Taylor's Series for one Variable

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots\cdots$$

This is called Taylor's series expansion of $f(x)$ about the point $x=a$

Maclaurin's Series

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Problems

① Obtain the first four terms of the Taylor's series of $\sin x$ about $x = \pi/4$

Sol:- $f(x) = \sin x$

$a = \pi/4$

$f(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$

$f'(x) = \cos x$

$f'(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$

$f''(x) = -\sin x$

$f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$

$f'''(x) = -\cos x$

$f'''(\pi/4) = -\cos \pi/4 = -1/\sqrt{2}$

$f^{iv}(x) = \sin x$

$f^{iv}(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$

$\vdots$

Taylor's series is

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

(2)

$$\sin x = \frac{1}{\sqrt{2}} + (x-\pi/4)\cdot \frac{1}{\sqrt{2}} + \frac{(x-\pi/4)^2}{2!}\cdot\left(-\frac{1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^4}{4!}\frac{1}{\sqrt{2}} + \cdots$$

$$= \frac{1}{\sqrt{2}} + (x-\pi/4)\frac{1}{\sqrt{2}} - \frac{(x-\pi/4)^2}{2\sqrt{2}} - \frac{(x-\pi/4)^3}{6\sqrt{2}} + \frac{(x-\pi/4)^4}{24\sqrt{2}} + \cdots$$

(2) Expand $\tan^{-1}x$ in powers of $(x-1)$

$a=1$

Sol:- $f(x) = \tan^{-1}x$ $\qquad f(1) = \tan^{-1}(1) = \pi/4$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$ $\qquad f'(1) = \frac{1}{1+1} = 1/2$

$f''(x) = (-1)(1+x^2)^{-2}\cdot(2x)$ $\qquad f''(1) = \frac{-2(1)}{(1+1)^2} = -1/2$

$f'''(x) = -2\left[(1+x^2)^{-2}\cdot 1 + x(-2)\cdot(1+x^2)^{-3}\cdot 2x\right]$

$f'''(1) = -2\left[(1+1)^{-2} + (-2)\cdot 1\cdot 2\cdot (1+1)^{-3}\right]$

$f'''(1) = 1/2$

Taylor's Series is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

$$\tan^{-1}x = \pi/4 + (x-1)\,\tfrac{1}{2} + \frac{(x-1)^2}{2!}\left(-\tfrac{1}{2}\right) + \frac{(x-1)^3}{3!}\cdot\tfrac{1}{2} + \cdots$$

$$\tan^{-1}x = \pi/4 + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + \cdots$$

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(5) Proove that $\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \cdots$ and hence evaluate $\log_e (1.1)$.

Sol:- $f(x) = \log_e x$

$a = 1$

$f(1) = \log_e 1 = 0$

$f'(x) = \frac{1}{x}$

$f'(1) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{x^2} = -x^{-2}$

$f''(1) = \frac{-1}{1} = -1$

$f'''(x) = -(-2)\cdot x^{-3}$

$= \frac{2}{x^3}$

$f'''(1) = \frac{2}{1} = 2$

⋮

Taylor's series is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$$\log x = 0 + (x-1)\cdot 1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}\cdot 2 + \cdots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{6} + \cdots$$

$$\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

put $x = 1.1$

$$\log_e 1.1 = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} + \cdots$$

$$= \underline{0.0953}$$

(4) using Maclaurin's series expand e^x upto four terms.

Sol:- $f(x) = e^x$ $\qquad f(0) = e^0 = 1$

$f'(x) = e^x$ $\qquad f'(0) = e^0 = 1$

$f''(x) = e^x$ $\qquad f''(0) = e^0 = 1$

$f'''(x) = e^x$ $\qquad f'''(0) = e^0 = 1$

Maclaurin's series is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$e^x = 1 + x \cdot 1 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 1 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(5) Obtain the Maclaurin's expansion of $f(x) = e^x \cos x$ upto the term containing x^3. (5)

