Schrödinger's Equation and Quantum Mechanics

Introduction: The Need for New Physics

- Prior to 1926, classical physics failed to explain behaviors at atomic scales.
- Schrödinger introduced a revolutionary equation replacing Newton's Second Law for quantum particles like electrons.
- His wave equation resembles classical heat and wave equations but includes the imaginary unit

$$i$$

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De Broglie Matter Waves

- In 1924, Louis de Broglie proposed matter behaves like waves, extending Einstein and Planck's photon energy relation

$$E = hf$$

to matter.
- Schrödinger was inspired by de Broglie's thesis, reinterpreting particles as wave phenomena.

Classical Wave Equation Basis

- Schrödinger started with the classical one-dimensional wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

- Solutions separate into spatial and temporal components, leading to sine and cosine functions with fixed frequencies determined by boundary conditions.
- Similar to vibrating strings, allowed modes are discrete, hinting at quantization in atom spectra.

Incorporation of Quantum Concepts

- Using de Broglie's relation linking wavelength to momentum

$$\lambda = \frac{h}{p}$$

, Schrödinger connected classical constants to kinetic and potential energy terms.
- Extended to three dimensions by including derivatives over

$$x, y, z$$

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Quantized Energy Solutions

- Schrödinger solved the wave equation for hydrogen, showing energy levels matched observed emission spectra.
- Energy quantization emerges naturally from wave boundary conditions.

Introduction of Imaginary Unit

$$i$$

- Time-dependent Schrödinger equation involves the imaginary unit multiplying the time derivative:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

- $$\Psi$$

  (wave function) must be complex-valued for this to hold.
- Using complex exponentials simplifies calculus and provides an elegant framework for evolving wave functions.

Physical Interpretation of the Wave Function

- Schrödinger's equation is linear, enabling superposition:
  - Summations of valid solutions are themselves valid solutions.
  - This explains interference patterns such as in the double-slit experiment.
- The amplitude

$$|\Psi|^2$$

  gives the probability density (Born rule).
- The phase (angle of the complex number) encodes interference information critical to wave behavior.

Numerical and Visual Examples

Step	Wave Function Value Ψ	Spatial 2nd Derivative $\frac{\partial^2 \Psi}{\partial x^2}$	Change Rate $\frac{\partial \Psi}{\partial t}$	Interpretation
Initial	1 + 0i	-2	$i \times 0.1 \times (-2) = -0.2i$	Rotation by 90° on complex plane
Time 1	1 - 0.2i	Update similarly	Update similarly	Wave function evolves circularly

- Wave packets illustrate spreading and interference in space and time.

Significance of Complex Numbers in Quantum Physics

- Although initially controversial, use of

$$i$$

  is essential for describing time evolution properly.
- Complex wave functions elegantly represent probabilistic amplitudes, capturing both magnitude and phase.
- Provide a compact mathematical framework to handle quantum interference effects.

Double-Slit Experiment Explained

- Electrons show interference patterns, contradicting classical particle predictions.
- Wave function representations show superposition and phase differences causing destructive and constructive interference.

Legacy and Philosophical Insights

- Schrödinger's equation bridges classical waves and quantum particles.
- Imaginary numbers were once considered 'impossible' but now are fundamental to physics.
- The wave function phase remains subtle but underpins fundamental quantum phenomena.

Summary

Schrödinger's groundbreaking wave equation, incorporating imaginary numbers, revolutionized physics by describing matter as complex-valued waves whose amplitudes yield probabilities and phases govern interference. Starting from classical wave analogies and building on de Broglie's insights, Schrödinger formulated a mathematical framework accurately describing atomic behavior and quantum phenomena like the double-slit interference. The introduction of the imaginary unit

$$i$$

enables elegant time-dependent evolution of wave functions, marking a profound shift from classical to quantum descriptions of nature. This formulation remains central to quantum mechanics and our understanding of the microscopic world.

Summary

- Schrödinger's equation replaced classical particle laws with wave descriptions including the imaginary unit

  $$i$$

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- Matter waves, inspired by de Broglie's hypothesis, describe quantized energy levels and interference.
- Complex-valued wave functions encode both probability amplitude and important phase information.
- The equation explains key quantum effects such as discrete atomic spectra and double-slit interference.
- Imaginary numbers provide an elegant and necessary mathematical tool for time evolution and interference in quantum mechanics.
- This development marked a fundamental paradigm shift in physics, showing nature fundamentally works with complex numbers rather than just real quantities.