SPARK

Date ______ Page ______

MODULE : 4.

LAPLACE TRANSFORM

- Let $f(t)$ be a real value of function defined for all $t > 0$, then laplace transform of $f(t)$ is denoted by $L[f(t)]$. It can be defined as $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \overline{f(s)}$ where s is a parameter

Laplace transform of some standard functions

$f(t) = a$ where a is constant.

1. By definition, $L[f(t)] = \int_0^\infty e^{-st} (f(t))\,dt$

$$L(a) = \int_0^\infty e^{-st} a\,dt$$

$$= a \left. \frac{e^{-st}}{-s} \right|_0^\infty$$

$$= \frac{a}{-s} \left[e^{-\infty} - e^{-0} \right]$$

$$= \frac{a}{-s} [0 - 1]$$

$$\boxed{L(a) = \frac{a}{s}}$$

$$\therefore L(1) = 1/s \quad ; \quad L(20) = 20/s.$$

2. $f(t) = e^{at}$

By definition, $L[f(t)] = \int_0^\infty e^{-st} (f(t))\,dt$.

$$= \int_0^\infty e^{-st} e^{at}\,dt.$$

$$= \int_0^\infty e^{-(s-a)t}\,dt.$$

$$= \left. \frac{e^{-(s-a)}}{-(s-a)} \right|_0^\infty$$

$$= \left[\frac{e^{-(\infty-a)}}{-(\infty-a)} - \frac{1}{-(s-a)} \right]$$

$$= \left[0 - \frac{1}{-(s-a)} \right]$$

$$\therefore \boxed{L[e^{at}] = \frac{1}{(s-a)}} \qquad \text{Similarly } \boxed{L[e^{-at}] = \frac{1}{s+a}}$$

3. $f(t) = \sinh at$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st} \sinh at \, dt$$

$$L(\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a) - (s-a)}{s^2 - a^2}\right]$$

$$L(\sinh at) = \frac{1}{2}\left[\frac{2a}{s^2 - a^2}\right]$$

$$\therefore \boxed{L(\sinh at) = \frac{a}{s^2 - a^2}}$$

4. $f(t) = \cosh at$

$$L[\cosh at] = \left[\frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

$$\boxed{L[\cosh at] = \frac{s}{s^2-a^2}}$$

5. $f(t) = \sin at.$

$$L[\sin at] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st} \sin at\,dt.$$

FOMULA :- $\int e^{at} \sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

FORMULA :- $\int e^{at} \cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$

$$= \int_0^\infty e^{-st} \sin at\,dt.$$

$$L[\sin at] = \frac{e^{-st}}{s^2+a^2}\left((-s)\sin at - a\cos at\right).\Big|_0^\infty$$

$$= \left[0 - \frac{1}{s^2+a^2}(0-a)\right]$$

$$\boxed{L[\sin at] = \frac{a}{s^2+a^2}}$$

6 $f(t) = \cos at$

$$L[\cos at] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cos at\, dt.$$

$$= \frac{e^{-st}}{s^2+a^2} \left((-s)\cos at + a \sin at \right) \Big|_0^\infty$$

$$= \left[0 - \frac{1}{s^2+a^2} \left[(-s) + 0 \right] \right]$$

$$\boxed{L[\cos at] = \frac{s}{s^2+a^2}}$$

* 7 $f(t) = t^n$ where n is a fraction and negative integer.

$$L(t^n) = \int_0^\infty e^{-st} t^n\, dt.$$

let $st = x$

$s\,dt = dx$.

if $t = \infty$; $x = \infty$

if $t = 0$; $x = 0$.

$$L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx.$$

$$\boxed{\therefore L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}}$$

SPARK Date ____ Page ____

$\Gamma(1/3)\,\Gamma(2/3) = 2\pi$

NOTE :- $\Gamma(n+1) = n\Gamma(n)$

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(3/2) = \sqrt{\pi}/2$

$\Gamma(5/2) = 3\sqrt{\pi}/4$

$\Gamma(7/2) = 15\sqrt{\pi}/8$

$\Gamma(-1/2) = -2\sqrt{\pi}$

* 8. $L(t^n) = f(t)$ where n is a positive integer.

$$L(t^n) = \int_0^\infty \underset{II}{e^{-st}}\, \underset{I}{t^n}\, dt$$

Integrate By parts

$$\int uv = u\int v\,dx - \int\left(\frac{d}{dx}u\int v\,dx\right)dx$$

$$= t^n \frac{e^{-st}}{-st}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-st}\, n t^{n-1}\, dt$$

$$= 0 + \frac{n}{s}\int_0^\infty \frac{e^{-st}}{t}\, t^{n-1}$$

$$L(t^n) = \frac{n}{s} L(t^{n-1})$$

$$L(t^{n-1}) = \frac{(n-1)}{s} L(t^{n-2})$$

$$L(t^{n-2}) = \frac{(n-2)}{s} L(t^{n-3}) \ldots\ldots$$

$$L(t^2) = \frac{2}{s} L[t^1]$$

$$L(t^1) = \frac{1}{s} L[t^0]$$

$$\therefore L[t^n] = \left[\frac{n}{s}\,\frac{(n-1)}{s}\,\frac{(n-1)}{s}\ldots\ldots\frac{2}{s}\,\frac{1}{s}\,L(t^0)\right]$$

$\int_0^\infty e^{-st} t^n dt$

$t^n \frac{e^{-st}}{-s}\Big|_0^\infty - \int n t^{n-1}\frac{e^{-st}}{-s}$

$0 + \frac{n}{s}\int t^{n-1} e^{-st}$

$L[t^n] = \frac{n}{s} L[t^{n-1}]$

$L[t^{n-1}] = \frac{(n-1)}{s} L[t^{n-2}]$

$L[t^{n-2}] = \frac{(n-2)}{s} L[t^{n-3}]$

$L[2] = \frac{2}{s} L[t^1]$

$\frac{n!}{s^n}\; L[1] = \frac{1}{s}\; \frac{n!}{s^n}$

$L[t^n] = \int_0^\infty e^{-st} f(t) = d\,t^n$

$st = x$

$s\,dt = dx$

$\int_0^\infty e^{-x} \frac{dx}{s}\left(\frac{x}{s}\right)^n$

$\frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n dx$

$\frac{\Gamma(n+1)}{s^{n+1}}$

SPARK

Date ______ Page ______

Find the Laplace transform.

1. $L\left[e^{2t} + 6e^{-6t} + 7e^{t}\right]$

$= \frac{1}{s-2} + \frac{6}{s+6} + \frac{7}{s-1}$

2. $f(t) = (2t+1)^2 + \cosh 3t$

$f(t) = 4t^2 + 1 + 4t + \frac{e^{3t} + e^{3t}}{2}$

$= \frac{4 \cdot 2!}{s^{2+1}} + \frac{1}{s} + \frac{4 \cdot 1!}{s^{1+1}} + \frac{s}{s^2 - 9}$

$f(t) = \frac{4 \cdot 2!}{s^3} + \frac{1}{s} + \frac{4 \cdot 1!}{s^2} + \frac{s}{s^2 - 3^2}$

3. $f(t) = \cosh^2 3t + \cos^2 3t$

$= \left(\frac{e^{3t} + e^{-3t}}{2}\right)^2 + \frac{1 + \cos 6t}{2}$

$= \frac{1}{4}\left[e^{6t} + e^{-6t} + 2e^{3t} \cdot e^{-3t}\right] + \frac{1}{2} + \frac{\cos 6t}{2}$

$L[f t] = \frac{1}{4}\left[\frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right] + \frac{1}{2s} + \frac{1}{2}\left(\frac{s}{s^2+6^2}\right)$

4. $f(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3$

$f(t) = \left(t^{1/2} + t^{-1/2}\right)^3$

$= t^{3/2} + t^{-3/2} + 3t\, t^{-1/2} + 3t^{1/2} t^{-1}$

$= t^{3/2} + t^{-3/2} + 3t^{1/2} + 3t^{-1/2}$

$$L[f(t)] = \frac{\Gamma(3/2+1)}{s^{3/2+1}} + \frac{\Gamma(-3/2+1)}{s^{-3/2+1}} + \frac{\Gamma(1/2+1)}{s^{1/2+1}} + \frac{\Gamma(-1/2+1)}{s^{-1/2+1}}$$

$$= \frac{\Gamma(5/2)}{s^{5/2}} + \frac{\Gamma(-1/2)}{s^{-1/2}} + \frac{\Gamma(3/2)}{s^{3/2}} + \frac{\Gamma(1/2)}{s^{1/2}}$$

$$= \frac{3\pi(5/2)}{4\, s^{5/2}} + \frac{(-2\pi)}{s^{-1/2}} + \frac{\sqrt{\pi}}{2s^{3/2}} + \frac{\sqrt{\pi}}{s^{1/2}}$$

$$L[f(t)] = \frac{3\pi}{4s^{5/2}} - \frac{2\pi}{s^{-1/2}} + \frac{\sqrt{\pi}}{2s^{3/2}} + \frac{\sqrt{\pi}}{s^{1/2}}$$

5. $f(t) = \sin t \ \sin 2t \ \sin 3t$

$$\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$L[\sin t \ \sin 2t \ \sin 3t] = -\frac{L}{2}[\cos 3t - \cos t]\sin 3t$$

$$= -\frac{L}{2}[\cos 3t \sin 3t - \cos t \sin 3t]$$

$$= -\frac{1}{4}L\left[(\sin 6t - \sin 0) - (\sin 4t + \sin 2t)\right]$$

$$= -\frac{1}{4}L\left[\sin 6t - \sin 4t - \sin 2t\right]$$

$$L[\sin t \sin 2t \sin 3t] = \frac{1}{4}\left[\frac{6}{s^2+6^2} - \frac{4}{s^2+4^2} - \frac{2}{s^2+2^2}\right].$$

6. $L[\cos t \cos 2t \cos 3t]$

$$= \frac{L}{2}[\cos 3t + \cos t] \cos 3t$$

$$= \frac{L}{2}[\cos 3t \cos 3t + \cos t \cos 3t]$$

$$= \frac{1}{4} L[\cos 6t + \cos 0 + \cos 4t + \cos 2t]$$

$$= \frac{1}{4} L[\cos 6t + 1 + \cos 4t + \cos 2t]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+6^2} + \frac{1}{s} + \frac{s}{s^2+4^2} + \frac{s}{s^2+2^2}\right]$$

