Fourier transforms and Z-transforms

Infinite Fourier transform (Complex Fourier transform)

The infinite FT or simply FT of a fn $f(x)$ is denoted as $F[f(x)]$ given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$\boxed{F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx}$$ provided the integral exists

Alternate form : $F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$

Inverse Fourier transform

The IFT of $F(u)$ denoted by $F^{-1}[F(u)]$ is given by

$$\boxed{F^{-1}[F(u)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)\, e^{-iux}\, dx}$$

Alternate form : $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u)\, e^{-iux}\, dx$

Properties of Fourier transform

① Linearity property

If $c_1, c_2, c_3 \ldots c_n$ are constants then

$$F[c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \cdots + c_n f_n(x)] =$$
$$c_1 F[f_1(x)] + c_2 F[f_2(x)] + \cdots + c_n F[f_n(x)]$$

Sol : $$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

LHS : $$F[\underbrace{c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)}] =$$
$$\int_{-\infty}^{\infty} [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]\, e^{iux}\, dx$$
$$= c_1 \int_{-\infty}^{\infty} f_1(x)\, e^{iux}\, dx + c_2 \int_{-\infty}^{\infty} f_2(x)\, e^{iux}\, dx +$$
$$\cdots + c_n \int_{-\infty}^{\infty} f_n(x)\, e^{iux}\, dx$$
$$= c_1 F[f_1(x)] + c_2 F[f_2(x)] + \cdots + c_n F[f_n(x)]$$
$$= RHS$$

② change of Scale property

If $F[f(x)] = F(u)$ then $F[f(ax)] = \frac{1}{a} F\left(\frac{u}{a}\right)$

By def : $$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$
$$F[f(ax)] = \int_{-\infty}^{\infty} f(ax)\, e^{iux}\, dx$$

put $\underbrace{ax = t}$ $\quad x \to \infty \quad t \to \infty$

$dx = dt/a \quad x \to -\infty \quad t \to -\infty$

$$F[f(ax)] = \int_{-\infty}^{\infty} f(t)\, e^{iu\left(\frac{t}{a}\right)} \frac{dt}{a} = \underbrace{\frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{u}{a}\right)t} dt}$$

$$\frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{u}{a}\right)t} dt = \frac{1}{a} F\left(\frac{u}{a}\right)$$

③ Shifting property

If $F[f(x)] = F(u)$ then $F[f(x-a)] = e^{iua} F(u)$

By def

$$F[f(x)] = F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$f(x)$

$f(x-a)$

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)\, e^{iux}\, dx$$

$t \to -\infty$ to ∞

put $(x-a) = t$

$dx = dt$

$e^{a+b} = e^a e^b$

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(t)\, e^{iu(t+a)}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{iut} \cdot \underbrace{e^{iua}}\, dt$$

$$= e^{iua} \underbrace{\int_{-\infty}^{\infty} f(t)\, e^{iut}\, dt}$$

$$\boxed{F[f(x-a)] = e^{iua} F(u)}$$

④ Modulation property

If $F[f(x)] = F(u)$ then $F[f(x)\cos ax] = \frac{1}{2}\left[F(u+a) + F(u-a)\right]$

By defn : FT

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$F[f(x)\cos ax] = \int_{-\infty}^{\infty} f(x)\cos ax\, e^{iux}\, dx$$

Note :

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right) e^{iux}\, dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)\, e^{iax}\, e^{iux}\, dx + \int_{-\infty}^{\infty} f(x)\, e^{-iax}\, e^{iux}\, dx\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)\, e^{i(u+a)x}\, dx + \int_{-\infty}^{\infty} f(x)\, e^{i(u-a)x}\, dx\right]$$

$$= \frac{1}{2}\left[F(u+a) + F(u-a)\right]$$

Fourier cosine and Sine transforms and their inverse transform

Transforms

Fourier cosine

✓ $F_c[f(x)] = F_c(u)$

$$= \int^{\infty} f(x) \cos ux \, dx$$

Alternate form

$$F_c(u) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos ux \, dx$$

Fourier Sine

✓ $F_s[f(x)] = F_s(u)$

$$= \int_0^{\infty} f(x) \sin ux \, dx$$

Alternate form

$$F_s(u) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin ux \, dx$$

Inverse Transforms

✓ $$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(u) \cos ux \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(u) \cos ux \, dx$$

✓ $$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(u) \sin ux \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(u) \sin ux \, dx$$

Note : 1) Linearity property

$$F_c[c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)] = c_1 F_c[f_1(x)] + c_2 F_c[f_2(x)] + \dots + c_n F_c[f_n(x)]$$

2) Change of scale ppt

If $F_c[f(x)] = F_c(u)$ then $F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{u}{a}\right)$

The 2 ppt continue to hold good in case of Fourier sine transform also.

