Unit 4

Beta Gamma functions and Laplace transforms I

1) Show that $\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}$

Put $a^2x^2 = t$

$x^2 = \frac{t}{a^2}$

$2x\,dx = \frac{dt}{a^2}$

$dx = \frac{dt}{2xa^2} = \frac{dt}{2a^2\sqrt{\frac{t}{a^2}}} = \frac{dt}{2a\sqrt{t}}$

$\int_0^\infty e^{-t}\frac{dt}{2a\sqrt{t}} =$

$$\boxed{\Gamma \text{ function} \Rightarrow 2\int_0^\infty e^{-x^2} x^{2n-1}\,dn = \sqrt{n}}$$

$\frac{1}{2a}\int_0^\infty e^{-t}\,t^{-1/2}\,dt = \frac{1}{2a}\int_0^\infty e^{-t}\,t^{-1/2}\,dt$

$= 1/2a \int_0^\infty e^{-t}\,t^{1/2-1}\,dt$

$$\boxed{\sqrt{1/2} = \sqrt{\pi}}$$

$= 1/2a\,\sqrt{1/2} = \sqrt{\pi}/2a$

$$\left[\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \right]$$

2) P.T $\displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx \times \int_0^\infty e^{-x^4} x^2\,dx = \frac{\pi}{4\sqrt{2}}$

$$\sqrt{n}\sqrt{1-n} = \frac{\pi}{\sin n\pi}$$

Consider $I_1 = \displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$

$$= \int_0^\infty e^{-x^2} x^{1/2} dx = \frac{1}{2}\left[2\int_0^\infty e^{-x^2} x^{2(1/4)-1}\,dx\right]$$

$$I_1 = \frac{1}{2}\sqrt{1/4}$$

Consider $I_2 = \displaystyle\int_0^\infty e^{-x^4} x^2\,dx$ Put $x^2 = t$, $2x\,dx = dt$

$$= \int_0^\infty e^{-t^2}\, t \,\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{4}\left[2\int_0^\infty e^{-t^2} t^{1/2}\,dt\right] = 1/4\left[2\int_0^\infty e^{-t^2} t^{2(3/4-1)}dt\right]$$

$$= 1/4\sqrt{3/4}$$

$$I_1 \times I_2 = 1/2\sqrt{1/4} \times 1/4\sqrt{3/4}$$

$$= 1/8\sqrt{1/4}\sqrt{1-1/4} = 1/8 \frac{\pi}{\sin \pi/4} = 1/8 \times \frac{\pi}{1/\sqrt{2}} = \frac{\sqrt{2}\pi}{8}$$

$$= \pi/4\sqrt{2}$$

3) PT $\displaystyle\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\,d\theta = \pi$

$$\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Consider $I_1 \displaystyle\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta$

$$= 1/2\left[2\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\,d\theta\right]$$

$2m - 1 = \frac{1}{2}$

$m = \frac{3}{4}$

$2n - 1 = 0$

$n = \frac{1}{2}$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{2}\frac{\sqrt{\frac{3}{4}}\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{4}+\frac{1}{2}}}$$

$$= \frac{\sqrt{\pi}}{2}\frac{\sqrt{\frac{3}{4}}}{\sqrt{\frac{5}{4}}}$$

$$I_2 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$$

$$= \frac{1}{2} \times \left[2\int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \, d\theta \right]$$

$2m = -\frac{1}{2} + 1$

$m = \frac{1}{4}$

$2n - 1 = 0$

$n = \frac{1}{2}$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{\frac{1}{2}\sqrt{\frac{1}{4}}\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{4}+\frac{1}{2}}} = \frac{\sqrt{\pi}}{2}\frac{\sqrt{\frac{1}{4}}}{\sqrt{\frac{3}{4}}}$$

$$I_1 \times I_2 = \frac{\sqrt{\pi}}{2}\frac{\sqrt{3/4}}{\sqrt{5/4}} \times \frac{\sqrt{\pi}}{2}\frac{\sqrt{\frac{1}{4}}}{\sqrt{3/4}} = \frac{\pi}{4} \times \frac{\sqrt{\frac{1}{4}}}{\frac{1}{4}\sqrt{\frac{1}{4}}} = \pi$$

4) PT $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}\, dx = \frac{\pi}{4\sqrt{2}}$

