Engineering Mathematics – II

Unit 1 – Differential – Equations – I

We have discussed several methods ~~like~~ for solving an O.D.E of 1st order & first degree. (homogeneous, linear, exact etc). In this chapter we discuss the method of solving linear differential equation of second & higher orders with constant co-efficients.

The general linear differential equation of n^{th} order

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = \phi(x)$$

where $a_1, a_2 \cdots a_n$ are constants. $\phi(x)$ is function of x only

$\phi(x) = 0$ is called homogeneous D.E

$\phi(x) \neq 0$ —"— non homogeneous D.E

we write
D-operator $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$ $\cdots D^n = \frac{d^n}{dx^n}$

Solution of Homogeneous linear differential eq

for quickly grasping the conceptual content of the method of solving a Homogeneous linear D.E we illustrate by taking 2nd order D.E as the same can be conveniently extended for equations of order two or more

(Refer Pg 472 grewal)

consider $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$ — (1)

$$(D^2 + a_1 D + a_2)\, y = 0 \quad \text{or} \quad f(D)y = 0$$

$$f(D) = D^2 + a_1 D + a_2$$

Theorem:- If y_1 & y_2 are only 2 solutions of the eq (1)

then $c_1y_1 + c_2y_2$ is also its solution

$$f(D)y_1 = 0 \quad \& \quad f(D)y_2 = 0$$

$$f(D)\,(c_1y_1 + c_2y_2) = 0$$

Since the general solution of a 2nd order D.E has to contain 2 arbitrary constant $c_1y_1 + c_2y_2$ is the general solution of (2) Let $y_c = c_1y_1 + c_2y_2$ is called the complimentary function & is the solution of homogeneous D.E

Solution of non homogeneous linear D.E

$$f(D)y = \phi(x)$$

if $y = y_p(x)$ be a particular solution

$$\therefore f(D)y_p = \phi(x)$$

now $f(D)(y_c + y_p) = f(D)y_c + f(D)y_p$

$$= 0 + \phi(x) = \phi(x)$$

y_p is called particular integral

$\therefore$ general solution or complete solution

$$y = y_c + y_p$$

Method of finding the complimentary function

As mentioned before let's consider 2nd order LDE

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$$

$$(D^2 + a_1 D + a_2)y = 0$$

$\therefore D^2 + a_1 D + a_2 = 0$ — Auxiliary equation

This being quadratic in D $\therefore$ 2 roots

(i) real & distinct, real & same, complex

Refer pg-472 grewal

Case 1 : real & different

$$(D-m_1)(D-m_2)y = 0$$

$$\frac{dy}{dx} - my = 0 \qquad \frac{dy}{dx} - m_2 y = 0$$

$$\frac{dy}{dx} = m_1 y$$

$$\frac{dy}{y} = m_1 x$$

$$\log y = m_1 x + K$$

$$y = e^{m_1 x} + K$$

$$\boxed{\therefore y = c_1 e^{m_1 x}}$$

$$|||^{ly} \quad \boxed{\therefore y = c_2 e^{m_2 x}}$$

$\therefore$ complete sol = $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case 2 roots are real & equal

$$m_1 = m_2 = m$$

$$(D-m)^2 y = 0 \qquad (D-m)(D-m)y = 0$$

put $(D-m)y = z$ $\quad (D-m)z = 0$

$$\frac{dz}{dx} - mz = 0 \qquad \frac{dz}{z} = m\,dx$$

$$z = c_1 e^{mx}$$

$$(D-m)y = c_1 e^{mx}$$

$$\frac{dy}{dx} - my = c_1 e^{mx}$$ — Linear D.E

$$e^{\int p\,dx} = e^{\int -m\,dx} \therefore = e^{-mx}$$

$$y e^{-mx} = \int c_1 e^{-mx} e^{mx}\,dx + c_2$$

$$y e^{-mx} = c_1 x + c_2$$

$$\underline{y = (c_1 x + c_2)\, e^{mx}}$$

<u>case 3</u> : roots are complex

$$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$$

$$\therefore \; G.S = \; y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha - i\beta x)}$$

$$y = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

$$= e^{\alpha x}\left(c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right)$$

$$= e^{\alpha x}\left((c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\right)$$

$$\therefore \; y = e^{\alpha x}\left(k_1\cos\beta x + k_2\sin\beta x\right)$$

illustrations

Roots of A·E	C·F (y_c)
1) 2, 3	$c_1 e^{2x} + c_2 e^{3x}$
2) 1, −1, 2, −2	$c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$
3) 3, 3	$(c_1 + c_2 x) e^{3x}$
4) 0, −2, −2, −2	$c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2) e^{-2x}$
5) $3 \pm 2i$	$e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$
6) $1 \pm 2i$, $1 + 2i$	$e^{x}((c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x)$

Solve

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$$

$$(D^3 + 6D^2 + 11D + 6)\, y = 0$$

AE $\quad D^3 + 6D^2 + 11D + 6 = 0$

Use calculator roots are $-1, -2, -3$

$\therefore$ Sol $\quad y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

Note: upto to degree three calculator gives the roots very easily so lets learn to get roots when calculator does not play a major role.

1) $(D^5 - D^4 - D + 1)\, y = 0$

$\boxed{\text{factorization}}$

AE $\quad m^5 - m^4 - m + 1 = 0$

$m^4(m-1) - 1(m-1) = 0$

$(m-1)\;(m^4 - 1) = 0$

$(m-1)\;(m^2-1)(m^2+1) = 0$

$m = 1, \pm 1, \pm i$

$\therefore$ roots $\; 1, 1, -1, \pm i$

$y = c_1 e^{-x} + (c_2 + c_3 x) e^{1x} + (c_4 \cos x + c_5 \sin x)$

2) $(4D^4 - 4D^3 - 23D^2 + 12D + 36)\, y = 0$

AE $\quad 4m^4 - 4m^3 - 23m^2 + 12m + 36$

(Synthetic division)

by inspection $m = 2$ (one root)

$$\begin{array}{c|ccccc} 2 & 4 & -4 & -23 & 12 & 36 \\ & 0 & 8 & 8 & -30 & -36 \\ \hline & 4 & 4 & -15 & -18 & \boxed{0} \end{array}$$

