① Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$ ①

Sol :- $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

A.E is $m^2 - 6m + 9 = 0$

$m = 3, 3$

$$C.F = (c_1 + c_2 x)e^{3x}$$

$$= c_1 \underbrace{e^{3x}}_{y_1} + c_2 \underbrace{x e^{3x}}_{y_2}$$

$y_1' = 3e^{3x}$

$y_2' = x \cdot 3e^{3x} + e^{3x}$

$y_2' = 3xe^{3x} + e^{3x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$= e^{3x} \cdot (3xe^{3x} + e^{3x}) - xe^{3x} \cdot 3e^{3x}$$

$$= \cancel{3xe^{6x}} + e^{6x} - \cancel{3xe^{6x}}$$

$$\boxed{W = e^{6x}}$$

$$A = -\int \frac{y_2 \phi(x)}{W} dx$$

$$A = -\int \frac{\cancel{x}e^{3x} \cdot \frac{e^{3x}}{x^{\cancel{2}}}}{e^{6x}} dx \Rightarrow A = -\int \frac{\cancel{e^{6x}}}{x\,\cancel{e^{6x}}} dx$$

$$A = -\int \frac{1}{x} dx$$

$$\underline{A = -\log x}$$

$$B = \int \frac{y_1 \phi(x)}{W} dx$$

$$= \int \frac{e^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}} dx$$

$$= \int \frac{1}{x^2} \frac{\cancel{e^{6x}}}{\cancel{e^{6x}}} dx$$

$$= \int \frac{1}{x^2} dx$$

$$B = -\frac{1}{x}$$

$$P.I = Ay_1 + By_2$$

$$P.I = (-\log x) e^{3x} + (-1/x) x e^{3x}$$

Sol

$$y = CF + PI$$

$$y = (c_1 + c_2 x) e^{3x} - (\log x) e^{3x} - e^{3x}$$

(2) Solve $y'' + a^2 y = \tan(ax)$ by method of variation of parameters

Sol :- $(D^2 + a^2) y = \tan(ax)$

A.E is $m^2 + a^2 = 0 \implies m^2 = -a^2$

$$m = \pm ai$$

$$C.F = c_1 \underbrace{\cos ax}_{y_1} + c_2 \underbrace{\sin ax}_{y_2}$$

$y_1' = -a\sin ax$ $\qquad$ $y_2' = a\cos ax$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W = (\cos ax)\cdot a\cos ax - (\sin ax)\cdot(-a\sin ax)$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$= a(\cos^2 ax + \sin^2 ax)$$

$$\boxed{W = a}$$

$$A = -\int \frac{y_2\,\phi(x)}{W}\,dx$$

$$A = -\int \frac{\sin ax \cdot \tan ax}{a}\,dx$$

$$= -\frac{1}{a}\int \sin ax \cdot \frac{\sin ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \left(\frac{1}{\cos ax} - \frac{\cos^2 ax}{\cos ax}\right)dx$$

$$A = -\frac{1}{a}\int (\sec ax - \cos ax)\,dx$$

$$A = -\frac{1}{a}\left[\frac{\log(\sec ax + \tan ax)}{a} - \frac{\sin ax}{a}\right]$$

$$= -\frac{1}{a^2}\left(\log(\sec ax + \tan ax) - \sin ax\right)$$

$$A = \frac{1}{a^2}\left[\sin ax - \log(\sec ax + \tan ax)\right]$$

$$B = \int \frac{y_1\,\phi(x)}{W}\,dx$$

$$B = \int \frac{\cos ax \cdot \tan ax}{a}\,dx$$

$$B = \frac{1}{a}\int \cancel{\cos ax} \cdot \frac{\sin ax}{\cancel{\cos ax}}\,dx$$

$$B = \frac{1}{a}\int \sin ax\,dx$$

$$B = \frac{1}{a}\left(-\frac{\cos ax}{a}\right)$$

$$B = -\frac{\cos ax}{a^2}$$

$$P.I = Ay_1 + By_2 \Rightarrow$$

$$\left(\frac{1}{a^2}\left(\sin ax - \log(\sec ax + \tan ax)\right)\right)\cos ax +$$

$$\left(-\frac{\cos ax}{a^2}\right) \cdot \sin ax$$

Sol

$$y = CF + PI$$

Solve by method of variation of parameters

(1) $y'' + y = \operatorname{cosec} x$

(2) $y'' - 2y' + 2y = e^{x} \tan x$

Partial Differential Equations

PDE :- Equations which contain one or more partial derivatives are called partial Differential Equations.

