(1) Solve $y''' - 3y'' + 3y' - y = 0$,

Given $y(0) = 2$, $y'(0) = 2$, $y''(0) = 10$

Sol :- $(D^3 - 3D^2 + 3D - 1)\, y = 0$

A.E is $m^3 - 3m^2 + 3m - 1 = 0$

$m = 1 \quad (1)^3 - 3(1)^2 + 3(1) - 1 = 0$

$$\begin{array}{c|cccc} m=1 & 1 & -3 & 3 & -1 \\ & 0 & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

$m^2 - 2m + 1 = 0$

$m^2 - m - m + 1 = 0$

$m(m-1) - 1(m-1) = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

Roots $= 1, 1, 1$

$y = (c_1 + c_2 x + c_3 x^2)\, e^x \longrightarrow (1)$

Given $y(0) = 2$, when $x = 0$, $y = 2$

(1) $\Rightarrow$

$2 = (c_1 + c_2(0) + c_3(0))\, e^0$

$\boxed{c_1 = 2}$

Diff (1) w.r.t 'x'

2

$$y' = (c_1 + c_2 x + c_3 x^2) e^x + e^x (0 + c_2 + c_3(2x)) \rightarrow (2)$$

Given $y'(0) = 2$ when $x=0$, $y'=2$

$$2 = (c_1 + c_2(0) + c_3(0)) e^0 + e^0 (c_2 + 0)$$

$$2 = c_1 + c_2 \quad \text{But } c_1 = 2$$

$$2 = 2 + c_2 \Rightarrow \boxed{c_2 = 0}$$

Diff (2) w.r.t 'x'

$$y'' = (c_1 + c_2 x + c_3 x^2) e^x + e^x (0 + c_2 + c_3(2x))$$

$$+ e^x (c_2 + c_3(2x)) + e^x (c_3(2))$$

Given $y''(0) = 10$,

$$10 = (c_1 + c_2(0) + c_3(0)) e^0 + e^0 (c_2 + c_3(0))$$

$$+ e^0 (c_2 + c_3(0)) + e^0 (2 c_3)$$

$$10 = (c_1) e^0 + 1 \cdot c_2 + c_2 + 2c_3$$

Sub c_1 & c_2

$$10 = 2 + 0 + 0 + 2c_3 \Rightarrow 2c_3 = 8$$

$$\boxed{c_3 = 4}$$

Sub c_1, c_2 and c_3 in ①

$y = (2 + 0 + 4x^2) e^{x}$

$y = (2 + 4x^2) e^{x}$

(2) Solve $y'' - 2y' + y = x, \quad y(0) = 0, y(1) = 3$

Sol :- $(D^2 - 2D + 1) y = x$

A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0, \Rightarrow m = 1, 1$

$CF = (c_1 + c_2 x) e^{x}$

$P.I = \frac{1}{D^2 - 2D + 1} x$

$= \frac{1}{1 + (D^2 - 2D)} x$

$= [1 + (D^2 - 2D)]^{-1} x$

$= (1 - (D^2 - 2D)) x$

$= x - D^2(x) + 2D(x)$

$= x - 0 + 2(1)$

$P.I = x + 2$

General sol is $y = CF + PI$

$\Rightarrow y = (c_1 + c_2 x) e^{x} + x + 2 \rightarrow ①$

Given $y(0)=0$

① $\Rightarrow$

$$0 = (c_1 + c_2(0))e^0 + 0 + 2$$

$$c_1 + 2 = 0 \Rightarrow \boxed{c_1 = -2}$$

Given $y(1) = 3$ when $x=1$, $y=3$

① $\Rightarrow$

$$3 = (c_1 + c_2(1))e^1 + 1 + 2$$

$$3 = (c_1 + c_2)e + 3$$

$$3 = 0$$

$$(c_1 + c_2)e = 0 \qquad \text{But } c_1 = -2$$

$$(-2 + c_2)e = 0$$

$$-2 + c_2 = 0 \Rightarrow \boxed{c_2 = 2}$$

Sub c_1 & c_2 in ①

$$y = (-2 + 2x)e^x + x + 2$$

Solve

① $y'' - 2y' + y = x$, $y(0) = 0$, $y'(0) = 2$

② $y'' + 4y' + 4y = 0$, $y(1) = 0$, $y'(0) = -1$

③ $(D^2 + D)y = 2 + 2x + x^2$, $y(0) = 8$, $y'(0) = -1$

④ $\frac{d^2y}{dx^2}$ $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$, $y(0) = 0$, $y'(0) = 1$

UNIT – III

Equations reducible to linear D.E with Constant Coefficients

(or)

