \epsilon(\alpha, \beta)E_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in \Delta$,
5. $[E_\alpha, E_\beta] = 0$ if $\alpha, \beta \in \Delta, \alpha + \beta \notin \Delta \cup 0$

The problem is to find non-zero $\epsilon(\alpha, \beta) \in \mathbb{F}$ such that $\mathfrak{g}$ with the four brackets above is a Lie algebra (i.e. skew-symmetry, Jacobi identity). Then automatically $\mathfrak{g}$ will be simple with the root system Δ, by our general criterion of simplicity.

Proposition 20.1. $\exists \epsilon : Q \times Q \to \pm 1$ *with the following properties:*

1. $\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)$
2. $\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$
3. $\epsilon(\alpha, \alpha) = (-1)^{(\alpha,\alpha)/2}$

Proof. Choose a set Π of simple roots $\{\alpha_1, \cdots, \alpha_r\}$ (so Π is a $\mathbb{Z}$-basis of Q). For each pair i, j, make a choice of $\epsilon(\alpha_i, \alpha_j)$ and $\epsilon(\alpha_j, \alpha_i)$ subject to the following constraints: $\epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) = (-1)^{(\alpha_i,\alpha_j)}$ (for $i \neq j$) and $\epsilon(\alpha_i, \alpha_i) = -1$. Now extend ϵ bi-multiplicatively to all pairs of elements in Q. Now we can verify that the relation $\epsilon(\alpha, \alpha) = (-1)^{(\alpha,\alpha)/2}$ works, where $\alpha = \sum_i k_i\alpha_i$:

$$\begin{aligned}
\epsilon(\alpha, \alpha) &= \prod_{i,j} \epsilon(\alpha_i, \alpha_j)^{k_ik_j} \\
&= \prod_i \epsilon(\alpha_i, \alpha_i)^{k_i^2} \prod_{i<j} \epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i)^{k_ik_j} \\
&= (-1)^{\sum_i k_i^2 \frac{(\alpha_i,\alpha_i)}{2}} \prod_{i<j} (-1)^{k_jk_i(\alpha_i,\alpha_j)} = (-1)^{\frac{(\alpha,\alpha)}{2}}
\end{aligned}$$

□

Remark. $\epsilon(\alpha, \alpha) = -1$ if α is a root. Further, if $\alpha, \beta \in Q$, we can extend the identity $\epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) = (-1)^{(\alpha_i,\alpha_j)}$ extends bi-multiplicatively to give $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$. Alternatively, note that $\epsilon(\alpha + \beta, \alpha + \beta) = \epsilon(\alpha, \alpha)\epsilon(\beta, \beta)\epsilon(\alpha, \beta)\epsilon(\beta, \alpha)$ gives us the following: $(-1)^{\frac{1}{2}(\alpha,\alpha)+\frac{1}{2}(\beta,\beta)+(\alpha,\beta)} = (-1)^{\frac{1}{2}(\alpha,\alpha)+\frac{1}{2}(\beta,\beta)}\epsilon(\alpha, \beta)\epsilon(\beta, \alpha)$, which implies $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$.

Theorem 20.2. *The brackets* (1) − (4) *above in* $\mathfrak{g}$*, with the form* ϵ *defined above, gives a simple Lie algebra of finite dimension with root system* $(V = \mathbb{R} \otimes_{\mathbb{Z}} Q, \Delta)$*.*

Proof. The skew-symmetry follows by the Remark, since $\epsilon(\alpha, \beta) = -\epsilon(\beta, \alpha)$ if $\alpha + \beta \in \Delta$. It now suffices to check the Jacobi identity when $a, b, c \in \mathfrak{h}$ or $E_\alpha(\alpha \in \Delta)$. If $a \in E_\alpha, b \in E_\beta, c \in E_\gamma$ and $\alpha + \beta + \gamma \notin \Delta \cup 0$, then the Jacobi identity trivially holds since all three terms are 0. For the same, it trivially holds when $a, b, c \in \mathfrak{h}$. Otherwise we have the following cases:

Case 1: $a, b \in \mathfrak{h}, c = E_\alpha$. Then we have $[a, [b, c]] = (\alpha, b)[a, E_\alpha] = (\alpha, b)(\alpha, a)E_\alpha$, $[b, [c, a]] = -[b, [a, E_\alpha]] = -(\alpha, b)(\alpha, a)E_\alpha, [c, [a, b]] = 0$, so they sum up to 0, as required.

Case 2: $a \in \mathfrak{h}, b = E_\alpha, c = E_\beta$. Then we have: $[a, [b, c]] = (\alpha+\beta, a)[b, c]; [b, [c, a]] = -(\beta, a)[b, c]; [c, [a, b]] = -(\alpha, a)[b, c]$, so they sum up to 0, as required.

Case 3: $a = E_\alpha, b = E_\beta, c = E_\gamma$, $\alpha + \beta + \gamma = 0$. Then we have:

1. $[E_\alpha, [E_\beta, E_\gamma]] = \epsilon(\beta, -\alpha - \beta)[E_\alpha, E_{-\alpha}] = -\epsilon(\beta, -\alpha)\epsilon(\beta, -\beta)\alpha$
2. $[E_\gamma, [E_\alpha, E_\beta]] = \epsilon(\alpha, \beta)[E_{-\alpha-\beta}, E_{\alpha+\beta}] = \epsilon(\alpha, \beta)(\alpha + \beta)$
3. $[E_\beta, [E_\gamma, E_\alpha]] = \epsilon(-\alpha - \beta, \alpha)[E_\beta, E_{-\beta}] = -\epsilon(-\alpha, \alpha)\epsilon(-\beta, \alpha)\beta$

To note that they sum to 0, observe the following:

$$\epsilon(\beta, -\beta)\epsilon(\beta, -\alpha)\alpha - \epsilon(-\alpha, \alpha)\epsilon(-\beta, \alpha)\beta + \epsilon(\alpha, \beta)(\alpha + \beta) = \epsilon(\beta, \alpha)\alpha + \epsilon(\beta, \alpha)\beta + \epsilon(\alpha, \beta)(\alpha + \beta) = 0$$

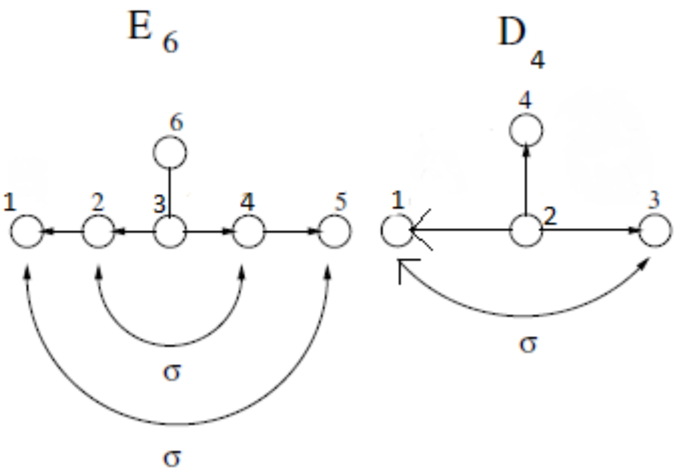
Exercise 20.1. Show that there are remaining two cases when $\alpha + \beta + \gamma \in \Delta$ (i) $\alpha = -\beta$ (ii) $(\alpha, \beta) = -1, (\beta, \gamma) = -1, (\alpha, \gamma) = 0$, and check the Jacobi identity in both of them.

Proof. (i) In this case, if $(\alpha, \gamma) = 0$, then since $\mathfrak{g}$ is simply laced, $\alpha + \gamma, \alpha - \gamma \notin \Delta$, so we have that $[E_\alpha, [E_{-\alpha}, E_\gamma]] = 0, [E_{-\alpha}, [E_\gamma, E_\alpha]] = 0, [E_\gamma, [E_\alpha, E_{-\alpha}]] = [E_\gamma, -\alpha] = (\alpha, \gamma)E_\gamma = 0$, so all three terms are 0.

WLOG, the other case is when $(\alpha, \gamma) = -1$ (since if $(\alpha, \gamma) = 1$ switch α with $-\alpha$), so since $\mathfrak{g}$ is simply laced, $\alpha + \gamma \in \Delta, \alpha - \gamma \notin \Delta$. Here we have that $[E_\alpha, [E_{-\alpha}, E_\gamma]] = 0, [E_{-\alpha}, [E_\gamma, E_\alpha]] = \epsilon(\gamma, \alpha)[E_{-\alpha}, E_{\alpha+\gamma}] = \epsilon(\gamma, \alpha)\epsilon(-\alpha, \alpha)\epsilon(-\alpha, \gamma)E_\gamma$ while $[E_\gamma, [E_\alpha, E_{-\alpha}]] = -[E_\gamma, \alpha] = (\alpha, \gamma)E_\gamma$. So it suffices to prove that $\epsilon(\gamma, \alpha)\epsilon(-\alpha, \alpha)\epsilon(-\alpha, \gamma) + (\alpha, \gamma) = 0$, which follows from the fact that $\epsilon(\gamma, \alpha)\epsilon(\alpha, \gamma) = (\alpha, \gamma) = -1$ in this case.

(ii) If no two of α, β, γ sum to 0, then using the fact that $(\alpha+\beta+\gamma, \alpha+\beta+\gamma) = 2$, one deduces that $(\alpha, \beta) + (\alpha, \gamma) + (\beta, \gamma) = -2$, so since none of them can be -2 (if $(\alpha, \beta) = -2, \alpha = -\beta$), after re-ordering $(\alpha, \beta) = -1, (\beta, \gamma) = -1, (\alpha, \gamma) = 0$. Then $[E_\alpha, [E_\beta, E_\gamma]] = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)E_{\alpha+\beta+\gamma}$; $[E_\beta, [E_\gamma, E_\alpha]] = 0; [E_\gamma, [E_\alpha, E_\beta]] = \epsilon(\alpha, \beta)\epsilon(\gamma, \alpha)\epsilon(\gamma, \beta)E_{\alpha+\beta+\gamma}$. So it suffices to show that we have: $\epsilon(\beta, \gamma)\epsilon(\alpha, \beta)\epsilon(\alpha, \gamma) + \epsilon(\alpha, \beta)\epsilon(\gamma, \alpha)\epsilon(\gamma, \beta) = 0$, which is true since $\epsilon(\alpha, \gamma) = \epsilon(\gamma, \alpha), \epsilon(\beta, \gamma) = -\epsilon(\gamma, \beta)$ in this case. □

Above was the simply-laced case. For the non-simply laced case, note by the following two exercises that each non-simply laced Lie algebra can be expressed as a sub-algebra of a simply-laced one. More precisely, type $B_r \subset D_{r+1}, C_r \subset A_{2r-1}, F_4 \subset E_6, G_2 \subset D_4$. To see this, put the following orientations on the Dynkin diagrams of E_6 and D_4 and define automorphisms of their respective Dynkin diagram (σ_2 for E_6 and σ_3 for D_4) switching the indicated vertices:



Exercise 20.2. Check that the map $E_i \to E_{\sigma(i)}, F_i \to F_{\sigma(i)}, H_i \to H_{\sigma(i)}$ defines an automorphism of $\tilde{\mathfrak{g}}(A)$, and hence of $\mathfrak{g}(A)$.

Proof. The relation $[H_{\sigma(i)}, H_{\sigma(j)}] = 0$ holds trivially, as does the relation $[E_i, F_j] = \sigma_{i,j} H_j$ (since the map σ is a bijection). It remains to check the relation $[H_{\sigma(i)}, E_{\sigma(j)}] = a_{ij} E_{\sigma(j)}$ and the analogous relation for the F's; this is exactly equivalent to $a_{ij} = a_{\sigma(i),\sigma(j)}$, which follows from the fact that σ is an automorphism of the Dynkin diagram and preserves the inner products of its roots. Since $\tilde{\mathfrak{g}}(A)$ has a unique maximal ideal, it is invariant under σ, hence σ induces an automorphism of $\mathfrak{g}(A)$. □

Exercise 20.3. For σ_2, in E_6 the eleemtns $\{X_1 + X_5, X_2 + X_4, X_3, X_6\}$ where $X = E, F$ or H lie in a fixed point sub-algebra $E_6^{\sigma_2}$ of σ_2 in E_6, and satisfy all Chevalley relations of $\tilde{\mathfrak{g}}(F_4)$. Likewise, for σ_3 and D_4, the elements $\{X_1 + X_3 + X_4, X_2\}$ satisfy all Chevalley relations of $\tilde{\mathfrak{g}}(G_2)$.

Proof. It is clear that the elements in question lie in the fixed point sub-algebra. In either case, the first Chevalley relations (that the Cartan subalgebra is abelian) is trivial. The third Chevalley relation (about the commutator of an E and an F) follows from the third Chevelley relation for E_6 and F_2, combined with the fact that in both sets $\{X_1{+}X_5, X_2{+}X_4, X_3, X_6\}$ and $\{X_1{+}X_3{+}X_4, X_2\}$, the indices of different elements are distinct. The second Chevalley relation is equivalent to saying that the in E_6, the four elements $\{X_1{+}X_5, X_2{+}X_4, X_3, X_6\}$ correspond (in terms of inner products) to the four simple roots of F_4; and that in D_4, the two elements $\{X_1 + X_3 + X_4, X_2\}$ correspond (in terms of inner products) to the two simple roots of G_2. Both of these assertions are trivial to verify. □

By these exercises, we have homomorphisms $\tilde{\mathfrak{g}}(F_4) \to \mathfrak{g}(E_6)^{\sigma_2}$, and $\tilde{\mathfrak{g}}(G_2) \to \mathfrak{g}(D_4)^{\sigma_3}$. This proves that $\mathfrak{g}(F_4)$ and $\mathfrak{g}(G_2)$ are finite dimensional, completing the proof. Soon we will show that in fact, $\mathfrak{g}(E_6)^{\sigma_3} = \mathfrak{g}(F_4), \mathfrak{g}(D_4)^{\sigma_3} = \mathfrak{g}(G_2)$. □

Using this explicit construction of simply-laced algebras, we can easily construct a symmetric invariant bilinear form (which is unique up to constant factor). We have a bilinear form $(\cdot,\cdot)$ on Q;

extend it by bilinearity to $\mathfrak{h}$. We let $(\mathfrak{h}, E_\alpha) = 0, (E_\alpha, E_\beta) = 0$ if $\alpha + \beta \neq 0, (E_\alpha, E_{-\alpha}) = -1$.

