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Robust H_∞ Static Output Feedback Control for TCP/AQM Routers Based on LMI Optimization

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Abstract: This paper proposes a new H_∞ static output feedback control method to address the congestion control problem in transmission control protocol networks using active queue management routers. Based on linear matrix inequality optimization, this method determines a static output feedback control law to minimize the H_∞ norm of the transfer function between the controlled queue length of the buffer and the exogenous disturbance affecting the available link bandwidth. A linear matrix inequality formulation is presented as a sufficient condition to guarantee the closed-loop system's asymptotic stability while maintaining disturbance rejection within a specified level, regardless of round-trip time delays. The proposed robust static output feedback control eliminates the need to measure or estimate all system states, thus simplifying practical implementation. The effectiveness of the proposed design method is demonstrated by applying it in a practical process, as illustrated through a numerical example.

Keywords: congestion control; TCP/AQM router; static output feedback; LMI; H_∞ control; time delay systems; Lyapunov–Krasovskii stability

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1. Introduction

The congestion control mechanism in TCP (Transmission Control Protocol) is considered one of the most essential issues due to the rapidly increasing demand for performance in internet applications [1–3]. To avoid congestion problems, various control principles for AQM (Active Queue Management) routers have been researched since the dynamics of TCP behavior were developed by stochastic differential equations in [4,5]. Recently, modern robust control theories [6–8] have been applied to AQM-based congestion control as well as classic control theories, such as PI (Proportional-Integral), PID (Proportional-Integral-Derivative) and pole placement [2,9–11].

H_∞ synthesis, especially, provides a mathematically analytic approach to minimize the effect of disturbances acting on the feedback system in view of the H_∞ norm. In this context, the available link capacity changes over time are considered as disturbances [12,13]. Thus, researchers have applied various H_∞ techniques for AQM routers with respect to memoryless state feedback [6,14] and dynamic output feedback control methods [15–17] for infinite dimensional systems with time delays and disturbances.

First, in previous studies such as [6,14], the memoryless state feedback H_∞ techniques were proposed to deal with RTT (Round Trip Time) delay and external disturbances. [6,14] have contributions that are relatively easier to design without high-order compensators and can be analyzed in practice. They also take into account the time-domain H_∞ design method, which can deal with time varying states. This time-domain approach has one of the most effective advantages: one can intuitively consider the disturbance and noise rejection ratio in the view of the H_∞ norm in the time domain. However, these existing full-state feedback methods require that all states be known at every time to form the control input. This may be a very strong requirement, since the queue size of the bottleneck router for the TCP/AQM system is only available in the feedback system [5,15].

On the other hand, based on dynamic output feedback, robust H_∞ compensators do not require the direct measurement of all states. The dynamic output feedback controller can be designed using only the queue length by observer–based methods, such as Kalman filters [17–21], or the dynamic controller, determined from the sensitivity weighting function in the frequency domain [10]. These approaches include the design of the observer or generally have high-order compensators, depending on the weighting function. However, this design complexity makes it difficult to implement in a real network process.

In this paper, the H_∞ design method of a SOF (Static Output Feedback) controller is hence proposed as an alternative controller for the AQM routers with time-varying delays, based on LMI (Linear Matrix Inequality) optimization [22,23]. The proposed SOF synthesis is appropriate to AQM-based congestion control since the control system does not require measuring the RTT and window size [22]. The determined controller guarantees the globally asymptotic stability and the attenuation level of disturbance effects independently on the delay time using the Lyapunov–Krasovskii stability condition [24]. The delay-independent sufficient conditions are derived as LMI frameworks for two cases, depending on whether there is an input delay or not, respectively. We consider the two-term feedback synthesis [24] to include the output and delayed output since it provides two degrees of freedom. The controller is a general approach, enabling the combination the instantaneous effect as well as the delay output, and thus it is flexibly available to the input feedback system only with the delay term, in practice.

The main contributions of the proposed method lie in: (i) the employment of SOF controller design for AQM-based congestion control problems with unknown time delay, in which the controller has the simplicity to implement and does not require measurements of all states; (ii) the robust stability and noise rejection with H_∞ norm bound level from exogenous inputs to controlled outputs, in which the unknown time delay is compensated by using appropriate Lyapunov–Krasovskii functionals; and (iii) the LMI optimization approach, to be able to deal with additional performances based on the relation between inputs and outputs, such as the minimum cost, norm bounded input, polytopic uncertainty, and so on [23].

This paper is organized as follows. In Section 2, the fluid model of the TCP congestion–avoidance algorithm and Lyapunov–Krasovskii stability analysis for unforced systems with time-delay are briefly introduced for the reader's comprehension. Section 3 presents LMI-base sufficient conditions for the H_∞ design of general linear time delay systems controlled by two term inputs. Section 4 reports the simulation results to illustrate the effectiveness of the proposed method in a practical process. Finally, the conclusion of the paper is detailed in Section 5.

