CHAPTER 6

GRAPHS

All the programs in this file are selected from

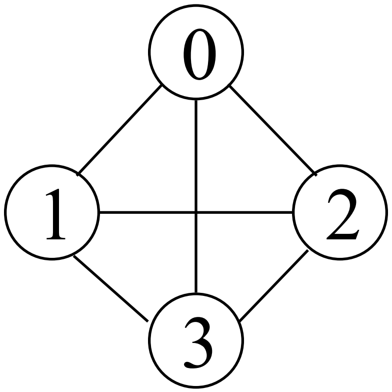
Ellis Horowitz, Sartaj Sahni, and Susan Anderson-Freed
"Fundamentals of Data Structures in C",
Computer Science Press, 1992.

Definition

- A graph G consists of two sets
  - a finite, nonempty set of vertices V(G)
  - a finite, possible empty set of edges E(G)
  - G(V,E) represents a graph
- An undirected graph is one in which the pair of vertices in a edge is unordered, $(v_0, v_1) = (v_1, v_0)$
- A directed graph is one in which each edge is a directed pair of vertices, $<v_0, v_1>\ !=\ <v_1, v_0>$

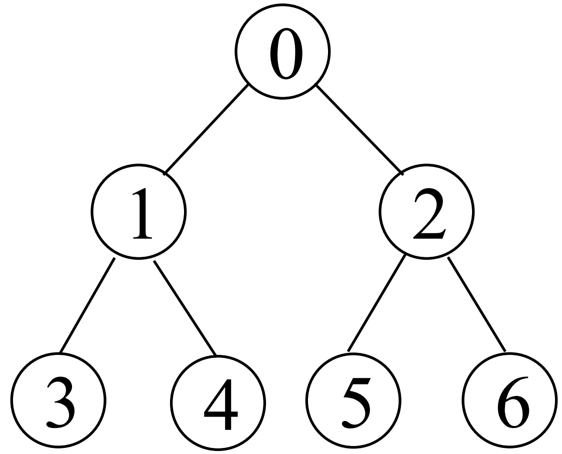
tail $\longrightarrow$ head

Examples for Graph



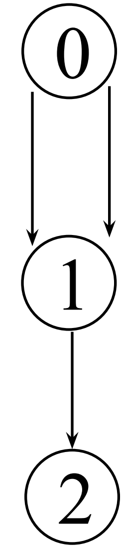
G_1

complete graph



G_2

incomplete graph



G_3

$V(G_1)=\{0,1,2,3\}$ $E(G_1)=\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$

$V(G_2)=\{0,1,2,3,4,5,6\}$ $E(G_2)=\{(0,1),(0,2),(1,3),(1,4),(2,5),(2,6)\}$

$V(G_3)=\{0,1,2\}$ $E(G_3)=\{<0,1>,<1,0>,<1,2>\}$

complete undirected graph: $n(n-1)/2$ edges

complete directed graph: $n(n-1)$ edges

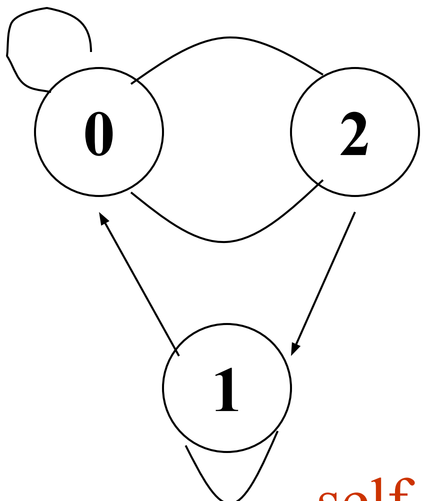
Complete Graph

- A complete graph is a graph that has the maximum number of edges
  - for undirected graph with n vertices, the maximum number of edges is $n(n-1)/2$
  - for directed graph with n vertices, the maximum number of edges is $n(n-1)$
  - example: G1 is a complete graph

Adjacent and Incident

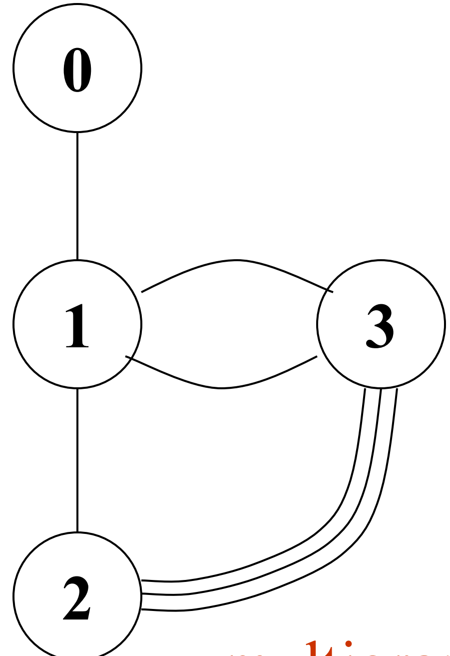
- If (v_0, v_1) is an edge in an undirected graph,
  - v_0 and v_1 are adjacent
  - The edge (v_0, v_1) is incident on vertices v_0 and v_1
- If $<v_0, v_1>$ is an edge in a directed graph
  - v_0 is adjacent to v_1, and v_1 is adjacent from v_0
  - The edge $<v_0, v_1>$ is incident on v_0 and v_1

***Figure 6.3:**Example of a graph with feedback loops and a multigraph (p.260)



self edge

(a)



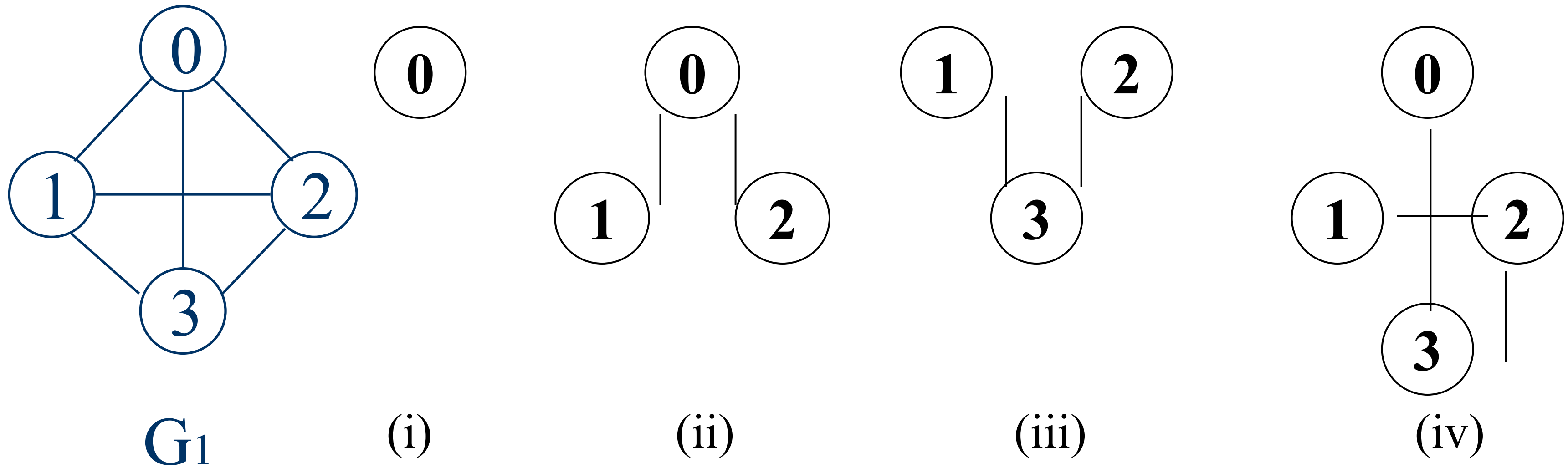
(b)

