Saathi

UNIT - III

(1) Symmetric matrix is a square matrix such that the matrix equals its transpose

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ is a symmetric matrix.

(2) Quadratic form is a homogenous expression of second degree in any number of variables

Example: $x^2 + 2xy + 3y^2$ is a quadratic form (of degree 2) in the variables x and y.

(3) If A is a symmetric matrix, then any two eigen vectors corresponding to distinct eigen values are orthogonal

Proof: Let λ_1 and λ_2 be distinct eigen values with associated eigen vectors v_1 and v_2.

We know that, $Av_1 = \lambda_1 v_1 \rightarrow (1)$ and

$Av_2 = \lambda_2 v_2 \rightarrow (2)$.

(1) $\cdot v_2$ and (2) $\cdot v_1$ implies that

$$v_2^T (Av_1) = \lambda_1 \langle v_2, v_1 \rangle \rightarrow (3)$$

$$(Av_2)^T v_1 = \lambda_2 \langle v_2, v_1 \rangle$$

$$\Rightarrow v_2^T A^T v_1 = \lambda_2 \langle v_2, v_1 \rangle \rightarrow (4)$$

Since $A = A^T$,

$$(4) \Rightarrow v_2^T A v_1 = \lambda_2 \langle v_2, v_1 \rangle \rightarrow (5)$$

From (3) and (5),

$$\lambda_1 \langle v_2, v_1 \rangle = \lambda_2 \langle v_2, v_1 \rangle$$

$$\Rightarrow (\lambda_1 - \lambda_2) \langle v_2, v_1 \rangle = 0$$

Since $\lambda_1 \neq \lambda_2$,

$$\langle v_2, v_1 \rangle = 0$$

$\Rightarrow v_1$ and v_2 are orthogonal.

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(7) $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$

$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{31}x_3x_1 + 2a_{23}x_2x_3$

$$\Rightarrow A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

$x^T = [x_1 \; x_2 \; x_3]$

$$\therefore x^TAx = [x_1 \; x_2 \; x_3] \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(8) $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$

i) for $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$,

$Q(x) = (-3)^2 - 8(-3)(1) - 5(1)^2$

$= 28$

ii) for $x = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$Q(x) = (2)^2 - 8(2)(-2) - 5(-2)^2$

$= 16$

iii) for $x = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$Q(x) = (1)^2 - 8(1)(-3) - 5(-3)^2$

$= -20$

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(4) Since A is symmetric, $A = A^T$

Consider, $(A^2)^T = (AA)^T = A^TA^T = AA = A^2 \rightarrow (1)$.

Consider, $(A^T)^2 = A^TA^T = AA = A^2 \rightarrow (2)$.

$\therefore$ from (1) and (2), $(A^2)^T = (A^T)^2 = A^2$

$\therefore A^2$ is also symmetric.

(5) If A is orthogonally diagonalizable, then

$D = P^{-1}AP$

$\Rightarrow D \cdot D = (P^{-1}AP)(P^{-1}AP)$

$\Rightarrow D^2 = P^{-1}A(PP^{-1})AP$

$\Rightarrow D^2 = P^{-1}AAP$

$\Rightarrow D^2 = P^{-1}A^2P$

Hence A^2 is also orthogonally diagonalizable.

(6) $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, find X^TAX

a) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ $\Rightarrow X^TAX = [x_1 \; x_2] \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= [4x_1 \; 3x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= [4x_1^2 + 3x_2^2]$

b). $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$ $\Rightarrow X^TAX = [x_1 \; x_2] \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= [3x_1 - 2x_2 \;\; -2x_1 + 7x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= [3x_1^2 - 4x_1x_2 + 7x_2^2]$.

$c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$

Example: Consider $v_1 = (1,3)$, $v_2 = (2,6)$

$c_1(1,3) + c_2(2,6) = (0,0)$

$\Rightarrow c_1 = 1,\ c_2 = -2$

$\therefore v_1$ and v_2 are linearly independent

(13) Basis: A subset B of vector space V is said to be the basis of V if: i) V is linearly independent
ii) B spans V.

Dimension: If a vector space V has a basis consisting of n number of elements then n is called as dimension of V.

(14) Standard basis that spans the vector space R^2 are:

$e_1 = (1,0)$

$e_2 = (0,1)$

(15) Standard basis that spans the vector space R^3 are.

$e_1 = (0,0,1)$

$e_2 = (0,1,0)$

$e_3 = (1,0,0)$

(9) $A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\Rightarrow x^TAx = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3$

$\Rightarrow x^TAx = 4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3 + 0$

a) $x = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \Rightarrow x^TAx = 4(16) + 2(-1)^2 + 5^2 + 6(-2) + 2(-5)$

$= 64 + 2 + 25 + -12 - 10$

$= 69$

b) $x = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \Rightarrow x^TAx = 4(1/3) + 2(1/3) + (1/3) + 6(1/3) + 2(1/3)$

$= 1/3\,[15] = 5$

(10) If x, v, w belong to V and scalars c, d belong to scalar field F, satisfy the commutative and associative operation of addition of vectors, and the associative and distributive operation of multiplication of vectors by scalars, then V is called vector space over the scalar field F.

(11) Let V be a vector space and we say $v_1, v_2 \dots v_n$ of a set of vectors in V is said to be linearly dependent if there exists a scalar $c_1, c_2 \dots c_n \in F$, not all zeroes such that $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$

Example: Let $v_1 = (1,3)$, $v_2 = (4,3) \in V$

$c_1(1,3) + c_2(4,3) = (0,0)$

$\Rightarrow c_1 + 4c_2 = 0,\ 3c_1 + 3c_2 = 0$

$\Rightarrow c_1 = c_2 = 0$

$\therefore v_1$ & v_2 are linearly dependent

(12) Let V be a vector space and we say $v_1, v_2 \dots v_n$ of a set of vectors in V is said to be linearly independent if there exists a scalar $c_1, c_2 \dots c_n \in F$, all zeroes, such that

(ii) Let $\begin{bmatrix}13\\6\end{bmatrix} = c_1\begin{bmatrix}1\\3\end{bmatrix} + c_2\begin{bmatrix}4\\1\end{bmatrix}$

$\Rightarrow c_1 + 4c_2 = 13 \Rightarrow c_2 = 3$ & $c_1 = 1$
$3c_1 + c_2 = 6$

$\Rightarrow \begin{bmatrix}13\\6\end{bmatrix} = 1\begin{bmatrix}1\\3\end{bmatrix} + 3\begin{bmatrix}4\\1\end{bmatrix}$

(iii) Let $\begin{bmatrix}-3\\3\\7\end{bmatrix} = c_1\begin{bmatrix}1\\-1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\1\\0\end{bmatrix} + c_3\begin{bmatrix}-1\\2\\1\end{bmatrix}$

$\Rightarrow c_1 + 2c_2 + -c_3 = -3$
$-c_1 + c_2 + 2c_3 = 3$
$2c_1 + 0c_2 + c_3 = 1$

$3c_1 - 5c_3 = 9$
$2c_1 + c_3 = 1 \rightarrow 5$
$c_1 = \frac{\cancel{8}4}{\cancel{10}5} = 0.1$ $\frac{0}{10}$
$0.3 + -0.5 + 3 =$

$\Rightarrow c_1 = -0.3$ & $c_2 = -0.5$ & $c_3 = 1.6$

$\Rightarrow \begin{bmatrix}-3\\3\\7\end{bmatrix} = -0.3\begin{bmatrix}1\\-1\\2\end{bmatrix} + -0.5\begin{bmatrix}2\\1\\0\end{bmatrix} + 1.6\begin{bmatrix}-1\\2\\1\end{bmatrix}$