2. Problem Statement

In this paper, the dynamic model of the TCP network using fluid-flow differential equation analysis is considered to avoid the congestion, described by the following nonlinear differential equations with input and state delays [4,5]:

$$\begin{cases} \dot{W}(t) = \frac{1}{\tau(t)} - \frac{W(t)}{2}\frac{W(t-\tau(t))}{\tau(t-\tau(t))}p(t-\tau(t)), \\ \dot{q}(t) = \begin{cases} -C(t) + \frac{N(t)}{\tau(t)}W(t), & q(t) > 0, \\ max\left\{0, -C(t) + \frac{N(t)}{\tau(t)}W(t)\right\}, & q(t) = 0, \end{cases} \\ \tau(t) = \frac{q(t)}{C(t)} + T_P, \end{cases} \tag{1}$$

where $W(t)$ is the average TCP window size (packets), $q(t)$ is the average queue length in the router (packets), $\tau(t)$ is the round-trip time (s), $C(t)$ is the available link capacity (packets/s), T_P is the propagation delay (s), $N(t)$ is the load factor (number of TCP sessions), and $0 \leq p(t) \leq 1$ is the probability of packet mark, which is used to reduce the sending rate and to maintain the bottleneck queue length before the buffer causes overflow. The queue length and window size have bounded the buffer capacity $\overline{q}$ and maximum window size $\overline{W}$, i.e., $q \in [0, \overline{q}]$ and $W \in [0, \overline{W}]$.

For the linear control, we follow the linearized model of Equation (1) at a given operating point $(q_0,\ W_0,\ p_0)$, as in Zheng et al. [6], taking $(q,\ W,\ p)$ as the states and the input as follows:

$$\begin{cases} \delta\dot{W}(t) = & -\frac{N}{\tau_0^2 C_0}[\delta W(t) + \delta W(t-\tau_0)] - \frac{1}{\tau_0^2 C_0}[\delta q(t) - \delta q(t-\tau_0)] \\ & -\frac{\tau_0 C_0^2}{2N^2}\delta p(t-\tau_0) + \frac{\tau_0 - T_P}{\tau_0^2 C_0}[\delta C(t) - \delta C(t-\tau_0)], \\ \delta\dot{q}(t) = & \frac{N}{\tau_0}\delta W(t) - \frac{1}{\tau_0}\delta q(t) - \frac{T_P}{\tau_0}\delta C(t). \end{cases} \tag{2}$$

where $\delta W(t) \triangleq W(t) - W_0$, $\delta q(t) \triangleq q(t) - q_0$, and $\delta p(t) \triangleq p(t) - p_0$. $\delta C(t) \triangleq C(t) - C_0$ is considered as a disturbance input to the system.

The linearized model with time delay can be represented as the following general state space representation with time-delay:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d) + Bu(t-h) + D\omega(t) \\ z(t) &= Lx(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3}$$

where $A = \begin{bmatrix} -\frac{N}{\tau_0^2 C_0} & -\frac{1}{\tau_0^2 C_0} \\ \frac{N}{\tau_0} & -\frac{1}{\tau_0} \end{bmatrix}$, $A_d = \begin{bmatrix} -\frac{N}{\tau_0^2 C_0} & \frac{1}{\tau_0^2 C_0} \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -\frac{\tau_0 C_0^2}{2N^2} \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} \frac{\tau_0 - T_P}{\tau_0^2 C_0} & -\frac{\tau_0 - T_P}{\tau_0^2 C_0} \\ -\frac{T_P}{\tau_0} & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $x(t) = \begin{bmatrix} \delta W(t) \\ \delta q(t) \end{bmatrix}$, $u(t) = \delta p(t)$, $\omega(t) = \begin{bmatrix} \delta C(t) \\ \delta C(t-\tau_0) \end{bmatrix}$, and $y(t) = \delta q(t)$.

In this paper, Assumption 1 holds when considering the delay independent stability of the nominal system. We propose a delay-independent robust control method since it is difficult to measure the delay time exactly in a network environment. The delay-dependent stability condition and assumption are introduced in Appendix A for the reader's comprehension.

3. H_∞ Control for Static Output Feedback System

In this section, we focus our attention on H_∞ synthesis for two cases of the general time delay systems with or without the input delay. The static output feedback controller is as follows:

$$u(t) = F_1 y(t) + F_2 y(t-d) \tag{4}$$

which combines the proportional terms of the output and the delayed output obtained from the real-time states and the delayed states, respectively. The objective is to determine the static output feedback gains F_1 and F_2, guaranteeing that the closed-loop system is asymptotically stable and simultaneously satisfies the following:

$$\| z(t) \|_2 < \gamma \| \omega(t) \|_2 \tag{5}$$

This ensures that the H_∞ norm of transfer function $T_{zw}(s)$ will not exceed a given positive constant level γ, where $\omega(t)$ represents the external input noise and $z(t)$ is the controlled output. In this paper, we select $\omega(t)$ to describe the disturbance $C(t)$ to the linearized system (2). For the reader's comprehension and readability, we will introduce the H_∞ control analysis in detail in Appendix B.

As for the H_∞ control, we have three special cases:

Case 1: If there exists no input delay h such that $h = 0$, the closed loop system has the states $x_c(t)$, $x_c(t-d)$, and $\omega(t)$.
Case 2: If there exists a positive input delay h different from the state delay d such that $h > 0$ and $h \neq d$, the closed loop system have the states $x_c(t)$, $x_c(t-d)$, $x_c(t-h)$, $x_c(t-d-h)$ and $\omega(t)$.

Case 3: If there exists a positive input delay h same as the state delay d such that $h > 0$ and $h = d$, the closed loop system have the states $x_c(t)$, $x_c(t-d)$, $x_c(t-2d)$, and $\omega(t)$.

For Case 1, we derive the LMI condition for the robust stability of the controlled system without input delay such that $h = 0$.

