UNIT-1

PR

JOSEPH FOURIER (1768-1830)

FOURIER SERIES :

Fourier series is an infinite series representation of periodic functions in terms of the trigonometric sine and cosine functions. It is very useful in the study of the heat conduction, mechanics, concentrations of chemicals and pollutants, electrostatics, accounts.

Taylor's Series and Maclaurin's series are the infinite series in ascending powers of $(x-a)$ and x which are valid only for continuous and differentiable functions.

Fourier series is possible NOT only for continuous functions but for periodic functions, functions discontinuous in their values and derivatives. Because of the periodic nature, Fourier series constructed for one period is valid for all values. Fourier series is very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as non-homogeneous terms.

PERIODIC FUNCTIONS:

A function $f(x)$ is said to be periodic of period T.
If $f(x+T) = f(x)$, for all real values x and $T>0$.

Ex: $\sin x$ and $\cos x$ are periodic functions of period 2π.

ie; $\sin(x+2\pi) = \sin x$

$\cos(x+2\pi) = \cos x$

$\tan x$ and $\cot x$ are periodic functions of period π.

ie, $\tan(x+\pi) = \tan x$

$\cot(x+\pi) = \cot x$.

NOTE:

1. The frequency of a periodic function in a given interval is the quotient of the length of the interval and the period of the function.
(The number of times the function repeats itself in the given interval).

2. A constant function is periodic for any period T.

3. A linear combination of periodic functions having period T is also periodic of period T.

Ex: If $h(x) = a\cos x + b\sin x$ then

$$h(x+2\pi) = a\cos(x+2\pi) + b\sin(x+2\pi)$$
$$= a\cos x + b\sin x = h(x).$$

DIRICHLETS CONDITIONS :

Suppose $f(x)$ is a real valued function which obeys the following conditions.

⟨i⟩ $f(x)$ is defined in an interval $(a, a+2l)$ and periodic function with period $2l$.

⟨ii⟩ $f(x)$ is continuous OR has only a finite number of discontinuities in the interval $(a, a+2l)$

⟨iii⟩ $f(x)$ has at the most a finite number of maxima OR minima in the interval $(a, a+2l)$.

Then $f(x)$ can be expanded as the infinite series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x$$

The series on the RHS is called the FOURIER series of $f(x)$ in the interval $(a, a+2l)$.

Further,

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)\,dx \longrightarrow (1)$$

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx \longrightarrow (2)$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx \longrightarrow (3)$$

$$n = 1, 2, \ldots \infty$$

are called the Fourier coefficients

The formulas (1), (2), (3) are called EULER's FORMULAE.

NOTE:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

3. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

4. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{If } f(-x) = f(x) \to \text{Even} \\ 0, & \text{If } f(-x) = -f(x) \to \text{Odd} \end{cases}$

5. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{If } f(2a-x) = f(x) \\ 0, & \text{If } f(2a-x) = -f(x) \end{cases}$

6. $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$.

NOTE:

$\sin(2n\pi+\theta) = \sin\theta$

$\cos(2n\pi+\theta) = \cos\theta$.

For $m \neq n$,

i) $\int_a^{a+2\pi} \cos mx \sin nx\,dx = 0$

ii) $\int_a^{a+2\pi} \sin mx \sin nx\,dx = 0 = \int_a^{a+2\pi} \cos mx \cdot \cos nx\,dx$.

iii) $\int_a^{a+2\pi} \cos mx\,dx = 0$

iv) $\int_a^{a+2\pi} \sin mx\,dx = 0$

For $m = n$,

i) $\int_a^{a+2\pi} \cos mx \cdot \cos nx\,dx = \int_a^{a+2\pi} \cos^2 mx\,dx = \pi$.

ii) $\int_a^{a+2\pi} \sin mx \cdot \sin nx\,dx = \int_a^{a+2\pi} \sin^2 mx\,dx = \pi$.

SPECIAL CASES:

Case (i) Suppose $a=0$, then $f(x)$ is defined over the interval $(0, 2l)$ and

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$n = 1, 2, 3, \ldots \infty$

Case (ii) Suppose $a=0$, $l=\pi$, then $f(x)$ is defined over the interval $(0, 2\pi)$ and

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$n = 1, 2, 3, \ldots \infty$

Case (iii) Suppose $a=-l$, then $f(x)$ is defined over the interval $(-l, l)$ and

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$n = 1, 2, 3, \ldots \infty$

Case (iv) Suppose $a=-\pi$, $l=+\pi$, then $f(x)$ is defined over the interval $(-\pi, \pi)$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$n = 1, 2, 3, \ldots \infty.$

NOTE:

(i) If $f(x)$ is an even function, then $f(x)\cos(n\pi/l)x$ is an even function and $f(x)\sin(n\pi/l)x$ is an odd function.

In this case, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x.dx$$

$$b_n = 0. \qquad , n = 1, 2, 3, \ldots\ldots \infty$$

(Hence if a periodic function $f(x)$ is even; its Fourier expansion contains only cosine terms and it is true in the case of $l = \pi$ as well.)

(ii) If $f(x)$ is an odd function, then $f(x)\cos\left(\frac{n\pi}{l}\right)x$ is an odd function and $f(x)\sin\left(\frac{n\pi}{l}\right)x$ is an even function.

In this case,

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = 0.$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$n = 1, 2, 3, \ldots \infty$$

(Hence, if a periodic function $f(x)$ is odd, its fourier expansion contains only sine terms and it is true in the case of $l=\pi$ as well.)

(iii) $\sin n\pi = 0$, $\quad \cos n\pi = (-1)^n = \begin{cases} 1 & \text{If } n \text{ is even} \\ -1 & \text{If } n \text{ is odd.} \end{cases}$

(iv) If $f(x)$ is discontinuous at $x=a$, then $f(x)$ can be expressed as $f(x) = \frac{1}{2}\left[f(a^+) + f(a^-)\right]$.

where, $f(a^+) = \lim_{h\to 0} f(a+h)$

and $f(a^-) = \lim_{h\to 0} f(a-h)$

Problems

[I] Obtain the Fourier expansion of the function $f(x) = x$ over the interval $(-\pi, \pi)$. Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. Given $f(x) = x$.

Sol: Here $\cancel{f(x)}$ $-f(-x) = -x \Rightarrow \cancel{f = f(x)}$. $f(x) = -f(x)$

Thus $f(x)$ is an odd function over the interval $(-\pi, \pi)$.

$\therefore \boxed{a_0 = 0}, \boxed{a_n = 0}$.

The fourier expansion of $f(x)$ is given by, $\cancel{f(x)}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \longrightarrow ①$$

when, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$.

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n} + 0\right] = -\frac{2}{n}(-1)^n$$

$$b_n = \frac{2}{n}(-1)^{n+1} \longrightarrow ②$$

Substituting ② in ①, we get,

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \longrightarrow ③$$ which is the required Fourier expansion.

Taking $x = \frac{\pi}{2}$ we get,

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right) = 1 - \frac{1}{2}(0) + \frac{1}{3}(-1) - \frac{1}{4}(0) + \frac{1}{5}(+1) \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

As the series has a period 2π, it represents the discontinuous functions called the SAW-TOOTHED wave form. It's important to note that the given function $y = x$ is continuous ~~fo~~ but the function represented by the series ③ has finite discontinuities at $x = \pm\pi, \pm 3\pi, \pm 5\pi$, etc.

2: Find the Fourier series expansion of $f(x) = x^2$ over the interval $(-\pi, \pi)$ and Hence dedace that

(i) $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (ii) $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Soln: Given $f(x) = x^2$, $\boxed{l = \pi}$

$\Rightarrow f(-x) = (-x)^2 = x^2 = f(x)$

Thus $f(x)$ is an even function over the interval $(-\pi, \pi)$

$\therefore \boxed{b_n = 0}$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- } (1)$$

Where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3\pi}\left[\pi^3 - 0\right]$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underset{u}{x^2}\, \underset{v}{\cos nx}\, dx$$

$$= \frac{2}{\pi} \left[x^2 \cancelto{0}{\left(\frac{\sin nx}{n}\right)} - (2x)\left(-\frac{\cos nx}{n^2}\right) + \cancelto{0}{(2)\left(-\frac{\sin nx}{n^3}\right)} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\left(x \cos nx\right)\right]_0^{\pi} = \frac{4}{\pi n^2}\left[\pi \cos n\pi - 0\right]$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}$$

Sustituting a_0 and a_n in $\boxed{1}$, we get

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \longrightarrow \boxed{2}$$

This is required Fourier Expansion.

(i) Put $x = \pi$ in $\boxed{2}$ we get

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \left[\because (-1)^{2n} = 1\right]$$

— o —

(ii) Put $x = 0$ in $\boxed{2}$, we get

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (1)$$

$$\Rightarrow \frac{-\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

— o —

3. Find the Fourier Series Expansion of $f(x) = |x|$ over the interval $(-\pi, \pi)$ and Hence deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Sol: Given $f(x) = |x|$, $\boxed{l = \pi}$

$\Rightarrow f(-x) = |-x| = |x| = f(x)$

Thus $f(x)$ is an Even function over the interval $(-\pi, \pi)$

$$\therefore \boxed{b_n = 0}$$

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} \quad \boxed{1}$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\pi^2 - 0\right]$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x| \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\cancel{\frac{\sin nx}{n}}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - \cos 0\right]$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

Substituting a_0 & a_n in III, we get

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos nx \quad — \boxed{2}$$

This is the required Fourier series.

Put $x = \pi$ in $\boxed{2}$, we get.

$$\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos n\pi$$

$$\Rightarrow \frac{\pi}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}(-1)^n$$

$$\Rightarrow \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{2n} - (-1)^n}{n^2}$$

$$\Rightarrow \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{\left[1 - (-1)^n\right]}{n^2}$$

$$= \frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

4. Find the Fourier series for $f(x) = 1 - x^2$ in the interval $-1 < x < 1$

Sol: Given $f(x) = 1 - x^2$

$\Rightarrow f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$

Thus $f(x)$ is even function over the interval $(-1, 1)$

$\therefore \boxed{b_n = 0}$ & $l = 1$

Thus, The Fourier series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right) x \quad \text{—} \ \boxed{I}$$

Where
$$a_0 = \frac{1}{l} \int_{-1}^{1} f(x)\, dx$$

$$= 2 \int_0^1 (1 - x^2)\, dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left[1 - \frac{1}{3}\right]$$

$$\boxed{a_0 = \frac{4}{3}}$$

$$a_n = \frac{1}{l} \int_{-1}^{1} f(x) \cos n\pi x\, dx$$

$$= \frac{2}{1} \int_0^1 (1 - x^2) \cos n\pi x\, dx$$

$$= 2\left[(1 - x^2)\cancel{\left(\frac{\sin n\pi x}{n\pi}\right)} - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\cancel{\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)}\right]_0^1$$

$$= \frac{-4}{\pi^2 n^2}\Big[x\cos n\pi x\Big]_0^1$$

$$= \frac{-4}{\pi^2 n^2}\Big[\cos n\pi - 0\Big] = \frac{-4}{\pi^2 n^2}(-1)^n$$

$$a_n = \frac{4}{\pi^2 n^2}(-1)^{n+1}$$

Substituting a_0 and a_n in II we get

$$1-x^2 = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi x.$$

This is the required Fourier Series.