L.D.E with Variable coefficients

1. Legendre's linear equation
2. Cauchy's linear equation

1. Legendre's linear equation:-

An equation of the form

$$a_0 (ax+b)^2 y'' + a_1 (ax+b) y' + a_2 y = \phi(x)$$

is called Legendre's linear equation of second order. Ex:- $(2x+1)^2 y'' + (2x+1) y' + 5y = e^x$

Procedure:-

put $t = \log_e (ax+b)$

$e^t = ax+b \Rightarrow ax+b = e^t$

$(ax+b)^2 y'' = a^2 D_1 (D_1 - 1) y$ where $D_1 = \frac{d}{dt}$

$(ax+b) y' = a D_1 y$ ly $(ax+b)^3 y''' = a^3 \cancel{D_1 (D_1-1)(D_1-2)} y$

$= a^3 D_1 (D_1 - 1)(D_1 - 2) y$

② Cauchy's linear equation

This is a particular case of Legendre's linear equation when $a = 1$, and $b = 0$

i.e $a_0 x^2 y'' + a_1 x y' + a_2 y = f(x)$

This is called Cauchy's linear equation of Second order.

Ex :- $x^2 y'' + 2xy' + 3y = \sin(\log x)$

Procedure Put $t = \log x \Rightarrow x = e^t$

$x^2 y'' = D_1(D_1 - 1) y$ where $D_1 = \frac{d}{dt}$

$x y' = D_1 y$

 lly $x^3 y''' = D_1(D_1 - 1)(D_1 - 2) y$

Problems on Cauchy's linear equation

① Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1+x)^2$

Sol :- $x^2 y'' - 3xy' + 4y = (1+x)^2$

Put $t = \log_e x \Rightarrow e^t = x$

$x^2y'' = D_1(D_1-1)y$ where $D_1 = \frac{d}{dt}$

$xy' = D_1 y$

then

$$D_1(D_1-1)y - 3D_1 y + 4y = (1+e^t)^2$$

$$(D_1^2 - D_1)y - 3D_1 y + 4y = (1+e^t)^2$$

$$(D_1^2 - D_1 - 3D_1 + 4)y = (1+e^t)^2$$

$$(D_1^2 - 4D_1 + 4)y = (1+e^t)^2$$

$$(D_1^2 - 4D_1 + 4)y = 1 + (e^t)^2 + 2e^t$$

$$(D_1^2 - 4D_1 + 4)\,y = 1 + e^{2t} + 2e^t$$

A.E is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0 \Rightarrow m = 2, 2$$

$$CF = (c_1 + c_2 t)\,e^{2t}$$

$$P.I = \frac{1}{D_1^2 - 4D_1 + 4}(1 + e^{2t} + 2e^t)$$

$$= \frac{1}{D_1^2 - 4D_1 + 4}\,1 + \frac{1}{D_1^2 - 4D_1 + 4}e^{2t} + \frac{1}{D_1^2 - 4D_1 + 4}2e^t$$

$$= \frac{1}{D_1^2 - 4D_1 + 4}e^{0t} + \frac{1}{D_1^2 - 4D_1 + 4}e^{2t} + 2\cdot\frac{1}{D_1^2 - 4D_1 + 4}e^t$$

put $D_1 = 0$, $\quad D_1 = 2$, $\quad D_1 = 1$

$$P.I = \cancel{\frac{1}{0-4(0)+4}e^{0t} + \frac{1}{(2)^2-4(2)+4}}$$

$$P.I = \frac{1}{0-0+4}e^{0t} + \frac{1}{2^2-4(2)+4}e^{2t} + 2\cdot\frac{1}{1^2-4(1)+4}e^{t}$$

$$= \frac{1}{4} + \underbrace{\frac{1}{8-8}}e^{2t} + 2\cdot e^{t}$$

$$= \frac{1}{4} + t\cdot\frac{1}{2D_1-4}e^{2t} + 2e^{t}$$

put $t = 2$

$$= \frac{1}{4} + t\cdot\underbrace{\frac{1}{2(2)-4}}e^{2t} + 2e^{t}$$

$$= \frac{1}{4} + t^2\cdot\frac{1}{2}e^{2t} + 2e^{t}$$

$$P.I = \frac{1}{4} + \frac{t^2e^{2t}}{2} + 2e^{t}$$

$$y = CF + PI$$

$$\underline{y =} (c_1 + c_2 t)\underline{e^{2t}} + \frac{1}{4} + \frac{t^2e^{2t}}{2} + 2e^{t}$$