Theorem 1. *If there exist symmetric and positive-definite matrices $X > 0$, $Q_1 > 0$ and $Q_T > 0$ such that the following LMIs hold simultaneously:*

$$\begin{bmatrix} AX + XA^T & XC^T & X & XL^T & A_d + BF_2C & D \\ CX & -\sigma & 0 & 0 & 0 & 0 \\ X & 0 & -Q_T & 0 & 0 & 0 \\ LX & 0 & 0 & -I & 0 & 0 \\ (A_d + BF_2C)^T & 0 & 0 & 0 & -Q_1 & 0 \\ D^T & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix} < 0 \tag{6}$$

$$\begin{bmatrix} AX + XA^T & B & X & XL^T & A_d + BF_2C & D \\ B^T & -\sigma & 0 & 0 & 0 & 0 \\ X & 0 & -Q_T & 0 & 0 & 0 \\ LX & 0 & 0 & -I & 0 & 0 \\ (A_d + BF_2C)^T & 0 & 0 & 0 & -Q_1 & 0 \\ D^T & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix} < 0 \tag{7}$$

where $Q_1 \leq \epsilon I$ and $Q_T < \epsilon^{-1} I$, and ϵ, σ and γ are positive scalars, then the controlled system of Case 1 is asymptotically stable with an H_∞ norm less than the specific level γ.

Proof. The closed-loop system of Case 1 becomes by the static output feedback control Equation (3) and the noise $\omega(t)$ as

$$\dot{x}_c(t) = (A + BF_1C)x_c(t) + (A_d + BF_2C)x_c(t-d) + D\omega(t).$$

Independent on the time delay, we define a Lyapunov–Krasovskii functional candidate $V(x_c(t))$ as:

$$V(x_c(t)) = x_c^T(t)Px_c(t) + \int_{t-d}^{t} x_c^T(\tau)Q_1x_c(\tau)d\tau. \tag{8}$$

We define a performance function for disturbance rejection in the view of the H_∞ norm as:

$$J_\infty = \int_0^\infty z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t)dt < 0. \tag{9}$$

Then, we establish the following sufficient condition, under which the controller Equation (3) stabilizes the system and guarantees the H_∞ norm bound γ for the attenuation of the disturbance effect:

$$J_\infty \leq \int_0^\infty z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(x_c(t))dt < 0 \tag{10}$$

where $\dot{V}(x_c(t))$ is the derivative along the trajectory of system (3).

The condition (10) is sufficiently satisfied at $t > 0$ if $z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(x_c(t)) < 0$ holds, which is equivalent to the LMI $x_n^T M x_n < 0$, where $x_n(t) = \begin{bmatrix} x(t) & x(t-d) & \omega(t) \end{bmatrix}^T$ and

$$M = \begin{bmatrix} P(A + BF_1C) + (A + BF_1C)^TP + Q_1 + L^TL & P(A_d + BF_2C) & PD \\ (A_d + BF_2C)^TP & -Q_1 & 0 \\ D^TP & 0 & -\gamma^2 \end{bmatrix}.$$

From the above LMI, we obtain that:

$$\dot{V}(x_c(t)) = \begin{bmatrix} x(t) \\ x(t-d) \end{bmatrix}^T \begin{bmatrix} P(A+BF_1C)+(A+BF_1C)^TP+Q_1 & P(A_d+BF_2C) \\ (A_d+BF_2C)^TP & -Q_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d) \end{bmatrix}$$

is negative, such that the closed loop system is internally stable when $\omega(t) \equiv 0$.

To determine static output feedback gains for H_∞ control design, the above LMI is equivalently obtained as the following ARI form by the Schur complement in the matrix M:

$$\Phi + PBF_1C + C^TF_1^TB^TP < 0 \tag{11}$$

where $\Phi = PA + A^TP + Q_1 + L^TL + P(A_d+BF_2C)Q_1^{-1}(A_d+BF_2C)^TP + \frac{1}{\gamma^2}PDD^TP$.

Eliminating the design parameter F_1 of ARI (11) by Lemma 1 in Appendix A, a solution matrix F_2 exists if and only if:

$$\begin{cases} C_\perp \Phi C_\perp^T < 0 \\ B_\perp^T P^{-1}\Phi P^{-1}B_\perp < 0 \end{cases} \tag{12}$$

where note that $(PB)_\perp = P^{-1}B_\perp$.

Using Lemma (1.2) in Appendix A, Equation (12) is equivalent holds if and only if there exists a scalar σ such that:

$$\begin{cases} \Phi + \sigma^{-1}C^TC < 0 \\ P^{-1}\Phi P^{-1} + \sigma^{-1}BB^T < 0 \end{cases}. \tag{13}$$

Pre and post-multiplying $X = P^{-1}$ to Equation (13), we rewrite it as:

$$\begin{cases} X\Phi X + \sigma^{-1}XC^TCX < 0 \\ X\Phi X + \sigma^{-1}BB^T < 0 \end{cases}. \tag{14}$$

Suppose that $\epsilon I \geq Q_1$ and $\epsilon^{-1}I > Q_T$, $\Phi < \Phi_T < 0$ is satisfied where $\Phi_T = \Phi - Q_1 + Q_T^{-1}$. Then, Equation (14) holds if:

$$\begin{cases} X\Phi_T X + \sigma^{-1}XC^TCX < 0 \\ X\Phi_T X + \sigma^{-1}BB^T < 0 \end{cases}. \tag{15}$$

Using the Schur complement, we find that Equations (6) and (7) hold if and only if Equation (15) is satisfied under Assumption 1. Thus, it is derived that if there exist symmetric, positive-definite matrices $X > 0$, $Q_1 > 0$ and $Q_T > 0$ to satisfy Equations (6) and (7) with respect to positive scalar ϵ, σ and γ, then they meet the asymptotic stability and the performance function (9), yielding H_∞ performance Equation (4). This completes the proof of Theorem 1. □

Remark 1. *Case 1, ϵ is smaller, the feasible set of Q_1 is smaller for F_2 and that of Q_T is larger for X in Equations (6) and (7). It is a kind of waterbed effect. We set the initial ϵ as sufficiently small so that the solution P of Lyapunov–Krasovskii equation has a large set for the stability of the controlled systems.*

Algorithm 1 is the proposed controller design procedure for Case 1 systems.