* 7. $L[\sin^3 3t]$

$$\sin^3 t = \frac{1}{4}[3\sin t - \sin 3t]$$

$$\cos^3 t = \frac{1}{4}[3\cos t + \cos 3t]$$

$$= L[\sin^3 3t]$$

$$= L\frac{1}{4}[3\sin 3t - \sin 9t].$$

$$= \frac{1}{4}\left[\frac{3 \cdot 3}{s^2+9} - \frac{9}{s^2+9^2}\right]$$

$$= \frac{1}{4}\left[\frac{9}{s^2+9} - \frac{9}{s^2+81}\right]$$

8. $L[e^{2t}\cosh 3t]$

$$= L\left[e^{2t}\left(\frac{e^{3t}+e^{-3t}}{2}\right)\right]$$

$$= \frac{1}{2}L\left[e^{2t}e^{3t} + e^{2t}e^{-3t}\right]$$

$$= \frac{1}{2}L\left[e^{5t} + e^{-t}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-5} + \frac{1}{s+1}\right]$$

9. $L[\sin^2 t \cosh 2t]$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$$= L\left[\frac{1+\cos 2t}{2}\cdot\left(\frac{e^{2t}+e^{-2t}}{2}\right)\right]$$

$$= \frac{1}{4}L\left[e^{2t}+e^{-2t} - e^{2t}\cos 2t - e^{-2t}\cos 2t\right]$$

$$= \frac{1}{4}L\left[e^{2t}+e^{-2t} - F(s-2) - F(s+2)\right] \quad (\because \text{property})$$

here $F(t) = \cos 2t$

$F(s) = \frac{s}{s^2+a^2}$

$f(s-2) = \frac{(s-2)}{(s-2)^2+2^2}$

$f(t) = \cos 2t$

$f(s) = \frac{s}{(s^2+2^2)}$

$f(s+2) = \frac{(s+2)}{(s+2)^2+2^2}$

$$= \frac{1}{4}\left[\frac{1}{s-2} + \frac{1}{s+2} - \frac{(s-2)}{(s-2)^2+2^2} - \frac{(s+2)}{(s+2)^2+2^2}\right]$$

Properties of Laplace transform

- $\boxed{\text{If } L[f(t)] = F(s) \text{ then } L[e^{at} f(t)] = F(s-a).}$ ↳ exponential value

10 $L[e^{2t}\cos^2 t]$

$= L\left[e^{2t}\left(\frac{1+\cos 2t}{2}\right)\right]$

$= \frac{1}{2}\left(L\left[e^{2t}\right] + L(e^{2t}\cos 2t)\right)$

$= \frac{1}{2}\left[\frac{1}{s-2} + F(s-2)\right].$

Here $f(t) = \cos 2t$

$F(s) = \frac{s}{s^2+2^2}.$

$f(s-2) = \frac{(s-2)}{(s-2)^2+2^2}.$

$= \frac{1}{2}\left[\frac{1}{(s-2)} + \frac{(s-2)}{(s-2)^2+2^2}\right]$

11 $L[e^{-t}(2\cos 5t - 3\sin 5t)]$

$= L[e^{-t}2\cos 5t - e^{-t}3\sin 5t]$

$= L[2e^{-t}\cos 5t] - L[3e^{-t}\sin 5t]$

$= 2F(s+1) - 3F(s+1).$

Het $f(t) = \cos 5t.$

$f(s) = \frac{s}{s^2+5^2}$

$f(s+1) = \frac{(s+1)}{(s+1)^2+5^2}$

$f(t) = \sin 5t$

$f(s) = \frac{5}{s^2+5^2}$

$f(s+1) = \frac{5}{(s+1)^2+5^2}.$

$= \frac{2(s+1)}{(s+1)^2+5^2} - \frac{3.5}{(s+1)^2+5^2}$

12. $L[e^{3t}\sin 5t \sin 3t]$

$= L\left[e^{3t} - \frac{1}{2}\left(\cos 8t - \cos 2t\right)\right]$

$= -\frac{1}{2} L\left[e^{3t}\cos 8t - e^{3t}\cos 2t\right].$

$= -\frac{1}{2} L\left[F_1(s-3) - F_2(s-3)\right]$

Here $f_1(t) = \cos 8t$ $\quad$ $f_2(t) = \cos 2t$

$f_1(s) = \frac{s}{s^2+8^2}$ $\quad$ $f_2(s) = \frac{s}{s^2+2^2}$

$f_1(s-3) = \frac{(s-3)}{(s^2-3)^2+8^2}$ $\quad$ $f_2(s-3) = \frac{(s-3)}{(s-3)^2+2^2}.$

$= -\frac{1}{2}\left[\frac{(s-3)}{(s-3)^2+8^2} - \frac{(s-3)}{(s-3)^2+2^2}\right]$

13 $L[\cosh t \sin^3 2t].$

$= L\left[\left(\frac{e^t+e^{-t}}{2}\right) \frac{1}{4}\left(3\sin 2t - \sin 6t\right)\right]$

$= \frac{1}{8} L\left[(e^t+e^{-t})(3\sin 2t - \sin 6t)\right]$

$= \frac{1}{8} L\left[3e^t\sin 2t + 3e^{-t}\sin 2t - e^t\sin 6t - e^{-t}\sin 6t\right]$

$$= \frac{1}{8} L \left[3 F_1(s-1) + 3 F_2(s+1) - F_3(s-1) - F_4(s+1) \right].$$

Here $f_1(t) = \sin 2t$.

$$f_1(s) = \frac{2}{s^2+2^2}.$$

$$f_1(s-1) = \frac{2}{(s-1)^2+2^2}$$

$$f_2(s+1) = \frac{2}{(s+1)^2+2^2}.$$

$f_3(t) = \sin 6t$

$$f_3(s) = \frac{6}{s^2+6^2}$$

$$f_3(s-1) = \frac{6}{(s-1)^2+6^2}$$

$$f_4(s+1) = \frac{6}{(s+1)^2+6^2}$$

$$= \frac{1}{8} \left[3 \cdot \frac{2}{(s-1)^2+2^2} + 3 \cdot \frac{2}{(s+1)^2+2^2} - \frac{6}{(s-1)^2+6^2} - \frac{6}{(s+1)^2+6^2} \right]$$

Q. 14. $L\left[\sqrt{t}\, e^{4t}\right]$

$$= L\left[e^{4t} \sqrt{t}\right]$$

$$= F(s-4).$$

Here $f(t) = \sqrt{t}$.

$$f(t) = t^{1/2}.$$

$$f(s) = \frac{\sqrt{(1/2+1)}}{s^{1/2+1}}$$

$$f(s) = \frac{\sqrt{(3/2)}}{s^{3/2}}$$

$$f(s) = \frac{\sqrt{\pi}}{2 s^{3/2}}$$

$$f(s) = \frac{\sqrt{\pi}}{2 s^{3/2}}$$

$$f(s-4) = \frac{\sqrt{\pi}}{2(s-4)^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2(s-4)^{3/2}}$$

15. $L[e^{-5t}\sin^2 3t]$

$$= L\left[e^{-5t}\left(\frac{1-\cos 6t}{2}\right)\right]$$

$$= \frac{1}{2} L\left[e^{-5t} - e^{-5t}\cos 6t\right]$$

$$= \frac{1}{2} L\left[e^{-5t} - F(s+5)\right]$$

Here $f(t) = \cos 6t$

$$f(s) = \frac{s}{s^2+6^2}$$

$$f(s+5) = \frac{(s+5)}{(s+5)^2+6^2}.$$

$$= \frac{1}{2}\left[\frac{1}{s+5} - \frac{(s+5)}{(s+5)^2+6^2}\right]$$

(b) $\boxed{\text{If } L[f(t)] = F(s) \text{ then } L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]}$

16. $L[t\cos at]$

$$= (-1)^1 \frac{d}{ds}[F(s)].$$

But $f(t) = \cos at$

$$f(s) = \frac{s}{s^2+a^2}.$$

$$= (-1)\frac{d}{s}\left[\frac{s}{s^2+a^2}\right]$$

SPARK

Date____ Page____

$$= (-1).\left[\frac{(s^2+a^2) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= (-1)\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= (-1)\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

17. $L[t^2 \sin at]$

Here $f(t) = \sin at$

$f(s) = \frac{a}{s^2+a^2}$

$$= (-1)^2 \frac{d^2}{ds^2}\left[\frac{a}{(s^2+a^2)}\right]$$

$$= (-1)^2 \cdot \frac{d}{ds}\left[\frac{(s^2+a^2)(0) - a(2s)}{(s^2+a^2)^2}\right]$$

$$= (-1)^2 \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$$

$$= (-1)^2\left[\frac{(s^2+a^2)^2(-2a) - (-2as)\,2(s^2+a^2)2s}{(s^2+a^2)^4}\right]$$

$$= 1\left[\frac{-2a(s^2+a^2)^2 + 8as^2(s^2+a^2)}{(s^2+a^2)^4}\right]$$

$$= \cancel{(s^2+a^2)}\left[\frac{-2a(s^2+a^2) + 8as^2}{(s^2+a^2)^{\cancel{4}}}\right]$$

$$= \frac{8as^2 - 2a(s^2+a^2)}{(s^2+a^2)^3}$$ OR $8as^2 - 2as^2 - 2a^3$ or $6as^2 - 2a^3$

