Problems on Type II

① Solve $y'' - 4y' + 13y = \cos 2x$

Sol:- $(D^2 - 4D + 13)y = \cos 2x$

A.E is

$m^2 - 4m + 13 = 0$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2}$$

$$m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$m = \frac{4 \pm \sqrt{-36}}{2}$$

$$m = \frac{4 \pm 6i}{2}$$

$$= \frac{\cancel{2}(2 \pm 3i)}{\cancel{2}}$$

$m = 2 \pm 3i$

$C.F = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

$$P.I = \frac{1}{D^2 - 4D + 13} \cos 2x$$

put $D^2 = -2^2$

$$= \frac{1}{(-2^2) - 4D + 13} \cos 2x$$

$$= \frac{1}{-4 - 4D + 13} \cos 2x$$

$$P.I = \frac{1}{9-4D} \cos 2x$$

$$= \frac{9+4D}{(9-4D)(9+4D)} \cos 2x$$

$$= \frac{(9+4D) \cos 2x}{9^2 - 16D^2}$$

Put $D^2 = -2^2$

$D^2 = -4$

$$= \frac{(9+4D) \cos 2x}{9^2 - 16(-4)}$$

$$= \frac{9 \cos 2x + 4D(\cos 2x)}{9^2 + 64}$$

$$= \frac{9 \cos 2x + 4 \cdot (-\sin 2x \cdot 2)}{81 + 64}$$

$$= \frac{9 \cos 2x - 8 \sin 2x}{145}$$

General solution is

$$y = C.F + P.I$$

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{9 \cos 2x - 8 \sin 2x}{145}$$

(3)

(2) Solve $(D^2-4D+3)y = \sin 3x \cdot \cos 2x$ *

Sol :- A.E is $m^2-4m+3=0$

$$m^2-3m-m+3=0$$

$$m(m-3)-1(m-3)=0$$

$$(m-1)(m-3)=0$$

$$m=1,\ m=3$$

$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

$$C.F = c_1e^{x}+c_2e^{3x}$$

$$P.I = \frac{1}{D^2-4D+3}\sin 3x\cos 2x$$

$$= \frac{1}{D^2-4D+3}\cdot\frac{1}{2}\left(\sin(3x+2x)+\sin(3x-2x)\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-4D+3}(\sin 5x+\sin x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x+\frac{1}{D^2-4D+3}\sin x\right]$$

Put $D^2=-5^2$ Put $D^2=-1^2$

$$= \frac{1}{2}\left[\frac{1}{-5^2-4D+3}\sin 5x+\frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-22-4D}\sin 5x+\frac{1}{2-4D}\sin x\right]$$

④

$$P.I = \frac{1}{2}\left[\frac{1}{-22-4D}\sin 5x + \frac{1}{2-4D}\sin x\right]$$

$D^2 = -a^2$

$= -(\ )^2$

$= (-1)^2$

$= -1^2$

$\cancel{D^2 = (-a)^2}$

$$= \frac{1}{2}\left[\frac{(-22+4D)\sin 5x}{(-22-4D)(-22+4D)} + \frac{(2+4D)\sin x}{(2-4D)(2+4D)}\right]$$

$$= \frac{1}{2}\left[\frac{-22\sin 5x + 4D(\sin 5x)}{(-22)^2 - (4D)^2} + \frac{2\sin x + 4D(\sin x)}{2^2 - (4D)^2}\right]$$

Put $D^2 = -5^2$

Put $D^2 = -1^2$

$$= \frac{1}{2}\left[\frac{-22\sin 5x + 4D(\sin 5x)}{484 - 16D^2} + \frac{2\sin x + 4D(\sin x)}{4 - 16D^2}\right]$$

$$= \frac{1}{2}\left[\frac{-22\sin 5x + 4(\cos 5x \cdot 5)}{484 - 16(-25)} + \frac{2\sin x + 4\cdot\cos x}{4 - 16(-1)}\right]$$

