MODULE-4

The Principle of Inclusion - Exclusion

If S is a finite set, then the number of element in S is called the order (or the size, or the cardinality) of S and is denoted by $|S|$. If A and B are subsets of S, then the order of $A \cup B$ is given by the formula

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Thus for determining the number of elements that are in $A \cup B$, we include all elements in A and B but exclude all elements common to A and B.

Principle of Inclusion - Exclusion for n sets

Let S be a finite set and $A_1, A_2 \ldots A_n$ be subsets of S. Then the Principle of inclusion - exclusion for $A_1, A_2, \ldots A_n$ states that

$$|A_1 \cup A_2 \cup A_3 \cdots \cup A_3| = \sum|A_i| - \sum|A_i \cap A_j| + \sum|A_i \cap A_j \cap A_k| + \cdots + (-1)^{n-1}|A_1 \cap A_2 \cdots \cap A_n|$$

Generalization

The Principle of inclusion - exclusion as given by expression $\bar{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$ gives the number of elements in S that satisfy none of the conditions $c_1, c_2, \ldots c_n$. The following expression determines the number of elements in S that satisfy exactly m of the n conditions $(0 \le m \le n)$;

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{n-m} \binom{n}{n-m} S_n$$

Problems 1. Out of 30 students in a hostel, 15 study History, 8 study Economics, and 6 study Geography. It is known that 3 students study all these subjects. Show that 7 or more students study none of these subjects.

⇒ Let 'S' denote the set of all students in the hostel, and A_1, A_2, A_3 denote the sets of students who study History, Economics and Geography, respectively.

Given, $S_1 = \sum |A_i| = 15 + 8 + 6 = 29$ and

$$S_3 = |A_1 \cap A_2 \cap A_3| = 3.$$

The number of students who do not study any of the three subjects is $|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}|$.

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_1 \cap A_2 \cap A_3|$$
$$= |S| - S_1 + S_2 - S_3$$
$$= 30 - 29 + S_2 - 3 = S_2 - 2.$$

where, $S_2 = \Sigma|A_i \cap A_j|$

we know that $(A_1 \cap A_2 \cap A_3)$ is a subset of $(A_i \cap A_j)$ for $i, j = 1, 2, 3$. Therefore, each of $|A_i \cap A_j|$, which are 3 in number, is greater than (or) equal to $|A_1 \cap A_2 \cap A_3|$.

$$S_2 = \Sigma|A_i \cap A_j| \geq 3|A_1 \cap A_2 \cap A_3| = 9.$$

$$\therefore |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| \geq 9 - 2 = 7.$$

(2) How many integers between 1 and 300 (inclusive) are (i) divisible by at least one of 5, 6, 8?
(ii) divisible by none of 5, 6, 8?

$\Rightarrow$

Let S denote, $S = \{1, 2, \ldots 300\}$ so that, $|S| = 300$. Also, let A_1, A_2, A_3 be subsets of S whose elements are divisible by 5, 6, 8 respectively.

(i) The number of elements of S that are divisible by at least one of 5, 6, 8 is,

$|A_1 \cup A_2 \cup A_3|$.

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} + |A_1 \cap A_2 \cap A_3|$$

W.k.T,

$|A_1| = 60$, $|A_2| = 50$, $|A_3| = 37$, $|A_1 \cap A_2| = 10$

$|A_1 \cap A_3| = 7$, $|A_2 \cap A_3| = 12$. $|A_1 \cap A_2 \cap A_3| = 2$.

$$|A_1 \cup A_2 \cup A_3| = (60+50+37) - (10+7+12) + 2 = 120.$$

Thus 120 elements of S are divisible by at least one of 5, 6, 8.

(ii). The number of elements of S that are divisible by none of 5, 6, 8 is,

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - |A_1 \cup A_2 \cup A_3| = 300 - 120 = 180.$$

(3) Find the number of nonnegative integer solutions of the equation.

$$x_1 + x_2 + x_3 + x_4 = 18$$

under the condition $x_i \leq 7$, for $i = 1, 2, 3, 4$

$\Rightarrow$

Let S denote the set of all nonnegative integer solutions of the given equation. The number of such solutions is, $C(4+18-1, 18) = C(21, 18)$

$$|S| = C(21, 18).$$

Let A_1 be the subset of S that contains the nonnegative integer solutions of the given equation under the conditions $x_1 > 7$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

$A_1 = \{ (x_1, x_2, x_3, x_4) \in S \mid x_1 > 7 \}$

Ill^{ly}, let, $A_2 = \{ (x_1, x_2, x_3, x_4) \in S \mid x_2 > 7 \}$

$A_3 = \{ (x_1, x_2, x_3, x_4) \in S \mid x_3 > 7 \}$

$A_4 = \{ (x_1, x_2, x_3, x_4) \in S \mid x_4 > 7 \}$.