Exercise 20.4. Check that this bilinear form is invariant.

Proof. It is sufficient to prove that $([a, b], c) = (a, [b, c])$ when a, b, c are each either in $\mathfrak{h}$ or of the form E_α. If $a, b, c \in \mathfrak{h}$ clearly both sides are 0. If $a, b \in \mathfrak{h}, c = E_\alpha$, then the LHS is clearly 0, while the RHS is also 0 since $(\mathfrak{h}, E_\alpha) = 0$; a similar situation happens if $b, c \in \mathfrak{h}, a = E_\alpha$. If $a, c \in \mathfrak{h}, b = E_\alpha$, then both sides are again 0 since $(\mathfrak{h}, E_\alpha) = 0$. If $a = E_\alpha, b = E_\beta, c \in \mathfrak{h}$, the LHS is 0 unless $\alpha + \beta = 0$, and $[b, c] \in \mathbb{F}E_\beta$, so the RHS is also 0 unless $\alpha + \beta = 0$. If $\alpha + \beta = 0$, then the LHS is $([E_\alpha, E_{-\alpha}], c) = -(\alpha, c)$, while the RHS is $(E_\alpha, [E_{-\alpha}, c]) = (\alpha, c)(E_\alpha, E_{-\alpha}) = (\alpha, c)$. By symmetry, the case where $c = E_\alpha, b = E_\beta, a \in \mathfrak{h}$ follows. If $a = E_\alpha, b \in \mathfrak{h}, c = E_\beta$, the LHS is $([E_\alpha, b], E_\gamma) = -(\alpha, b)(E_{\alpha, E_\gamma})$, and the RHS is $(E_\alpha, [b, E_\gamma]) = (b, \gamma)(E_\alpha, E_\gamma)$. Clearly both quantities are equal if $\alpha + \gamma = 0$, and both quantities are 0 otherwise. The final case is when $a = E_\alpha, b = E_\beta, c = E_\gamma$; here both sides are clearly 0 unless $\alpha + \beta + \gamma = 0$. If this quantity is 0, then the LHS is $-\epsilon(\alpha, \beta)$, while the RHS is $\epsilon(\beta, -\alpha - \beta) = -\epsilon(\beta, \alpha) = \epsilon(\alpha, \beta)$, where in the last equality we use the fact that $\alpha + \beta$ is a root. $\square$

Next we define the compact form $\mathfrak{g}_C$ of $\mathfrak{g}$ when $\mathbb{F} = \mathbb{C} \supset \mathbb{R}$. Suppose $\mathfrak{g}$ is simply-laced, and $\mathfrak{g}_\mathbb{R} = \mathfrak{h}_\mathbb{R} \oplus (\bigoplus_{\alpha \in \Delta} \mathbb{R}E_\alpha)$ be the Lie algebra over $\mathbb{R}$. Define an automorphism $\omega_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$ by letting it act as -1 on $\mathfrak{h}$, and let $\omega_\mathbb{R}(E_\alpha) = E_{-\alpha}$.

Exercise 20.5. Check that this is an automorphism.

Proof. It suffices to prove that $\omega([a, b]) = [\omega(a), \omega(b)]$ when a, b are either in $\mathfrak{h}$ or of the form E_α. If both a, b are in $\mathfrak{h}$, then both sides are 0. If $a \in \mathfrak{h}, b = E_\alpha$, then the LHS is $\omega((\alpha, a)E_\alpha) = (\alpha, a)E_{-\alpha}$, while the RHS is $[-a, E_{-\alpha}] = (\alpha, a)E_{-\alpha}$. Finally, if $a = E_\alpha, b = E_\beta$ then both sides are 0 unless $\alpha + \beta$ is a root; if it is a root then both sides are clearly $\epsilon(\alpha, \beta)E_{\alpha+\beta}$ since $\epsilon(\alpha, \beta) = \epsilon(-\alpha, -\beta)$. $\square$

Now extend $\omega_\mathbb{R}$ from $\mathfrak{g}_\mathbb{R}$ to $\mathfrak{g} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{g}_\mathbb{R}$ to be an anti-linear automorphism ω, by $\omega(\lambda a) = \bar{\lambda}\omega(a)$.

Definition 20.1. The fixed point set of ω is a Lie algebra over $\mathbb{R}$, $\mathfrak{g}_\mathbb{C}$, called the compact form of $\mathfrak{g}$.

Exercise 20.6. If $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then $\mathfrak{g}_\mathbb{C} = \mathfrak{su}_n = \{A \in \mathfrak{sl}_n(\mathbb{C}) | A = -\bar{A}^t\}$, and $\omega(A) = -\bar{A}^t$.

Proof. In this case, it is clear that $E_{\alpha_i - \alpha_j} = E_{ij}$ if $i < j$, and $-E_{ij}$ if $i > j$ (this is to fulfill the condition $[E_\alpha, E_{-\alpha}] = -\alpha$). Then it is clear that the automorphism $\omega_\mathbb{R}$ sends A to $-A^t$, and consequently ω sends A to $-\bar{A}^t$, as required. $\square$

Proposition 20.3. *The restriction of the invariant symmetric bilinear form $(\cdot, \cdot)$ from $\mathfrak{g}$ to $\mathfrak{g}_c$ is negative definite.*

Proof. We can write $\mathfrak{g}_c = i\mathfrak{h}_\mathbb{R} + \sum_{\alpha \in \Delta_+} \mathbb{R}(E_\alpha + E_{-\alpha}) + \sum_{\alpha \in \Delta_+} i\mathbb{R}(E_\alpha - E_{-\alpha})$ and these 3 spaces are orthogonal to each other. It remains to show that it is negative-definite one each space. This is true because $(ih, ih) = -(h, h) < 0$; $(E_\alpha + E_{-\alpha}, E_\alpha + E_{-\alpha}) = -2 < 0, (i(E_\alpha - E_{-\alpha}), i(E_\alpha - E_{-\alpha})) = -2 < 0$. $\square$

Finally, the restriction of the invariant bilinear form (Killing form) from $\mathfrak{g}(E_6)$ or $\mathfrak{g}(D_4)$ to $\mathfrak{g}^{\sigma_i}$ is non-degenerate, hence $\mathfrak{g}^{\sigma_i}$ is semi-simple and thus simple. To see this, just take $\mathfrak{g}_c \cap \mathfrak{g}^{\sigma_i}$, where $\mathfrak{g} = E_6$ or D_4. Since the Killing form is negative definite on $\mathfrak{g}_c$, it is negative definite on $\mathfrak{g}_c \cap \mathfrak{g}^{\sigma_i}$, and thus also on its complexification $\mathfrak{g}^{\sigma_i}$. It follows that $E_6^{\sigma_2} = F_4, D_4^{\sigma_3} = G_2$.

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Lecture 21 — The Weyl Group of a Root System

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Let V be a finite dimensional Euclidean vector space, i.e. a real vector space with a positive definite symmetric bilinear form $(.\,,.)$. Let $a \in V$ be a nonzero vector and denote by r_a the orthogonal reflection relative to a, i.e. $r_a(a) = -a, r_a(v) = v$ if $(a, v) = 0$.

Formula 21.1. $r_a(v) = v - \frac{2(a,v)}{(a,a)}a$.

Exercise 21.1. Prove: (a) $r_a \in \mathcal{O}_v(\mathbb{R})$, i.e. $(r_a u, r_a v) = (u, v), u, v \in V$. (b) $r_a = r_{-a}$ and $r_a^2 = 1$. (c) $detr_a = -1$. (d) If $A \in \mathcal{O}_v(\mathbb{R})$, then $Ar_aA^{-1} = r_{A(a)}$.

Proof. For (a), $(r_a u, r_a v) = (u - \frac{2(a,u)}{(a,a)}a, v - \frac{2(a,v)}{(a,a)}a) = (u, v) - \frac{2(a,u)(a,v)}{(a,a)} - \frac{2(a,v)(a,u)}{(a,a)} + 4\frac{(a,u)(a,v)(a,a)}{(a,a)^2} = (u, v)$.

For (b), since $r_a(a) = -a$, we have $r_{-a}(-a) = a$. Also, for any v perpendicular to a, it is also perpendicular to $-a$, thus $r_{-a}(v) = v$. Thus r_a acts the same way as r_{-a} on the entire space, so $r_a = r_{-a}$. Next, $r_a^2(a) = r_a(-a) = a$. Also, for v perpendicular to a, $r_a^2(v) = r_a(v) = v$. Thus, r_a^2 acts the same way as 1 does, so $r_a^2 = 1$.

For (c), r_a fixes the space (of dimension dimV-1) perpendicular to a, thus it has 1 as an eigenvalue of multiplicity $dimV - 1$. The other eigenvalue is -1, corresponding to eigenvector a. Thus $detr_a$ equals product of all eigenvalues, which is -1.

For (d), notice that $Ar_aA^{-1}(A(a)) = Ar_a(a) = A(-a) = -A(a)$. Also, for any v perpendicular to $A(a)$, $(A^{-1}(v), a) = (v, A(a)) = 0$ as $A \in \mathcal{O}$. Thus $Ar_aA^{-1}(v) = A(A^{-1}(v)) = v$. Thus Ar_aA^{-1} acts the same way as $r_{A(a)}$, so they are equal. □

Definition 21.1. Let (V, Δ) be a root system. Let W be the subgroup of $\mathcal{O}_v(\mathbb{R})$, generated by all r_α, where $\alpha \in \Delta$. The group W is called the Weyl group of the root system (V, Δ) (and of the corresponding semisimple lie algebra $\mathfrak{g}$).

Proposition 21.1. *(a) $w(\Delta) = \Delta$ for all $w \in W$. (b) W is a finite subgroup of the group $\mathcal{O}_v(\mathbb{R})$.*

Proof. For (a), it suffice to show that $r_\alpha(\beta)(= \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha) \in \Delta$ if $\alpha, \beta \in \Delta$. First, r_α is nonsingular as it has determinant -1. Recall the string property of (V, Δ): $\{\beta - k\alpha | k \in \mathbb{Z}\} \cap (\Delta \cup 0) = \{\beta - p\alpha, ..., \beta + q\alpha\}$, where $p, q \in \mathbb{Z}_+, p - q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. Hence $p \geq \frac{2(\alpha,\beta)}{(\alpha,\alpha)}, q \geq -\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. So if $(\alpha, \beta) \leq 0$, by the string property, we can add α to β at least $-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ times and if $(\alpha, \beta) \geq 0$ we can subtract α from β at least $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ times, which exactly means that $\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Delta \cup \{0\}$. But it can't be 0 since r_α is nonsingular.

(b) is clear since Δ spans V, so if $w \in W$ fixes all elements of Δ, it must be 1, so W embeds in the group of permutations of the finite set Δ by (a). Therefore W is finite. □

Remark 21.1. (a) shows the string property of the root system implies that Δ is W-invariant. One can show converse is true: if we replace string property by W-invariance of Δ and that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$, then we get an equivalent definition of a root system. One has to check only for the case $dimV = 2$. (see Serre)

Fix $f \in V^*$ which doesn't vanish on Δ, let $\Delta_+ = \{\alpha \in \Delta | f(\alpha) > 0\}$ be the subset of positive roots and let $\Pi = \{\alpha_1, ..., \alpha_r\} \in \Delta_+$ be the set of simple roots ($r = dimV$). Then the reflections $s_i = r_{\alpha_i}$ are called simple reflections.

Theorem 21.2. *(a) $\Delta_+ \backslash \{\alpha_i\}$ is s_i-invariant. (b) if $\alpha \in \Delta_+ \backslash \Pi$, then there exists i such that $hts_i(\alpha) < ht\alpha$ ($ht \sum_i k_i \alpha_i = \sum_i k_i$). (c) If $\alpha \in \Delta_+ \backslash \Pi$, then there exists a sequence of simple reflections $s_{i_1}, ...s_{i_k}$ such that $s_{i_1}, ...s_{i_k}(\alpha) \in \Pi$ and also $s_{i_j}, ...s_{i_k}(\alpha) \in \Delta_+$ for all $1 \leq j \leq k$. (d) The group W is generated by simple reflections.*

Proof. (a) For a positive root $\alpha, s_i(\alpha) = \alpha - n\alpha_i$, where n is an integer. If $\alpha \neq \alpha_i$, then all coefficients in the decomposition of simple root remain positive, except possibly for the coefficient of α_i. But in a positive root α, all coefficients are nonnegative. Hence if α is not simple, it must have positive coefficient in front of some simple roots other than α_i (other positive integer multiples of α_i are not in Δ by definition of root system), thus $s_i(\alpha)$ should remain positive.

(b) $s_i(\alpha) = \alpha - \frac{2(\alpha,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$. Now if $hts_i(\alpha) \geq ht\alpha$ for all i, then $(\alpha, \alpha_i) \leq 0$ for all i. But then $(\alpha, \alpha) = \sum_i a_i(\alpha, \alpha_i) \leq 0$. Since $\alpha = \sum_i a_i\alpha_i, a_i \geq 0$. So $\alpha = 0$, a contradiction.

