MODULE-5 Laplace Transform

Def^n If $f(t)$ is a real valued function defined for all $t \geq 0$, then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt,$$ provided the integral exist.

$L\{f(t)\} = \bar{f}(s)$ [as solving the integral we will left with s only]

$\Rightarrow L^{-1}\{\bar{f}(s)\} = f(t)$. is called inverse L.T.

Note Linear Property

$$L\{c_1 f_1(t) \pm c_2 f_2(t)\} = c_1 L\{f_1(t)\} \pm c_2 L\{f_2(t)\}$$

where c_1, c_2 are constants

Laplace transform of discontinuous function

Suppose $f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & , a < t < b \\ f_3(t) & - t > b. \end{cases}$

then $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ (by dy)

$$= \int_0^a e^{-st} f_1(t)\, dt + \int_a^b e^{-st} f_2(t)\, dt + \int_b^\infty e^{-st} f_3(t)\, dt.$$

Que Find the Laplace Transform of

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$$

Sol $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^2 \underset{\text{II}}{e^{-st}}\, \underset{\text{I}}{t^2}\, dt + \int_2^3 e^{-st}(t-1)\, dt + \int_3^\infty e^{-st}\, 7\, dt$$

$$= t^2 \frac{e^{-st}}{-s} - 2t \frac{e^{-st}}{(-s)^2} + \frac{2e^{-st}}{(-s)^3} \Big|_0^2 + \frac{e^{-st}}{-s}(t-1) - \frac{e^{-st}}{(-s)^2} \Big|_2^3$$

$$+ 7 \frac{e^{-st}}{-s} \Big|_3^\infty$$

$$= -\frac{4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} + \frac{2e^{-2s}}{-s^3} + \frac{2}{s^3} + \frac{2e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} + \frac{7}{s}$$

$$= \frac{2}{s^3} - \frac{e^{-2s}}{s^3}\left(2 + 3s + 3s^2\right) + \frac{e^{-3s}}{s^2}\left(5s - 1\right)$$

Que 2. Find L.T of $f(t)$, where

$f(t) = |t-1| + |t+1|$, $t \geq 0$.

Sol If $t=1$, $f(t) = 2$

If $0<t<1$, $f(t) = -(t-1)+(t+1)$
$= 2$

If $t>1$, $f(t) = 2t$.

$$\therefore f(t) = \begin{cases} 2 & 0<t\leq 1 \\ 2t & t>1 \end{cases}$$

$$\therefore L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 2e^{-st} dt + \int_1^\infty 2t e^{-st} dt$$

$$= \frac{2e^{-st}}{-s}\Big|_0^1 + 2\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{(-s)^2}\right]_1^\infty$$

$$= \frac{2}{s}\left(1 - \frac{e^{-s}}{s}\right)$$

Que Find $L\{f(t)\}$, $f(t) = \begin{cases} \sin\left(t-\frac{\pi}{3}\right), & t>\frac{\pi}{3} \\ 0, & t<\frac{\pi}{3} \end{cases}$

Sol $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^{2\pi/3} e^{-st} \sin\left(t - \frac{\pi}{3}\right) dt + \int_{2\pi/3}^{\infty} 0 \cdot dt$$

$$= \frac{e^{-st}}{(-s)^2 + 1^2} \left[-s \sin\left(t - \frac{\pi}{3}\right) - \cos\left(t - \frac{\pi}{3}\right) \right] \Bigg|_0^{2\pi/3}$$

$$= \frac{e^{-\frac{\pi s}{3}}}{s^2+1} \left[-1 \right] - \frac{1}{s^2+1} \left[-s \right.$$

$$= \int_0^{\pi/3} e^{-st}\, 0 \cdot dt + \int_{\pi/3}^{\infty} e^{-st} \sin\left(t - \frac{\pi}{3}\right) dt$$

$$= \frac{e^{-st}}{(-s)^2 + 1^2} \left[-s \sin\left(t - \frac{\pi}{3}\right) - \cos\left(t - \frac{\pi}{3}\right) \right] \Bigg|_{\pi/3}^{\infty}$$

$$\left[\because \int e^{at} \sin bt = \frac{e^{at}}{a^2+b^2} \left[a \sin bt - b \cos bt \right] \right.$$

$$\left. \int e^{at} \cos bt = \frac{e^{at}}{a^2+b^2} \left[a \cos bt + b \sin bt \right] \right]$$

$$= \frac{e^{-\infty}}{s^2+1} \left\{ -1 \right] - \frac{e^{-\frac{\pi s}{3}}}{s^2+1} \left[-1 \right\} = 0 + \frac{e^{-\pi s/3}}{s^2+1} = \frac{e^{-\pi s/3}}{s^2+1}$$

Ans

Laplace Transform of some standard functions

① $L\{a\}$, where a is constant

Proof $L\{a\} = \int_0^{\infty} e^{-st} \cdot a\,dt = a \frac{e^{-st}}{-s}\Big|_0^{\infty}$

$= 0 - \frac{a e^{-0s}}{-s} = \frac{a}{s}.$

$$\boxed{L\{a\} = \frac{a}{s}}$$

$$\therefore \boxed{L\{1\} = \frac{1}{s}}$$

② $L\{e^{at}\} = \frac{1}{s-a}$

③ $L\{\cosh at\} = \frac{s}{s^2-a^2}$

Pf: $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$

$= \frac{1}{2}\,\frac{2s}{s^2-a^2} = \frac{s}{s^2-a^2}$

(4) $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

(5) $L\{\cos at\} = \dfrac{s}{s^2+a^2}, \quad s>0$

Proof: $L\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$

$$= \frac{e^{-st}}{s^2+a^2}\left[-s\cos at + a\sin at\right]\Big|_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-s) = \frac{s}{s^2+a^2}$$

(6) Similarly $L\{\sin at\} = \dfrac{a}{s^2+a^2}, \quad s>0$

(7) $L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$

Proof $L\{t^n\} = \int_0^\infty e^{-st}t^n\,dt$

Put $st = x$, $dt = \dfrac{dx}{s}$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n\,dx$$

which is gamma function

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

[Def^n of gamma

$\Gamma(n+1) = n!$ if n is a +ve integer

$= n\Gamma n$ if n is +ve fraction

OR

$$\Gamma n = \begin{cases} (n-1)! & \text{+ve integer} \\ (n-1)\Gamma(n-1) & \text{for +ve fraction} \\ \dfrac{\Gamma(n+1)}{n} & \text{for -ve fraction} \end{cases}$$

$$\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$]

Que or eg.