5

$$P.I = \frac{1}{2}\left[\frac{-22\sin 5x + 20\cos 5x}{484+400} + \frac{2\sin x + 4\cos x}{20}\right]$$

$$= \frac{1}{2}\left[\frac{-22\sin 5x + 20\cos 5x}{884} + \frac{2\sin x + 4\cos x}{20}\right]$$

$$P.I = \frac{-22\sin 5x + 20\cos 5x}{1768} + \frac{2\sin x + 4\cos x}{40}$$

General Sol. is

$$y = c.f + p.I$$

$$y = c_1 e^{x} + c_2 e^{3x} + \left(\frac{-22\sin 5x + 20\cos 5x}{1768} + \frac{2\sin x + 4\cos x}{40}\right)$$

③ Solve $(D^2 - 3D + 5)y = \frac{\sin^2 x}{4}$

Sol:- A.E is $m^2 - 3m + 5 = 0$

$$m = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 * 1 * 5}}{2}$$

$$m = \frac{3 \pm \sqrt{9-20}}{2}$$

$$m = \frac{3 \pm \sqrt{-11}}{2} \Rightarrow m = \frac{3 \pm \sqrt{11}\, i}{2}$$

$$C.F = e^{\frac{3x}{2}}\left(c_1 \cos \frac{\sqrt{11}}{2}x + c_2 \sin \frac{\sqrt{11}}{2}x\right)$$ (6)

$$P.I = \frac{1}{D^2 - 3D + 5} \frac{\sin^2 x}{4}$$

$$= \frac{1}{D^2 - 3D + 5} \cdot \frac{1}{4} \cdot \left(\frac{1 - \cos 2x}{2}\right)$$

$$= \frac{1}{8} \cdot \frac{1}{D^2 - 3D + 5}\left(\underline{1} - \underline{\cos 2x}\right)$$

$$= \frac{1}{8} \cdot \frac{1}{D^2 - 3D + 5}\left(\underline{e^{0x}} - \underline{\cos 2x}\right)$$

$$= \frac{1}{8}\left[\frac{1}{D^2 - 3D + 5} e^{0x} - \frac{1}{D^2 - 3D + 5} \cos 2x\right]$$

put $D = 0$, put $D^2 = -2^2$

$$= \frac{1}{8}\left[\frac{1}{0 - 0 + 5} e^{0x} - \frac{1}{-4 - 3D + 5} \cos 2x\right]$$

$$= \frac{1}{8}\left[\frac{1}{5}(1) - \frac{1}{\underline{1 - 3D}} \cos 2x\right]$$

$$= \frac{1}{8}\left[\frac{1}{5} - \frac{(1 + 3D)\cos 2x}{(1 - 3D)(1 + 3D)}\right]$$

$$= \frac{1}{8}\left[\frac{1}{5} - \frac{\cos 2x + 3D(\cos 2x)}{1^2 - (3D)^2}\right]$$

$$P.I = \frac{1}{8}\left[\frac{1}{5} - \frac{(\cos 2x + 3(-\sin 2x)\cdot 2)}{1-9D^2}\right]$$ (7)

Put $D^2 = -2^2$

$$= \frac{1}{8}\left[\frac{1}{5} - \frac{\cos 2x - 6\sin 2x}{1-9(-4)}\right]$$

$$= \frac{1}{8}\left[\frac{1}{5} - \frac{\cos 2x - 6\sin 2x}{37}\right]$$

$$P.I = \frac{1}{40} - \frac{(\cos 2x - 6\sin 2x)}{296}$$

General sol is

$$y = C.F + P.I$$

$$y = e^{3x/2}\left(c_1 \cos\frac{\sqrt{11}}{2}x + c_2 \sin\frac{\sqrt{11}}{2}x\right) + \frac{1}{40} - \frac{(\cos 2x - 6\sin 2x)}{296}$$

