Notes:Applications of Double and Triple Integrals

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1. *Find the area of the circle $x^2 + y^2 = a^2$ by using double integral.*

Solution

Since, the circle is symmetric about the coordinates axes, area of the circle is 4 times the area *OAB* as shown in Figure.

For the region *OAB*, y varies from 0 to $\sqrt{a^2 - x^2}$ and x varies from 0 to a.

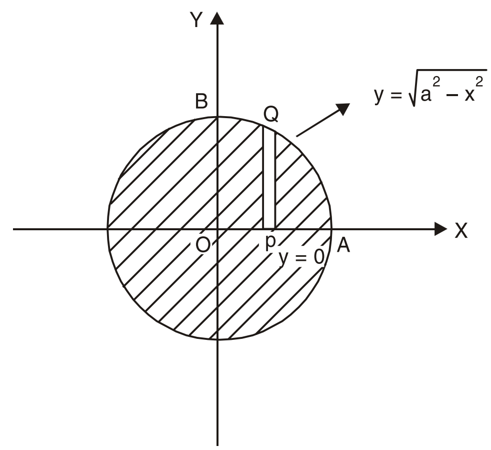


Fig. 3.16

$$\therefore \text{ Area of the circle } = 4\int_0^a \int_{y=0}^{\sqrt{a^2-x^2}} dy\, dx$$

$$= 4\int_0^a [y]_{y=0}^{\sqrt{a^2-x^2}}\, dx$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \pi a^2 \text{ sq. units}$$

2. *Find by double integration the area enclosed by the curve $r = a\,(1 + \cos\theta)$ between $\theta = 0$ and $\theta = \pi$.*

Solution Area $= \iint r\, dr\, d\theta$

where r varies from 0 to $a\,(1 + \cos\theta)$ and θ varies from 0 to π

$$\int\sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

i.e., $$A = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_{r=0}^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2 (1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \left\{2\cos^2\left(\frac{\theta}{2}\right)\right\}^2 d\theta$$

$$= 2a^2 \int_0^{\pi} \cos^4\left(\frac{\theta}{2}\right) d\theta$$

Put $\theta/2 = \phi,\ d\theta = 2d\phi$

and ϕ varies from 0 to $\pi/2$

$$\therefore \quad A = 2a^2 \int_0^{\pi/2} \cos^4\phi \cdot 2d\phi$$

$$= 4a^2 \int_0^{\pi/2} \cos^4\phi \cdot d\phi$$

$$= 4a^2 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{(by the reduction formula)}$$

Area, $A = 3\pi a^2/4$ sq. units.

3. *Find the value of* $\iiint z\, dx\, dy\, dz$ *where V is the hemisphere* $x^2 + y^2 + z^2 = a^2,\ z \ge 0$.

Solution

Let $$I = \iiint_V z\, dx\, dy\, dz$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} z\, dz\, dy\, dx$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$= \frac{1}{2} \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(a^2 - x^2 - y^2\right) dy\, dx$$

$$= \frac{1}{2} \int_{x=-a}^{a} \left[\left(a^2 - x^2\right) y - \frac{y^3}{3}\right]_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \cdot \frac{4}{3} \int_{-a}^{a} \left(a^2 - x^2\right)^{3/2} dx$$

$$= \frac{2}{3} \cdot 2 \int_{0}^{a} \left(a^2 - x^2\right)^{3/2} dx$$

Put $\quad x = a \sin \theta$

$dx = a \cos \theta \, d\theta$

θ varies from 0 to $\pi/2$

$$= \frac{4}{3} \int_{\theta=0}^{\pi/2} \left(a^2 \cos^2 \theta\right)^{3/2} a \cos \theta \, d\theta$$

$$= \frac{4a^4}{3} \int_{0}^{\pi/2} \cos^4 \theta \, d\theta$$

$$= \frac{4a^4}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \qquad \text{(By applying reduction formula)}$$

$$= \frac{\pi a^4}{4}$$

Thus, $\quad I = \dfrac{\pi a^4}{4}.$

4. *Using multiple integrals find the volume of the ellipsoid* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