$L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$

Sol $= \frac{1}{s-2} + 4 \cdot \frac{3!}{s^4} - \frac{2 \times 3}{s^2+9} + \frac{3 \times s}{s^2+9}$

$= \frac{1}{s-2} - \frac{24}{s^4} \cancel{+} \frac{6}{s^2+9} + \frac{3s}{s^2+9}$

Que 2 $L\{t\sqrt{t} + 4t^3 + 3^t\}$

$= L\{t^{3/2}\} + 4L\{t^3\} + L\{e^{t\log 3}\}$

$= \dfrac{\Gamma\frac{5}{2}}{s^{5/2}} + 4 \cdot \dfrac{3!}{s^4} + \dfrac{1}{s-\log 3}$

$= \dfrac{\frac{3}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} + \dfrac{24}{s^4} + \dfrac{1}{s-\log 3}$

$= \dfrac{3}{4}\dfrac{\sqrt{\pi}}{s^{5/2}} + \dfrac{24}{s^4} + \dfrac{1}{s-\log 3}$

Que $L\{\cos(2t+3) + \cos 7t \cos 3t\}$

Sol $L\{\cos 2t \cos 3 - \sin 2t \sin 3\} + \frac{1}{2} L\{2\cos 7t \cos 3t\}$

$\Rightarrow \cos 3\, L\{\cos 2t\} - \sin 3\, L\{\sin 2t\} + \frac{1}{2} L\{\cos 10t + \cos 4t\}$

$\Rightarrow \cos 3 \cdot \dfrac{s}{s^2+4} - \dfrac{2\sin 3}{s^2+4} + \dfrac{1}{2}\dfrac{s}{s^2+10^2} + \dfrac{1}{2}\dfrac{s}{s^2+4^2}$

Que $L\{\sin 3t \sin 2t \sin t\}$.

$\Rightarrow \frac{1}{2} L\{(2\sin 3t \sin 2t) \sin t\}$

$= \frac{1}{2} L\{(\cos t - \cos 5t) \sin t\}$

$= \frac{1}{2} L\{\cancel{\sin^2 t} - \sin \sin \sin\}$

$= \frac{1}{2}\left[L\{\cos t \sin t\} - L\{\cos 5t \sin t\}\right)$

$= \frac{1}{2}\left[\frac{1}{2}L\{\sin 2t\} - \frac{1}{2}L\{\sin 6t - \sin 4t\}\right)$

$= \frac{1}{4}\left[L\{\sin 2t\} - L\{\sin 6t\} + L\{\sin 4t\}\right)$

$= \frac{1}{4}\left[\frac{2}{s^2+4} - \frac{6}{s^2+36} + \frac{4}{s^2+16}\right)$

$= \frac{1}{2}\left[\frac{1}{s^2+4} - \frac{3}{s^2+36} + \frac{2}{s^2+16}\right)$

Properties of Laplace Transform

First Shifting Property

1) If $L\{f(t)\} = \bar{f}(s)$, then $L\{e^{at} f(t)\} = \bar{f}(s-a)$

Pf By def^n $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$

$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$= \bar{f}(s-a)$

e.g. $L\{t^n\} = \frac{n!}{s^{n+1}}$, $L\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$

$L\{\sin bt\} = \frac{b}{s^2+b^2}$, $L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

$L\{\cos bt\} = \frac{s}{s^2+b^2}$, $L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$.

$L\{\sinh bt\} = \frac{b}{s^2-b^2}$, $L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$.

Que $L\{t^3e^{-2t}\} =$

$$L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$$

$$L\{e^{-2t}t^3\} = \frac{3!}{(s+2)^4}.$$

($\sin 3x = 3\sin x - 4\sin^3 x$)

ue $L\{e^{-3t}\sin^3 2t\}$

ol $\sin^3 2t = \frac{1}{4}\left[3\sin 2t - \sin 6t\right]$

$$L\{\sin^3 2t\} = \frac{1}{4}\left[3\cdot\frac{2}{s^2+4} - \frac{6}{s^2+36}\right]$$

$$L\{e^{-3t}\sin^3 2t\} = \frac{6}{4}\left[\frac{1}{(s+3)^2+4} - \frac{1}{(s+3)^2+36}\right]$$

Que $L\{\cosh 2t \cos 2t\}$

Sol $L\left\{\frac{e^{2t}+e^{-2t}}{2}\cos 2t\right\}$

$$= \frac{1}{2}\left[\frac{s}{(s-2)^2+4} + \frac{s}{(s+2)^2+4}\right]$$

Property II When function is multiply by t^n

Proof:- If $L\{f(t)\} = \bar{f}(s)$, then

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

Property III When function is divided by t

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$,

provided integral exists

$$L\left\{\frac{f(t)}{t^2}\right\} = \int_s^\infty\left[\int_s^\infty \bar{f}(s).\,ds\right]ds$$

Prop-IV Laplace transform of Integrals

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(t)\,dt\right\} = \dfrac{\bar{f}(s)}{s}$.

Que Find $L\{t\sin^2 t\}$

Sol $L\{t\sin^2 t\} = L\left\{t\left(\dfrac{1-\cos 2t}{2}\right)\right\}$

$= \dfrac{1}{2}L\{t\} - L\left\{\dfrac{t\cos 2t}{2}\right\}$

$= \dfrac{1}{2}\left[\dfrac{1}{s^2}\right] - \dfrac{1}{2}L\{t\cos 2t\}$ —— (1)

Now $L\{\cos 2t\} = \dfrac{s}{s^2+4} = \bar{f}(s)$

$L\{t\cos 2t\} = (-1)^1 \dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right)$

$= (-1)\dfrac{(s^2+4) - 2s^2}{(s^2+4)^2}$

$= \dfrac{s^2-4}{(s^2+4)^2}$

$\therefore$ (1) becomes

$L\{t\sin^2 t\} = \dfrac{1}{2}\left[\dfrac{1}{s^2} - \dfrac{s^2-4}{(s^2+4)^2}\right]$

