rock recently stockpiled on roadbed above in preparation for new construction. In a few days, the spring thaw will soak the ground and send torrents of water through the culvert. The engineer speculates on possible designs in terms of the conditions and restrictions imposed by available equipment, materials, and time. Identify the strategic or tactical nature of each decision faced by the engineer, and discuss the factors that should be considered (including sensitivity, if appropriate).

1.12 Answer the same question as just posed in Prob. 1.11, but for the following scenario: The owner of a wholesale distribution center seeks to improve her delivery service in order to meet competition. To do so, she can buy or rent more trucks, subcontract deliveries, open additional outlets, and/or improve the handling facilities. Capital is limited, and the outlook for increased volume is uncertain. First she must decide if any action is needed. If it is, she must select the most useful alternative.

CHAPTER 2

TIME VALUE OF MONEY

If you would know the value of money, go and try to borrow some.

Benjamin Franklin, *Poor Richard's Almanack*, 1758

Interest is the cost of using capital. Its history extends as far back as the recorded transactions of humanity. In earliest times, before money was coined, capital was represented by wealth in the form of personal possessions, and interest was paid in kind. For example, a loan of seed to a neighbor before planting was returned after harvest with an additional increment. We can surmise that the concept of interest in its modern sense arose from such loans for productive purposes.

By the time the Greek and Roman empires were in their ascendancies, interest rates were somewhat standardized and occasionally legislated. The amount charged for loans to the most reliable borrowers was around 10 percent, ranging from 4 percent in first-century Rome to about 50 percent for grain loans in Egypt during the same period.[†]

The concept of interest has not changed much through the centuries, but the modern credit structure differs markedly from that of antiquity. Lending or investing was relatively inconvenient in ancient days because transactions were made directly between individuals. There were no banking organizations to act as intermediaries and no credit instruments in the money market. Governments were rarely able to float loans since they could not pledge the private resources of their people. Also, governments had not discovered the practice of deficit financing.

[†]S. Homer, *History of Interest Rates*, Rutgers, New Brunswick, N.J., 1963.

Today, there are many credit instruments, and most people use them. Business and government are the biggest borrowers. Businesses seek the use of capital goods to increase productivity. Governments borrow against future tax revenues to finance highways, welfare programs, and public services. Households also borrow to make purchases in excess of their current cash resources. Such borrowers, and the corresponding lenders, must acknowledge the time value of their commitments.

The following example reveals the significance of interest in economic transactions and confirms the importance of understanding how interest operates, whether it pertains to personal finances or to professional practices.

EXAMPLE 2.1 **Impact of Interest Rates and Loan Periods**

The purchase of a home is the largest investment most people make. By the summer of 1993, many people had refinanced their homes because interest rates dipped to the lowest levels in some 20 years. The two tables below vividly portray the impact of interest rates and loan periods. A shorter repayment period at a given interest rate or a lower interest rate for a given loan period gives a significant saving.

A $100,000 loan at 8.5% interest for four repayment periods

Repayment period, years	**Monthly payment, $**	**Total interest, $**
15	984.51	77,210
20	867.57	108,217
25	804.96	141,485
30	768.63	176,707

A $100,000 loan for 30 years at four interest rates

Interest rate, %	**Monthly payment, $**	**Total interest, $**
7.5	699.21	151,722
8.5	768.63	176,707
9.5	841.15	202,806
10.0	877.28	215,812

A further interesting historical example relates to New York City lending $1 million to the nation's capital during the War of 1812. During the period of New York's financial crises in 1975, it was suggested that Washington might be billed for the original loan and the accumulated interest. At 6 percent interest compounded annually, that $1 million loan would have increased to an $11.2 billion debt.

2.1 INTEREST AND THE TIME VALUE OF MONEY

Nearly everyone is directly exposed to interest transactions occasionally and is indirectly affected regularly. Credit cards are a mainstay of commerce; they have an interest load for delayed payments. Key parts of a contract for purchasing an automobile or home are the interest stipulations.

The rate of interest paid on municipal bonds directly affects tax rates for property in the affected area. Businesses borrow to expand or simply maintain operations, and the cost of their borrowing must be repaid from more profitable operations enabled by loans. All this borrowing taken together adds up to an enormous debt, and it all has interest charges.

To fully appreciate interest charges, one must comprehend the reasons for the charges, understand how they are calculated, and realize their effect on cash flows. Interest represents the earning power of money. It is the premium paid to compensate a lender for the administrative cost of making a loan, the risk of nonrepayment, and the loss of use of the loaned money. A borrower pays interest charges for the opportunity to do something now that otherwise would have to be delayed or would never be done. *Simple interest I* is a charge directly proportional to the capital (principal *P*) loaned at rate *i* for *N* periods, so that $I = PiN$. *Compound interest* includes charges for the accumulated interest as well as the amount of unpaid principal.

A *nominal interest rate r* of 8 percent compounded quarterly, e.g., indicates an interest charge of 2 percent per quarter compounded four times per year. If *m* is the number of compounding periods per year, we will see that the equivalent *effective interest rate*, or *actual annual interest* earned or paid, *i* from a nominal rate is

$$\boxed{i_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1} \tag{2.1}$$

Continuous interest i_∞ is the effective interest rate as *m* approaches infinity, and its equivalent effective interest rate is

$$\boxed{i_\infty = e^r - 1} \tag{2.2}$$

as will be shown in Sec. 2.4.3.

Time-value mechanics involve the use of compound-interest factors to translate payments of various amounts occurring at various times to a single equivalent payment. Interest factors are symbolized by notations based on interest *i*, number of periods *N*, *P* = present worth, *F* = future worth, and *A* = annuity payment. An ordinary annuity is a series of equal payments, at equal intervals, with the first payment at the end of the first period. When payments in an annuity increase by a constant increment *G* each period, an equivalent ordinary annuity is determined through use of an arithmetic gradient factor.

Seven discrete interest factors are commonly used to evaluate cash flows and convert them to summary statements that define alternative uses for capital. These factors are represented by functional symbols that assist calculations; the values of the factors for different interest rates and numbers of compounding periods are tabulated in App. D. Also, they can be computed by using CHEER, the computer program supplied with this text.

The fundamental concepts of interest and the basic constructs of interest calculation introduced in this chapter are the foundations for discounted cash flow applications to be developed in the next four chapters.

2.2 REASONS FOR INTEREST

The significance of interest can be shown through a historical purchase that most of us are familiar with, the Manhattan Island purchase.

EXAMPLE 2.2 **All You Have to Do to Be Rich Is Live Long Enough**

The famous purchase of Manhattan Island from the Indians for $24 is often referred to as an exceptional bargain. This incident reputedly occurred in 1626, when Peter Minuit of the Dutch West India Company bought the rights to the island from local residents. Was it a bargain? For the sake of argument, suppose that the Indians could have invested the money at a reasonable interest rate of 6 percent compounded annually. Over the years since then, the original $24 investment would have grown as shown in the following table:

Year	Value of original $24 investment compounded at 6% per year, $
1626	24.00
1676	442.08
1726	8,143.25
1776	149,999.92
1826	2,763,021.69
1876	50,895,285.85
1926	937,499,017.25
1976	17,268,876,530.40
1996	55,383,626,485.92

Whether Minuit or the Indians got the better deal depends on the perspective taken. If they could have invested at a higher rate, say the 18 percent return earned on money market funds in 1981, the Indians' theoretical investment would have been worth, in 1981, just over

$790,000,000,000,000,000,000,000

or $790 septillion. On the other hand, at the 6 percent rate, a value closer to 1993 rates, its value would have increased to slightly more than $27 billion by 1984, when the assessed value of taxable real estate in Manhattan was about $24 billion.

The effect of interest should be evident from Example 2.2. The reasons for this effect become more apparent when we examine the uses of capital. In our economic environment, capital is the basic resource. It can be converted to production goods, consumer goods, or services. It has the power to earn and to satisfy wants.

From a lender's viewpoint, capital is a fluid resource. Capital can be spent on goods expected to produce a profit or on personal satisfaction. It can be hoarded or given away. It can also be loaned. If it is loaned, the lender will normally expect some type of compensation. The common compensation is interest. Interest compensates for the administrative expense of making the loan, for the risk that the loan will not be repaid, and for the loss of earnings that would have been obtained if the money had been invested for productive purposes.

From a borrower's viewpoint, a loan is both an obligation and an opportunity. A borrower must expect to repay the loan. Failure to repay leads to a damaged reputation, loss of possessions, and other consequences. The loan offers an opportunity to do something immediately that would otherwise have to be delayed. In some cases the objective would no longer exist after a delay. To take advantage of an existing course of action or to fulfill a current need, the borrower agrees to pay a certain amount in addition to the sum immediately received. This premium is the interest paid to avoid waiting for the money.

Implied in both the lender's and borrower's viewpoints is the earning power of money. For money to earn something, the owner or user must wait (waiting-earning is obviously opposed to the spending-owing use of money to gratify immediate desires). Interest payments have been likened to the reward for waiting, but it is more appropriate for engineering economists to view interest as the productive gain from the efficient use of the money resource. The prevailing interest rate is essentially a measure of the productivity to expect from the resource. An owner of money can lend it at the prevailing rate and wait to be repaid the original amount plus an extra increment. Equivalently, the borrower could reloan the money at a higher rate to acquire a gain larger than the amount to be repaid, or the money could be converted to productive goods that would be expected to earn more than the amount needed to repay the loan. In both cases the prevailing interest rate sets the minimum level of expected productivity, and both cases involve time between receipt and return of the loan to secure the earnings—the *time value of money*.

2.3 SIMPLE INTEREST

In the very unusual circumstance when a *simple interest rate* is quoted, the interest earned is directly proportional to the capital involved in the loan. Expressed as a formula, the interest earned *I* through several time periods is found by

$$I = PiN$$

where P = present amount or principal
i = interest rate per period
N = number of interest periods (usually years)

Since the principal, or amount borrowed, P is a fixed value, the annual interest charged is constant. Therefore, the total amount a borrower is obligated to pay a lender is

$$F = P + I = P + PiN = P(1 + iN)$$

where F is a future sum of money to be paid. When N is not a full year, there are two ways to calculate the simple interest earned during the period of the loan. When *ordinary simple interest* is used, the year is divided into twelve 30-day periods, or a year is considered to have 360 days. In *exact simple interest*, a year has precisely the calendar number of days, and N is the fraction of the number of days the loan is in effect that year.

An example of simple interest as the rental cost of money is a loan of \$1000 for 2 months at 10 percent. With ordinary simple interest, the amount to be repaid is

$$F = P(1 + iN)$$

where N is $\frac{2}{12}$ year, giving

$$F = \$1000\,(1 + 0.01667) = \$1016.67$$

With exact simple interest when the two months are January and February in a nonleap year, the future sum is

$$F = P\left[1 + (i)\left(\frac{31 + 28}{365}\right)\right]$$

and

$$F = \$1000\,(1 + 0.01616) = \$1016.16$$

2.4 COMPOUND INTEREST

Again, assume a loan of \$1000, this time for 2 years at an interest rate of 10 percent compounded annually; the pattern of interest compounding is shown in Table 2.1.

TABLE 2.1

Future value of \$1000 loan when interest is due on both the principal and unpaid interest

Year	Amount owed at beginning of year, \$	Interest on amount owed, \$	Amount owed at end of year, \$
1	1000	1000 × 0.10 = 100	1000 + 100 = 1100
2	1100	1100 × 0.10 = 110	1100 + 110 = 1210

The amount to be repaid with simple interest is

$$\$1000\,[1 + 0.1(2)] = \$1200$$

Thus, the amount to be repaid for the given loan is now \$10 greater for compound interest than for simple interest (\$1210 − \$1200). The \$10 difference accrues from the interest charge on the \$100 earned during the first year that was not accounted for in the simple-interest calculation. The formula for the calculations in Table 2.1, using previously defined symbols, is

$$\begin{matrix}\text{Compound} \\ \text{amount} \\ \text{due in} \\ \text{2 years}\end{matrix} = \begin{matrix}\text{amount} \\ \text{borrowed}\end{matrix} + \begin{matrix}\text{year 1} \\ \text{interest}\end{matrix} + \begin{pmatrix}\text{amount} \\ \text{borrowed} \\ \text{plus year 1} \\ \text{interest due}\end{pmatrix}\begin{pmatrix}\text{interest} \\ \text{rate}\end{pmatrix}$$

$$\begin{aligned} F_2 &= P + Pi + (P + Pi)i \\ &= P(1 + i + i + i^2) \\ &= P(1 + i)^2 \\ &= \$1000(1 + 0.10)^2 \\ &= \$1000(1.21) = \$1210 \end{aligned}$$

The key equation in the development above is $F_2 = P(1 + i)^2$. Generalized for any number of interest periods N, this expression becomes

$$\boxed{F_N = P(1 + i)^N} \tag{2.3}$$

and $(1 + i)^N$ is known as the *compound-amount factor*. It is one of several interest factors derived in this chapter for which numerical values are tabulated in App. D.

