(1) Evaluate $\int_0^\infty e^{-3t}\, t \sin t \, dt$

Sol :- $L\{\sin t\} = \frac{1}{s^2+1}$

$$L\{t \sin t\} = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\left[\frac{(s^2+1)\frac{d}{ds}(1) - 1\cdot\frac{d}{ds}(s^2+1)}{(s^2+1)^2}\right]$$

$$= -\left[\frac{0 - (2s)}{(s^2+1)^2}\right]$$

$$L\{t \sin t\} = \frac{2s}{(s^2+1)^2}$$

$$\int_0^\infty e^{-3t}\, t \sin t \, dt = \frac{2s}{(s^2+1)^2}$$

$$\int_0^\infty e^{-st} f(t)\, dt$$

Put $s = 3$

$$= \frac{2(3)}{(3^2+1)^2}$$

$$= \frac{6}{100}$$

$$= \underline{3/50}$$

2) Evaluate $\int_0^\infty t\,e^{-3t} \cos 2t \, dt$

Sol :-

$$L\{\cos 2t\} = \frac{s}{s^2+2^2}$$

$$L\{t \cdot \cos 2t\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= -\left[\frac{(s^2+4)\frac{d}{ds}(s) - s\frac{d}{ds}(s^2+4)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{(s^2+4)\cdot 1 - s \cdot 2s}{(s^2+4)^2}\right]$$

$$= -\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$L\{t \cos 2t\} = \frac{s^2-4}{(s^2+4)^2}$$

$$\int_0^\infty e^{-3t}\, t \cos 2t \, dt = \frac{s^2-4}{(s^2+4)^2}$$

Put $s=3$

$$= \frac{3^2-4}{(3^2+4)^2}$$

$$= \frac{5}{(13)^2} = \frac{5}{169}$$

③ Evaluate $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt$

Sol :-
$$\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = \int_0^\infty e^{-0t}\left(\frac{e^{-t}-e^{-3t}}{t}\right)dt$$

$$f(t) = e^{-t}-e^{-3t}$$

$$L\{e^{-t}-e^{-3t}\} = \frac{1}{s+1} - \frac{1}{s+3}$$

$$L\left\{\frac{e^{-t}-e^{-3t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+3}\right)ds$$

$$= \left[\log(s+1) - \log(s+3)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+1}{s+3}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \log\left(\frac{s+1}{s+3}\right) - \log\left(\frac{s+1}{s+3}\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{\cancel{s}(1+1/s)}{\cancel{s}(1+3/s)}\right) - \log\left(\frac{s+1}{s+3}\right)$$

$$= \log 1 - \log\left(\frac{s+1}{s+3}\right)$$

$$= 0 - \log\left(\frac{s+1}{s+3}\right) = -\log\left(\frac{s+1}{s+3}\right)$$

$$= \log\left(\frac{s+3}{s+1}\right)$$

$$L\left\{\frac{e^{-t}-e^{-3t}}{t}\right\} = \log\left(\frac{s+3}{s+1}\right)$$

$$\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-3t}}{t}\right)dt = \log\left(\frac{s+3}{s+1}\right)$$

Put $s=0$

$$= \log\left(\frac{0+3}{0+1}\right)$$

$$= \underline{\underline{\log 3}}$$

1. Evaluate $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt$

2. Evaluate $\int_0^\infty t\,e^{-2t}\sin 2t\,dt$

(F) If $L\{f(t)\} = F(s)$ Show that

(i) $L\{\sinh at\, f(t)\} = \frac{1}{2}\left[F(s-a) - F(s+a)\right]$

(ii) $L\{\cosh at\, f(t)\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$

Sol :- (i) LHS $L\{\sinh at\, f(t)\} = L\left\{\left(\frac{e^{at}-e^{-at}}{2}\right)f(t)\right\}$

$$= \frac{1}{2}\left[L\left(e^{at}f(t)\right) - L\left(e^{-at}f(t)\right)\right]$$

(i) $= \frac{1}{2}\left[F(s-a) - F(s+a)\right]$

RHS

ii) $L\{\cosh at\, f(t)\} = L\left\{\left(\frac{e^{at}+e^{-at}}{2}\right) f(t)\right\}$

