1. Examine $f(x,y) = x^3 + y^3 - 3axy$ for extreme values.

Sol:- $f(x,y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 + 0 - 3ay$ $\qquad f_y = 3y^2 - 3ax$

$f_x = 3x^2 - 3ay$

For max (or) min

$f_x = 0$ & $f_y = 0$

$3x^2 - 3ay = 0$ $\qquad 3y^2 - 3ax = 0$

$x^2 - ay = 0$ $\qquad y^2 - ax = 0$

$x^2 = ay \rightarrow (1)$ $\qquad y^2 = ax \rightarrow (2)$

Squaring

$x^4 = a^2y^2$ $\qquad (\because$ From (2) $y^2 = ax)$

$x^4 = a^2(ax)$

$x^4 = a^3 x$

$x^4 - a^3 x = 0$

$x(x^3 - a^3) = 0$

$x = 0,$ $\quad x^3 - a^3 = 0$

$x^3 = a^3$

$x = a$

If $x = 0$

$y^2 = 0$

$\Rightarrow y = 0$

$(0,0)$

If $x = a$

$y^2 = a \cdot a \Rightarrow y^2 = a^2$

$y = a$

Stationary points $\quad (0,0) \quad (a,a)$

$$r = \frac{\partial^2 f}{\partial x^2} = f_{xx} = 6x$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -3a$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

	$(0,0)$	(a,a)
$r = 6x$	$r = 0$	$6a$
$s = -3a$	$-3a$	$-3a$
$t = 6y$	$t = 0$	$6a$
$rt - s^2$	$0 - (-3a)^2$ $= -9a^2$	$36a^2 - (-3a)^2$ $= 36a^2 - 9a^2$ $= 27a^2$

at $(0,0)$ $rt - s^2 < 0$
$(0,0)$ is saddle point

at (a,a) $r > 0$ $(a > 0)$, $rt - s^2 > 0$
f has minimum value.

at (a,a) $r < 0$ $(a < 0)$ $rt - s^2 > 0$
f has maximum value.

Maximum & Min values

(2) Show that $f(x,y) = xy(1-x-y)$ is maximum at the point $(1/3, 1/3)$ (5)

Sol:- $f(x,y) = xy(1-x-y)$

$$f = xy - x^2y - xy^2$$

$$f_x = y - 2xy - y^2 \qquad f_y = x - x^2 - 2xy$$

$$r = f_{xx} = 0 - 2y - 0 = -2y \qquad t = f_{yy} = 0 - 0 - 2x$$
$$t = -2x$$

$$s = f_{xy} = 1 - 2x - 2y$$

at $(1/3, 1/3)$ $\quad r = -2(1/3) = -2/3 < 0$

at $(1/3, 1/3)$ $\quad s = 1 - 2(1/3) - 2(1/3)$
$$= 1 - 2/3 - 2/3$$
$$= \frac{3-2-2}{3}$$
$$= -1/3$$

at $(1/3, 1/3)$ $\quad t = -2(1/3) = -2/3$

$$rt - s^2 = (-2/3)(-2/3) - (-1/3)^2$$
$$= \frac{4}{9} - 1/9 \Rightarrow rt - s^2 = 3/9$$
$$rt - s^2 = 1/3 > 0$$

$r < 0$, $rt - s^2 > 0$

$\therefore$ $f(x,y)$ has max at $(1/3, 1/3)$

3. Discuss the maxima and minima of $f(x,y) = x^3y^2(1-x-y)$

<u>Sol</u> :-

$$f(x,y) = x^3y^2(1-x-y)$$

$$f = x^3y^2 - x^4y^2 - x^3y^3$$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 2x^3y - 2x^4y - 3x^3y^2$$

for max (or) min

$f_x = 0$. $\quad f_y = 0$

$f_x = 0 \Rightarrow$

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(3 - 4x - 3y) = 0$$

$x^2y^2 = 0 \quad 3 - 4x - 3y = 0$

$3 - 4x - 3y = 0$

$x = 0$, $y = 0$. $\Rightarrow 4x + 3y = 3 \quad \text{—} (1)$

$f_y = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$

$$x^3y(2 - 2x - 3y) = 0$$

$x^3y = 0 \quad 2 - 2x - 3y = 0$

$x = 0, y = 0 \quad 2x + 3y = 2 \rightarrow (2)$

$\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}$ $\left.\begin{array}{l} 4x+3y=3 \\ 2x+3y=2 \end{array}\right\}$