5. If $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$ show that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right]$$

in the interval $(0, 2)$

Sol: Given $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$

Here $f(x)$ is defined over the interval $(0, 2l)$ where $\boxed{l = 1}$

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{— [1]}$$

where

$$a_0 = \frac{1}{l}\int_0^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{\pi}{2}(1-0) + \pi\left[\left(4-\frac{4}{2}\right) - \left(2-\frac{1}{2}\right)\right]$$

$$= \frac{\pi}{2} + \pi\left[\cancel{2} - \cancel{2} + \frac{1}{2}\right] = \frac{\pi}{2} + \frac{\pi}{2}$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{l}\int_0^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left[x\cancel{\left(\frac{\sin n\pi x}{n\pi}\right)} - (1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \pi\left[(2-x)\cancel{\left(\frac{\sin n\pi x}{n\pi}\right)} - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \pi\left[\frac{1}{n^2\pi^2}(\cos n\pi - \cos 0)\right] + \pi\left[-\frac{1}{n^2\pi^2}(\cos 2n\pi - \cos n\pi)\right]$$

$$= \cancel{\frac{1}{n^2\pi}(-1)^n} - \frac{1}{n^2\pi} - \frac{1}{n^2\pi}(-1)^{2n} + \cancel{\frac{1}{n^2\pi}(-1)^n}$$

$$= \frac{2(-1)^n}{n^2\pi} - \frac{2}{n^2\pi}$$

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l}\int_0^2 f(x) \sin n\pi x \, dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx.$$

$$= \pi\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \pi\left[(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \pi\left[\frac{-1}{n\pi}(\cos n\pi - 0)\right] + \pi\left[\frac{-1}{n\pi}\left(0 - \cos n\pi\right)\right]$$

$$= -\frac{1}{n}(-1)^n \cancel{+\frac{1}{n}} + \frac{1}{n}(-1)^n$$

$$\boxed{b_n = 0}$$

Substituting a_0, a_n, b_n in III we get

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos n\pi x$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[-\frac{2}{1^2}\cos\pi x + 0 - \frac{2}{3^2}\cos 3\pi x + 0 - \frac{2}{5^2}\cos 5\pi x \cdots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} \cdots\right]$$

6. obtain the Fourier Series expansion of the function $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & \text{in } 0 < x < \pi \end{cases}$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Sol: Given $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & \text{in } 0 < x < \pi \end{cases}$

Here $f(-x) = 1 - \frac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

and $f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

Thus $f(x)$ is an Even function in the interval $(-\pi, \pi)$

$$\therefore \boxed{b_n = 0}$$

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——— II}$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left[x - \frac{x^2}{\pi} \right]_0^{\pi}$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(\frac{-2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-2}{\pi n^2}\left(\cos n\pi - \cos 0\right)\right]$$

$$a_n = \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

Substituting a_0 and a_n in III, we get

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos nx.$$

Put $x=0$, we get

$$1 = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}(1).$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}+\cdots$$

$$\therefore \boxed{\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots}$$

7. Find the Fourier series expansion of $f(x) = x + x^2$ over the interval $-\pi \le x \le \pi$. Hence deduce the following

(i) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - + \cdots$

(ii) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots$

(iii) $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

Sol: Given $f(x) = x + x^2$

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- [1]}$$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\cancel{(x+x^2)\left(\frac{\sin nx}{n}\right)}^{0} - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + \cancel{(2)\left(\frac{-\sin nx}{n^3}\right)}^{0}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\Big[(1+2x)\cos nx\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\left[(1+2\pi)(-1)^n - (1-2\pi)(-1)^n\right] \qquad \begin{aligned}&\cos(-n\pi)\\&= \cos n\pi\end{aligned}$$

$$= \frac{1}{\pi n^2}\left[\cancel{(-1)^n} + 2\pi(-1)^n - \cancel{(-1)^n} + 2\pi(-1)^n\right]$$

$$= \frac{1}{\pi n^2}\left[4\pi(-1)^n\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - \cancel{\left(\frac{-\sin nx}{n^2}\right)(1+2x)}^{0} + (2)\left(\frac{+\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(x+x^2)\frac{\cos nx}{n} + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(x+x^2)\frac{\cos nx}{n} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{(\pi+\pi^2)\cdot\frac{(-1)^n}{n} - (-\pi+\pi^2)\frac{(-1)^n}{n}\right\} + 2\cancel{\left\{\frac{(-1)^n}{n^3} - \frac{(-1)^n}{n^3}\right\}}^{0}\right]$$

$$\cancel{= -\frac{1}{\pi}\Big[}$$

$$= \frac{(-1)^n}{\pi n}\left[-\left\{\pi+\pi^2 - (-\pi+\pi^2)\right\}\right]$$

$$= \frac{(-1)^n}{\pi n}\left[-\left\{\pi + \cancel{\pi^2} + \pi - \cancel{\pi^2}\right\}\right]$$

$$= -\frac{2(-1)^n}{n} = \frac{2}{n}(-1)^{n+1}$$

$$\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

Substituting the values of a_0, a_n & b_n in III, we get

$$x+x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \qquad \text{—— } \boxed{2}$$

8. Expand $f(x) = x \sin x$ as a Fourier series in the interval $(-\pi, \pi)$, Hence deduce that

(i) $\frac{\pi}{2} = 1 + \frac{2}{1\cdot3} - \frac{2}{3\cdot5} + \frac{2}{5\cdot7} \cdots$ (ii) $\frac{\pi-2}{4} = \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \cdots$

Sol: Given $f(x) = x \sin x$

$$\Rightarrow f(-x) = -x \cdot \sin(-x) = x \sin x = f(x)$$

Thus $f(x)$ is an Even function over the Interval $(-\pi, \pi)$

$$\therefore \quad \boxed{b_n = 0}$$

The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} \boxed{1}$$

Where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin x \, dx$$

$$= \frac{2}{\pi} \left[x(-\cos x) - (-\cancel{\sin x}) \right]_0^{\pi}$$

$$= -\frac{2}{\pi} \left[\pi \cos \pi - 0 \right] = -2 \cos \pi = -2(-1)$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \left[\frac{1}{2} \left\{ \sin(n+1)x - \sin(n-1)x \right\} \right] dx$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \left[\sin(n+1)x - \sin(n-1)x \right] dx$$

For $n=1$, $a_1 = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$

$$= \frac{1}{\pi} \left[x\left(\frac{-\cos 2x}{2}\right) - (1)\cancel{\left(\frac{-\sin 2x}{4}\right)}^{\,0} \right]_0^{\pi}$$

$$= \frac{-1}{2\pi} \left[\pi \cos 2\pi - 0 \right] = -\frac{1}{2}(1) = -\frac{1}{2}$$

$$\boxed{a_1 = -\frac{1}{2}}$$

For $n \geq 2$, $a_n = \frac{1}{\pi} \left[x \left\{ \frac{-\cos(n+1)x}{n+1} - \frac{-\cos(n-1)x}{(n-1)} \right\} \right.$

$$\left. - (1) \left\{ \cancel{\frac{-\sin(n+1)x}{(n+1)^2}}^{\,0} - \cancel{\left(\frac{-\sin(n-1)x}{(n-1)^2} \right)}^{\,0} \right\} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{x \cos(n+1)x}{(n+1)} + \frac{x \cos(n-1)x}{(n-1)} \right]_0^{\pi}$$

$-:(-1)^{1}$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}\left\{\pi \cos(n+1)\pi - 0\right\} + \frac{1}{n-1}\left\{\pi \cos(n-1)\pi - 0\right\}\right]$$

$$= -\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}$$

$$= \frac{1}{n+1}(-1)^{n+2} + \frac{1}{n-1}(-1)^{n-1}$$

$$= \frac{1}{n+1}(-1)^{n}\cdot(-1)^{2} + \frac{1}{n-1}(-1)^{n}\cdot(-1)^{-1}$$

$(-1)^{-1} = \frac{1}{(-1)^{1}} = \frac{1}{-1} = -1$

$$= (-1)^{n}\left[\frac{1}{n+1} + \frac{1}{n-1}(-1)\right]$$

$$= (-1)^{n}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= (-1)^{n}\left[\frac{n-1-(n+1)}{n^{2}-1^{2}}\right] = \frac{(-1)^{n}\cdot(-2)}{n^{2}-1}$$

$$\boxed{a_n = \frac{2(-1)^{n+1}}{(n+1)(n-1)}}, \quad n \geq 2$$

substituting a_0 & a_n in III we get

$$x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^{2}-1}\cos nx \quad \text{---} \boxed{2}$$

$$\Rightarrow x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{(n-1)(n+1)}\cos nx \quad \text{---} \boxed{2}$$

(i) Put $x = \pi/2$ in $\boxed{2}$, we get

$$\frac{\pi}{2}\times 1 = 1 - 0 + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{(n-1)(n+1)}\cos\left(\frac{n\pi}{2}\right)$$

$\rightarrow$

$$\frac{\pi}{2} = 1 + 2\left[\frac{-1}{1\cdot 3}\cancel{\cos\pi}^{-1} + \frac{1}{2\cdot 4}\cancel{\cos\left(\frac{3\pi}{2}\right)}^{0} + \frac{-1}{3\cdot 5}\cancel{\cos 2\pi}^{+1} + \frac{1}{4\cdot 6}\cancel{\cos\left(\frac{5\pi}{2}\right)}^{0} + \frac{-1}{5\cdot 7}\cancel{\cos 3\pi}^{-1}\cdots\right]$$

$$\boxed{\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots}$$

(ii) Consider $\frac{\pi - 2}{4} = \frac{1}{2}\left(\frac{\pi-2}{2}\right)$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 1\right]$$

$$= \frac{1}{2}\left[\frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots\right] \text{ from (i)}$$

$$\therefore \boxed{\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots}$$

9. If $f(x) = \begin{cases} 0 & \text{for } -\pi < x \le 0 \\ \sin x & \text{for } 0 < x < \pi \end{cases}$

Prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2mx}{4m^2 - 1}$.