multigraph:
multiple occurrences
of the same edge

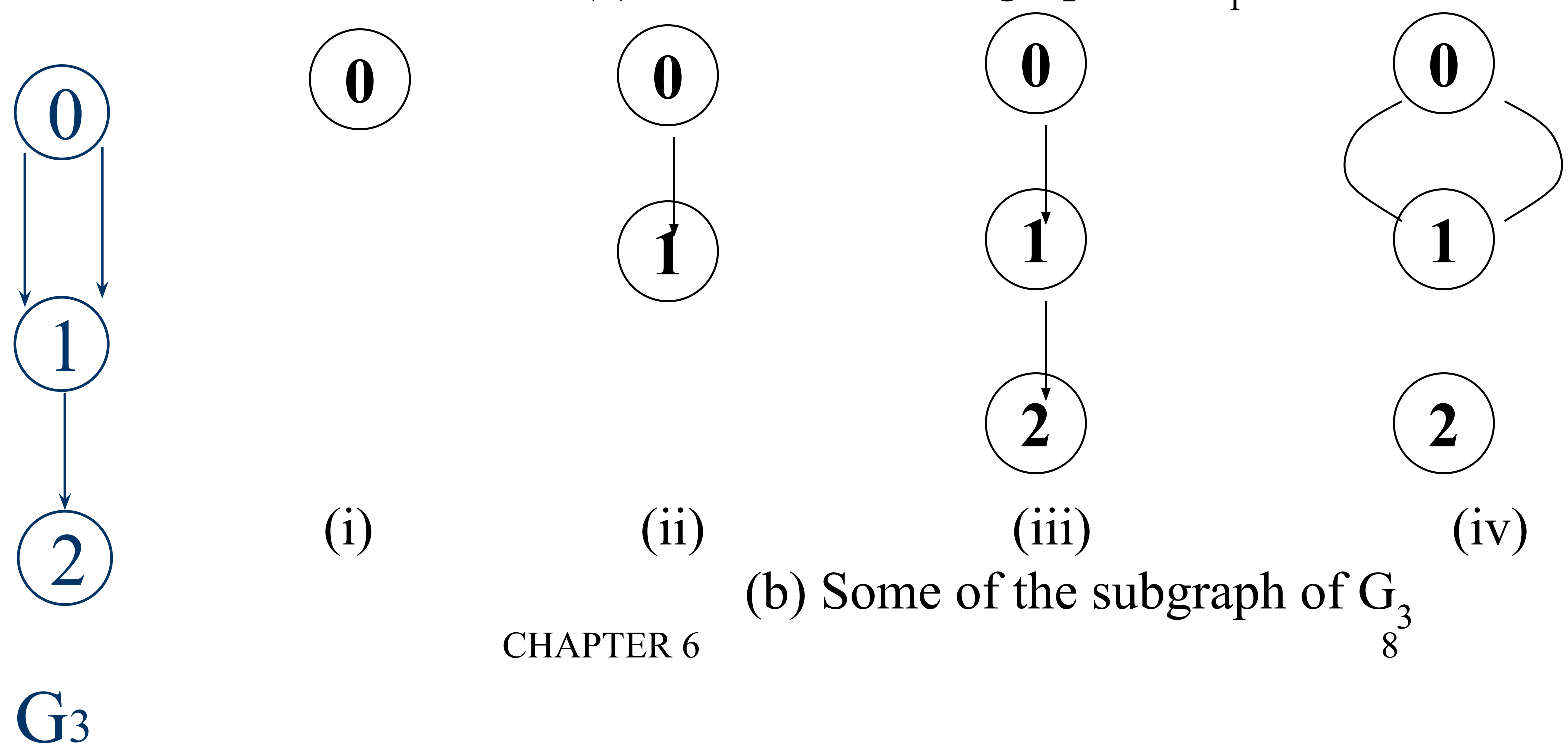
Subgraph and Path

- A subgraph of G is a graph G' such that V(G') is a subset of V(G) and E(G') is a subset of E(G)
- A path from vertex v_p to vertex v_q in a graph G, is a sequence of vertices, $v_p, v_{i1}, v_{i2}, \ldots, v_{in}, v_q$, such that $(v_p, v_{i1}), (v_{i1}, v_{i2}), \ldots, (v_{in}, v_q)$ are edges in an undirected graph
- The length of a path is the number of edges on it

Figure 6.4: subgraphs of G_1 and G_3 (p.261)



(a) Some of the subgraph of G_1

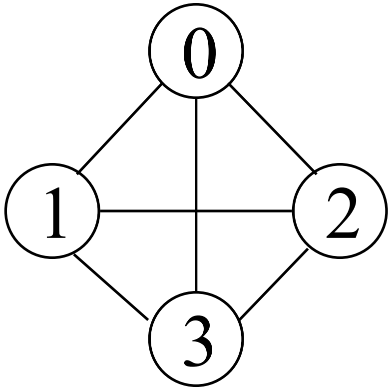


(b) Some of the subgraph of G_3

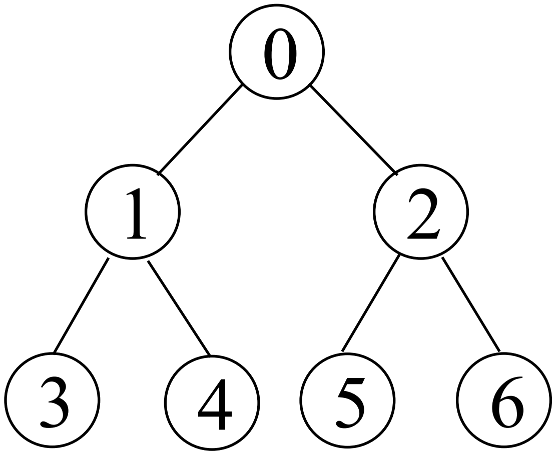
Simple Path and Style

- A simple path is a path in which all vertices, except possibly the first and the last, are distinct
- A cycle is a simple path in which the first and the last vertices are the same
- In an undirected graph G, two vertices, v_0 and v_1, are connected if there is a path in G from v_0 to v_1
- An undirected graph is connected if, for every pair of distinct vertices v_i, v_j, there is a path from v_i to v_j

connected



G_1



G_2

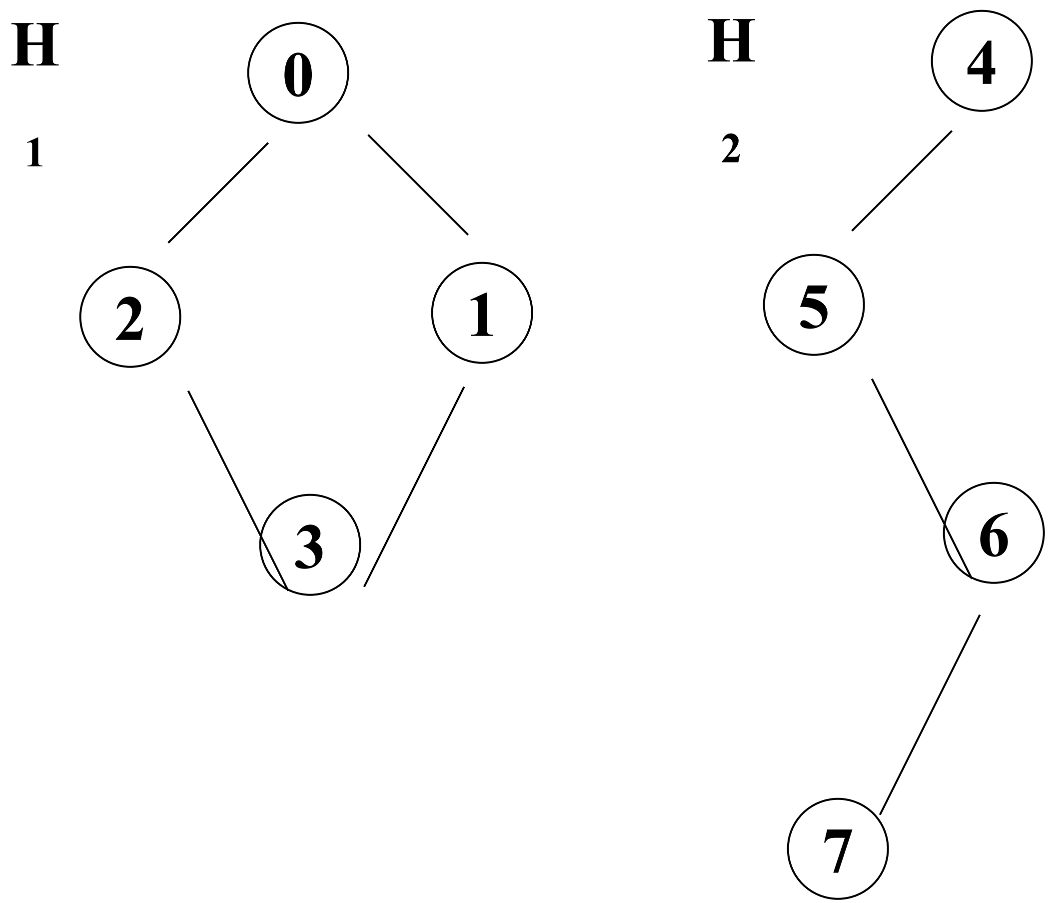
tree (acyclic graph)

Connected Component

- A connected component of an undirected graph is a maximal connected subgraph.
- A tree is a graph that is connected and acyclic.
- A directed graph is strongly connected if there is a directed path from v_i to v_j and also from v_j to v_i.
- A strongly connected component is a maximal subgraph that is strongly connected.