∴ The required solution, $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$

Let us set $y_1 = x_1 - 8$. Then, $x_1 > 7$. (i.e) $x \geq 8$).

corresponds to $y_1 \geq 0$, when written in terms of

y_1, $y_1 + x_2 + x_3 + x_4 = 10$.

The number of nonnegative integer solutions

of this equation is $C(4+10-1, 10) = C(13, 10)$.

$|A_1| = C(13, 10)$.

Ill^{ly}, $|A_2| = |A_3| = |A_4| = C(13, 10)$

Let us take $y_1 = x_1 - 8$, $y_2 = x_2 - 8$. then $x_1 > 7$

and $x_2 > 7$ correspond to $y_1 \geq 0$ and $y_2 \geq 0$.

When written in terms of y_1 and y_2,

$$y_1 + y_2 + x_3 + x_4 = 2.$$

The number of non-negative integer solutions of this Equation is $C(4+2-1,2) = C(5,2)$.

$|A_1 \cap A_2|$. $\therefore |A_1 \cap A_2| = C(5,2)$

$|A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4|$

$= C(5,2)$.

The given equation, more than two x_i's cannot be greater than 7 simultaneously.

$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4|$

$= |A_2 \cap A_3 \cap A_4| = 0$,

and, $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$.

Accordingly, we find that

$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k|$

$+ |A_1 \cap A_2 \cap A_3 \cap A_4|$

$= C(21,18) - \binom{4}{1} \times C(13,10) + \binom{4}{2} \times C(5,2) - 0 + 0$

$= 1330 - (4 \times 286) + (6 \times 30) = \underline{366}$.

(4) In how many ways 5 number of a's, 4 number of b's and 3 number of c' can be arranged so that all the identical letters are not in a single book?

⇒ The given letters are $5+4+3=12$ in number. Of which 5 are a's 4 are b's and 3 are c's. If S is the set of all permutations (arrangements) of these letters, we've,

$$|S| = \frac{12!}{5!\,4!\,3!}$$

Let A_1 be the set of arrangements of the letters where the 5 a's are in a single block. The number of such arrangements is,

$$|A_1| = \frac{8!}{4!\,3!}$$

$|||^{ly}$, if A_2 is the set of arrangements where the 4 b's are in a single block, and A_3 is the set of arrangements where the 3 c's are in a single block, we've,

$$|A_2| = \frac{9!}{5!\,3!} \quad \text{and} \quad |A_3| = \frac{10!}{5!\,4!}$$

Likewise,

$$|A_1 \cap A_2| = \frac{5!}{3!}, \quad |A_1 \cap A_3| = \frac{6!}{4!}, \quad |A_2 \cap A_3| = \frac{7!}{5!}$$

$$|A_1 \cap A_2 \cap A_3| = 3!$$

The required number of arrangements is,

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - \{|A_1| + |A_2| + |A_3|\} + \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} - |A_1 \cap A_2 \cap A_3|.$$

$$= \frac{12!}{5!\,4!\,3!} - \left\{\frac{8!}{4!\,3!} + \frac{9!}{5!\,3!} + \frac{10!}{5!\,4!}\right\} + \left\{\frac{5!}{3!} + \frac{6!}{4!} + \frac{7!}{5!}\right\} - 3!$$

$$= 27720 - (280 + 504 + 1260) + (20 + 30 + 42) - 6$$

$$= 25762.$$

⑤ In how many ways can the 26 letters of the English alphabet be permuted so that none of the patterns CAR, DOG, PUN (or) BYTE occurs?

Let S denote the set of all permutations of the 26 letters. Then $|S| = 26!$

Let A_1 be the set of all permutations in which CAR appears. This word, CAR consists of three letters which form a single block.