Put

$x^2 = \sin\theta$

$dx \cdot 2x = d\theta \cos\theta$

$dx = \frac{\cos\theta\, d\theta}{2x}$

Put

$x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\, d\theta$

$dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$

$$dx = \frac{\cos\theta \, d\theta}{2\sqrt{\sin\theta}}$$

$$I_1 = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \frac{\cos\theta \, d\theta}{2\sqrt{\sin\theta}} = \int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta$$

$$= \frac{1}{2}\left[2\left[\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta \, d\theta\right]\right.$$

$$2m-1 = 1/2 \qquad n = 1/2$$

$$2m = 1 + 1/2 = 3/2$$

$$m = 3/4$$

$$= \frac{1}{2}\beta\left(3/4, 1/2\right) = \frac{1/2 \, \sqrt{3/4}\sqrt{1/2}}{\sqrt{3/4 + 1/2}} = \frac{\sqrt{\pi}}{2} \frac{\sqrt{3/4}}{\sqrt{5/4}}$$

$$\int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \frac{\sec^2\theta \, d\theta}{2\sqrt{\tan\theta}} = \int_0^{\pi/4} \frac{\sec\theta \, d\theta}{2\sqrt{\frac{\sin\theta}{\cos\theta}}}$$

$$= \int_0^{\pi/4} \frac{1}{\cos\theta} \frac{d\theta \sqrt{\cos\theta}}{2\sqrt{\sin\theta}}$$

$$= \int_0^{\pi/4} \frac{d\theta}{2\sqrt{\sin\theta}\sqrt{\cos\theta}} = \frac{2}{2}\int_0^{\pi/2} \frac{d\theta}{(\sin\theta)^{1/2}(\cos\theta)^{1/2}}$$

5) $\int_0^1 x^m \left[\log\left(1/x\right)\right]^n dx = \frac{\sqrt{n+1}}{(m+1)^{n+1}}$

Put $\log\left(1/x\right) = t$

$1/x = e^t$ $\quad x = e^{-t}$

$dx = -dt\, e^{-t}$

$x = 0,\ t = \infty$

$x = 1,\ t = 0$

$I = \int_\infty^0 e^{-tm} t^n (-e^{-t} dt) = \int_0^\infty t^n e^{-(m+1)t} dt$

$(m+1)t = z$

$dt = \frac{dz}{m+1}$

$= \int_0^\infty \frac{z^n}{(m+1)^n} e^{-z} \left(\frac{dz}{m+1}\right)$

$= \int_0^\infty \frac{z^n e^{-z}}{(m+1)^{n+1}} dz = \frac{1}{(m+1)^{n+1}} \int_0^\infty z^n e^{-z} dz$

$= \frac{\sqrt{n+1}}{(m+1)^{n+1}}$

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Laplace transforms - $\mathcal{L}\{t\} = 1/s$

If $f(t)$ is any real valued function, defined for all $t > 0$, the Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ and is defined by $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Note -
We can observe that RHS is a function of s since after integrating wrt t b/w the limits $t = 0$ to ∞, t eliminates and it remains in terms of s.

$\therefore \mathcal{L}\{f(t)\} = F(s)$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t)$

Here $f(t)$ is called inverse Laplace transform of $F(s)$

Function of exponential order -
A function $f(t)$ is said to be of exponential order,

$\lim_{t \to \infty} f(t)\, e^{-at} = 0$

Laplace transform of some std functions -

$f(t) = 1$

$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\, dt = \left(\frac{e^{-st}}{-s}\right)_0^\infty = 1/s$

$\Rightarrow \mathcal{L}\{a\} = a/s$

where a is a constant

$f(t) = t$

$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

$= 1/s^2$

$f(t) = e^{-at}$

$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt = \int_0^\infty e^{-(s+a)t}\, dt$

$= \left[\dfrac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \dfrac{1}{s+a}$

IIIly $f(t) = e^{at}$

$\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

$f(t) = \cosh at$

$f(t) = \cosh at = \dfrac{e^{at} + e^{-at}}{2}$

$\mathcal{L}(\cosh at) = \mathcal{L}\left\{\dfrac{e^{at} + e^{-at}}{2}\right\}$

$= 1/2 \left\{\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}\right\}$

$= \dfrac{1}{2}\left[\dfrac{1}{s-a} + \dfrac{1}{s+a}\right]$

$= 1/2 \left[\dfrac{2s}{s^2-a^2}\right] = \dfrac{s}{s^2-a^2}$

$\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

$f(t) = \cos at$

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}}{(-s)^2 + a^2} \left(-s\cos at + a\sin at \right) \right]_0^\infty$$

$$= \left[\frac{1}{s^2 + a^2} \right] \left(0 - 1\left(-s(1) + 0 \right) \right) = \frac{s}{s^2 + a^2}$$