$4m^3 + 4m^2 - 15m - 18 = 0$

roots, $\; 2, 2, -\frac{3}{2}, -\frac{3}{2}$

repeated roots) (note How it comes in calculator)

use calculator

$y = (c_1 + c_2 x) e^{2x} + (c_3 + c_4 x) e^{-3x/2}$

3) $(D^4+64)y=0$

$\boxed{\text{Use formula}}$
$a^2+b^2=(a+b)^2-2ab$

AE $m^4+64=0$

$(m^2)^2+(8)^2=0$

$(m^2+8)^2-(4m)^2=0$

$(m^2+8-4m)(m^2+8-4m)=0$

$\therefore m^2-4m+8=0 \qquad m^2+4m+8=0$

(use calculator) $2\pm 2i \qquad -2\pm 2i$

$$y=e^{2x}(c_1\cos 2x+c_2\sin 2x)$$
$$+e^{-2x}(c_3\cos 2x+c_4\sin 2x)$$

Note
if initial or boundary conditions are given then use it to find the arbitrary constants
(will be done later)

Inverse differential operator & the particular integral (Refer pg - 475 grewal)

we have $D=\frac{d}{dx}$ $\quad D[G(x)]=F(x)$

$\int F(x) = G(x)$ - inverse operator

i.e. $\frac{1}{D}F(x)=G(x)$

consider the equation $\frac{d^n y}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+\dots+k_n y=\phi(x)$

case 1 $\phi(x) = e^{ax}$

Since $De^{ax} = ae^{ax}$, $D^2e^{ax} = a^2e^{ax}$ - - - -

- - - $D^n e^{ax} = a^n e^{ax}$

$$(D^n + k_1 D^{n-1} - - - + k_n) e^{ax} = (a^n + k_1 a^{n-1} - - + k_n) e^{ax}$$

$$\frac{f(D)}{f(D)} e^{ax} \quad \Big| \; f(D) = e^{ax} = f(a) e^{ax}$$

$$\therefore \frac{1}{f(D)} e^{ax} = \frac{1}{f(D)} f(a) e^{ax} . \quad \text{or } e^{ax} = \frac{f(a)}{f(D)} e^{ax}$$

$$\div f(a)$$

$$\boxed{\therefore \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}}$$

note
$$\frac{1}{D-a} F(x) = e^{ax} \int F(x) e^{-ax} dx$$

if $f(a) = 0$ a is the root of A.E

$(D-a)$ is factor of $f(D)$

$$f(D) = (D-a)\,\phi(D)$$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax} \int e^{ax} e^{-ax} dx$$

$$= x \frac{1}{\phi(a)} e^{ax}$$

$$\boxed{\therefore \frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}}$$

$f(D) = 0$

Illy $f'(a) = 0$ $\quad \frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$

Case 2: $\phi(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$D(\sin(ax+b)) = a\cos(ax+b)$$

$$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$$

in general $(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$

$$(D^2)^r \sin(ax+b) = (-a^2)^r \sin(ax+b)$$

$$\therefore \frac{1}{f(D^2)} f(D^2) \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2) \sin(ax+b)$$

$$\therefore \sin(ax+b) = f(-a^2) \frac{1}{f(D^2)} \sin(ax+b)$$

$\div f(-a^2)$

$$\boxed{\therefore \frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b)}$$

$f(-a^2) = 0$

$$\frac{1}{f(D^2)} \sin(ax+b) = x \frac{1}{f'(a^2)} \sin(ax+b)$$

$$\text{i.e. } \frac{1}{f(D^2)} \sin(ax+b) = \text{IP of } \frac{1}{f(D^2)} e^{i(ax+b)}$$

$$= \text{IP of } x \frac{1}{f'(D^2)} e^{i(ax+b)}$$

III^{ly} for $\cos(ax+b)$

Case 3: $\phi(x) = x^m$ $\qquad PI = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$

expand $[f(D)]^{-1}$ in ascending power of D as far as the term D^m & operate on x^m.

alternate Method

PI is found by division.
By writing $\phi(x)$ in descending powers of x
& $f(D)$ in ascending powers of D. divide completely
quotient is P.I.

case 4: $\phi(x) = V e^{ax}$ $\quad V = g(x)$

$u = g_1(x)$ $\quad D(e^{ax} u) = e^{ax} Du + a e^{ax} u = (D+a) u$

$$D^2(e^{ax} u) = e^{ax} (D+a)^2 u$$

$$\vdots$$

$$D^n(e^{ax} u) = e^{ax} (D+a)^n u$$

$$\therefore \frac{1}{f(D)} f(D) e^{ax} u = \frac{1}{f(D)} \left[e^{ax} f(D+a) u \right]$$

$$e^{ax} u = \frac{1}{f(D)} \left[e^{ax} f(D+a) u \right]$$

now $\quad f(D+a) u = V \quad \therefore u = \frac{1}{f(D+a)} V$

$$\therefore e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} e^{ax} V$$

$$\boxed{\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V}$$

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<u>Case 5:</u> $x V$ $\quad V = g(x)$

or $x^n V$

Let $u = g_1(x)$

$$D^n(xu) = x D^n u + n D^{n-1} u = x(D^n u) + \frac{d}{dD}(D^n u)$$

$$f(D)\, x u = x f(D) u + f'(D) u$$

$$u = \frac{1}{f(D)} V$$

$$f(D)\, x \frac{1}{f(D)} V = x V + f'(D) + \frac{1}{f(D)} V$$

$$x \frac{1}{f(D)} V = \frac{1}{f(D)} x V + \frac{1}{f(D)} f'(D) \frac{1}{f(D)} V$$

$$\boxed{\frac{1}{f(D)} x V = \left[x - \frac{f'(D)}{f(D)}\right] \frac{V}{f(D)}}$$

<u>Note</u> 1) $\cosh x = \frac{e^x + e^{-x}}{2}$ $\quad \sinh x \; \frac{e^x - e^{-x}}{2}$