3) <u>Modulation ppt</u>

If $F_s[f(x)] = F_s(u)$ & $F_c[f(x)] = F_c(u)$ then

i) $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(u+a) + F_s(u-a)]$

*ii) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(u-a) - F_c(u+a)]$

iii) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(u+a) + F_c(u-a)]$

*iv) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(u+a) - F_s(u-a)]$

Problems

① Find the FT of $f(x) = \begin{cases} a^2 - x^2 & |x| < a \\ 0 & |x| > a \end{cases}$

and hence evaluate

$$\int_0^{\infty} \frac{\sin x - x\cos x}{x^3}\, dx$$

Sol: Def of FT : $F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$

$|x| < a$

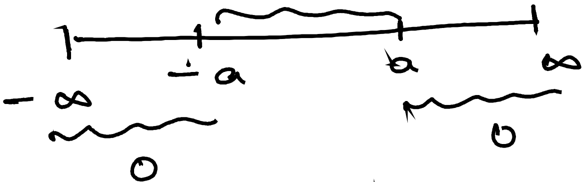
$-a < x < a$

$\Rightarrow x < a$ & $x > -a$

$|x| > a$

$-a > x > a$

$\Rightarrow x > a$ & $x < -a$



$$= \int_{-a}^{a} \underbrace{(a^2 - x^2)}_{u}\, \underbrace{e^{iux}}_{v}\, dx$$

$$= \left[(a^2 - x^2)\left[\frac{e^{iux}}{iu}\right] - (-2x)\left[\frac{e^{iux}}{(iu)^2}\right] + (-2)\left[\frac{e^{iux}}{(iu)^3}\right] \right]_{-a}^{a}$$

$$= \left[0 + 2a\frac{e^{iua}}{-u^2} - 2\frac{e^{iua}}{-iu^3} - \left(0 + (-a)\frac{e^{-iua}}{-u^2} - 2\frac{e^{-iua}}{-iu^3}\right)\right]$$

$i^2 = -1$

$i^3 = i^2 \cdot i$
$= -i$

$$= \left[0 + 2a \frac{e^{iua}}{i^2u^2} - 2\frac{e^{iua}}{i^3u^3} - \left(0 + 2(-a)\frac{e^{-iua}}{i^2u^2} - 2\frac{e^{-iua}}{i^3u^3}\right)\right]$$

$i^2 = -1$

$i^3 = -i$

$$= -\frac{2ae^{iua}}{u^2} + \frac{2e^{iua}}{iu^3} - \frac{2ae^{-iua}}{u^2} - \frac{2e^{-iua}}{iu^3}$$

$$= \frac{-2a}{u^2}\left[e^{iua} + e^{-iua}\right] + \frac{2}{iu^3}\cdot\left[e^{+iua} - e^{-iua}\right]$$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$= \frac{-2a}{u^2}\left[2\cos au\right] + \frac{2}{\cancel{i}u^3}\left[2\cancel{i}\sin au\right]$$

$$= -\frac{4a}{u^2}\cos au + \frac{4}{u^3}\sin au$$

$$\boxed{F(u) = \frac{4}{u^3}\left[-au\cos au + \sin au\right]}$$

W.K.T Inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$= \frac{1}{2\pi}\, 4\int_{-\infty}^{\infty} \frac{-au\cos au + \sin au}{u^3}\, e^{-iux}\, du$$

Take $x = 0$

$$f(0) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-au\cos au + \sin au}{u^3}\,du$$

$$a^2 = \frac{2}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{-au\cos au + \sin au}{u^3}}_{f(u)}\,du$$

Replace u to $-u$ $\Longrightarrow$ $\cancel{f(u)} = \dfrac{-a(-u)\cos a(-u) + \sin a(-u)}{-u^3}$