The set A_1 therefore consists of all permutations which contain this single block and the 23 remaining letters. $\therefore |A_1| = 24!$

$|||^{ly}$, if A_2, A_3, A_4 are the set of all permutations which contain DOG, PUN and BYTE respectively, we've, $|A_2| = 24!$ $|A_3| = 24!$ $|A_4| = 23!$

Likewise, $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = (26-6+2)! = 22!$

$|A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4| = (26-7+2) = 21!$

$|A_1 \cap A_2 \cap A_3| = (26-9+3)! = 20!$

$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = (26-10+3)!$
$= 19!$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = (26-13+4)! = 17!$

$\therefore$ The required number of permutations is given by, $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k|$
$+ |A_1 \cap A_2 \cap A_3 \cap A_4|$

$= 26! - (3 \times 24! + 23!) + (3 \times 22! + 3 \times 21!)$
$- (20! + 3 \times 19!) + 17!$

⑥ In how many ways can one arrange the letters in the word CORRESPONDENTS so that

i) there is no pair of consecutive identical letters?

ii) there are exactly two pairs of consecutive identical letters?

iii) there are at ~~least~~ least three pairs of consecutive identical letters?

$\Rightarrow$ In the word CORRESPONDENTS, there occur one each of C, P, D and T and two each of

O, R, E, S, N. If S is the set of all permutations of these 14 letters, we've,

$$|S| = \frac{14!}{(2!)^5}$$

Let A_1, A_2, A_3, A_4, A_5 be the sets of permutations in which O's, R's, E's, S's, N's appear in pairs respectively,

Then, $|A_i| = \frac{13!}{(2!)^4}$ for $i = 1, 2, 3, 4, 5$

Also, $|A_i \cap A_j| = \frac{12!}{(2!)^3}$, $|A_i \cap A_j \cap A_k| = \frac{11!}{(2!)^2}$

$|A_i \cap A_j \cap A_k \cap A_p| = \frac{10!}{(2!)}$ $|A_1 \cap A_2 \cap A_3 \cdots \cap A_5| = 9!$

From these,

$$S_0 = N = |S| = \frac{14!}{(2!)^5}, \quad S_1 = C(5,1) \times \frac{13!}{(2!)^4}$$

$$S_2 = C(5,2) \times \frac{12!}{(2!)^3}, \quad S_3 = C(5,3) \times \frac{11!}{(2!)^2}$$

$$S_4 = C(5,4) \times \frac{10!}{2!}, \quad S_5 = C(5,5) \times 9!$$

Accordingly, the number of permutations where there is no pair of consecutive identical letter is,

$$E_0 = S_0 - \binom{1}{1} S_1 + \binom{2}{2} S_2 - \binom{3}{3} S_3 + \binom{4}{4} S_4 - \binom{5}{5} S_5$$

$$= \frac{14!}{(2!)^5} - \binom{5}{1} \times \frac{13!}{(2!)^4} + \binom{5}{2} \times \frac{12!}{(2!)^3} - \binom{5}{3} \times \frac{11!}{(2!)^2} + \binom{5}{4} \times \frac{10!}{2!} - \binom{5}{5} \times 9!$$

The number of permutations where there are exactly two pairs of consecutive identical letters,

$$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 - \binom{5}{3} S_5$$

$$= \binom{5}{2} \times \frac{12!}{(2!)^3} - \binom{3}{1}\binom{5}{3} \times \frac{11!}{(2!)^2} + \binom{4}{2}\binom{5}{4} \times \frac{10!}{2!} - \binom{5}{3}\binom{5}{5} \times 9!$$

the number of permutations where there are at least three pairs of consecutive identical letter is

$$L_3 = S_3 - \binom{3}{2} S_4 + \binom{4}{2} S_5$$

$$= \binom{5}{3} \times \frac{11!}{(2!)^2} - \binom{3}{2}\binom{5}{4} \times \frac{10!}{2!} + \binom{4}{2}\binom{5}{5} \times 9!$$

Derangements:-

A permutation of n distinct objects in which none of the objects is in its natural place is called a derangement.

Formula for d_n

The following is the formula for d_n for $n \geq 1$:

$$d_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}.$$

$$= n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

For example, $D_2 = 2!\left[1 - \frac{1}{1!} + \frac{1}{2!}\right] = 1.$

$$D_3 = 3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 6\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right)$$

$$D_3 = 2.$$

$$D_4 = 9,\ D_5 = 44,\ D_6 = 265,\ D_7 = 1854.$$

① Evaluate d_5, d_6, d_7, d_8

$$\Rightarrow\ d_5 = (5!)\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}$$

$$= (120)\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44$$

$$d_6 = (6!)\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right\}$$

$$= (720)\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} - \frac{1}{720}\right) = 265$$

$$d_7 \approx \left[7! \times e^{-1}\right] \approx \left[5040 \times 0.3679\right] \approx 1854$$

$$d_8 \approx \left[8! \times e^{-1}\right] \approx \left[40320 \times 0.3679\right] \approx 14833$$

$L_m = S_m - {}^{m+1}C_m S_{m+1} +$

② From the set of all permutations of n distinct objects, one permutation is chosen at random. What is the probability that it is not a derangement?

