Module-2.

Solutions of Simultaneous D.E of I order

(1) Suppose if we get a linear D.Eq in which 2 (or) more dependent variable & a single independent variable, these equations are called "simultaneous eqn"

ex

$$f_1(D)x + f_2(D)y = \phi_1(t)$$
$$g_1(D)x + g_2(D)y = \phi_2(t)$$

x & y are function of an independent variable "t"

KSC exe:

(19) Solve $\frac{dx}{dt} + y = e^t$; $\frac{dy}{dt} - x = e^{-t}$.

$Dx + y = e^t \quad \times D \qquad D^2x + Dy = e^t \quad (\because D(e^t) = e^t)$

$$\begin{array}{r} Dy - x = e^{-t} \\ \underline{- \quad + \quad\quad -} \\ D^2x + x = e^t - e^{-t} \end{array}$$

$$(D^2+1)x = e^t - e^{-t}$$

A. Eqn is $m^2+1=0 \Rightarrow m = \pm i$

$\therefore x_c = c_1\cos t + c_2 \sin t$.

$$x_p = \frac{e^t - e^{-t}}{D^2+1} = \frac{e^t - e^{-t}}{2} = \sinh t$$

$\therefore x = x_c + x_p = c_1 \cos t + c_2 \sin t + \sinh t$.

$$\frac{dx}{dt} + y = e^t$$

$$y = e^t - \frac{d}{dt}\left(c_1 \cos t + c_2 \sin t + \sinh t\right)$$

$$= e^t + c_1 \sin t - c_2 \cos t - \cosh t.$$

$$= e^t - \cosh t + c_1 \sin t - c_2 \cos t$$

$$= e^t - \frac{\left(e^t + e^{-t}\right)}{2} + c_1 \sin t - c_2 \cos t$$

$$= \frac{e^t - e^{-t}}{2} + c_1 \sin t - c_2 \cos t$$

$$= \sinh t + c_1 \sin t - c_2 \cos t.$$

Solve $\frac{dx}{dt} = 2y$; $\frac{dy}{dt} = 2z$ $\frac{dz}{dt} = 2x.$

Diff $dx/dt = 2y$ w.r.to t

$$\frac{d^2x}{dt^2} = 2 \cdot \frac{dy}{dt} = 2 \cdot 2z = 4z$$

again Diff w.r.to t

$$\frac{d^3x}{dt^3} = 4 \cdot \frac{dz}{dt} = 4 \cdot 2x.$$

$$\therefore \left(D^3 - 8\right) x = 0.$$

A.E $(D-2)\left(D^2 + 2D + 4\right)$ ($D=2$ is a factor find $D-2 \overline{\big) D^3 - 8}$ D^2+2D+4)

$$\Rightarrow D = 2, \quad -1 \pm i\sqrt{3}.$$

Solution is $x = c_1 e^{2t} + e^{-t}(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t)$

<u>Now to find y</u> :

$$2y = \frac{dx}{dt} \Rightarrow y = \frac{1}{2}\frac{dx}{dt}$$

$$y = \frac{1}{2}\Big(2c_1 e^{2t} + e^{-t}(-\sqrt{3}c_2 \sin\sqrt{3}t + \sqrt{3}c_3 \cos\sqrt{3}t) + (c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t)\,(-e^{-t})\Big)$$

$$y = c_1 e^{2t} + \frac{1}{2}e^{-t}\Big(\cos\sqrt{3}t\,(\sqrt{3}c_3 - c_2) - \sin\sqrt{3}t\,(\sqrt{3}c_2 + c_3)\Big)$$

From IInd equation.

