KAN Tutorial Slides

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Kolmogorov-Arnold Representation Theorem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $f : \Omega \to \mathbb{R}$ be a continuous function; i.e. $f \in C(\Omega)$.

Then there exist continuous univariate functions

$$\Phi_q : \mathbb{R} \to \mathbb{R}, \quad q = 1, \dots, 2d+1;$$

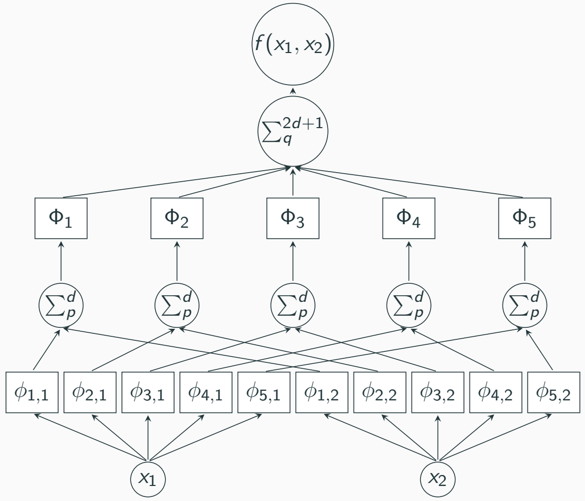
and continuous univariate functions

$$\phi_{pq} : \mathbb{R} \to \mathbb{R}, \quad p = 1, \dots, d; \quad q = 1, \dots, 2d+1;$$

such that for every $\mathbf{x} = (x_1, \dots, x_d) \in \Omega$,

$$f(\mathbf{x}) = \sum_{q=1}^{2d+1} \Phi_q \left(\sum_{p=1}^{d} \phi_{pq}(x_p) \right).$$

Kolmogorov–Arnold Representation Theorem

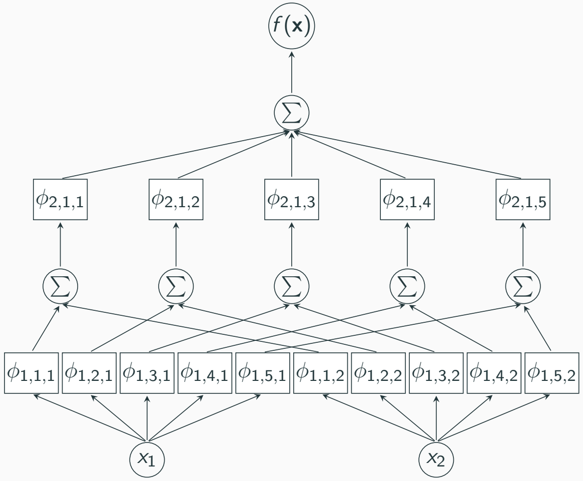


The theorem states that any $f(x_1, x_2)$ can be written as a sum of univariate compositions.

The diagram shows this expression visually: each block represents a component of the decomposition.

Together, they form a **Kolmogorov–Arnold Network (KAN)**.

Kolmogorov–Arnold Networks

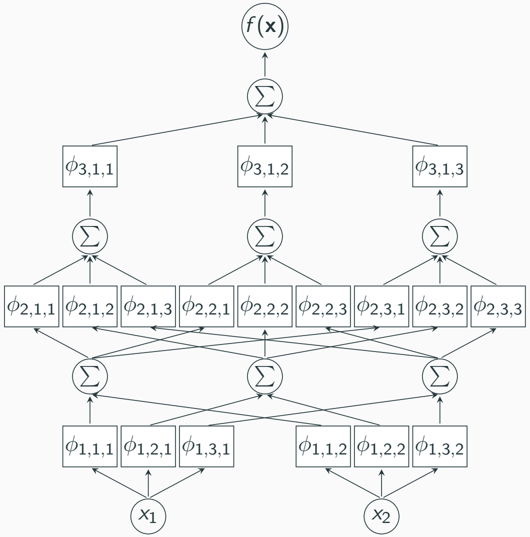


In a network setting, each univariate function is written as $\phi_{l,p,q}$, where:

- l: layer depth
- p: output node index
- q: input node index

This network is a **KAN [2,5,1]**: it has 2 inputs, one hidden layer with 5 nodes, and 1 output.

Kolmogorov–Arnold Networks



$f(\mathbf{x})$
Σ
$\phi_{3,1,1}$ $\phi_{3,1,2}$ $\phi_{3,1,3}$
Σ Σ Σ
$\phi_{2,1,1}$ $\phi_{2,1,2}$ $\phi_{2,1,3}$ $\phi_{2,2,1}$ $\phi_{2,2,2}$ $\phi_{2,2,3}$ $\phi_{2,3,1}$ $\phi_{2,3,2}$ $\phi_{2,3,3}$
Σ Σ Σ
$\phi_{1,1,1}$ $\phi_{1,2,1}$ $\phi_{1,3,1}$ $\phi_{1,1,2}$ $\phi_{1,2,2}$ $\phi_{1,3,2}$
x_1 x_2

KAN [2,3,3,1] – two inputs, two hidden layers of 3, one output.

