Introduction: The Laplace Equation $\nabla^2 u = 0$ in various orthogonal curvilinear Coordinate Systems is of great importance in many physical and Engineering problems.

The solution of Laplace's Equation in cylindrical and spherical system leads to two important ordinary Equations namely Bessel differential Equation and legendre differential Equation respectively.

The Series solution of the Bessel's differential Equation is a special functions known as the Bessel's function. The special polynomial function that occurs in the process of solving in Series the legendre's differential Equation is known as legendre polynomial.

NOTE: legendre polynomials are likely to occur in problems showing spherical symmetry. They are obtained by power Series method.

Bessel's ODE and Bessel's Functions are likely to occur in the problem showing cylindrical Symmetry. They are obtained by Frobenius method.

power Series method is a standard method for Solving linear ODES with variable coefficients. It gives solution in the form of power Series. These Series Can be used for computing values, graphing curves, proving formulas, and Exploring properties of the Solutions.

The Extension of power Series method is called Frobenius method.

Solution of Laplace Equation in cylindrical System leading to Bessel differential Equation.

proof: The Coordinates (ρ, ϕ, z) are called the cylindrical coordinates and the relationship with the Cartesian coordinates (x, y, z) is given by $x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$

The Laplace Equation $\nabla^2 f = 0$ in the cylindrical system.

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 \quad \text{——— (1)}$$

we shall solve this by the method of seperation of variables.

let $f = f_1 f_2 f_3$ be the soln of (1). where

$f_1 = f_1(\rho)$ $f_2 = f_2(\phi)$ $f_3 = f_3(z)$ sub in (1)

$$\frac{\partial^2 (f_1 f_2 f_3)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial (f_1 f_2 f_3)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}(f_1 f_2 f_3) + \frac{\partial^2}{\partial z^2}(f_1 f_2 f_3) = 0$$

$$f_2 f_3 \frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho} f_2 f_3 \frac{df_1}{d\rho} + \frac{1}{\rho^2} f_1 f_3 \frac{d^2 f_2}{d\phi^2} + f_1 f_2 \frac{d^2 f_3}{dz^2} = 0$$

$\div f_1 f_2 f_3$

$$\frac{1}{f_1}\frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho f_1}\frac{df_1}{d\rho} + \frac{1}{\rho^2 f_2}\frac{d^2 f_2}{d\phi^2} + \frac{1}{f_3}\frac{d^2 f_3}{dz^2} = 0$$

$$\frac{1}{f_1}\frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho f_1}\frac{df_1}{d\rho} + \frac{1}{\rho^2 f_2}\frac{d^2 f_2}{d\phi^2} = -\frac{1}{f_3}\frac{d^2 f_3}{dz^2}$$

The LHS is a function of ρ, ϕ and RHS is a function of z. $\therefore$ they must be equal to a constant.

$$\frac{1}{f_3}\frac{d^2 f_3}{dz^2} = 1$$

$$\frac{1}{f_1}\frac{d^2 f_1}{d\rho^2} + \frac{1}{\rho f_1}\frac{df_1}{d\rho} + \frac{1}{\rho^2 f_2}\frac{d^2 f_2}{d\phi^2} = -1$$

xply ρ^2

$$\frac{\rho^2}{f_1}\frac{d^2 f_1}{d\rho^2} + \frac{\rho}{f_1}\frac{df_1}{d\rho} + \frac{1}{f_2}\frac{d^2 f_2}{d\phi^2} = -\rho^2 \Rightarrow \frac{\rho^2}{f_1}\frac{d^2 f_1}{d\rho^2} + \frac{\rho}{f_1}\frac{df_1}{d\rho} + \rho^2 = \frac{-1}{f_2}\frac{d^2 f_2}{d\phi^2}$$

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Again LHS is a function of ρ and RHS is function ϕ

$\therefore$ we must Equate to a Constant $-\frac{1}{f_2}\frac{d^2 f_2}{d\phi^2} = n^2$

$$\frac{\rho^2}{f_1}\frac{d^2 f_1}{d\rho^2} + \frac{\rho}{f_1}\frac{df_1}{d\rho} + \rho^2 = n^2$$

$$\frac{\rho^2}{f_1}\frac{d^2 f_1}{d\rho^2} + \frac{\rho}{f_1}\frac{df_1}{d\rho} + \rho^2 - n^2 = 0.$$

$\times$ ply f_1

$$\rho^2\frac{d^2 f_1}{d\rho^2} + \rho\frac{df_1}{d\rho} + (\rho^2 - n^2)f_1 = 0$$

Sub $\rho = x$ $f_1 = y$

$$\boxed{x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0}$$

This is Bessel's differential Equation of order n in the standard form originating from the Laplace Equation in cylindrical system

Solution of Laplace Equation in spherical System leading to legendre Differential Equation

Soln: The Coordinates (r, θ, ϕ) are called the spherical polar Coordinates and the relationship with the Cartesian Coordinates (x, y, z) is given by $x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$

The Laplace Equation $\nabla^2 f = 0$ in the spherical System is given by

$$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = 0 \quad \text{(1)}$$

we shall solve this by the method of seperation of variables.

let $f = f_1 f_2 f_3$ be the soln. where $f_1 = f_1(r)$ $f_2 = f_2(\theta)$ $f_3 = f_3(\phi)$

Subsh ①

$$\frac{\partial^2}{\partial r}(f_1 f_2 f_3) + \frac{2}{r}\frac{\partial}{\partial r}(f_1 f_2 f_3) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}(f_1 f_2 f_3)\right)$$

$$+\frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial\phi}(f_1 f_2 f_3) = 0$$

$$f_2 f_3 \frac{d^2 f_1}{dr} + \frac{2}{r} f_2 f_3 \frac{df_1}{dr} + \frac{f_1 f_3}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{df_2}{d\theta}\right)$$

$$+\frac{f_1 f_2}{r^2 \sin^2\theta}\frac{d^2 f_3}{d\phi^2} = 0$$

÷ by $f_1 f_2 f_3$

$$\frac{1}{f_1}\frac{d^2 f_1}{dr^2} + \frac{2}{r f_1}\frac{df_1}{dr} + \frac{1}{r^2 \sin\theta f_2}\left[\sin\theta \frac{d^2 f_2}{d\theta^2} + \frac{df_2}{d\theta}\cos\theta\right]$$

$$+\frac{1}{r^2 \sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2} = 0$$

$$\left[\frac{1}{f_1}\frac{d^2 f_1}{dr^2} + \frac{2}{r f_1}\frac{df_1}{dr}\right] + \left[\frac{1}{r^2 f_2}\frac{d^2 f_2}{d\theta^2} + \frac{\cot\theta}{r^2 f_2}\frac{df_2}{d\theta}\right] + \frac{1}{r^2 \sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2} = 0$$

× ply r^2

$$\left[\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2} + \frac{2r}{f_1}\frac{df_1}{dr}\right] + \left[\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2} + \frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right] + \frac{1}{\sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2} = 0$$

× ply $\sin^2\theta$

$$\sin^2\theta\left[\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2} + \frac{2r}{f_1}\frac{df_1}{dr}\right] + \sin^2\theta\left[\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2} + \frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right] = -\frac{1}{f_3}\frac{d^2 f_3}{d\phi^2}$$

∴ LHS is a function of r, θ and RHS is a function of ϕ ②
they must be equal to constant.

$$\frac{1}{f_3}\frac{d^2 f_3}{d\phi^2} = 0$$ Equation ② On dividing by $\sin^2\theta$

$$\left[\frac{r^2}{f_1}\frac{d^2f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}\right]+\left[\frac{1}{f_2}\frac{d^2f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right]=0$$

$$\left[\frac{1}{f_2}\frac{d^2f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right]=-\left[\frac{r^2}{f_1}\frac{d^2f_1}{dr}+\frac{2r}{f_1}\frac{df_1}{dr}\right]$$

Again LHS is a function of θ and RHS is a function of r, with the result they must be Equal to a constant.