*Figure 6.5: A graph with two connected components (p.262)

connected component (maximal connected subgraph)

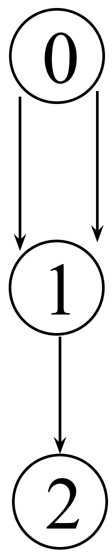


G_4 (not connected)

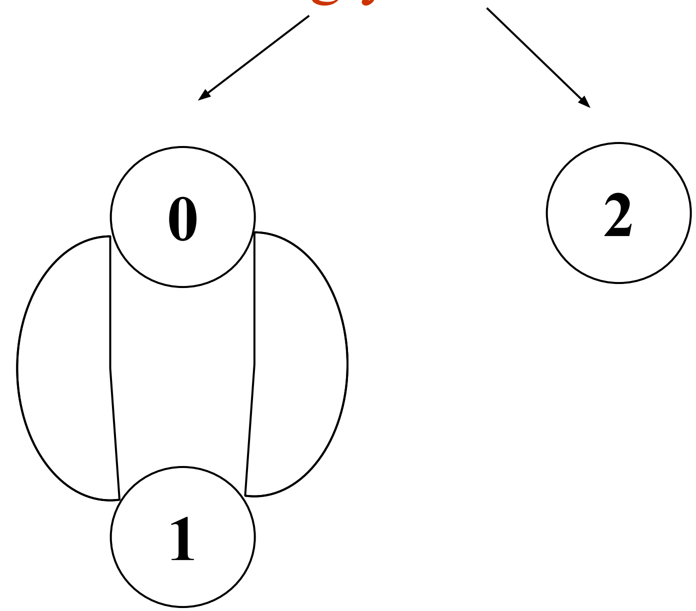
***Figure 6.6:** Strongly connected components of G_3 (p.262)

not strongly connected

strongly connected component
(maximal strongly connected subgraph)



G_3



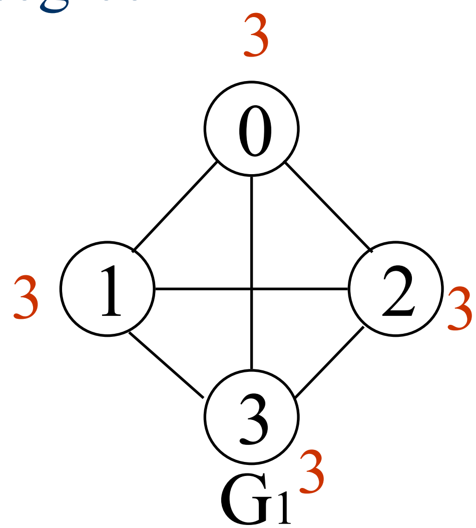
Degree

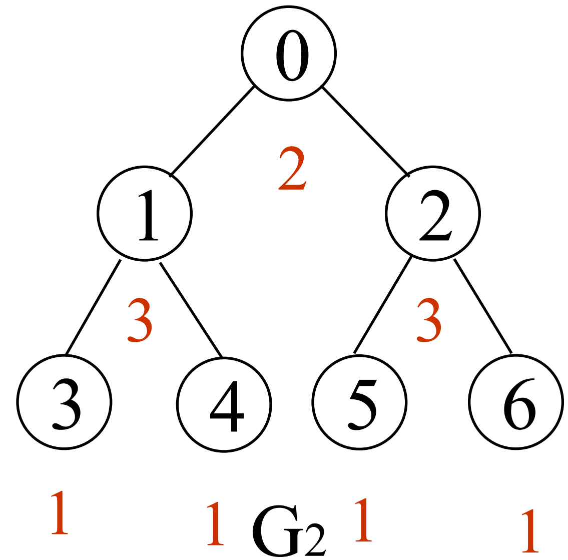
- The degree of a vertex is the number of edges incident to that vertex
- For directed graph,
  - the in-degree of a vertex v is the number of edges that have v as the head
  - the out-degree of a vertex v is the number of edges that have v as the tail
  - if d_i is the degree of a vertex i in a graph G with n vertices and e edges, the number of edges is

$$e = \left(\sum_{0}^{n-1} d_i\right) / 2$$

undirected graph

degree

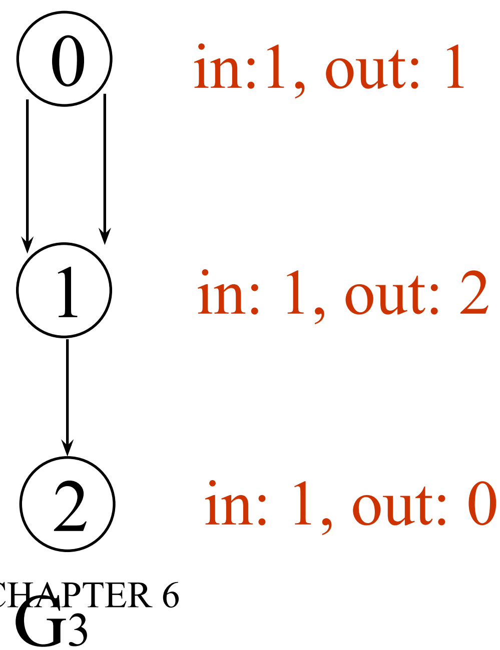




directed graph

in-degree

out-degree



ADT for Graph

structure Graph is

objects: a nonempty set of vertices and a set of undirected edges, where each edge is a pair of vertices

functions: for all $graph \in Graph$, v, v_1 and $v_2 \in Vertices$

Graph Create()::=return an empty graph

Graph InsertVertex(*graph*, *v*)::= return a graph with *v* inserted. *v* has no incident edge.

Graph InsertEdge(*graph*, v_1,v_2)::= return a graph with new edge between v_1 and v_2

Graph DeleteVertex(*graph*, *v*)::= return a graph in which *v* and all edges incident to it are removed

Graph DeleteEdge(*graph*, v_1, v_2)::=return a graph in which the edge (v_1, v_2) is removed

Boolean IsEmpty(*graph*)::= if (*graph*==*empty graph*) return TRUE else return FALSE

List Adjacent(*graph*,*v*)::= return a list of all vertices that are adjacent to *v*

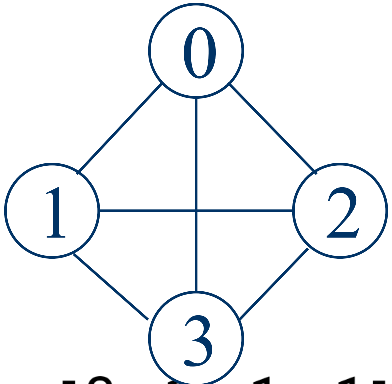
Graph Representations

- Adjacency Matrix
- Adjacency Lists
- Adjacency Multilists

Adjacency Matrix

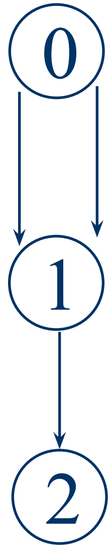
- Let G=(V,E) be a graph with n vertices.
- The adjacency matrix of G is a two-dimensional n by n array, say adj_mat
- If the edge (v_i, v_j) is in E(G), adj_mat[i][j]=1
- If there is no such edge in E(G), adj_mat[i][j]=0
- The adjacency matrix for an undirected graph is symmetric; the adjacency matrix for a digraph need not be symmetric

Examples for Adjacency Matrix



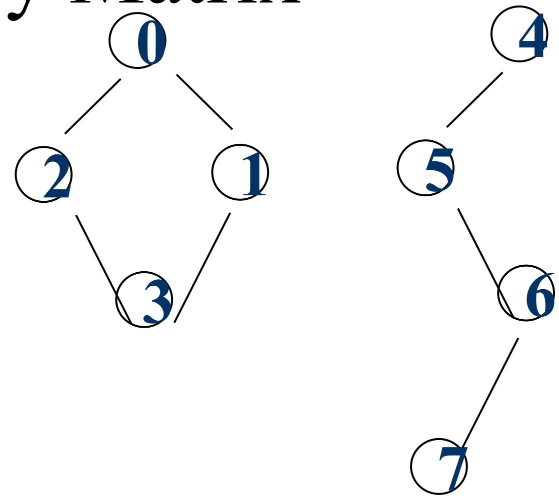
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