→ The number of permutations of n distinct objects is $n!$. The number of derangements of these objects is d_n.

∴ The probability that a permutation chosen is not a derangements,

$$P = 1 - \frac{d_n}{n!} = 1 - \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^n}{n!}$$

$L_m = S_m - {}^{m+1}C_m S_{m+1} + {}^{m+2}C_m S_{m+2}$

③ In how many ways can the integers 1, 2, 3 ... 10 be arranged in a line so that no even integer is in its natural place.

Let A_1 be the set of all permutations of the given integers where 2 is in its natural place. A_2 be the set of all permutations in which 4 is in its natural place, and so on.

The number of permutations where no even integer is in its natural place is $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5|$. This is given by,

$|\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_5| = |S| - S_1 + S_2 - S_3 + S_4 - S_5$

we note that $|S| = 10!$

Now, the permutations in A_1 are all of the form $b_1 b_3 b_4 \ldots b_{10}$ where $b_1 b_3 b_4 \ldots b_{10}$ is a permutation of $1, 3, 4, 5, \ldots 10$. As such, $|A_1| = 9!$

$|||^{ly}$, $|A_2| = |A_3| = |A_4| = |A_5| = 9!$

So that, $S_1 = \sum |A_i| = 5 \times 9! = C(5,1) \times 9!$

The permutations in $A_1 \cap A_2$ are all of the form $b_1 2 b_3 4 b_5 b_6 \ldots b_{10}$ where $b_1 b_3 b_5 b_6 \ldots b_{10}$ is a permutation of $1, 3, 5, 6, \ldots 10$. As such, $|A_1 \cap A_2| = 8!$

$|||^{ly}$, each of $|A_i \cap A_j| = 8!$ are there are $C(10,2)$ such terms, $S_2 = \sum |A_i \cap A_j| = C(5,2) \times 8!$

Likewise, $S_3 = C(5,3) \times 7!$, $S_4 = C(5,4) \times 6!$

$S_5 = C(5,5) \times 5!$

Accordingly, Expression (i) gives the required number as, $|\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_5| = 10! - C(5,1) \times 9! + C(5,2) \times 8! - C(5,3) \times 7! + C(5,4) \times 6! - C(5,5) \times 5! = 2170680$

(4) Prove that, for any positive integer n,

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_k$$

⇒ For any positive integer n, the total number of permutations of 1,2,3.. n is n!. In each such permutations there exist k (where $0 \le k \le n$) elements which are in their natural positions called fixed elements, and n-k elements which are ~~not~~ in their original positions. The k element can be chosen in $\binom{n}{k}$ ways. and the remaining n-k elements can then be choosen in d_{n-k} ways. Hence there are $\binom{n}{k} d_{n-k}$ permutations of 1,2,3.. n with k fixed elements and n-k deranged elements. As k varies from 0 to n, we count all of the n! permutations of 1,2,3.... n.

Thus, $n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-1}$

$$= \binom{n}{0} d_n + \binom{n}{1} d_{n-1} + \binom{n}{2} d_{n-2} + \dots + \binom{n}{n} d_0$$

$$= \sum_{k=0}^{n} \binom{n}{n-k} d_k = \sum_{k=0}^{n} \binom{n}{k} d_k.$$

ROOK POLYNOMIALS

Consider a board that resembles a full chess board or a part of a chess board. Let n be the number of squares present in the board. Pawns are placed in the squares of the board such that not more than one pawn occupies a square.

Then, according to the Pigeonhole Principle, not more than n pawns can be used. Two pawns placed on a board having 2 (or) more squares are said to capture (or take) each other if they (pawns) are in the same row or in the same column of the board. For $2 \le k \le n$, let r_k denote the number of ways in which k pawns can be placed on a board such that no two pawns capture each other - that is, no two pawns are in the same row or in the same column of the board.

Then the polynomial; $1 + r_1 x + r_2 x^2 + \dots + r_n x^n$ is called the rook polynomial for the board considered. If the board is given the name c, then the polynomial is denoted by $r(C, x)$.

