If a function $f(t)$ is defined by 1

$$f(t) = \begin{cases} f_1(t), & \text{for } t \le a \\ f_2(t). & \text{for } t > a \end{cases}$$

Verify that $f(t) = f_1(t) + \{f_2(t) - f_1(t)\} H(t-a)$

Sol:- By definition of $H(t-a)$ (or) $U(t-a)$

$$H(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$

Multiplying by $f_2(t) - f_1(t)$ on b.s

$$\{f_2(t) - f_1(t)\} H(t-a) = \begin{cases} 0, & t \le a \\ f_2(t) - f_1(t), & t > a \end{cases}$$

Adding $f_1(t)$ on both sides

$$f_1(t) + \{f_2(t) - f_1(t)\} H(t-a) = \begin{cases} f_1(t) & , t \le a \\ \cancel{f_1(t)} + f_2(t) - \cancel{f_1(t)}, & t > a \end{cases}$$

$$= \begin{cases} f_1(t). & t \le a \\ f_2(t) & t > a \end{cases}$$

$$f_1(t) + \{f_2(t) - f_1(t)\} H(t-a) = f(t)$$

Note:- If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t) & a < t \le b \\ f_3(t) & t > b \end{cases}$

then

$$f(t) = f_1(t) + (f_2(t) - f_1(t))\, H(t-a) + (f_3(t) - f_2(t))\, H(t-b)$$

Problems

① Express $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & t > 2 \end{cases}$ in terms of Heaviside's function and hence find its Laplace transform.

Sol:- $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & t > 2 \end{cases}$

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}\, H(t-a)$$

$$f(t) = t^2 + (4t - t^2)\, H(t-2)$$

$$L\{f(t)\} = L\{t^2\} + L\{(4t - t^2)\, H(t-2)\} \rightarrow (1)$$

(3)

$$L\{t^2\} = \frac{2!}{s^{2+1}}$$

$$L\{t^2\} = \frac{2!}{s^3} \longrightarrow (2)$$

$$L\{(4t-t^2)\, H(t-2)\}$$

$$L\{f(t-a)\, H(t-a)\}$$

$$a = 2$$

$$f(t-2) = 4t - t^2 \qquad \text{put } t = t+2$$

$$f(t+\cancel{2}-\cancel{2}) = 4(t+2) - (t+2)^2$$

$$f(t) = 4t+8 - (t^2 + 4t + 4)$$

$$= \cancel{4t} + 8 - t^2 - \cancel{4t} - 4$$

$$= 4 - t^2$$

$$L\{4-t^2\} = L\{4\} - L\{t^2\}$$

$$= \frac{4}{s} - \frac{2!}{s^3}$$

$$= \frac{4}{s} - \frac{2}{s^3}$$

$$L\{(4t-t^2)\, H(t-2)\} = e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right) \longrightarrow (3)$$

Sub ② & ③ in ①

$$L\{f(t)\} = \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)$$

② Express $f(t) = \begin{cases} \sin t, & 0 < t < \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$ interms of Heaviside's function and hence find its Laplace transform

Sol :- $f(t) = \begin{cases} \sin t, & 0 < t < \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$

$$f(t) = \sin t + (\cos t - \sin t)\, H(t - \pi/2)$$

$$L\{f(t)\} = L\{\sin t\} + L\{(\cos t - \sin t)\, H(t - \pi/2)\} \rightarrow ①$$

$$L\{\sin t\} = \frac{1}{s^2+1} \rightarrow ②$$

$$L\{(\cos t - \sin t)\, H(t-\pi/2)\}$$

$a = \pi/2$

$f(t - \pi/2) = \cos t - \sin t$ put $t = t + \pi/2$

$f(t + \pi/2 - \pi/2) = \cos(t + \pi/2) - \sin(t + \pi/2)$

(5)

$$f(t) = -\sin t - \cos t$$

$$L\{-\sin t - \cos t\} = -L\{\sin t\} - L\{\cos t\}$$

$$= -\frac{1}{s^2+1} - \frac{s}{s^2+1}$$

$$= -\frac{(s+1)}{s^2+1}$$

$$L\{(\cos t - \sin t)\, H(t - \pi/2)\} = e^{-\frac{\pi s}{2}}\left(\frac{-(s+1)}{s^2+1}\right)$$

$$= -e^{-\frac{\pi s}{2}}\frac{(s+1)}{s^2+1} \rightarrow \boxed{3}$$

Sub (2) & (3) in (1)

$$L\{f(t)\} = \frac{1}{s^2+1} - \frac{e^{-\frac{\pi s}{2}}(s+1)}{s^2+1}$$

$$L\{f(t)\} = \frac{1 - e^{-\frac{\pi s}{2}}(s+1)}{s^2+1}$$

3. Express $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$ in terms of unit step function and hence find its Laplace transform.