Ex :- 1) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

2) $\frac{\partial^2 z}{\partial x \partial t} = e^{t} \sin x$

Order of PDE : Order of PDE is same as that of order of the highest partial derivative in the equation.

Degree of PDE : Degree is the degree of highest order derivative

Notations

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2}$

Solving PDE by direct integration

Problems

(1) Solve $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y$

Sol:- $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

Integrating w.r.t 'x'

$$\frac{\partial z}{\partial y} = \int x^2 y \, dx$$

$$\frac{\partial z}{\partial y} = y \cdot \frac{x^3}{3} + f(y)$$

Integrating w.r.t 'y'

$$z = \int y \frac{x^3}{3} dy + \int f(y) dy + g(x)$$

$$z = \frac{x^3}{3}\cdot\frac{y^2}{2} + h(y) + g(x)$$

where $\int g(y)\,dy = h(y)$

(2) Solve $\frac{\partial^2 z}{\partial x^2} = 2y^2$ by direct integration.

Sol:- $\frac{\partial^2 z}{\partial x^2} = 2y^2$ $\qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 2y^2$

Integrating w.r.t 'x'

$$\frac{\partial z}{\partial x} = 2y^2\int dx + f(y)$$

$$\frac{\partial z}{\partial x} = 2y^2 x + f(y)$$

Integrating w.r.t 'x'

$$z = 2y^2\int x\,dx + \int f(y)\,dx$$

$$z = \cancel{2}y^2\cdot\frac{x^2}{\cancel{2}} + f(y)\cdot x + g(y)$$

$$z = x^2y^2 + x\cdot f(y) + g(y)$$

(3) Solve $\frac{\partial^2 z}{\partial x^2} = \sin(xy)$

Sol:- $\frac{\partial^2 z}{\partial x^2} = \sin(xy)$ $\qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \sin(xy)$

Integrating w.r.t 'x'

$$\frac{\partial z}{\partial x} = \frac{-\cos(xy)}{y} + f(y)$$

Integrating w.r.t 'x'

$$z = \int -\frac{\cos(xy)}{y}\,dx + \int f(y)\,dx + g(y)$$

$$z = -\frac{1}{y}\cdot\frac{\sin(xy)}{y} + f(y)\cdot x + g(y)$$

$$z = \frac{-\sin(xy)}{y^2} + x\cdot f(y) + g(y)$$

4. Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2$ by direct integration.

Sol:—

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2 \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{y}{x} + 2$$

Integrating w.r.t 'x'

$$\frac{\partial z}{\partial y} = \int \frac{y}{x}\,dx + \int 2\,dx + f(y)$$

$$\frac{\partial z}{\partial y} = y\cdot\int \frac{1}{x}\,dx + 2\int dx + f(y)$$

$$\frac{\partial z}{\partial y} = y\cdot\log x + 2x + f(y)$$

Integrating w.r.t 'y'

$$z = \log x\int y\,dy + 2x\int dy + \int f(y)\,dy + g(x)$$

$$z = \log x\cdot\frac{y^2}{2} + 2xy + g(y) + g(x)$$

where $\int f(y)\,dy = g(y)$

5. Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$. Given $u = 0$ when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ when $x = 0$

Sol :- $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = e^{-t}\cos x$

(or)

$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = e^{-t}\cos x$

Integrating w.r.t 'x'

$$\frac{\partial u}{\partial t} = e^{-t}\int \cos x \, dx + f(t)$$

$$\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t) \longrightarrow (1)$$

Given $\frac{\partial u}{\partial t} = 0$ when $x = 0$

(1) $\Rightarrow$ $0 = e^{-t}\sin 0 + f(t)$

$$0 = 0 + f(t)$$

$$\boxed{f(t) = 0}$$

Sub $f(t) = 0$ in (1)

$$\frac{\partial u}{\partial t} = e^{-t}\sin x$$

Integrating w.r.t 't'

