① A rectangular box open at the top is to have volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

Sol:- Let x, y, z respectively be the length, breadth and height of the rectangular box.

Let v be the volume and its surface area be S.

Volume $= V = xyz = 32$ cubic ft

Total Surface area $=$

$$= 2(xy + yz + zx) - xy$$

$$S = xy + 2yz + 2zx$$

$$F = S + \lambda V$$

$$F = (xy + 2yz + 2zx) + \lambda(xyz)$$

$$F_x = (y + 2z) + \lambda(yz)$$

$$F_y = (x + 2z) + \lambda(xz)$$

$$F_z = (2y + 2x) + \lambda(xy)$$

$$F_x = 0, \quad F_y = 0, \quad F_z = 0$$

②

$f_x = 0$

$$\Rightarrow (y+2z) + \lambda yz = 0$$

$$\lambda = \frac{-(y+2z)}{yz} \checkmark$$

$$F_y = 0 \Rightarrow (x+2z) + \lambda xz = 0$$

$$\lambda = \frac{-(x+2z)}{xz} \checkmark$$

$$F_z = 0 \Rightarrow (2y+2x) + \lambda xy = 0$$

$$\lambda = \frac{-(2y+2x)}{xy} \checkmark$$

$$-\frac{(y+2z)}{yz} = -\frac{(x+2z)}{xz} = -\frac{(2y+2x)}{xy}$$

$$\cancel{-}\frac{(y+2z)}{y\cancel{z}} = \frac{\cancel{-}(x+2z)}{x\cancel{z}}$$

$$\cancel{xy} + 2xz = \cancel{xy} + 2yz$$

$$\cancel{2}x\cancel{z} = \cancel{2}y\cancel{z}$$

$$\boxed{x=y}$$

$$\cancel{-}\frac{(x+2z)}{\cancel{x}z} = \frac{\cancel{-}(2y+2x)}{\cancel{x}y}$$

$$xy + \cancel{2yz} = \cancel{2yz} + 2xz$$

$$\cancel{x}y = 2\cancel{x}z \Rightarrow \boxed{y=2z}$$

$x = y = 2z$

$x = 2z$

$z = x/2$

Volume $= V = xyz = 32$

$x \cdot x \cdot x/2 = 32$

$x^3 = 64 \Rightarrow \boxed{x = 4}$

$y = 4, \quad z = x/2 = 4/2 = 2$

$\underline{x = 4}, \quad y = \underline{4}, \quad \underline{z = 2}$

(2) Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$

Sol :- $F = x^m y^n z^p + \lambda(x + y + z)$

$F_x = m x^{m-1} y^n z^p + \lambda(1)$

$F_y = x^m n \cdot y^{n-1} \cdot z^p + \lambda(1)$

$F_z = p \cdot z^{p-1} x^m y^n + \lambda(1)$

$F_x = 0, \quad F_y = 0 \quad F_z = 0$

$m \cdot x^{m-1} y^n z^p + \lambda = 0 \Rightarrow \lambda = -m x^{m-1} y^n z^p$

$F_y = 0 \quad x^m \cdot n \cdot y^{n-1} z^p + \lambda = 0 \Rightarrow \lambda = -x^m \cdot n y^{n-1} z^p$

$F_z = 0 \quad p z^{p-1} x^m y^n + \lambda = 0 \Rightarrow \lambda = -p z^{p-1} x^m y^n$

$$-m x^{m-1} y^n z^p = -n x^m y^{n-1} z^p = -p x^m y^n z^{p-1}$$

$$\cancel{-} m \underline{x^{m-1}} \underline{y^n} \cancel{z^p} = \cancel{-} n \underline{\underline{x^m}} y^{n-1} \cancel{z^p}$$

$$m y = n \cdot x \implies \boxed{y = \frac{nx}{m}}$$

$$\cancel{-} n \cancel{x^m} \underline{y^{n-1}} \underline{z^p} = \cancel{-} p \cancel{x^m} \underline{y^n} z^{p-1}$$