$$z = \frac{1}{2}\,\frac{dy}{dt}$$

$$z = \frac{1}{2}(2c_1 e^{2t}) + \frac{1}{4}\Big(-e^{-t}\big(\cos\sqrt{3}t\,(\sqrt{3}c_3 - c_2) - \sin\sqrt{3}t\,(\sqrt{3}c_2 + c_3)\big) + e^{-t}\big(-\sqrt{3}\sin\sqrt{3}t\,(\sqrt{3}c_3 - c_2) - \sqrt{3}\cos\sqrt{3}t\,(\sqrt{3}c_2 + c_3)\big)\Big)$$

$$z = c_1 e^{2t} + \frac{1}{4}e^{-t}\Big(\cos\sqrt{3}t\,(-2c_2 - 2\sqrt{3}c_3) - (2\sqrt{3}c_2 - 2c_3)\sin\sqrt{3}t\Big)$$

~~Method of~~

$\frac{dx}{dt} + 2y + \sin t = 0$ $\frac{dy}{dt} - 2x - \cos t = 0.$

$x = 0$ & $y = 1$ when $t = 0.$

Soln:

$Dx + 2y = -\sin t \times 2$

$Dy - 2x = \cos t \times D$

$2Dx + 4y = -2\sin t$

$D^2 y - 2Dx = -\sin t$

$(D^2 + 4)\, y = -3\sin t$

A-E $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$C.F = c_1 \cos 2t + c_2 \sin 2t$

$P.I = \frac{-3\sin t}{D^2+4} = \frac{-3\sin t}{-1+4} = -\sin t$

$y = c_1 \cos 2t + c_2 \sin 2t - \sin t.$ — (1)

$dy/dx = -2c_1 \sin 2t + 2c_2 \cos 2t - \cos t.$

& $= -2c_1 \sin 2t + 2c_2 \cos 2t - 2\cos t$

$\therefore 2x = Dy - \cos t =$

$x = -c_1 \sin 2t + c_2 \cos 2t - \cos t$ — (2)

when $t = 0,\ x = 0,\ y = 1$

$\boxed{1 = c_1}$, $0 = c_2 - 1 \Rightarrow \boxed{c_2 = 1}$

$\therefore x = \cos 2t - \sin 2t - \cos t$

$y = \cos 2t + \sin 2t - \sin t.$

Solution of Legendre & Cauchy linear D. Eqn.

Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + y = \log x \frac{\sin(\log x)+1}{x}$.

Solution

Put $x = e^t \quad \therefore \quad t = \log x \qquad \frac{dt}{dx} = 1/x$

$\therefore \; x\frac{dy}{dx} = Dy \qquad \left(\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = Dy \cdot \frac{1}{x} \right)$

$x^2 \frac{d^2y}{dx^2} = D(D-1)y.$

$\therefore$ the given eqn becomes

$$\left(D(D-1) - 3D + 1 \right) y = t \, \frac{\sin t + 1}{e^t}$$

$\left(D^2 - 4D + 1\right) y = e^{-t} . t\,(\sin t + 1)$ $\rightarrow$ linear with constant coeff.

A.E is $D^2 - 4D + 1 = 0$; $D = 2 \pm \sqrt{3}$

$$C.F = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} = e^{2t}\left(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}\right)$$

$$P.I = \frac{e^{-t} . t\,(\sin t + 1)}{D^2 - 4D + 1} = e^{-t}\left(\frac{t}{D^2 - 6D + 6} + \frac{t \sin t}{D^2 - 6D + 6} \right)$$

$$P.I = e^{-t}\left(P_1 + P_2 \right)$$ (put $D = D-1$ since e^{-t})

$P_1 = \frac{t}{D^2 - 6D + 6} = t/6 + 1/6$ by division method

$= \frac{t+1}{6}.$

$$\begin{array}{r|l} & \boxed{t/6 + 1/6} \\ D^2 - 6D + 6 & t \\ & t - 1 \\ \hline & 1 \\ & 1 \\ \hline & 0 \end{array}$$

$$P_2 = I\cdot P \text{ of } \frac{e^{it}\cdot t}{D^2-6D+6} = I\cdot P\, e^{it} \cdot \frac{t}{(D+i)^2-6(D+i)+6}$$

$$= I\cdot P \text{ of } e^{it} \frac{t}{D^2+(2i-6)D+(5-6i)t}$$

$$= I\cdot P \text{ of } e^{it}\left(\frac{t}{5-6i} - \frac{2i-6}{61}\right) \quad \text{(by division method)}$$