2) $x^n \cos ax$ & $x^n \sin ax$

$$e^{iax} = \cos ax + i \sin ax$$

$x^n \cos ax = R.P\,(e^{iax} x^n)$. $\quad x^n \sin ax = I.P(e^{iax} x^n)$

3) $a^x = e^{\log a^x} = e^{(\log a) x}$

illustrations

1) $y'' - 4y = \sinh^2 x$

$(D^2 - 4)y = \sinh^2 x$

$AE = D^2 - 4 = 0$

$m = \pm 2 \qquad CF = c_1 e^{2x} + c_2 e^{-2x}$

$$PI = \frac{1}{D^2-4}\left(\frac{e^{x} - e^{-x}}{2}\right)^2$$

$$\frac{1}{4}\frac{e^{2x}}{D^2-4} + \frac{1}{4}\frac{e^{-2x}}{D^2-4} - \frac{1}{2}\frac{1}{D^2-4}$$

$$= \frac{1}{4} x \frac{e^{2x}}{2D} + \frac{1}{4} x \frac{e^{2x}}{2D} - \frac{1}{2}\left(\frac{1}{-4}\right)$$

$$= \frac{1}{16} x e^{2x} - \frac{1}{16} x e^{-2x} + \frac{1}{8}$$

$$GS = CF + PI = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{16}\left(x e^{2x} - x e^{-2x} + 2\right)$$

2) $\frac{d^2y}{dx^2} + y = \sin^2 x \cos^2 x$

$AE \quad m^2 + 1 = 0 \quad m = \pm i$

$CF = c_1 \cos x + c_2 \sin x$

$$PI = \frac{\sin^2 \cos^2 x}{D^2+1} = \frac{1}{D^2+1}\left(\frac{\sin 2x}{2}\right)^2$$

$$\frac{1}{4}\frac{1}{D^2+1}\frac{1}{2}(1 - \cos 4x) = \frac{1}{8}\frac{1}{D^2+1} - \frac{1}{8}\frac{\cos 4x}{D^2+1}$$

$= \frac{1}{8} + \frac{1}{120}\cos 4x$

(7)

3) $(D^2+3D+2)y = 2x^2+4x+1$

$AE = m^2+3m+2=0$

$-1, -2 \qquad CF = c_1e^{-x}+c_2e^{-2x}$

$PI = \frac{2x^2+4x+1}{2+3D+D^2}$

$$
\begin{array}{r|l}
 & x^2-x+1 \\
\hline
2+3D+D^2 & 2x^2+4x+1 \\
 & 2x^2+6x+2 \\
 & (-)\ (-)\ (-) \\
\hline
 & -2x-1 \\
 & -2x-3 \\
 & (+)\ (+) \\
\hline
 & +2 \\
 & \underline{(-)\,2} \\
 & 0
\end{array}
$$

$\therefore\ y = c_1e^{-x}+c_2e^{-2x}$
$+ x^2-x+1$

4) $y''-2y'+y = xe^x\sin x$

$AE \quad m^2-2m+1=0 \quad m=1;1$

$CF = (c_1+c_2x)e^x$

$y_p = \frac{e^x(x\sin x)}{(D-1)^2} \qquad D\to D+1$

$e^x\left(\frac{x\sin x}{D^2}\right) \longrightarrow$ can be done $\int\left\{\int x\sin x\,dx\right\}dx$

$e^x\left(x-\frac{2D}{D^2}\right)\frac{\sin x}{D^2} = e^x\left(x-\frac{2}{D}\right)\frac{\sin x}{-1}$

$e^x(-x\sin x-2\cos x)$

$$y = (c_1 + c_2 x)e^{-x} - e^{x}(x\sin x + 2\cos x)$$

5) Solve $y'' - 4y' + 4y = 8x^2 e^{2x}\cos 2x$

AE $m^2 - 4m + 4 = 0 \qquad m = 2, 2$

$$CF = (c_1 + c_2 x)e^{2x}$$

$$y_p = \frac{8x^2 e^{2x}\cos 2x}{(D-2)^2} = \qquad D \to D+2$$

$$e^{2x}\left(\frac{8x^2\cos 2x}{D^2}\right) \rightarrow \text{can be done } \int\left\{\int 8x^2\cos 2x\, dx\right\}dx$$

$$y_p = e^{2x}\ \text{Real part}\ \frac{8x^2 e^{2ix}}{D^2} \qquad D \to D + 2i$$

$$e^{2x}\, e^{2ix}\, \frac{8x^2}{D^2 + 4iD - 4}$$

$$-4 + 4iD + D^2 \Big)\ 8x^2 \ \Big(\ -2x^2 - 4ix + 3$$

$$\begin{array}{r} 8x^2 \\ 8x^2 - 16ix - 4 \\ \hline 16ix + 4 \\ 16ix + 16 \\ \hline -12 \\ -12 \\ \hline 0 \end{array}$$

$$-2x^2 - 4ix + 3$$

$$y_p = e^{2x}\ R.P\,(\cos 2x + i\sin 2x)(-2x^3 + 3 - 4ix)$$

$$y_p = e^{2x}(\cos 2x(3 - 2x^2) + 4x\sin 2x)$$

$$\therefore\ y = (c_1 + c_2 x)e^{2x} + e^{2x}[\cos 2x(3 - 2x^2) + 4x\sin 2x]$$

Method of variation of parameters

This Method is general & applied to all form $y''+py'+qy=X$ — (1)

p, q & X are functions of x

$$PI = -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx \quad -(2)$$

where y_1 & y_2 are solutions of $y''+py'+qy=0$ — (3)

$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \rightarrow$ wronskian of y_1, y_2

proof Let CF of (1) be $c_1y_1+c_2y_2$

replace c_1 c_2 are $u(x)$ & $v(x)$

$$y = uy_1+vy_2 \quad -(4)$$

Diff wrt x $\quad y' = uy_1'+vy_2'+u'y_1+v'y_2$ — (6)

Let $u'y_1+v'y_2=0$ — (5)

$\therefore y' = uy_1'+vy_2'$

Diff (4) & sub in (1) $\quad \therefore y_1$ & y_2 Satisfy (3)

$\therefore$ solving (6) & (7)

$$u' = -\frac{y_2X}{W} \qquad v' = \frac{y_1X}{W}$$

$\therefore$ we get (2)

Method of undetermined co-efficients

to find PI we assume trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution depends on the form of X.

This method is suitable only when the eq is with constant coefficient & X or $\phi(x)$ is in some particular forms.