$f(-u)$

$$f(-u) = \frac{au\cos au - \sin au}{-u^3}$$

$$f(-u) = \frac{-au\cos au + \sin au}{u^3}$$

$f(u) = f(-u)$ $f(u)$ is even fn

$$\therefore a^2 = \frac{2}{\pi}\, 2\int_0^{\infty} \frac{-au\cos au + \sin au}{u^3}\,du$$

$$\frac{\pi a^2}{4} = \int_0^{\infty} \frac{-au\cos au + \sin au}{u^3}\,du$$

take $au = x \Rightarrow du = dx/a$ $\quad x \to 0$ to ∞

$$\frac{\pi a^2}{4} = \int_0^{\infty} \frac{-x\cos x + \sin x}{(x/a)^3}\cdot\frac{dx}{a}$$

$$\pi/_4\, \cancel{a^2} = \frac{a^3}{\cancel{a}}\int_0^{\infty} \frac{\sin x - x\cos x}{x^3}\,dx$$

$$\pi/4 = \int_0^{\infty} \frac{\sin x - x\cos x}{x^3}\,dx$$

② Find F.T of $f(x) = \left\{ \begin{matrix} 1 & |x|<a \\ 0 & |x|>a \end{matrix} \right\}$

Hence evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$

$f(x)=1$

$-a<x<a$

$x=0$

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\,dx$$

$$= \int_{-a}^{a} 1 \cdot e^{iux}\,dx = \left.\frac{e^{iux}}{iu}\right]_{-a}^{a}$$

$$= \frac{1}{iu}\left(e^{iua} - e^{-iua}\right]$$

$$= \frac{1}{\cancel{i}u}\, 2\cancel{i}\sin au = \frac{2\sin au}{u}$$

using Inverse F T

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin au}{u}\, e^{-iux}\,du$$

$x=0$

$$1 = \frac{1}{2\pi}(2)\int_{-\infty}^{\infty} \frac{\sin au}{u}\,du$$

$$1 = \frac{1}{\pi}\, 2\int_0^{\infty} \frac{\sin au}{u}\,du$$

$$1 = \frac{2}{\pi}\int_0^{\infty} \frac{\sin au}{u}\,du$$

Replace u to $-u$

$f(u) = f(-u)$

$\therefore f(u)$ is even

put $au = t$ $\quad du = dt$ $\quad t \to 0$ to ∞

$$\pi/2 = \int_0^{\infty} \frac{\sin t}{(t/a)}\, dt/a = \frac{\cancel{a}}{\cancel{a}}\int_0^{\infty} \frac{\sin t}{t}\, dt$$

$$\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2 \qquad \Big|\ \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

③ Find F.T of $f(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

& hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$

Sol :

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\,dx$$

$|x| \le 1$

$-1 \le x \le 1$

$x \le 1 \qquad x > -1$

$1-|x|$

$-\infty \quad -1 \quad +1 \quad \infty$

$0 \qquad 0$

$|x| > 1$

$-1 > x > 1$

$x > 1 \qquad x < -1$

$1-|x| \qquad 1-|x|$

$1-(-x) \qquad 1-x$

$1+x \qquad 1-x$

$-1 \quad 0 \quad 1$

$$= \int_{-1}^{1} f(x) e^{iux}\,dx$$

$$= \int_{-1}^{0} f(x) e^{iux}\,dx + \int_0^1 f(x) e^{iux}\,dx$$

$$= \int_{-1}^{0} (1+x) e^{iux}\,dx + \int_0^1 (1-x) e^{iux}\,dx$$

$$= \left[(1+x)\frac{e^{iux}}{iu} - (1)\frac{e^{iux}}{(iu)^2}\right]_{-1}^{0} + \left[(1-x)\frac{e^{iux}}{iu} - (-1)\frac{e^{iux}}{(iu)^2}\right]_0^1$$

$$= \left[\frac{1}{iu} - \frac{1}{i^2u^2} - \left(0 - \frac{e^{-iu}}{i^2u^2}\right)\right] + \left[0 + \frac{e^{iu}}{i^2u^2} - \left(\frac{1}{iu} + \frac{1}{i^2u^2}\right)\right]$$

$$= \cancel{\frac{1}{iu}} + \frac{1}{u^2} - \frac{e^{-iu}}{u^2} - \frac{e^{iu}}{u^2} - \cancel{\frac{1}{iu}} + \frac{1}{u^2}$$

$$= \frac{2}{u^2} - \frac{1}{u^2}\left[e^{-iu} + e^{iu}\right]$$

$$= \frac{2}{u^2} - \frac{1}{u^2}\left[2\cos u\right] = \frac{2}{u^2}\left[1 - \cos u\right]$$