$f(t) = \sin at$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[\frac{e^{-st}}{(-s)^2 + a^2} \right]_0^\infty \left[a \right] = \frac{a}{s^2 + a^2}$$

$f(t) = t^n$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

Put $st = x$

$t = x/s$

$dt = \frac{dx}{s}$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-x} \frac{x^n}{s^n} \left(\frac{dx}{s} \right)$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx \Rightarrow \mathcal{L}\{t^n\} = \frac{1}{s^{n+1}} \Gamma(n+1)$$

WKT $\Gamma(n+1) = n!$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n \in \mathbb{Z}^+$$

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Properties of Laplace transform -

i) Shifting property

1) If $\mathcal{L}\{f(t)\} = F(s)$, then P.T $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$$

1ly $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

2) If $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds}\{F(s)\}$$

WKT

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

diff eq① wrt 's' on b.s

$$\frac{d}{ds}[F(s)] = \int_0^\infty e^{-st}(-t) f(t)\, dt$$

$$-\frac{d}{ds}[F(s)] = \int_0^\infty e^{-st} [t f(t)]\, dt$$

$$-d/ds\,[F(s)] = \mathcal{L}\{t\, f(t)\}$$

3) If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ ①

Integrate wrt to s btn s to ∞

$$\int_{s=s}^\infty F(s)\,ds = \int_{s=s}^\infty \int_{t=0}^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_{t=0}^\infty \left(\int_{s=s}^\infty e^{-st}\,ds\right) f(t)\,dt$$

$$= \int_{t=0}^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,ds$$

$$= \int_{t=0}^\infty \left[-1/t\right]\left(-e^{-st}\right) f(t)\,dt$$

$$= \int_{t=0}^\infty e^{-st}\left(\frac{f(t)}{t}\right) dt$$

$$\int_{s=s}^\infty F(s)\,ds = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

4) If $\mathcal{L}\{f(t)\} = F(s)$ P.T $\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$

Let $F(t) = \int_0^t f(t)\,dt$

$F(0) = 0$; $F'(t) = f(t)$

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \mathcal{L}\{F(t)\}$$

$$= \int_0^\infty e^{-st} . F(t)\,dt$$

$$= \left[F(t)\, e^{-st}/-s\right]_0^\infty - \left[\int F'(t)\left(e^{-st}/-s\right)\right]_0^\infty$$

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = 0 + \frac{1}{s}\left[\int_0^{\infty} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{s}\,\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

2) If $\mathcal{L}\{f(t)\} = F(s)$, then P.T Laplace transform of

$$\mathcal{L}\{\sinh at\ f(t)\} \quad \frac{1}{2}\left[F(s-a) - F(s+a)\right]$$

$$\mathcal{L}\{\cosh at\ f(t)\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$$

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$\mathcal{L}\{\sinh at\ f(t)\} = \mathcal{L}\left\{\left(\frac{e^{at} - e^{-at}}{2}\right) f(t)\right\}$$

$$= \frac{1}{2}\left\{\mathcal{L}\, e^{at} f(t) - \mathcal{L}\left\{e^{-at} f(t)\right\}\right\}$$

$$= \frac{1}{2}\left[F(s-a) - F(s+a)\right]$$

III^{ly} $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

$$\mathcal{L}\{\cosh at\ f(t)\} = \mathcal{L}\left\{\frac{e^{at} + e^{-at}}{2} f(t)\right\}$$

$$= \frac{1}{2}\left\{\mathcal{L} e^{at} f(t) - \mathcal{L} e^{-at} f(t)\right\}$$

$$= \frac{1}{2}\left[F(s-a) + F(s+a)\right]$$

3) If $\mathcal{L}\{f(t)\} = F(s)$ the P.T $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}[F(s)]$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right] \qquad -①$$

if $n=1$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Diff wrt s on b.s

$$d/ds\,[F(s)] = \int_0^\infty e^{-st}(-t)\,f(t)\,ds$$

$$-\frac{dF(s)}{ds} = \int_0^\infty e^{-st}\,[\,t\,f(t)\,]\,dt$$

$$(-1)^1\,d/ds\,F(s) = \mathcal{L}\{t\,f(t)\} \quad -②$$

$P(n)$ is true for $n=1$

Let us assume that $P(n)$ is $n=k$

$$\mathcal{L}\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k} F(s)$$

Now wrt $P(n)$ is true for $k+1$

$$\mathcal{L}\{t^{k+1} f(t)\} = \mathcal{L}\{t\,\{t^k f(t)\}\}$$

$$= (-1)^1\, d/ds\, \mathcal{L}\{t^k f(t)\}$$

$$= (-1)^1\, d/ds\left\{(-1)^k\, d^k/ds^k\,[F(s)]\right\}$$

$$\mathcal{L}\{t^{k+1} f(t)\} = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}[F(s)]$$