Sol :- $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$

$$f(t) = f_1(t) + (f_2(t) - f_1(t))\, H(t-a)$$

$$L\{f(t)\} = L\{f_1(t)\} + L\{(f_2(t) - f_1(t))\, H(t-a)\}$$

$$L\{f(t)\} = L\{(t-1)\} + L\{((3-t)-(t-1))\, H(t-2)\} \longrightarrow (1)$$

$$L\{(t-1)\} = L\{t\} - L\{1\}$$

$$L\{(t-1)\} = \frac{1}{s^2} - \frac{1}{s} \longrightarrow (2)$$

$$L\{((3-t)-(t-1))\, H(t-2)\}$$

$a=2$

$$f(t-2) = 3-t-t+1 = 4-2t$$

Put $t = t+2$

$$f(t+2-2) = 4-2(t+2)$$

$$f(t) = \cancel{4} - 2t - \cancel{4}$$

$$= -2t$$

$$L\{-2t\} = -2 \cdot \frac{1}{s^2}$$

$$L\{(4-2t)\, H(t-2)\} = e^{-2s}\left(-\frac{2}{s^2}\right)$$

$$= \cdot\, \frac{2e^{-2s}}{s^2} \rightarrow (3)$$

Sub (2) & (3) in (1)

$$L\{f(t)\} = \frac{1}{s^2} - \frac{1}{s} - \frac{2e^{-2s}}{s^2}$$

(4) Express $f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ t, & 1 < t \leq 2 \\ t^2, & t > 2 \end{cases}$

interms of Heaviside's function and hence find its Laplace transform.

$$f(t) = f_1(t) + (f_2(t) - f_1(t))\, H(t-a) + (f_3(t) - f_2(t))\, H(t-b)$$

$$f(t) = 1 + (t-1)\, H(t-1) + (t^2 - t)\, H(t-2)$$

$$L\{f(t)\} = L\{1\} + L\{(t-1)\, H(t-1)\} + L\{(t^2-t)\, H(t-2)\} \longrightarrow 1$$

$$L\{1\} = \frac{1}{s} \rightarrow 2$$

$$L\{(t-1)\, H(t-1)\}$$

$a = 1$

$$f(t-a) = (t-1)$$

put $t = t+1$

$$f(t+1-1) = (t+\cancel{1}-\cancel{1})$$

$$f(t) = t$$

$$L\{t\} = \frac{1}{s^2}$$

$$L\{(t-1)\, H(t-1)\} = e^{-s} \cdot \frac{1}{s^2} \longrightarrow 3$$

$$L\{(t^2-t)\, H(t-2)\}$$

$a = 2$

$$f(t-2) = t^2 - t$$

put $t = t+2$

$$f(t) = (t+2)^2 - (t+2)$$
$$= t^2 + 4t + 4 - t - 2$$
$$= t^2 + 3t + 2$$
$$L\{t^2+3t+2\} = \frac{2}{s^3} + 3\cdot\frac{1}{s^2} + \frac{2}{s}$$
$$L\{(t^2-t)\; H(t-2)\} = e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right) \longrightarrow (4)$$

Sub (2), (3) & (4) in (1)

$$L\{f(t)\} = \frac{1}{s} + e^{-s}\cdot\frac{1}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$$

(5) Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$$

after expressing interms of Heaviside's function.

Sol :- $f(t) = \sin t + (\sin 2t - \sin t)\, H(t-\pi)$ 10

$+ (\sin 3t - \sin 2t)\, H(t-2\pi)$

$L\{f(t)\} = L\{\sin t\} + L\{(\sin 2t - \sin t)\, H(t-\pi)\}$

$+ L\{(\sin 3t - \sin 2t)\, H(t-2\pi)\} \rightarrow (1)$

$L\{\sin t\} = \frac{1}{s^2+1} \rightarrow (2)$

$\underline{L\{(\sin 2t - \sin t)\, H(t-\pi)\}}$

$f(t-\pi) = \sin 2t - \sin t$ $\quad a = \pi$

Put $t = t+\pi$

$f(t+\pi-\pi) = \sin 2(t+\pi) - \sin(t+\pi)$

$= \sin(2t+2\pi) - \sin(\pi+t)$

$= \sin 2t - (-\sin t)$

$= \sin 2t + \sin t$

$L\{\sin 2t + \sin t\} = \frac{2}{s^2+2^2} + \frac{1}{s^2+1}$

$L\{(\sin 2t - \sin t)\, H(t-\pi)\} = e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right) \rightarrow (3)$

(11)

$$L\{(\sin 3t - \sin 2t)\, H(t-2\pi)\}$$

$$a = 2\pi$$

$$f(t-2\pi) = \sin 3t - \sin 2t$$

Put $t = t+2\pi$

$$f(t+2\pi-2\pi) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$$

$$= \sin(3t+6\pi) - \sin(2t+4\pi)$$

$$f(t) = \sin 3t - \sin 2t$$

$$L\{\sin 3t - \sin 2t\} = \frac{3}{s^2+9} - \frac{2}{s^2+4}$$

$$L\{(\sin 3t - \sin 2t)\, H(t-2\pi)\} = e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right) \rightarrow (4)$$

Sub (2), (3) & (4) in (1)

$$L\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} - \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)$$

6. Express $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 1, & \pi < t \le 2\pi \\ \sin t, & t > 2\pi \end{cases}$

in terms of Heaviside's function and hence find its Laplace transform.

