1) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2+2x+4$

Sol :- $(D^2+D)\,y = x^2+2x+4$

A.E is $m^2+m=0$

$m(m+1)=0 \Rightarrow m=0,\ m=-1$

$C.F = c_1 e^{0x} + c_2 e^{-x}$

$P.I = \frac{1}{D^2+D}\, x^2+2x+4$

$= \frac{1}{D(D+1)}\, x^2+2x+4$

$= \frac{1}{D}\cdot(1+D)^{-1}\, x^2+2x+4$

$= \frac{1}{D}\cdot\left[1-D+D^2+\cdots\right](x^2+2x+4)$

$= \frac{1}{D}\left[(x^2+2x+4) - D(x^2+2x+4) + D^2(x^2+2x+4) + \cdots\right]$

$= \frac{1}{D}\left[(x^2+2x+4) - (2x+2) + (2)\right]$

$= \frac{1}{D}\left[x^2+\cancel{2x}+4-\cancel{2x}-\cancel{2}+\cancel{2}\right]$

$= \frac{1}{D}\left[x^2+4\right]$

$$P.I = \frac{1}{D}(x^2+4)$$

$$= \int (x^2+4)\,dx$$

$$= \frac{x^3}{3} + 4x$$

General sol is

$$y = CF + PI$$

$$y = c_1 e^{0x} + c_2 e^{-x} + \frac{x^3}{3} + 4x$$

② Solve $y'' - 2y' + 2y = x^2 + x$

Sol:- $(D^2 - 2D + 2)y = x^2 + x$

A.E is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot1\cdot2}}{2}$$

$$m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{2 \pm \sqrt{-4}}{2}$$

$$m = \frac{2 \pm 2i}{2}$$

$$m = 1 \pm i$$

$$CF = e^{x}(c_1 \cos x + c_2 \sin x)$$

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$$P.I = \frac{1}{D^2-2D+2}(x^2+x)$$

$$= \frac{1}{2\left[1+\frac{D^2-2D}{2}\right]}(x^2+x)$$

$$= \frac{1}{2}\left(1+\frac{D^2-2D}{2}\right)^{-1}(x^2+x)$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2-2D}{2}\right)+\left(\frac{D^2-2D}{2}\right)^2+\cdots\right](x^2+x)$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{\cancel{2}D}{\cancel{2}}+\frac{1}{4}\left(D^4+4D^2-4D^3\right)+\cdots\right](x^2+x)$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+D+\frac{1}{\cancel{4}}\cancel{4}D^2\right](x^2+x)$$

$$P.I = \frac{1}{2}\left[1-\frac{D^2}{2}+D+D^2\right](x^2+x)$$ (4)

$$= \frac{1}{2}\left[1+D+\frac{1}{2}D^2\right](x^2+x)$$

$$= \frac{1}{2}\left[(x^2+x)+D(x^2+x)+\frac{1}{2}D^2(x^2+x)\right]$$

$$= \frac{1}{2}\left[x^2+x+(2x+1)+\frac{1}{\cancel{2}}\cdot\cancel{2}\right]$$

$$= \frac{1}{2}\left[x^2+x+2x+2\right]$$

$$P.I = \frac{1}{2}\left[x^2+3x+2\right]$$

General sol is

$$y = CF + PI$$

$$y = e^{x}(c_1\cos x + c_2\sin x) + \frac{1}{2}(x^2+3x+2)$$

Type IV

P.I of the form $\frac{e^{ax}\cdot V}{f(D)}$,

where V is a function of x.

