Numerical solutions of ODEs of 1st order:

In this chapter we consider some of the numerical methods to solve 1st order ODEs. Generally the value of the dependent variable (y) will be given at the initial value of x of the soln of of the ODE $\frac{dy}{dx} = f(x, y)$.

Such a DE is referred to as an Initial Value Problem (IVP).

Here we study first Picard's & Taylor's series methods. These methods give the soln in the form of a series. So they are called series solutions for the DEs.

1. Picard's method of Successive Approximations:-

Consider the DE of 1st order, $\frac{dy}{dx} = f(x, y)$, given $y(x_0) = y_0$ $\longrightarrow (1)$

(1) can be written as $dy = f(x, y)\, dx \Rightarrow y = y_0$ when $x = x_0$

On integrating the LHS between the limits y & y_0 & the RHS between the limits x & x_0 we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \Rightarrow y - y_0 = \int_{x_0}^{x} f(x, y)\, dx.$$

$$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x, y)\, dx.$$

The first approximation y_1 in this method is obtained by taking $y = y_0$ on the RHS and writing the corresponding y on LHS as y_1 $\Rightarrow y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$.

$|||^{ly}$ taking $y = y_1$ on RHS, we obtain y_2 on LHS as $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$

Continuing like this we get $\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx.}$

This is the iterative formula used in the Picard's method to obtain the successive approximations.

Problem 1 :- Using Picard's method, find a solution upto 5th approximation of the d.e $\frac{dy}{dx} = y+x$ ∋ $y(0)=1$. Verify the answer.

Sol :- We've $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = 1 + \int_0^x (1+x)\,dx = 1 + x + \frac{x^2}{2}$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx = 1 + \int_0^x \left(\left(1+x+\frac{x^2}{2}\right) + x\right)dx = 1 + x + x^2 + \frac{x^3}{6}$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx = 1 + \int_0^x \left(\left(1+x+x^2+\frac{x^3}{6}\right) + x\right)dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)\,dx = 1 + \int_0^x \left(\left(1+x+x^2+\frac{x^3}{3}+\frac{x^4}{24}\right)+x\right)dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

$$y_5 = y_0 + \int_{x_0}^{x} f(x, y_4)\,dx = 1 + \int_0^x \left(\left(1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{120}\right)+x\right)dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

This y_5 is the soln upto 5th approximation

Also $\frac{dy}{dx} - y = x$ is the Leibnitz linear in x

$$IF = e^{\int -dx} = e^{-x}$$

So the soln is $y \cdot IF = \int Q \cdot IF\,dx$

$$\Rightarrow y e^{-x} = \int x e^{-x}\,dx \quad \Rightarrow y e^{-x} = x(-e^{-x}) - \int (-e^{-x})\,dx$$

$$\Rightarrow y e^{-x} = -x e^{-x} - e^{-x} + c.$$

$$\Rightarrow y = c e^{x} - x - 1$$

$$\Rightarrow y\big|_{x=0} = c e^{0} - 0 - 1 = c - 1 = 1$$

$$\Rightarrow c - 1 = 1 \quad \Rightarrow \boxed{c = 2}$$

$$\therefore y = 2e^{x} - x - 1$$

(2) Use Picard's method to obtain the third approximations to the solution of $\frac{dy}{dx}+y=e^x$, $y(0)=1$ and hence find $y(0.2)$. Find also the true value of the solution of the given eqn.

Soln:- The formula is $y = 1+\int_0^x (e^x - y)\,dx$

$$\Rightarrow y_1 = e^x - x$$

$$y_2 = 1+\frac{x^2}{2}$$

$$y_3 = e^x - x - \frac{x^3}{6} \text{ is the required soln.}$$

$\Rightarrow y(0.2) = 1.0200694$.

The exact soln is $y \cdot e^x = \int e^x \cdot e^x \, dx + C \Rightarrow y = \frac{e^x}{2} + ce^{-x}$; $C = \frac{1}{2}$

So $y = \cosh x$ so $y\big|_{x=0.2} = 1.0200668$.

(3) Employ Picard's method to find the soln of the DE $\frac{dy}{dx} = x^2+y^2$, given that $y=0$ when $x=0$. Hence find $y(0.1)$ correct to 4 decimal places

Soln: The formula is $y = \int_0^x (x^2+y^2)\,dx \Rightarrow y_1 = \frac{x^3}{3}$; $y_2 = \frac{x^3}{3}+\frac{x^7}{63}$

$\Rightarrow y(0.1) = 0.00033$

(4) Given $\frac{dy}{dx} = xe^y$, $y(0)=0$ determine $y(0.1)$, $y(0.2)$ and $y(1)$ using Picard's method. Compare the soln with exact solution

soln:- $y = e^{\frac{x^2}{2}} - 1$; $y(0.1) = 0.005$; $y(0.2) = 0.0202$

$y(1) = 0.6487$.