Que $L\{te^{-2t}\sin 4t\}$

Sol $L\{\sin 4t\} = \dfrac{4}{s^2+4}$

$$L\{t\sin 4t\} = (-1)^1 \frac{d}{ds}\frac{4}{s^2+4}$$

$$= -4\,\frac{-2s}{(s^2+4)^2}$$

$$= \frac{8s}{(s^2+4)^2}$$

$$L\{e^{-2t}\,t\sin 4t\} = \frac{8(s+2)}{\left[(s+2)^2+4\right]^2}.$$

Que $L\left\{\dfrac{\cos 2t - \cos 3t}{t}\right\}$

Sol $L\{\cos 2t - \cos 3t\} = \dfrac{s}{s^2+4} - \dfrac{s}{s^2+9}$

$$L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \int_s^\infty \frac{s}{s^2+4} - \frac{s}{s^2+9}\,ds$$

$$= \frac{1}{2}\left[\log(s^2+4) - \log(s^2+9)\right]\Big|_s^\infty$$

$$= \frac{1}{2} \log \frac{s^2+4}{s^2+9} \Big|_s^{\infty}$$

$$= \frac{1}{2} \log \frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}} \Big|_s^{\infty} = \frac{1}{2}\log 1 - \frac{1}{2} \log \frac{s^2+4}{s^2+9}$$

$$= \frac{1}{2} \log \frac{s^2+9}{s^2+4} \quad \text{Ans}$$

Que $L\{ \frac{\sin^2 t}{t} \}$

Sol $L\{ \frac{1-\cos 2t}{2t} \}$

Now $L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$

$$\frac{1}{2} L\{ \frac{1-\cos 2t}{t} \} = \frac{1}{2} \left[\int_s^{\infty} \frac{1}{s} - \frac{s}{s^2+4} \, ds \right]$$

$$= \frac{1}{2} \log s - \frac{1}{4} \log (s^2+4) \Big|_s^{\infty}$$

$$= \frac{1}{4} \log \frac{s^2}{s^2+4} \Big|_s^{\infty} = \frac{1}{4} \log \frac{s^2+4}{s^2}$$

Que $L\{t^2 \sin 3t\}$

sol $L\{\sin 3t\} = \frac{9}{s^2+9}$

$$L\{t^2 \sin 3t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{9}{s^2+9}\right)$$

$$= -9 \frac{d}{ds} \frac{2s}{(s^2+9)^2}$$

$$= -18\left[\frac{(s^2+9)^2 - s\,2(s^2+9).2s}{(s^2+9)^4}\right]$$

$$= -18\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right]$$

$$= -18\left[\frac{-3s^2+9}{(s^2+9)^3}\right]$$

$$= \frac{54[s^2-3)}{(s^2+9)^3}$$

Que $L\{\int_0^t e^{-t}\cos t\, dt\}$

$L\{\cos t\} = \frac{s}{s^2+1}$

$L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}$

$L\{\int_0^t e^{-t}\cos t\, dt\} = \frac{1}{s}\frac{(s+1)}{(s+1)^2+1}$

Que $L\left\{\int_0^t e^{2t}\frac{\sin at}{t}\,dt\right\}$

Sol $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= \tan^{-1}\frac{s}{a}\Big|_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{a}$$

$$= \cot^{-1}\frac{s}{a}$$

$$L\left\{e^{2t}\frac{\sin at}{t}\right\} = \cot^{-1}\frac{(s-2)}{a}$$

$$L\left\{\int_0^t e^{2t}\frac{\sin at}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}\left(\frac{s-2}{a}\right) \quad \text{Ans}$$

Que $L\left\{e^{-4t}\int_0^t t\sin 3t\,dt\right\}$

Sol $L\{\sin 3t\} = \frac{3}{s^2+9}$

$$L\{t\sin 3t\} = (-1)^1\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{6}{(s^2+9)^2}$$

$$L\left\{ e^{-4t}\int_0^t t\sin 3t\,dt \right\} = \frac{6}{\left((s+4)^2+9\right)^2} \quad \text{Ans}$$

Que Evaluate

(a) $\int_0^\infty e^{-3t}\, t\sin t\, dt$. — (1)

Sol We know

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \quad — (2)$$

Comp (1) & (2).

$s = 3,\ f(t) = t\sin t$.

$$\therefore \int_0^\infty e^{-3t}\, t\sin t\, dt = L\{t\sin t\}\Big|_{s=3}$$

$$= (-1)^1 \frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}\Big|_{s=3}$$

$$= \frac{6}{100} = 0.06 \text{ or } \frac{3}{50}$$

(b) Evaluate $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt$ — (1)

Sol ~~Comp with def^n~~

$L\{e^{-t} - e^{-3t}\} = \frac{1}{s+1} - \frac{1}{s+3}$

$L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\} = \int_s^\infty \frac{1}{s+1} - \frac{1}{s+3}\, ds$

$= \log \frac{s+3}{s+1}$

Now $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ ②

Comp. ① & ②,

$s = 0,\ f(t) = \frac{e^{-t} - e^{-3t}}{t}$,

$\therefore \int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt = L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\}\Big|_{s=0}$

$= \log\left(\frac{s+3}{s+1}\right)\Big|_{s=0}$

$= \log 3$

Que: Show that $\int_0^\infty e^{-t} \frac{\sin t}{t}\, dt = \frac{\pi}{4}$.

Sol ~~L{sin t}~~ Comp with def^n

$L\{f(t)\} = \int_0^\infty e^{-t} \frac{\sin t}{t}\, dt = \int_0^\infty e^{-st} f(t)\, dt$

$s = 1$. $f(t) = \frac{\sin t}{t}$.

$\therefore$ $L\{\sin t\} = \frac{1}{s^2+1}$

$L\{\frac{\sin t}{t}\} = \int_s^\infty \frac{1}{s^2+1}\, ds = \tan^{-1} s \Big|_s^\infty$

$= \cot^{-1} s \Big|_{s=1}$

$= \cot^{-1} 1 = \frac{\pi}{4}$ Ans

Que $L\{\sin\sqrt{t}\}$

Sol $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin\sqrt{t} = \sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots$

$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!} L\{t^{3/2}\} + \frac{L\{t^{5/2}\}}{5!} - \cdots$

$= \frac{\Gamma\frac{3}{2}}{s^{3/2}} - \frac{1}{3!} \frac{\Gamma\frac{5}{2}}{s^{5/2}} + \frac{1}{5!} \frac{\Gamma\frac{7}{2}}{s^{7/2}}$

$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} - \frac{1}{6} \frac{\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} + \frac{1}{5!} \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{7/2}}$

$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\sqrt{\pi}}{s^{7/2}\,64} - \cdots$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} \cdots\right]$$