Solution

The volume (V) is 8 times in the first octant (V_1)

i.e., $\quad V = 8V_1 = 8 \iiint dz\, dy\, dx$

z varies from 0 to $c\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}$

y varies from 0 to $(b/a)\ \sqrt{a^2 - x^2}$

x varies from 0 to a

$$V = 8V_1 = 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2 - x^2}} \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2-x^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2-x^2}} \frac{1}{b}\sqrt{b^2\left\{1-\left(\frac{x^2}{a^2}\right)\right\}-y^2}\, dy\, dx$$

We shall use $\int \sqrt{\alpha^2 - y^2}\, dy = \dfrac{y\sqrt{\alpha^2 - y^2}}{2} + \dfrac{\alpha^2}{2}\sin^{-1}\left(\dfrac{y}{\alpha}\right)$

where $\alpha^2 = b^2\ \{1 - x^2/a^2\} = b^2\ (a^2 - x^2)/a^2$

$$\therefore \qquad V = \frac{8c}{b} \int_{x=0}^{a} \int_{y=0}^{\alpha} \sqrt{\alpha^2 - y^2}\, dy\, dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \left[\frac{y\sqrt{\alpha^2 - y^2}}{2} + \frac{\alpha^2}{2}\sin^{-1}\left(\frac{y}{\alpha}\right)\right]_0^{\alpha} dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} 0 + \frac{\alpha^2}{2}\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \frac{\pi}{2} \cdot \frac{1}{2}\frac{b^2}{a^2}\left(a^2 - x^2\right) dx$$

$$= \frac{2bc\pi}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2bc\pi}{a^2} \cdot \frac{2a^3}{3} = \frac{4\pi\, abc}{3}$$

Thus the required volume $(V) = \dfrac{4\pi abc}{3}$ cubic units.

Gamma and Beta Functions

1. THE GAMMA FUNCTION

The gamma function may be regarded as a generalization of n! (n-factorial), where n is any positive integer to x!, where x is any real number. (With limited exceptions, the discussion that follows will be restricted to positive real numbers.) Such an extension does not seem reasonable, yet, in certain ways, the gamma function defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt \qquad (1)$$

meets the challenge. This integral has proved valuable in applications. However, because it cannot be r presented through elementary functions, establishment of its properties take some effort. Some of the important ones are outlined below.

The gamma function is convergent for x > 0. It follows from eq.(1) that

From (1): $\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$

Integrating by parts

$$\Gamma(x+1) = \left[t^x\left(\frac{e^{-t}}{-1}\right)\right]_0^\infty + x\int_0^\infty e^{-t}t^{x-1}\, dt$$

$$= \{0-0\} + x\Gamma(x)$$

$$\therefore\ \Gamma(x+1) = x\Gamma(x) \qquad (2)$$

This is a fundamental recurrence relation for gamma functions. It can also be written as

$$\Gamma(x) = (x-1)\Gamma(x-1).$$

A number of other results can be derived from this as follows: If $x = n$, a positive integer, i.e. if $n \geq 1$, then

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n). \\
&= n(n-1)\Gamma(n-1) \qquad \text{since } \Gamma(n) = (n-1)\Gamma(n-1) \\
&= n(n-1)(n-2)\Gamma(n-2) \qquad \text{since } \Gamma(n-1) = (n-2)\Gamma(n-2) \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= n(n-1)(n-2)(n-3)\ldots 1\Gamma(1) \\
&= n!\Gamma(1)
\end{aligned}$$

But $\Gamma(1) = \int_0^\infty t^0 e^{-t} dt = [-e^{-t}]_0^\infty = 1$

$$\Rightarrow \qquad \Gamma(n+1) = n! \qquad (3)$$

Example:

$\Gamma(7) = 6! = 720, \qquad \Gamma(8) = 7! = 5040, \qquad \Gamma(9) = 40320$

We can also use the recurrence relation in reverse

$\Gamma(x+1) = x\Gamma(x) \qquad \Rightarrow \qquad \Gamma(x) = \frac{\Gamma(x+1)}{x}$

What happens when $x = \frac{1}{2}$? We will investigate.

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\,dt$$

Putting $t = u^2$, $dt = 2u\,du$, then

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty u^{-1} e^{-u^2} 2u\,du = 2\int_0^\infty e^{-u^2}\,du.$$

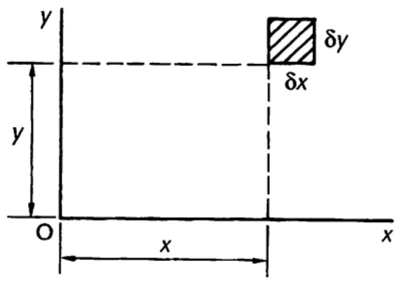
Unfortunately, $\int_0^\infty e^{-u^2}\,du$ cannot easily be determined by normal means. It is, however, important, so we have to find a way of getting round the difficulty.