G_1



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

G_2



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

G_4

symmetric

undirected: $n^2/2$
directed: n^2

Merits of Adjacency Matrix

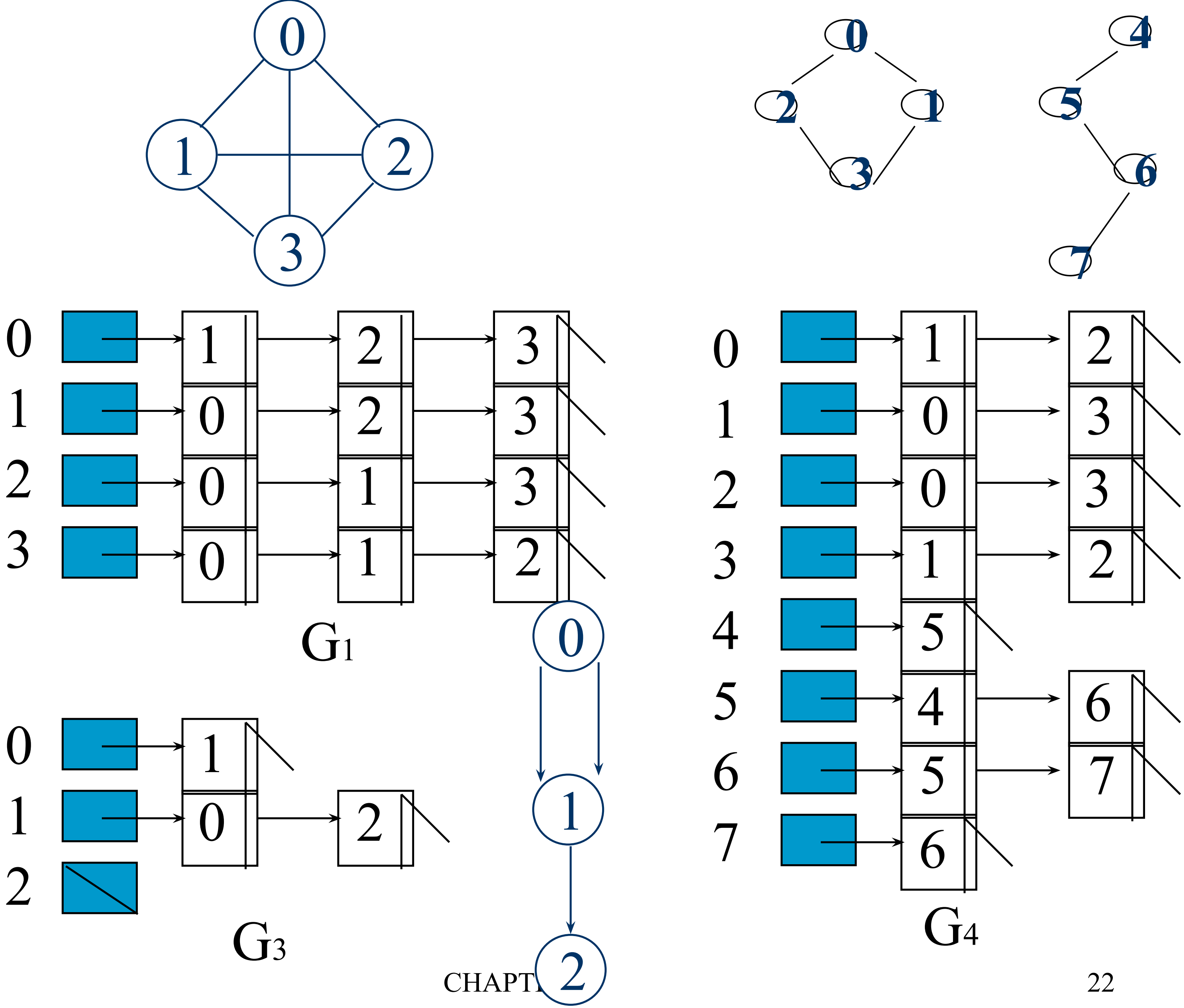
- From the adjacency matrix, to determine the connection of vertices is easy
- The degree of a vertex is $\sum_{j=0}^{n-1} adj\_mat[i][j]$
- For a digraph, the row sum is the out_degree, while the column sum is the in_degree

$$ind(vi) = \sum_{j=0}^{n-1} A[j,i] \quad outd(vi) = \sum_{j=0}^{n-1} A[i,j]$$

Data Structures for Adjacency Lists

Each row in adjacency matrix is represented as an adjacency list.

```
#define MAX_VERTICES 50
typedef struct node *node_pointer;
typedef struct node {
    int vertex;
    struct node *link;
};
node_pointer graph[MAX_VERTICES];
int n=0; /* vertices currently in use *
```

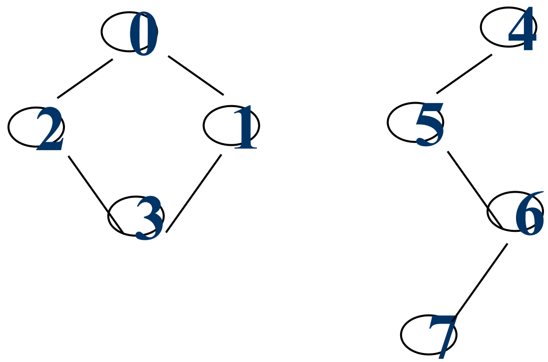


An undirected graph with n vertices and e edges ==> n head nodes and 2e list nodes

Interesting Operations

- degree of a vertex in an undirected graph
  - # of nodes in adjacency list
- # of edges in a graph
  - determined in O(n+e)
- out-degree of a vertex in a directed graph
  - # of nodes in its adjacency list
- in-degree of a vertex in a directed graph
  - traverse the whole data structure

Compact Representation



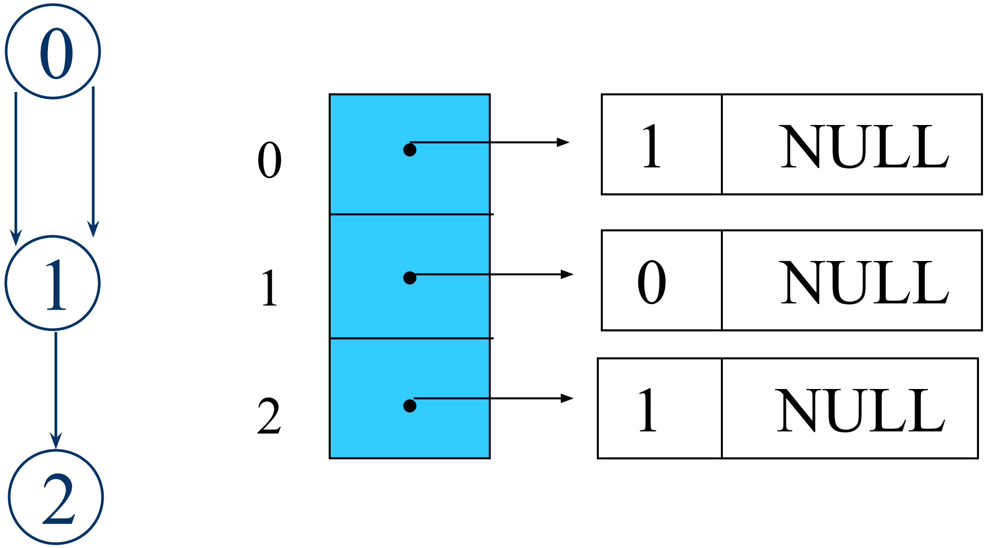
node[0] … node[n-1]: starting point for vertices

node[n]: n+2e+1

node[n+1] … node[n+2e]: head node of edge

[0]	9		[8]	23		[16]	2
[1]	11	0	[9]	1	4	[17]	5
[2]	13		[10]	2	5	[18]	4
[3]	15	1	[11]	0		[19]	6
[4]	17		[12]	3	6	[20]	5
[5]	18	2	[13]	0		[21]	7
[6]	20		[14]	3	7	[22]	6
[7]	22	3	[15]	1			

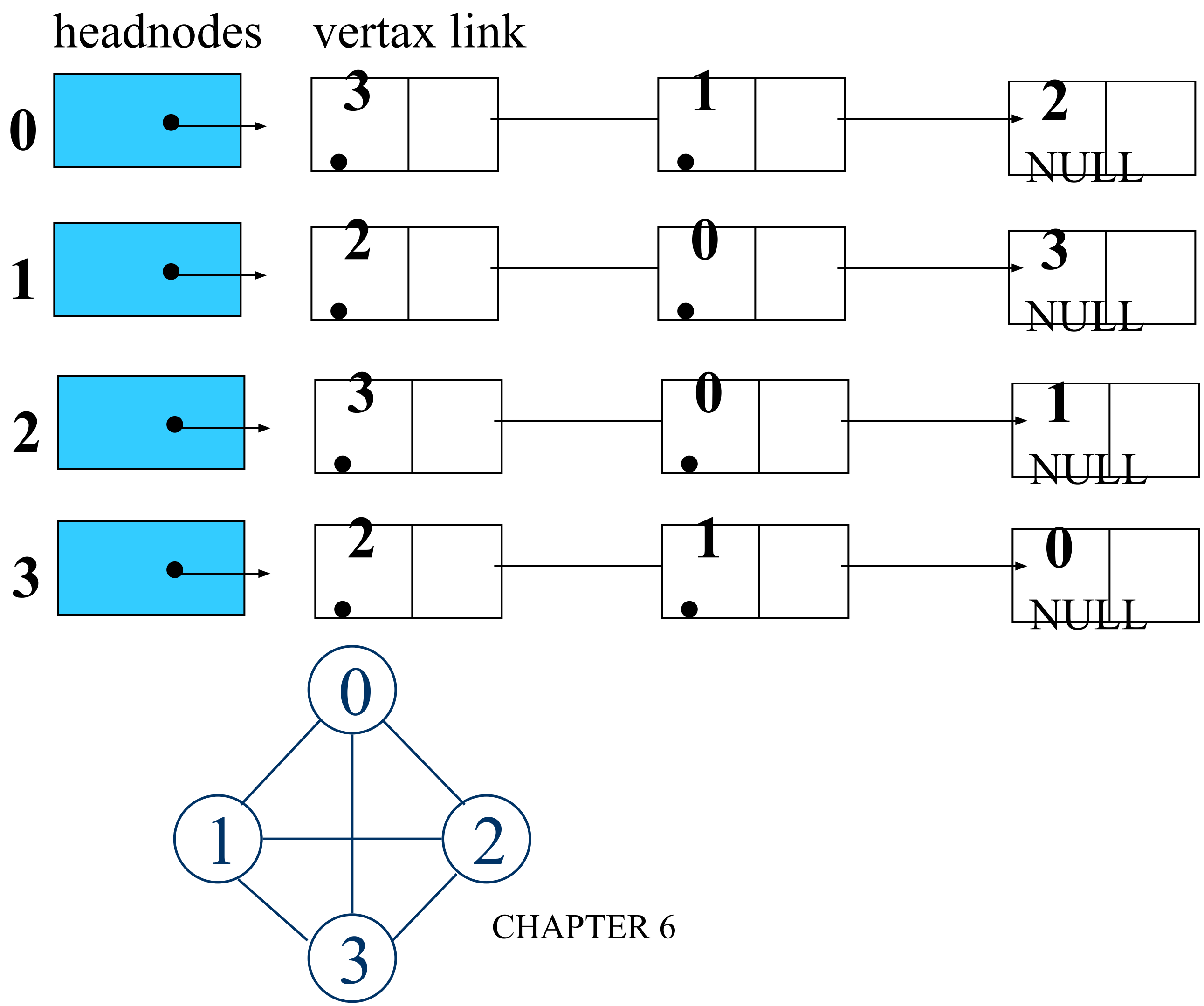
Figure 6.10: Inverse adjacency list for G_3