<u>Que</u> $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Sol $\cos\sqrt{t} = 1 - \frac{(\sqrt{t})^2}{2!} + \frac{(\sqrt{t})^4}{4!} \cdots$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} - \frac{(\sqrt{t})^2}{\sqrt{t}\,2!} + \frac{(\sqrt{t})^4}{4!} \cdots$$

$$= (t)^{-1/2} - \frac{(t)^{1/2}}{2!} + \frac{(t)^{3/2}}{4!}$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} - \frac{\Gamma\frac{3}{2}}{2!\,s^{3/2}} + \frac{\Gamma\frac{5}{2}}{s^{5/2}\,4!}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} - \frac{\frac{1}{2}\sqrt{\pi}}{2\,s^{3/2}} + \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{4!\,s^{5/2}} \cdots$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{4}{s} + \frac{1}{32s^2} \cdots\right]$$

Periodic functions

Laplace transform of Periodic function

If $f(t)$ is a periodic function with period T i.e. $f(t) = f(t+T)$, then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

<u>Proof</u> Given $f(t+T) = f(t)$. as $f(t)$ is periodic

By defn

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

Put $t = u+T$

$dt = du$.

when $t = T$, $u = 0$

$t = \infty$, $u = \infty$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+T)} f(u+T)\,du.$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \int_0^\infty e^{-su} f(u)\,du$$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT} L\{f(t)\}$$

$$\Rightarrow (1-e^{-sT})\, L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt$$

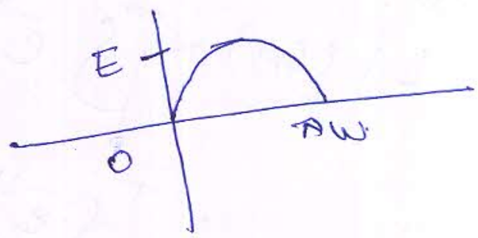
$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

where $f(t) = f(t+T)$

Que Find the L.T. of full wave rectifier ~~f(t) = E sin wt~~
$f(t) = E\sin\omega t$, $0<t<\frac{\pi}{\omega}$, having period $\frac{\pi}{\omega}$

Sol

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$



Here period $= \frac{\pi}{\omega}$

$$= \frac{1}{1-e^{-s\frac{\pi}{\omega}}} \int_0^{\pi\omega} e^{-st} E\sin\omega t\,dt$$

$$= \frac{E}{1-e^{-\frac{s\pi}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t - \omega\cos\omega t\right)\right]_0^{\pi\omega}$$

$$= \frac{E}{1-e^{s\pi/\omega}} \left[\frac{e^{-s\frac{\pi}{\omega}}}{s^2+\omega^2}(\omega) - \frac{1}{s^2+\omega^2}(\omega)\right]$$

$$= \frac{E\omega\left(1+e^{-\pi s/\omega}\right)}{(s^2+\omega^2)\left(1-e^{-s\pi/\omega}\right)}$$

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Que 2 $f(t) = \begin{cases} t & 0 < t < \pi \\ \pi - t & \pi < t < 2\pi \end{cases}$

Draw the graph of $f(t)$ as periodic function and hence find L.T or Show that $L\{f(t)\} = \frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right)$

Sol

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt \quad \text{for periodic } f^n$$

Here $T = 2\pi$

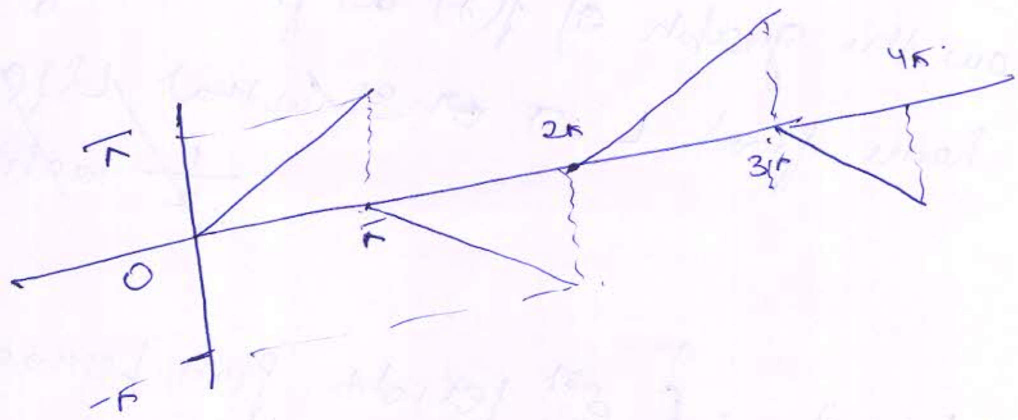
$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} f(t)e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} t e^{-st} + \int_{\pi}^{2\pi}(\pi-t)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\left(\frac{te^{-st}}{-s} - \frac{e^{-st}}{(-s)^2}\right)_0^{\pi} + \left(\frac{(\pi-t)e^{-st}}{-s} - \frac{(-1)e^{-st}}{(-s)^2}\right)_{\pi}^{2\pi}\right.$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{\pi e^{-\pi s}}{-s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2}\right]$$

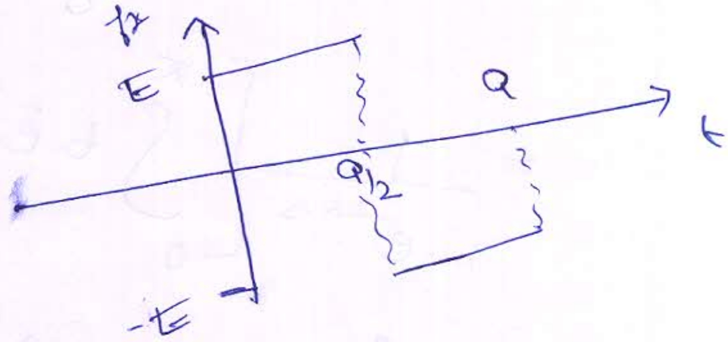
$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left(e^{-2\pi s} - 2e^{-\pi s} + 1\right)\right]$$



