LAPLACE TRANSFORMS

INTRODUCTION

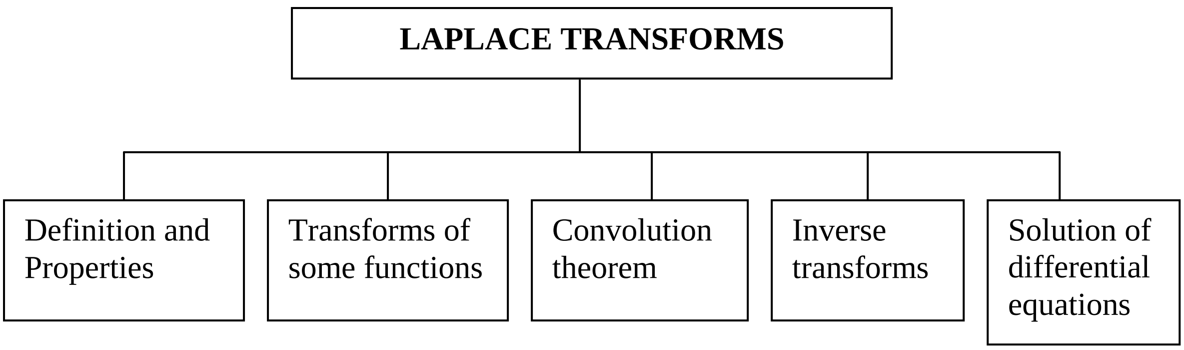
- Laplace transform is an integral transform employed in solving physical problems.
- Many physical problems when analysed assumes the form of a differential equation subjected to a set of initial conditions or boundary conditions.
- By initial conditions we mean that the conditions on the dependent variable are specified at a single value of the independent variable.
- If the conditions of the dependent variable are specified at two different values of the independent variable, the conditions are called boundary conditions.
- The problem with initial conditions is referred to as the Initial value problem.
- The problem with boundary conditions is referred to as the Boundary value problem.

Example 1 : The problem of solving the equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=x$ with conditions y(0) = y' (0) = 1 is an initial value problem

Example 2 : The problem of solving the equation $3\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\cos x$ with y(1)=1, y(2)=3 is called Boundary value problem.

Laplace transform is essentially employed to solve initial value problems. This technique is of great utility in applications dealing with mechanical systems and electric circuits. Besides the technique may also be employed to find certain integral values also. The transform is named after the French Mathematician P.S. de' Laplace (1749 – 1827).

The subject is divided into the following sub topics.



Definition :

Let f(t) be a real-valued function defined for all $t \geq 0$ and s be a parameter, real or complex. Suppose the integral $\int_0^\infty e^{-st} f(t)dt$ exists (converges). Then this integral is called the Laplace transform of f(t) and is denoted by Lf(t).

Thus,

$$Lf(t) = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

We note that the value of the integral on the right hand side of (1) depends on s. Hence Lf(t) is a function of s denoted by F(s) or $\overline{f}(s)$.

Thus,

$$Lf(t) = F(s) \qquad (2)$$

Consider relation (2). Here f(t) is called the Inverse Laplace transform of F(s) and is denoted by L^{-1} [F(s)].

Thus,

$$L^{-1}[F(s)] = f(t) \qquad (3)$$

Suppose f(t) is defined as follows :

$$f(t) = \begin{cases} f_1(t), & 0 < t < a \\ f_2(t), & a < t < b \\ f_3(t), & t > b \end{cases}$$

Note that f(t) is piecewise continuous. The Laplace transform of f(t) is defined as

$$Lf(t) = \int_0^\infty e^{-st} f(t)$$

$$= \int_0^a e^{-st} f_1(t)dt + \int_a^b e^{-st} f_2(t)dt + \int_b^\infty e^{-st} f_3(t)dt$$

NOTE : In a practical situation, the variable t represents the time and s represents frequency. Hence the Laplace transform converts the time domain into the frequency domain.

Basic properties

The following are some basic properties of Laplace transforms :

1. **Linearity property** : For any two functions f(t) and ϕ(t) (whose Laplace transforms exist) and any two constants a and b, we have

$$L\,[a\,f(t) + b\,\phi(t)] = a\,L\,f(t) + b\,L\phi(t)$$

Proof :- By definition, we have

$$L[af(t)+b\phi(t)] = \int_0^\infty e^{-st}\left[af(t)+b\phi(t)\right]dt \quad = a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st}\phi(t)dt$$

$$= a\,L\,f(t) + b\,L\phi(t)$$

This is the desired property.

In particular, for a=b=1, we have

$$L\,[\,f(t) + \phi(t)] = L\,f(t) + L\phi(t)$$

and for a = -b = 1, we have

$$L\,[\,f(t) - \phi(t)] = L\,f(t) - L\phi(t)$$

2. **Change of scale property** : If L f(t) = F(s), then L[f(at)] = $\frac{1}{a}F\left(\frac{s}{a}\right)$, where a is a positive constant.

Proof :- By definition, we have

$$Lf(at) = \int_0^\infty e^{-st} f(at)dt \qquad (1)$$

Let us set at = x. Then expression (1) becomes,

$$L\,f(at) = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)dx \qquad = \frac{1}{a}F\left(\frac{s}{a}\right)$$

This is the desired property.

3. **Shifting property** :- Let a be any real constant. Then

$$L\,[e^{at}f(t)] = F(s-a)$$

Proof :- By definition, we have

$$L\,[e^{at}f(t)] = \int_0^\infty e^{-st}\left[e^{at} f(t)\right]dt \qquad = \int_0^\infty e^{-(s-a)} f(t)dt$$

$$= F(s-a)$$

This is the desired property. Here we note that the Laplace transform of e^{at} f(t) can be written down directly by changing s to s-a in the Laplace transform of f(t).

TRANSFORMS OF SOME FUNCTIONS

1. Let a be a constant. Then

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \frac{1}{s-a}, \quad s > a$$

Thus,

$$L(e^{at}) = \frac{1}{s-a}$$

In particular, when a=0, we get

$$L(1) = \frac{1}{s}, \quad s > 0$$

By inversion formula, we have

$$L^{-1}\frac{1}{s-a} = e^{at} L^{-1}\frac{1}{s} = e^{at}$$

2. $$L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right) \qquad = \frac{1}{2}\int_0^\infty e^{-st}\left[e^{at} + e^{-at}\right]dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} + e^{-(s+a)t} dt\right]$$

Let $s > |a|$. Then,

$$L(\cosh at) = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty \qquad = \frac{s}{s^2 - a^2}$$

Thus,

$$L\,(\cosh at) = \frac{s}{s^2 - a^2}, \quad s > |a|$$

and so

$$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

3. L (sinh at) = $L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{a}{s^2-a^2}$, s > |a|

Thus,

L (sinh at) = $\frac{a}{s^2-a^2}$, s > |a|

and so,

$$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{\sinh at}{a}$$

4. L (sin at) = $\int_0^\infty e^{-st} \sin at$ dt

Here we suppose that s > 0 and then integrate by using the formula

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

Thus,

L (sinh at) = $\frac{a}{s^2+a^2}$, s > 0

and so

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sinh at}{a}$$

5. L (cos at) = $\int_0^\infty e^{-st} \cos at dt$

Here we suppose that s>0 and integrate by using the formula

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

Thus,

L (cos at) = $\frac{s}{s^2+a^2}$, s > 0

and so

$$L^{-1}\frac{s}{s^2+a^2} = \cos at$$

6. Let n be a constant, which is a non-negative real number or a negative non-integer. Then

$$L(t^n) = \int_0^\infty e^{-st} t^n dt$$

Let s > 0 and set st = x, then

$$L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

The integral $\int_0^\infty e^{-x} x^n dx$ is called gamma function of (n+1) denoted by $\Gamma(n+1)$. Thus

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

In particular, if n is a non-negative integer then $\Gamma(n+1)$ =n!. Hence

$$L(t^n) = \frac{n!}{s^{n+1}}$$

and so

$$L^{-1} \frac{1}{s^{n+1}} = \frac{t^n}{\Gamma(n+1)} \quad \text{or} \quad \frac{t^n}{n!} \quad \text{as the case may be}$$

TABLE OF LAPLACE TRANSFORMS

f(t)	F(s)
1	$\frac{1}{s}$, s > 0
e^{at}	$\frac{1}{s-a}$, s > a
coshat	$\frac{s}{s^2-a^2}$, s > |a|
sinhat	$\frac{a}{s^2-a^2}$, s > |a|
sinat	$\frac{a}{s^2+a^2}$, s > 0
cosat	$\frac{s}{s^2+a^2}$, s > 0
t^n, n=0,1,2,…..	$\frac{n!}{s^{n+1}}$, s > 0
t^n, n > -1	$\frac{\Gamma(n+1)}{s^{n+1}}$, s > 0

Application of shifting property :-

The shifting property is

If L f(t) = F(s), then L [e^{at}f(t)] = F(s-a)

Application of this property leads to the following results :

1. $L(e^{at}\cosh bt) = [L(\cosh bt)]_{s\to s-a} = \left(\frac{s}{s^2-b^2}\right)_{s\to s-a} = \frac{s-a}{(s-a)^2-b^2}$

Thus,

L(e^{at}coshbt) $= \frac{s-a}{(s-a)^2-b^2}$

and

$$L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at}\cosh bt$$

2. $L(e^{at}\sinh bt) = \dfrac{a}{(s-a)^2 - b^2}$

and

$$L^{-1}\frac{1}{(s-a)^2 - b^2} = e^{at}\sinh bt$$

3. $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}$

and

$$L^{-1}\frac{s-a}{(s-a)^2 + b^2} = e^{at}\cos bt$$

4. $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2 - b^2}$

and

$$L^{-1}\frac{1}{(s-a)^2 - b^2} = \frac{e^{at}\sin bt}{b}$$

5. $L(e^{at}t^n) = \dfrac{\Gamma(n+1)}{(s-a)^{n+1}}$ or $\dfrac{n!}{(s-a)^{n+1}}$ as the case may be

Hence

$$L^{-1}\frac{1}{(s-a)^{n+1}} = \frac{e^{at}t^n}{\Gamma(n+1)} \quad \text{or} \quad \frac{n!}{(s-a)^{n+1}} \quad \text{as the case may be}$$

Examples :-

1. Find Lf(t) given $f(t) = \begin{cases} t, & 0 < t < 3 \\ 4, & t > 3 \end{cases}$

Here

$$Lf(t) = \int_0^\infty e^{-st} f(t)dt = \int_0^3 e^{-st} t\,dt + \int_3^\infty 4e^{-st}dt$$

Integrating the terms on the RHS, we get

$$Lf(t) = \frac{1}{s}e^{-3s} + \frac{1}{s^2}(1 - e^{-3s})$$

This is the desired result.

2. Find Lf(t) given f(t) = $\begin{cases} \sin 2t, & 0 < t \le \pi \\ 0, & t > \pi \end{cases}$

Here

$$Lf(t) = \int_0^{\pi} e^{-st} f(t)dt + \int_{\pi}^{\infty} e^{-st} f(t)dt \qquad = \int_0^{\pi} e^{-st} \sin 2t dt$$

$$= \left[\frac{e^{-st}}{s^2+4} \{-s\sin 2t - 2\cos 2t\} \right]_0^{\pi} \qquad = \frac{2}{s^2+4}\left[1 - e^{-\pi s}\right]$$

This is the desired result.

3. Evaluate : (i) L(sin3t sin4t)

(ii) $L(\cos^2 4t)$

(iii) $L(\sin^3 2t)$

(i) Here

$$L(\sin 3t \sin 4t) = L\left[\frac{1}{2}(\cos t - \cos 7t)\right]$$

$$= \frac{1}{2}\left[L(\cos t) - L(\cos 7t)\right], \text{ by using linearity property}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+49}\right] = \frac{24s}{(s^2+1)(s^2+49)}$$

(ii) Here

$$L(\cos^2 4t) = L\left[\frac{1}{2}(1+\cos 8t)\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+64}\right]$$

(iii) We have

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

For θ=2t, we get

$$\sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$$

so that

$$L(\sin^3 2t) = \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right] = \frac{48}{(s^2+4)(s^2+36)}$$

This is the desired result.

4. Find L(cost cos2t cos3t)

Here

$$\cos 2t \cos 3t = \frac{1}{2}[\cos 5t + \cos t]$$

so that

$$\text{cost cos2t cos3t} = \frac{1}{2}[\cos 5t \cos t + \cos^2 t]$$

$$= \frac{1}{4}[\cos 6t + \cos 4t + 1 + \cos 2t]$$

Thus

$$L(\text{cost cos2t cos3t}) = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{1}{s} + \frac{s}{s^2+4}\right]$$

5. Find $L(\cosh^2 2t)$

We have

$$\cosh^2\theta = \frac{1+\cosh 2\theta}{2}$$

For θ = 2t, we get

$$\cosh^2 2t = \frac{1+\cosh 4t}{2}$$

Thus,

$$L(\cosh^2 2t) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-16}\right]$$

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6. Evaluate (i) $L(\sqrt{t})$ (ii) $L\left(\frac{1}{\sqrt{t}}\right)$ (iii) $L(t^{-3/2})$

We have $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$

(i) For n= $\frac{1}{2}$, we get

$$L(t^{1/2}) = \frac{\Gamma(\frac{1}{2}+1)}{s^{3/2}}$$

Since $\Gamma(n+1) = n\Gamma(n)$, we have $\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

Thus, $L(\sqrt{t}) = \dfrac{\sqrt{\pi}}{2s^{3/2}}$

(ii) For n = $-\dfrac{1}{2}$, we get

$$L(t^{-1/2}) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

(iii) For n = $-\dfrac{3}{2}$, we get

$$L(t^{-3/2}) = \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{-1/2}} = -2\sqrt{\pi s}$$

7. Evaluate : (i) $L(t^2)$ (ii) $L(t^3)$

We have,

$$L(t^n) = \frac{n!}{s^{n+1}}$$

(i) For n = 2, we get

$$L(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$$

(ii) For n=3, we get

$$L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$$

8. Find $L[e^{-3t}(2\cos 5t - 3\sin 5t)]$

Given =

$2L(e^{-3t}\cos 5t) - 3L(e^{-3t}\sin 5t)$

$= 2\dfrac{(s+3)}{(s+3)^2+25} - \dfrac{15}{(s+3)^2+25}$, by using shifting property

$= \dfrac{2s-9}{s^2+6s+34}$, on simplification

9. Find L[coshat sinhat]

Here

$$L[\cosh at \sin at] = L\left[\frac{\left(e^{at}+e^{-at}\right)}{2}\sin at\right]$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{a(s^2+2a^2)}{[(s-a)^2+a^2][(s+a)^2+a^2]}$$, on simplification

10. Find L(cosht $\sin^3$ 2t)

Given

$$L\left[\left(\frac{e^t+e^{-t}}{2}\right)\left(\frac{3\sin 2t-\sin 6t}{4}\right)\right]$$

$$= \frac{1}{8}\left[3\cdot L\left(e^t\sin 2t\right)-L(e^t\sin 6t)+3L(e^{-t}\sin 2t)-L(e^{-t}\sin 6t)\right]$$

$$= \frac{1}{8}\left[\frac{6}{(s-1)^2+4}-\frac{6}{(s-1)^2+36}+\frac{6}{(s+1)^2+4}-\frac{6}{(s+1)^2+36}\right]$$

$$= \frac{3}{4}\left[\frac{1}{(s-1)^2+4}-\frac{1}{(s-1)^2+36}+\frac{1}{(s+1)^2+24}-\frac{1}{(s+1)^2+36}\right]$$

11. Find $L(e^{-4t}t^{-5/2})$

We have

L(t^n) = $\frac{\Gamma(n+1)}{s^{n+1}}$ Put n= -5/2. Hence

L($t^{-5/2}$) = $\frac{\Gamma(-3/2)}{s^{-3/2}}=\frac{4\sqrt{\pi}}{3s^{-3/2}}$ Change s to s+4.

Therefore, $L(e^{-4t}t^{-5/2})=\frac{4\sqrt{\pi}}{3(s+4)^{-3/2}}$

Transform of t^n f(t)

Here we suppose that n is a positive integer. By definition, we have

F(s) = $\int_0^\infty e^{-st}f(t)dt$

Differentiating 'n' times on both sides w.r.t. s, we get

$$\frac{d^n}{ds^n}F(s)=\frac{\partial^n}{\partial s^n}\int_0^\infty e^{-st}f(t)dt$$

Performing differentiation under the integral sign, we get

$$\frac{d^n}{ds^n}F(s)=\int_0^\infty (-t)^n e^{-st} f(t)dt$$

Multiplying on both sides by $(-1)^n$, we get

$$(-1)^n \frac{d^n}{ds^n}F(s)=\int_0^\infty (t^n f(t)e^{-st}dt = L[t^n f(t)],\ \text{by definition}$$

Thus,

$$L[t^n f(t)]= (-1)^n \frac{d^n}{ds^n}F(s)$$

This is the transform of t^n f(t).

Also,

$$L^{-1}\left[\frac{d^n}{ds^n}F(s)\right]=(-1)^n t^n f(t)$$

In particular, we have

$$L[t\ f(t)] = -\frac{d}{ds}F(s),\ \text{for n=1}$$

$$L[t^2\ f(t)]=\frac{d^2}{ds^2}F(s),\ \text{ for n=2, etc.}$$

Also,

$$L^{-1}\left[\frac{d}{ds}F(s)\right]=-tf(t) \quad \text{and}$$

$$L^{-1}\left[\frac{d^2}{ds^2}F(s)\right]=t^2 f(t)$$

Transform of $\frac{f(t)}{t}$

We have , $F(s) = \int_0^\infty e^{-st} f(t)dt$

Therefore,

$$\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right]ds \qquad = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt \qquad = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt = L\left(\frac{f(t)}{t}\right)$$

Thus,

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)ds$$

This is the transform of $\frac{f(t)}{t}$

Also,

$$L^{-1}\int_s^\infty F(s)ds = \frac{f(t)}{t}$$

Examples :

1. Find L[te^{-t} sin4t]

We have,

$$L[e^{-t}\sin 4t] = \frac{4}{(s+1)^2+16}$$

So that,

$$\text{L}[\text{te}^{-t}\ \text{sin4t}] = 4\left[-\frac{d}{ds}\left\{\frac{1}{s^2+2s+17}\right\}\right]$$

$$= \frac{8(s+1)}{(s^2+2s+17)^2}$$

2. Find L(t^2 sin3t)

We have

$$\text{L(sin3t)} = \frac{3}{s^2+9}$$

So that,

$$\text{L}(\text{t}^2\ \text{sin3t}) = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) \quad = -6\frac{d}{ds}\frac{s}{(s^2+9)^2} \quad = \frac{18(s^2-3)}{(s^2+9)^3}$$

3. Find $L\left(\frac{e^{-t}\sin t}{t}\right)$

We have

$$L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1}$$

Hence $L\left(\frac{e^{-t}\sin t}{t}\right) = \int_0^\infty \frac{ds}{(s+1)^2+1} = \left[\tan^{-1}(s+1)\right]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) \quad = \cot^{-1}(s+1)$$

4. Find $L\left(\frac{\sin t}{t}\right)$. Using this, evaluate $L\left(\frac{\sin at}{t}\right)$

We have

$$L(\sin t) = \frac{1}{s^2+1}$$

So that

$$Lf(t) = L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = F(s)$$

Consider

$$L\left(\frac{\sin at}{t}\right) = a\,L\left(\frac{\sin at}{at}\right) = aLf(at)$$

$$= a\left[\frac{1}{a}F\left(\frac{s}{a}\right)\right], \text{ in view of the change of scale property}$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

5. Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

We have

$$L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

So that

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]ds \qquad = \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\underset{s\to\infty}{Lt}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 + \log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right] \qquad = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

6. Prove that $\int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$

We have

$$\int_0^\infty e^{-st} t \sin t dt = L(t \sin t) \quad = -\frac{d}{ds} L(\sin t) \quad = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

Putting s = 3 in this result, we get

$$\int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$$

This is the result as required.

ASSIGNMENT

I Find L f(t) in each of the following cases :

1. $f(t) = \begin{cases} e^t, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$

2. $f(t) = \begin{cases} 1, & 0 \le t \le 3 \\ t, & t > 3 \end{cases}$

3. $f(t) = \begin{cases} \dfrac{t}{a}, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

II. Find the Laplace transforms of the following functions :

4. $\cos(3t + 4)$
5. $\sin 3t \sin 5t$
6. $\cos 4t \cos 7t$
7. $\sin 5t \cos 2t$
8. $\sin t \sin 2t \sin 3t$
9. $\sin^2 5t$
10. $\sin^2(3t+5)$
11. $\cos^3 2t$
12. $\sinh^2 5t$
13. $t^{5/2}$
14. $\left(\sqrt{t} + \dfrac{1}{\sqrt{t}}\right)^3$
15. 3^t
16. 5^{-t}
17. $e^{-2t} \cos^2 2t$
18. $e^{2t} \sin 3t \sin 5t$
19. $e^{-t} \sin 4t + t \cos 2t$
20. $t^2 e^{-3t} \cos 2t$

21. $\dfrac{1-e^{-2t}}{t}$

22. $\dfrac{e^{-at}-e^{-bt}}{t}$

23. $\dfrac{\sin^2 t}{t}$

24. $\dfrac{2\sin 2t \sin 5t}{t}$

III. Evaluate the following integrals using Laplace transforms :

25. $\int_0^{\infty} te^{-2t} \sin 5t dt$

26. $\int_0^{\infty} e^{-5t} t^3 \cos t dt$

27. $\int_0^{\infty} \left(\dfrac{e^{-at}-e^{-bt}}{t} \right) dt$

28. $\int_0^{\infty} \dfrac{e^{-t} \sin t}{t} dt$

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Transforms of the derivatives of f(t)

Consider

$$L\, f'(t) = \int_0^\infty e^{-st} f'(t)dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt \text{, by using integration by parts}$$

$$= \left\lfloor \underset{t\to\infty}{Lt} (e^{-st} f(t) - f(0)\right\rfloor + sLf(t)$$

= 0 - f (0) + s Lf(t)

Thus

L $f'(t)$ = s Lf(t) – f(0)

Similarly,

L $f''(t)$ = s^2 L f(t) – s f(0) - $f'(0)$

In general, we have

$$Lf^n(t) = s^n Lf(t) - s^{n-1} f(0) - s^{n-2} f'(0) - .... - f^{n-1}(0)$$

Transform of $\int_0^t$ f(t)dt

Let ϕ (t) = $\int_0^t f(t)dt$. Then $\phi(0) = 0$ and $\phi'(t) = f(t)$

Now,

$$L\, \phi(t) = \int_0^\infty e^{-st}\phi(t)dt$$

$$= \left[\phi(t)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \phi'(t)\frac{e^{-st}}{-s}dt$$

$$= (0-0) + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

Thus,

$$L\int_0^t f(t)dt = \frac{1}{s}Lf(t)$$

Also,

$$L^{-1}\left[\frac{1}{s}Lf(t)\right] = \int_0^t f(t)dt$$

Examples :

1. By using the Laplace transform of sinat, find the Laplace transform of cosat.

Let

f(t) = sin at, then Lf(t) = $\dfrac{a}{s^2+a^2}$

We note that

$$f'(t) = a\cos at$$

Taking Laplace transforms, we get

$$Lf'(t) = L(a\cos at) = aL(\cos at)$$

or L(cosat) = $\dfrac{1}{a}Lf'(t) = \dfrac{1}{a}\left[sLf(t) - f(0)\right]$

$$= \frac{1}{a}\left[\frac{sa}{s^2+a^2} - 0\right]$$

Thus

L(cosat) = $\dfrac{s}{s^2+a^2}$

This is the desired result.

2. Given $L\left[2\sqrt{\dfrac{t}{\pi}}\right] = \dfrac{1}{s^{3/2}}$, show that $L\left[\dfrac{1}{\sqrt{\pi t}}\right] = \dfrac{1}{\sqrt{s}}$

Let f(t) = $2\sqrt{\dfrac{t}{\pi}}$, given L[f(t)] = $\dfrac{1}{s^{3/2}}$

We note that, $f'(t) = \dfrac{2}{\sqrt{\pi}}\dfrac{1}{2\sqrt{t}} = \dfrac{1}{\sqrt{\pi t}}$

Taking Laplace transforms, we get

$$Lf'(t) = L\left[\frac{1}{\sqrt{\pi t}}\right]$$

Hence

$$L\left[\frac{1}{\sqrt{\pi t}}\right] = Lf'(t) = sLf(t) - f(0)$$

$$= s\left(\frac{1}{s^{3/2}}\right) - 0$$

Thus

$$L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}$$

This is the result as required.

3. Find $L\int_0^t \left(\dfrac{\cos at - \cos bt}{t}\right)dt$

Here

$$Lf(t) = L\left(\frac{\cos at - \cos bt}{t}\right) = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Using the result

$$L\int_0^t f(t)dt = \frac{1}{s}Lf(t)$$

We get,

$$L\int_0^t \left(\frac{\cos at - \cos bt}{t}\right)dt = \frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

4. Find $L\int_0^t te^{-t}\sin 4t dt$

Here

$$L\left[te^{-t}\sin 4t\right] = \frac{8(s+1)}{(s^2+2s+17)^2}$$

Thus

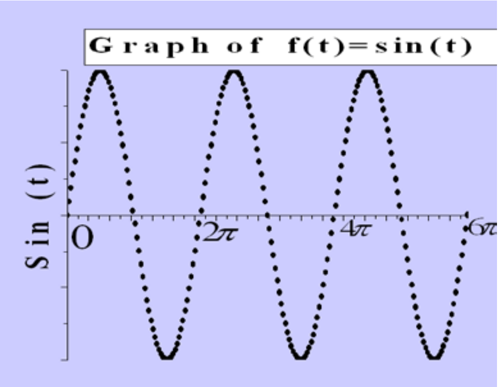
$$L\int_0^t te^{-t}\sin 4t dt = \frac{8(s+1)}{s(s^2+2s+17)^2}$$

Transform of a periodic function

A function f(t) is said to be a periodic function of period T > 0 if f(t) = f(t + nT) where n=1,2,3,….. The graph of the periodic function repeats itself in equal intervals.