Set $\frac{r^2}{f_1}\frac{d^2f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}=n(n+1)$.

$$\frac{1}{f_2}\frac{d^2f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}=-n(n+1)$$

$$\frac{d^2f_2}{d\theta^2}+\cot\theta\frac{df_2}{d\theta}+n(n+1)f_2=0 \quad \text{— (4)}$$

now by taking $x=\cos\theta$ we shall convert the differential Equation given by (4) in terms of f_2 and 'x' follows

$$\frac{df_2}{d\theta}=\frac{df_2}{dx}\cdot\frac{dx}{d\theta}=\frac{df_2}{dx}(-\sin\theta)=-\sin\theta\frac{df_2}{dx} \quad \text{— (5)}$$

$$\frac{d^2f_2}{d\theta^2}=\frac{d}{d\theta}\left(\frac{df_2}{d\theta}\right)=\frac{d}{d\theta}\left(-\sin\theta\frac{df_2}{dx}\right)$$

$$=-\sin\theta\frac{d}{d\theta}\left(\frac{df_2}{dx}\right)+\frac{df_2}{dx}(-\cos\theta)$$

$$=-\sin\theta\frac{d}{dx}\left(\frac{df_2}{dx}\right)\frac{dx}{d\theta}-\frac{df_2}{dx}(\cos\theta)$$

$$=-\sin\theta\frac{d^2f_2}{dx^2}(-\sin\theta)-\frac{df_2}{dx}(\cos\theta)$$

$$\frac{d^2f_2}{d\theta^2}=\sin^2\theta\frac{d^2f_2}{dx}-\cos\theta\frac{df_2}{dx} \quad \text{— (6)}$$

Hence (4) as a consequence of (5) and (6) become

$$\sin^2\theta \frac{d^2f_2}{dx^2} - \cos\theta \frac{df_2}{dx} + \cot\theta \left(-\sin\theta \frac{df_2}{dx}\right) + n(n+1)f_2 = 0$$

$$\sin^2\theta \frac{d^2f_2}{dx^2} - \cos\theta \frac{df_2}{dx} - \cos\theta \frac{df_2}{dx} + n(n+1)f_2 = 0$$

$$(1-\cos^2\theta)\frac{d^2f_2}{dx^2} - 2\cos\theta \frac{df_2}{dx} + n(n+1)f_2 = 0$$

$$x = \cos\theta$$

$$(1-x^2)\frac{d^2f_2}{dx^2} - 2x\frac{df_2}{dx} + n(n+1)f_2 = 0$$

This Equation can be written in the following Standard from

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0}$$

This is legende's differential Equation in the Standard from orignating from the Laplace Equation in the spherical system.

Series solution of Bessel's Differential Equation leading to Bessel functions:

Proof: The Bessel differential Equation of order n is in the Form

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \quad \text{---} \ (1)$$

where n is non negative real const.

We Employ frobenius method to solve this Equation as we have the coefficient of $y'' = x^2 = P_0(x)$ (say) and $P_0(x)=0$ at $x=0$

We assume the solution of (1) in the Form

$$y=\sum_{r=0}^{\infty} a_r x^{k+r} \quad \text{---} \ (2)$$

$$y'=\sum_{r=0}^{\infty} a_r (k+r)\, x^{k+r-1}.$$

$$y''=\sum_{r=0}^{\infty} a_r (k+r)(k+r-1)\, x^{k+r-2}$$

$(1) \Rightarrow$

$$x^2\sum_{r=0}^{\infty} a_r (k+r)(k+r-1)x^{k+r-2}+x\sum_{r=0}^{\infty} a_r (k+r)x^{k+r-1}+(x^2-n^2)\sum_{r=0}^{\infty} a_r x^{k+r}=0$$

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1)x^{k+r}+\sum_{r=0}^{\infty} a_r(k+r)x^{k+r}+\sum_{r=0}^{\infty} a_r x^{k+r+2}-n^2\sum_{r=0}^{\infty} a_r x^{k+r}=0$$

Collecting the coefficient of first, second and fourth terms together.

$$\sum_{r=0}^{\infty} a_r x^{k+r}\left[\underline{(k+r)}(k+r-1)+\underline{k+r}-n^2\right]+\sum_{r=0}^{\infty} a_r x^{k+r+2}=0$$

take $k+r$ common

$$\sum_{r=0}^{\infty} a_r x^{k+r}\left[(k+r)^2-n^2\right]+\sum_{r=0}^{\infty} a_r x^{k+r+2}=0.$$

We shall Equate the coefficient of lowest degree term in x, that is x^k to zero

$a_0 (k^2 - n^2)$ | we take $a_0 \neq 0$ becoze we assumed soln as $\sum_{r=0}^{\infty} a_r x^{k+r} = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} \ldots$ if $a_0 = 0$ $= a_1 x^{k+1} \ldots$

Setting $a_0 \neq 0$ we have $k^2 - n^2 = 0 \Rightarrow k^2 = n^2 \Rightarrow k = \pm n$

also we need to independently Equate the Coefficient of x^{k+1}. $a_1 [(k+1)^2 - n^2] = 0$

$\Rightarrow a_1 = 0$ $\quad (k+1)^2 - n^2 = 0 \quad (k+1)^2 = n^2$ & $k+1 = \pm n$

which cannot be accepted as we have already becoz already we have $\boxed{k = \pm n}$ now can we have $k = \pm n$ $k+1 = \pm n$ (not pos.

Next we shall equate the coeffiont of x^{k+r} $(r \geq 2)$ to zero.

$$a_r [(k+r)^2 - n^2] + a_{r-2} = 0.$$

$$\boxed{a_r = \frac{-a_{r-2}}{[(k+r)^2 - n^2]}} \quad (r \geq 2) \quad \text{--- (3)}$$

$k = +n$ in (3) becomes.

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} = \frac{-a_{r-2}}{2nr + r^2}$$

put $r = 2, 3, 4 \ldots$

$$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)} ; \quad a_3 = \frac{-a_1}{6n+9} = 0 \quad a_1 = 0.$$

III laly $a_5, a_7 \ldots$ are all zero. $a_1 = 0 = a_3 = a_5 = a_7$

$$a_4 = \frac{-a_2}{8n+16} = \frac{-a_2}{8(n+2)} = \frac{a_0}{8(n+2)4(n+1)} = \frac{a_0}{32(n+1)(n+2)} \ldots$$

$$y = x^k (a_0 + a_1 x + a_2 x^2 + \ldots)$$

als $k = +n$ be denoted by y_1.

$$y_1 = x^n \left[a_0 - \frac{a_0}{4(n+1)} x^2 + \frac{a_0}{32(n+1)(n+2)} x^4 \ldots \right]$$

$$y_1 = a_0 x^n \left[1 - \frac{1}{2^2 (n+1)} x^2 + \frac{x^4}{2^5 (n+1)(n+2)} \ldots \right] \quad \text{--- (4)}$$

$\because$ ~~被~~ $k=-n$.

$$y_2 = a_0 x^{-n}\left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} \cdots\right] \quad (5)$$

The complet soln of (1) is given by $y = Ay_1 + By_2$ where A and B are arbitrary

we shall standardize solution (4) by choosing

$$a_0 = \frac{1}{2^n \Gamma n+1}$$

$$y_1 = \frac{1\, x^n}{2^n \Gamma n+1}\left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)2} \cdots\right]$$

$$y_1 = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma n+1}\left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)} \times \frac{1}{2} \cdots\right]$$

$$= \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma n+1} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma n+1 (n+1)(n+2)} \times \frac{1}{2}\right]$$

Using the property of gama.

$$\Gamma n = (n-1)\Gamma n-1$$

$$\Gamma n+2 = (n+1)\Gamma n+1$$

$$\Gamma n+3 = (n+2)\Gamma n+2 = (n+2)(n+1)\Gamma n+1$$

$$y_1 = \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma n+1} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma n+2} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma n+3 \cdot 2} \cdots\right]$$

$$y_1 = \left(\frac{x}{2}\right)^n\left[\frac{(-1)^0}{\Gamma n+1\ 0!}\left(\frac{x}{2}\right)^0 - \left(\frac{x}{2}\right)^2 \frac{(-1)^1}{\Gamma n+2\ 1!} + \frac{(-1)^2}{\Gamma n+3\ 2!}\left(\frac{x}{2}\right)^4 \cdots\right]$$

$$\left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\, r!}\left(\frac{x}{2}\right)^{2r}$$

$$\boxed{J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}}$$

Further, the solution $k=-n$ (inrespect of y_2) be denoted by $J_{-n}(x)$

hence the general solution of the Bessel's Equation is given

$$\boxed{y = a\,J_n(x) + b\,J_{-n}(x)}$$

Equation reducible to the form of Bessel's Equation

Consider the differential Eq. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0$ — (1)

we shall show that the Equation is reducible to form of Bessel Equation

put $t = \lambda x$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \lambda\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\times\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\lambda\frac{dy}{dt}\right) = \frac{d}{dt}\left(\lambda\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \lambda^2\frac{d^2y}{dt^2}$$

Sub $x = t/\lambda$ in (1)

$$\frac{t^2}{\lambda^2}\cdot\lambda^2\frac{d^2y}{dt^2} + \frac{t}{\lambda}\lambda\frac{dy}{dt} + (t^2 - n^2)y = 0.$$

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$$

This is the form of Bessel differential Equation whose solution is given by $y = a J_n(t) + b J_{-n}(t)$

$y = a J_n(\lambda x) + b J_{-n}(\lambda x)$ is the solution of (1).

properties of Bessel's function.