$$u = \sin x \int e^{-t}\, dt + g(x)$$

$$u = \sin x \cdot \frac{e^{-t}}{(-1)} + g(x)$$

$$u = -\sin x \, e^{-t} + g(x) \longrightarrow (2)$$

Given $u=0$, when $t=0$

$$0 = -e^{0}\sin x + g(x)$$

$$g(x) = \sin x$$

Sub $g(x)$ in (2)

$$u = -e^{-t}\sin x + \sin x$$

$$u = \sin x\,(1-e^{-t})$$

(6) Solve $\frac{\partial^2 z}{\partial x^2} = x+y$. Given $z=y^2$ when $x=0$ and $\frac{\partial z}{\partial x} = 0$ when $x=2$

Sol:- $\frac{\partial^2 z}{\partial x^2} = x+y$

Integrating w.r.t 'x'

$$\frac{\partial z}{\partial x} = \int x\,dx + \int y\,dx + f(y)$$

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} + y\cdot x + f(y) \rightarrow (1)$$

Given when $x=2$, $\frac{\partial z}{\partial x} = 0$

$(1) \Rightarrow$

$$0 = \frac{2^2}{2} + y\cdot 2 + f(y)$$

$$0 = 2 + 2y + f(y)$$

$$\boxed{f(y) = -(2+2y)}$$

Sub $f(y)$ in (1) $\Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} + xy - (2+2y)$$

integrating w.r.t 'x'

$$z = \frac{1}{2} \cdot \int x^2 dx + y \int x\, dx - 2\int dx - 2y\int dx + g(y)$$

$$z = \frac{1}{2} \cdot \frac{x^3}{3} + y \cdot \frac{x^2}{2} - 2x - 2xy + g(y) \longrightarrow (2)$$

Given $z = y^2$, when $x = 0$

$$y^2 = 0 + 0 - 0 - 0 + g(y)$$

$$\boxed{g(y) = y^2}$$

Sub $g(y)$ in (2)

$$z = \frac{x^3}{6} + \frac{x^2 y}{2} - 2x - 2xy + y^2$$

(7) Solve by direct integration given

$\frac{\partial^2 u}{\partial x \partial y} = \frac{x}{y} + a$ given that $u = 0$ when

$x = 0$ and $\frac{\partial u}{\partial x} = x$ when $y = 1$

Sol :- $\frac{\partial^2 u}{\partial x \partial y} = \frac{x}{y} + a$

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{x}{y} + a$

Integrating w.r.t 'y'

$$\frac{\partial u}{\partial x} = x\int \frac{1}{y}\,dy + a\int dy + f(x)$$

$$\frac{\partial u}{\partial x} = x.\log y + ay + f(x) \rightarrow (1)$$

Given $\frac{\partial u}{\partial x} = x$. when $y = 1$

$(1) \Rightarrow$

$$x = x.\log 1 + a(1) + f(x)$$

$$x = 0 + a + f(x)$$

$$\boxed{f(x) = x - a}$$

Sub $f(x)$ in (1)

$$\frac{\partial u}{\partial x} = x.\log y + ay + x - a$$

Integrating w.r.t 'x'

$$u = \log y\int x\,dx + ay\int dx + \int x\,dx - a\int dx + g(y)$$

$$u = \log y . \frac{x^2}{2} + ayx + \frac{x^2}{2} - ax + g(y) \rightarrow (2)$$

Given $u = 0$ when $x = 0$

(2) ⇒

$$0 = \log y \cdot 0 + 0 + 0 - 0 + g(y)$$

$$\boxed{g(y) = 0}$$

Sub $g(y) = 0$ in (2)

$$u = \log y \cdot \frac{x^2}{2} + axy + \frac{x^2}{2} - ax$$

Solve by direct integration

(1) $\log_e\left(\frac{\partial^2 z}{\partial x \partial y}\right) = x + y$

(or) $\frac{\partial^2 z}{\partial x \partial y} = e^{x+y}$

(2) $\frac{\partial^2 u}{\partial x \partial y} = e^{y}$

(3) $\frac{\partial^2 z}{\partial x \partial y} = e^{y} \cos x$

(4) $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ Given

$\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$

$z = 0$ when $x = 0$

(5) $\frac{\partial^2 z}{\partial x^2} = 6x,$