Limit of a complex function – A function $\omega = f(z)$ is said to tend to limit l as z approaches a point z_0, if for every real ε, we can find a positive real δ s.t.

$$|f(z) - l| < \varepsilon \quad \text{for} \quad |z - z_0| < \delta$$

Continuity of $f(z)$ – A function $\omega = f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$

* Any function $f(z)$ is continuous in any region R of the z-plane, if it is continuous at every point of that region

Derivative of $f(z)$ – A complex function $f(z)$ is said to be differentiable at a point a if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \text{ exists}$$

$$\therefore \quad f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

or if $z - a = \delta z$

then
$$f'(a) = \lim_{\delta z \to 0} \frac{f(a + \delta z) - f(a)}{\delta z}$$

Theorem – If a complex function $f(z)$ is differentiable at a point a, then it is continuous at that point 'a' but not vice versa.

Proof: Let $f(z)$ be differentiable at a. Then $\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = f'(a)$

$$\therefore \lim_{z \to a} \{f(z) - f(a)\} = \lim_{z \to a} \left[\frac{f(z) - f(a)}{z - a} (z - a)\right]$$

$$= \lim_{z \to a} \frac{f(z) - f(a)}{(z - a)} \cdot \lim_{z \to a} (z - a) = f'(a) \cdot 0 = 0$$

$$\therefore \lim_{z \to a} f(z) = f(a) \qquad \therefore f(z) \text{ is cont. at } a.$$

Prove that the function $f(z) = |z|^2$ is continuous everywhere but not differentiable except at the origin.

Soln. $f(z) = |z|^2 = z\bar{z} = |x+iy|^2 = x^2+y^2$ $\therefore$ it is polynomial

There fore continuous everywhere

We have $f'(z)$

$$= \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{|z+\delta z|^2 - |z|^2}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(z+\delta z)(\bar{z}+\delta \bar{z}) - z\bar{z}}{\delta z} = \lim_{\delta z \to 0} \frac{z\,\delta\bar{z} + \delta z . \bar{z} + \delta z\,\delta\bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} \left\{ \bar{z} + \delta\bar{z} + z\frac{\delta\bar{z}}{\delta z} \right\} = \lim_{\delta z \to 0} \left(\bar{z} + z\frac{\delta \bar{z}}{\delta z} \right) \quad (\because \delta z \to 0 \Rightarrow \delta\bar{z} \to 0)$$

Let $\delta z = r(\cos\theta + i\sin\theta)$ & $\delta\bar{z} = r(\cos\theta - i\sin\theta)$

$$\frac{\delta\bar{z}}{\delta z} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} = (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)^{-1}$$

$$= (\cos\theta - i\sin\theta)(\cos\theta - i\sin\theta) = (\cos\theta - i\sin\theta)^2$$

$$\therefore \frac{\delta\bar{z}}{\delta z} = \cos 2\theta - i\sin 2\theta$$

Now for different values for θ, $\frac{\delta\bar{z}}{\delta z}$ will have different values.

$\therefore \lim_{\delta z \to 0} \frac{\delta\bar{z}}{\delta z}$ does not tend to a unique limit when

$z \neq 0$

$\therefore f'(z)$ is not unique.

Analytic function – A function $f(z)$, which is single valued and possesses a unique derivative w.r.t. z at all points of a region R, is called an analytic function of z in that region. It is also called a regular or an holomorphic function.

Entire function – A function which is analytic everywhere in the complex plane.

Singular points – A point at which an analytic function fails to have a derivative.

Theorem – The necessary and sufficient conditions for the function $w = f(z) = u(x,y) + i\,v(x,y)$ to be analytic in a region R, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are continuous functions of x & y in R.

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

Proof – Necessary condition – Let $w = f(z) = u(x,y) + i v(x,y)$ be analytic in a region R, then $\frac{dw}{dz} = f'(z)$ exists uniquely at every point of that region.

Let δx and δy be the increments in x & y respectively.

Let δu, δv and δz be the corresponding increments in u, v and z respectively. Then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i \frac{\delta v}{\delta z} \right)$$

Since the function $\omega = f(z)$ is analytic in the region R, the limit (1) must exist independent of the manner in which $\delta z \to 0$ i.e. along δx and $\delta y \to 0$

First let $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$

($\because$ $z = x+iy$, $z+\delta z = (x+\delta x) + i(y+\delta y)$ & $\delta z = \delta x + i\delta y$)

$\therefore$ from eq^n (1) $$f'(z) = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{—— (2)}$$

Now let $\delta z \to 0$ along a line $\parallel$ to y-axis so that $\delta x = 0$ and $\delta z = i\delta y$

$\therefore$ from (1) $$f'(z) = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\delta y} + \frac{i\delta v}{i\delta y} \right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{—— (3)}$$

By (2) & (3) $$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating real and imaginary parts

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}}$$

These are Cauchy Riemann's eq^n (C-R eq^n).