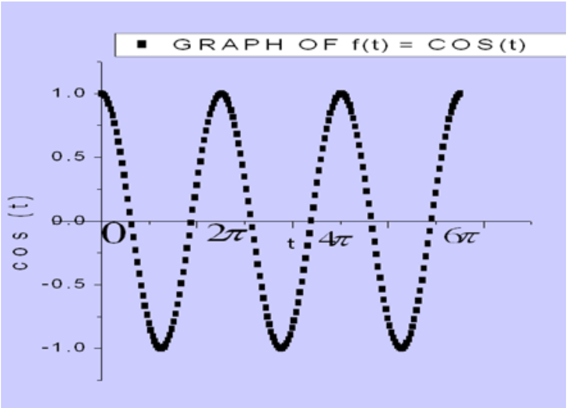
For example, sint, cost are periodic functions of period 2π since sin(t + 2nπ) = sin t, cos(t + 2nπ) = cos t.

The graph of f(t) = sin t is shown below :



Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on.

The graph of f(t) = cos t is shown below :



Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on.

Formula :

Let f(t) be a periodic function of period T. Then

$$Lf(t) = \frac{1}{1-e^{-ST}} \int_0^T e^{-st} f(t) dt$$

Proof :

By definition, we have

$$\text{L f(t)} = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-su} f(u) du$$

$$= \int_0^T e^{-su} f(u) du + \int_T^{2T} e^{-su} f(u) du + ... + \int_{nT}^{(n+1)T} e^{-su} f(u) du + ... + \infty$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-su} f(u) du$$

Let us set u = t + nT, then

$$\text{L f(t)} = \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-s(t+nT)} f(t+nT) dt$$

Here

f(t+nT) = f(t), by periodic property

Hence

$$Lf(t) = \sum_{n=0}^{\infty}(e^{-sT})^n \int_0^T e^{-st} f(t)dt$$

$$= \left[\frac{1}{1-e^{-ST}}\right]\int_0^T e^{-st} f(t)dt,$$ identifying the above series as a geometric series.

Thus

$$\text{L f(t)} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt$$

This is the desired result.

Examples:-

1. For the periodic function f(t) of period 4, defined by f(t) = $\begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}$

find L f(t)

Here, period of f(t) = T = 4

We have,

$$\text{L f(t)} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt \qquad = \left[\frac{1}{1-e^{-4s}}\right]\int_0^4 e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-4s}}\left[\int_0^2 3te^{-st}dt + \int_2^4 6e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-4s}}\left[3\left\{\left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^2 - \int_0^2 1.\frac{e^{-st}}{-s}dt\right\} + 6\left(\frac{e^{-st}}{-s}\right)_2^4\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{3(1-e^{-2s}-2se^{-4s})}{s^2}\right]$$

Thus,

$$\text{Lf(t)} = \frac{3(1-e^{-2s}-2se^{-4s})}{s^2(1-e^{-4s})}$$

2. A periodic function of period $\frac{2\pi}{\omega}$ is defined by

$$f(t) = \begin{cases} E\sin\omega t, & 0 \le t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega} \end{cases}$$

where E and ω are positive constants. Show that

$$\text{L f(t)} = \frac{E\omega}{(s^2+w^2)(1-e^{-\pi s/w})}$$

Here

T = $\dfrac{2\pi}{\omega}$. Therefore

$$L\ f(t) = \frac{1}{1-e^{-s(2\pi/\omega)}}\int_0^{2\pi/\omega} e^{-st} f(t)dt \quad = \frac{1}{1-e^{-s(2\pi/\omega)}}\int_0^{\pi/\omega} Ee^{-st}\sin\omega t dt$$

$$= \frac{E}{1-e^{-s(2\pi/\omega)}}\left[\frac{e^{-st}}{s^2+\omega^2}\{-s\sin\omega t-\omega\cos\omega t\}\right]_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{-s(2\pi/\omega)}}\frac{\omega(e^{-s\pi/\omega}+1)}{s^2+\omega^2}$$

$$= \frac{E\omega(1+e^{-s\pi/\omega})}{(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega})(s^2+\omega^2)}$$

$$= \frac{E\omega}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}$$

This is the desired result.

3. A periodic function f(t) of period 2a, a>0 is defined by

$$f(t) = \begin{cases} E, & 0 \le t \le a \\ -E, & a < t \le 2a \end{cases}$$

show that

$$L\ f(t) = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

Here T = 2a. Therefore

$$L\ f(t) = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a Ee^{-st}dt + \int_a^{2a} -Ee^{-st}dt\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left[\left(1-e^{-sa}\right)+(e^{-2as}-e^{-as})\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left[\left(1-e^{-as}\right)^2\right] = \frac{E(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{E}{s}\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

This is the result as desired.

-- 29.04.05 -------------------------------------

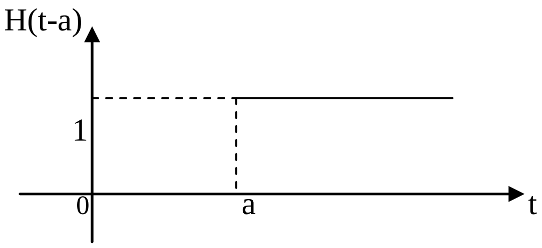
Step Function :

In many Engineering applications, we deal with an important discontinuous function H(t-a) defined as follows :

$$H(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$

where a is a non-negative constant.

This function is known as the unit step function or the Heaviside function. The function is named after the British electrical engineer Oliver Heaviside. The function is also denoted by u(t-a). The graph of the function is shown below:



Note that the value of the function suddenly jumps from value zero to the value 1 as $t \to a$ from the left and retains the value 1 for all t>a. Hence the function H(t-a) is called the unit step function.

In particular, when a=0, the function H(t-a) become H(t), where

$$H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$$

Transform of step function

By definition, we have

$$L[H(t-a)] = \int_0^\infty e^{-st} H(t-a)dt$$

$$= \int_0^a e^{-st} 0dt + \int_a^\infty e^{-st}(1)dt$$

$$= \frac{e^{-as}}{s}$$

In particular, we have L H(t) = $\frac{1}{s}$

Also, $L^{-1}\left[\frac{e^{-as}}{s}\right] = H(t-a)$ and $L^{-1}\left(\frac{1}{s}\right) = H(t)$

Heaviside shift theorem

Statement :-
L [f(t-a) H(t-a)] = e^{-as} Lf(t)

Proof :- We have

L [f(t-a) H(t-a)] = $\int_0^\infty f(t-a)H(t-a)e^{-st}dt$

$= \int_a^\infty e^{-st} f(t-a)dt$

Setting t-a = u, we get

L[f(t-a) H(t-a)] = $\int_0^\infty e^{-s(a+u)} f(u)du$

= e^{-as} L f(t)

This is the desired shift theorem.
Also,
L^{-1} [e^{-as} L f(t)] = f(t-a) H(t-a)

Examples :

1. Find L[e^{t-2} + sin(t-2)] H(t-2)

Let
f(t-2) = [e^{t-2} + sin(t-2)]
Then
f(t) = [e^t + sint]
so that
L f(t) = $\frac{1}{s-1} + \frac{1}{s^2+1}$
By Heaviside shift theorem, we have

L[f(t-2) H(t-2)] = e^{-2s} Lf(t)
Thus,
$$L[e^{(t-2)} + \sin(t-2)]H(t-2) = e^{-2s}\left[\frac{1}{s-1} + \frac{1}{s^2+1}\right]$$

2. Find L(3t^2 +2t +3) H(t-1)

Let
f(t-1) = 3t^2 +2t +3

so that

$$f(t) = 3(t+1)^2 + 2(t+1) + 3 = 3t^2 + 8t + 8$$

Hence

$$Lf(t) = \frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s}$$

Thus

$$L[3t^2 + 2t + 3]\, H(t-1) = L[f(t-1)\, H(t-1)]$$
$$= e^{-s} L f(t)$$
$$= e^{-s}\left[\frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s}\right]$$

3. Find $Le^{-t} H(t-2)$

Let $f(t-2) = e^{-t}$, so that, $f(t) = e^{-(t+2)}$

Thus,

$$L\, f(t) = \frac{e^{-2}}{s+1}$$

By shift theorem, we have

$$L[f(t-2)H(t-2)] = e^{-2s} Lf(t) = \frac{e^{-2(s+1)}}{s+1}$$

Thus

$$L\left[e^{-t} H(t-2)\right] = \frac{e^{-2(s+1)}}{s+1}$$

4. Let $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$

Verify that

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]H(t-a)$$

Consider

$$f_1(t) + [f_2(t) - f_1(t)]H(t-a) = f_1(t) + \begin{cases} f_2(t) - f_1(t), & t > a \\ 0, & t \le a \end{cases}$$

$$= \begin{cases} f_2(t), & t > a \\ f_1(t), & t \le a \end{cases} = f(t), \text{ given}$$

Thus the required result is verified.

5. Express the following functions in terms of unit step function and hence find their Laplace transforms.

1. $f(t) = \begin{cases} t^2, & 1 < t \le 2 \\ 4t, & t > 2 \end{cases}$

2. $\begin{cases} \cos t, & 0 < t < \pi \end{cases}$

f(t) =

sint, t > π

1. Here,

$f(t) = t^2 + (4t-t^2)\, H(t-2)$

Hence,

$$L\, f(t) = \frac{2}{s^3} + L(4t - t^2)H(t-2) \qquad \text{(i)}$$

Let

$\phi(t-2) = 4t - t^2$

so that

$\phi(t) = 4(t+2) - (t+2)^2 = -t^2 + 4$

Now,

$$L\phi(t) = -\frac{2}{s^3} + \frac{4}{s}$$

Expression (i) reads as

$$L\, f(t) = \frac{2}{s^3} + L[\phi(t-2)H(t-2)]$$

$$= \frac{2}{s^3} + e^{-2s} L\phi(t)$$

$$= \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)$$

This is the desired result.

2. Here

$f(t) = \cos t + (\sin t - \cos t)H(t-\pi)$

Hence,

$$L\, f(t) = \frac{s}{s^2+1} + L(\sin t - \cos t)H(t-\pi) \qquad \text{(ii)}$$

Let

$\phi(t-\pi) = \sin t - \cos t$

Then

$\phi(t) = \sin(t+\pi) - \cos(t+\pi) = -\sin t + \cos t$

so that

$$L\, \phi(t) = -\frac{1}{s^2+1} + \frac{s}{s^2+1}$$

Expression (ii) reads as

$$L\, f(t) = \frac{s}{s^2+1} + L[\phi(t-\pi)H(t-\pi)]$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\phi(t)$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s-1}{s^2+1}\right]$$

-------------------------------- 3.05.04 --------------------------
CONVOLUTION

The convolution of two functions f(t) and g(t) denoted by f(t) $*$ g(t) is defined as

$$f(t) * g(t) = \int_0^t f(t-u)g(u)du$$

Property : f(t) $*$ g(t) = g(t) $*$ f(t)

Proof :- By definition, we have

$$f(t) * g(t) = \int_0^t f(t-u)g(u)du$$

Setting t-u = x, we get

$$f(t) * g(t) = \int_t^0 f(x)g(t-x)(-dx)$$

$$= \int_0^t g(t-x)f(x)dx = g(t) * f(t)$$

This is the desired property. Note that the operation $*$ is commutative.

Convolution theorem :-

L[f(t) $*$ g(t)] = L f(t) . L g(t)

Proof :- Let us denote

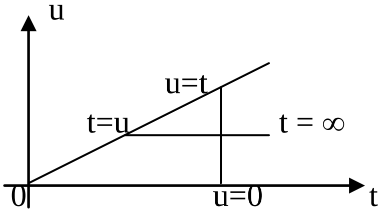
$$f(t) * g(t) = \phi(t) = \int_0^t f(t-u)g(u)du$$

Consider

$$L[\phi(t)] = \int_0^\infty e^{-st}[\int_0^t f(t-u)g(u)]dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(t-u)g(u)du \qquad (1)$$

We note that the region for this double integral is the entire area lying between the lines u =0 and u = t. On changing the order of integration, we find that t varies from u to ∞ and u varies from 0 to ∞.



Hence (1) becomes

$$L[\phi(t)] = \int_{u=0}^\infty \int_{t=u}^\infty e^{-st} f(t-u)g(u)dtdu$$

$$= \int_0^\infty e^{-su} g(u) \left\{ \int_u^\infty e^{-s(t-u)} f(t-u) dt \right\} du$$

$$= \int_0^\infty e^{-su} g(u) \left\{ \int_0^\infty e^{-sv} f(v) dv \right\} du \quad , \quad \text{where } v = t-u$$

$$= \int_0^\infty e^{-su} g(u) du \int_0^\infty e^{-sv} f(v) dv$$

$$= L\, g(t) \,.\, L\, f(t)$$

Thus

$$L\, f(t) \,.\, L\, g(t) = L[f(t) * g(t)]$$

This is desired property.

Examples :

1. Verify Convolution theorem for the functions f(t) and g(t) in the following cases :

(i) f(t) = t, g(t) = sint (ii) f(t) =t, g(t) = e^t

(i) Here,

$$f * g = \int_0^t f(u) g(t-u) du \quad = \int_0^t u \sin(t-u) du$$

Employing integration by parts, we get

$$f * g = t - \sin t$$

so that

$$L\,[f * g] = \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)} \qquad (1)$$

Next consider

$$L\, f(t) \,.\, L\, g(t) = \frac{1}{s^2} \cdot \frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)} \qquad (2)$$

From (1) and (2), we find that

$$L\,[f * g] = L\, f(t) \,.\, L\, g(t)$$

Thus convolution theorem is verified.

(ii) Here

$$f * g = \int_0^t u e^{t-u} du$$

Employing integration by parts, we get

$$f * g = e^t - t - 1$$

so that

$$L[f * g] = \frac{1}{s-1} - \frac{1}{s^2} - \frac{1}{s} = \frac{1}{s^2(s-1)} \qquad (3)$$

Next

$$L\, f(t) \,.\, L\, g(t) = \frac{1}{s^2} \cdot \frac{1}{s-1} = \frac{1}{s^2(s-1)} \qquad (4)$$

From (3) and (4) we find that

$L[f * g] = L\, f(t) \,.\, L\, g(t)$

Thus convolution theorem is verified.

2. By using the Convolution theorem, prove that

$$L\int_0^t f(t)dt = \frac{1}{s} Lf(t)$$

Let us define g(t) = 1, so that g(t-u) = 1

Then

$$L\int_0^t f(t)dt = L\int_0^t f(t)g(t-u)dt = L[f * g]$$

$$= L\, f(t) \,.\, L\, g(t) = L\, f(t) \,.\, \frac{1}{s}$$

Thus

$$L\int_0^t f(t)dt = \frac{1}{s} Lf(t)$$

This is the result as desired.

3. Using Convolution theorem, prove that

$$L\int_0^t e^{-u}\sin(t-u)du = \frac{1}{(s+1)(s^2+1)}$$

Let us denote, $f(t) = e^{-t}$ $g(t) = \sin t$, then

$$L\int_0^t e^{-u}\sin(t-u)du = L\int_0^t f(u)g(t-u)du = L\, f(t) \,.\, L\, g(t)$$

$$= \frac{1}{(s+1)} \cdot \frac{1}{(s^2+1)} = \frac{1}{(s+1)(s^2+1)}$$

This is the result as desired.

ASSIGNMENT

1. By using the Laplace transform of coshat, find the Laplace transform of sinhat.

2. Find (i) $L\int_0^t \frac{\sin t}{t}dt$ (ii) $L\int_0^t e^{-t}\cos t dt$ (iii) $L\int_0^t e^{-t}\cos t dt$

(iv) $L\int_0^t \left(\frac{\sin t}{t}\right)e^t dt$ (v) $L\int_0^t t^2 \sin at dt$

3. If $f(t) = t^2$, $0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$, find L f(t)

4. Find L f(t) given $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a - t, & a < t \le 2a \end{cases}$ $f(2a+t) = f(t)$

5. Find L f(t) given $f(t) = \begin{cases} 1, & 0 < t < \frac{a}{2} \\ -1, & \frac{a}{2} < t < a \end{cases}$ $f(a+t) = f(t)$

6. Find the Laplace transform of the following functions :
(i) $e^{t-1}H(t-2)$ (ii) $t^2 H(t-2)$ (iii) $(t^2 + t + 1) H(t+2)$
(iv) $(e^{-t}\sin t) H(t - \pi)$

7. Express the following functions in terms of unit step function and hence find their Laplace transforms :

(i) $f(t) = \begin{cases} 2t, & 0 < t \le \pi \\ 1, & t > \pi \end{cases}$ (ii) $f(t) = \begin{cases} t^2, & 2 < t \le 3 \\ 3t, & t > 3 \end{cases}$

(iii) $f(t) = \begin{cases} \sin 2t, & 0 < t \le \pi \\ 0, & t > \pi \end{cases}$ (iv) $f(t) = \begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$

8. Let

$$f(t) = \begin{cases} f_1(t), & t \le a \\ f2(t), & a < t \le b \\ f3(t), & t > b \end{cases}$$

Verify that

$$f(t) = f1(t) + [f2(t) - f1(t)]H(t-a) + [f3(t) - f2(t)] H(t-b)$$

9. Express the following function in terms of unit step function and hence find its Laplace transform.

$$f(t) = \begin{cases} \text{Sint}, & 0 < t \le \pi \\ \text{Sin2t}, & \pi < t \le 2\pi \\ \text{Sin3t}, & t > 2\pi \end{cases}$$

10. Verify convolution theorem for the following pair of functions:
   (i) f(t) = cosat, g(t) = cosbt
   (ii) f(t) = t, g(t) = t e^{-t}
   (iii) f(t) = e^t g(t) = sint

11. Using the convolution theorem, prove the following:
   (i) $L\int_0^t (t-u)e^{u-1}\cos u du = \frac{s}{(s+1)^2(s^2+1)}$
   (ii) $L\int_0^t (t-u)ue^{-au} du = \frac{1}{s^2(s+a)^2}$

INVERSE LAPLACE TRANSFORMS

Let L f(t) = F(s). Then f(t) is defined as the inverse Laplace transform of F(s) and is denoted by

L^{-1} F(s). Thus L^{-1} F(s) = f(t).

Linearity Property

Let L^{-1} F(s) = f(t) and L^{-1} G(s) = g(t) and a and b be any two constants. Then

$$L^{-1}[a\,F(s) + b\,G(s)] = a\,L^{-1}F(s) + b\,L^{-1}G(s)$$

Table of Inverse Laplace Transforms

F(s)	$f(t) = L^{-1}F(s)$
$\frac{1}{s}, s > 0$	1
$\frac{1}{s-a}, s > a$	e^{at}
$\frac{s}{s^2+a^2}, s > 0$	Cos at
$\frac{1}{s^2+a^2}, s > 0$	$\frac{\text{Sin at}}{a}$
$\frac{1}{s^2-a^2}, s > \lvert a\rvert$	$\frac{\text{Sin h at}}{a}$
$\frac{s}{s^2-a^2}, s > \lvert a\rvert$	Cos h at
$\frac{1}{s^{n+1}}, s > 0$ $n = 0, 1, 2, 3, \ldots$	$\frac{t^n}{n!}$
$\frac{1}{s^{n+1}}, s > 0$ $n > -1$	$\frac{t^n}{\Gamma(n+1)}$

Examples

1. Find the inverse Laplace transforms of the following:

$$(i)\frac{1}{2s-5} \qquad (ii)\frac{s+b}{s^2+a^2} \qquad (iii)\frac{2s-5}{4s^2+25}+\frac{4s-9}{9-s^2}$$

Here

$$(i)\ \ L^{-1}\frac{1}{2s-5}=\frac{1}{2}L^{-1}\frac{1}{s-5/2}=\frac{1}{2}e^{\frac{5t}{2}}$$

$$(ii)\ \ L^{-1}\frac{s+b}{s^2+a^2}=L^{-1}\frac{s}{s^2+a^2}+b\,L^{-1}\frac{1}{s^2+a^2}=\cos at+\frac{b}{a}\sin at$$

$$(iii)\ \ L^{-1}\left[\frac{2s-5}{4s^2+25}+\frac{4s-8}{9-s^2}\right]=\frac{2}{4}L^{-1}\frac{s-5/2}{s^2+\frac{25}{4}}-4L^{-1}\frac{s-9/2}{s^2-9}$$

$$=\frac{1}{2}\left[\cos\frac{5t}{2}-\sin\frac{5t}{2}\right]-4\left[\cos h3t-\frac{3}{2}\sin h3t\right]$$

Evaluation of L^{-1} F(s – a)

We have, if L f(t) = F(s), then $L[e^{at} f(t)] = F(s-a)$, and so

$$L^{-1} F(s-a) = e^{at} f(t) = e^{at} L^{-1} F(s)$$

Examples

1. Evaluate: $L^{-1}\dfrac{3s+1}{(s+1)^4}$

$$\text{Given} = L^{-1}\frac{3(s+1-1)+1}{(s+1)^4}=3\,L^{-1}\frac{1}{(s+1)^3}-2\,L^{-1}\frac{1}{(s+1)^4}$$

$$=3e^{-t}\,L^{-1}\frac{1}{s^3}-2e^{-t}\,L^{-1}\frac{1}{s^4}$$

Using the formula

$$L^{-1}\frac{1}{s^{n+1}}=\frac{t^n}{n!}\quad and\ taking\ n=2\ \ and\ \ 3,\ we\ get$$

$$Given=\frac{3e^{-t}t^2}{2}-\frac{e^{-t}t^3}{3}$$

2. Evaluate: $L^{-1}\dfrac{s+2}{s^2-2s+5}$

$$\text{Given} = L^{-1}\frac{s+2}{(s-1)^2+4} = L^{-1}\left[\frac{(s-1)+3}{(s-1)^2+4}\right] = L^{-1}\frac{s-1}{(s-1)^2+4} + 3L^{-1}\frac{1}{(s-1)^2+4}$$

$$= e^t\, L^{-1}\frac{s}{s^2+4} + 3\, e^t\, L^{-1}\frac{1}{s^2+4}$$

$$= e^t \cos 2t + \frac{3}{2} e^t \sin 2t$$

$$\text{Evaluate}: L^{-1}\frac{2s+1}{s^2+3s+1}$$

$$\text{Given} = 2L^{-1}\frac{\left(s+\frac{3}{2}\right)-1}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}} = 2\left[L^{-1}\frac{\left(s+\frac{3}{2}\right)}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}} - L^{-1}\frac{1}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}}\right]$$

$$= 2\left[e^{\frac{-3t}{2}} L^{-1}\frac{s}{s^2-\frac{5}{4}} - e^{\frac{-3t}{2}} L^{-1}\frac{1}{s^2-\frac{5}{4}}\right]$$

$$= 2e^{\frac{-3t}{2}}\left[\cos h\frac{\sqrt{5}}{2}t \frac{2}{\sqrt{5}}\sin h\frac{\sqrt{5}}{2}t\right]$$

$$4.\ \text{Evaluate}: \quad \frac{2s^2+5s-4}{s^3+s^2-2s}$$

we have

$$\frac{2s^2+5s-4}{s^3+s^2-2s} = \frac{2s^2+5s-4}{s(s^2+s-2)} = \frac{2s^2+5s-4}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

Then

$$2s^2+5s-4 = A(s+2)(s-1) + Bs(s-1) + Cs(s+2)$$

For s = 0, we get A = 2, for s = 1, we get C = 1 and for s = -2, we get B = -1. Using these values in (1), we get

$$\frac{2s^2+5s-4}{s^3+s^2-2s} = \frac{2}{s} - \frac{1}{s+2} + \frac{1}{s-1}$$

Hence

$$L^{-1}\frac{2s^2+5s-4}{s^2+s^2-25} = 2 - e^{-2t} + e^t$$

$$5.\ \text{Evaluate}: L^{-1}\frac{4s+5}{(s+1)^2+(s+2)}$$

Let us take

$$\frac{4s+5}{(s+1)^2+(s+2)}=\frac{A}{(s+1)^2}+\frac{B}{s+1}+\frac{C}{s+2}$$

Then

4s + 5 = A(s + 2) + B(s + 1) (s + 2) + C (s + 1)^2

For s = -1, we get A = 1, for s = -2, we get C = -3

Comparing the coefficients of s^2, we get B + C = 0, so that B = 3. Using these values in (1), we get

$$\frac{4s+5}{(s+1)^2+(s+2)}=\frac{1}{(s+1)^2}+\frac{3}{(s+1)}-\frac{3}{s+2}$$

Hence

$$L^{-1}\frac{4s+5}{(s+1)^2+(s+2)}=e^{-t}L^{-1}\frac{1}{s^2}+3e^{-t}L^{-1}\frac{1}{s}-3e^{-2t}L^{-1}\frac{1}{s}$$

$$=te^{-t}+3e^{-t}-3e^{-2t}$$

$$5.\ \text{Evaluate}: L^{-1}\frac{s^3}{s^4-a^4}$$

Let

$$\frac{s^3}{s^4-a^4}=\frac{A}{s-a}+\frac{B}{s+a}+\frac{Cs+D}{s^2+a^2} \qquad (1)$$

Hence

s^3 = A(s + a) (s^2 + a^2) + B (s-a)(s^2+a^2)+(Cs + D) (s^2 – a^2)

For s = a, we get A = ¼; for s = -a, we get B = ¼; comparing the constant terms, we get D = a(A-B) = 0; comparing the coefficients of s^3, we get
1 = A + B + C and so C = ½. Using these values in (1), we get

$$\frac{s^3}{s^4-a^4}=\frac{1}{4}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]+\frac{1}{2}\frac{s}{s^2+a^2}$$