(5) Solve $y' = 1+2xy$, $y(0)=0$ by Picard's method

Soln:- $y = x + \frac{2x^3}{3} + \frac{4x^5}{15}$;

(6) Given $\frac{dy}{dx} = x - y$, $y(0) = 1$, find $y(0.1)$, $y(0.2)$ using Picard's method. Soln: $y = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{120}$; 0.83746

Note:- This Picard's method of successive approximations, cannot be applied to find a soln of every 1st order DE. For some of them it is difficult to obtain a series form as the integrand may not be integrable. Therefore some more methods are available to solve this type of DEs. Let us demonstrate this limitation with the following example.

(7) Find the ~~solve~~ value of $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0) = 1$ using Picard's method at $x = 0.1$ & $x = 0.2$

Soln:- By Picard's method, $y_1 = y_0 + \int_{x_0}^{x} \left(\frac{y-x}{y+x}\right) dx$

$$= 1 + \int_0^x \left(\frac{y_0 - x}{y_0 + x}\right) dx$$

$$= 1 + \int_0^x \frac{1-x}{1+x}\, dx.$$

$$= 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right) dx$$

$$= 1 + \left(-x + 2\log(1+x)\right)_0^x$$

$$= 1 + \left(-x + 2\log(1+x)\right)$$

Since further integration is not possible, further approximations are not possible. So we may not get a soln to the required accuracy.

Taylor's Series Method:-

To solve a first order differential equation $\frac{dy}{dx} = f(x, y)$ by this method, we consider the Taylor's series expansion of $y(x)$ w.r.t the point (x_0). ie in powers of $(x - x_0)$.

ie $y(x) = y(x_0) + (x - x_0)\frac{y'(x_0)}{1!} + (x - x_0)^2 \cdot \frac{y''(x_0)}{2!} + (x - x_0)^3 \frac{y'''(x_0)}{3!} + \cdots$

Problems:- Use Taylor's method to find approximate value of $y(1 \cdot 1)$ and $y(1 \cdot 3)$ for the DE $y' = xy^{1/3}$, $y(1) = 1$. Compare the numerical solution obtained with exact solution.

Soln:- Given $f(x, y) = xy^{1/3} = y'$; $y'|_{x_0} = 1$

$y'' = x \cdot \frac{1}{3} y^{-2/3} \cdot y' + y^{1/3} = \frac{x^2}{3} y^{-1/3} + y^{1/3}$; $y''|_{x_0} = \frac{4}{3}$

$y''' = \frac{x^2}{3} \cdot \left(\frac{-1}{3} y^{-4/3}\right) \cdot y' + \frac{2x}{3} \cdot y^{-1/3} + \frac{x}{3} y^{-2/3} y' =$

$= -\frac{x^2}{9} y^{-4/3} + \frac{2x}{3} y^{-1/3} + \frac{x}{3} y^{2/3}$; $y'''|_{x_0} = \frac{-1}{9} + \frac{2}{3} + \frac{1}{3} = \frac{8}{9}$

Now substituting in Taylor's series, we get

$y(x) = y(x_0) + (x - x_0) \frac{y'(x_0)}{1!} + (x - x_0)^2 \frac{y''(x_0)}{2!} + (x - x_0)^3 \frac{y'''(x_0)}{3!} + \cdots$

taking $x = 1 \cdot 1$

$= 1 + (0 \cdot 1)(1) + (0 \cdot 1)^2 \cdot \frac{(4/3)}{2!} + (0 \cdot 1)^3 \frac{(8/9)}{6} + \cdots$

$= 1 + 0 \cdot 1 + 0 \cdot 0066 + 0 \cdot 000148 = 1 \cdot 1067$.

Illy $y(1 \cdot 2) = y(x_1) + (x_2 - x_1) \frac{y'(x_1)}{1!} + (x_2 - x_1)^2 \frac{y''(x_1)}{2!} + (x_2 - x_1)^3 \, y'''(x_1)/3! + \cdots$

$\Rightarrow \quad y_1' = x_1 y_1^{1/3} = (1.1)(1.1067)^{1/3} = 1.13782$

$y_1'' = \frac{1}{3} x_1^2 y_1^{-1/3} + y_1^{1/3} = \frac{1}{3}(1.1)^2 (1.1067)^{-1/3} + (1.1067)^{1/3}$