<u>Sufficient Condition</u>– Let $f(z) = u+iv$ be single valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of a region R and satisfying C-R eq^n.

i.e. $u_x = v_y$ & $u_y = -v_x$

Now we have to show that $f(z)$ is analytic i.e. $f'(z)$ exists at every point of the region R.

By Taylor's theorem

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

(omitting second & higher degree terms of δx & δy)

$$= \left[u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right)\right] + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)\right]$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Now $f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{-\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\delta y \qquad (\because i^2 = -1)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z$$

$$\Rightarrow \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \quad f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$\therefore$ $f'(z)$ exists because u_x & v_x exist.

<u>Conjugate function</u> – Let $f(z) = u(x,y) + iv(x,y)$ is an analytic funcⁿ then u & v are called conjugate function.

⇒ C-R equations in Polar form –

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof - A complex no. z can be written as $re^{i\theta}$

$\therefore \quad f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is unique.

Let δz be the increment in z corresponding to the increments δr, $\delta \theta$ in r & θ respectively.

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i \lim_{\delta z \to 0} \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \quad \text{— (1)}$$

$\because \quad z = re^{i\theta}$

$$\delta z = \frac{\partial z}{\partial r}\delta r + \frac{\partial z}{\partial \theta}\delta\theta = \frac{\partial(re^{i\theta})}{\partial r}\delta r + \frac{\partial(re^{i\theta})}{\partial \theta}\delta\theta$$

i.e. $\quad \delta z = e^{i\theta}\delta r + ire^{i\theta}\delta\theta$

Now since $\delta z \to 0$ i.e. now two cases will arise -

Case (i) - Let $\delta\theta = 0$ so that $\delta z = e^{i\theta}\delta r$ and $\delta z \to 0$ imply $\delta r \to 0$, put in eq^n (1)

$$f'(z) = \lim_{\delta r \to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta}\delta r} + i \lim_{\delta r \to 0} \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta}\delta r}$$

i.e. $\quad f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{— (2)}$

Case-(ii) - Let $\delta r = 0$ so that $\delta z = ire^{i\theta}\delta\theta$ and $\delta z \to 0$ imply $\delta\theta \to 0$, now eq^n (1)

$$f'(z) = \lim_{\delta\theta \to 0} \frac{u(r, \theta+\delta\theta) - u(r,\theta)}{ire^{i\theta}\delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r, \theta+\delta\theta) - v(r,\theta)}{ire^{i\theta}\delta\theta}$$

$$= \frac{1}{ire^{i\theta}}\left[\lim_{\delta\theta \to 0} \frac{u(r, \theta+\delta\theta) - u(r,\theta)}{\delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r, \theta+\delta\theta) - v(r,\theta)}{\delta\theta}\right]$$

$$f'(z) = \frac{1}{ire^{i\theta}}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right] = \frac{1}{re^{i\theta}}\left[\frac{1}{i}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right]$$

$$f'(z) = \frac{1}{re^{i\theta}}\left[-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right] = e^{-i\theta}\left[\frac{-i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right]$$

$$= e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right] \quad \text{--- (3)}$$

from eq^n (2) and (3)

$$e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right]$$

Equating the coefficients.

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \& \quad \boxed{\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}}$$

Que 1 - Prove that $\sinh z$ is analytic and find its derivative.

Soln - Here $f(z) = u + iv = \sinh(x+iy)$

$$= \sinh x \cos y + i \cosh x \sin y$$
$$= u + iv$$

Now for C-R eq^s.

$$\frac{\partial u}{\partial x} = \cosh x \cos y \quad ; \quad \frac{\partial u}{\partial y} = -\sinh x \sin y \quad \text{--- (i)}$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y \quad ; \quad \frac{\partial v}{\partial y} = \cosh x \cos y \quad \text{--- (ii)}$$

from eq^n (i) & (ii), we can check

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$\therefore$ C-R eq^s are satisfied.

$\because$ $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions

$\therefore$ u_x, v_x, u_y, v_y are also continuous

$\therefore$ $f(z)$ is analytic function everywhere

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cosh x \cos y + i \sinh x \sin y$

$$= \cosh(x+iy) = \cosh z$$

Que 2 Show that $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin and that its derivative is $\frac{1}{z}$.

