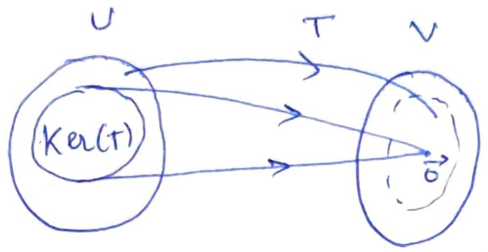
Kernel & range of linear Transformation

Let $T: U \to V$ be a linear transformation.

Then,

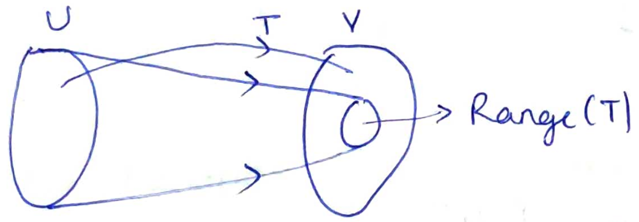
* The set of vectors in U that are mapped into the zero vector of V is called the kernel of the linear transformation 'T' is denoted by ker (T)



Note:

Ker (T) ⊂ U & is called Null space of linear transformation. T

* The dimension kernal - T is denoted by dim (ker (T)) is called the Nullity of linear transformation T.



* The set of vectors in V that are the images of all vectors in U is called the Range of linear transformation 'T' & is denoted by <u>Range (T)</u>

Note:

Range (T) ⊂ V and the dimension of Range (T), denoted by dim (Range (T)) = Rank of its matrix of the linear transformation T.

Rank - Nullity theorem

Let $T: U \to V$ be a linear transformation, then dimension of domain (T) = dim (Ker (T)) + dim (Range (T))

$$\dim(\text{domain}(T)) = \dim(\ker(T)) + \dim(\text{Range}(T))$$

$$\dim(\text{domain}(T)) = \dim(\text{Ker}(T)) + \dim(\text{Range}(T))$$

$$\dim(U) = \text{Nullity} + \text{Rank}$$

Problems :-

1) determine the kernel, Range of L.T for the following

(a) $T: R^2 \to R^3$ where $T(x, y) = (3x, x-y, y)$

Also verify Rank nullity theorem

$\text{Ker}(T) = ?$, Range (T) = ?

$\dim(U) = ?$

Nullity = ? Rank = ?

$R^2 \xrightarrow{2\times3} R^3$

$A \to 3\times2$

Standard basis of $R^3 = \{(1,0)(0,1)\}$

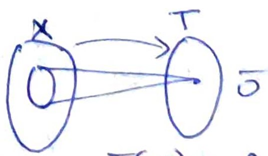
$T(1,0) = (3, 1, 0)$

$T(0,1) = (0, -1, 1)$

$$\cancel{A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \qquad A = \begin{bmatrix} 3 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}_{3\times2}$$

$\dim(\text{dom}(T)) = \dim(U)$ = no of colums in A = 2

$$A^T = \begin{bmatrix} 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$



$T(X) = AX$

in echelon form

$$= \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$\rho(A^T) = 2$

$\rho(A^T) = \rho(A) = 2$

$\dim(\text{range}(T))$ = rank = 2

$$\text{Range} = \text{span}\{(1, \tfrac{1}{3}, 0) + (0, 1, -1)\} \quad \text{i.e span}(A^T)$$

$$= \{c_1, \tfrac{c_1}{3} + c_2, -c_2\}$$

$$\text{Ker}(T) = TX = A\vec{X} = \vec{0}$$

fig (a)

$$\begin{bmatrix} 3 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x = 0$

$\boxed{x = 0}$

$x - y = 0$

$\boxed{y = 0}$

$$\text{Ker}(T) = \{0, 0\}$$

$\dim(\text{Ker}(T))$ = no. of free parametres $= 0$

Nullity $= 0$

Rank Nullity thm :

$$\dim(V) = \text{Rank} + \text{nullity}$$

$$2 = 2 + 0$$

$$2 = 2$$

(b) $T: R^3 \to R^2$ where $T(x, y, z) = (x+y, z)$ and verify Rank Nullity theorem.

Standard Basis of $R^3 = \{(1,0,0), (0,1,0)\ (0,0,1)\}$

$$T(1,0,0) = (1,0)$$

$$T(0,1,0) = (1,0)$$

$$T(0,0,1) = (0,1)$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2\times 3}$$

$\dim(U) =$ no of columns $= 3$

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

reduced to echloen form

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3 \quad R_3 \rightarrow \quad \text{and then} \quad R_3 \rightarrow R_3 - R_1$$

$\rho(A^T) = 2$

$\rho(A) = \rho(A^T) = 2$

Range $= \text{span}\{(c_1(1,0) + c_2(0,1) + c_3(0,0)\}$

$= \{c_1, c_2\}$

$\text{Ker}(T)$: $TX = AX = \vec{0}_U \; \vec{0}_V$

$$\underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$\left.\begin{array}{l} x + y = 0 \\ z = 0 \end{array}\right\}$ 2 independent equations

Free parameter $=$ no of unknow $-$ no of indipendent eq

$= 3 - 2 = 1$

$z = 0$, $x = k$, ~~$y = k$~~ $y = -k$

$\text{Ker}(T) = \{k, -k, 0\}$

$\dim(\text{Ker}(T)) =$ Nullity $= 1$

Rank - nullity the

$\dim(U) =$ Nulity $+$ Rank

$= 1 + 2$

$3 = 3$

2) Determine the kernel & Range of Transformation T defined by each of following matrices & hence verify Rank Nullity thm.