(c) Just apply (b) finitely many times until we get a simple root.

(d) Denote by W' the subgroup of W, generated by simple reflections. By (c), for any $\alpha \in \Delta_+$ there exists $w \in W'$, such that $w(\alpha) = \alpha_j \in \Pi$. Hence by Ex.21.1(d), $r_\alpha = w^{-1}s_jw$, which lies in W'. So W' contains all reflections r_α with $\alpha \in \Delta$. But $r_{-\alpha} = r_\alpha$, so W' contains all reflections, hence $W' = W$. □

Example 21.1. $\Delta_{A_r} = \{\epsilon_i - \epsilon_j | 1 \leq i, j \leq r + 1, i \neq j\} \subset V = \{\sum_{i=1}^{r+1} a_i\epsilon_i | \sum_i a_i = 0, a_i \in \mathbb{R}\} \subset \mathbb{R}^{r+1} = \oplus_{i=1}^{r+1}\mathbb{R}\epsilon_i$, with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$.

We have $r_{\epsilon_i-\epsilon_j}(\epsilon_s) = \epsilon_s - \frac{2(\epsilon_s,\epsilon_i-\epsilon_j)}{2}(\epsilon_i - \epsilon_j) = \begin{cases} \epsilon_s & \text{if} \quad s \neq i, s \neq j \\ \epsilon_j & \text{if} \quad s = i \\ \epsilon_i & \text{if} \quad s = j \end{cases}$ So $r_{\epsilon_i-\epsilon_j}$ is transposition of ϵ_i, ϵ_j, so $W_{A_r} = S_{r+1}$.

Exercise 21.2. Compute Weyl Group for root systems of type B_r, C_r, D_r. In particular show that for B_r and C_r they are isomorphic, but not isomorphic to D_r.

Proof. $\Delta_{B_r} = \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i, j \leq r, i \neq j\} \cup \{\pm\epsilon_i\} \subset \mathbb{R}^r = \oplus_{i=1}^r\mathbb{R}\epsilon_i$, with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. It is easy to see that, as in the example, $r_{\epsilon_i-\epsilon_j}$ switches ϵ_i with ϵ_j, r_{ϵ_i} switches the sign of ϵ_i and the other r_α are generated by the previous two. Thus the Weyl Group consisting of all elements that permute r elements as well as switch some of their signs. So it is the semidirect product group $\mathbb{Z}_2^r \rtimes S_r$. The root system for C_r is just that for B_r with $\pm\epsilon_i$ replaced by $\pm2\epsilon_i$. So of course the reflections are exactly the same, so Weyl Groups for B_r and C_r are the same.

$\Delta_{D_r} = \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i, j \leq r, i \neq j\} \subset \mathbb{R}^r = \oplus_{i=1}^r\mathbb{R}\epsilon_i$, with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. Now $r_{\epsilon_i-\epsilon_j}$ acts as before, and $r_{\epsilon_i+\epsilon_j}$ switches ϵ_i to $-\epsilon_j$ and ϵ_j to $-\epsilon_i$. Thus Weyl Group is the group of all elements that permute r elements as well as switching an even number of their signs. Thus it is $\mathbb{Z}_2^{r-1} \rtimes S_r$, and it is not isomorphic to that of B_r and C_r. □

Definition 21.2. Consider the open (in usual topology) set $V - \cup_{\alpha\in\Delta} T_\alpha$ in V (T_α is the hyperplane perpendicular to α). The connected components of the set are called open chambers, the closures are called closed chambers. $C = \{v \in V | (\alpha_i, v) > 0, \alpha_i \in \Pi\}$ is called the fundemental chamber; $\overline{C} = \{v | (\alpha_i, v) \geq 0, \alpha_i \in \Pi\}$ is called closed fundemental chamber.

Exercise 21.3. Show that the open fundemental chamber is a chamber.

Proof. We first show that C is connected. Given any $u, v \in C$, we have $(\alpha_i, u) > 0$ for all $\alpha_i \in \Pi$. Now given any $\alpha \in \Delta_+, \alpha$ is a linear comobination of α_i with positive coefficients. Thus $(\alpha, u) > 0$. Similarly, $(\alpha, v) > 0$. Now any point on the straight segment connecting u and v has the form $xu + (1-x)v$ for some $0 \leq x \leq 1$. Thus it is easy to see that $(\alpha, x) > 0$. Thus u, v are in a single component, i.e. an open chamber (we only considered α being a positive root. But if it is negative, we get similarly that $(\alpha, u), (\alpha, v)$ and (α, x) are all negative).

Now, take any element u' not in C, then by definition $(\alpha_i, u') \leq 0$ for some $\alpha_i \in \Pi$. Then obviously u' and $u \in C$ are not in the same component as they are separated by the hyperplane T_{α_i}. Thus C is an entire component, i.e. an open chamber. □

Theorem 21.3. *(a) W permutes all chambers transitively, i.e. for any two chambers C_i and C_j there exists $w \in W$ such that $w(C_i) = C_j$. (b) Let Δ_+ and Δ'_+ be two subsets of positive roots, defined by linear functions f and f'. Then there exists $w \in W$ such that $w(\Delta_+) = \Delta'_+$. In particular, the Cartan matrix of (V, Δ) is independent of the choice of f.*

Proof. (a) Choose a segment connecting points in C_i, C_j, which doesn't intersect $\cup_{\alpha,\beta\in\Delta}(T_\alpha \cap T_\beta)$. Let's move along the segment until we hit a hyperplane T_α. Then replace C_i by $r_\alpha C_i$. After finitely many steps we hit the chamber C_j.

(b) a linear function f on V can be written as f_a, where $f_a(v) = (a, v)$ for fixed $a \in V$. f doesn't vanish on Δ means that $a \notin \cup_{\alpha\in\Delta} T_\alpha$, so $f = f_a$ with a in some open chamber. If we move a around this chamber, the set Δ_+, defined by f remains unchanged. Hence all the subsets of positive roots in Δ are labelled by open chamber and if $w(C) = C'$, then for the corresponding sets of positive roots Δ_+ and Δ'_+ we get that $w(\Delta_+) = \Delta'_+$. □

Definition 21.3. Let $s_1, ...s_r$ be the simple reflections in W (they depend on choice of Δ_+). Any $w \in W$ can be written as a product $w = s_{i_1}...s_{i_t}$ due to Theorem 21.2(d). Such a decomposition with minimal possible number of factors t is called a reduced decomposition and in this case $t = l(w)$ is called the length of w. Note: $detw = (-1)^{l(w)}$ since $dets_i = -1$. E.g. $l(1) = 0, l(s_i) = 1, l(s_i, s_j) = 2$ if $i \neq j$, but $= 0$ if $i = j$ since $s_i^2 = 1$.

Lemma 21.4. *(Exchange Lemma) Suppose that $s_{i_1}...s_{i_{t-1}}(\alpha_{i_t}) \in \Delta_-, \alpha_{i_t} \in \Pi$. Then the expression $w = s_{i_1}...s_{i_t}$ is not reduced. More precisely, $w = s_{i_1}...s_{i_{m-1}}s_{i_{m+1}}...s_{i_{t-1}}$ for some $1 \leq m \leq t-1$.*

Proof. Consider the roots $\beta_k = s_{i_{k+1}}...s_{i_{t-1}}(\alpha_{i_t})$ for $0 \leq k \leq t-1$. Then $\beta_0 \in \Delta_-$ and $\beta_{t-1} = \alpha_{i_t} \in \Delta_+$. Hence there exists $1 \leq m \leq t-1$ such that $\beta_{m-1} \in \Delta_-, \beta_m \in \Delta_+$. But $\beta_{m-1} = s_{i_m}\beta_m$. Hence by Theorem 21.2(a), $\beta_m = \alpha_{i_m} \in \Pi$. Let $\overline{w} = s_{i_{m+1}}...s_{i_{t-1}}$, by Ex.21.1(d), it follows $\overline{w}s_{i_t}\overline{w}^{-1} = s_{i_m}$, or $\overline{w}s_{i_t} = s_{i_m}\overline{w}$. The result follows by multiplying both sides by $s_{i_1}...s_{i_m}$ on the left. □

Corollary 21.5. *W acts simply transitively on chambers, i.e. if $w(C) = C$, then $w = 1$.*

Proof. In the contrary case, $w(C) = C$ for some $w \neq 1$, hence $w(\Delta_+) = \Delta_+$ for Δ_+ corresponding to C. Take a reduced expression $w = s_{i_1}...s_{i_t}, t \geq 1$. Then $w(\alpha_{i_t}) = s_{i_1}...s_{i_{t-1}}(-\alpha_{i_t}) \in \Delta_+$. Hence $s_{i_1}...s_{i_{t-1}}(\alpha_{i_t}) \in \Delta_-$. Hence by Exchange Lemma, $s_{i_1}...s_{i_t}$ is nonreduced. Contradiction.

Transitivity is proved in Theorem 21.3 (a). □

18.745 Introduction to Lie Algebras	November 30, 2010

Lecture 22 — The Universal Enveloping Algebra

Prof. Victor Kac — *Scribe: Aaron Potechin*

Definition 22.1. Let g be a Lie algebra over a field F. An enveloping algebra of g is a pair (φ, U), where U is a unital associative algebra and $\varphi : g \to U_-$ is a Lie algebra homomorphism, where U_- stands for U with the bracket $[a, b] = ab - ba$.

Example 22.1. Let $\varphi : g \to \text{End } V$ be a representation of g in a vector space V. Then the pair $(\text{End } V, \varphi)$ is an enveloping algebra of g.

Definition 22.2. The universal enveloping algebra of g is an enveloping algebra $(\Phi, U(g))$ which has the following universal mapping property: for any enveloping algebra (φ, U) of g there exists a unique associative algebra homomorphism $f : U(g) \to U$ such that $\varphi = f \circ \Phi$.

Exercise 22.1. Prove that the universal enveloping algebra is unique (if it exists).

Solution: Assume $(\Phi_1, U_1(g))$ and $(\Phi_2, U_2(g))$ are both universal enveloping algebras of g. Then by the universal mapping property of the universal enveloping algebra, we have unique maps $f_{11}, f_{12}, f_{21}, f_{22}$ such that for $i, j \in \{1, 2\}$, $\Phi_i = f_{ij} \circ \Phi_j$. Now $\Phi_i = id \circ \Phi_i$, so by uniqueness, $f_{ii} = id$. $\Phi_i = f_{ij} \circ f_{ji} \circ \Phi_i$, so by uniqueness $f_{ij} \circ f_{ji} = f_{ii} = id$. $f_{12} = f_{21}^{-1}$, so $(\Phi_1, U_1(g))$ and $(\Phi_2, U_2(g))$ are isomorphic, as needed. □

Existence of universal enveloping algebras:
Let $T(g)$ be the free unital associative algebra on a basis $a_1, a_2, \cdots$ of g and let $J(g)$ be the two-sided ideal of $T(g)$ generated by the elements $a_i a_j - a_j a_i - [a_i, a_j]$. Then $U(g) = T(g)/J(g)$.
Define $\Phi : g \to U(g)_-$ by letting $\Phi(a_i) =$ the image of a_i in $U(g)$ and extending linearly.

Remark: $T(g)$ is called the tensor algebra over the vector space g.
$T(g) = F \oplus g \oplus (g \otimes g) \oplus (g \otimes g \otimes g) \oplus \cdots$ with the concatenation product.
Good linear algebra textbooks: Artin, Vinberg.

Exercise 22.2. Prove that $(\Phi, U(g))$ is the universal enveloping algebra (i.e the universality property holds).

Solution: First, we will show that $(\Phi, U(g))$ is an enveloping algebra. $T(g)$ is a unital associative algbera, so $U(g) = T(g)/J(g)$ is a unital associative algebra. To check that Φ is a Lie algebra homomorphism, by linearity, it suffices to check that for all i, j, $\Phi([a_i, a_j]) = [\Phi(a_i), \Phi(a_j)]$. But this is clear, as $\Phi([a_i, a_j]) - [\Phi(a_i), \Phi(a_j)] = [a_i, a_j] - a_i a_j - a_j a_i$ is in $J(g)$, so it is zero in $U(g)$.

Now let (U, φ) be another enveloping algebra. Define the map $f : T(g) \to U$ by taking $f(\prod_{k=1}^{l} a_{i_k}) = \prod_{k=1}^{l} \varphi(a_{i_k})$ and extending linearly. This is clearly a well-defined unital associative algebra homomorphism.
Now for all i, j, $f([a_i, a_j] - a_i a_j - a_j a_i) = \varphi([a_i, a_j]) - \varphi(a_i)\varphi(a_j) - \varphi(a_j)\varphi(a_i) = 0$ because $\varphi : g \to U_-$ is a Lie algebra homomorphism. Thus, $J(g) \subseteq \ker(f)$, so $f : U(g) \to U$ is a well-defined unital associative algebra homomorphism.

For all i, $(f \circ \Phi)(a_i) = f(a_i) = \varphi(a_i)$, so by linearity $\varphi = f \circ \Phi$.

Assume that $f' : U(g) \to U$ is another unital associative algebra homomorphism with $\varphi = f' \circ \Phi$. Then $f'(1) = f(1) = 1$ and for all i, $f(a_i) = \varphi(a_i) = (f' \circ \Phi)(a_i) = f'(a_i)$. $1, a_1, a_2, \cdots$ generate $U(g)$ as a unital associative algebra, so $f = f'$. Thus, f is unique, as needed. $\square$

Corollary 22.1. *Any representation $\pi : g \to End\ V$ extends uniquely to a homomorphism of associative algebras $U(g) \to End\ V$ (so that $a_i \mapsto \pi(a_i)$).*

Theorem 22.2. *Poincaré-Birkhoff-Witt (PBW) theorem: Let $a_1, a_2, \cdots$ be a basis of g. Then the monomials* (*) $a_{i_1} a_{i_2} \cdots a_{i_s}$ with $i_1 \leq i_2 \leq \cdots \leq i_s$ form a basis of $U(g)$.