Taking inverse transforms, we get

$$L^{-1}\frac{s^3}{s^4-a^4}=\frac{1}{4}\left[e^{at}+e^{-at}\right]+\frac{1}{2}\cos at$$

$$=\frac{1}{2}\left[\cos hat+\cos at\right]$$

6. Evaluate: $L^{-1}\dfrac{s}{s^4+s^2+1}$

Consider

$$\frac{s}{s^4+s^2+1}=\frac{s}{\left(s^2+s+1\right)+\left(s^2-s+1\right)}=\frac{1}{2}\left[\frac{2s}{\left(s^2+s+1\right)\left(s^2-s+1\right)}\right]$$

$$=\frac{1}{2}\left[\frac{\left(s^2+s+1\right)-\left(s^2-s+1\right)}{\left(s^2+s+1\right)\left(s^2-s+1\right)}\right]=\frac{1}{2}\left[\frac{1}{\left(s^2-s+1\right)}-\frac{1}{\left(s^2+s+1\right)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

Therefore

$$L^{-1}\frac{s}{s^4+s^2+1}=\frac{1}{2}\left[e^{\frac{1}{2}t}L^{-1}\frac{1}{s^2+\frac{3}{4}}-e^{-\frac{1}{2}t}L^{-1}\frac{1}{s^2+\frac{3}{4}}\right]$$

$$=\frac{1}{2}\left[e^{\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}-e^{-\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right]$$

$$=\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\sin h\left(\frac{t}{2}\right)$$

Evaluation of $L^{-1}[e^{-as}\ F(s)]$

We have, if Lf(t) = F(s), then L[f(t-a) H(t-a) = e^{-as} F(s), and so

$L^{-1}[e^{-as} F(s)] = f(t-a) H(t-a)$

Examples

(1) $Evaluate: L^{-1}\dfrac{e^{-ss}}{(s-2)^4}$

Here

$$a=5,\ \ F(s)=\frac{1}{(s-2)^4}$$

$$Therefore\ \ f(t)=L^{-1}F(s)=L^{-1}\frac{1}{(s-2)^4}=e^{2t}L^{-1}\frac{1}{s^4}\quad =\frac{e^{2t}t^3}{6}$$

Thus

$$L^{-1}\frac{e^{-5s}}{(s-2)^4}\ =f(t-a)\,H(t-a)$$

$$=\frac{e^{2(t-5)}(t-5)^3}{6}H(t-5)$$

(2) $Evaluate: L^{-1}\left[\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{se^{-2\pi s}}{s^2+4}\right]$

$$Given=\ f_1(t-\pi)H(t-\pi)+\ f_2(t-2\pi)H(t-2\pi)\qquad (1)$$

$$Here\ \ f_1(t)=L^{-1}\frac{1}{s^2+1}=\sin t$$

$$f_2(t)=L^{-1}\frac{s}{s^2+4}=\cos 2t$$

Now relation (1) *reads as*

$$Given\ =\ \sin(t-\pi)\,H(t-\pi)+\cos 2(t-2\pi)\,H(t-2\pi)$$

$$=-\cos t\ H(t-\pi)+\cos(2t)\,H(t-2\pi)$$

Inverse transform of logarithmic and inverse functions

We have, if $\mathrm{L\,f(t) = F(s)}$, then $\mathrm{L}[\mathrm{tf}(t)] = -\dfrac{d}{ds}F(s)$ *Hence.*

$$L^{-1}\left(-\frac{d}{ds}F(s)\right) = tf(t)$$

Examples

(1) *Evaluate* : $L^{-1}\ \log\left(\dfrac{s+a}{s+b}\right)$

$$Let\ F(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$$

$$Then\ -\frac{d}{ds}F(s) = -\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$So\ that\quad L^{-1}\left[-\frac{d}{ds}F(s)\right] = -\left[e^{-at} - e^{-bt}\right]$$

$$or\quad t\ f(t) = e^{-bt} - e^{-at}$$

$$Thus\qquad f(t) = \frac{e^{-bt} - e^{-at}}{b}$$

(2) Evaluate $L^{-1} \quad \tan^{-1}\left(\frac{a}{s}\right)$

$$Let\ F(s) = \tan^{-1}\left(\frac{a}{s}\right)$$

$$Then\ -\frac{d}{ds}F(s) = \left[\frac{a}{s^2+a^2}\right]$$

$$or \quad L^{-1}\left[-\frac{d}{ds}F(s)\right] = \sin at \quad so\ that$$

$$or \quad t\ f(t) = \sin at$$

$$f(t) = \frac{\sin at}{a}$$

Inverse transform of $\left[\frac{F(s)}{s}\right]$

$$Since L\int_0^t f(t)dt = \frac{F(s)}{s}, we\ have,$$

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)dt$$

Examples

(1) $Evaluate:\ L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

$$Let\ us\ denote\ F(s) = \frac{1}{s^2+a^2} \quad so\ that$$

$$f(t) = L^{-1}F(s) = \frac{\sin at}{a}$$

$$Then\ L^{-1} = \frac{1}{s(s^2+a^2)} = L^{-1}\frac{F(s)}{s} = \int_0^t \frac{\sin at}{a}dt$$

$$= \frac{(1-\cos at)}{a^2}$$

(2) *Evaluate*: $L^{-1}\left[\dfrac{1}{s^2(s+a)^2}\right]$

we have $L^{-1}\dfrac{1}{(s+a)^2} = e^{-at}t$

Hence $L^{-1}\dfrac{1}{s(s+a)^2} = \int_0^t e^{-at}t\,dt$

$= \dfrac{1}{a^2}\left[1 - e^{-at}(1+at)\right]$, on integration by parts.

Using this, we get

$$L^{-1}\frac{1}{s^2(s+a)^2} = \frac{1}{a^2}\int_0^t \left[1 - e^{-at}(1+at)\right]dt$$

$$= \frac{1}{a^3}\left[at\left(1+e^{-at}\right) + 2\left(e^{-at} - 1\right)\right]$$

NEXT CLASS 24.05.05

Inverse transform of F(s) by using convolution theorem

We have, if L(t) = F(s) and Lg(t) = G(s), then

$L[f(t) * g(t)] = Lf(t)\cdot Lg(t) = F(s)\,G(s)$ *and so*

$$L^{-1}[F(s)\,G(s)] = f(t) * g(t) = \int_0^t f(t-u)g(u)du$$

This expression is called the convolution theorem for inverse Laplace transform

Examples

Employ convolution theorem to evaluate the following :

(1) $L^{-1}\dfrac{1}{(s+a)(s+b)}$

Let us denote $F(s)=\dfrac{1}{s+a},\ G(s)=\dfrac{1}{s+b}$

Taking the inverse, we get $f(t) = e^{-at},\ g(t) = e^{-bt}$

Therefore, by convolution theorem,

$$L^{-1}\frac{1}{(s+a)(s+b)}=\int_0^t e^{-a(t-u)}e^{-bu}\,du \qquad = e^{-at}\int_0^t e^{(a-b)u}\,du$$

$$= e^{-at}\left[\frac{e^{(a-b)t}-1}{a-b}\right] \qquad = \frac{e^{-bt}-e^{-at}}{a-b}$$

(2) $L^{-1}\dfrac{s}{(s^2+a^2)^2}$

Let us denote $F(s)=\dfrac{1}{s^2+a^2},\ G(s)=\dfrac{s}{s^2+a^2}$ Then

$f(t) = \dfrac{\sin at}{a},\ g(t) = \cos at$

Hence by convolution theorem,

$$L^{-1}\frac{s}{(s^2+a^2)^2}=\int_0^t \frac{1}{a}\sin a(t-u)\cos au\,du$$

$$=\frac{1}{a}\int_0^t \frac{\sin at+\sin(at-2au)}{2}du, \qquad \text{by using compound angle formula}$$

$$=\frac{1}{2a}\left[u\sin at-\frac{\cos(at-2au)}{-2a}\right]_0^t=\frac{t\sin at}{2a}$$

(3) $L^{-1}\ \dfrac{s}{(s-1)(s^2+1)}$

Here

$$F(s)=\frac{1}{s-1},\ G(s)=\frac{s}{s^2+1}$$

Therefore

$$f(t) = e^t,\ g(t) = \sin t$$

By convolution theorem, we have

$$L^{-1}\frac{1}{(s-1)(s^2+1)}=\int e^{t-u}\sin u\,du \qquad = e^t\left[\frac{e^{-u}}{2}(-\sin u-\cos u)\right]_0^t$$

$$=\frac{e^t}{2}\left[e^{-t}(-\sin t-\cos t)-(-1)\right]=\frac{1}{2}\left[e^t-\sin t-\cos t\right]$$

Assignment

By employing convolution theorem, evaluate the following :

(1) $L^{-1}\ \dfrac{1}{(s+1)(s^2+1)}$

(2) $L^{-1}\ \dfrac{s}{(s+1)^2(s^2+1)}$

(3) $L^{-1}\ \dfrac{1}{(s^2+a^2)^2}$

(4) $L^{-1}\ \dfrac{s^2}{(s^2+a^2)(s^2+b^2)},\ a\neq b$

(5) $L^{-1}\ \dfrac{1}{s^2(s+1)^2}$

(6) $L^{-1}\ \dfrac{4s+5}{(s-1)^2(s+2)}$

LAPLACE TRANSFORM METHOD FOR DIFFERENTIAL EQUATIONS

As noted earlier, Laplace transform technique is employed to solve initial-value problems. The solution of such a problem is obtained by using the Laplace Transform of the derivatives of function and then the inverse Laplace Transform.

The following are the expressions for the derivatives derived earlier.

$$L[f'(t)] = s\,L\,f(t) - f(o)$$

$$L[f''(t) = s^2\,L\,f(t) - s\,f(o) - f'(o)$$

$$L\,[f'''(t) = s^3\,L\,f(t) - s^2\,f(o) - s\,f'(o) - f''(o)$$

Examples

1) Solve by using Laplace transform method

$$y' + y = t\,e^{-t},\ y(o) = 2$$

Taking the Laplace transform of the given equation, we get

$$\left[sL\ y(t)-y(o)\right]+L\ y(t)=\frac{1}{(s+1)^2}$$

Using the given condition, this becomes

$$(s+1)L\ y(t)-2=\frac{1}{(s+1)^2}$$

so that

$$L\ y(t)=\frac{2s^2+4s+3}{(s+1)^3}$$

Taking the inverse Laplace transform, we get

$$Y(t)=L^{-1}\frac{2s^2+4s+3}{(s+1)^3}$$

$$=L^{-1}\left[\frac{2(s+1-1)^2+4(s+1-1)+3}{(s+1)^3}\right]$$

$$=L^{-1}\left[\frac{2}{s+1}+\frac{1}{(s+1)^3}\right]$$

$$=\frac{1}{2}e^{-t}\left(t^2+4\right)$$

This is the solution of the given equation.

Solve by using Laplace transform method :

(2) $y''+2y'-3y=\sin t,\quad y(o)=y'(o)=0$

Taking the Laplace transform of the given equation, we get

$$\left[s^2Ly(t)-sy(o)-y'(o)\right]+2\left[s\ Ly(t)-y(o)\right]-3\ L\ y(t)=\frac{1}{s^2+1}$$

Using the given conditions, we get

$$L\ y(t)\left[s^2+2s-3\right]=\frac{1}{s^2+1}$$

or

$$L\ y(t)=\frac{1}{(s-1)(s+3)(s^2+1)}$$

or

$$y(t)=L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]$$

$$=L^{-1}\left[\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{s^2+1}\right]$$

$$=L^{-1}\left[\frac{1}{8}\ \frac{1}{s-1}-\frac{1}{40}\ \frac{1}{s+3}+\frac{-\frac{s}{10}-\frac{1}{5}}{s^2+1}\right]$$ by using the method of partial sums,

$$=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}(\cos t+2\sin t)$$

This is the required solution of the given equation.

3) Employ Laplace Transform method to solve the integral equation.

$$\mathrm{f(t)}=1+\int_0^t f(u)\sin(t-u)du$$

Taking Laplace transform of the given equation, we get

$$L\ f(t)=\frac{1}{s}+L\int_0^t f(u)\sin(t-u)du$$

By using convolution theorem, here, we get

$$L\ f(t)=\frac{1}{s}+Lf(t)\cdot L\sin t=\frac{1}{s}+\frac{L\ f(t)}{s^2+1}$$

Thus

$$L\ f(t)=\frac{s^2+1}{s^3}\quad or\quad f(t)=L^{-1}\left(\frac{s^2+1}{s^3}\right)=1+\frac{t^2}{2}$$

This is the solution of the given integral equation.

(4) A particle is moving along a path satisfying, the equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$ where x denotes the displacement of the particle at time t. If the initial position of the particle is at x = 20 and the initial speed is 10, find the displacement of the particle at any time t using Laplace transforms.

Given equation may be rewritten as

$$x''(t) + 6x'(t) + 25x(t) = 0$$

Here the initial conditions are $x(o) = 20,\ x'(o) = 10$.
Taking the Laplace transform of the equation, we get

$$Lx(t)\left[s^2 + 6s + 25\right] - 20s - 130 = 0 \quad or$$

$$Lx(t) = \frac{20s + 130}{s^2 + 6s + 25}$$

so that

$$x(t) = L^{-1}\left[\frac{20s+130}{(s+3)^2+16}\right] \quad = L^{-1}\left[\frac{20(s+3)+70}{(s+3)^2+16}\right]$$

$$= 20\ L^{-1}\frac{s+3}{(s+3)^2+16} + 70\ \ L^{-1}\frac{1}{(s+3)^2+16}$$

$$= 20\ e^{-3t}\cos 4t + 35\,\frac{e^{-3t}\ \sin 4t}{2}$$

This is the desired solution of the given problem.

(5) A voltage Ee^{-at} is applied at t = 0 to a circuit of inductance L and resistance R. Show that the current at any time t is $\frac{E}{R - aL}\left[e^{-at} - e^{-\frac{Rt}{L}}\right]$

The circuit is an LR circuit. The differential equation with respect to the circuit is

$$L\frac{di}{dt} + Ri = E(t)$$

Here L denotes the inductance, i denotes current at any time t and E(t) denotes the E.M.F.
It is given that $E(t) = E\,e^{-at}$. With this, we have

Thus, we have

$$L\frac{di}{dt} + Ri = Ee^{-at} \quad or$$

$$Li'(t) + R\, i(t) = Ee^{-at}$$

$$\mathrm{L}[\mathrm{L_T}\ \mathrm{i'(t)}] + R[\mathrm{L_T}\ \mathrm{i'(t)}] = E\ \mathrm{L_T}\left(\mathrm{e}^{-at}\right) \qquad or$$

Taking Laplace transform (L_T) on both sides, we get

$$L[s\ L_T\ i(t) - i(o)] + R[L_T\ i(t)] = E\frac{1}{s+a}$$

Since i(o) = o, we get $\quad L_T\ i(t)[sL + R] = \dfrac{E}{s+a} \qquad or$

$$L_T\ i(t) = \frac{E}{(s+a)+(sL+R)}$$

Taking inverse transform $\mathrm{L_T^{-1}}$, we get $\quad i(t) = L_T^{-1}\dfrac{E}{(s+a)(sL+R)}$

$$= \frac{E}{R-aL}\left[L_T^{-1}\frac{1}{s+a} - L\ \ L_T^{-1}\frac{1}{sL+R}\right]$$

Thus

$$i(t) = \frac{E}{R-aL}\left[e^{-at} - e^{-\frac{Rt}{L}}\right]$$

This is the result as desired.

(6) Solve the simultaneous equations for x and y in terms of t given $\dfrac{\mathrm{dx}}{\mathrm{dt}} + 4y = 0$,

$\dfrac{\mathrm{dy}}{\mathrm{dt}} - 9x = 0$ with x(o) = 2, y(o) = 1.

Taking Laplace transforms of the given equations, we get

$$[\mathrm{s\ Lx(t) - x(o)}] + 4Ly(t) = 0$$

$$-9\ L\ x(t) + [s\ Ly(t) - y(o)] = 0$$

Using the given initial conditions, we get

$$s\ L\ x(t) + 4\ L\ y(t) = 2$$

$$-9\ L\ x(t) + 5\ L\ y(t) = 1$$

Solving these equations for Ly(t), we get

$$L\ y(t)\ = \frac{s+18}{s^2+36}$$

so that

$$y(t) = L^{-1}\left[\frac{s}{s^2+36}+\frac{18}{s^2+36}\right]$$

$$= \cos 6t + 3\sin 6t \qquad (1)$$

Using this in $\frac{dy}{dt} - 9x = 0$, we get

$$x(t) = \frac{1}{9}\left[-6\sin 6t + 18\cos 6t\right]$$

or

$$x(t) = \frac{2}{3}\left[3\cos 6t - \sin 6t\right] \qquad (2)$$

(1) and (2) together represents the solution of the given equations.

Assignment

Employ Laplace transform method to solve the following initial – value problems

1) $y''' + 2y'' - y' - 2y = 0 \quad given \;\; y(0) = y'(0) = 0, \;\; y''(0) = 6$

2) $y'' + 5y' + 6y = e^{2t}, \;\; y(0) = 2, \; y'(0) = 1$

3) $y'' + 4y' + 3y = e^{-t}, \;\; y(0) = 1 = y'(0)$

4) $y'' + 3y' + 2y = 2t^2 + 2t + 2, \;\; y(0) = 2, \; y'(0) = 0$

5) $y'' + 9y = \cos 2t, \;\; y(0) = 1, \;\; y\left(\frac{\pi}{2}\right) = -1$

6) $y'' + 2y' + y = t, \;\; y(0) = -3, \; y(1) = -1$

7) $y'' + y = H(t-1), \;\; y(0) = 0, \;\; y'(0) = 1$

8) $f(t) = 1 + 2\int_0^t f(t-u)e^{-2u}du$

9) $f'(t) = t + \int_0^t f(t-u)\cos u \, du, \;\; f(0) = 4$

10) A particle moves along a line so that its displacement x from a fixed point at any time t is governed by the equation $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 80\sin 5t$. If the particle is initially at rest, find the displacement at any time t.

11) In an L - R circuit, a voltage $E \sin \omega t$ is applied at $t = 0$. If the current is zero initially, find the current at any time.

Solve the following simultaneous differential equations for x and y in terms of t :

12) $\frac{dx}{dt}+y=\sin t,\quad \frac{dy}{dt}+x=\cos t,\quad x(0)=2,\ y(0)=0$

13) $\frac{dx}{dt}-y=e^{t},\quad \frac{dy}{dt}+x=\sin t,\quad x(0)=1,\ y(0)=0$

14) $\frac{dx}{dt}=2x-3y,\quad \frac{dy}{dt}=y-2x,\quad x(0)=8,\ y(0)=3$

15) $\frac{dx}{dt}+3y=2x,\quad \frac{dy}{dt}+2x=y,\quad x(0)=8,\ y(0)=3$

LAPLACE TRANSFORMS

INTRODUCTION

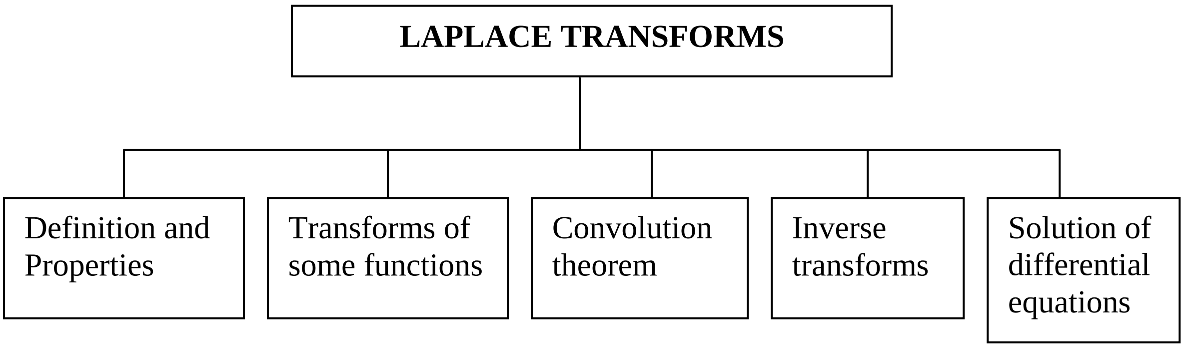
- Laplace transform is an integral transform employed in solving physical problems.
- Many physical problems when analysed assumes the form of a differential equation subjected to a set of initial conditions or boundary conditions.
- By initial conditions we mean that the conditions on the dependent variable are specified at a single value of the independent variable.
- If the conditions of the dependent variable are specified at two different values of the independent variable, the conditions are called boundary conditions.
- The problem with initial conditions is referred to as the Initial value problem.
- The problem with boundary conditions is referred to as the Boundary value problem.

Example 1 : The problem of solving the equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=x$ with conditions y(0) = y' (0) = 1 is an initial value problem

Example 2 : The problem of solving the equation $3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \cos x$ with y(1)=1, y(2)=3 is called Boundary value problem.

Laplace transform is essentially employed to solve initial value problems. This technique is of great utility in applications dealing with mechanical systems and electric circuits. Besides the technique may also be employed to find certain integral values also. The transform is named after the French Mathematician P.S. de' Laplace (1749 – 1827).

The subject is divided into the following sub topics.



Definition :

Let f(t) be a real-valued function defined for all t ≥ 0 and s be a parameter, real or complex. Suppose the integral $\int_0^\infty e^{-st} f(t)dt$ exists (converges). Then this integral is called the Laplace transform of f(t) and is denoted by Lf(t).

Thus,

$$Lf(t) = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

We note that the value of the integral on the right hand side of (1) depends on s. Hence Lf(t) is a function of s denoted by F(s) or $\overline{f}(s)$.

Thus,

$$Lf(t) = F(s) \qquad (2)$$

Consider relation (2). Here f(t) is called the Inverse Laplace transform of F(s) and is denoted by L^{-1} [F(s)].

Thus,

$$L^{-1}[F(s)] = f(t) \qquad (3)$$

Suppose f(t) is defined as follows :

$$f(t) = \begin{cases} f_1(t), & 0 < t < a \\ f_2(t), & a < t < b \\ f_3(t), & t > b \end{cases}$$

Note that f(t) is piecewise continuous. The Laplace transform of f(t) is defined as

$$Lf(t) = \int_0^\infty e^{-st} f(t)$$

$$= \int_0^a e^{-st} f_1(t)dt + \int_a^b e^{-st} f_2(t)dt + \int_b^\infty e^{-st} f_3(t)dt$$

NOTE : In a practical situation, the variable t represents the time and s represents frequency. Hence the Laplace transform converts the time domain into the frequency domain.

Basic properties

The following are some basic properties of Laplace transforms :

1. **Linearity property** : For any two functions f(t) and ϕ(t) (whose Laplace transforms exist) and any two constants a and b, we have

$$L[a f(t) + b\,\phi(t)] = a\,L f(t) + b\,L\phi(t)$$

Proof :- By definition, we have

$$L[af(t)+b\phi(t)] = \int_0^\infty e^{-st}[af(t)+b\phi(t)]dt \quad = a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st}\phi(t)dt$$

$$= a\, L\, f(t) + b\, L\phi(t)$$

This is the desired property.

In particular, for a=b=1, we have

$$L\,[\,f(t) + \phi(t)] = L\,f(t) + L\phi(t)$$

and for a = -b = 1, we have

$$L\,[\,f(t) - \phi(t)] = L\,f(t) - L\phi(t)$$

2. **Change of scale property** : If L f(t) = F(s), then L[f(at)] = $\frac{1}{a}F\left(\frac{s}{a}\right)$, where a is a positive constant.

Proof :- By definition, we have

$$Lf(at) = \int_0^\infty e^{-st} f(at)dt \qquad (1)$$

Let us set at = x. Then expression (1) becomes,

$$L\,f(at) = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)dx \qquad = \frac{1}{a}F\left(\frac{s}{a}\right)$$

This is the desired property.

3. **Shifting property** :- Let a be any real constant. Then

$$L\,[e^{at}f(t)] = F(s-a)$$

Proof :- By definition, we have

$$L\,[e^{at}f(t)] = \int_0^\infty e^{-st}\left[e^{at} f(t)\right]dt \qquad = \int_0^\infty e^{-(s-a)} f(t)dt$$

$$= F(s-a)$$

This is the desired property. Here we note that the Laplace transform of e^{at} f(t) can be written down directly by changing s to s-a in the Laplace transform of f(t).