Evaluation of $\int_0^\infty e^{-x^2}\,dx$

Let $I = \int_0^\infty e^{-x^2}\,dx$, then also $I = \int_0^\infty e^{-y^2}\,dy$

$$\therefore\ I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

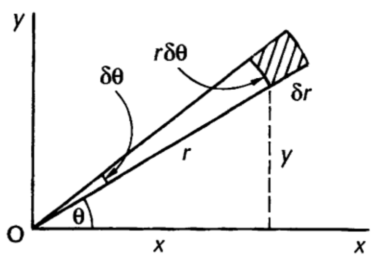
$\delta a = \delta x\,\delta y$ represents an element of area in the x–y plane and the integration with the stated limits covers the whole of the first quadrant.



Converting to polar coordinates, the element of area $\delta a = r\,\delta\theta\,\delta r$. Also, $x^2 + y^2 = r^2$

$$\therefore\ e^{-(x^2+y^2)} = e^{-r^2}$$

For the integration to cover the same region as before,



the limits of r are $r = 0$ to $r = \infty$
the limits of θ are $\theta = 0$ to $\theta = \pi/2$.

$$\therefore\ I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{e^{-r^2}}{2}\right]_0^\infty d\theta$$

$$= \int_0^{\pi/2}\left(\frac{1}{2}\right)d\theta = \left[\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$\therefore\ I = \frac{\sqrt{\pi}}{2}$$

$$\therefore\ \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \qquad (5)$$

Before that diversion, we had established that

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du$$

We now know that $\int_0^\infty e^{-u^2}\,du = \dfrac{\sqrt{\pi}}{2}$ $\quad\therefore\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

From this, using the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, we can obtain the following

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\ \Gamma(\tfrac{1}{2}) = \tfrac{1}{2}(\sqrt{\pi}) \qquad \therefore\ \Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\ \Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right) \qquad \therefore\ \Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4}$$

Negative values of x

Since $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$, then as $x \to 0$, $\Gamma(x) \to \infty$ $\ \therefore\ \Gamma(0) = \infty$.

The same result occurs for all negative integral values of x – which does not follow from the original definition, but which is obtainable from the recurrence relation.

Because at $x = -1$, $\quad \Gamma(-1) = \dfrac{\Gamma(0)}{-1} = \infty$

$x = -2$, $\quad \Gamma(-2) = \dfrac{\Gamma(-1)}{-2} = \infty$ etc.

Also, at $x = -\tfrac{1}{2}$, $\quad \Gamma(-\tfrac{1}{2}) = \dfrac{\Gamma(\tfrac{1}{2})}{-\tfrac{1}{2}} = -2\sqrt{\pi}$

and at $x = -\tfrac{3}{2}$, $\quad \Gamma(-\tfrac{3}{2}) = \dfrac{\Gamma(-\tfrac{1}{2})}{-\tfrac{3}{2}} = \dfrac{4}{3}\sqrt{\pi}$

So we have

(a) For n a positive integer

$\Gamma(n+1) = n\Gamma(n) = n!$

$\Gamma(1) = 1;\quad \Gamma(0) = \infty;\quad \Gamma(-n) = \pm\infty$

(b) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi};\qquad \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$

$\Gamma(\tfrac{3}{2}) = \dfrac{\sqrt{\pi}}{2};\qquad \Gamma(-\tfrac{3}{2}) = \dfrac{4}{3}\sqrt{\pi}$

$\Gamma(\tfrac{5}{2}) = \dfrac{3\sqrt{\pi}}{4};\qquad \Gamma(-\tfrac{5}{2}) = -\dfrac{8}{15}\sqrt{\pi}$

$\Gamma(\tfrac{7}{2}) = \dfrac{15\sqrt{\pi}}{8};\qquad \Gamma(-\tfrac{7}{2}) = \dfrac{16}{105}\sqrt{\pi}$

Example:

Evaluate $\int_0^\infty x^7 e^{-x}\mathrm{d}x$.