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18. $L[t^3 \cos at]$

Here $f(t) = \cos at$

$$f(s) = \frac{s}{s^2+a^2}$$

$$= (-1)^3 \frac{d^3}{ds^3}\left[\frac{s}{s^2+a^2}\right].$$

$$= -1 \frac{d^2}{ds^2}\left[\frac{(s^2+a^2) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= -1 \frac{d^2}{ds^2}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= -1 \frac{d^2}{ds^2}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= -1 \frac{d}{ds}\left[\frac{(s^2+a^2)^2(-2s) - (a^2-s^2)2(s^2+a^2)(2s)}{(s^2+a^2)^4}\right]$$

$$= -1 \frac{d}{ds}\left[\frac{-2s(s^2+a^2)^2 - 4s(a^2-s^2)(s^2+a^2)}{(s^2+a^2)^4}\right]$$

$$= -1 \frac{d}{ds} \cancel{(s^2+a^2)}\left[\frac{-2s(s^2+a^2) - 4s(a^2-s^2)}{(s^2+a^2)^{\cancel{4}}}\right]$$

$$= -1 \frac{d}{ds}\left[\frac{-2s^3 - 2a^2s - 4sa^2 + 4s^3}{(s^2+a^2)^3}\right]$$

$$= -1 \frac{d}{ds}\left[\frac{2s^3 - 6a^2s}{(s^2+a^2)^3}\right].$$

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$$= (-1) \left[\frac{(s^2+a^2)^3 (6s^2-6a^2) - (2s^3-6a^2s)\, 3(s^2+a^2)^2 \cdot 2s}{(s^2+a^2)^6} \right]$$

$$= (-1) \cancel{(s^2+a^2)^2} \left[\frac{(s^2+a^2)(6s^2-6a^2) - (2s^3-6a^2s)\, 6s}{\cancel{(s^2+a^2)^6}} \right]$$

$$= (-1) \left[\frac{(s^2+a^2)(6s^2-6a^2) - 12s^4 + 36a^2s}{(s^2+a^2)^4} \right]$$

$$= (-1) \left[\frac{6s^4 - \cancel{6s^2a^2} + \cancel{6a^2s^2} - 6a^4 - 12s^4 + 36a^2s^2}{(s^2+a^2)^4} \right]$$

$$= (-1) \left[\frac{6s^4 - 6a^4 - 12s^4 + 36a^2s^2}{(s^2+a^2)^4} \right]$$

$$= (-1) \left[\frac{36a^2s^2 - 6a^4 - 6s^4}{(s^2+a^2)^4} \right]$$

$$= \frac{6a^4 + 6s^4 - 36a^2s^2}{(s^2+a^2)^4}$$

19 $L[t^2 \cos 3t]$

Here $f(t) = \cos 3t$

$$f(s) = \frac{s}{s^2+9}.$$

$$= (-1)^2 \frac{d^2}{ds^2} \left[\frac{s}{s^2+9} \right].$$

$$= \frac{d}{ds} \left[\frac{(s^2+9) - s(2s)}{(s^2+9)^2} \right]$$

$$= \frac{d}{ds}\left[\frac{s^2+9-2s^2}{(s^2+9)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{9-s^2}{(s^2+9)^2}\right]$$

$$= \left[\frac{-(s^2+9)^2(2s) - (9-s^2)\,2(s^2+9)(2s)}{(s^2+9)^2}\right]$$

$$= \left[\frac{-2s(s^2+9)^2 - 4s(9-s^2)(s^2+9)}{(s^2+9)^2}\right]$$

$$= \cancel{(s^2+9)}\left[\frac{-2s(s^2+9) - 4s(9-s^2)}{\cancel{(s^2+9)}^4}\right]$$

$$= \left[\frac{-2s^3-18s-36s+4s^3}{(s^2+9)^3}\right]$$

$$= \frac{2s^3-54s}{(s^2+9)^3}$$

20. $L[t^3 \cos 4t]$

Here $f(t) = \cos 4t$

$$f(s) = \frac{s}{s^2+16}.$$

$$= \frac{6(4)^4+6s^4-36(16)^2s^2}{(s^2+16)^4}$$ (∵ From Prob No 18)

$$= \frac{1536+6s^4-1296s^2}{(s^2+16)^4}$$

$\bullet$ $\boxed{\text{If } L[f(t)] = F(s) \text{ then } L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds.}$

Q1. $L\left[\frac{e^{at}-e^{bt}}{t}\right]$

Here $f(t) = e^{at}-e^{bt}$.

$f(s) = \frac{1}{s-a} - \frac{1}{s-b}$

$\Rightarrow L\left[\frac{e^{at}-e^{bt}}{t}\right] = \int_s^\infty \frac{1}{s-a} - \frac{1}{s-b}\,ds.$

$= \log(s-a) - \log(s-b)\Big|_s^\infty$ $\qquad \log 1 = 0$

$= \log(s-a) \div \log(s-b)$ $\qquad \log\infty = 0.$

$= \log\left(\frac{s-a}{s-b}\right)\Big|_s^\infty$

$= \log\left(\frac{\infty-a}{\infty-b}\right) - \log\left(\frac{s-a}{s-b}\right)$

$= 0 - \log\left(\frac{s-a}{s-b}\right)$

$L\left[\frac{e^{at}-e^{bt}}{t}\right] = \log\left(\frac{s-b}{s-a}\right)$

Q2. $L\left[\frac{\cos at - \cos bt}{t}\right]$

Here $f(t) = \cos at - \cos bt$

$f(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$s^2+a^2 = u.$

$= \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\,ds.$ $\qquad 2s\,ds = du.$

$s\,ds = \frac{du}{2}.$

UL :- ∞; $\rightarrow$ dont chang the limits

LL :- s;

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FORMULA :- $\int \frac{2x}{x^2+a^2} = \log(x^2+a^2)$.

$$= \int_s^\infty \frac{s}{s^2+a^2}\, ds - \int_s^\infty \frac{s}{s^2+b^2}\, ds.$$

Multiply & divide by 2

$$= \frac{1}{2} \int_s^\infty \frac{2s}{s^2+a^2}\, ds - \int_s^\infty \frac{2s}{s^2+b^2}\, ds$$

$$\frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$\frac{1}{2} \log\left(\frac{s^2+a^2}{s^2+b^2}\right)_s^\infty \rightarrow \frac{1}{2} \log\left[0 - \frac{s^2+a^2}{s^2+b^2}\right]$$

$$= -\frac{1}{2} \log\left(\frac{s^2+a^2}{s^2+b^2}\right) \text{ or } \frac{1}{2} \log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

23 $L\left[\frac{\sin at}{t}\right]$

Here $f(t) = \sin at$

$f(s) = \frac{a}{s^2+a^2}$

$$= \int_s^\infty \frac{a}{s^2+a^2}\, ds$$

$\int \tan^{-1} x = \frac{1}{1+x^2}$.

$= a \log(s^2+a^2)$

$$= \cancel{a} \cdot \frac{1}{\cancel{a}} \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

$= \tan^{-1}\infty - \tan^{-1}(s/a)$.

$\pi/2 - \tan^{-1}(s/a)$

24 $L\left[\frac{\sin^2 t}{t}\right]$

$f(t) = \cancel{\cos^2 t}\ \sin^2 t$

$f(t) = \frac{1-\cos 2t}{2}.$

$f(s) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+4}\ ds.$

$= \frac{1}{2}\left[\int_s^\infty \frac{1}{s} - \int_s^\infty \frac{s}{s^2+4}\right] ds.$

$= \frac{1}{2}\left[\log s\Big|_s^\infty - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+2^2}\right]$

$= \frac{1}{2}\log s\Big|_s^\infty - \frac{1}{4}\log(s^2+4).\Big|_s^\infty$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

$= \frac{1}{2}\left[\log s - \log(s^2+4)^{1/2}\right]_s^\infty$

$= \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)\Big|_s^\infty$

$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$

25 $L\left[\frac{2\sin t \ \sin 3t}{t}\right]$

$$f(t) = \frac{2\sin t \ \sin 3t}{t}.$$

$$f(t) = 2\left[-\frac{1}{2}\left[\cos(4t) - \cos(2t)\right]\right]$$

$$f(t) = -\left[\cos 4t - \cos 2t\right]$$

$$f(t) = -\left[\cos 4t - \cos 2t\right]$$

$$f(s) = +\frac{s}{s^2+4} - \frac{s}{s^2+16}$$

$$= \int_0^\infty \frac{s}{s^2+4} - \frac{s}{s^2+16}\, ds$$

Multiply and ÷ by 2.

$$= \frac{1}{2}\int_0^\infty \frac{2s}{s^2+4} - \frac{2s}{s^2+16}.$$

$$= \frac{1}{2}\left[\log(s^2+4) - \log(s^2+16)\right]_0^\infty$$

$$= \frac{1}{2}\log\frac{(s^2+4)}{(s^2+16)}\Bigg|_0^\infty$$

$$= \frac{1}{2}\left[\log\frac{(s^2+16)}{(s^2+4)}\right]$$

26 Laplace Transform of periodic function

If $f(t)$ is said to be periodic function of period $T>0$ then $f(t+nt) = f(t)$.
where $n = 1, 2, 3 \ldots$

Ex:- $\cos\theta$ and $\sin\theta$ are two periodic function of the period

6m * Theorem :-

$$L\left[f(t)\right] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

then $f(t)$ is a periodic function of $T>0$.

Proof :-

W.K.T $L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt$ ($\because$ By definition).

$$= \int_0^\infty e^{-su} f(u)\, du.$$

$$= \int_0^T e^{-su} f(u)\, du + \int_T^{2T} e^{-su} f(u)\, du + \int_{2T}^{3T} e^{-su} f(u)\, d$$

$$+ \ldots + \int_{nT}^{(n+1)T} e^{-su} f(u)\, du + \ldots$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-su} f(u)\, dt$$

put $u = t + nT$

$du = dt$.

L.L if $u = nT$; $nT = t + nT$

$\boxed{t = 0}$

U.L if $u = (n+1)T$; $(n+1)T = t + nT$

$nT + T = t + nT$

$\boxed{t = T}$

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$$= \sum_{n=0}^{\infty} \int_0^T e^{-s(t+nT)} f(t+nT)\, dt.$$

Since $f(t)$ is periodic function ; $f(t+nT) = f(t)$.

$$= \sum_{n=0}^{\infty} \int_0^T e^{-st} e^{-snT} f(t)\, dt$$

$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)\, dt.$$

$$\sum_{n=0}^{\infty} e^{-snT} = 1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots$$

$$= 1 + e^{-sT} + (e^{-sT})^2 + (e^{-sT})^3 + \cdots$$

$$= a + ar + ar^2 + ar^3 + \cdots$$

$$= \frac{a}{1-r} \quad \text{(since sum to infinity by GS)}$$

$$\therefore \sum_{n=0}^{\infty} e^{-snT} = \frac{1}{1-e^{-sT}}$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

1. Given $f(t) = t^2$, $0 < t < 2$ & $f(t+2) = f(t)$ then find $L[f(t)]$ (periodic fun)

The given function is a periodic function with period $T = 2$

Where $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} \, t^2 \, dt$$

$$= \frac{1}{1-e^{-2s}} \left[-t^2 \frac{e^{-st}}{s} - 2t\frac{e^{-st}}{s^2} + \frac{2e^{-st}}{-s^2} \right]_0^2$$

$$= \frac{1}{1-e^{-2s}} \left[-4\frac{e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{4} \right] - \left[0 + 0 - \frac{2}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{4} + \frac{2}{s^2} \right]$$

2. $f(t) = \begin{cases} E & 0 < t < a/2 \\ -E & a/2 < t < a. \end{cases}$ Given that $f(t+a) = f(t)$ show that $L[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{4}\right)$

Here $f(t+a) = f(t)$ which is a periodic function with $T = a$.