$$F(u) = \frac{2}{u^2}\left(2\sin^2 u/2\right) = \frac{4\sin^2(u/2)}{u^2}$$

Taking FIT

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\,\frac{\sin^2(u/2)}{u^2}\,e^{-iux}\,du$$

put $x = 0$

$$1 = \frac{1}{2\pi}\cdot 4\,2\int_{-\infty}^{\infty} \frac{\sin^2(u/2)}{u^2}\,du$$

$$\pi/2 = 2\int_0^{\infty} \frac{\sin^2(u/2)}{u^2}\,du$$

Take $u/2 = t$ $\quad du = 2dt \quad t \to 0 \text{ to } \infty$

$$\pi/2 = 2\int_0^{\infty} \frac{\sin^2 t}{(2t)^2}\,2dt = \frac{4}{4}\int_0^{\infty} \frac{\sin^2 t}{t^2}\,dt$$

$$\int_0^{\infty} \frac{\sin^2 t}{t^2}\,dt = \pi/2 \;//$$

* Find Complex F·T of $f(x) = e^{-a^2x^2}$ where a is a +ve constant. Hence deduce that $e^{-x^2/2}$ is self reciprocal in respect of Complex F·T

Note : Self reciprocal : $F[e^{-x^2/2}] = K e^{-u^2/2}$

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2} e^{iux}\, dx = \int_{-\infty}^{\infty} e^{-a^2\left(x^2 - \frac{iux}{a^2}\right)} dx$$

- completing square

$$\int_{-\infty}^{\infty} e^{-a^2\left(x^2 - \frac{iux}{a^2} + \left(\frac{iu}{2a^2}\right)^2 - \left(\frac{iu}{2a^2}\right)^2\right)} dx$$

Constant term $= \frac{1}{2}[\text{coeff of } x]^2$

$$= \int_{-\infty}^{\infty} e^{-a^2\left[\left\{x - \frac{iu}{2a^2}\right\}^2 - \left(\frac{iu}{2a^2}\right)^2\right]} dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left\{x - \frac{iu}{2a^2}\right\}^2} e^{\frac{-u^2 \cdot a^2}{4a^4}} dx$$

$$= e^{-u^2/4a^2} \int_{-\infty}^{\infty} e^{-a^2\left(x - \frac{iu}{2a^2}\right)^2} dx$$

put $a\left(x-\frac{iu}{2a^2}\right)=t$ $t\to-\infty$ to ∞

$dx = \frac{dt}{a}$

$$F(u) = \int e^{-u^2/4a^2}\int_{-\infty}^{\infty} e^{-t^2}\,\frac{dt}{a}$$

$$= \frac{e^{-u^2/4a^2}}{a}\underbrace{\int_{-\infty}^{\infty} e^{-t^2}\,dt}$$

Note : $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$

$$= \frac{e^{-u^2/4a^2}}{a}\sqrt{\pi}$$

$e^{-x^2/2}$

$F(u)= k\,e^{-u^2/2}$

put $a^2 = 1/2$

$$= \frac{e^{-u^2/4(1/2)}\sqrt{\pi}}{a} = \frac{\sqrt{\pi}}{a}\,e^{-u^2/2}$$

$$F(u) = k\,e^{-u^2/2} \qquad k = \sqrt{\pi}/a$$

$\therefore$ $e^{-x^2/2}$ is self reciprocal wrt complex F.T.

$\therefore$

* Obtain Fourier cosine transform of

$$f(x) = \begin{cases} 4x & 0 < x < 1 \\ 4-x & 1 < x < 4 \\ 0 & x > 4 \end{cases}$$

<u>Sol</u> :

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$= \int_0^1 4x \cos ux + \int_1^4 (4-x) \cos ux + \int_4^\infty 0 \, dx .$$

$$= \left[4x \frac{\sin ux}{u} - (4)\left[-\frac{\cos ux}{u^2}\right]\right]_0^1 +$$

$$\left[(4-x)\frac{\sin ux}{u} - (-1)\left[-\frac{\cos ux}{u^2}\right]\right]_1^4 + 0$$