Algorithm 1: Design algorithm for Case 1

STEP 1. Set the initial positive and sufficiently small ϵ_{int} to have enough large solution set of Equations (6) and (7)
STEP 2. Given positive scalar ϵ_{int}, solve the linear objective function minimization problem
$$min\ \gamma_{int}$$
$$\gamma_{int} > 0,\ \sigma > 0,\ X > 0,\ Q_1 > 0,\ and\ Q_T > 0.$$
subject to Equations (6) and (7) to obtain F_2.
STEP 3. If the problem is infeasible, relax the values of ϵ_{int} to be larger so that the solution set of F_2 becomes larger, and go to STEP 2. If it is feasible, go to STEP 4.
STEP 4. Compute P, the inverse matrix of X.
STEP 5. Solve the linear objective function minimization problem:
$$min\ \gamma_{min}$$
$$\gamma_{min} > 0,\ Q > 0,\ P > 0$$
subject to $\begin{bmatrix} P(A+BF_1C)+(A+BF_1C)^TP+Q+L^TL & P(A_d+BF_2C) & PD \\ (A_d+BF_2C)^TP & -Q & 0 \\ D^TP & 0 & -\gamma_{min}^2 \end{bmatrix} < 0$
to obtain F_1 with the factors P and F_2 obtained in STEP 2 and 4, respectively.
Obtained in STEP 2, P and F_2 guarantee the controlled systems with the H_∞ norm bounded by γ_{int} regardless the control factor F_1. We developed STEP 5 to determine F_1 to guarantee a better noise rejection ratio of the closed-loop system from the new variable γ_{min}, Q and P.

For Case 2, we now consider the system with positive input delay $h > 0$ and $h \neq d$ where the input $u(t)$ is the same as Equation (3), which consists of the delay and proportional terms.

Theorem 2. *If there exist symmetric and positive-definite matrices $X > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$ and $Q_T > 0$ such that the following LMIs hold simultaneously:*

$$\begin{bmatrix} AX+XA^T & X & XL^T & A_d & BF_1C & BF_2C & D \\ X & -Q_T & 0 & 0 & 0 & 0 & 0 \\ LX & 0 & -I & 0 & 0 & 0 & 0 \\ A_d^T & 0 & 0 & -Q_1 & 0 & 0 & 0 \\ C^TF_1^TB^T & 0 & 0 & 0 & -Q_2 & 0 & 0 \\ C^TF_2^TB^T & 0 & 0 & 0 & 0 & -Q_3 & 0 \\ D^T & 0 & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix} < 0 \tag{16}$$

and $\sum_{i=1}^{3} Q_i \leq \epsilon I,\ Q_T \leq \epsilon^{-1} I$ where ϵ and γ are positive scalar, then the controlled system of Case 2 is asymptotically stable with the H_∞ norm less than the specific level γ.

Proof. The closed-loop system of Case 2 is derived from the static output feedback control Equation (3) and the noise $\omega(t)$ as follows:

$$\dot{x}_c(t) = Ax_c(t) + A_dx_c(t-d) + BF_1Cx_c(t-h) + BF_2Cx_c(t-d-h) + D\omega(t)$$

Independent of the time delay, define a Lyapunov–Krasovskii functional candidate

$$\begin{aligned} V(x_c(t)) &= x_c^T(t)Px_c(t) + \int_{t-d}^{t} x_c^T(\tau)Q_1x_c(\tau)d\tau \\ &+ \int_{t-h}^{t} x_c^T(\tau)Q_2x_c(\tau)d\tau + \int_{t-d-h}^{t} x_c^T(\tau)Q_3x_c(\tau)d\tau \end{aligned} \tag{17}$$

in order to guarantee the H_∞ norm bound γ for the attenuation of the disturbance effect, the condition (10) is sufficiently satisfied at $t > 0$ if $z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(x_c(t)) < 0$

holds, which is equivalent to the LMI $x_z^T M_z x_z < 0$, where $x_Z(t) = x(t)\ x(t-d)\ x(t-h)\ x(t-d-h)\ \omega(t)\ ^T$ and

$$M_z = \begin{bmatrix} PA + A^T P + \sum_{i=1}^{3} Q_i + L^T L & PA_d & PB_h F_1 C & PB_h F_2 C & PD \\ (PA_d)^T & -Q_1 & 0 & 0 & 0 \\ (PB_h F_1 C)^T & 0 & -Q_2 & 0 & 0 \\ (PB_h F_2 C)^T & 0 & 0 & -Q_3 & 0 \\ D^T P & 0 & 0 & 0 & -\gamma^2 \end{bmatrix} \tag{18}$$

From the above LMI, we obtain that the derivative $\dot{V}(x_c(t))$ along the trajectory of system (3) is negative, such that the closed loop system is internally stable when $\omega(t) \equiv 0$.