Deduce the following

(a) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$

(b) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{\pi - 2}{4}$

Sol: Given $f(x) = \begin{cases} 0 & \text{for } -\pi < x \le 0 \\ \sin x & \text{for } 0 < x < \pi \end{cases}$

The fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- [1]}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \sin x\,dx\right]$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = -\frac{1}{\pi}\left[\cos\pi - \cos 0\right]$$

$$= -\frac{1}{\pi}\left[-1 - 1\right] = \frac{2}{\pi}$$

$$\boxed{a_0 = \frac{2}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\int_0^{\pi}\left\{\sin(n+1)x - \sin(n-1)x\right\}\right]dx$$

$$a_n = \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

For

$\underline{n=1,}$

$$a_1 = \frac{1}{2\pi}\left[\int_0^{\pi}\sin 2x\, dx\right]$$

$$= \frac{1}{2\pi}\left(-\frac{\cos 2x}{2}\right)_0^{\pi}$$

$$= -\frac{1}{4\pi}\left(\cos 2\pi - \cos 0\right) = -\frac{1}{4\pi}(1-1) = 0$$

$$\boxed{a_1 = 0}$$

For

$\underline{n \geq 2}$

$$a_n = \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n+1}\left\{\cos(n+1)\pi - \cos 0\right\} + \frac{1}{n-1}\left\{\cos(n-1)\pi - \cos 0\right\}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n+1} + \frac{1}{n-1}(-1)^{n-1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n+1}(-1)^{n+2} + \frac{1}{n+1} + \frac{1}{n-1}(-1)^{n}\cdot(-1)^{-1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^{n}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n}}{n-1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^{n}\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^{n}\left\{\frac{\cancel{n}-1-\cancel{n}-1}{(n+1)\cdot(n-1)}\right\} + \frac{\cancel{n}-1-\cancel{n}-1}{(n+1)(n-1)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2(-1)^{n}}{(n+1)(n-1)} - \frac{2}{(n+1)(n-1)}\right]$$

$$= \frac{-1}{\pi(n+1)(n-1)}\left[(-1)^{n}+1\right]$$

$$\therefore \boxed{a_n = \frac{-\left[1+(-1)^{n}\right]}{\pi(n+1)(n-1)}}, \quad n \geq 2$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x \sin nx \, dx$$

$\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

$$b_n = \frac{-1}{2\pi}\int_{0}^{\pi}\left[\cos(n+1)x - \cos(n-1)x\right]dx$$

$\underline{\underline{n=1}}$

$$b_1 = \frac{-1}{2\pi}\int_{0}^{\pi}(\cos 2x - 1)\,dx$$

$$= \frac{-1}{2\pi}\left[\cancel{\frac{\sin 2x}{2}}^{\,0} - x\right]_{0}^{\pi} = \frac{1}{2\pi}(x)_{0}^{\pi} = \frac{1}{2}$$

For $n \geq 2$,

$$b_n = \frac{-1}{2\pi}\left[\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right]_0^{\pi}$$

$$\boxed{b_n = 0}$$

Substituting the above values in [II], we get

$$f(x) = \frac{1}{\pi} + \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{(n+1)(n-1)}\cos nx + \frac{1}{2}\sin x$$

(or)

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{(n+1)(n-1)}\cos nx$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1+(-1)^{2m}}{(2m+1)(2m-1)}\cos(2m)x$$

$n = 2m$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1} \quad \text{--- } \boxed{2}$$

$(-1)^{2m} = 1$

Rewriting (2), we get

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{(2m-1)(2m+1)} \quad \text{--- } \boxed{3}$$

(i) Put $x = 0$ in [3], we get

$$0 = \frac{1}{\pi} + 0 - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)(2m+1)}$$

$$\Rightarrow -\frac{1}{\pi} = -\frac{2}{\pi}\left[\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots}$$

(ii)

Put $x = \pi/2$ in $\boxed{3}$, we get $\quad (\because \sin \pi/2 = 1)$

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos m\pi}{(2m-1)(2m+1)}$$

$$\Rightarrow \frac{1}{1} - \frac{1}{\pi} - \frac{1}{2} = -\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)(2m+1)}$$

$$\Rightarrow \frac{2\pi - 2 - \pi}{2\pi} = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)(2m+1)}$$

$$\Rightarrow \frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

10. obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & , 0 < x < \pi \end{cases}$$

and Hence deduce that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Sol: Given $f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & , 0 < x < \pi \end{cases}$ & $l = \pi$

The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—(I)}$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -\pi\, dx + \int_{0}^{\pi} x\, dx \right]$$

$$= \frac{1}{\pi} \left[\left(-\pi x\right)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{0}^{\pi} \right]$$

$\int_a^b f(x)dx = f(b) - f(a)$

$$= \frac{1}{\pi} \left[-\pi\left(0+\pi\right) + \frac{\pi^2}{2} \right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right]$$

$$\boxed{a_0 = -\pi/2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx \, dx + \int_{0}^{\pi} x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\cancel{\left(\frac{\sin nx}{n}\right)}_{-\pi}^{0} + \left\{x\cancel{\left(\frac{\sin nx}{n}\right)} - \left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi n^2}\left[\cos nx\right]_0^{\pi}$$

$\cos(n\pi) - \cos 0$

$(-1)^n - 1$

$$\boxed{a_n = \frac{(-1)^n - 1}{\pi n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx \, dx + \int_{0}^{\pi} x \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{-\cos nx}{n}\right) - \cancel{\left(\frac{-\sin nx}{n^2}\right)}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left(\cos 0 - \cos n\pi\right) - \frac{1}{n}\left(\pi \cos n\pi - 0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\{1-(-1)^n\} - \frac{\pi}{n}(-1)^n\right]$$

$$= \frac{1}{n}\left[1 - (-1)^n - (-1)^n\right]$$

$$\boxed{b_n = \frac{1 - 2(-1)^n}{n}}$$

Substituting the above values in III, we got

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1-2(-1)^n}{n}\sin nx \longrightarrow \boxed{2}$$

At $\underline{x=0}$, $f(x)$ is discontinuous, Then $f(x)$ can be written as

$(0^-, x<0, 0^+, x>0)$

$x<0$, $f(0^-) = -\pi$

$x>0$, $f(0^+) = 0$

$$f(x) = \frac{1}{2}\left[f(0^{-}) + f(0^{+})\right]$$

$$= \frac{1}{2}\left[-\pi + 0\right]$$

$$= -\frac{\pi}{2}$$

From Eqn $\boxed{2}$, we get

$$\frac{-\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}(1) + 0$$

$$\Rightarrow \frac{-\pi}{2} + \frac{\pi}{4} = \frac{1}{\pi}\left[\frac{-2}{1^2} + 0 + \frac{-2}{3^2} + 0 + \frac{-2}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{-\pi}{4} = \frac{1}{\pi}\left[\frac{-2}{1^2} - \frac{2}{3^2} - \frac{2}{5^2} - \cdots\right]$$

$$\Rightarrow \frac{-\pi}{4} = \frac{-2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$$

11. obtain the Fourier Expansion of $f(x) = \begin{cases} -K, & -\pi < x < 0 \\ K, & 0 < x < \pi \end{cases}$

and Hence deduce that $\boxed{\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}}$

Sol: Given $f(x) = \begin{cases} -K, & -\pi < x < 0 \\ K, & 0 < x < \pi \end{cases}$ & $l = \pi$

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow \boxed{1}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\,dx + \int_{0}^{\pi}(K)\,dx\right]$

$$= \frac{K}{\pi}\left[-(x)_{-\pi}^{0} + (x)_{0}^{\pi}\right]$$

$$= \frac{K}{\pi}\left[-\{0-(-\pi)\} + (\pi - 0)\right]$$

$$= \frac{K}{\pi}\left[-\pi + \pi\right] = 0$$

$\therefore \boxed{a_0 = 0}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\cos nx\,dx + \int_{0}^{\pi}(K)\cos nx\,dx\right]$$

$$= \frac{K}{\pi}\left[-\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right]$$

$\boxed{a_n = 0}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\,\sin nx\,dx + \int_{0}^{\pi}(K)\,\sin nx\,dx\right]$$

$$= \frac{K}{\pi}\left[-\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{K}{\pi n}\left[\left(\cos 0 - \cos n\pi\right) - \left(\cos n\pi - \cos 0\right)\right]$$

$$= \frac{K}{\pi n}\left[1 - (-1)^n - (-1)^n + 1\right]$$

$$b_n = \frac{K}{\pi n}\left[2 - 2(-1)^n\right] = \frac{2K}{\pi n}\left[1-(-1)^n\right],$$

Substituting a_0, a_n, & b_n in III we get

$$f(x) = \frac{2K}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx \quad \text{---}\ \boxed{2}$$

At $x = \frac{\pi}{2}$, $f(x)$ is continuous

$\therefore f(x) = K$

$0 < x < \pi$

Substituting in [2], we get

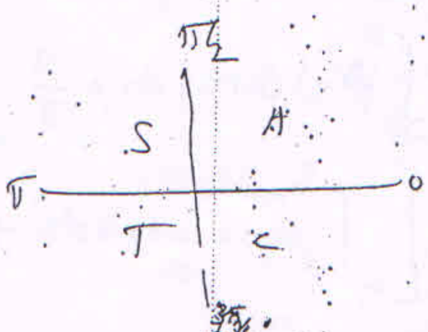
$$0 = \frac{2\cancel{k}}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$K = \frac{2K}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{2} = \frac{2}{1}\sin\frac{\pi}{2} + 0 + \frac{2}{3}\sin\frac{3\pi}{2} + 0 + \frac{2}{5}\sin\left(\frac{5\pi}{2}\right) + 0 \cdots$$

$$\frac{\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$



[2] Expand $f(x) = |\cos x|$ as a Fourier Series in the interval $(-\pi, \pi)$

Sol: Given $f(x) = |\cos x|$, $\boxed{l = \pi}$

$\Rightarrow f(-x) = |\cos(-x)| = |\cos x| = f(x)$

Thus $f(x)$ is an even function over the interval $(-\pi, \pi)$

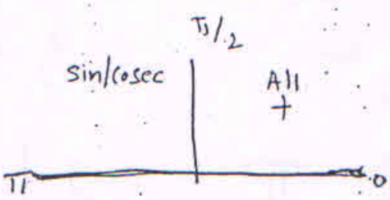
$$\therefore \boxed{b_n = 0}$$

The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} \boxed{1}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx$$



$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\sin x\right)_0^{\pi/2} - \left(\sin x\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\sin \pi/2 - 0\right) - \left(\overset{1\,0}{\cancel{\sin\pi}} - \sin \pi/2\right)\right]$$

$$\boxed{a_0 = \frac{4}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx\,dx + \int_{\pi/2}^{\pi} -\cos x \cos nx\,dx\right]$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{1}{2}\left\{(\cos(n+1)x + \cos(n-1)x)\,dx\right\} - \int_{\pi/2}^{\pi} \frac{1}{2}\left\{\cos(n+1)x + \cos(n-1)x\right\}dx\right]$$

For $n=1$,

$$a_1 = \frac{2}{\pi}\times\frac{1}{2}\left[\int_0^{\pi/2}(\cos 2x + 1)\,dx - \int_{\pi/2}^{\pi}(\cos 2x + 1)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\cancel{\frac{\sin 2x}{2}}^{\,0} + x\right)_0^{\pi/2} - \left(\cancel{\frac{\sin 2x}{2}}^{\,0} + x\right)_{\pi/2}^{\pi}\right]$$

$$a_1 = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right] = 0$$

For $n \geq 2$

$$a_n = \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{(n-1)}\right\}_0^{\pi/2} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - \left\{0 - \frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi/2}{n+1}\right\}\right]$$

$$= \frac{1}{\pi}\left[2\cdot\frac{\sin(n+1)\pi/2}{n+1} + \frac{2\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right]$$

$\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right) = \cos\frac{n\pi}{2}$

$\sin\left(\frac{n\pi}{2}-\pi/2\right) = -\cos n\pi/2$

$$= \frac{2\cos n\pi/2}{\pi}\left[\frac{\cancel{n}-\cancel{1} - \cancel{n}-\cancel{1}}{n^2-1}\right]$$

$$a_n = \frac{-4\cos n\pi/2}{\pi(n^2-1)}, \quad n \geq 2$$

Substituting the above values in III, we get

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{\cos n\pi/2}{n^2-1}\cos nx$$