Sol :- $f(x) = e^x \cos x$

$f(0) = e^0 \cdot \cos 0 = 1$

$f'(x) = e^x(-\sin x) + \cos x \, e^x$

$= e^x(\cos x - \sin x)$

$f'(0) = e^0(\cos 0 - \sin 0)$

$= 1(1-0)$

$= 1$

$f''(x) = e^x(-\sin x - \cos x) + (\cos x - \sin x) e^x$

$= e^x(-\sin x - \cancel{\cos x} + \cancel{\cos x} - \sin x)$

$= e^x(-2\sin x)$

$= -2e^x \sin x$

$f''(0) = e^0(-\sin 0 - \cos 0) + e^0(\cos 0 - \sin 0)$

$= 1(0-1) + 1(1-0)$

$= 0$

$f'''(x) = -2[e^x \cos x + \sin x \cdot e^x]$

$f'''(0) = -2[e^0 \cos 0 + \sin 0 \cdot e^0]$

$= -2(1)$

$= -2$

Maclaurin's series is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$e^x \cos x = 1 + x \cdot 1 + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \cdots$$

$$e^x \cos x = 1 + x \cdot 1 + 0 + \cancel{\frac{x^3}{3!}} \cdot \frac{x^3}{3!}(-2) + \cdots$$

$$e^x \cos x = 1 + x - \frac{x^3}{3} + \cdots$$

⑥ Using Maclaurin's series expand $\sqrt{1+\sin x}$ upto the term containing x^4

Sol:- $f(x) = \sqrt{1+\sin x}$

$$f(x) = \sqrt{\sin^2 x/2 + \cos^2 x/2 + 2 \cdot \sin x/2 \cos x/2}$$

$$f(x) = \sqrt{(\sin x/2 + \cos x/2)^2}$$

$$f(x) = \sin x/2 + \cos x/2$$

$$f(0) = \sin 0 + \cos 0 = 0+1 = 1$$

$$f'(x) = \cos x/2 \cdot 1/2 - \sin x/2 \cdot 1/2 = 1/2 [\cos x/2 - \sin x/2]$$

$$f'(0) = 1/2 [\cos 0 - \sin 0] = 1/2(1-0) = 1/2$$

$$f''(x) = 1/2 [\sin x/2 \cdot \tfrac{1}{2} - \cos x/2 \cdot 1/2] = 1/2 \cdot 1/2 [-\sin x/2 - \cos x/2]$$

$$f''(0) = 1/4 [-\sin 0 - \cos 0] = 1/4(-1) = -1/4$$

$$f'''(x) = \frac{1}{4}\left[-\cos\frac{x}{2}\cdot\frac{1}{2} - \left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2}\right]$$

$$= \frac{1}{8}\left[-\cos\frac{x}{2} + \sin\frac{x}{2}\right]$$

$$f'''(0) = \frac{1}{8}\left[-\cos 0 + \sin 0\right]$$

$$= -\frac{1}{8}$$

$$f^{iv}(x) = \frac{1}{8}\left[-\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2} + \cos\frac{x}{2}\cdot\frac{1}{2}\right]$$

$$= \frac{1}{8}\cdot\frac{1}{2}\left[\sin\frac{x}{2} + \cos\frac{x}{2}\right]$$

$$= \frac{1}{16}\left[\sin\frac{x}{2} + \cos\frac{x}{2}\right]$$

$$f^{iv}(0) = \frac{1}{16}\left[\sin 0 + \cos 0\right]$$

$$= \frac{1}{16}(1)$$

$$= \frac{1}{16}$$

Maclaurin's Series is

$$f(x) = f(0) + x\cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$\sqrt{1+\sin x} = 1 + x\cdot\frac{1}{2} + \frac{x^2}{2!}\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\left(-\frac{1}{8}\right) + \frac{x^4}{4!}\left(\frac{1}{16}\right) + \cdots$$

$$\underline{\sqrt{1+\sin x}} = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \cdots$$

7. Expand $\log(1+\sin x)$ in powers of x upto the term containing x^3

Sol:- $f(x) = \log(1+\sin x)$ $\qquad f(0) = \log(1+\sin 0) = 0$

$$f'(x) = \frac{1}{1+\sin x} \cdot \cos x = \frac{\cos x}{1+\sin x}$$

$$f'(0) = \frac{\cos 0}{1+\sin 0} = \frac{1}{1+0} = 1$$

$$f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$f''(x) = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-\cancel{(1+\sin x)}}{(1+\sin x)^{\cancel{2}}}$$

$$= \frac{-1}{1+\sin x}$$

$$f''(0) = \frac{-1}{1+\sin 0} = \frac{-1}{1} = -1$$

$$f''(x) = \frac{-1}{1+\sin x} = -(1+\sin x)^{-1}$$

$$f'''(x) = (-1)(-1)(1+\sin x)^{-2} \cdot \cos x$$

$$= \frac{\cos x}{(1+\sin x)^2}$$

$$f'''(0) = \frac{\cos 0}{(1+\sin 0)^2} = \frac{1}{1} = 1$$

$\vdots$

Maclaurin's series is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\log(1+\sin x) = 0 + x \cdot 1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \cdots$$