Why go deeper?

- **Theory:** Any continuous f admits a shallow KAN $[n,\ 2n+1,\ 1]$.
- **Practice:** Deeper KANs can model non-continuous functions. Depth improves expressivity.

B-Splines

$\phi_{l,p,q}$ can be chosen from any family of continuous univariate functions. A common choice is the **B-spline** family.

A B-spline of degree k is defined as:

$$B_k(x) = \sum_{i=1}^{n-k-1} P_i N_{i,k}(x)$$

where n is the number of control points (length of the knot vector),
$N_{i,k}$ are the basis functions of degree k,
and P_i are the basis function weights.

B-Splines

The basis functions follow the standard **Cox–de Boor recursive definition**:

$$N_{i,0}(x) = \begin{cases} 1, & t_i \leq x < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(x) = \frac{x - t_i}{t_{i+k} - t_i} N_{i,k-1}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(x), \quad k > 0$$

where $t_i \in [t_1, t_n]$ is the **knot vector**, a non-decreasing sequence of real numbers.

B-Splines as KAN Edges

All univariate functions share the same spline degree k and knot vector length n. Each $\phi_{l,p,q}$ combines a basis function (similar to residual connections) with a B-spline expansion:

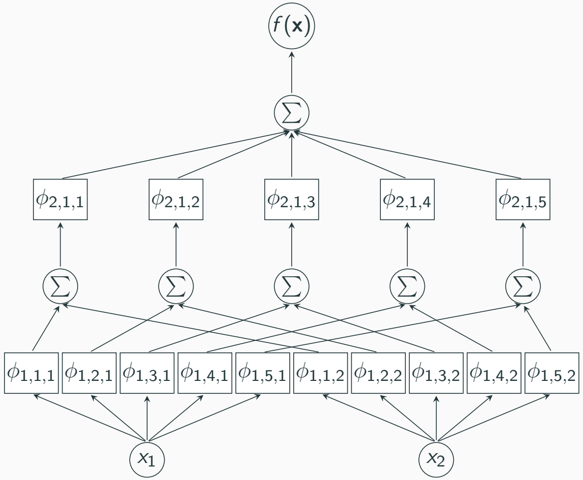
$$\phi(x) = w_b\, b(x) + \sum_{i=1}^{n-k-1} P_i\, N_{i,k}(x)$$

Here, w_b is the learnable weight of the basis function, and the spline coefficients P_i scale the individual B-spline functions directly.

We choose the basis as:

$$b(x) = \mathrm{SiLU}(x) = \frac{x}{1+e^{-x}}$$

KAN Parameters



$f(\mathbf{x})$
$\sum$
$\phi_{2,1,1}$ $\phi_{2,1,2}$ $\phi_{2,1,3}$ $\phi_{2,1,4}$ $\phi_{2,1,5}$
$\phi_{1,1,1}$ $\phi_{1,2,1}$ $\phi_{1,3,1}$ $\phi_{1,4,1}$ $\phi_{1,5,1}$ $\phi_{1,1,2}$ $\phi_{1,2,2}$ $\phi_{1,3,2}$ $\phi_{1,4,2}$ $\phi_{1,5,2}$
x_1
x_2

Hyperparameters

- n: number of control points.
- k: B-spline degree.

Learnable parameters

(for each edge)

- t_i: knot vectors, $i \in [1, n]$.
- P_i: B-spline weights, $i \in [1, n - k - 1]$.
- w_b: basis weight.

Capabilities of KANs with B-Splines

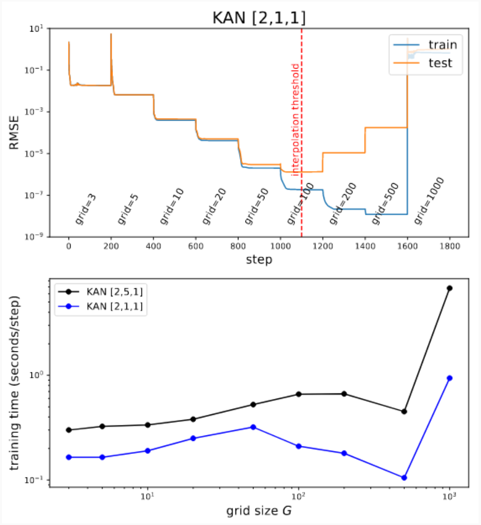
- **Grid extension**: progressively increase model capacity by refining the spline grid without retraining from scratch.
- **Continual learning**: local support ensures new information affects only nearby regions, reducing catastrophic forgetting.
- **Sparsity**: regularization and pruning remove redundant components, simplifying the model without major accuracy loss.
- **Symbolic regression**: univariate structure enables conversion of learned functions into interpretable closed-form expressions.

Grid Extension

Grid extension refines a trained KAN by adding more spline knots without restarting training.