Thus by definition,

$$r(C, x) = 1 + \underset{r_1 x}{nx} + r_2 x^2 + \dots + r_n x^n \quad \text{(1)}$$

While defining this polynomial, it has been assumed that $n \geq 2$. In the trivial case where $n=1$ (that is, in the case where a board contains only one square), $r_2, r_3 \ldots$ are identically zero and the rook polynomial $r(C, x)$ is defined by,

$$r(C, x) = 1 + x \quad \text{—} (2)$$

The expressions (1) & (2) can be put in the following combined form which holds for a board C with $n \geq 1$ squares

$$r(C, x) = 1 + r_1 x + r_2 x^2 + \ldots + r_n x^n \quad - (3)$$

Here, $r_1 = n =$ number of squares in the board.

Example:

(1) consider the board containing 6 squares,

1	2	
		3
4	5	6

For this board $r_1 = 6$. We observed that 2 non-capturing rooks can have the following positions:

(1, 3), (1, 5), (1, 6), (2, 3), (2, 4), (2, 6), (3, 4), (3, 5).

These positions are 8 in number. $\therefore r_2 = 8$.

Next, 3 mutually non-capturing rooks can be placed only in the following two positions: $(1,3,5), (2,3,4)$. Thus $r_3 = 2$. We find that four (or) more mutually non-capturing rooks cannot be placed on the board. Thus $r_4 = 0$, $r_5 = 0$, $r_6 = 0$. Accordingly, for this board, the rook polynomial is,

$$r(C, x) = 1 + 6x + 8x^2 + 2x^3$$

② Consider the board containing 8 squares (marked 1 to 8).

1	2	3
4		5
6	7	8

For this board, $r_1 = 8$.

In this board, the positions of 2 non-capturing rooks are: $(1,5)$, $(1,7)$, $(1,8)$, $(2,4)$, $(2,5)$, $(2,6)$, $(2,8)$, $(3,4)$, $(3,6)$, $(3,7)$, $(4,7)$, $(4,8)$, $(5,6)$, $(5,7)$.

These are 14 number: $\therefore r_2 = 14$. The positions of 3 mutually non-capturing rooks are $(1,5,7)$, $(2,4,8)$, $(2,5,6)$, $(3,4,7)$.

These are 4 in number, $\therefore r_3 = 4$

We check that the board has no positions for more than 3 mutually non-capturing rocks.

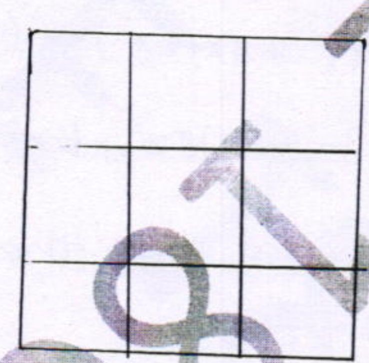
Hence, $r_4 = r_5 = r_6 = r_7 = r_8 = 0$.

Thus, for this board, the rook polynomial is,

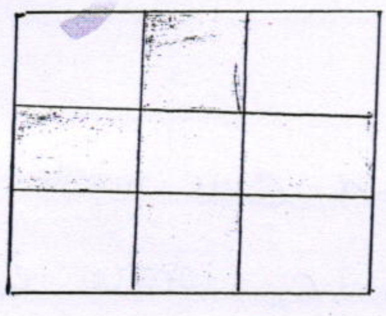
$$r(C, x) = 1 + 8x + 14x^2 + 4x^3.$$

(3) Find the rook polynomial for the 3x3 board by using the expansion formula.

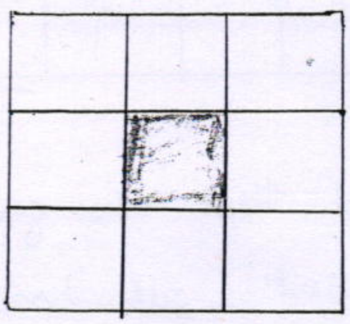
=>



The 3x3 board. Let us mark the square which is at the centre of the board. The boards D and E appear as shown below (the shaded parts are the deleted parts).



D



E

For the board D, we find that $r_1 = 4, r_2 = 2, r_3 = r_4 = 0$

$$r(D, x) = 1 + 4x + 2x^2$$

The board E is the same as the one considered (3x3)

As such for this board,

$$r(E, x) = 1 + 8x + 14x^2 + 4x^3$$

Now, the expansion formula gives

$$r(C_{3\times3}, x) = x\, rD(x) + r(E, x)$$
$$= x(1 + 4x + 2x^2) + (1 + 8x + 14x^2 + 4x^3)$$
$$= 1 + 9x + 18x^2 + 6x^3$$

(4) Find the rook polynomial for the board shown below (shaded part)

1	2			
3	4			
			5	6
			7	8
		9	10	11

a) We note that the given board C is made up of two disjoint sub-boards C_1 and C_2, where C_1 is the 2x2 board with squares numbered 1 to 4 and C_2

is the board with squares numbered 5 to 11.