(a)
$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 1 & -1 & 8 & 7 \\ 2 & -4 & 6 & 10 \end{bmatrix}_{3\times 4}$$

$$A = \begin{bmatrix} 1 & -2 & 3 & 5 \\ 1 & -1 & 8 & 7 \\ 2 & -4 & 6 & 10 \end{bmatrix}$$

$\dim(U) =$ No of col $= 4$

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -4 \\ 3 & 8 & 6 \\ 5 & 7 & 10 \end{bmatrix}$$

$$\cancel{A^T}\, A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 5R_1 \end{matrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = \rho(A^T) = 2$$

$$\text{Range} = \text{span}(A^T) = \{c_1(1,1,2) + c_2(0,1,0) + c_3(0,0,0) + c_4(0,0,0)\}$$

$$\text{Range} = \{c_1,\ c_1 + c_2,\ 2c_1\}$$

ker (T) : $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 1 & -1 & 8 & 7 \\ 2 & -4 & 6 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - 2y + 3z + 5w = 0$

Same $x - y + 8z + 7w = 0$

$2x - 4y + 6z + 10w = 0$

no of independent equ = 2

no of unknown = 4

Free parameters = $4 - 2 = 2$

Nullity = 2

R - N Thm

dim(U) = Rank + Nullity

$4 = 2 + 2$

$4 = 4$

$$[A : B] = \left[\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 1 & -1 & 8 & 7 & 0 \\ 2 & -4 & 6 & 10 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 1 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 \qquad \rho(A:B) = 2$

infinitely many solutions

$x - 2y + 3z + 5w = 0$

$y + 5z + 2w = 0$

$z = k_1 \quad , \quad w = k_2 \qquad y = -(5k_1 + 2k_2)$

$$x + 10k_1 + 4k_2 + 3k_1 + 5k_2 = 0$$

$$x = -13k_1 - 9k_2$$

$$\ker(T) = \cancel{\{(-13} \{-(13k_1 + 9k_2), -(5k_1 + 2k_2), k_1, k_2\}$$

Fourier Series

Infinite Series

If $\{U_n\}$ is a sequence of real numbers then the expression $U_1 + U_2 + U_3 + \dots + U_n$ is called infinite series.

The Infinite series is denoted by $\cancel{\Sigma U_n} =$

$$\boxed{\Sigma U_n = U_1 + U_2 + U_3 + \dots + U_n}$$

Series of positive terms

A series ΣU_n is said to be a series of positive terms if $U_n > 0 \ \forall \ n \in N$.

Alternating series

A series ΣU_n in which the terms are alternate positive & negative is called Alt series.

$$\Sigma U_n = U_1 - U_2 + U_3 - U_4 + \dots$$

If $\Sigma U_n = U_1 + U_2 + \dots + U_n + \dots$ is a infinite series

then $S_1 = U_1 \rightarrow$ [1st partial sum of ΣU]

$S_2 = \cancel{U_2}^{U_1+U_2}$ [2nd partial sum of ΣU]

~~$S_n = U_n \rightarrow$ [~~

$S_n = U_1 + U_2 + \dots + U_n$ [nth partial sum of ΣU]

Nature of Series

Let ΣU_n be a series and $\{S_n\}$ be the corresponding sequence of partial sums.

(1) series ΣU_n is convergent

if sequence $\{S_n\}$ is convergent

$$\boxed{\lim_{n\to\infty} \{S_n\} = l \quad \text{is finite \& unique}}$$

Then we say that $\Sigma U_n \rightarrow$ converges to l

(2) The series ΣU_n is divergent if $\{S_n\}$ is divergent

ie $$\boxed{\lim_{n\to\infty} \{S_n\} = +\infty \text{ or } -\infty}$$

(3) (a) The series ΣU_n is said to oscillate finitely if the sequence $\{S_n\}$ oscillate finitely

ie $\lim_{n\to\infty} S_n$ is finite but not unique

(b) The series ΣU_n is said to oscillate infinitely if the sequence $\{S_n\}$ oscillates infinitely.

$$\boxed{\lim_{n\to\infty} S_n = +\infty \text{ or } -\infty}$$

---

(1) eg: consider the series $1^3 + 2^3 + \dots + n^3 + \dots$

$$S_n = 1^3 + 2^3 + \dots n^3$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4} = \infty$$

$\Sigma U_n \rightarrow$ diverges since $\lim_{n\to\infty} \{S_n\} = \infty$

$$\Sigma S_n = 1^3 + 2^3 + \cdots n^3$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4}$$

(2) $\Sigma(-1)^n$

$$= -1+1-1+1-1+1-\cdots$$

$$S_n = (-1)^n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (-1)^n = 0 \text{ or } -1$$

$\therefore$ ΣU_n oscillate finitely

Note:

If a series ΣU_n is convergent then $\boxed{\lim_{n\to\infty} U_n = 0}$

geometric Series

In series, $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^{n-1}$

1) The convergent if $-1 < x < 1$, let $|x| < 1$

2) divergent if $x \geq 1$

3) osicillatory if $x < -1$

P- series or Harmonic Series

The Series $\Sigma \frac{1}{n^P} = \frac{1}{1^P} + \frac{1}{2^P} + \cdots$ is called p series or Harmonic series.

1) If P series is convergent if $P > 1$

(2) and divergent if $P \leq 1$