(19) (i) $(1,-3), (2,-5)$
$\rightarrow$ Let $(x,y) \in R^2$
$\Rightarrow (x,y) = c_1(1,-3) + c_2(2,-5)$
$\Rightarrow x = c_1 + 2c_2$
$y = -3c_1 - 5c_2$
$AX = B$
where $A = \begin{bmatrix}1 & 2\\-3 & -5\end{bmatrix}$, $X = \begin{bmatrix}c_1\\c_2\end{bmatrix}$, $B = \begin{bmatrix}x\\y\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1 & 2\\-3 & -5\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$

(16) (i) $A = \begin{bmatrix}-1 & 2\\2 & -4\end{bmatrix} \Rightarrow |A| = 4-4 = 0 \Rightarrow |A| = 0$
$\Rightarrow \{(-1,2),(2,-4)\}$ are linearly dependent

(ii) $A = \begin{bmatrix}3 & 1\\9 & 3\end{bmatrix} \Rightarrow |A| = 9-9 = 0 \Rightarrow |A| = 0$
$\Rightarrow \{(3,1),(9,3)\}$ are linearly dependent

(iii) $A = \begin{bmatrix}-2 & 3\\6 & -9\end{bmatrix} \Rightarrow |A| = 18-18 = 0 \Rightarrow |A| = 0$
$\Rightarrow \{(-2,3),(6,9)\}$ are linearly dependent

(iv) $A = \begin{bmatrix}1 & 5\\0 & 0\end{bmatrix} \Rightarrow |A| = 0-0 = 0 \Rightarrow |A| = 0$
$\Rightarrow \{(1,5),(0,0)\}$ are linearly dependent

(17) (i) $A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \Rightarrow |A| = 1-0 = 1 \neq 0$
$\Rightarrow$ The given vectors are linearly independent

(ii) $A = \begin{bmatrix}1 & 2\\3 & 2\end{bmatrix} \Rightarrow |A| = 2-6 = -4 \neq 0$
$\Rightarrow$ The given vectors are linearly independent

(iii) $A = \begin{bmatrix}-1 & 3\\2 & 5\end{bmatrix} \Rightarrow |A| = -5-6 = -11 \neq 0$
$\Rightarrow$ The given vectors are linearly independent

(iv) $A = \begin{bmatrix}2 & -4\\5 & 3\end{bmatrix} \Rightarrow |A| = 6+20 = 26 \neq 0$
$\Rightarrow$ The given vectors are linearly independent

(18) (i) $(-1,7) : (1,-1)\ (2,4)$

$\begin{bmatrix}-1\\7\end{bmatrix} = c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}c_1+2c_2\\-c_1+4c_2\end{bmatrix}$

$\Rightarrow c_1 + 2c_2 = -1 \Rightarrow c_2 = 1$ & $c_1 = -3$
$-c_1 + 4c_2 = 7$

$\Rightarrow \begin{bmatrix}-1\\7\end{bmatrix} = -3\begin{bmatrix}1\\-1\end{bmatrix} + 1\begin{bmatrix}2\\4\end{bmatrix}$

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$\therefore (1,1)$ & $(-2,1)$ span entire R^2

(iii) $(-3,1)$, $(3,-1)$

→ Let $(x,y) \in R^2$

$(x,y) = c_1(-3,1) + c_2(3,-1)$

$\Rightarrow x = -3c_1 + 3c_2$

$y = c_1 - c_2$

$$\Rightarrow \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $[A:B] = \begin{bmatrix} -3 & 3 & : & x \\ 1 & -1 & : & y \end{bmatrix}$

$R_2' \to 3R_2 + R_1$

$[A:B] \sim \begin{bmatrix} -3 & 3 & : & x \\ 0 & 0 & : & 3y+x \end{bmatrix}$

$\rho(A:B) \neq \rho(A) \quad \forall\ 3y+x \neq 0$.

$\therefore$ System is inconsistent for $(x,y) \in R^2$.

$\therefore$ the given vectors donot span entire R^2

(iv) $(3,2)$, $(1,1)$, $(1,0)$

→ Let $(x,y) \in R^2$

$\Rightarrow x = 3c_1 + c_2 + c_3$

$y = 2c_1 + c_2 + 0c_3$

$\therefore A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} x \\ y \end{bmatrix}$, $X = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

Let $[A:B] = \begin{bmatrix} 3 & 1 & 1 & : & x \\ 2 & 1 & 0 & : & y \end{bmatrix}$

$R_2' \to 3R_2 - 2R_1$

$\Rightarrow [A:B] \sim \begin{bmatrix} 3 & 1 & 1 & : & x \\ 0 & 1 & -2 & : & 3y-2x \end{bmatrix}$

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Augmented matrix $[A:B] = \begin{bmatrix} 1 & 2 & : & x \\ -3 & -5 & : & y \end{bmatrix}$

$R_2' \to R_2 + 3R_1$

$[A:B] \sim \begin{bmatrix} 1 & 2 & : & x \\ 0 & 1 & : & y+3x \end{bmatrix}$

$\forall\ y+3x \neq 0$, $\rho(A) = \rho(A:B) = 2$

$\therefore$ System is consistent $\forall\ (x,y) \in R^2$, $y+3x \neq 0$.

$\therefore$ for any $(a,b) \in R^2$, it can be expressed as a linear combination of $(1,-3)$ and $(2,-5)$

$\therefore (1,-3)$ and $(2,-5)$ span the entire R^2

(ii) $(1,1)$, $(-2,1)$

→ Let $(x,y) \in R^2$

$(x,y) = c_1(1,1) + c_2(-2,1)$

$\Rightarrow x = c_1 - 2c_2$

$y = c_1 + c_2$

Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $B = \begin{bmatrix} x \\ y \end{bmatrix}$

We know that, $AX = B$.

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$[A:B] \Rightarrow \begin{bmatrix} 1 & -2 & : & x \\ 1 & 1 & : & y \end{bmatrix}$

$R_2' \to R_2 - R_1$

$[A:B] \sim \begin{bmatrix} 1 & -2 & : & x \\ 0 & 3 & : & y-x \end{bmatrix}$

$\forall\ y \neq x$, $\rho[A:B] = \rho(A)$

$\therefore$ System is consistent $\forall\ (x,y) \in R^2$, $x \neq y$.

For any $(x,y) \in R^2$, it can be expressed as lin. comb of $(1,1)$, $(-2,1)$.

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(ii) Same as before.

$c_1 = -45, \quad c_2 = 14.$

(iii) $(1,-4), (3,1)$

$x = c_1 + 3c_2$

$y = -4c_1 + c_2$

$$[A:B] = \left[\begin{array}{cc:c} 1 & 3 & x \\ -4 & 1 & y \end{array}\right]$$

$R_2' \to R_2 + 4R_1$

$$\sim \left[\begin{array}{cc:c} 1 & 3 & x \\ 0 & 13 & y+4x \end{array}\right] \to (1)$$

$\rho[A:B] = \rho(A) \ \forall \ 4x+y \neq 0.$

$\therefore$ System is consistent

$\therefore (1,-4), (3,1)$ spans entire R^2.

Now, $(-3,5) = (c_1 + 3c_2, \ -4c_1 + c_2)$

$$(1) \Rightarrow \left[\begin{array}{cc:c} 1 & 3 & -3 \\ 0 & 13 & -15 \end{array}\right]$$

$c_1 + 3c_2 = -3$

$0c_1 + 13c_2 = -15$

$\Rightarrow c_1 = \frac{6}{13}, \quad c_2 = -\frac{15}{13}$

$-3 + 3 \cdot \frac{15}{13}$

$\frac{6}{13}$

(2) (i) $\{(1,3), (-1,2)\}$

Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$

$|A| = 2 + 3 = 5 \neq 0.$

∴ ~~It is linearly independent~~

Let $(a,b) \in R^2$

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\Rightarrow a = c_1 - c_2$

$b = 3c_1 + 2c_2$

Let $a = b = 0$,

$\rho(A:B) = \rho(A)$

$\therefore$ System is consistent

$\therefore$ All $(x,y) \in R^2$ can be expressed as the lin. comb. of $(3,2), (1,1), (1,0)$.