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} (f(t)) \, dt.$$

$$= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t) \, dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} f(t) \, dt + \int_{a/2}^a e^{-st} f(t) \, dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} E \, dt + \int_{a/2}^a e^{-st} (-E) \, dt \right]$$

$$= \frac{E}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} \, dt - \int_{a/2}^a e^{-st} \, dt \right]$$

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$$= \frac{E}{1-e^{-as}} \left[\frac{e^{-st}}{-s}\Big|_0^{a/2} - \frac{e^{-st}}{-s}\Big|_{a/2}^{a} \right]$$

$$= \frac{-E}{s(1-e^{-as})} \left[e^{-st}\Big|_0^{a/2} - e^{-st}\Big|_{a/2}^{a} \right]$$

$$= \frac{-E}{s(1-e^{-as})} \left[(e^{-sa/2} - 1) - (e^{-as} - e^{-sa/2}) \right]$$

$$= \frac{-E}{s(1-e^{-as})} \left[e^{-sa/2} - 1 - e^{-as} + e^{-sa/2} \right]$$

$$= \frac{-E}{s(1-e^{-as})} \left[2e^{-sa/2} - 1 - e^{-as} \right] \quad (\times \text{ by } -).$$

$$= \frac{E}{s(1-e^{-as})} \left[e^{-as} + 1 - 2e^{-sa/2} \right]$$

$(a-b)^2 = a^2+b^2-2ab$

$= (e^{-as/2})^2 + 1^2 - 2e^{-as/2} \cdot 1$

$= e^{-2as/2} + 1^2 - 2e^{-as/2} \cdot 1$

$= e^{-as} + 1 - 2e^{-as/2}$

(multiply & ÷ by 2 for denominator).
$$= \frac{E}{s(1-e^{-as})} \left[(e^{-sa/2} - 1)^2 \right]$$

$$= \frac{E}{s(1^2 - (e^{-as/2})^2)} (1 - e^{-sa/2})^2.$$

$$= \frac{E}{s\,[1+e^{-as/2})\cancel{(1-e^{-as/2})}} \cancel{(1-e^{-as/2})^2}$$

$\frac{2as}{2}$

$$= \frac{E}{s} \frac{(1-e^{-sa/2})}{(1+e^{-sa/2})}. \quad \times \text{ by } \frac{e^{as/4}}{e^{as/4}}.$$

$$= \frac{E}{s} \frac{(e^{sa/4} - e^{-as/4})}{(e^{as/4} + e^{-as/4})}.$$

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$$= \frac{E}{s} \tanh\left(\frac{as}{4}\right)$$

3. $f(t) = \begin{cases} t & 0<t<a \\ 2a-t & a<t<2a \end{cases}$ given that $f(t+2a) = f(t)$

S.T $L[f(t)] = \frac{1}{s^2}\left[\tanh\left(\frac{as}{2}\right)\right]$

Its a periodic function of period $T = 2a$.

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{a} e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st}\, t\,dt + \int_0^{2a} e^{-st}(2a-t)\,dt.\right]$$

$\Rightarrow \int_0^a e^{-st}\, t\,dt = t\int e^{-st}dt - \int\left(\frac{d}{dx} t \int e^{-st}\right)dt$

$$= t\,\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}. \Rightarrow \boxed{t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}}$$

$\Rightarrow \int_0^{2a} e^{-st}(2a-t)\,dt = (2a-t)\frac{e^{-st}}{-s} - \int (-1).\frac{e^{-st}}{+s}\,dt$

$$= \boxed{(2a-t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2}}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^a + \left[\frac{(2a-t)e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right]_a^{2a}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}\right] + \left[0 + \frac{e^{-2as}}{s^2} - \left(\frac{ae^{-sa}}{-s} + \frac{e^{-sa}}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\cancel{\frac{-ae^{-as}}{s}} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \cancel{\frac{ae^{-sa}}{s}} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[-2e^{-as} + 1 + e^{-2as}\right].$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[1^2 - 2e^{-as}\cdot 1 + (e^{-as})^2\right]$$

$$= \frac{1}{s^2[(1^2)-(e^{-as})^2]}(1-e^{-as})^2$$

$$= \frac{1}{s^2[(1+e^{-as})\cancel{(1-e^{-as})}]}(1-e^{-as})^{\cancel{2}}$$

$$= \frac{1}{s^2}\frac{(1-e^{-as\times\frac{1}{2}})}{(1+e^{-as})} \times \frac{e^{as/2}}{e^{as/2}}$$

$$= \frac{1}{s^2}\left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right).$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

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Date______ Page______

7. $f(t) = \begin{cases} E\sin\omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$ given that $f(t+2\pi/\omega) = f(t)$ then find $L[f(t)]$

Here, its a periodic function of period $T = 2\pi/\omega$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-(2\pi/\omega)s}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} f(t)\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t)\,dt\right].$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} E\sin\omega t\,dt + \cancel{\int_{\pi/\omega}^{2\pi/\omega} e^{-st}\, 0\,dt}\right].$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t\,dt.$$

FORMULA $\int e^{at} \sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}.$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\left[-s\sin\omega(\pi/\omega) - \omega\cos\omega(\pi/\omega)\right]\right] - \left[\frac{1}{s^2+\omega^2}(-s\sin 0 - \omega\cos 0)\right].$$

$$= \frac{E}{(1)^2-(e^{-\pi s/\omega})^2} \left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}(0-\omega(-1))\right] - \left[\frac{1}{s^2+\omega^2}(0-\omega)\right]$$

$$= \frac{E\omega \; [e^{-s\pi/\omega} + 1]}{(1+e^{-s\pi/\omega})(1-e^{-s\pi/\omega})(s^2+\omega^2)}$$

$$= \frac{E\omega}{(s^2+\omega^2)(1-e^{-s\pi/\omega})}$$

5.

⇒ **Unit step function or Heaviside function.**

Its denoted by $U(t-a)$ or $H(t-a)$ and it can be defined as

$$U(t-a) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases}$$

Laplace of unit step function

By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$.

$$L[U(t-a)] = \int_0^\infty e^{-st} U(t-a)\, dt$$

$$= \int_0^a e^{-st} (0)\, dt + \int_a^\infty e^{-st} (1)\, dt.$$

$$= \int_a^\infty e^{-st}\, dt.$$

$$= \left. \frac{e^{-st}}{-s} \right|_a^\infty$$

$$= \left[0 + \frac{e^{-sa}}{s} \right]$$

$L[u(t+3)] = e^{3s}/s$.

$$\therefore \boxed{L[u(t-a)] = \frac{e^{-sa}}{s}}$$

Ex $L[u(t-2)] = e^{-2s}$

Properties

- $L[f(t-a) \cdot u(t-a)] = e^{-as} F(s)$. where $F(s) = L[f(t)]$.

- If $f(t) = \begin{cases} f_1(t) & 0<t<a \\ f_2(t) & t>a \end{cases}$ then we can express this function in terms of unit step function as

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a).$$

- If $f(t) = \begin{cases} f_1(t) & 0<t<a \\ f_2(t) & a<t<b \\ f_3(t) & t>b. \end{cases}$ then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$.

1. $L[(t^2+2t+1)\, u(t-1)]$.

$= e^{-1s} f(s)$.

Here $f(t-1) = t^2+2t+1 = (t+1)^2$

$f(t) = ?$

Replace t with $(t+1)$

$f(t+1-1) = (t+1+1)^2$.

$f(t) = (t+2)^2$

$f(t) = t^2+4+4t$

$f(s) = \dfrac{2!}{s^3} + \dfrac{4}{s} + \dfrac{4\,1!}{s^2}$

$\therefore e^{-s}\left[\dfrac{2}{s^3} + \dfrac{4}{s} + \dfrac{4}{s^2}\right]$

2. $L\left[(e^{t-1} + \sin(t-1))\, u(t-1) \right] = e^{-1s} f(s).$

$f(t-1) = e^{t-1} + \sin(t-1)$

Replace t with $t+1$.

$f(t) = e^{t} + \sin(t+1-1)$

$f(t) = e^{t} + \sin t$

$f(s) = \dfrac{1}{s-1} + \dfrac{1}{s^2+1^2}.$

$\therefore e^{-s}\left[\dfrac{1}{s-1} + \dfrac{1}{s^2+1^2} \right]$

3. $L\left[(t^3+t^2+t+1)\, u(t+1) \right] = e^{s} F(s)$

$(a-b)^3 = a^3 - b^3 - 3a^2b + 3b^2$

$f(t+1) = t^3 + t^2 + t + 1$

Replace t with $(t-1)$

$f(t) = (t-1)^3 + (t-1)^2 + (t-1) + 1$

$f(t) = t^3 - 1^3 - 3t^2 + 3t + t^2 + 1 - 2t + t - \cancel{1} + \cancel{1}$

$f(t) = t^3 - 2t^2 + 2t.$

$f(s) = \dfrac{3!}{s^4} - \dfrac{2.2!}{s^3} + \dfrac{2}{s^2}$

$\therefore e^{s}\left[\dfrac{6}{s^4} - \dfrac{4}{s^3} + \dfrac{2}{s^2} \right]$

Date ______ Page ______

4. $L[(\cos t + \cos 2t)u(t-\pi)] = e^{-s\pi}F(s)$.

$f(t-\pi) = \cos t + \cos 2t$

Replace t with $f(t+\pi)$.

$f(t) = \cos(t+\pi) + \cos 2(2t+\pi)$

$f(t) = -\cos t + \cos 2t$.

$f(t) = -\cos t + \cos 2t$

$f(s) = -\frac{s}{s^2+1^2} + \frac{s}{s^2+4}$

$= e^{-s\pi}\left[\frac{-s}{s^2+1} + \frac{s}{s^2+4}\right]$

SMQ 5. Express $f(t) = \begin{cases} 1 & \overset{0}{0} < t < \overset{a}{1} \\ t & \overset{a}{1} < t < \overset{b}{2} \\ t^2 & t > 2 \end{cases}$ in term of unit step function else find $L[f(t)]$.

$a = 1,\ b = 2$

$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

$f(t) = 1 + [t-1]u(t-a) + [t^2 - t]u(t-b)$

$f(t) = 1 + (t-1)u(t-1) + (t^2-t)u(t-2)$

$t^2+4+2t \cdot 2$

$L[f(t)] = \frac{1}{s} + e^{-s}F_1(s) + e^{-2s}F_2(s)$

- $e^{-s}F_1(s)$

$f(t-1) = t-1$

Replace t with $t+1$

$f(t) = t+1-1$

$f(t) = t$

$f(s) = \frac{1}{s^2}$

- $e^{-2s}F_2(s)$.

$f(t-2) = t^2 - t$.

Reple t with $t+2$.