Since C_1 is the 2×2 board, we've

$$r(C_1, x) = 1+4x+2x^2$$

we note that C_2 is the same as the board considered (3×3 board). we've,

$$r(C_2, x) = 1+7x+10x^2+2x^3$$

∴ The product formula yields the rook polynomials for the given board as,

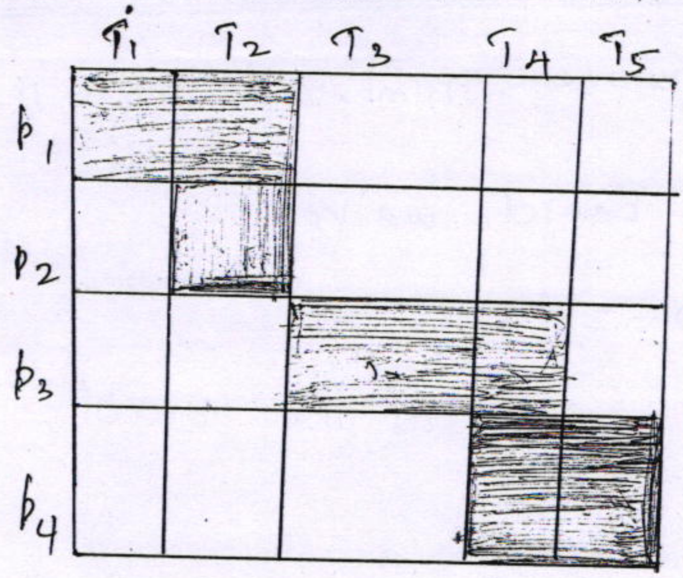
$$r(C, x) = r(C_1 x) \times r(C_2 x)$$

$$= (1+4x+2x^2)(1+7x+10x^2+2x^3)$$

$$= 1+11x+40x^2+56x^3+28x^4+4x^5$$

5 Four persons P_1, P_2, P_3, P_4 who arrive late for a dinner party find that only one chair at each of five tables T_1, T_2, T_3, T_4 and T_5 is vacant. P_1 will not sit at T_1 or T_2, P_2 will not sit at T_2, P_3 will not sit at T_3 or T_4, and P_4 will not sit at T_4 (or) T_5. Find the number of ways they can occupy the vacant chairs.

⇒ Consider the board shown below, representing the



situation. The shaded in the first row indicate that tables T_1 and T_2 are forbidden for p_1 and so on.

For the board made up of shaded squares in the above figure. the rook polynomial is given by,

$$r(C, x) = 1 + 7x + 16x^2 + 13x^3 + 3x^4.$$

Thus, here, $r_1 = 7$, $r_2 = 16$, $r_3 = 13$, $r_4 = 2$.

$$S_0 = 5! = 120, \quad S_1 = (5-1)! \times r_1 = 168,$$

$$S_2 = (5-2)! \times r_2 = 96, \quad S_3 = (5-3)! \times r_3 = 26$$

$$S_4 = (5-4)! \times r_4 = 3.$$

Consequently, the number of ways in which the four persons can occupy the chair is

$$S_0 - S_1 + S_2 - S_3 + S_4 = 120 - 168 + 96 - 26 + 3$$

$$= 25.$$

Recurrence Relations

First-order Recurrence Relations:-

We consider for solution recurrence relations of the form,

$$a_n = ca_{n-1} + f(n), \quad \text{for } n \geq 1 \quad —(1)$$

where c is a known constant and $f(n)$ is a known function. Such a relation is called a linear recurrence relation of first-order with constant co-efficient. If $f(n) \equiv 0$, the relation is called homogeneous; otherwise, It is called non-homogeneous.