$\therefore (3,2), (1,1), (1,0)$ span entire R^2

(iv) Let $(x,y) \in R^2$

$x = c_1 - 2c_2$

$y = 4c_1 + 0c_2$

$AX = B$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Augmented matrix: $[A:B] = \left[\begin{array}{cc:c} 1 & -2 & x \\ 4 & 0 & y \end{array}\right]$

$R_2' \to R_2 - 4R_1$

$$\sim \left[\begin{array}{cc:c} 1 & -2 & x \\ 0 & 8 & y-4x \end{array}\right] \to (1)$$

$\rho(A:B) = \rho(A) \ \forall \ (x,y) \in R^2$

$\therefore$ System is consistent.

$\Rightarrow (1,4)$ & $(-2,0)$ spans entire R^2

Now $(-3,5) = c_1(1,4) + c_2(-2,0)$

$\Rightarrow x = -3, \ y = 5$

$$(1) \Rightarrow \left[\begin{array}{cc:c} 1 & -2 & -3 \\ 0 & 8 & 17 \end{array}\right]$$

$\Rightarrow c_1 - 2c_2 = -3$

$0 + 8c_2 = 17$

$\Rightarrow c_1 = 5/4, \quad c_2 = 17/8$

$\therefore (-3,5) = \frac{5}{4}(1,4) + \frac{17}{8}(-2,0)$

$$a = c_1 + 0c_2 + c_3$$
$$b = c_1 + c_2 + 0c_3$$
$$c = c_1 + 2c_2 + c_3$$

$$\Rightarrow [A:B] \quad \begin{bmatrix} 1 & 0 & 1 & : & a \\ 1 & 1 & 0 & : & b \\ 1 & 2 & 1 & : & c \end{bmatrix}$$

$R_2' \to R_2 - R_1, \quad R_3' \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & a \\ 0 & 1 & -1 & : & b-a \\ 0 & 2 & 0 & : & c-a \end{bmatrix}$$

$c - a - 2b - 2a$

$-3a - 2b + c$

$R_3' \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & a \\ 0 & 1 & -1 & : & b-a \\ 0 & 0 & 2 & : & -3a-2b+c \end{bmatrix}$$

$\therefore$ System is consistent

$\therefore$ v_1, v_2, v_3 spans entire $\mathbb{R}^3$

$\therefore$ v_1, v_2, v_3 are the bases of $\mathbb{R}^3$

(iv) & (v) similarly done.

then $\quad c_1 - c_2 = 0 \quad \Rightarrow c_1 = c_2 = 0$
$\quad 3c_1 + 2c_2 = 0$

$\therefore$ It is linearly independent

Any set of 2 linearly dependent vectors in $\mathbb{R}^2$ spans $\mathbb{R}^2$.

Therefore given set of vectors is a basis for $\mathbb{R}^2$

(ii) $\{\underset{v_1}{(2,6)}, \underset{v_2}{(4,1)}\}$

$\rightarrow$ Let $A = \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix}$ & $c_1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ 1 \end{bmatrix} = (0,0)$

Let $\Rightarrow 2c_1 + 4c_2 = 0. \quad \Rightarrow c_1 = c_2 = 0.$
$\quad 6c_1 + c_2 = 0.$

$\therefore$ It is linearly independent.

Let $(a,b) \in \mathbb{R}^2$.

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad a = 2c_1 + 4c_2$
$\quad b = 6c_1 + c_2$

$\Rightarrow c_1 = \dfrac{-a-4b}{22}, \quad c_1 = \dfrac{3a-b}{11}$

$\therefore$ v_1 & v_2 spans entire $\mathbb{R}^2$

$\therefore$ v_1 and v_2 are basis of $\mathbb{R}^2$.

(iii) $\{\underset{v_1}{(1,1,1)}, \underset{v_2}{(0,1,2)}, \underset{v_3}{(3,0,1)}\}$

$\rightarrow$ Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$|A| = 1 + 3(1) = 4 \neq 0$

$\therefore$ It is linearly independent

Let $(a,b,c) \in \mathbb{R}^3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

22. Linear transformation:

Let U and V be two vector space. The mapping T from U to V ($T: U \to V$) is called a linear transformation iff.

i) $T(U+V) = T(U) + T(V)$

ii) $T(cU) = cT(U)$

where: $U, V \in U$ and $c \to$ scalar

A linear transformation $T: U \to V$ is called linear map of U.

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23. Given: $T: R^3 \to R^2$ i) $T(x,y,z) = (3x, y^2)$

ii) $T(x,y,z) = (x+2, 4y)$

To prove: Not linear

i) $T(x,y,z) = (3x, y^2)$

Let $u = (x_1, y_1, z_1) \in R^3$ $c \to$ scalar

$v = (x_2, y_2, z_2)$

• $T(U+V) = T(U) + T(V)$

LHS $= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (3(x_1 + x_2), (y_1 + y_2)^2)$ — ①

RHS $= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$= (3x_1, y_1^2) + (3x_2, y_2^2)$

$= (3(x_1 + x_2), y_1^2 + y_2^2)$ — ②

LHS $\neq$ RHS. i.e., $T(U+V) \neq T(U) + T(V)$ for the transformation $T(x,y,z) = (3x, y^2)$. Hence it is not linear.

23. ii) $T(x, y, z) = (x+2, 4y)$

let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ $\in R^3$ $\quad c \to$ scalar

<u>condition 1:</u> $T(u+v) = T(u) + T(v)$

$LHS = T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (x_1+x_2+2, 4(y_1+y_2))$ -①

$RHS = T(u) + T(v)$

$= T(x_1, y_1, z_1) + T(x_2 y_2 z_2)$

$= (x_1+2, 4y_1) + (x_2+2, 4y_2)$

$= (x_1+x_2+4, 4(y_1+y_2))$ -②

① ≠ ② i.e., $T(u+v) \neq T(u) + T(v)$ for the transformation $T(x, y, z) = (x+2, 4y)$. Hence it is <u>not linear</u>.

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24. <u>Given:</u> $T: R^2 \to R$ $\quad T(x, y) = x + a$

$a \neq 0$

TO <u>prove</u>: $T(x, y) = x + a$ is Not linear.

let $u = (x_1, y_1)$, $v = (x_2, y_2)$ $\in R^2$

<u>condition 1:</u> $T(u+v) = T(u) + T(v)$

$LHS = T(x_1+x_2, y_1+y_2) = x_1+x_2+a$ -①

RHS = $T(U) + T(V)$

$= T(x_1, y_1) + T(x_2, y_2)$

$= x_1 + a + x_2 + a$

$= x_1 + x_2 + 2a$ - (2)

Since $a \neq 0$, (1) $\neq$ (2)

$\therefore T(U+V) = T(U) + T(V)$ fails for the transformation $T(x,y) = x + a$. Hence it is not linear.

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25. Given: $T: R^3 \to R^2$ $T(x,y,z) = (xy, z)$

To prove: T is not linear.

let $U = (x_1, y_1, z_1)$ $\in R^3$
$V = (x_2, y_2, z_2)$

condition 1: $T(U+V) = T(U) + T(V)$

LHS $= T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= ((x_1 + x_2)(y_1 + y_2), (z_1 + z_2))$ - (1)

RHS $= T(U) + T(V)$

$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$= (x_1 y_1, z_1) + (x_2 y_2, z_2)$

$(x_1 y_1 + x_2 y_2, (z_1 + z_2))$ - (2)

(1) $\neq$ (2)

Hence: $T(U+V) = T(U) + T(V)$ fails for the transformation $T(x,y,z) = (xy, z)$. Hence it is not linear.