$J_{-n}(x) = (-1)^n J_n(x)$ where n is a positive integer

proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{+n+2r} \frac{1}{\overline{|+n+r+1}\, r!}$ —— (1)

$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\overline{|-n+r+1}\, r!}$ —— (2)

in (2) $\overline{|-n+r+1} = \overline{|r-(n-1)}$ is of the form $\overline{|(-K)}$ for

$K = 0, 1, 2 \ldots (n-2)$ and $\Gamma(0)$ for $n = (n-1)$

noting that $\overline{|(-k)} \to \infty$ n $\frac{1}{\overline{|-k}} \to 0$, k being integer.

we can say that $\frac{1}{\overline{|r-(n-1)}} \to 0$ $r = 0, 1, 2 \ldots (n-1)$

consider $J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\overline{|-n+r+1}\, r!}$ —— (3)

to get the formula in the form $J_n(x)$ that is $+n$

let $r-n = s$ or $r = s+n$ so that $r = n$, $s = 0$

$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{-n+2(s+n)} \frac{1}{\overline{|-n+\cancel{s+n+1}}\, s+n!}$

$\sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{n+2s} \frac{1}{\overline{|s+1}\, (s+n)!}$

using property of gamma function $\overline{|s+1} = s!$

and $s+n! = \overline{|s+n+1}$

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$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{n+2s} \frac{1}{\Gamma(n+s+1)\, s!}$$

comparing with (1) we observe that summation in the RHS is $J_n(x)$

Thus we have proved $\underline{J_{-n}(x) = (-1)^n J_n(x)}$.

propeely 2: $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$ when n is positive integer.

proof: we have $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \left(-\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$= \sum_{r=0}^{\infty} (-1)^r (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$(-1)^n \sum_{r=0}^{\infty} \left[(-1)^3\right]^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$(-1)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$J_n(-x) = (-1)^n J_n(x)$$

$$(-1)^n J_n(x) = J_{-n}(x)$$

$$\boxed{J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)}$$

Recurrence relations / Recurrence Formulae

We derive recurrence relations relating to Bessel function of different orders from the basic definition of $J_n(x)$

1. $2n J_n(x) = x\left[J_{n+1}(x) + J_{n-1}(x)\right]$

proof:

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$2n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2n}{\Gamma(n+r+1)\, r!}$$

We shall write $2n = 2(n+r) - 2r$.

$$2nJ_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2(n+r)}{\Gamma(n+r+1)\, r!} - \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2r}{\Gamma(n+r+1)\, r!}$$

we can write $\Gamma(n+r+1) = (n+r)\,\Gamma(n+r)$

$r! = r(r-1)!$

$$2nJ_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{\cancel{2}}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{\cancel{2}}{\Gamma(n+r)\,(n+r)\, r!} - \sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{\cancel{2}}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{\cancel{2}}{\Gamma(n+r+1)\,(r-1)!}$$

$\Gamma(n+r)\,(n+r)\, r!$

$$= x\sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n-1+r+1)} - x\sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)\,(r-1)!}$$

rewriting,

$$= x\sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} - x\sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)\,(r-1)!}.$$

Put $r-1 = s$ $\Rightarrow$ $r = s+1$ in the second term of RHS

$$2nJ_n(x) = x\sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} - x\sum_{s=0}^{\infty} (-1)^{s+1} \left(\frac{x}{2}\right)^{n+2s+1} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$= x\sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} + x\sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$[J_{n-1}(x) + J_{n+1}(x)]$

(2) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Proof:

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{} \left(\frac{x}{2}\right)^{n+2r-1} \times \frac{1}{2} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$n+2r = n+r+r$$

$$2J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{\cancel{n+r}}{\cancel{(n+r)}\Gamma(n+r)\, r!} + \sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{\cancel{r}}{\Gamma(n+r+1)\, \cancel{r}(r-1)!}$$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n+r)\, r!} + \sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)\, (r-1)!}$$

Put $r-1 = s$ in 2nd term.

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n+r)\, r!} + \sum_{s=0}^{\infty} (-1)^{s+1} \left(\frac{x}{2}\right)^{n+2s+1} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} - \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$= \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

(3) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma_{n+r+1}\, r!}$

$$x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r x^{2n+2r} \frac{1}{2^{n+2r}\Gamma_{n+r+1}\, r!}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r (2n+2r)\, x^{2n+2r-1} \frac{1}{2^{n+2r}(n+r)\Gamma_{n+r}\, r!}$$

$$= \sum_{r=0}^{\infty} (-1)^r 2(n+r)\, x^n \cdot x^{n+2r-1} \frac{1}{2^{n+2r}(n+r)\Gamma_{n+r}\, r!}$$

$$x^n \sum_{r=0}^{\infty} (-1)^r x^{n+2r-1} \frac{1}{2^{n+2r-1}\Gamma_{n+r}\, r!}$$

$$x^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma_{n-1+r+1}\, r!}$$

$$\boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)}$$

(4) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma_{n+r+1}\, r!}$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} (-1)^r x^{-n}(x^n)\, x^{2r} \frac{1}{2^{n+2r}\Gamma_{n+r+1}\, r!}$$

$$= \sum_{r=0}^{\infty} (-1)^r x^{2r} \frac{1}{2^{n+2r}\Gamma_{n+r+1}\, r!}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r\, 2r\, x^{2r-1} \frac{1}{2^{n+2r}\Gamma_{n+r+1}\, r!}$$

xply ÷ x^n w

$$\sum_{r=1}^{\infty} (-1)^r \cancel{r}\, x^{-n} x^{n+2r-1} \frac{1}{2^{n+2r-1}\Gamma_{n+r+1}\,(r-1)\cancel{r}!}$$

put $r-1=s$ or $r=s+1$

$$x^{-n}\sum_{s=0}^{\infty}(-1)^{s+1}\left(\frac{x}{2}\right)^{n+2s+2-1}\frac{1}{\Gamma(n+1+s+1)\,s!}$$

$$-x^{-n}\sum_{s=0}^{\infty}(-1)^{s}\left(\frac{x}{2}\right)^{n+1+2s}\frac{1}{\Gamma(n+1+s+1)\,s!}$$

$$\boxed{\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)}$$

(3) $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

proof. $$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_n'(x)=\sum_{r=0}^{\infty}(-1)^r(n+2r)\left(\frac{x}{2}\right)^{n+2r-1}\cdot\frac{1}{2}\,\frac{1}{\Gamma(n+r+1)\,r!}$$

$$xJ_n'(x)=\sum_{r=0}^{\infty}(-1)^r(n+2r)\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!}$$

let us write $n+2r=2(n+r)-n$.

$$xJ_n'(x)=\sum_{r=0}^{\infty}(-1)^r\,2\cancel{(n+r)}\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\cancel{(n+r)}\Gamma(n+r)\,r!}+\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{-n}{\Gamma(n+r+1)\,r!}$$

$$=\sum_{r=0}^{\infty}(-1)^r\,2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{\Gamma(n+r)\,r!}-n\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!}$$

$$=x\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{\Gamma(n-1+r+1)\,r!}-nJ_n(x)$$

$$\boxed{xJ_n'(x)=xJ_{n-1}(x)-nJ_n(x)}$$

(6) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r n \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} + \sum_{r=0}^{\infty} (-1)^r 2r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$= n J_n(x) + \sum_{r=1}^{\infty} (-1)^r 2 \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, (r-1)!}$$

put $r-1 = s$ & $r = s+1$

$$= n J_n(x) + \sum_{s=0}^{\infty} (-1)^{s+1} 2 \left(\frac{x}{2}\right)^{n+2s+2} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$= n J_n(x) - \sum_{s=0}^{\infty} (-1)^s 2\left(\frac{x}{2}\right) \left(\frac{x}{2}\right)^{n+2s+1} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$= n J_n(x) - x \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$\boxed{x J_n'(x) = n J_n(x) - x J_{n+1}(x)}$$

problems

prove that :

(a) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ (b) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

by def :

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

Put $n = 1/2$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{1/2+2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

$$\sqrt{\frac{x}{2}} \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

$$\sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma 3/2} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma 5/2\; 1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma 7/2\; 2!} - - - \right] \quad (1)$$

We know that $\Gamma 1/2 = \sqrt{\pi}$.