$= \frac{1}{2}\left[L\{e^{at} f(t)\} + L\{e^{-at} f(t)\}\right]$

$= \frac{1}{2}\left[F(s-a) + F(s+a)\right]$

RHS

Find

② $L\left\{\int_0^t \int_0^t (t\sin t)\, dt\right\}$

Sol:-

$L\left\{\int_0^t \int_0^t f(t)\, dt\right\} = \frac{1}{s^2} \cdot F(s)$

$L\{\sin t\} = \frac{1}{s^2+1}$

$L\left\{\int_0^t f(t)\, dt\right\} = \frac{1}{s} F(s)$

$L\{t \cdot \sin t\} = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$= -\left[\frac{(s^2+1)\frac{d}{ds}(1) - \cdot \frac{d}{ds}(s^2+1)}{(s^2+1)^2}\right]$

$F(s) = -\left[\frac{0 - (2s)}{(s^2+1)^2}\right] = \frac{+2s}{(s^2+1)^2}$

$$L\left\{\int_0^t \int_0^t t \sin t \, dt\right\} = \frac{1}{s^2} \cdot \frac{2s}{(s^2+1)^2}$$

(3) Find $L\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3\right\}$

$$\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3 = (\sqrt{t})^3 + 3 \cdot (\sqrt{t})^2 \frac{1}{\sqrt{t}} + 3 \cdot \sqrt{t}\left(\frac{1}{\sqrt{t}}\right)^2 + \left(\frac{1}{\sqrt{t}}\right)^3$$

$$= t^{3/2} + 3 \cdot t^{1/2} + 3 \cdot t^{-1/2} + t^{-3/2}$$

$$L\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{t^{3/2}\right\} + 3 L\left\{t^{1/2}\right\} + 3 \cdot L\left\{t^{-1/2}\right\} + L\left\{t^{-3/2}\right\}$$

$$\boxed{\begin{aligned} \Gamma(n+1) &= n \cdot \Gamma(n) \\ \Gamma(n) &= \frac{\Gamma(n+1)}{n} \end{aligned}}$$

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{\Gamma(3/2+1)}{s^{3/2+1}} + 3 \cdot \frac{\Gamma(1/2+1)}{s^{1/2+1}} + 3 \cdot \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} + 3 \cdot \frac{\Gamma(-3/2+1)}{s^{-3/2+1}}.$$

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$$= \frac{3/2 \, \Gamma 3/2}{s^{5/2}} + 3 \cdot \frac{1/2 \, \Gamma 1/2}{s^{3/2}} + 3 \cdot \frac{\Gamma 1/2}{s^{1/2}} + \cdot \frac{\Gamma -1/2}{s^{-1/2}}$$

$$= \frac{3}{2} \cdot \frac{\Gamma 1 + 1/2}{s^{5/2}} + \frac{3}{2} \cdot \frac{\Gamma 1/2}{s^{3/2}} + 3 \cdot \frac{\Gamma 1/2}{s^{1/2}} + \frac{\Gamma -1/2}{s^{-1/2}}$$

$$= \frac{3}{2} \cdot \frac{1/2 \cdot \Gamma 1/2}{s^{5/2}} + 3/2 \, \frac{\Gamma 1/2}{s^{3/2}} + \frac{3 \cdot \Gamma 1/2}{s^{1/2}} + \frac{\frac{\Gamma -1/2+1}{-1/2}}{s^{-1/2}}$$

$$\left(\because \Gamma n+1 = n \, \Gamma n \right.$$
$$\left. \Gamma n = \frac{\Gamma n+1}{n} \right)$$
$$\Gamma 1/2 = \sqrt{\pi}$$

$$= \frac{3}{4} \frac{\sqrt{\pi}}{s^{5/2}} + \frac{3}{2} \cdot \frac{\sqrt{\pi}}{s^{3/2}} + \frac{3 \cdot \sqrt{\pi}}{s^{1/2}} + \frac{(-2)\, \Gamma 1/2}{s^{-1/2}}$$

$$= \frac{3}{4} \frac{\sqrt{\pi}}{s^{5/2}} + \frac{3}{2} \frac{\sqrt{\pi}}{s^{3/2}} + \frac{3 \sqrt{\pi}}{\sqrt{s}} - \frac{2 \sqrt{\pi}}{1/\sqrt{s}}$$