Proof. Easy part: the monomials(*) span $U(g)$.
Proof is by induction on the pair (s, N), where s is the degree of the monomial and N is the number of inversions, i.e. number of pairs i_m, i_n for which $m < n$ but $i_m > i_n$, lexicographically ordered, i.e. $((s, N) > (s', N')$ if $s > s'$ or $s = s'$ and $N > N')$
For $N = 0$ there is nothing to prove.
If $N \geq 1$, then in the monomial we have $a_{i_t} a_{i_{t+1}}$ where $i_t > i_{t+1}$, as otherwise the monomial is already in our set of monomials(*).
But we have the relation $a_{i_t} a_{i_{t+1}} = a_{i_{t+1}} a_{i_t} + [a_{i_t}, a_{i_{t+1}}]$, so that in $U(g)$:
$a_{i_1} a_{i_2} \cdots a_{i_s} = a_{i_1} \cdots a_{i_{t+1}} a_{i_t} \cdots a_{i_s} + a_{i_1} \cdots a_{i_{t-1}} [a_{i_t}, a_{i_{t+1}}] a_{i_{t+2}} \cdots a_{i_s}$. The first term is a monomial of degree s and $N - 1$ inversions, and the second term is the sum of monomials with degree $s - 1$. Thus, by the inductive hypothesis, each of these monomials is generated by the monomials(*), so $a_{i_1} a_{i_2} \cdots a_{i_s}$ is also generated by these monomials, as needed.

The hard part: why are the monomials(*) linearly independent?
Let B_s be the vector space over F with basis $b_{i_1} \cdots b_{i_s}$ with $i_1 \leq i_2 \leq \cdots i_s$.
Take $B_0 = F$ and let $B = \oplus_{s \geq 0} B_s$. We shall construct a linear map $f : T(g) \to B$ such that $J(g) \subseteq \ker(f)$ and $f(a_{i_1} \cdots a_{i_s}) = (b_{i_1} \cdots b_{i_s})$ if $i_1 \leq i_2 \leq \cdots \leq i_s$.
This will induce a linear map $f : U(g) = T(g)/J(g) \to B$. Hence the monomials(*) are linearly independent because $b_{i_1} \cdots b_{i_s}$, $i_1 \leq i_2 \leq \cdots i_s$ are linearly independent.

Construction: $f(1) = 1$, $f(a_{i_1} \cdots a_{i_s}) = (b_{i_1} \cdots b_{i_s})$ if $i_1 \leq i_2 \leq \cdots \leq i_s$, and
(1) $f(a_{i_1} \cdots a_{i_t} a_{i_{t+1}} \cdots a_{i_s}) = f(a_{i_1} \cdots a_{i_{t+1}} a_{i_t} \cdots a_{i_s}) + f(a_{i_1} \cdots a_{i_{t-1}} [a_{i_t}, a_{i_{t+1}}] a_{i_{t+2}} \cdots a_{i_s})$
if $i_t > i_{t+1}$

By induction on (s, N), we can use the inversion (1) to reduce $f(a_{i_1} \cdots a_{i_s})$ to a sum of terms of the form $f(a_{j_1} \cdots a_{j_{s'}})$, where $j_1 \leq j_2 \leq \cdots j_{s'}$. We just need to check that the final expression is independent of which sequence of inversions we choose. We do this by induction on (s, N).

Case 1:
$a_{i_1} \cdots a_{i_s} = a_{i_1} \cdots a_{i_t} a_{i_{t+1}} \cdots a_{i_r} a_{i_{r+1}} \cdots a_{i_s}$, where $i_t > i_{t+1}$ and $i_r > i_{r+1}$
Using the left inversion first gives us
$f(a_{i_1} \cdots a_{i_{t+1}} a_{i_t} \cdots a_{i_r} a_{i_{r+1}} \cdots a_{i_s}) + f(a_{i_1} \cdots [a_{i_t}, a_{i_{t+1}}] \cdots a_{i_r} a_{i_{r+1}} \cdots a_{i_s})$
By the inductive hypothesis, we may use any sequence of inversions (1) to evaluate this, so using the right inversion on each term, we get
$f(a_{i_1} \cdots a_{i_{t+1}} a_{i_t} \cdots a_{i_{r+1}} a_{i_r} \cdots a_{i_s}) + f(a_{i_1} \cdots a_{i_{t+1}} a_{i_t} \cdots [a_{i_r}, a_{i_{r+1}}] \cdots a_{i_s}) +$
$f(a_{i_1} \cdots [a_{i_t}, a_{i_{t+1}}] \cdots a_{i_{r+1}} a_{i_r} \cdots a_{i_s}) + f(a_{i_1} \cdots [a_{i_t}, a_{i_{t+1}}] \cdots [a_{i_r}, a_{i_{r+1}}] \cdots a_{i_s})$

Using the right inversion first gives us
$f(a_{i_1}\cdots a_{i_t}a_{i_{t+1}}\cdots a_{i_{r+1}}a_{i_r}\cdots a_{i_s})+f(a_{i_1}\cdots a_{i_t}a_{i_{t+1}}\cdots[a_{i_r},a_{i_{r+1}}]\cdots a_{i_s})$.
By the inductive hypothesis, we may use any sequence of inversions (1) to evaluate this, so using the left inversion on each term, we get
$f(a_{i_1}\cdots a_{i_{t+1}}a_{i_t}\cdots a_{i_{r+1}}a_{i_r}\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_t},a_{i_{t+1}}]\cdots a_{i_{r+1}}a_{i_r}\cdots a_{i_s})+$
$f(a_{i_1}\cdots a_{i_{t+1}}a_{i_t}\cdots[a_{i_r},a_{i_{r+1}}]\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_t},a_{i_{t+1}}]\cdots[a_{i_r},a_{i_{r+1}}]\cdots a_{i_s})$
These expressions are the same, so we get the same result whether we begin with the left inversion or the right inversion.

Case 2: Inversions overlap
$a_{i_1}\cdots a_{i_s}=a_{i_1}\cdots a_{i_t}a_{i_{t+1}}a_{i_{t+2}}\cdots a_{i_s}$ with $i_t>i_{t+1}>i_{t+2}$.

Exercise 22.3. Show that we get the same result whether we start with the inversion on $a_{i_t}a_{i_{t+1}}$ or the inversion on $a_{i_{t+1}}a_{i_{t+2}}$.

Solution: First using the left inversion gives us
$f(a_{i_1}\cdots a_{i_{t+1}}a_{i_t}a_{i_{t+2}}\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_t},a_{i_{t+1}}]a_{i_{t+2}}\cdots a_{i_s})$
Using the inversion on $a_{i_t}a_{i_{t+2}}$ in the first term, we get
$f(a_{i_1}\cdots a_{i_{t+1}}a_{i_{t+2}}a_{i_t}\cdots a_{i_s})+f(a_{i_1}\cdots a_{i_{t+1}}[a_{i_t},a_{i_{t+2}}]\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_t},a_{i_{t+1}}]a_{i_{t+2}}\cdots a_{i_s})$
Using the inversion on $a_{i_{t+1}}a_{i_{t+2}}$ in the first term, we get
$f(a_{i_1}\cdots a_{i_{t+2}}a_{i_{t+1}}a_{i_t}\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_{t+1}},a_{i_{t+2}}]a_{i_t}\cdots a_{i_s})+f(a_{i_1}\cdots a_{i_{t+1}}[a_{i_t},a_{i_{t+2}}]\cdots a_{i_s})+$
$f(a_{i_1}\cdots[a_{i_t},a_{i_{t+1}}]a_{i_{t+2}}\cdots a_{i_s})$

First using the right inversion gives us
$f(a_{i_1}\cdots a_{i_t}a_{i_{t+2}}a_{i_{t+1}}\cdots a_{i_s})+f(a_{i_1}\cdots a_{i_t}[a_{i_{t+1}},a_{i_{t+2}}]\cdots a_{i_s})$
Using the inversion on $a_{i_t}a_{i_{t+2}}$ in the first term, we get
$f(a_{i_1}\cdots a_{i_{t+2}}a_{i_t}a_{i_{t+1}}\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_t},a_{i_{t+2}}]a_{i_{t+1}}\cdots a_{i_s})+f(a_{i_1}\cdots a_{i_t}[a_{i_{t+1}},a_{i_{t+2}}]\cdots a_{i_s})$
Using the inversion on $a_{i_t}a_{i_{t+1}}$ in the first term, we get
$f(a_{i_1}\cdots a_{i_{t+2}}a_{i_{t+1}}a_{i_t}\cdots a_{i_s})+f(a_{i_1}\cdots a_{i_{t+2}}[a_{i_t},a_{i_{t+1}}]\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_t},a_{i_{t+2}}]a_{i_{t+1}}\cdots a_{i_s})+$
$f(a_{i_1}\cdots a_{i_t}[a_{i_{t+1}},a_{i_{t+2}}]\cdots a_{i_s})$

Now look at equation (1).
$f(a_{i_1}\cdots a_{i_t}a_{i_{t+1}}\cdots a_{i_s})=f(a_{i_1}\cdots a_{i_{t+1}}a_{i_t}\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_t},a_{i_{t+1}}]\cdots a_{i_s})$ if $i_t>i_{t+1}$
By skew-symmetry, this gives

$$\begin{aligned}
f(a_{i_1}\cdots a_{i_{t+1}}a_{i_t}\cdots a_{i_s}) &= f(a_{i_1}\cdots a_{i_t}a_{i_{t+1}}\cdots a_{i_s})-f(a_{i_1}\cdots[a_{i_t},a_{i_{t+1}}]\cdots a_{i_s})\\
&= f(a_{i_1}\cdots a_{i_t}a_{i_{t+1}}\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_{t+1}},a_{i_t}]\cdots a_{i_s})
\end{aligned}$$

Thus, (1) holds regardless of whether $i_t>i_{t+1}$ or $i_{t+1}>i_t$. By the inductive hypothesis, we may use this relation freely for monomials of dimension $s-1$ and it will not change the result. By linearity, we have that for any $b,c\in g$
$f(a_{i_1}\cdots bc\cdots a_{i_{s-1}})-f(a_{i_1}\cdots cb\cdots a_{i_{s-1}})=f(a_{i_1}\cdots[b,c]\cdots a_{i_{s-1}})$.
Applying this, we find that the difference between the first expression and the second expression above is
$f(a_{i_1}\cdots[[a_{i_{t+1}},a_{i_{t+2}}],a_{i_t}]\cdots a_{i_s})+f(a_{i_1}\cdots[a_{i_{t+1}},[a_{i_t},a_{i_{t+2}}]]\cdots a_{i_s})+f(a_{i_1}\cdots[[a_{i_t},a_{i_{t+1}}],a_{i_{t+2}}]\cdots a_{i_s})$
Using skew-symmetry, this is
$f(a_{i_1}\cdots[[a_{i_{t+1}},a_{i_{t+2}}],a_{i_t}]\cdots a_{i_s})+f(a_{i_1}\cdots[[a_{i_{t+2}},a_{i_t}],a_{i_{t+1}}]\cdots a_{i_s})+f(a_{i_1}\cdots[[a_{i_t},a_{i_{t+1}}],a_{i_{t+2}}]\cdots a_{i_s})$

which is 0 by the Jacobi identity.
Thus, we get the same result regardless of which inversion we start with. □

By induction, for any monomial $a_{i_1} \cdots a_{i_s}$, the evaluation of $f(a_{i_1} \cdots a_{i_s})$ is independent of the sequence of inversions(1) that we use, so $f(a_{i_1} \cdots a_{i_s})$ is well-defined. Thus, $f : T(g) \to B$ is well-defined.

It remains to show that $J(g) \subseteq \ker(f)$. By linearity, it suffices to show that for all i, j, for all $A, B \in T(g)$, $f(A(a_i a_j - a_j a_i - [a_i, a_j])B) = 0$. If $i > j$, this is just the equation (1). If $i < j$, the eqation (1) gives
$f(Aa_j a_i B) = f(Aa_i a_j B) + f(A[a_j, a_i]B)$. Using skew-symmetry,
$f(Aa_i a_j B) = f(Aa_j a_i B) - f(A[a_j, a_i]B) = f(Aa_j a_i B) + f(A[a_i, a_j]B)$.
$f(A(a_i a_j - a_j a_i - [a_i, a_j])B) = 0$, as needed. This completes the proof. □

The Casimir Element of $U(g)$

We assume that $\dim g < \infty$ and g carries a non-degenerate symmetric invariant bilinear form $(\cdot, \cdot)$ (e.g. g is semi-simple and $(a, b) = K(a, b)$)

Choose a basis $\{a_i\}$ of g and let b_i be the dual basis i.e. $(a_i, b_j) = \delta_{ij}$. The Casimir element is the following element of $U(g)$: $\Omega = \sum_{i=1}^{\dim g} a_i b_i$.

Exercise 22.4. Show that Ω is independent of the choice of the basis $\{a_i\}$.