$= 1.4243$

$y_1''' = 0.9297.$

On substitution in Taylor series, we get

$y_2 = 1.1067 + 0.113782 + 0.00712 + 0.00015495$

$= 1.2278.$

|||ly $y_3 = y(1.3)$ is given by $1.3639.$

The analytical soln is; $\frac{dy}{y^{1/3}} = x\,dx$ on separating variables

$\frac{3}{2} y^{2/3} = \frac{x^2}{2} + c$

$\Rightarrow \frac{3}{2}(1)^{2/3} = \frac{1}{2} + c \quad \Rightarrow \quad \frac{3}{2} = \frac{1}{2} + c \quad \Rightarrow \quad c = 1.$

$\therefore$ The soln is $\frac{3}{2} y^{2/3} = \frac{x^2}{2} + 1$

or $y^{2/3} = \frac{x^2+2}{3}$

So $y(1.1) = \left[\frac{(1.1)^2+2}{3}\right]^{\frac{3}{2}} \Rightarrow y(1.1) = 1.1068$

$y(1.2) = \left[\frac{(1.2)^2+2}{3}\right]^{3/2} = \frac{1.1467}{((x^2+1)/3)}$; $= 1.2278$

& $y(1.3) = \left[\frac{(1.3)^2+2}{3}\right]^{3/2} = \frac{1.23}{1(23)}$; $= 1.364$

(2) Use Taylor's Series method to find the approximate Value of y when $x = 0.1$ given $y(0)=1$ and $y' = 3x+y^2$

<u>Soln</u>:-

$y' = 3x + y^2 \Rightarrow y_0' = 3x_0 + y_0^2 = 1$

$\Rightarrow y'' = 3 + 2yy' \Rightarrow y_0'' = 3 + 2y_0 y_0' = 5$

$\Rightarrow y''' = 2yy'' + 2(y')^2 \Rightarrow y_0''' = 2y_0 y_0'' + 2(y_0')^2 = 12$

$\Rightarrow y^{iv} = 2yy''' + 2y'y'' + 4y'y'' \Rightarrow y_0^{iv} = 54$

Given $x_0 = 0$; $y_0 = 1$; $h = 0.1$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$= 1 + (0.1)1 + \frac{(0.1)^2}{2} \cdot 5 + \frac{(0.1)^3}{6} \cdot 12 + \frac{(0.1)^4}{24} \cdot 54 + \cdots$$

$$= 1.127.$$

(3) Find by Taylor's series method the Value of y at $x = 0.1$ to five places of decimal form $y' = x^2y - 1$, $y(0)=1$

<u>Soln</u>:- Here $x_0 = 0$; $y_0 = 1$; $h = 0.1$.

$y' = x^2 y - 1 \Rightarrow y_0' = x_0^2 y_0 - 1 = -1$

$y'' = 2xy + x^2 y' \Rightarrow y_0'' = 2x_0 y_0 + x_0^2 y_0' = 0$

$y''' = 2xy' + 2y + x^2 y'' + 2xy' \Rightarrow y_0''' = 2$

$y^{iv} = 2xy'' + 2y' + 2y' + x^2 y''' + 2xy'' + 2xy'' + 2y' = -6$

$$y_1 = y(0.1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$$

$$= -1 + (0.1)(-1) + \frac{(0.1)^2}{2} \cdot 0 + \frac{(0.1)^3}{6} \cdot 2 + \frac{(0.1)^4}{24} \cdot (-6) = 0.9003$$

$$= 0.9003.$$

(4) Solve $\frac{dy}{dx} = xy + 1$ and $y(0) = 1$ using Taylor's Series method and Compute $y(0.1)$

Soln :- $x_0 = 0$; $y_0 = 1$; $h = 0.1$

$y' = xy + 1$; $y_0' = 1$

$y'' = \cancel{x}y' + y$; $y_0'' = 1$

$y''' = xy'' + y' + y'$; $y_0''' = 2$

$y^{iv} = xy''' + y'' + 2y''$; $y_0^{iv} = 3$

$y(0.1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$

$= 1.1053$.

(5) Solve the following first order differential equations using Taylor's Series method

i) $y' = x^2y - 1$, $y(0) = 1$. Compute $y(0.3)$ Soln: 0.97

ii) $y' = y - x^2$, $y(0) = 1$ in $0 \le x \le 0.2$ upto 3rd approximation

Soln: $1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{60}$;

$y(0.1) = 1.1018$

iii) $y' = x + y^2 + 1$, $y(0) = 0$, obtain the series approximations upto the fifth degree terms. Soln :- $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{20}$

iv) Solve $y' = x - y^2$, $y(0) = 1$ using Taylor's Series method and Compute $y(0.1)$ and $y(0.2)$

Soln: $y = y_0 + \frac{h}{1!} \cdot y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$

$y(0.1) = 0.91381$; $y(0.2) = 0.8512$

Modified Euler's method

This method is an enhancement of Euler's method. In Modified Euler's method, we take the average of the slopes at (x_0, y_0) & $(x_1, y_1^{(0)})$, ie two points where as in the Euler's method, the slope is considered at only one point.

