1 Given $y = Ke^{-2x} + 3x$, find the member of orthogonal trajectory passing through $(0,3)$

Sol:- Given $y = Ke^{-2x} + 3x$ — (1)

Diff w r t x

$$\frac{dy}{dx} = K \cdot e^{-2x}(-2) + 3$$

$$\frac{dy}{dx} = -2(y-3x) + 3$$

From (1)

$Ke^{-2x} = y - 3x$

$$\frac{dy}{dx} = -2y + 6x + 3$$

$$\frac{dy}{dx} = 6x - 2y + 3$$

Put $\frac{dy}{dx} = -\frac{dx}{dy}$

$$-\frac{dx}{dy} = 6x - 2y + 3$$

$$\frac{dx}{dy} = 2y - 6x - 3$$

$$\frac{dx}{dy} + 6x = 2y - 3$$ linear in y

$P = 6$, $Q = 2y - 3$

$$\text{If} = e^{\int p\,dy} = e^{\int 6\,dy} = e^{6y}$$

2

Sol

$$x\,(IF) = \int Q\,(IF)\,dy + c$$

$$x \cdot e^{6y} = \int (2y-3)\, e^{6y}\,dy + c$$

$$x\, e^{6y} = (2y-3)\frac{e^{6y}}{6} - (2)\cdot\frac{e^{6y}}{36} + c$$

$$x \cdot e^{6y} = \frac{2y-3}{6}\, e^{6y} - \frac{2}{36}\, e^{6y} + c$$

$$x = \frac{2y-3}{6}\,\frac{\cancel{e^{6y}}}{\cancel{e^{6y}}} - \frac{2}{36}\,\frac{\cancel{e^{6y}}}{\cancel{e^{6y}}} + \frac{c}{e^{6y}}$$

$$x = \frac{2y-3}{6} - \frac{2}{36} + c\,e^{-6y} \rightarrow (2)$$

Given point (0,3)

i.e $y=3$, when $x=0$

$$0 = \frac{2(3)-3}{6} - \frac{1}{18} + c\,e^{-6(3)}$$

$$0 = \frac{3}{6} - \frac{1}{18} + c\,e^{-18}$$

$$= \frac{9-1}{18} + c\,e^{-18}$$

$$0 = \frac{8}{18} + c\,e^{-18}$$

$$0 = \frac{4}{9} + c\,e^{-18} \Rightarrow c\,e^{-18} = -\frac{4}{9}$$

$$c = -\frac{4}{9}\,\frac{1}{e^{-18}}$$

$$\boxed{c = -\frac{4}{9}\,e^{18}}$$

Sub c in (2), we get

$$x = \frac{2y-3}{6} - \frac{1}{18} + \left(\frac{-4}{9}e^{18}\right)e^{-6y}$$

$$x = \frac{2y-3}{6} - \frac{1}{18} - \frac{4}{9}e^{18-6y}$$

Problems on O.T for polar curves

(1) Find the O.T of the family $r = a(1+\sin\theta)$

Sol :-

Given $r = a(1+\sin\theta)$

$$\log r = \log(a(1+\sin\theta))$$

$$\log r = \log a + \log(1+\sin\theta)$$

Diff w.r.t 'θ'

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\sin\theta}\cdot(\cos\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta}$$

Put $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$\frac{1}{\cancel{r}}\left(-r^{\cancel{2}}\frac{d\theta}{dr}\right) = \frac{\cos\theta}{1+\sin\theta}$$

$$-r\frac{d\theta}{dr} = \frac{\cos\theta}{1+\sin\theta}$$

④

$$-\frac{dr}{r} = \frac{1+\sin\theta}{\cos\theta}\,d\theta$$

$$\frac{dr}{r} + \frac{1+\sin\theta}{\cos\theta}\,d\theta = 0$$

$$\int \frac{dr}{r} + \int \frac{1+\sin\theta}{\cos\theta}\,d\theta = 0$$

$$\log r + \int\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)d\theta = 0$$