13. Expand $f(x) = e^{-ax}$ as a Fourier Series in the interval $(-\pi, \pi)$. Hence deduce that

$$\operatorname{Cosech}\pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

Sol: Given $f(x) = e^{-ax}$

The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— I}$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax}\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-ax}}{-a} \right]_{-\pi}^{\pi}$$

$$= -\frac{1}{\pi a} \left[e^{-a\pi} - e^{a\pi} \right]$$

$$= \frac{1}{a\pi} \left[e^{a\pi} - e^{-a\pi} \right]$$

$$\boxed{a_0 = \frac{2 \sinh a\pi}{a\pi}}$$

$$\sinh a\pi = \frac{e^{a\pi} - e^{-a\pi}}{2}$$

$$\cosh a\pi = \frac{e^{a\pi} + e^{-a\pi}}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}\left\{-a\cos nx + \cancel{n\sin nx}^{\,0}\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{-a}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{-a}{\pi\cdot(a^2+n^2)}\left[e^{-a\pi}\cos n\pi - e^{a\pi}\cos n\pi\right]$$

$$= \frac{-a(-1)^n}{\pi(a^2+n^2)}\left[e^{-a\pi} - e^{a\pi}\right]$$

$$= \frac{a(-1)^n}{\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$\boxed{a_n = \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(\cancel{-a\sin nx}^{\,0} - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$b_n = \frac{-n(-1)^n}{\pi(a^2+n^2)}\left(e^{-a\pi} - e^{a\pi}\right)$$

$$\therefore\ b_n = \frac{2\sinh a\pi\ n(-1)^n}{\pi\ (a^2+n^2)}$$

Substituting the values of a_0, a_n & b_n in Eqn III, we get

$$e^{-ax} = \frac{\sinh a\pi}{a\pi} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh a\pi}{a^2+n^2}\cos nx$$

$$+ \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx. \quad \text{—} \boxed{2}$$

Put $x=0$ and $a=1$ in $\boxed{2}$, we get

$$\cancel{e^{-x}}\ 1 = \frac{\sinh\pi}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sinh\pi}{n^2+1}(1) + 0$$

$$\Rightarrow 1 = \frac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\right)$$

$$\Rightarrow \frac{\pi}{\sinh\pi} = 1 + \cancel{2}\left(\frac{-1}{\cancel{2}}\right) + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{\sinh\pi} = 0 + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

(or)

$$\operatorname{cosech}\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}.$$

14 Expand the function

$$f(x) = \begin{cases} 1+2x & \text{in } -3 < x \le 0 \\ 1-2x & \text{in } 0 \le x < 3 \end{cases}$$

as a Fourier series and deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Sol: The given $f(x)$ is an even function in the given interval $(-l, l) = (-3, 3)$. Its Fourier coefficients are $b_n \equiv 0$, and

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_0^3 (1-2x)\,dx = \frac{2}{3}\left(3 - 2\times\frac{3^2}{2}\right) = -4.$$

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{n\pi}{3}x\,dx = \frac{2}{3}\int_0^3 (1-2x)\cos\left(\frac{n\pi}{3}\right)x\,dx$$

$$= \frac{2}{3}\left[(1-2x)\left\{\frac{\sin(n\pi/3)x}{(n\pi/3)}\right\} - (-2)\left\{-\frac{\cos(n\pi/3)x}{(n\pi/3)^2}\right\}\right]_0^3$$

$$= -\frac{4}{3(n\pi/3)^2}(\cos n\pi - 1) = \frac{12}{n^2\pi^2}\left\{1-(-1)^n\right\}$$

Hence, the required Fourier expansion is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{3}\right)x = -2 + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{\{1-(-1)^n\}}{n^2}\cos\left(\frac{n\pi}{3}\right)x$$

Deduction: Setting $x=0$ in this and nothing that $f(0)=1$, we get

$$1 = -2 + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{\{1-(-1)^n\}}{n^2} \quad \text{or} \quad 3 = \frac{12}{\pi^2}\left\{\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\right\}$$

or $\quad 3 = \frac{24}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \quad$ or $\quad \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

15 Find the Fourier expansion for the function

$$f(x) = \begin{cases} 2-x, & 0<x<4 \\ x-6, & 4<x<8 \end{cases}$$

Sol:- Here, the interval is $(0, 2l) = (0, 8)$ so that $l = 4$

$$\therefore a_0 = \frac{1}{4}\int_0^8 f(x)\,dx = \frac{1}{4}\left[\int_0^4 (2-x)\,dx + \int_4^8 (x-6)\,dx\right]$$

$$= \frac{1}{4}\left\{\left[-\frac{(2-x)^2}{2}\right]_0^4 + \left[\frac{(x-6)^2}{2}\right]_4^8\right\}$$

$$= \frac{1}{8}\left\{\left[-(-2)^2 + 2^2\right] + \left[2^2 - (-2)^2\right]\right\} = 0.$$

$$a_n = \frac{1}{4}\int_0^8 f(x)\cos\left(\frac{n\pi}{4}\right)x\,dx = \frac{1}{4}\left[\int_0^4 (2-x)\cos\left(\frac{n\pi}{4}\right)x\,dx + \int_4^8 (x-6)\cos\left(\frac{n\pi}{4}\right)x\,dx\right]$$

$$= \frac{1}{4}\left\{\left[(2-x)\left(\frac{4}{n\pi}\right)\sin\left(\frac{n\pi}{4}\right)x - (-1)\left(\frac{4}{n\pi}\right)^2\left(-\cos\frac{n\pi}{4}x\right)\right]_0^4\right.$$

$$\left. + \left[(x-6)\left(\frac{4}{n\pi}\right)\sin\left(\frac{n\pi}{4}\right)x - \left(\frac{4}{n\pi}\right)^2\left(-\cos\frac{n\pi}{4}x\right)\right]_4^8\right\}$$

$$= \frac{1}{4}\left\{-\left(\frac{4}{n\pi}\right)^2\left[\cos n\pi - 1\right] + \left(\frac{4}{n\pi}\right)^2(\cos 2n\pi - \cos n\pi)\right\}.$$

$$= \frac{8}{n^2\pi^2}(1-\cos n\pi) = \frac{8}{n^2\pi^2}\left\{1-(-1)^n\right\}$$

$$b_n = \frac{1}{4}\int_0^8 f(x)\sin\left(\frac{n\pi}{4}\right)x\,dx = \frac{1}{4}\left[\int_0^4 (2-x)\sin\left(\frac{n\pi}{4}\right)x\,dx + \int_4^8 (x-6)\sin\left(\frac{n\pi}{4}\right)x\,dx\right]$$

$$= \frac{1}{4}\left\{\left[(2-x)\left\{-\frac{4}{n\pi}\cos\left(\frac{n\pi}{4}\right)x\right\}-(-1)\left\{-\left(\frac{4}{n\pi}\right)^2\sin\left(\frac{n\pi}{4}\right)x\right\}\right]_0^4\right.$$

$$\left.+\left[(x-6)\left\{-\frac{4}{n\pi}\cos\left(\frac{n\pi}{4}\right)x\right\}-\left\{-\left(\frac{4}{n\pi}\right)^2\sin\left(\frac{n\pi}{4}\right)x\right\}\right]_4^8\right\}$$

$$= \frac{1}{4}\left\{(-2)\left(-\frac{4}{n\pi}\cos n\pi\right)-2\left(-\frac{4}{n\pi}\right)+2\left(-\frac{4}{n\pi}\cos 2n\pi\right)+2\left(-\frac{4}{n\pi}\cos n\pi\right)\right\} = 0$$

Hence, the required Fourier expansion is

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{4}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{4}\right)x.$$

$$= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\{1-(-1)^n\}}{n^2}\cos\left(\frac{n\pi}{4}\right)x.$$

$\boxed{16}$ Obtain the Fourier expansion of,

$$f(x) = \begin{cases} l-x, & 0<x<l \\ 0, & l\le x<2l \end{cases}$$

Over the interval $(0, 2l)$. Hence deduce the following:

(a) $\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\ldots$ (b) $\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

Sol: The Fourier coeffients for the given $f(x)$ are

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx = \frac{1}{l}\int_0^{l}(l-x)dx = \frac{1}{l}\left[lx-\frac{1}{2}x^2\right]_0^l = \frac{l}{2},$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\int_0^{l}(l-x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left\{\left[(l-x)\left(\frac{l}{n\pi}\right)\sin\left(\frac{n\pi}{l}\right)x-\left(\frac{l}{n\pi}\right)^2\cos\left(\frac{n\pi}{l}\right)x\right]_0^l\right\}$$

$$= \frac{1}{l} \cdot \frac{l^2}{n^2\pi^2} \left\{ -\cos n\pi + 1 \right\} = \frac{1}{n^2\pi^2} \left\{ 1 - (-1)^n \right\}$$

and $$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{1}{l} \int_l^{2l} (l - x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l} \left\{ \left[(l-x) \left\{ -\frac{l}{n\pi} \cos\left(\frac{n\pi}{l}\right) x \right\} + \left\{ -\left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi}{l}\right) x \right\} \right]_0^l \right\}$$

$$= \frac{1}{l} \cdot \frac{l^2}{n\pi} = \frac{l}{n\pi}$$

Therefore, the Fourier expansion of the given $f(x)$ is

$$f(x) = \frac{l}{4} + \frac{l}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \cos\left(\frac{n\pi}{l}\right) x + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{l}\right) x \longrightarrow ①$$

Deductions :

(a) Taking $x = l/2$ in ① and noting that $f(l/2) = l - (l/2) = l/2$, we get

$$\frac{l}{2} = \frac{l}{4} + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \text{ or } \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right).$$

or $$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$$

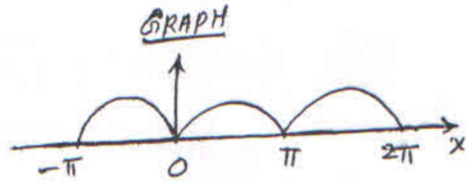
(b) Taking $x = l$ in ① and noting that $f(l) = 0$, we get

$$0 = \frac{l}{4} + \frac{l}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \cdot (-1)^n = \frac{l}{4} + \frac{l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}$$

or $$\frac{1}{4} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} = \frac{1}{\pi^2} \left\{ \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots \right\} = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

or $$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

17 Expand as a Fourier series $f(x) = |\sin x|$, $-\pi \le x \le \pi$.

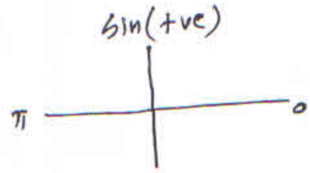


Sol: Here $f(x) = |\sin x|$ & $l = \pi$

$\Rightarrow f(-x) = |\sin(-x)| = |\sin x| = f(x)$

Thus $f(x)$ is Even fun over $(-\pi, \pi)$

$\therefore \boxed{b_n = 0}$.