Soln – Here $f(z) = u + iv = \log z = \log(x+iy)$

Let $x = r\cos\theta$ & $y = r\sin\theta$

$$\therefore \quad f(z) = \log(x+iy) = \log(re^{i\theta})$$

$$= \log r + \log e^{i\theta} = \log r + i\theta$$

$$= \log\left(\sqrt{x^2+y^2}\right) + i\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$= u + iv$$

Now for C-R eq^{n}s

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \; ; \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \; ; \quad \frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad = \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ C-R eq^{n}s are satisfied, except when $x^2 + y^2 = 0$
i.e. $x = 0,\ y = 0$

$\therefore$ $f(z)$ is analytic everywhere except at the origin.

Also
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x - iy}{x^2+y^2} = \frac{x^2+y^2}{(x^2+y^2)(x+iy)} =$$

$$= \frac{1}{x+iy} = \frac{1}{z}.$$

Que 3 Prove that z^n is analytic. Hence find its derivative.

Soln –
$$f(z) = z^n = (x+iy)^n = (re^{i\theta})^n = r^n . e^{in\theta}$$

$$= r^n(\cos n\theta + i\sin n\theta) = u + iv$$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta \; , \quad \frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$$

$$\frac{\partial u}{\partial \theta} = -nr^n \sin n\theta, \qquad \frac{\partial v}{\partial \theta} = nr^n \cos n\theta$$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$\therefore$ C-R eq^{ns} are satisfied.

Hence $f(z)$ is analytic if $f'(z)$ or $\frac{dw}{dz}$ exists for all finite values of z

$$\therefore \quad \frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$$

$$= (\cos\theta - i\sin\theta) \cdot nr^{n-1}(\cos n\theta + i\sin n\theta)$$

$$= nr^{n-1}\left[\cos(n-1)\theta + i\sin(n-1)\theta\right]$$

This exists for all finite values of r including 0, except when $r=0$ and $n \leq 1$

Imp* Note or $f'(z) = e^{-i\theta}(u_r + iv_r)$ or $e^{-i\theta}\left(\frac{1}{r}v_\theta - \frac{i}{r}u_\theta\right)$

for polar form

$$\therefore \; f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(nr^{n-1}\cos n\theta + inr^{n-1}\sin n\theta)$$

$$= nr^{n-1}e^{-i\theta}(\cos n\theta + i\sin n\theta)$$

$$= nr^{n-1} \cdot e^{-i\theta} \cdot e^{in\theta} = nr^{n-1}e^{i(n-1)\theta}$$

$$f'(z) = n(re^{i\theta})^{n-1} = nz^{n-1}$$

Que. 4 Show that an analytic function with constant modulus is constant.

Soln. Let $f(z) = u + iv$

$$\therefore \; |f(z)| = |u+iv| = \text{constant}$$

$$\Rightarrow \quad \sqrt{u^2+v^2} = \text{constant} = c \text{ (say)}$$

$$\Rightarrow \quad u^2+v^2 = c^2 \quad \text{—— (i)}$$

Diff. partially w.r.t. x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{—— (ii)}$$

Again diff partially w.r.t y

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad \text{—— (iii)}$$

Using C-R eqⁿ in (iii)

$$u\left(-\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x}\right) = 0$$

$$\Rightarrow \quad v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} = 0 \quad \text{—— (iv)}$$

Squaring and adding (ii) & (iv)

$$u^2+v^2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} = 0$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad \left\{\because\ u^2+v^2 \neq 0 \text{ see eq}^n \text{ (i)}\right\}$$

$$|f'(z)|^2 = 0 \qquad \because f'(z) = u_x + iv_x$$

$$|f'(z)| = 0$$

$\Rightarrow$ $f(z)$ is constant

Que. Prove that the function $f(z) = z|z|$ is not analytic anywhere.

Que. Show that $f(z) = \begin{cases} e^{-z^{-4}}, & z \neq 0 \\ 0, & 0 \end{cases}$ is not analytic at $z=0$, although C-R eqⁿˢ are satisfied at that point.