We recognise this as the standard form of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t \qquad \text{with the variables changed.}$$

It is often convenient to write the gamma function as

$$\Gamma(v) = \int_0^\infty x^{v-1}e^{-x}\mathrm{d}x$$

Our example then becomes

$$I = \int_0^\infty x^7 e^{-x}\mathrm{d}x = \int_0^\infty x^{v-1}e^{-x}\,\mathrm{d}x \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

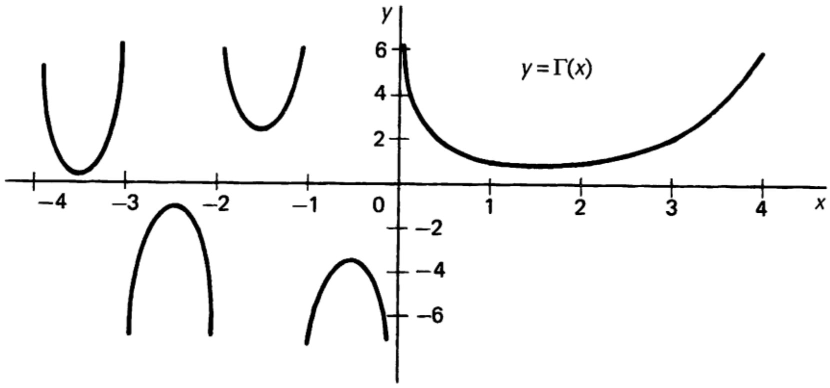
$$\text{i.e. } \int_0^\infty x^7 e^{-x}\,dx = \Gamma(8) = 7! = 5040$$

Graph of $y = \Gamma(x)$

Values of $\Gamma(x)$ for a range of positive values of x are available in tabulated form in various sets of mathematical tables. These, together with the results established above, enable us to draw the graph of $y = \Gamma(x)$.

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
$\Gamma(x)$	∞	1·772	1·000	0·886	1·000	1·329	2·000	3·323	6·000

x	−0·5	−1·5	−2·5	−3·5
$\Gamma(x)$	−3·545	2·363	−0·945	0·270



Example:

Evaluate $\int_0^\infty x^3 e^{-4x}\,dx$.

If we compare this with $\Gamma(v) = \int_0^\infty x^{v-1} e^{-x}\,dx$, we must reduce the power of e to a single variable, i.e. put $y = 4x$, and we use this substitution to convert the whole integral into the required form.

$y = 4x \quad \therefore\ dy = 4\,dx \qquad$ Limits remain unchanged.

The integral now becomes ....

$$\boxed{I = \int_0^\infty \left(\frac{y}{4}\right)^3 e^{-y}\,\frac{dy}{4}}$$

$$\therefore\ I = \frac{1}{4^4}\int_0^\infty y^3 e^{-y}\,dy = \frac{1}{4^4}\Gamma(v) \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

$$\int_0^\infty y^{v-1}e^{-y}dy = \int_0^\infty y^3 e^{-y}dy \qquad \therefore\ v = 4$$

$$\therefore\ I = \frac{1}{4^4}\,\Gamma(4) = \ldots\ldots\ldots\ldots$$

$$I = \frac{1}{256}\Gamma(4) = \frac{1}{256}(3!) = \frac{6}{256} = \frac{3}{128}$$

2. THE BETA FUNCTION

The beta function is a two-parameter composition of gamma functions that has been useful enough in application to gain its own name.

The beta function $\mathrm{B}(m, n)$, is defined by

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x \tag{1}$$

which converges for $m > 0$ and $n > 0$.

Putting $(1-x) = u \quad \therefore\ x = 1-u \quad \therefore\ \mathrm{d}x = -\,\mathrm{d}u$

Limits: when $x = 0,\ u = 1;\quad$ when $x = 1,\ u = 0$

$$\therefore\ \mathrm{B}(m, n) = -\int_1^0 (1-u)^{m-1}\,u^{n-1}\,\mathrm{d}u = \int_0^1 (1-u)^{m-1}\,u^{n-1}\mathrm{d}u$$

$$= \int_0^1 u^{n-1}(1-u)^{m-1}\mathrm{d}u = \mathrm{B}(n, m)$$

$$\therefore\ \mathrm{B}(m, n) = \mathrm{B}(n, m) \tag{2}$$

Alternative form of the beta function

We had

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$$

If we put $x = \sin^2\theta$, the result then becomes ....