$f(t) = (t+2)^2 - (t+2)$.

$f(t) = t^2 + 4 + 4t - t - 2$.

$f(t) = t^2 + 3t + 2$

$f(s) = \frac{2!}{s^3} + \frac{3 \cdot 1}{s^2} + \frac{2}{s}$.

$$L[f(t)] = \frac{1}{s} + e^{-s}\left[\frac{1}{s^2}\right] + \left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right] e^{-2s}$$

6. Express $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 1 & \pi < t < 2\pi \\ \sin t & t > 2\pi \end{cases}$

Here $a = \pi$; $b = 2\pi$

$f(t) = \cos t + (1 - \cos t)\, u(t - \pi) + (\sin t - 1)\, u(t - 2\pi)$.

$f(t) = \cos t + (1 - \cos t)\, u(t - \pi) + (\sin t - 1)\, u(t - 2\pi)$

$\rightarrow$ $L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s} F_1(s) + e^{-2\pi s} F_2(s)$.

- $f(t - \pi) = 1 - \cos t$.
  Replace t with $(t + \pi)$
  $f(t) = 1 - \cos(t + \pi)$
  $f(t) = 1 + \cos t$.
  $f(s) = \frac{1}{s} + \frac{s}{s^2+1}$

- $f(t - 2\pi) = \sin t - 1$
  Replace t with $(t + 2\pi)$
  $f(t) = \sin(t + 2\pi) - 1$
  $f(t) = \sin t - 1$
  $f(s) = \frac{1}{s^2+1} - \frac{1}{s}$

$\therefore$ $L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{1}{s} + \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s}\right]$

7. express $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi < t < 2\pi \\ \cos 3t & t > 2\pi \end{cases}$

Here $a = \pi$;
$b = 2\pi$;

Date______ Page______

$f(t) = \cos t + (\cos 2t - \cos t)\,u(t-\pi) + (\cos 3t - \cos 2t)\,u(t-2\pi)$

$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s} F_1(s) + e^{-2\pi s} F_2(s)$

- $f(t-\pi) = \cos 2t - \cos t$ $\qquad f(t+\pi-\pi)$

  Replace t with $(\pi+t)$

  $f(t) = \cos 2(\pi+t) - \cos(t+\pi)$

  $f(t) = \cos(2\pi + 2t) - \cos(\pi+t)$

  $f(t) = \cos 2t + \cos t$

  $f(s) = \frac{s}{s^2+4} + \frac{s}{s^2+1}$

- $f(t-2\pi) = \cos 3t - \cos 2t$

  Replace t by $(t+2\pi)$

  $f(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$

  $f(t) = \cos(6\pi + 3t) - \cos(2\pi + 2t)$

  $f(t) = \cos 3t - \cos 2t$

  $f(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$.

- $L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$

8 Express $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ \sin 2t & \pi < t < 2\pi \\ \cos 3t & t > 2\pi \end{cases}$

Similary to Prob 7^{th}.

$L[f(t)] = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right) - e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)$

Inverse Laplace Transform.

1. $L[a] = \frac{1}{s} \Rightarrow L^{-1}\left(\frac{1}{s}\right) = a.$

2. $L[e^{at}] = \frac{1}{s-a} \Rightarrow L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

3. $L[e^{-at}] = \frac{1}{s+a} \Rightarrow L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

4. $L[\sin at] = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

5. $L[\cos at] = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

6. $L[\sinh at] = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{\sinh at}{a}.$

7. $L[\cosh at] = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at.$

8. $L[t^n] = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

9. $L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2} \Rightarrow L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{e^{at}\sin bt}{b}.$

10. $L[e^{at}\cos bt] = \frac{(s-a)}{(s-a)^2+b^2} \Rightarrow L^{-1}\left(\frac{(s-a)}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

Find the inverse laplace transform.

1. $L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$

2. $L^{-1}\left[\frac{1}{s^4}\right] = \frac{t^3}{3!}$

3. $L^{-1}\left[\frac{s}{s^2+25}\right] = \cos 5t$

4. $L^{-1}\left[\frac{1}{s^2+36}\right] = \frac{\sin 6t}{6}$

5. $L^{-1}\left[\frac{3}{s^2+3^2}\right] = \sin 3t$

6. $L^{-1}\left[\frac{4s-5}{s^2+25}\right]$

$= L^{-1}\left[4 \cdot \frac{s}{s^2+25} - \frac{5}{s^2+25}\right]$

$= 4\cos 5t - \sin 5t$

7. $L^{-1}\left[\frac{2s+3}{s^2-9} + \frac{3s+5}{s^2-25}\right]$.

$L^{-1}\left[2 \cdot \frac{s}{s^2-9} + \frac{3}{s^2-9} + \frac{3 \cdot s}{s^2-25} + \frac{5}{s^2-25}\right]$

$= 2\cosh 3t + \sinh 3t + 3\cosh 5t + \sinh 5t$

8. $L^{-1}\left[\frac{1}{s+2} + \frac{3}{2s-2} + \frac{5s}{2s^2-4}\right]$

$= e^{-2t} + \frac{3}{2} L^{-1}\left[\frac{1}{s-1}\right] + \frac{5}{2}\left[\frac{s}{s^2-2}\right]$

$= e^{-2t} + \frac{3}{2} e^{t} + \frac{5}{2} \cos\sqrt{2}\, t$

9. $L^{-1}\left[\frac{1}{3s^2+16}\right]$

$= \frac{1}{3} L^{-1}\left[\frac{1}{s^2+16/3}\right] = \frac{1}{3} L^{-1}\left[\frac{1}{s^2+(4/\sqrt{3})^2}\right]$

$= \frac{1}{3} \frac{\sin(4/\sqrt{3})t}{4/\sqrt{3}}$

10. $L^{-1}\left[\frac{2s-5}{8s^2-50} + \frac{4s-5}{9-s^2}\right]$

$= L^{-1}\left[\frac{2s}{8s^2-50} - \frac{5}{8s^2-50} + \frac{4s}{9-s^2} - \frac{5}{9-s^2}\right]$

$= \frac{2}{8} L^{-1}\left[\frac{s}{s^2-(\sqrt{50}/\sqrt{8})^2}\right] - \frac{5}{8} L^{-1}\left[\frac{1}{s^2-(5/2)^2}\right] + (-4)\left[\frac{s}{s^2-9}\right] + 5\left[\frac{1}{s^2-9}\right]$

$= \frac{2}{8} \cosh\left(\frac{5}{2}\right)t - \frac{5}{8} \frac{\sinh(5/2)t}{5/2} - 4\cosh 3t + 5\frac{\sinh 3t}{3}$

Date______ Page______

11. $L^{-1}\left[\frac{(s+2)^3}{s^6}\right]$

$$L^{-1}\left[\frac{s^3+8+3s^2\cdot 2+3\cdot 2^2 s}{s^6}\right]$$

$$L^{-1}\left[\frac{s^3+8+6s^2+12s}{s^6}\right].$$

$$L^{-1}\left[\frac{1}{s^3}+\frac{8}{s^6}+\frac{6s}{s^4}+\frac{12}{s^5}\right]$$

$$= \frac{t^2}{2!}+8\cdot\frac{t^5}{5!}+6\cdot\frac{t^3}{3!}+12\cdot\frac{t^4}{4!}\Bigg]$$

10) Property:

$L[e^{at}f(t)] = F(s-a)$

$$\boxed{\therefore\ L^{-1}[F(s-a)] = e^{at}f(t). \quad \text{OR} = e^{at}L^{-1}[F(s)]}$$

12 $L^{-1}\left[\frac{3s+1}{(s+1)^4}\right]$

$$= L^{-1}\left[\frac{3s+\overbrace{3-2}^{1}}{(s+1)^4}\right] = L^{-1}\left[\frac{3(s+1)-2}{(s+1)^4}\right].$$

Using property

$$= e^{-t}L^{-1}\left[\frac{3s}{s^4}-\frac{2}{s^4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{3}{s^3}-\frac{2}{s^4}\right]$$

$$= e^{-t}\left[3\cdot\frac{t^2}{2!}-2\cdot\frac{t^3}{3!}\right]$$

13. $L^{-1}\left[\dfrac{s}{(s+3)^5}\right]$

$= L^{-1}\left[\dfrac{s+3-3}{(s+3)^5}\right]$

$= L^{-1}\left[\dfrac{(s+3) - 3}{(s+3)^5}\right]$

Using property

$= e^{-3t} L^{-1}\left[\dfrac{s}{s^5} - \dfrac{3}{s^5}\right]$

$= e^{-3t} L^{-1}\left[\dfrac{1}{s^4} - \dfrac{3}{s^5}\right]$

$= e^{-3t}\left[\dfrac{t^3}{3!} - 3.\dfrac{t^4}{4!}\right]$

14. $L^{-1}\left[\dfrac{s+3}{(s+3)^2+36}\right]$

$= e^{-3t} L^{-1}\left[\dfrac{s}{s^2+36}\right]$

$= e^{-3t} \cos 6t$

15. $L^{-1}\left[\dfrac{s^2}{(s-a)^3}\right]$

SPARK

Date ______ Page ______

Computing the squares

16. $L^{-1}\left[\frac{s+2}{s^2+2s+5}\right]$

Divide the coeff by 2 & square it and then add and minus

$D \Rightarrow s^2+2s+5+1-1$

$(s^2-2s+1)+4$

$(s-1)^2+4$

$(s-1)^2+2^2$

Nume $\rightarrow s+2+1-1 = (s-1)+3$.