Soln.

$$f(z) = e^{-z^{-4}} = e^{-(x+iy)^{-4}}$$

$$= e^{-\frac{1}{(x+iy)^4} \times \frac{(x-iy)^4}{(x-iy)^4}} = e^{-\left\{\frac{(x-iy)^4}{(x^2+y^2)^4}\right\}}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left[x^4+y^4-6x^2y^2-4ixy(x^2-y^2)\right]}$$

$$f(z) = u+iv = e^{-(x^4+y^4-6x^2y^2)/(x^2+y^2)^4}\left[\cos\left\{\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right\} + i\sin\left\{\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right\}\right]$$

At $z=0$ $$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{e^{-x^{-4}} - 0}{x} = \lim_{x\to 0} \frac{1}{x} e^{-x^{-4}}$$

$$= \lim_{x\to 0} \frac{1}{x e^{x^{4}}} = \lim_{x\to 0} \frac{1}{x\left\{1+\frac{1}{x^4}+\frac{1}{2x^8}+\cdots\right\}} = \lim_{x\to 0} \frac{1}{\left\{x+\frac{1}{x^3}+\frac{1}{2x^7}+\cdots\right\}}$$

$$= 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{-e^{-y^{-4}}}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \left(\frac{0}{y}\right) = 0$$

$\therefore$ C-R eq^{ns} are satisfied at $z=0$

But $$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{e^{-z^{-4}}}{z}$$

$$= \lim_{r\to 0} \frac{e^{-(re^{i\pi/4})^{-4}}}{re^{i\pi/4}} \qquad (\text{if } z\to 0 \text{ along } z=re^{i\pi/4})$$

$$= \infty$$

Que. Examine the nature of the function

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}, \quad z\neq 0$$

$$f(0) = 0$$

in the region including the origin

Soln: $u + iv = \dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}}, \quad z \neq 0$

At the origin $\dfrac{\partial u}{\partial x} = \lim_{x \to 0} \dfrac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \dfrac{0-0}{x} = 0$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0$$

Sly, $\dfrac{\partial v}{\partial x} = 0 = \dfrac{\partial v}{\partial y}$

$\therefore$ C-R equations are satisfied at the origin

But $f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}} - 0 \right) \cdot \dfrac{1}{x+iy}$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{m^5 x^7}{x^4 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^3}{1 + m^{10} x^6} = 0$$

Again let $z \to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

$\therefore f'(0)$ does not exist $\therefore f(z)$ is not analytic at origin.

Que: If $f(z) = \dfrac{x^3 y (y - ix)}{x^6 + y^2}, \quad z \neq 0$

$\qquad\qquad 0 \quad , \quad z = 0$

Prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner & also find $f(z)$ is not analytic at $z = 0$.

Harmonic function

A function of x, y which has continuous partial derivatives of the first and second orders and satisfies Laplace's eqn. is called Harmonic function

In cartesian form - If $w = f(z) = f(x, y)$ ~~$z(x, y)$~~ is the function

Then $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Then $f(x, y)$ is called harmonic function

In polar form - $$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$$

Harmonic property -

If $f(z) = u + iv$ is an analytic function then u and v are both harmonic functions.

OR

The real and imaginary parts of an analytic function are harmonic.

Proof - Let $f(z) = u + iv$ be an analytic in some region of z- plane, then u & v satisfy C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—(i)} \quad \& \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \quad \text{—(ii)}$$

Now we have to for both u & v

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Diff eqn (i) & (ii) w.r.t. x & y respectively

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{—(3)} \quad \& \quad \frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial y \partial x} \quad (4)$$

Adding (3) & (4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Sly Diff (i) & (ii) w.r.t. y & x respectively

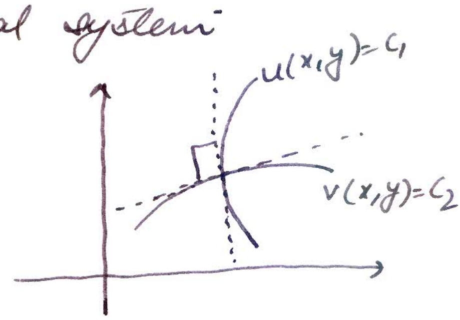
$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \text{—(5)} \qquad \& \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad (6)$$

Eq^n (5) − (6) $\Rightarrow$ $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

$\because$ u and v both satisfy Laplace's eq^n $\therefore$ u & v are harmonic functions.

* Here u and v are called conjugate harmonic functions.