$$\log r + \int (\sec\theta + \tan\theta)\,d\theta = 0$$

$$\log r + \int \sec\theta\,d\theta + \int \tan\theta\,d\theta = 0$$

$$\log r + \log(\sec\theta + \tan\theta) + \log(\sec\theta) = \log c$$

$$\log\left[r(\sec\theta + \tan\theta)\cdot\sec\theta\right] = \log c$$

$$r((\sec\theta + \tan\theta)\sec\theta) = c$$

$$r\left(\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\cdot\frac{1}{\cos\theta}\right) = c$$

$$r\left(\frac{1+\sin\theta}{\cos^2\theta}\right) = c$$

$$r\left(\frac{1+\sin\theta}{1-\sin^2\theta}\right) = c$$

$$r\left(\frac{\cancel{1+\sin\theta}}{\cancel{(1+\sin\theta)}(1-\sin\theta)}\right) = c$$

$$\frac{r}{1-\sin\theta} = c \Rightarrow \underline{r = c(1-\sin\theta)}$$

② Find the orthogonal trajectories of the family $r^n \cos n\theta = a^n$ ⑤

Sol:— $r^n \cos n\theta = a^n$

$\log(r^n \cos n\theta) = \log a^n$

$\log r^n + \log \cos n\theta = \log a^n$

$n \log r + \log \cos n\theta = n \log a$

Diff w.r.t 'θ'

$n \cdot \frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\cos n\theta} \cdot (-\sin n\theta) \cdot n = 0$

$\div n$ $\quad \frac{1}{r}\frac{dr}{d\theta} - \tan n\theta = 0$

$\frac{1}{r}\frac{dr}{d\theta} = \tan n\theta$

Put $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$\frac{1}{\cancel{r}}\left(-r^{\cancel{2}} \frac{d\theta}{dr}\right) = \tan n\theta$

$-r \frac{d\theta}{dr} = \tan n\theta$

$\frac{dr}{-r} = \frac{1}{\tan n\theta} d\theta$

$\frac{dr}{r} + \frac{d\theta}{\tan n\theta} = 0$

$\frac{dr}{r} + \cot n\theta \, d\theta = 0$

$\int \frac{dr}{r} + \int \cot n\theta \, d\theta = \log c$

$$\log r + \frac{\log(\sin n\theta)}{n} = \log c$$

$$\frac{n \log r + \log(\sin n\theta)}{n} = \log c$$

$$n \log r + \log(\sin n\theta) = n \log c$$

$$\log r^n + \log(\sin n\theta) = \log c^n$$

$$\log(r^n \sin n\theta) = \log c^n$$

$$c^n = r^n \sin n\theta$$

(3) Show that the orthogonal trajectories of the family of cardiods $r = a\cos^2\theta/2$ is the family of cardiods $r = b\sin^2\theta/2$ another family of the

Sol :-

$$r = a\cos^2\theta/2$$

$$\log r = \log(a\cos^2\theta/2)$$

$$\log r = \log a + \log\cos^2\theta/2$$

$$\log r = \log a + 2\log\cos\theta/2$$

$$\log r = \log a +$$

Diff w.r.t 'θ'

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \cancel{2}\cdot\frac{1}{\cos\theta/2}\cdot(-\sin\theta/2)\cdot\frac{1}{\cancel{2}}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan\theta/2$$

Put $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$\frac{1}{\cancel{r}}\left(-r^{2}\frac{d\theta}{dr}\right) = -\tan\theta/2$$

$$\frac{dr}{r} = \frac{1}{\tan\theta/2}\,d\theta$$

$$\frac{dr}{r} = \cot\theta/2\,d\theta$$

$$\int\frac{dr}{r} = \int\cot\theta/2\,d\theta + c$$

$$\int\frac{dr}{r} - \int\cot\theta/2\,d\theta = c$$

$$\int\frac{dr}{r} - \int\frac{\cos\theta/2}{\sin\theta/2}\,d\theta = c$$

$$\log r - \frac{\log(\sin\theta/2)}{1/2} = \log c$$

$$\log r - 2\log\sin\theta/2 = \log c$$

$$\log r - \log\sin^{2}\theta/2 = \log c$$

$$\log\left(\frac{r}{\sin^{2}\theta/2}\right) = \log c$$

$$c = \frac{r}{\sin^{2}\theta/2}$$

$$r = c\sin^{2}\theta/2$$

$$r = b\sin^{2}\theta/2$$

④ Find the orthogonal trajectories of the family of curves $r = \frac{2a}{1+\cos\theta}$ ⑧