The formula of Euler's method is $y_{n+1} = y_n + h f(x_n, y_n)$

Iterative formula for the Euler's Modified method is

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$$

Problem 1:- Using modified Euler's method find $y(0.2)$ and $y(0.4)$ given $y' = y + e^x$, $y(0) = 0$.

Soln:- $x_0 = 0$; $y_0 = 0$; $h = 0.2$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 0 + 0.2(y_0 + e^{x_0})$

$$y_1^{(0)} = 0.2(0 + e^0) = 0.2$$

Now $x_1 = 0.2$ and $f(x_1, y_1^{(0)}) = f(0.2, 0.2) = 0.2 + e^{0.2} = 1.4214$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 0 + 0.1\left[0 + e^0 + 0.2 + e^{0.2}\right]$$

$$= 0.24214$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 0 + 0.1\left[0 + e^0 + 0.24214 + e^{0.2}\right]$$

$$= 0.1\left[0 + 1 + 0.24214 + 1.2214\right]$$

$$= 0.2463$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right] = 0 + \frac{0.2}{2}\left[1 + 0.2463 + e^{0.2}\right]$$

$$= 0.2468$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 0 + \frac{0.2}{2}\left[1 + 0.2463 + e^{0.2}\right] = 0.2468.$$

Problem 2:- Given $\frac{dy}{dx} = -xy^2$, $y(0) = 2$. Compute $y(0.2)$ in steps of 0.1 using modified Euler's method.

Soln:- $x_0 = 0$; $y_0 = 2$; $h = 0.2$

By Euler's formula, $y_1 = y_0 + h f(x_0, y_0)$

$$= 2 + 0.1\left[-x_0 y_0^2\right] = 2$$

Take it as $y_1^{(0)}$;

So $$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right] = 2 + \frac{0.1}{2}\left[(-x_0 y_0^2) + (-x_1 y_1^2)\right]$$

$$= 1.98$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right] = 2 + \frac{0.1}{2}\left[(-x_0 y_0^2) + (-x_1 y_1^{(1)^2})\right]$$

$$= 1.9804$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right] = 2 + \frac{0.1}{2}\left[-x_0 y_0^2 + (-x_1 y_1^{(2)^2})\right]$$

$$= 1.9804.$$

$\because y_1^{(2)} = y_1^{(3)} = 1.9804$. $y_1 = 1.9804$

$x_1 = 0.1$; $y_1 = 1.9804$; $x_2 = 0.2$ & $h = 0.1$;

$$y_2^{(0)} = y_1 + h f(x_1, y_1) = 1.9804 + (0.1)\left(-(0.1)(1.9804)^2\right) = 1.94118$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right] = 1.9804 + \frac{0.1}{2}\left[-0.3922 + (-0.2)(1.94118)^2\right]$$

$= 1.9231$

$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] = 1.9804 + \frac{0.1}{2}\left[-0.3922 + (-0.2)(1.9231)^2\right]$

$= 1.9804 + 0.05\left[-0.3922 + (-0.2)(3.6983)\right] = 1.9238$

$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] = 1.9238$

$\because y_2^{(2)} = y_2^{(3)} = 1.9238$

$\therefore y_2 = y(0.2) = 1.9238$

Prob 3:- Given $y' = x + \sin y$, $y(0) = 1$, compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Modified Euler's method.

Soln:- $x_0 = 0$; $y_0 = 1$; $h = 0.2$; $f(x, y) = x + \sin y$

By Euler's formula, $y_1 = y_0 + h f(x_0, y_0)$

$= 1 + 0.2\left[x_0 + \sin y_0\right]$

$= 1 + 0.2\left[0 + \sin 1\right] = 1.163$

we take it as $y_1^{(0)} = 1.163$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1 + \frac{0.2}{2}\left[\sin 1 + 1.12\right]$

$= 1.1961$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1 + \frac{0.2}{2}\left[\sin 1 + 1.1961\right]$

$= 1.2038$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 1 + \frac{0.2}{2}\left[\sin 1 + 1.2038\right]$

$= 1.20452$

$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] = 1 + \frac{0.2}{2}\left[\sin 1 + 1.20452\right]$

$= 1.2046$

Prob 4:- Solve the DE $\frac{dy}{dx} = 2+\sqrt{xy}$, $y(1)=1$, by Modified Euler's method and obtain y at $x=2$ in steps of 0.2.