$$n \cdot z = p y$$

$$\boxed{z = \frac{py}{n}}$$

Sub y in z

$$z = \frac{p\left(\frac{nx}{m}\right)}{n}$$

$$\boxed{z = \frac{px}{m}}$$

Given

$$x + y + z = a$$

$$x + \frac{nx}{m} + \frac{px}{m} = a$$

$$\frac{mx + nx + px}{m} = a$$

$$mx + nx + px = ma$$

$$x(m + n + p) = ma \implies \boxed{x = \frac{ma}{m+n+p}}$$

$$y = \frac{nx}{m}$$

$$y = \frac{n}{\cancel{m}}\left(\frac{a\cancel{m}}{m+n+p}\right)$$

$$\boxed{y = \frac{an}{m+n+p}}$$

$$z = \frac{px}{m}$$

$$z = \frac{p}{\cancel{m}} \cdot \frac{a\cancel{m}}{m+n+p}$$

$$\boxed{z = \frac{ap}{m+n+p}}$$

$$\text{Max value} = x^m y^n z^p$$

$$= \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$= m^m n^n p^p \left(\frac{a^{m+n+p}}{(m+n+p)^{m+n+p}}\right)$$

3. Find the maximum and minimum distances of the point $(1,2,3)$ from the sphere $x^2+y^2+z^2 = 56$

Sol:- let $A(1,2,3)$ be the point

let $P(x,y,z)$ be any point on the sphere.

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$$AP = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$$

$$AP^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$$

$$U = (x-1)^2 + (y-2)^2 + (z-3)^2$$

$$x^2 + y^2 + z^2 = 56$$

$$\phi(x,y,z) =$$

$$F = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda(x^2+y^2+z^2)$$

$$F_x = 2(x-1) + \lambda(2x)$$

$$F_y = 2(y-2) + \lambda(2y)$$

$$F_z = 2(z-3) + \lambda(2z)$$

$$F_x = 0, \quad F_y = 0, \quad F_z = 0$$

$$2(x-1) + \lambda(2x) = 0 \implies (x-1) + \lambda x = 0$$

$$\lambda x = 1 - x$$

$$\lambda = \frac{1-x}{x}$$

$$2(y-2) + \lambda(2y) = 0 \implies y - 2 + \lambda y = 0$$

$$\lambda y = 2 - y$$

$$\lambda = \frac{2-y}{y}$$

$$2(z-3) + \lambda(2z) = 0 \implies z - 3 + \lambda z = 0$$

$$\lambda = \frac{3-z}{z}$$

$$\frac{1-x}{x} = \frac{2-y}{y} = \frac{3-z}{z}$$

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$$\frac{1-x}{x} = \frac{2-y}{y} \Rightarrow \frac{1}{x} - \cancel{1} = \frac{2}{y} - \cancel{1}$$

$$\frac{1}{x} = \frac{2}{y}$$

$$\boxed{y = 2x}$$

$$\frac{1-x}{x} = \frac{3-z}{z} \Rightarrow \frac{1}{x} - \cancel{1} = \frac{3}{z} - \cancel{1}$$

$$\frac{1}{x} = \frac{3}{z} \Rightarrow \boxed{z = 3x}$$

Sub y & z in $x^2 + y^2 + z^2 = 56$

$$x^2 + (2x)^2 + (3x)^2 = 56$$

$$x^2 + 4x^2 + 9x^2 = 56$$

$$14x^2 = 56 \Rightarrow x^2 = 4$$

$$\boxed{x = \pm 2}$$

$$y = 2x \Rightarrow y = \pm 4$$

$$z = 3x \Rightarrow z = \pm 6$$

points

$P_1 (2, 4, 6)$ $\quad$ $P_2 (-2, -4, -6)$

$$A.P_1 = \sqrt{(1-2)^2 + (2-4)^2 + (3-6)^2}$$

$$= \sqrt{1 + 4 + 9}$$

$$AP_1 = \sqrt{14} \text{ units}$$

$$AP_2 = \sqrt{(1-(-2))^2 + (2-(-4))^2 + (3+6)^2}$$

$$= \sqrt{9 + 36 + 81}$$

$$= \sqrt{126}$$

$$= \sqrt{14 \times 9}$$

$$= 3\sqrt{14} \text{ units}$$

Minimum distance $= \sqrt{14}$ units

Maximum distance $= 3\sqrt{14}$ units

(4) Divide 24 into three parts such that the continued product of the first, square of the second and cube of the third may be maximum.