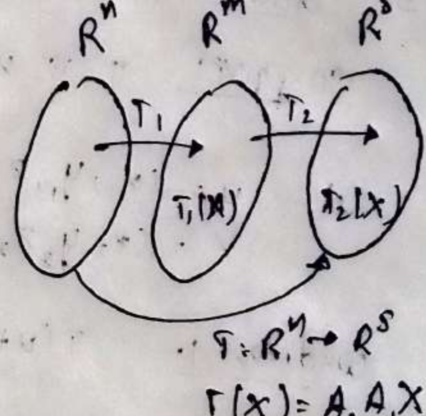
26. <u>Composite transformation</u>:

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^s$ both being matrix transformation defined by $T_1(x) = A_1X$ and $T_2(x) = A_2X$ where A_1 is matrix of order $m \times n$ and A_2 is $s \times m$.

These two transformations can be combined to a single transformation $T : R^n \to R^s$ which is called composite transformation defined by

$$T(x) = T_2(T_1(x))$$
$$= T_2(A_1X)$$
$$= A_2A_1X.$$

$$\boxed{\begin{aligned} T &= T_2 \circ T_1 = A_2A_1 \\ T(x) &= A_2A_1X. \end{aligned}}$$



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27. <u>Reflection about x-axis</u>: consider the operator $T : R^2 \to R^2$ defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ that maps a point into the mirror image in x-axis and hence T is called reflection in x-axis.

In matrix form, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$T(x) = A(x)$$

$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \to$ Standard matrix of reflection in x-axis

23. <u>Given</u>: Angles of rotation

i) $\pi/2$ ii) $-\pi/2$ iii) π iv) $3\pi/2$

To <u>find</u>: Matrix and image of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

<u>Sol</u>: <u>Rotation transformation about origin:</u>

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

i) $\theta = \pi/2$ $\quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

↓
matrix

$$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

ii) $\theta = -\pi/2$ $\quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

iii) $\theta = \pi$ $\quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

iv) $\theta = -3\pi/2 \simeq \pi/2$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

---

29. <u>Standard matrix of transformation</u> :

Let $T : R^n \to R^m$ be a linear transformation define vectors $e_1, e_2 \cdots e_n$ of R^n as follows:

$e_1 = [1, 0, 0 \ldots]$ and $T(e_1) = [a_{11}\ a_{21} \cdots a_{m1}]$

$e_2 = [0, 1, 0 \ldots]$

$\vdots$

$e_n = [0, 0 \cdots 1]$

Then the matrix $A = [T(e_1), T(e_2) \cdots T(e_n)]$ is standard matrix of T where the vectors $e_1, e_2 \cdots e_n$ are called standard basis of R^n.

Eg: For R^2, $e_1 = [1, 0]$, $e_2 = [0, 1]$ are standard basis.

$$T(e_1) = T\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

29. a) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ x-y \end{bmatrix}$ on R^2.

Standard basis $e_1 = [1, 0]$

$e_2 = [0, 1]$

$\therefore T(e_1) = T\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T(e_2) = T\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

standard matrix $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ //

---

b) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$

$T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\therefore A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ //

$T(e_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

---

c) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-5y \\ 3y \end{bmatrix}$

$T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\therefore A = \begin{bmatrix} 2 & -5 \\ 0 & 3 \end{bmatrix}$ //

$T(e_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

---

d) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ -3x \end{bmatrix}$

$T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ $\therefore A = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$ //

$T(e_2) = T[0 \ldots] = [2 \ldots]$

30. Reflection about y-axis:

Operator T from $R^2 \to R^2$ defined by the transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$ maps a point into the mirror image in y-axis and T is called reflection in y-axis.

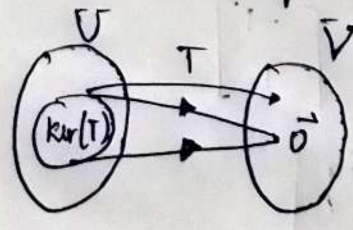
$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

---

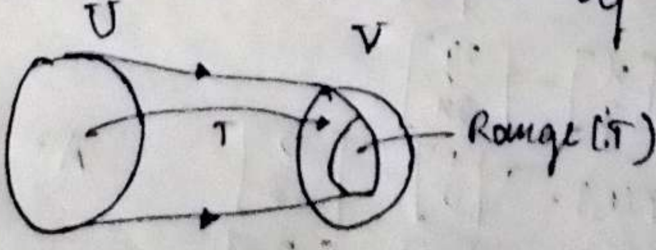
31. Kernel and Range of a linear transformation:

Let $T : U \to V$ be a linear transformation.

- The set of vectors in U that are mapped into zero vectors of V is called kernel of linear transformation T, denoted by Ker (T).



- The set of vectors in V that are the images of all vectors in U is called the Range of linear transformation T, denoted by Range (T).



32. <u>Coordinate vectors:</u>

Let V be a vector space with basis $B = \{v_1, v_2 \dots v_n\}$ and let $v \in V$, then there exists scalars $c_1, c_2 \dots c_n$ such that

$$\boxed{v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n}$$ //

The column vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = v_B$ is called coordinate vector of v relative to the given basis B. (denoted by v_B)

The scalars $c_1, c_2 \dots c_n$ are called coordinates of v relative to basis B.

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33. <u>Procedure to find transition matrix for change of basis:</u>

Let $B = \{v_1, v_2 \dots v_n\}$

$B' = \{v_1', v_2' \dots v_n'\}$ be basis for a vector space V.

A vector v in V will have coordinate vectors v_B and $v_{B'}$ relative to bases B and B'.

The relation b/w them (u_B and $u_{B'}$) is given by:

$$\boxed{u_{B'} = P u_B} \quad \text{or} \quad \boxed{u_B = P^{-1} u_{B'}}$$

where $P = \left[(u_1)_{B'}, (u_2)_{B'} \cdots (u_n)_{B'} \right]$

is called change of basis matrix or transition matrix.

---

34. $u = (4,5)$

a) $B = \{(1,0), (0,1)\}$

$u_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = ?$

$u = c_1 u_1 + c_2 u_2$

$u = c_1(1,0) + c_2(0,1)$

$(4,5) = (c_1, c_2)$

$\boxed{\therefore c_1 = 4 \\ c_2 = 5}$ //

$\therefore u_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ //

b) $B' = \{(2,1), (-1,1)\}$

$u = c_1 u_1 + c_2 u_2$

$(4,5) = c_1(2,1) + c_2(-1,1)$

$4 = 2c_1 - c_2$

$5 = c_1 + c_2$

$\therefore u_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ //

$\therefore 9 = 3c_1$

$\boxed{c_1 = 3}$ // $\quad \boxed{\therefore c_2 = 2}$ //

35. To find: v_B in R^3

a) $v = (4,0,-2)$ $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$v = c_1 v_1 + c_2 v_2 + c_3 v_3$

$(4,0,-2) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$(4,0,-2) = (c_1, c_2, c_3)$

$\therefore c_1 = 4$
$c_2 = 0$
$c_3 = -2$

$v_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ //

b) $v = (6,-3,1)$ $B = \{(1,-1,0)\ (2,1,-1)\ (2,0,0)\}$

$v = c_1 v_1 + c_2 v_2 + c_3 v_3$

$(6,-3,1) = c_1(1,-1,0) + c_2(2,1,-1) + c_3(2,0,0)$

$\therefore 6 = c_1 + 2c_2 + 2c_3$
$-3 = -c_1 + c_2$
$1 = -c_2$

$\boxed{\therefore c_2 = -1}$

$\therefore -3 = -c_1 - 1$

$\boxed{c_1 = 2}$ //

$\therefore 6 = 2 - 2 + 2c_3$

$\boxed{c_3 = 3}$ //

$\therefore v_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ //

II Apply / Analyze:

1. Given: $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

Characteristic equation for 3rd order matrix:

$$\lambda^3 - (\Sigma d)\lambda^2 + (\Sigma m_d)\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - (6+6+5)\lambda^2 + (29+29+32)\lambda - 144 = 0$$

$$\Rightarrow \lambda^3 - 17\lambda^2 + 90\lambda - 144 = 0$$

$$\boxed{\lambda = 8, 3, 6}$$

$$[A - \lambda I][X] = [0]$$

$$\begin{bmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

case i: $\lambda = 8$

same $\left\langle \begin{array}{l} -2x_1 - 2x_2 - x_3 = 0 \\ -2x_1 - 2x_2 - x_3 = 0 \end{array} \right.$

$-x_1 - x_2 - 3x_3 = 0$

Free factor $= 3 - 2$

$= 1$

let $\boxed{x_1 = k}$

$\therefore$ $\cancel{-2k} - \cancel{2x_2} - x_3 = 0$

$\underline{(+)\ \cancel{-2k} \overset{(+)}{-} \cancel{2x_2} \overset{(+)}{-} 6x_3 = 0}$

$5x_3 = 0$ $\qquad \boxed{x_3 = 0}$

$\therefore$ $-k - x_2 = 0$

$\boxed{x_2 = -k}$

$X_1 = [k \quad -k \quad 0]^T$

when $k = 1$

$\boxed{X_1 = [1 \quad -1 \quad 0]^T}$

<u>case ii:</u> $\lambda = 3$

$$3x_1 - 2x_2 - x_3 = 0$$
$$-2x_1 + 3x_2 - x_3 = 0$$
$$-x_1 - x_2 - 2x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & -1 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{5} = \frac{-x_2}{-5} = \frac{x_3}{5}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k \text{ (say, 1)}$$

$$\boxed{\therefore X_2 = [1 \quad 1 \quad 1]^T}$$ //

<u>case iii)</u> $\lambda = 6$

$$-2x_2 - x_3 = 0$$
$$-2x_1 - x_3 = 0$$
$$-x_1 - x_2 - x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & -1 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{2} = \frac{-x_2}{-2} = \frac{x_3}{-4}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2} = k \text{ (say 1)}$$

$$\boxed{\therefore X_3 = [1 \quad 1 \quad -2]^T}$$ //

$\therefore$ Modal matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

To normalize:

$$||x|| = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore ||x_1|| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$||x_2|| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$||x_3|| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$$\therefore N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} //$$

- The diagonal elements will be in the order of eigen values considered by you.

2. $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ characteristic eq: $(\lambda-7)^2(\lambda+2)=0$

Sol: $(\lambda-7)^2(\lambda+2)=0$

$$\boxed{\therefore \lambda = -2, 7, 7}$$

$[A-\lambda I][X]=[0]$

$$\begin{bmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

case (i): $\lambda = -2$

$$5x_1 - 2x_2 + 4x_3 = 0$$
$$-2x_1 + 8x_2 + 2x_3 = 0$$
$$4x_1 + 2x_2 + 5x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -2 & 4 \\ 8 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & 4 \\ -2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -2 \\ -2 & 8 \end{vmatrix}}$$

$$\frac{x_1}{-36} = \frac{-x_2}{18} = \frac{x_3}{36}$$

$$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{2} = k \text{ (say 1)}$$

$$\boxed{\therefore X_1 = [-2 \quad -1 \quad 2]^T}$$

Case (ii): $\lambda = 7$

$$-4x_1 - 2x_2 + 4x_3 = 0$$

same $\left\{ \begin{array}{l} -2x_1 - x_2 + 2x_3 = 0 \\ 4x_1 + 2x_2 - 4x_3 = 0 \end{array} \right.$

free parameter $= 3 - 1 = 2$

$x_1 = k_1, \quad x_2 = k_2$

$$\therefore \quad -4k_1 - 2k_2 + 4x_3 = 0$$

$$4x_3 = 4k_1 + 2k_2$$

$$x_3 = k_1 + \frac{k_2}{2}$$

$$\therefore \quad X_2 = \begin{bmatrix} k_1 \\ k_2 \\ k_1 + \frac{k_2}{2} \end{bmatrix}$$

$$X_2 = \begin{bmatrix} k_1 \\ 0 \\ k_1 \end{bmatrix} + \begin{bmatrix} 0 \\ k_2 \\ \frac{k_2}{2} \end{bmatrix}$$

$$= k_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{X_2} + \frac{k_2}{2} \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{X_3}$$

By Gram-Schmidt orthogonalization

$$Z = X_3 - \left(\frac{X_3 \cdot X_2}{X_2 \cdot X_2} \right) X_2$$

scalar product

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \left(\frac{1}{2} \right) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 2 \\ 1/2 \end{bmatrix} = Z$$

$\therefore$ Modal matrix $P = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & z \end{array} \\ \begin{bmatrix} -2 & 1 & -1/2 \\ -1 & 0 & 2 \\ 2 & 1 & 1/2 \end{bmatrix} \end{array}$

- x_3 can also be chosen instead of x_2

Normal matrix N.

$||x_1|| = \sqrt{4+1+4} = 3$

$||x_2|| = \sqrt{1+1} = \sqrt{2}$

$||z|| = \sqrt{\frac{1}{4}+4+\frac{1}{4}} = \frac{3}{\sqrt{2}}$

$$N = \begin{bmatrix} -2/3 & +1/\sqrt{2} & -1/3\sqrt{2} \\ -1/3 & 0 & 2\sqrt{2}/3 \\ 2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \end{bmatrix}$$

$\therefore D = N^T A N$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} //$$

---

3. a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)^2 - 1 = 0$

$\Rightarrow 9 + \lambda^2 - 6\lambda - 1 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$

$\Rightarrow \boxed{\lambda = 2, 4} //$

$\therefore [A - \lambda I][x] = [0]$

$$\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>case (i):</u> $\lambda = 2$

$\text{same} \left\langle \begin{array}{l} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{array} \right.$

Free parameter $= 2 - 1 = 1$

let $x_1 = k$

$\therefore x_2 = -k$

$\therefore X_1 = [k \quad -k]^T$

$X_1 = [1 \quad -1]^T$ when $k = 1$

<u>case (ii):</u> $\lambda = 4$

$\text{same} \left\langle \begin{array}{l} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{array} \right.$

$\therefore$ Free parameter $= 2 - 1 = 1$

let $x_1 = k$

$\therefore x_2 = k$

$X_2 = [k \quad k]^T$

$X_2 = [1 \quad 1]^T$ when $k = 1$.

$\therefore P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\|X_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\|X_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$\therefore \quad D = N^T A N$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}_{//}$$

---

3. b) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$ $\quad |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2 - 25 = 0$

$\Rightarrow 1 + \lambda^2 - 2\lambda - 25 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 24 = 0$

$\boxed{\lambda = -4, 6}_{//}$

$[A - \lambda I][X] = [0]$

$$\begin{bmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case (i): $\lambda = -4$

same $\left\langle \begin{array}{l} 5x_1 + 5x_2 = 0 \\ 5x_1 + 5x_2 = 0 \end{array} \right.$ $\quad$ Free Parameter $= 2 - 1 = 1$

let $x_1 = k$

then $x_2 = -k$

$\therefore X_1 = [k \ \ -k]^T$

$X_1 = [1 \ \ -1]^T_{//}$ when $k = 1$

Case (ii): $\lambda = 6$

$$\text{Same} \left\langle \begin{array}{l} -5x_1 + 5x_2 = 0 \\ 5x_1 - 5x_2 = 0 \end{array} \right.$$

$F.P = 2 - 1 = 1$

let $x_1 = k$

Then $x_2 = k$

$$X_2 = [k \quad k]^T$$

$$X_2 = [1 \quad 1]^T \quad \text{when } k=1.$$

$$\therefore P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$||X_1|| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$||X_2|| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = N^T A T$$

$$= \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix} //$$

3.c) $A = \begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$; $|A - \lambda I| = 0$

$$\begin{vmatrix} 16-\lambda & -4 \\ -4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 16 - \lambda - 16\lambda + \lambda^2 - 16 = 0$.