Soln:- 5.051

Prob 5:- Solve numerically $y' = y+e^x$, $y(0)=0$ for $x=0.2, 0.4$ by Modified Euler's method

Soln:- 0.24214, 0.59116

Prob 6:- Using Modified Euler's method, obtain $y(0.25)$ given $y' = 2xy$, $y(0)=1$

Soln:- 1.0625

Prob 7:- Given that $\frac{dy}{dx} = x^2+y^2$, $y(0)=1$, determine $y(0.1)$ and $y(0.2)$ using Modified Euler's method.

Soln:- 1.17266, 1.25066

Prob 8:- If $\frac{dy}{dx} = x+\sqrt{y}$, use Modified Euler's method to approximate y when $x=0.6$ in steps of 0.2 given $y(0)=1$

Soln:- 1.8861

Prob 9:- Using Modified Euler's method, find an approximate value of y when $x=0.3$ given $y'=x+y$, $y(0)=1$

Soln:- 1.4004

Prob 10:- Solve $y' = x^2+y$ using Modified Euler's method to approximate y when $x=0.02, 0.04$ and 0.06 with $h=0.02$

Soln:- 1.0202, 1.0408, 1.0619

Runge-Kutta Methods :- (4th order Classical method)

In this method, we solve the differential equations

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ by $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Where $k_1 = h\,f(x_0, y_0)$; $k_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$; $k_4 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{3})$

Problem 1: Apply fourth order RK method, to find an approximate value of y when $x = 0.2$ in steps of 0.1 given that $y' + y = 0$, $y(0) = 1$.

Soln:- Here $x_0 = 0$; $y_0 = 0$; $h = 0.1$; $y' = f(x,y) = -y$

So $y_1 = y(0.1) = y_0 + k$ where $k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

& $k_1 = h\,f(x_0, y_0) = (0.1)(-y_0) = -0.1$.

$k_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1[f(0.05, 0.95)]$

$= 0.1[-0.95] = -0.095$

$k_3 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1[f(0.05, 0.9525)]$.

$= 0.1(-0.9525) = -0.09525$

$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.1[f(0.05, 0.90475)]$

$= (0.1)(-0.90475) = -0.090475$

So $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.9048$

Now $x_1 = x_0 + h = 0.1$; $y_1 = 0.9048$; $h = 0.1$

$y_2 = y(0.2) = y_1 + k$ where $k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

& $k = h\,f(x_1, y_1) = 0.1[-y_1] = 0.1(-0.9048) = -0.09048$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1\left(-\left(y_1 + \frac{k_1}{2}\right)\right) = 0.1(-0.85959)$

$= -0.085959$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1\left(-\left(y_1 + \frac{k_2}{2}\right)\right) = 0.1(-0.86185)$

$= -0.086185$

$k_4 = h f(x_1 + h, y_1 + k_3) = 0.1\left(-(y_1 + k_3)\right) = 0.1(-0.81865)$

$= -0.081865$

So $y_2 = y_1 + k = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.81873.$

$\therefore y(0.2) = 0.81873.$

Problem 2:- Apply 4th order RK Method to find $y(0.1)$ & $y(0.2)$ given that $y' = xy + y^2$, $y(0) = 1$;

Soln:- Here $x_0 = 0$; $y_0 = 1$; $h = 0.1$; $f(x, y) = y' = xy + y^2$

$y_1 = y(0.1) = y_0 + k$ where $k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k_1 = h f(x_0, y_0) = 0.1(x_0 y_0 + y_0^2) = (0.1)(0 + 1) = 0.1$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1155$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1122$

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.1 f(x_0 + h, y_0 + k_3) = 0.1248$

$y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + 0.1133 = 1.1133$

$y(0.2) = y_1 + k$ where $k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k_1 = h f(x_1, y_1) = 0.1[x_1 y_1 + y_1^2] = 0.1351$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1\left[(x_1 + \frac{h}{2})(y_1 + \frac{k_1}{2}) + (y_1 + \frac{k_1}{2})^2\right] = 0.1571$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1599$

$k_4 = h f(x_1 + h, y_1 + k_3) = 0.1876$

$\therefore\; y_2 = y(0\cdot2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1\cdot1133 + k$

$= 1\cdot1133 + \frac{1}{6}(0\cdot1351 + 0\cdot3142 + 0\cdot3198 + 0\cdot1876)$

$\therefore y_2 = y(0\cdot2) = 1\cdot2728$

Prob 3 :- Solve $y' = x - y$ given that $y(1) = 0\cdot4$. Find $y(1\cdot2)$ using 4th order RK Method.