Sol:- Let x, y, z be the numbers

$$x+y+z = 24$$

$$U = x y^2 z^3$$

$$F = U + \lambda \phi$$

$$F = xy^2z^3 + \lambda(x+y+z)$$

$$F_x = y^2z^3 + \lambda(1)$$

$$F_y = 2xyz^3 + \lambda(1)$$

$$F_z = 3xy^2z^2 + \lambda(1)$$

$F_x = 0 \Rightarrow \lambda = -y^2z^3$

$F_y = 0 \Longrightarrow \lambda = -2xyz^3$

$F_z = 0 \Longrightarrow -3xy^2z^2 = \lambda$

$$-y^2z^3 = -2xyz^3 = -3xy^2z^2$$

$$\cancel{-}y^{\cancel{2}}\cancel{z^3} = \cancel{-}2x\cancel{y}\cancel{z^3}$$

$$\boxed{y = 2x}$$

$$\cancel{-}2\cancel{x}\cancel{y}z^{\cancel{3}} = \cancel{-}3\cancel{x}y^{\cancel{2}}\cancel{z^2}$$

$$2z = 3y$$

$$2z = 3\cdot(2x)$$

$$\boxed{z = 3x}$$

Sub y & z in $x+y+z=24$

$$x + 2x + 3x = 24$$

$$6x = 24 \Rightarrow \boxed{x = 4}$$

$$y = 2x \Rightarrow \boxed{y = 8}$$

$$z = 3x \Rightarrow \boxed{z = 12}$$

Numbers are 4, 8, 12

$$U = xy^2z^3$$

$$U = 4(8)^2(12)^3$$

$$U = 442368$$

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Show that the volume of the greatest parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{8abc}{3\sqrt{3}}$

Sol:- $\phi = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

let $2x, 2y, 2z$ be the length, breadth and height

volume $= V = 8xyz$

$$F = \underline{8xyz} + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

$$F_x = 8yz + \lambda\left(\frac{2x}{a^2}\right)$$

$$F_y = 8xz + \lambda\left(\frac{2y}{b^2}\right)$$

$$F_z = 8xy + \lambda\left(\frac{2z}{c^2}\right)$$

$F_x = 0, \quad F_y = 0, \quad F_z = 0$

$$8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \Rightarrow \lambda\left(\frac{2x}{a^2}\right) = -8yz$$

$$= \frac{-8yza^2}{2x}$$

$$\lambda = -4yza^2/x$$

$$8xz + \lambda\left(\frac{2y}{a^2}\right) = 0 \Rightarrow \lambda\left(\frac{2y}{b^2}\right) = -8xz$$

$$\lambda = \frac{-8xzb^2}{2y}$$

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$$8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \implies \lambda\left(\frac{2z}{c^2}\right) = -8xy$$

$$\lambda = \frac{-8xyc^2}{2z}$$

$$\lambda = \frac{-4xyc^2}{z}$$

$$\frac{-4a^2yz}{x} = \frac{-4b^2xz}{y} = \frac{-4xyc^2}{z}$$

$$\frac{\cancel{-4}a^2y\cancel{z}}{x} = \frac{\cancel{-4}b^2x\cancel{z}}{y} \qquad \frac{\cancel{-4}a^2\cancel{y}z}{x} = \frac{\cancel{-4}x\cancel{y}c^2}{z}$$

$$a^2y^2 = b^2x^2 \qquad a^2z^2 = c^2x^2$$

$$\boxed{y^2 = \frac{b^2x^2}{a^2}} \qquad \boxed{z^2 = \frac{c^2x^2}{a^2}}$$

Sub y^2 & z^2 in $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{\cancel{b^2}x^2/a^2}{\cancel{b^2}} + \frac{\cancel{c^2}x^2/a^2}{\cancel{c^2}} = 1$$

$$3\frac{x^2}{a^2} = 1 \implies x^2 = a^2/3$$

$$\boxed{x = a/\sqrt{3}}$$

Sub $x^2 = a^2/3$ in y^2

$$y^2 = \frac{b^2 a^2/3}{a^2}$$

$$y^2 = b^2/3 \implies \boxed{y = b/\sqrt{3}}$$

Sub x^2 in z^2

$$z^2 = \frac{x^2 c^2}{a^2} = \frac{c^2 a^2/3}{a^2}$$

$$z^2 = c^2/3$$

$$\boxed{z = c/\sqrt{3}}$$

$$\text{Volume} = 8xyz$$

$$= 8 \cdot a/\sqrt{3} \cdot b/\sqrt{3} \cdot c/\sqrt{3}$$

$$\text{Volume} = \frac{8abc}{3\sqrt{3}}$$

6. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Sol :- Let $2x, 2y, 2z$ be the length, breadth and height

$$\text{Volume} = V = (2x)(2y)(2z)$$

$$V = 8xyz$$

Sphere eqn is $x^2 + y^2 + z^2 = r^2$

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$f = 8xyz + \lambda(x^2+y^2+z^2-r^2)$