2.4.1 Nominal Interest Rates

Interest rates are normally quoted on an annual basis. However, agreements may specify that interest will be compounded several times per year: monthly, quarterly, semiannually, etc. For example, 1 year divided into four quarters with interest at 2 percent per quarter is typically quoted as *8 percent compounded quarterly.* Stated in this fashion, the 8 percent rate is called a *nominal annual interest rate.* The future value at the end of 1 year for $200 earning interest at 8 percent compounded quarterly is developed as

$$F_{3\text{ mo}} = P + Pi = \$200 + \$200(0.02) = \$200 + \$4 = \$204$$

$$F_{6\text{ mo}} = \$204 + \$204(0.02) = \$204 + \$4.08 = \$208.08$$

$$F_{9\text{ mo}} = \$208.08 + \$208.08(0.02) = \$208.08 + \$4.16 = \$212.24$$

$$F_{12\text{ mo}} = \$212.24 + \$212.24(0.02) = \$212.24 + \$4.24 = \$216.48$$

The result of the nominal interest rate is to produce a higher value than might be expected from 8 percent compounded annually. At 8 percent compounded annually, the $200 mentioned above would earn in 1 year

$$F_{12\text{ mo}} = \$200 + \$200(0.08) = \$216$$

which is $0.48 less than the amount accrued from the nominal rate of 8 percent compounded quarterly. An interest rate of 1.5 percent per month is also a nominal interest rate that could appear quite reasonable to the uninitiated. Using the compound-amount factor to calculate how much would have to be repaid on a 1-year loan of $1000 at a nominal interest rate of 18 percent compounded monthly (1.5 percent per period with 12 interest periods per year) gives

$$F_{12\text{ mo}} = \$1000(1 + 0.015)^{12} = \$1195.62$$

This can be compared with the future value of the same loan at 18 percent compounded semiannually (9 percent per period with two interest periods per year):

$$F_{2\text{ periods}} = \$1000(1.09)^2 = \$1188.10$$

Thus, more frequent compounding with a nominally stated annual rate does indeed increase the future worth.

2.4.2 Effective Interest Rates

Confusion about the actual interest earned is eliminated by stating the charge as an *effective interest rate.* Efforts to protect borrowers from exotic

statements of interest charges was the thrust behind the national truth-in-lending law passed in 1973. The effective annual interest rate is simply the ratio of the interest charge for 1 year to the principal (amount loaned or borrowed). For the 1-year loan of $1000 at a nominal annual interest rate of 18 percent compounded monthly,

$$\text{Effective annual interest rate} = \frac{F-P}{P} = \frac{\$1196 - \$1000}{\$1000} = \frac{\$196}{\$1000}(100\%) = 19.6\%$$

For the same loan at 18 percent compounded semiannually,

$$\text{Effective annual interest rate} = \frac{\$1188 - \$1000}{\$1000} = \frac{\$188}{\$1000}(100\%) = 18.8\%$$

The effective annual interest rate can be obtained without reference to the principal. Based on the same reasoning utilized previously and by using Eq. (2.1), the effective annual interest rate i_{eff} for a nominal interest rate r of 18 percent compounded semiannually is

$$i_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{0.18}{2}\right)^2 - 1$$
$$= (1 + 0.09)^2 - 1 = 1.188 - 1$$
$$= 0.188 \text{ or } 18.8\%$$

This means that a nominal annual interest rate of 18 percent compounded semiannually is equivalent to a compound-interest rate of 18.8 percent on an annual basis.

2.4.3 Continuous Compounding

The ultimate limit for the number of compounding periods in 1 year is called *continuous compounding.* Under this accrual pattern, m approaches infinity as interest compounds continuously, moment by moment. The effective interest rate for continuous compounding for a nominal interest rate r is developed as follows:

The interest periods are made infinitesimally small:

$$i_\infty = \lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^m - 1$$

The right side of the equality is rearranged to include r in the exponent:

$$\left(1+\frac{r}{m}\right)^m - 1 = \left[\left(1+\frac{r}{m}\right)^{m/r}\right]^r - 1$$

The term inside the square brackets is recognized as the value of the mathematical symbol e [$e = 2.718$ is the value of $(1 + 1/n)^n$, as n approaches infinity]:

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m/r} = e$$

By substitution,

$$i_\infty = \lim_{m\to\infty}\left[\left(1+\frac{r}{m}\right)^{m/r}\right]^r - 1 = e^r - 1$$

As an example of continuous compounding, when the nominal interest rate is $r = 18.232$ percent,

$$i_\infty = e^r - 1 = e^{0.18232} - 1 = 0.20 \text{ or } 20\%$$

and, correspondingly, when the effective annual interest rate is 22.1 percent,

$$0.221 = e^r - 1$$

$$1.221 = e^r$$

$$r = \ln 1.221 = 0.20 = 20\%$$

The most obvious computational effect of using continuous interest is that it produces a larger future amount than does the same rate compounded discretely; the effective interest when $r = 20$ percent instead of $i = 20$ percent realizes a 10.5 percent increase in value:

$$\frac{0.221 - 0.20}{0.20}(100\%) = 10.5\%$$

The rationale for using continuous interest in economic analyses is that the cash flow in certain situations is best approximated by a continuous pattern; i.e., cash transactions tend to be spread out over 1 year in a more or less even distribution rather than being concentrated at particular dates. Some mathematical models are also facilitated by the assumption of continuous compounding rather than periodic compounding.

In actual practice, however, interest rates are seldom quoted on a continuous basis, and the vast majority of organizations use discrete compounding periods in their economic studies. The reason for this is probably custom or the familiarity that makes it easier to understand periodic interest charges. Accounting practices that categorize receipts and disbursements as end-of-year values and financial experiences with annual tax, insurance, or mortgage payments contribute to thinking in terms of discrete periods. Yet continuous and discrete compounding are both only approximations of true cash flow, because cash neither flows like a free stream of water nor gushes like a geyser at given intervals. Receipts and disbursements are often irregular in amount and in timing.

In the following discussion of the time value of money and in subsequent chapters on economic comparison methods, end-of-year compounding is utilized. Tables of interest factors are provided in App. D for discrete compounding and in App. E for continuous compounding. *Discrete compounding* is implied in all examples and exercises in this text unless specified otherwise.

2.6 TIME-VALUE EQUIVALENCE

Two things are *equivalent* when they produce the same effect. The effective interest rate computed for a nominally stated interest rate is an equivalent expression of the interest charge. Both interest charges produce the same effect on an investment. In considering time-value conversion, the equivalent numerical values of money are determined, not values with equivalent *purchasing power*. The amount of goods that can be purchased with a given sum of money varies up and down as a function of special localized circumstances and nationwide or worldwide economic conditions. Inflation is discussed completely in Chap. 10, but Sec. 3.1.1 will briefly introduce the topic. In this chapter about time-value mechanics, attention is directed toward calculations based on the *earning power* of money, which relates time and earnings to locate time-equivalent money amounts.

If \$1000 were sealed and buried today, it would have a cash value of \$1000 when it was dug up 2 years from now. Regardless of changes in buying power, the value remains constant because the earning power of the money was forfeited. It was observed earlier that \$1000 deposited at 10 percent interest compounded annually has a value of $\$1000(1 + 0.10)^2 = \$1210$ after 2 years. Therefore, \$1000 today is equivalent to \$1210 in 2 years from now, if it earns at a prevailing rate of 10 percent compounded yearly. Similarly, to have \$1000 in 2 years from now, one need only deposit

$$\$1000\,\frac{1}{(1+0.10)^2} = \$826.45$$

today. In theory, then, if 10 percent is an acceptable rate of return, an investor will be indifferent between having \$826.45 in hand or having a trusted promise to receive \$1000 in 2 years.

Consider other facets of the *earning power of money.* The $1000 could have been used to pay two equal annual $500 installments. The buried $1000 could be retrieved after 1 year, an installment paid, and the remaining $500 reburied until the second payment became due. If the $1000 is instead deposited at 10 percent, $1100 would be available at the end of the first year. After the first $500 installment was paid, the remaining $600 would draw interest until the next payment. Paying the second $500 installment would leave

$$\$600\,(1.10) - \$500 = \$660 - \$500 = \$160$$

in the account. Because of the earning power of money, the initial deposit could have been reduced to $868 to pay out $500 at the end of each of the 2 years:

First year: $\$868\,(1.10) - \$500 = \$955 - \$500 = \$455$

Second year: $\$455\,(1.10) = \$500 =$ second installment

Thus, $868 is equivalent to $500 received 1 year from now plus another $500 received 2 years from now:

First year: $\dfrac{\$500}{1.10} = \$455$

Second year: $\$455 + \dfrac{\$500}{(1.10)^2} = \$455 + \dfrac{\$500}{1.21}$

$$= \$455 + \$413 = \$868$$

The concept of equivalence is the cornerstone for time-value-of-money comparisons. To have a precise meaning, income and expenditures must be identified with time as well as with amount. A decision between alternatives having receipts and disbursements spread over a period of time is made by comparing the equivalent outcomes of the alternatives at a given date.

Figure 2.1 shows the translation of $1000 at time 0 (now) to equivalent alternative expressions of cash flow. Figure 2.1 shows that

- $1000 today is equivalent to $1791 received 10 years from now.
- $1000 today is equivalent to $237.40 received at the end of each year for the next 5 years.
- $1000 today is equivalent to $317.70 received at the end of years 6, 7, 8, 9, and 10.
- $237.40 received at the end of each year for the next 5 years is equivalent to a lump sum of $1791 received 10 years from now.
- $317.70 received at the end of years 6, 7, 8, 9, and 10 is equivalent to $1791 in 10 years from now.

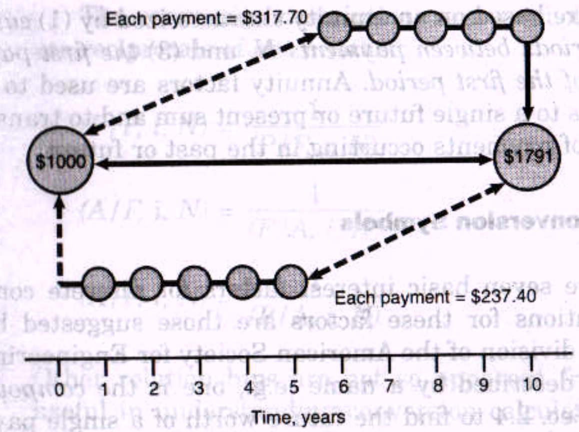


FIGURE 2.1 Equivalence outcomes with an interest rate of 6 percent compounded annually.

- $237.40 received at the end of each year for the next 5 years is equivalent to $317.70 at the end of years 6, 7, 8, 9, and 10.

Equivalence is at the heart of making engineering economic decisions. How to perform the economic evaluations, which is key to much of engineering decision making, is a major thrust of this book.

Figure 2.1 graphically shows the equivalence between cash flows, or amounts, at different times. This is similar to the popular engineering economy representation schema called *cash flow diagramming.* We will represent the cash flow relations in the development of compound-interest factors in the next section in a manner similar to Fig. 2.1. Following the compound-interest factor development, in Sec. 2.7 we formally discuss cash flow diagramming. The reason for delaying the formal discussion is that some concepts will be introduced that might cloud the compound-interest factor development.

2.6 COMPOUND-INTEREST FACTORS

Cash flow is translated to a given time by determining either its present worth or its future worth. A present-worth calculation converts a single future sum or a series of future values to an equivalent amount at an earlier date. This date is not necessarily the present. Future-worth calculations convert values occurring at any time to an equivalent amount at a later date.

Equivalent values could be determined by calculating the compound amount of each sum for each period. This tedious routine is avoided by using compound-interest tables for different present- and future-worth factors. There are two basic types of factors. The one we have already considered, in Sec. 2.4, converts a single amount to a present or future value. The other type is for a series of uniform values called an *annuity.* The tables in

App. D are based on an annuity characterized by (1) *equal payments A*, (2) *equal periods between payments N*, and (3) *the first payment occurring at the end of the first period*. Annuity factors are used to convert a series of payments to a single future or present sum and to translate single sums to a series of payments occurring in the past or future.

2.6.1 Conversion Symbols

There are seven basic interest factors for discrete compounding. Names and notations for these factors are those suggested by the engineering economy division of the American Society for Engineering Education. Each factor is described by a name (e.g., one is the *compound-amount factor*, used in Sec. 2.4 to find the future worth of a single payment) and a notational form where a functional symbol suggests the use of the interest factor, as in $(F/P, i\%, N)$, again for the compound-amount factor that is used to find F given P, given a specific interest rate $i\%$ and N. Time-value conversions and associated factors are summarized in Table 2.2.

The symbols for the first six time-value conversions are abbreviations for the equivalent values sought (future worth F, present worth P, or uniform series amounts A) and the data given (F, P, or A with its associated interest rate $i\%$ and number of compounding periods N). The arithmetic gradient conversion factor is used to convert a constantly increasing or decreasing series to a uniform series of amounts A that can then be an input to other interest factors. In the equation

$$F = \$1000\,(F/P, 10, 2)$$

\$1000 is the known present amount, the interest rate i is 10 percent, and F is the equivalent future worth after two periods ($N = 2$). The whole symbol stands for the numerical expression $(1 + 0.10)^2$, and the numerical value is found in App. D. To find the value for $(F/P, 10, 2)$, look for 2 in the N column of the 10 percent table, and then read across to the compound-amount factor column to find 1.2100.

TABLE 2.2

Interest factors for discrete cash flow with end-of-period compounding

Factor	To find	Given	Symbol
Compound amount	Future worth F	Present amount P	$(F/P, i\%, N)$
Present worth	Present worth P	Future amount F	$(P/F, i\%, N)$
Sinking fund	Annuity amounts A	Future amount F	$(A/F, i\%, N)$
Series compound amount	Future worth F	Annuity amounts A	$(F/A, i\%, N)$
Capital recovery	Annuity amounts A	Present amount P	$(A/P, i\%, N)$
Series present worth	Present worth P	Annuity amounts A	$(P/A, i\%, N)$
Arithmetic gradient conversion	Annuity amounts A	Uniform change in amount G	$(A/G, i\%, N)$

The conversion descriptions and symbols indicate that certain factors are reciprocals of one another:

$$(F/P, i, N) = \frac{1}{(P/F, i, N)}$$

$$(A/F, i, N) = \frac{1}{(F/A, i, N)}$$

$$(A/P, i, N) = \frac{1}{(P/A, i, N)}$$

Other relationships are not so apparent from the abbreviations but are useful in understanding conversion calculations. The following equalities will be verified during the development of the conversion symbols:

$$(F/P, i, N) \times (P/A, i, N) = (F/A, i, N)$$

$$(F/A, i, N) \times (A/P, i, N) = (F/P, i, N)$$

$$(A/F, i, N) + i = (A/P, i, N)$$

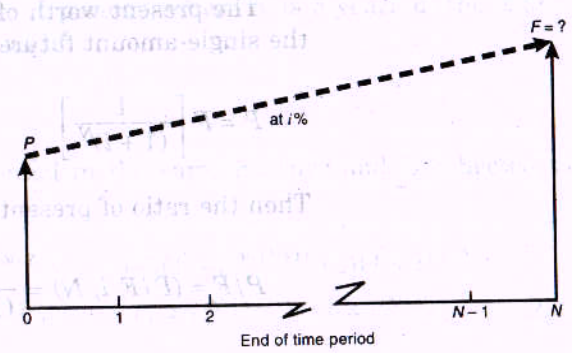
2.6.2 Development of Interest Formulas

A better understanding of the conversion process is achieved by studying the development of the interest factor formulas. The symbols employed in the following discussion of the seven interest factors are the same as those described previously: i = interest rate per period and N is the number of compounding periods. Remember that these factors are for discrete compounding and their numerical values are tabulated in App. D; corresponding factors for continuous compounding are presented in Sec. 2.8.3.