$\therefore L^{-1}\left[\frac{(s-1)+3}{(s-1)^2+2^2}\right]$

$(s-1)\rightarrow s$ & x|y by e^t

$= e^t L^{-1}\left[\frac{s+3}{s^2+2^2}\right]$

$= e^t L^{-1}\left[\frac{s}{s^2+2^2}+\frac{3}{s^2+2^2}\right]$

$= e^t\left[\cos 2t + \frac{3\sin 2t}{2}\right]$

17. $L^{-1}\left[\frac{1}{s^2-5s+13}\right]$

Deno $\rightarrow s^2-5s+13+(5/2)^2-(5/2)^2$

$s^2-5s+13+25/4-25/4$

$(s^2-5s+25/4)+13-25/4$

$(s-5/2)^2+27/4$

$(s-5/2)^2+(\sqrt{27}/2)^2$

$L^{-1}\left[\frac{1}{(s-5/2)^2+(\sqrt{27}/2)^2}\right]$

$e^{5/2t} L^{-1}\left[\frac{1}{s^2+(\sqrt{27}/2)^2}\right]$

$e^{5/2t}\frac{\sin \sqrt{27}/2\, t}{\sqrt{27}/2}$

18. $L^{-1}\left[\frac{s-2}{s^2-s+2}\right]$

Deno $\rightarrow$ $s^2-s+2+1/4-1/4$

$(s^2-s+1/4)+2-1/4$

$(s-1/2)^2+7/4$

$(s-1/2)^2+(\sqrt{7}/2)^2$

Nume $\rightarrow$ $s-2+1/2-1/2$

$(s-1/2)-3/2$

$= L^{-1}\left[\frac{(s-1/2)-3/2}{(s-1/2)^2+(\sqrt{7}/2)^2}\right]$

$= e^{1/2 t} L^{-1}\left[\frac{s-3/2}{s^2+(\sqrt{7}/2)^2}\right]$

$= e^{1/2 t}\left[\cos\sqrt{7}/2\, t - \frac{3/2 \sin\sqrt{7}/2\, t}{\sqrt{7}/2}\right]$

19. $L^{-1}\left[\frac{2s}{s^2+2s+5}\right]$

Deno $=$ $s^2+2s+5+1-1$

$(s^2+2s+1)+5-1$

$(s+1)^2+2^2$

Nume : $2[s+1-1]$

$2(s+1)-1$.

$= 2L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+2^2}\right]$

$= 2e^{-t} L^{-1}\left[\frac{s-1}{s^2+4}\right]$

$$= 2e^{-t} L^{-1}\left[\frac{s}{s^2+2^2} - \frac{1}{s^2+2^2}\right]$$

$$= 2e^{-t}\left[\cos 2t - \frac{\sin 2t}{2}\right].$$

Inverse Laplace Transforms by Partial Fractions.

- $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$

- $\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$

- $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

1. $L^{-1}\left[\frac{1}{(s-1)(s-2)(s-3)}\right]$

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$1 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2).$$

$s = 1$ $\quad 1 = A(-1)(-2) + 0 + 0.$

$2A = 1$

$\boxed{A = 1/2}.$

$s = 2.$ $\quad 1 = A(0)(-1) + B(1)(-1) + C(1)(0)$

$1 = -B$

$\boxed{B = -1}$

$s = 3$ $\quad 1 = A(0) + B(0) + C(2)(1)$

$1 = 2C$

$\boxed{C = 1/2}$

$\therefore L^{-1}\left[\frac{1}{(s-1)(s-2)(s-3)}\right] = L^{-1}\left[\frac{1/2}{s-1} - \frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}\right].$

$= \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}.$

2. $L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right]$

$\frac{1}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+1)}$

$s=1$ $\quad 1 = A(s^2+1) + (Bs+C)(s-1)$

$1 = A(2) + (Bs+C)0.$

$\boxed{A = 1/2}$

$s=0$ $\quad 1 = A(1) + (0+C)(-1)$

$1 = A - C$

$1 - 1/2 = -C$

$\boxed{C = -1/2}$

$s = -$ Compare the coefficient of s^2 on B.S

$0 = A + B$

$A = -B$

$\boxed{B = -1/2}$

$\therefore L^{-1}\left[\frac{1/2}{(s-1)} + \frac{-1/2\,s + 1/2}{(s^2+1)}\right]$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{(s-1)} - \frac{s+1}{(s^2+1)}\right].$$

$$= \frac{1}{2}\left[e^t - L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{1}{2}\left[e^t - \cos t - \sin t\right]$$

Q. 3. $L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right]$

s^2+2s+5.
cant split.

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = \frac{A}{s-1} + \frac{Bs+C}{(s^2+2s+5)}$$

$s = 1$.

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1).$$

$$5+3 = A(1+2+5) + (B+C)(0).$$

$$8 = 8A$$

$$\boxed{A = 1}$$

$s = 0$.

$$0+3 = A(0+0+5) + (0+C)(-1).$$

$$3 = 5A - C$$

$$3 = 5 - C$$

$$C = 5-3$$

$$\boxed{C = 2}$$

Compare the coefficient of s^2 on B.S.

$$0 = A + B$$

$$B = -A$$

$$\boxed{B = -1}$$

$= L^{-1}\left[\frac{1}{(s-1)} + \frac{-s+2}{(s^2+2s+5)}\right]$

$= e^t + L^{-1}\left[\frac{-s+2}{s^2+2s+5}\right]$

$= e^t - L^{-1}\left[\frac{s-2}{s^2+2s+5}\right]$

Deno $\rightarrow$ $s^2+2s+5+1-1$

$(s^2+2s+1)+5-1$

$(s+1)^2+2^2$

Nume $\rightarrow$ $s-2+1-1$

$(s+1)-2-1$

$(s+1)-3$

$= e^t - \cancel{e}^t L^{-1}\left[\frac{(s+1)-3}{(s+1)^2+2^2}\right]$

$= e^t - e^{-t} L^{-1}\left[\frac{s-3}{s^2+2^2}\right]$

$= e^t - e^{-t} L^{-1}\left[\frac{s}{s^2+2^2} - \frac{3}{s^2+2^2}\right]$

$= e^t - e^{-t}\left[\cos 2t - \frac{3\sin 2t}{2}\right]$

SPARK

Date ______ Page ______

Q. 4. $L^{-1}\left[\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]$

$s^3-6s^2+11s-6$

$s=1,2,3$

Deno $\Rightarrow$ $s^3-6s^2+11s-6$

$(s-1)(s-2)(s-3)$.

Num $\Rightarrow$ No changes.

$$L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right] = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}.$$

$s=2$. $2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$.

$8-12+5 = 0 + B(1)(-1) + 0$

$1 = -B$

$\boxed{B=-1}$

$s=1$ $2-6+5 = A(-1)(-2) + 0 + 0$

$1 = 2A$

$\boxed{A = 1/2}$

$s=0$. $5 = A(-2)(-3) + B(-1)(-3) + C(-1)(-2)$.

$5 = 6A + 3B + 2C$.

$5 = \cancel{6}^{3}(1/\cancel{2}) - 3 + 2C$

$5 = 3-3+2C$

$5 = 2C$

$\boxed{C = 5/2}$

$$= L^{-1}\left[\frac{1}{2}\frac{1}{(s-1)} - \frac{1}{(s-2)} + \frac{5}{2}\frac{1}{(s-3)}\right].$$

$$= \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$$

5. $L^{-1}\left[\dfrac{2s^2+5s-4}{s^3+s^2-2s}\right]$

Deno :- s^3+s^2-2s $\quad (s = 1, -2, 0)$.

$s(s-1)(s+2)$

$L^{-1}\left[\dfrac{2s^2+5s-4}{s^3+s^2-2s}\right] = \dfrac{A}{s} + \dfrac{B}{(s-1)} + \dfrac{C}{(s+2)}$

$s=1$ $\quad 2s^2+5s-4 = A(s-1)(s+2) + Bs(s+2) + Cs(s-1)$

$2+5-4 = 0 + B(3) + C(0)$

$3 = 3B$

$\boxed{B = 1}$

$s=-2.$ $\quad 8-10-4 = 0+0-2C(-3)$

$-6 = 6C$

$\boxed{C = -1}$

$s=0.$ $\quad 0+0-4 = A(-1)(2) + 0(2) + 0(-1).$

$-4 = -2A$

$\boxed{A = 2}$

$= L^{-1}\left[\dfrac{2}{s} + \dfrac{1}{(s-1)} - \dfrac{1}{(s+2)}\right]$

$= 2 + e^t - e^{-2t}$

SPARK

Date ______ Page ______

6. $L^{-1}\left[\frac{s}{s^4+4a^4}\right]$

$(a+b)^2 = a^2+b^2+2ab$

$a^2+b^2 = (a+b)^2-2ab$

Deno :- $s^4+4a^4 = (s^2+2a^2)^2 - 4a^2s^2$

$= (s^2+2a^2)^2 - (2as)^2$

$= (s^2+2a^2+2as)(s^2+2a^2-2as)$

Num :- $(s^2+2a^2+2as)-(s^2+2a^2-2as) = 4as$

$\therefore \frac{1}{4a} L^{-1}\left[\frac{4as}{s^4+4a^4}\right] = \frac{1}{4a} L^{-1}\left[\frac{(s^2+2a^2+2as)-(s^2+2a^2-2as)}{(s^2+2a^2+2as)(s^2+2a^2-2as)}\right]$

$= \frac{1}{4a} L^{-1}\left[\frac{1}{(s^2+2a^2-2as)} - \frac{1}{(s^2+2a^2+2as)}\right]$

Div :- s^2+2a^2-2as

$s^2-2as+2a^2+a^2-a^2$

$(s-a)^2+a^2$

s^2+2a^2+2as

$s^2+2as+2a^2+a^2-a^2$

$(s+a)^2+a^2$

$= \frac{1}{4a} L^{-1}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$

$= \frac{1}{4a}\left[e^{at}\frac{\sin at}{a} - e^{-at}\frac{\sin at}{a}\right]$

$= \frac{1}{4a}\frac{\sin at}{a}\left[e^{at}-e^{-at}\right]$

$= \frac{1}{2a^2}\sin at \sinh at.$

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7. $L^{-1}\left[\dfrac{s}{s^4+s^2+1}\right]$

$s^4+1+2s^2-s^2$

Deno :- $s^4+s^2+1 = (s^2+1)^2 - s^2$

$= (s^2+1+s)(s^2+1-s)$.

Num :- $s = (s^2+1+s) - (s^2+1-s) \Rightarrow \cancel{s^2}+\cancel{1}+s-\cancel{s^2}-\cancel{1}+s = 2s$

So multiply & ÷ by 2 to num

$= \dfrac{1}{2} L^{-1}\left[\dfrac{2s}{s^4+s^2+1}\right]$

$= \dfrac{1}{2} L^{-1}\left[\dfrac{(s^2+1+s)-(s^2+1-s)}{(s^2+1+s)(s^2+1-s)}\right]$

$= \dfrac{1}{2} L^{-1}\left[\dfrac{1}{(s^2+1-s)} - \dfrac{1}{(s^2+1+s)}\right]$

Deno :- $s^2+1-s = s^2-s+1+1/4-1/4$

$= (s-1/2)^2 + 3/4$.