$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

(8) Expand $\tan^{-1}x$ in ascending powers of x upto the first three non zero terms and hence show that $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} + \cdots\right)$

$y = f(x) = \tan^{-1} x$

$y(0) = f(0) = \tan^{-1}(0) = 0$

$y_1 = f'(x) = \frac{1}{1+x^2}$

$y_1(0) = f'(0) = \frac{1}{1+0} = 1$

$y_1 = \frac{1}{1+x^2}$

$(1+x^2) y_1 = 1$

Diff w r t 'x'

$(1+x^2) \cdot y_2 + y_1 \cdot 2x = 0$

$f''(0) =$

$(1+0) y_2(0) + y_1(0) \cdot 2(0) = 0$

$y_2(0) + 0 = 0$

$\boxed{y_2(0) = 0}$

Diff 'w r t 'x'

$(1+x^2) \cdot y_3 + y_2 \cdot 2x + 2(x \cdot y_2 + y_1 \cdot 1) = 0$

at $x = 0$

$f'''(0) =$

$(1+0) y_3(0) + y_2(0) \cdot 2(0) +$

$2(0 \cdot y_2(0) + y_1(0)) = 0$

$y_3(0) + 0 + 2(0+1) = 0$

$\boxed{y_3(0) = -2}$

$(1+x^2) y_3 + 4 x y_2 + 2 y_1 = 0$

Diff w r t 'x'

$(1+x^2) y_4 + y_3 \cdot 2x + 4(x \cdot y_3 + y_2 \cdot 1) + 2 \cdot y_2 = 0$

$(1+x^2)y_4 + \underline{6xy_3} + \underline{6y_2} = 0$

~~[illegible]~~ at $x=0$

$(1+0)y_4(0) + 6[0\cdot y_3(0)] + 6\cdot y_2(0) = 0$

$y_4(0) + 0 + 0 = 0$

$\boxed{y_4(0) = 0}$

$(1+x^2)\cdot y_5 + y_4\cdot 2x + 6[x\cdot y_4 + y_3\cdot 1] + 6\cdot y_3 = 0$

at $x=0$

$(1+0)\ y_5(0) + 2[y_4(0)\cdot 0] + 6[0\cdot y_4(0) + y_3(0)] + 6y_3(0) = 0$

$y_5(0) + 0 + 6(0+(-2)) + 6(-2) = 0$

$y_5(0) - 12 - 12 = 0$

$\boxed{y_5(0) = 24}$

Maclaurin's series is

$f(x) = f(0) + x\cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$\tan^{-1}x = 0 + x\cdot 1 + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(24) + \cdots$

$\tan^{-1}x = 0 + x + 0 + (-2)\frac{x^3}{3!} + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(24) + \cdots$

at

$= x - \frac{2x^3}{6} + \frac{24x^5}{120} + \cdots$

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

Put $x = 1$ on both sides

$\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} + \cdots$

$\pi/4 = 1 - 1/3 + 1/5 + \cdots$

$\pi = 4(1 - 1/3 + 1/5 + \cdots)$

(9) Expand $e^{a\sin^{-1}x}$ in ascending powers of x upto the term containing x^4.

Sol:- $y = e^{a\sin^{-1}x}$ $\qquad y(0) = e^{a\sin^{-1}(0)} = e^0 = 1$

$y_1 = e^{a\sin^{-1}x} \cdot a \cdot \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2} \cdot y_1 = a \cdot e^{a\sin^{-1}x}$