$P(n)$ is true for $n=k+1$

$$)^n \frac{d}{ds^n}[F(s)]$$

4) Obtain Laplace transform of foll functions -

i) $\mathcal{L}\{(t^5+1)^2\}$ ii) $\mathcal{L}\{\sin 6t\,\sin 2t\}$

iii) $\mathcal{L}\{(\sqrt{t} + 1/\sqrt{t})^3\}$

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i) $\mathcal{L}\{(t^5+1)^2\} = \mathcal{L}\{t^{10}+1+2t^5\}$

$$= \mathcal{L}\{t^{10}\} + \mathcal{L}(1) + \mathcal{L}\{2t^5\}$$

$$= \frac{10!}{s^{11}} + \frac{1!}{s} + \frac{2\times 5!}{s^6}$$

ii) $\sin 6t \sin 2t = \frac{1}{2}\left[\cos(6t-2t) - \cos(6t+2t)\right]$

$$\mathcal{L}(\sin 6t \sin 2t) = \frac{1}{2}\left[\mathcal{L}(\cos 4t) - \mathcal{L}(\cos 8t)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+16} - \frac{s}{s^2+64}\right]$$

$64 - 16$

$\frac{5}{64}$ $\frac{16}{28}$

$$= \frac{28s}{(s^2+16)(s^2+64)}$$

using $\mathcal{L}\{t^n\} = \frac{\Gamma{n+1}}{s^{n+1}}$ $\qquad n\Gamma n = \Gamma{n+1}$

iii) $\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3 = (\sqrt{t})^3 + \left(\frac{1}{\sqrt{t}}\right)^3 + 3\sqrt{t}\left(\frac{1}{\sqrt{t}}\right)^2 + 3(\sqrt{t})^2\left(\frac{1}{\sqrt{t}}\right)$

$$= t^{3/2} + t^{-3/2} + 3t^{-1/2} + 3t^{1/2}$$

$$\mathcal{L}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3 = \mathcal{L}\{t^{3/2}\} + \mathcal{L}\{t^{-3/2}\} + \mathcal{L}\{3t^{-1/2}\} + \mathcal{L}\{3t^{1/2}\}$$

$$= \frac{\Gamma{5/2}}{s^{5/2}} + \frac{\Gamma{-1/2}}{s^{-1/2}} + \frac{3\,\Gamma{-1/2+1}}{s^{1/2}} + \frac{3\,\Gamma{3/2}}{s^{3/2}}$$

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Obtain Laplace transform of the following -

i) $\sin^3 3t$ ii) $t^2\cos 2t$ iii) $\frac{\sin 2t}{t}$ iv) Evaluate $\int_0^\infty t^3 e^{-t}\,dt$

v) $t^3\cosh 4t$ vi) $\sin\sqrt{t}$ and $\cos\sqrt{t}$ viii) $\int_0^t te^{-t}\cos t\,dt$

WKT

i) $\sin^3 3t$ $3\sin t - 4\sin^3 t = \sin 3t$

$$4\sin^3 t = 3\sin t - \sin 3t$$

$t \to 3t$

$$\sin^3 3t = \frac{1}{4}\left[3\sin 3t - \sin 9t\right]$$

$$L\{\sin^3 3t\} = \frac{3}{4}L\{\sin 3t\} - \frac{1}{4}\{\sin 9t\}$$

$$= \frac{3}{4}\left[\frac{3}{s^2+9}\right] - \frac{1}{4}\left[\frac{9}{s^2+81}\right] = \frac{9}{4}\left[\frac{1}{s^2+9} - \frac{1}{s^2+81}\right]$$

ii) $L\{t^2\cos 2t\} = (-1)^2\frac{d^n F(s)}{ds^2}$ $L\{t^n f(t)\} = (-1)^n\frac{dF(s)}{ds^n}$