$\Gamma n = (n-1)\Gamma n-1$.

$\Gamma 3/2 = \Gamma 1/2+1 = 1/2\, \Gamma 1/2 = 1/2 \sqrt{\pi} = \frac{\sqrt{\pi}}{2}$

$\Gamma 5/2 = \Gamma 3/2+1 = 3/2 \cdot 1/2 \sqrt{\pi} = 3/4 \sqrt{\pi}$

$\Gamma 3/2 = \Gamma(1/2+1)$
$= 1/2\, \Gamma 1/2$
$= 1/2 \sqrt{\pi}$

$\Gamma 7/2 = 5/2 \cdot 3/2 \cdot 1/2 \sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$ Sub all these value in (1)

$$J_{1/2}(x) = \sqrt{x/2} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4} \cdot \frac{4}{3\sqrt{\pi}} + \frac{x^4}{\cancel{16}} \frac{8}{15\sqrt{\pi}\, 2!} - - - \right.$$

$$\sqrt{x/2\pi} \left[2 - \frac{x^2}{3} + \frac{x^4}{30} - - - \right]$$

(x) pwr ÷ by 2

$$\sqrt{x/2\pi} \cdot \frac{2}{x} \left[x - \frac{x^3}{6} + \frac{x^5}{60} - - - \right]$$

we have taken $2/x$ as a common factor keeping in view

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$\boxed{J_{1/2} = \sqrt{\frac{2}{\pi x}}\sin x}$$

also put $n = -1/2$ in (1)

$$J_{-1/2}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-1/2+2r}\frac{1}{\Gamma(r+1/2)\, r!}$$

$$\left(\frac{x}{2}\right)^{-1/2}\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{1}{\Gamma(r+1/2)\, r!}$$

$$\sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma 1/2} - \left(\frac{x}{2}\right)^2\frac{1}{\Gamma 3/2\; 1!} + \left(\frac{x}{2}\right)^4\frac{1}{\Gamma 5/2\; 2!} - \cdots\right.$$

$$\sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4}\frac{2}{\sqrt{\pi}\; 1!} + \frac{x^4}{16}\frac{4}{3\sqrt{\pi}\; 2!} - \cdots\right.$$

$$\sqrt{\frac{2}{x\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$\boxed{J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x}$$

② obtain the expressions for $J_{1/2}(x)$ and $J_{-1/2}(x)$. Then use a suitable recurrence relation to deduce for $J_{3/2}(x)$ & $J_{-3/2}(x)$

Soln $\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

let us consider the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \quad \text{—①}$$

put $n = 1/2$ in ①

$$J_{-1/2}(x) + J_{3/2}(x) = \frac{1}{x} J_{1/2}(x)$$

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x\right] = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x - x\cos x}{x}\right]$$

put $n = -1/2$

$$J_{-3/2}(x) + J_{1/2}(x) = -\frac{1}{x} J_{-1/2}(x)$$

$$J_{-3/2}(x) = -\left[J_{1/2}(x) + \frac{1}{x} J_{-1/2}(x)\right]$$

$$= -\left[\sqrt{\frac{2}{\pi x}} \sin x + \frac{1}{x}\sqrt{\frac{2}{\pi x}} \cos x\right]$$

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left[\frac{x\sin x + \cos x}{x}\right]$$

(3) Show that $J_{3/2}(x)\sin x - J_{-3/2}(x)\cos x = \sqrt{\frac{2}{\pi x^3}}$

Soln:- We know $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x - x\cos x}{x}\right]$

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{x\sin x + \cos x}{x}\right]$$

$$J_{3/2}(x)\sin x - J_{-3/2}(x)\cos x = \sqrt{\frac{2}{\pi x}} \times \frac{1}{x}\left[\sin^2 x - x\sin x\cos x + x\sin x\cos x + \cos^2 x\right]$$

$$= \sqrt{\frac{2}{\pi x} \times \frac{1}{x}} = \frac{2}{\pi x^3}$$

(4) prove the following results

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$$

$$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right]$$

Soln: from the recurrence relation we have

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

$$n = 3/2$$

$$J_{1/2}(x) + J_{5/2}(x) = \frac{3}{x}J_{3/2}(x)$$

$$J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$$

$$J_{5/2}(x) = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x - x\cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x - 3x\cos x - x^2\sin x}{x^2}\right]$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{(3-x^2)\sin x}{x^2} - \frac{3\cos x}{x}\right]$$

$n = -3/2$

$$J_{-5/2}(x) + J_{-1/2}(x) = -\frac{3}{x} J_{-3/2}(x)$$

$$J_{-5/2}(x) = -\frac{3}{x} J_{-3/2}(x) - J_{-1/2}(x)$$

$$= -\frac{3}{x} \cdot -\sqrt{\frac{2}{\pi x}} \left[\frac{x \sin x + \cos x}{x}\right] - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{3x \sin x + 3\cos x - x^2 \cos x}{x^2}\right]$$

$$\boxed{J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3\sin x}{x} + \frac{3 - x^2}{x^2} \cos x\right]}$$

(5) Starting from the expressions of $J_{1/2}(x)$ and $J_{-1/2}(x)$ in the standard form prove the following results

(a) $J'_{1/2}(x)\, J_{-1/2}(x) - J'_{-1/2}(x)\, J_{1/2}(x) = \frac{2}{\pi x}$

(b) $\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(2x)\, dx = \frac{1}{\sqrt{\pi}}$

Soln

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x .$$

$$J'_{1/2}(x) = \sqrt{\frac{2}{\pi}} \left[\frac{1}{\sqrt{x}} \cos x + \sin x \times -\frac{1}{2} x^{-3/2}\right]$$

$$J'_{1/2}(x) = \sqrt{\frac{2}{\pi}} \left[\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x\sqrt{x}}\right]$$

$$J'_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\cos x - \frac{\sin x}{2x}\right).$$

$$J'_{-1/2}(x) = \sqrt{\frac{2}{\pi}} \left[\frac{1}{\sqrt{x}}(-\sin x) + \cos x \times -\frac{1}{2} x^{-3/2}\right]$$

$$= -\sqrt{\frac{2}{\pi x}} \left[\sin x + \frac{\cos x}{2x}\right]$$

$J'_{1/2}(x)\, J_{-1/2}(x) - J'_{-1/2}(x)\, J_{1/2}(x)$

$$\sqrt{\frac{2}{\pi x}}\left[\cos x - \frac{\sin x}{2x}\right] \times \sqrt{\frac{2}{\pi x}}\cos x + \sqrt{\frac{2}{\pi x}}\sin x$$

This prove

$$\sqrt{\frac{2}{\pi x}} \times \sqrt{\frac{2}{\pi x}}\left[\cos^2 x + \sin^2 x\right] = \frac{2}{\pi x} \quad //$$

(b) $$J_{1/2}(2x) = \sqrt{\frac{2}{\pi(2x)}}\sin 2x = \sqrt{\frac{1}{\pi x}}\sin 2x = \frac{1}{\sqrt{\pi x}}\sin 2x$$

$$\sqrt{x}\, J_{1/2}(2x) = \sqrt{x}\,\frac{1}{\sqrt{\pi x}}\sin 2x = \frac{1}{\sqrt{\pi}}\sin 2x$$

$$\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(2x)\,dx = \int_0^{\pi/2} \frac{1}{\sqrt{\pi}}\sin 2x\,dx$$

$$= \frac{1}{\sqrt{\pi}}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= -\frac{1}{2\sqrt{\pi}}\left(\cos \pi - \cos 0\right) = \frac{1}{\sqrt{\pi}}$$

$$\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(2x)\,dx = \frac{1}{\sqrt{\pi}}.$$

$\boxed{14/01/2015}$

(6) Starting from the Series expression for $J_n(x)$ prove that $J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$ Hence deduce that

$$J_4(x) = \frac{8}{x}\left(\frac{6}{x^2} - 1\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

Soln: $$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x) \quad \text{--- (1)}$$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

put $n=3$ $$J_4(x) = \frac{2\times 3}{x} J_3(x) - J_2(x). \quad \text{--- (2)}$$

put $n=1,2$ in ① to obtain $J_2(x)$ and $J_3(x)$.