$k\,\phi(x)$ or $\phi(x)$	Assumption of y_p	Restrictions
e^{mx}	$a e^{mx}$	m is not a root of AE
x^n (poly) $a_0+a_1x+\cdots+a_nx^n$	$a_0+a_1x+\cdots+a_nx^n = Q_n(x)$	0 is not a root of AE
$\sin nx$ or $\cos nx$ or both	$a\cos nx + b\sin nx$	$\pm in$ are not roots of the AE
$e^{mx}\cos nx$ or $e^{mx}\sin nx$ or both	$e^{mx}(a\cos nx + b\sin nx)$	$m \pm in$ are not roots of AE
$x\cos nx$ or $x\sin nx$	$(a+bx)\cos nx + (c+dx)\sin nx$	$\pm in$ are not root of AE
in case of x^2 i.e $x^2\cos nx$	$(a+bx+cx^2)\cos nx + (d+ex+fx^2)\sin nx$	$\pm in$ are not roots of AE
$P_n(x)\, e^{ax}$	$Q_n(x)\, e^{ax}$	m is not the root of AE

$K\,\phi(x)$ or $\phi(x)$	Assumption of y_p	Restrictions
K (constant)	A	0 is not the root of DE
$e^{ax} P_n(x) \times \begin{Bmatrix} \cos bx \\ \sin bx \end{Bmatrix}$	$e^{ax} Q_n(x) \{\cos bx + \sin bx\}$	$a \pm ib$ not root of DE

Note:- if any part of assumed y_p is a part of C.F then we multiply by x^m

ie in cases of restrictions the assumed y_p is multiplied by x, x^2, x^3 ----- according as the real root / complex root pair is repeated once, twice, thrice etc

eg

Roots	given $\phi(x)$	Assume y_p	Remark
$\pm 2, \pm 3$	$3e^{2x} - 4e^{x}$	$a x e^{2x} - b e^{x}$	2 is a root but 1 is not root of DE
$0, 1, 2$	$2x^2 - 4x + 5$	$x(a + bx + cx^2)$	0 is root of DE
$0, 1, \pm 3i$	$2\cos 3x$	$x(a\cos 3x + b\sin 3x)$	$\pm 3i$ is a root of DE
$2, 3, \pm i$	$x^2 \sin 2x$	$(a + bx + cx^2)\cos 2x + (d + ex + fx^2)\sin 2x$	$\pm 2i$ not a root of DE
$\pm 2i$	$12x^2$	$a + bx + cx^2$	0 is not a root of DE

1) solve by Method of undetermined coefficient

$$(D^3+D^2-4D+4)\,y = 3e^{-x}-4x-6$$

$$AE = m^3+m^2-4m-4=0$$

$$\cancel{m^2 \pm m} \quad -1,-2,2 \text{ roots}$$

$$y_c = c_1e^{-x}+c_2e^{-2x}+c_3e^{2x}$$

$$\text{Asume } PI = y_p = axe^{-x}+b+cx$$ $\left\{\begin{array}{l}\text{Since } -1 \\ \text{is root } \& \\ 0 \text{ is not} \\ \text{root of AE}\end{array}\right.$

$$y_p' = a(-xe^{-x}+e^{-x})+c$$

$$y_p'' = a(xe^{-x}-2e^{-x})$$

$$y_p''' = a(-xe^{-x}+3e^{-x}) \quad \text{Sub in quesn}$$

$$-axe^{-x}+3ae^{-x}+axe^{-x}-2ae^{-x}+4axe^{-x}$$

$$-4ae^{-x}-4c-4axe^{-x}-4(-4axe^{-x}$$

$$-4b-4cx = 3e^{-x}-4x-6$$

equating the co-efficient

$$-4c=-4 \qquad -3a=3$$

$$c=1 \qquad a=-1$$

$$-4(b+c) = -6$$

$$b+c=\frac{3}{2} \qquad b=\frac{1}{2}$$

$$y_p = -xe^{-x}+\frac{1}{2}+x$$

$$\therefore\ y = c_1e^{-x}+c_2e^{2x}+c_3e^{-2x}$$

Solve by Variation of parameters (10)

$y'' + y = \log(\cos x)$

$AE = m^2 + 1 = 0 \quad m = \pm i$

$y_{c_f} = c_1 \cos x + c_2 \sin x$

$\therefore y = A \cos x + B \sin x$ $\quad$ $y_1 = \cos x$, $y_2 = \sin x$

$W = 1$

$A = \int \frac{-y_2 \phi(x)}{W}$ $\quad$ $B = \int \frac{y_1 \phi(x)}{W}$

$A = \int - \sin x \cdot \log(\cos x)\, dx$

$\cos x = t \quad -\sin x\, dx = dt$

$\int \log t\, dt$

$t \log t - t + k_1$

$A = \cos x(\log(\cos x) - \cos x + k)$

$B = \int \cos x \log(\cos x)\, dx$

$\log(\cos x) \sin x - \int \sin x \frac{1}{\cos x}(-\sin x\, dx)$

$\log(\cos x) \sin x + \int \frac{\sin^2 x}{\cos x}\, dx$

$$= \log(\cos x)\sin x + \int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$B = \log(\cos x)\sin x + \int \sec x - \cos x\,dx$$

$$\therefore B = \log(\cos x)\sin x + \log|\sec x + \tan x| - \sin x + k_2$$

$$\therefore y = A\cos x + B\sin x$$

$$y = (\cos x \log(\cos x) - \cos x + k_1)\cos x + (\log(\cos x)\sin x + \log(\sec x + \tan x) - \sin x + k_2)\sin x$$

$$\therefore y = \log(\log x) + k_1\cos x + k_2\sin x + \sin x\log(\sec x + \tan x) - 1$$

we have done different types of problems Now we ~~se~~ will do the Mixed Types of problems.

$(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$ Ex 13.1 pg 474 grewal (11)

Synthetic division

$4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0$ - Auxilary equation

$m = -1$ $\quad 4 + 8 - 7 - 11 + 6 = 0$

root

$$\begin{array}{r|rrrrr} -1 & 4 & -8 & -7 & 11 & 6 \\ & 0 & -4 & 12 & -5 & -6 \\ \hline & 4 & -12 & 5 & 6 & \lfloor 0 \end{array}$$

$4m^2 - 12m^2 + 5m + 6 = 0$

by inspection $m = 2$ is root

$32 - 48 + 10 + 6 = 0$

$$\begin{array}{r|rrrr} 2 & 4 & -12 & 5 & 6 \\ & 0 & 8 & -8 & -6 \\ \hline & 4 & -4 & -3 & \lfloor 0 \end{array}$$

$4m^2 - 4m - 3 = 0$

$2m(2m - 3) + 1(2m - 3) = 0$

$m = -\frac{1}{2}, \ 3/2$

$\therefore$ roots are $-1, 2, -1/2, 3/2$

$\therefore$ g.s $\Rightarrow$ $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-x/2} + c_4 e^{3x/2}$