Additional sample applications of the interest factors are provided in the review exercises at the end of this chapter.

2.6.2.1 Compound-amount factor (single payment)

Use: To find F, given P.
Symbols: $(F/P, i\%, N)$
Formula: $F = P(1 + i)^N$
$= P(F/P, i, N)$



The effect of compound interest on an investment was demonstrated in previous examples. The future worth of a present amount when interest is accumulated at a specific rate i for a given number of periods N, where F_1 is the future worth at the end of the first period and F_N is the future worth at the end of N periods, is found by using the development of Eq. (2.3) given earlier:

$$F_1 = P + Pi = P(1+i)$$

$$F_2 = F_1(1+i) = P(1+i)(1+i) = P(1+i)^2$$

$$F_3 = F_2(1+i) = F_1(1+i)^2 = P(1+i)(1+i)^2 = P(1+i)^3$$

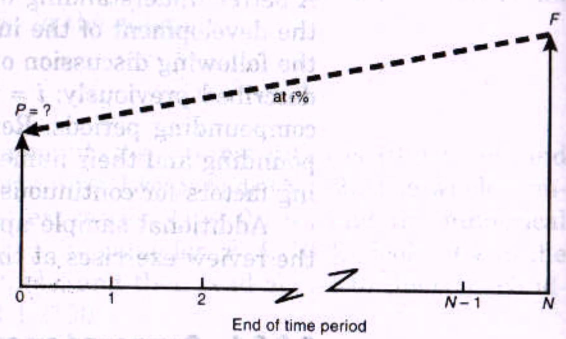
$$F_N = P(1+i)^N$$

The ratio of future worth to present amount is then expressed as

$$F/P = (F/P, i, N) = (1+i)^N$$

2.6.2.2 Present-worth factor (single payment)

Use: To find P, given F.
Symbols: $(P/F, i\%, N)$
Formula: $P = F[1/(1+i)^N] = F(P/F, i, N)$



The present worth of a sum N periods in the future is P. Rearranging the single-amount future-value formula, Eq. (2.3), gives

$$P = F\left[\frac{1}{(1+i)^N}\right] \tag{2.4}$$

Then the ratio of present worth to future value is

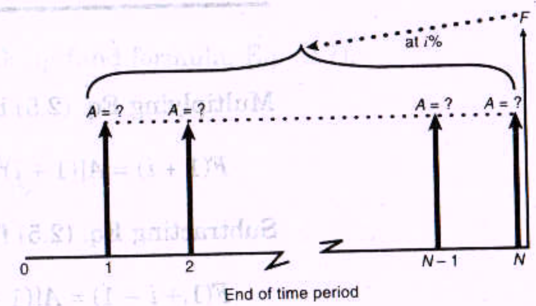
$$P/F = (P/F, i, N) = \frac{1}{(1+i)^N}$$

That the present-worth factor is simply the reciprocal of the compound-amount factor is confirmed by applying it to the data given in Table 2.1, where the future worth of \$1000 at 10 percent compounded annually was shown to be \$1210. Equivalently, $P = F(P/F, i, N)$ is the expression for present worth when the future worth is known. The numerical value of the present-worth factor $(P/F, 10, 2)$ is found in the 10 percent table of App. D at $N = 2$ to be 0.82645. Then

$$P = \$1210(0.82645) = \$1000$$

2.6.2.3 Sinking fund factor (uniform series)

Use: To find A, given F.
Symbols: $(A/F, i\%, N)$
Formula: $A = F\{i/[(1+i)^N - 1]\} = F(A/F, i, N)$



A fund established to accumulate a given future amount through the collection of a uniform series of payments is called a *sinking fund*. Each payment has a constant value A and is made at the end of an interest period.

The growth pattern of a sinking fund is illustrated in Table 2.3. Each end-of-year payment A is equal to \$1000, and payments continue for 5 years. Interest is 8 percent compounded annually. It is assumed that each payment begins to draw interest as soon as it is deposited in the sinking fund account. Thus, the first payment draws interest for 4 years, and the last payment receives no interest.

A more general expression for the future worth of an annuity develops from the use of symbols to represent the values in Table 2.3. The first payment, earning interest for $N - 1$ periods, where N is 5 years in the example, increases to a future worth of

$$F = A(1+i)^{N-1}$$

Each of the payments is treated in the same manner and is collected to obtain the total amount F:

$$F = A(1+i)^{N-1} + A(1+i)^{N-2} + \cdots + A(1+i)^{N-(N+1)} + A(1+i)^{N-N}$$

$$F = A[(1+i)^{N-1} + (1+i)^{N-2} + \cdots + (1+i)^{N-(N+1)} + (1+i)^{N-N}] \tag{2.5}$$

TABLE 2.3

Compound amount of a uniform series of payments

Time of payment (end of year)	Amount A of payment, $	Future worth at end of each year, $
1	1000	$1000(1.08)^4 = 1360$
2	1000	$1000(1.08)^3 = 1260$
3	1000	$1000(1.08)^2 = 1166$
4	1000	$1000(1.08)^1 = 1080$
5	1000	$1000(1.08)^0 = 1000$
		Annuity value F at the end of year 5 = 5866

Multiplying Eq. (2.5) by $1 + i$ gives

$$F(1+i) = A[(1+i)^N + (1+i)^{N-1} + \cdots + (1+i)^{N-(N+2)} + (1+i)^1] \qquad (2.6)$$

Subtracting Eq. (2.5) from Eq. (2.6) results in

$$F(1+i-1) = A[(1+i)^N - 1]$$

and solving for A gives

$$A = F\left[\frac{i}{(1+i)^N - 1}\right] \qquad (2.7)$$

The sinking fund factor can now be expressed as

$$(A/F, i, N) = \frac{i}{(1+i)^N - 1} \qquad (2.8)$$

Now, applying the sinking fund factor to the data in Table 2.3, we have

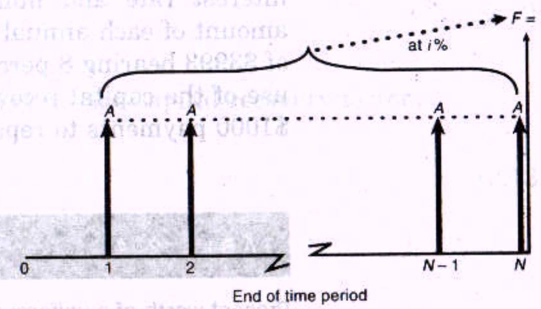
$$A = \$5866(A/F, 8, 5)$$
$$= \$5866(0.17046) = \$1000$$

The sinking fund factor that we have just developed for a uniform series will now make it very easy for us to determine the series compound-amount factor that is used for finding F given A.

2.6.2.4 Series compound-amount factor (uniform series)

Use: To find F, given A
Symbols: $(F/A, i\%, N)$
Formula:

$$F = A\,\frac{(1+i)^N - 1}{i}$$
$$= A(F/A, i, N)$$



From the development of the sinking fund formula, Eq. (2.7),

$$Fi = A[(1+i)^N - 1]$$

which is expressed in terms of F as

$$F = A\left[\frac{(1+i)^N - 1}{i}\right]$$

Then the series compound-amount factor to use in calculating the future worth of an annuity is

$$(F/A, i, N) = \frac{(1+i)^N - 1}{i}$$

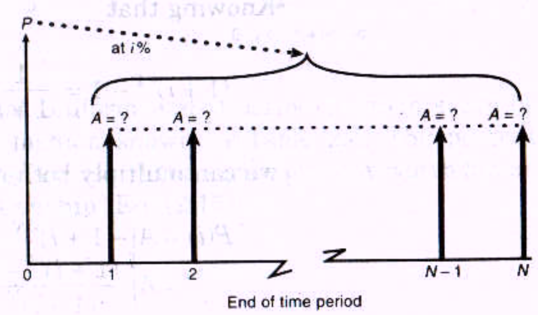
The future worth of the annuity composed of five annual payments of $1000, each invested at 8 percent compounded annually, as was shown in Table 2.3, is

$$F = \$1000(F/A, 8, 5)$$
$$= \$1000(5.86660) = \$5866.60$$

2.6.2.5 Capital recovery factor (uniform series)

Use: To find A, given P.
Symbols: $(A/P, i\%, N)$
Formula:

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$$
$$= P(A/P, i, N)$$



The capital recovery factor is used to determine the amount of each future annuity payment required to dissipate a given present value when the interest rate and number of payments are known. For instance, the amount of each annual payment made for 5 years in order to repay a debt of \$3993 bearing 8 percent annual interest can be determined through the use of the capital recovery factor. Table 2.4 shows that it would take five \$1000 payments to repay the \$3993 debt.

TABLE 2.4

Present worth of a uniform series of payments

Time of payment (end of year)	Amount A of payment, \$	Present worth of payments at beginning of year 1, \$
1	1000	$1000(1.08)^{-1} = 926$
2	1000	$1000(1.08)^{-2} = 857$
3	1000	$1000(1.08)^{-3} = 794$
4	1000	$1000(1.08)^{-4} = 735$
5	1000	$1000(1.08)^{-5} = 681$
		Present worth *P* of 5-year annuity = 3993

Using symbols to represent the conversions shown in Table 2.4, we find that the present worth of an annuity is

$$P = A[(1+i)^{-1} + (1+i)^{-2} + \cdots + (1+i)^{-N+1} + (1+i)^{-N}] \tag{2.9}$$

Multiplying both sides of Eq. (2.9) by $(1+i)^{-1}$ gives

$$P(1+i)^{-1} = A[(1+i)^{-2} + (1+i)^{-3} + \cdots + (1+i)^{-N} + (1+i)^{-N-1}] \tag{2.10}$$

Subtracting Eq. (2.9) from (2.10) results in

$$P[(1+i)^{-1} - 1] = A[(1+i)^{-N-1} - (1+i)^{-1}] \tag{2.11}$$

Knowing that

$$(1+i)^{-1} - 1 = \frac{1}{1+i} - \frac{1+i}{1+i} = \frac{-i}{1+i}$$

we can multiply both sides of Eq. (2.11) by $-(1+i)$ to get

$$P(i) = A[-(1+i)^{-N-1}(1+i) + (1+i)^{-1}(1+i)] = A\left[\frac{(1+i)^N - 1}{(1+i)^N}\right]$$

and

$$A = P\,\frac{i(1+i)^N}{(1+i)^N - 1} \tag{2.12}$$

from which comes the expression for the capital recovery factor

$$(A/P, i, N) = \frac{i(1+i)^N}{(1+i)^N - 1} \tag{2.13}$$

As applied to the data in Table 2.4 where $P = \$3993$,

$$A = \$3993\,(A/P, 8, 5)$$
$$= \$3993(0.25046) = \$1000$$

The relationship between the capital recovery factor [Eq. (2.13)] and the sinking fund factor [Eq. (2.8)] is as follows:

$$(A/P, i, N) = (A/F, i, N) + i \tag{2.14}$$

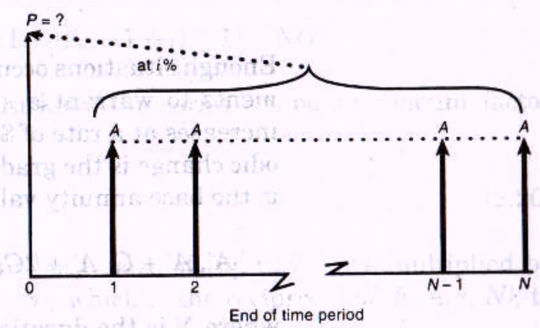
This can be shown by substituting the factors from Eqs. (2.8) and (2.13) into Eq. (2.14):

$$\frac{i(1+i)^N}{(1+i)^N - 1} = \frac{i}{(1+i)^N - 1} + i = \frac{i + i(1+i)^N - i}{(1+i)^N - 1} = \frac{i(1+i)^N}{(1+i)^N - 1} \tag{2.15}$$

2.6.2.6 Series present-worth factor (uniform series)

Use: To find *P*, given *A*.
Symbols: $(P/A, i\%, N)$
Formula:

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = A(P/A, i, N)$$



The present value of a series of uniform end-of-period payments can be calculated in the cumbersome fashion shown in Table 2.4. The present worth is more readily determined by use of the series present-worth factor. Expressing the known relationship [Eq. (2.15)]

$$A = P\,\frac{i(1+i)^N}{(1+i)^N - 1}$$

in terms of P yields

$$P = A\frac{(1+i)^N - 1}{i(1+i)^N} = A(P/A, i, N) \tag{2.16}$$

which is the time-value expression of the present worth of an annuity.