Que Given $f(t) = \begin{cases} E & 0 < t < a/2 \\ -E & a/2 < t < a \end{cases}$

Show then $L\{f(t)\} = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$

Sol



For periodic function

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \left(\int_0^{a/2} E e^{-st}\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt \right)$$

$$= \frac{E}{1-e^{-as}\, s} \left[-e^{-as/2} + 1 + e^{-as} - e^{-as/2} \right]$$

$$= \frac{E}{s(1-e^{-as})} \left(1 - 2e^{-as/2} + e^{-as}\right)$$

$$= \frac{E \left(1-e^{-as/2}\right)^2}{s\left(1-e^{-as}\right)} = \frac{E\left(1-e^{-as/2}\right)^2}{s\left(1-e^{-as/2}\right)\left(1+e^{-as/2}\right)}$$

$$= \frac{E}{s} \left(\frac{1-e^{-as/2}}{1+e^{-as/2}} \right),$$

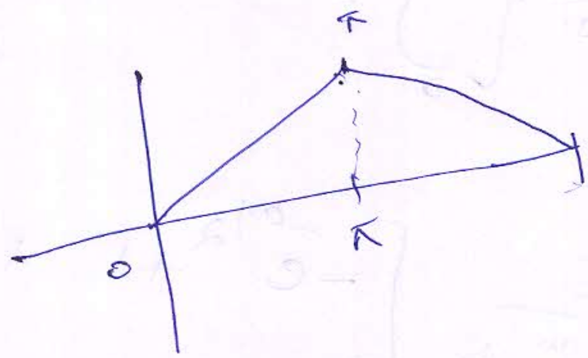
Multiplying Num and denom. by $e^{as/4}$

$$= \frac{E}{s} \left(\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}} \right) = \frac{E}{s} \tanh\left(\frac{as}{4}\right)$$

Ans

Que If $f(t) = \begin{cases} t & , 0 < t < \pi \\ 2\pi - t & \pi < t < 2\pi \end{cases}$

Show that $L\{f(t)\} = \frac{1}{s^2} \tanh\left(\frac{\pi s}{2}\right)$



Change of scale property

If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Proof $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s)$

$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$

Put $at = u$
$dt = \frac{du}{a}$

$= \int_0^\infty e^{-su/a} \cdot f(u) \frac{du}{a}$

$= \frac{1}{a} \int_0^\infty e^{-(s/a)u} f(u)\, du = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

UNIT STEP FUNCTION

Def^n : The unit step function $u(t-a)$ or Heaviside function $H(t-a)$ is

$$u(t-a) \text{ or } H(t-a) = \begin{cases} 0 , & t \leq a \\ 1 , & t > a. \end{cases}$$

Laplace Transform of unit step function

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-as} \cdot 1\, dt$$

$$\boxed{L\{u(t-a)\} = \frac{e^{-as}}{s}}$$

Particularly when $a = 0$

$$\boxed{L\{u(t)\} = \frac{1}{s}} \qquad u(t-0) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

Second Shifting property

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,\bar{f}(s).$$

OR

$$L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

Proof

~~L{f(t+a~~ $u(t-a) = \begin{cases} 0, & t \le a \\ 1 & t > a \end{cases}$

$$f(t-a)\,u(t-a) = \begin{cases} 0, & t \le a \\ f(t-a) & t > a \end{cases}$$

$$L\{f(t-a)\,u(t-a)\} = \int_0^a 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t - a = v$

$dt = dv$

when $t = a,\ v = 0$

$t = \infty,\ v = \infty$

$$= \int_0^\infty e^{-s(v+a)} f(v)\,dv$$

$$= e^{-as}\int_0^\infty e^{-sv} f(v)\,dv$$

$$= e^{-as}\bar{f}(s) = e^{-as}L\{f(t)\}$$

$$L\{f(t-a)u(t-a)\} = e^{-as}\bar{f}(s) = e^{-as}L\{f(t)\}$$

Que Find $L\{e^{t-2}u(t-2)\}$

Sol $L\{f(t-2)u(t-2)\} = e^{-2s}\bar{f}(s)$

$$= e^{-2s}L\{f(t)\}$$

$$f(t-2) = e^{t-2}$$

$$f(t) = e^{t}$$

$$L\{f(t)\} = \frac{1}{s-1} = \bar{f}(s)$$

$$\therefore L\{e^{t-2}u(t-2)\} = e^{-2s}\frac{1}{s-1}$$

Que 2 $L\{\{e^{t-1}+\sin(t-1)\}u(t-1)\}$

Sol $L\{e^{t-1}u(t-1)\} + L\{\sin(t-1)u(t-1)\}$

$$= e^{-s}L\{e^{t}\} + e^{-s}L\{\sin t\}$$

$$= e^{-s}\left\{\frac{1}{s-1} + \frac{1}{s^2+1}\right\}$$

Qus $L\{(t^2+2t-1)\,u(t-3)\}$

Sol $L\{f(t-3)\,u(t-3)\}$

$f(t-3) = t^2+2t-1$ Replacing $t \to t+3$

$f(t) = (t+3)^2 + 2(t+3) - 1$

$= t^2 + 8t + 14.$

$L\{f(t)\} = L\{t^2+8t+14\}$

$= \frac{2!}{s^3} + \frac{8}{s^2} + \frac{14}{s} = F(s)$

$L\{f(t^2+2t-1)u(t-3)\} = e^{-3s}\left(\frac{2}{s^3}+\frac{8}{s^2}+\frac{14}{s}\right)$

Qus $L\{(\sin t + \cos t)\,u(t-\frac{\pi}{2})\}$

Sol $f(t-\frac{\pi}{2}) = \sin t + \cos t$

$f(t) = \sin(t+\frac{\pi}{2}) + \cos(t+\frac{\pi}{2})$

$= \cos t - \sin t$

$\therefore L\{(\sin t + \cos t)\,u(t-\frac{\pi}{2})\} = e^{-\frac{\pi}{2}s}\, L\{\cos t - \sin t\}$

$= e^{-\frac{\pi}{2}s}\left(\frac{s-1}{s^2+1}\right)$

Exercise

Que $L\{(t^2-6t+9)e^{-(t-3)}u(t-3)\}$

$= L\{(t-3)^2 e^{-(t-3)} u(t-3)\}$

$f(t) = t^2 e^{-t}$

$L\{f(t)\} = \frac{2}{(s+1)^3}$

$\therefore L\{\quad\} = e^{-3s} \frac{2}{(s+1)^3}$ Ans

To express discontinuous function $f(t)$ in terms of unit step function

(1) $f(t) = \begin{cases} f_1(t) & t \le a \\ f_2(t) & t > a \end{cases}$

$f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a)$

OR

$= f_1(t)[u(t-a) - u(t-a)] + f_2(t)[u(t-a)]$

(2) $f(t) = \begin{cases} f_1(t) & t \le a \\ f_2(t) & a < t \le b \\ f_3(t) & t > b \end{cases}$