TRANSFORMS OF SOME FUNCTIONS

3. Let a be a constant. Then

$$L(e^{at}) = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt$$

$$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \frac{1}{s-a}, \quad s > a$$

Thus,

$$L(e^{at}) = \frac{1}{s-a}$$

In particular, when a=0, we get

$$L(1) = \frac{1}{s}, \quad s > 0$$

By inversion formula, we have

$$L^{-1}\frac{1}{s-a} = e^{at}L^{-1}\frac{1}{s} = e^{at}$$

4. $L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right) \qquad = \frac{1}{2}\int_0^\infty e^{-st}\left[e^{at}+e^{-at}\right]dt$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} + e^{-(s+a)t}dt\right]$$

Let $s > |a|$. Then,

$$L(\cosh at) = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty \qquad = \frac{s}{s^2-a^2}$$

Thus,

$$L(\cosh at) = \frac{s}{s^2-a^2}, \quad s > |a|$$

and so

$$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

3. $L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{a}{s^2-a^2}, \quad s > |a|$

Thus,

$$L(\sinh at) = \frac{a}{s^2-a^2}, \quad s > |a|$$

and so,

$$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{\sinh at}{a}$$

4. L (sin at) = $\int_0^\infty e^{-st} \sin at \ dt$

Here we suppose that s > 0 and then integrate by using the formula

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2}[a \sin bx - b \cos bx]$$

Thus,

L (sinh at) = $\frac{a}{s^2 + a^2}$, s > 0

and so

$$L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{\sinh at}{a}$$

5. L (cos at) = $\int_0^\infty e^{-st} \cos at dt$

Here we suppose that s>0 and integrate by using the formula

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2}[a \cos bx + b \sin bx]$$

Thus,

L (cos at) = $\frac{s}{s^2 + a^2}$, s > 0

and so

$$L^{-1} \frac{s}{s^2 + a^2} = \cos at$$

6. Let n be a constant, which is a non-negative real number or a negative non-integer. Then

L(t^n) = $\int_0^\infty e^{-st} t^n dt$

Let s > 0 and set st = x, then

$$L(t^n) = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

The integral $\int_0^\infty e^{-x} x^n dx$ is called gamma function of (n+1) denoted by $\Gamma(n+1)$. Thus

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

In particular, if n is a non-negative integer then $\Gamma(n+1)$ =n!. Hence

$$L(t^n) = \frac{n!}{s^{n+1}}$$

and so

$$L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\Gamma(n+1)} \quad \text{or} \quad \frac{t^n}{n!} \quad \text{as the case may be}$$

TABLE OF LAPLACE TRANSFORMS

f(t)	F(s)
1	$\frac{1}{s}$, s > 0
e^{at}	$\frac{1}{s-a}$, s > a
coshat	$\frac{s}{s^2-a^2}$, s > |a|
sinhat	$\frac{a}{s^2-a^2}$, s > |a|
sinat	$\frac{a}{s^2+a^2}$, s > 0
cosat	$\frac{s}{s^2+a^2}$, s > 0
t^n, n=0,1,2,…..	$\frac{n!}{s^{n+1}}$, s > 0
t^n, n > -1	$\frac{\Gamma(n+1)}{s^{n+1}}$, s > 0

Application of shifting property :-

The shifting property is

If L f(t) = F(s), then L [e^{at}f(t)] = F(s-a)

Application of this property leads to the following results :

1. $L(e^{at}\cosh bt) = [L(\cosh bt)]_{s\to s-a} = \left(\frac{s}{s^2-b^2}\right)_{s\to s-a} = \frac{s-a}{(s-a)^2-b^2}$

Thus,

L(e^{at}coshbt) $= \frac{s-a}{(s-a)^2-b^2}$

and

$$L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at}\cosh bt$$

2. $L(e^{at}\sinh bt) = \dfrac{a}{(s-a)^2 - b^2}$

and

$$L^{-1}\frac{1}{(s-a)^2 - b^2} = e^{at}\sinh bt$$

3. $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}$

and

$$L^{-1}\frac{s-a}{(s-a)^2 + b^2} = e^{at}\cos bt$$

4. $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2 - b^2}$

and

$$L^{-1}\frac{1}{(s-a)^2 - b^2} = \frac{e^{at}\sin bt}{b}$$

5. $L(e^{at}t^n) = \dfrac{\Gamma(n+1)}{(s-a)^{n+1}}$ or $\dfrac{n!}{(s-a)^{n+1}}$ as the case may be

Hence

$$L^{-1}\frac{1}{(s-a)^{n+1}} = \frac{e^{at}t^n}{\Gamma(n+1)} \quad \text{or} \quad \frac{n!}{(s-a)^{n+1}} \quad \text{as the case may be}$$

Examples :-

1. Find Lf(t) given $f(t) = \begin{cases} t, & 0 < t < 3 \\ 4, & t > 3 \end{cases}$

Here

$$Lf(t) = \int_0^\infty e^{-st} f(t)dt = \int_0^3 e^{-st} t dt + \int_3^\infty 4e^{-st} dt$$

Integrating the terms on the RHS, we get

$$Lf(t) = \frac{1}{s}e^{-3s} + \frac{1}{s^2}(1 - e^{-3s})$$

This is the desired result.

2. Find Lf(t) given f(t) = $\begin{cases} \sin 2t, & 0 < t \leq \pi \\ 0, & t > \pi \end{cases}$

Here

$$Lf(t) = \int_0^{\pi} e^{-st} f(t)dt + \int_{\pi}^{\infty} e^{-st} f(t)dt \qquad = \int_0^{\pi} e^{-st} \sin 2t dt$$

$$= \left[\frac{e^{-st}}{s^2+4} \{-s\sin 2t - 2\cos 2t\} \right]_0^{\pi} \qquad = \frac{2}{s^2+4}\left[1 - e^{-\pi s}\right]$$

This is the desired result.

3. Evaluate : (i) L(sin3t sin4t)

(iv) L($\cos^2 4t$)

(v) L($\sin^3 2t$)

(i) Here

$$L(\sin 3t \sin 4t) = L\left[\frac{1}{2}(\cos t - \cos 7t)\right]$$

$$= \frac{1}{2}\left[L(\cos t) - L(\cos 7t)\right], \text{ by using linearity property}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+49}\right] = \frac{24s}{(s^2+1)(s^2+49)}$$

(ii) Here

$$L(\cos^2 4t) = L\left[\frac{1}{2}(1+\cos 8t)\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+64}\right]$$

(iii) We have

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

For θ=2t, we get

$$\sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$$

so that

$$L(\sin^3 2t) = \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right] = \frac{48}{(s^2+4)(s^2+36)}$$

This is the desired result.

4. Find L(cost cos2t cos3t)

Here

$$\cos 2t \cos 3t = \frac{1}{2}[\cos 5t + \cos t]$$

so that

$$\cos t \cos 2t \cos 3t = \frac{1}{2}[\cos 5t \cos t + \cos^2 t]$$

$$= \frac{1}{4}[\cos 6t + \cos 4t + 1 + \cos 2t]$$

Thus

$$L(\cos t \cos 2t \cos 3t) = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{1}{s} + \frac{s}{s^2+4}\right]$$

5. Find $L(\cosh^2 2t)$

We have

$$\cosh^2\theta = \frac{1+\cosh 2\theta}{2}$$

For θ = 2t, we get

$$\cosh^2 2t = \frac{1+\cosh 4t}{2}$$

Thus,

$$L(\cosh^2 2t) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-16}\right]$$

6. Evaluate (i) $L(\sqrt{t})$ (ii) $L\left(\frac{1}{\sqrt{t}}\right)$ (iii) $L(t^{-3/2})$

We have $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$

(i) For $n = \frac{1}{2}$, we get

$$L(t^{1/2}) = \frac{\Gamma(\frac{1}{2}+1)}{s^{3/2}}$$

Since $\Gamma(n+1) = n\Gamma(n)$, we have $\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

Thus, $L(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}}$

(ii) For n = $-\frac{1}{2}$, we get

$$L(t^{-1/2}) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

(iii) For n = $-\frac{3}{2}$, we get

$$L(t^{-3/2}) = \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{-1/2}} = -2\sqrt{\pi s}$$

7. Evaluate : (i) $L(t^2)$ (ii) $L(t^3)$

We have,

$$L(t^n) = \frac{n!}{s^{n+1}}$$

(i) For n = 2, we get

$$L(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$$

(ii) For n=3, we get

$$L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$$

8. Find $L[e^{-3t}(2\cos 5t - 3\sin 5t)]$

Given =

$2L(e^{-3t}\cos 5t) - 3L(e^{-3t}\sin 5t)$

$= 2\frac{(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}$, by using shifting property

$= \frac{2s-9}{s^2+6s+34}$, on simplification

9. Find L[coshat sinhat]

Here

$$\text{L[coshat sinat]} = L\left[\frac{\left(e^{at}+e^{-at}\right)}{2}\sin at\right]$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{a(s^2+2a^2)}{[(s-a)^2+a^2][(s+a)^2+a^2]} \text{ , on simplification}$$

10. Find L(cosht $\sin^3$ 2t)

Given

$$L\left[\left(\frac{e^t+e^{-t}}{2}\right)\left(\frac{3\sin 2t-\sin 6t}{4}\right)\right]$$

$$= \frac{1}{8}\left[3\cdot L\left(e^t\sin 2t\right)-L(e^t\sin 6t)+3L(e^{-t}\sin 2t)-L(e^{-t}\sin 6t)\right]$$

$$= \frac{1}{8}\left[\frac{6}{(s-1)^2+4}-\frac{6}{(s-1)^2+36}+\frac{6}{(s+1)^2+4}-\frac{6}{(s+1)^2+36}\right]$$

$$= \frac{3}{4}\left[\frac{1}{(s-1)^2+4}-\frac{1}{(s-1)^2+36}+\frac{1}{(s+1)^2+24}-\frac{1}{(s+1)^2+36}\right]$$

11. Find $L(e^{-4t}t^{-5/2})$

We have

$\text{L}(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$ Put n= -5/2. Hence

$\text{L}(t^{-5/2}) = \dfrac{\Gamma(-3/2)}{s^{-3/2}} = \dfrac{4\sqrt{\pi}}{3s^{-3/2}}$ Change s to s+4.

Therefore, $L(e^{-4t}t^{-5/2}) = \dfrac{4\sqrt{\pi}}{3(s+4)^{-3/2}}$

Transform of t^n f(t)

Here we suppose that n is a positive integer. By definition, we have

$$\text{F(s)} = \int_0^\infty e^{-st}f(t)dt$$

Differentiating ‘n’ times on both sides w.r.t. s, we get

$$\frac{d^n}{ds^n}F(s)=\frac{\partial^n}{\partial s^n}\int_0^\infty e^{-st}f(t)dt$$

Performing differentiation under the integral sign, we get

$$\frac{d^n}{ds^n}F(s)=\int_0^\infty (-t)^n e^{-st}f(t)dt$$

Multiplying on both sides by $(-1)^n$, we get

$$(-1)^n\frac{d^n}{ds^n}F(s)=\int_0^\infty (t^n f(t)e^{-st}dt = L[t^n f(t)],\ \text{by definition}$$

Thus,

$$L[t^n f(t)]= (-1)^n\frac{d^n}{ds^n}F(s)$$

This is the transform of t^n f(t).

Also,

$$L^{-1}\left[\frac{d^n}{ds^n}F(s)\right]=(-1)^n t^n f(t)$$

In particular, we have

$$L[t\, f(t)] = -\frac{d}{ds}F(s),\ \text{for } n=1$$

$$L[t^2 f(t)]=\frac{d^2}{ds^2}F(s),\ \ \text{for } n=2,\ \ \text{etc.}$$

Also,

$$L^{-1}\left[\frac{d}{ds}F(s)\right]=-tf(t)\quad \text{and}$$

$$L^{-1}\left[\frac{d^2}{ds^2}F(s)\right]=t^2 f(t)$$

Transform of $\frac{f(t)}{t}$

We have , F(s) = $\int_0^\infty e^{-st}f(t)dt$

Therefore,

$$\int_s^\infty F(s)ds=\int_s^\infty\left[\int_0^\infty e^{-st}f(t)dt\right]ds \qquad =\int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt \qquad = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt = L\left(\frac{f(t)}{t}\right)$$

Thus,

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)ds$$

This is the transform of $\dfrac{f(t)}{t}$

Also,

$$L^{-1}\int_s^\infty F(s)ds = \frac{f(t)}{t}$$

Examples :

1. Find L[te^{-t} sin4t]

We have,

$$L[e^{-t}\sin 4t] = \frac{4}{(s+1)^2+16}$$

So that,

$$L[te^{-t}\sin 4t] = 4\left[-\frac{d}{ds}\left\{\frac{1}{s^2+2s+17}\right\}\right]$$

$$= \frac{8(s+1)}{(s^2+2s+17)^2}$$

2. Find L(t^2 sin3t)

We have

$$L(\sin 3t) = \frac{3}{s^2+9}$$

So that,

$$L(t^2\sin 3t) = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) \quad = -6\frac{d}{ds}\frac{s}{(s^2+9)^2} \quad = \frac{18(s^2-3)}{(s^2+9)^3}$$

3. Find $L\left(\dfrac{e^{-t}\sin t}{t}\right)$

We have

$$L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1}$$

Hence $L\left(\frac{e^{-t}\sin t}{t}\right) = \int_0^{\infty}\frac{ds}{(s+1)^2+1} = \left[\tan^{-1}(s+1)\right]_s^{\infty}$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) \quad = \cot^{-1}(s+1)$$

4. Find $L\left(\frac{\sin t}{t}\right)$. Using this, evaluate $L\left(\frac{\sin at}{t}\right)$

We have

$$L(\sin t) = \frac{1}{s^2+1}$$

So that

$$Lf(t) = L\left(\frac{\sin t}{t}\right) = \int_s^{\infty}\frac{ds}{s^2+1} = \left[\tan^{-1}s\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s = F(s)$$

Consider

$$L\left(\frac{\sin at}{t}\right) = a\,L\left(\frac{\sin at}{at}\right) = aLf(at)$$

$$= a\left[\frac{1}{a}F\left(\frac{s}{a}\right)\right], \text{ in view of the change of scale property}$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

5. Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

We have

$$L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

So that

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^{\infty}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]ds \qquad = \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\underset{s\to\infty}{Lt}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 + \log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right] \quad = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

6. Prove that $\int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$

We have

$$\int_0^\infty e^{-st} t \sin t dt = L(t \sin t) \quad = -\frac{d}{ds} L(\sin t) \quad = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

Putting s = 3 in this result, we get

$$\int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$$

This is the result as required.

ASSIGNMENT

I Find L f(t) in each of the following cases :

1. $f(t) = \begin{cases} e^t, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$

2. $f(t) = \begin{cases} 1, & 0 \le t \le 3 \\ t, & t > 3 \end{cases}$

3. $f(t) = \begin{cases} \dfrac{t}{a}, & 0 \le t < a \\ 2, & t \ge a \end{cases}$

II. Find the Laplace transforms of the following functions :

4. $\cos(3t + 4)$
5. $\sin 3t \sin 5t$
6. $\cos 4t \cos 7t$
7. $\sin 5t \cos 2t$
8. $\sin t \sin 2t \sin 3t$
9. $\sin^2 5t$
10. $\sin^2(3t+5)$
11. $\cos^3 2t$
12. $\sinh^2 5t$
13. $t^{5/2}$
14. $\left(\sqrt{t} + \dfrac{1}{\sqrt{t}}\right)^3$
15. 3^t
16. 5^{-t}
17. $e^{-2t} \cos^2 2t$
18. $e^{2t} \sin 3t \sin 5t$
19. $e^{-t} \sin 4t + t \cos 2t$
20. $t^2 e^{-3t} \cos 2t$

21. $\dfrac{1-e^{-2t}}{t}$

22. $\dfrac{e^{-at}-e^{-bt}}{t}$

23. $\dfrac{\sin^2 t}{t}$

24. $\dfrac{2\sin 2t \sin 5t}{t}$

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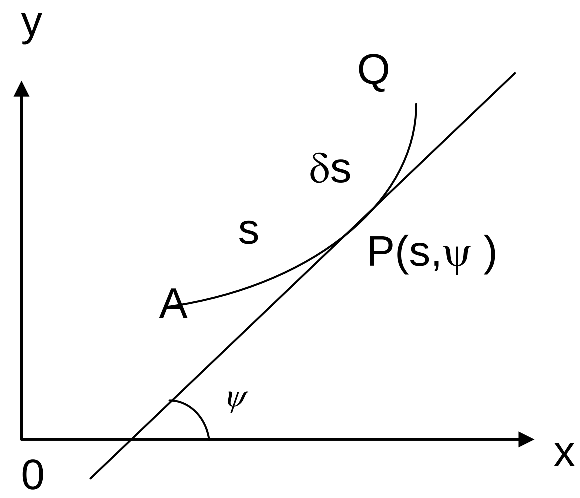
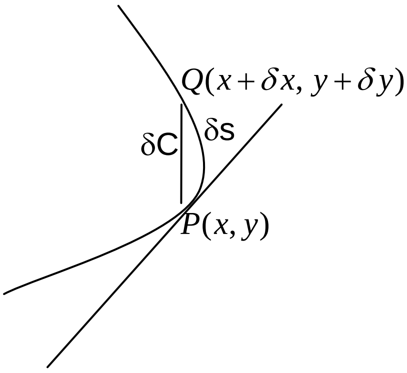
DERIVATIVES OF ARC LENGTH

Consider a curve C in the XY plane. Let A be a fixed point on it. Let P and Q be two neighboring positions of a variable point on the curve C. If ‘s’ is the distance of P from A measured along the curve then ‘s’ is called the arc length of P. Let the tangent to C at P make an angle ψ with X-axis. Then (s,ψ) are called the intrinsic co-ordinates of the point P. Let the arc length AQ be s + δs. Then the distance between P and Q measured along the curve C is δs. If the actual distance between P and Q is δC. Then δs=δC in the limit Q → P along C.



$$i.e. \quad \underset{Q\to P}{Lt} \frac{\delta s}{\delta C} = 1$$

Cartesian Form:



Let $y = f(x)$ be the Cartesian equation of the curve C and let $P(x, y)$ *and* $Q(x+\delta x, y+\delta y)$ be any two neighboring points on it as in fig.

Let the arc length $PQ = \delta s$ and the chord length $PQ = \delta C$. Using distance between two points formula we have $PQ^2 = (\delta C)^2 = (\delta x)^2 + (\delta y)^2$

$$\therefore \left(\frac{\delta C}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \quad or \quad \frac{\delta C}{\delta x} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

$$\Rightarrow \frac{\delta s}{\delta x} = \frac{\delta s}{\delta C} \cdot \frac{\delta C}{\delta x} = \frac{\delta s}{\delta C}\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

We note that $\delta x \to 0$ as $Q \to P$ along C, also that when $Q \to P, \dfrac{\delta s}{\delta C} = 1$

$\therefore$ When $Q \to P$ i.e. when $\delta x \to 0$, from (1) we get

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \to (1)$$

Similarly we may also write

$$\frac{\delta s}{\delta y} = \frac{\delta s}{\delta C} \cdot \frac{\delta C}{\delta y} = \frac{\delta s}{\delta C}\sqrt{1 + \left(\frac{\delta x}{\delta y}\right)^2}$$

and hence when $Q \to P$ this leads to

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \quad \to (2)$$

Parametric Form: Suppose $x = x(t)$ *and* $y = y(t)$ is the parametric form of the curve C.

Then from (1)

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}=\frac{1}{dx/dt}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\therefore \frac{ds}{dt}=\frac{ds}{dx}\cdot\frac{dx}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \quad \rightarrow (3)$$

Note: Since ψ is the angle between the tangent at P and the X-axis,

we have $\frac{dy}{dx}=\tan\psi$

$$\Rightarrow \frac{ds}{dx}=\sqrt{1+(y')^2}=\sqrt{1+\tan^2\psi}=Sec\psi$$

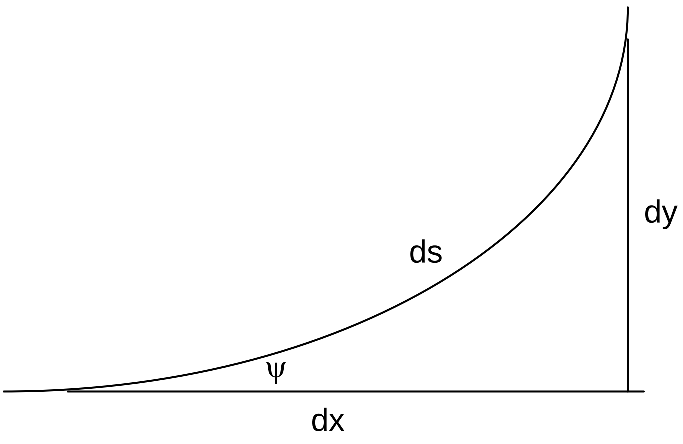
Similarly

$$\frac{ds}{dy}=\sqrt{1+\frac{1}{(y')^2}}=\sqrt{1+\frac{1}{\tan^2\psi}}=\sqrt{1+\cot^2\psi}=Co\sec\psi$$

i.e. $\text{Cos }\psi=\frac{dx}{ds}$ *and* $Sin\,\psi=\frac{dy}{ds}$

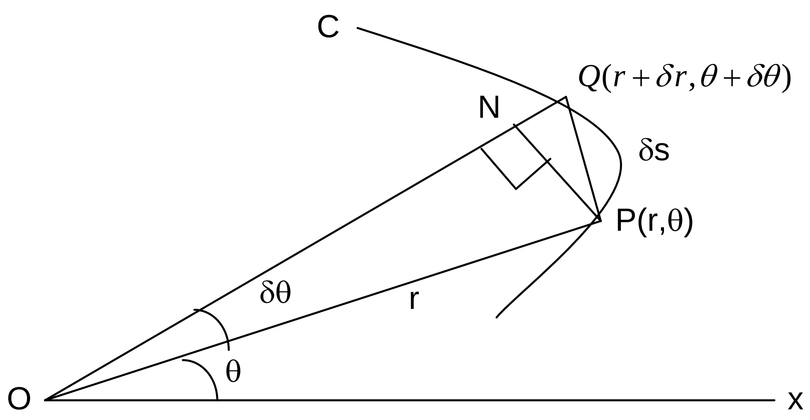
$$\therefore \left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2=1\Rightarrow (ds)^2=(dx)^2+(dy)^2$$

We can use the following figure to observe the above geometrical connections among dx, dy, ds *and* ψ.



Polar Curves :

Suppose $r = f(\theta)$ is the polar equation of the curve C and $P(r,\theta)$ *and* $Q(r+\delta r, \theta+\delta\theta)$ be two neighboring points on it as in figure:



Consider PN $\perp$ OQ.

In the right-angled triangle OPN, We have $Sin\ \delta\theta = \frac{PN}{OP} = \frac{PN}{r} \Rightarrow PN = r\ Sin\ \delta\theta = r\delta\theta$

since $\text{Sin}\delta\theta = \delta\theta$ when $\delta\theta$ *is very small*.

From the figure we see that, $Cos\ \delta\theta = \frac{ON}{OP} = \frac{ON}{r} \Rightarrow ON = r\ Cos\ \delta\theta = r(1) = r$

$\because \cos\delta\theta = 1\ when\ \ \delta\theta \to 0$

$\therefore NQ = OQ - ON = (r+\delta r) - r = \delta r$

$$From\ \Delta PNQ,\ PQ^2 = PN^2 + NQ^2\ i.e,\ (\delta C)^2 = (r\delta\theta)^2 + (\delta r)^2$$

$$\Rightarrow \frac{\delta C}{\delta\theta} = \sqrt{r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2} \quad \therefore \frac{\delta S}{\delta\theta} = \frac{\delta S}{\delta C} \cdot \frac{\delta C}{\delta\theta} = \frac{\delta S}{\delta C}\sqrt{r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2}$$

We note that $when\ Q \to P\ along\ the\ curve,\ \delta\theta \to 0\ also\ \dfrac{\delta S}{\delta C} = 1$

$$\therefore when\ Q \to P,\ \frac{dS}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \to (4)$$

$$Similarly,\quad (\delta C)^2 = (r\delta\theta)^2 + (\delta r)^2 \Rightarrow \frac{\delta C}{\delta r} = \sqrt{1 + r^2\left(\frac{\delta\theta}{\delta r}\right)^2}$$

$$and\quad \frac{\delta S}{\delta r} = \frac{\delta S}{\delta C} \cdot \frac{\delta C}{\delta r} = \frac{\delta S}{\delta C}\sqrt{1 + r^2\left(\frac{\delta\theta}{\delta r}\right)^2}$$

$$\therefore when\ Q \to P,\ we\ get\ \frac{dS}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \to (5)$$

Note:

We know that Tan $\phi = \mathrm{r}\dfrac{\mathrm{d}\theta}{\mathrm{dr}}$

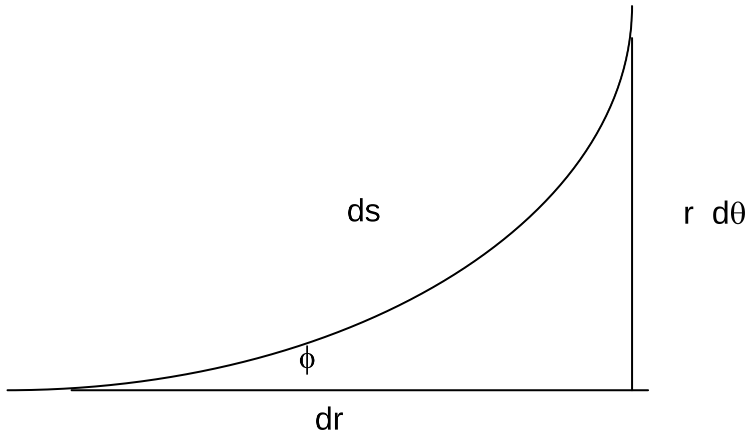
$$\therefore \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r^2 Cot^2\phi} = r\sqrt{1 + Cot^2\phi} = rCo\sec\phi$$

Similarly

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} = \sqrt{1 + Tan^2\phi} = Sec\phi$$

$$\therefore \frac{dr}{ds} = Cos\phi \quad and \quad \frac{d\theta}{ds} = \frac{1}{r}Sin\phi$$

The following figure shows the geometrical connections among ds, dr, dθ and ϕ



Thus we have :

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2},\quad \frac{ds}{dy}=\sqrt{1+\left(\frac{dx}{dy}\right)^2},\quad \frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\frac{ds}{dr}=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\quad and\quad \frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

Example 1: $\frac{ds}{dx}$ and $\frac{ds}{dy}$ for the curve $x^{2/3}+y^{2/3}=a^{2/3}$

$$x^{2/3}+y^{2/3}=a^{2/3}\Rightarrow\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0\quad\Rightarrow y'=\frac{x^{\frac{-1}{3}}}{y^{\frac{-1}{3}}}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Hence $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{y^{2/3}}{x^{2/3}}}=\sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}=\sqrt{\frac{a^{2/3}}{x^{2/3}}}=\left(\frac{a}{x}\right)^{1/3}$

Similarly

$$\frac{ds}{dy}=\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+\frac{x^{2/3}}{y^{2/3}}}=\sqrt{\frac{x^{2/3}+y^{2/3}}{y^{2/3}}}=\sqrt{\frac{a^{2/3}}{y^{2/3}}}=\left(\frac{a}{y}\right)^{1/3}$$