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$J_3(x) = \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

Sub in ②

$$J_4(x) = \frac{6}{x}\left[\frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= J_1(x)\left[\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right] + J_0(x)\left[1 - \frac{24}{x^2}\right]$$

$$\boxed{J_4(x) = \frac{8}{x}\left[\frac{6}{x^2} - 1\right] J_1(x) + \left[1 - \frac{24}{x^2}\right] J_0(x).}$$

⑤ prove that $J_0'(x) = -J_1(x)$

Soln: we know that $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

put $n=0$

$$\frac{d}{dx}\left[J_0(x)\right] = -J_1(x)$$

$$\boxed{J_0'(x) = -J_1(x)}$$ or $J_1(x) = -J_0'(x)$

⑧ prove that $\int J_3(x)\,dx + J_2(x) + \frac{2}{x} J_1(x) = 0$

$$\int J_3(x)\,dx = -J_2(x) - \frac{2}{x} J_1(x) + C$$

Soln: consider the recurrence relation

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x).$$

Inte both side

$$\int x^{-n} J_{n+1}(x) = -x^{-n} J_n(x) \quad \text{—①}$$

let us write $J_3(x) = x^2 \left[x^{-2} J_3(x) \right]$

$\therefore \int J_3(x)\,dx = \int x^2 \left[x^{-2} J_3(x) \right] dx.$

Inte RHS by parts

$\int u v\,dx = u v - \int v\,du.$ $u = x^2$ $v = \int x^{-2} J_3(x)$

$$\int J_3(x)\,dx = x^2 \int x^{-2} J_3(x)\,dx - \int \left[\int x^{-2} J_3(x) \right] 2x\,dx \quad \text{——} (2)$$

From (1) we have $n=2$ $\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x)$

Sub in (2)

$$\int J_3(x)\,dx = x^2 \left[-x^{-2} J_2(x) \right] - \int -\left\{ x^{-2} J_2(x) \right\} 2x\,dx.$$

$$\int J_3(x)\,dx = -J_2(x) + 2 \int x^{-1} J_2(x)\,dx.$$

From (1) we have $n=1$ $\int x^{-1} J_2(x)\,dx = -x^{-1} J_1(x)$

$$\int J_3(x)\,dx = -J_2(x) + 2 \left[-x^{-1} J_1(x) \right]$$

$$\boxed{\int J_3(x)\,dx = -J_2(x) - \frac{2}{x} J_1(x) + c}$$ where c being the constant of x

(9) Starting from a recurrence relation show that

(i) $4 J_n'' = J_{n-2} - 2J_n + J_{n+2}$

(ii) $8 J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}.$

Sol: From the recurrence Relation we have

$$J_n'(x) = \frac{1}{2} \left[J_{n-1}(x) - J_{n+1}(x) \right]$$

$$2J_n'(x) = \left[J_{n-1}(x) - J_{n+1}(x) \right] \quad \text{——} (1)$$

Diff w.r.t x we have

$$2J_n''(x) = J'_{n-1}(x) - J'_{n+1}(x).$$

xply (2):

$$4 J_n''(x) = 2J'_{n-1}(x) - 2J'_{n+1}(x).$$

we use (1) in RHS of this Equation replacing n by $(n-1)$ and also by $(n+1)$

Use ① in the RHS of this Equation replacing n by $(n-1)$ and also by $n+1$

$$4J_n'' = (J_{n-2} - J_n) - (J_n - J_{n+2})$$

$$4J_n'' = J_{n-2} - 2J_n - J_{n+2} \quad \text{as required}$$

Diff this w.r.t x

$$4J_n''' = J'_{n-2} - 2J_n' - J'_{n+2}$$

xply by ②

$$8J_n''' = 2J'_{n-2} - 2(2J_n') - 2J'_{n+2}$$

Using ① in RHS by replacing 'n' by $(n-2)$ & $n+2$.

$$8J_n''' = (J_{n-3} - J_{n-1}) - 2(J_{n-1} - J_{n+1}) + (J_{n+1} - J_{n+3})$$

$$\boxed{8J_n''' = J_{n-3} - 3J_{n-1} + J_{n+3} + 3J_{n+1}} \quad \text{as required}$$

(10) Prove that $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

Soln: we know that $8J_n''' = J_{n-3} - 3J_{n-1} + J_{n+3} + 3J_{n+1}$

put $n=0$

$$8J_0''' = J_{-3} - 3J_{-1} + 3J_1 - J_3$$

We know that $J_{-n} = (-1)^n J_n$.

$$8J_0''' = -J_3 + 3J_1 + 3J_1 - J_3$$

$$8J_0''' = 6J_1 - 2J_3$$

$$4J_0''' = 3J_1 - J_3$$

We know that $-J_1 = \frac{d}{dx}(J_0)$ or $J_1 = -J_0'$

$$4J_0''' = -3J_0' - J_3$$

$$\boxed{4J_0'''(x) + 3J_0'(x) + J_3(x) = 0}$$

11. Show that $\frac{d}{dx}\left[x J_n J_{n+1}\right] = x\left[J_n^2 - J_{n+1}^2\right]$

Soln

$$\frac{d}{dx}\left[x J_n J_{n+1}\right] = x\left[J_n J_{n+1}' + J_{n+1} J_n'\right] + J_n J_{n+1}.$$

$$= x J_n J_{n+1}' + x J_{n+1} J_n' + J_n J_{n+1} \quad \text{—} (1)$$

Using the recurrence relations

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{——} (2)$$

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x) \quad \text{——} (3)$$

Replacing n by $n+1$ in (3) becomes

$$x J_{n+1}'(x) = x J_n(x) - (n+1) J_{n+1}(x) \quad \text{——} (4)$$

Sub (4) and (2) in the RHS of (1) we get

$$J_n\left[x J_n(x) - (n+1) J_{n+1}\right] + J_{n+1}\left[n J_n(x) - x J_{n+1}(x)\right] + J_n J_{n+1}(x)$$

$$= x J_n^2(x) - \cancel{n J_{n+1} J_n} - J_n \cancel{J_{n+1}} + \cancel{n J_{n+1} J_n(x)} - x J_{n+1}^2 + J_n \cancel{J_{n+1}}$$

$$= x J_n^2(x) - x J_{n+1}^2 = x\left[J_n^2(x) - J_{n+1}^2\right]$$

Thus we have proved that

$$\boxed{\frac{d}{dx}\left[x J_n J_{n+1}\right] = x\left[J_n^2(x) - J_{n+1}^2\right]}$$

(12). prove that $\lim_{x\to 0} \frac{J_n(x)}{x^n} = \frac{1}{2^n \Gamma(n+1)}$; $(n > -1)$

Soln: we know that $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)} - (-1)^1\left(\frac{x}{2}\right)^{n+3} + \text{— —}$$

$$\cancel{= \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2^2\Gamma(n+2)} + \frac{x^4}{2^4\Gamma(n+3)} + \frac{x^6}{2^6\Gamma(n+4)} - - -\right.}$$

$$\cancel{\left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}}\left[1 - x^2\right.$$

$$= \frac{x^n}{2^n}\frac{1}{\Gamma(n+1)}\left[1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2^4(n+2)(n+1)}-\frac{x^6}{2^6(n+1)(n+2)(n+3)}\cdots\right.$$

$$\Rightarrow \lim_{x\to 0}\frac{J_n(x)}{x^n} = \lim_{x\to 0}\frac{x^n}{x^n 2^n\Gamma(n+1)}\left[1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2^4(n+1)(n+2)}-\cdots\right.$$

$$\boxed{\lim_{x\to 0}\frac{J_n(x)}{x^n} = \frac{1}{2^n\Gamma(n+1)}}$$

(13) Show that $\int_0^x x^{-n} J_{n+1}(x)\,dx = \frac{1}{2^n\Gamma(n+1)} - \frac{J_n(x)}{x^n}$

