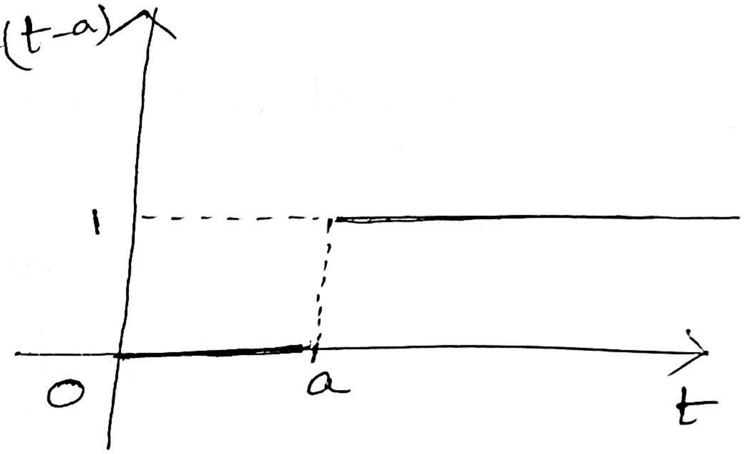
Unit-step function (or) Heaviside's function

The unit step function (or) Heaviside's function defined as follows

$$H(t-a) \text{ (or) } U(t-a) = \begin{cases} 0, & \text{for } t \le a \\ 1, & \text{for } t > a \end{cases}$$

where 'a' is non-negative constant



Note:- if $a = 0$

$$H(t-a) = H(t)$$

$$H(t) = \begin{cases} 0, & \text{for } t \le 0 \\ 1, & \text{for } t > 0 \end{cases}$$

Properties

(1) $$L\{H(t-a)\} = \frac{e^{-as}}{s}.$$

<u>Proof</u> :- $L\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\,dt$

$\because L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

But $H(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$

$$L\{H(t-a)\} = \int_0^a e^{-st} H(t-a)\,dt + \int_a^\infty e^{-st} H(t-a)\,dt$$

$$= \cancelto{0}{\int_0^a e^{-st}(0)\,dt} + \int_a^\infty e^{-st} \cdot 1\,dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-as}\right]$$

$$= -\frac{1}{s}\left[0 - e^{-as}\right]$$

$$L\{H(t-a)\} = \frac{e^{-as}}{s}$$

(2) Heaviside's Shifting theorem

$$L\{f(t-a)\,H(t-a)\} = e^{-as} L\{f(t)\}$$

Proof:- $L\{f(t-a)\,H(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,H(t-a)\,dt$

But $L\{H(t-a)\} = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$

$$= \int_0^a e^{-st} f(t-a)\cdot H(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)\,H(t-a)\,dt$$

$$= \cancel{\int_0^a e^{-st}\cdot f(t-a)\cdot 0\,dt} + \int_a^\infty e^{-st} f(t-a)\cdot 1\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

put $t-a=u$

$t = u+a$

$dt = du$

If $t=a$, $u=0$

$t=\infty$, $u=\infty$

$$= \int_0^\infty e^{-s(u+a)} f(u)\, du$$

$$= \int_0^\infty e^{-su} \cdot e^{-as} \cdot f(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\, du \qquad L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= e^{-as} \cdot L\{f(t)\}$$

$$L\{f(t-a)\, H(t-a)\} = \underline{e^{-as} \cdot L\{f(t)\}}$$

$$\text{(or)} = \underline{e^{-as} F(s)} \qquad \because \underline{F(s) = L\{f(t)\}}$$

Problems

① find $L\{(t^3 + t^2 + t + 1)\, u(t+1)\}$

Sol:- $L\{f(t-a)\, U(t-a)\}$

$a = -1$

$L\{f(t+1) \cdot U(t+1)\}$

$f(t+1) = t^3 + t^2 + t + 1$

put $t = t-1$

$f(t-1+1) = (t-1)^3 + (t-1)^2 + (t-1) + 1$

$$f(t) = \begin{cases} (t-1)^3 + (t-1)^2 + (t-1) + 1 \\ \end{cases}$$

$$f(t) = t^3 - 3t^2 + 3t - \cancel{1} \quad + t^2 + \cancel{1} - 2t + t - \cancel{1} + \cancel{1}$$

$$f(t) = t^3 - 2t^2 + 2t$$

$$L\{f(t)\} = \frac{3!}{s^4} - 2 \cdot \frac{2!}{s^3} + 2 \cdot \frac{1}{s^2}$$

$$L\{(t^3 + t^2 + t + 1)\ u(t+1)\} = e^{-as} \cdot \underbrace{L\{f(t)\}}$$

$$= e^{-(-1)s}\left(\frac{3!}{s^4} - \frac{2 \cdot 2!}{s^3} + \frac{2}{s^2}\right)$$

$$= e^{s}\left(\frac{6}{s^4} - \frac{4}{s^3} + \frac{2}{s^2}\right)$$

② Find $L\{\underline{\sin t}\ u(t-\pi)\}$

Sol:- $L\{f(t-a)\ u(t-a)\}$

$a = \pi$

$$f(t-\pi) = \sin t \qquad t = t+\pi$$

$$f(t + \cancel{\pi} - \cancel{\pi}) = \sin(t+\pi)$$

$$f(t) = \sin(\pi + t)$$

$$f(t) = -\sin t$$

$L\{f(t)\} = L\{-\sin t\}$

$= -\dfrac{1}{s^2+1}$

$L\{f(t-a)\ U(t-a)\} = e^{-as} \cdot L\{f(t)\}$

$L\{\sin t \cdot U(t-\pi)\} = e^{-\pi s}\left(-\dfrac{1}{s^2+1}\right)$

$= -\dfrac{e^{-\pi s}}{s^2+1}$

③ find $L\{4\sin(t-3)\ H(t-3)\}$

Sol:- $L\{f(t-a)\ H(t-a)\}$

$a = 3$

$f(t-3) = 4\sin(t-3)$

put $t = t+3$

$f(t+3-3) = 4\sin(t+3-3)$

$f(t) = 4\sin t$

$L\{f(t)\} = 4\,L\{\sin t\}$

$= 4 \cdot \dfrac{1}{s^2+1}$

$L\{f(t)\} = \dfrac{4}{s^2+1}$

$$L\{4\sin(t-3)\, u(t-3)\} = e^{-3s} \cdot \frac{4}{s^2+1}$$

(4) find the Laplace transform of $(1-e^{2t})\, u(t+1)$

Sol:- $L\{(1-e^{2t})\, u(t+1)\}$

$a=-1$

$L\{f(t-a)\, u(t-a)\}$

$f(t+1) = 1-e^{2t}$

put $t = t-1$

$f(t-1+1) = 1-e^{2(t-1)}$

$f(t) = 1-e^{2t-2}$

$f(t) = 1-e^{2t} \cdot e^{-2}$

$L\{f(t)\} = L\{1-e^{2t} \cdot e^{-2}\}$

$= L\{1\} - e^{-2} \cdot L\{e^{2t}\}$

$L\{f(t)\} = \frac{1}{s} - e^{-2} \cdot \frac{1}{s-2}$

$L\{(1-e^{2t})\, u(t+1)\} = e^{-(-1)\cdot s}\left(\frac{1}{s} - \frac{1}{e^2}\, \frac{1}{s-2}\right)$