Determine in-degree of a vertex in a fast way.

Figure 6.13: Alternate order adjacency list for G_1 (p.268)

Order is of no significance.



Adjacency Multilists

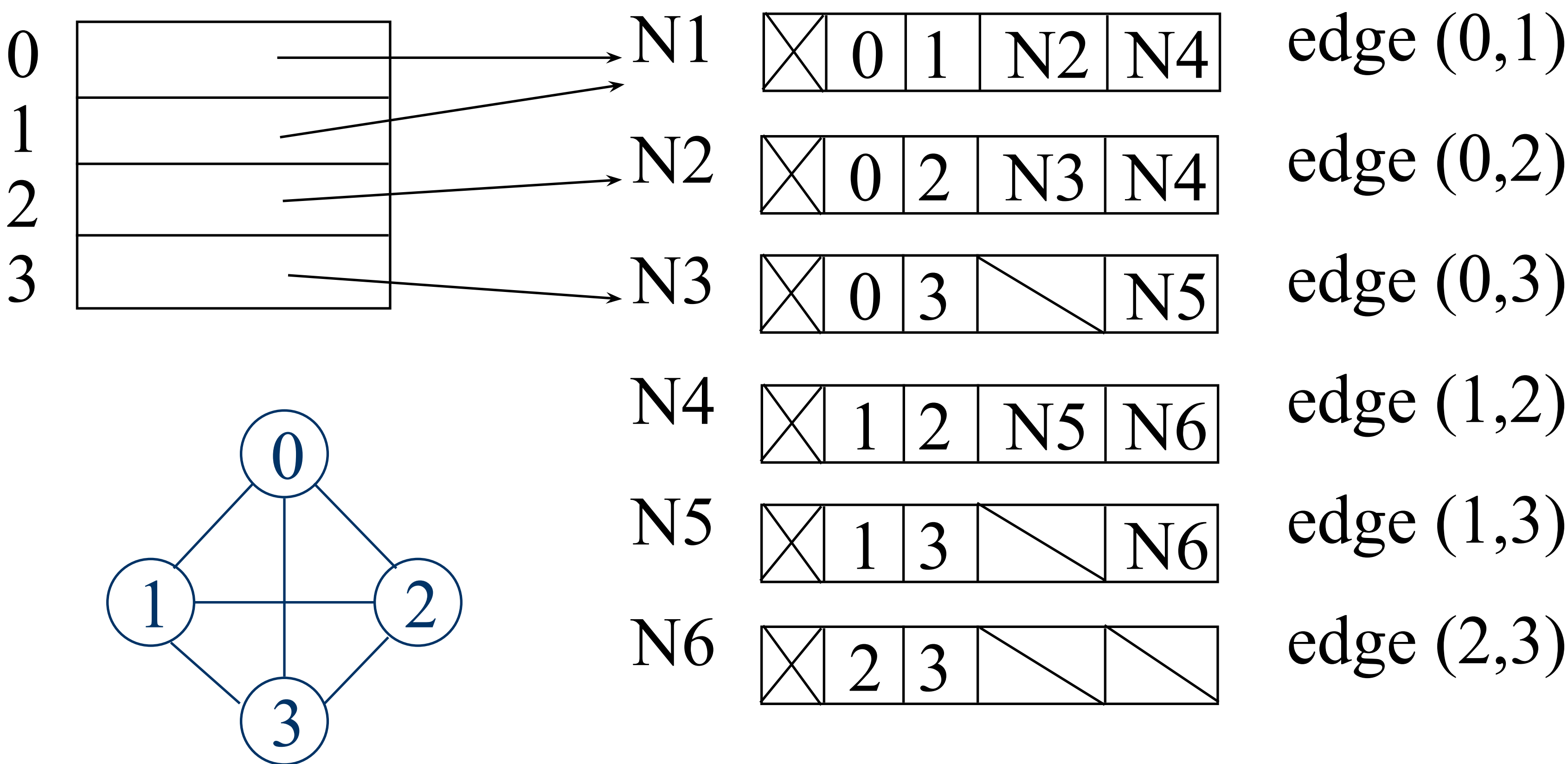
- An edge in an undirected graph is represented by two nodes in adjacency list representation.
- Adjacency Multilists
  - –lists in which nodes may be shared among several lists.
    (an edge is shared by two different paths)

marked	vertex1	vertex2	path1	

path2

Example for Adjacency Multlists

Lists: vertex 0: M1->M2->M3, vertex 1: M1->M4->M5
vertex 2: M2->M4->M6, vertex 3: M3->M5->M6



six edges

Adjacency Multilists

```
typedef struct edge *edge_pointer;
typedef struct edge {
    short int marked;
    int vertex1, vertex2;
    edge_pointer path1, path2;
};
edge_pointer graph[MAX_VERTICES];
```

marked	vertex1	vertex2	path1	

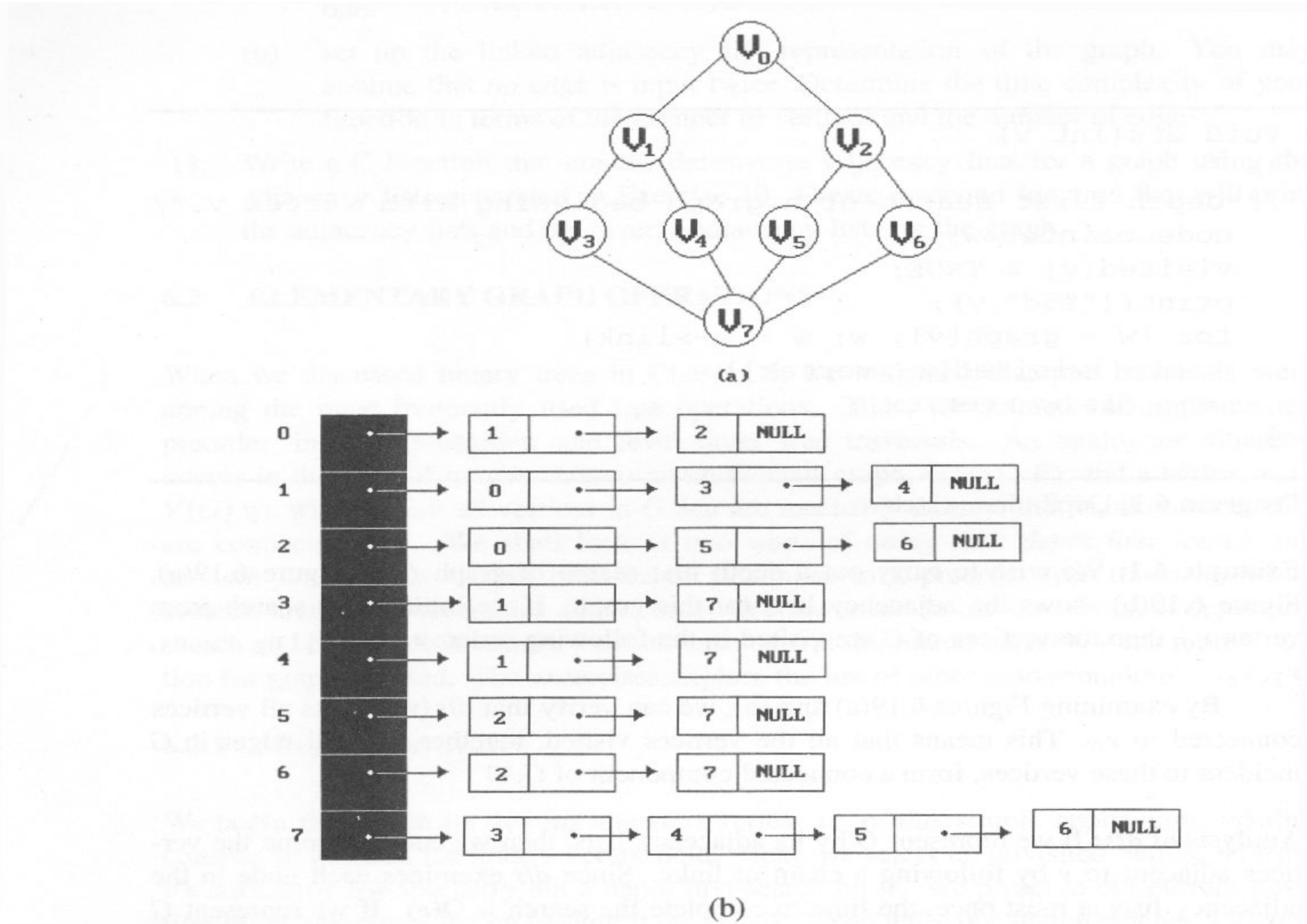
path2

Some Graph Operations

- Traversal
  Given G=(V,E) and vertex v, find all $w \in V$, such that w connects v.
  - Depth First Search (DFS)
    preorder tree traversal
  - Breadth First Search (BFS)
    level order tree traversal
- Connected Components
- Spanning Trees