Sol: we have recurrence relation

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

$$\int_0^x x^{-n}J_{n+1}(x) = -\left[x^{-n}J_n(x)\right]_0^x$$

$$\int_0^x x^{-n}J_{n+1}(x) = -x^{n}J_n(x) + \left[x^{-n}J_n(x)\right]_{x=0} \quad \text{——(1)}$$

The second term in the RHS is in indeterminat form 0/0 and hence we use the concept of limit

$$\lim_{x\to 0}\frac{J_n(x)}{x^n} = \lim_{x\to 0}\frac{1}{x^n}\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\, r!}$$

$$= \lim_{x\to 0}\frac{1}{x^n}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{2^{n+2r}}x^n\frac{1}{\Gamma(n+r+1)\, r!}$$

$$= \lim_{x\to 0}\sum_{r=0}^{r}(-1)^r\frac{x^{2r}}{2^{n+2r}\Gamma(n+r+1)\, r!}$$

$$\lim_{x\to 0}x^{-n}J_n(x) = \lim_{x\to 0}\left[\frac{1}{2^n\Gamma(n+1)} - \frac{x^2}{2^{n+2}\Gamma(n+2)\,1!} + \frac{x^4}{2^{n+4}\Gamma(n+3)\,2!}-\cdots\right.$$

Verify that $y = x^n J_n(x)$ is a solution of the differential Equation $xy'' + (1-2n)y' + xy = 0$

Soln by data $y = x^n J_n(x)$

$$y' = x^n J_n'(x) + n x^{n-1} J_n(x)$$

$$y'' = x^n J_n''(x) + n x^{n-1} J_n'(x) + n(n-1) x^{n-2} J_n(x) + n x^{n-1} J_n'(x)$$

$$y'' = x^n J_n''(x) + 2n x^{n-1} J_n'(x) + n(n-1) x^{n-2} J_n(x)$$

$$xy'' + (1-2n) y' + xy = 0$$

$$x\left[x^n J_n''(x) + 2n x^{n-1} J_n'(x) + n(n-1) x^{n-2} J_n(x)\right] + (1-2n)\left[x^n J_n'(x) + n x^{n-1} J_n(x)\right] + x x^n J_n(x) = 0$$

$$x^{n+1} J_n''(x) + \cancel{2n x^n J_n'(x)} + n(n-1) x^{n-1} \cancel{J_n(x)} + \underline{x^n J_n'(x)} + n x^{n-1} J_n(x) - \cancel{2n x^n J_n'(x)} - 2n^2 x^{n-1} J_n(x) + \underline{x^{n+1} J_n(x)} = 0$$

$$x^{n+1} J_n''(x) + x^n J_n'(x) + x^{n+1} J_n(x) - n^2 x^{n-1} J_n(x).$$

$$x^{n-1}\left[x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x)\right]$$

$x^{n-1} \cdot 0$ $\quad \because J_n(x)$ is a soluln of $x^2y'' + xy' + (x^2-n^2)y = 0$

Thus we have $y = x^n J_n(x)$ is a soln of the Equ.

2. Verify that $y = \sqrt{x}\, J_{3/2}(x)$ is a solution of differential Equation $x^2 y'' + (x^2 - 2) y = 0$

(2) prove that $\frac{d}{dx}\left[x\left[J_n'(x)\, J_{-n}(x) - J_{-n}'(x)\, J_n(x)\right]\right]=0$

Soln: we know that $J_n(x)$ and $J_{-n}(x)$ are solutions of the Bessel's Equation

$x^2y''+xy'+(x^2-n^2)y=0$

if $u=J_n(x)$ and $v=J_{-n}(x)$

$x^2u''+xu'+(x^2-n^2)u=0$ — (1)

$x^2v''+xv'+(x^2-n^2)v=0$ — (2)

(x) (1) by v and (2) by u

$x^2vu''+xvu'+(x^2-n^2)uv=0$

$x^2v''u+xuv'+(x^2-n^2)uv=0$

on subtracting and divide by x

$x(vu''-v''u)+(vu'-v'u)=0$

$\frac{d}{dx}\left[x(vu'-v'u)\right]=0$

$\frac{d}{dx}\left[x\left[J_{-n}(x)\, J_n'(x) - J_{-n}'(x)\, J_n(x)\right]\right]=0$

(3) obtain the solution of the following equation in

$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{1}{9x^2}\right)y=0$

Soln xply by x^2.

$x^2y''+xy'+(x^2-1/9)y=0$ — (1)

we have Bessel's Equation.

$x^2y''+xy'+(x^2-n^2)y=0$

The soln is given by $y=aJ_n(x)+bJ_{-n}(x)$ — (2)

comparing (1) & (2) $n^2=1/9 \quad n=\pm 1/3$

Thus the soln of this equation.

$y=aJ_{1/3}(x)+bJ_{-1/3}(x)$

Show that the solution of the Equation

$x^2y''+xy'+(x^2-1/4)y=0$ is $c_1\frac{\sin x}{\sqrt{x}}+c_2\frac{\cos x}{\sqrt{x}}$.

Soln: $x^2y''+xy'+(x^2-n^2)y=0$ we have $n^2=1/4$ $\therefore n=\pm 1/2$

Soln $y=a\,J_{1/2}(x)+b\,J_{-1/2}(x)$

But $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

hence $y=a\sqrt{\frac{2}{\pi x}}\sin x+b\sqrt{\frac{2}{\pi x}}\cos x$

$y=\left(a\sqrt{2/\pi}\right)\frac{\sin x}{\sqrt{x}}+b\sqrt{2/\pi}\,\frac{\cos x}{\sqrt{x}}$

hence $C_1=a\sqrt{2/\pi}$ $C_2=b\sqrt{2/\pi}$ c_1 & c_2 are arbitrary

Orthogonal property of Bessel Functions

If α and β are two distinct roots of $J_n(x)=0$ then

$$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx=\begin{cases}0 & \text{if } \alpha\neq\beta\\ \frac{1}{2}\left[J_n'(\alpha)\right]^2=\frac{1}{2}\left[J_{n+1}(\alpha)\right]^2 & \text{if } \alpha=\beta\end{cases}$$

proof: we know that $J_n(\lambda x)$ is a solution of the Equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-n^2)y=0$$

$u=J_n(\alpha x)$ and $v=J_n(\beta x)$ the associated differential Equation.

$$x^2u''+xu'+(\alpha^2x^2-n^2)u=0 \quad \text{—}①$$

$$x^2v''+xv'+(\beta^2x^2-n^2)v=0 \quad \text{—}②$$

$\otimes$ply ① by $\frac{v}{x}$ ② by $\frac{u}{x}$. we obtain.

$$xvu''+vu'+\alpha^2uvx-\frac{n^2uv}{x}=0$$

$$xuv''+uv'+\beta^2uvx-\frac{n^2uv}{x}=0$$

$$\frac{d}{dx}\left\{x(vu'-uv')\right\}=(\beta^2-\alpha^2)xuv$$

Integrate both sides w.r.t x 0/1

$$\left[x(vu' - uv')\right]_{x=0}^{1} = (\beta^2-\alpha^2)\int_{x=0}^{1} x\,u\,v\,dx.$$

$$vu' - uv'\Big/_{x=1} - 0 = (\beta^2-\alpha^2)\int_0^1 x\,u\,v\,dx \quad \text{—③}$$

Since $u = J_n(\alpha x)$, $v = J_n(\beta x)$ we have $u' = \alpha J_n'(\alpha x)$

$v' = \beta J_n'(\beta x)$.

and as a consequence of these ③ becomes

$$J_n(\beta x)\,\alpha J_n'(\alpha x) - J_n(\alpha x)\,\beta J_n'(\beta x)\Big]_{x=1} = (\beta^2-\alpha^2)\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx$$

$$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \frac{1}{\beta^2-\alpha^2}\Big[\alpha J_n(\beta x)\,J_n'(\alpha x) - \beta J_n(\alpha x)\,J_n'(\beta x)\Big]. \quad \text{—④}$$