Put $D = D+a$

$$P.I = e^{ax}\cdot\frac{V}{f(D+a)}$$

$\frac{e^{2x}}{V}$

$e^{2x}\cos 2x$

$D = D+2$

Problems

(1) solve $(D^2 - 2D + 4)y = e^x \cdot \cos x$

Sol:- A.E is

$$m^2 - 2m + 4 = 0$$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 4}}{2}$$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$m = \frac{2 \pm \sqrt{-12}}{2}$$

$$m = \frac{2 \pm \sqrt{-(4 \times 3)}}{2}$$

$$m = \frac{2 \pm 2\sqrt{3}i}{2}$$

$$m = \frac{\cancel{2}(1 \pm \sqrt{3}i)}{\cancel{2}}$$

$$cf = e^x(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$$

$$P.I = \frac{1}{D^2 - 2D + 4} e^x \cdot \cos x \qquad (a = 1)$$

Put $D = D + 1$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$P.I = e^{x} \frac{1}{D^2+\cancel{2D}+1-\cancel{2D}-2+4} \cos x$$

$$= e^{x} \cdot \frac{1}{D^2+3} \cos x$$

$$= e^{x} \cdot \frac{1}{D^2+3} \cos x$$

$a=1$ $\cos ax$

Put $D^2 = -a^2$
$D^2 = -1$

$$= e^{x} \cdot \frac{1}{-1+3} \cos x$$

$$P.I = \frac{e^{x} \cdot \cos x}{2}$$

General solution is

$$y = CF + PI$$

$$y = e^{x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right) + \frac{e^{x}\cos x}{2}$$

(2) Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}(1+x)$

Sol:- $(D^2 - 7D + 6)y = e^{2x}(1+x)$

A.E is $m^2 - 7m + 6 = 0$

$m^2 - 6m - m + 6 = 0$

$m(m-6) - 1(m-6) = 0$

$(m-1)(m-6) = 0$

$m = 1,\ m = 6$

$C.F = c_1 e^{x} + c_2 e^{6x}$

$$P.I = \frac{1}{D^2 - 7D + 6} e^{2x}(1+x)$$

$\underline{a = 2}$

Put $D = D + 2$

$$= e^{2x} \cdot \frac{1}{(D+2)^2 - 7(D+2) + 6}(1+x)$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 7D - 14 + 6}(1+x)$$

$=$

$$= e^{2x} \cdot \frac{1}{D^2 - 3D - 4}(1+x)$$

$$= e^{2x} \cdot \frac{1}{-4\left(1 + \frac{D^2 - 3D}{4}\right)}(1+x)$$

$$= -\frac{e^{2x}}{4}\left(1 - \frac{D^2 - 3D}{4}\right)^{-1}(1+x)$$

$$P.I = -\frac{e^{2x}}{4}\left[1 + \frac{D^2-3D}{4} + \cdots\right](1+x)$$

$(1-x)^{-1} = 1+x+\cdots$

$$= -\frac{e^{2x}}{4}\left[(1+x) + \underbrace{\frac{1}{4}D^2(1+x)} - \frac{3}{4}D(1+x)\right]$$

$$= -\frac{e^{2x}}{4}\left[1+x + 0 - \frac{3}{4}(1)\right]$$

$$= -\frac{e^{2x}}{4}\left(x+1-3/4\right]$$

$$P.I = -\frac{e^{2x}}{4}\left(x+1/4\right)$$

General solution is

$y = CF + P.I$

$$y = c_1e^{x} + c_2e^{6x} - \frac{e^{2x}}{4}\left(x+1/4\right)$$

3 Solve $(D^2+4D+5)y = x^2 e^{2x}$

Sol:- A.E is $m^2+4m+5=0$

$$m = \frac{-4 \pm \sqrt{4^2-4 \cdot 1 \cdot 5}}{2}$$

$$m = \frac{-4 \pm \sqrt{-4}}{2}$$

$$m = \frac{-4 \pm 2i}{2}$$

$$m = \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$$cf = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$$P.I = \frac{1}{D^2+4D+5} x^2 e^{2x}$$