The above LMI $M_z < 0$ is equivalent to the following ARI form by the Schur complement in the matrix M_z:

$$\begin{aligned} PA + A^T P + & \sum_{i=1}^{3} Q_i + L^T L + PA_d Q_1^{-1} A_d{}^T P + PBF_1 C Q_2^{-1} C^T F_1^T B^T P \\ & + PBF_2 C Q_3^{-1} C^T F_2^T B^T P + \frac{1}{\gamma^2} PDD^T P < 0 \end{aligned} \tag{19}$$

Assuming $\sum_{i=1}^{3} Q_i \leq \epsilon I < Q_T^{-1}$, pre and post-multiplying the condition (19) by $X = P^{-1}$, we can rewrite Equation (19) as the following sufficient condition:

$$\begin{aligned} AX + XA^T + & XQ_T^{-1} X + XL^T LX + A_d Q_1^{-1} A_d^T + BF_1 C Q_2^{-1} C^T F_1^T B^T \\ & + BF_2 C Q_3^{-1} C^T F_2^T B^T + \tfrac{1}{\gamma^2} DD^T < 0 \end{aligned} \tag{20}$$

Given $\epsilon > 0$, It can be rearranged equivalently as the LMI (17) form by Schur complement.

For Case 3, if the input delay h is the same as the state delay d, the LMI condition of theorem 2 can be simplified as:

$$\begin{bmatrix} AX + XA^T & X & XL^T & A_d + BF_1 C & BF_2 C & D \\ X & -Q_T & 0 & 0 & 0 & 0 \\ LX & 0 & -I & 0 & 0 & 0 \\ A_d^T + C^T F_1^T B^T & 0 & 0 & -Q_1 & 0 & 0 \\ C^T F_2^T B^T & 0 & 0 & 0 & -Q_2 & 0 \\ D^T & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix} < 0 \tag{21}$$

and $\sum_{i=1}^{2} Q_i \leq \epsilon I,\ Q_T \leq \epsilon^{-1} I$. □

Remark 2. *The validity of Theorem 2 remains unchanged regardless of whether $h < d$ or $h > d$. Since the Lyapunov–Krasovskii functional candidate $V(x_c(t))$ does not change with respect to the delay time h if $h \neq d$, the LMI condition Equation (16) holds regardless of the input delay time $h > 0$, which differs from d.*

Algorithm 2 is the proposed controller design procedure for Case 2 and 3 systems.

Algorithm 2: Design algorithm for Case 2 and Case 3

STEP 1. STEP 1. Set the initial positive and sufficiently small ϵ_{int} to have enough solution set of Equations (6) and (7).
STEP 2. Solve the linear objective function minimization problem

$$\min_{\gamma > 0,\ P > 0, Q_i > 0, Q_T > 0} \gamma$$

subject to Equation (16) (for Case 3, Equation (21)), $\sum_{i=1}^{3} Q_i \leq \epsilon_{int} I$ and $Q_T \leq \epsilon_{int}^{-1} I$ to obtain F_1 and F_2, given positive ϵ_{int}.
STEP 3. If the problem is infeasible, relax the values of ϵ_{int} to be larger and go to STEP 2. If it is feasible, stop the procedure.

In Theorem 1 and 2, the stability of the closed-loop system and the disturbance rejection level γ in view of H_∞ synthesis are guaranteed independently of the state and input time delays. This is an important feature of the proposed method for congestion control because the performance and stability are guaranteed, regardless of how large round-trip delay time occurs. The proposed design algorithm to the AQM routers for the congestion control of TCP networks is implicated in the next section, based on the dynamic model of TCP.

4. Experimental Studies

In this paper, the same example in paper [6] is simulated in order to show the effectiveness of the proposed method. The paper [6] developed the static state feedback controller to minimize the H_∞ norm of the transfer function with time delays from the exogenous input to the desired output, based on the time domain approach. Hence, it is more reasonable to compare the performances of the proposed static output feedback H_∞ synthesis with those of the memoryless state feedback H_∞ synthesis in [6] than other dynamic output feedback control methods [10,14–21]. The proposed method was also compared with RED (Random Early Detection), which is one of the most popular existing control methods for AQM routers [3].

Now consider the following TCP dynamics in terms of network parameters when a load of $N = 1200$ TCP flows, $T_P = 0.2$ s, and $C_o = 25,000$ packets/s such that the link bandwidth is 100 Mb/s if the packet size is 500 Bytes. The operating points are fixed as $q_o = 500$ packets; $W_o = 5$ packets; $p_o = 0.0952$. Other parameters are set to be $\overline{q} = 800$ packets, $\overline{W} = 20$ and $T_p = (200 + 40\xi)$ ms, where ξ is a random variable uniformly distributed over the interval $[-0.5, 0.5]$. In the simulation, the initial window size and initial queue length of the router are chosen to be zero. In all the simulations, the same disturbance profile on the available link bandwidth is used for design methods.

The proposed static output feedback controller was designed based on the same network environment as reference [10] and the linearized model Equation (2). The performance of the controller was evaluated in MATLAB/Simulink based on the nonlinear differential equation model (1) of AQM routers, which is a more fundamental model. This equation-based simulator provides graphical and numerical results to verify the effectiveness of the proposed control method and to design the controller for the systems.

The performance and effectiveness of the proposed control method were verified additionally, using NS-2 with the dumbbell network topology, which is one of the most general packet-based network simulators. All parameters for NS-2 experiments are the same as mentioned above. Through NS-2 experimental results, not only the performance of the proposed controller but also differences in operational characteristics depending on the simulators can be checked. Since these simulation results reflect the applicability of the proposed control method, we can more practically estimate the performance of the AQM router.

