✓

$$L\{F(t)\} = f(s)$$
$$= \int_0^\infty e^{-st} F(t)\, dt$$

1. $F(t)$ is piece-wise continuous $[0,b]$
2. $F(t)$ has an exponential order

$\underbrace{e^{-at} F(t)}_{\neq 0}$ must be finite as $t \to \infty$.

Result :- (Laplace Transform of derivatives)

If $L\{F(t)\} = f(s)$ then

$$L\{F^{(n)}(t)\} = s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \cdots - F^{n-1}(0)$$

Proof :- By the def of L.T

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

put $F(t) = F'(t)$, we get

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$$
$$= \int_0^\infty e^{-st} \frac{d}{dt}\{F(t)\}\, dt$$

$$= \lim_{K\to\infty} \int_0^K e^{-st}\, d(F(t))$$

$$= \lim_{K\to\infty} \left[\left\{ e^{-st} F(t) \right\}_0^K - \int_0^K -s e^{-st} F(t)\, dt \right]$$

$$= \lim_{K\to\infty} \left[e^{-sK} F(K) - F(0) + s \int_0^K e^{-st} F(t)\, dt \right]$$

$$= \underbrace{\lim_{K\to\infty} e^{-sK} F(K)} - F(0) + s \lim_{K\to\infty} \int_0^K e^{-st} F(t)\, dt$$

Since $F(t)$ has an exponential order

$$\lim_{K\to\infty} e^{-sK} F(K) = 0$$

$$L\{F'(t)\} = -F(0) + s \int_0^\infty e^{-st} F(t)\, dt$$

$$\boxed{L\{F'(t)\} = s L\{F(t)\} - F(0)} \longleftarrow (1)$$

$$\therefore \boxed{L\{F'(t)\} = s f(s) - F(0)}$$

Again replace $F(t)$ by $F'(t)$ in (1)

$$\therefore \quad L\{F''(t)\} = s L\{F'(t)\} - F'(0)$$

$$= s\{ s f(s) - F(0)\} - F'(0)$$

$$\boxed{L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)}$$

Similary,

$$L\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{n-1}(0).$$

NOTE :-

If $L\{F(t)\} = f(s)$ Then

(i) $L\{F'(t)\} = s f(s) - F(0)$

(ii) $L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

(iii) $L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$

① Find $L\{\sin\sqrt{t}\}$.

We know that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Put $x = \sqrt{t}$

$$\therefore \sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots$$

$$\therefore L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right\}$$

$$= L\{t^{1/2}\} - \frac{1}{3!} L\{t^{3/2}\} + \frac{1}{5!} L\{t^{5/2}\} - \cdots$$

$L\{t^n\} = \frac{n!}{s^{n+1}}$, n is a +ve integer

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

$\Gamma(n+1) = n\Gamma(n)$ $\quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$\Gamma(n+1) = n!$, if n is a +ve integer

$$L\{\sin\sqrt{t}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}} - \frac{1}{3!}\frac{\Gamma(\frac{3}{2}+1)}{s^{3/2+1}} + \frac{1}{5!}\frac{\Gamma(\frac{5}{2}+1)}{s^{5/2+1}} - \cdots$$

$$= \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{5/2}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{7/2}} - \cdots$$

$$= \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}}\left[1 - \frac{\frac{3}{2}}{3!\,s} + \frac{\frac{5}{2}\cdot\frac{3}{2}}{5!\,s^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{\cancel{3}}{\cancel{3}\cdot2\cdot2\cdot s} + \frac{\cancel{5}\cdot\cancel{3}}{\cancel{5}\cdot4\cdot\cancel{3}\cdot2\cdot1\cdot2\cdot2\cdot s^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{(1/4s)}{1!} + \frac{(1/4s)^2}{2!} - \cdots\right]$$

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = e^{-x}$$

$$\boxed{L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s}}$$

(2) If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$ then find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

Let $F(t) = \sin\sqrt{t}$ and $f(s) = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$

Now $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$. Also $F(0) = 0$

$\therefore$ By Laplace Transform of derivatives

$$L\{F'(t)\} = s f(s) - F(0)$$

$$L\left\{\frac{\cos\sqrt{t}}{\cancel{2}\sqrt{t}}\right\} = s \cdot \frac{\sqrt{\pi}}{\cancel{2} s^{3/2}} e^{-1/4s} - 0$$

$$\boxed{L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}} e^{-1/4s}}$$

(3) If $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$ then find $L\left\{\frac{1}{\sqrt{t\pi}}\right\}$.

Let $F(t) = 2\sqrt{\frac{t}{\pi}}$ and $f(s) = \frac{1}{s^{3/2}}$

Now $F'(t) = \frac{\cancel{2}}{\sqrt{\pi}} \frac{1}{\cancel{2}\sqrt{t}} = \frac{1}{\sqrt{t\pi}}$

Also $F(0) = 0$

By Laplace Transform of derivatives

$$L\{F'(t)\} = s f(s) - F(0)$$

$$\therefore \quad L\left\{\frac{1}{\sqrt{t\pi}}\right\} = s \cdot \frac{1}{s^{3/2}} - 0$$

$$= 1/s^{1/2} = 1/\sqrt{s}$$

$$\therefore \boxed{\text{If } L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}} \text{ then } L\left\{\frac{1}{\sqrt{t\pi}}\right\} = \frac{1}{s^{1/2}}}$$

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