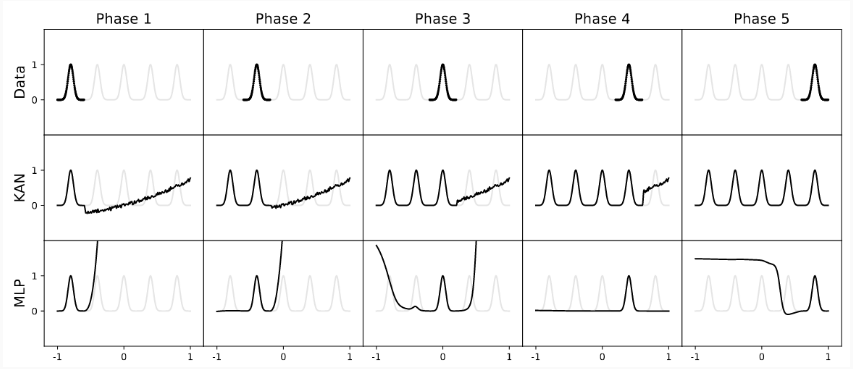
- Train on a coarse grid first.
- Add knots to increase resolution and capacity.
- Initialize new coefficients by least-squares fitting.
- Continue training to improve accuracy.

Test loss often improves until the parameter count roughly matches the number of data points.



Continual Learning

Because B-splines have **local support**, updates to $\phi(x)$ in one region of the input space affect only nearby points. This locality mitigates **catastrophic forgetting**, a common issue in MLPs where learning new data can overwrite previously acquired knowledge.

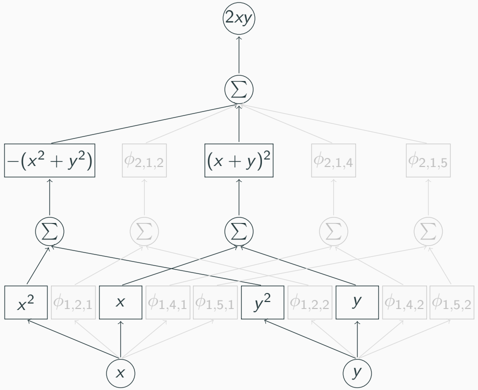


Sparsity

Sparsity removes unnecessary components, revealing the essential structure of our target function.

- **Regularization** drives many spline weights toward zero.
- Irrelevant $\phi_{l,p,q}$ can be pruned after training.
- The result is a compact, interpretable network.

Sparsity helps towards **interpretability**.



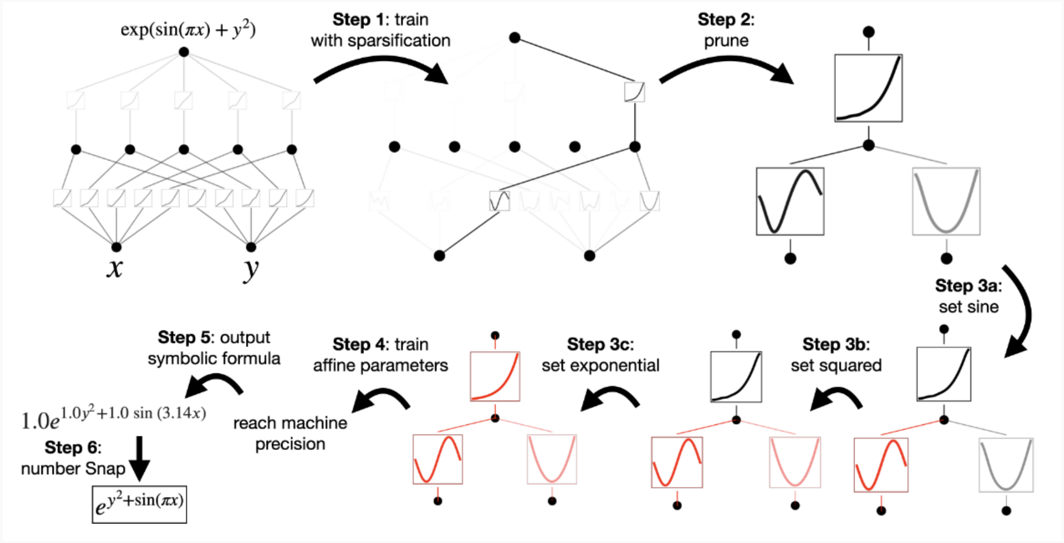
Symbolic Regression

KANs provide an interpretable path from neural models to closed-form expressions:

- Each learned $\phi_{l,p,q}$ is a univariate function, which can often be approximated by simple analytic forms (e.g., sin, exp, log).
- After training, these functions are "snapped" to symbolic templates via affine fitting, producing human-readable equations.
- The resulting network can be viewed as a composition graph of symbolic functions approximating $f(x)$.

This makes KANs suitable not only for prediction but also for **discovering interpretable laws** from data.

KAN Train Steps



$\exp(\sin(\pi x) + y^2)$
x
y
Step 1: train with sparsification
Step 2: prune
Step 3a: set sine
Step 3b: set squared
Step 3c: set exponential
Step 4: train affine parameters
reach machine precision
Step 5: output symbolic formula
$1.0e^{1.0y^2+1.0\sin(3.14x)}$
Step 6: number Snap
$\boxed{e^{y^2+\sin(\pi x)}}$