The relation (1) can to solved in a trivial way. First, we note that this relation may be rewritten as (by changing n to $n+1$)

$$a_{n+1} = ca_n + f(n+1), \quad \text{for } n \geq 0 \quad — (2)$$

For, $n = 0, 1, 2, 3, \ldots$ this relation yields, respectively.

$$a_1 = ca_0 + f(1)$$

$$a_2 = ca_1 + f(2) = c\{ca_0 + f(1)\} + f(2)$$

$$= c^2 a_0 + cf(1) + f(2)$$

$$a_3 = ca_2 + f(3) = c\{c^2 a_0 + cf(1) + f(2)\} + f(3)$$

$$= c^3 a_0 + c^2 f(1) + c f(2) + f(3).$$

and so on. Examining these, we obtain, by induction

$$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \cdots + c f(n-1) + f(n)$$

$$= c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k), \quad \text{for } n \geq 1 \quad - (3)$$

This is the general solution of the recurrence relation (2) which is equivalent to the relation (1).

If $f(n) \equiv 0$. that is if the recurrence relation is homogeneous, the solution (3) becomes

$$a_n = c^n a_0 \quad \text{for} \quad n \geq 1 \quad - (4)$$

The solutions (3) and (4) yield particular solutions if a_0 is specified. The specified value of a_0 is called the initial condition.

Problems:

1. Solve the recurrence relation $a_n = n a_{n-1}$ for $n \geq 1$ given that $a_0 = 1$

$\Rightarrow$ From the given relation, we find that,

$$a_1 = 1 \times a_0, \quad a_2 = 2a_1 = (2 \times 1) a_0,$$

$$a_3 = 3 \times a_2 = (3 \times 2 \times 1) a_0,$$

$$a_4 = 4 \times a_3 = (4 \times 3 \times 2 \times 1) a_0 \quad \text{and so on.}$$

Evidently, the general solution is (by induction)

$$a_n = (n!) a_0 \quad \text{for} \quad n \geq 1.$$

Using the given initial condition $a_0 = 1$, $\therefore \boxed{a_n = n!}$

2. Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 3^n$ for $n \geq 1$ given that $a_0 = 2$.

$\Rightarrow$ The given relation may be rewritten as

$$a_n = 3^n a_0 \quad a_{n+1} = 3a_n + 5 \times 3^{n+1} \text{ for } n \geq 0$$

$$= 3a_n + f(n+1) \quad \text{where } f(n) = 5 \times 3^n.$$

The general solution for this relation is,

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k)$$

$$= 3^n a_0 + 3^{n-1} f(1) + 3^{n-2} f(2) + 3^{n-3} f(3) + \dots + 3^0 f(n).$$

Substituting for a_0 and $f(n)$, $n = 1, 2, \dots n$ in this we get

$$a_n = 2 \times 3^n + 3^{n-1} \times (5 \times 3^1) + 3^{n-2} \times (5 \times 3^2) + 3^{n-3} \times (5 \times 3^3)$$
$$+ \dots + 3^0 \times (5 \times 3^n)$$

$$= 2 \times 3^n + 5 \times (3^n + 3^n + 3^n + \dots + 3^n) \quad (n \text{ terms})$$

$$= 2 \times 3^n + 5 \times (n 3^n)$$

$$= (2 + 5n) 3^n$$

This is the required solution.

3. Find the recurrence relation and the initial condition for the sequence

2, 10, 50, 250 . Hence find the general term of the sequence.

$\Rightarrow$ The given sequence is $\langle a_r \rangle$, where $a_0 = 2$, $a_1 = 10$

$a_2 = 10$, $a_2 = 50$, $a_3 = 250 \dots$

$a_1 = 5a_0$, $a_2 = 5a_1$, $a_3 = 5a_2$ and so on.

From these, we readily note that the recurrence relation for the given sequence is $a_n = 5a_{n-1}$ for $n \geq 1$ with $a_0 = 2$ as the initial condition.

This solution of this relation is,

$$a_n = 5^n a_0 = 5^n \times 2.$$

This is the general term of the given sequence.

4. Suppose that there are $n \geq 2$ persons at a party and that each of these persons shakes hands (exactly once) with all of the other persons present. Using a recurrence relation, find the number of hand shakes.

=) let a_{n-2} denotes the number of hand shakes among the $n \geq 2$ persons present. (If $n=2$, the number of handshakes is 1; that is $a_0 = 1$). If a new person joins the party, he will shake hands with each of the n persons already present. Thus, the number of hand shakes increases by n when the number of persons changes to n+1 from n. Thus,

$$a_{(n+1)-2} = a_{n-2} + n \quad \text{for } n \geq 2$$

(or) $a_{m+1} = a_m + (m+2)$ for $m \geq 0$, where $m = n-2$

setting $f(m) = m+1$,

$$a_{m+1} = a_m + f(m+1) \quad \text{for } m \geq 0$$

The general solution of this nonhomogeneous recurrence relation is,

$$a_m = (1^m \times a_0) + \sum_{k=1}^{m} 1^{m-k} f(k) = a_0 + \sum_{k=1}^{m} (k+1)$$

Since, $a_0 = 1$, this becomes,

$$a_m = 1 + \{2 + 3 + 4 + \cdots + m + (m+1)\}$$

$$= \frac{1}{2}(m+1)(m+2) \quad \text{for } m \geq 0.$$

(or) $a_{n-2} = \frac{1}{2}(n-1)n$ for $n \geq 2$

This is the number of handshakes in the party when $n \geq 2$ persons are present.