$f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a)$
$+ [f_3(t) - f_2(t)]\, u(t-b)$

(3) $f(t) = \begin{cases} f_1(t) & a < t < b \\ f_2(t) & b < t < c \end{cases}$

Sol: $f(t) = f_1(t)\, u(t-a) + \left(f_2(t) - f_1(t)\right) u(t-b)$
$- f_2(t)\, u(t-c)$

OR

$f(t) = f_1(t)\left[u(t-a) - u(t-b) + f_2(t)\middle| u(t-b) - u(t-c)\right]$

Que Express the following function in term of Heaviside unit step function and hence find their L.T

Sol (a) $f(t) = \begin{cases} t^2 & 0 < t < 4 \\ 5 & t > 4 \end{cases}$

Sol (a) $f(t) = t^2 + (5-t^2)\, u(t-4)$

$L\{f(t)\} = L\{t^2\} + L\{(5-t^2)\, u(t+4)\}$

$L\{t^2\} = \frac{2}{s^3}$

$f(t-4) = 5 - t^2$

$f(t) = 5 - (t+4)^2 = -(t^2 + 8t + 11)$

$$\therefore L\{f(t)\} = \frac{2}{s^3} - e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} - \frac{11}{s}\right) \quad \text{Ans}$$

<u>Que</u> $f(t) = \begin{cases} \cos t & 0 \leq t \leq \pi \\ 1 & \pi < t \leq 2\pi \\ \sin t & t > 2\pi \end{cases}$

Sol $f(t) = \cos t + (1 - \cos t)\, u(t-\pi) + (\sin t - 1)\, u(t - 2\pi)$

$L\{f(t)\} = L\{\cos t\} + L\{(1-\cos t)\, u(t-\pi)\} + L\{(\sin t - 1)\, u(t-2\pi)\}$

Now $L\{(1-\cos t)\, u(t-\pi)\} =$

$$e^{-\pi s} L\{1 - \cos(t+\pi)\}$$

$$= e^{-\pi s} L\{1 + \cos t\}$$

$$= e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right)$$

$$L\{(\sin t - 1)\, u(t-2\pi)\} = e^{-2\pi s} L\{\sin(t+2\pi) - 1\}$$

$$= e^{-2\pi s} L\{\sin t - 1\}$$

$$= e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)$$

$$\therefore L\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)$$

Que 3 $f(t) = \begin{cases} t^2, & 1 < t \le 2 \\ 4t, & t > 2 \end{cases}$

Sol $f(t) = t^2 u(t-1) +$

Que $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ -t+3 & 2 < t < 3 \end{cases}$

Sol $f(t) = (t-1)\left[u(t-1) - u(t-2)\right] + (-t+3)\left(u(t-2) - u(t-3)\right)$

$= (t-1)u(t-1) + (-t+3-t+1)u(t-2) + (-t+3)\cdot u(t-3)$

$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$

$L\{f(t)\} = e^{-s}\cdot\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2}$

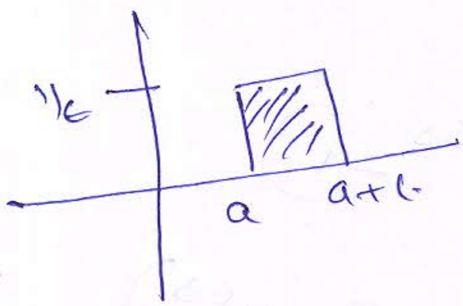
$= \left(e^{-s} - 2e^{-2s} + e^{-3s}\right)\frac{1}{s^2}$

UNIT IMPULSIVE FUNCTION

The unit impulsive function or Dirac delta function $\delta(t-a)$ is defined as follows:

$$\delta(t-a) = \lim_{\epsilon \to 0} \delta_\epsilon(t-a), \quad a \geq 0$$

where
$$\delta_\epsilon(t-a) = \begin{cases} 1/\epsilon & \text{if } a \leq t \leq a+\epsilon \\ 0 & \text{, otherwise} \end{cases}$$



[Used in earthquakes, hammer, where time is less and outcome is more.]

OR

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & , t \neq a \end{cases}$$

Such that
$$\int_0^\infty \delta(t-a)\,dt = 1 \quad (a \geq 0)$$

$\boxed{\text{The area is } 1}$

...sive function

$$L\{\delta_\epsilon(t-a)\} = \int_0^\infty e^{-st}\,\delta_\epsilon(t-a)\,dt$$

$$= \int_a^{a+\epsilon} e^{-st}\,\frac{1}{\epsilon}\,dt$$

$$= \frac{e^{-s(a+\epsilon)} - e^{-as}}{-s\epsilon}$$

$$= \frac{e^{-as}}{-s\epsilon}\left(e^{-\epsilon s} - 1\right)$$

$$\ldots\delta(t-a)\} = \lim_{\epsilon \to 0} \frac{e^{-as}}{s}\left(\frac{1-e^{-\epsilon s}}{\epsilon}\right) \quad 0/0 \text{ form}$$

$$= \frac{e^{-as}}{\cancel{s}}\;\frac{-e^{-\epsilon s}\cdot(-\cancel{s})}{1}$$

$$\boxed{\ldots-a) = e^{-as}}$$

$$\ldots\{\delta(t)\} = e^{0} = 1$$

$$\boxed{L\{\delta(t)\} = 1}$$

$$\cancel{L^{-1}\{\delta(t)\} = L^{-1}\{1}$$

$$\boxed{L^{-1}\{1\} = \delta(t)}$$

Result:-

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

Proof $\int_0^\infty f(t)\,\delta(t-a)\,dt = \lim_{\epsilon\to 0}\int_0^\infty f(t)\,\delta_\epsilon(t-a)\,dt$

$$= \lim_{\epsilon\to 0}\int_a^{a+\epsilon} f(t)\cdot\frac{1}{\epsilon}\,dt$$

$$= \frac{(a+\epsilon-a)}{\epsilon} f(c). \quad a \le c \le a+\epsilon$$

[By ~~Cauchy~~ Mean value thm
If f is continuous in $[a,b]$, then
$\int_a^b f(x)\,dx = (b-a) f(c)$, $a < c < b$]

$$= \lim_{\epsilon\to 0} f(c), \quad a \le c \le a+\epsilon$$

$$= f(a)$$

$$\boxed{\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)}$$

Que $L\{2\delta(t-1) + 4(\delta(t+3))\}$

Sol $2L\{\delta(t-1)\} + 4L\{\delta(t+3)\}$

$= 2e^{-s} + 4e^{3s}$

Que $L\{\sinh 3t\, \delta(t-2)\}$

$= L\{\frac{e^{3t}-e^{-3t}}{2}\delta(t-2)\}$

$= \frac{1}{2}L\{e^{3t}\delta(t-2) - e^{-3t}\delta(t-2)\}$

$= \frac{1}{2}\left[e^{-2(s-3)} - e^{-2(s+3)}\right]$ [By IST shifting prop]