$= (s-1/2)^2 + (\sqrt{3}/2)^2$

Deno :- $s^2+1+s = s^2+s+1+1/4-1/4$

$= (s+1/2)^2 + 3/4$

$= (s+1/2)^2 + (\sqrt{3}/2)^2$

$\frac{2}{4}$

$= \dfrac{1}{2} L^{-1}\left[\dfrac{1}{(s-1/2)^2+(\sqrt{3}/2)^2} - \dfrac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}\right]$

$= \dfrac{1}{2} L^{-1}\left[\dfrac{e^{1/2t}}{s^2+(\sqrt{3}/2)^2} - \dfrac{e^{-1/2t}}{s^2+(\sqrt{3}/2)^2}\right]$

$= \dfrac{1}{\cancel{2}}\left[e^{1/2t}\dfrac{\sin(\sqrt{3}/2)t}{\sqrt{3}/\cancel{2}} - e^{-1/2t}\dfrac{\sin(\sqrt{3}/2)t}{\sqrt{3}/\cancel{2}}\right]$

$$= \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}\right)t \left[e^{1/2 t} - e^{-1/2 t} \right]$$

$\lim \frac{e^t - e^{-t}}{2} = \sinh x$

(multiply & ÷ by 2)

$$= \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}\right)t \sinh\left(\frac{t}{2}\right)$$

Convolution Theorem

We know that $L[f(t)] = F(s)$ or $L^{-1}[F(s)] = f(t)$

and $L[g(t)] = G(s)$ or $L^{-1}[G(s)] = G(t)$.

Then $L^{-1}[F(s)\,G(s)] = \int_0^t f(u)\,g(t-u)\,du = \int_0^t f(t-u)\,g(u)\,du$.

6M* 1. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ by convolution Theorem

$$L^{-1}\left[\frac{s}{(s^2+a^2)} \times \frac{1}{(s^2+a^2)}\right]$$

Here $F(s) = \frac{s}{(s^2+a^2)}$ and $G(s) = \frac{1}{(s^2+a^2)}$

But $f(t) = \cos at$ $\quad g(t) = \frac{\sin at}{a}$.

By C.T

$$L^{-1}[F(s)G(s)] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos au \, \frac{\sin a(t-u)}{a}\, du.$$

$$= \frac{1}{a} \int_0^t \underset{\cos a}{\cos au}\ \underset{\sin b}{\sin a(t-u)}\, du.$$

$$\boxed{\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right].}$$

$$= \frac{1}{2a}\int_0^t \sin(au + at - au) - \sin(au - at + au)\,du.$$

$$= \frac{1}{2a}\int_0^t \sin at - \sin(2au - at)\,du.$$

$$= \frac{1}{2a}\left[u \sin at + \frac{\cos(2au - at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[\left(t \sin at + \frac{\overset{\cos(2at-at)}{\cos(at)}}{2a}\right) - \left(0 + \frac{\cos 2a(0) - \cos at}{2a}\right)\right]$$

$$= \frac{1}{2a}\left[t \sin at + \frac{\cos(at)}{2a} - \frac{\cos at}{2}\right]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t \sin at}{2a}$$

*. 2/ $L^{-1}\left[\frac{1}{(s-)(s^2+1)}\right]$

Here $F(s) = \frac{1}{(s-1)}$ $\qquad G(s) = \frac{1}{(s^2+1)}$

$f(t) = e^t$ $\qquad g(t) = \frac{\sin t}{1}.$

By CT

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_0^t f(t-u)\, g(u)\, du.$$

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$$= \int_0^t e^{(t-u)} \sin u \, du$$

const

$$= \int_0^t e^t \cdot e^{-u} \sin u \, du$$

$$= e^{+t} \int_0^t e^{-u} \sin u \, du.$$

FORMULA :- $\int e^{at} \sin bt \, dt = \dfrac{e^{at}}{a^2+b^2}(a \sin bt - b \cos bt)$	$a=-1$; $b=1$

$$= e^t \left[\frac{e^{-u}}{1+1} (-\sin u - \cos u) \right]_0^t$$

$$= \frac{e^t}{2} \left[e^{-t}(-\sin t - \cos t) - 1\,(0-1) \right]$$

$$= \frac{e^t}{2} \left[-e^{-t} \sin t - e^{-t} \cos t + 1 \right]$$

$$= -\frac{\sin t}{2} - \frac{\cos t}{2} + \frac{e^t}{2}$$

3. $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)} \right]$

$L^{-1}\left[\dfrac{s}{(s^2+a^2)} \times \dfrac{s}{(s^2+b^2)} \right]$

Here $F(s) = \dfrac{s}{(s^2+a^2)}$ $\qquad G(s) = \dfrac{s}{s^2+b^2}$

$f(t) = \cos at$ $\qquad g(t) = \cos bt$

By C. T

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t \cos a(t-u) \cos bu \, du$$

$$\boxed{\cos a \cos b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]}$$

$$= \frac{1}{2}\int_0^t \cos a(t-u)+bu - \cos(at-au-bu)\, du$$

$$= \frac{1}{2}\int_0^t \cos(at-au+bu) - \cos(at-au-bu)\, du$$

$$= \frac{1}{2}\left[\frac{\sin(at-au+bu)}{(b-a)} - \frac{\sin(at-au-bu)}{(-a-b)}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin(at-at+bt)}{(b-a)} - \frac{\sin(at-at-bt)}{(-a-b)}\right) - \left(\frac{\sin(at)}{b-a} + \frac{\sin(at)}{(a+b)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin bt}{b-a} + \frac{\sin bt}{a+b} - \frac{\sin at}{b-a} - \frac{\sin at}{a+b}\right]$$

$$= \frac{1}{2}\left[\sin bt\left(\frac{1}{b-a} - \frac{1}{a+b}\right) - \sin at\left(\frac{1}{b-a} + \frac{1}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\sin bt\left(\frac{a+b-b+a}{b^2-a^2}\right) - \sin at\left(\frac{a+b+b-a}{b^2-a^2}\right)\right]$$

$$= \frac{1}{2}\left[\sin bt\left(\frac{2a}{b^2-a^2}\right) - \sin at\left(\frac{2b}{b^2-a^2}\right)\right]$$

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Date ______ Page ______

4. $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$.

$L^{-1}\left[\frac{s+2}{(s^2+4s+5)} \times \frac{1}{(s^2+4s+5)}\right]$.

$F(s) = \frac{s+2}{(s^2+4s+5)}$. $\quad G(s) = \frac{1}{s^2+4s+5}$

Deno :- $s^2+4s+5+4-4$

$(s^2+4s+4)+5-4$

$(s+2)^2+(1)^2$

Deno :- s^2+4s+5

$(s+2)^2+(1)^2$

$G(s) = \frac{1}{(s+2)^2+(1)^2}$

Num :- $s+2$ · (already in $s+2$).

$F(s) = \frac{s+2}{(s+2)^2+(1)^2}$

$g(t) = e^{-2t}\frac{1}{s^2+1^2}$

$f(t) = e^{-2t}\left(\frac{s}{s^2+1^2}\right)$

$g(t) = e^{-2t}\sin t$.

$f(t) = e^{-2t}\cos t$

By CT :-

$L^{-1}\left[F(s) * G(s)\right] = \int_0^t f(u)\, g(t-u)\, du.$

$= \int_0^t e^{-2u}\cos u \; e^{-2(t-u)}\sin(t-u)\, du.$

$= \int_0^t e^{-2u}\cos u \; e^{-2t}\cdot e^{2u}\sin(t-u)\, du.$

$= e^{-2t}\int_0^t \cos u\,\sin(t-u)\, du.$

FORMULA :- $\cos A\sin B = \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$

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$$= \frac{e^{-2t}}{2}\int_0^t \sin(u+t-u) - \sin(u-t+u)\,du$$

$$= \frac{e^{-2t}}{2}\int_0^t \sin t - \sin(2u-t)\,du$$

$$= \frac{e^{-2t}}{2}\left[u\sin t + \frac{\cos(2u-t)}{2}\right]_0^t$$

$$= \frac{e^{-2t}}{2}\left[\left(t\sin t + \frac{\cos(t)}{2}\right) - \left(0 + \frac{\cos(-t)}{2}\right)\right]$$

$$= \frac{e^{-2t}}{2}\left[t\sin t + \frac{\cos t}{2} - \frac{\cos t}{2}\right]$$

$$= \frac{e^{-2t}}{2}\left[t\sin t\right]$$

5 $L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$

$F(s) = \frac{1}{s^3}$ $\qquad$ $G(s) = \frac{1}{s^2+1}$

$F(s) = \frac{1}{s^{2+1}}$ $\qquad$ $g(t) = \sin t$

$f(t) = \frac{t^2}{2!} = \frac{t^2}{2}.$

BY CT :

$L^{-1}[F(s)G(s)] = \int_0^t f(u)\,g(t-u)\,du.$

$$= \int_0^t \frac{u^2}{2} \sin(t-u)\, du \qquad \int uv = u\int v - \int\left(\frac{d}{d}u \int v\right) du$$

$$= \frac{1}{2}\int_0^t u^2 \sin(t-u)\, du.$$

$$= \frac{1}{2}\left[\frac{-u^2\cos(t-u)}{-1} - \int 2u \cdot u^2 \cos(t-u)\, du\right]$$

$$= \frac{1}{2}\left[u^2\cos(t-u) - 2\int u^3 \cos(t-u)\, du\right]$$

$$= \frac{1}{2}\left[u^2\cos(t-u) - \frac{2u\sin(t-u)}{-1} + 2\left[\frac{\cos(t-u)}{(-1)(-1)}\right]\right]_0^t$$

$$= \frac{1}{2}\left[u^2\cos(t-u) + 2u\sin(t-u) \mp 2\cos(t-u)\right]_0^t$$

$$= \frac{1}{2}\left[\left(t^2\cos(0) + 2t\sin(0) - 2\cos(0)\right) - \left(u^2\cos t(0) + 2(0) - 2\cos t\right)\right]$$

$$= \frac{1}{2}\left[t^2 - 2 - u^2\cos t(0) + 2\cos t\right]$$

$$= \frac{1}{2}\left[t^2 - 2 + 2\cos t\right]$$

$$\boxed{\int uv = uv^1 - u_1 v'' + u_2 v''' - \cdots}$$

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6 $L^{-1}\left[\dfrac{s}{(s+2)(s^2+9)}\right]$.