Example 2: *Find* $\frac{ds}{dx}$ for the curve $y=a\log\left(\frac{a^2}{a^2-x^2}\right)$

$$y=a\log a^2-a\log\left(a^2-x^2\right)\Rightarrow\frac{dy}{dx}=-a\left(\frac{-2x}{a^2-x^2}\right)=\frac{2ax}{a^2-x^2}$$

$$\therefore \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{4a^2x^2}{\left(a^2-x^2\right)}} = \sqrt{\frac{\left(a^2-x^2\right)+4a^2x^2}{\left(a^2-x^2\right)^2}} = \sqrt{\frac{\left(a^2+x^2\right)^2}{\left(a^2-x^2\right)^2}} = \frac{a^2+x^2}{a^2-x^2}$$

Example 3: If x = ae^{t}Sint, y = ae^{t}Cost, find $\frac{ds}{dt}$

$$x = ae^t \text{Sint} \Rightarrow \frac{dx}{dt} = ae^t\text{Sint} + ae^t\text{Cost}$$

$$y = ae^t \text{Cost} \Rightarrow \frac{dy}{dt} = ae^t\text{Cost} - ae^t\text{Sint}$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{a^2e^{2t}\left(Cos\,t+Sin\,t\right)^2+a^2e^{2t}\left(Cos\,t-Sin\,t\right)^2}$$

$$= ae^t\sqrt{2\left(Cos^2t+Sin^2t\right)} = a\sqrt{2}\,e^t \qquad \because (a+b)^2+(a-b)^2 = 2\left(a^2+b^2\right)$$

Example 4: If x = a$\left[\text{Cos t + log Tan }\frac{t}{2}\right]$, y = a Sin t, find $\frac{ds}{dt}$

$$\frac{dx}{dt} = a\left[-Sin\,t+\frac{Sec^2\,{}^t\!/\!_2}{2\cdot Tan\,{}^t\!/\!_2}\right] = a\left[-Sin\,t+\frac{1}{2Sin\frac{t}{2}Cos\frac{t}{2}}\right]$$

$$= a\left[-Sin\,t+\frac{1}{Sin\,t}\right] = a\frac{\left(1-Sin^2t\right)}{Sin\,t} = \frac{a\,Cos^2t}{Sin\,t} = a\,Cost\cdot Cot\,t$$

$$\frac{dy}{dt} = a\,Cos\,t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{a^2Cos^2t\,Cot^2t+a^2Cos^2t} = \sqrt{a^2Cos^2t\left(Cot^2t+1\right)}$$

$$= \sqrt{a^2Cos^2t\cdot Co\sec^2 t} = \sqrt{a^2Cot^2t} = a\,Cot\,t$$

Example 5: If x = a Cos^{3}t, y = Sin^{3}t, find $\frac{ds}{dt}$

$$\frac{dx}{dt} = -3\,a\,Cos^2t\,Sin\,t,\ \frac{dy}{dt} = 3\,a\,Sin^2t\,Cos\,t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{9a^2Cos^4t\,Sin^2t+9a^2Sin^4t\,Cos^2t}$$

$$= \sqrt{9a^2 Cos^2 t\ Sin^2 t \left(Cos^2 t + Sin^2 t\right)} = 3\ a\ Sin\ t\ Cos\ t$$

Exercise:

Find $\frac{ds}{dt}$ for the following curves:

(i) x=a(t + Sint), y=a(1-Cost)

(ii) x=a(Cost + t Sint), y= aSint

(iii) x= a log(Sect + tant), y= a Sect

Example 6: If $r^2 = a^2 \, \text{Cos} \, 2\theta$, Show that $r \dfrac{ds}{d\theta}$ is constant

$$r^2 = a^2 Cos 2\theta \Rightarrow 2r\frac{dr}{d\theta} = -2a^2 Sin 2\theta \quad \Rightarrow \frac{dr}{d\theta} = \frac{-a^2}{r} Sin 2\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + \frac{a^4}{r^2} Sin^2 2\theta} = \frac{1}{r}\sqrt{r^4 + a^4 Sin^2 2\theta}$$

$$\therefore r\frac{ds}{d\theta} = \sqrt{r^4 + a^4 Sin^2 2\theta} = \sqrt{a^4 Cos^2 2\theta + a^4 Sin^2 2\theta}$$

$$= a^2\sqrt{Cos^2\theta + Sin^2 2\theta} = a^2 = \text{Constant} \qquad \therefore r\frac{ds}{d\theta} \text{ is constant for } r^2 = a^2 Cos 2\theta$$

Example 7: For the curve $\theta = \text{Cos}^{-1}\left(\dfrac{r}{k}\right) - \dfrac{\sqrt{k^2 - r^2}}{r}$, Show that $r\dfrac{ds}{dr}$ is constant.

$$\frac{d\theta}{dr} = \frac{-1}{\sqrt{1 - \frac{r^2}{k^2}}} \cdot \frac{1}{k} - \frac{r\left(\frac{-2r}{2\sqrt{k^2 - r^2}}\right) - \sqrt{k^2 - r^2}(1)}{r^2} = \frac{-1}{\sqrt{k^2 - r^2}} + \frac{r^2 + \left(k^2 - r^2\right)}{r^2\sqrt{k^2 - r^2}}$$

$$= \frac{-1}{\sqrt{k^2 - r^2}} + \frac{k^2}{r^2\sqrt{k^2 - r^2}} = \frac{-r^2 + k^2}{r^2\sqrt{k^2 - r^2}} \qquad = \frac{\sqrt{k^2 - r^2}}{r}$$

$$\therefore \frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)2} = \sqrt{1 + r^2\frac{\left(k^2 - r^2\right)}{r^2}} = \frac{\sqrt{r^2 + k^2 - r^2}}{r} = \frac{k}{r}$$

Hence $\quad r\dfrac{ds}{dr} = k$ (constant)

Example 8: For a polar curve $r = f(\theta)$ show that $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{r^2 - p^2}}, \dfrac{ds}{d\theta} = \dfrac{r^2}{p}$

We know that $Cos\,\phi = \dfrac{dr}{ds}$ *and* $\dfrac{d\theta}{ds} = \dfrac{1}{r} Sin\,\phi$

$$\therefore \frac{dr}{ds} = Cos\varphi = \sqrt{1 - Sin^2\phi} = \sqrt{1 - \frac{p^2}{r^2}} = \frac{\sqrt{r^2 - p^2}}{r} \qquad \because p = rSin\varphi$$

$$\therefore \frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}$$

$$Also \quad \frac{ds}{d\theta} = \frac{r}{Sin\,\phi} = \frac{r}{p/r} = \frac{r^2}{p}$$

Exercise:

Find $\frac{ds}{dr}$ *and* $\frac{ds}{d\theta}$ for the following curves:

(i) r=a(1+ Cosθ)

(ii) r^m=a^{m}Cosmθ

(iii) $r =$ aSec$^2(\frac{\theta}{2})$

Mean Value Theorems

Recapitulation

Closed interval: An interval of the form $a \le x \le b$, that includes every point between a and b and also the end points, is called a closed interval and is denoted by $[a,b]$.

Open Interval: An interval of the form $a < x < b$, that includes every point between a and b but not the end points, is called an open interval and is denoted by (a,b)

Continuity: A real valued function $f(x)$ is said to be continuous at a point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

The function $f(x)$ is said to be continuous in an interval if it is continuous at every point in the interval.

Roughly speaking, if we can draw a curve without lifting the pen, then it is a continuous curve otherwise it is discontinuous, having discontinuities at those points at which the curve will have breaks or jumps.

We note that all elementary functions such as algebraic, exponential, trigonometric, logarithmic, hyperbolic functions are continuous functions. Also the sum, difference, product of continuous functions is continuous. The quotient of continuous functions is continuous at all those points at which the denominator does not become zero.

Differentiability: A real valued function $f(x)$ is said to be differentiable at point x_0 if

$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists uniquely and it is denoted by $f'(x_0)$.

A real valued function f(x) is said to be differentiable in an interval if it is differentiable at every in the interval or if $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists uniquely. This is denoted by $f'(x)$. We say that either $f'(x)$ exists or f(x) is differentiable.

Geometrically, it means that the curve is a smooth curve. In other words a curve is said to be smooth if there exists a unique tangent to the curve at every point on it. For example a circle is a smooth curve. Triangle, rectangle, square etc are not smooth, since we can draw more number of tangents at every corner point.

We note that if a function is differentiable in an interval then it is necessarily continuous in that interval. The converse of this need not be true. That means a function is continuous need not imply that it is differentiable.

Rolle's Theorem: (French Mathematician Michelle Rolle 1652-1679)

Suppose a function $f(x)$ satisfies the following three conditions:

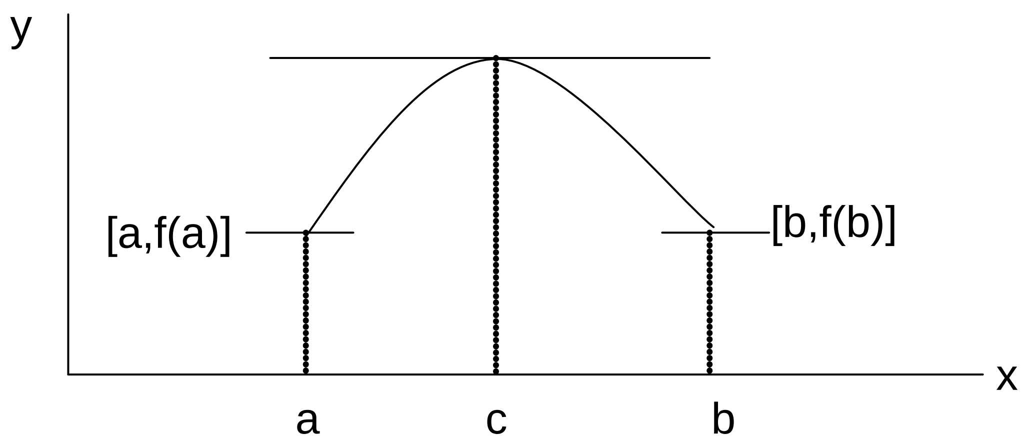
(i) $f(x)$ is continuous in a closed interval $[a,b]$

(ii) $f(x)$ is differentiable in the open interval (a,b)

(iii) $f(a) = f(b)$

Then there exists at least one point c in the open interval (a,b) such that $f'(c) = 0$

Geometrical Meaning of Rolle's Theorem: Consider a curve $f(x)$ that satisfies the conditions of the Rolle's Theorem as shown in figure:



As we see the curve $f(x)$ is continuous in the closed interval $[a,b]$, the curve is smooth i.e. there can be a unique tangent to the curve at any point in the open interval (a,b) and also $f(a) = f(b)$. Hence by Rolle's Theorem there exist at least one point c belonging to (a,b) such that $f'(c) = 0$. In other words there exists at least one point at which the tangent drawn to the curve will have its slope zero or lies parallel to x-axis.

Notation:

Often we use the statement**:** there exists a point c belonging to (a,b) such that …….

We present the same in mathematical symbols as: $\exists$ c $\in (a,b)$ **:** $f'(c) = 0$

From now onwards let us start using the symbolic notation.

Example 1: Verify Rolle's Theorem for $f(x) = x^2$ in $[-1,1]$

First we check whether the conditions of Rolle's theorem hold good for the given function:

(i) $f(x) = x^2$ is an elementary algebraic function, hence it is continuous every where and so also in $[-1,1]$.

(ii) $f'(x) = 2x$ exists in the interval (-1, 1) i.e. the function is differentiable in (-1, 1).

(iii) Also we see that $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$ i.e., $f(-1) = f(1)$

Hence the three conditions of the Rolle's Theorem hold good.
$\therefore$ By Rolle's Theorem $\exists$ $c \in (-1,1)$ **:** $f'(c) = 0$ that means 2c = 0 $\Rightarrow$ c= 0 $\in (-1,1)$

Hence Rolle's Theorem is verified.

Example 2: Verify Rolle's Theorem for $f(x) = x(x+3)e^{-x/2}$ in [-3, 0]

$f(x)$ is a product of elementary algebraic and exponential functions which are continuous and hence it is continuous in $[-3, 0]$

$$f'(x) = (2x+3)e^{-x/2} - \frac{1}{2}(x^2+3x)e^{-x/2}$$
$$= [(2x+3) - \frac{1}{2}(x^2+3x)]e^{-x/2}$$
$$= -\frac{1}{2}[x^2 - x - 6]e^{-x/2} = -\frac{1}{2}(x-3)(x+2)e^{-x/2} \text{ exists in (-3, 0)}$$

$f(-3) = 0$ and also $f(0) = 0 \therefore f(-3) = f(0)$

That is, the three conditions of the Rolle's Theorem hold good.

$\therefore \exists$ c $\in (-3,0)$ **:** $f'(c) = 0$

$$\therefore -\frac{1}{2}(c-3)(c+2)e^{-c/2} = 0 \Rightarrow c = 3, -2, \infty$$

Out of these values of c, since $-2 \in (-3, 0)$, the Rolle's Theorem is verified.

Example 3: Verify Rolle's Theorem for $f(x) = (x-a)^m (x-b)^n$ in [a, b] where a <b and a, b>0.

$f(x)$ is a product of elementary algebraic functions which are continuous and hence it is continuous in [a, b]

$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$$
$$= [m(x-b) + n(x-a)](x-a)^{m-1}(x-b)^{n-1}$$
$$= [x(m+n) - (mb+na)](x-a)^{m-1}(x-b)^{n-1} \text{ exists in } (a,b)$$

$f(a) = 0$ and also $f(b) = 0 \therefore f(a) = f(b)$

Hence the three conditions of the Rolle's Theorem hold good.
$\therefore \exists\ c \in (a,b)\ :\ f'(c) = 0$

$$\therefore [c(m+n) - (mb+na)](c-a)^{m-1}(c-b)^{n-1} = 0 \Rightarrow c = \frac{mb+na}{m+n}, a, b$$

Out of these values of c, since $c = \dfrac{mb+na}{m+n} \in (a,b)$, the Rolle's Theorem is verified.

Example 4: Verify Rolle's Theorem for $f(x) = \log\dfrac{x^2+ab}{x(a+b)}$ in [a, b]

$f(x) = \log(x^2+ab) - \log x - \log(a+b)$ is the sum of elementary logarithmic functions which are continuous and hence it is continuous in [a, b]

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} \text{ exists in } (a,b)$$

$$f(a) = \log\frac{a^2+ab}{a(a+b)} = \log\frac{a^2+ab}{a^2+ab} = \log 1 = 0$$

and similarly $f(b) = \log\dfrac{b^2+ab}{b(a+b)} = \log\dfrac{b^2+ab}{b^2+ab} = \log 1 = 0$

Hence the three conditions of the Rolle's Theorem hold good. $\therefore \exists$ c$\in (a,b)\ :\ f'(c) = 0$

$$f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} = 0 \Rightarrow \frac{2c^2-c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 - ab = 0 \Rightarrow c = \sqrt{ab}$$

Since $c = \sqrt{ab} \in (a,b)$, the Rolle's Theorem is verified.

Exercise: Verify Rolle's Theorem for

(i) $f(x) = e^x(\sin x - \cos x)$ in $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$,

(ii) $f(x) = x(x-2)e^{x/2}$ in $[0,2]$

,

(iii) $f(x) = \dfrac{\sin 2x}{e^{2x}}$ in $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$.

Lagrange's Mean Value Theorem (LMVT):
(Also known as the First Mean Value Theorem)
(Italian-French Mathematician J. L. Lagrange 1736-1813)

Suppose a function $f(x)$ satisfies the following two conditions:

(i) $f(x)$ is continuous in a closed interval $[a,b]$

(ii) $f(x)$ is differentiable in the open interval (a,b)

Then there exists at least one point c in the open interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Proof: Consider the function $\phi(x)$ defined by $\phi(x)=f(x)-kx$ where k is a constant to be found such that $\varphi(a)=\phi(b)$.

Since $f(x)$ is continuous in the closed interval $[a,b]$, $\phi(x)$ which is a sum of continuous functions is also continuous in the closed interval $[a,b]$

$\phi'(x)=f'(x)-kx\to(1)$ exists in the interval (a,b) as $f(x)$ is differentiable in (a,b).

We have $\phi(a)=f(a)-ka$ and $\phi(b)=f(b)-kb$

$$\therefore \phi(a)=\phi(b)\Rightarrow f(a)-ka=f(b)-kb\Rightarrow k=\frac{f(b)-f(a)}{b-a}\to(2)$$

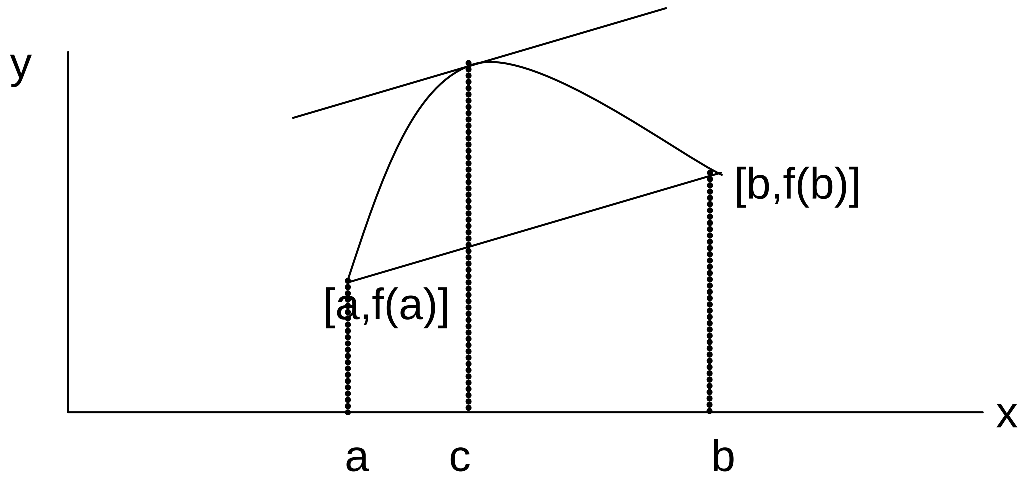
This means that when k is chosen as in (2) we will have $\phi(a)=\phi(b)$
Hence the conditions of Rolle's Theorem hold good for $\phi(x)$ in $[a,b]$
$\therefore$ By Rolle's Theorem $\exists$ $c\in(a,b)$ **:** $\phi'(c)=0$

$\phi'(c)=0\Rightarrow f'(c)-k=0\Rightarrow k=f'(c)\to(3)$

From (2) and (3), we infer that $\exists$ $c\in(a,b)$ **:** $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Thus the Lagrange's Mean Value Theorem (LMVT) is proved.

Geometrical Meaning of Lagrange's Mean Value Theorem: Consider a curve $f(x)$ that satisfies the conditions of the LMVT as shown in figure:



From the figure, we observe that the curve $f(x)$ is continuous in the closed interval $[a,b]$; the curve is smooth i.e. there can be a unique tangent to the curve at any point in the open interval (a,b). Hence by LMVT there exist at least one point c belonging to (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$. In other words there exists at least one point at which the tangent drawn to the curve lies parallel to the chord joining the points $[a, f(a)]$ and $[b, f(b)]$.

Other form of LMVT: The Lagrange's Mean Value Theorem can also be stated as follows:

Suppose $f(x)$ is continuous in the closed interval $[a,a+h]$ and is differentiable in the open interval $(a,a+h)$ then $\exists\theta\in(0,1): f(a+h)=f(a)+hf'(a+\theta h)$

When $\theta\in(0,1)$ we see that $a+\theta h\in(a,a+h)$ i.e., here $c=a+\theta h$

Using the earlier form of LMVT we may write that $f'(a+\theta h)=\dfrac{f(a+h)-f(a)}{(a+h)-a}$

On simplification this becomes $f(a+h)=f(a)+hf'(a+\theta h)$.

Example 5: Verify LMVT for $f(x)=(x-1)(x-2)(x-3)$ in $[0,4]$

$f(x)=(x-1)(x-2)(x-3)=x^3-6x^2+11x-6$ is an algebraic function hence it is continuous in $[0,4]$

$f'(x)=3x^2-12x+11$exists in $(0,4)$ i.e., $f(x)$ is differentiable in $(0,4)$.
i.e., both the conditions of LMVT hold good for $f(x)$ in$[0,4]$.

Hence $\exists c\in(0,4): f'(c)=\dfrac{f(4)-f(0)}{4-0}$

i.e., $3c^2-1c+11=\dfrac{(3)(2)(1)-(-1)(-2)(-3)}{4-0}$

i.e, $3c^2-12c+11=3\Rightarrow 3c^2-12c+8=0\Rightarrow c=2\pm\dfrac{2}{\sqrt{3}}\in(0,4)$

Hence the LMVT is verified.

Example 6: Verify LMVT for $f(x)=\log_e x$ in $[1,e]$

$f(x)=\log_e x$ is an elementary logarithmic function hence continuous in $[1,e]$.

$f'(x)=\dfrac{1}{x}$ exists in $(1,e)$ or $f(x)$ is differentiable in$(1,e)$.

i.e., both the conditions of LMVT hold good for $f(x)$ in $[1,e]$

Hence $\exists c\in(1,e): f'(c)=\dfrac{f(e)-f(1)}{e-1}\Rightarrow\dfrac{1}{c}=\dfrac{\log e-\log 1}{e-1}$

$$\Rightarrow\frac{1}{c}=\frac{1}{e-1}\Rightarrow c=e-1\in(1,e)$$

Hence the LMVT is verified.

Example 7: Verify LMVT for $f(x)=\sin^{-1}x$ in $[0,1]$

We find $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$ exists in $(0,1)$, and hence $f(x)$ is continuous in $[0,1]$.

i.e., both the conditions of LMVT hold good for $f(x)$ in $[0,1]$

Hence $\exists c\in(0,1): f'(c)=\dfrac{f(1)-f(0)}{1-0}\Rightarrow\dfrac{1}{\sqrt{1-c^2}}=\dfrac{\sin^{-1}1-\sin^{-1}0}{1}$

$$\Rightarrow\frac{1}{\sqrt{1-c^2}}=\frac{\pi}{2}\Rightarrow\sqrt{1-c^2}=\frac{2}{\pi}\Rightarrow c^2=\frac{\pi^2-4}{\pi^2}$$

$$\therefore c=\frac{\sqrt{\pi^2-4}}{\pi}=0.7712\in(0,1)$$

Hence the LMVT is verified.

Example 8: Find θ of LMVT for $f(x)=ax^2+bx+c$ or

Verify LMVT by finding θ for $f(x)=ax^2+bx+c$

$f(x)=ax^2+bx+c$ is an algebraic function hence continuous in $[a,a+h]$.

$f'(x)=2ax+b$ exists in $(a,a+h)$

i.e., both the conditions of LMVT hold good for $f(x)$ in $[a,a+h]$

Then the second form of LMVT $\exists\theta\in(0,1): f(a+h)=f(a)+hf'(a+\theta h)\to(1)$

Here ; $f(a+h)=a(a+h)^2+b(a+h)+c=a^3+ah^2+2a^2h+ab+bh+c\to(2)$

$f(a)=a^3+ab+c\to(3)$; $f'(a+\theta h)=2a(a+\theta h)+b=2a^2+2a\theta h+b\to(4)$

Using (2), (3) and (4) in (1) we have:

$$a^3+ah^2+2a^2h+ab+bh+c=a^3+ab+c+h(2a^2+2a\theta h+b)\Rightarrow\theta=\frac{1}{2}\in(0,1)$$

Hence the LMVT is verified.

Example 9: Find θ of LMVT for $f(x)=\log_e x$ in $\left[e,e^2\right]$

$f(x)=\log_e x$ is an elementary logarithmic function hence continuous in $\left[e,e^2\right]$

$f'(x)=\dfrac{1}{x}$ exists in $\left(e,e^2\right)$.

Hence by the second form of LMVT

$$\exists\theta\in(0,1): f(e^2)=f(e)+(e^2-e)f'\left[e+\theta(e^2-e)\right]$$

$$\Rightarrow \log e^2=\log e+\frac{(e^2-e)}{e+\theta(e^2-e)}$$

$$i.e.,\ 2\log_e e=\log_e e+\frac{e(e-1)}{e\left[1+\theta(e-1)\right]}\Rightarrow 2-1=\frac{(e-1)}{\left[1+\theta(e-1)\right]}$$

$$\Rightarrow 1+\theta(e-1)=e-1\Rightarrow \theta=\frac{e-2}{e-1}\in(0,1)$$

Hence the LMVT is verified.

Example 10: If $x>0$, Apply Mean Value Theorem to show that

$(i)\ \dfrac{x}{1+x}<\log_e(1+x)<x$ and $0<\dfrac{1}{\log_e(1+x)}-\dfrac{1}{x}<1$

Consider $f(x)=\log_e(1+x)$ in $[0,x]$. We see that $f(x)$ is an elementary logarithmic function hence continuous in $[0,x]$. Also $f'(x)=\dfrac{1}{1+x}$ exists in $(0,x)$.

$\therefore$ by LMVT $\exists\theta\in(0,1): f(x)=f(0)+xf'(0+\theta x)$

$$i.e., \log(1+x)=\log 1+\frac{x}{1+\theta x}\Rightarrow \log(1+x)=\frac{x}{1+\theta x}\rightarrow(1)$$

Since $x>0$ and $0<\theta<1$, we have $0<\theta x<x\Rightarrow 1<1+\theta x<1+x$

$$\Rightarrow 1>\frac{1}{1+\theta x}>\frac{1}{1+x}\Rightarrow\frac{1}{1+x}<\frac{1}{1+\theta x}<1\Rightarrow\frac{x}{1+x}<\frac{x}{1+\theta x}<x\rightarrow(2)$$

From (1) and (2) we get $\dfrac{x}{1+x}<\log_e(1+x)<x$

Also from (1) we have, $\dfrac{1}{\log(1+x)}=\dfrac{1+\theta x}{x}\Rightarrow\dfrac{1}{\log(1+x)}-\dfrac{1}{x}=\theta$

Since $0<\theta<1$, we can write $0<\dfrac{1}{\log(1+x)}-\dfrac{1}{x}<1.$

Example 11: Apply Mean Value Theorem to show that

$$\frac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{b-a}{\sqrt{1-b^2}},\, where\ 0<a<b$$

Consider $f(x)=\sin^{-1}x$ in $[a,b]$

We see that $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$ exists in (a,b)

Hence $f(x)$ is differentiable in (a,b) and also it is continuous in $[a,b]$.