$$L\{f(t)\} = F(s)$$

$$L\{\cos 2t\} = \frac{\phi}{s^2+4} = F(s)$$

$$\Rightarrow L\{t^2\cos 2t\} = \frac{d^2\left[\frac{\phi}{s^2+4}\right]}{d\phi^2} = d/ds\left[\frac{(s^2+4) - \phi(2s)}{(s^2+4)^2}\right]$$

$$= d/ds\left[\frac{4-s^2}{(\phi^2+4)^2}\right] = \left[\frac{-2s(s^2+4)^2 - 2(s^2+4)(2s)(4-s^2)}{(\phi^2+4)^4}\right]$$

$$= \frac{2s(s^2-12)}{(s^2+4)^3}$$

iii) $\frac{\sin 2t}{t}$ Using $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_{s=s}^{\infty} F(s)\,ds$

$\mathcal{L}\left\{\frac{\sin 2t}{t}\right\} = \int_{s=s}^{\infty} \mathcal{L}\{\sin 2t\}\,ds$ $F(s) = \mathcal{L}\{f(t)\}$

$= \int_{s=s}^{\infty} \frac{2}{s^2+4}\,ds = \frac{1}{2} \times 2 \times \tan^{-1}\left(\frac{s}{2}\right)_s^{\infty}$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$

$= \cot^{-1}\left(\frac{1}{2}\right)$

iv) $f(t) = t^3 \cosh 4t$

$= t^3\left(\frac{e^{4t} + e^{-4t}}{2}\right)$

WKT

$\mathcal{L}\{e^{at} f(t)\} = F(s-a) = F(s)_{s \to s-a}$

$\mathcal{L}\{t^3 \cosh 4t\} = \frac{1}{2}\left[\mathcal{L}\{t^3 e^{4t}\} + \mathcal{L}\{t^3 e^{-4t}\}\right]$

$= \frac{1}{2}\left[\left(\frac{3!}{s^4}\right)_{s \to s-4} + \left(\frac{3!}{s^4}\right)_{s \to s+4}\right]$

$= \frac{3!}{2}\left[\frac{1}{(s-4)^4} + \frac{1}{(s+4)^4}\right]$ using $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

vi) $\sin\sqrt{t}$

WKT

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \dots$

$\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \dots$

$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!} L\{t^{3/2}\} + \frac{1}{5!} L\{t^{5/2}\} - \dots$

$L\{t^{1/2}\} = \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{1/2\,\Gamma(1/2)}{s^{3/2}} = \sqrt{\pi}/2s^{3/2}$

$L\{t^{3/2}\} = \frac{\Gamma(3/2+1)}{s^{3/2+1}} = \frac{3/2\,\Gamma(3/2)}{s^{5/2}} = \frac{3/2 \times 1/2\sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}$

$L\{t^{5/2}\} = \frac{5/2 \times 3/2 \times 1/2\,\Gamma(1/2)}{s^{7/2}} = \frac{15\sqrt{\pi}}{8s^{7/2}}$

$\Rightarrow$

$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} - \left(\frac{3\sqrt{\pi}}{4s^{5/2}}\right)\frac{1}{3!} + \left(\frac{15\sqrt{\pi}}{8s^{7/2}}\right)1/5!$

$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{\frac{1}{4s}}{1!} + \frac{\left(\frac{1}{4s}\right)^2}{2!} \dots\right]$

$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$

$L\{\int_0^t te^{-t}\cos t\,dt\}$

$L\{\int_0^t f(t)dt\} = \frac{F(s)}{s}$ —①

$L\{\int_0^t \underbrace{te^{-t}\cos t\,dt}_{f(t)}\} = \frac{L\{te^{-t}\cos t\}}{s}$ —②

③

Consider $L\{te^{-t}\cos t\} = L\{e^{-t}(t\cos t)\} = L\{t\cos t\}_{s\to s+1}$

Consider $\mathcal{L}\{t\cos t\} = (-1)^1 \frac{d}{ds}\left[\mathcal{L}\{\cos t\}\right]$

$$= \frac{d}{ds}\left[\frac{s}{s^2+1}\right] = -\left[\frac{1(s^2+1) - s(2s)}{(s^2+1)^2}\right]$$