<u>Orthogonal System</u> - Every analytic function $f(z) = u + iv$ defines two families of curves $u(x,y) = c_1$ & $v(x,y) = c_2$ which form an orthogonal system



Consider the two families of curves

$$u(x,y) = c_1 \quad \text{—(i)} \qquad \& \qquad v(x,y) = c_2 \quad \text{—(ii)}$$

Diff eq^n (i) w.r.t. x

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = m_1$$

Sly $$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = m_2$$

$$\therefore \quad m_1 m_2 = \frac{-\partial u/\partial x}{\partial u/\partial y} \times \frac{-\partial v/\partial x}{\partial v/\partial y} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} \quad \{\because \text{C-R eq}^{ns}$$

$$m_1 m_2 = -1$$

$\therefore$ u & v form an orthogonal system

Determination of conjugate function -

If $f(z) = u+iv$, where u & v are conjugate function then we find v, if u is given in the following way

$$v = v(x,y)$$

$$\therefore \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\Rightarrow \quad dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \text{(by C-R) \& (}u\text{ is given)}$$

$$\Rightarrow \quad M = -\frac{\partial u}{\partial y} \quad \& \quad N = \frac{\partial u}{\partial x}$$

$$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\because \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \quad -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

, (which is true as u being an harmonic function)

$\therefore$ dv is exact

Now we can integrate it, to get v.

and then can find $f(z) = u+iv$

*Sly If v is given then for u

$$du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

Milne's Thomson Method--

Que. Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic conjugate.

Soln.
$$u = \frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\times 2x}{(x^2+y^2)} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Also $$\frac{\partial u}{\partial y} = \frac{1}{2(x^2+y^2)} \times 2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Now $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is a harmonic function.

Let $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$

$$= \left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy$$

$$dv = \frac{x\,dy - y\,dx}{x^2+y^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

After integrating

$$v = \tan^{-1}\left(\frac{y}{x}\right) + C$$

<u>Milne Thomson Method</u> – we directly construct $f(z)$ in terms of z without first finding out v when u is given or u when v is given.

$$z = x+iy \Rightarrow \bar{z} = x-iy$$

$$\Rightarrow x = \frac{1}{2}(z+\bar{z}) \;\&\; y = \frac{1}{2i}(z-\bar{z})$$

$$\therefore \quad f(z) = u(x,y) + iv(x,y)$$

$$= u\left\{\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right\} + iv\left\{\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right\} \quad \text{—(1)}$$

Eq^n(1) is an identity in z & $\bar{z}$. Put $\bar{z} = z$ we get

$$f(z) = u(z,0) + iv(z,0) \quad \text{—— (2)}$$

Now $f(z) = u + iv$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(C-R equations)}$$

$$= \phi_1(x,y) - i\phi_2(x,y)$$

Now $f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

Integrating above equation

$$f(z) = \int\{\phi_1(z,0) - i\phi_2(z,0)\}dz + C$$

Sly if $v(x,y)$ is given, then

$$f(z) = \int[\psi_1(z,0) + i\psi_2(z,0)]dz + C$$

~~Metho~~

<u>Case 1</u> – when only real part $u(x,y)$ is given

(a) Find $\frac{\partial u}{\partial x}$

(b) $\frac{\partial u}{\partial x} = \phi_1(x,y)$

(c) Find $\frac{\partial u}{\partial y}$, and assume it $\phi_2(x,y)$

(d) Find $\phi_1(z,0)$ & $\phi_2(z,0)$ by replacing x by z & y by 0.

(e) Find $f(z)$ by

$$f(z) = \int\{\phi_1(z,0) - i\phi_2(z,0)\}dz + C$$

Case-II When only imaginary part $v(x,y)$ is given

(a) find $\frac{\partial v}{\partial y}$, and put it equal to $\varphi_1(x,y)$

(b) $\frac{\partial v}{\partial x}$ & take $\frac{\partial v}{\partial x} = \varphi_2(x,y)$

(c) find $\varphi_1(z,0)$ & $\varphi_2(z,0)$ by replacing x by z & y by 0.

(d) Now $f(z) = \int[\varphi_1(z,0) + i\varphi_2(z,0)]\,dz + C$

Case III When $u-v$ is given

(a) $f(z) = u + iv$

(b) $if(z) = iu - v$

(c) Add (a) & (b) $(1+i)f(z) = (u-v) + i(u+v)$

or $F(z) = U + iV$

Here $U = u - v$ & $V = u + v$

Now U is given, so same procedure as (Case I)

$$\therefore \quad F(z) = \int\{\phi_1(z,0) - i\phi_2(z,0)\}\,dz + C$$

$$(1+i)f(z) = \quad ''$$

$$f(z) = \frac{1}{(1+i)}\{ \quad '' \quad \}$$

Case-IV When $u+v$ is given

(a) $f(z) = u + iv$

(b) $if(z) = iu - v$

& $F(z) = U + iV$

Now V is given

$$\therefore \quad F(z) = \int[\varphi_1(z,0) + i\varphi_2(z,0)]\,dz + C$$

and in last $f(z) = \frac{F(z)}{(1+i)}$

Que. 1 Find the analytic function $w = u + iv$ if $v = \log(x^2+y^2) + x - 2y$

Soln: $v = \log(x^2+y^2) + x - 2y$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2 = \varphi_1(x,y)$$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1 = \varphi_2(x,y)$$