$\sqrt{1-x^2} \cdot y_1 = a \cdot y$ $\rightarrow$ squaring

$(1-x^2)\, y_1^2 = a^2 y^2$

$$(1-x^2)y_1^2 = y^2a^2$$

at $x=0$

$$(1-0)\,y_1^2(0) = (1)\,a^2$$

$$y_1^2(0) = a^2 \quad\Longrightarrow\quad \underline{y_1(0) = a}$$

$$(1-x^2)y_1^2 = y^2a^2$$

Diff w.r.t 'x'

$$(1-x^2)\,2y_1\cdot y_2 + y_1^2(-2x) = \underline{2yy_1a^2}$$

at $x=0$

$$(1-0)\,2\cdot y_1(0)\cdot y_2(0) - 2(0)\,y_1^2(0) = 2\,y(0)\,y_1(0)\,a^2$$

$$2\cdot a\cdot y_2(0) - 0 = 2\cdot 1\cdot a\,a^2$$

$$\cancel{2a}\,y_2(0) = \cancel{2}\,a^{\cancel{3}\,2}$$

$$\boxed{y_2(0) = a^2}$$

$$(1-x^2)\,2y_1y_2 - 2xy_1^2 = 2yy_1a^2$$

$$\cancel{2y_1}\left[(1-x^2)y_2 - x\cdot y_1\right] = \cancel{2y_1}\left[ya^2\right]$$

$$(1-x^2)y_2 - xy_1 = ya^2$$

Diff w.r.t 'x'

$$(1-x^2)y_3 + y_2\cdot(-2x) - (x\cdot y_2 + y_1\cdot 1) = a^2\cdot y_1$$

at $x=0$,

$$(1-0)\,y_3(0) - 2(0)\,y_2(0) - 0\cdot y_2(0) - y_1(0) = a^2\cdot y_1(0)$$

$$y_3(0) - y_1(0) = a^2 y_1(0)$$

$$y_3(0) - a = a^2 \cdot a$$

$$\boxed{y_3(0) = a^3 + a}$$

$$(1-x^2) y_3 - 3x y_2 - y_1 = y_1 a^2$$

Diff w.r.t 'x'

$$(1-x^2) y_4 + y_3(-2x) - 3(x \cdot y_3 + y_2 \cdot 1) - y_2 = y_2 a^2$$

at $x=0$

$$(1-0) y_4(0) - 2(0) y_3(0) - 3(0) y_3(0) - 3 y_2(0) - y_2(0) = \cancel{4} y_2(0) a^2$$

$$y_4(0) - 0 - 0 - 3 \cdot a^2 - \cancel{a}a^2 = a^2 \cdot a^2$$

$$y_4(0) - 4a^2 = a^4$$

$$\boxed{y_4(0) = a^4 + 4a^2}$$

Maclaurin's Series is

$$f(x) = f(0) + (x)f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$e^{a\sin^{-1}x} = 1 + x \cdot a + \frac{x^2}{2!} a^2 + \frac{x^3}{3!} a(1+a^2) + \frac{x^4}{4!}(a^4+4a^2) + \cdots$$

10. Expand $(\sin^{-1}x)^2$ in powers of x using Maclaurin's series upto the term containing x^6.

Sol :- $y = f(x) = (\sin^{-1}x)^2$ $\qquad y(0) = (\sin^{-1}0)^2 = 0$

$$y_1 = 2 \cdot \sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$y_1 \cdot \sqrt{1-x^2} = 2 \cdot \sin^{-1}x$$

Squaring on b.s

$$y_1^2(1-x^2) = 4(\sin^{-1}x)^2$$

$$y_1^2(1-x^2) = 4 \cdot y$$

at $x=0$

$$y_1^2(0)(1-0) = 4\, y(0) \Rightarrow y_1^2(0) = 4 \cdot 0$$

$$\boxed{y_1(0) = 0}$$

$$(1-x^2)y_1^2 = 4y$$

Diff w.r.t x

$$(1-x^2)\, 2y_1y_2 \quad y_1^2(-2x) = 4y_1$$

$$\cancel{2y_1}\left[(1-x^2)y_2 - xy_1\right] = \cancel{4y_1}\, 2$$

$$(1-x^2)y_2 - xy_1 = 2$$

at $x=0$

$$(1-0)\,y_2(0) - 0 = 2$$

$$\boxed{y_2(0) = 2}$$

Continue

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1. Obtain first four terms of the Taylor's series of $\cos x$ about $x = \pi/3$

2. Expand $\sin^{-1} x$ in powers of x upto second degree term.

3. Expand $\log(1+\sin(2x))$ in powers of x upto term containing x^4

4. Using Maclaurin's series expand $\log(\sec x)$ upto the term containing x^6