(4) Solve $(D^2+16)y = e^{-3x} + \cos 4x$

Sol:- A.E is $m^2+16=0$

$$m^2 = -16 \Rightarrow m = \pm 4i$$

$$C.F = c_1\cos 4x + c_2 \sin 4x$$

$$P.I = \frac{1}{D^2+16}\left(e^{-3x} + \cos 4x\right)$$

$$= \frac{1}{D^2+16} e^{-3x} + \frac{1}{D^2+16} \cos 4x$$

put $D = -3$ put $D^2 = -4^2$

$$= \frac{1}{(-3)^2+16} e^{-3x} + \frac{1}{-16+16} \cos 4x$$

$$= \frac{1}{25} e^{-3x} + x \cdot \frac{1}{2D} \cos 4x$$

$$= \frac{1}{25} e^{-3x} + \frac{x}{2} \underbrace{\frac{1}{D} \cos 4x}$$

$\frac{1}{D} = \int$

$$= \frac{1}{25} e^{-3x} + \frac{x}{2} \int \cos 4x \, dx$$

$$= \frac{1}{25} e^{-3x} + \frac{x}{2} \cdot \frac{\sin 4x}{4}$$

$$P.I = \frac{1}{25} e^{-3x} + \frac{x \sin 4x}{8}$$

General solution is

$$y = CF + P.I$$

$$y = C_1 \cos 4x + C_2 \sin 4x + \frac{e^{-3x}}{25} + \frac{x \sin 4x}{8}$$

⑤ Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = \underbrace{e^{-x}}_{I} + \underbrace{\sin 2x}_{II}$; ⑨

Sol:- $(D^3 + 2D^2 + D)y = e^{-x} + \sin 2x$

A.E is $m^3 + 2m^2 + m = 0$

$m(m^2 + 2m + 1) = 0$

$m = 0, \quad m^2 + 2m + 1 = 0$

$(m+1)^2 = 0 \implies m = -1, -1$

Roots $= 0, -1, -1$

$C.F = c_1 e^{0x} + (c_2 + c_3 x)e^{-x}$

$P.I = \frac{1}{D^3 + 2D^2 + D}(e^{-x} + \sin 2x)$

$= \frac{1}{D^3 + 2D^2 + D} e^{-x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$

Put $D = -1$ Put $D^2 = -2^2$

$= \frac{1}{(-1)^3 + 2(-1)^2 + (-1)} e^{-x} + \frac{1}{(D^2)D + 2D^2 + D} \sin 2x$

Put $D^2 = -4$

$= \frac{1}{-1 + 2 - 1} e^{-x} + \frac{1}{-4D + (-4) \cdot 2 + D} \sin 2x$

$= x \cdot \frac{1}{3D^2 + 4D + 1} e^{-x} + \frac{1}{-3D - 8} \sin 2x$

$$P.I = x \cdot \frac{1}{3D^2+4D+1} e^{-x} + \frac{1}{-3D-8} \sin 2x$$

$$= x \cdot \frac{1}{3D^2+4D+1} e^{-x} + \frac{(-3D+8)\sin 2x}{(-3D-8)(-3D+8)}$$

$$= x \cdot \frac{1}{3D^2+4D+1} e^{-x} + \frac{-3D(\sin 2x) + 8\sin 2x}{(-3D)^2 - 8^2}$$

$$= x \cdot \frac{1}{\underline{3D^2+4D+1}} e^{-x} + \frac{-3\cdot(\cos 2x)\cdot 2 + 8\sin 2x}{9D^2 - 64}$$

put $D = -1$ put $D^2 = -4$

$$= x \cdot \frac{1}{3(-1)^2 + 4(-1) + 1} e^{-x} + \frac{-6\cdot\cos 2x + 8\sin 2x}{9(-4) - 64}$$

$$= x \cdot \frac{1}{\underbrace{4-4}} e^{-x} + \frac{-6\cos 2x + 8\sin 2x}{-36-64}$$

$$= x^2 \cdot \frac{1}{6D+4} e^{-x} + \frac{-6\cos 2x + 8\sin 2x}{(-100)}$$

put $D = -1$

$$= x^2 \cdot \frac{1}{6(-1)+4} e^{-x} - \frac{(8\sin 2x - 6\cos 2x)}{100}$$

$$P.I = -\frac{x^2}{2} e^{-x} - \frac{8\sin 2x - 6\cos 2x}{100}$$

General Solution is

$y = CF + PI$

$y = c_1 e^{0x} + (c_2 + c_3 x) e^{-x} + \frac{x^2 e^{-x}}{2} - \frac{(8\sin x - 6\cos 2x)}{100}$