The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \longrightarrow \text{II}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\, dx = \frac{2}{\pi}\int_0^{\pi} \sin x\, dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{4}{\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{1}{\cancel{2}}\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right] dx.$$

For $n=1$, $a_1 = \frac{1}{\pi}\int_0^{\pi} (\sin 2x - \sin 0)\, dx = 0$

For $n \geq 2$, $a_n = \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{-\cos(n-1)x}{(n-1)}\right]_0^{\pi}$

$$a_n = \frac{-2}{\pi(n^2-1)}\left[1 - (-1)^{n-1}\right], \quad n \geq 2$$

substituting in II, we get

$$\boxed{|\sin x| = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{-2\left[1-(-1)^{n-1}\right]}{\pi(n^2-1)} \cos nx} \qquad (\because a_1 = 0)$$

is the required Fourier series

18 Expand $f(x) = \sqrt{1-\cos x}$ in $-\pi < x < \pi$ as a Fourier Series

Sol: Here $f(x) = \sqrt{1-\cos x}$ & $l = \pi$

$\Rightarrow f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$

Thus $f(x)$ is EVEN function over $(-\pi, \pi)$

$\therefore \boxed{b_n = 0}$

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \longrightarrow \boxed{1}$$

$\sqrt{1-\cos x} = \sqrt{2\sin^2\left(\frac{x}{2}\right)}$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sqrt{1-\cos x}\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\,\sin\left(\frac{x}{2}\right)dx$$

$$= \frac{2\sqrt{2}}{\pi}\left[-\frac{\cos\left(\frac{x}{2}\right)}{\left(\frac{1}{2}\right)}\right]_0^{\pi} = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sqrt{2}\,\sin\left(\frac{x}{2}\right)\cos nx\,dx$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^{\pi}\frac{1}{2}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{\cos(2n+1)x}{(2n+1)} - \frac{-\cos(2n-1)x}{(2n-1)}\right]_0^{\pi}$$

$$a_n = \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

Substituting in $\boxed{1}$ we get

$$\boxed{\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx}$$ is the

Required Fourier Series

HALF RANGE FOURIER SERIES

Sometimes it is required to expand a function $f(x)$ over the half-interval $(0, l)$ (or) $(0, \pi)$ in a Fourier series of Period $2l$.

If it is required to expand $f(x)$ in the interval $(0,l)$ then it is immaterial what the function may be outside the range $0<x<l$. change it arbitrarily in $(l,0)$. If we extend the function $f(x)$ by reflecting it in the y-axis so that $f(-x)=f(x)$ Then the extended function is even for which $b_n=0$. The Fourier Series expansion will contain only ~~sine~~ Cosine terms.

If we extend the function $f(x)$ by reflecting it in the origin so that $\cancel{f(-x)=f(x)}$ $f(-x)=-f(x)$ then extended function is odd for which $a_0=0=a_n$. The Fourier expansion will contain only sine terms.

Thus A function $f(x)$ defined over the interval $0<x<l$ has two distinct Half-Range series.

The Half Range Fourier Cosine Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$

The Half-Range Sine Series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x$$

Where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi}{l}\right)x \, dx$

The above results also holds good for $l=\pi$ also.

Problems on Half-Range Series

II. Expand $(x-1)^2$ as Half-Range Fourier sine and cosine series for $0 \le x \le 1$, Hence Show that

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

Sol: Given $f(x) = (x-1)^2$, $\boxed{l=1}$

(i) Half-Range sine Series :

The Half-Range Fourier Sine Series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \longrightarrow \text{III}$$

where
$$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x \, dx$$

$$= 2\int_0^1 (x-1)^2 \sin n\pi x \, dx$$

$$= 2\left[(x-1)^2\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(2(x-1)\right)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[-\frac{1}{n\pi}\left\{0-1\right\} + \frac{2}{n^3\pi^3}\left\{(-1)^n - 1\right\}\right]$$

$$b_n = 2\left[\frac{1}{n\pi} + \frac{2}{n^3\pi^3}\left\{(-1)^n - 1\right\}\right]$$

Substituting the value of b_n in III, we get

$$(x-1)^2 = 2\sum_{n=1}^{\infty}\left[\frac{1}{n\pi} + \frac{2}{n^3\pi^3}\left\{(-1)^n - 1\right\}\right]\sin n\pi x$$

(ii) Half-Range Cosine Series:

The Half-Range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \text{——} \boxed{\text{III}}$$

where

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx$$

$$= 2\int_0^1 (x-1)^2\,dx$$

$$= 2\left.\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}\left[0-(-1)\right] = \frac{2}{3}$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx$$

$$= 2\int_0^1 (x-1)^2 \cos n\pi x\,dx$$

$$= 2\left[\cancel{(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right)}^{0} - \left(2(x-1)\right)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (2)\cancel{\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)}^{0}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left[(x-1)\cos n\pi x\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left[0 - (-1)(1)\right] = \frac{4}{n^2\pi^2}$$

Substituting in III, we get

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2} \rightarrow \boxed{2}$$

Deductions:

(i) Put $x=0$ in $\boxed{2}$, we get

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

(ii) Put $x=1$ in $\boxed{2}$, we get

$$0 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Rightarrow \boxed{\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}}$$

** $\boxed{2}$ Expand $f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \end{cases}$ as the Fourier Series of Sine terms.

Sol: Given $f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \end{cases}$ $\boxed{l = 1}$

The Half-Range Fourier Sine Series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{—} \boxed{1}$$

Where $b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin n\pi x\,dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$= 2\left[\left\{\left(\frac{1}{4}-x\right)\left(\frac{-\cos n\pi x}{n\pi}\right)-(-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right\}_0^{1/2}\right.$$

$$\left.+\left\{\left(x-\frac{3}{4}\right)\left(\frac{-\cos n\pi x}{n\pi}\right)-(1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right\}_{1/2}^{1}\right]$$

$$= 2\left[\frac{-1}{n\pi}\left\{-\frac{1}{4}\cos\frac{n\pi}{2}-\frac{1}{4}(1)\right\}-\frac{1}{n^2\pi^2}\left\{\sin\frac{n\pi}{2}\right\}\right.$$

$$\left.-\frac{1}{n\pi}\left\{\frac{1}{4}(-1)^n-\left(-\frac{1}{4}\right)\cos\frac{n\pi}{2}\right\}+\frac{1}{n^2\pi^2}\left[0-\sin\frac{n\pi}{2}\right]\right]$$

$$= 2\left[\frac{-1}{4n\pi}\left\{-\cancel{\cos\frac{n\pi}{2}}-1+(-1)^n+\cancel{\cos\frac{n\pi}{2}}\right\}\right.$$

$$\left.\cancel{\frac{1}{n^2\pi^2}\{}-\frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{-2}{n\pi}\left[\frac{1}{4}\left\{(-1)^n-1\right\}+\frac{2}{n\pi}\sin\frac{n\pi}{2}\right]$$

$$\therefore b_n = \frac{2}{n\pi}\left[\frac{1}{4}\left\{1-(-1)^n\right\}-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right]$$

Substituting in III, we get

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{1}{4}\left\{1-(-1)^n\right\}-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right]\sin n\pi x$$

3. Find the Cosine Half Range Series for $f(x) = x\sin x$ in $0 < x < \pi$, Hence deduce that

$$\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

(or)

$$\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots$$

Sol: Given $f(x) = x\sin x$, $l = \pi$

The Half-Range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \rightarrow \boxed{1}$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} x\left\{\frac{1}{2}(\sin(n+1)x - \sin(n-1)x\right\}dx\right.$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} x\left\{\sin(n+1)x - \sin(n-1)x\right\}\right]dx$$

$n=1,\quad a_1 = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$

$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\cancel{\frac{\sin 2x}{4}}\right)\right]_0^{\pi}$

$= \frac{1}{\pi}\left[-\frac{1}{2}\left(\pi \cos 2\pi - 0\right)\right] = \frac{1}{\pi}\left(-\frac{1}{2}\pi\right)$

$\boxed{a_1 = -\frac{1}{2}}$

$n \geq 2,\quad a_n = \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} - \frac{-\cos(n-1)x}{(n-1)}\right\} - \right.$

$\left. (1)\left\{\cancel{\frac{-\sin(n+1)x}{(n+1)^2}} - \cancel{\frac{-\sin(n-1)x}{(n-1)^2}}\right\}\right]_0^{\pi}$

$\cancel{= \frac{1}{\pi}\left[-\frac{1}{n+1}\left\{\pi(-1)^{n+1} - 0\right\}\right.}$

$\cos(n+1)\pi = (-1)^{n+1}$

$= \frac{1}{\pi}\left[-\frac{1}{n+1}\left\{\pi(-1)^{n+1} - 0\right\} + \frac{1}{n-1}\left\{\pi(-1)^{n-1} - 0\right\}\right]$

$= -\frac{1}{n+1}\left[(-1)^n \cdot (-1)^1\right] + \frac{1}{n-1}\left[(-1)^n \cdot (-1)^{-1}\right]$

$= (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$

$= (-1)^n\left[\frac{n-1 - (n+1)}{(n-1)(n+1)}\right] = (-1)^n\left[\frac{-2}{(n-1)(n+1)}\right]$

$a_n = \frac{2(-1)^{n+1}}{(n-1)(n+1)}$

4 For the functions $f(x) = x(l-x)$, obtain the half-range cosine series and the half-range sine series over the interval $(0, l)$.

Sol:- The half-range cosine series over the interval $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x$$

Here, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l (lx - x^2)\,dx = \frac{2}{l}\left(l\cdot\frac{l^2}{2} - \frac{l^3}{3}\right) = \frac{l^2}{3}$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx = \frac{2}{l}\int_0^l (lx - x^2)\cos\left(\frac{n\pi}{l}\right)x\,dx$.

$$= \frac{2}{l}\left[(lx - x^2)\left\{\frac{l}{n\pi}\sin\left(\frac{n\pi}{l}\right)x\right\} - (l-2x)\left\{-\frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi}{l}\right)x\right\} + (-2)\left\{-\frac{l^3}{n^3\pi^3}\sin\left(\frac{n\pi}{l}\right)x\right\}\right]_0^l$$

$$= \frac{2}{l}\left[l\left(-\frac{l^2}{n^2\pi^2}\cos n\pi\right) + l\left(-\frac{l^2}{n^2\pi^2}\right)\right] = \frac{-2l^2}{n^2\pi^2}\left\{(-1)^n + 1\right\}.$$

$$\therefore f(x) = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{[1+(-1)^n]}{n^2}\cos\left(\frac{n\pi}{l}\right)x$$

The half-range sine series over the interval $(0, l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x.$$

Here, $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx = \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi}{l}\right)x\,dx$.

$$= \frac{2}{l}\left[(lx - x^2)\left\{-\frac{l}{n\pi}\cos\left(\frac{n\pi}{l}\right)x\right\} - (l-2x)\left\{-\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\right)x\right\} + (-2)\left\{\frac{l^3}{n^3\pi^3}\cos\left(\frac{n\pi}{l}\right)x\right\}\right]_0^l$$

$$= \frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right] = \frac{4l^2}{n^3\pi^3}\left\{1 - (-1)^n\right\}$$

Substituting in III, we get

$$x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{(n-1)(n+1)} \cos nx \qquad — \boxed{2}$$

Put $x = \frac{\pi}{2}$ in (2) we get

$$\frac{\pi}{2} \times 1 = 1 - 0 + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(n-1)(n+1)} \cos \frac{n\pi}{2}$$

$$\frac{\pi}{2} - 1 = 2\left[\frac{-1}{1\cdot 3}(-1) + \frac{1}{2\cdot 4}(0) + \frac{-1}{3\cdot 5}(1) + \frac{1}{4\cdot 6}(0) + \frac{-1}{5\cdot 7}(-1) + \cdots\right]$$

$$\frac{\pi - 2}{2} = 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\right]$$

$$\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

x'ly by 2

$$\frac{\pi}{2} - 1 = \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots$$

$$\boxed{\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots}$$

$$\therefore\ f(x) = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n^3} \sin\left(\frac{n\pi}{l}\right)x.$$

5 Find the half-range Fourier cosine and sine series for the function

$$f(x) = \begin{cases} kx & \text{in } 0 \le x \le l/2 \\ k(l-x) & \text{in } l/2 < x \le l. \end{cases}$$

where k is a non-integer positive constant.