$\Rightarrow \quad \lambda^2 - 17\lambda = 0$

$\Rightarrow \quad \lambda(\lambda - 17) = 0$

$\boxed{\therefore \lambda = 0, 17}$ //.

$$\begin{bmatrix} 16-\lambda & -4 \\ -4 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case (i): $\lambda = 0$

same < $16x_1 - 4x_2 = 0$; $-4x_1 + x_2 = 0$

F.P $= 2 - 1 = 1$

let $x_1 = k$

$\therefore 16k - 4x_2 = 0$

$x_2 = 4k$

$\therefore X_1 = [k \quad 4k]^T$

$X_1 = [1 \quad 4]^T$ when $k = 1$

Case (ii): $\lambda = 17$

same < $-x_1 - 4x_2 = 0$; $-4x_1 - 16x_2 = 0$

F.P $= 2 - 1 = 1$

let $x_1 = k$

$\therefore x_2 = \frac{-k}{4}$ $\quad \therefore X_2 = [4 \quad -1]^T$ when $k = 4$.

$\therefore P = \begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix}$ $\therefore$ $||x_1|| = \sqrt{1+16} = \sqrt{17}$

$||x_2|| = \sqrt{16+1} = \sqrt{17}$

$N = \begin{bmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ 4/\sqrt{17} & -1/\sqrt{17} \end{bmatrix}$

$D = N^T A N$

$= \begin{bmatrix} 0 & 0 \\ 0 & 17 \end{bmatrix}$

3.d. $A = \begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$ $|A - \lambda I| = 0$

$\begin{vmatrix} -7-\lambda & 24 \\ 24 & 7-\lambda \end{vmatrix} = 0$

$\Rightarrow -49 - \cancel{7\lambda} + \cancel{7\lambda} + \lambda^2 - 576 = 0$

$\Rightarrow \lambda^2 - 625 = 0$

$\Rightarrow (\lambda - 25)(\lambda + 25) = 0$

$\boxed{\therefore \lambda = -25, 25}$

- Find P
- Find N

$D = N^T A N = \begin{bmatrix} -25 & 0 \\ 0 & 25 \end{bmatrix}$ //

---

3. e. $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $\lambda^3 - (\Sigma d)\lambda^2 + (\Sigma m_d)\lambda - |A| = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + (-4)\lambda - (-20) = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 - 4\lambda + 20 = 0$

$\boxed{\lambda = -2, 2, 5}$

- Find P
- Find N

$D = N^T A N = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ //

3.f. $A = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}$

$$\lambda^3 - (\Sigma d)\lambda^2 + (\Sigma m_d)\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - 21\lambda^2 + 111\lambda - 91 = 0.$$

$$\Rightarrow \boxed{\therefore \lambda = 1, 7, 13}$$ //

- Find P
- find N

$D = N^{T}AN$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix}$ //

4. Given: $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$

To verify: $V_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are eigen vectors of A.

$\lambda^3 - 12\lambda^2 + 21\lambda - 10 = 0$

$\boxed{\lambda = 1, 1, 10}$

$[A - \lambda I][X] = [0]$

$$LHS = \begin{bmatrix} 5-\lambda & -4 & -2 \\ -4 & 5-\lambda & 2 \\ -2 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10+2\lambda-8-2 \\ 8+10-2\lambda+2 \\ 4+4+2-\lambda \end{bmatrix} = \begin{bmatrix} 2\lambda-20 \\ 20-2\lambda \\ 10-\lambda \end{bmatrix}$$

For $\lambda = 10$, $\begin{bmatrix} 2\lambda-20 \\ 20-2\lambda \\ 10-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = RHS.$

Illy
$$= \begin{bmatrix} 5-\lambda & -4 & -2 \\ -4 & 5-\lambda & 2 \\ -2 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda-4 \\ -4+5-\lambda \\ -2+2 \end{bmatrix} = \begin{bmatrix} 1-\lambda \\ 1-\lambda \\ 0 \end{bmatrix}$$

For $\lambda = 1$, $\begin{bmatrix} 1-\lambda \\ 1-\lambda \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = RHS$

$\therefore V_1$ and V_2 are eigen vectors of A.

5(iii) $\{f, g, h\}$, $f(x) = x^2+3x-1$

$g(x) = x+3$

$h(x) = 2x^2-x+1$

To check for linear dependency:

$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0x^2+0x+0$

$c_1(x^2+3x-1) + c_2(x+3) + c_3(2x^2-x+1) = 0x^2+0x+0$

$x^2(c_1+2c_3) + x(3c_1+c_2-c_3) + (-c_1+3c_2+c_3)$

$= 0x^2+0x+0$

$\therefore$ $c_1+2c_3=0$

$3c_1+c_2-c_3=0$

$-c_1+3c_2+c_3=0$

$$\begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -1 & 3 & 1 \end{vmatrix} = 24 \neq 0$$

$\therefore$ $\{f, g, h\}$ are linearly independent.

6.(i) $(1,2,3)$ $(-1,-1,0)$ $(2,5,4)$

v_1 v_2 v_3

Let $(x,y,z) \in R^3$

$(x,y,z) = c_1(1,2,3) + c_2(-1,-1,0) + c_3(2,5,4)$

$(x,y,z) = (c_1-c_2+2c_3, 2c_1-c_2+5c_3, 3c_1+4c_3)$

$\therefore$ $x = c_1-c_2+2c_3$

$y = 2c_1-c_2+5c_3$ $z = 3c_1+4c_3$

$$\therefore [A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 2 & -1 & 5 & y \\ 3 & 0 & 4 & z \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 0 & 1 & 1 & y-2x \\ 0 & 3 & -2 & z-3x \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - 3R_2 \\ R_3 \to R_3 \left(-\frac{1}{5}\right) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 0 & 1 & 1 & y-2x \\ 0 & 0 & 1 & \frac{3x-3y+z}{-5} \end{array}\right]$$

$$\therefore c_3 = \frac{3x-3y+z}{-5}$$

$$c_2 = \frac{-7x+2y+z}{5}$$

$$c_1 = \frac{4x-4y+3z}{5}$$

$$\therefore \left(\frac{4x-4y+3z}{5}\right)(1,2,3) + \left(\frac{-7x+2y+z}{5}\right)(-1,-1,0)$$

$$+ \left(\frac{3x-3y+z}{-5}\right)(2,5,4) \text{ spans } R^3.$$

For $(1,3,-2)$

$$c_3 = \frac{3-9-2}{-5} = \frac{8}{5}$$

$$c_2 = \frac{-7+6-2}{5} = \frac{-3}{5}$$

$$c_1 = \frac{4-12-6}{5} = \frac{-14}{5}$$

$$\boxed{\therefore (1,3,-2) = \frac{-14}{5}(1,2,3) - \frac{3}{5}(-1,-1,0) + \frac{8}{5}(2,5,4)}$$

and ii) Both are same

7. (ii) $(1,-2,3)$ $(-2,4,1)$ $(-4,8,9)$

To check for linear dependency

$$c_1(1,-2,3) + c_2(-2,4,1) + c_3(-4,8,9) = 0$$

$$\therefore \quad c_1 - 2c_2 - 4c_3 = 0$$
$$-2c_1 + 4c_2 + 8c_3 = 0$$
$$3c_1 + c_2 + 9c_3 = 0$$

$$\therefore \begin{vmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ 3 & 1 & 9 \end{vmatrix} = 1(36-8) + 2(-18-24) - 4(-2-12)$$

$$= 28 - 84 + 56$$

$$= \boxed{0}$$

$\therefore$ The given set of vectors is <u>linearly dependent.</u>

To express one vector as L.C of other.