The reciprocal relationship between the capital recovery factor and the series present-worth factor is demonstrated by the data from Table 2.4:

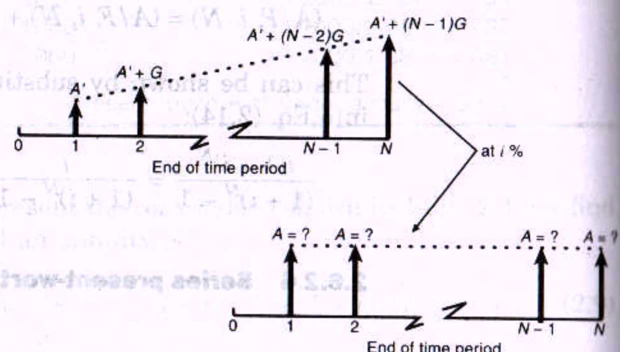
$$P = \$1000(P/A, 8, 5)$$
$$= \$1000(3.99271) = \$3992.71$$

which indicates the equivalence of having \$3992.71 in hand and a firm contract to receive five year-end payments of \$1000 each when the interest rate is 8 percent.

2.6.2.7 Arithmetic gradient conversion factor (to uniform series)

Use: To find A, given G.
Symbols: $(A/G, i\%, N)$
Formula:

$$A = G\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right]$$
$$= G(A/G, i, N)$$



Enough situations occur in which series of payments increase at equal increments to warrant a special conversion factor. A series of payments that increases at a rate of \$200 per year is illustrated in Fig. 2.2. The \$200 periodic change is the gradient G, and the payment at the end of the first period is the base annuity value A'. The pattern of an arithmetic gradient is then

$$A', A' + G, A' + 2G, \ldots, A' + (N-1)G$$

where N is the duration of the series ($N = 5$ in Fig. 2.2).

A uniformly increasing series can be evaluated by calculating F or P for each individual payment and summing the results. The calculation time is reduced by converting the series to an equivalent annuity of equal payments A. The formula for this translation is developed by separating the series shown in Fig. 2.2 into two parts: a base annuity designated A' and an arithmetic gradient series increasing by G each period. The future worth of the G values in Fig. 2.2 is calculated as

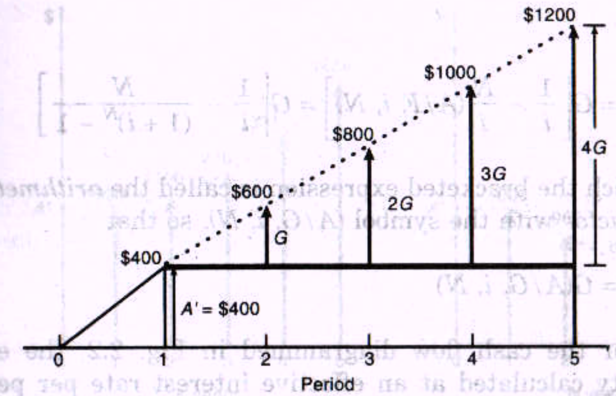


FIGURE 2.2 Uniform gradient series for five periods.

$$F = \$200\,(F/P, i, 3) + \$400\,(F/P, i, 2) + \$600\,(F/P, i, 1) + \$800$$
$$= G(1+i)^3 + 2G(1+i)^2 + 3G(1+i) + 4G \tag{2.17}$$

Multiplying Eq. (2.17) by $1 + i$ gives

$$F(1+i) = G(1+i)^4 + 2G(1+i)^3 + 3G(1+i)^2 + 4G(1+i) \tag{2.18}$$

Subtracting Eq. (2.18) from Eq. (2.17) yields

$$F - F - Fi = -G(1+i)^4 - G(1+i)^3 - G(1+i)^2 - G(1+i) + 4G \tag{2.19}$$

Knowing that for this case $4G = (N-1)G$ and multiplying both sides of Eq. (2.19) by −1, we get

$$Fi = G[(1+i)^4 + (1+i)^3 + (1+i)^2 + (1+i) + 1] - NG$$

We have, inside the square brackets, the series compound-amount factor $(F/A, i, 5)$ so that

$$Fi = G(F/A, i, N) - NG \tag{2.20}$$

To convert F to an annuity, both sides of Eq. (2.20) are multiplied by the sinking fund factor $(A/F, i, N)$, which is the reciprocal of $(F/A, i, N)$, to get

$$Fi(A/F, i, N) = G - NG(A/F, i, N)$$

Since $A = F(A/F, i, N)$, we get

$$A = \frac{G}{i} - \frac{NG}{i}(A/F, i, N)$$

or

$$A = G\left[\frac{1}{i} - \frac{N}{i}(A/F,\ i,\ N)\right] = G\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right] \qquad (2.21)$$

in which the bracketed expression is called the *arithmetic gradient conversion factor* with the symbol $(A/G, i, N)$, so that

$$A = G(A/G,\ i,\ N)$$

For the cash flow diagrammed in Fig. 2.2, the equivalent uniform annuity calculated at an effective interest rate per period of 10 percent is

$$\begin{aligned} A &= A' + G(A/G,\ i,\ N) \\ &= \$400 + \$200(A/G,\ 10,\ 5) \\ &= \$400 + \$200(1.81013) = \$762 \end{aligned}$$

which means that five end-of-period payments of \$762 are equivalent to five payments starting at \$400 and which increase by \$200 each period.

The gradient factor may also be applied to a pattern of payments that decrease by a constant increment each period. The formula would then be

$$A = A' - G(A/G,\ i,\ N)$$

As an example, assume that an endowment was originally set up to provide a \$10,000 first payment with payments decreasing by \$1000 each year during the 10-year endowment life. What constant annual payment for 10 years would be equivalent to the original endowment plan if $i = 8$ percent? We have

$$\begin{aligned} A &= \$10{,}000 - \$1000(A/G,\ 8,\ 10) \\ &= \$10{,}000 - \$1000(3.87131) \\ &= \$6128.69 \end{aligned}$$

2.6.2.8 Development of geometric series formulas

A *geometric series* is a nonuniform progression that grows or declines at a *constant percentage rate* per period. The most familiar example is the effect of inflation or deflation on a cash flow stream. For instance, the rate of growth during a period of inflation could be 10 percent per period. An item priced at \$1000 during the first year would then increase by 10 per-

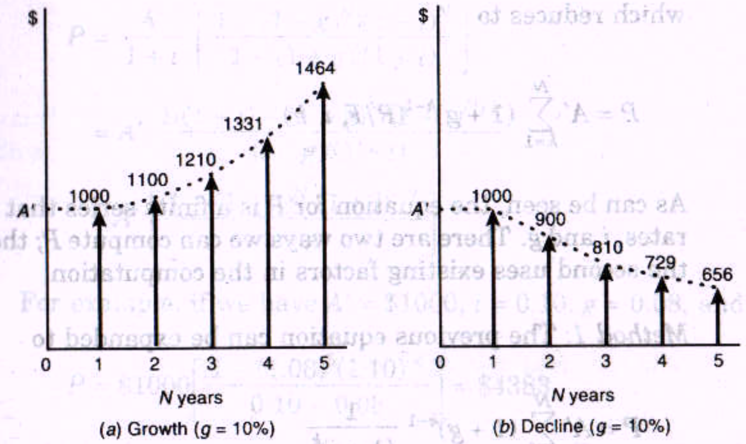


FIGURE 2.3 Geometric growth and decline patterns.

cent per year, as shown in Fig. 2.3*a*. A decline of 10 percent per year, starting from the same \$1000 figure, would take the pattern shown in Fig. 2.3*b*.

We could determine the present worth of the geometric growth pattern in Fig. 2.3*a* in a cumbersome manner, using the appropriate $(P/F, i, N)$ values:

$$\begin{aligned} P = {} & \$1000(P/F,\ i,\ 1) + \$1100\ (P/F,\ i,\ 2) + \$1210(P/F,\ i,\ 3) \\ & + \$1331(P/F,\ i,\ 4) + \$1464(P/F,\ i,\ 5) \end{aligned} \qquad (2.22)$$

If the interest rate i is 8 percent and, as we know, the geometric growth rate g is 10 percent, Eq. (2.22) becomes

$$\begin{aligned} P = {} & \$1000[(P/F,\ 0.08,\ 1) + (1 + 0.10)\ (P/F,\ 0.08,\ 2) \\ & + (1 + 0.10)^2\ (P/F,\ 0.08,\ 3) + (1 + 0.10)^3\ (P/F,\ 0.08,\ 4) \\ & + (1 + 0.10)^4\ (P/F,\ 0.08,\ 5)] \end{aligned} \qquad (2.23)$$

Substituting the appropriate App. D values for $(P/F,\ 0.08,\ N)$ into Eq. (2.23), we get

$$\begin{aligned} P = {} & \$1000[0.92593 + 1.1(0.85734) + 1.21(0.79383) + 1.331(0.73503) \\ & + 1.464(0.68058)] = \$4804 \end{aligned}$$

Now, let us generalize Eq. (2.22) for any A', P, N, g, and i:

$$\begin{aligned} P = {} & A'(P/F,\ i,\ 1) + A'(1+g)\ (P/F,\ i,\ 2) + A'(1+g)^2\ (P/F,\ i,\ 3) + \cdots \\ & + A'(1+g)^{N-1}(P/F,\ i,\ N) \end{aligned}$$

which reduces to

$$P = A' \sum_{k=1}^{N} (1+g)^{k-1}(P/F,\ i,\ k)$$

As can be seen, the equation for P is a finite series that is a function of two rates, i and g. There are two ways we can compute P; the first is direct, and the second uses existing factors in the computation.

Method 1: The previous equation can be expanded to

$$P = A' \sum_{k=1}^{N} (1+g)^{k-1} \frac{1}{(1+i)^k}$$

$$= \frac{A'}{1+g} \sum_{k=1}^{N} \left(\frac{1+g}{1+i}\right)^k$$

We can see that if $i = g$,

$$P = \frac{NA'}{1+g} = \frac{NA'}{1+i}$$

When $i \neq g$, we can simplify the finite series for P by

$$P = \frac{A'}{1+g} \sum_{k=1}^{N} \left(\frac{1+g}{1+i}\right)^k$$

Letting $(1+g)/(1+i) = x$, we get

$$P = \frac{A'}{1+g} \sum_{k=1}^{N} x^k$$

$$= \frac{A'}{x(1+i)}(x + x^2 + \cdots + x^N)$$

$$= \frac{A'}{1+i}(1 + x^1 + \cdots + x^{N-1})$$

Multiplying $1 + x^1 + \cdots + x^N$ by x, subtracting the result from the previous equation, and solving for the closed form of the series give

$$P = \frac{A'}{1+i}\left(\frac{1-x^N}{1-x}\right)$$

Substituting $(1+g)/(1+i)$ back for x results in

$$P = \frac{A'}{1+i}\left[\frac{1-(1+g)^N/(1+i)^N}{1-(1+g)/(1+i)}\right]$$

$$= A' \frac{1/(1+i) - (1+g)^N/(1+i)^{N+1}}{(i-g)/(1+i)}$$

$$= A'\left[\frac{1-(1+g)^N(1+i)^{-N}}{i-g}\right]$$

For example, if we have $A' = \$1000$, $i = 0.10$, $g = 0.08$, and $N = 5$,

$$P = \$1000\left[\frac{1-(1.08)^5(1.10)^{-5}}{0.10-0.08}\right] = \$4383$$

Similarly, if the values for i and g were reversed:

$$P = \$1000\left[\frac{1-(1.10)^5(1.08)^{-5}}{0.08-0.10}\right] = \$4804$$

Method 2. We can compute the present worth of the geometric gradient by using a pseudo interest rate that depends on the relationship between i and g.

Case 1. $g > i$
Form $i' = (1+g)/(1+i) - 1$ and use i' in

$$P = \frac{A'}{1+i}(F/A,\ i',\ N)$$

For the earlier example with $g = 0.10$ and $i = 0.08$, we have

$$i' = \frac{1.10}{1.08} - 1 = 0.0185$$

This requires interpolation from the tables at the back of the book; or CHEER, the computer program ancillary to this text, can be used to find $(F/A, 1.85, 5)$ directly, as we will see in the review exercises at the end of this chapter:

$$P = \frac{\$1000}{1.08}(5.1585) = \$4804$$

This is exactly what we expected from our earlier direct calculation.

Case 2. $g < i$
Form $i' = (1+i)/(1+g) - 1$ and use i' in

$$P = \frac{A'}{1+g}(P/A, i', N)$$

If we reversed the values for i and g in the previous example, we would have i' the same as before since the larger of g or i is always in the numerator; $i' = 0.0185$.

$$P = \frac{\$1000}{1.08}(4.73405) = \$4383$$

2.6.2.9 Midyear accounting convention

Usually, cash flows are assumed to occur at the end of a time period, say, at the end of a year or at the end of a quarter. This might require the accumulation of multiple incomes or disbursements that occur during a time period and the use of the accumulated values at the end of the period without considering the effects of interest. The errors in using the end-of-period convention are generally insignificant. The federal government requires that an asset be placed in service at the middle of a purchase year, which forces an asset with a life of 5 years, e.g., to be depreciated over 6 years. Therefore, an asset depreciated over 5 years will be assumed to have costs over 6 years. In a straight-line depreciation method, the capital cost depreciation would be split in the 6 years as follows: 10, 20, 20, 20, 20, and 10 percent at time periods 1 through 6, respectively. Further comments will be given on the midyear assumption in Chap. 9, which covers the implications of income taxes on engineering economic decisions.

2.7 CASH FLOW DIAGRAMS

Figure 2.1 and diagrams with the compound-interest factor development showed how equivalence between incomes and payments made at various times can be clarified by a graphical display. Figure 2.1 is not really representative of a realistic engineering economy situation because several equivalent situations are shown on the diagram. Be that as it may, representing engineering decision data in graphical form can greatly enhance the understanding of the problem. Cash flow diagrams are tools to help the decision maker to understand and solve such problems.

