1. *Given* $U(z) = \dfrac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$, *find* u_n

>> $U(z)$ or $\bar{u}(z) = \dfrac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$ by data.

We shall factorize the denominator first.

$$z^3 - 5z^2 + 8z - 4 = (z^3 - 5z^2 + 4z) + (4z - 4)$$
$$= z(z^2 - 5z + 4) + 4(z - 1)$$
$$= z(z-1)(z-4) + 4(z-1)$$
$$= (z-1)(z^2 - 4z + 4)$$
$$= (z-1)(z-2)^2$$

We have $\bar{u}(z) = \dfrac{4z^2 - 2z}{(z-1)(z-2)^2}$

We have $Z_T^{-1}\left[\dfrac{z}{z-1}\right] = 1$, $Z_T^{-1}\left[\dfrac{z}{z-2}\right] = 2^n$, $Z_T^{-1}\left[\dfrac{2z}{(z-2)^2}\right] = 2^n \cdot n$

We resove $\bar{u}(z)$ as follows.

$$\bar{u}(z) = \frac{4z^2 - 2z}{(z-1)(z-2)^2} = A \cdot \frac{z}{z-1} + B \cdot \frac{z}{z-2} + C \cdot \frac{2z}{(z-2)^2} \quad \ldots (1)$$

or $$\frac{4z^2 - 2z}{(z-1)(z-2)^2} = \frac{Az(z-2)^2 + Bz(z-1)(z-2) + 2Cz(z-1)}{(z-1)(z-2)^2}$$

or $$4z - 2 = A(z-2)^2 + B(z-1)(z-2) + 2C(z-1)$$

Put $z = 1 : 2 = A(1) \quad \therefore \quad A = 2$

Put $z = 2 : 6 = 2C(1) \quad \therefore \quad C = 3$

Equating the coefficient of z^2 on both sides we have,

$A + B = 0 \quad \therefore \quad B = -2$

Substituting the values of A, B, C in (1) and taking inverse we have

$$Z_T^{-1}[\bar{u}(z)] = 2\,Z_T^{-1}\left[\frac{z}{z-1}\right] - 2\,Z_T^{-1}\left[\frac{z}{z-2}\right] + 3\,Z_T^{-1}\left[\frac{2z}{(z-2)^2}\right]$$
$$= 2.1 - 2.2^n + 3 \cdot 2^n \cdot n$$

Thus $Z_T^{-1}[\bar{u}(z)] = u_n = 2 - 2^{n+1} + 3n \cdot 2^n$

Thus, $y_n = 2\left[\left(4\right) \quad \left(4\right)\right]$

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41. *Solve :* $u_{n+2} - 3u_{n+1} + 2u_n = 1$ *by using Z-transforms.*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) - 3Z_T(u_{n+1}) + 2Z_T(u_n) = Z_T(1)$$

ie., $$z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-1}$$

ie., $$[z^2 - 3z + 2]\bar{u}(z) - u_0(z^2 - 3z) - u_1 z = \frac{z}{z-1}$$

ie., $$[(z-1)(z-2)]\bar{u}(z) = u_0(z^2 - 3z) + u_1 z + \frac{z}{z-1}$$

or $$\bar{u}(z) = u_0 \cdot \frac{z^2 - 3z}{(z-1)(z-2)} + u_1 \cdot \frac{z}{(z-1)(z-2)} + \frac{z}{(z-1)^2(z-2)}$$

ie., $$\bar{u}(z) = u_0 \cdot p(z) + u_1 \cdot q(z) + r(z)\ (say) \qquad \ldots(1)$$

We shall find the inverse Z-transform of $p(z)$, $q(z)$ and $r(z)$

Consider, $p(z) = \dfrac{z^2 - 3z}{(z-1)(z-2)}$

Let $$\frac{p(z)}{z} = \frac{z-3}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

or $$z - 3 = A(z-2) + B(z-1)$$

Put $z = 1 :\ -2 = A(-1) \qquad \therefore\ A = 2$

Put $z = 2 :\ -1 = B(1) \qquad \therefore\ B = -1$

$\therefore$ $$Z_T^{-1}[p(z)] = 2Z_T^{-1}\left[\frac{z}{z-1}\right] - Z_T^{-1}\left[\frac{z}{z-2}\right]$$

ie., $$Z_T^{-1}[p(z)] = 2.1 - 2^n = 2 - 2^n \qquad \ldots(2)$$

Consider $q(z) = \dfrac{z}{(z-1)(z-2)}$ *(Refer Problem-26)*

$$Z_T^{-1}[q(z)] = 2^n - 1 \quad \ldots$$

Consider $r(z) = \dfrac{z}{(z-1)^2(z-2)}$

Let $\dfrac{z}{(z-1)^2(z-2)} = C \cdot \dfrac{z}{z-1} + D \cdot \dfrac{z}{(z-1)^2} + E \cdot \dfrac{z}{z-2}$