A. Matlab/Simulink

Associating Case (2) with the input delay studied in Section 3, the state-space model equation is calculated as:

$$A = \begin{bmatrix} -0.9917 & -0.0008 \\ 5454.5 & -4.5455 \end{bmatrix}, \ A_d = \begin{bmatrix} -0.9917 & 0.0008 \\ 0 & 0 \end{bmatrix}, \ B_h = \begin{bmatrix} -47.7431 \\ 0 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & 0 \\ -0.9091 & 0 \end{bmatrix}$$

from the linearized state-space model Equation (3), where the matrix L is chosen as $\begin{bmatrix} 0 & 1 \end{bmatrix}$ to consider the queue length in the router as the exogenous output. Regarding the characteristics of the system, F_2 does not exist and the state delay d is equal to the input delay h.

Obtained by the proposed LMI optimization, the static output feedback gain F_1 is 1.00×10^{-5}, which minimizes the disturbance index γ_{opt} as 0.4621. The result of controlled queue length and TCP window size are described in Figures 1 and 2, respectively, comparing them with the state feedback H_∞ control proposed in [6] and the RED control developed in [4]. Figure 3 shows the control input (mark probability) of each control methods.

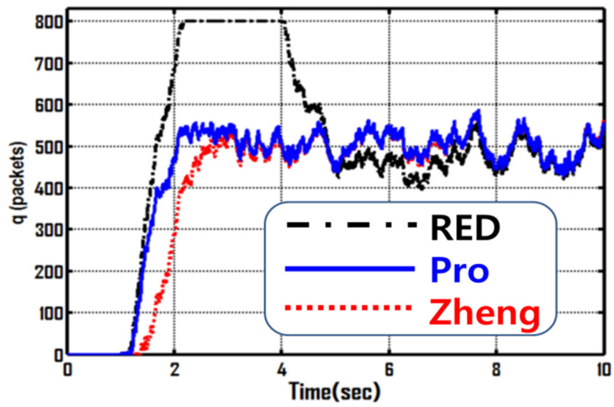

Figure 1. Comparison of queue lengths by Matlab/Simulink.

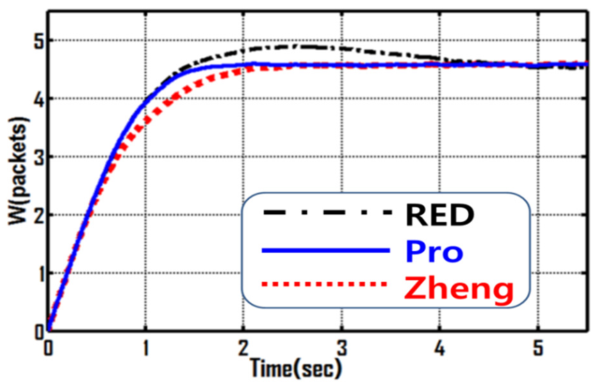

Figure 2. Comparison of window sizes by Matlab/Simulink.

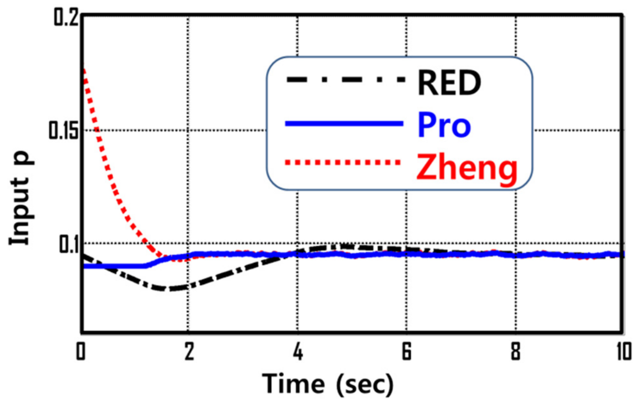

Figure 3. Comparison of control input p (Mark probability) by Matlab/Simulink.

Figure 1 illustrates that the performance of the controller proposed is fast and well suited to the desired operating position q_o without the buffer saturation, while the RED method causes the buffer saturation at the start. The percent overshoots of queue size of the proposed and Zheng's H_∞ methods are less than 20%, such that the steady state responses maintain between 600 and 400 packets during 10 s even if the round-trip time is with uniformly distributed random noise, shown in Figure 4.

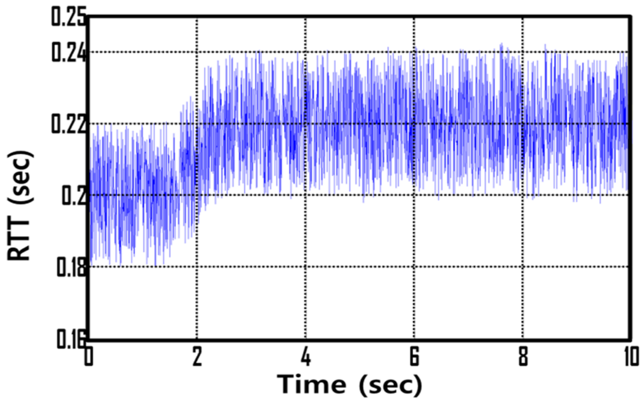

Figure 4. Round-trip time with random noise by Matlab/Simulink.