Sol is

$$y = \underline{(c_1 + c_2(\log x))x^2 + \frac{1}{4} + \frac{(\log x)^2 x^2}{2} + 2\cdot x}$$

where $t = \log x$

$x = e^{t}$

(2) Solve $x^2y'' - xy' + 2y = x\sin(\log x)$

Sol :- Put $t = \log x$, $\Rightarrow x = e^t$

$xy' = D_1 y$ where $\underline{D_1 = \frac{d}{dt}}$

$x^2 y'' = D_1(D_1 - 1)y$

then $D_1(D_1-1)y - D_1 y + 2y = e^t \sin t$

$(D_1^2 - D_1)y - D_1 y + 2y = e^t \sin t$

$(D_1^2 - 2D_1 + 2)y = e^t \sin t$ ✓

A.E is $m^2 - 2m + 2 = 0$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\cdot1\cdot2}}{2}$$

$$m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{2 \pm \sqrt{-4}}{2}$$

$$m = \frac{2 \pm 2i}{2} \Rightarrow m = 1 \pm i$$

$$CF = e^t(c_1 \cos t + c_2 \sin t)$$

$$P.I = \frac{1}{D_1^2 - 2D_1 + 2} e^t \sin t$$

$$P.I = e^{t} \cdot \frac{1}{D_1^2 - 2D_1 + 2} \sin t$$

Put $D_1 = D_1 + 1$ $\quad a = 1$

$$= e^{t} \cdot \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} \sin t$$

$$= e^{t} \frac{1}{D_1^2 + \cancel{2D_1} + 1 - \cancel{2D_1} - \cancel{2} + \cancel{2}} \sin t$$

$$= e^{t} \cdot \frac{1}{D_1^2 + 1} \sin t$$

Put $D_1^2 = -1^2$

$$= e^{t} \cdot \frac{1}{-1+1} \sin t$$

$$= e^{t} \cdot t \cdot \frac{1}{2D_1} \sin t$$

$$= \frac{te^{t}}{2} \int \sin t \, dt \qquad \frac{1}{D_1} = \int$$

$$= \frac{te^{t}}{2} (-\cos t)$$

$$P.I = -\frac{te^{t} \cos t}{2}$$

$$y = CF + PI$$

$$y = e^{t}(c_1 \cos t + c_2 \sin t) - \frac{te^{t} \cos t}{2}$$

Sol is

$$y = x\left(c_1 \cos(\log x) + c_2 \sin(\log x)\right) - \frac{\log x \cdot x \cdot \cos(\log x)}{2}.$$

(3) Solve $x^2 y'' + xy' + y = \log x \cdot \sin(\log x)$

Sol:- put $\log x = t \Rightarrow x = e^t$

~~then~~ $xy' = D_1 y$

$x^2 y'' = D_1(D_1 - 1)y$

then $D_1(D_1 - 1)y + D_1 y + y = t \sin t$

$(D_1^2 - \cancel{D_1} + \cancel{D_1} + 1)y = t \sin t$

$(D_1^2 + 1)y = t \sin t$ ✓

A.E is $m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm i$

$CF = c_1 \cos t + c_2 \sin t$

$$P.I = \frac{1}{D_1^2 + 1} t \sin t \rightarrow (1)$$

$$= \left(t - \frac{2D_1}{D_1^2 + 1}\right) \frac{\sin t}{D_1^2 + 1}$$

$\left[\left(x - \frac{f'(D)}{f(D)}\right) \frac{V}{f(D)}\right.$ Type V

Put $D_1^2 = -1^2$

$$P.I = \left(t - \frac{2D_1}{D_1^2+1}\right)\frac{\sin t}{\underbrace{-1+1}}$$

$$= \left(t - \frac{2D_1}{D_1^2+1}\right) t \cdot \frac{\sin t}{2D_1} \qquad \frac{1}{D_1}\sin t = \int \sin t\, dt$$

$$= \frac{1}{2}\left[t - \frac{2D_1}{D_1^2+1}\right] t(-\cos t)$$

$$P.I = \frac{1}{2}\left[-t^2\cos t + \frac{2D_1(t\cos t)}{D_1^2+1}\right]$$

$$= \frac{1}{2}\left[-t^2\cos t + \frac{2(t(-\sin t)+\cos t\cdot 1)}{D_1^2+1}\right]$$

$$= \frac{1}{2}\left[-t^2\cos t - \frac{2t\sin t + 2\cos t}{D_1^2+1}\right]$$

$$= \frac{1}{2}\left[-t^2\cos t - \underbrace{\frac{2t\sin t}{D_1^2+1}}_{P.I} + \frac{2\cos t}{D_1^2+1}\right] \rightarrow (\text{From } ①)$$