$F(s) = \dfrac{1}{s+2}$ $\quad G(s) = \dfrac{s}{s^2+9}$

$f(t) = e^{-2t}$ $\quad g(t) = \cos 3t$

Using CT

$$L^{-1}[F(s)\,G(s)] = \int_0^t f(u)\,g(t-u)\,du \quad \text{or} \quad \int_0^t f(t-u)\,g(u)$$

$$= \int_0^t e^{-2u}\cos 3(t-u)\,du \quad \text{or} \quad \int_0^t e^{-2(t-u)}\cos 3u$$

$$= \int_0^t e^{-2t}\cdot e^{2u}\cos 3u\,du$$

$$= e^{-2t}\int_0^t e^{2u}\cos 3u\,du.$$

FORMULA: $\displaystyle\int e^{au}\cos bu = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu)$ $\quad a=2;\ b=$

$$= e^{-2t}\left[\frac{e^{2u}}{13}(2\cos 3u + 3\sin 3u)\right]_0^t$$

$$= \frac{e^{-2t}}{13}\left[e^{2t}(2\cos 3t + 3\sin 3t) - e^0(2+0)\right]$$

$$= \frac{2\cos 3t}{13} + \frac{3\sin 3t}{13} - \frac{2e^{-2t}}{13}$$

Solution of Linear Differential Equation using Laplace Transform

Here y', y'', y''' are the successive derivative;

there $L[y'] = s. Ly(t) - y(0)$

$L(y'') = s^2. Ly(t) - sy(0) - y'(0)$

$L(y''') = s^3 Ly(t) - s^2 y(0) - sy'(0) - y''(0)$

$L(y^{iv}) = s^4 Ly(t) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0)$

1. $y'' + 3y' + 2y = 0$ given that $y(0) = 0$ and $y'(0) = 2$.

Apply Laplace on B.S.

$L(y'' + 3y' + 2y) = L(0)$.

$L(y'')(t) + 3L(y')(t) + 2L(y)(t) = 0$.

$[s^2 Ly(t) - sy(0) - y'(0)] + 3[s Ly(t) - y(0)] + 2Ly(t) = 0$.

But $y(0) = 0$ & $y'(0) = 2$

$[s^2 Ly(t) - 2 + 3s Ly(t) + 2Ly(t) = 0$.

$Ly(t)[s^2 + 3s + 2] = 2$.

$$Ly(t) = \frac{2}{s^2+3s+2}$$

$$y(t) = L^{-1}\left[\frac{2}{s^2+3s+2}\right]$$

$$y(t) = L^{-1}\left[\frac{2}{(s+1)(s+2)}\right]$$

Partial Fraction

$$\therefore \frac{2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$s = -2$ $\quad 2 = A(s+2) + B(s+1)$

$2 = A(0) + B(-1)$

$\boxed{B = -2}$

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$s=-1$ $\quad 2 = A(1) + B(0)$

$$\boxed{A = 2}$$

$$y(t) = L^{-1}\left[\frac{2}{s+1} + \frac{-2}{s+2}\right]$$

$$y(t) = 2e^{-t} - 2e^{-2t}$$

2. $y'' + 2y' - 3y = \sin t \quad$ G.T $y(0) = 0$ & $y'(0) = 0$.

Apply Laplace on B.S

$$L[y'' + 2y' - 3y] = L[\sin t]$$

$$Ly''(t) + 2Ly'(t) - 3Ly(t) = L[\sin t]$$

$$[s^2 Ly(t) - sy(0) - y'(0)] + 2[sLy(t) - y(0)] - 3Ly(t) = L(\sin t)$$

$$s^2 Ly(t) - \cancel{sy(0)} - \cancel{y'(0)} + 2sLy(t) - \cancel{2y(0)} - 3Ly(t) = L[\sin t]$$

$$s^2 Ly(t) + 2sLy(t) - 3Ly(t) = L[\sin t]$$

$$Ly(t)[s^2 + 2s - 3] = L[\sin t]$$

$$Ly(t)[s^2 + 2s - 3] = \frac{1}{s^2+1}$$

$$Ly(t) = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$Ly(t) = \frac{1}{(s^2+1)(s-1)(s+3)}$$

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Date ______ Page ______

$$y(t) = L\left[\frac{1}{(s^2+1)(s-1)(s+3)}\right]$$

$$y(t) = L\left[\frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+3}\right].$$

$s = 1$ $\quad 1 = As+B\,(s-1)(s+3) + C\,(s^2+1)(s+3) + D(s^2+1)(s-1)$

$1 = A+B\,(0)(3) + C\,(2)(4) + D(0)$

$1 = 8C$

$\boxed{C = 1/8}$

$s = -3$ $\quad 1 = As+B\,(0) + C(0) + D\,(9+1)(-4)$

$1 = D(10)(-4).$

$1 = -40D$

$\boxed{D = -1/40}$

$s =$ compare the coeffcient of s^3 on B.S

$0 = A + C + D.$

$0 = A + 1/8 - 1/40.$

$\boxed{A = -1/10}$

compare the coefficient of s^2 on B.S

$0 = B + 3C - D + 2A$

$0 = B + 3(1/8) + 1/40 + 2(1/10).$

$\boxed{B = -1/5}$

$$y(t) = L^{-1}\left[-\frac{1}{10}\frac{s}{s^2+1} - \frac{1}{5}\frac{1}{s^2+1} + \frac{1}{8}\frac{1}{s-1} - \frac{1}{40}\frac{1}{s+3}\right]$$

$$y(t) = -\frac{1}{10}\cos t - \frac{1}{5}\sin t + \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t}$$

3 $y'' + 4y' + 4y = e^{-t}$ given that $y(0)=0$ & $y'(0)=0$.

$L[y''+4y'+4y] = L[e^{-t}]$

$Ly''(t) + 4Ly'(t) + 4Ly(t) = L[e^{-t}]$

$[s^2 Ly(t) - sy(0) - y'(0)] + 4[s.Ly(t) - y(0)] + 4Ly(t) = \frac{1}{s+1}$

$s^2 Ly(t) + 4s Ly(t) + 4Ly(t) = \frac{1}{s+1}$

$Ly(t)[s^2+4s+4] = \frac{1}{s+1}$.

$(s+2)^2 = s^2+4+4s$

$Ly(t) = \frac{1}{(s^2+4s+4)(s+1)}$.

$y(t) = L^{-1}\left[\frac{1}{(s+2)^2(s+1)}\right]$

$y(t) = L^{-1}\left[\frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{C}{s+1}\right]$

$s=-2$ $\quad 1 = A(s+2)(s+1) + B(s+1) + C(s+2)^2$.

$1 = A(0) + B(-1) + C(0)$

$\boxed{B=-1}$

$s=-1$ $\quad 1 = A(0) + B(0) + C(1)^2$

$\boxed{C=1}$

$s=0$ $\quad 1 = A(2)(1) + B(1) + C(4)$.

$1 = 2A - (-1)(1) + 1(4)$

$1 = 2A + 1 + 4$

$1 = 2A + 5$

$-4 = 2A$

$\boxed{A=-2}$

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Date ______ Page ______

$$y(t) = L^{-1}\left[\frac{-2}{s+2} + \frac{(-1)}{(s+2)^2} + \frac{1}{s+1} \right]$$

$$y(t) = -2e^{-2t} - e^{-2t}t + e^{-t}$$

4. $y''' + 2y'' - y' - 2y = 0 \qquad y(0) = 0 = y'(0) \quad \& \quad y''(0) = 6$

$$L[y''' + 2y'' - y' - 2y] = L[0]$$

$$[s^3 Ly(t) - s^2\cancel{y(0)} - s\cancel{y'(0)} - y''(0)] + 2[s^2 Ly(t) - s\cancel{y(0)} - \cancel{y'(0)}]$$
$$-[s\,Ly(t) - y(0)] - 2yL(t) = 0$$

$$s^3 Ly(t) + 2s^2 Ly(t) - s\,Ly(t) - 2yL(t) = 0$$

$$\cancel{s^3} Ly(t)[s^3 + 2s^2 - s] - 2]-6 = 0$$

$$Ly(t) = \frac{6}{(s^3 + 2s^2 - s - 2)} \qquad s = 1, -1, -2$$

$$Ly(t) = \frac{6}{(s-1)(s+1)(s+2)} \Rightarrow y(t) = L^{-1}\left[\frac{6}{(s-1)(s+1)(s+2)}\right.$$

$$\frac{6}{(s-1)(s+1)(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$$

$s = 1$

$$6 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$$
$$6 = A(2)(3) + B(0) + C(0)$$
$$6 = A(6)$$
$$\boxed{A = 1}$$

$s = -2$

$$6 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$$
$$6 = A(0) + B(0) + C(-3)(-1)$$
$$\boxed{C = 2}$$

$s=0$ $\quad 6 = A(1)(2) + B(-1)(2) + C(-1)(1)$

$6 = 2A - 2B - C$

$6 = \cancel{2} - 2B - \cancel{2}$

$\boxed{B = -3}$

$$Ly(t) = \left[\frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}\right]$$

$$y(t) = e^{t} - 3e^{-t} + 2e^{-2t}$$

5. $(D^3 - 3D^2 + 3D - 1)\,y(t) = t^2 e^t$ where $y(0)=1;\ y'(0)=0;\ y''(0)=-2$.

$(y''' - 3y'' + 3y' - 1)\,y(t) = L(t^2 e^t)$ $\quad t^2 = \frac{n!}{s^{n+1}}$

$L[y''' - 3y'' + 3y' - y](t) = L[t^2 e^t]$

$[s^3 Ly(t) - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2 Ly(t) - sy(0) - y'(0)]$
$+ 3[s\,Ly(t) - y(0)] - Ly(t) = \frac{2}{(s-1)^3}$

$s^3 Ly(t) + 2 - 3s^2 Ly(t) + 3s\,Ly(t) - Ly(t) = \frac{2}{(s-1)^3}$

$Ly(t)[s^3 - 3s^2 + 3s - 1] + 2 - s^2 + 3s - 3 = \frac{2}{(s-1)^3}$

$Ly(t)\,[s^3 - 3s^2 + 3s - 1] - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$

$Ly(t)\,[s^3 - 3s^2 + 3s - 1] = \frac{2}{(s-1)^3} + s^2 - 3s + 1$

$Ly(t) = \frac{\cancel{s^2 + 3s + 3}\ 2}{(s-1)^3(s^3 - 3s^2 + 3s - 1)} + \frac{s^2 - 3s + 1}{s^3 - 3s^2 + 3s - 1}$

SPARK

Date______ Page______

$$L\,y(t) = \frac{2}{(s-1)^6} + \frac{s^2-3s+1}{(s-1)^3}$$

$$y(t) = L^{-1}\left[\frac{2}{(s-1)^6}\right] + L^{-1}\left[\frac{s^2-3s+1}{(s-1)^3}\right].$$

$$y(t) = \frac{2e^t t^5}{5!} + e^t L^{-1}\left[\frac{s^2-3s+1}{s^3}\right]$$

$$y(t) = \frac{2e^t t^5}{5!} + e^t\left(1-3t+\frac{t^2}{2}\right)$$