Sol:- The given $f(x)$ is defined over the interval $(0, l)$.

For this function,

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left\{\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right\} = \frac{2k}{l}\cdot\frac{l^2}{4} = \frac{1}{2}kl.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx = \frac{2k}{l}\left\{\int_0^{l/2} x\cos\left(\frac{n\pi}{l}\right)x\,dx + \int_{l/2}^{l}(l-x)\cos\left(\frac{n\pi}{l}\right)x\,dx\right\}$$

$$= \frac{2k}{l}\left\{\left[x\left\{\frac{1}{n\pi}\sin\left(\frac{n\pi}{l}\right)x\right\} - \left\{-\frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi}{l}\right)x\right\}\right]_0^{l/2}\right.$$

$$\left. + \left[(l-x)\left\{\frac{1}{n\pi}\sin\left(\frac{n\pi}{l}\right)x\right\} + \left\{-\frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi}{l}\right)x\right\}\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left\{\frac{l}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right) - \frac{l}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2}\right.$$

$$\left. - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right\}$$

$$= \frac{2kl}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$$

and $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx = \frac{2k}{l}\left\{\int_0^{l/2} x\sin\left(\frac{n\pi}{l}\right)x\,dx + \int_{l/2}^{l}(l-x)\sin\left(\frac{n\pi}{l}\right)x\,dx\right\}$

$$= \frac{2k}{l}\left\{\left[x\left\{-\frac{1}{n\pi}\cos\left(\frac{n\pi}{l}\right)x\right\} - \left\{-\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\right)x\right\}\right]_0^{l/2}\right.$$

$$\left. + \left[(l-x)\left\{-\frac{1}{n\pi}\cos\left(\frac{n\pi}{l}\right)x\right\} + \left\{-\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\right)x\right\}\right]_{l/2}^{l}\right.$$

$$= \frac{2k}{l}\left\{-\frac{l}{2}\cdot\frac{l}{n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l}{2}\cdot\frac{1}{n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2}$$

The half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x$$

$$= \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}\cos\frac{n\pi}{l}x$$

The half-range sine series is

$$\cancel{f(x) = \sum_{n=1}^{\infty}}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x = \frac{4kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n\pi}{2}\right)\sin\frac{n\pi}{l}x$$

COMPLEX FORM OF FOURIER SERIES

The Fourier series of a periodic function of period $2l$ defined in $(a, a+2l)$ is given by

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{—}\boxed{1}$$

We know that $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ & $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

Eqn $\boxed{1}$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}}\right) + \sum_{n=1}^{\infty} \frac{b_n}{2i}\left(e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}}\right) + \sum_{n=1}^{\infty} \frac{-b_n i}{2}\left(e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{2}\left(a_n - i b_n\right) e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} \frac{1}{2}\left(a_n + i b_n\right) e^{\frac{-in\pi x}{l}}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} C_{-n} e^{-\frac{in\pi x}{l}} \quad \text{—}\boxed{2}$$

Where $C_0 = \frac{a_0}{2}$, $C_n = \frac{1}{2}(a_n - ib_n)$ & $C_{-n} = \frac{1}{2}(a_n + ib_n)$

$\cancel{C_0 = \frac{1}{2l}\int_a^{a+2l} f(x)\,dx}$

$$\therefore C_n = \frac{1}{2l}\left[\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx - i\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{2l}\left[\int_a^{a+2l} f(x)\left\{\cos\left(\frac{n\pi x}{l}\right) - i\sin\left(\frac{n\pi x}{l}\right)\right\}\right]dx$$

$$C_n = \frac{1}{2l}\left[\int_a^{a+2l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx\right] \quad \text{———}\boxed{3} \quad n = 1, 2, 3, \ldots$$

and $$C_{-n} = \frac{1}{2l}\left[\int_a^{a+2l} f(x)\, e^{\frac{in\pi x}{l}}\,dx\right] \quad \text{———}\boxed{4} \quad n = 1, 2, 3, \ldots$$

$$C_0 = \frac{1}{2l}\int_a^{a+2l} f(x)\,dx \quad \text{———}\boxed{5}$$

Equation's (3), (4) & (5) can be put in the common form as

$$C_n = \frac{1}{2l}\int_a^{a+2l} f(x)\, e^{\frac{-in\pi x}{l}}\,dx. \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\text{———}\boxed{6}$$

~~Using $\boxed{6}$~~,

In view of $\boxed{6}$, EX$\boxed{2}$ becomes

$$\left.\begin{aligned} f(x) &= C_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=-\infty}^{-1} c_n e^{\frac{in\pi x}{l}} \\ &\boxed{\text{OR}} \\ f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \end{aligned}\right\} \quad \boxed{7}$$

The Series on the RHS of Expression $\boxed{7}$ is called the complex form $\boxed{\text{or}}$ Exponential form of Fourier Series $f(x)$ over the interval $(a, a+2l)$.

The coefficients C_n, $n = 0, \pm 1, \pm 2, \ldots$ are called Complex OR Exponential Fourier Coefficients.

The above results hold good ~~for~~ the Intervals $(0, 2l)$ $(0, 2\pi)$ $(-l, l)$ $(-\pi, \pi)$ as well.

[2] Obtain the complex form of Fourier series for the function $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ \sin x & \text{for } 0 < x < \pi \end{cases}$

Hence deduce the corresponding Trigonometric series.

Sol: Given $f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ \sin x & , 0 < x < \pi \end{cases}$

The complex form of Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \longrightarrow \text{①} \quad (\because l = \pi)$$

where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \{a \sin bx - b \cos bx\}$$

$$= \frac{1}{2\pi} \int_0^{\pi} e^{-inx} \sin x \, dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{-inx}}{(-in)^2 + 1^2} \{-in \sin x - 1 \cdot \cos x\} \right]_0^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{-inx}}{1-n^2} \{-in \overset{0}{\cancel{\sin x}} - \cos x\} \right]_0^{\pi} \quad \text{for } n \neq \pm 1$$

$$= \frac{1}{2\pi(1-n^2)} \left[-\{e^{-in\pi} \cos \pi - e^0 \cos 0\} \right]$$

$$C_n = +\frac{1}{2\pi(1-n^2)} \left[(-1)^n + 1 \right] = \frac{-\left[1 + (-1)^n\right]}{2\pi(n^2-1)} \; ; n \neq \pm 1$$

For $n=1$, $C_1 = \frac{1}{2\pi} \int_0^{\pi} e^{-ix} \sin x \, dx = -\frac{i}{4}$

$n=-1$, $C_{-1} = \frac{1}{2\pi} \int_0^{\pi} e^{ix} \sin x \, dx = \frac{i}{4}$

~~Obtain~~ Substituting the above values in ① we get

$$\therefore f(x) = c_0 + c_1 e^{ix} + \sum_{n=2}^{\infty} c_n e^{inx} + c_{-1} e^{-ix} + \sum_{n=-\infty}^{-2} c_n e^{inx}.$$

$$f(x) = \frac{1}{\pi} - \frac{i}{4} e^{ix} + \frac{i}{4} e^{-ix} - \frac{1}{2\pi}\left\{ \sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1} e^{inx} + \sum_{n=-\infty}^{-2} \frac{1+(-1)^n}{n^2-1} e^{inx} \right\} \longrightarrow \boxed{2}$$

Deduction :

Since $f(x)$ is purely real, the LHS of Expansion $\boxed{2}$ must be equal to the real part of the RHS.

$$f(x) = \frac{1}{\pi} - \frac{1}{4}(-\sin x) + \frac{1}{4}(\sin x) - \frac{1}{2\pi}\left\{ \sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1} \cos nx + \sum_{n=-\infty}^{-2} \frac{1+(-1)^n}{n^2-1} \cos nx \right\}$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{1}{2\pi}\left\{ \sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1} \cos nx + \underbrace{\sum_{n=2}^{\infty} \frac{1+(-1)^{-n}}{(-n)^2-1} \cos(-n)x}_{} \right\}$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{1}{2\pi}\, 2 \sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1} \cos nx$$

(OR)

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2mx}{(2m)^2-1}$$; putting $2m = n$

$1 + (-1)^{2m} = 1 + 1 = 2$

Problems

1. Find the complex form of the Fourier Series of $f(x) = e^{-x}$ in $-1 \le x \le 1$

Sol: Given $f(x) = e^{-x}$, $-1 \le x \le 1$

The complex form of the Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x} \qquad (\because l = 1)$$

$$\text{where } c_n = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x}\,dx \qquad \left| c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\pi x}\,dx \right.$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos n\pi = (-1)^n$

$\sin n\pi = 0$

$$= -\frac{1}{2}\left[\frac{e^{-(1+in\pi)} - e^{(1+in\pi)}}{(1+in\pi)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{1} e^{in\pi} - e^{-1} e^{-in\pi}}{1+in\pi} \times \frac{1-in\pi}{1-in\pi}\right]$$

$$= \frac{1}{2}\left[\frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)(1-in\pi)}{1^2 - i^2 n^2\pi^2}\right]$$

$$c_n = \left(\frac{e^{1} - e^{-1}}{2}\right)\frac{(1-in\pi)(-1)^n}{1^2+n^2\pi^2} = \frac{(-1)^n(1-in\pi)}{1^2+n^2\pi^2}\sinh 1$$

$$\therefore \boxed{e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-in\pi)}{1^2+n^2\pi^2}\sinh 1 \cdot e^{in\pi x}}$$ is the required Complex form.

$\boxed{3}$ Obtain the Fourier series in the complex form for $f(x) = x$ in $-\pi < x < \pi$. Hence deduce the corresponding trigonometric series.