$$= I\cdot P \text{ of } \frac{e^{it}(5+6i)}{(5-6i)(5+6i)}\left(t - \frac{2i-6}{5-6i}\right)$$

$$\left(\because (5+6i)(5-6i) = 61\right)$$

$$= I\cdot P\ \frac{5+6i}{61}\cdot(\cos t + i\sin t)\left(t - \frac{2i-6}{5-6i}\right)$$

$$= I\cdot P\ \frac{1}{61}\left(5\cos t - 6\sin t\right) + i\,(5\sin t + 6\cos t)\left(t + \frac{42+2\,6i}{61}\right)$$

$$P_2 = \frac{26}{3721}\left(5\cos t - 6\sin t\right) + \frac{1}{61}\left(5\sin t + \cos t\right)\left(t + \frac{42}{61}\right)$$

$$\therefore y = P_1 + P_2 \quad //$$

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$$(5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 8(5+2x)^2.$$

Soln

$$(5+2x) = e^t$$

$$(5+2x)\frac{dy}{dx} = 2Dy \qquad (D = d/dt)$$

$$(5+2x)^2 \frac{d^2y}{dx^2} = 4(D-1)Dy$$

Given equation becomes $\therefore (D^2 - 4D + 2)y = 2e^{2t}$ (Simplify !)

Solution is

$$y = C.F + P.I$$

$$D^2 - 4D + 2 = 0 \qquad D = 2 \pm \sqrt{2}$$

$$\therefore C.F = c_1 e^{(2+\sqrt{2})t} + c_2 e^{(2-\sqrt{2})t}$$

$$P.I = \frac{2e^{2t}}{D^2-4D+2} = e^{2t} \frac{2}{4-8+2}$$

$$P.I = -e^{2t}$$

$$y = c_1 e^{(2+\sqrt{2})t} + c_2 e^{(2-\sqrt{2})t} - e^{2t}$$

$$\cancel{c_1 (5+2x)^2}$$

$$= e^{2t}\left(c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}\right) - e^{2t}$$

$$y = (5+2x)^2 \left(c_1 (5+2x)^{\sqrt{2}} + c_2 (5+2x)^{-\sqrt{2}}\right) - 5 + 2x$$

VTU2009

General ~~exe~~ $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(1+x))$

Soln

Put $(1+x) = e^t$; $(1+x)\frac{dy}{dx} = Dy$; $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$

when $D = d/dt$.

$D(D-1)y + Dy + y = 2\sin t$

$(D^2+1)y = 2\sin t$

$D^2+1 = 0 \Rightarrow D = \pm i$

$C.F = c_1\cos t + c_2 \sin t$

$P.I = \frac{2\sin t}{D^2+1}$ put $D^2 = -1$

Dr becomes zero

$P.I = 2t \cdot \frac{1}{2D}\sin t$

$= 2t\frac{1}{2}\int \sin t\,dt = -t\cos t$

$y = c_1\cos t + c_2\sin t - t\cos t$

again put $t = \log(1+x)$

$y = c_1\cos(\log(1+x)) + c_2\sin(\log(1+x))$

$- \log(1+x)\cos(\log(1+x))$

VTU HW $(3x-2)^2y'' - 3(3x-2)y' = 9(3x-2)\sin\log(3x-2)$

Put $e^t = 3x-2 \Rightarrow y'(3x-2) = 3Dy$; $y''(3x-2)^2 = 3^2D(D-1)y$

Ans. $y_c = c_1 + c_2e^{2t} = c_1 + c_2(3x-2)^2$

$y_P = e^t\frac{\sin t}{-2} = -\frac{(3x-2)}{2}\sin(\log(3x-2))$

Solve $2p^3 - (2x + 4\sin x - \cos x)p^2$

$- (x\cos x - 4x\sin x + \sin 2x)\, p + x\cdot \sin 2x = 0$

Solution:

Put $p = x$

$2x^3 - 2x^3 - x^2 4\cancel{\sin x} + x^2\cos x - x^2\cos x + 4x^2\cancel{\sin x} - x\cancel{\sin 2x}$