Solve

① $(D^3 + D^2 - D) y = 2\cos^2 x$

② $y'' + 9y = \cos 2x \cdot \cos x$

③ $(D^2 + 2D + 1) y = e^{-3x} - \cos^2 x$

④ $(D^3 + D^2 + D + 1) y = \sin 2x + 2\cos 3x$

⑤ $(D^2 + 16) y = e^{-x} + 2\cos 4x$

Type - III

P.I of the form $\frac{1}{f(D)} x^m$ (polynomial in x)

Standard Binomial expansions

(1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

(2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

(3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

(4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Ex:- $(D^2+D+1)y = x$

$$P.I = \frac{1}{D^2+D+1}x$$

$$= \frac{1}{1+\underbrace{(D+D^2)}}x$$

$$= \left(1+(D+D^2)\right)^{-1} x$$

$(1+x)^{-1} = 1 - x + x^2 + \cdots$

$$= \left[1 - (D+D^2) + (D+D^2)^2 + \cdots\right] x$$

$$= x - (D+D^2)x + $$

$$= x - D(x) + D^2(x)$$

$$= x - 1 + 0 + 0 - \cdots$$

① Solve $(D^2-2D+1)y = x$

Sol:- A.E is

$$m^2-2m+1 = 0$$

$$(m-1)^2 = 0 \Rightarrow m = 1, 1$$

$$C.F = (c_1+c_2x)e^x$$

$$P.I = \frac{1}{D^2-2D+1}x$$

$$= \frac{1}{1+(D^2-2D)}x = \left[1+(D^2-2D)\right]^{-1} x$$

$$P.I = [1+(D^2-2D)]^{-1} x$$

$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

$$= (1-(D^2-2D)+(D^2-2D)^2+\cdots) x$$

$$= [1-(D^2-2D)+(D^2-2D)^2+\cdots] x^1$$

$$= (1-(D^2-2D)+\cdots) x^1$$

$D^2, D^3 \cdots$ becomes zero

$$= x-(D^2-2D)x+\cdots$$

$$= x-D^2(x)+2D(x)+\cdots$$

$$= x-0+2(1)$$

$$P.I = x+2$$

General solution is

$$y = CF + P.I$$

$$y = (c_1+c_2x)e^x + (x+2)$$

② Solve $(D^2+D+1)y = x^3$ *

Sol :- A.E is $m^2+m+1=0$

$$m = \frac{-1 \pm \sqrt{1^2-4(1)(1)}}{2} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$m = \frac{-1 \pm \sqrt{3}i}{2}$$

$$CF = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

$$P.I = \frac{1}{D^2+D+1}x^3$$

$$= \frac{1}{1+(D+D^2)}x^3$$

$$= \left[1+(D+D^2)\right]^{-1}x^3$$

$$(1+x)^{-1} = 1 - x + x^2 + \cdots$$

$$= \left[1 - (D+D^2) + (D+D^2)^2 - (D+D^2)^3 + \cdots\right]x^3$$

$$= \left[1 - (D+D^2) + (D^2 + D^4 + 2D^3) - (D^3 + D^6 + 3D^4 + 3D^5) + \cdots\right]x^3$$