Solution: From linear algebra, to go from one basis to another, it is sufficient to use the following three operations:
1. $a'_i = a_i$ if $i \neq j$, $a'_j = ca_j$ where $c \in F$, $c \neq 0$
2. $a'_i = a_i$ if $i \neq j$, $i \neq k$. $a'_j = a_k$, $a'_k = a_j$
3. $a'_i = a_i$ if $i \neq j$. $a'_j = a_j + ca_k$, where $c \in F$

To show that Ω is independent of the choice of the basis $\{a_i\}$, it is sufficient to show that Ω is invariant under these three operations. For these three operations, the dual basis changes as follows:
1. $b'_i = b_i$ if $i \neq j$, $b'_j = \frac{1}{c} a_j$
2. $b'_i = b_i$ if $i \neq j$, $i \neq k$. $b'_j = b_k$, $b'_k = b_j$
3. $b'_i = b_i$ if $i \neq k$. $b'_k = b_k - cb_j$

In all three cases, it is easily verified that $\Omega' = \sum_{i=1}^{\dim g} a'_i b'_i = \sum_{i=1}^{\dim g} a_i b_i = \Omega$. □

Lemma on dual basis:

Lemma 22.3. *For any $a \in g$ write*
(2) $[a, a_i] = \sum_j \alpha_{ij} a_j$, (3) $[a, b_i] = \sum_j \beta_{ij} b_j$, $\alpha_{ij}, \beta_{ij} \in F$. Then $\alpha_{ij} = -\beta_{ji}$.

Proof. Taking the inner product of (2) with b_j and of (3) with a_j, we get
$([a, a_i], b_j) = \alpha_{ij}$ and $([a, b_i], [a_j]) = \beta_{ij}$. We also have
$([a, a_i], b_j) = (a, [a_i, b_j])$ and $([a, b_i], [a_j]) = (a, [b_i, a_j]) = -(a, [a_j, b_i])$, hence $\alpha_{ij} = -\beta_{ji}$. □

Definition 22.3. Let g be a Lie algbera, and V be a g-module.
(We used the language of a representation π of g in V, notation $\pi(g)V$, $g \in g$, $v \in V$. A little more

convenient is the equivalent language of a g-module V, notation: gV)
A 1-cocycle of g with coefficients in a g-module V is a linear map $f : g \to V$ such that
(4) $f([a, b]) = af(b) - bf(a)$

Example: Trivial 1-cocycle: for $v \in V$, let $f_v(a) = av$.

Exercise 22.5. Show that $f_v : g \to V$ is a 1-cocycle.

Solution: f_v is clearly linear, and $f([a, b]) = [a, b](v) = a(b(v)) - b(a(v)) = af(b) - bf(a)$, as needed. Thus, f_v is a 1-cocycle of g. □

Denote by $Z^1(g, V)$ the space of all 1-cocycles of g with coefficients in V. Then by exercise 22.5, trivial cocycles form a subspace denoted by $B^1(g, V)$.

Definition 22.4. $H^1(g, V) = Z^1(g, v)/B^1(g, V)$ is called the first coboundary.

Note that $H^1(g, V) = 0$ just means that any 1-cocycle of g, i.e. any linear map $f : g \to V$ satisfying (4) is trivial, i.e. of the form $f = f_v$ for some V.

Theorem 22.4. *If g is a semi-simple Lie algebra over a field F of characteristic 0 and V is a finite-dimensional g-module, then $H^1(g, V) = 0$*

Lemma 22.5. *If $\{a_i\}$ and $\{b_j\}$ are dual bases of g and f is a 1-cocycle of g with values in V then for any $a \in g$ we have $a(\sum_i a_i f(b_i)) = \Omega f(a)$*

Exercise 22.6. Prove this using Lemma 1.

Solution: The equation (4) gives that for all $a, b \in g$, $af(b) = bf(a) + f([a, b])$.

$$\begin{aligned}
a(\sum_i a_i f(b_i)) &= \sum_i a(a_i f(b_i)) \\
&= \sum_i a(f([a_i, b_i])) + \sum_i a(b_i f(a_i)) \\
&= \sum_i f([a, [a_i, b_i]]) + \sum_i [a_i, b_i](f(a)) + \sum_i a(b_i f(a_i))
\end{aligned}$$

By the Jacobi identity, for all $a, b, c \in g$, $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$, so
$[a, [b, c]] = [b, [a, c]] - [c, [a, b]]$.

$$\begin{aligned}
\sum_i f([a, [a_i, b_i]]) &= \sum_i f([a_i, [a, b_i]]) - \sum_i f([b_i, [a, a_i]]) \\
&= \sum_i \sum_j f([a_i, \beta_{ij} b_j]) - \sum_i \sum_j f([b_i, \alpha_{ij} a_j]) \\
&= \sum_i \sum_j \beta_{ij} f([a_i, b_j]) + \sum_j \sum_i \alpha_{ji} f([a_i, b_j]) = 0 \text{ by Lemma 1}
\end{aligned}$$

$$
\begin{aligned}
a(\sum_i a_i f(b_i)) &= \sum_i [a_i, b_i](f(a)) + \sum_i a(b_i f(a_i)) \\
&= \sum_i a_i b_i (f(a)) - \sum_i b_i a_i (f(a)) + \sum_i a(b_i f(a_i)) \\
&= \Omega f(a) - \sum_i b_i a(f(a_i)) - \sum_i b_i f([a_i, a]) + \sum_i a(b_i f(a_i)) \\
&= \Omega f(a) + \sum_i b_i f([a, a_i]) + \sum_i a(b_i f(a_i)) - \sum_i b_i a(f(a_i)) \\
&= \Omega f(a) + \sum_i \sum_j b_i f(\alpha_{ij} a_j) + \sum_i [a, b_i] f(a_i) \\
&= \Omega f(a) + \sum_i \sum_j \alpha_{ij} b_i f(a_j) + \sum_i \sum_j \beta_{ij} b_j f(a_i) = \Omega f(a) \text{ by Lemma 1}
\end{aligned}
$$

$\square$

18.745 Introduction to Lie Algebras	December 2, 2010

Lecture 23 — Decomposition of Semisimple Lie Algebras

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Notation 23.1. First, we recall some facts from lecture 22. Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module.

- $Z^1(g, v) = \{f : \mathfrak{g} \mapsto V | f([a, b]) = af(b) - bf(a)\}$ is the space of 1-cocycles.
- $Z^1(\mathfrak{g}, v)$ contains the subspace $B^1(\mathfrak{g}, V) = \{f_v | f_v(a) = av\}$ of trivial 1-cocycles.
- $H^1(\mathfrak{g}, v) = Z^1(\mathfrak{g}, V)/B^1(\mathfrak{g}, V)$ is the first cohomology.
- $\Omega = \sum_j a_j b_j$ and is called the Casamir operator.

From now on, $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. We will prove that $H^1(\mathfrak{g}, V) = 0$ for any $\mathfrak{g}$-module V. This will be used to prove the Weyl's Complete Reducibility Theorem and Levi's Theorem.

The following exercise follows from the definitions and will be used to prove $H^1(\mathfrak{g}, V)$ vanishes.

Exercise 23.1. $H^1(\mathfrak{g}, V_1 \oplus V_2) = H^1(\mathfrak{g}, V_1) \oplus H^1(\mathfrak{g}, V_2)$, where the V_i are $\mathfrak{g}$-modules.

Proof. [Solution] First we show that $Z(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$. It is clear that $Z(\mathfrak{g}, V_1 \oplus V_2) \supset Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$. Furthermore, every 1-cocycle $\varphi \in Z(\mathfrak{g}, V_1 \oplus V_2)$ can be decomposed as $\pi_1 \circ \varphi \oplus \pi_2 \circ \varphi \in Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)$.

It is also clear that $B(\mathfrak{g}, V_1 \oplus V_2) = B(\mathfrak{g}, V_1) \oplus B(\mathfrak{g}, V_2)$ since $\varphi_{v_1 \oplus v_2} = \varphi_{v_1} \oplus \varphi_{v_2}$.

Therefore $H^1(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1 \oplus V_2)/B(\mathfrak{g}, V_1 \oplus V_2) = Z(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)/B(\mathfrak{g}, V_1) \oplus B(\mathfrak{g}, V_2) = Z(\mathfrak{g}, V_1)/B(\mathfrak{g}, V_1) \oplus Z(\mathfrak{g}, V_2)/B(\mathfrak{g}, V_2) = H^1(\mathfrak{g}, V_1) \oplus H^1(\mathfrak{g}, V_2)$. □

We will also use the following lemma from lecture 22 and a corollary.

Lemma 23.1. *If $\mathfrak{g}$ is a Lie algebra with an invariant non-degenerate bilinear form $(.,.)$, a_i, b_i are dual basis of $\mathfrak{g}$ (i.e $(a_i, b_j) = \delta_{ij}$) and $f \in Z^1(\mathfrak{g}, V)$:*

$$a \sum_j a_j f(b_j) = \Omega(f(a)), \forall a \in \mathfrak{g}.$$

Corollary 23.2. *The Casamir operator commutes with the action of $\mathfrak{g}$ on any $\mathfrak{g}$-module V i.e. $a(\Omega(v)) = \Omega(av)$ for any $a \in \mathfrak{g}, v \in V$.*

Proof. Take $f = f_v$. The equation in the lemma becomes:

$$a(\Omega(v)) = a \sum_j a_j b_j(v) = \Omega(a(v))$$

□

Theorem 23.3 (Vanishing Theorem). *If V is a finite dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, V) = 0$.*

Proof. By Corollary 23.2, Ω commutes with the action of $\mathfrak{g}$ on V. This means the generalized eigenspace decomposition for Ω acting on $V = \oplus_{\lambda\in\mathbb{F}} V_\lambda$ is $\mathfrak{g}$-invariant i.e. all the subspaces V_λ are $\mathfrak{g}$-invariant. Let $V' = \oplus_{\lambda\neq 0} V_\lambda$, so that $V = V_0 \oplus V'$. As both V_0 and V' are $\mathfrak{g}$-invariant, they both are $\mathfrak{g}$-submodules of V.

By Exercise 23.1 $H^1(\mathfrak{g}, V) = H^1(\mathfrak{g}, V_0) \oplus H^1(\mathfrak{g}, V')$. We will prove both of these terms equal 0. First, consider $H^1(\mathfrak{g}, V_0)$.

Let $f : \mathfrak{g} \mapsto V_0$ be a 1-cocycle.

Let $\mathfrak{g}_0 = \{a \in \mathfrak{g} | aV = 0\}$. This is the kernel of the representation $\mathfrak{g} \mapsto EndV$ and therefore an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, the ideal $\mathfrak{g}_0$ is also semisimple. Since $\mathfrak{g}_0$ is semisimple, $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$. For $a, b \in \mathfrak{g}_0$, $f[a, b]) = af(b) - bf(a) = 0 - 0 = 0$. $f[\mathfrak{g}_0, \mathfrak{g}_0] = 0$ and thus $f[\mathfrak{g}_0] = 0$. This means that $\bar{f} : \bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}_0 \mapsto V$ is a well-defined map.

Without loss of generality we may assume that $\mathfrak{g} = \bar{\mathfrak{g}}$, for any 1-cocycle of $\mathfrak{g}$ induces one on $\bar{\mathfrak{g}}$ and vice versa.

We now specify an invariant bilinear form on $\mathfrak{g}$. Take for $(.,.)$ on $\mathfrak{g}$ the trace form of $\mathfrak{g}$ on V_0: $(a, b)_V = tr_{V_0} ab$. By Cartan's criterion, this trace form is non-degenerate as $\bar{\mathfrak{g}}$ is a semisimple Lie algebra by assumption and the representation $\bar{\mathfrak{g}} \mapsto V_0$ is faithful by the construction of $\bar{\mathfrak{g}}$.

$$tr_V(\Omega) = \sum_i tr_V a_i b_i = \sum_i (a_i, b_i)_V = dim\mathfrak{g},$$

where a_i, b_i are dual basis.

$\Omega|_{V_0}$ is nilpotent and therefore $tr_V(\Omega) = 0$. This implies $\mathfrak{g} = 0$ which implies $H^1(\mathfrak{g}, V_0) = 0$.

Now to prove that $H^1(\mathfrak{g}, V') = 0$,

For any 1-cocycle $f : \mathfrak{g} \mapsto V'$, by Lemma 23.1, $a(m) = \Omega(f(a))$ where $m = \sum_i a_i(f(b_i)) \in V'$. But $\Omega|_{V'}$ is an invertible operator and $\Omega a = a\Omega$, so that $f(a) = \Omega^{-1}am = a\Omega^{-1}m$. $f = f_{\Omega^{-1}m}$ is a trivial 1-cocycle and $H^1(\mathfrak{g}, V') = 0$. □

Remark 23.1. A couple notes on the proof. When V is finite dimensional, $V = V_0 \oplus V'$ over any field $\mathbb{F}$ and operator Ω. Where $\Omega|_{V_0}$ is nilpotent and $\Omega|V'$ is invertible. We do not need to assume that $\mathbb{F}$ is algebraically closed.

The assumption of semisimplicity was used to prove a non-degenerate invariant bilinear form exists. Does the converse hold, that if $H^1(V, \mathfrak{g}) = 0$ then $\mathfrak{g}$ is semisimple?

With the Vanishing Theorem one can prove these theorems about semisimple Lie algebras.

Theorem 23.4 (Weyl Complete Reducibility Theorem). *If $char\mathbb{F} = 0$ and V is a finite dimensional module over a semisimple Lie algebra $\mathfrak{g}$, then for any submodule U of V there exists a complementary submodule U' so that $V = U \oplus U'$.*

Note that since V is finite dimensional it follow from this theorem that V is isomorphic to the direct sum of irreducible $\mathfrak{g}$-modules.

Theorem 23.5 (Levi's Theorem). *If $\mathfrak{g}$ is a finite dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0, and $R(\mathfrak{g})$ is the radical then $\mathfrak{g}$ contains a subalgebra s complementary to $R(\mathfrak{g})$ Furthermore, $\mathfrak{g} = s \ltimes R(\mathfrak{g})$.*

Definition 23.1. s is called a *Levi factor*.