<u>Sol:</u> Here $f(x) = x$, $l = \pi$

complex form of Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx} dx = \frac{i(-1)^n}{n}$; $n \neq 0$

$$\therefore \quad x = \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{n}[\cos nx + i\sin nx] \quad ; n \neq 0$$

Since LHS of this expansion is real, this must be equal to the real part of RHS.

$$x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n}(-\sin nx)$$

$$= \sum_{n=-\infty}^{-1} \frac{(-1)^n}{n}(-\sin nx) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(-\sin nx)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{-n}}{(-n)}\{-\sin(-n)x\} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(-\sin nx)$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(-\sin nx)$$

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$ which is the required trigonometric series

$\Big| \; (-1)^{-n} = (-1)^n$

4. Obtain the complex form of the Fourier series for the function

$$f(x) = \begin{cases} -k & \text{in } -\pi < x < 0 \\ k & \text{in } 0 < x < \pi \end{cases}$$

$$c_n = \frac{1}{2\pi}\left\{ \int_{\pi}^{0} -k e^{-inx}\,dx + \int_{0}^{\pi} k e^{-inx}\,dx \right\}$$

$$c_n = \frac{k\{1-(-1)^n\}}{\pi i n} \quad , \ (n \neq 0)$$

$$f(x) = \frac{k}{i\pi} \sum_{n=-\infty}^{\infty} \frac{1-(-1)^n}{n} e^{inx} \qquad (n \neq 0) \ .$$

Problems on Fourier series :

PR

I

1 Obtain the Fourier expansion of the function $f(x) = x$ over the interval $(-\pi, \pi)$. Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{n}(-1)^{n+1}$, Put $x = \frac{\pi}{2}$.

2 Find the Fourier series for the function $f(x) = x^2$ in the interval $(-\pi, \pi)$. Deduce the following

(i) $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (ii) $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ (iii) $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$, $b_n = 0$

(i) put $x = \pi$ (ii) put $x = 0$ (iii) Adding (i) & (ii)

3 Find the Fourier Series for the expansion $f(x) = x + x^2$ over the interval $-\pi \le x \le \pi$. Hence deduce the following

(i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - + \cdots$ (ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

$a_0 = \frac{2}{3}\pi^2$, $a_n = \frac{4}{n^2}(-1)^n$, $b_n = \frac{2}{n}(-1)^{n+1}$

(i) $x = 0$ (ii) $x = \pi$, $x = -\pi$, Adding both cases

(iii) Adding (i) & (ii)

4 Find the Fourier Series for the function $f(x) = |x|$ in $-\pi \le x \le \pi$

Hence deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$a_0 = \pi$, $a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$, $b_n = 0$, Put $x = \pi$

5 Expand $f(x) = x \sin x$ as a Fourier series in the interval $(-\pi, \pi)$

Hence deduce the following

(i) $\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots$ (ii) $\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - + \cdots$

$a_0 = 2$, $a_1 = -\frac{1}{2}$, $a_n = \frac{2(-1)^{n-1}}{(n-1)(n+1)}$, $(n \geq 2)$

(i) $x = \pi/2$, (ii) $\frac{\pi-2}{4} = \frac{1}{2}\left(\frac{\pi}{2} - 1\right) =$

6 Expand $f(x) = \sqrt{1 - \cos x}$ in a Fourier series over the interval $(-\pi, \pi)$

$a_0 = \frac{4\sqrt{2}}{\pi}$, $a_n = \frac{-4\sqrt{2}}{\pi(4n^2 - 1)}$

7 Expand $f(x) = |\cos x|$ in a Fourier series over the interval $(-\pi, \pi)$

$a_0 = \frac{4}{\pi}$, $a_1 = 0$, $a_n = \frac{-4\cos(n\pi/2)}{\pi(n^2 - 1)}$, $(n \geq 2)$

8 obtain the Fourier Expansion of the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 + (2x/\pi) & , -\pi < x \leq 0 \\ 1 - (2x/\pi) & , 0 \leq x < \pi \end{cases}$$

Deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

$a_0 = 0$, $a_n = \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$, put $x = 0$

9 Expand $f(x) = e^{-ax}$ as a Fourier series in the interval $(-\pi, \pi)$

Hence prove that $\operatorname{cosech} \pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$

Also deduce that Fourier expansions of e^{ax}, $\cosh ax$ and $\sinh ax$ over the interval $(-\pi, \pi)$

$a_0 = \frac{2}{a\pi} \sinh a\pi$, $a_n = \frac{2a(-1)^n}{\pi(a^2+n^2)} \sinh a\pi$, $b_n = \frac{2n(-1)^n}{\pi(a^2+n^2)} \sinh a\pi$ $\boxed{2}$

Deductions:

(i) changing a to $-a$, we get the Fourier expansion of e^{ax}.

(ii) $e^{ax} + e^{-ax} \to \cosh ax$, $e^{ax} - e^{-ax} \to \sinh ax$

(iii) put $x = 0$ & $a = 1$ we get $\operatorname{cosech}\pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$.

$\boxed{10}$ If $f(x) = \begin{cases} 0 & \text{for } -\pi < x \le 0 \\ \sin x & \text{for } 0 < x < \pi \end{cases}$

Prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2mx}{4m^2 - 1}$

Deduce the following

(i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$ (ii) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi - 2}{4}$

$a_0 = \frac{2}{\pi}$, $a_1 = 0$, $a_n = -\frac{[1 + (-1)^n]}{\pi(n^2-1)}$, $(n \geq 2)$

$b_1 = \frac{1}{2}$, $b_n = 0$, $(n \geq 2)$, on Expansion we get $f(x)$

Deductions: (i) put $x = 0$, $f(x) = 0$

(ii) put $x = \pi/2$, $f(x) = 1$

$\boxed{11}$ Expand the function $f(x) = x(2\pi - x)$ in Fourier series over the interval $(0, 2\pi)$. Hence deduce the following

(i) $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (ii) $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ (iii) $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$a_0 = \frac{4}{3}\pi^2$, $a_n = -\frac{4}{n^2}$, $b_n = 0$

(i) Put $x = 0$

(ii) Put $x = \pi$

(iii) Adding (i) & (ii)

12 Find the Fourier expansion of the function $f(x)=\left(\frac{\pi-x}{2}\right)^2$ over the interval $(0, 2\pi)$. Hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$a_0 = \frac{\pi^2}{6}$, $a_n = \frac{1}{n^2}$, $b_n = 0$, Put $x=0$

13 Obtain the Fourier series for $f(x) = e^{-ax}$, $a>0$ in the interval $(0, 2\pi)$

$a_0 = \frac{1}{a\pi}(1-e^{-2a\pi})$, $a_n = \frac{a(1-e^{-2a\pi})}{\pi(n^2+a^2)}$, $b_n = \frac{n(1-e^{-2a\pi})}{\pi(n^2+a^2)}$.

14 Obtain the Fourier expansion of $f(x) = x\sin x$ in the interval $(0, 2\pi)$. Deduce the following

(i) $\frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$ (ii) $\frac{\pi}{2} = 1 - 2\sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{n^2-1}$

$a_0 = -2$, $a_1 = -\frac{1}{2}$, $a_n = \frac{2}{n^2-1}$, $(n \geq 2)$, $b_1 = \pi$, $b_n = 0$ $(n\geq 2)$

Deductions: (i) put $x=0$
(ii) put $x = \pi/2$.

15 Find the Fourier expansion of the function $f(x)$ defined by

$$f(x) = \begin{cases} x & , 0 \leq x \leq \pi \\ 2\pi - x & , \pi < x \leq 2\pi \end{cases}$$

Deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$a_0 = \pi$, $a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$, $b_n = 0$

Put $x = \pi$

16. An alternating current after passing through a rectifier is given by [3]

$$I = \begin{cases} I_0 \sin x, & 0 < x \le \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

where I_0 is a constant, Develop I in a Fourier Series

$$I = I_0 \left\{ \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \right\}$$

17. Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$

$$a_0 = \frac{2}{l}(\sinh L), \quad a_n = \frac{2l(-1)^n}{l^2 + n^2\pi^2}(\sinh l), \quad b_n = \frac{2n\pi(-1)^n}{l^2 + n^2\pi^2}(\sinh l).$$

18. Expand the function $f(x) = \begin{cases} 1 + 2x & \text{in } -3 < x \le 0 \\ 1 - 2x & \text{in } 0 \le x < 3 \end{cases}$

as a Fourier series and deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

$$a_0 = -4, \quad a_n = \frac{12}{n^2\pi^2}\left[1 - (-1)^n\right], \quad b_n = 0, \quad (\text{even fun})$$

Put $x = 0$

19. Find the Fourier series that represents the function $f(x) = 1 + \sin x$ in the interval $-1 < x < 1$

$$a_0 = 2, \quad a_n = 0, \quad b_n = \frac{(-1)^{n+1}}{n^2\pi^2 - 1}(2n\pi) \sin 1$$

20. Find the Fourier expansion $f(x) = \begin{cases} 1 + (4/3)x, & -3/2 < x \le 0 \\ 1 - (4/3)x & 0 \le x < 3/2 \end{cases}$

over $(-3/2, 3/2)$

$$f(x) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\left[1 - (-1)^n\right]}{n^2} \cos\left(\frac{2n\pi}{3}\right)x.$$

21 When a sinusoidal voltage $E\sin\omega t$ is passed through a half-wave rectifier which clips the Negative portion of the wave, the resulting periodic function is given by

$$u(t) = \begin{cases} 0 , & -\pi/\omega < t < 0 \\ E\sin\omega t , & 0 < t < \pi/\omega \end{cases}$$

Develop this function in a Fourier series.

$$a_0 = \frac{2E}{\pi} , \quad a_1 = 0 , \quad a_n = -\frac{E\left[1-(-1)^{n-1}\right]}{\pi(n^2-1)} \ (n \geq 2)$$

$$b_1 = \frac{E}{2} , \quad b_n = 0 \ (n \geq 2).$$

22 Obtain the Fourier expansion of the function $f(x) = 2x - x^2$ over the interval $(0, 2l)$

Deduce the expansions of this $f(x)$ over the intervals.

(i) $(0, 2)$ (ii) $(0, 3)$

$$a_0 = \frac{4}{l}\left(1 - \frac{2}{3}l\right) , \quad a_n = -\frac{4l^2}{n^2\pi^2} , \quad b_n = \frac{4l(l-1)}{n\pi}$$

(i) take $l = 1$ (ii) Take $l = 3/2$.

23 Obtain the Fourier series for the function

$$f(x) = \begin{cases} \pi x , & \text{in } 0 \leq x \leq 1 \\ \pi(2-x) & \text{in } 1 \leq x \leq 2 \end{cases} \quad \text{over the interval } (0,2)$$

Deduce that $\dfrac{\pi^2}{8} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$. $\boxed{l=1}$

$$a_0 = \pi , \quad a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] , \quad b_n = 0$$

Put $x = 1$

24. Find the Fourier expansion for the function $\boxed{4}$

$$f(x) = \begin{cases} 2-x, & 0 < x < 4 \\ x-6, & 4 < x < 8 \end{cases} \qquad \boxed{l = 4}$$

$$a_0 = 0, \quad a_n = \frac{8}{n^2\pi^2}\left[1-(-1)^n\right], \quad b_n = 0$$

25 Obtain the Fourier expansion of $f(x) = \begin{cases} l-x, & 0<x<l \\ 0, & l \le x < 2L \end{cases}$

over the interval $(0, 2l)$. Hence deduce the following

(i) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$

(ii) $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$$a_0 = \frac{l}{2}$$

$$a_n = \frac{l}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$b_n = \frac{l}{n\pi}$$

~~26~~ Deductions (i) Put $x = l/2$ (ii) put $x = l$

Discontinuous functions :

26 Obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$ and deduce that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$a_0 = -\frac{\pi}{2}, \quad a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right], \quad b_n = \frac{1}{n}\left[1-2(-1)^n\right]$$

Deduction :

At $x = 0$, $f(x) = \frac{1}{2}\left[f(0^-) + f(0^+)\right] = \frac{-\pi}{2}$

27 Obtain the Fourier expansion of $f(x) = \begin{cases} -K, & -\pi < x < 0 \\ K, & 0 < x < \pi \end{cases}$

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \dots$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

HALF-RANGE FOURIER SERIES & COMPLEX FORM.