2) $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$ ex 13.1 general

$AE = (D^4 + 8D^2 + 16)\, y = 0$

$m^4 + 8m^2 + 16 = 0$

$(m^2+4)^2 = 0$

$(m^2+4)\,(m^2+4) = 0$

$m^2+4=0 \quad m = \pm 2i$

$\therefore$ roots are $m = \pm 2i, \pm 2i$ (repeated complex roots)

$\therefore$ g.s $y = (C_1 + C_2x)\cos 2x + (\cancel{C_3} + C_4x)\sin 2x$

3) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

AE $m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$m = 1, 1, 1$

g.s $y = (C_1 + C_2x + C_3x^2)\, e^x$

Ex 13.2 pg 486
given

(12)

1) $(D^2-1)y = x\sin x + (1+x^2)e^x$

AE $m^2-1=0 \quad m=\pm 1$

$CF = c_1e^x + c_2e^{-x}$

$PI = \dfrac{(1+x^2)e^x + x\sin x}{D^2-1} = \underbrace{\dfrac{(1+x^2)e^x}{D^2-1}}_{(1)} + \underbrace{\dfrac{x\sin x}{D^2-1}}_{(2)}$

(1) $\dfrac{(1+x^2)e^x}{D^2-1} = e^x\left\{\dfrac{1+x^2}{(D+1)^2-1}\right\}$

$= e^x\left\{\dfrac{1}{D^2+2D}(1+x^2)\right\}$

$$\begin{array}{r|l} & \frac{x^3}{6} - x^2/4 + 3x/4 \\ \hline 2D+D^2 & x^2+1 \\ & x^2+x \\ & (-)\quad(-) \\ \hline & -x+1 \\ & -x-\frac{1}{2} \\ & (+)\quad(+) \\ \hline & 3/2 \\ & 3/2 \\ \hline & 0 \end{array}$$

$PI = e^x\left\{\dfrac{x^3}{6} - \dfrac{x^2}{4} + \dfrac{3x}{4}\right\}$

(2) $\dfrac{x\sin x}{D^2-1}$ $\qquad \dfrac{xV}{f(D)} = \left[x - \dfrac{f'(D)}{f(D)}\right]\dfrac{V}{f(D)}$

$\left[x - \dfrac{2D}{D^2-1}\right]\dfrac{\sin x}{D^2-1}$

$$\left[x - \frac{2D}{D^2-1}\right] \frac{\sin x}{-1-1} = -\frac{x\sin x}{2} + \frac{\cos x}{D^2-1} \qquad D^2=-1$$

$$-\frac{x\sin x}{2} + \frac{\cos x}{-2} = -\frac{1}{2}(x\sin x + \cos x)$$

$GS \Rightarrow y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{x e^{3x}}{12}(2x^2 - 3x + 9)$$
$$-\frac{1}{2}(x\sin x + \cos x)$$

2) $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$$

$$m^2 - 6m + 9 = 0$$
$$(m-3)^2 = 0 \qquad m = 3, 3$$
$$y_1 = (c_1 + c_2 x)e^{3x}$$

$$y_p = \frac{6e^{3x}}{D^2-6D+9} + 7\frac{e^{-2x}}{D^2-6D+9} - \frac{\log 2\, e^{0}}{D^2-6D+9}$$

$$= \frac{6e^{3x}}{3^2-18+9} + 7\frac{e^{-2x}}{(-2)^2-6(-2)+9} - \frac{\log 2\, e^{0}}{0-0+9}$$

$Dr=0$

$$\frac{6x e^{3x}}{2D-6} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$$

$\llcorner Dr=0$

$$6\,\frac{x^2 e^{3x}}{2} = 3x^2 e^{3x}$$

$$\therefore\ G.S = (c_1 + c_2 x)e^{3x} + 3x^2 e^{3x} + \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$$

3) $y'' + 16y = x \sin 3x$

$$(D^2+16)y = x\sin 3x$$

$$m^2 + 16 = 0 \qquad m = \pm 4i$$

$$y_c = c_1 \cos 4x + c_2 \sin 4x$$

$$y_p = \frac{x \sin 3x}{D^2+16}$$

$$\left[x - \frac{f'(D)}{f(D)}\right]\frac{V}{f(D)} \qquad D^2 = -9$$

$$= \left(x - \frac{2D}{D^2+16}\right)\frac{\sin 3x}{D^2+16} = \frac{x \sin x}{7} - \frac{6 \cos x}{7(D^2+16)}$$

$$= \frac{x \sin 3x}{7} - \frac{6 \cos 3x}{7(-9+16)}$$

$$y_p = \frac{x \sin 3x}{7} - \frac{6 \cos 3x}{49}$$

$$\therefore\ G.S = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{49}(7x \sin 3x - 6\cos 3x)$$

4) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$

AE $m^3 + 2m^2 + m = 0$

$m(m+1)^2 = 0$ $\quad$ $m = 0, -1, -1$

$CF = C_1 + (C_2 + C_3 x) e^{-x}$

$$PI = \underbrace{\frac{1}{D^3+2D^2+D} e^{-x}}_{1} + \underbrace{\frac{\sin 2x}{D^3+2D^2+D}}_{2}$$

1) $\frac{e^{-x}}{D^3+2D^2+D} = \frac{e^{-x}}{-1+2-1}$ $\quad D=0$

$\frac{x e^{-x}}{3D^2+4D+1} = \frac{x e^{-x}}{3-4+1}$ $\quad D=0$

$\frac{x^2 e^{-x}}{6D+4} = \frac{x^2 e^{-x}}{-6+4} = -\frac{1}{2} e^{-x} x^2$

2) $\frac{\sin 2x}{D^3+2D^2+D}$ $\quad D^2 = -4$

$\frac{\sin 2x}{D^2(D)+2D^2+D} = \frac{\sin 2x}{-4D-8+D} = \frac{\sin 2x}{-3D-8}$