Second-order Homogeneous Recurrence Relations

We now consider a method of solving recurrence relations of the form

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0 \quad \text{for } n \geq 2 \qquad (1)$$

where c_n, c_{n-1} and c_{n-2} are real constants with $c_n \neq 0$. A relation of this types is called a second-order linear homogeneous recurrence relation with constant co-efficients.

$$\boxed{c_n k^2 + c_{n-1} k + c_{n-2} = 0} \qquad -(2)$$

Thus, $a_n = ck^n$ is a solution of (1) if k satisfies the quadratic equation (2). This quadratic equation is called the auxiliary equation or the characteristic equation for the relation (1).

Case (1) The two roots k_1 and k_2 of equation (2) are real and distinct. Then we take,

$$a_n = Ak_1^n + Bk_2^n \quad \text{--- (3)}$$

where A and B are arbitrary real constants, as the general solution of the relation (1).

Case (2) The two roots k_1 and k_2 of equation (2) are real and equal, with k as the common value. Then we take,

$$a_n = (A + Bn)k^n \quad \text{--- (4)}$$

where A and B are arbitrary real constants, as the general solution of the relation (1).

Case : (3) The two roots k_1 and k_2 of equation (2) are complex. Then k_1 and k_2 are complex conjugates of each other, so that if $k_1 = p + iq$, then $k_2 = p - iq$ and we take,

$$a_n = r^n (A\cos n\theta + B\sin n\theta) \quad \text{--- (5)}$$

where A and B are arbitrary complex constants, $r = |k_1| = |k_2| = \sqrt{p^2+q^2}$ and $\theta = \tan^{-1}(q/p)$ as the general solution of the relation (1).

(1) Solve the recurrence relation

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad \text{for } n \geq 2,$$

given that $a_0 = 5$, $a_1 = 12$.

=> The characteristic equation for the given relation is, $k^2 - 6k + 9 = 0$, (or) $(k-3)^2 = 0$

whose roots are $k_1 = k_2 = 3$. Therefore, the general solution for a_n is,

$$a_n = (A + Bn)3^n$$

where A and B are arbitrary constants.

Using the given initial conditions $a_0 = 5$ and $a_1 = 12$ in Equation, we get $5 = A$ and $12 = 3(A+B)$.

Solving these we get, $A = 5$, and $B = -1$.

Putting these values in Equation, we get,

$$a_n = (5-n)3^n$$

This is the solution of the given relation, under the given initial conditions.

(2) Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}), \quad \text{for } n \geq 2.$$

given that $a_0 = 1$ and $a_1 = 2$.

=> For the given relation, the characteristic equation is $k^2 - 2k + 2 = 0$.

The roots are,

$$K = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

Therefore, the general solution for a_n is,

$$a_n = r^n [A \cos n\theta + B \sin n\theta].$$

where A and B are arbitrary constants,

$r = |(1 \pm i)| = \sqrt{2}$, and $\tan\theta = 1$, $\theta = \pi/4$.

$$a_n = (\sqrt{2})^n \left[A \cos \frac{n\pi}{4} + B \sin \frac{n\pi}{4}\right]$$

Using the given initial conditions $a_0 = 1$ and $a_1 = 2$

we get, $1 = A$ and $2 = (\sqrt{2})\left[A \cos \frac{\pi}{4} + B \sin \frac{\pi}{4}\right]$

$$= A + B.$$

$A = 1$, $B = 1$ Putting these values of A and B.

$$\therefore a_n = (\sqrt{2})^n \left[\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}\right].$$

This is the solution of the given relation under under the given conditions.

(3) If $a_0 = 0$, $a_1 = 1$, $a_2 = 4$ and $a_3 = 37$ satisfy the recurrence relation

$$a_{n+2} + b a_{n+1} + c a_n = 0 \quad \text{for } n \geq 0$$

determine the constant b and c and then solve the relation for a_n.