Unit -5
NP Complete problems

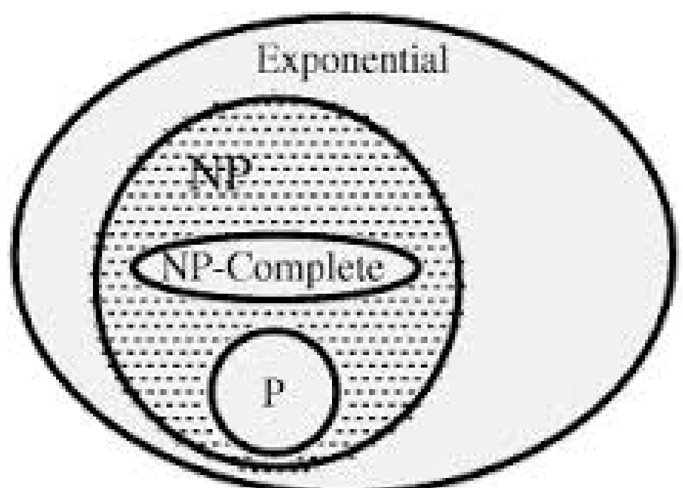
NP and P

- **What is NP?**
- NP is the set of all decision problems (question with yes-or-no answer) for which the 'yes'-answers can be **verified** in polynomial time ($O(n^k)$ where n is the problem size, and k is a constant) by a deterministic Turing machine. Polynomial time is sometimes used as the definition of *fast* or *quickly*.
- **What is P?**
- P is the set of all decision problems which can be **solved** in polynomial time by a deterministic Turing machine. Since it can solve in polynomial time, it can also be verified in polynomial time. Therefore P is a subset of NP.

NP-Complete

- **What is NP-Complete?**
- A problem x that is in NP is also in NP-Complete if and only if every other problem in NP can be quickly (ie. in polynomial time) transformed into x. In other words:
- x is in NP, and
- Every problem in NP is

reducible to x



NP-Hard

- **What is NP-Hard?**
- NP-Hard are problems that are at least as hard as the hardest problems in NP. Note that NP-Complete problems are also NP-hard. However not all NP-hard problems are NP (or even a decision problem), despite having 'NP' as a prefix. That is the NP in NP-hard does not mean 'non-deterministic polynomial time'. Yes this is confusing but its usage is entrenched and unlikely to change.

Polynomial time reductions

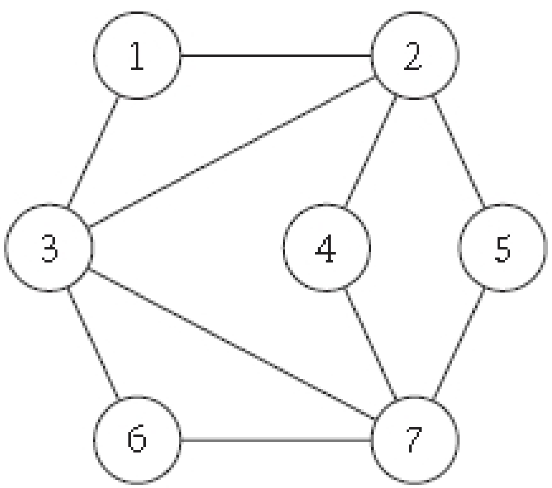
- “Problem *X is at least* as hard as problem *Y.” We will formalize this through the notion of reduction:* we will show that a particular problem *X is at least as hard as some other* problem *Y by arguing that, if we had a “black box” capable of solving X,* then we could also solve *Y. (In other words, X is powerful enough to let us* solve *Y.)*

Polynomial time reductions

- Suppose we had a *black box that could solve instances of a problem X; if we write down the* input for an instance of *X, then in a single step, the black box will return the* correct answer. We can now ask the following question:
- *(∗) Can arbitrary instances of problem Y be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to a black box that solves problem X?*
- If the answer to this question is yes, then we write *Y ≤P X; we read this as "Y is polynomial-time reducible to X," or "X is at least as hard as Y (with* respect to polynomial time)."

Independent set and vertex cover

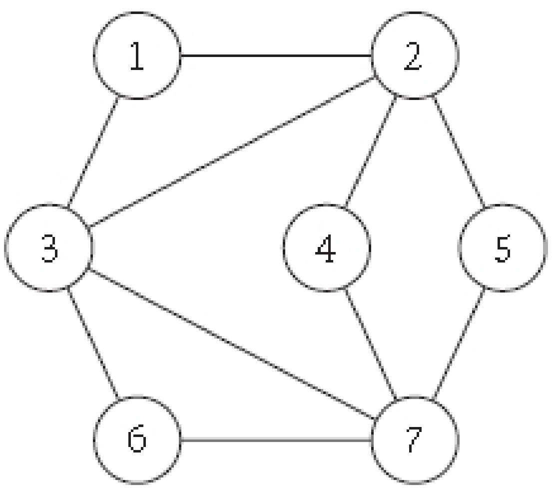
- in a graph *$G = (V, E)$, we say a set of nodes*
- *$S \subseteq V$ is independent if no two nodes in S are joined by an edge.*



Given a graph G and a number k, does G contain an independent set of
size at least k?

Independent set and vertex cover

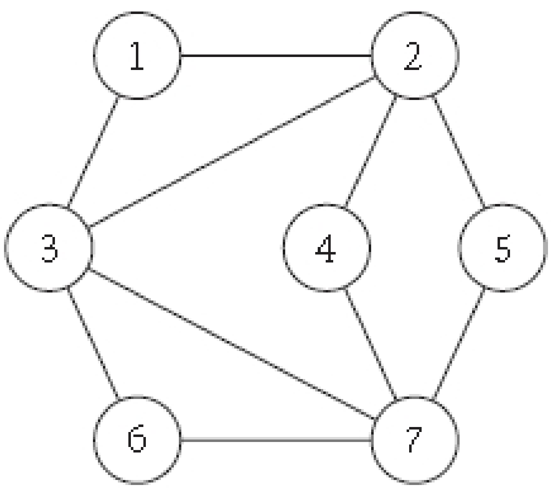
- *Vertex Cover. Given a graph $G = (V, E)$, we say that a set* of nodes $S \subseteq V$ *is a vertex cover if every edge* $e \in E$ *has at least one end in S.*



the set of nodes {1, 2, 6, 7} is a vertex cover of size 4, while the set {2, 3, 7} is a vertex cover of size 3.

Independent set and vertex cover

- We don’t know how to solve either Independent Set or Vertex Cover in polynomial time; but what can we say about their relative difficulty?



V={1,2,3,4,5,6,7}
E=…….
Indepent set S={1,4,5,6}
Vertex cover = {2,3,7}

Independent set and vertex cover

- *Let G = (V, E) be a graph. Then S is an independent set if and only if its complement V − S is a vertex cover.*
- **Proof.**
- **First, suppose that *S is an independent set. Consider an arbitrary edge*** *e = (u, v). Since S is independent, it cannot be the case that both u and v are*
- in *S; so one of them must be in V − S. It follows that every edge has at least*
- one end in *V − S, and so V − S is a vertex cover.*

- Conversely, suppose that *V − S is a vertex cover. Consider any two nodes*
- *u and v in S. If they were joined by edge e, then neither end of e would lie*
- in *V − S, contradicting our assumption that V − S is a vertex cover. It follows*
- that no two nodes in *S are joined by an edge, and so S is an independent set.*