$f_x = 8yz + \lambda(2x)$

$f_y = 8xz + \lambda(2y)$

$f_z = 8xy + \lambda(2z)$

$f_x = 0$, $f_y = 0$, $f_z = 0$

$8yz + \lambda(2x) = 0$

$\lambda 2x = -8yz$

$\lambda = \frac{-4yz}{x}$

$8xz + \lambda(2y) = 0$

$2\lambda y = -8xz$

$\lambda = \frac{-4xz}{y}$

$8xy + 2\lambda z = 0$

$2\lambda z = -8xy$

$\lambda = \frac{-4xy}{z}$

$-\frac{4yz}{x} = -\frac{4xz}{y} = -\frac{4xy}{z}$

$\cancel{-4}\frac{yz}{x} = \cancel{-4}\frac{xz}{y}$

$y^2\cancel{z} = \cancel{z}x^2$

$\boxed{y = x}$ ✓

$\cancel{-4}\frac{xz}{y} = \cancel{-4}\frac{xy}{z}$

$z^2 = y^2$

$\boxed{z = y}$

$x = y = z$

$\therefore$ It is a cube

$x^2+y^2+z^2 = r^2$

$x^2+x^2+x^2 = r^2 \implies 3x^2 = r^2$

$x^2 = r^2/3 \implies \boxed{x = r/\sqrt{3}}$

$y = r/\sqrt{3}$ & $z = r/\sqrt{3}$

Max. volume = $8xyz$

$$= 8 \frac{r}{\sqrt{3}} \cdot \frac{r}{\sqrt{3}} \cdot \frac{r}{\sqrt{3}}$$

$$= \frac{8r^3}{3\sqrt{3}}$$

7) In a plane triangle ABC find the maximum value of $\cos A \cos B \cos C$

Sol:- let $U = \cos A \cos B \cos C$

$$\phi = A+B+C = \pi$$

$$F = U + \lambda \phi$$

$$F = \cos A \cos B \cos C + \lambda (A+B+C-\pi)$$

$$F_A = -\sin A \cos B \cos C + \lambda (1)$$

$$F_B = -\sin B \cos A \cos C + \lambda (1)$$

$$F_C = -\sin C \cos A \cos B + \lambda (1)$$

$$F_A = 0 \Rightarrow \lambda = \sin A \cos B \cos C$$

$$F_B = 0 \Rightarrow \lambda = \cos A \sin B \cos C$$

$$F_C = 0 \Rightarrow \lambda = \cos A \cos B \sin C$$

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$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$

$\sin A \cos B \cancel{\cos C} = \cos A \sin B \cancel{\cos C}$

$\sin A \cos B - \cos A \sin B = 0$

$\sin(A-B) = 0$

$A - B = 0$ (or) $\underline{A - B = \pi}$ $\times$ Not possible

$\Rightarrow$ $\boxed{A = B}$.

$\cancel{\cos A} \sin B \cos C = \cancel{\cos A} \cos B \sin C$

$\sin B \cos C - \cos B \sin C = 0$

$\sin(B-C) = 0$ (or) $B - C = \pi$

$\underline{B - C = \pi}$ $\times$ Not possible

$B - C = 0$

$\boxed{B = C}$

$\underline{A = B = C} = \pi/3$

Maximum value of $\cos A \cos B \cos C =$

$\cos \pi/3 \cdot \cos \pi/3 \cos \pi/3$

$= 1/2 \cdot 1/2 \cdot 1/2$

$= \underline{1/8}$

Solve

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1. Find the extreme values of $x^2+y^2+z^2, = 1$ when $xy+yz+zx = p$

2. Find the minimum value of x^2yz^3 Subject to $2x+y+3z = a$

3. A rectangular box open at the top is to have volume of 108 cubic ft. find the dimension of the box. if its total surface area is minimum.

UNIT – II

D.E
- O.D.E – ordinary D.E
- P.D.E – partial D.E

ODE

Ex 1) $y'' + 2y' = e^x$ – order = 2

2) $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \log x$ 2

P.DE 1) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = 0$

Order of D.E order of D.E is the order of highest derivative occuring in it

Degree of DE The degree of DE is the degree of the highest derivative after simplification $\left(\frac{d^2y}{dx^2}\right)^2 + 2\frac{dy}{dx} = 0$

order = 2, Degree = 2