$= -\frac{\sin 2x}{3D+8} \times \frac{3D-8}{3D-8} = -\left\{ \frac{3D-8 \ (\sin 2x)}{9D^2-64} \right.$

$$= -\frac{1}{9(-4)-64}\,(6\cos 2x - 8\sin 2x)$$

$$\frac{1}{100}\,(6\cos 2x - 8\sin 2x)$$

$$\therefore G.S = \quad y = c_1 + (c_2 + c_3 x)e^{-x}$$

$$+ \frac{1}{2}x^2 e^{-x} + \frac{1}{50}(3\cos 2x - 4\sin 2x)$$

5) $y'' + 2y = x^2 e^{3x} + e^{x}\cos 2x$

$AE = D^2 + 2 = 0 \qquad m^2 + 2 = 0 \quad m = \pm i\sqrt{2}$

$CF = A\cos\sqrt{2}x + B\sin\sqrt{2}x$

$$PI = \frac{1}{D^2+2}\cdot x^2 e^{3x} + \frac{1}{D^2+1}\,e^{x}\cos 2x$$

(1) (2)

$$(1) \Rightarrow e^{3x}\,\frac{1}{(D+3)^2+2}\,x^2$$

$$e^{3x}\,\frac{x^2}{D^2+6D+11} \qquad \therefore (1)\ \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right)$$

$$\begin{array}{r|l}
 & \dfrac{x^2}{11} - \dfrac{12}{11^2}x + \dfrac{50}{11^3} \\
11 + 6D + D^2 & x^2 \\
 & \dfrac{x^2}{11} + \dfrac{12x}{11} + \dfrac{2}{11} \\
 & (-)\quad(-)\quad(-) \\
 & -\dfrac{12}{11}x - \dfrac{2}{11} \\
 & -\dfrac{12}{11}x - \dfrac{72}{11^2} \\
 & (+)\quad(+) \\
 & \dfrac{50}{121} \\
 & \dfrac{50}{121} \\
 & 0
\end{array}$$

(2) $\frac{1}{D^2+2} e^x \cos 2x$

$$= e^x \left(\frac{1}{(D+1)^2+2} \cos 2x \right)$$

$$= e^x \frac{1}{D^2+2D+3} \cos 2x \qquad D^2 = -4$$

$$= e^x \frac{1}{2D-1} \cos 2x = e^x \frac{1}{2D-1} \frac{(2D+1)}{(2D+1)} \cos 2x$$

$$= e^x \frac{(2D+1)\cos 2x}{4D^2-1} = \frac{e^x}{-17}\left(2(-\sin 2x)\cdot 2 + \cos 2x \right)$$

$$= -\frac{e^x}{17}\left(-4\sin 2x + \cos 2x \right)$$

$$\therefore y = A\cos\sqrt{2}x + B\sin\sqrt{2}x + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121} \right)$$

$$+ \frac{e^x}{17}\left(4\cos 2x - \cos 2x \right)$$

6) $\frac{d^4y}{dx^4} - y = \cos x \cosh x$

AE $m^4 - 1 = 0 \qquad (m^2-1)(m^2+1) = 0$

$m = \pm 1 \quad \pm i$

$CF = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

$$PI = \frac{\cos x \left(\frac{e^x + e^{-x}}{2} \right)}{D^4 - 1} = \frac{1}{2}\left\{ \frac{e^x \cos x}{D^4-1} + \frac{e^{-x}\cos x}{D^4-1} \right\}$$

$$= \frac{1}{2}\left\{ e^{x} \frac{\cos x}{((D+1)^2-1)((D+1)^2+1)} + e^{-x} \frac{1}{((D-1)^2-1)((D-1)^2+1)} \cos x \right] \quad 15$$

$$= \frac{1}{2}\left\{ e^{x} \frac{\cos x}{(D^2+2D)(D^2+2D+2)} + e^{-x} \frac{\cos x}{(D^2-2D)(D^2-2D+2)} \right]$$

$$D^2 = -1$$

$$= \frac{1}{2}\left\{ e^{x} \frac{\cos x}{(2D-1)(2D+1)} + e^{-x} \frac{1}{(2D+1)(2D-1)} \cos x \right]$$

$$\frac{1}{2}\left\{ e^{x} \frac{\cos x}{4D^2-1} + e^{-x} \frac{1}{4D^2-1} \cos x \right]$$

$$\frac{1}{2}\left(e^{x}\left(-\frac{1}{5}\right)\cos x + e^{-x}\left(-\frac{1}{5}\right)\cos x \right] = -\frac{1}{5}\cos x \cosh x$$

G.S $\quad y = c_1 e^{x} + c_2 e^{-x} + (c_3 \cos x + c_4 \sin x)$

$$-\frac{1}{5}\cos x \cosh x$$

d) $\quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$

$(D^2+3D+2)y = 4\cos^2 x$

$m^2+3m+2=0 \qquad m = -1, -2$

$y \cdot CF = c_1 e^{-x} + c_2 e^{-2x}$

$4\cos^2 x = 2(1+\cos 2x)$

$$PI = \underbrace{\frac{2}{D^2+3D+2}}_{1} + \underbrace{\frac{2\cos 2x}{D^2+3D+2}}_{2}$$

(1) $\dfrac{2e^{0x}}{D^2+3D+2} = \dfrac{2}{0+0+2} = 1$

(2) $\dfrac{2\cos 2x}{D^2+3D+2} = 2\left[\dfrac{\cos 2x}{-4+3D+2}\right] = 2\,\dfrac{\cos 2x}{3D-2}$

$2\,\dfrac{(3D+2)}{(3D-2)(3D+2)}\cos 2x = \dfrac{2(-6\sin 2x+2\cos 2x)}{9D^2-4}$

$= \dfrac{3\cos 2x - \cos 2x}{10}$

$\therefore\ y = c_1 e^{-x} + c_2 e^{-2x} + 1 + \dfrac{3\sin 2x - \cos 2x}{10}$

(8) $y'' + 16y = x\sin 3x$

$(D^2+16)y = 0 \qquad A.E = m^2+16 = 0$

$m = \pm 4i \qquad y(CF) = c_1\cos 4x + c_2\sin 4x$

$\dfrac{x\sin nx}{D^2+16} = \left[x - \dfrac{2D}{D^2+16}\right]\dfrac{\sin 3x}{D^2+16} \qquad D^2 = -9$

$= \left[x - \dfrac{2D}{D^2+16}\right]\dfrac{\sin 3x}{7}$

$= \dfrac{x\sin 3x}{7} - \dfrac{6\cos 3x}{49}$

$\therefore\ y = c_1\cos 4x + c_2\sin 4x + \dfrac{1}{49}(7x\sin 3x - 6\cos 3x)$

Solve by variation of parameters   ex 13.3 Pg 490 grewal

(16)