$$= \left[1 - D - \cancel{D^2} + \cancel{D^2} + 2D^3 - D^3 \cdots\right)x^3$$

$$\left(1 - D + D^3\right)x^3$$

$$= x^3 - D(x^3) + D^3(x^3)$$

$D(x^3) = 3x^2$

$D^2(x^3) = 6x$

$D^3(x^3) = 6$

$$P.I = x^3 - 3x^2 + 6$$

General sol

$$y = CF + PI$$

$$y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + x^3 - 3x^2 + 6$$

(3) Solve $(2D^2+2D+3)y = x^2+2x-1$

Sol :- A.E is $2m^2+2m+3=0$

$$m = \frac{-2 \pm \sqrt{2^2-4\cdot 2\cdot 3}}{2(2)}$$

$$m = \frac{-2 \pm \sqrt{4-24}}{4}$$

$$m = \frac{-2 \pm \sqrt{-20}}{4}$$

$$m = \frac{-2 \pm \sqrt{-(4\times 5)}}{4}$$

$$m = \frac{-2 \pm 2\sqrt{5}i}{4} \Rightarrow m = \frac{\cancel{2}(-1 \pm \sqrt{5}i)}{\cancel{4}\,2}$$

$$m = \frac{-1 \pm \sqrt{5}i}{2}$$

$$C.F = e^{-x/2}\left(c_1 \cos\frac{\sqrt{5}}{2}x + c_2 \sin\frac{\sqrt{5}}{2}x\right)$$

$$P.I = \frac{1}{2D^2+2D+3}(x^2+2x-1)$$

$$= \frac{1}{3\left(1+\frac{2D^2+2D}{3}\right)}(x^2+2x-1)$$

$$= \frac{1}{3}\left(1+\frac{2D^2+2D}{3}\right)^{-1}(x^2+2x-1)$$

$$= \frac{1}{3}\left(1+\frac{2D^2+2D}{3}\right)^{-1}(x^2+2x-1)$$

$$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$$

$$= \frac{1}{3}\left[1-\left(\frac{2D^2+2D}{3}\right)+\left(\frac{2D^2+2D}{3}\right)^2-\cdots\right](x^2+2x-1)$$

$$= \frac{1}{3}\left[1-\frac{2D^2}{3}-\frac{2D}{3}+\left(\frac{1}{9}\left(4D^4+4D^2+8D^3\right)+\cdots\right]\right.$$
$$(x^2+2x-1)$$

$$= \frac{1}{3}\left[1-\frac{2D^2}{3}-\frac{2D}{3}+\frac{4}{9}D^4+\frac{4}{9}D^2+\frac{8}{9}D^3+\cdots\right](x^2+2x-1)$$

$$= \frac{1}{3}\left[1-\frac{2D^2}{3}-\frac{2D}{3}+\frac{4}{9}D^2\right](x^2+2x-1)$$

$$= \frac{1}{3}\left[(x^2+2x-1)-\frac{2}{3}D^2(x^2+2x-1)-\frac{2}{3}D(x^2+2x-1)\right.$$
$$\left.+\frac{4}{9}D^2(x^2+2x-1)\right]$$

$$= \frac{1}{3}\left[x^2+2x-1-\frac{2}{3}(2)-\frac{2}{3}(2x+2)+\frac{4}{9}(2)\right]$$

$$= \frac{1}{3}\left[x^2+2x-1-\frac{4}{3}-\frac{4x}{3}-\frac{4}{3}+\frac{8}{9}\right]$$

$$= \frac{1}{3}\left[x^2 + \underbrace{2x - \frac{4x}{3}} - 1 - \frac{4}{3} - \frac{4}{3} + \frac{8}{9}\right]$$

$$= \frac{1}{3}\left[x^2 + \frac{6x-4x}{3} + \frac{(-9-12-12+8)}{9}\right]$$

$$P.I = \frac{1}{3}\left[x^2 + \frac{2x}{3} - \frac{25}{9}\right]$$

$$= \frac{x^2}{3} + \frac{2x}{9} - \frac{25}{27}$$

General sol is

$$y = CF + PI$$

$$y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{5}}{2}x + c_2 \sin\frac{\sqrt{5}}{2}x\right) + \frac{x^2}{3} + \frac{2x}{9} - \frac{25}{27}$$