Theorem 23.6 (Mal'cev Thceorem). *All Levi factors s in $\mathfrak{g}$ are conjugate to each other by an automorphsim of $\mathfrak{g}$.*

Remark 23.2. The proofs use projection operators in a vector space V. A projector P of a subspace U of V is an endomorphism of V such that: $P(V) \subset U$ and $P(u) = u$ for $u \in U$.

Note that if P_0 is a projection operator from V to U, then any other projector of V to U has the form $P = P_0 + A$ where $A(V) \subset U, A(u) = 0$.

Proof: Weyl Complete Reducibility Theorem. Consider the space End V with the following $\mathfrak{g}$-module structure: $a(A) = \pi(a)A - A\pi(a)$. where $a \in \mathfrak{g}, A \in \text{End } V$ $\pi : \mathfrak{g} \mapsto \text{End } V$ is the representation of $\mathfrak{g}$ on V. This satisfies the properties of a $\mathfrak{g}$-module by the Jacobi identity.

Fix a projector $P_0 : V \mapsto U$ and consider the corresponding trivial 1-cocycle. $f_{P_0} = a(P_0) = \pi(a)P_0 - P_0\pi(a)$. Let $M \subset EndV$ be the following subspace: $M = \{A|A(V) \subset U, A(U) = 0\}$. Let $A \in M$ then

$(\pi(a)A - A\pi(a))v = \pi(a)u - Av' \in U.$

$(\pi(a)A - A\pi(a))u = 0 - Au' = 0.$

We know $f_{P_0} : \mathfrak{g} \mapsto EndV$, but in fact $f_{P_0} : \mathfrak{g} \mapsto M$ since

$(\pi(a)P_0 - P_0\pi(a))v = \pi(a)u - P_0v' \in U$

$(\pi(a)P_0 - P_0\pi(a))u = \pi(a)u - \pi(a)u = 0$

Therefore, $f_{P_0} \in Z^1(\mathfrak{g}, M)$.

By the Vanishing Theorem, $H^1(\mathfrak{g}, M) = 0$, so f_{P_0} is a trivial 1-cocycle and$f_{P_0}(a) = A\pi(a) - \pi(a)A$ for some $A \in M$.Rearranging this gives: $(\pi(a))(P_0 - A) = (P_0 - A)(\pi(a))$ In other words, $P_0 - A$ is a new projector P of V with $a(P_0 - A) = 0$

Letting $U' = \ker P$ we get $V = U \oplus U'$ where U' is $\mathfrak{g}$-invariant, since the projector P commutes with $\mathfrak{g}$.

U and U' are our complementary g-submodules. $\square$

The proof of Levi's Theorem follows the same method.

Proof: Levi's Theorem. We prove it by induction on $dim\mathfrak{g}$. The case $dim\mathfrak{g} = 1$ is obvious.

Case 1 $R(\mathfrak{g})$ is not abelian. Consider the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g}/[R(\mathfrak{g}), R(\mathfrak{g})]$, where $dim\bar{\mathfrak{g}} < dim\mathfrak{g}$ and we apply the inductive assumption. $\bar{\mathfrak{g}} = \bar{s} \ltimes R(\bar{\mathfrak{g}})$. Let $\mathfrak{g}_1$ be the premiage of $\bar{s}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_1 + R(\mathfrak{g})$ with $dim\mathfrak{g}_1 < dim\mathfrak{g}$. By the inductive assumption, $\mathfrak{g}_1 = s \ltimes R(\mathfrak{g}_1)$. and $\mathfrak{g} = s \ltimes (R(\mathfrak{g}_1) + R(\mathfrak{g}))$.

Exercise 23.2. $R(\mathfrak{g}_1) + R(\mathfrak{g})$ is a solvable ideal of $\mathfrak{g}$, since $R(\mathfrak{g})$ is a maximal solvable ideal of $\mathfrak{g}$ so $R(\mathfrak{g}_1) \subset R(\mathfrak{g})$, show that g=$R(g) \ltimes s$.

Proof. [Solution]$R(\mathfrak{g}_1) + R(\mathfrak{g})$ is an ideal of $\mathfrak{g}$: Write an arbitrary element of $R(\mathfrak{g}_1) + R(\mathfrak{g})$ as $a + b$, where $a \in R(\mathfrak{g}_1)$ and $b \in R(\mathfrak{g})$. Write an arbitrary element of $\mathfrak{g}$ as $c+d$, where $c \in \mathfrak{g}_1$ and $d \in R(\mathfrak{g})$. Then $[a + b, c + d] = [a, c] + [a, d] + [b, c + d]$.

We have $[a, c] \in R(\mathfrak{g}_1)$ because $R(\mathfrak{g}_1)$ is an ideal of $\mathfrak{g}_1$, $[a, d], [b, c + d] \in R(\mathfrak{g})$ because $R(\mathfrak{g})$ is an ideal. Furthermore, $R(\mathfrak{g}_1) + R(\mathfrak{g})$ is solvable. Assume $R(\mathfrak{g}_1)^{(m)} = 0$, and $R(\mathfrak{g})^{(n)} = 0$,then $(R(\mathfrak{g}_1) + R(\mathfrak{g}))^{(m)} \subseteq R(\mathfrak{g}_1)^{(m)} + R(\mathfrak{g}) = R(\mathfrak{g})$. Hence, $(R(\mathfrak{g}_1) + R(\mathfrak{g}))^{(mn)} = 0$. Thus, we have a solvable ideal of $\mathfrak{g}$ which contains $R(\mathfrak{g})$, so we conclude that our ideal is in fact $R(\mathfrak{g})$, and $R(\mathfrak{g}_1) \subset R(\mathfrak{g})$. Hence we conclude

$$\mathfrak{g} = s \ltimes R(\mathfrak{g}).$$

□

Case 2 $R(\mathfrak{g})$ is abelian. Construct the following $\mathfrak{g}$-module structure on the space End $\mathfrak{g}$: $a(m) = (\mathbf{ad}\ a)m - m(\mathbf{ad}\ a), m \in End\mathfrak{g}$.

Let $\tilde{M} = \{m \in \text{End } \mathfrak{g} | m(\mathfrak{g}) \subset R(\mathfrak{g}), m(R(\mathfrak{g})) = 0\}$. Since $R(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, $\tilde{M}$ is a $\mathfrak{g}$-submodule of End $\mathfrak{g}$. Next let $\tilde{R} = \{\mathbf{ad}\ a | a \in R(\mathfrak{g})$, which is also a $\mathfrak{g}$-module.

Finally, put $M = \tilde{M}/\tilde{R}$. This is a $\mathfrak{g}$-module as it is the quotient of two g-modules.

Exercise 23.3. Show that M is an s-module and that $R(\mathfrak{g})$ acts on M trivally.

Proof. [Solution] $R(\mathfrak{g})$ acts trivially on M because $R(\mathfrak{g})$ is abelian. $[R(\mathfrak{g}), \tilde{M}] = 0$.Therefore, M is an s-module because we let $\tilde{s}$ be the preimage of s in $\tilde{M}$ and define $s(m) = \tilde{s}(m)$. As $R(\mathfrak{g})$ acts trivially, this is well-defined. □

Now fix a projector P_0 of $\mathfrak{g}$ on $R(\mathfrak{g})$.

Exercise 23.4. Prove that $f(a) = (\mathbf{ad}\ \tilde{a})P_0 - P_0(\mathbf{ad}\ \tilde{a})$, where $\tilde{a}$ is the preimage of $a \in \mathfrak{g}$ under the canonical map $\mathfrak{g} \mapsto \mathfrak{g}/Rad(\mathfrak{g}) = s$, is a well-defined 1-cocycle of s in $M = \tilde{M}/\tilde{R}$.

Proof. [Solution] First we need to show that f is well defined on M. For any $r \in \tilde{R}$, $(\mathbf{ad}\ (\tilde{a}+r))P_0 - P_0(\mathbf{ad}\ (\tilde{a}+r)) = (\mathbf{ad}\ \tilde{a})P_0 + (\mathbf{ad}\ r)P_0 - P_0(\mathbf{ad}\ \tilde{a} + \mathbf{ad}\ r) = (\mathbf{ad}\ \tilde{a})P_0 - P_0(\mathbf{ad}\ \tilde{a}) + 0 - (\mathbf{ad}\ r)$ This is because P_0 is the identity on $R(\mathfrak{g})$ and $R(\mathfrak{g})$ is abelian, so $\mathbf{ad}\ r(R(\mathfrak{g}) = 0$. f is well-defined on M.

To show that it is a 1-cocycle we compute the necessary identity: $f[a, b] = (\mathbf{ad}\ [\tilde{a}, \tilde{b}])P_0 - P_0(\mathbf{ad}\ [\tilde{a}, \tilde{b}])$. Here we use that the fact that the pullback map respects the Lie bracket.

$a(f(b)) - b(f(a)) = (\mathbf{ad}\ a)f(b) - f(b)(\mathbf{ad}\ a) - (\mathbf{ad}\ b)f(a) + f(a)(\mathbf{ad}\ b) = (\mathbf{ad}\ a)((\mathbf{ad}\ \tilde{b})P_0 - P_0(\mathbf{ad}\ \tilde{b})) - ((\mathbf{ad}\ \tilde{b})P_0 - P_0(\mathbf{ad}\ \tilde{b}))(\mathbf{ad}\ a) - (\mathbf{ad}\ b)((\mathbf{ad}\ \tilde{a})P_0 - P_0(\mathbf{ad}\ \tilde{a})) + ((\mathbf{ad}\ \tilde{a})P_0 - P_0(\mathbf{ad}\ \tilde{a}))(\mathbf{ad}\ b) = (\mathbf{ad}\ [\tilde{a}, \tilde{b}])P_0 - P_0(\mathbf{ad}\ [\tilde{a}, \tilde{b}]$ □

Applying the vanishing of $H^1(s, M)$, we conclude that for $a \in s$, $f(a) = f_m(a)$ for some $m \in M$. Let $\tilde{m}$ be a preimage of m in $\tilde{M}$ from the canonical map and put $P = P_0 - \tilde{m}$. P is a projector of $\mathfrak{g}$ on $R(\mathfrak{g})$ for which $P(s) = 0$.

It is immediate to check that $[P, \mathbf{ad}\ \mathfrak{g}] = [P, \mathbf{ad}\ R(\mathfrak{g})] \subset \mathbf{ad}\ R(\mathfrak{g})$. This gives two cases depending if P and $\mathfrak{g}$ commute or not:

Case 1 $[P, \mathbf{ad}\ \mathfrak{g}] = 0$

Exercise 23.5. Given that $[P, \mathbf{ad}\ \mathfrak{g}] = 0$, prove that ker P is an ideal of $\mathfrak{g}$ so $\mathfrak{g} = \ker P \ltimes R(\mathfrak{g})$.

Proof. [Solution] P is an endomorphism, so ker P is clearly a subalgebra. We need to show that $[\mathfrak{g}a] \in \ker P$. For $a \in \ker P$, $P([\mathfrak{g}, a]) = P(\mathbf{ad}\ \mathfrak{g}(a)) = \mathbf{ad}\ \mathfrak{g}P(a)$, since $[P, \mathbf{ad}\ \mathfrak{g}] = 0$, and $P(a) = 0$ as $a \in \ker P$, so $P([\mathfrak{g}, a]) = \mathbf{ad}\ \mathfrak{g}(0) = 0$ and $[\mathfrak{g}a] \in \ker P$.

$\mathbf{ad}\ : R(\mathfrak{g}) \mapsto \mathrm{End} \ker P$ since ker P is an ideal of $\mathfrak{g} \supset R(\mathfrak{g})$. Therefore, we make $\ker P \ltimes R(\mathfrak{g})$. $\mathfrak{g} = \ker P \oplus R(\mathfrak{g})$ as vector spaces. The following is a Lie algebra homomorphism: $(k, r) \mapsto k + r \in \mathfrak{g}$ They clearly respect the opperations. The bracket is defined in $\ker P \ltimes R(\mathfrak{g})$ to agree with it in $\mathfrak{g}$. It is an isomorphism of vector spaces. Thus it is also a Lie algebra isomorphism. □

Case 2 $[P, \mathbf{ad}\ \mathfrak{g}] \neq 0$. Consider the subalgebra $g_1 = \{a \in g | [P, \mathbf{ad}\ a] = 0\}$. It is a subalgebra of $\mathfrak{g}$ such that $dim\mathfrak{g}_1 < dim\mathfrak{g}$. By the inductive assumption, $g_1 = s \ltimes Rad(g_1)$. Now we use that $[P, \mathbf{ad}\ \mathfrak{g}] \subset \mathbf{ad}\ R(\mathfrak{g})$. In other words, $[P, \mathbf{ad}\ b] = \mathbf{ad}\ r_b$, for any $b \in \mathfrak{g}$ there exists such $r_b \in R(\mathfrak{g})$. Since $(\mathbf{ad}\ r_b)P = 0$ as P is a projector with these properties) and $P\mathbf{ad}\ r_b = \mathbf{ad}\ r_b$, we get $[P, \mathbf{ad}\ r_b] = \mathbf{ad}\ r_b$. So $[P, \mathbf{ad}\ (b - r_b)] = 0$. This means $b - r_b \in \mathfrak{g}_1$. $\mathfrak{g} = \mathfrak{g}_1 + R(\mathfrak{g})$ as vector spaces. Apply the proof of Ex 23.2 we deduce that $\mathfrak{g} = s \ltimes R(g)$. □

This proves Levi's Theorem.