During the construction of a cash flow diagram, the structure of a problem often becomes distinct. It is usually advantageous to first define the time frame over which cash flows occur. This establishes the horizontal scale, which is divided into time periods, often in years. Receipts and disbursements are then located on the time scale in adherence to problem specifications. Individual outlays or receipts are designated by vertical lines; relative magnitudes can be suggested by the heights of lines, but exact scaling wastes time. Whether a cash flow is positive or negative (pos-

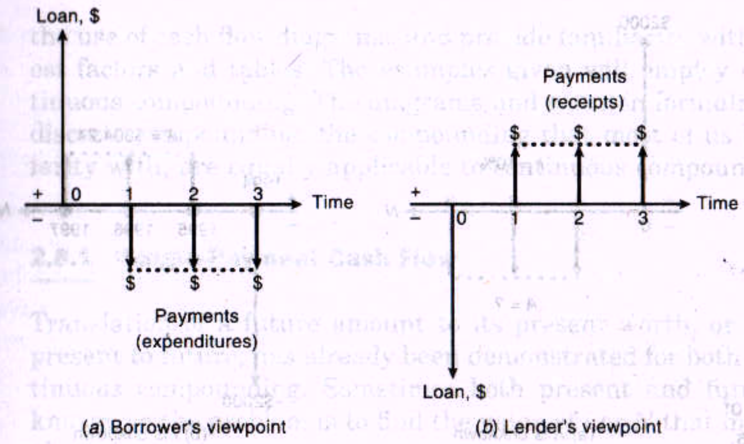


FIGURE 2.4 Perspectives for cash flow diagrams.

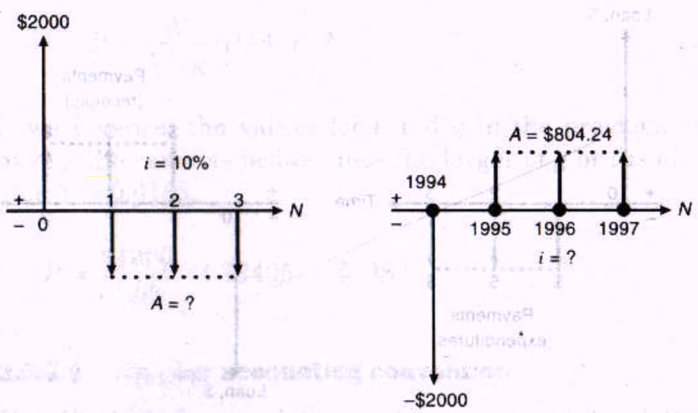
itive above the axis and negative below) depends on whose viewpoint is portrayed.

Figure 2.4a and b represents the same transaction: a loan paid off in three installments. From the borrower's viewpoint in Fig. 2.4a, the receipt of the loan is a positive inflow of cash, whereas subsequent installment payments represent negative outflows. Flows are reversed when viewed from the lender's perspective in Fig. 2.4b.

Although cash flow diagrams are simply graphical representations of income and outlay, they should exhibit as much information as possible. It is useful to show the interest rate, and it may be helpful to designate the unknown that must be solved for in a problem. Figure 2.4 is redrawn in Fig. 2.5 to represent specific problems. In Fig. 2.5a, the three equal payments are represented by a convenient convention that indicates a 3-year annuity in which the payment size A is unknown. Amount A may be circled to indicate the needed solution. The given interest rate (i = 10 percent) is entered in a conspicuous space, and the amount of the loan is placed on the arrow at time 0. The style of a cash flow diagram is important only insofar as it contributes to clarity.

In Fig. 2.5b, numbered years are replaced with dates, and the size of A is given. The problem is to find the interest rate that makes the annuity equivalent to the loan value. Sometimes it may clarify the situation to put in dashed lines for arrows that represent cash flows of unknown magnitude. This or a similar tactic is especially useful when a problem comprises several separate cash flow segments, each of which must be replaced by an equivalent value; these, in turn, are converted to a single sum representing all the segments.

The obvious requirements for cash flow diagramming are completeness, accuracy, and legibility. The measure of a successful diagram is that someone else can understand the problem fully from it. If it passes this test, you are unlikely to confuse yourself. We will utilize the concepts of



(a) A is unknown (b) i is unknown

FIGURE 2.5
Different versions of cash flow diagrams.

cash flow diagramming in the following sections (and in later chapters) where a series of examples will be presented.

2.8 CALCULATION OF TIME-VALUE EQUIVALENCES

Concepts concerning the time value of money will now become working tools. Compound-interest factors for both discrete and continuous interest will be applied to a variety of cash flows. The purpose of the calculations is to develop skills in converting cash flow patterns to equivalent sums that are more useful in comparing investments. The use of cash flow diagrams will be emphasized to portray receipts and disbursements associated with an economic situation.

The purpose of time-value calculations is to translate receipts and disbursements of various amounts occurring at various times to a cash flow pattern that assists an economic evaluation, which will generally be for *engineering decision making* in this text. The translation is essentially mechanical in the same way as a vector is routinely decomposed into component forces; rules of geometry direct vector operations, and time-value relations direct cash flow translations. Although errors in translation can arise from carelessness, mistakes due to incorrect problem formulations are the ones to guard against.

The notable statements in a discounted cash flow problem are the elements P, F, A, N, and i. Generally, three of the elements are known for each cash flow, and the problem entails solving for a fourth element. Several money-time translations may be required in one solution.

A variety of typical cash flow patterns are treated in the following pages. Examples are presented to put cash flow problems in a realistic setting, show different perspectives for the same type of problem, demonstrate

the use of cash flow diagrams, and provide familiarity with the use of interest factors and tables. The examples given will employ discrete and continuous compounding. The diagrams and solution formulations common to discrete compounding, the compounding that most of us have some familiarity with, are equally applicable to continuous compounding.

2.8.1 Single-Payment Cash Flow

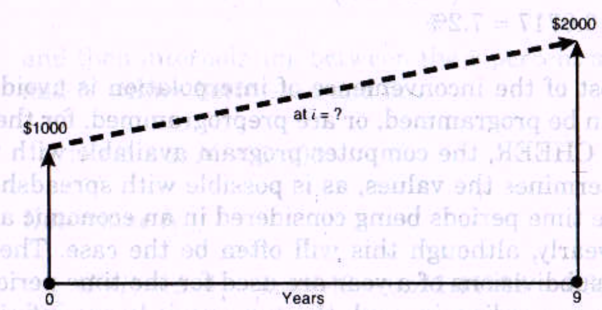
Translation of a future amount to its present worth, or the reverse from present to future, has already been demonstrated for both discrete and continuous compounding. Sometimes both present and future amounts are known, so the problem is to find the value of i or N that makes them equivalent.

EXAMPLE 2.3 **Unknown Interest Rate**

At what annual interest rate will $1000 invested today be worth $2000 in 9 years?

Solution

Given P = $1000, F = $2000, and N = 9 years, find i.



$$\frac{F}{P} = (F/P, i, 9)$$

$$\frac{\$2000}{\$1000} = 2 = (F/P, i, 9)$$

$$i = 8\%$$

The interest rate i is determined by locating the interest rate at which the single-payment compound-amount factor is equal to 2.0 and N = 9 (a reciprocal relation using a present-worth factor could serve as well). The numerical value of i is found by leafing through the pages of interest rates and noting the appropriate factor values for the given number of periods. Another way to find i directly is

$$(F/P, i, 9) = (1+i)^9 = 2 \qquad i = 2^{1/9} - 1 = 0.08$$

When tables are used rather than CHEER to determine compound-interest factor values, it is often necessary to interpolate when N or i is unknown. The error introduced by linear interpolation is relatively insignificant for most practical applications. If the investment period for Example 2.3 had been 10 years instead of 9, the interest rate calculation would have been

$$(F/P, i, 10) = F/P = \frac{\$2000}{\$1000} = 2.0$$

At i = 7 percent, $(F/P, 7, 10) = 1.96715$; at i = 8 percent, $(F/P, 8, 10) = 2.15892$. Then, by interpolation,

interpolation

$$\begin{aligned} i &= 0.07 + 0.01\left(\frac{2.0000 - 1.96715}{2.15892 - 1.96715}\right) \\ &= 0.07 + 0.01\left(\frac{0.03285}{0.19177}\right) \\ &= 0.0717 = 7.2\% \end{aligned}$$

Most of the inconvenience of interpolation is avoided with calculators that can be programmed, or are preprogrammed, for the interest factor formulas. CHEER, the computer program available with this text, very simply determines the values, as is possible with spreadsheet applications.

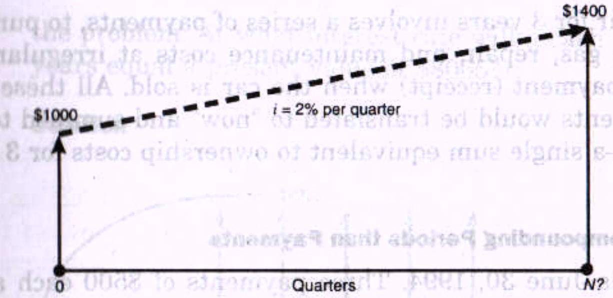
The time periods being considered in an economic analysis do not have to be yearly, although this will often be the case. The ultimate situation where subdivisions of a year are used for the time periods is where continuous compounding is used; the time periods are infinitesimally small. In engineering economic analyses, the time periods often occur quarterly, or four times per year.

EXAMPLE 2.4 **Unknown Number of Interest Periods**

A loan of \$1000 is made today under an agreement that \$1400 will be received in payment sometime in the future. When should the \$1400 be received if the loan is to earn interest at a rate of 8 percent compounded quarterly?

Solution

Given $P = \$1000$, $F = \$1400$, $i = r/m$ = 8 percent/4 = 2 percent, find N in quarters:



$$\frac{F}{P} = \frac{\$1400}{\$1000} = 1.4$$

By interpolation, N = 17 quarters, or 4 years 3 months.

Example 2.4 is slightly deceptive because it involves a nominal interest rate. The problem is clarified by noting in the cash flow diagram that the time scale is in quarters of a year and that i is given as the rate per quarter. The problem could also have been solved by converting the nominal rate to its equivalent effective annual interest rate

$$i_{\text{eff}} = \left(1 + \frac{0.08}{4}\right)^4 - 1 = 1.082 - 1 = 0.082$$

and then interpolating between the 8 percent and 9 percent interest tables and N values between 4 and 5:

$$(F/P, 8.2, N) = 1.400$$

Determine N.

$$(F/P, 8.2, 4) = 1.36049 + 0.2(1.41158 - 1.36049) = 1.37071$$

$$(F/P, 8.2, 5) = 1.46933 + 0.2(1.53862 - 1.46933) = 1.48319$$

$$\begin{aligned} N &= 4 + 1\left(\frac{1.4000 - 1.37071}{1.48319 - 1.37071}\right) \\ &= 4.26 \text{ years} \end{aligned}$$

The result differs from 4 years 3 months only by minor roundoff errors in the interest factors.

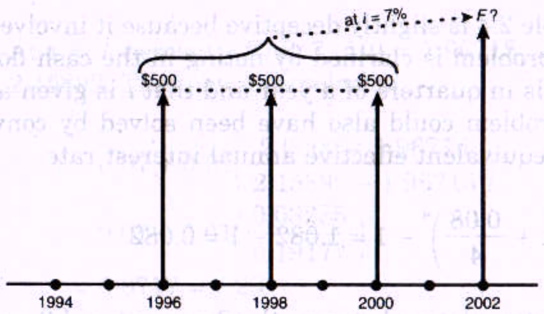
2.8.2 Multiple-Payment Cash Flows

Practical problems customarily involve both single payments and annuities. For instance, to determine the equivalent present worth (cost) of owning a car for 3 years involves a series of payments, to purchase the car and provide gas, repair, and maintenance costs at irregular intervals, and a single payment (receipt) when the car is sold. All these receipts and disbursements would be translated to "now" and summed to find the present worth—a single sum equivalent to ownership costs for 3 years.

EXAMPLE 2.5 **More Compounding Periods than Payments**

"Now" is June 30, 1994. Three payments of $500 each are to be received every 2 years, starting 2 years from now, and deposited in a bank where they will earn interest at 7 percent per year. How large will the bank account be on June 30, 2002?

Solution



$$\begin{aligned} F &= \$500\,(F/P,\,7,\,6) + \$500\,(F/P,\,7,\,4) + \$500\,(F/P,\,7,\,2) \\ &= \$500\,(1.50073 + 1.31080 + 1.14490) \\ &= \$500\,(3.95643) = \$1978 \end{aligned}$$

The equal payments in Example 2.5 do not constitute an ordinary annuity because there are fewer payments than there are compounding periods. Therefore, each payment must be translated individually to the 2002 date and added to the worth of the other payments at that date to obtain the equivalent future worth of all three payments.

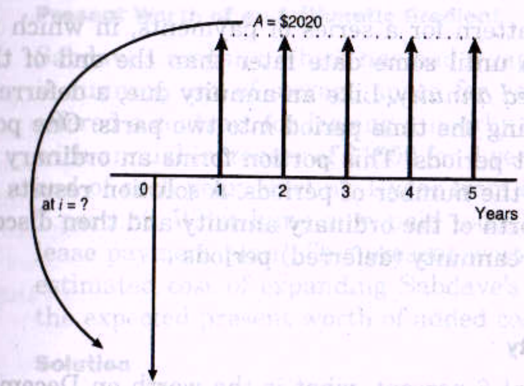
Frequently in engineering economic decision making, a company will require that a minimum specified interest rate be achievable before investing in a new process or product. This means that the interest rate in the analysis will be the unknown.

EXAMPLE 2.6 **Annuity with an Unknown** ***i***

A numerically controlled milling machine that can be purchased for $8065 is estimated to reduce production costs annually by $2020. The machine will operate for 5 years, at which time it will have no resale value. What rate of return will be earned on the investment? (Alternative statement of

the problem: At what interest rate will a cash flow of $2020 per year for 5 years equal a present value of $8065?)