$$(1,-2,3) = c_1(-2,4,1) + c_2(-4,8,9)$$

$$\therefore \quad -2c_1 - 4c_2 = 1$$
$$4c_1 + 8c_2 = -2$$
same

$$c_1 + 9c_2 = 3$$

$$\therefore \quad 2c_1 + 4c_2 = -1$$
$$\underline{\overset{(-)}{2c_1} \overset{(-)}{+} 18c_2 = \overset{(-)}{6}}$$
$$-14c_2 = -7$$
$$\boxed{c_2 = +1/2}$$

$$\therefore \quad c_1 = 3 - 9/2$$
$$\boxed{c_1 = -3/2}$$

$\therefore\ (1,-2,3) = -\frac{3}{2}(-2,4,1) + \frac{1}{2}(-4,8,9)$

8. iii) $\{(2,-t), (2t+6, 4t)\}$

For linear dependency, $c_1 v_1 + c_2 v_2 = 0$

$\therefore\ c_1(2,-t) + c_2(2t+6, 4t) = 0$

Comparing.

$2c_1 + c_2(2t+6) = 0$

$-tc_1 + c_2(4t) = 0$

$$\begin{vmatrix} 2 & 2t+6 \\ -t & 4t \end{vmatrix} = 0$$

$$\Rightarrow\ t \begin{vmatrix} 2 & 2t+6 \\ -1 & 4 \end{vmatrix} = 0$$

$\boxed{\therefore t=0}$ //

$8 + 2t + 6 = 0$

$2t = -14$

$\boxed{t=-7}$ //

$\therefore$ For $t=0$ and -7, the given set is linearly dependent.

9. ii) $f(x) = x^2 + x - 3$

$g(x) = x^2 - x + 1$

$h(x) = x^2 + x - 1$

For basis, – Must span P_2

– Linearly independent

• $ax^2+bx+c \in P_2$

$$ax^2+bx+c = c_1 f(x) + c_2 g(x) + c_3 h(x)$$

$$\therefore ax^2+bx+c = x^2(c_1+c_2+c_3) + x(c_1-c_2+c_3) + (-3c_1+c_2-c_3)$$

$\therefore$ $c_1 + c_2 + c_3 = a$

$c_1 - c_2 + c_3 = b$

$-3c_1 + c_2 - c_3 = c$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 1 & b \\ -3 & 1 & -1 & c \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & 0 & b-a \\ 0 & 4 & 2 & c+3a \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & 0 & b-a \\ 0 & 0 & 2 & a+2b+c \end{array}\right] \quad \begin{array}{l} R_2 \to R_2\left(-\frac{1}{2}\right) \\ R_3 \to R_3\left(\frac{1}{2}\right) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & \frac{b-a}{-2} \\ 0 & 0 & 1 & \frac{a+2b+c}{2} \end{array}\right]$$

$\therefore c_3 = \frac{a+2b+c}{2} \qquad c_2 = \frac{a-b}{2}$

$$c_1 = \frac{2a}{2} - \frac{(a-b)}{2} - \frac{(a+2b+c)}{2}$$

$$c_1 = \frac{-b-c}{2}$$

$$\therefore -\left(\frac{b+c}{2}\right)(x^2+x-3) + \left(\frac{a-b}{2}\right)(x^2-x+1)$$

$+\left(\frac{a+2b+c}{2}\right)(x^2+x-1)$ spans P_2.

To check for linear dependency.

$$c_1+c_2+c_3=0$$
$$c_1-c_2+c_3=0$$
$$-3c_1+c_2-c_3=0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -3 & 1 & -1 \end{vmatrix}$$

$$= 1(1-1) - 1(-1+4) + 1(1-4)$$

$$= 0 - 3 - 3$$

$$= -6 \neq 0$$

$\therefore$ ~~Then~~ $\{f, g, h\}$ are linearly independent.

$\therefore$ $\{f, g, h\}$ is basis over the vector space P_2 of polynomials of degree ≤ 2.

21.
(i) $\{(1,3),(-1,2)\}$

$$\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 2+3 = 5 \neq 0$$ Linearly independent

$$\begin{bmatrix} a \\ b \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$ C_1, C_2 are scalars $\in R$

$$C_1 - C_2 = a$$
$$3C_1 + 2C_2 = b$$

$$[A:B] = \begin{bmatrix} 1 & -1 & : & a \\ 3 & 2 & : & b \end{bmatrix}$$ $R_2' \to R_2 - 3R_1$

$$= \begin{bmatrix} 1 & -1 & : & a \\ 0 & 5 & : & b-3a \end{bmatrix}$$ $2-3(-1)$

$\rho(A) = \rho(A:B)$ $\forall (a,b) \in R^2$

There exists C_1, C_2 $\forall (a,b) \in R^2$

Hence $(1,3), (-1,2)$ spans R^2

Bases

(ii) $\{(2,6),(4,1)\}$

$$\begin{vmatrix} 2 & 6 \\ 4 & 1 \end{vmatrix} = 2 - 24 = -22 \neq 0$$ Linearly independent

$$\begin{bmatrix} a \\ b \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + C_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$2c_1 + 4c_2 = a$
$6c_1 + c_2 = b$

$$\cancel{\begin{bmatrix} a \\ b \end{bmatrix}} = [A:B] = \begin{bmatrix} 2 & 4 & : & a \\ 6 & 1 & : & b \end{bmatrix} \quad R_2' \to R_2 - 3R_1$$

$$= \begin{bmatrix} 2 & 4 & : & a \\ 0 & -11 & : & b-3a \end{bmatrix} \quad \begin{matrix} 6-3(2) \\ 1-3(4) \end{matrix}$$

$\rho(A) = \rho(A:B) \quad \forall (a,b) \in R^2$

There exists c_1 and c_2 $\forall (a,b) \in R^2$

Hence $(2,6),(4,1)$ spans R^2

Bases

(iii) $\{(1,1,1),(0,1,2),(3,0,1)\}$ $\quad c_1, c_2, c_3$ are scalars $\in R$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = 4 \neq 0 \quad \text{Linearly independent}$$

$$c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$c_1 + 3c_3 = a$
$c_1 + c_2 = b$
$c_1 + 2c_2 + c_3 = c$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 1 & 1 & 0 & b \\ 1 & 2 & 1 & c \end{array}\right] \quad \begin{array}{l} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 0 & 1 & -3 & b-a \\ 0 & 2 & -2 & c-a \end{array}\right] \quad R_3' \to R_3 - 2R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 0 & 1 & -3 & b-a \\ 0 & 0 & 4 & c+a-2b \end{array}\right]$$

$-2-2(-3)$

$c-a-2(b-a)$

$c-a-2b+2a$

$\rho(A) = \rho(A:B) \quad \forall (a,b,c) \in R^3$

There exists $c_1, c_2, c_3 \ \forall (a,b,c) \in R^3$

These vectors span R^3

Bases

(iv) $\{(1,2,3), (2,4,1), (3,0,0)\}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 0 & 0 \end{vmatrix} = -30 \neq 0$$ Linearly independent

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 + 2C_2 + 3C_3 = a$$
$$2C_1 + 4C_2 = b$$
$$3C_1 + C_2 = c$$