Now $\varphi_1(z,0) = -2$, $\varphi_2(z,0) = \frac{2z}{z^2} + 1 = \frac{2}{z} + 1$

Now Milne's Thomson Method

$$f(z) = \int [\varphi_1(z,0) + i\varphi_2(z,0)]\,dz + C$$

$$= \int \left\{-2 + i\left(\frac{2}{z} + 1\right)\right\} dz + C$$

$$= -2z + i(2\log z + z) + C$$

$$w = (i-2)z + 2i\log z + C$$

Que. 2 $u = \log\sqrt{x^2+y^2}$, $f(1) = 2i$

Soln $\frac{\partial u}{\partial x} = \frac{1 \times 2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \times \frac{2y}{(x^2+y^2)} = \frac{y}{x^2+y^2}$$

Now $\phi_1(z,0) = \frac{z}{z^2+0} = \frac{1}{z}$, $\phi_2(z,0) = 0$

By Milne's Thomson formula

$$f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\}\,dz + C$$

$$f(z) = \int \frac{1}{z}\,dz + C = \log z + C \quad -(i)$$

Now given $f(1) = 2i$, put in (i)

$$2i = \log 1 + C$$

$$C = 2i$$

$$\therefore f(z) = \log z + 2i$$

Que. If $u = e^x(x\cos y - y\sin y)$ is a harmonic function, find $f(z)$ s.t. $f(1) = e$

Soln. Hint:
$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = \phi_2(x,y)$$

$$\phi_1(z,0) = e^z(1+z) \quad \& \quad \phi_2(z,0) = 0$$

$$f(z) = ze^z + c \qquad \& \quad c = 0 \Rightarrow f(z) = ze^z$$

Que. Find the analytic func. $f(z)$ whose imaginary part is $\left(r - \frac{k^2}{r}\right)\sin\theta$, $r \neq 0$. Hence find the real part of $f(z)$ and prove that it is harmonic.

Soln
$$v = \left(r - \frac{k^2}{r}\right)\sin\theta$$

$$v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta \quad ; \quad v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$$

Now we have $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$f'(z) = e^{-i\theta}\left(\frac{1}{r}v_\theta + iv_r\right) \qquad (\because \text{C-R eq}^{ns})$$

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}\left(r - \frac{k^2}{r}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right]$$

$$= e^{-i\theta}\left[\left(1 - \frac{k^2}{r^2}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right]$$

$$= e^{-i\theta}\left[(\cos\theta + i\sin\theta) - \frac{k^2}{r^2}(\cos\theta - i\sin\theta)\right]$$

$$= e^{-i\theta}\left[e^{i\theta} - \frac{k^2}{r^2}e^{-i\theta}\right] = 1 - \frac{k^2}{r^2e^{2i\theta}} = 1 - \frac{k^2}{z^2}$$

$$\therefore f'(z) = 1 - \frac{k^2}{z^2}$$

Integrating $f(z) = z + \frac{k^2}{z} + c$

Now we have to find $u(r, \theta)$ from $f(z)$ in polar form $(z = re^{i\theta})$

$$u + iv = re^{i\theta} + \frac{k^2}{re^{i\theta}} + c$$

$$= r(\cos\theta + i\sin\theta) + \frac{k^2}{r}(\cos\theta - i\sin\theta) + c$$

$$u + iv = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta \qquad (\& \; c = 0)$$

$$\therefore \quad u = \left(r + \frac{k^2}{r}\right)\cos\theta$$

Now we have to prove u is harmonic function

for this we need to prove

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{—— (i)}$$

$$u_r = \left(1 - \frac{k^2}{r^2}\right)\cos\theta \quad \& \quad u_\theta = -\left(r + \frac{k^2}{r}\right)\sin\theta$$

$$u_{rr} = \frac{+2k^2}{r^3}\cos\theta \quad \& \quad u_{\theta\theta} = -\left(r + \frac{k^2}{r}\right)\cos\theta$$

Put in (i)

$$\frac{2k^2}{r^3}\cos\theta + \frac{1}{r}\left(1 - \frac{k^2}{r^2}\right)\cos\theta + \frac{1}{r^2}\left\{-\left(r + \frac{k^2}{r}\right)\right\}\cos\theta$$

$$= 0$$

$\therefore$ u is harmonic.