Expand the ~~following~~ function $f(x) = x(\pi - x)$ over the interval $(0, \pi)$ in half-Range Fourier cosine series and in half-Range Fourier Sine series. Deduce the following

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ (iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{12}$.

$a_0 = \frac{\pi^2}{3}$, $a_n = -\frac{2}{n^2}\left[1 + (-1)^n\right]$, $x(\pi - x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n^2} \cos nx. \; -(*)$

$b_n = \frac{4}{n^3 \pi}\left[1 - (-1)^n\right]$, $x(\pi - x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin nx. \; -(**)$

Put $x = 0$, Put $x = \pi/2$, $\}$ in $(*)$ Put $x = \pi/2$ in $(**)$

Expand (a) $f(x) = \cos x$ in half-range sine series, and (b) $f(x) = \sin x$ in half-range cosine series, over the interval $(0, \pi)$.

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{2}{\pi}\left[\frac{\sin^2 x}{2}\right]_0^{\pi} = 0.$$

$$b_n = \frac{2n[1+(-1)^n]}{\pi(n^2-1)}, \quad f(x) = \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{n[1+(-1)^n]}{(n^2-1)} \sin nx$$

$$a_0 = \frac{4}{\pi}, \quad a_1 = 0$$

$$a_n = -\frac{2[1+(-1)^n]}{\pi(n^2-1)}, \quad f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{[1+(-1)^n]}{(n^2-1)} \cos nx$$

Obtain the half range cosine series and the half range sine series for the functions $f(x) = x \sin x$, over the interval $(0, \pi)$

$$b_1 = \frac{\pi}{2}, \quad b_n = -\frac{4n\{1+(-1)^n\}}{\pi(n-1)^2(n+1)^2}$$

$$f(x) = \frac{\pi}{2}\sin x - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{n\{1+(-1)^n\}}{(n-1)^2(n+1)^2} \sin nx$$

4. Find (a) the half range cosine series, and (b) the half-range sine series representing the function.

$$f(x) = \begin{cases} x, & 0 < x \le \pi/2 \\ \pi - x, & \pi/2 \le x < \pi \end{cases}$$

$$a_0 = \frac{\pi}{2}, \quad a_n = \cancel{-\frac{2}{\pi}\left\{\frac{\pi}{2n}\sin(n\pi/2) + \right.} = -\frac{2}{\pi n^2}\left\{1 + (-1)^n - 2\cos(n\pi/2)\right.$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right\}\cos nx$$

5. Obtain the half-range Fourier cosine series for the functions given below, over the interval $(0, \pi)$

$$f(x) = \begin{cases} \pi/3 & \text{for } 0 < x < (\pi/3) \\ 0 & \text{for } (\pi/3 < x < (2\pi/3)) \\ -\pi/3 & \text{for } (2\pi/3) < x < \pi \end{cases}$$

Hence deduce that

$$\frac{\pi}{2} = \sum_{n=1}^{\infty}\frac{1}{n}\left\{\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right\}$$

$$a_0 = 0, \quad a_n = \frac{2}{3n}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\cos nx$$

<u>Deduction</u>. Therefore, at $x = \pi/3$, we have

$$\frac{1}{2}\left\{f(x^+) + f(x^-)\right\} = \frac{1}{2}\left(0 + \frac{\pi}{3}\right) = \frac{\pi}{6}.$$

$$\frac{\pi}{2} = \sum_{n=1}^{\infty}\frac{1}{n}\left\{\sin\frac{2n\pi}{3} + \sin\frac{3n\pi}{3} + \sin\frac{n\pi}{3}\right\} = \sum_{n=1}^{\infty}\frac{1}{n}\left\{\sin\frac{2n\pi}{3} + \sin\frac{n\pi}{3}\right\}$$

6 Expand $f(x) = (x-1)^2$ as half-range Fourier sine and cosine series for $0 \le x \le 1$. Hence deduce the following:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ (c) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$$b_n = \frac{2}{n\pi}\left\{1 + \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]\right\}$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + \frac{2}{n^2\pi^2}\{(-1)^n - 1\}\right]\sin n\pi x$$

$$a_0 = \frac{2}{3}, \quad a_n = \frac{4}{n^2\pi^2}, \quad f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

Deductions - (a) Taking $x = 0$

(b) Next, putting $x = 1$

(c) Adding results (vii) and (viii)

7 Expand the function $f(x)$ defined by

$$f(x) = \begin{cases} (1/4) - x & \text{for } 0 < x < 1/2 \\ x - (3/4) & \text{for } 1/2 < x < 1 \end{cases}$$

in a half-range sine series

$$b_n = \frac{1}{2n\pi}\{1 - (-1)^n\} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{1}{2n\pi}\left[1 - (-1)^n\right] - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\sin n\pi x$$

8 For the function $f(x) = x(1-x)$, obtain the half range cosine series and the half-range sine series over the interval $(0, 1)$.

$a_0 = \frac{l^2}{3}$

$a_n = \frac{2l^2}{n^2\pi^2}\left\{(-1)^n + 1\right\}$, $f(x) = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{[1+(-1)^n]}{n^2}\cos\left(\frac{n\pi}{l}\right)x$

$b_n = \frac{4l^2}{n^3\pi^3}\left\{1-(-1)^n\right\}$

$f(x) = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\sin\left(\frac{n\pi}{l}\right)x$

$\boxed{9}$ Find the half-range Fourier cosine and sine series for the function ;

$$f(x) = \begin{cases} kx & \text{in } 0 \le x \le l/2 \\ k(l-x) & \text{in } l/2 < x \le l \end{cases}$$

where k is a non-integer positive constant.

$a_0 = \frac{1}{2}kl$, $\quad a_n = \frac{2kl}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$

$b_n = \frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x = \frac{4kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n\pi}{2}\right)\sin\frac{n\pi}{l}x$$

$\boxed{10}$ Obtain the complex Fourier series for the function $f(x) = e^{ax}$, where a is a real constant, over the interval $(-l, l)$.

$$C_n = \frac{(al + in\pi)(\sinh al)}{(a^2l^2 + n^2\pi^2)}(-1)^n$$

$$e^{ax} = \sum_{n=-\infty}^{\infty} C_n e^{i(n\pi/l)x} = (\sinh al)\sum_{n=-\infty}^{\infty}\frac{(-1)^n(al+in\pi)}{(a^2l^2+n^2\pi^2)}e^{i(n\pi/l)x}$$

$l = \pi$

$$e^{ax} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{(a^2+n^2)} e^{inx}$$

$$a = \pm 1 \; ; \quad e^{\pm x} = \pm \frac{(\sinh \pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(\pm 1 + in)}{(1+n^2)} e^{inx}$$

11 Obtain the complex Fourier series for the function $f(x) = x$ over the interval $(-\pi, \pi)$. Hence deduce the corresponding trigonometric Fourier series.

$$C_0 = 0 \; , \qquad C_n = \frac{i}{n}(-1)^n$$

$$x = \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{n} e^{inx} \; , \quad n \neq 0$$

Deduction, $x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

12 Obtain the complex Fourier series for $f(x) = \cos ax$, where a is non-integer real constant, over the interval $(-\pi, \pi)$. Deduce the corresponding Trigonometric Fourier Series.

$$C_n = \frac{(a \sin a\pi)(-1)^n}{\pi(a^2 - n^2)} \; , \quad f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} = \frac{a \sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)}{a^2 - n^2} e^{inx}$$

Deduction, $\cos ax = \frac{\sin a\pi}{a\pi} + \frac{2a \sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 - n^2} \cos nx$

13 Obtain the complex Fourier series for the function

$$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$$

where k is a real constant, over the interval $(-\pi, \pi)$. Deduce the corresponding trigonometric Fourier series.

$$c_0 = 0 \quad , \quad c_n = -\frac{ik}{\pi n}\{1-(-1)^n\}$$

$$f(x) = -\frac{ik}{\pi}\sum_{n=-\infty}^{\infty} \frac{\{1-(-1)^n\}}{n} e^{inx}, \quad n \neq 0.$$

<u>Deduction</u>, $f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n} \sin nx.$

$\boxed{14}$ Obtain the complex Fourier series expansion for the given function.

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ \sin x & \text{for } 0 < x < \pi. \end{cases}$$

Hence deduce the corresponding trigonometric Fourier series.

$$c_n = \frac{1}{2\pi(1-n^2)}\left(e^{-in\pi}+1\right) = \frac{1}{2\pi(n^2-1)}\{(-1)^n+1\} \quad n \neq 1.$$

$$c_1 = -\frac{i}{4} \quad , \quad c_{-1} = \frac{i}{4}$$

$$f(x) = \frac{1}{\pi} - \frac{i}{4}e^{ix} + \frac{i}{4}e^{-ix} - \frac{1}{2\pi}\left\{\sum_{n=2}^{\infty}\frac{(-1)^n+1}{n^2-1}e^{inx} + \sum_{n=-\infty}^{-2}\frac{(-1)^n+1}{n^2-1}e^{inx}\right\}$$

<u>Deduction</u>, $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{(2m)^2-1}$

$\boxed{15}$ Obtain the complex Fourier series for the function.

$$f(x) = \begin{cases} 0 & \text{for } 0 < x < l \\ a & \text{for } l < x < 2l \end{cases}$$

over the interval $(0, 2l)$. Deduce the corresponding trigonometric Fourier series.

$$C_0 = \frac{a}{2}, \quad C_n = \frac{ia}{2n\pi}\left\{1-(-1)^n\right\} \quad n \neq 0$$

$$f(x) = \frac{a}{2} + \frac{ia}{2\pi}\sum_{n=-\infty}^{\infty} \frac{\left\{1-(-1)^n\right\}}{n} e^{i(n\pi/l)x}, \quad n \neq 0$$

Deduction, $f(x) = \dfrac{a}{2} - \dfrac{a}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{\left\{1-(-1)^n\right\}}{n} \sin\left(\frac{n\pi}{l}\right)x$.

16 Obtain the complex Fourier series for the function $f(x) = (x \sin x)$ over the interval $(0, 2\pi)$. Hence deduce the corresponding trigonometric Fourier series.

$$C_1 = -\left(\frac{1}{4} + \frac{\pi}{2}i\right), \quad C_n = \frac{1}{n^2-1}$$

$$f(x) = -1 - \left(\frac{1}{4} + \frac{\pi}{2}i\right)e^{ix} - \left(\frac{1}{4} - \frac{\pi}{2}i\right)e^{-ix} + \sum_{n=2}^{\infty} \frac{1}{n^2-1} e^{inx} + \sum_{n=-\infty}^{-2} \frac{1}{n^2-1} e^{inx}$$

Deduction, $f(x) = -1 - \frac{1}{2}\cos x + \pi \sin x + 2\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} \cos nx$