$x=0,\quad 2x+3y=2 \implies y=2/3 \qquad (0, 2/3)$

$y=0 \quad 2x+3y=2 \implies 2x=2,\ x=1 \qquad (1, 0)$

$x=0. \quad 4x+3y=3 \implies 3y=3,\ y=1 \qquad (0, 1)$

$y=0 \quad 4x+3y=3 \implies 4x=3,\ x=3/4 \qquad (3/4, 0)$

Solving ① & ②

$$\begin{array}{r} 4x+3y=3 \\ 2x+3y=2 \\ (-) \\ \hline 2x=1 \end{array} \implies x=1/2$$

Sub $x=1/2$ in $2x+3y=2$

$\cancel{2}(1/\cancel{2})+3y=2$

$3y=1 \implies y=1/3$

$(1/2, 1/3)$

Stationary points

$(0,0)\quad (0, 2/3)\quad (1,0)\quad (0,1)\quad (3/4, 0)\quad (1/2, 1/3)$

$r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$

$= 6xy^2(1 - 2x - y)$

$s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$= x^2y(6 - 8x - 9y)$

$t = f_{yy} = 2x^3 - 2x^4 - 6x^3y$

$= 2x^3(1 - x - 3y)$

	$(0,0)$	$(0,2/3)$	$(1,0)$	$(0,1)$	$(3/4,0)$	$(1/2,1/3)$
$r = 6xy^2(1-2x-y)$	0	0	0	0	0	$-1/9$
$s = x^2y(6-8x-9y)$	0	0	0	0	0	$-1/12$
$t = 2x^3(1-x-3y)$	0	0	0	0	0	$-1/8$
$rt - s^2$	0	0	0	0	0	$1/144$

at $(1/2, 1/3)$ $\quad r < 0, \quad rt - s^2 > 0$

$\therefore$ f has max value at $(1/2, 1/3)$

at all remaining points $rt - s^2 = 0$

$\therefore$ further investigation is required.

Max value $f(1/2, 1/3) = (1/2)^3 (1/3)^2 (1 - 1/2 - 1/3)$

$= 1/432$

4) Find the extreme values of

$$f(x,y) = \sin x \sin y \sin(x+y),$$

$$0 < x < \pi/2,\quad 0 < y < \pi/2$$

Sol:- $f(x,y) = \sin x \sin y \sin(x+y)$

$f_x = \sin y\left[\sin x \cdot \cos(x+y) + \sin(x+y)\cdot\cos x\right]$

$f_x = \sin y\left[\sin(x + x+y)\right]$

$f_x = \sin y\left[\sin(2x+y)\right]$

$f_y = \sin x\left[\sin y \cdot \cos(x+y) + \sin(x+y)\cdot\cos y\right]$

$f_y = \sin x\left[\sin(x+2y)\right]$

For max (or) min

$f_x = 0$

$\sin y\left(\sin(2x+y)\right) = 0$

$\sin y = 0$

$y = 0.$

(or)