Therefore, by LMVT $\exists c\in(a,b): f'(c)=\dfrac{f(b)-f(a)}{b-a}\Rightarrow\dfrac{1}{\sqrt{1-c^2}}=\dfrac{\sin^{-1}b-\sin^{-1}a}{b-a}\to(1)$

Since $a<c<b\Rightarrow a^2<c^2<b^2\Rightarrow -a^2>-c^2>-b^2\Rightarrow 1-a^2>1-c^2>1-b^2$

$$\therefore \frac{1}{1-a^2}<\frac{1}{1-c^2}<\frac{1}{1-b^2}\ or\ \frac{1}{\sqrt{1-a^2}}<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-b^2}}\to(2)$$

From (1) and (2), we have

$$\frac{1}{\sqrt{1-a^2}}<\frac{\sin^{-1}b-\sin^{-1}a}{b-a}<\frac{1}{\sqrt{1-b^2}}\Rightarrow\frac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{b-a}{\sqrt{1-b^2}}$$

Exercise: Verify the Lagrange's Mean Value Theorem for

(i) $f(x)=x(x-1)(x-2)$ in $\left[0,\dfrac{1}{2}\right]$

(ii) $f(x)=Tan^{-1}x$ in $[0,1]$

Mean Value Theorems

Cauchy's Mean Value Theorem (CMVT):
(French Mathematician A. L. Cauchy 1789-1857)

Suppose two functions $f(x)$ and $g(x)$ satisfies the following conditions:

(i) $f(x)$ and $g(x)$ are continuous in a closed interval $[a,b]$

(ii) $f(x)$ and $g(x)$ are differentiable in the open interval (a,b)

(iii) $g'(x) \neq 0$ *for all x*.[*for all can be denoted by the symbol* $\forall$]

Then there exists at least one point c in the open interval (a,b)
such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Proof: Consider the function $\phi(x)$ defined by $\phi(x) = f(x) - k\,g(x)$ where k is a constant to be found such that $\varphi(a) = \phi(b)$.

Since $f(x)$ and $g(x)$ are continuous in the closed interval $[a,b]$, $\phi(x)$ which is a sum of continuous functions is also continuous in the closed interval $[a,b]$

$\phi'(x) = f'(x) - k\,g'(x) \rightarrow (1)$ exists in the interval (a,b) as $f(x)$ and $g(x)$ are differentiable in (a,b).

We have $\phi(a) = f(a) - kg(a)$ and $\phi(b) = f(b) - k\,g(b)$

$$\therefore \phi(a) = \phi(b) \Rightarrow f(a) - kg(a) = f(b) - k\,g(b) \Rightarrow k = \frac{f(b)-f(a)}{g(b)-g(a)} \rightarrow (2)$$

This means that when k is chosen as in (2) we will have $\phi(a) = \phi(b)$
Hence the conditions of Rolle's Theorem hold good for $\phi(x)$ in $[a,b]$
$\therefore$ By Rolle's Theorem $\exists$ $c \in (a,b)$ **:** $\phi'(c) = 0$

$$\phi'(c) = 0 \Rightarrow f'(c) - k\,g'(c) = 0 \Rightarrow k = \frac{f'(c)}{g'(c)} \rightarrow (3)$$

From (2) and (3), we infer that $\exists$ $c \in (a,b)$ **:** $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Thus the Cauchy's Mean Value Theorem (CMVT) is proved.

Example 12: Verify the Cauchy's MVT for $f(x)=x^2$ and $g(x)=x^4$ in $[a,b]$

$f(x)=x^2$ and $g(x)=x^4$ are algebraic polynomials hence continuous in $[a,b]$

$f'(x)=2x$ and $g'(x)=4x^3$ exist in (a,b)

also we see that $g'(x)\neq 0$ *for all* $x\in(a,b)$ since $0<a<b$

i.e., the conditions of CMVT hold good for $f(x)$ and $g(x)$ in$[a,b]$.

Hence $\exists$ $c\in(a,b)$: $\dfrac{f'(c)}{g'(c)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

i.e., $\dfrac{2c}{4c^3}=\dfrac{b^2-a^2}{b^4-a^4}\Rightarrow\dfrac{1}{2c^2}=\dfrac{1}{b^2+a^2}\Rightarrow c=\sqrt{\dfrac{b^2+a^2}{2}}\in(a,b)$

Hence the CMVT is verified.

Example 13: Verify the Cauchy's MVT for $f(x)=\log x$ and $g(x)=\dfrac{1}{x}$ in $[1,e]$

$f(x)=\log x$ and $g(x)=\dfrac{1}{x}$ are elementary logarithmic and rational algebraic functions that are continuous in $[1,e]$

$f'(x)=\dfrac{1}{x}$ and $g'(x)=\dfrac{-1}{x^2}$ exist in $(1,e)$

also we see that $g'(x)\neq 0$ *for all* $x\in(1,e)$

i.e., the conditions of CMVT hold good for $f(x)$ and $g(x)$ in$(1,e)$.

Hence $\exists\, c\in(1,e)$: $\dfrac{f'(c)}{g'(c)}=\dfrac{f(e)-f(1)}{g(e)-g(1)}$

i.e., $\dfrac{1/c}{-1/c^2}=\dfrac{\log e-\log 1}{\frac{1}{e}-1}\Rightarrow -c=\dfrac{1}{\frac{1}{e}-1}\Rightarrow c=\dfrac{e}{e-1}\in(1,e)$

Hence the CMVT is verified.

Example 14: Verify the Cauchy's MVT for $f(x)=e^x$ and $g(x)=e^{-x}$ in $[a,b]$

$f(x)=e^x$ and $g(x)=e^{-x}$ are elementary exponential functions that are continuous in $[a,b]$

$f'(x)=e^{x}$ and $g'(x)=-e^{-x}$ exist in (a,b)

also we see that $g'(x)\neq 0$ *for all* $x\in(a,b)$, since $0<a<b$.

i.e., the conditions of CMVT hold good for $f(x)$ and $g(x)$ in (a,b).

Hence $\exists\, c\in(a,b)$ **:** $\dfrac{f'(c)}{g'(c)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

i.e., $\dfrac{e^{c}}{-e^{-c}}=\dfrac{e^{b}-e^{a}}{e^{-b}-e^{-a}}\Rightarrow -e^{2c}=\dfrac{e^{b}-e^{a}}{\left[\dfrac{1}{e^{b}}-\dfrac{1}{e^{a}}\right]}\Rightarrow -e^{2c}=\dfrac{e^{b}-e^{a}}{\left[\dfrac{e^{a}-e^{b}}{e^{a+b}}\right]}$

$$e^{2c}=e^{a+b}\Rightarrow c=\frac{a+b}{2}\in(a,b)$$

Hence the CMVT is verified.

Example 15: Verify the Cauchy's MVT for $f(x)=Sin\ x$ and $g(x)=Cos\ x$ in $\left[0,\dfrac{\pi}{2}\right]$

$f(x)=Sin\ x$ and $g(x)=Cos\ x$ are elementary trignometric functions that are continuous in $\left[0,\dfrac{\pi}{2}\right]$.

$f'(x)=Cos\ x$ and $g'(x)=-Sin\ x$ exist in $\left(0,\dfrac{\pi}{2}\right)$

also we see that $g'(x)\neq 0$ *for all* $x\in\left(0,\dfrac{\pi}{2}\right)$.

i.e., the conditions of CMVT hold good for $f(x)$ and $g(x)$ in $\left(0,\dfrac{\pi}{2}\right)$.

Hence $\exists\, c\in\left(0,\dfrac{\pi}{2}\right)$ **:** $\dfrac{f'(c)}{g'(c)}=\dfrac{f(\frac{\pi}{2})-f(0)}{g(\frac{\pi}{2})-g(0)}$

$$\Rightarrow \frac{Cos\ c}{-Sin\ c}=\frac{Sin(\frac{\pi}{2})-Sin(0)}{Cos(\frac{\pi}{2})-Cos(0)}\Rightarrow Cot\ c=1\Rightarrow c=\frac{\pi}{4}\in\left(0,\frac{\pi}{2}\right)$$

Hence the CMVT is verified.

Exercise: Verify the Cauchy's Mean Value Theorem for

(i) $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ in $\left[\dfrac{1}{4}, 1\right]$

(ii) $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x}$ in $[a, b]$

(iii) $f(x) = Sin\ x$ and $g(x) = Cos\ x$ in $[a, b]$

Taylor's Mean Value Theorem:
(Generalised Mean Value Theorem):
(English Mathematician Brook Taylor 1685-1731)

Suppose a function $f(x)$ satisfies the following two conditions:

(i) $f(x)$ and it's first (n-1) derivatives are continuous in a closed interval $[a, b]$

(ii) $f^{(n-1)}(x)$ is differentiable in the open interval (a, b)

Then there exists at least one point c in the open interval (a, b) such that

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{\lfloor 2}f''(a)+\frac{(b-a)^3}{\lfloor 3}f'''(a)+\ldots..$$

$$\ldots..+\frac{(b-a)^{n-1}}{\lfloor n-1}f^{(n-1)}(a)+\frac{(b-a)^n}{\lfloor n}f^{(n)}(c)\rightarrow(1)$$

Taking $b=a+h$ and for $0<\theta<1$, the above expression (1) can be rewritten as

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+\ldots.+\frac{h^{n-1}}{\lfloor n-1}f^{(n-1)}(a)+\frac{h^n}{\lfloor n}f^{(n)}(a+\theta h)\rightarrow(2)$$

Taking b=x in (1) we may write

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor 2}f''(a)+\frac{(x-a)^3}{\lfloor 3}f'''(a)+\ldots+\frac{(x-a)^{n-1}}{\lfloor n-1}f^{(n-1)}(a)+R_n\rightarrow(3)$$

$$Where\quad R_n=\frac{(x-a)^n}{\lfloor n}f^{(n)}(c)\rightarrow \text{Re}\,mainder\ term\ after\ n\ terms$$

When $n\rightarrow\infty$, we can show that $|R_n|\rightarrow 0$, thus we can write the Taylor's series as

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor 2}f''(a)+\ldots+\frac{(x-a)^{n-1}}{\lfloor n-1}f^{(n-1)}(a)+\ldots.$$
$$=f(a)+\sum_{n=1}^{\infty}\frac{(x-a)^n}{\lfloor n}f^{(n)}(a)\rightarrow(4)$$

Using (4) we can write a Taylor's series expansion for the given function f(x) in powers of (x-a) or about the point 'a'.

Maclaurin's series:
(Scottish Mathematician Colin Maclaurin 1698-1746)

When a=0, expression (4) reduces to a Maclaurin's expansion given by

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0)+\ldots+\frac{x^{n-1}}{\lfloor n-1}f^{(n-1)}(0)+\ldots.$$
$$=f(0)+\sum_{n=1}^{\infty}\frac{x^n}{\lfloor n}f^{(n)}(0)\rightarrow(5)$$

Example 16: Obtain a Taylor's expansion for $f(x)=Sin\,x$ in the ascending powers of $\left(x-\frac{\pi}{4}\right)$ up to the fourth degree term.

The Taylor's expansion for $f(x)$ about $\frac{\pi}{4}$ is

$$f(x)=f(\frac{\pi}{4})+(x-\frac{\pi}{4})f'(\frac{\pi}{4})+\frac{(x-\frac{\pi}{4})^2}{\lfloor 2}f''(\frac{\pi}{4})+\frac{(x-\frac{\pi}{4})^3}{\lfloor 3}f'''(\frac{\pi}{4})+\frac{(x-\frac{\pi}{4})^4}{\lfloor 4}f^{(4)}(\frac{\pi}{4})\ldots\ldots\rightarrow(1)$$

$$f(x)=Sin\,x\Rightarrow f\left(\frac{\pi}{4}\right)=Sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\;;\qquad f'(x)=Cos\,x\Rightarrow f'\left(\frac{\pi}{4}\right)=Cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f''(x)=-Sin\,x\Rightarrow f''\left(\frac{\pi}{4}\right)=-Sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f'''(x)=-Cos\,x\Rightarrow f'''\left(\frac{\pi}{4}\right)=-Cos\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f^{(4)}(x)=Sin\,x\Rightarrow f^{(4)}\left(\frac{\pi}{4}\right)=Sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

Substituting these in (1) we obtain the required Taylor's series in the form

$$f(x)=\frac{1}{\sqrt{2}}+(x-\frac{\pi}{4})(\frac{1}{\sqrt{2}})+\frac{(x-\frac{\pi}{4})^2}{\lfloor 2}(-\frac{1}{\sqrt{2}})+\frac{(x-\frac{\pi}{4})^3}{\lfloor 3}(-\frac{1}{\sqrt{2}})+\frac{(x-\frac{\pi}{4})^4}{\lfloor 4}(\frac{1}{\sqrt{2}})\ldots\ldots$$

$$f(x)=\frac{1}{\sqrt{2}}\left[1+(x-\frac{\pi}{4})-\frac{(x-\frac{\pi}{4})^2}{\lfloor 2}+\frac{(x-\frac{\pi}{4})^3}{\lfloor 3}-\frac{(x-\frac{\pi}{4})^4}{\lfloor 4}+\ldots\right]$$

Example 17: Obtain a Taylor's expansion for $f(x)=\log_e x$ up to the term containing $(x-1)^4$ and hence find $\log_e(1.1)$.

The Taylor's series for $f(x)$ about the point 1 is

$$f(x)=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{\lfloor 2}f''(1)+\frac{(x-1)^3}{\lfloor 3}f'''(1)+\frac{(x-1)^4}{\lfloor 4}f^{(4)}(1)\ldots\ldots\rightarrow(1)$$

Here $f(x)=\log_e x\Rightarrow f(1)=\log 1=0\,;\qquad f'(x)=\frac{1}{x}\Rightarrow f'(1)=1$

$$f''(x)=-\frac{1}{x^2}\Rightarrow f''(1)=-1;\qquad f'''(x)=\frac{2}{x^3}\Rightarrow f'''(1)=2$$

$$f^{(4)}(x)=-\frac{6}{x^4}\Rightarrow f^{(4)}(1)=-6\ \text{ etc.,}$$

Using all these values in (1) we get

$$f(x) = \log_e x = 0 + (x-1)(1) + \frac{(x-1)^2}{\underline{|2}}(-1) + \frac{(x-1)^3}{\underline{|3}}(2) + \frac{(x-1)^4}{\underline{|4}}(-6) \ldots.$$

$$\Rightarrow \log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \ldots.$$

Taking x=1.1 in the above expansion we get

$$\Rightarrow \log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} \ldots. = 0.0953$$

Example 18: Using Taylor's theorem Show that

$$\log_e(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \quad for\ 0<\theta<1, x>0$$

Taking n=3 in the statement of Taylor's theorem, we can write

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{\lfloor 2} f''(a) + \frac{x^3}{\lfloor 3} f'''(a+\theta x) \rightarrow (1)$$

Consider $f(x) = \log_e x \quad \Rightarrow f'(x) = \frac{1}{x};\ f''(x) = -\frac{1}{x^2}\ and\ f'''(x) = \frac{2}{x^3}$

Using these in (1), we can write,

$$\log(a+x) = \log a + x\left(\frac{1}{a}\right) + \frac{x^2}{\lfloor 2}\left(-\frac{1}{a^2}\right) + \frac{x^3}{\lfloor 3}\left(\frac{2}{(a+\theta x)^3}\right) \rightarrow (2)$$

For a=1 in (2) we write,

$$\log(1+x) = \log 1 + x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3} = x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3}$$

Since $x>0\ and\ \theta>0,\ (1+\theta x)^3 > 1\ and\ therefore \frac{1}{(1+\theta x)^3} < 1$

$$\therefore\ \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Example 19: Obtain a Maclaurin's series for $f(x) = Sin\ x$ up to the term containing x^5.

The Maclaurin's series for f(x) is

$$f(x) = f(0) + x\ f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{(4)}(0) + \frac{x^5}{\lfloor 5} f^{(5)}(0) \ldots\ldots \to (1)$$

Here $f(x) = Sin\ x \Rightarrow f(0) = Sin\ 0 = 0$ $\quad f'(x) = Cos\ x \Rightarrow f'(0) = Cos\ 0 = 1$

$f''(x) = -Sin\ x \Rightarrow f''(0) = -Sin\ 0 = 0$ $\quad f'''(x) = -Cos\ x \Rightarrow f'''(0) = -Cos\ 0 = -1$

$f^{(4)}(x) = Sin\ x \Rightarrow f^{(4)}(0) = Sin\ 0 = 0$ $\quad f^{(5)}(x) = Cos\ x \Rightarrow f^{(5)}(0) = Cos\ 0 = 1$

Substituting these values in (1), we get the Maclaurin's series for $f(x) = Sin\ x$ as

$$f(x) = Sin\ x = 0 + x\,(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^4}{\lfloor 4}(0) + \frac{x^5}{\lfloor 5}(1) \ldots\ldots \Rightarrow Sin\ x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots\ldots$$

Note:

As done in the above example, we can find the Maclaurin's series for various functions, for ex:

$$(i)\quad Cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} \ldots\ldots \qquad (ii)\quad e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \frac{x^6}{\lfloor 6} \ldots\ldots$$

$$(iii)\quad (1+x)^m = 1 + mx + \frac{m(m-1)}{\lfloor 2} x^2 + \frac{m(m-1)(m-2)}{\lfloor 3} x^3 + \frac{m(m-1)(m-2)(m-3)}{\lfloor 4} x^4 \ldots\ldots$$

Taking $m = -1$ in (iii) we can get $\quad (1+x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \ldots\ldots$

Replacing x by (-x) in this we get $\quad (1-x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \ldots\ldots$

We use these expansions in the study of various topics.

Indeterminate Forms:

While evaluating certain limits, we come across expressions of the form $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0 \, and \, 1^\infty$ which do not represent any value. Such expressions are called Indeterminate Forms.

We can evaluate such limits that lead to indeterminate forms by using L'Hospital's Rule (French Mathematician 1661-1704).

L'Hospital's Rule:

If f(x) and g(x) are two functions such that

(i) $\lim_{x\to a} f(x) = 0 \; and \; \lim_{x\to a} g(x) = 0$

(ii) $f'(x) \; and \; g'(x) \; exist \; and \; g'(a) \neq 0$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

The above rule can be extended, i.e, if

$f'(a) = 0 \; and \; g'(a) = 0 \; then \; \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = ...$

Note:

1. We apply L'Hospital's Rule only to evaluate the limits that in $\frac{0}{0}, \frac{\infty}{\infty}$ forms. Here we differentiate the numerator and denominator separately to write $\frac{f'(x)}{g'(x)}$ and apply the limit to see whether it is a finite value. If it is still in $\frac{0}{0} \; or \; \frac{\infty}{\infty}$ form we continue to differentiate the numerator and denominator and write further $\frac{f''(x)}{g''(x)}$ and apply the limit to see whether it is a finite value. We can continue the above procedure till we get a definite value of the limit.

2. To evaluate the indeterminate forms of the form $0\times\infty, \infty-\infty,$ we rewrite the functions involved or take L.C.M. to arrange the expression in either $\frac{0}{0} \; or \; \frac{\infty}{\infty}$ and then apply L'Hospital's Rule.

3. To evaluate the limits of the form $0^0, \infty^0 \; and \; 1^\infty$ i.e, where function to the power of function exists, call such an expression as some constant, then take logarithm on both

sides and rewrite the expressions to get $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.

4. We can use the values of the standard limits like

$$\lim_{x\to 0}\frac{\sin x}{x}=1;\ \lim_{x\to 0}\frac{\tan x}{x}=1;\ \lim_{x\to 0}\frac{x}{\sin x}=1;\ \lim_{x\to 0}\frac{x}{\tan x}=1;\ \lim_{x\to 0}\cos x=1;\ etc$$

Evaluate the following limits:

Example 1: Evaluate $\lim_{x\to 0}\frac{\sin x-x}{\tan^3 x}$

$$\lim_{x\to 0}\frac{\sin x-x}{\tan^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x-1}{3\tan^2 x\sec^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x}{6\tan x\sec^4 x+6\tan^3 x\sec^2 x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-\cos x}{6\sec^6 x+24\tan^2 x\sec^4 x+18\tan^2 x\sec^4 x+12\tan^4 x\sec^2 x}=-\frac{1}{6}$$

Method 2:

$$\lim_{x\to 0}\frac{\sin x-x}{\tan^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\sin x-x}{x^3}}{\left(\frac{\tan x}{x}\right)^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin x-x}{x^3}\left(\frac{0}{0}\right)\quad\because \lim_{x\to 0}\frac{\tan x}{x}=1$$

$$=\lim_{x\to 0}\frac{\cos x-1}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x}{6x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}$$

Example 2: Evaluate $\lim_{x\to 0}\frac{a^x-b^x}{x}$

$$\lim_{x\to 0}\frac{a^x-b^x}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{a^x\log a-b^x\log b}{1}=\log a-\log b=\log\frac{a}{b}$$

Example 3: Evaluate $\lim_{x\to 0}\frac{x\sin x}{\left(e^x-1\right)^2}$

$$\lim_{x\to 0}\frac{x\sin x}{\left(e^x-1\right)^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin x+x\cos x}{2\left(e^x-1\right)e^x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{2\left[e^x.e^x+(e^x-1)e^x\right]}=\frac{1+1-0}{2[1+0]}=\frac{2}{2}=1$$

Example 4: Evaluate $\lim_{x\to 0}\frac{x\,e^x-\log(1+x)}{x^2}$

$$\lim_{x\to 0}\frac{x\,e^x-\log(1+x)}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+x\,e^x-\frac{1}{1+x}}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+e^x+x\,e^x+\frac{1}{(1+x)^2}}{2}=\frac{1+1+0+1}{2}=\frac{3}{2}$$

Example 5: Evaluate $\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}$

$$\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sinh x+\sin x}{\sin x+x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cosh x+\cos x}{\cos x+\cos x-x\sin x}=\frac{1+1}{1+1-0}=\frac{2}{2}=1$$

Example 6: Evaluate $\lim_{x\to 0}\frac{\cos x-\log(1+x)-1+x}{\sin^2 x}$

$$\lim_{x\to 0}\frac{\cos x-\log(1+x)-1+x}{\sin^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x-\frac{1}{1+x}+1}{\sin 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\cos x+\frac{1}{(1+x)^2}}{2\cos 2x}=\frac{-1+1}{2}=0$$

Example 7: Evaluate $\lim_{x\to 1}\frac{x^x-x}{x-1-\log x}$

$$\lim_{x\to 1}\frac{x^x-x}{x-1-\log x}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)-1}{1-\frac{1}{x}}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)^2+x^{x-1}}{\frac{1}{x^2}}=\frac{1+1}{1}=2$$

$$\textit{Since } y=x^x\Rightarrow \log y=x\log x\Rightarrow \frac{1}{y}y'=1+\log x\Rightarrow y'=y(1+\log x)$$

$$\textit{and then } \frac{d}{dx}(x^x)=x^x(1+\log x)$$

Example 8: Evaluate $\lim_{x\to \frac{\pi}{4}}\frac{\sec^2 x-2\tan x}{1+\cos 4x}$

$$\lim_{x\to \frac{\pi}{4}}\frac{\sec^2 x-2\tan x}{1+\cos 4x}\left(\frac{0}{0}\right)=\lim_{x\to \frac{\pi}{4}}\frac{2\sec^2 x\tan x-2\sec^2 x}{-4\sin 4x}\left(\frac{0}{0}\right)=\lim_{x\to \frac{\pi}{4}}\frac{2\sec^4 x+4\sec^2 x\tan^2 x-4\sec^2 x\tan x}{-16\cos 4x}$$

$$=\frac{2\left(\sqrt{2}\right)^4+4\left(\sqrt{2}\right)^2(1)^2-4\left(\sqrt{2}\right)^2}{16}=\frac{8}{16}=\frac{1}{2}$$

Example 9: Evaluate $\lim_{x\to a}\dfrac{\log(\sin x.\cos ec\ a)}{\log(\cos a.\sec x)}$

$$\lim_{x\to a}\frac{\log(\sin x.\cos ec\ a)}{\log(\cos a.\sec x)}\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{\left[\dfrac{\cos x\cos ec\ a}{\sin x.\cos ec\ a}\right]}{\left[\dfrac{\sec x\tan x.\cos a}{\cos a.\sec x}\right]}=\lim_{x\to a}\frac{\cot x}{\tan x}=\cot^2 a$$

Example 10: Evaluate $\lim_{x\to 0}\dfrac{e^x+e^{-x}-2\cos x}{x\sin x}$

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x-e^{-x}+2\sin x}{\sin x+x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x}{\cos x+\cos x-x\sin x}=\frac{1+1+2}{1+1-0}=2$$

Example 11: Evaluate $\lim_{x\to 0}\dfrac{x\cos x-\log(1+x)}{x^2}$

$$\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x-x\sin x-\dfrac{1}{1+x}}{2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-\sin x-\sin x-x\cos x+\dfrac{1}{(1+x)^2}}{2}=\frac{-0-0-0+1}{2}=\frac{1}{2}$$

Example 12: Evaluate $\lim_{x\to 0}\dfrac{\log(1-x^2)}{\log\cos x}$

$$\lim_{x\to 0}\frac{\log(1-x^2)}{\log\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\left[\dfrac{-2x}{1-x^2}\right]}{\left(\dfrac{-\sin x}{\cos x}\right)}=\lim_{x\to 0}\frac{2x\cos x}{(1-x^2)\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\cos x-2x\sin x}{(1-x^2)\cos x-2x\sin x}=\frac{2-0}{1-0}=2$$

Example 13: Evaluate $\lim_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3\sin^2 x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6\sin x\cos^2 x-3\sin^3 x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6\cos^3 x-12\sin^2 x\cos x-9\sin^2 x\cos x}=\frac{0+2+1}{6-0-0}=\frac{3}{6}=\frac{1}{2}$$

Method 2:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\dfrac{\tan x-\sin x}{x^3}}{\left(\dfrac{\sin x}{x}\right)^3}=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\left(\frac{0}{0}\right)\quad\because\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$= \lim_{x\to 0} \frac{\sec^2 x - \cos x}{3x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{2\sec^2 x \tan x + \sin x}{6x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x + \cos x}{6} = \frac{0+2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

Example 14: Evaluate $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$

$$\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x} = \lim_{x\to 0} \frac{\frac{\tan x - x}{x^3}}{\left(\frac{\tan x}{x}\right)} = \lim_{x\to 0} \frac{\tan x - x}{x^3}\left(\frac{0}{0}\right) \qquad \because \lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6} = \frac{0+2}{6} = \frac{1}{3}$$

Example 15: Evaluate $\lim_{x\to 0} \frac{e^{ax} - e^{-ax}}{\log(1+bx)}$

$$\lim_{x\to 0} \frac{e^{ax} - e^{-ax}}{\log(1+bx)}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{ae^{ax} + ae^{-ax}}{b/(1+bx)} = \frac{a+a}{b} = \frac{2a}{b}$$

Example 16: Evaluate $\lim_{x\to 0} \frac{a^x - 1 - x\log a}{x^2}$

$$\lim_{x\to 0} \frac{a^x - 1 - x\log a}{x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{a^x \log a - \log a}{2x}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{a^x (\log a)^2}{2} = \frac{1}{2}(\log a)^2$$

Example 17: Evaluate $\lim_{x\to 0} \frac{e^x - \log(e+ex)}{x^2}$

$$\lim_{x\to 0} \frac{e^x - \log(e+ex)}{x^2} = \lim_{x\to 0} \frac{e^x - \log e(1+x)}{x^2} = \lim_{x\to 0} \frac{e^x - \log e - \log(1+x)}{x^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^x - \frac{1}{1+x}}{2x}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{e^x + \frac{1}{(1+x)^2}}{2} = \frac{1+1}{2} = 1$$

Exercise: Evaluate the following limits.