18.745 Introduction to Lie Algebras	December 7, 2010
Lecture 24 — Finite dimensional $\mathfrak{g}$-modules over a s.s. Lie algebra.	
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1 Finite dimensional representations of semisimple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional semisimpe Lie algebra, over an algebraically closed field $\mathbb{F}$ of characteristic 0. Choose a Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and a subset of positive roots $\Delta_+ \subset \mathfrak{h}^*$. Let

$$\mathfrak{g} = \mathfrak{N}_- \oplus \mathfrak{h} \oplus \mathfrak{N}_+$$

be the triangular decomposition. Recall that $\mathfrak{N}_+$ (resp. $\mathfrak{N}_-$) is generated by the vectors $E_1, \dots, E_r$ (resp. $F_1, \dots, F_r$) or, equivanlenty, that $\mathfrak{N}_\pm = \oplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm\alpha}$. Let us define

$$\mathfrak{b} \doteqdot \mathfrak{h} \oplus \mathfrak{N}_+.$$

$\mathfrak{b}$ is called a Borel subalgebra. Note that

$$[\mathfrak{b}, \mathfrak{b}] = \mathfrak{N}_+. \tag{1}$$

Indeed, $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{N}_+$, follows immediately by the definition of $\mathfrak{b}$, while $[\mathfrak{b}, \mathfrak{b}] \supset \mathfrak{N}_+$ follows from the fact that $[h, \mathfrak{g}_\alpha] \neq 0$, if $\alpha(h) \neq 0$ and such h always exists, since $\alpha \neq 0$. As $\mathfrak{N}_+$ is a nilpotent subalgebra, we see that $\mathfrak{b}$ is a solvable subalgebra. Moreover, $\mathfrak{b}$ is a maximal solvable subalgebra (and all such subalgebras are conjugated).
Since by Weyl's complete reducibility theorem, every finite dimensional $\mathfrak{g}$-module is a direct sum of irreducible ones, it suffices to study finite dimensional, irreducible $\mathfrak{g}$-modules.

Proposition 1.1. *Let V be afinite dimensional, irreducile $\mathfrak{g}$-module. Then $\exists \Lambda \in \mathfrak{h}^*$ and $0 \neq v_\Lambda \in V$ s.t. the following three properties hold:*

i) $hv_\Lambda = \Lambda(h)v_\Lambda$, $\forall h \in \mathfrak{h}^*$;

ii) $\mathfrak{N}_+ v_\Lambda = 0$;

iii) $\mathfrak{U}(\mathfrak{g})v_\Lambda = V$.

It follows immediately that property iii) is equivalent to the following property

$$iii)' \; \mathfrak{U}(\mathfrak{N}_-)$$

Proof. By Lie's Theorem, $\mathfrak{b}$ has an eigenvector $0 \neq v \in V$ so that $\forall b \in \mathfrak{b}$, $\tilde{\Lambda}(b)v$, for some $\tilde{\Lambda} \in \mathfrak{h}^*$. But, by the property illustrated in (1), we see that $\tilde{\Lambda}(\mathfrak{N}_+) = 0$, since $\tilde{\Lambda}([b_1, b_2]) = \Lambda(b_1)\Lambda(b_2) - \Lambda(b_2)\Lambda(b_1)$. Let $\Lambda = \tilde{\Lambda}_{|\mathfrak{h}} \in \mathfrak{h}^*$, then *i)* and *ii)* hold and *iii)* follows from the irreducibility of the $\mathfrak{g}$-module V, since $\mathfrak{U}(g)v_\lambda$ (we are identifying $v_\Lambda = v$) is a non-zero submodule of V (it contains v_Λ since $Id \in \mathfrak{U}(\mathfrak{g})$). ∎

Definition 1.1. A $\mathfrak{g}$-module V (not necessarily finite dimensional) with the property that $\exists\Lambda \in \mathfrak{h}^*$ and $0 \neq v_\Lambda \in V$ such that properties $i), ii), iii)$ from the previous proposition hold, is called highest weight module with heighest weight Λ and v_λ is called a heighest weight vector.

Let $\Delta_+ = \{\beta_1, \dots, \beta_r\}$ be the set of positive roots for $\mathfrak{g}$. Choose root vectors $E_{\beta_i} \in \mathfrak{N}_+,\ E_{-\beta_i} \in \mathfrak{N}_-$ and let $h_1, \dots, H_n$ be a nasis for $\mathfrak{h}$, then vectors $E_{\beta_i}, E_{-\beta_i}\ (i = 1, \dots, N),\ h_j\ (j = 1, \dots, n)$ form a basis for $\mathfrak{g}$. By PBW theorem, monomials of the form

$$E_{-\beta_1}^{m_1} \dots E_{-\beta_N}^{m_N} H_1^{s_1} \dots H_r^{s_r} E_{\beta_1}^{n_1} \dots E_{\beta_N}^{n_N},\ m_i, n_j, s_k \in \mathbb{Z}_+.$$

In particular

Definition 1.2. For an arbitrary $\mathfrak{g}$-module V, let h be an element of $\mathfrak{h}^*$, we denote $V_\lambda = \{v \in V \mid hv = \lambda(h)v,\ \forall h \in \mathfrak{h}\}$ the weight space for $\mathfrak{h}$ attached to λ. A non-zero vector $v \in V_\lambda$ is called singular of weight λ if $\mathfrak{N}_+ v = 0$.

Example 1.1. Any $\Lambda \in \mathfrak{h}^*$ is a singular weight of a highest weight $\mathfrak{g}$-module with highest weight Λ.

Notation 1.1. Given $\Lambda \in \mathfrak{h}^*$, let $D(\Lambda) = \{\Lambda - \sum_{i=1}^r k_i\alpha_i : k_i \in \mathbb{Z}_+\} \subset \mathfrak{h}^*$, where $\Pi = \{\alpha_1,\ \alpha_2,\ \dots\ \alpha_r\}$ is the set of simple roots of $\mathfrak{g}$.

Theorem 1.2. *Let V be a highest weight $\mathfrak{g}$-module with highest weight $\Lambda \in \mathfrak{h}^*$. Then,*

(a) $V = \bigoplus_{\lambda \in D(\Lambda)} V_\lambda$

(b) $V_\Lambda = \mathbb{F}v_\Lambda$ *and* $\dim V_\lambda < \infty$

(c) V is an irreducible $\mathfrak{g}$-module if and only if $\mathbb{F}^* v_\Lambda$ are the only singular vectors.

(d) V contains a unique proper maximal submodule.

(e) If v is a singular vector with weight λ, then $\Omega(v) = (\lambda+2\rho, \lambda)v$. Here $(\cdot,\cdot)$ is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ and Ω is the corresponding Casimir operator, and $2\rho = \sum_{\alpha \in \Delta_+} \alpha$.

(f) $\Omega|_V = (\Lambda + 2\rho, \Lambda) Id_V$

(g) If λ is a singular weight, then $(\lambda + \rho, \lambda + \rho) = (\Lambda + \rho, \Lambda + \rho)$.

Proof: Pf a, b)

By iii), $V = U(\mathfrak{n}_-)v_\Lambda = \sum \mathbb{F}E_{-\beta_1}^{m_1} \dots . E_{-\beta_N}^{m_N} v_\Lambda \in V_{\Lambda - \sum_1^N m_i\beta_i \in D(\Lambda)}$, proving a) and b).

Pf c)

We know

$$(*) \qquad U = \oplus_{\lambda \in D(\Lambda)} (U \cap V_\lambda)$$

for a submodule U by a previous lecture. So choose $\lambda \in D(\Lambda)$ to be of minimal height with $U \cap V_\lambda \neq 0$. Then $E_\alpha v = 0$ for any $v \in U \cap V_\lambda$, so v is a singular vector. And if v is a singular vector of weight λ, then $U(\mathfrak{g})v = U(\mathfrak{n}_-)v$ which is a proper submodule of V unless $\lambda = \Lambda$.

Pf d)

The sum of proper submodules of V is again a proper submodule because it does not contain v_Λ. Thus this sum is a unique maximal submodule.

Pf e)

Take a a basis $\{E_{\beta_i}, E_{-\beta_i}, H_i\}$ and its dual $\{E_{-\beta_i}, E_{\beta_i}, H^i\}$ and compute Casimir operator $\Omega = \sum_1^r H_i H^i + \sum_1^N E_{\beta_i} E_{-\beta_i} + E_{-\beta_i} E_{\beta_i} = \sum_1^r H_i H^i + 2\sum_1^N E_{-\beta_i} E_{\beta_i} + 2\nu^{-1}\alpha$. Apply this to a singular vector v_λ to get

$$\Omega v_\lambda = \sum_1^r \lambda(H_i)\lambda(H^i) v_\lambda + \sum_1^N (\lambda, \beta_i) v_\lambda + 0$$

The right hand side is $(\lambda, \lambda) + 2(\lambda, \rho))$.

Pf f)

$\Omega v_\Lambda = (\Lambda + 2\rho, \Lambda) v_\Lambda$ by e) and since Ω commutes with $U(\mathfrak{g})$ we get $\Omega(E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N} v_\Lambda v_\lambda) = (\Lambda + 2\rho, \Lambda) E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N} v_\Lambda$

Pf g)

follows from f) and e).

Pf h)

If λ is singular weight, then $(\lambda + 2\rho, \lambda) = (\Lambda + 2\rho, \Lambda)$ by g). This describes a compact set in which the singular weights must lie. But $\lambda \in D(\Lambda)$, a discrete set. As the intersection of a discrete set and compact set is finite, we have that the singular weights must be finite in number.

A **Verma module** $M(\Lambda)$ is highest weight module with highest weight Λ such that any other module with highest weight Λ is quotient of $M(\Lambda)$. We construct $M(\Lambda)$ as $U(\mathfrak{g})/U(\mathfrak{g})(\mathfrak{n}_+; h - \Lambda(h), h \in \mathfrak{h})$

By Theorem 1 d), $M(\Lambda)$ has unique maximum submodule $J(\Lambda)$ such that $L(\Lambda) = M(\Lambda)/J(\Lambda)$ is unique highest weight module with highest weight Λ.

Theorem 1.3. *(a) For any $\Lambda \in \mathfrak{h}*$, there exists a Verma module $M(\Lambda)$, unique up to isomorphism.*

(b) $M(\Lambda)$ has unique irreducible quotient $L(\Lambda)$

(c) $M(\Lambda)] = M(\Lambda')$ (resp. $L(\Lambda) = L(\Lambda')$) iff $\Lambda = \Lambda'$

(d) $E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N}^{m_N} v_\lambda$ form basis of $M(\Lambda)$.

Proof: a), b), c) are clear. d) follows from the PBW theorem because $E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N}^{m_N}$ never lies in $J(\Lambda)$.

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Lecture 25 —Dimensions and Characters of Semisimple Lie Algebras

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Let $\mathfrak{g}$ be as in the last lecture - finite dimesional semisimple lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra, and $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta_+ \subset \Delta$, as before, a system of simple roots. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$, $\mathfrak{b} = \mathfrak{h}_+ + \mathfrak{n}_+$, with $\mathfrak{b}$ - a Borel subalgebra, and $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}_+$. Let $(\cdot, \cdot)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, let $\rho = \frac{1}{2}\Sigma_{\alpha\in\Delta_+}\alpha$. Let $\{E_i, H_i, F_i\}$ be the Chevalley generators satisfying $H_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i,\alpha_i)}$, $E_i \in \mathfrak{g}_\alpha$, $F_i \in \mathfrak{g}_{-\alpha}$ and such that $< E_i, H_i, F_i >$ form the standard basis of $\mathrm{sl}_2(\mathbb{F})$. Define the subset $P_+ = \{\lambda \in \mathfrak{h}^* | \lambda(H_i) \in \mathbb{Z}_+ \text{ for all } i = 1, \ldots r\}$

Theorem 25.1. *(Cartan) The $\mathfrak{g}$-modules $\{L(\Lambda)\}_{\Lambda\in P_+}$ are, up to isomorphism, all irreducible finite-dimensional $\mathfrak{g}$-modules. (Recall from previous lectures that $L(\Lambda)$ is the irreducible heigest weight module with heighest weight λ.)*

Theorem 25.2. *(H.Weyl Dimension formula) If $\Lambda \in P_+$, then* $\dim L(\Lambda) = \prod\limits_{\alpha\in\Delta_+} \frac{(\Lambda+\rho,\alpha)}{(\rho,\alpha)}$

Example 25.1. $\mathfrak{g} = sl_2(\mathbb{F}) =< E, F, H >$. Then all Verma modules $M(\Lambda)$, where $\Lambda \in \mathfrak{h}^* = \mathbb{F}$ since $\mathfrak{h}^* = \mathbb{F}H$, have basis $F^J v_\Lambda, j \in \mathbb{Z}_+$. By the key sl_2 lemma, $M(\Lambda)$ is irreducible unless $\Lambda = \Lambda(H) \in \mathbb{Z}_+$ In the latter case (by the same lemma) $EF^{\Lambda+1}v_\Lambda = 0$, hence $F^{\Lambda+1}v_\Lambda$ is a singular vector. So $M(\Lambda)$ is not irreducible. But $L(\Lambda) = M(\Lambda)/\mathcal{U}(sl_2(\mathbb{F})F^{\Lambda+1}v_\Lambda$ is irreducible since $F^j v_\Lambda$, $0 \le j \le m = \Lambda(H)$ are independent. So by fundamental sl_2 lemma, $\dim(L(\Lambda)) = m + 1$

Proof of Theorem 25.1. Recall that $\sigma_i =< E_i, F_i, G_i >\simeq sl_2(F)$ and $v_\Lambda \in L(\Lambda)$ satisfies $E_i(v_\Lambda) = 0, H_i(v_\Lambda) = \Lambda(H_i)v_\Lambda$. Hence by fundamental sl_2 lemma, $\Lambda(H_i) \in Z_+$. So $\dim L(\Lambda) < \infty \Rightarrow \Lambda \in P_+$. Coversely, if $\Lambda \in P_+$, then by Theorem 2, $\dim(\Lambda) < \infty$. □

Lemma 25.1. *Recall $\rho = \frac{1}{2}\sum\limits_{\alpha\in\Delta_+}\alpha$. Then $\rho(H_i) = 1$*