$+ x\cancel{\sin 2x} = 0$

$\Rightarrow (p-x)$ is a factor

$\therefore$ The given eqn becomes

$(p-x)\left(2p^2 - (4\sin x - \cos x)\, p - \sin 2x\right) = 0$

$\Rightarrow (p-x)\left(2p(p - 2\sin x) + \cos x\,(p - 2\sin x)\right) = 0$

$\Rightarrow (p-x)\,(p - 2\sin x)\,(2p + \cos x) = 0$

$\Rightarrow \quad p - x = 0 \Rightarrow p = x \Rightarrow \frac{dy}{dx} = x$

$\therefore \quad \boxed{y = x^2/2 + c}$

Now $\quad p - 2\sin x = 0 \quad \Rightarrow \quad \frac{dy}{dx} - 2\sin x = 0$

$\Rightarrow dy - 2\sin x\, dx = 0$

Integ $\quad \boxed{y + 2\cos x = c.}$

$2p + \cos x = 0$

$\Rightarrow 2\,dy + \cos x\,dx = 0$

$\therefore 2y + \sin x = c$

$\therefore$ General solution is

$(2y - x^2 - y)\,(y + 2\cos x - c)\,(2y + \sin x - c) = 0$.

—— x ——

Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

$(p + 2x)p^2 - y^2p\,(p + 2x) = (p + 2x)(p^2 - y^2p)$

$\Rightarrow (p + 2x)\,p\,(p - y^2) = 0$

$p + 2x = 0$	$p = 0$,	$p - y^2 = 0$
$dy + 2x\,dx = 0$	$\Rightarrow \frac{dy}{dx} = 0$	$\frac{dy}{dx} - y^2 = 0$
$y + x^2 = c$	$\Rightarrow y = c$	$\Rightarrow \frac{dy}{dx} = y^2$
		$\frac{dy}{y^2} = dx$
		Integ
		$\Rightarrow$
		$-\frac{1}{y} = x + c$.

$\boxed{\therefore \text{Solution is } (y + x^2 - c)(y - c)(xy + cy + 1) = 0}$

Solve

{Grewal exe KSC}

$$p + \cos y \sin px = \sin y \cos px$$

$$p = \sin y \cos px - \cos y \sin px$$

$$p = \sin(y - px)$$

$$\Rightarrow \sin^{-1} p = y - px$$

$$\Rightarrow y = \sin^{-1} p + px = px + \sin^{-1} p$$

(clairaut's form.)

$$\therefore \boxed{y = cx + \sin^{-1} c}$$

Solve $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0.$

Soln $p^2x^2 - 2pxy + y^2 - p^2a^2 - b^2 = 0$

(ie) $(y - px)^2 = p^2a^2 + b^2$

$\therefore y = px \pm \sqrt{p^2a^2 + b^2}$ is in clairaut's form.

$\therefore$ General solution is $y = cx \pm \sqrt{c^2a^2 + b^2}$

Grewal
exe

$(y-px)(p-1)=p$; solve

$\Rightarrow y-px = \frac{p}{p-1}$

$\Rightarrow y = px + \frac{p}{p-1}$ is in Clairaut's form

$\therefore$ General solution is $y = cx + \frac{c}{c-1}$ — (1)

Differentiate w.r.to c

$$0 = x + \frac{(c-1)-c}{(c-1)^2}$$

$$0 = x - \frac{1}{(c-1)^2}$$

$$\Rightarrow x = \frac{1}{(c-1)^2} \Rightarrow c-1 = \pm \frac{1}{\sqrt{x}}$$

$$\Rightarrow \boxed{c = 1 \pm \frac{1}{\sqrt{x}}}$$

Put c in G. Soln (eq (1))

$$y = \left(1 \pm \frac{1}{\sqrt{x}}\right)x + \frac{(1 \pm 1/\sqrt{x})}{\left(\cancel{1} \pm \frac{1}{\sqrt{x}} - \cancel{1}\right)}$$