1) $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$

AE $m^2 + 1 = 0$ $m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

G.S $y = A(x)\cos x + B(x)\sin x$

$A(x)\, y_1 + B(x)\, y_2$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$A = \int \frac{-y_2\,\phi(x)}{W}\, dx = -\int \frac{\sin x}{1} \frac{1}{1+\sin x}\, dx$

$A = -\int \frac{\sin x + 1 - 1}{1+\sin x}\, dx = \int -1 + \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx$

$A = -x + \int \frac{1-\sin x}{\cos^2 x}\, dx$

$A = -x + \int (\sec^2 x - \sec x \tan x)\, dx$

$A = -x + \tan x - \sec x + k_1$

$B = \int \frac{y_1\,\phi(x)}{W} = \int \frac{\cos x}{1+\sin x} \frac{(1-\sin x)}{(1-\sin x)}\, dx$

$= \int \frac{(1-\sin x)\cos x}{\cos^2 x} = \int \frac{1-\sin x}{\cos x}\, dx$

$$\int (\sec x - \tan x)\,dx \;-\;$$

$$\log|\sec x + \tan x| + \log|\cos x| + k_2$$

$\sec x = \frac{1}{\cos x}$

$\tan x = \frac{\sin x}{\cos x}$

$$B = \log\left(\frac{1+\sin x}{\cos x}\right) + \log|\cos x| + k_2$$

$$\therefore B = \log(1+\sin x) - \cancel{\log|\cos x|} + \cancel{\log|\cos x|} + k_2$$

$$B = \log(1+\sin x) + k_2$$

$$\therefore y = (-x + \tan x - \sec x + k_1)\cos x$$
$$+ (\log(1+\sin x) + k_2)\sin x$$

$$\therefore y = k_1\cos x + k_2\sin x - (x\cos x + 1)$$
$$+ \sin x \log(1+\sin x)$$

(2) $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$$ ex 13.2 pg 486 grewal

$$(D^2 + 3D + 2)y = e^{e^x}$$

AE $\quad m^2 + 3m + 2 = 0 \qquad m = -1, -2$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

g.s $\quad y = A e^{-x} + B e^{-2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$$A = \int -\frac{y_2\,\phi(x)}{W}\,dx = -\int \frac{e^{-2x}e^{e^x}}{-e^{-3x}}\,dx$$

(17)

$$= \int e^{x} e^{e^x}\,dx = \boxed{\begin{array}{l} e^x = t \\ e^x\,dx = dt \end{array}}$$

$$= \int e^t\,dt = e^t + k_1$$

$$A = e^{e^x} + k_1$$

$$B = \int \frac{y_1\,\phi(x)}{W}\,dx = \int \frac{e^{-x}e^{e^x}}{-e^{-3x}}$$

$$= -\int e^{2x} e^{e^x}\,dx$$

$$= -\int e^x e^x e^{e^x}\,dx = -\int t\,e^t\,dt$$

$$= B = -(te^t - e^t) + k_2$$

$$\therefore B = e^{e^x}(1 - e^x) + k_2$$

$$\therefore y = (e^{e^x} + k_1)e^{-x} + \left(e^{e^x}(1 - e^x) + k_2\right)e^{-2x}$$

$$\therefore y = k_1 e^{-x} + k_2 e^{-2x} + e^{-2x}e^{e^x}$$

(3)

$y'' - 2y' + 2y = e^x \tan x$

AE $m^2 - 2m + 2 = 0$ $\quad m = 1 \pm i$

$cf = e^x (c_1 \cos x + c_2 \sin x)$

$y = A e^x \cos x + B e^x \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(-\sin x + \cos x) & e^x(\sin x + \cos x) \end{vmatrix}$$

$= e^{2x}(\cos x \sin x + \cos^2 x) - e^{2x}(-\sin^2 x + \sin x \cos x)$

$= e^{2x}(\cos x \sin x + \cos^2 x + \sin^2 x - \sin x \cos x)$

$W = e^{2x}$.

$$A = \int \frac{-y_2 \phi(x)}{W} dx = -\int \frac{e^x \sin x \, e^x \tan x}{e^{2x}} dx$$

$$A = -\int \sin x \frac{\sin x}{\cos x} dx = -\int \frac{\sin^2 x}{\cos x} dx$$

$$A = -\int \frac{1 - \cos^2 x}{\cos x} dx = -\int \sec x - \cos x \, dx$$

$A = -\{\log(\sec x + \tan x) - \sin x\}$

$A = \sin x - \log(\sec x + \tan x) + k_1$

$$B = \int \frac{y_1 \phi(x)}{W} dx = \int \frac{e^x \cos x \, e^x \tan x}{e^{2x}} dx$$

$\int \sin x \, dx = -\cos x + k_2$

$R_g = (\sin x - \log(\sec x + \tan x) + k_1)\, e^x \cos x$ (18)

$+ (-\cos x)\, e^x \sin x$

$y = e^x (K_1 \cos x + K_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$

(4) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{1}{1+e^{-x}}$ Ex 13.3 pg 490 general

$AE = m^2 - 3m + 2 = 0$

$m = 1, 2$

$y_c = c_1 e^x + c_2 e^{2x}$

$y = A e^x + B e^{2x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$

$\phi(x) = \frac{1}{1+e^{-x}} = \frac{1}{1+\frac{1}{e^x}} = \frac{e^x}{e^x+1}$

$A = \int \frac{-y_2 \phi(x)}{W} dx = -\int e^{2x} \frac{e^x}{1+e^x} \frac{1}{e^{3x}} dx$

$A = -\int \frac{1}{1+e^x} dx \quad e^x = t \quad e^x dx = dt \quad dx = \frac{dt}{t}$