(3) $\mathcal{L}\{te^{-t}\cos t\} = \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to s+1}$

$$= \frac{(s+1)^2-1}{\left[(s+1)^2+1\right]^2}$$

(5) $\Rightarrow$ $\mathcal{L}\left\{\int_0^t te^{-t}\cos t\, dt\right\} = \frac{1}{s}\left[\frac{(s+1)^2-1}{\left[(s+1)^2+1\right]^2}\right]$

$$= \frac{s+2}{(s^2+2s+2)^2}$$

Evaluate i) $\int_0^\infty t^3 e^{-t} dt$ ii) $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$

i) $\mathcal{L}\{t^3\} = \int_0^\infty e^{-st} t^3 dt$

$$\frac{3!}{s^{3+1}} = \int_0^\infty e^{-st} t^3 dt$$

Put $s = 1$

$$\frac{6}{1^4} = \int_0^\infty e^{-t} t^3 dt = 6$$

ii) $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$

$$\mathcal{L}\left\{\frac{e^{-t}-e^{-3t}}{t}\right\} = \int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-3t}}{t}\right) dt$$

$$\int_{s=s}^{\infty} \mathcal{L}\{e^{-t} - e^{-3t}\}\, ds = \int_0^{\infty} e^{-st}\left(\frac{e^{-t} - e^{-3t}}{t}\right) dt$$

$$\int_0^{\infty} e^{-st}\left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = \int_{s=s}^{\infty}\left[\frac{1}{s+1} - \frac{1}{s+3}\right] ds$$

$$= \Big[\log(s+1) - \log(s+3)\Big]_{s=s}^{\infty}$$

$$= \log\left[\frac{s+1}{s+3}\right]_{s=s}^{\infty} = \log\left[\frac{s(1+1/s)}{s(1+3/s)}\right]_{s}^{\infty}$$

$$= \log(1) - \log\left(\frac{1+1/s}{1+3/s}\right)$$

$$\int_0^{\infty} e^{-st}\left(\frac{e^{-t} - e^{-3t}}{t}\right) = 0 - \log\left(\frac{s+1}{s+3}\right) = \log\left(\frac{s+3}{s+1}\right)$$

Put $s = 0$

$$\int_0^{\infty}\left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = \underline{\log 3}$$

If $f(t)$ and it's $n-1$ derivatives are continuous then P.T Laplace transform of $\mathcal{L}\{f^n(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0)$

$$- s^{n-2} f'(0) \cdots\cdots - f^{n-1}(0) \quad (1)$$

Let $P(n) : \mathcal{L}\{f^n(t)\}$

Consider $\mathcal{L}\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t)\, dt$

$$= \Big[e^{-st} f(t)\Big]_0^{\infty} - \int_0^{\infty} e^{-st}(-s) f(t)\, dt$$

$$\mathcal{L}\{f'(t)\} = \big[0 - f(0)\big] + s\int_0^{\infty} e^{-st} f(t)\, dt$$

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

$\therefore$ eq① is true for $n=1$

Assume that eq ① is true for $n=k$

$\mathcal{L}\{f^{k}(t)\} = s^{k}\mathcal{L}\{f(t) - s^{k-1}f(0) - s^{k-2}f(0) \dots - f^{k-1}(0)$ —②

Now WPT eq ① is true for $n=k+1$

$$\mathcal{L}\{f^{k+1}(t)\} = \int_0^\infty e^{-st} f^{k+1}(t)\,dt$$

$$= \left[e^{-st} f^{k}(t)\right]_0^\infty - \int_0^\infty e^{-st}(-s)\, f^{k}(t)\,dt$$

$$= \left[0 - f^{k}(0)\right] + s\int_0^\infty e^{-st} f^{k}(t)\,dt$$

$$= s\mathcal{L}\{f^{k}(t)\} - f^{k}(0)$$

$$= s\left[s^{k}\mathcal{L}\{f(t) - s^{k-1}f(0) - s^{k-2}f'(0) \dots f^{k-1}(0)\right] - f^{k}(0)$$

$\therefore$ eq① is true for $n=k+1$

write $f^{k}(0) = f^{(k-1)+1}(0)$

Note - if $n=1$, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

if $n=2$, $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

if $n=3$, $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$

Given that →

If $\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-1/4s}$

then S.T $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-1/4s}$

Let $f(t) = \sin\sqrt{t}$

$f'(t) = \cos\sqrt{t} \times 1/2\sqrt{t}$

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\mathcal{L}\{\sin\sqrt{t}\} - \sin(0)$

$1/2\ \mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \cancel{s} \times \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-1/4s} = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-1/4s}$