$$= e^{2x} \cdot \frac{1}{D^2+4D+5} x^2$$

put $D = D+2$

$$= e^{2x} \cdot \frac{1}{(D+2)^2+4(D+2)+5} x^2$$

$$= e^{2x} \cdot \frac{1}{D^2+4D+4+4D+8+5} x^2$$

$$P.I = e^{2x} \cdot \underbrace{\frac{1}{D^2+8D+17}}\, x^2$$

$$= e^{2x} \cdot \frac{1}{17\left(1+\frac{D^2+8D}{17}\right)} x^2$$

$$= \frac{e^{2x}}{17}\left(1+\frac{D^2+8D}{17}\right)^{-1} x^2$$

$(1+x)^{-1} = 1 - x + x^2 - \cdots$

$$= \frac{e^{2x}}{17}\left[1-\left(\frac{D^2+8D}{17}\right)+\left(\frac{D^2+8D}{17}\right)^2 - \cdots\right] x^2$$

$$= \frac{e^{2x}}{17}\left[x^2 - \frac{1}{17}D^2(x^2) - \frac{8}{17}D(x^2) + \frac{1}{289}\left(D^4 + \underbrace{64D^2} + \frac{16D^3}{17}\right)x^2\right]$$

$$= \frac{e^{2x}}{17}\left(x^2 - \frac{1}{17}D^2(x^2) - \frac{8}{17}D(x^2) + \frac{1}{289}64D^2(x^2)\right)$$

$$= \frac{e^{2x}}{17}\left(x^2 - \frac{1}{17}(2) - \frac{8}{17}(2x) + \frac{64}{289}(2)\right)$$

$$= \frac{e^{2x}}{17}\left(x^2 - \frac{2}{17} - \frac{16x}{17} + \frac{128}{289}\right)$$

(11)

$$P.I = \frac{e^{2x}}{17}\left(x^2 - \frac{16x}{17} - \underbrace{\frac{2}{17} + \frac{128}{289}}\right)$$

$$= \frac{e^{2x}}{17}\left(x^2 - \frac{16x}{17} + \frac{128-34}{289}\right)$$

$$P.I = \frac{e^{2x}}{17}\left(x^2 - \frac{16x}{17} + \frac{94}{289}\right)$$

Gen sol is

$$y = CF + PI$$

$$y = e^{-2x}(c_1 \cos x + c_2 \sin x) + \frac{e^{2x}}{17}\left(x^2 - \frac{16x}{17} + \frac{94}{289}\right)$$

(4) Solve $(D^2-4)y = 8xe^x$

Sol:- A.E is $m^2-4=0$

$$m = \pm 2$$

$$CF = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I = \frac{1}{D^2-4} 8xe^x \qquad a=1$$

$$= 8e^x \cdot \frac{1}{D^2-4} x \qquad \text{Put } D = D+1$$

$$= 8e^x \frac{1}{(D+1)^2-4} x$$

$$P.I = 8e^{x} \cdot \frac{1}{D^2+2D+1-4} x$$

$$= 8e^{x} \cdot \frac{1}{D^2+2D-3} x$$

$$= 8e^{x} \cdot \frac{1}{-3\left(1-\frac{D^2+2D}{3}\right)} x$$

$$= -\frac{8e^{x}}{3}\left(1-\frac{D^2+2D}{3}\right)^{-1} x$$

$$(1-x)^{-1} = 1+x+x^2+\cdots$$

$$= -\frac{8e^{x}}{3}\left(1+\frac{D^2+2D}{3}\right) x$$

$$= -\frac{8e^{x}}{3}\left(x+\frac{1}{3}D^2(x)+\frac{2}{3}D(x)\right)$$

$$= -\frac{8e^{x}}{3}\left(x+0+\frac{2}{3}(1)\right)$$

$$P.I = -\frac{8e^{x}}{3}\left(x+\frac{2}{3}\right)$$

Gen Sol is

$$y = CF + PI$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{8e^{x}}{3}\left(x+\frac{2}{3}\right)$$

Miscellaneous

① Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

Sol:- A.E is $\cancel{D}$ $(m^2 - 3m + 2) = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0$

$m = 1, \; m = 2$

CF $\quad y = c_1 e^{x} + c_2 e^{2x}$

$$P.I = \frac{1}{D^2 - 3D + 2}\left(xe^{3x} + \sin 2x\right)$$

$$= \frac{1}{D^2 - 3D + 2} xe^{3x} + \frac{1}{D^2 - 3D + 2}\sin 2x$$

(Type IV) (Type II)