Solution



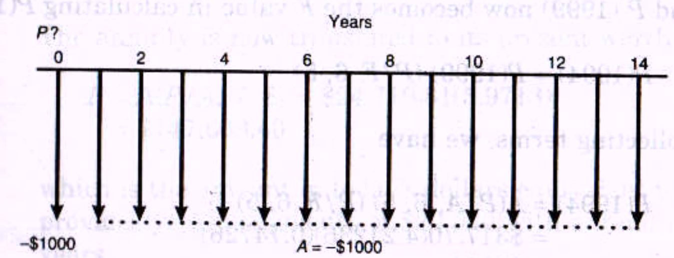
$$\frac{P}{A} = \frac{\$8065}{\$2020} = 3.993 = (P/A,\,i,\,5) \qquad \text{and} \qquad i = 8\%$$

A series of payments made at the beginning instead of the end of each period is sometimes referred to as an *annuity due*. Rather than create a special factor for this annuity pattern, we divide the series into two parts. If the first payment is translated separately, the remaining payments fit the pattern for an ordinary annuity beginning at the time of the first payment. The present worth of the series is the sum of the first payment plus the product of one payment times the series present-worth factor, where N is the number of payments minus 1.

EXAMPLE 2.7 **Annuity Due**

What is the present worth of a series of 15 year-end payments of $1000 each, when the first payment is due today and the interest rate is 5 percent?

Solution



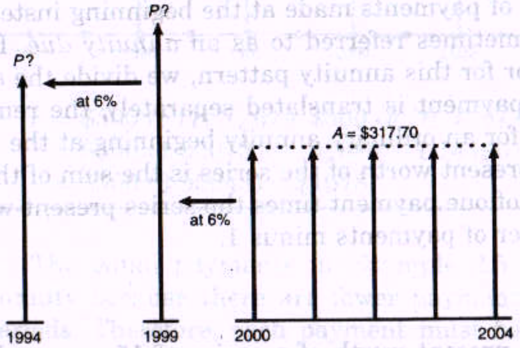
$$P = A + A(P/A, 5, 14)$$
$$= \$1000 + \$1000(9.89864)$$
$$= \$1000 + \$9899$$
$$= \$10{,}899$$

Another pattern for a series of payments, in which the first payment does not begin until some date later than the end of the first period, is called a *deferred annuity*. Like an annuity due, a deferred annuity is evaluated by dividing the time period into two parts. One portion is the number of payment periods. This portion forms an ordinary annuity. The second portion is the number of periods. A solution results from determining the present worth of the ordinary annuity and then discounting this value through the preannuity (deferred) periods .

EXAMPLE 2.8 **Deferred Annuity**

With interest at 6 percent, what is the worth on December 31, 1994, of a series of year-end payments of \$317.70 made from the years 2000 through 2004?

Solution



Starting from the known values of A, i, and N, we have at the end of 1999:

$$P(1999) = A(P/A, 6, 5)$$

and P (1999) now becomes the F value in calculating $P(1994)$:

$$P(1994) = P(1999)\ (P/F, 6, 5)$$

Collecting terms, we have

$$P(1994) = A(P/A, 6, 5)\ (P/F, 6, 5)$$
$$= \$317.70(4.21236)(0.74726)$$
$$= \$1000$$

The results of Example 2.8 may be recognized as one of the equivalent outcomes presented without proof in Fig. 2.1.

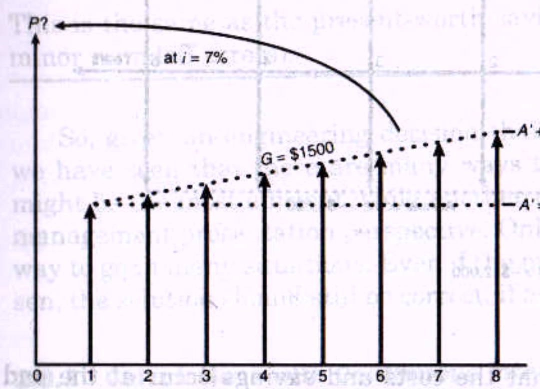
Many engineering economy problems require the use of an arithmetic gradient due to expected annual increases in costs or incomes.

EXAMPLE 2.9 **Present Worth of an Arithmetic Gradient**

Sabdave Company, which has had a rapid growth in business, finds that it is running out of storage space for raw materials. Jenna Company has offered a contract for leasing a nearby storage facility at \$20,000 per year with annual increases of \$1500 for 8 years. Payments are to be made at the end of each year, starting 1 year from now. The prevailing interest rate is 7 percent. What lump sum paid today would be equivalent to the 8-year lease payment plan? This present worth could then be compared with the estimated cost of expanding Sabdave's existing storage facility (including the expected present worth of added costs in running the facility).

Solution

The base annuity A' and the gradient G per period are shown on the cash flow diagram:



The first step in solving for the present worth of the lease payment plan is to convert the increasing annual payments to a uniform series:

$$A = A' + G(A/G, 7, 8) = \$20{,}000 + \$1500(3.14654)$$
$$= \$20{,}000 + \$4719.81$$
$$= \$24{,}719.81$$

The annuity is now translated to its present worth as

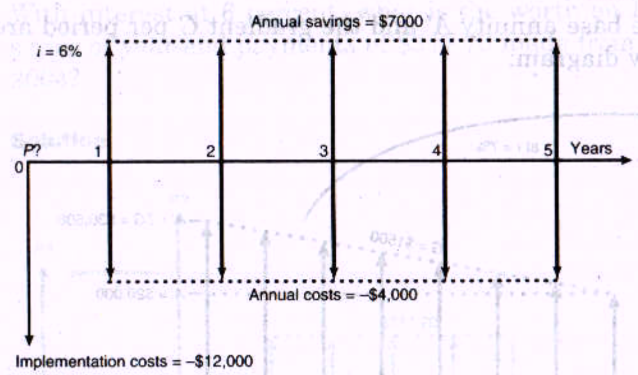
$$P = A(P/A, 7, 8) = \$24{,}719.81(5.9713)$$
$$= \$147{,}609.40$$

which is the amount in today's dollars equivalent to the lease contract that provides yearly payments of \$20,000 with annual increases of \$1500 for 8 years.

More extensive economic situations usually include income, possibly savings, and costs. Such situations are evaluated by calculating the net outcome at a certain time. A cash flow diagram incorporates receipts and disbursements by displaying income above the time line and outlays below the line. Other payment categories can be handled similarly.

EXAMPLE 2.10 **Income and Outlay**

The management of an expanding manufacturing firm is considering a proposal from a consulting group to introduce a new method of training inexperienced machine tool operators. The consultants claim that their program will produce savings of $7000 per year over the planned 5-year life of the project. Immediate costs to implement the program are $12,000, and annual training expenses will be $4000. The firm uses 6 percent annual interest for cost comparisons. Should the firm accept the proposal, assuming all other factors are negligible?



Solution

Assuming that the costs and savings occur at the end of a year, we find that

$$A = \text{annual savings} - \text{annual costs} = \$7000 - \$4000 = \$3000$$

For the proposal to be acceptable, the net return must be greater than the $12,000 initial cost. By translating the 5-year annuity to the present time, we compare the initial cost and the present worth of savings P directly:

$$P = A(P/A, 6, 5) = \$3000(4.21236) = \$12,637$$

The indicated total net savings exceed the initial cost by

$$\$12,637 - \$12,000 = \$637$$

which gives very little leeway for error in cost or savings estimates.

The same conclusion results from taking a different approach with the same data. From the capital recovery formula, the annual return on the gross savings A required for 5 years to meet a present obligation of $12,000 is

$$A = P(A/P, 6, 5) = \$12,000(0.23740) = \$2849$$

Comparing the required return with the expected annual gross savings shows annual net savings of

$$\$3000 - \$2849 = \$151$$

a marginal saving at best.

We have seen two approaches to the problem, and usually there are several ways to tackle such problems. The solutions, though, are identical. This can be seen by determining the present worth of the $151 expected to be saved annually over 5 years:

$$P = \$151(P/A, 6, 5) = \$151(4.21236) = \$636.07$$

This is the same as the present-worth saving we determined earlier (with minor roundoff errors).

So, given an engineering decision that requires an economic analysis, we have seen that there are many ways to attack the analysis. One way might be the most efficient while another might be the most logical from a management presentation perspective. Only experience will reveal the best way to go in many situations. Even if the most expedient method is not chosen, the solution should still be correct, if assumptions and data are correct.

2.8.3 Calculations with Continuous Interest

In our earlier discussion of nominal interest rates (Sec. 2.4.1), it was apparent that for a specific nominal rate, the effective interest rate gets larger as the compounding interval is shortened. The effective interest rate i for a nominal interest rate of 18 percent ($r = 0.18$) compounded semiannually ($m = 2$) was shown to be

$$i_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{0.18}{2}\right)^2 - 1 = 0.188 \text{ or } 18.8\%$$

and $i_\infty = e^r - 1$ when the interest periods are infinitesimally small. For the same 18 percent nominal interest compounded continuously,

$$i_\infty = e^{0.18} - 1 = 1.1972 - 1 = 0.1972 \text{ or } 19.72\%$$

Using continuous compounding in place of discrete compounding leads to an increase in the force of interest on the time value of money. A single payment earning continuous interest has a future value of

$$F = Pe^{rN}$$

where the continuous-interest compound factor e^{rN} relates to the $(1 + i)^N$ used in the F/P factor for discrete compounding.

EXAMPLE 2.11 **Comparison of Continuous and Discrete Compounding**

A sum of money doubles in size after 9 years, when it is invested at 8 percent interest compounded annually:

$$F = P(1 + 0.08)^9$$
$$2P = P(1.999)$$

At what rate of continuous compounding will an amount double in only one-half the time taken at the 8 percent effective rate?

Solution

It is given that $F = 2P$ at $N = 9/2 = 4.5$ years. Substituting these values into the future-worth equation and using continuous compounding, we get

$$F = 2P = P(e^{rN}) \quad \text{and} \quad 2 = e^{4.5r}$$

And r can be found in the right-hand equation by taking natural logarithms of both sides:

$$\ln 2 = 4.5r \ln e$$

and

$$0.693 = 4.5r \qquad r = \frac{0.693}{4.5}(100\%) = 15.4\%$$

2.8.3.1 Continuous-compounding, discrete cash flow

There are two versions of continuous interest. The *discrete cash flow* form applies continuous compounding to a payment whenever it is received, but the payment is assumed to be received in one lump sum. The other form assumes that the cash flow is continuous. The future-worth formula $F = Pe^{rN}$ and its reciprocal $P = Fe^{-rN}$ are discrete cash flows compounded continuously.

Continuous-compounding factors with *discrete payments* can be similarly developed from the end-of-period annuity formulas. First, recall from the discussion of effective interest that e^{rN} corresponds to $(1 + i)^N$ and that the effective interest rate is

$$i_\infty = e^r - 1$$

Substitution of these expressions into the sinking fund formula yields the fact that the discrete cash flow, end-of-period compounding, sinking fund formula is equal to the discrete cash flow, continuous-compounding, *sinking fund* formula when $i = e^r - 1$:

$$A = F\left[\frac{i}{(1+i)^N - 1}\right] = F\left(\frac{e^r - 1}{e^{rN} - 1}\right)$$

and the reciprocal of the expression in parentheses is the series compound-amount factor for continuous compounding.

By similar reasoning, the capital recovery formula with continuous compounding and discrete payments is

$$A = P\left[\frac{e^{rN}(e^r - 1)}{e^{rN} - 1}\right]$$

and the reciprocal of the bracketed expression is the continuous-compounding series present-worth factor.

In every formula the A, F, and P values resulting from computations using either end-of-period or continuous compounding are identical for discrete payments when the continuous interest rate is equivalent to the effective interest rate. This relation is apparent when numbers are substituted into the sinking fund formulas displayed above. Letting $i = 22.1$ percent, which corresponds to a nominal continuous interest rate of 20 percent $(0.221 = e^{0.2} - 1)$, and applying the equivalent interest rates in the two sinking fund formulas with $N = 2$ gives

End-of-period compounding		Continuous compounding
$A = F\left(\frac{0.221}{1.221^2 - 1}\right)$	$=$	$F\left(\frac{e^{0.2} - 1}{e^{(0.2)(2)} - 1}\right)$
$= F\left(\frac{0.221}{1.491 - 1}\right)$	$=$	$F\left(\frac{1.221 - 1}{1.492 - 1}\right)$
0.45	$=$	0.45

which shows that the factors are equal when $i = e^r - 1$ and payments are discrete.

EXAMPLE 2.12 **Continuous Compounding of a Discrete-Payment Annuity**

At the end of each year, a single payment of \$1766 is deposited in an account that earns 6 percent compounded continuously. What is the amount in the account immediately after the fifth payment?

Solution

Given $A = \$1766$, $r = 0.06$, and $N = 5$, we calculate F as

$$F = A\left(\frac{e^{rN} - 1}{e^{r} - 1}\right) = \$1766\left(\frac{e^{(0.06)(5)} - 1}{e^{(0.06)} - 1}\right)$$

$$= \$1766\left(\frac{1.3499 - 1}{1.0618 - 1}\right) = \$1766(5.6618)$$

$$= \$9999$$

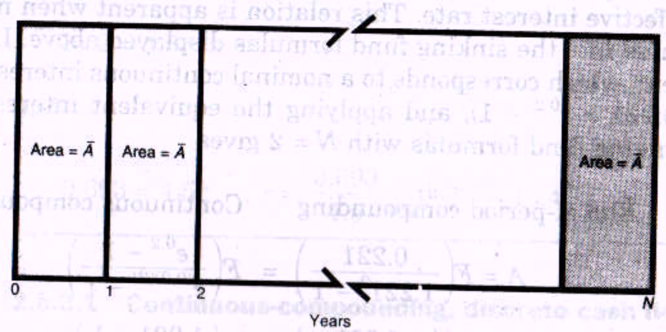
This future worth is not notably higher than that earned by annual compounding of the same annuity:

$$F = \$1766(F/A, 6, 5) = \$1766(5.63709) = \$9955$$

2.8.3.2 Continuous-compounding, continuous cash flow

The other version of continuous compounding occurs when the total payment for 1 year is received in continuous, small, equal payments during that year. We will let $\bar{A}$ designate the total amount of each payment per year, which continues for N years at a nominal interest rate r per year.