or $1 = C(z-1)(z-2) + D(z-2) + E(z-1)^2$

Put $z = 1 : \ 1 = D(-1) \qquad \therefore \quad D = -1$

Put $z = 2 : \ 1 = E(1) \qquad \therefore \quad E = 1$

Equating the coefficient of z^2 on both sides we get,

$$C + E = 0 \qquad \therefore \quad C = -1$$

Now, $Z_T^{-1}[r(z)] = -Z_T^{-1}\left[\dfrac{z}{z-1}\right] - Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right] + Z_T^{-1}\left[\dfrac{z}{z-2}\right]$

ie., $Z_T^{-1}[r(z)] = -1 - n + 2^n \quad \ldots (4)$

With reference to (1) we have,

$$Z_T^{-1}[u(z)] = u_0 \cdot Z_T^{-1}[p(z)] + u_1 Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the R.H.S we have,

$$Z_T^{-1}[\overline{u}(z)] = u_0(2 - 2^n) + u_1(2^n - 1) - 1 - n + 2^n$$

ie., $u_n = (2u_0 - u_1 - 1) + (-u_0 + u_1 + 1)\,2^n - n$

Let us denote $c_1 = 2u_0 - u_1 - 1$ and $c_2 = -u_0 + u_1 + 1$ where c_1 and c_2 are arbitrary constants.

Thus $u_n = c_1 + c_2 \cdot 2^n - n$ is the required solution.

48. *Find the impulse response of a system described by* $y_{n+1} + 2y_n = \delta_n$; $y_0 = 0$ *by applying Z-transforms.*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+1}) + 2Z_T(y_n) = Z_T(\delta_n)$$

ie., $$z[\bar{y}(z) - y_0] + 2\bar{y}(z) = 1$$

ie., $(z+2)\,\bar{y}(z) = 1$ or $\bar{y}(z) = \dfrac{1}{z+2}$

ie., $\bar{y}(z) = \dfrac{1}{z\left(1+\dfrac{2}{z}\right)} = \dfrac{1}{z}\left(1+\dfrac{2}{z}\right)^{-1}$

Now, $\bar{y}(z) = \dfrac{1}{z}\left\{1-\dfrac{2}{z}+\left(\dfrac{2}{z}\right)^{2}-\left(\dfrac{2}{z}\right)^{3}+\cdots\right\}$

$$\bar{y}(z) = \frac{1}{z}-2\left(\frac{1}{z^2}\right)+2^2\left(\frac{1}{z^3}\right)-2^3\left(\frac{1}{z^4}\right)+\cdots$$

$$\bar{y}(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\,2^{n-1}\left(\frac{1}{z^n}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\,2^{n-1}\,z^{-n}$$

ie., $Z_T(y_n) = \sum_{n=1}^{\infty}(-2)^{n-1}\,z^{-n}$

Thus $y_n = (-2)^{n-1}$ where $n \geq 1$

16. Solve the equation using Z - Transforms : $u_{n+2} - 5u_{n+1} + 6u_n = H_n$ with $u_0 = 0, u_1 = 1$, where H_n is a unit step sequence.

Soln :

$$u_{n+2} - 5u_{n+1} + 6u_n = H_n$$

$$Z_T[u_{n+2}] - 5Z_T[u_{n+1}] + 6Z_T[u_n] = Z_T[H_n]$$

$$z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] - 5z[U(z) - \cancel{z_0}] + 6U(z) = \frac{z}{z-1}$$

$$U(z)(z^2 - 5z + 6) - z = \frac{z}{z-1} \Rightarrow U(z)(z^2-5z+6) = \frac{z}{z-1} + z$$

$$= \frac{\cancel{z} + z^2 \cancel{-z}}{z-1}$$

$$U(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$(z^2-5z+6)U(z) = \frac{z^2}{(z-1)}$$

$$\frac{U(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$\Rightarrow U(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$\frac{z}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$z = 2$, $\quad 2 = -B \Rightarrow B = \cancel{-1/2}\ \underline{\underline{-2}}$

$z = 3$, $\quad 3 = 2C \Rightarrow C = \underline{\underline{3/2}}$

$z = 1$, $\quad 1 = 2A \Rightarrow A = \underline{\underline{1/2}}$

$$u_n = Z_T^{-1}\left[\frac{z}{2(z-1)} - \frac{2z}{z-2} + \frac{3z}{2(z-3)}\right]$$

$$= \frac{1}{2} - 2 \cdot 2^n + \frac{3}{2} \cdot 3^n = \underline{\underline{\frac{1}{2} - 2^{n+1} + \frac{3^{n+1}}{2}}}$$