(Not necessary to write in the exam)

$$= e^{3x} \cdot \frac{1}{D^2 - 3D + 2} x + \frac{1}{D^2 - 3D + 2}\sin 2x$$

Put $D = D + 3$

$$= e^{3x} \frac{1}{(D+3)^2 - 3(D+3) + 2} x + \frac{1}{D^2 - 3D + 2}\sin 2x$$

(14)

$$= e^{3x}\cdot\frac{1}{D^2+\cancel{9}+6D-3D-\cancel{9}+2}x+\frac{1}{D^2-3D+2}\sin 2x$$

Put $D^2=-2^2$

$$= e^{3x}\cdot\frac{1}{D^2+3D+2}x+\frac{1}{-4-3D+2}\sin 2x$$

$$= e^{3x}\cdot\frac{1}{2\left(1+\frac{D^2+3D}{2}\right)}x+\frac{1}{-2-3D}\sin 2x$$

$$= \frac{e^{3x}}{2}\left(1+\frac{D^2+3D}{2}\right)^{-1}x+\frac{-2+3D}{(-2-3D)(-2+3D)}\sin 2x$$

$$= \frac{e^{3x}}{2}\left(1-\left(\frac{D^2+3D}{2}\right)\right)x+\frac{3D-2}{(-2)^2-(3D)^2}\sin 2x$$

$$= \frac{e^{3x}}{2}\left(x-\frac{D^2}{2}(x)-\frac{3}{2}D(x)\right)+\frac{(3D-2)\sin 2x}{4-9D^2}$$

$$= \frac{e^{3x}}{2}\left(x-0-\frac{3}{2}(1)\right)+\frac{3D(\sin 2x)-2\sin 2x}{4-9D^2}$$

Put $D^2=-2^2$

$$= \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right)+\frac{3\cos 2x\cdot 2-2\sin 2x}{4-9(-4)}$$

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$$P.I = \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) + \frac{6\cos 2x - 2\sin 2x}{40}$$

General sol is

$$y = CF + PI$$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}\left(x-3/2\right) + \frac{6\cos 2x - 2\sin 2x}{40}$$

② Solve $\frac{d^2y}{dx^2} + y = e^{-x} + x^3 + e^{x}\sin x$

Sol:- $(D^2+1)y = e^{-x} + x^3 + e^{x}\sin x$

A.E is $m^2+1 = 0$

$m = \pm i$

$$cf = c_1\cos x + c_2\sin x$$

$$P.I = \frac{1}{D^2+1}\left(e^{-x} + x^3 + e^{x}\sin x\right)$$

$$= \frac{1}{D^2+1}e^{-x} + \frac{1}{D^2+1}x^3 + \frac{1}{D^2+1}e^{x}\sin x$$

Type I Type III Type IV

$$= \frac{1}{D^2+1}e^{-x} + (1+D^2)^{-1}x^3 + e^{x}\cdot\frac{1}{D^2+1}\sin x$$

Put $D = D+1$

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$$P.I = \frac{1}{D^2+1} e^{-x} + (1+D^2)^{-1} x^3 + e^{x} \cdot \frac{1}{(D+1)^2+1} \sin x$$

put $D=-1$

$$= \frac{1}{(-1)^2+1} e^{-x} + \left(1 - D^2 + D^4 \cdots\right) x^3 + \frac{e^{x}}{D^2+2D+1+1} \sin x$$

$$= \frac{1}{2} e^{-x} + \left(x^3 - D^2(x^3) + \cdots\right) + \frac{e^{x}}{D^2+2D+2} \sin x$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + \frac{e^{x}}{D^2+2D+2} \sin x \quad a=1$$

put $D^2 = -1^2$

$$= \frac{e^{-x}}{2} + x^3 - 6x + \frac{e^{x}}{-1+2D+2} \sin x$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + \frac{e^{x}}{2D+1} \sin x$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^{x} \cdot \frac{(2D-1)}{(2D+1)(2D-1)} \sin x$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + \frac{e^{x}(2D-1)\sin x}{4D^2-1^2}$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^{x} \cdot \frac{2D(\sin x) - \sin x}{4D^2-1^2}$$

put $D = -1^2$

$$P.I = \frac{e^{-x}}{2} + x^3 - 6x + e^{x} \cdot \frac{2\cos x - \sin x}{-4-1}$$

$$P.I = \frac{e^{-x}}{2} + x^3 - 6x + e^{x} \cdot \left(\frac{2\cos x - \sin x}{-5}\right)$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + \frac{e^{x}(\sin x - 2\cos x)}{5}$$