(or)

Find the O.T's of family of confocal and co-axial parabolas $r = \frac{2a}{1+\cos\theta}$

Sol:- Given $r = \frac{2a}{1+\cos\theta}$

$$\log r = \log\left(\frac{2a}{1+\cos\theta}\right)$$

$$\log r = \log 2a - \log(1+\cos\theta)$$

Diff w.r.t 'θ'

$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{1+\cos\theta}(-\sin\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{2}\sin\theta/2\ \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \tan\theta/2$$

Put $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$\frac{1}{\cancel{r}}\left(-r^{\cancel{2}}\frac{d\theta}{dr}\right) = \tan\theta/2$$

$$\frac{dr}{r} = -\frac{1}{\tan\theta/2}\,d\theta$$

$$\frac{dr}{r} + \cot\theta/2\,d\theta = 0$$

$$\int \frac{dr}{r} + \int \cot \theta/_2 \, d\theta = \log c$$

$$\log r + 2 \log \sin \theta/_2 = \log c$$

$$\log r + \log \sin^2 \theta/_2 = \log c$$

$$\log (r \sin^2 \theta/_2) = \log c$$

$$c = r \sin^2 \theta/_2$$

$$c = r \cdot \frac{1}{2}(1 - \cos\theta)$$

$$r = \frac{2c}{1-\cos\theta}$$

Solve

① Show that the O.T is of the family of curves $r = a\cos\theta$ is $r = b\sin\theta$.

② Show that the O.T's of the family of $r(1-\cos\theta) = a$ is the family of $r(1+\cos\theta) = b$.

③ If $\frac{dr}{d\theta} = r\cot(\theta/_2)$ is the differential equation of the family of cardioids $r = a(1-\cos\theta)$. then find its orthogonal trajectory

Simple electric Circuits

Element	Symbols
Current	i
Resistance	R
Inductance	L
Capacitance	C
emf	E
quantity of electricity	q
time	t

Relations

1) $i = \frac{dq}{dt}$ (or) $q = \int i\, dt$
2) voltage drop across the resistance $= Ri$
3) voltage drop across the inductance $= L\frac{di}{dt}$
4) voltage drop across the capacitance $= q/C$

Kirchoff's law

(1) The algebraic sum of the voltage drops around any closed circuit is equal to the resultant emf in the circuit

(2) The algebraic sum of current flowing into any node is zero.

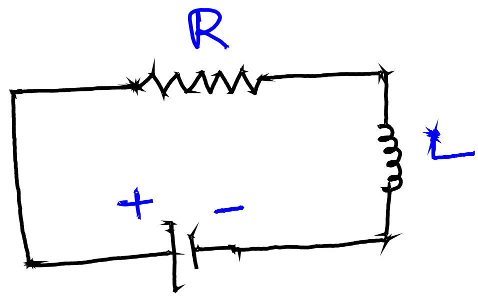
L-R Series Circuit

Let i be the current flowing in the circuit containing resistance R and inductance L in series with voltage source E at any time t

By Kirchoff's Law

$$L\frac{di}{dt} + Ri = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$



LCR circuit

Let i be the current in the circuit containing the resistance R, inductance L, and capacitance C in series with voltage source E at any time t

By Kirchoff's law

$$Ri + L\frac{di}{dt} + \frac{q}{c} = E$$

$$L\frac{di}{dt} + Ri + \frac{q}{c} = E$$

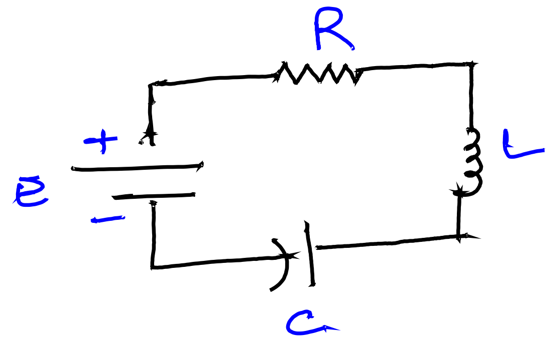
$$L\frac{di}{dt} + Ri + \frac{q}{C} = E$$

$$\frac{di}{dt} + \frac{R}{L}i + \frac{q}{LC} = \frac{E}{L} \rightarrow (1) \checkmark$$