Soln :- Here $x_0 = 1$; $y_0 = 0\cdot4$; $x_1 = 1\cdot1$; $x_2 = 1\cdot2$; $h = 0\cdot1$

$y_1 = y(1\cdot1) = y_0 + k = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = 0\cdot06$; $k_2 = 0\cdot062$; $k_3 = 0\cdot0619$; $k_4 = 0\cdot06381$

$y_1 = y_0 + k = 0\cdot4 + \frac{1}{6}(0\cdot06 + 2(0\cdot062) + 2(0\cdot0619) + 0\cdot06381)$

$= 0\cdot4619$

$y_2 = y_1 + k = 0\cdot4619 + \frac{1}{6}(0\cdot70191 + 2(0\cdot03371) + 2(0\cdot67124) + 0\cdot06686)$

$= 0\cdot825$

Prob 4 :- Solve $\frac{dy}{dx} = xy$ using 4th order RK method for $x = 0\cdot2$ given $y(0) = 1$, taking $h = 0\cdot2$

Soln :- $x_0 = 0$; $y_0 = 1$; $h = 0\cdot2$; $y' = f(x,y) = xy$ $\quad x_1 = 0\cdot2$

$y_1 = y_0 + k = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0 + 2(0\cdot02) + 2(0\cdot0202) + 0\cdot0408)$

$= 1\cdot0202.$

Prob 5 :- Using 4th order RK Method to solve $y' = \frac{y-x}{y+x}$ at $x = 0\cdot2$, taking $h = 0\cdot2$ with $y(0) = 1$.

Soln :- Here $x_0 = 0$, $y_0 = 1$; $h = 0\cdot2$; $x_1 = 0\cdot2$; $y' = f(x,y) = \frac{y-x}{y+x}$

$y_1 = y(0\cdot2) = y_0 + k = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0\cdot2 + (0\cdot1666 + 0\cdot16619)2 + 0\cdot0707) = 1\cdot15607.$

Predictor- Corrector Methods :-

The earlier methods are single step methods. ie to find y_{n+1} we use only y_n. But in these Predictor - Corrector methods, to y_{n+1} we need not just y_n but also some of the earlier values of y like y_{n-1}, y_{n-2}, y_{n-3} etc., ∴ These methods are called multi-step methods. In this chapter, we discuss two such methods called Milne's method & Adams-Bashforth method. They both have two formulae called Predictor & corrector.

First we shall discuss Milne's method :-

In this method, the predictor is given by

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$ and the Corrector is given by

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

which can be generally written as

$$\boxed{\begin{aligned} y_{n+1}^{(P)} &= y_{n-3} + \frac{4h}{3}\left[2y_{n-2}' - y_{n-1}' + 2y_n'\right] \\ y_{n+1}^{(c)} &= y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right] \end{aligned}}$$

Problem :- Use Milne's method to find $y(0.8)$ and $y(1.0)$ from $y' = 1+y^2$; $y(0) = 0$. Find the initial values $y(0.2)$, $y(0.4)$ & $y(0.6)$ from RK Method

Soln :- Here $x_0 = 0$; $y_0 = 0$; $h = 0.2$; $x_1 = 0.2$; $y' = 1+y^2$

$$y_1 = y_0 + k = y_0 + \frac{1}{6}\left(0.2 + 0.404 + 0.40408 + 0.20816\right)$$

$= 0.2027;$

So $y(0.2) = 0.2027$

$x_1 = 0.2;\ y_1 = 0.2027;\ h = 0.2$

$y(0.4) = y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.2027 + \frac{1}{6}\left(0.2082 + 2(0.2188) + 2(0.2195) + 0.2356\right]$

$= 0.4228$

$x_2 = 0.4;\ y_2 = 0.4228;\ h = 0.2$

$y(0.6) = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_3 = y(0.6) = 0.4228 + \frac{1}{6}(1.5678) = 0.6841$

So using Predictor, $y_4^p = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$= 0 + \frac{4}{3}(0.2)\left[2(1.0411) - 1.1787 + 2(1.4681)\right] = 1.0239$

$\because\ y_1' = 1 + y_1^2 = 1 + (0.2027)^2 = 1.0411$

$y_2' = 1 + y_2^2 = 1 + (0.4228)^2 = 1.1787$

$y_3' = 1 + y_3^2 = 1 + (0.6841)^2 = 1.4681$

Now $y_4' = 1 + y_4^2 = 1 + (1.0239)^2 = 2.0484$

The corrector $y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4'^{p})$

$= 0.4228 + \frac{0.2}{3}(1.1787 + 4(1.4681) + 2.0484)$

$\Rightarrow y(0.8) = 0.4228 + 0.6066 = 1.0294$

<u>To find $y(1.0)$</u>: The Predictor $y_5^p = y_1 + \frac{4h}{3}(2y_2' - y_3' + 2y_4')$