2) $\mathcal{L}\{2\sqrt{t/\pi}\} = 1/s^{3/2}$ then S.T $\mathcal{L}\left\{\frac{1}{\sqrt{\pi t}}\right\} = 1/\sqrt{s}$

Let $f(t) = \frac{2\sqrt{t}}{\sqrt{\pi}} \Rightarrow f'(t) = 2/\sqrt{\pi} \times \frac{1}{2\sqrt{t}} = 1/\sqrt{\pi t}$

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

$\mathcal{L}\left\{\frac{1}{\sqrt{\pi t}}\right\} = s \times 1/s^{3/2} = 1/\sqrt{s}$

Evaluate the following -

1) $\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\}$ 2) $\mathcal{L}\left\{\frac{\sin t \sin 3t}{t}\right\}$

3) $\mathcal{L}\left\{t^2 \sin 3t \, e^{-2t}\right\}$

i) $\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_{s=s}^{\infty} \mathcal{L}\{\cos at - \cos bt\} \, ds$

$s^2 + a^2 = t$

$2s\,ds = dt$

$$= \frac{1}{2}\int_{s=s}^{\infty} \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2} \, ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_{s=s}^{\infty}$$

$$= \frac{1}{2}\log\left[\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty}$$

$$= \frac{1}{2}\log\left[s^2\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

ii) $\mathcal{L}\left\{\frac{\sin t \sin 3t}{t}\right\} = \mathcal{L}\left\{\frac{-(\cos 4t - \cos 2t)}{2t}\right\}$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds$

$$= \int_s^{\infty} \mathcal{L}\left(-(\cos 4t - \cos 2t)\right) ds$$

$$= \int_s^{\infty} \mathcal{L}\left(-\cos 4t + \cos 2t\right) ds$$

$$= \int_s^{\infty} \frac{-s}{s^2+4^2} + \frac{s}{s^2+2^2} \, ds$$

$= \frac{1}{2} \times \frac{1}{2} \left[\log \left(\frac{s^2 + b^2}{s^2 + a^2} \right) \right]$

iii) $\mathcal{L}\{ t^2 \sin 3t \, e^{-2t} \}$

$$\mathcal{L}\{ t^n f(t) \} = (-1)^n \frac{d^n F(s)}{ds^n}$$

OR

$$\mathcal{L}\{ e^{-at} f(t) \} = F(s+a)$$

$\Rightarrow \mathcal{L}\{ t^2 \sin 3t \, e^{-2t} \} = (-1)^2 \frac{d^2 F(s)}{ds^2} = \frac{d^2 F(s)}{ds^2} = \frac{d^2 F(s)}{ds^2}$

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-2t}$$

$$= \int_0^\infty e^{-st} \sin 3t \, t^2 \, dt$$

$$\boxed{s = 2}$$

$$\frac{d}{ds}\left[F(s) \right] = \int_0^\infty e^{-st} (-t) f(t) \, dt$$

$$-\frac{d}{ds}\left[F(s) \right] = \mathcal{L}\{ t f(t) \} = (-1)^1 \frac{d}{ds} F(s)$$

$\int F(s) \, ds$

iv) $\mathcal{L}\{ t^2 \sin 3t \, e^{-2t} \}$

$$= \mathcal{L}\{ \underbrace{t^2 \sin 3t}_{f(t)} \, e^{-2t} \} = F(s+2)$$

$$F(s) = \mathcal{L}\{ f(t) \} = \mathcal{L}\{ t^2 \sin 3t \}$$

$$= (-1)^2 \frac{d^2}{ds^2} \left[\frac{3}{s^2 + a^2} \right] = \frac{d}{ds} \left[\frac{-3}{(s^2+9)^2} 2s \right]$$

$$F(s) = \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2 - s2(s^2+a^2)(2s)}{(s^2+9)^4}\right]$$

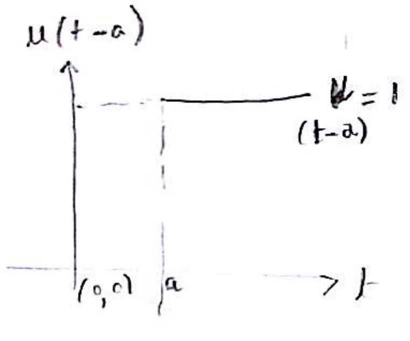
$$= \frac{-6(s^2+9)[s^2+9-4s^2]}{(s^2+9)^3}$$

$$F(s) = \frac{-6(9-3s^2)}{(s^2+9)^3}$$

$$\Rightarrow \mathcal{L}\{t^2 \sin 3t\, e^{-2t}\} = \frac{-6(9-3(s+2)^2)}{((s+2)^2+9)^3}$$