*Figure 6.19: Graph *G* and its adjacency lists (p.274)

depth first search: v0, v1, v3, v7, v4, v5, v2, v6



breadth first search: v0, v1, v2, v3, v4, v5, v6, v7

Depth First Search

```
#define FALSE 0
#define TRUE 1
short int visited[MAX_VERTICES];
```

```
void dfs(int v)
{
  node_pointer w;
  visited[v]= TRUE;
  printf("%5d", v);
  for (w=graph[v]; w; w=w->link)
    if (!visited[w->vertex])
      dfs(w->vertex);
}
```

Data structure
adjacency list: $O(e)$
adjacency matrix: $O(n^2)$

Breadth First Search

```
typedef struct queue *queue_pointer;
typedef struct queue {
     int vertex;
     queue_pointer link;
};
void addq(queue_pointer *,
          queue_pointer *, int);
int deleteq(queue_pointer *);
```

Breadth First Search *(Continued)*

```
void bfs(int v)
{
  node_pointer w;
  queue_pointer front, rear;
  front = rear = NULL;
  printf("%5d", v);
  visited[v] = TRUE;
  addq(&front, &rear, v);
```

adjacency list: O(e)
adjacency matrix: $O(n^2)$

```
  while (front) {
    v= deleteq(&front);
    for (w=graph[v]; w; w=w->link)
      if (!visited[w->vertex]) {
        printf("%5d", w->vertex);
        addq(&front, &rear, w->vertex);
        visited[w->vertex] = TRUE;
      }
  }
}
```

Connected Components

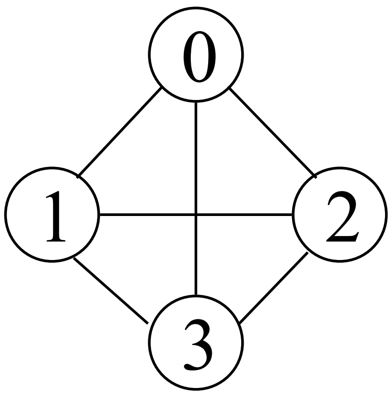
```
void connected(void)
{
    for (i=0; i<n; i++) {
        if (!visited[i]) {
            dfs(i);
            printf("\n");
        }
    }
}
```

adjacency list: $O(n+e)$
adjacency matrix: $O(n^2)$

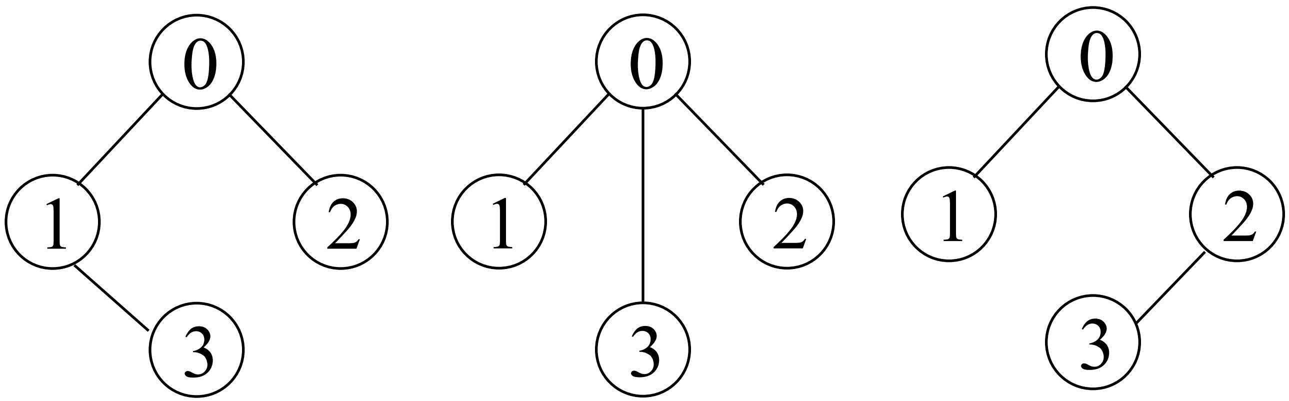
Spanning Trees

- When graph G is connected, a depth first or breadth first search starting at any vertex will visit all vertices in G
- A spanning tree is any tree that consists solely of edges in G and that includes all the vertices
- E(G): T (tree edges) + N (nontree edges)
  where T: set of edges used during search
  N: set of remaining edges

Examples of Spanning Tree



G_1

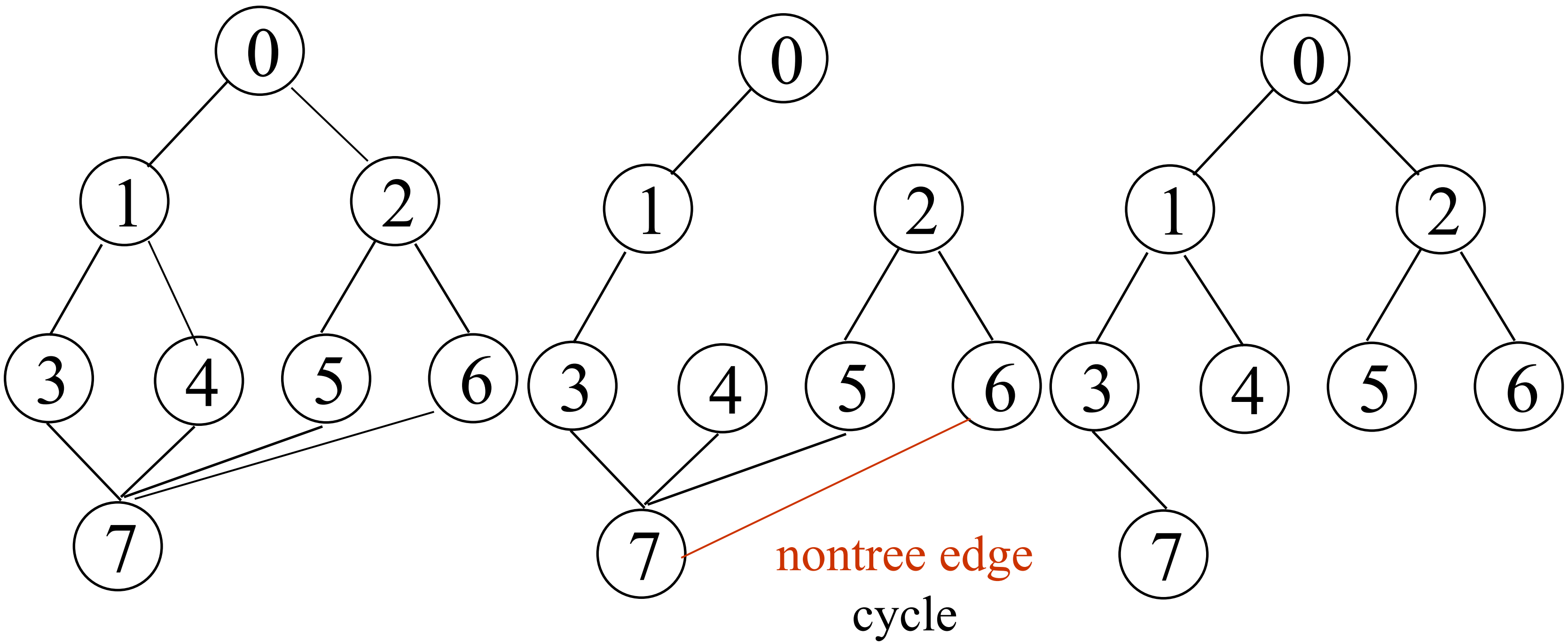


Possible spanning trees

Spanning Trees

- Either dfs or bfs can be used to create a spanning tree
  - When dfs is used, the resulting spanning tree is known as a depth first spanning tree
  - When bfs is used, the resulting spanning tree is known as a breadth first spanning tree
- While adding a nontree edge into any spanning tree, this will create a cycle