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & a \\ 2 & 4 & 0 & : & b \\ 3 & 1 & 0 & : & c \end{bmatrix} \quad \begin{matrix} R_2' \to R_2 - 2R_1 \\ R_3' \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & : & a \\ 0 & 0 & -6 & : & b-2a \\ 0 & -5 & -9 & : & c-3a \end{bmatrix} \quad \begin{matrix} 4-2(2) \\ 0-2(3) \\ 1-3(2) \end{matrix}$$

$$R_2 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & 2 & 3 & : & a \\ 0 & -5 & -9 & : & c-3a \\ 0 & 0 & -6 & : & b-2a \end{bmatrix}$$

$$\rho(A) = \rho(A:B) \quad \forall (a,b,c) \in R^3$$

C_1, C_2, C_3 exists $\forall (a,b,c) \in R^3$

These vectors span R^3

Bases

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(v) $\{(1,2,2),(-1,0,1),(-3,1,-1)\}$

$$\begin{vmatrix} 1 & 2 & 2 \\ -1 & 0 & 1 \\ -3 & 1 & -1 \end{vmatrix} = -11 \neq 0$$ Linearly independent

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$$

$c_1 - c_2 - 3c_3 = a$

$2c_1 + 0c_2 + c_3 = b$

$2c_1 + c_2 - c_3 = c$

$$[A:B] = \begin{bmatrix} 1 & -1 & -3 & : & a \\ 2 & 0 & 1 & : & b \\ 2 & 1 & -1 & : & c \end{bmatrix},$$

$R_2' \to R_2 - 2R_1$

~~$R_3' \to R_3 - R_2$~~,

$R_3' \to R_3 - 2R_1$

$$= \begin{bmatrix} 1 & -1 & -3 & : & a \\ 0 & 2 & 7 & : & b-2a \\ 0 & 3 & 5 & : & c-2a \end{bmatrix}$$

$0-2(-1)$

$1-2(-3)$

$1-2(-1)$

$-1-2(-3)$

$$= \begin{bmatrix} 1 & -1 & -3 & : & a \\ 0 & 2 & 7 & : & b-2a \\ 0 & 0 & -11 & : & 2a-3b+2c \end{bmatrix}$$

$R_3' \to 2R_3 - 3R_2$

$10-3(7)$

$2(c-2a)-3(b-2a)$

$2c-4a-3b+6a$

$\rho(A) = \rho(A:B) \ \forall (a,b,c) \in R^3$

There exists $c_1, c_2, c_3 \ \forall (a,b,c) \in R^3$

Hence these vectors span R^3 Bases //

23.

(i) $T(x,y,z) = (3x, y^2)$ $\quad \alpha, \beta \in \cancel{(x,y,z)}$ R^3

Let $\alpha = (x_1, y_1, z_1)$ $\quad \beta = (x_2, y_2, z_2)$

$T(\alpha) = (3x_1, y_1^2)$

$T(\beta) = (3x_2, y_2^2)$

$c_1\alpha + c_2\beta = c_1(x_1, y_1, z_1) + c_2(x_2, y_2, z_2)$

$= (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2, c_1z_1 + c_2z_2)$

$T(c_1\alpha + c_2\beta) = (3(c_1x_1 + c_2x_2), (c_1y_1 + c_2y_2)^2)$

$= (3(c_1x_1 + c_2x_2), c_1^2y_1^2 + c_2^2y_2^2 + 2c_1c_2y_1y_2)$

$= c_1(3x_1, c_1y_1 + 2c_2y_1y_2) + c_2(3x_2 + c_2y_2)$

$\neq c_1T(\alpha) + c_2T(\beta)$

Not linear

(ii) $T(x,y,z) = (x+2, 4y)$

Let $\alpha = (x_1, y_1, z_1)$ $\quad \beta = (x_2, y_2, z_2)$

$T(\alpha) = (x_1 + 2, 4y_1)$

$T(\beta) = (x_2 + 2, 4y_2)$

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$C_1\alpha + C_2\beta = C_1(x_1, y_1, z_1) + C_2(x_2, y_2, z_2)$
$= (C_1x_1 + C_2x_2, C_1y_1 + C_2y_2, C_1z_1 + C_2z_2)$

$T(C_1\alpha + C_2\beta) = (C_1x_1 + C_2x_2 + 2, 4(C_1y_1 + C_2y_2))$

$\cancel{= C_1(x_1 +} \neq C_1T(\alpha) + C_2T(\beta)$

Not linear

24. $T : R^2 \to R$

$T(x, y) = x + a$ $\qquad \alpha, \beta \in \cancel{(x, y)}$ R^2

Let $\alpha = (x_1, y_1)$ $\beta = (x_2, y_2)$

$T(\alpha) = x_1 + a$
$T(\beta) = x_2 + a$

$C_1\alpha + C_2\beta = C_1(x_1, y_1) + C_2(x_2, y_2)$
$= (C_1x_1 + C_2x_2, C_1y_1 + C_2y_2)$

$T(C_1\alpha + C_2\beta) = C_1x_1 + C_2x_2 + a$
$\neq C_1T(\alpha) + C_2T(\beta)$

Not linear

25. $T(x,y,z) = (xy, z)$ $\quad \alpha, \beta \in R^3$

Let $\alpha = (x_1, y_1, z_1)$ $\quad \beta = (x_2, y_2, z_2)$

$T(\alpha) = (x_1 y_1, z_1)$

$T(\beta) = (x_2 y_2, z_2)$

$c_1\alpha + c_2\beta = c_1(x_1, y_1, z_1) + c_2(x_2, y_2, z_2)$

$= (c_1 x_1 + c_2 x_2, c_1 y_1 + c_2 y_2, c_1 z_1 + c_2 z_2)$

$T(c_1\alpha + c_2\beta) = ((c_1 x_1 + c_2 x_2)(c_1 y_1 + c_2 y_2), c_1 z_1 + c_2 z_2)$

$= (c_1^2 x_1 y_1 + c_1 c_2 x_1 y_2 + c_1 c_2 x_2 y_1 + c_2^2 x_2 y_2, c_1 z_1 + \ldots)$

$= c_1(c_1 x_1 y_1 + c_2 x_2 y_1, z_1) + c_2(c_1 x_1 y_2 + c_2 x_2 y_2, z_2)$

$\neq c_1 T(\alpha) + c_2 T(\beta)$

Not linear

28.

a) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\theta = \frac{\pi}{2}$ $\quad T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\theta = -\frac{\pi}{2}$

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b) $T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$

c) $T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-2\\-1\end{bmatrix}$

$\theta = \pi$

d) $T\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0 & +1\\-1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$

$\theta = -\frac{3\pi}{2}$

29.

a) $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}2x\\x-y\end{bmatrix}$ $\begin{bmatrix}x\\y\end{bmatrix} \in R^2$

Std Bases for R^2 $e_1 = (1,0)$ $e_2 = (0,1)$

Std Matrix $A = [T(e_1) \quad T(e_2)]$

$\overbrace{T\begin{bmatrix}0\\1\end{bmatrix}}^{T(e_2)} = \begin{bmatrix}0\\-1\end{bmatrix}$ $\overbrace{T\begin{bmatrix}1\\0\end{bmatrix}}^{T(e_1)} = \begin{bmatrix}2\\1\end{bmatrix}$

$A = \begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}$

b) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$

Std bases for R^2

$e_1 = (1,0)$ $e_2 = (0,1)$

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Std Matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

c) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-5y \\ 3y \end{bmatrix}$

Std. bases for R^2

$e_1 = (1,0)$ $e_2 = (0,1)$

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

Std Matrix $A = \begin{bmatrix} 2 & -5 \\ 0 & 3 \end{bmatrix}$

d)

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d) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2y \\ -3x \end{bmatrix}$

Std bases for R^2

$e_1 = (1, 0)$ $e_2 = (0, 1)$

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$