(i) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$ $\qquad$ (ii) $\lim_{x\to 0} \frac{\log(1+x^3)}{\sin^3 x}$

(iii) $\lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$ (iv) $\lim_{x\to\frac{\pi}{2}}\frac{\log\sin x}{(x-\frac{\pi}{2})^2}$

(v) $\lim_{x\to 0}\frac{\cosh x+\log(1-x)-1+x}{x^2}$ (vi) $\lim_{x\to\frac{\pi}{2}}\frac{\sin x\sin^{-1}x}{x^2}$

(vii) $\lim_{x\to 0}\frac{e^{2x}-(1+x)^2}{x\log(1+x)}$

Limits of the form $\left(\frac{\infty}{\infty}\right)$:

Example 18: Evaluate $\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}$

$$\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{(2\cos 2x/\sin 2x)}{(\cos x/\sin x)}=\lim_{x\to 0}\frac{2\cot 2x}{\cot x}=\lim_{x\to 0}\frac{2\tan x}{\tan 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x}{2\sec^2 2x}=\frac{2}{2}=1$$

Example 19: Evaluate $\lim_{x\to 0}\frac{\log x}{\cos ecx}$

$$\lim_{x\to 0}\frac{\log x}{\cos ecx}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{1\backslash x}{-\cos ec\,x.\cot x}=\lim_{x\to 0}\frac{-\sin^2 x}{x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-2\sin 2x}{\cos x-x\sin x}=\frac{0}{1-0}=0$$

Example 20: Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\tan x}$

$$\lim_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\tan x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\frac{\pi}{2}}\frac{-\tan x}{\sec^2 x}=-\lim_{x\to\frac{\pi}{2}}\frac{-\tan x}{\sec^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{-\sin x\cos x}{1}=\frac{-0}{1}=0$$

Example 21: Evaluate $\lim_{x\to 1}\frac{\log(1-x)}{\cot\pi x}$

$$\lim_{x\to 1}\frac{\log(1-x)}{\cot\pi x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 1}\frac{-1/(1-x)}{-\pi\cos ec^2\pi x}=\lim_{x\to 1}\frac{\sin^2\pi x}{\pi(1-x)}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{2\pi\sin\pi x\cos\pi x}{-\pi}=\frac{0}{-\pi}=0$$

Example 22: Evaluate $\lim_{x\to 0}\log_{\tan 2x}\tan 3x$

$$\lim_{x\to 0}\log_{\tan 2x}\tan 3x=\lim_{x\to 0}\left(\frac{\log\tan 3x}{\log\tan 2x}\right)\left(\frac{\infty}{\infty}\right)\qquad\because\log_b a=\frac{\log_e a}{\log_e b}$$

$$= \lim_{x\to 0}\left(\frac{3\sec^2 3x/\tan 3x}{2\sec^2 2x/\tan 2x}\right) = \lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right) = \lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{6/\sin 6x}{4/\sin 4x}\right) = \lim_{x\to 0}\left(\frac{6\sin 4x}{4\sin 6x}\right)\left(\frac{0}{0}\right) = \lim_{x\to 0}\left(\frac{24\cos 4x}{24\cos 6x}\right) = \frac{24}{24} = 1$$

Example 23: Evaluate $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}$

$$\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}\left(\frac{\infty}{\infty}\right) = \lim_{x\to a}\frac{1/(x-a)}{e^x/(e^x-e^a)} = \lim_{x\to a}\frac{(e^x-e^a)}{e^x(x-a)}\left(\frac{0}{0}\right) = \lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x} = \frac{e^a}{e^a} = 1$$

Exercise: Evaluate the following limits.

$(i)\ \lim_{x\to 0}\frac{\log\tan x}{\log x}$ $\qquad (ii)\ \lim_{x\to 0}\frac{\log\sin x}{\cot x}$

$(iii)\ \lim_{x\to 0}\frac{\cot 2x}{\cot 3x}$ $\qquad (iv)\ \lim_{x\to\frac{\pi}{2}}\frac{\sec x}{\tan 3x}$ $\qquad (v)\lim_{x\to 0}\log_{\sin x}\sin 2x$

Limits of the form $(0\times\infty)$**:** To evaluate the limits of the form $(0\times\infty)$, we rewrite the given expression to obtain either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

Example 24: Evaluate $\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x$

$$\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x\ \ (0\times\infty\ \ form) = \lim_{x\to\infty}\frac{(a^{\frac{1}{x}}-1)}{\left(\frac{1}{x}\right)}\left(\frac{0}{0}\right) = \lim_{x\to\infty}\frac{a^{\frac{1}{x}}(\log a)\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to\infty}a^{\frac{1}{x}}(\log a) = a^0\log a = \log a$$

Example 25: Evaluate $\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$

$$\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x\ \ (0\times\infty\ \ form) = \lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)}{\cot x}\left(\frac{0}{0}\right) = \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\cos ec^2 x} = \frac{0}{1} = 0$$

Example 26: Evaluate $\lim_{x\to 1}\sec\frac{\pi}{2x}.\log x$

$$\lim_{x\to 1} \sec\frac{\pi}{2x}.\log x \;\left(\infty\times 0\;\; form\right) = \lim_{x\to 1}\frac{\log x}{\cos\dfrac{\pi}{2x}}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{1/x}{-\dfrac{\pi}{2}\left(\sin\dfrac{\pi}{2x}\right)\left(\dfrac{-1}{x^2}\right)} = \lim_{x\to 1}\frac{2x}{\pi\sin\dfrac{\pi}{2x}} = \frac{2}{\pi}$$

Example 27: Evaluate $\lim\limits_{x\to 0} x\log\tan x$

$$\lim_{x\to 0} x\log\tan x \;\left(0\times\infty\;\; form\right) = \lim_{x\to 0}\frac{\log\tan x}{(1/x)}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{\sec^2 x/\tan x}{\left(\dfrac{-1}{x^2}\right)} = \lim_{x\to 0}\frac{-x^2}{\sin x.\cos x}$$

$$= \lim_{x\to 0}\frac{-2x^2}{\sin 2x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-4x}{2\cos 2x} = \frac{0}{2} = 0$$

Example 28: Evaluate $\lim\limits_{x\to 1}(1-x^2)\tan\dfrac{\pi x}{2}$

$$\lim_{x\to 1}(1-x^2)\tan\frac{\pi x}{2}\;\left(0\times\infty\;\; form\right) = \lim_{x\to 1}\frac{1-x^2}{\cot\dfrac{\pi x}{2}}\left(\frac{0}{0}\right) = \lim_{x\to 1}\frac{-2x}{-\dfrac{\pi}{2}\cos ec^2\dfrac{\pi x}{2}} = \frac{2}{\left(\dfrac{\pi}{2}\right)} = \frac{4}{\pi}$$

Example 29: Evaluate $\lim\limits_{x\to 0}\tan x.\log x$

$$\lim_{x\to 0}\tan x.\log x\left(0\times\infty\;\; form\right) = \lim_{x\to 0}\frac{\log x}{\cot x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{1/x}{-\cos ec^2 x} = \lim_{x\to 0}\frac{-\sin^2 x}{x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-\sin 2x}{1} = \frac{0}{1} = 0$$

Limits of the form $(\infty-\infty)$**:** To evaluate the limits of the form $(\infty-\infty)$, we take L.C.M. and rewrite the given expression to obtain either $\left(\dfrac{0}{0}\right)$ or $\left(\dfrac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

Example 30: Evaluate $\lim\limits_{x\to 0}\left[\dfrac{1}{x}-\cot x\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]=\lim_{x\to 0}\left[\frac{1}{x}-\frac{\cos x}{\sin x}\right]\left(\infty-\infty\ form\right)$$

$$=\lim_{x\to 0}\left[\frac{\sin x - x\cos x}{x\sin x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\cos x-\cos x+x\sin x}{\sin x + x\cos x}\right]$$

$$=\lim_{x\to 0}\left[\frac{x\sin x}{\sin x + x\cos x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\sin x + x\cos x}{\cos x+\cos x - x\sin x}\right]=\frac{0+0}{1+1-0}=0$$

Example 31: Evaluate $\lim_{x\to\frac{\pi}{2}}\left[\sec x-\tan x\right]$

$$\lim_{x\to\frac{\pi}{2}}\left[\sec x-\tan x\right]=\lim_{x\to\frac{\pi}{2}}\left[\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right]\left(\infty-\infty\ form\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{1-\sin x}{\cos x}\right]\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}}\left[\frac{-\cos x}{-\sin x}\right]=\frac{0}{1}=0$$

Example 32: Evaluate $\lim_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]$

$$\lim_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]\left(\infty-\infty\ form\right)=\lim_{x\to 1}\left[\frac{(x-1)-x\log x}{(x-1)\log x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\left[\frac{1-1-\log x}{\frac{x-1}{x}+\log x}\right]=\lim_{x\to 1}\left[\frac{-\log x}{1-\frac{1}{x}+\log x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 1}\left[\frac{-1/x}{\frac{1}{x^2}+\frac{1}{x}}\right]=\frac{-1}{1+1}=\frac{-1}{2}$$

Example 33: Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]\left(\infty-\infty\ form\right)=\lim_{x\to 0}\left[\frac{(e^x-1)-x}{x(e^x-1)}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{e^x-1}{(e^x-1)+xe^x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{e^x}{e^x+e^x+xe^x}\right]=\frac{1}{1+1+0}=\frac{1}{2}$$

Example 34: Evaluate $\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]$

$$\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right](\infty-\infty\ \ form)=\lim_{x\to 0}\left[\frac{x-\sin x}{x\sin x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{1-\cos x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)==\lim_{x\to 0}\left[\frac{\sin x}{\cos x+\cos x-x\sin x}\right]=\frac{0}{1+1}=0$$

Example 35: Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{1-\frac{1}{1+x}}{2x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\frac{1}{(1+x)^2}}{2}\right]=\frac{1}{2}$$

Example 36: Evaluate $\lim_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]$

$$\lim_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]=\lim_{x\to 0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right](\infty-\infty\ \ form)=\lim_{x\to 0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{a\sin\frac{x}{a}-x\cos\frac{x}{a}}{x\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{a.\frac{1}{a}\cos\frac{x}{a}-\cos\frac{x}{a}+\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]$$

$$=\lim_{x\to 0}\left[\frac{\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\frac{1}{a}\sin\frac{x}{a}+\frac{x}{a}.\frac{1}{a}.\cos\frac{x}{a}}{\frac{1}{a}\cos\frac{x}{a}+\frac{1}{a}\cos\frac{x}{a}-\frac{x}{a^2}\sin\frac{x}{a}}\right]=\frac{0+0}{\frac{1}{a}+\frac{1}{a}-0}=0$$

Exercise: Evaluate the following limits.

(i) $\lim_{x\to a}\left(2-\frac{x}{a}\right)\cot(x-a)$ $\quad$ (ii) $\lim_{x\to 0}\left(\cos ecx-\cot x\right)$

(iii) $\lim_{x\to\frac{\pi}{2}}\left[x\tan x-\frac{\pi}{2}\sec x\right]$ $\quad$ (iv) $\lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)$

(v) $\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{x\tan x}\right]$ $\quad$ (vi) $\lim_{x\to\frac{\pi}{2}}\left[2x\tan x-\pi\sec x\right]$

Example 37: Find the value of 'a' such that $\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}$ is finite. Also find the value of the limit.

Let A= $\lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{2\cos 2x + a\cos x}{3x^2} = \frac{2+a}{0} \neq \textit{finite}$

We can continue to apply the L'Hospital's Rule, if 2+a=0 i.e., a= -2 .

$For\ a = -2,$

$$A = \lim_{x\to 0}\frac{2\cos 2x - 2\cos x}{3x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = -1$$

$\therefore$ *The given* lim *it will have a finite value when* $a = -2$ *and it is* -1.

Example 38: Find the values of 'a' and 'b' such that $\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3} = \frac{1}{3}$.

Let $A = \lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{(1-a\cos x)+ax\sin x + b\cos x}{3x^2} = \frac{1-a+b}{0} \neq \textit{finite}$

We can continue to apply the L'Hospital's Rule, if 1-a+b=0 i.e., a-b = 1 .

$For\ a - b = 1,$

$$A = \lim_{x\to 0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{2a\sin x + ax\cos x - b\sin x}{6x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{3a\cos x - ax\sin x - b\cos x}{6} = \frac{3a-b}{6} = \textit{finite}$$

This finite value is given as $\frac{1}{3}$. *i.e.,* $\frac{3a-b}{6} = \frac{1}{3} \Rightarrow 3a - b = 2$

Solving the equations $a - b = 1$ *and* $3a - b = 2$ *we obtain* $a = \frac{1}{2}$ *and* $b = -\frac{1}{2}$.

Example 39: Find the values of 'a' and 'b' such that $\lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2} = 1$**.**

$$Let\ A = \lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2} = \frac{a-b}{0} \neq finite$$

We can continue to apply the L'Hospital's Rule, if a-b=0, since the denominator=0 .

$For\ a - b = 0$**,**

$$A = \lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{a\sinh x + b\sin x}{2x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{a\cosh x + b\cos x}{2} = \frac{a+b}{2}$$

$$But\ this is\ given\ as\ 1.\ \therefore a+b=2$$

$Solving\ the\ equations\ a-b=0\ and\ a+b=2\ we\ obtain\ a=1\ and\ b=1.$

Limits of the form $0^0, \infty^0$ *and* 1^∞ **:** To evaluate such limits, where function to the power of function exists, we call such an expression as some constant, then take logarithm on both sides and rewrite the expressions to get $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.

Example 40: Evaluate $\lim_{x\to 0} x^x$

$$Let\ A = \lim_{x\to 0} x^x\ (0^0\ \ form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0}\log x^x = \lim_{x\to 0} x.\log x\ (0\times\infty\ \ form) = \lim_{x\to 0}\frac{\log x}{1/x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{1/x}{(-1/x^2)} = \lim_{x\to 0}\frac{-x}{1} = \frac{0}{1} = 0$$

$$\log_e A = 0 \Rightarrow A = e^0 = 1 \quad \therefore \lim_{x\to 0} x^x = 1$$

Example 41: Evaluate $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

$$Let\ A=\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}\ (1^{\infty}\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to 0}\log(\cos x)^{\frac{1}{x^2}}=\lim_{x\to 0}\frac{1}{x^2}\log\cos x\ (\infty\times 0\ \ form)=\lim_{x\to 0}\frac{\log\cos x}{x^2}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-\tan x}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sec^2 x}{2}=\frac{-1}{2}$$

$$\log_e A=-\frac{1}{2}\Rightarrow A=e^{\frac{-1}{2}}=\frac{1}{\sqrt{e}}\quad \therefore \lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}=\frac{1}{\sqrt{e}}$$

Example 42: Evaluate $\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}$

$$Let\ A=\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}\ (\infty^0\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to\frac{\pi}{2}}\log(\tan x)^{\cos x}=\lim_{x\to\frac{\pi}{2}}\cos x\log(\tan x)\ (0\times\infty\ \ form)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\log\tan x}{\sec x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x/\tan x}{\sec x.\tan x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin^2 x}=\frac{0}{1}=0$$

$$\log_e A=0\Rightarrow A=e^0=1\quad \therefore \lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}=1$$

Example 43: Evaluate $\lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}}$

$$Let\ A=\lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}}\ (1^{\infty}\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to 0}\log(\frac{\tan x}{x})^{\frac{1}{x^2}}=\lim_{x\to 0}\frac{1}{x^2}\log(\frac{\tan x}{x})\ (\infty\times 0\, form)$$

$$=\lim_{x\to 0}\frac{\log(\frac{\tan x}{x})}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\sin x.\cos x}-\frac{1}{x}}{2x}=\lim_{x\to 0}\frac{\frac{2}{\sin 2x}-\frac{1}{x}}{2x}=\lim_{x\to 0}\frac{2x-\sin 2x}{2x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2-2\cos 2x}{4x\sin 2x + 4x^2 \cos 2x}\left(\frac{0}{0}\right) == \lim_{x\to 0} \frac{-4\sin 2x}{4\sin 2x + 16x\cos 2x - 8x^2 \sin 2x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-8\cos 2x}{24\cos 2x - 48x\sin 2x - 16x^2 \cos 2x} = \frac{-8}{24} = \frac{-1}{3}$$

$$\log_e A = -\frac{1}{3} \Rightarrow A = e^{-\frac{1}{3}} \quad \therefore \lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}} = e^{-\frac{1}{3}}$$

Example 44: Evaluate $\lim_{x\to 0}(a^x + x)^{\frac{1}{x}}$

$$Let\ A = \lim_{x\to 0}(a^x + x)^{\frac{1}{x}}\ (1^\infty\ \ form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0} \log(a^x + x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{1}{x}\log(a^x + x)\ (\infty \times 0\ form)$$

$$= \lim_{x\to 0} \frac{\log(a^x + x)}{x}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{(a^x \log a + 1)/(a^x + x)}{1}$$

$$= \log a + 1 = \log a + \log e = \log ae$$

$$\therefore \log_e A = \log ea \Rightarrow A = ea \qquad Hence \lim_{x\to 0}(a^x + x)^{\frac{1}{x}} = ea.$$

Example 45: Evaluate $\lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}$

$$Let\ A = \lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}\ (1^\infty\ \ form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to a} \log(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}} = \lim_{x\to a} \tan\frac{\pi x}{2a}.\log(2-\frac{x}{a})\left(\infty \times 0\ \ form\right)$$

$$= \lim_{x\to a} \frac{\log(2-\frac{x}{a})}{\cot\frac{\pi x}{2a}}\left(\frac{0}{0}\right) = \lim_{x\to a}\left(\frac{\frac{(-1/a)}{2-\frac{x}{a}}}{-\frac{\pi}{2a}\cos ec^2\frac{\pi x}{2a}}\right) = \lim_{x\to a} \frac{2}{\pi}.\frac{\sin^2\frac{\pi x}{2a}}{2-\frac{x}{a}} = \frac{2}{\pi}$$

$$\therefore \log_e A = \frac{2}{\pi} \Rightarrow A = e^{\frac{2}{\pi}} \qquad Hence \lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}} = e^{\frac{2}{\pi}}.$$

Exercise: Evaluate the following limits.

$(i)\ \lim_{x\to 0}(\cos ax)^{\frac{b}{x^2}}$ $\qquad (ii)\ \lim_{x\to 0}\left(\frac{1+\cos x}{2}\right)^{\frac{1}{x^2}}$

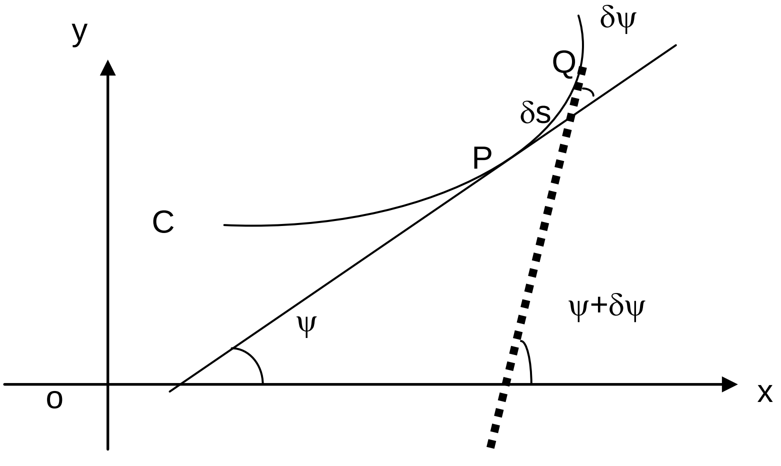
$(iii)\ \lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ $\qquad (iv)\ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$(v)\ \lim_{x\to 0}(\sin x)^{\tan x}$ $\qquad (iv)\ \lim_{x\to 0}(1+\sin x)^{\cot x}$

$(vii)\ \lim_{x\to 0}(\cos x)^{\operatorname{cosec}^2 x}$ $\qquad (viii)\ \lim_{x\to \frac{\pi}{4}}(\tan x)^{\tan 2x}$

$(ix)\ \lim_{x\to \infty}\left(\frac{ax+1}{ax-1}\right)^{x}$ $\qquad (x)\ \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

CURVATURE



Consider a curve C in XY-plane and let P, Q be any two neighboring points on it. Let arc AP=s and arc PQ=δs. Let the tangents drawn to the curve at P, Q respectively make angles ψ and ψ+δψ with X-axis i.e., the angle between the tangents at P and Q is δψ. While moving from P to Q through a distance'δs', the tangent has turned through the angle 'δψ'. This is called the bending of the arc PQ. Geometrically, a change in ψ represents the bending of the curve C and the ratio $\frac{\delta\psi}{\delta s}$ represents the ratio of bending of C between the point P & Q and the arc length between them.

$\therefore$ Rate of bending of Curve at P is

$$\frac{d\psi}{ds} = \underset{Q \to P}{Lt} \frac{\delta\psi}{\delta s}$$

This rate of bending is called the curvature of the curve C at the point P and is denoted by κ (kappa). Thus $\kappa = \frac{d\psi}{ds}$

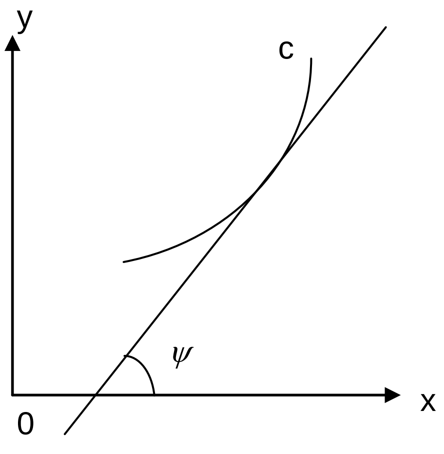
We note that the curvature of a straight line is zero since there exist no bending i.e. κ=0, and that the curvature of a circle is a constant and it is not equal to zero since a circle bends uniformly at every point on it.

If $\kappa \neq 0$, then $\frac{1}{\kappa}$ is called the radius of curvature and is denoted by ρ (rho - Greek letter).

$$\therefore \rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$$

Radius of curvature in Cartesian form :

Suppose y = f(x) is the Cartesian equation of the curve considered in figure.



we have $y' = \frac{dy}{dx} = Tan\,\psi \Rightarrow y'' = \frac{d^2y}{dx^2} = Sec^2\,\psi \cdot \frac{d\psi}{dx} = \left(1+Tan^2\psi\right)\frac{d\psi}{ds}\cdot\frac{ds}{dx}$

But we know that $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\therefore \frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right] \cdot \frac{d\psi}{ds} \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{\mathrm{d}s}{\mathrm{d}\psi} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{\left[1+(y')^2\right]^{3/2}}{y''}$$

This is the expression for radius of curvature in Cartesian form.

NOTE: We note that when y'=∞, we find ρ using the formula $\rho = \dfrac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)}$

Example 9: Find the radius of curvature of the curve $x^3+y^3 = 2a^3$ at the point (a, a).

$$x^3+y^3 = 2a^3 \Rightarrow 3x^2+3y^2 \cdot y' = 0 \quad \Rightarrow y' = -\frac{x^2}{y^2} \ \ hence\ at\ (a,a),\ \ y' = -1$$

$$\therefore y'' = -\left[\frac{y^2(2x)-x^2(2y)y'}{y^4}\right],\ hence\ at\ (a,a),\ y'' = -\left[\frac{2a^3+2a^3}{a^4}\right] = -\frac{4}{a}$$

$$\therefore\ \rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{\left[1+(-1)^2\right]^{3/2}}{-4/a}\ \ i.e.,\ |\rho| = \frac{a}{4}\cdot 2\sqrt{2} \quad = \quad \frac{a}{\sqrt{2}}$$

Example 10: Find the radius of curvature for $\sqrt{x}+\sqrt{y} = \sqrt{a}$ at the point where it meets the line y=x.