Since α, and β are distinct roots of $J_n(x) = 0$ we have $J_n(\alpha) = 0$ & $J_n(\beta) = 0$. with the RHS of ④ becomes 0 provided $\beta^2-\alpha^2 \neq 0$ or $\beta \neq \alpha$. [If $\beta = \alpha$ we get $0/0$]

Thus we proved if $\alpha \neq \beta$

$$\boxed{\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = 0} \quad \text{—⑤}$$

we shall discuss the case $\alpha = \beta$

The RHS of ⑤ becomes an an indeterminate form of the type $\frac{0}{0}$ when $\alpha = \beta$.

we shall Evaluate by taking limits on both sides as $\beta \to \alpha$ keeping α fixed, by applying L' Hospital's Rule.

i.e $$\lim_{\beta\to\alpha}\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \lim_{\beta\to\alpha}\frac{1}{\beta^2-\alpha^2}\Big[\alpha J_n(\beta)\,J_n'(\alpha) - \beta J_n(\alpha x)\,J_n'(\beta x)\Big]$$

$\because \alpha$ fixed we must have $J_n(\alpha) = 0$ as α is root of $J_n(x) = 0$.

$$\therefore \lim_{\beta\to\alpha} \int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \lim_{\beta\to\alpha} \frac{1}{\beta^2-\alpha^2} \{\alpha J_n(\beta) J_n'(\alpha)\}.$$

$$= \lim_{\beta\to\alpha} \frac{1}{2\beta} \{\alpha J_n'(\beta) J_n'(\alpha)\}$$

The Nr and dr differentiated seperately w.r.t β by L'Hospital.

$$\int_0^1 x[J_n(\alpha x)]^2 dx = \frac{1}{2\alpha} \alpha J_n'(\alpha) J_n'(\alpha) = \frac{1}{2}[J_n'(\alpha)]^2$$

$$\int_0^1 x J_n^2(\alpha x) dx = \frac{1}{2}[J_n'(\alpha)]^2. \quad \text{——— } 6$$

Further we have the recurrence relation $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}$

$$J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha)$$

$\because J_n(\alpha)=0$ we obtain $J_n'(\alpha) = -J_{n+1}(\alpha)$

Eq 6 becomes:

$$\int_0^1 x J_n^2(\alpha x)\, dx = \frac{1}{2}[J_{n+1}(\alpha)]^2.$$

This result is known as Lommel integral formula

NOTE: The orthogonal property is also presented in the form.

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{a^2}{2} J_{n+1}(\alpha a)]^2 & \text{if } \alpha = \beta \end{cases}$$

Series solution of Legendre's Differential Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{--- (1)}$$

The coefficient of $y'' = (1-x^2) = P_0(x)$ and $P_0(x) \neq 0$ at $x=0$

we Employ power series method

$$y = \sum_{r=0}^{\infty} a_r x^r \quad \text{--- (2)}$$

$$y' = \sum_{r=0}^{\infty} a_r r x^{r-1} \qquad y'' = \sum_{r=0}^{\infty} a_r r(r-1)x^{r-2}$$

$$(1) \Rightarrow (1-x^2)\sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - 2x\sum_{r=0}^{\infty} a_r r x^{r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^r - 2\sum_{r=0}^{\infty} a_r r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

we Equate various powers of x to Zero.

we first Equate x^{-2} & x^{-1} available only in the first summation to zero.

Coeff of x^{-2} $\quad a_0\, 0(-1) = 0 \Rightarrow a_0 \neq 0$.

x^{-1} $\quad a_1 (1)(0) = 0 \Rightarrow a_1 \neq 0$.

Next we shall Equate coefficient of x^r $(r \geq 0)$

$$a_{r+2}(r+2)(r+1) - a_r r(r-1) - 2a_r r + n(n+1)a_r = 0$$

$$a_{r+2}(r+2)(r+1) = a_r\left[r(r-1) + 2r - n(n+1)\right]$$

$$\boxed{a_{r+2} = \frac{-\left[n(n+1) - r^2 - r\right]}{(r+2)(r+1)}} \quad \text{--- (3)}$$

put $r = 0, 1, 2, 3$ --- (3)

$$a_2 = \frac{-n(n+1)}{2}a_0\ ; \quad a_3 = \frac{-(n^2+n-2)}{6}a_1 = -\frac{(n-1)(n-2)}{6}a_1$$

$$a_4 = \frac{-(n^2+n-6)}{12}a_2 = -\frac{(n-2)(n+3)}{12} \cdot \frac{-n(n+1)}{2}a_0.$$

$$a_4 = -\frac{(n^2+n-6)}{12} \cdot a_2 = -\frac{(n-2)(n+3)}{12} \times -\frac{n(n+1)}{2} a_0$$

$$a_4 = \frac{n(n+1)(n-2)(n+3)}{24} a_0.$$

$$a_5 = \frac{(n-1)(n+2)(n-3)(n+4)}{120} a_1 \text{ and so on } - -$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{ - - -}$$

$$y = (a_0 + a_2 x + a_4 x^2 - -) + (a_1 x + a_3 x^3 + a_5 x^5 - -)$$

$$y = a_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4 - - \right]$$

$$+ a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} x^5 - - \right] \quad (4)$$

Let $U(x)$ & $V(x)$ be two infinit seri of (4)

$$\boxed{y = a_0 U(x) + a_1 V(x)}$$

Rodrigue's Formula.

we derive a formula for the legendre polynomials $P_n(x)$ in the form

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$ Known as Rodrigue's formula

proof: let $u = (x^2-1)^n$

we shall first establish that the n^{th} derivative of u, that is u_n is a soln of legendre's differential Equation

$$(1-x^2) y'' - 2x y' + n(n+1) y \quad \text{—— (1)}$$

Diff u w.r.t 'x'

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1} 2x.$$

$$(x^2-1) u_1 = 2nx (x^2-1)^n$$

$$(x^2-1) u_1 = 2nx u.$$

Diff w.r.t x.

$$(x^2-1) u_2 + 2x u_1 = 2n (x u_1 + u).$$

$$(uv)_n = u v_n + n u_1 v_{n-1} + \frac{n(n-1)}{2!} u_2 v_{n-2} - - - -$$

$$(x^2-1) u_2 /_n + 2 [x u_1]_n = 2n [x u_1]_n + 2n u_n.$$

$$(x^2-1) u_{n+2} + n 2x u_{n+1} + \frac{n(n-1)}{2} 2 u_n + 2 [x u_{n+1} + n 1 u_n]$$

$$= 2n [x u_{n+1} + n.1 u_n] + 2n u_n.$$

$$(x^2-1) u_{n+2} + 2nx \cancel{u_{n+1}} + (n^2-n) u_n + 2x u_{n+1} + \cancel{2n u_n}$$

$$= 2nx \cancel{u_{n+1}} + 2n^2 u_n + \cancel{2n u_n}.$$

$$(x^2-1) u_{n+2} + 2x u_{n+1} - n^2 u_n - n u_n = 0$$

$$(x^2-1) u_{n+2} + 2x u_{n+1} - n u_n (n+1) = 0$$

$$(1-x^2) u_{n+2} - 2x u_{n+1} + n(n+1) u_n = 0$$

This can be put in the form

$$(1-x^2)\,U_n'' - 2x\,U_n' + n(n+1)\,U_n = 0 \quad \text{—(2)}.$$

compare (2) with (1) we conclude U_n is a solution of the legendre's differential Equation it may be observed that u is a polynomial of degree $2n$ and hence U_n will be polynomial of degree n.

also $P_n(x)$ which satisfies the legendre differential Equation is also a polynomial of degree n. hence U_n must be the same as $P_n(x)$ but for some constant fact K

$$P_n(x) = K U_n = K[(x^2-1)^n]_n$$

$$P_n(x) = K[(x-1)^n(x+1)^n]_n$$

$$P_n(x) = K\Big[(x-1)^n\{(x+1)^n\}_n + n\,n(x-1)^{n-1}\{(x+1)^n\}_{n-1}$$

$$+ \frac{n(n-1)}{2}\,n(n-1)(x-1)^{n-2}\{(x+1)^n\}_{n-2}$$

$$\cdots \{(x-1)^n\}_n (x+1)^n\Big] \quad \text{—(3)}$$

it should be observed that if $Z = (x-1)^n$

$$Z_1 = n(x-1)^{n-1}, \quad Z_2 = n(n-1)(x-1)^{n-2} \ldots \text{etc}.$$

$$Z_n = n(n-1)(n-2) \ldots 2.1\,(x-1)^{n-n}$$

$$Z_n = n!\,(x-1)^0 = n!$$

$\{(x-1)^n\}_n = n!$, we proceed to find K by choosing a suitable value of x

put $x=1$ in (3) the RHS all the terms in RHS becomes zero except last term which $n!\,(1+1)^n$
$n!\,2^n$

$P_n(1) = K n! 2^n$ $P_n(1) = 1$

$$\boxed{K = \frac{1}{n! 2^n}}$$

$$\boxed{P_n(x) = K u_n = \frac{1}{n! 2^n} \{(x^2-1)^n\}_n}$$

Thus we have proved Rodrigue's formula

Problems : ① using Rodrigue's formula obtain expression for $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ $P_4(x)$ and $P_5(x)$, hence Express x^2, x^3, x^4, x^5 in terms legende polynomial.