Proof. Consider the reflection from the Weyl group corresponding to α_i:

$$r_{\alpha_i}\rho = r_{\alpha_i}(\frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha\in\Delta\setminus\{\alpha_i\}}\alpha) = \rho - \alpha_i$$

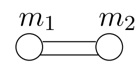
But $r_{\alpha_i}(\lambda) = \lambda - \lambda(H_i)\alpha_i$ hence $\rho(H_i) = 1$. □

Example 25.2. $\mathfrak{g} = \mathrm{sl}_2$, Then $\Lambda(H) = m$ means that $\Lambda = m\rho$. So

$$\dim L(\Lambda) = \frac{(m+1)(\alpha,\alpha)}{(\alpha,\alpha)} = m + 1$$

Example 25.3. $\mathfrak{g} = \mathrm{sl}_3$. We have $\Pi = \{\alpha_1, \alpha_2\}$, $(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\rho = \alpha_1 + \alpha_2$, $\rho(\alpha_i) = 1$, $\Lambda = m_1\Lambda_1 + m_2\Lambda_2$, where $(\Lambda_i, \alpha_j) = \delta_{ij}$. By Cartan's theorem, $\dim L(\Lambda) < \infty$ iff $m_1, m_2 \in \mathbb{Z}_+$. We compute $(\Lambda+\rho,\alpha_1)=m_1+1$, $(\Lambda+\rho,\alpha_2)=m_2+1$, and $(\Lambda+\rho,\alpha_1+\alpha_2)=m_1+m_2+1$, so we have

$$\mathrm{dim}L(\Lambda)=\frac{(m_1+1)(m_2+1)(m_1+m_2+1)}{2}$$

In general, we may write $\Lambda=\Sigma_i k_i\Lambda_i$, where $\Lambda_i(H_j)=\delta_{ij}$. Then $\mathrm{dim}L(\Lambda)<\infty$ iff $k_i\in\mathbb{Z}_+$. These k_i are called *labels* of the heighest weight. They are depicted on the Dynkin diagram: 

We'll deduce Weyl's Dimensional formula from the Weyl character formula

Definition 25.1. Let M be a $\mathfrak{g}$-module which is $\mathfrak{h}$-diagonalizable, let $M=\bigoplus\limits_{\lambda\in\mathfrak{h}^*}M_\lambda$, then

$$ch(M)=\sum_\lambda(\mathrm{dim}M_\lambda)e^\lambda$$

Here $e^\lambda e^\mu=e^{\lambda+\mu}, e^0=1$

Theorem 25.3. *(Weyl Character Formula) Let* $R=\prod\limits_{\alpha\in\Delta_+}(1-e^{-\alpha})$. *if* $\Lambda\in P_+$ *then*

$$e^\rho R chL(\Lambda)=\sum_{w\in W}(\mathrm{det}w)e^{w(\Lambda+\rho)} \qquad (*)$$

We'll first derive Theorem2 from Theorem3, then prove Theorem3.
Given $\mu\in h^*$, consider the following linear map from linear comibnation of the e^λ to functions of t that $F_\mu(e^\lambda)=e^{t(\lambda,\mu)}$.Apply F_μ to both sides of (*), we have

$$\begin{aligned} e^{t(\rho,\rho)}\prod(1-e^{-t(\rho,\alpha)})F_\rho chL(\Lambda) &= \sum_{w\in W}(\mathrm{det}w)e^{t(w(\Lambda+\rho),w^{-1}(\rho))} \\ &= \sum_{w\in W}(\mathrm{det}w)e^{t(w(\Lambda+\rho),w(\rho))} \\ &= F_{\Lambda+\rho}\sum_{w\in W}(\mathrm{det}w)e^{w(\rho)} \end{aligned}$$

Now we can value the sum.Note that $L(0)$ is the trivial 1-dim $\mathfrak{g}$-module, hence $chL(0)=1$. Therefore (*) implies

$$e^\rho R=\sum_{w\in W}(\mathrm{det}w)e^{w(\rho)} \qquad (**)$$

The above equality becomes

$$e^{t(\rho,\rho)}(F_\rho ch(L(\Lambda)))=\frac{F_{\Lambda+\rho}e^\rho R}{\prod\limits_{\alpha\in\Delta_+}(1-e^{-t(\rho,\alpha)})}=\prod_{\alpha\in\Delta_+}\frac{(1-e^{-t(\Lambda+\rho,\alpha)})}{(1-e^{-t(\rho,\alpha)})}$$

As $t\to 0$, $e^{t(\rho,\rho)}\to 1$. hence

$$LHS=F_\rho ch(L(\Lambda))=\sum_\lambda \mathrm{dim}L(\Lambda)e^{t(\rho,\Lambda)}=\mathrm{dim}L(\Lambda)$$

And by L'Hospitals rule,

$$\lim_{t\to 0}RHS=\lim_{t\to 0}\prod_{\alpha\in\Delta_+}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}\frac{e^{-t(\lambda+\rho,\alpha)}}{e^{-t(\rho,\alpha)}}=RHS \text{ of Weyl dimension fomula}$$

Proof of the Weyl's character formula. □

Lemma 25.2. *If $\Lambda(H_i) \in \mathbb{Z}_+$, then $chL(\Lambda)$ is r_i-invariant.*

Proof. By the key sl_2 lemma, $E_i F_i^{\Lambda(H_i)+1} v_\Lambda = 0$, $E_j F_i^{\Lambda(H_i)+1} v_\Lambda = 0$ for $j \neq i$ since E_j and F_i commute. So $F_i^{\Lambda(H_i)+1} v_\Lambda$ is a singular vector of $L(\Lambda)$ and since $L(\Lambda)$ is irreducible, so it has no singular weights other than Λ, we conclude that $F_i^{\Lambda(H_i)+1} v_\Lambda = 0$.

But $L(\Lambda) = \mathcal{U}(\mathfrak{g}) v_\Lambda$, hence for $v \in L(\Lambda)$ we conclude that $F_i^N v = 0$ for $N \gg 0$ Also obviously $E_i^N v = 0$ for $N \gg 0$.

It follows that any $v \in L(\Lambda)$ lies in a sl_2-invariant finite dimensional subspace. Hence by Weyl's complete reducibility theorem $L(\Lambda)$ is a direct sum of irreducible sl_2-modules. So it suffices to prove that the character of a finite dimensional irreducible sl_2-modules is r_{α_i}-invariant. Note that

$$chL(m\rho) = e^{m\rho} + e^{(m-2)\rho} + \ldots + e^{-m\rho}$$

Since $r_{\alpha_i}(\rho) = \rho - r_{\alpha_i}$, the character is r_{α_i}-invariant. Hence the lemma holds for $L(\Lambda)$ as well. □

Lemma 25.3. $R\mathrm{ch}M(\Lambda) = e^\Lambda$

Proof. We know from last lecture that vectors $E_{-\beta_1}^{m_1} \ldots E_{-\beta_N}^{m_N}$ form a basis of $M(\Lambda)$. Hence

$$\begin{aligned} chM(\Lambda) &= \sum_{(m_1,\ldots m_N)\in\mathbb{Z}_+^N} e^{\Lambda - m_1\beta_1 \ldots - m_N\beta_N} \\ &= e^\Lambda \sum_{(m_1,\ldots m_N)\in\mathbb{Z}_+^N} e^{\Lambda - m_1\beta_1 \ldots - m_N\beta_N} \\ &= \frac{e^\Lambda}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})} \end{aligned} \tag{1}$$

by geometric progression. □

Lemma 25.4. $w(e^\rho R) = (\mathrm{det} w) e^\rho R$ *for any* $w \in W$

Proof. Since W is generated by r_{α_i}, it suffices to check $r_{\alpha_i}(e^\rho R) = -e^\rho R$. Indeed, we can rewrite R as

$$R = (1 - e^{\alpha_i}) \prod_{\alpha\in\Delta_+\backslash\alpha_i} (1 - e^{-\alpha})$$

Note that $\prod_{\alpha\in\Delta_+\backslash\alpha_i} (1 - e^{-\alpha})$ is r_{α_i}-invariant, so by Lemma 1

$$\begin{aligned} r_{\alpha_i}(e^\rho R) &= e^{\rho-\alpha_i}(1 - e^{\alpha_i}) \prod_{\alpha\in\Delta_+\backslash\alpha_i} (1 - e^{-\alpha}) \\ &= e^\rho(e^{-\alpha_i} - 1) \prod_{\alpha\in\Delta_+\backslash\alpha_i} (1 - e^{-\alpha}) \\ &= -e^\rho R \end{aligned} \tag{2}$$

as wanted. □

Lemma 25.5. *Let $\Lambda \in \mathfrak{h}_{\mathbb{R}}^*$ and let V be a highest weight module with highest weight Λ. Let $D(\Lambda) = \{\Lambda - \Sigma k_i\alpha_i, k_i \in \mathbb{Z}_+\}$. Then $chV = \Sigma_{\lambda \in B(\Lambda)} a_\lambda \; chL(\lambda)$, where $B(\Lambda) = \{\lambda \in D(\Lambda) | (\Lambda + \rho, \Lambda + \rho) = (\lambda + \rho, \lambda + \rho)$ and $a_\Lambda = 1, a_\lambda \in \mathbb{Z}_+\}$.*

Proof. Proof is by induction on $\dim V = \sum\limits_{\lambda \in B(\Lambda)} \dim V_\lambda < \infty$ due to Theorem2(h) from last lecture that $|B(\Lambda)| < \infty$ is finite. If $\sum\limits_{\lambda \in B(\Lambda)} = 1$ then Λ is the only singular weight hence by Theorem2(c) from last lecture that $V = L(\Lambda)$ so $chV = chL(\Lambda)$. If there another singular vector $v_\lambda, \lambda \neq \Lambda$, let $U = \mathcal{U}(\mathfrak{g})v_\lambda$ and consider the following exact sequence of $\mathfrak{g}$-modules

$$0 \to U \to V \to V/U \to 0$$

Then $ch(V) = ch(U) + ch(U/V)$, now we apply the induction assumption to each of the terms. □

Lemma 25.6. *In the assumptions of Lemma 5 and $V = L(\Lambda)$ is irreducible, we have $chV = \Sigma_{\lambda \in B(\lambda)} b_\lambda chM_\lambda$, where $b_\Lambda = 1$ and $b_\lambda \in Z$.*

Proof. By Lemma 5 we have for any $\mu \in B(\Lambda)$: $chM(\mu) = \sum\limits_{\lambda \in B(\mu)} a_{\lambda,\mu} chL(\lambda)$. Let $B(\Lambda) = \{\Lambda = \lambda_1, \ldots, \lambda_r\}$. Order them in such a way that $\lambda_i - \lambda_j \notin \{\sum\limits_i k_i\alpha_i | k_i \in \mathbb{Z}_+\}$ if $i > j$. We get a system of equations $chM_{\lambda_j} = \sum\limits_i a_{ij} chL(\lambda_i)$, where $a_{ii} = 1, a_{ij} = 0$ for $i > j$. So the matrix a_{ij} of this system is upper triangular matrix of integers with 1's on the diagonal and so its inverse, which expresses ch$L(\Lambda)$'s in terms of ch$M(\mu)$'s for $\mu \in B(\Lambda)$ is a matrix of integers with ones on the diagonal as well, and we are done. □

Proof. Proof of Theorem 3. With Lemma 6, $chL(\Lambda) = \sum\limits_{\lambda \in B(\lambda)} b_\lambda \mathrm{ch}M(\lambda)$, where $b_\Lambda = 1$. Multiply both sides by $e^\rho R$ we get from Lemma 3 that

$$e^\rho R chL(\Lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda e^{\lambda + \rho} \tag{!}$$

By Lemma 2 $L(\Lambda)$ is W-invariant, hence by Lemma 4, $e^\rho R chL(\Lambda)$ is W-anti-invariant (i.e. multiplied by the determinant). Hence the left hand side of the equation is anti-invariant, and therefore so is the right hand side. Hence using any simple transitivity of W on weyl chambers we can rewrite (!) as follows:

$$e^\rho R chL(\Lambda) = \sum_{w \in W} (\det w) e^{w(\Lambda + \rho)} + \sum_{\lambda \in B(\Lambda) \setminus \{\Lambda\}, \lambda + \rho \in P_+} b_\lambda \sum_{w \in W} (\det w) e^{w(\lambda + \rho)}$$

So it remains to show that the second term in this sum is 0, i.e. we need to prove that $\{\lambda \in B(\Lambda) \setminus \{\Lambda\}$ s.t. $\lambda + \rho \in P_+\} = \varnothing$. Note that $\lambda \in B(\Lambda)$ is of the form $\Lambda - \sum\limits_i k_i\alpha_i, k_i \in Z_+$. Since $B(\Lambda) \subset D(\Lambda)$ and also $(\lambda + \rho, \lambda + \rho) = (\Lambda + \rho, \Lambda + \rho)$. Hence

$$\begin{aligned} 0 &= (\Lambda + \rho, \Lambda + \rho) - (\lambda + \rho, \lambda + \rho) \\ &= (\Lambda - \lambda, \lambda + \Lambda + 2\rho) \\ &= (\sum_i k_i\alpha_i, \Lambda) + (\sum_i k_i\alpha_i, \lambda) + 2(\sum_i k_i\alpha_i, \rho) \end{aligned}$$

since $(\Lambda, \alpha_i) = \frac{2\Lambda(H_i)}{(\alpha_i, \alpha_i)} \geq 0$ and similarly $(\lambda + \rho, \alpha_i) \geq 0$, and $(\rho, \alpha) = (\alpha_i, \alpha_i)/2 > 0$ This gives a contradiction so we complete the proof. $\square$