$y = \pi$

$\sin(2x+y) = 0$

$2x+y = 0$

(or) $2x+y = \pi$

$f_y = 0$

$\sin x \cdot \sin(x+2y) = 0$

$\sin x = 0$

$x = 0$ (or) π

$\sin(x+2y) = 0$

$x+2y = 0$

(or) $x+2y = \pi$

$(0,0)\quad(\pi,\pi)$

$$\begin{aligned} 2x+y&=\pi \\ x+2y&=\pi \end{aligned} \Rightarrow \begin{aligned} 2x+y&=\pi \\ (-) \quad 2x+4y&=2\pi \\ \hline 3y&=\pi \end{aligned}$$

$$\Rightarrow \boxed{y=\pi/3}$$

Sub $y=\pi/3$ in $2x+y=\pi$

$$2x+\pi/3=\pi$$

$$2x=\pi-\pi/3$$

$$2x=2\pi/3$$

$$\boxed{x=\pi/3}$$

✓✓

$(\pi/3, \pi/3) \in (0<x<\pi/2, \; 0<y<\pi/2)$

$$\begin{aligned} x+2y&=\pi \\ 2x+y&=0 \end{aligned} \Rightarrow \begin{aligned} 2x+4y&=2\pi \\ 2x+y&=0 \\ \hline 3y&=2\pi \end{aligned} \Rightarrow y=2\pi/3$$

Sub $y=2\pi/3$ in $x+2y=\pi$

$$x+2(2\pi/3)=\pi$$

$$x=\pi-4\pi/3$$

$$\boxed{x=-\pi/3}$$

$(-\pi/3, 2\pi/3)$ ✗

Stationary point $(\pi/3, \pi/3)$

$r = f_{xx} = \sin y \cdot \cos(2x+y) \cdot 2$

$s = f_{xy} = \sin y \cdot \cos(2x+y) + \sin(2x+y) \cdot \cos y$

$t = f_{yy} = \sin x \cdot \cos(x+2y) \cdot 2$

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at $(\pi/3, \pi/3)$

$$r = \sin \pi/3 \cos\left(2\pi/3 + \pi/3\right) \cdot 2$$

$$\frac{\sqrt{3}}{\cancel{2}} \cdot \cos \pi \cdot \cancel{2}$$

$$= \sqrt{3}(-1)$$

$$= -\sqrt{3}$$

$$s = \sin \pi/3 \cdot \cos\left(2\pi/3 + \pi/3\right) + \sin\left(2\pi/3 + \pi/3\right) \cdot \cos \pi/3$$

$$= \frac{\sqrt{3}}{2} \cdot \cos\pi + \sin \pi \cdot \cos \pi/3$$

$$= \frac{\sqrt{3}}{2}(-1) + 0 \cdot$$

$$= -\frac{\sqrt{3}}{2}$$

$$t = 2 \sin \pi/3 \cdot \cos\left(\pi/3 + 2\pi/3\right)$$

$$= \cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}} \cdot \cos \pi$$

$$= \sqrt{3}(-1)$$

$$= -\sqrt{3}$$

at $(\pi/3, \pi/3)$ $\quad r < 0,$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2$$

$$= 3 - \frac{3}{4} \cdot \qquad rt - s^2 > 0$$

$$= 9/4 > 0$$

Max value $= ?$

$\therefore$ at $(\pi/3, \pi/3)$ f has max value.

Solve

1. Find the maximum and minimum values of $f(x,y) = x^3+y^3+3x-12y+20$

2. Show that $f(x,y) = x^3+y^3-3xy+1$ is minimum at $(1,1)$

Lagrange's method of undetermined multipliers

This is a method of obtaining the stationary values of a function of several variables subject to a specified condition.

Consider a function $u(x,y,z)$ of three independent variables x, y, z. Suppose x,y,z are subjected to a condition (constraint) of the form

$\phi(x,y,z) = c$, c is a constant

Form

$$F = U + \lambda\phi$$

$$F = U(x,y,z) + \lambda(\phi(x,y,z))$$

Find F_x, F_y, F_z

$F_x = 0$, $F_y = 0$, $F_z = 0$

we solve for x, y, z and λ from the above set of equations along with $\phi(x,y,z) = c$. The values of $U(x,y,z)$ are the stationary values.