If we have the case of equal payments per period, the flow pattern is as follows:



If the compounding and flow are both discrete, we have a future value after N years of

$$F = \frac{A[(1 + i)^N - 1]}{i}$$

With continuous cash flows, we are allowing $\bar{A}$ to be broken into a large number (say, m) of equal cash flows spaced at equal points within the year, so that

$$F = \frac{\bar{A}}{m}\,\frac{(1 + r/m)^{mN} - 1}{r/m} = \bar{A}\,\frac{(1 + r/m)^{mN} - 1}{r}$$

where r is the nominal annual interest rate. From our earlier continuous-interest computations, we know that

$$\left(1 + \frac{r}{m}\right)^m = e^r \qquad \text{as } m \to \infty$$

so

$$F = \bar{A}\left(\frac{e^{rN} - 1}{r}\right)$$

From a classical calculus approach, this is the same as

$$F = \bar{A}\int_{m=0}^{N} e^{rm}\, dm = \bar{A}\left(\frac{e^{rN} - 1}{r}\right)$$

The expression in parentheses is called the *continuous-compounding series compound-amount factor for continuous uniform payments.*

If we wanted to determine $P/\bar{A}$, we could start from the completely discrete case:

$$P = A\left[\frac{(1 + i)^N - 1}{i(1 + i)^N}\right]$$

and form the continuous equivalent,

$$P = \frac{\bar{A}}{m}\,\frac{(1 + r/m)^{mN} - 1}{r/m(1 + r/m)^{mN}}$$

As m approaches infinity, this reduces to

$$P = \bar{A}\left(\frac{e^{rN} - 1}{re^{rN}}\right)$$

TABLE 2.5

Symbols and formulas for continuous compounding of a continuous flow. Here $\bar{A}$ designates total amount accumulated in small equal payments during 1 year, N represents the total number of years for compounding, and the continuous interest rate is r per year

Application formula	Equation
$(\bar{A}/F, r\%, N)$	$\bar{A} = F\left(\frac{r}{e^{rN}-1}\right)$
$(F/\bar{A}, r\%, N)$	$F = \bar{A}\left(\frac{e^{rN}-1}{r}\right)$
$(\bar{A}/P, r\%, N)$	$\bar{A} = P\left(\frac{re^{rN}}{e^{rN}-1}\right)$
$(P/\bar{A}, r\%, N)$	$P = \bar{A}\left(\frac{e^{rN}-1}{re^{rN}}\right)$

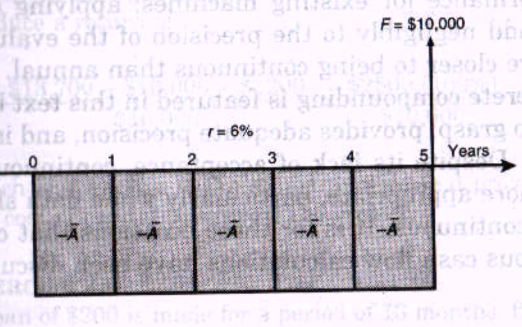
Functional notations and formulas for four basic continuous-compounding factors for continuous-flow payments are given in Table 2.5.

The assumption of a continuous flow of disbursements and income throughout a year is rare as compared with the end-of-year payment pattern. However, the continuous-flow assumption is more revealing and applicable than is continuous compounding of discrete payments, because the main reason for using continuous interest in an economic evaluation is to determine the effects of continuous cash flows, not just the continuous interest on discrete payments. Consequently, tables are provided in App. E for continuous-compounding, continuous-flow interest factors. These factors follow the functional notations given in Table 2.5 and are applied to continuous cash flows according to the same logic and procedures that govern discrete compound-interest calculations.

EXAMPLE 2.13 Continuous Compounding of a Continuous-Payment Annuity

A savings plan offered by a company allows employees to set aside part of their daily wages and have the money earn 6 percent compounded continuously. What annual amount withdrawn from pay will accumulate $10,000 in 5 years? (Alternative statement of the problem: What continuous-flow annual annuity will yield a future value of $10,000 in 5 years when compounded continuously at 6 percent?)

Solution



$$\bar{A} = F(\bar{A}/F, 6, 5)$$

$$= F\left(\frac{r}{e^{rN}-1}\right)$$

$$= F\left(\frac{0.06}{e^{0.3}-1}\right) = \$10{,}000(0.1715) = \$1715$$

The effect of continuous compounding of continuous flow can be judged by comparing Example 2.12 with this example (where Example 2.12 accumulated $9999, a slight rounding difference from this example's $10,000).

The amount of money to be held back per year with a continuous-payment annuity as contrasted to continuous compounding of a discrete-payment annuity is $51 less, where A from Example 2.12 is $1766:

$$A - \bar{A} = \$1766 - \$1715 = \$51$$

2.8.3.3 Role of continuous compounding in engineering economic studies

Some savings institutions advertise continuous compounding as an inducement to savers. The intent is to attract investors by paying higher effective interest than the competitors while still adhering to the nominal interest rate set by regulation. This is an example of discrete continuous compounding.

Actual flows of funds in industry, minute-by-minute incoming receipts and outgoing disbursements, are essentially continuous. Since continuous cash flow closely approximates the pattern of business transactions, it would seem reasonable to compound that flow pattern in evaluating business proposals. It seldom is reasonable. Engineering economic studies primarily rely on discrete compounding because they typically focus on aggregate payments that are assumed to take place on specified dates. If we were investigating a possible new machine, estimates of cash flow for the future use of the new machine would be even less precise than assessments of performance for existing machines; applying continuous compounding would add negligibly to the precision of the evaluation, even though cash flows are closer to being continuous than annual.

Discrete compounding is featured in this text because it is conceptually easier to grasp, provides adequate precision, and is the most widely applied method. Despite its lack of acceptance, continuous compounding is sometimes more appropriate, particularly when data show that the cash flow is indeed continuous. It is for these occasions that continuous-compounding, continuous cash flow calculations have been discussed.

2.9 REVIEW EXERCISES AND DISCUSSIONS

EXERCISE 1

The money earned from making a loan is evident in the contract (sometimes in fine print). A loan of $10,000 for 1 year at an interest rate of 10 percent earns the lender $10,000 × 0.10 = $1000. A borrower usually does not know in advance exactly how much will be gained from a loan to buy productive goods. It is often impossible to segregate precisely the receipts due to a certain production operation when that operation is a small part of a much larger production system. In such cases an evaluation study may be made of the amount that production costs of the system are decreased by improvements to the given operation, assuming the operation must be performed to maintain the total process. Then the earnings are in the form of "cost savings" in the system, which are compared with the investment cost of acquiring and using assets to improve the operation.

Assume a machine is purchased for $10,000 with the loan mentioned above. The machine will be completely worn out by the end of the year, and its operating costs will be $100 per month more than the costs of the present operation. How large a cost reduction must be provided by the machine for its purchase to earn a 15 percent return for the borrower?

SOLUTION 1

The costs involved include repayment of the loan plus interest charges for the loan, extra operating costs, and investment earnings resulting from the purchase of the machine:

Loan repayment (purchase price of machine)	$10,000
Interest paid on loan for 1 year = $10,000 × 0.10	1,000
Additional operating cost incurred = $100/month × 12	1,200
15% earnings on $10,000 borrowed = $10,000 × 0.15	1,500
Necessary cost reduction to support investment	$13,700

If the cost reduction turned out to be only $13,700 − $1500 = $12,200, it would cover only expenses and nothing would be gained from the machine's purchase. However, if the company could afford $10,000 of its own money for the machine rather than borrowing it, a cost reduction of $12,200 will yield a 10 percent return on the investment, which is the rate the company could earn by simply lending its money at the "going" interest rate of 10 percent. A cost reduction of $13,700 will produce a return of

$$\frac{\$13{,}700 - \$10{,}000 - \$1200}{\$10{,}000} = \frac{\$2500\ (100\%)}{\$10{,}000} = 25\%$$

which is attractive to the company because it provides an added 15 percent beyond the cost of capital to support the project.

EXERCISE 2

A loan of $200 is made for a period of 13 months, from January 1 to January 31 the following year, at a simple interest rate of 10 percent. What future amount is due at the end of the loan period?

SOLUTION 2

Using ordinary simple interest, we find that the total amount to be repaid after 13 months is

$$\begin{aligned} F &= P + PiN \\ &= \$200 + \$200(0.10)\left(1 + \tfrac{1}{12}\right) \\ &= \$200 + \$200(0.1083) \\ &= \$200 + \$21.67 = \$221.67 \end{aligned}$$

If exact simple interest is used, the future value (assuming the year in question is not a leap year) is

$$\begin{aligned} F &= P + PiN \\ &= \$200 + \$200(0.10)\left(1 + \tfrac{31}{365}\right) \\ &= \$200 + \$200(0.10849) = \$221.70 \end{aligned}$$

EXERCISE 3

A credit plan charges interest at the rate of 18 percent compounded monthly. What is the effective interest rate?

SOLUTION 3

The nominal 18 percent rate constitutes monthly charges of 1.5 percent. From this statement we know that $r = 0.18$ and $m = 12$, so the effective interest rate can be calculated as

$$\begin{aligned} i_{\text{eff}} &= \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{0.18}{12}\right)^{12} - 1 \\ &= 1.015^{12} - 1 = 1.1956 - 1 = 0.1956 \text{ or } 19.56\% \end{aligned}$$

The same result can be obtained by recognizing that

$$1.015^{12} - 1 = (F/P, 1.5, 12) - 1$$

Then the tables in App. D can be used to find the value of the compound-amount factor at $i = 1.5$ percent and $N = 12$:

$$(F/P, 1.5, 12) = 1.19562$$

so

$$1.19562 - 1 = 0.1956 \text{ or } 19.56\%$$

EXERCISE 4
How much would a person have had to invest 1 year ago to have $2500 available today, when the investment earned interest at the nominal rate of 12 percent compounded monthly?

SOLUTION 4
It is first necessary to convert the nominal rate to its corresponding periodic rate: 12 percent compounded monthly means that an investment earns 1 percent per month. Next, it must be recognized that today's worth is a future worth in terms of when investment P was made, 1 year previously. It is known that $F = \$2500$, $i = 1$ percent, and $N = 12$ (12 months have passed since the original investment); therefore,

$$P = F(P/F, 1, 12) = \$2500(0.88745) = \$2219$$

EXERCISE 5
What annual year-end payment must be made each year to have $20,000 available 5 years from now? The compound annual interest rate is 6 percent.

SOLUTION 5
The 5-year annuity is a sinking fund that has a value at maturity of $F = \$20,000$. The necessary annual deposits equal

$$A = F(A/F, 6, 5) = \$20,000(0.17740) = \$3548$$

EXERCISE 6
If you deposit $10,000 today, what equal amounts can you withdraw at the end of each quarter for the next 4 years, when the nominal interest rate is 10 percent?

SOLUTION 6
The withdrawals form an annuity where $N = 16$ and $i = 10/4$ percent. Given $P = \$10,000$, the capital invested is recovered by payments of

$$A = \$10,000(A/P, 2.5, 16) = \$10,000(0.07660) = \$766$$

EXERCISE 7
An ambitious saver plans to deposit $2000 in a money market account starting 1 year from now and wants to increase annual deposits by $1000 each year for the following 6 years. Assuming that deposits earn 9 percent annually, determine what equal-payment annuity would accumulate the same amount over the 7-year period.

SOLUTION 7
The first step in using the arithmetic gradient conversion factor is to identify the base annuity A' and the gradient G. Annuity A' is the first payment of $2000, and G is the amount by which the payments increase each year, or $1000. Then

$$A = A' + G(A/G, i, N) = \$2000 + \$1000(A/G, 9, 7)$$
$$= \$2000 + \$1000(2.6574) = \$2000 + \$2657 = \$4657$$

Thus seven equal payments of $4657 are equivalent to seven payments increasing by $1000 increments from $2000 for the first one to $8000 for the last one.

EXERCISE 8
The ambitious saver in Exercise 7 has changed plans. Instead of increasing each deposit by $1000, each deposit will be raised by 20 percent over the previous one. What equal-payment annuity will accumulate the some amount over the 7-year period, when deposits earn 9 percent annually?