Because if $x = \sin^2\theta,\quad \mathrm{d}x = 2\sin\theta\,\cos\theta\,\mathrm{d}\theta$.

When $x = 0,\ \theta = 0$; when $x = 1,\ \theta = \pi/2$. $1 - x = 1 - \sin^2\theta = \cos^2\theta$

$$\therefore\ \mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\,\cos\theta\,\mathrm{d}\theta$$

$$\therefore\ \mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,\mathrm{d}\theta \tag{3}$$

Relation between the gamma and Beta Functions

If m and n are positive integers

$$\mathrm{B}(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Also, we have previously established that, for n a positive integer,

$$n! = \Gamma(n+1)$$

$$\therefore\ (m-1)! = \Gamma(m) \quad \text{and} \quad (n-1)! = \Gamma(n)$$

and also $(m+n-1)! = \Gamma(m+n)$

$$\therefore\ \mathrm{B}(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

The relation $\mathrm{B}(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ holds good even when m and n are not necessarily integers.

Application of gamma and beta functions

The use of gamma and beta functions in the evaluation of definite integrals depends largely on the ability to change the variables to Express the integral in the basic form of the beta function

$$\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$$

or its trigonometrical form $2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,\mathrm{d}\theta$.

Example:

Evaluate $I=\int_0^1 x^5(1-x)^4\,\mathrm{d}x$.

Compare this with $\mathrm{B}(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$

Then $m-1=5 \quad \therefore\ m=6$ and $n-1=4 \quad \therefore\ n=5$

$$\therefore\ I=\mathrm{B}(6, 5)=\ldots\ldots\ldots\ldots$$

$$\boxed{I=\mathrm{B}(6, 5)=\frac{5!\,4!}{10!}=\frac{1}{1260}}$$

Example:

Evaluate $I=\int_0^1 x^4\sqrt{1-x^2}\mathrm{d}x$.

Comparing this with $\mathrm{B}(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$

we see that we have x^2 in the root, instead of a single x.

Therefore, put $x^2=y \quad \therefore\ x=y^{\frac{1}{2}} \quad \mathrm{d}x=\frac{1}{2}y^{-\frac{1}{2}}\mathrm{d}y$

The limits remain unchanged. $\therefore\ I=\ldots\ldots\ldots\ldots$

$$\boxed{I=\tfrac{1}{2}\mathrm{B}\left(\tfrac{5}{2},\tfrac{3}{2}\right)}$$

Because

$$I=\int_0^1 y^2(1-y)^{\frac{1}{2}}\,\frac{1}{2}y^{-\frac{1}{2}}\,\mathrm{d}y=\frac{1}{2}\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}}\,\mathrm{d}y$$

$m-1=\frac{3}{2} \quad \therefore\ m=\frac{5}{2}$ and $n-1=\frac{1}{2} \quad \therefore\ n=\frac{3}{2}$

$\therefore\ I=\frac{1}{2}\mathrm{B}\left(\frac{5}{2},\frac{3}{2}\right)$

Expressing this in gamma functions

$$\boxed{I=\frac{1}{2}\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}}$$

From our previous work on gamma functions

$$\Gamma\left(\tfrac{3}{2}\right)=\frac{\sqrt{\pi}}{2};\quad \Gamma\left(\tfrac{5}{2}\right)=\frac{3\sqrt{\pi}}{4};\quad \Gamma(4)=3!$$

$$I=\frac{1}{2}\cdot\frac{(3\sqrt{\pi}/4)(\sqrt{\pi}/2)}{3!}=\frac{\pi}{32}.$$

Now you can work through this one in much the same way. There are no tricks.

Curvilinear Coordinates

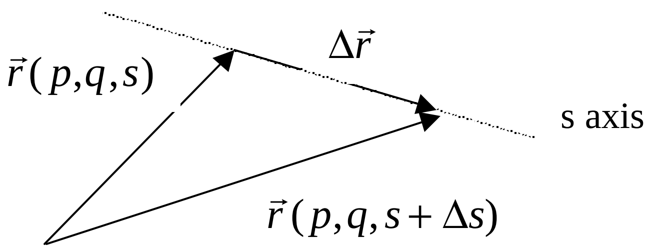
In this section, the concepts of unit vectors and scale factors in orthogonal curvilinear coordinate systems are introduced. The corresponding expressions for gradient and divergence are given.