Unit Step function -

Unit Step function is denoted by $u(t-a)$ or $H(t-a)$ and is defined as $u(t-a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a \end{cases}$



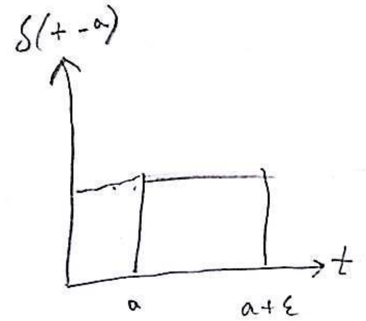
- is also called as heavy sides function.

If $a = 0$, $u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$

Unit impulse function OR Dirac Delta function -

The unit impulse function denoted by $\delta(t-a)$ and is given by $\lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)$ if $a \geq 0$

$$\delta(t-a) = \begin{cases} 1/\varepsilon, & a \leq t \leq a+\varepsilon \\ 0, & \text{other cases} \end{cases}$$



P.T $\mathcal{L}\{u(t-a)\} = e^{-as}/s$

$$\mathcal{L}\{u(t-a)\} = \int_{t=0}^{\infty} e^{-st} u(t-a)\, dt$$

$$= \int_0^a 0 e^{-st}\, dt + \int_a^{\infty} e^{-st} (1)\, dt$$

$$= \left(e^{-st}/-s \right)_a^{\infty} = -1/s \left(0 - e^{-as} \right)$$

$$\mathcal{L}\{u(t-a)\} = e^{-as}/s$$

if $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{u(t-a) f(t-a)\}$

$$= e^{-as} F(s)$$

$$\mathcal{L}\{u(t-a) f(t-a)\} = \int_0^{\infty} e^{-st} u(t-a) f(t-a)\, dt$$

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$$= \int_{t=0}^{a} e^{-st} f(t-a)\,(0)\,dt + \int_{t=a}^{\infty} e^{-st} f(t-a)\cdot 1\,dt$$

$t - a = u$

$dt = du$

$t = a + u$

$$= \int_{u=0}^{\infty} e^{-s(u+a)} f(u)\,du = \int_{0}^{\infty} e^{-su} e^{-as} f(u)\,du$$

$$= e^{-as}\left[\int_{u=0}^{\infty} e^{-su} f(u)\,du\right] = e^{-as}\,\mathcal{L}\{f(u)\}$$

$$\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as} F(s)$$

Note- if $a = 0$, then $\mathcal{L}\{H(t)\} = \frac{e^{-0s}}{s} = 1/s$

Prove that $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

$$\mathcal{L}\{\delta(t-a)\} = \int_{0}^{\infty} e^{-st} f(t-a)\,dt$$

$$= \int_{0}^{\infty} e^{-st}\left[\lim_{\epsilon \to 0} \delta_\epsilon (t-a)\right] dt$$

$$= \lim_{\epsilon \to 0} \int_{0}^{b}\left[e^{-st}\,\delta_\epsilon (t-a)\,dt\right]$$

$$= \lim_{\epsilon \to 0}\left[\int_{0}^{a} e^{-st}(0) + \int_{a}^{a+\epsilon} e^{-st}(1/\epsilon)\,dt + \int_{a+\epsilon}^{\infty} e^{-st}(0)\,dt\right.$$

$$= \lim_{\epsilon \to 0} \left[\int_a \frac{e^{-st}}{\epsilon} dt \right]$$

$$= \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} \right) \left(\frac{e^{-st}}{-s} \right)_a^{a+\epsilon}$$

$$= \lim_{\epsilon \to 0} \left(\frac{1}{-s\epsilon} \left[e^{-s(a+\epsilon)} - e^{-as} \right) \right.$$

$$= e^{-as} \lim_{\epsilon \to 0} - \left[\frac{e^{-s\epsilon} - 1}{s\epsilon} \right]$$

$$= e^{-as} \lim_{\epsilon \to 0} \left[- \frac{e^{-s\epsilon}(-s)}{s} \right] = e^{-as}$$

i.e. $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

$$\mathcal{L}\{\delta(t)\} = e^{0} = 1.$$

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