SOLUTION 8
The annual 20 percent change is a geometric gradient where $g = 0.20$. Given $A' = \$2000$ and $N = 7$, the present worth of the series is

$$P = A'\left[\frac{1 - (1+g)^N(1+i)^{-N}}{i - g}\right]$$
$$= \$2000\left[\frac{1 - (1.2)^7(1.09)^{-7}}{0.09 - 0.2}\right] = \$17,457$$

The equivalent uniform annuity is now

$$A = \$17,457(A/P, 9, 7) = \$17,457(0.19869) = \$3458.53$$

We could have found P by using

$$P = \frac{A'}{1+i}(F/A, i', 7)$$

where

$$i' = \frac{1.20}{1.09} - 1 = 0.101$$

CHEER, this text's computer program, can be used to find any value of the discrete-compounding interest factors. To find $(F/A, 10.1, 7)$, we access the screens shown in Fig. 2.6. The first screen (Fig. 2.6a) is the initial screen obtained when

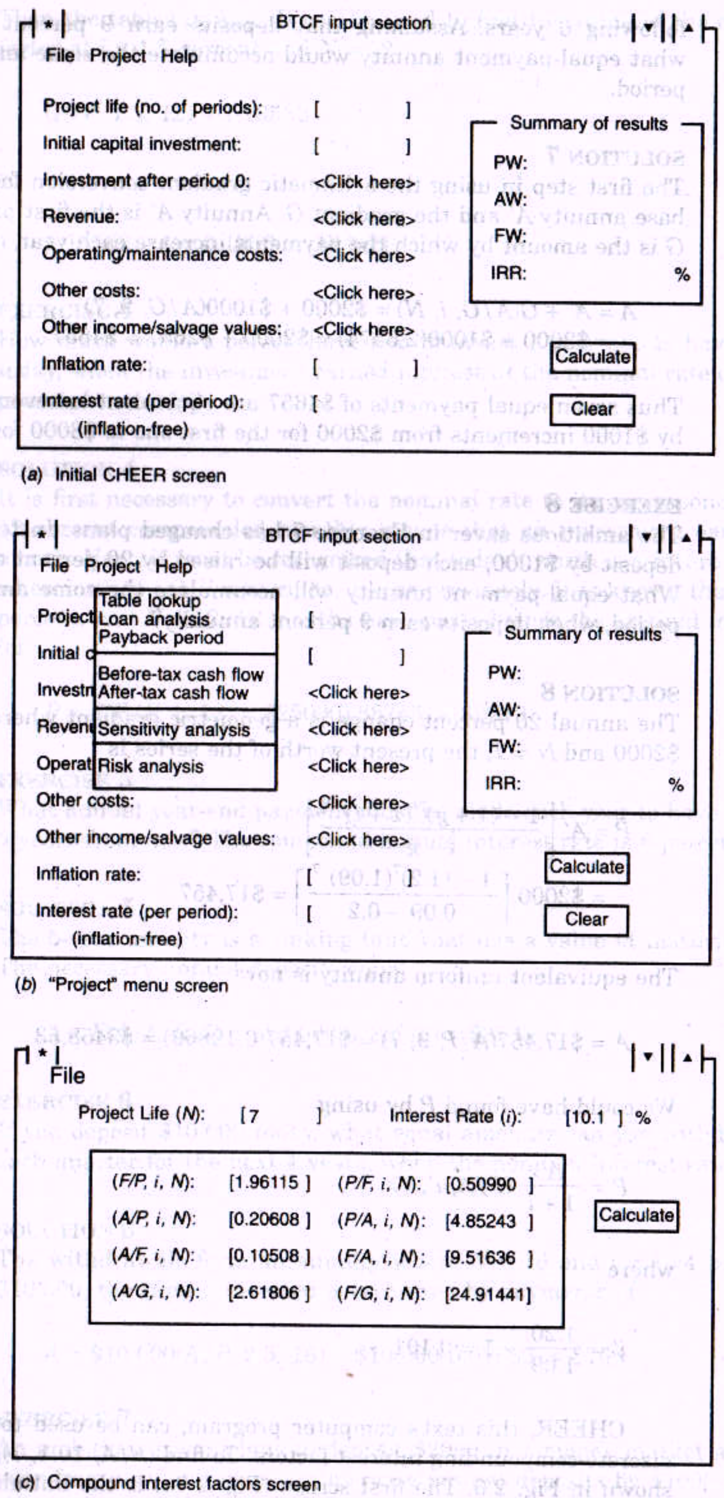


FIGURE 2.6 Using CHEER to find compound-interest factors.

CHEER is accessed. The top menu file shows File, Project, and Help. We access Project, and we get a menu superimposed over the initial menu as shown in Fig. 2.6*b*. The top menu item is Table Lookup which, when accessed, brings up the third screen (Fig. 2.6*c*). We type in 7 for the project life and 10.1 for the interest rate, and we click Calculate. The screen shows that we get eight factor values. We can see that $(F/A, 10.1, 7)$ is 9.51636. It is a very simple matter to change the N or i values to determine any value that we need. We will not spend very much time on CHEER in the chapters prior to Chap. 5 since the reader needs to fully understand the material before relying on a computer program to assist or completely solve a problem. The reader will find that CHEER will greatly speed things in looking up compound-interest factors, especially when interpolation is called for.

Now we can complete our problem solution:

$$P = \frac{\$2000}{1.09}(9.51636) = \$17{,}461$$

This is the same value we found earlier except for minor rounding error.

EXERCISE 9

Receipts from an investment will decline by \$150 each year for 5 years from a level of \$1000 at the end of the first year. For an interest rate of 7 percent, calculate a constant annual series amount that is equivalent to the gradient over the following 6-year period.

SOLUTION 9

The base amount ($A' = \$1000$) is decreased by a uniform amount each year ($G = -\$150$). Given $i = 7$ percent and $N = 6$,

$$\begin{aligned} A &= \$1000 - \$150(A/G, 7, 6) = \$1000 - \$150(2.30322) \\ &= \$654.67 \end{aligned}$$

EXERCISE 10

A manufacturing firm in a foreign country has agreed to pay \$25,000 in royalties at the end of each year for the next 5 years for the use of a patented product design. If the payments are left in the foreign country, interest on the retained funds will be paid at an annual rate of 15 percent.

What total amount will be available in 5 years under these conditions?

How large would the uniform annual payments have to be if the patent owners insisted that a minimum of \$175,000 be accumulated in the account by the end of 5 years?

SOLUTION 10

The annual payments form an annuity. Knowing that A = \$25,000 per period, i = 15 percent per period, and there are five periods, we calculate the future worth F as

$$F = A(F/A, 15, 5) = \$25{,}000(6.74238) = \$168{,}558$$

If the patent owners insisted on an accumulated value of \$175,000, the five end-of-year royalty payments would have to be

$$A = F(A/F, 15, 5) = \$175{,}000(0.14832) = \$25{,}956$$

EXERCISE 11

The inventor of an automatic coin-operated newsstand called Automag believes that the economic evaluation of the invention should be based on continuous compounding of continuous cash flow because income from the Automag will be essentially continuous and disbursements for services and purchases of materials (newspapers, paperbacks, magazines, etc.) will occur regularly and frequently. The expected life of Automag is 5 years. Annual income should average \$132,000 at a good location, and the total expenses for servicing materials are expected to average \$105,000 per year. What initial price could be paid for Automag (delivered and ready to operate) to allow a buyer to earn 15 percent on the investment if the income and disbursements are accurate?

SOLUTION 11

Net annual receipts from the Automag are expected to average

$$\$132{,}000 - \$105{,}000 = \$27{,}000$$

Since the cash flow is almost continuous, it is appropriate to apply continuous interest discounting. After recognition that the acceptable price of an Automag is the present-worth equivalent of a continuous annuity resulting from net receipts over 5 years, we get the value of the $P/\bar{A}$ factor from App. E for $N = 5$ and $r = 13.976$ percent, which corresponds to an effective interest rate of 15 percent:

$$P = \bar{A}(P/\bar{A}, 13.976, 5) = \$27{,}000(3.59773) = \$97{,}139$$

2.10 PROBLEMS

Problems 2.1 through 2.11 are adapted from Mathematics of Finance *by L. L. Smail (McGraw-Hill, New York). This college text was published in 1925, and it shows that basic interest problems have not changed much over the years.*

2.1 What sum must be loaned at 8 percent simple interest to earn \$350 in years?

2.2 How long will it take \$800 to yield \$72 in simple interest at 4 percent?

2.3 At what rate will \$65.07 yield \$8.75 in simple interest in 3 years 6 months?

2.4 How long will it take any sum to triple itself at 5 percent simple-interest rate?

2.5 Find the ordinary and exact simple interest on \$5000 at 5 percent for days.

2.6 If the interest on a certain sum for 3 months is \$63.87 at 5 percent simple interest, what would it be at 6 percent?

2.7 Find the compound amount of \$100 for 4 years at 6 percent compounded annually.

2.8 What is the compound amount of \$750 for 5 years at 6 percent compounded quarterly?

2.9 Accumulate a principal of \$1000 for 5 years 9 months at a nominal rate of 12 percent compounded monthly. How much interest is earned?

2.10 Find the difference between the amount of \$100 at simple interest and at compound interest for 5 years at 5 percent.

2.11 Find the compound amount of \$5000 at 6 percent for 4, 8, and 12 years, and compare the results. Does doubling the time double the amount of interest earned?

2.12 Determine the effective interest rate for a nominal annual rate of 6 percent that is compounded

(*a*) Semiannually
(*b*) Quarterly
(*c*) Monthly
(*d*) Daily

2.13 A personal loan of \$1000 is made for a period of 18 months at an interest rate of $1\frac{1}{2}$ percent per month on the unpaid balance. If the entire amount owed is repaid in a lump sum at the end of that time, determine

(*a*) The effective annual interest rate
(*b*) The total amount of interest paid

2.14 What nominal annual interest rate compounded monthly yields an effective annual rate of 19.56 percent?

2.15 A loan of \$5000 is scheduled to be repaid in equal monthly installments over $2\frac{1}{2}$ years. The nominal interest rate is 6 percent. How large is each payment?

2.16 How much will a piece of property have to increase in value over the next 5 years if it is to earn 10 percent per year on the purchase price?

2.17 How much less would it cost to pay off a \$3000 loan in 1 year with 12 equal payments when interest is 12 percent compounded monthly, as opposed to making a single payment when the effective interest rate is 12 percent?

2.18 Fred borrowed \$1000 from Friendly Finance Company at a nominal annual rate of 18 percent compounded monthly. Determine how long Fred will be required to pay off his debt if he makes the following monthly payments.

(*a*) \$175.54
(*b*) \$ 49.93
(*c*) \$ 15.00

2.19 Service records for a specific piece of production equipment indicate that a replacement machine will have first-year maintenance costs of approximately \$1000 and that these costs will increase by \$200 per year for each additional year of service. Assuming the equipment is to be in service for 10 years and using an interest rate of 15 percent, determine the maximum amount that should be paid for a lifetime maintenance contract at the time the equipment is purchased.

2.20 Maintenance costs on a selected piece of production equipment are expected to be $1000 the first year of operation and will probably increase at a rate of 20 percent per year throughout a 10-year service life. Using an interest rate of 15 percent compute the present worth of the expected costs.

2.21 Compare the answers obtained in Prob. 2.19 with those obtained in Prob. 2.20. What accounts for the significant difference in present-worth values?

2.22 The net income from a newly purchased piece of construction equipment is expected to be $12,000 the first year and to decrease by $1500 each year as maintenance costs increase. The equipment will be used for 4 years. What annual annuity will produce an equivalent income, when the interest rate is 8 percent?

2.23 Traffic flow over a new bridge is expected to be 1 million vehicles the first year it is in use. The average traffic rate is expected to increase by 5 percent per year.

- (a) Find the expected number of vehicles using the bridge in the 20th year of service.
- (b) Determine the total expected number of vehicles using the bridge during the 20-year period.
- (c) Comment on the relationship between conventional interest computations in financial analyses and this normal engineering problem.

2.24 Assuming that a toll of $1 per vehicle is charged for use of the bridge in Prob. 2.23, determine the present worth of all projected toll collections, using an interest rate of 8 percent.

2.25 The amount of $1200 per year is to be paid into an account over each of the next 5 years. Using a nominal interest rate of 12 percent per year, determine the total amount that the account will contain at the end of the fifth year under the following conditions:

- (a) Deposits made at the first of each year with simple interest
- (b) Deposits made at the end of each year, with interest compounded annually
- (c) Deposits made at the end of each month, with interest compounded annually
- (d) Deposits made at the end of each month, with interest compounded monthly
- (e) Deposits made at the end of each year, with interest compounded monthly
- (f) Deposits made at the end of each year, with interest compounded continuously
- (g) Deposits made continuously, with interest compounded annually
- (h) Deposits made continuously and interest compounded continuously

2.26 Today's price for materials used in a production process is expected to hold constant for this year at $100,000. Find the present worth of 5 years' supply for the same amount of material used each year, when the interest rate is 8 percent, if the price changes at a constant annual rate of

- (a) $g = -5$ percent
- (b) $g = 0$ percent
- (c) $g = 5$ percent

2.27 A building site for a new gasoline station was purchased 10 years ago for $50,000. The site has recently been sold for $120,000. Disregarding any taxes, determine the rate of interest obtained on the initial investment.

2.28 The rights to a patent have been sold under an agreement in which annual year-end payments of $10,000 are to be made for the next 10 years. What is the current worth of the annuity at an interest rate of 7 percent?

2.29 A deferred annuity is to pay $500 per year for 10 years with the first payment coming 6 years from today. Determine the present worth of the annuity, using an interest rate of 12 percent.

2.30 An inventor has been offered $12,000 per year for the next 5 years and $6000 annually for the following 7 years for the exclusive rights to an invention. At what price could the inventor afford to sell the rights to earn 10 percent, disregarding taxes?

2.31 A company 3 years ago borrowed $40,000 to pay for a new machine tool, agreeing to repay the loan in 100 monthly payments at an annual nominal interest rate of 12 percent compounded monthly. The company now wants to pay off the loan. How much would this payment be, assuming no penalty costs for early payout?

2.32 If the population of a certain suburb is 35,000 at the end of 1994 and the average annual rate of increase is estimated at 7 percent, what should its population be at the end of the year 2003 if the 7 percent growth rate remains constant?

2.33 Derive the equation for calculating the future worth of a series of discrete payments when interest is compounded continuously. Start from the expression

$$F = A + Ae^{r} + Ae^{2r} + \cdots + Ae^{(N-2)r} + Ae^{(N-1)r}$$

and follow the procedure used in developing the sinking fund factor for end-of-period compounding.

2.34 Net receipts from a continuously producing oil well add up to $185,000 over 1 year. What is the present worth of the well if it maintains steady output until it runs dry in 8 years, assuming $r = 8$ percent?

2.35 You have a chance to buy a new car with a list price of $12,000. You have to pay $2000 down, and the dealer will finance the remainder at an nominal annual rate of 6 percent, compounded monthly for 5 years.

- (a) Determine the amount of your monthly payment.
- (b) How much total interest will you pay over 5 years?

2.36 A recently developed citrus orchard will come into full bearing after 6 years. Starting in the seventh year and continuing through a productive life of 20 years, the orchard is expected to produce an average net yield of $80,000 per year. What is the equitable present cash value of the investment, if money is worth 7 percent per annum?

2.37 A company is planning to buy an inspection device (coordinate-measuring machine) for $45,000. The expected life of the device is 5 years, and the expected annual operating costs and taxes are $600 for the first year with an added increase per year of $100 for years 2 through 5. Maintenance costs will be zero in the first 2