But $i = \frac{dq}{dt}$

$$\frac{d}{dt}\left(\frac{dq}{dt}\right) + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E}{L}$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E}{L} \rightarrow (2)$$



Note :- if $q=0$, $t=0$
$i=0$

Problems

(1) The charge $+q$ on a plate of condenser of capacity C, charge through a resistance R by a steady voltage E satisfy the D.E $R\frac{dq}{dt} + \frac{q}{C} = E$ if $q=0$ at $t=0$ then show that $q = CE\left(1 - e^{-t/RC}\right)$ and also find the current flowing into the plate

Sol :- Given $R\frac{dq}{dt} + \frac{q}{c} = E$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R} \quad (\text{linear in } q)$$

$$P = \frac{1}{RC}, \quad Q = \frac{E}{R}$$

$$I.F = e^{\int P\,dt} = e^{\int \frac{1}{RC}dt}$$

$$= e^{1/RC \int dt}$$

$$I.F = e^{t/RC}$$

Sol :-

$$q\,(IF) = \int Q\,(IF)\,dt + K$$

K = constant

$$q\,e^{t/RC} = \int \frac{E}{R} \cdot e^{t/RC}\,dt + K$$

$$q\,e^{t/RC} = \frac{E}{R} \cdot \frac{e^{t/RC}}{1/RC} + K$$

$$q\,e^{t/RC} = \frac{EC}{\cancel{R}}\,e^{t/RC} + K$$

$$q = \frac{EC \cdot e^{t/RC} + K}{e^{t/RC}}$$

$$q = EC + K e^{-t/RC} \rightarrow (1)$$

$q = EC + K e^{-t/RC}$

Given $q = 0$ at $t = 0$

$0 = EC + K \cdot e^{0}$

$K + EC = 0 \Rightarrow \boxed{K = -EC}$

Sub K in ①

$q = EC - EC e^{-t/RC}$

$q = EC(1 - e^{-t/RC})$ ✓

$i = \frac{dq}{dt}$

$= \frac{d}{dt}\left[EC(1 - e^{-t/RC})\right]$ ✓

$= EC \cdot \left(0 - e^{-t/RC} \cdot \frac{-1}{RC}\right)$

$= \frac{E\cancel{C}}{R\cancel{C}} e^{-t/RC}$

$$\boxed{i = \frac{E}{R} e^{-t/RC}}$$

(2) A voltage Ee^{-at} applied at $t=0$ to a Circuit of inductance L and resistance R. then show that the Current at a time t is $\frac{E}{R-aL}\left(e^{-at} - e^{-Rt/L}\right)$. (15)

Sol :- By Kirchoff's law The D.E. for L.R Series circuit is

$$L\frac{di}{dt} + Ri = Ee^{-at}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$$

$$P = \frac{R}{L}, \qquad Q = \frac{E}{L}e^{-at}$$

$$I.F = e^{\int P\,dt} = e^{\int \frac{R}{L}dt}$$

$$= e^{R/L\int dt}$$

$$= e^{Rt/L}$$

Sol.