$y_4' = 1 + y_4^2 = 1 + (1.0294)^2 = 2.05966$

$y_5 = 0.2027 + \frac{4}{3}(0.2)\left[2(1.787) - 1.4681 + 2(2.05966)\right]$

$= 0.2027 + 1.3356 = 1.5383$

$y(1.0) = 1.5383.$

The corrector $y_5^c = y_3 + \frac{h}{3}(y_3' + 4y_4' + y_5')$

where $y_5' = 1 + y_5^2 = 1 + (1.5383)^2 = 3.3664$

$y_5 = y(1.0) = 0.6841 + \frac{0.2}{3}[1.4681 + 4(2.05966) + 3.3664]$

$= 0.6841 + 0.87154 = 1.5556.$

Prob2:- Find the soln of $\frac{dy}{dx} = x - y$ at $x = 0.4$, $y(0) = 1$ using Milne's method. Use Euler's modified method to evaluate $y(0.1)$, $y(0.2)$ and $y(0.3)$.

Soln:- Here $x_0 = 0$, $y_0 = 1$; $h = 0.1$

By Euler's Modified method $y_1 = y(0.1) = 0.995$

$y_2 = y(0.2) = 0.8371$

$y_3 = y(0.3) = 0.7812.$

Using y_0, y_1, y_2 and y_3, we've $y_1' = x_1 - y_1 = -0.8095$

$y_2' = x_2 - y_2 = -0.6371$

$y_3' = x_3 - y_3 = -0.1812$

By Milne's Predictor, $y_4^p = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$= 1 + \frac{4(0.1)}{3}[-1.619 + 0.6371 - (0.1812)2]$

$= 1 - 0.15508 = 0.84492$

$y_4' = y_4'^p = x_4 - y_4 = 0.4 - 0.84492 = -0.44492$

$y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4'^p) = 0.8371 + \frac{0.1}{3}(-0.6371 - 0.7248 - 0.44492$

$= 0.8371 - 0.06023 = 0.7769.$

$\Rightarrow y(0.4) = y_4 = 0.7769$

Prob3 :- Use Milne's method to find $y(0.3)$ from $y' = x^2 + y^2$ $y(0) = 1$. Find the initial values $y(-0.1)$, $y(0.1)$ & $y(0.2)$ from Taylor's series method.

Soln :- Here $x_0 = 0$; $y_0 = 1$; $h = 0.1$; $f(x,y) = x^2 + y^2$;

$y(-0.1) = 0.9087$; $y(0.1) = 1.1113$; $y(0.2) = 1.2506$.

$y_0' = 1$; $y_1' = 1.2449$; $y_2' = 1.6040$;

The Predictor $y_3^p = y_{-1} + \frac{4h}{3}\left[2y_0' - y_1' + 2y_2'\right]$

$$= 0.9087 + \frac{0.4}{3}\left(2 - 1.2449 + 3.2080\right) = 1.4371$$

$y_3' = (0.3)^2 + (1.4371)^2 = 2.1552$

The corrector $y_3^c = y_1 + \frac{h}{3}\left[y_1' + 4y_2' + y_3'\right]$

$$= 1.1113 + \frac{0.1}{3}\left[1.2449 + 6.4160 + 2.1552\right]$$

$$= 1.4385;$$

$\therefore y_3 = y(0.3) = 1.4385.$

Prob4 :- Use Milne's method to solve $y' = \frac{2y}{x}$ with $y(1) = 2$ compute $y(2)$ by Milne's method. Find the starting values using Runge-Kutta method taking $h = 0.25$

Soln :- 8.00

Prob5 :- Use Milne's Predictor - Corrector method to find the soln of $y' + \frac{y}{x} = \frac{1}{x^2}$ at 1.4 given $y(1) = 1$; $y(1.1) = 0.996$; $y(1.2) = 0.986$; $y(1.3) = 0.972$.

Soln :- 0.949.

Adams-Bashforth Method:-

In this method the Predictor is $y_4^{(P)} = y_3 + \frac{h}{24}\left(55y_3' - 59y_2' + 37y_1' - 9y_0'\right)$

and the corrector is $y_4^{(C)} = y_3 + \frac{h}{24}\left(9y_4' + 19y_3' - 5y_2' + y_1'\right)$.

y_1, y_2, y_3 ..... are obtained using any of the earlier methods like Picard's, Taylor's, Euler's, Modified Euler's or RK Methods and then evaluate the corresponding derivatives y_1', y_2', y_3' and use them to find the predictor y_4^P and using the predictor, we find the corrector.