The method of obtaining stationary values of $f(x,y,z)$ by this process described above is called Lagrange's method of undetermined multipliers

(1) Find the minimum value of $x^2+y^2+z^2$ when $x+y+z = 3a$

Sol:- $U = x^2+y^2+z^2$ $\qquad \phi \Rightarrow x+y+z = 3a$

let $F = U + \lambda\phi = (x^2+y^2+z^2) + \lambda(x+y+z)$

$f_x = 2x + \lambda \cdot 1$

$F_y = 2y + \lambda \cdot 1$ $\quad f_x = 0, \quad F_y = 0, \quad F_z = 0$

$F_z = 2z + \lambda \cdot 1$

$2x + \lambda = 0, \quad 2y + \lambda = 0, \quad 2z + \lambda = 0$

$\lambda = -2x, \quad \lambda = -2y, \quad \lambda = -2z$

$-\cancel{2}x = -\cancel{2}y = -\cancel{2}z = \lambda$

$\lambda = \underline{x = y = z}$

But $x + y + z = 3a$

$\therefore$

$x + x + x = 3a$

$3x = 3a \Rightarrow x = a$

$y = a, \quad z = a$

Minimum Value of $x^2 + y^2 + z^2$ is

$a^2 + a^2 + a^2 = \underline{3a^2}$

② The temperature T at any point (x,y,z) in space $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2+y^2+z^2=1$ (13)

Sol :-

$$U(x,y,z) = 400xyz^2$$

$$\phi(x,y,z) = x^2+y^2+z^2 = 1$$

$$F = U + \lambda\phi$$

$$F = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$$

$$F_x = 400 \cdot yz^2 + \lambda \cdot 2x$$

$$F_y = 400\,xz^2 + \lambda \cdot 2y$$

$$F_z = 400\,xy(2z) + \lambda \cdot 2z$$

$F_x = 0$, $F_y = 0$, $F_z = 0$

$$400yz^2 + \lambda \cdot 2x = 0 \Rightarrow \cancel{2}\lambda x = -\overset{200}{\cancel{400}}yz^2$$

$$\lambda = \frac{-200yz^2}{x}$$

$$400xz^2 + \lambda 2y = 0 \Rightarrow \cancel{2}\lambda y = -\overset{200}{\cancel{400}}xz^2$$

$$\lambda = \frac{-200xz^2}{y}$$

$$800xyz + \lambda \cdot 2z = 0$$

$$\cancel{2}\lambda \cancel{z} = -\overset{400}{\cancel{800}}xy\cancel{z}$$

$$\lambda = -400xy$$

$$\frac{-200yz^2}{x} = \frac{-200xz^2}{y} = -400xy$$

$$\frac{\cancel{-200}y\cancel{z^2}}{x} = \frac{\cancel{-200}x\cancel{z^2}}{y}$$

$$\frac{y}{x} = \frac{x}{y}$$

$$y^2 = x^2 \Rightarrow \underline{y = x}$$

$$\frac{\cancel{-200}\cancel{x}z^2}{y} = \overset{2}{\cancel{-400}}\cancel{x}y$$

$$z^2 = 2y^2 \qquad \because y = x$$

$$z^2 = 2x^2$$

$$z = \sqrt{2}x$$

But $x^2+y^2+z^2 = 1$

$$x^2 + (x)^2 + (\sqrt{2}x)^2 = 1 \Rightarrow x^2 + x^2 + 2x^2 = 1$$

$$4x^2 = 1$$

$$\Rightarrow x^2 = 1/4 \Rightarrow x = 1/2$$

$x = 1/2$

$y = x \Rightarrow y = 1/2$

$z = \sqrt{2}\,x = \sqrt{2} \cdot 1/2$

$= \frac{1}{\sqrt{2}}$

$x = 1/2, \quad y = 1/2, \quad z = 1/\sqrt{2}$

The maximum (highest) temperature

$T = 400xyz^2$

$T = 400 * 1/2 \cdot 1/2 \cdot (1/\sqrt{2})^2$

$T = 400 \cdot 1/2 \cdot 1/2 \cdot 1/2$

$T = \underline{\underline{50}}$