$$P.I = \frac{1}{2}\left[-t^2\cos t - 2P.I + \frac{2\cos t}{D_1^2+1}\right] \quad \text{put } D_1^2 = -1$$

$$= \frac{1}{2}\left[-t^2\cos t - 2P.I + \frac{2\cos t}{\underbrace{-1+1}}\right]$$

$$= \frac{1}{2}\left[-t^2\cos t - 2P.I + t\,\cancel{2}\cdot\frac{1}{\cancel{2}D_1}\cos t\right]$$

$$= \frac{1}{2}\left[-t^2\cos t - 2P.I + t\sin t\right]$$

$$P.I = -\frac{t^2\cos t}{2} - P.I + \frac{t}{2}\sin t$$

$$2P.I = -\frac{t^2\cos t}{2} + \frac{t}{2}\sin t$$

$$P.I = -\frac{t^2}{4}\cos t + \frac{t \sin t}{4}$$

$$y = CF + PI$$

$$y = c_1 \cos t + c_2 \sin t - \frac{t^2}{4}\cos t + t \sin t$$

Sol is

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \log x \sin(\log x)$$

4) Solve $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x\log x$

Sol:- $x^3y'''+3x^2y''+xy'+y=x\log x$

Put $t=\log x \Rightarrow x=e^t$

$xy'=D_1y$

$x^2y''=D_1(D_1-1)y$

$x^3y'''=D_1(D_1-1)(D_1-2)y$

then $D_1(D_1-1)(D_1-2)y+3D_1(D_1-1)y+D_1y+y=e^t t$

$D_1(D_1^2-3D_1+2)y+3(D_1^2-D_1)y+D_1y+y=te^t$

$(D_1^3-\cancel{3D_1^2}+\cancel{2D_1}+\cancel{3D_1^2}-\cancel{3D_1}+\cancel{D_1}+1)y=te^t$

$(D_1^3+1)y=te^t$

A.E is $m^3+1=0$

$m=-1$ satisfies

$$\begin{array}{r|rrrr} m=-1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$m^2-m+1=0$

$$m^2 - m + 1 = 0$$

$$m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2}$$

$$m = \frac{1 \pm \sqrt{1-4}}{2}$$

$$m = \frac{1 \pm \sqrt{3}\, i}{2}$$

$$CF = e^{t/2}\left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t\right) + c_3 e^{-t}$$

$$P.I = \frac{1}{D_1^3 + 1}\, t e^t$$

$$= e^t \cdot \frac{1}{D_1^3 + 1}\, t \qquad \begin{array}{l} a = 1 \\ \text{Put } D_1 = D_1 + 1 \end{array}$$

$$= e^t \cdot \frac{1}{(D_1 + 1)^3 + 1}\, t$$

$$= e^t \cdot \frac{1}{D_1^3 + 3D_1^2 + 3D_1 + 1 + 1}\, t$$

$$\cancel{= e^t \cdot \frac{1}{1 + (D_1^3 + 3D_1^2 + 3D_1)}\, t}$$

$$= e^t \cdot \frac{1}{D_1^3 + 3D_1^2 + 3D_1 + 2}\, t$$

$$P.I = e^t \cdot \frac{1}{2\left(1 + \frac{D_1^3 + 3D_1^2 + 3D_1}{2}\right)} t$$

$$= \frac{e^t}{2}\left(1 + \frac{D_1^3 + 3D_1^2 + 3D_1}{2}\right)^{-1} t$$

$$= \frac{e^t}{2}\left(1 - \left(\frac{D_1^3 + 3D_1^2 + 3D_1}{2}\right)\right) t$$

$$= \frac{e^t}{2}\left(t - \frac{D_1^3(t)}{2} - \frac{3}{2} D_1^2(t) - \frac{3}{2} D_1(t)\right]$$

$$= \frac{e^t}{2}\left(t - \frac{3}{2}(1)\right)$$

$$P.I = \frac{e^t}{2}\left(t - \frac{3}{2}\right)$$

$$\boxed{\begin{aligned} e^t &= x \\ (e^t)^{1/2} &= x^{1/2} \\ e^{-t} &= x^{-1} \\ &= 1/x \end{aligned}}$$

$$y = CF + PI$$

$$y = e^{t/2}\left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t\right) + c_3 e^{-t} + \frac{e^t}{2}(t - 3/2)$$

Sol is

$$y = x^{1/2}\left(c_1 \cos \frac{\sqrt{3}}{2}(\log x) + c_2 \sin \frac{\sqrt{3}}{2}(\log x)\right) + c_3 \cdot \frac{1}{x} + \frac{x}{2}\left(\log x - 3/2\right)$$