DFS VS BFS Spanning Tree



DFS Spanning

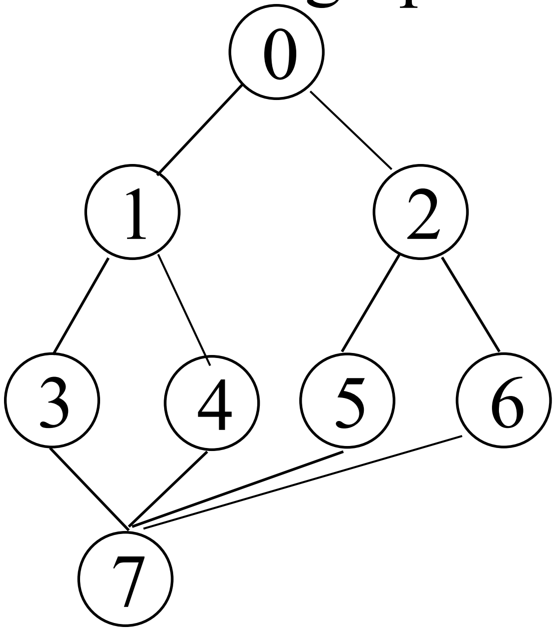
BFS Spanning

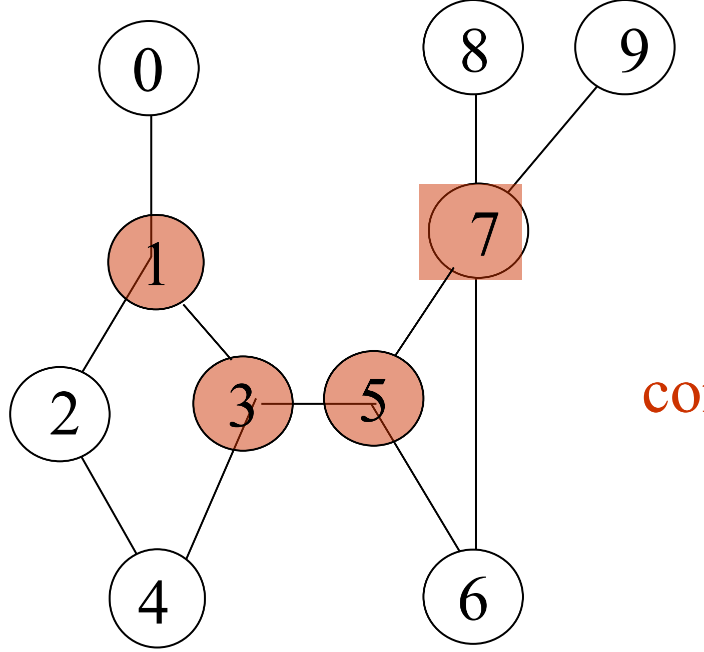
A spanning tree is a minimal subgraph, G', of G such that V(G')=V(G) and G' is connected.

Any connected graph with n vertices must have at least n-1 edges.

A biconnected graph is a connected graph that has no articulation points.