$= \frac{1}{2}\, e^{-2s}\left(\frac{e^{6}-e^{-6}}{2}\right)$

Que $L\{(t+1)^2\delta(t-a)\}$

$= L\{t^2\delta(t-a)\} + L\{2t\,\delta(t-a)\} + L\{\delta(t-a)\}$

$= (-1)^2\frac{d^2}{ds^2}L\{\delta(t-a)\} + 2(-1)^1\frac{d}{ds}L\{\delta(t-a)\} + e^{-as}$

$= (-1)^2\frac{d^2}{ds^2}e^{-as} + 2(-1)\frac{d}{ds}e^{-as} + e^{-as}$

$= a^2 e^{-as} + 2a e^{-as} + e^{-as}$

$= (a+1)^2 e^{-as}$

OR

Ans $L\{(t+1)^2 \delta(t-a)\} = \int_0^\infty e^{-st}(t+1)^2 \delta(t-a)$

Now $\int_0^\infty f(t)\,\delta(t-a) = f(a)$.

$$\boxed{\therefore \int_0^\infty e^{-st}(t+1)^2 \delta(t-a) = e^{-as}(a+1)^2}$$

Que $L\left\{ \dfrac{2\left(\delta(t-1) + 6\,\delta(t-2)\right)}{t} \right\}$

$= \int_s^\infty 2e^{-s} + 6e^{-2s}\, ds$

$= \left. \dfrac{2e^{-s}}{-1} + \dfrac{6e^{-2s}}{-2} \right|_s^\infty$

$= 2e^{-s} + 3e^{-2s}$

Que $\int_0^\infty \sin 2t\, \delta\left(t - \frac{\pi}{4}\right) = \sin 2\frac{\pi}{4}$

$= \sin \frac{\pi}{2} = 1$

Que $L\{\sin 2t\, \delta(t-2)\}$

$= \int_0^\infty e^{-st} \sin 2t\, \delta(t-2)$

$= e^{-2s} \sin 4$

Inverse Laplace Transform

If $L\{f(t)\} = \bar{f}(s)$, then $f(t)$ is called inverse L.T of $\bar{f}(s)$ and is denoted by $L^{-1}\{\bar{f}(s)\}$.

Formulas

(1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$ (2) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(3) $L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = e^{at}\frac{t^n}{(n-1)!}$

(4) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma n}$ (5) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

(6) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$ (7) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

(8) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

(9) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = e^{at}\frac{1}{b}\sin bt$

(10) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

(11) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cosh bt$

Que $L^{-1}\left\{\frac{(s+2)^3}{s^6}\right\}$

Sol $L^{-1}\left\{\frac{s^3+8+6s^2+12s}{s^6}\right\}$

$= L^{-1}\left\{\frac{1}{s^3}\right\} + 8L^{-1}\left\{\frac{1}{s^6}\right\} + 6L^{-1}\left\{\frac{1}{s^4}\right\} + 12L^{-1}\left\{\frac{1}{s^5}\right\}$

$= \frac{t^2}{2!} + \frac{8t^5}{5!} + \frac{6t^3}{3!} + \frac{12t^4}{4!}$

Property

If $L\{f(t)\} = \bar{f}(s)$, then

$L\{f(t-a)\,u(t-a)\} = e^{-as}\bar{f}(s)$

$\therefore L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)\,u(t-a)$

Que $L^{-1}\left\{\frac{1+e^{-3s}}{s^2}\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}$

$= \frac{t}{1!} + L^{-1}\left\{e^{-3s}\frac{1}{s^2}\right\}$

$= \frac{t}{1!} + (t-3)\,u(t-3)$

Que $L^{-1}\left\{\frac{s+5}{s^2-6s+13}\right\}$

Sol $L^{-1}\left\{\frac{s+5}{s^2-6s+9+4}\right\}$

$$= L^{-1}\left\{\frac{s-3+8}{(s-3)^2+2^2}\right\}$$

$$= L^{-1}\left\{\frac{s-3}{(s-3)^2+2^2}\right\} + 8\,L^{-1}\left\{\frac{1}{(s-3)^2+2^2}\right\}$$

$$= e^{3t}\cos 2t + 8\,\frac{e^{3t}\sin 2t}{2}$$

$$= e^{3t}\left[\cos 2t + 4\sin 2t\right]$$

Qu $L^{-1}\left\{\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right\}$

Sol $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

On solving

$A = 1/2,\ B = -1,\ C = 5/2$

$$L^{-1}\left\{\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right\} = L^{-1}\left\{\frac{1/2}{s-1}\right\} + L^{-1}\left\{\frac{-1}{s-2}\right\} + \left\{\frac{5/2}{s-3}\right\}$$

$$= \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$$

Que $L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}$

Sol $= L^{-1}\left\{\frac{s}{(s^2+1)^2-s^2}\right\}$

$= L^{-1}\left\{\frac{s}{(s^2+1-s)(s^2+1+s)}\right\}$

$= \frac{1}{2} L^{-1}\left\{\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right\}$

$= \frac{1}{2} e^{\frac{1}{2}t} \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) - e^{-t/2} \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right)$

$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \ \sinh\left(\frac{t}{2}\right)$ Ans

Property

If $L\{f(t)\} = \bar{f}(s)$, i.e. $L^{-1}\{\bar{f}(s)\} = f(t)$.

then $L^{-1}\{-\bar{f}'(s)\} = t f(t)$

In general $L^{-1}\left\{\frac{d^n}{ds^n}\bar{f}(s)\right\} = (-1)^n t^n f(t)$

Que $L^{-1}\left\{ \log\left(\frac{s^2+1}{s(s+1)}\right)\right\}$

Sol Let $L^{-1}\left\{ \log(s^2+1) - \log s - \log(s+1)\right\} = f(t)$.