Scale factors and unit vectors

Suppose that the position vector of a moving point depends on three scalar parameters, *p, q,* and *s,* called the coordinates of the point

$$\vec{r} = \vec{r}(p, q, s)$$

For example, in a Cartesian coordinate system *p, q,* and *s* would correspond to the coordinates *x, y,* and *z*, while in a cylindrical polar coordinate system they could correspond to *r,* q , and *z.* In general, *p, q,* and *s* are said to form a *curvilinear coordinate system* (because when two of the coordinates are held fixed, and the third is varied, the point moves along a curvy line)



If one of the coordinates, *s*, is changed by an infinitesimal amount, Δs (with *p* and *q* held constant) then the position vector of the point will change by the infinitesimal amount

$$\Delta \vec{r} = \vec{r}(p,q,s+\Delta s) - \vec{r}(p,q,s)$$

$\Delta\vec{r}$ points in the direction that the point moves when only *s* is changed. We call this direction the *s* axis at the point. If the change in *s* is positive then the point moves along the positive *s* axis at the point *(p,q,s).*

Consider, now, the following partial derivative

$$\frac{\partial \vec{r}}{\partial s} = \lim_{\Delta s \to 0}\left[\frac{\Delta\vec{r}}{\Delta s}\right] = \lim_{\Delta s \to 0}\left[\frac{\vec{r}(q,r,s+\Delta s) - \vec{r}(q,r,s)}{\Delta s}\right]$$

Since Δs is a scalar quantity then $\frac{\P\, \vec{r}}{\P\, s}$ and $\Delta\vec{r}$ point in the same direction.

Hence, $\frac{\P\, \vec{r}}{\P\, s}$ is also a vector along the *s* axis. A unit vector along the *s* axis can be generated by dividing the partial derivative by its length, h_s , defined as

$$h_S^2 = \frac{\partial \vec{r}}{\partial s} . \frac{\partial \vec{r}}{\partial s}$$

The unit vector becomes

$$\hat{e}_S = \frac{1}{h_S} \frac{\partial \vec{r}}{\partial s}$$

or,

$$\hat{e}_S = \frac{\dfrac{\partial \vec{r}}{\partial s}}{\sqrt{\dfrac{\partial \vec{r}}{\partial s} . \dfrac{\partial \vec{r}}{\partial s}}}$$

The quantity h_s is called the scale factor in the s direction. Physically h_s is the distance moved by the point per unit change in its s coordinate.

Scale factors and unit vectors in the p, and q, directions can be similarly defined, to gives a triad of unit vectors defining the direction of the coordinate axes at a point in space. Note that for a general system of coordinates each unit vector will point in different directions at different points in space.

If the coordinates p, q, and s are defined in such a way that the three unit vectors are *mutually normal* everywhere, then p, q, and s are said to form an *orthogonal* curvilinear coordinate system. The system will be orthogonal if all of the following are true.

$$\frac{\partial \vec{r}}{\partial p} . \frac{\partial \vec{r}}{\partial q} = 0 \quad ; \quad \frac{\partial \vec{r}}{\partial p} . \frac{\partial \vec{r}}{\partial s} = 0 \quad ; \quad \frac{\partial \vec{r}}{\partial q} . \frac{\partial \vec{r}}{\partial s} = 0$$

The first of these can be written

$$\begin{aligned} \frac{\partial \vec{r}}{\partial p} . \frac{\partial \vec{r}}{\partial q} &= \frac{\partial \left(x \hat{i} + y \hat{j} + z \hat{k} \right)}{\partial p} . \frac{\partial \left(x \hat{i} + y \hat{j} + z \hat{k} \right)}{\partial q} \\ &= \frac{\partial x}{\partial p} \frac{\partial x}{\partial q} + \frac{\partial y}{\partial p} \frac{\partial y}{\partial q} + \frac{\partial z}{\partial p} \frac{\partial z}{\partial q} = 0 \end{aligned}$$

Gradient and Divergence in Orthogonal Curvilinear Coordinate Systems:

We state here, without proof, that the gradient and divergence in the *orthogonal* curvilinear coordinate system (p,q,s) are given by the following expressions:

$$\vec{\nabla} T = \frac{1}{h_p}\frac{\partial T}{\partial p}\hat{e}_p + \frac{1}{h_q}\frac{\partial T}{\partial q}\hat{e}_q + \frac{1}{h_s}\frac{\partial T}{\partial s}\hat{e}_s$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{h_p h_q h_s}\left[\frac{\partial}{\partial p}\left(h_q h_s F_p\right) + \frac{\partial}{\partial q}\left(h_p h_s F_q\right) + \frac{\partial}{\partial s}\left(h_p h_q F_s\right)\right]$$

These expressions, in conjunction with the appropriate scale factors, permit the heat conduction equation to be expressed in any orthogonal curvilinear coordinate system.