Sol

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

we shall put $n = 0, 1, 2, 3, 4, 5$

$$P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2-1)^0 = 1$$

$$\boxed{P_0(x) = 1}$$

$$P_1(x) = \frac{1}{2^1 1!} \frac{d^1}{dx^1} (x^2-1)^1 = \frac{1}{2^1 1} \frac{d^1}{dx} (x^2-1)$$

$$= \frac{1}{2} (2x) = x$$

$$\boxed{P_1(x) = x}$$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2-1)^2 = \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1).$$

$$= \frac{1}{8} \frac{d}{dx} (4x^3 - 4x)$$

$$= \frac{1}{8} (12x^2 - 4) = \frac{1}{2} (3x^2 - 1)$$

$$\boxed{P_2(x) = \frac{1}{2} (3x^2 - 1)}$$

$$P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2-1)^3$$

$$x^6 - 1 - 3x^4 + 3x^2)$$

$$= \frac{1}{8 \times 6} \frac{d^3}{dx^3} \left((x^2)^3 - 1 - 3x^4 + 3x^2\right)$$

$$\frac{1}{48} \frac{d^2}{dx^2} \left[6x^5 - 12x^3 + 6x\right]$$

$$\frac{1}{48} \frac{d}{dx} \left[30x^4 - 36x^2 + 6\right]$$

$$\frac{1}{48} \left[120x^3 - 72x\right] = \frac{24}{48} \left[5x^3 - 3x\right]$$

$$\boxed{P_3(x) = \frac{24}{48} \left[5x^3 - 3x\right]} = \frac{1}{2} \left[5x^3 - 3x\right]$$

$$P_4(x) = \frac{1}{2^4 4!} \frac{d^4}{dx^4} (x^2-1)^4.$$

$$(x^2-1)^4 = (x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$$

$$(x-a)^n = x^n - nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 - nC_3 x^{n-3} a^3 - - (-1)^n a^n$$

We shall use $\frac{d^n}{dx^n}(x^m) = \frac{m!}{m-n!} x^{m-n}$. $m > n$. $\boxed{n = 4}$

$$P_4(x) = \frac{1}{16 \times 24} \left[\frac{8!}{4!} x^4 - 4 \times \frac{6!}{2!} \times x^2 + 6 \times \frac{4!}{0!} x^0\right]$$

$$= \frac{1}{16 \times 24} \left[1680x^4 - 1440x^2 + 144\right]$$

$$P_4(x) = \frac{48}{16 \times 24} \left[35x^4 - 30x^2 + 3\right]$$

$$P_4(x) = \frac{1}{8} \left[35x^4 - 30x^2 + 3\right]$$

$$\frac{1}{2^5\,5!}\frac{d^5}{dx^5}(x^2-1)^5$$

$$= \frac{1}{32\times 120}\frac{d^5}{dx^5}\left(x^{10}-5x^8+10x^6-10x^4+5x^2-1\right)$$

$$\frac{1}{32\times 120}\left(\frac{10!}{5!}x^5 - 5\frac{8!}{3!}x^3 + 10\times\frac{6!}{1!}x\right)$$

$$= \frac{1}{32\times 120}\left(30240x^5 - 33600x^3 + 7200x\right)$$

$$\boxed{P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)}$$

We now express x^2, x^3, x^4, x^5 in terms legendre' polynomial

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$2P_2(x) = 3x^2-1$$

$$3x^2 = 2P_2(x)+1.$$

$$x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$$

Next consider $P_3(x) = \frac{1}{2}\left[5x^3-3x\right]$

$$2P_3(x) = 5x^3-3x$$

$$5x^3 = 2P_3(x)+3x = 2P_3(x)+3P_1(x)$$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

Next consider $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$$8P_4(x) = 35x^4 - 30x^2 + 3$$

$$35x^4 = 8P_4(x) + 30x^2 - 3 = 8P_4(x) + 10\times 3x^2 - 3$$

$$= 8P_4(x) + \cancel{20P_2(x)}\ 10\left[2P_2(x)+1\right]-3$$

$$= 8P_4(x) + 20P_2(x) + 10 - 3$$

$$35x^4 = 8P_4(x) + 20P_2(x) + 7 \;//$$

$$x^4 = \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$8P_5(x) = 63x^5 - 70x^3 + 15x.$$

$$63x^5 = 8P_5(x) + 70x^3 - 15x$$

$$63x^5 = 8P_5(x) + 14[2P_3(x) + 3P_1(x)] - 15P_1(x)$$

$$63x^5 = 8P_5(x) + 28P_3(x) + 27P_1(x)$$

$$x^5 = \frac{8}{63} P_5(x) + \frac{4}{9} P_3(x) + \frac{3}{7} P_1(x)$$

② if $x^3 + 2x^2 - x + 1 = aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x)$

find a, b, c, d.

Soln $f(x) = x^3 + 2x^2 - x + 1.$

$$f(x) = \left[\frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)\right] + 2\left[\frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)\right] - P_1(x) + P_0(x)$$

$$f(x) = \frac{2}{5} P_3(x) + \frac{4}{3} P_2(x) - \frac{2}{5} P_1(x) + \frac{5}{3} P_0(x)$$

hence we $aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x)$

$$= \frac{5}{3} P_0(x) - \frac{2}{5} P_1(x) + \frac{4}{3} P_2(x) + \frac{2}{5} P_3(x)$$

$$a = \frac{5}{3} \quad b = -\frac{2}{5} \quad c = \frac{4}{3} \quad d = \frac{2}{5}$$

③ ① $P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$

② $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

Sol

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$P_2(\cos\theta) = \frac{1}{2}\left[\frac{3}{2}(1+\cos 2\theta) - 1\right]$$

$$= \frac{1}{4}\left[3+3\cos 2\theta - 2\right]$$

$$P_2(\cos\theta) = \frac{1}{4}\left[1+3\cos 2\theta\right]$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

$$P_3(\cos\theta) = \frac{1}{2}\left[5\cdot\frac{1}{4}(\cos 3\theta + 3\cos\theta) - 3\cos\theta\right]$$

$$= \frac{1}{8}\left[5\cos 3\theta + 15\cos\theta - 12\cos\theta\right]$$

$$P_3(\cos\theta) = \frac{1}{8}\left[3\cos\theta + 5\cos 3\theta\right]$$

(8) prove that $\int_{-1}^{+1} x^3 P_4(x)\,dx = 0$

Soln: Rodrigue's formula $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$

∴ put $n=4$ $\quad P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$

$$P_4(1) = \frac{1}{8}\left[35 - 30 + 3\right] = 1.$$

$$\int_{-1}^{1} x^3 P_4(x)\,dx = \int_{-1}^{1} x^3 \cdot \frac{1}{8}\left[35x^4 - 30x^2 + 3\right] dx$$

$$= \frac{1}{8}\int_{-1}^{1}\left(35x^7 - 30x^5 + 3x^3\right) dx$$

$$\frac{1}{8}\left[35\frac{x^8}{8} - 30\frac{x^6}{6} + \frac{3x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{8}\left[\frac{35}{8}(1-1) - \frac{1}{5}(1-1) + \frac{3}{4}(1-1)\right] = 0$$

$$\boxed{\int_{-1}^{1} x^3 P_4(x)\,dx = 0}$$