Sol:— $f(t) = \cos t + (1-\cos t)\, H(t-\pi) + (\sin t - 1)\, H(t-2\pi)$

$$L\{f(t)\} = L\{\cos t\} + L\{(1-\cos t)\, H(t-\pi)\} + L\{(\sin t-1)\, H(t-2\pi)\} \rightarrow (1)$$

$$L\{\cos t\} = \frac{s}{s^2+1} \rightarrow (2)$$

$\underline{L\{(1-\cos t)\, H(t-\pi)\}}$ $\quad a = \pi$

$f(t-\pi) = 1-\cos t$

Put $t = t+\pi$

$f(t+\pi-\pi) = 1-\cos(t+\pi)$

$f(t) = 1 - (-\cos t)$

$= 1+\cos t$

$$L\{1+\cos t\} = \frac{1}{s} + \frac{s}{s^2+1}$$

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$$L\{(1-\cos t)\,H(t-\pi)\} = e^{-\pi s}\left(\frac{1}{s}+\frac{s}{s^2+1}\right) \rightarrow (3)$$

$$L\{(\sin t-1)\,H(t-2\pi)\}$$

$a = 2\pi$

$f(t-2\pi) = \sin t - 1$

Put $t = t+2\pi$

$f(t+2\pi-2\pi) = \sin(t+2\pi) - 1$

$f(t) = \sin t - 1$

$$L\{\sin t - 1\} = \frac{1}{s^2+1} - \frac{1}{s}$$

$$L\{(\sin t-1)\,H(t-2\pi)\} = e^{-2\pi s}\left(\frac{1}{s^2+1}-\frac{1}{s}\right) \rightarrow (4)$$

Sub (2), (3) & (4) in (1)

$$L\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s}+\frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1}-\frac{1}{s}\right)$$

Express the following functions interms of Heaviside's function and hence find its Laplace transform.

1. $$f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t & 1 < t \le 2 \\ 0 & t > 2 \end{cases}$$

2. $$f(t) = \begin{cases} \cos t, & 0 < t \le \pi \\ \cos 2t, & \pi < t \le 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$$

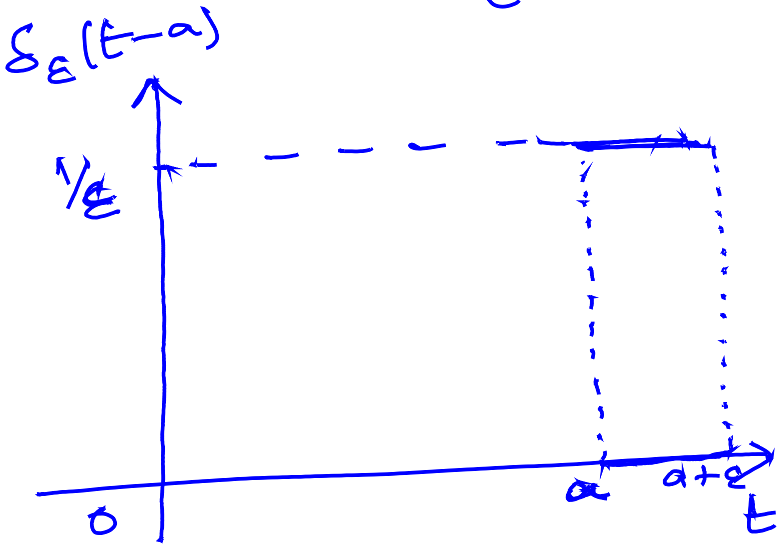
3. $$f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 4t, & 2 < t < 4 \\ 8 & t > 4 \end{cases}$$

Unit impulse function (or) Dirac- Delta function

The unit impulse function (or) Dirac delta function is denoted by $\delta(t-a)$ and is defined as

$$\delta(t-a) = \underset{\varepsilon \to 0}{Lt} \; \delta_\varepsilon(t-a), \quad a \geq 0$$

where
$$\delta_\varepsilon(t-a) = \begin{cases} 1/\varepsilon, & a \leq t \leq a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$$



Note:- $L\{\delta(t-a)\} = e^{-as}$

If $a=0$, $L\{\delta(t)\} = 1$

(1) Find $L\{2\delta(t-1)+3\delta(t-2)+4\delta(t+3)\}$

Sol:- $2L\{\delta(t-1)\} + 3L\{\delta(t-2)\} + 4L\{\delta(t+3)\}$

$$= 2\cdot e^{-s} + 3\cdot e^{-2s} + 4e^{3s}$$

$L\{\delta(t-a)\} = e^{-as}$

(2) Find $L\{t^3\delta(t-3)\}$

$$L\{\delta(t-3)\} = e^{-3s}$$

$$L\{t^3\cdot\delta(t-3)\} = (-1)^3\frac{d^3}{ds^3}(e^{-3s})$$

$$= -\frac{d^2}{ds^2}(-3e^{-3s})$$

$$= -\frac{d}{ds}(9e^{-3s})$$

$$= -(-27)e^{-3s}$$

$$= 27e^{-3s}$$