Cartesian Coordinates:

Consider, first, the simple Cartesian coordinate system.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

This coordinate system has the unique property that the respective unit vectors point in the same direction at all points in space, i.e. the unit vectors are independent of x, y, and z.

A typical scale factor becomes

$$h_x^2 = \frac{\partial\vec{r}}{\partial x}.\frac{\partial\vec{r}}{\partial x}$$

where

$$\frac{\partial\vec{r}}{\partial x} = \frac{\partial\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{\partial x}$$

Since the unit vectors are independent of x this becomes

$$\frac{\partial\vec{r}}{\partial x} = \hat{i}$$

and

$$h_x = 1$$

Similarly,

$$h_y = 1 \quad ; \quad h_z = 1$$

Cylindrical Polar Coordinate System:

A cylindrical polar coordinate system is defined as follows

$$x = r\cos q$$
$$y = r\sin q$$
$$\mathrm{z} = \mathrm{z}$$

and the position vector becomes

$$\vec{r} = \mathrm{r}\ \cos q\ \hat{\mathrm{i}} + \mathrm{r}\ \sin q\ \hat{\mathrm{j}} + \mathrm{z}\hat{\mathrm{k}}$$

The scale factor in the q direction is calculated as follows;

$$\begin{aligned} \mathrm{h}_q &= \left| \frac{\partial}{\partial q}\left(\mathrm{r}\cos q\ \hat{\mathrm{i}} + \mathrm{r}\sin q\ \hat{\mathrm{j}} + \mathrm{z}\hat{\mathrm{k}}\right) \right| \\ &= \left| -\mathrm{r}\sin q\ \hat{\mathrm{i}} + \mathrm{r}\cos q\ \hat{\mathrm{j}} \right| \\ &= \mathrm{r}\sqrt{\cos^2 q + \sin^2 q} = \mathrm{r} \end{aligned}$$

The remaining two scale factors are 1.

To test for orthogonality, say between the q and r unit vectors, calculate

$$\frac{\partial x}{\partial q}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial q}\frac{\partial y}{\partial r} + \frac{\partial \mathrm{z}}{\partial q}\frac{\partial \mathrm{z}}{\partial r} =$$
$$\frac{\partial(r\cos q)}{\partial q}\frac{\partial(r\cos q)}{\partial r} + \frac{\partial(r\sin q)}{\partial q}\frac{\partial(r\sin q)}{\partial r} + \frac{\partial \mathrm{z}}{\partial q}\frac{\partial \mathrm{z}}{\partial r} =$$
$$(-r\sin q)(\cos q) + (r\cos q)(\sin q) + (0)(1) = 0$$

Then repeat for the other unit vectors.

The gradient of T can be written as

$$
\begin{aligned}
\vec{\nabla} T &= \frac{1}{h_r}\frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{h_\theta}\frac{\partial T}{\partial \theta}\hat{e}_\theta + \frac{1}{h_z}\frac{\partial T}{\partial z}\hat{e}_z \\
&= \frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{e}_\theta + \frac{\partial T}{\partial z}\hat{k}
\end{aligned}
$$

The Divergence can be written as

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{F} &= \frac{1}{h_r h_\theta h_z}\left[\frac{\partial}{\partial r}\left(h_\theta h_z F_r\right) + \frac{\partial}{\partial \theta}\left(h_r h_z F_\theta\right) + \frac{\partial}{\partial z}\left(h_r h_\theta F_z\right)\right] \\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_r\right) + \frac{\partial}{\partial \theta}\left(F_\theta\right) + \frac{\partial}{\partial z}\left(rF_z\right)\right]
\end{aligned}
$$

Combining the above two expressions the Laplacian can be written as

$$
\begin{aligned}
\nabla^2 T &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial T}{\partial z}\right)\right] \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}
\end{aligned}
$$