On the line y = x, $\sqrt{x}+\sqrt{x} = \sqrt{a} \quad i.e\ 2\sqrt{x} = \sqrt{a} \quad or \quad x = \frac{a}{4}$

i.e., We need to find ρ at $\left(\frac{a}{4},\frac{a}{4}\right)$

$$\sqrt{x}+\sqrt{y} = \sqrt{a} \Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y' = 0\ \ i.e\ \ y' = -\sqrt{\frac{y}{x}},\ \ hence\ at\ \left(\frac{a}{4},\frac{a}{4}\right), y' = -1$$

$$Also,\ y'' = -\left[\frac{\sqrt{x}\,\dfrac{1}{2\sqrt{y}}\cdot y' - \sqrt{y}\,\dfrac{1}{2\sqrt{x}}}{x}\right]$$

$$\therefore at\left(\frac{a}{4},\frac{a}{4}\right), y'' = -\left[\frac{\sqrt{\dfrac{a}{4}}\,\dfrac{1}{2\sqrt{\dfrac{a}{4}}}\cdot(-1) - \sqrt{\dfrac{a}{4}}\,\dfrac{1}{2\sqrt{\dfrac{a}{4}}}}{\dfrac{a}{4}}\right] = -\frac{(-\frac{1}{2}-\frac{1}{2})}{\frac{a}{4}} = -\frac{(-1)}{\frac{a}{4}} = \frac{4}{a}$$

$$\therefore \rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{\left[1+(-1)^2\right]^{3/2}}{4/a} = \frac{a}{4}2\sqrt{2} = \frac{a}{\sqrt{2}}$$

Example 11: Show that the radius of curvature for the curve y = 4 Sin x - Sin 2x

at x = $\pi/2$ is $5\sqrt{5}/4$

y = 4 Sin x - Sin 2x $\Rightarrow y' = 4\,Cos\ x - 2\,Cos\ 2x$

$\therefore$ when x $= \pi/2$, $y' = 4Cos\,\pi/2 - 2Cos\pi = 0 - 2(-1) = 2$

Also, $y'' = -4\,Sin\ x + 4\,Sin\ 2x$ and when x $= \pi/2$, $y'' = -4\,Sin\ \pi/2 + 4\,Sin\ \pi = -4$

$$\therefore \rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{\left[1+2^2\right]^{3/2}}{-4} \Rightarrow |\rho| = \frac{5\sqrt{5}}{4}$$

Example 12: Find the radius of curvature for $xy^2 = a^3 - x^3$ at (a, 0).

$$xy^2 = a^3 - x^3 \Rightarrow y^2 + 2xy\ y' = -3x^2$$

$$\therefore y' = \frac{-3x^2 - y^2}{2xy}\ and\ at\ (a,0),\ \ y' = \infty$$

In such cases we write $\dfrac{dx}{dy} = \dfrac{2xy}{-3x^2 - y^2}$ $and\ \ at(a,0), \dfrac{dx}{dy} = 0$

Also $\frac{dx}{dy} = \frac{-2xy}{3x^2 + y^2} \Rightarrow \frac{d^2x}{dy^2} = \left[\frac{(3x^2 + y^2)\left(2\frac{dx}{dy}y + 2x\right) - 2xy\left(6x\frac{dx}{dy} + 2y\right)}{(3x^2 + y^2)^2} \right]$

$$\therefore\ At(a, 0),\ \frac{d^2x}{dy^2} = \left[\frac{(3a^2 + 0)(0 + 2a) - 0}{(3a^2 + 0)^2} \right] = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$$

$$\therefore \rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{[1 + o^2]^{3/2}}{-2/3a}\ or \qquad |\rho| = \frac{3a}{2}$$

Exercise:

Find the radius of curvature for the following curves:

(i) $x^3 + y^3 = 3axy\ at \left(\frac{3a}{2}, \frac{3a}{2}\right)$

(ii) $y^2 = \frac{a^2(a - x)}{x}\ at\ (a, 0)$

(iii) $a^2y = x^3 - a^3\ at\ the\ point\ where\ it\ crosses\ x - axis$

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ at\ the\ ends\ of\ the\ major\ axis.$

(v) $a^2y = x^3 - a^3\ at\ the\ point\ (a, 0).$

(vi) $y^2 = 2x(3 - x^2)\ at\ the\ point\ where\ the\ tangents\ are\ parallel\ to\ x - axis.$

(vii) $x^2y = a(x^2 + y^2)\ at\ (-2a, 2a).$

(viii) $For\ the\ curve\ y = \frac{ax}{a + x}\ show\ that \left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$

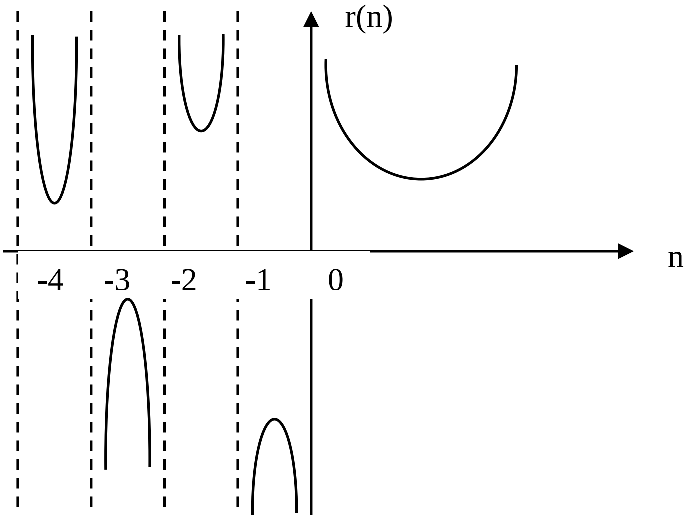
M.G.Geetha

INTEGRAL CALCULUS
Gamma and Beta Functions

Definition:

The improper integral $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ is defined as the gamma function.

Here n is a real number called the parameter of the function. Γ(n) exists for all real values of n except 0, -1, -2., ………..the graph of which is shown below :



Recurrence formula

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0$$

$$= \left\{ \frac{x^n}{n} e^{-x} \right\}_0^\infty + \int_0^\infty \frac{x^n}{n} e^{-x} dx, \quad \text{integrating by parts.}$$

Applying the definition of Gamma function and using $\frac{x^n}{e^x} \to 0$ as $x \to \infty$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \text{.........................(1)}$$

$$or \quad \Gamma(n+1) = n\Gamma(n) \text{.................(2)}$$

Note:

(1) Γ(n) is not convergent when n = 0, -1, -2, …….

(2) If Γ(n) is known for 0 < n < 1, then its value for 1 < n < 2 can be found using equation (2). Also, its values for -1 < n < 0 can be got using equation (1)

(3) If n is a +ve integer, using the recurrence relation and Γ(1) = 1, we get

$$\begin{aligned}\Gamma(n) &= (n - 1)\,\Gamma(n - 1)\\ &= (n - 1)(n - 2)\,\Gamma(n - 2) \text{ and so on}\\ &= (n - 1)(n - 2)(n - 3) \ldots\ldots 1\\ &= (n - 1)!\end{aligned}$$

Problems:

(1) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

By definition, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$= \int_0^\infty e^{-t^2} t^{2n-1} dt \quad using \quad x = t^2$$

$$\therefore \Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x^2} dx \quad = \int_0^\infty e^{-y^2} dy$$

$$\therefore \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dxdy$$

Converting Cartesian (x, y) to polar (r, θ) system, we get

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} (2r) dr d\theta$$

$$= \int_0^{\pi/2} \left[e^{-r^2}\right]_0^{\infty} d\theta, \qquad \because \ -\int 2re^{-r^2} dr = e^{-r^2}$$

$$= 2\int_0^{\pi/2} d\theta, \qquad \because e^{-r^2} \to 0 \text{ as } r \to \infty$$

$$= \pi$$

Taking square roots on both sides, we get $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(2) Show that $\int_0^{\infty} e^{-x^4} dx = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$

Let $x^4 = y \Rightarrow 4x^3 dx = dy \Rightarrow dx = \frac{1}{4} y^{-3/4} dy$

$$\therefore \int_0^{\infty} e^{-x^4} dx = \frac{1}{4}\int_0^{\infty} e^{-y} y^{-3/4} dy = \frac{1}{4}\Gamma\left(\frac{1}{4}\right), \text{ using the definition.}$$

(3) Find $\int_0^1 (x \ln x)^5 dx$

using $\ln x = -y$, we get $x = e^{-y}$, $dx = -e^{-y} dy$

and y ranges from ∞ to 0 when $x = 0$ to 1

$$\therefore \int_0^{\infty} (x \ln x)^5 dx = -\int_0^{\infty} y^5 e^{-6y} dy, \text{ interchanging the lower and the upper limits}$$

$$= \int_0^{\infty} e^{-t} \frac{t^5 dt}{6^6}, \text{ choosing } 6y = t$$

$$= -\frac{1}{6^6}\Gamma(6) = -\frac{5!}{6^6}$$

(4) Prove that $\int_0^a \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}} = a\sqrt{\pi}$

using $\ln\left(\frac{a}{x}\right) = t$, we get $\frac{a}{x} = e^t$

$dx = -ae^{-t} dt$ and $t = \infty$ to 0 when $x = 0$ to a

$$\therefore \int_0^a \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}} = \int_0^\infty t^{-1/2} a.e^{-t} dt \quad = a\Gamma\left(\frac{1}{2}\right) = a\sqrt{\pi}$$

(5) Show that $\displaystyle\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$.

Hence show that $\displaystyle\int_0^1 \frac{\log x}{\sqrt{x}} dx = -4$ when 'n' is a positive integer and m > -1.

Let logx = - t $\Rightarrow$ dx = -e^{-t} dt and t = ∞ to 0 when x = 0 to 1.

$$\therefore \int_0^1 x^m (\log x)^n dx = \int_0^\infty e^{-mt} (-t)^n e^{-t} dt = (-1)^n \int_0^\infty t^n e^{-(m+1)t} dt$$

using $(m+1)t = y$, $\displaystyle\int_0^1 x^m (\log x)^n dx \; = (-1)^n \int_0^\infty \frac{y^n}{(m+1)^n} e^{-y} \frac{dy}{m+1}$, $\because m > -1$

$$\therefore \int_0^1 x^m (\log x)^n dx = \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n dy$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} n!, \quad \because \Gamma(n+1) = n!$$

choosing $n = 1$ and $m = -\frac{1}{2}$, we get the required result.

Beta function

Definition: Beta function, denoted by B(m,n) is defined by

$B(m,n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1} dx$, where m and n are positive real numbers.

By a property of definite integral, $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1} dx = \int_0^1 (1-x)^{m-1} x^{n-1} dx$

Therefore we have **B(m,n) = B(n,m)**

Alternate Expressions for B(m,n)

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \text{..........................}(1)$$

(i) Let $x = \frac{1}{1+t} \Rightarrow dx = -\frac{1}{(1+t)^2}dt, \quad 1 - x = \frac{t}{1+t}$

and $t = \infty$ to 0 when $x = 0$ to 1.

$\therefore (1)$ reduces to $B(m,n) = \int_0^\infty \left(\frac{1}{1+t}\right)^{m-1}\left(\frac{1}{1+t}\right)^{n-1}\left(\frac{1}{(1+t)^2}\right)dt$

$B(m,n)$ is also $= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt$ using $B(m,n) = B(n,m)$

(ii) Let $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta d\theta$ and $\theta = 0$ to $\frac{\pi}{2}$ when $x = 0$ to 1

Then (1) reduces to $2\int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2m-1}\theta d\theta$

$\therefore B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$

Relation between Gamma and Beta functions.

Prove that $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof:

We know that $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}dx$ *and* $\Gamma(m) = 2\int_0^\infty e^{-y^2}x^{2m-1}dy$

Then $\Gamma(m)\,\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2n-1}y^{2m-1}dxdy$

$= 4\int_0^\infty e^{-r^2}r^{2(m+n)-1}dr \int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$ in polar coordinates

$= \Gamma(m+n)\,B(m,n)$ by definitions.

Note:

Using $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}, \quad 0 < n < 1$

We get $\Gamma(n)\,\Gamma(1-n) = \Gamma(1)\,B(1-n,n)$ using the above relation

But $B(1-n,n)=\int_0^\infty \frac{x^{n-1}}{1+x}dx$ using definition of Beta function.

Therefore $\Gamma(n)\,\Gamma(1-n) = \pi/\sin n\pi, \quad 0 < n < 1$

Problems

(1) Show that $\int_0^1 \frac{5x}{\sqrt{1-x^5}}dx = B\left(\frac{2}{5},\frac{1}{2}\right)$

Letting $x^5 = t$, we get $x = t^{1/5}, dx = \frac{1}{5}t^{-4/5}dt$ and $t = 0$ to 1 when $x = 0$ to 1

$$\therefore \int_0^1 \frac{5x}{\sqrt{1-x^5}}dx = \int_0^1 5t^{1/5}(1-t)^{-1/2}\frac{1}{5}t^{-4/5}dt$$

$$= \int_0^1 t^{-3/5}(1-t)^{-1/2}dt$$

$$= B\left(\frac{2}{5},\frac{1}{2}\right) \text{ by definition.}$$

(2) Evaluate $\int_0^\infty \frac{x^{10}(1-x^8)}{(1+x)^{30}}dx - \int_0^\infty \frac{x^{18}}{(1+x)^{30}}dx$

$= B(11,19) - B(19,11) = 0$, using $B(m,n) = B(n,m)$.

(3) Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$

$$\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$$= \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right), \quad \text{by definition}$$

(4) Show that $\int_0^n \left(1-\frac{x}{n}\right)^n x^{t-1}dx = n^t B(t,n+1), \quad t>0, n>1$

Put $\frac{x}{n} = y \Rightarrow dx = n\,dy$ and $y = 0$ to 1 when $x = 0$ to n

$$\therefore \int_0^n \left(1-\frac{x}{n}\right)^n x^{t-1}dx = \int_0^1 (1-y)^n n^{t-1} y^{t-1} n dy$$

$$= n^t \int_0^1 (1-y)^n y^{t-1} dy = n^t B(t, n+1) \quad \text{by definition.}$$

(5) Prove that $B(n,n) = \frac{1}{2^{2n-1}} \mathbf{B}\left(n, \frac{1}{2}\right)$

We know that $\mathrm{B}(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1} 2\theta}{2^{2n-1}} d\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \int_0^{\pi} \frac{\sin^{2n-1}\phi}{2^{2n-1}} d\phi, \quad \text{putting } 2\theta = \phi$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1}\phi}{2^{2n-1}} d\phi, \quad \sin\phi \text{ is even in } (0,\pi) \text{ as } \sin(\pi - \phi) = \sin\phi$$

Applying the definition of Beta function, *we get* $\mathrm{B}(n,n) = \frac{1}{2^{2n-1}} \mathbf{B}\left(n, \frac{1}{2}\right)$

(6) Prove the **Duplication formula**

$$\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$$

Using the previous result, $\mathrm{B}\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{1}{2^{2n}} \mathrm{B}\left(n+\frac{1}{2}, \frac{1}{2}\right)$ ....(1)

Also $$\frac{B\left(n+\frac{1}{2}, n+\frac{1}{2}\right)}{B\left(n+\frac{1}{2}, \frac{1}{2}\right)} = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\Gamma(n+1)}{\Gamma(2n+1)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \text{.....................(2)}$$

From (1) and (2) we get $\frac{\Gamma\left(n+\frac{1}{2}\right) n\,\Gamma(n)}{2n\,\Gamma(2n)\sqrt{\pi}} = \frac{1}{2^{2n}}$

$$\therefore \Gamma(n).\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$$

(7) Prove that $B(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

By one of the definitions of Beta function, we have

$$B(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$=\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx+\int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$= I_1 + I_2$ (say)

Substituting $x=\frac{1}{y}$ in I_2, we get

$$I_2=\int_1^0 \frac{y^{m+n}}{y^{n-1}(1+y)^{m+n}}\left(-\frac{1}{y^2}\right)dy=\int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

Hence $B(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

Conclusion: We evaluated double integrals using change of order of integration. We used change of variables for double and triple integrals to simplify the integrals in certain cases. We saw how Gamma and Beta functions were useful in evaluating complicated integrals.

Multiple Choice:

1. The value of $\int_0^1\int_1^2 xy^2 dydx$ is

(a) 1/2

(b) 7/3

(c) 7/6

(d) 7/2

2. The value of $\int_0^1\int_x^{x^2} xdx\, dy$ is

(a) -1/12

(b) -1/4

(c) 0

(d) -1/15

3. The value of $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$ is

(a) 1

(b) 0

(c) 2

(d) 8

4. The value of $\Gamma(-10)$ is

(a) 10!

(b) 9!

(c) -9!

(d) Not defined

5. The value of $\Gamma(-5/2)$ is

(a) $-2\sqrt{\pi}$

(b) $\frac{4}{3}\sqrt{\pi}$

(c) $-\frac{8}{15}\sqrt{\pi}$

(d) Not defined

6. $\int_{0}^{\infty}\frac{x^9\left(1-x^8\right)dx}{(1+x)^{26}}$ simplifies to

(a) 0

(b) $\frac{(10!18!)}{28!}$

(c) $\frac{(9!17!)}{27!}$

(d) $\frac{(2,9!17!)}{27!}$

7. $\int_{0}^{\pi/2}\sqrt{\cot\theta}\,d\theta$ is equal to

(a) $\frac{\pi}{2}$

(b) 2π

(c) $\sqrt{2}\,\pi$

(d) $\pi/\sqrt{2}$

8. The value of B(1/2, 1/2) is

(a) $\sqrt{\pi}$

(b) π

(c) $2\sqrt{\pi}$

(d) 1

Questions & Answers

(1) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x-y} \frac{dxdydz}{(1+x+y+z)^3}$

Ans: Let this integral be I, then

$$I = \int_{x-0}^{1} \left(\int_{y-0}^{1-x} \left(\int_{z-0}^{1-x-y} \frac{dz}{(1+x+y+z)^3} \right) dy \right) dx$$

$$= \int_{x-0}^{1} \int_{y-0}^{1-x} \left[\frac{-1}{2(1+x+y+z)^3} \right]_{z-0}^{1-x-y} dydx$$

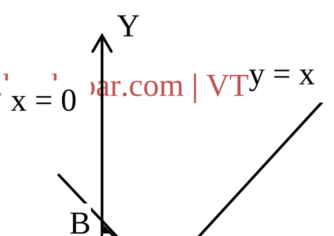
$$= \int_{x-0}^{1} \int_{y-0}^{1-x} \left[\frac{-1}{2(1+x+y+z)^2} \right] dx$$

$$= \frac{1}{2}\left(\ln 2 - \frac{5}{8} \right)$$

(2) Change the order of integration and hence evaluate the integral

$$\int_0^1 \int_x^{2-x} \frac{x}{y} dxdy$$

Ans: The region of integration is the shaded portion shown in the figure below



To get the limits for x first, we need to divide the shaded area into two parts AMB and AMO where the x values are 0 to 2-y and 0 to y respectively. The respective y values are 1 to 2 and 0 to 1.

The given integral is now

$$= \int_{y-0}^{1} \int_{x-0}^{y} \frac{x}{y} dxdy + \int_{y-1}^{2} \int_{x-0}^{2-y} \frac{x}{y} dxdy$$

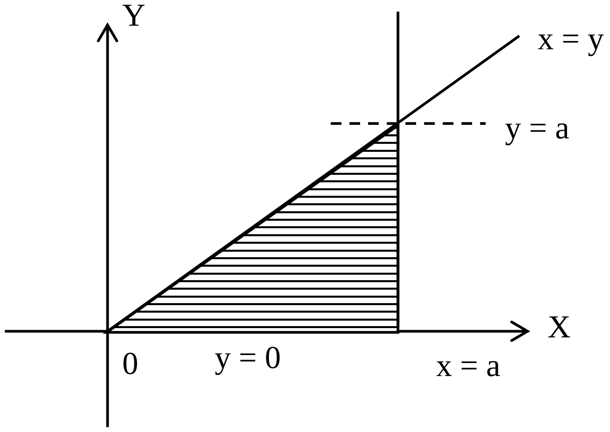
$$= \int_{y-0}^{1} \frac{y}{2} dy + \int_{y-1}^{2} \left(\frac{2}{y} + \frac{y}{2} - 2 \right) dy$$

= 2 ln 2 -1.

(3) Change to polar coordinates and hence evaluate

$$\int_0^a \int_y^a \frac{x}{\sqrt{x^2 + y^2}} dxdy$$

Ans: The region of integration is as shown below



We see that θ ranges from 0 to π/4 in the shaded region. R ranges from 0 to x = a i.e., a sec θ.

The given integral

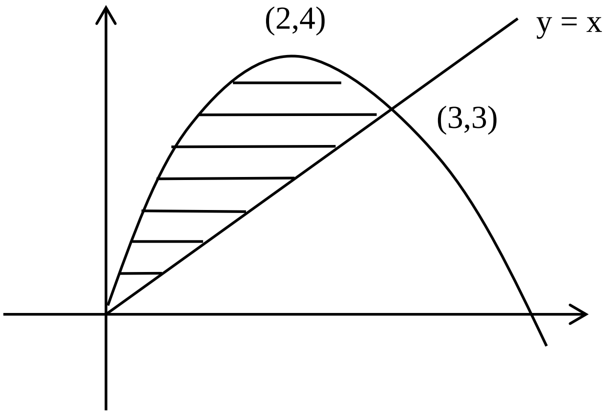
$$= \int_0^{\pi/4} \int_0^{a\sec\theta} r^2 \cos^2\theta \, dr \, d\theta$$

$$= \int_0^{\pi/4} \frac{a^3}{3} \sec\theta \, d\theta = \frac{a^3}{3} \left[\ln(\sec\theta + \tan\theta)\right]_0^{\pi/4}$$

$$= \frac{a^3}{3} \ln\left(\sqrt{2} + 1\right)$$

(4) Find the area between the parabola $y = 4x - x^2$ and the line $y = x$, using double integration.

Ans: The required area is $\iint_r dxdy$



$$= \int_{x-0}^{3} \int_{y-x}^{4x-x^2} dxdy = \int_{x-0}^{3} \left(3x - x^2\right) dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}$$

Practice Questions

1. Evaluate the following integrals

(i) $\int_0^3 \int_1^2 x(1 + x + y) dxdy$

(ii) $\int_0^{\pi/2} \int_0^a r^2 \sin\theta drd\theta$

(iii) $\int_0^1 \int_x^{\sqrt{x}} \left(x^2 + y^2\right) dydx$

(iv) $\int_1^a \int_1^b \frac{1}{xy} dydx$

(v) $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dxdydz$

(vi) $\int_{1}^{3}\int_{1/x}^{1}\int_{0}^{\sqrt{xy}} xyz\,dz\,dy\,dx$

(vii) $\int_{0}^{\pi/2}\int_{0}^{a\cos\theta}\int_{0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

2. Change the order of integration and hence evaluate

(i) $\int_{0}^{1}\int_{\sqrt{y}}^{2-y} xy\,dx\,dy$

(ii) $\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

(iii) $\int_{0}^{a}\int_{x^2/a}^{2a-x} xy\,dy\,dx$

(iv) $\int_{0}^{a}\int_{0}^{x} \frac{\cos y}{\sqrt{(a-x)(a-y)}}\,dy\,dx$

3. Change to respective coordinate systems and hence evaluate

(i) $\int_{0}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ to polar system

(ii) $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dy\,dx$ to polar system

(iii) $\int_{0}^{1}\int_{x}^{\sqrt{2x-x^2}} x^2+y^2\,dy\,dx$ to polar system

(iv) $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ to spherical polar

(v) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2-y^2}}^{1} \frac{dzdydx}{\sqrt{x^2+y^2+z^2}}$ to spherical polar

(vi) $\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=\sqrt{x^2+y^2}}^{2} xyzdxdydz$ to cylindrical polar

4. Find the area between y = 2 - x and circle $x^2 + y^2 = 4$
5. Find the area bounded between the parabola $y^2 = 4ax$ and $x^2 = 4ay$
6. Show that the area of one loop of the lemniscates $r^2 = a^2 \cos 2\theta$ is $a^2/2$.
7. Find the area of one petal of the rose $r = a \sin 3\theta$.
8. Find the area of the circle $r = a \sin\theta$ outside the cardioide $r = a(1 - \cos\theta)$
9. Find the volume of the paraboloid of revolution $x^2 + y^2 = 4z$ cut off by the plane z =4.
10. Find the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = R^2$
11. Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$.
12. Find the volume cut off the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$

13. Evaluate $\int_0^\infty x^2 e^{-x^4} dx$

14. *Find* $\int_0^1 \sqrt[3]{\log\left(1/x\right)}\, dx$

15. Evaluate $\int_0^2 \frac{3x^4}{16}\left(8-x^3\right)^{-1/3} dx$

16. *Find* $\int_0^{\pi/2} \tan^p\theta\, d\theta,\ 0<p<1$

17. Show that $2a^n \int_0^\infty t^{2n-1} e^{-at^2} dt = \Gamma(n)$

18. Show that $\int_0^\infty e^{-x^2 \log a} dx,\ a>0$ is $\frac{1}{2}\sqrt{\frac{\pi}{\log a}}$

19. If m and n are real constants > -1, prove that

$$\int_0^1 x^m \left\{\log\left(\frac{1}{x}\right)\right\}^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

20. Show that $\int_0^\infty x^{n-1} \cos ax \, dx = \frac{1}{a^n}\Gamma(n)\cos\frac{n\pi}{2}$

Hint: Write cos ax = Real part of e^{iax}

21. Show that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{1}{n}B\left(\frac{1}{n},\frac{1}{2}\right)$

22. Show that $B(m,n) = \frac{1}{2^{m+n-1}}\int_{-1}^{1}(1+x)^{m-1}(1-x)^{n-1}dx$

Hence deduce that $\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,dx = \pi$

23. Show that $\int_0^\infty xe^{-x^8}dx \int_0^\infty x^2e^{-x^4}dx = \frac{\pi}{16\sqrt{2}}$

24. Show that $\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = (b-a)^{m+n-1}B(m,n)$