Between 0 to 8 s, the meaningful responses of the queue size and control input are listed in Table 1. The RED and proposed methods have similar rise time (80%) performances of 1.56 and 1.67 s, respectively. However, the RED method shows the worst settling time performance and buffer saturation. The settling times of control input p are measured for the 1% steady state error of the operating point $p_o = 0.0952$. The proposed method shows the fastest settling time 1.56 s in Figure 3. All of the input values remain at the operating point $p_o = 0.0952$ after 7.5 s, such that all of the queue sizes are shown as similar performances. Although the disturbance rejection level, 0.4621, achieved by the proposed controller is slightly larger than the level of 0.1719 by Zheng's H_∞ controller, the proposed controller is easier to design and implement in practice because it does not require the estimation of all states from the output.

Table 1. Comparison of performances.

	Overshoot of q Size (%)	Rise Time of q Size (s)	Settling Time of Input p (s)	γ
RED	Saturated	1.56	6.32	-
Pro	20	1.67	1.56	0.4621
Zheng	20	2.15	1.94	0.1719

From the results, it is observed that the proposed method yields lower overshoot than the RED and yields almost the same rise time as the RED and a shorter rise time than Zheng's H_∞ synthesis, while the proposed has a worse disturbance rejection H_∞ level than Zheng's H_∞ synthesis.

B. NS-2

In this section, we compare the control performance of the proposed static output feedback H_∞ controller with the RED method using the NS-2 (version 2.35) simulator [25]. The values of the network parameters are described at the beginning of this section, such as link bandwidth and packet size. The simulations are 50 s long, and the data of the queue length are collected throughout the simulation at every sampling time.

Figure 5 shows the responses of the queue in buffer controlled by the proposed and RED methods. Zheng's method cannot be well defined and performed by NS-2 simulator since it requires the full state information at each time, so we compared only the results of the proposed method and the RED method.

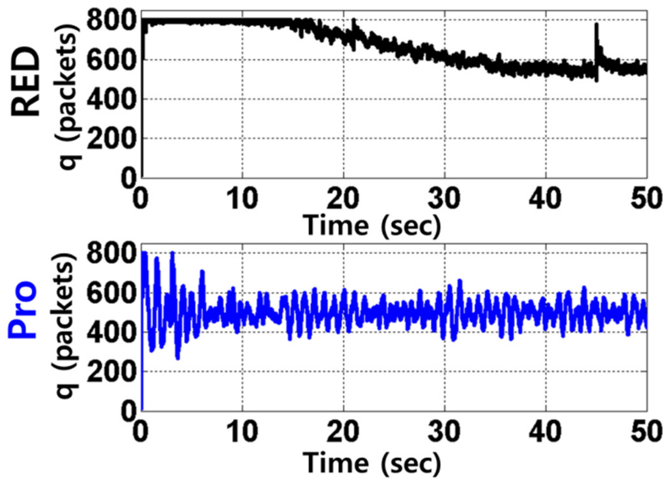

Figure 5. Comparison of queue lengths by NS-2.

From Figure 5, we can observe that, by using the proposed H_∞ congestion controller, a stable response is maintained even in the situation with disturbance (the available link capacity) and time delay (round-trip time), while the queue response of the RED method shows the buffer saturation until 17 s. The prevention of the buffer overflow is shown continuously after 3 s in the result of the proposed method.

The proposed controller works better with the maintenance at the desired queue length of 500 packets in the router, even if the controller yields a more fluctuating response than the RED method.

The queue size distributions in the router are described in Figure 6. The queue length of the RED method is distributed in the area larger than the desired queue length in the router because of the steady state error. The distribution near the maximum buffer size of 800 packets, especially, is caused by the buffer overflow at the starting time. On the other hand, the proposed method provides a well-balanced distribution performance, indicated by the bell shape without packet loss in Figure 6, despite the varying round-trip time delay and external disturbance (the available link capacity).

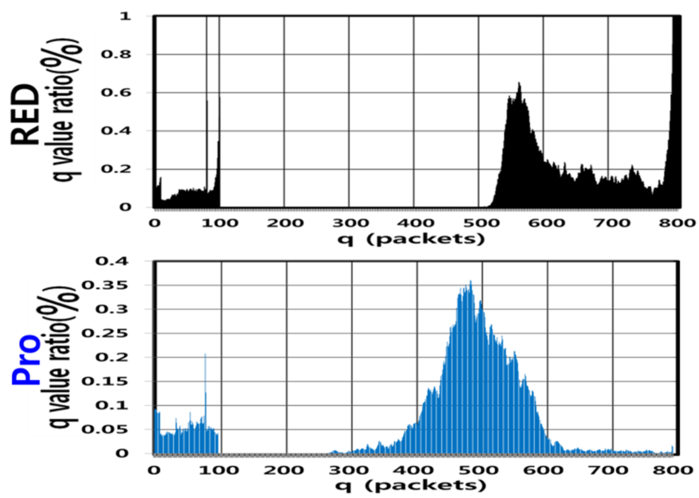

Figure 6. Comparison of queue value distributions by NS-2.

5. Conclusions

In this paper, we have developed a new H_∞ design method for AQM routers to avoid congestion in the TCP Internet Protocol, based on the LMI approach for static output feedback systems with time delay. The control gain is designed to internally stabilize the feedback system with time delay and to minimize the H_∞ norm of the transfer function between the controlled output and the external disturbance. In our proposed design approach, the static output feedback control gain is directly obtained from the derived sufficient conditions by solving a linear optimization problem with LMI constraints. This guarantees the disturbance attenuation level and the internal stability of the system.