$$i(IF) = \int Q(IF)\,dt + k$$

$$i\left(e^{Rt/L}\right) = \int \frac{E}{L}e^{-at}\cdot e^{Rt/L}\,dt + k$$

$$i\,e^{Rt/L} = \frac{E}{L}\int e^{(R/L-a)t}\,dt + k$$

$$i \cdot e^{Rt/L} = \frac{E}{L} \frac{e^{(R/L - a)t}}{\left(\frac{R}{L} - a\right)} + K$$

$$= \frac{E}{\cancel{L}} \cdot \frac{e^{(R/L - a)t}}{\left(\frac{R - aL}{\cancel{L}}\right)} + K$$

$$i \cdot e^{Rt/L} = \frac{E}{R - aL} e^{(\frac{R}{L} - a)t} + K$$

$$i = \frac{E}{R - aL} \frac{e^{(R/L - a)t}}{e^{Rt/L}} + \frac{K}{e^{Rt/L}}$$

$$i = \frac{E}{R - aL} \frac{\cancel{e^{Rt/L}} \cdot e^{-at}}{\cancel{e^{Rt/L}}} + K e^{-Rt/L}$$

$$i = \frac{E}{R - aL} e^{-at} + K \cdot e^{-Rt/L} \rightarrow (1)$$

Given $i = 0$, at $t = 0$

$$0 = \frac{E}{R - aL} e^{0} + K e^{0}$$

$$K + \frac{E}{R - aL} = 0$$

$$\boxed{K = -\frac{E}{R - aL}} \rightarrow (2)$$

Sub (2) in (1)

$$i = \frac{E}{R-aL} e^{-at} - \frac{E}{R-aL} e^{-Rt/L}$$

$$i = \frac{E}{R-aL} \left[e^{-at} - e^{-Rt/L} \right]$$

$$i = \frac{E}{R-aL} \left[e^{-at} - e^{-Rt/L} \right]$$

(3) The equation of an L-R Circuit is given by $L\frac{di}{dt} + Ri = 10\sin t$. if $i = 0$ at $t = 0$ express i as a function of t.

Sol :- $L\frac{di}{dt} + Ri = 10\sin t$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{10}{L} \sin t$$

$$P = \frac{R}{L}, \qquad Q = \frac{10}{L} \sin t$$

$$I.F = e^{\int p\,dt} = e^{\frac{R}{L}\int dt} = e^{Rt/L}$$

Sol

$$i(IF) = \int Q\,(IF)\,dt + c$$

$$i\, e^{Rt/L} = \int \frac{10}{L} \sin t \cdot e^{Rt/L}\,dt + K$$

$$i \cdot e^{Rt/L} = \frac{10}{L}\int e^{Rt/L} \sin t \, dt + k$$

$$\int e^{ax} \sin(bx + c)\, dx = \frac{e^{ax}}{a^2+b^2}\left(a \sin(bx+c) - b\cos(bx+c)\right)$$

$$i \cdot e^{Rt/L} = \frac{10}{L}\left[\frac{e^{Rt/L}}{\left(\frac{R}{L}\right)^2+1}\left(\frac{R}{L}\sin t - \cos t\right)\right] + K \longrightarrow (1)$$

at $t=0$ $\quad i=0$

$$0 = \frac{10}{L}\left(\frac{e^0}{\left(\frac{R}{L}\right)^2+1}\left(\frac{R}{L}\sin 0 - \cos 0\right)\right) + K$$

$$= \frac{10}{L}\left(\frac{1}{\frac{R^2}{L^2}+1}\right)(0-1) + K$$

$$= \frac{10}{\cancel{L}} \cdot \frac{-1}{\frac{R^2+L^2}{L^{\cancel{2}}}} + K$$

$$0 = \frac{-10L}{R^2+L^2} + K$$

$$\boxed{K = \frac{10L}{R^2+L^2}} \longrightarrow (2)$$

Sub (2) in (1), we get

i.e. $e^{Rt/L} = \frac{10L}{\cancel{10L}}\left(\frac{e^{Rt/L}}{R^2+L^2}\left(\frac{R}{L}\sin t - \cos t\right) + \frac{10L}{R^2+L^2}\right.$ 19

$$i = \frac{10L}{R^2+L^2} \frac{\cancel{e^{Rt/L}}\left(\frac{R}{L}\sin t - \cos t\right)}{\cancel{e^{Rt/L}}} + \frac{10L}{R^2+L^2} \cdot \frac{1}{e^{Rt/L}}$$

$$i = \frac{10L}{R^2+L^2}\left(\frac{R}{L}\sin t - \cos t\right) + \frac{10L\, e^{-Rt/L}}{R^2+L^2}$$