Problem 1:- Apply Adams-Bashforth method and find y at $x = 4.4$ given $5xy' + y^2 - 2 = 0$ & $y = 1$ at $x = 4$ initially by generating other values using Taylor's series expansion.

Soln.: Using the Taylor's Series expansion, we find

$y(4.1) = 1.0049$ $\quad y(4.3) = 1.0142$.

$y(4.2) = 1.0097$ $\quad$ Here $x_0 = 4$; $y_0 = 1$; $h = 0.1$.

The predictor for y_4, $y_4^P = y_3 + \frac{h}{24}\left(55y_3' - 59y_2' + 37y_1' - 9y_0'\right)$

Here $y_0' = \frac{2-1^2}{5.4} = 0.05$; $y_1' = \frac{2-(1.0049)^2}{5\cdot(4.1)} = 0.0483$

$y_2' = \frac{2-(1.0097)^2}{5\cdot(4.2)} = 0.0467$; $y_3' = \frac{2-(1.0142)^2}{5\cdot(4.3)} = 0.0452$

So $y_4^P = 1.0142 + \frac{0.1}{24}\left[55(0.0452) - 59(0.0467) + 37(0.0483) - 9(0.05)\right]$

$= 1.0187$

The corrector is $y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$.

$= 1.0142 + \frac{0.1}{24}\left[9(0.0437) + 19(0.0452) - 5(0.0467) + 0.0485\right]$

$= 1.0186$.

$$\boxed{\text{Here } y_4' = \frac{2-(0.0187)^2}{5\cdot(4\cdot4)} = 0.0437.}$$

$\therefore\ y_4 = y(4.4) = 1.0186$.

Prob2:- Solve $y' + y + xy^2 = 0$ with $y_0 = 1;\ y_1 = 0.9008;\ y_2 = 0.8066$
$y_3 = 0.722$ w.r.t $x_0 = 0;\ x_1 = 0.1;\ x_2 = 0.2;\ x_3 = 0.3$ respectively.
find y when $x = 0.4$ using Adams-Bashforth method

Soln:- Here $x_0 = 0 \quad x_1 = 0.1 \quad x_2 = 0.2 \quad x_3 = 0.3$

$y_0 = 1 \quad y_1 = 0.9008 \quad y_2 = 0.8066 \quad y_3 = 0.722$.

Now the Predictor $y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$

Here $y_0' = -(y_0 + x_0 y_0^2) = -1$

$y_1' = -(y_1 + x_1 y_1^2) = -0.9819$

$y_2' = -(y_2 + x_2 y_2^2) = -0.9367$

$y_3' = -(y_3 + x_3 y_3^2) = -0.8784$.

$\therefore\ y_4^P = 0.722 + \frac{0.1}{24}\left[55(-0.8784) - 59(-0.9367) + 37(-0.9819) - 9(-1)\right]$

$y_4^{(P)} = 0.6371;$

$\Rightarrow\ y_4' = -(y_4 + x_4 {y_4^{p}}^2) = -0.7995$

Now the corrector $y_4^{(c)}$ is $y_4^c = y_3 + \frac{h}{24}\left(9y_4' + 19y_3' - 5y_2' + y_1'\right)$

$$= 0.722 + \frac{0.1}{24}\left[9(-0.7995) + 19(-0.8784) - 5(-0.9367) - 9(-1)\right]$$

$$y_4^{(c)} = 0.6379$$

W.r.t this $y_4^{(c)}$, $y_4' = -\left[0.6379 + (0.4)(0.6379)^2\right] = -0.8007$

Using this again in the predictor we get

$$y_4^{(c)} = 0.722 + \frac{0.1}{24}\left[9(-0.8007) + 19(-0.8784) - 5(-0.9367) - 9(-1)\right]$$

$$= 0.6379$$

$$\Rightarrow y_4 = y(0.4) = 0.6379.$$

Problem 3: Using Adams Bashforth method find $y(1.4)$ given $y' = (2x^2 + y)/2$, given $y(1) = 2$. Find $y(1.1)$, $y(1.2)$, $y(1.3)$ using Taylor's series expansion of order 4.

Soln: 3.0793

Problem 4: Using Adams Bashforth method find $y(1.4)$ given $y' + (y/x) = 1/x^2$, given $y(1) = 1$; $y(1.1) = 0.996$; $y(1.2) = 0.986$; $y(1.3) = 0.972$.

Soln: 0.949.

Problem 5: Use Taylor series expansion of order 4 and find $y(0.1)$, $y(0.2)$, & $y(0.3)$ and then solve $y' + y = x^2$; $y(0) = 1$ using Adams-Bashforth method & Milne's method

Soln: 0.6897.