Que. If $f(z) = u + iv$ is analytic and φ is any differentiable func. of x & y, prove that

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \left\{\left(\frac{\partial\varphi}{\partial u}\right)^2 + \left(\frac{\partial\varphi}{\partial v}\right)^2\right\}|f'(z)|^2$$

Soln:- Here $f(z) = u + iv$, and φ is a fun of x & y then by the chain rule of partial diff.

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial x} \quad —(1) \qquad \& \qquad \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial y} \quad —(2)$$

$(1)^2 + (2)^2$

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 = \left(\frac{\partial \varphi}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2\left(\frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x}\times\frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial x}\right)$$

$$+ \left(\frac{\partial \varphi}{\partial u}\right)^2\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial xy} * \frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$

$$= \left(\frac{\partial \varphi}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2\left(\frac{\partial u}{\partial y}\right)^2 + \cancel{2\left(-\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial \varphi}{\partial v}\cdot\frac{\partial u}{\partial y}\right)}$$

$$+ \left(\frac{\partial \varphi}{\partial u}\right)^2\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2\left(\frac{\partial u}{\partial x}\right)^2 + \cancel{2\left(\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial \varphi}{\partial v}\cdot\frac{\partial u}{\partial x}\right)}$$

$$= \left(\frac{\partial \varphi}{\partial u}\right)^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \left(\frac{\partial \varphi}{\partial v}\right)^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]\left[\left(\frac{\partial \varphi}{\partial u}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2\right]$$

$$= \left[\left(\frac{\partial \varphi}{\partial u}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2\right]|f'(z)|^2$$

Que: 2 Prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re f(z)|^2 = 2|f'(z)|^2$$

Soln. We have $f(z) = u + iv$

$$u = Re f(z) \Rightarrow |Re(f(z))|^2 = |u|^2 = u^2$$

Now $\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(u^2) = 2\left\{u\frac{\partial^2 u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2\right\}$$

Sly $\frac{\partial^2}{\partial y^2}(u^2) = 2\left\{u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right\}$

LHS: $\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re\, f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2$

$$= 2\left\{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\}$$

$$= 2\left\{u \cdot 0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} = 2|f'(z)|^2$$

Que: P.T. - $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$

Application of analytic functions to flow problems -

Since the real and imaginary parts of an analytic func. satisfy the Laplace's equation in two variables, these conjugate functions give solutions to a no. of field and flow problems.

Now consider the two dimensional irrotational motion of an incompressible fluid, in planes parallel to xy-plane. Let $\vec{V}$ be the velocity of a fluid particle, then it can be expressed as

$$\vec{V} = v_x \hat{i} + v_y \hat{j} \quad \text{——(i)}$$

Since the motion is irrotational, then $\exists$ a scalar func^n $\phi(x,y)$ s.t.

$$\vec{V} = \nabla\phi(x,y) = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} \quad \text{——(ii)}$$

From (i) & (ii) $\quad v_x = \frac{\partial\phi}{\partial x}$ & $v_y = \frac{\partial\phi}{\partial y}$

The scalar function $\phi(x,y)$ which gives the velocity components v_x and v_y is called velocity potential

$\because$ The fluid is incompressible we have

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V} = 0$$

i.e.

$$\left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j\right)\cdot\left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j\right) = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$\Rightarrow$ ϕ is a harmonic function.

$\phi(x,y)$ is the real part of an analytic func^n.

$$w = f(z) = \phi(x,y) + i\psi(x,y)$$

$\psi(x,y)$ is the conjugate harmonic func.

Que. If $\omega = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, find the function ϕ.

Soln. Here $\omega = \phi + i\psi$

For C-R eq^n. we need to satisfy

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{—(i)} \qquad \& \qquad \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x} \quad \text{—(ii)}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) = -2y - \frac{2xy}{(x^2+y^2)^2} \quad \text{—(1)}$$

Integrating (1) w.r.t. x we get $\phi = -2xy + \frac{y}{x^2+y^2} + f(y)$ — (2)

(for this take $x^2+y^2 = t$

$2x\,dx = dt \Rightarrow y\int \frac{dt}{t^2}$

$= -y \times \frac{1}{t}$)

Now from eq^n (ii)

$$\frac{\partial \phi}{\partial y} = \frac{-\partial}{\partial x}\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) = -2x + \frac{x^2-y^2}{(x^2+y^2)^2} \cancel{+ f(y)} \quad \text{—(A)}$$

~~Now integrating it w.r.t y~~

~~$\phi = -2xy$~~

Diff eq^n (2) partially w.r.t y

$$\frac{\partial \phi}{\partial y} = -2x + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = -2x + \frac{x^2-y^2}{(x^2+y^2)^2} + f'(y) \quad \text{—(B)}$$

Comparing (A) and (B)

$$f'(y) = 0 \Rightarrow f(y) = \text{constant}$$

$$\therefore \phi = -2xy + \frac{y}{x^2+y^2} + c$$

Que. If $f(z) = u(r,\theta) + iv(r,\theta)$ be an analytic func^n. If $u = -r^3\sin 3\theta$ then construct the corresponding analytic func^n $f(z)$ in terms of z.