$$= \frac{4}{u}\sin u + \frac{4}{u^2}\cos u - 0 + \frac{4}{u^2}(1) +$$

$$0 - \frac{\cos 4u}{u^2} - \frac{3}{u}\sin 3u + \frac{\cos u}{u^2}$$

$$\Rightarrow \frac{4}{u}\sin u + \frac{4}{u^2}\cos u - \frac{4}{u^2} - \frac{\cos 4u}{u^2} - \frac{3}{u}\sin u$$

$$+ \frac{\cos u}{u^2}$$

$$F_c(u) = \frac{\sin u}{u} + \frac{5}{u^2}\cos u - \frac{1}{u^2}(4 + \cos 4u)$$

* Find Fourier Sine transform of $\frac{e^{-ax}}{x}$ $(a>0)$

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$F_s(u) = \int_0^\infty \frac{e^{-ax}}{x} \sin ux \, dx \quad \text{—— (1).}$$

Leibnitz rule under integral sign

$$f(u) = \int_0^\infty g(x)\, h(u)\, dx$$

$$\frac{d}{du} f(u) = \int_0^\infty g(x) \frac{\partial}{\partial u} h(u)\, dx$$

$F_s(0)$ | $\frac{d}{dx} \sin ax$

$\cos ax \cdot a$

$$\frac{d}{du} F_s(u) = \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial u} (\sin ux)\, dx$$

$$F_s'(u) = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos ux \cdot (\cancel{x})\, dx$$

$$= \int_0^\infty e^{-ax} \cos ux \, dx$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$$

$$= \left[\frac{e^{-ax}}{(-a)^2+u^2} \left[-a \cos ux + u \sin ux\right]\right]_0^\infty$$

$$= \frac{1}{a^2+u^2} \left[\cancel{e^{-\infty}} \; 0 - 1(-a)\right]$$

$$\underline{F_s'(u)} = a/a^2+u^2$$

$e^{-\infty} = 0$

Integrate wrt u.

$$F_s(u) = \int \frac{a}{a^2+u^2}\,du = a\int \frac{du}{a^2+u^2}$$

$$= a \cdot \frac{1}{a} \tan^{-1}(u/a) + c$$

$$F_s(u) = \tan^{-1}(u/a) + c \quad \text{--- (2)}$$

To find c

put $u=0$ in (2) put $u=0$ in (1)

$$F_s(0) = \tan^{-1} 0 + c$$

$$0 = 0 + c$$

$$c = 0$$

$$F_s(u) = \underline{\underline{\tan^{-1}(u/a)}}$$

Convolution of two functions

Def : If $f(x)$ and $g(x)$ are any 2 fns then the convolution of 2 fns $f(x)$ & $g(x)$ is denoted by

$$\boxed{f(x) * g(x) \;\; \text{(OR)} \;\; (f*g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du}$$

$*$ → convolution operator

Property : Prove that Convolution is Commutative

i.e, $f(x) * g(x) = g(x) * f(x)$

Sol Df :

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \qquad u \to -\infty \text{ to } \infty$$

put $x - u = t$, $-du = dt$ $\qquad t \to +\infty$ to $-\infty$

$$= \int_{+\infty}^{-\infty} f(x-t)\, g(t)\, (-dt)$$

$$= \int_{-\infty}^{\infty} g(t)\, f(x-t)\, dt \qquad g(x) * f(x)$$

$$= \int_{-\infty}^{\infty} \cancel{g(t) * f(t)}$$

$$\boxed{f(x) * g(x) = g(x) * f(x)}$$

Convolution theorem for Fourier transform

The F.T of Convolution of $f(x)$ & $g(x)$ is the product of their F.T

ie $F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)]$

Parseval's identity

Fourier transform

a) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\,G(u)\,du = \int_{-\infty}^{\infty} f(x)g(x)\,dx$$

b) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(u)|^2\,du = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

Fourier Sine transform

a) $$\frac{2}{\pi}\int_{0}^{\infty} F_s(u)\,G_s(u)\,du = \int_{0}^{\infty} f(x)g(x)\,dx$$

b) $$\frac{2}{\pi}\int_{0}^{\infty} |F_s(u)|^2\,du = \int_{0}^{\infty} |f(x)|^2\,dx$$

Fourier Cosine transform

a) $$\frac{2}{\pi}\int_{0}^{\infty} F_c(u)\,G_c(u)\,du = \int_{0}^{\infty} f(x)g(x)\,dx$$

b) $$\frac{2}{\pi}\int_{0}^{\infty} |F_c(u)|^2\,du = \int_{0}^{\infty} |f(x)|^2\,dx$$