$$\Rightarrow y = x \pm \sqrt{x} \pm \frac{\cancel{\sqrt{x}}(\sqrt{x} \pm 1)}{\cancel{\sqrt{x}}}$$

is the singular soln

—x—

Solve & find the Singular Solution of

$y^2(1+p^2) = a^2.$

Solution :

$$p^2 = 1 - a^2/y^2 = \frac{y^2-a^2}{y^2}$$

$$p = \frac{\sqrt{y^2-a^2}}{y}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-a^2}}{y}$$

$$\frac{y\,dy}{\sqrt{y^2-a^2}} = dx$$

Substitute $y^2-a^2 = t^2$

$\Rightarrow \cancel{2}t\,dt = \cancel{2}y\,dy$

$$\therefore \frac{\cancel{t}\,dt}{\cancel{t}} = dx$$

Integrating. $t = x + c$

(ie) $\sqrt{y^2-a^2} = x+c$

$\therefore$ Gen Solution is $(y^2-a^2) = (x+c)^2$

Differentiating partially w.r.to c

$2(x+c) = 0 \Rightarrow x+c = 0 \Rightarrow \boxed{c=-x}$

$\therefore$ Singular solution is $y^2 - a^2 = 0 \Rightarrow \boxed{y = \pm a}$

Solve

$$xp^2 - 2yp + ax = 0$$

We can solve this for y

(If we solve this for p we get

$$p = \frac{y \pm \sqrt{y^2 - ax^2}}{x}$$ & we have to put $y = vx$ & proceed)

$$y = \frac{ax + xp^2}{2p} = \frac{ax}{2p} + \frac{xp}{2}.$$

Differentiating both sides

$$\frac{dy}{dx} = \frac{a}{2p} + \frac{x}{2} \cdot \frac{(0 - a)}{p^2}\frac{dp}{dx} + \frac{p}{2} + \frac{x}{2} \cdot \frac{dp}{dx}$$

$$p = \frac{a}{2p} - \frac{ax}{2p^2}\frac{dp}{dx} + \frac{p}{2} + \frac{x}{2}\frac{dp}{dx}$$

$$\Rightarrow \frac{p}{2} - \frac{a}{2p} = \frac{dp}{dx}\left(\frac{x}{2} - \frac{ax}{2p^2}\right)$$

$$\frac{p^2 - a}{2p}\left(1 - x/p \cdot \frac{dp}{dx}\right) = 0$$

$$\Rightarrow \left(1 - x/p\, \frac{dp}{dx}\right) = 0 \quad \Rightarrow \frac{dx}{x} - \frac{dp}{p} = 0$$

~~Integ~~

② Integrating

$$\log x - \log p = c$$

$$\Rightarrow p = cx.$$

eliminating p.

$$y = \frac{ax}{2 \times cx} + x \cdot \frac{cx}{2}$$

$$\boxed{2y = cx^2 + a/c}$$

(ie) $\boxed{2cy = c^2x^2 + a}$ is the general solution

Solve $\sin y \cos^2 x = \cos^2 y\, p^2 + p \sin x \cdot \cos x\, \cos y$

by reducing to clairauts form, put $\sin y = u$, $\sin x = v$.

<u>Solution</u>:

$$\sin y \cdot \cos^2 x = \cos^2 y\, p^2 + \sin x \cdot \cos x \cdot \cos y\, p$$

$\div$ by $\cos^2 x$

$$\sin y = \frac{\cos^2 y}{\cos^2 x} \cdot p^2 + p \sin x \cdot \frac{\cos y}{\cos x}$$

$$P = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$p = \frac{\cos x}{\cos y} \cdot \frac{du}{dx}$$

$\therefore$ eqn becomes

$$u = \left(\frac{du}{dv}\right)^2 + v\left(\frac{du}{dv}\right) \text{ is in}$$

clairauts form. $u = v \cdot p + p^2$

$\therefore$ General solution is

$$u = cv + c^2.$$

(ie) $\sin y = c \cdot \sin x + c^2$

is the soln.