ie $L^{-1}\{ \bar{f}(s)\} = f(t)$

$$L^{-1}\left\{ \frac{d}{ds}\left(\log(s^2+1) - \log s - \log(s+1)\right)\right\} = -t f(t)$$

$$= L^{-1}\left\{ \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\} = -t f(t)$$

$$= \left(2\cos t - 1 - e^{-t}\right) = -t f(t)$$

$$f(t) = \frac{e^{-t} + 1 - 2\cos t}{t}.$$

Convolution thm

Defn of convolution. The convolution of two function $f(t)$ & $g(t)$ is defined as

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

Convolution thm

If $L^{-1}\{\bar{f}(s)\} = f(t)$, $L^{-1}\{\bar{g}(s)\} = g(t)$.

then $L^{-1}\{\bar{f}(s).\bar{g}(s)\} = f(t) * g(t)$

$$= \int_0^t f(u)\,g(t-u)\,du \text{ or}$$

$$= \int_0^t f(t-u)\,g(u)\,du$$

Proof We will show

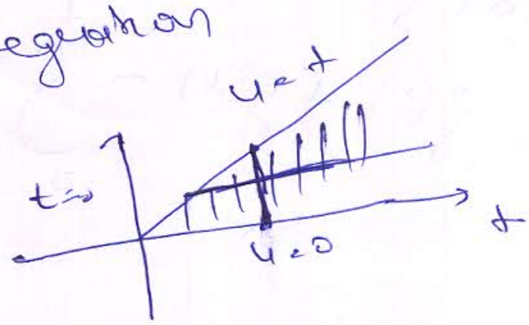
$$L\left\{\int_0^t f(u)\,g(t-u)\,du\right\} = \bar{f}(s).\bar{g}(s)$$

L.H.S

$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)\,g(t-u)\,du\right]dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt$$

Changing order of integration



$$= \int_0^{\infty}\int_u^{\infty} e^{-st} f(u) g(t-u)\, dt\, du$$

Put $t-u=v$

$dt = dv$

If $t=u,\ v=0 \,/\, t=\infty,\ v=\infty$

$$= \int_0^{\infty}\int_0^{\infty} e^{-s(u+v)} f(u) g(v)\, dv\, du$$

$$= \int_0^{\infty} e^{-su} f(u)\, du \int_0^{\infty} e^{-sv} g(v)\, dv$$

$$= \bar{f}(s)\, \bar{g}(s)$$

Hence Proved

Que Find $L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

Sol $\bar{f}(s) = \frac{1}{s^2+1}$, $\bar{g}(s) = \frac{1}{s^2+1}$

$L^{-1}\{\bar{f}(s)\} = \sin t = f(t)$, $L^{-1}\{\bar{g}(s)\} = \sin t = g(t)$

$$\therefore L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \int_0^t \sin u \sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left[\cos(2u-t) - \cos t\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(2u-t)}{2} - u\cos t\right]$$

$$= \frac{1}{2}\left[\sin t - t\cos t\right]$$

Que $L^{-1}\left\{\frac{s+1}{(s^2+2s+g)^2}\right\}$

$$= L^{-1}\left\{\frac{s+1}{(s^2+2s+2)(s^2+2s+2)}\right\}$$

$$g(s) = \frac{s+1}{(s+1)^2+1}, \quad f(s) = \frac{1}{s^2+2s+2}$$

$L^{-1}\{g(s)\} = e^{-t}\cos t = g(t)$, $L^{-1}\{f(s)\} = e^{-t}\sin t = f(t)$

$$L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\} = \int_0^t e^{-u}\sin u\, e^{-(t-u)}\cos(t-u)\,du$$

$$= \frac{e^{-t}}{2}\int_0^t \sin t + \sin(2u-t)\,du$$

$$= \frac{e^{-t}}{2}\left[t\sin t - \frac{\cos t}{2} + \frac{\cos t}{2}\right]$$

$$= \frac{te^{t}\sin t}{2}$$

Laplace Transform of derivatives

If $L\{f(t)\} = \bar{f}(s)$, then

$L\{f'(t)\} = sL\{f(t)\} - f(0)$

$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

Que Solve. $x''(t) - 2x'(t) + x = e^t$, $x(0) = 2$, $x'(0) = -1$, using L.T

Sol $x''(t) - 2x'(t) + x = e^t$

$L\{x''(t)\} - 2L\{x'(t)\} + L\{x\} = L\{e^t\}$

$$\left[s^2 L\{x\} - s\,x(0) - x'(0)\right] - 2\left(s\bar{x}(s) - x(0)\right) + L\{x(t)\} = \frac{1}{s-1}$$

$$(s^2 - 2s + 1)\,\bar{x}(s) - 2s - 5 = \frac{1}{s-1}$$

$$L\{x(t)\} = \frac{1}{(s-1)(s^2-2s+1)} + \frac{2s-5}{(s^2-2s+1)}$$

$$= \frac{1}{(s-1)^3} + \frac{2(s-1)-3}{(s-1)^2}$$

$$x(t) = e^t\left(\frac{t^2}{2!} + 2 - 3t\right)$$

Que Solve

$$\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t$$

$x = 2$, $y = 0$ when $t = 0$

Sol

$$x'(t) + y = \sin t$$

$$x + y'(t) = \cos t$$

Taking Laplace on b.s

$L\{x'(t)\} + L\{y\} = L\{\sin t\}$

$$s\,L\{x\} - x(0) + L\{y\} = \frac{1}{s^2+1}$$

$L\{x\} + L\{y'(t)\} = \cos t$

$$L\{x\} + sL\{y\} - y(0) = \frac{s}{s^2+1}$$

$$\therefore \quad s\bar{x}(s) + \bar{y}(s) = \frac{1}{s^2+1} + 2 \quad \text{---(1)}$$

$$\bar{x} + s\bar{y} = \frac{s}{s^2+1} \quad \text{---(2)}$$

Multiply ① by s and subtract ②

$$(s^2-1)\,\bar{x} = 2s$$

$$\bar{x} = \frac{2s}{s^2-1}$$

$$\boxed{x = L^{-1}\left\{\frac{2s}{s^2-1}\right\} = 2\cosh t}$$

$$y = \sin t - \frac{dx}{dt}$$

$$\boxed{y = \sin t - 2\sinh t}$$