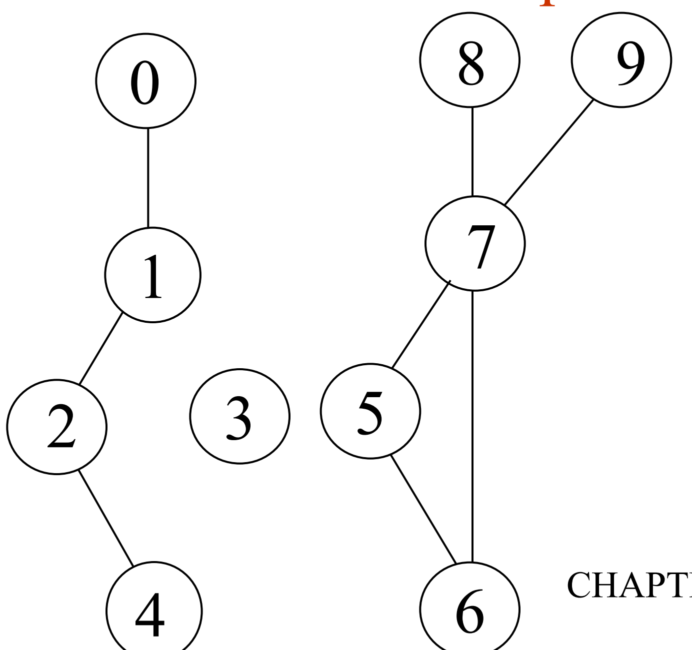
biconnected graph



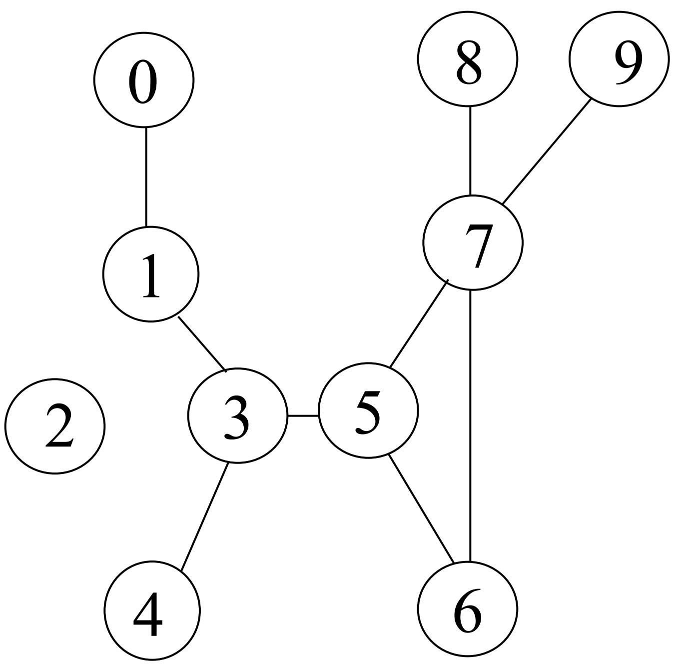


connected graph

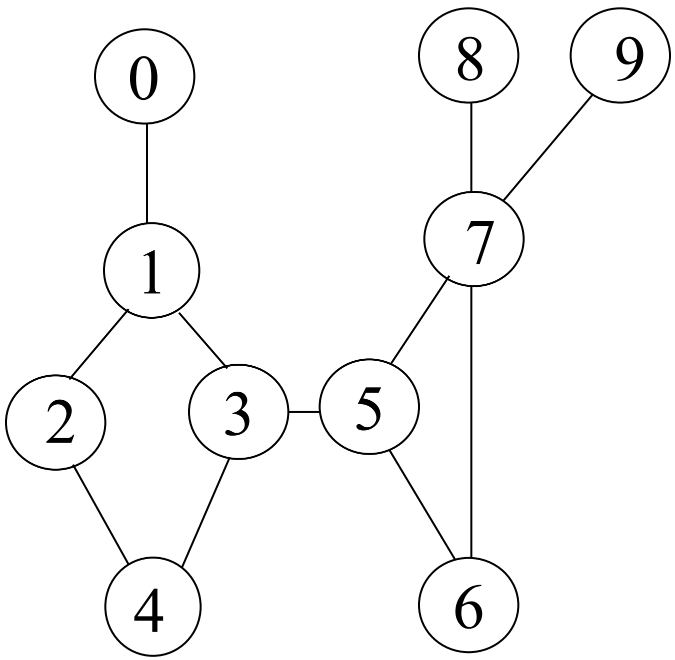
two connected components

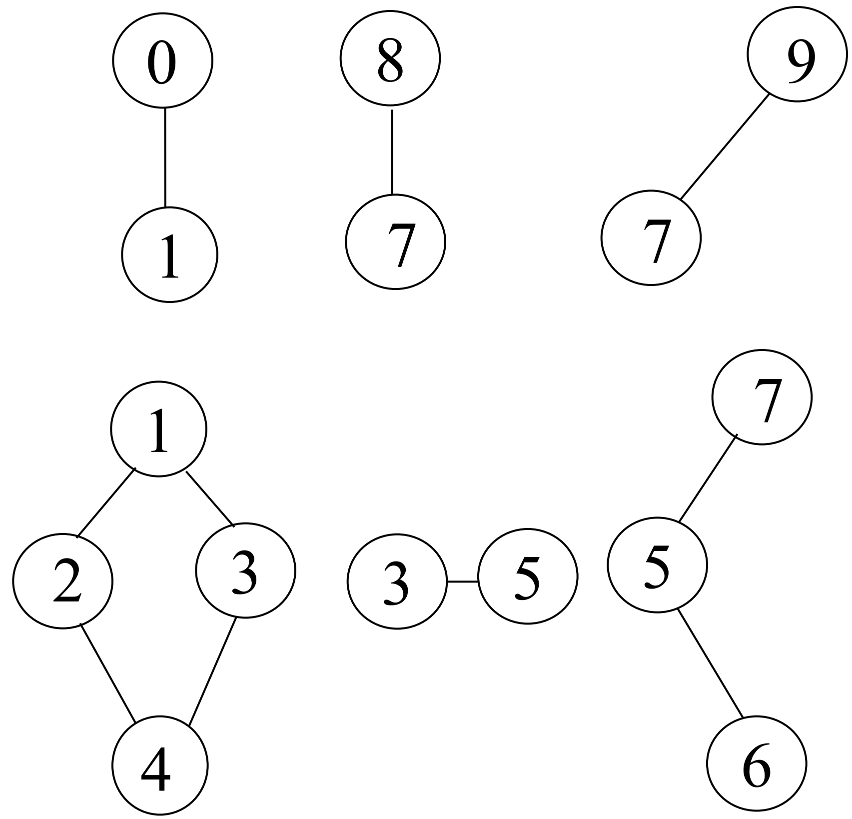


one connected graph



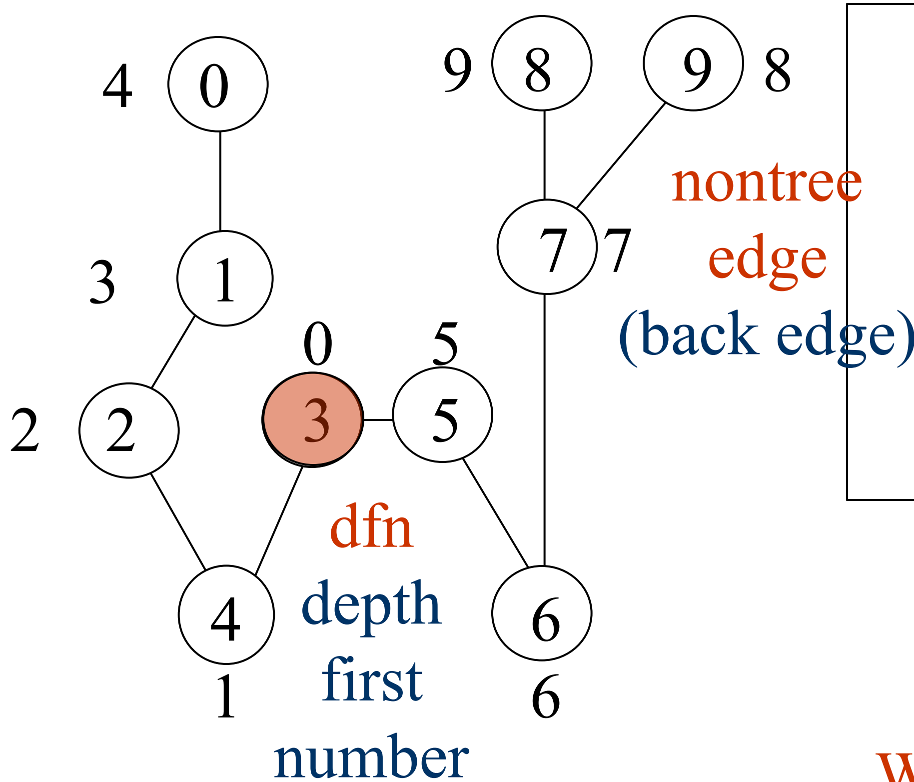
biconnected component: a maximal connected subgraph H (no subgraph that is both biconnected and properly contains H)



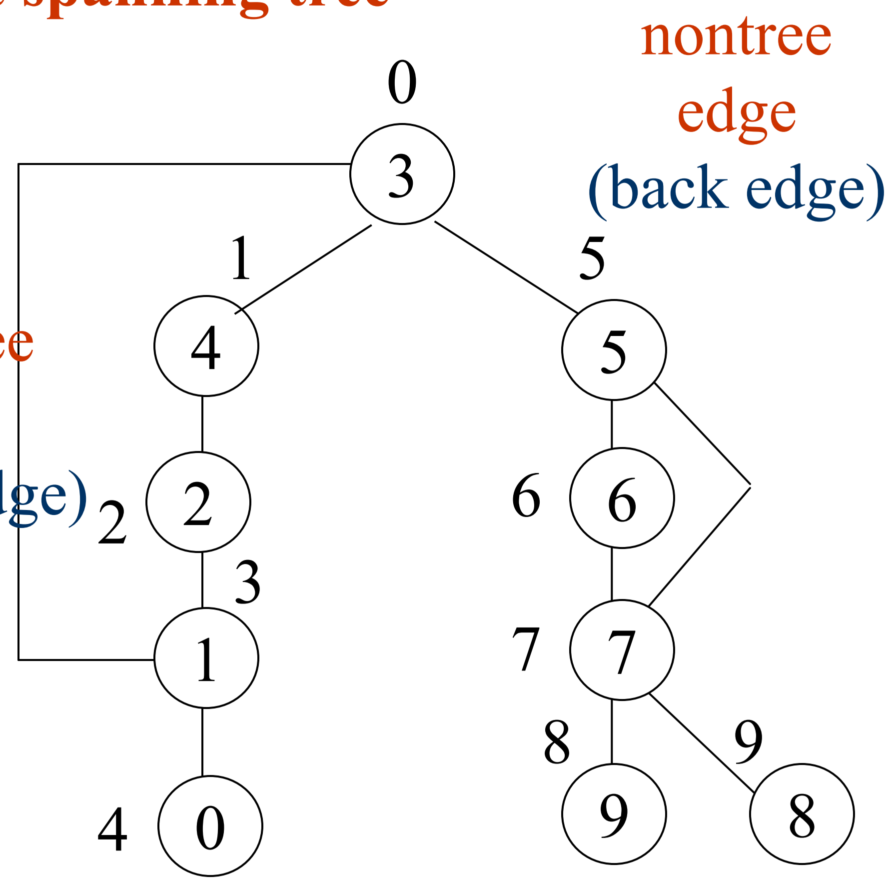


biconnected components

Find biconnected component of a connected undirected graph by **depth first spanning tree**



(a) depth first spanning tree



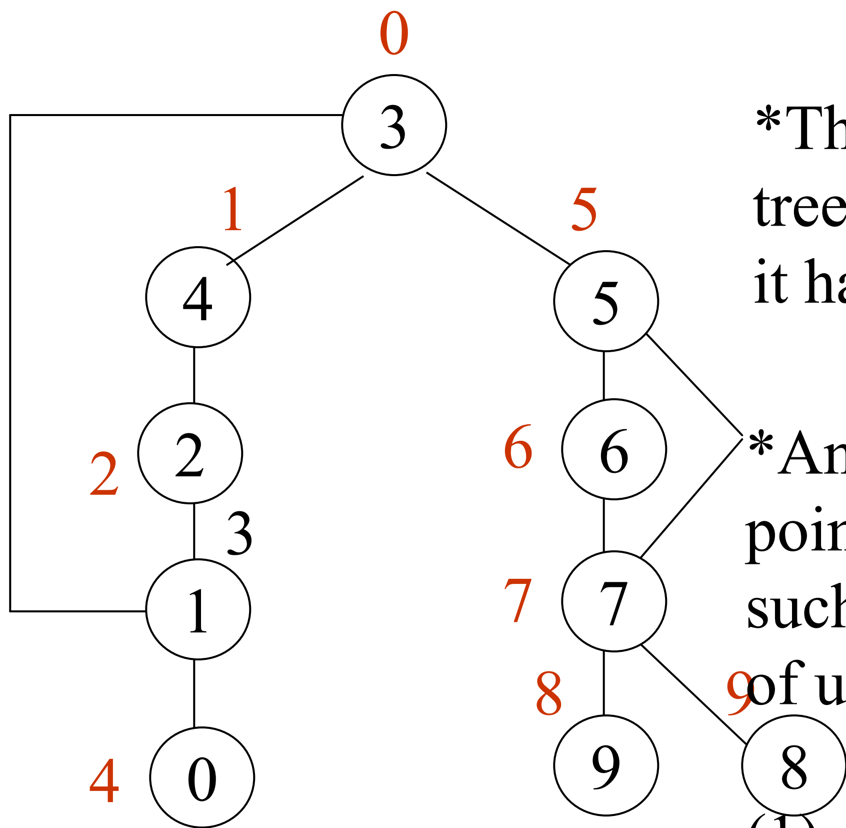
Why is cross edge impossible?

(b)

If u is an ancestor of v then dfn(u) < dfn(v).

***Figure 6.24**: *dfn* and *low* values for *dfs* spanning tree with *root* =3(p.281)

Vertax	0	1	2	3	4	5	6	7	8	9
dfn	4	3	2	0	1	5	6	7	9	8
low	4	0	0	0	0	5	5	5	9	8



*The root of a depth first spanning tree is an articulation point iff it has at least two children.

*Any other vertex u is an articulation point iff it has at least one child w such that we cannot reach an ancestor of u using a path that consists of (1) only w (2) descendants of w (3) single back edge.

low(u)=min{dfn(u),
min{low(w)|w is a child of u},
min{dfn(w)|(u,w) is a back edge}

u: articulation point
low(child) ≥ dfn(u)