Soln. $u = -r^3 \sin 3\theta$

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta \quad ; \quad \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

we have

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

$$= \left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right) dr + \left(r\frac{\partial u}{\partial r}\right) d\theta \quad \text{(by C-R eq}^n\text{s)}$$

$$= (\cancel{-}3r^2\cos 3\theta)\, dr - 3r^3 \sin 3\theta \, d\theta$$

$$dv = d(r^3 \cos 3\theta)$$

After integrating

$$v = r^3 \cos 3\theta + c$$

$$\therefore f(z) = u + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta + ic$$

$$= i^2 r^3 \sin 3\theta + ir^3 \cos 3\theta + ic$$

$$= ir^3(\cos 3\theta + i\sin 3\theta) + c$$

$$\therefore f(z) = ir^3 e^{3i\theta} + c = i(re^{i\theta})^3 + c = iz^3 + c$$

Que. In a two dimension fluid flow if the velocity potential $\phi = e^{-x}\cos y + xy$, find the stream function

Soln. $\phi = e^{-x}\cos y + xy$

$$\frac{\partial \phi}{\partial x} = -e^{-x}\cos y + y \; ; \; \frac{\partial \phi}{\partial y} = -e^{-x}\sin y + x$$

By C-R eq^n. $\phi_x = \psi_y$ & $\phi_y = -\psi_x$

$$\therefore \frac{\partial \psi}{\partial y} = -e^{-x}\cos y + y \quad \& \quad \frac{\partial \psi}{\partial x} = e^{-x}\sin y - x$$

Integrating

$$\psi = \int(-e^{-x}\cos y + y)\,dy + f(x) \quad \&$$

$$\psi = \int(e^{-x}\sin y - x)\,dx + g(y)$$

$\therefore \quad \varphi = -e^{-x}\sin y + \frac{y^2}{2} + f(x)$

$\& \quad \varphi = -e^{-x}\sin y - \frac{x^2}{2} + g(y)$

After comparing we choose

$$f(x) = \frac{-x^2}{2} \quad \& \quad g(y) = \frac{y^2}{2}$$

$\therefore \quad \varphi = -e^{-x}\sin y - \frac{x^2}{2} + \frac{y^2}{2}$

Que: If $u - v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u + iv$ is an analytic fun of $z = x+iy$, find $f(z)$ in terms of z

Soln. $\quad f(z) = u + iv \quad \Rightarrow \quad if(z) = iu - v$

Adding $\quad (1+i)f(z) = (u-v) + i(u+v)$

$$F(z) = U + iV$$

Now $\quad U = u - v = (x-y)(x^2+4xy+y^2)$

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2 = \phi_1(x,y)$$

$$\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2 = \phi_2(x,y)$$

Now put $x = z$ $\&$ $y = 0$

$$F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]dz + C$$

$$= \int \{3z^2 - i(3z^2)\}dz + C$$

$$F(z) = (1-i)z^3 + C$$

$$(1+i)F(z) = (1-i)z^3 + C$$

$$f(z) = \left(\frac{1-i}{1+i}\right)z^3 + \left(\frac{C}{1+i}\right)$$

$$f(z) = \frac{-2i}{\cancel{1+i}\,2}z^3 + C_1 = -iz^3 + C_1$$

Que. If $u+v = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ & $f(z)=u+iv$, find $f(z)$

in terms of z.

Soln Here $V = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$

$$\frac{\partial v}{\partial x} = \frac{2\cos 2x\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = \psi_2(x,y)$$

$$\frac{\partial v}{\partial y} = \frac{-2\sin 2x\sinh 2y}{(\cosh 2y - \cos 2x)^2} = \psi_1(x,y)$$

Now $x = z$ & $y = 0 \Rightarrow \psi_1(z,0) = 0$, $\psi_2(z,0) = -\operatorname{cosec}^2 z$

$$f(z) = \int(\psi_1(z,0)+i\psi_2(z,0))\,dz + C$$

$$= \int -i\operatorname{cosec}^2 z\,dz + C = i\cot z + C$$

Ans. $f(z) = \frac{1}{2}(1+i)\cot z + C_1$ where $C_1 = \dfrac{C}{1+i}$