$A = -\int \frac{1}{t+1} \frac{dt}{t}$ partial fraction or direct

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\therefore A = -\int \frac{1}{t} - \frac{1}{t+1}\, dt = \log(t+1) - \log t$$

$$= \log\left(\frac{t+1}{t}\right)$$

$$A = \log\left(\frac{e^{x}+1}{e^{x}}\right) + K_1$$

$$B = \int \frac{y_1\, \phi(x)}{w}\, dx = \int e^{x} \frac{e^{x}}{1+e^{x}} \frac{1}{e^{3x}}\, dx$$

$$B = \int \frac{dx}{e^{x}(e^{x}+1)} \qquad e^{x} = t$$

$$\therefore B = \int \frac{dt}{t^2(t+1)}$$

$$\frac{1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1} \quad \text{partial fractions}$$

$$A = -1 \quad B = 1 \quad C = 1$$

$$\therefore \int \frac{1}{t^2(t+1)} = -\int \frac{dt}{t} + \int \frac{dt}{t^2} + \int \frac{dt}{t+1}$$

$$B = -\log t - \frac{1}{t} + \log(t+1) = \log\left(\frac{t+1}{t}\right) - \frac{1}{t}$$

$$B = \left(\log\left(\frac{e^{x}+1}{e^{x}}\right) - \frac{1}{e^{x}} + K_2\right)$$

y/z

$$y = \underline{K_1 e^{x}} + K_2 e^{2x} + \log(1+e^{-x})(e^{x}+e^{2x}) \underline{- e^{x}}$$ (19)

add

$$y = K_1' e^{x} + K_2 e^{2x} + \log(1+e^{-x})(e^{x}+e^{2x})$$

$K_1' = (K_1 - 1)$

Solve by method of undetermined co-efficients

pg 490 ex 13.3 grewal

1) $(D^2 - 2D)\, y = e^{x} \sin x$

$m^2 - 2m = 0 \qquad m = 0, 2$

$CF = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$

$PI = e^{x}(K_1 \sin x + K_2 \cos x) = y_p \text{ (say)}$

$$y_p' = e^{x}(K_1 \cos x - K_2 \sin x) + e^{x}(K_1 \sin x + K_2 \cos x)$$

$$y_p' = e^{x}((K_1 - K_2)\sin x + (K_1 + K_2)\cos x)$$

$$y_p'' = e^{x}((K_1 - K_2)\cos x - (K_1 + K_2)\sin x) + e^{x}((K_1 - K_2)\sin x + (K_1 + K_2)\cos x)$$

$$y_p'' = e^{x}(K_1 - K_2 + K_1 + K_2)\cos x + (-K_1 - K_2 + K_1 - K_2)\sin x$$

$$y_p'' = e^{x}(2K_1 \cos x - 2K_2 \sin x)$$

$$y_p'' - 2y_p' = e^{x} \sin x$$

$= e^x(2k_1\cos x - 2k_2\sin x) - 2e^x((k_1 - k_2)\sin x$

$+ (k_1 + k_2)\cos x) = e^x \sin x$

$e^x((2k_1 - 2k_1 - 2k_2)\cos x + (-2k_2 - 2k_1 + 2k_2)\sin x)$

$= e^x \sin x$

$-2k_2 e^x \cos x - 2k_1 e^x \sin x = e^x \sin x$

$-2k_2 = 0 \qquad -2k_1 = 1$

$k_2 = 0 \qquad k_1 = -\frac{1}{2}$

$PI = e^x\left(-\frac{1}{2}\sin x + 0 \cdot \cos x\right) = -\frac{1}{2}e^x \sin x$

$\therefore y = c_1 + c_2 e^{2x} - \frac{1}{2}e^x \sin x.$

2) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

DE $\quad m^2 + m - 2 = 0 \qquad m = 1, -2$

$CF = c_1 e^x + c_2 e^{-2x}$

$PI = k_1 x + k_0 + k_2 \sin x + k_3 \cos x = y_p$ (say)

$y_p' = k_1 + k_2 \cos x - k_3 \sin x$

$y_p'' = -k_2 \sin x - k_3 \cos x$

$y_p'' + y_p' - 2y_p = x + \sin x$

$-k_2 \sin x - k_3 \cos x + k_1 + k_2 \cos x - k_3 \sin x$

$-2(k_1 x + k_0 + k_2 \sin x + k_3 \cos x) = x + \sin x$ (20)

$(-3k_2 - k_3)\sin x + (-3k_3 + k_2)\cos x - 2k_1 x$

$+ k_1 - 2k_0 = x + \sin x$

equating the co-efficients

(1) $-3k_2 - k_3 = 1$ $\qquad -2k_1 = 1$

(2) $-3k_3 + k_2 = 0$ $\qquad k_1 = \frac{-1}{2}$

$k_1 - 2k_0 = 0$

$-\frac{1}{2} - 2k_0 = 0$ $\qquad k_0 = \frac{-1}{4}$

$-3k_2 - k_3 = 1$

$k_2 = 3k_3$

$-3(3k_3) - k_3 = 1$ $\qquad -10k_3 = 1$

$k_3 = -\frac{1}{10}$ $\qquad k_2 = \frac{-3}{10}$

$\therefore PI = -\frac{1}{2}x - \frac{1}{4} - \frac{3}{10}\sin x - \frac{1}{10}\cos x$

G.S $\qquad y = c_1 e^{x} + c_2 e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{3}{10}\sin x$

$- \frac{1}{10}\cos x$

3) $y'' + 4y = e^{-x} + x^2$

$m^2 + 4 = 0 \qquad m = \pm 2i$

$y_{c_f} = c_1 \cos 2x + c_2 \sin 2x$

PI let $y_p = k_3 e^{-x} + k_2 x^2 + k_1 x + k_0$

$y_p' = -k_3 e^{-x} + 2k_2 x + k_1$

$y_p'' = k_3 e^{-x} + 2k_2$

$\therefore \quad y_p'' + 4y_p = e^{-x} + x^2$

$k_3 e^{-x} + 2k_2 + 4(k_3 e^{-x} + k_2 x^2 + k_1 x + k_0)$
$= e^{-x} + x^2$

$5k_3 e^{-x} + 4k_2 x^2 + 4k_1 x + 2k_2 + 4k_0 = e^{-x} + x^2$

equating the co-efficients in the equation we get

$y_p'' + 4y_p = e^{-x} + x^2$

$k_3 e^{-x} + 2k_2 + 4(k_3 e^{-x} + k_2 x^2 + k_1 x + k_0)$
$= e^{-x} + x^2$

$5k_3 = 1 \qquad 4k_2 = 1 \qquad k_1 = 0 \qquad 2k_2 + 4k_0 = 0$

$k_3 = \frac{1}{5} \qquad k_2 = \frac{1}{4} \qquad 2\left(\frac{1}{4}\right) + 4k_0 = 0$

$k_0 = -\frac{1}{8}$

$PI = \frac{1}{5} e^{-x} + \frac{1}{4} x^2 - \frac{1}{8}$

G.S $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^{-x} + \frac{1}{4} x^2 - \frac{1}{8}$