General solution is

$$y = CF + PI$$

$$y = c_1\cos x + c_2\sin x + \frac{e^{-x}}{2} + x^3 - 6x + \frac{e^{x}(\sin x - 2\cos x)}{5}$$

③ Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{3x} + e^{-2x}\sin 2x$

Sol:- $(D^2+5D+6)y = e^{3x} + e^{-2x}\sin 2x$

A.E is $m^2 + 5m + 6 = 0$

$$m^2 + 3m + 2m + 6 = 0$$

$$m(m+3) + 2(m+3) = 0$$

$$(m+2)(m+3) = 0$$

$$m = -2, \quad m = -3$$

$$CF = c_1 e^{-2x} + c_2 e^{-3x}$$

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$$P.I = \frac{1}{D^2+5D+6}\left(e^{3x} + e^{-2x}\sin 2x\right)$$

$$= \frac{1}{D^2+5D+6} e^{3x} + \frac{1}{D^2+5D+6} e^{-2x}\sin 2x$$

Type I

Type IV

Put $D = 3$

$$= \frac{1}{3^2+5(3)+6} e^{3x} + e^{-2x}\frac{1}{D^2+5D+6}\sin 2x$$

Put $D = D-2$

$$= \frac{1}{30} e^{3x} + e^{-2x}\frac{1}{(D-2)^2+5(D-2)+6}\sin 2x$$

$$= \frac{e^{3x}}{30} + e^{-2x}\frac{1}{D^2+4-4D+5D-10+6}\sin 2x$$

$$= \frac{e^{3x}}{30} + e^{-2x}\cdot\frac{1}{D^2+D}\sin 2x$$

Put $D^2 = -2^2$

$$= \frac{e^{3x}}{30} + e^{-2x}\cdot\frac{1}{-4+D}\sin 2x$$

$$= \frac{e^{3x}}{30} + e^{-2x}\frac{(D+4)\sin 2x}{(-4+D)(D+4)}$$

$$PI = \frac{e^{3x}}{30} + e^{-2x} \frac{D(\sin 2x) + 4\sin 2x}{D^2 - 4^2}$$

Put $D^2 = -4$

$$= \frac{e^{3x}}{30} + e^{-2x} \frac{(\cos 2x)\cdot 2 + 4\sin 2x}{-4-16}$$

$$= \frac{e^{3x}}{30} + e^{-2x} \frac{2\cos 2x + 4\sin 2x}{(-20)}$$

$$PI = \frac{e^{3x}}{30} - \frac{e^{-2x}}{20}(2\cos 2x + 4\sin 2x)$$

$$PI = \frac{e^{3x}}{30} - \frac{e^{-2x}(\cos 2x + 2\sin 2x)}{10}$$

General Sol is

$$y = cf + PI$$

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{3x}}{30} - \frac{e^{-2x}(\cos 2x + 2\sin 2x)}{10}$$

Solve

1. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x + x^2$

2. $y'' + 2y = x^2 e^{3x} + e^{x}\cos 2x$

3. $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

4. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{x} + x + \sin 2x$

5. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^{x} + x + \cos 2x$

$$PI_1 = \frac{1}{D^2+5D+6} e^{-2x} \cdot \sin 2x$$

$$= (\qquad)$$

$$PI_2 = \frac{1}{D^2+5D+6} x^2$$

$$= (\qquad)$$

$$PI = \underbrace{PI_1 + PI_2}$$

$$y = CF + PI$$