Stable Matching Problem

Stable Matching Problem

- The problem itself—the *Stable Matching Problem*—has several origins.
- The Stable Matching Problem originated, in part, in 1962, when David Gale and Lloyd Shapley, two mathematical economists, asked the question: Could one design a college admissions process, or a job recruiting process, that was *self-enforcing*?

Stable Matching Problem

- Gale and Shapley asked: Given a set of preferences among employers and applicants, can we assign applicants to employers so that for every employer *E*, and every applicant *A* who is not scheduled to work for *E*, at least one of the following two things is the case?

(i) *E* prefers every one of its accepted applicants to *A*; or

(ii) *A* prefers her current situation over working for employer *E*.

- If this holds, the outcome is stable
- Gale and Shapley proceeded to develop a striking algorithmic solution to this problem

Stable Matching Problem

Formulating the Problem

- Each applicant is looking for a single company, but each company is looking for many applicants.
- It is useful, at least initially, to eliminate these complications and arrive at a more “bare-bones” version of the problem:
    - Each of *n* applicants applies to each of *n* companies, and each company wants to accept a *single* applicant.

Stable Matching Problem

Formulating the Problem

- Gale and Shapley - observe that this special case can be viewed as the problem of devising a system by which each of *n* men and *n* women can end up getting married: our problem naturally has the analogue of two “genders”—the applicants and the companies
- and in the case we are considering, everyone is seeking to be paired with exactly one individual of the opposite gender.

Stable Matching Problem

So consider a set $M = \{m_1, \ldots, m_n\}$ of n men, and a set $W = \{w_1, \ldots, w_n\}$ of n women. Let $M \times W$ denote the set of all possible ordered pairs of the form (m, w), where $m \in M$ and $w \in W$. A *matching* S is a *set* of ordered pairs, each from $M \times W$, with the property that each member of M and each member of W appears in at most one pair in S. A *perfect matching* S' is a matching with the property that each member of M and each member of W appears in *exactly* one pair in S'.

Stable Matching Problem

Formulating the Problem

- A perfect matching corresponds simply to a way of pairing off the men with the women - there is neither singlehood nor polygamy.

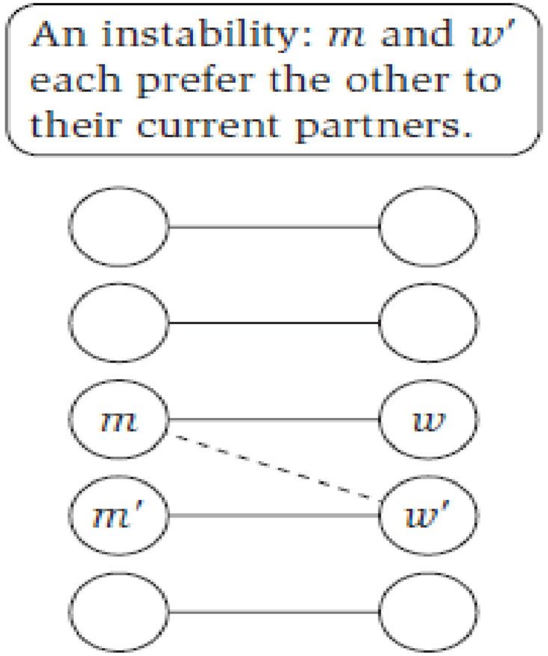

Figure 1.1 Perfect matching S with instability (m, w').

Stable Matching Problem

Our goal, then, is a set of marriages with no instabilities. We'll say that a matching S is *stable* if (i) it is perfect, and (ii) there is no instability with respect to S. Two questions spring immediately to mind:

- Does there exist a stable matching for every set of preference lists?
- Given a set of preference lists, can we efficiently construct a stable matching if there is one?

Stable Matching Problem

First, suppose we have a set of two men, $\{m, m'\}$, and a set of two women, $\{w, w'\}$. The preference lists are as follows:

m prefers w to w'.
m' prefers w to w'.

w prefers m to m'.
w' prefers m to m'.

Stable Matching Problem

agreement: the men agree on the order of the women, and the women agree on the order of the men. There is a unique stable matching here, consisting of the pairs (m, w) and (m', w'). The other perfect matching, consisting of the pairs (m', w) and (m, w'), would not be a stable matching, because the pair (m, w) would form an instability with respect to this matching. (Both m and w would want to leave their respective partners and pair up.)

Stable Matching Problem

the preferences are

m prefers w to w'.
m' prefers w' to w.
w prefers m' to m.
w' prefers m to m'.

In this second example, there are two different stable matchings. The matching consisting of the pairs (m, w) and (m', w') is stable, because both men are as happy as possible, so neither would leave their matched partner. But the matching consisting of the pairs (m', w) and (m, w') is also stable, for the complementary reason that both women are as happy as possible. This is an important point to remember as we go forward—it's possible for an instance to have more than one stable matching.