The proposed static output feedback control method is straightforward to design and easily applicable to the AQM router system, as it does not require designing an observer to estimate the unmeasurable states of the system. Instead, it relies on storing and measuring only the queue length over time. Furthermore, the LMI-based H_∞ approach enables not only robust control to attenuate the effects of disturbance inputs on the controlled output, but also consideration of various practical aspects of the LMI approach. These aspects include minimum cost, norm-bounded input, polytopic uncertainty, and more.

The effectiveness of the proposed approach has been verified by implementing it to the congestion control of TCP/AQM routers and comparing it to the other control methods via the differential equation-based and packet-based simulation tools, Matlab/Simulink and NS-2, respectively.

The developed method provides the sufficient condition of not depending on the system's time delays, thus establishing delay-independent H_∞ control. Consequently, we expect that further research can develop the necessary and sufficient conditions to satisfy a given disturbance rejection level and overcome the conservatism of the proposed delay-independent control such that the delay-dependent control can be used as the less conservative design method.

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Data Availability Statement: The datasets presented in this article are not readily available because the data are part of an ongoing study.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

TCP	Transmission Control Protocol
AQM	Active Queue Management
PI	Proportional-Integral
PID	Proportional-Integral-Derivative
RTT	Round Trip Time
SOF	Static Output Feedback
LMI	Linear Matrix Inequality
ARI	Algebraic Riccati Inequality
RED	Random Early Detection

Appendix A

Assumption A1. *$\lambda(A) \in \mathcal{C}^-$ where λ and $\mathcal{C}^-$ denote the set of eigenvalues and the proper left.*

Corollary A1. *Subject to Assumption 1, the time delay system without input and disturbance* [24]

$$\dot{x}(t) = Ax(t) + A_d x(t-d)$$

where $d > 0$ is an unknown constant delay and is globally asymptotically stable, independent of the time delay if one of the following two equivalent conditions holds.

(1) There exist symmetric positive-definite matrices P and Q with appropriate dimensions satisfying the algebraic Riccati inequality (ARI)

$$PA + A^T P + PA_d Q^{-1} A_d^T P + Q < 0$$

(2) There exist symmetric positive-definite matrices P and Q with appropriate dimensions satisfying the linear matrix inequality (LMI)

$$\begin{bmatrix} PA + A^T P & PA_d \\ A_d^T P & -Q \end{bmatrix} < 0.$$

Lemma A1. **([23]).** *Given matrices* $G = G^T$*,* U*, and* V*,*
(1) There exists a matrix X satisfying:

$$G + UXV^T + VX^T U^T < 0$$

if and only if $U_\perp^T G U_\perp < 0$ *and* $V_\perp^T G V_\perp < 0$*, where* $U_\perp$*,* $V_\perp$ *are the orthogonal complements of* U*,* V*, respectively.*
(2) $U_\perp^T G U_\perp < 0$ *holds if and only if there exists a scalar* σ *such that* $G + \sigma^{-1} UU^T < 0$*.*

Assumption A2. $\lambda(A + A_d) \in \mathcal{C}^-$ *for the delay-dependent stability condition.*

Since A_d is negative-semi definite for AQM router, $\lambda(A + A_d) \in \mathcal{C}^-$ is sufficiently satisfied if Assumption 1 is met.

Corollary A2. *Subject to Assumption 2, the time delay system without input and disturbance [16]*

$$\dot{x}(t) = Ax(t) + A_d x(t - d)$$

globally and asymptotically stable, dependent on the time delay if the following LMI conditions hold

$$\begin{bmatrix} P(A + A_d) + (A + A_d)^T P + d(Q_0 + Q_1) & dPA_d A & dPA_d A_d \\ (dPA_d A)^T & -dQ_0 & 0 \\ (dPA_d A_d)^T & 0 & -dQ_1 \end{bmatrix} < 0$$

Appendix B

H_∞ Control Problem:

The H_∞ control problem is to find a proper and real rational controller $F(s)$ that internally stabilizes the system plant $G(s)$ and minimizes the H_∞ norm of the transfer matrix $T_{zw}(s)$ from exogenous input $\omega(t)$ to $z(t)$, where $T_{zw}(s) = C_z(sI - A)^{-1} B_\omega + D_{z\omega}$ [26].

$\omega(t)$ is the exogenous signal to be injected into the system, such as reference signal, disturbance signal, measurement noise, etc. $z(t)$ is the controlled output such as system output, error signal, etc., that can be selected arbitrarily by the designers as a meaningful signal to be minimized from the exogenous signal $\omega(t)$ [26].

Clearly, the infinite norm of a scalar transfer function $T_{zw}(s)$ is the peak value on the Bode magnitude plot of $|T_{zw}(s)|$. This means that error signals are minimized by the controller in the sense of worst-case effect on the energy of exogenous controlled output $z(t)$ [27].

In this paper, we focus our attention on optimal H_∞ control using the LMI method among the many solutions to obtain the controller to minimize the H_∞ norm.

From the bounded-real lemma (Kalman–Yakubovich–Popov lemma), we have $\| T_{zw}(s) \|_\infty < \gamma$ if and only if there exists positive definite P such that:

$$\begin{bmatrix} PA + A^T P & PB_\omega \\ B_\omega^T P & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_z^T \\ D_{z\omega}^T \end{bmatrix} \begin{bmatrix} C_z & D_{z\omega} \end{bmatrix} < 0.$$

This LMI condition for H_∞ optimal control problem can be derived from the so-called dissipation inequality [27]

$$z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(x(t)) < 0.$$

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