ORTHOGONALITY

Definition 1 :

Let u be an n-dimensional vector, The length of u (OR) norm of u is written as $\|u\|$ and is defined by

$$\|u\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \quad ; \text{ where } u = (u_1, u_2, \dots u_n)$$

Ex : i) If $u = (-3, -2)$, $\|u\| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$

ii) If $u = (1, -2, -5)$, $\|u\| = \sqrt{(1)^2 + (-2)^2 + (-5)^2} = \sqrt{30}$

NOTE: i) $\|u\| \geq 0$

ii) $\|u\| = 0$ Iff u is the zero vector.

Definition 2 :

Two vectors u and v in R^n are orthogonal to each other if $u^T v = 0 = v^T u$.

If $u^T u = 0$, u is orthogonal to itself.

NOTE: Two vectors u and v are orthogonal if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ (Pythagorean theorem).

Definition 3 :

If u and v are n-dimensional vectors then the inner product (or) scalar product (or) dot product of u and v is denoted by $\langle u, v \rangle$ (or) $u \cdot v$ (or) $u^T v$ and is defined by

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n .$$

THEOREM: If non-zero vectors $v_1, v_2, \ldots, v_K$ are mutually orthogonal then these vectors are linearly independent.

Proof: Given (i) $v_i \neq 0 \Rightarrow \|v_i\| > 0 \quad \forall i$.

(ii) $v_i^T v_j = 0$, $i \neq j$

Consider $c_1 v_1 + c_2 v_2 + \cdots + c_K v_K = 0$

$\Rightarrow v_1^T (c_1 v_1 + c_2 v_2 + \cdots + c_K v_K) = v_1^T 0$

$\Rightarrow c_1 (v_1^T v_1) + c_2 (v_1^T v_2) + \cdots + c_K (v_1^T v_K) = 0$

$\Rightarrow c_1 (v_1^T v_1) = 0$ (using (ii))

$\Rightarrow c_1 \|v_1\|^2 = 0$

$\Rightarrow \boxed{c_1 = 0}$ (using (i))

Similarly, $c_2 = 0 = c_3 = 0 \cdots = c_K$.

$\therefore \{v_1, v_2, \ldots v_K\}$ are Linearly independent.

Note: The converse of above theorem may not true

Ex: The vectors $\{(1,1)$ and $(1,2)\}$ are linearly independent

But they are not orthogonal ($u^T v \neq 0$).

Note: Let u, v and w be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(i) $u \cdot v = v \cdot u$

(ii) $(u+v) \cdot w = u \cdot w + v \cdot w$

(iii) $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

Orthogonal sets :

A set of vectors $\{u_1, u_2, \dots u_n\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal.

i.e. $u_i \cdot u_j = 0$ whenever $i \neq j$.

Ex: Let $\left\{ u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$ then

$$u_1 \cdot u_2 = (3)(-1) + (1)(2) + (1)(1) = 0$$
$$u_1 \cdot u_3 = (3)(-\tfrac{1}{2}) + (1)(-2) + (1)(7/2) = 0$$
$$u_2 \cdot u_3 = (-1)(-\tfrac{1}{2}) + (2)(-2) + (1)(\tfrac{7}{2}) = 0$$

$\therefore \{u_1, u_2, u_3\}$ forms an orthogonal set.

Orthogonal basis :

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
(vectors of W are mutually orthogonal)

Note : Let $\{u_1, u_2, \dots u_n\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $y \in W$, the weights in the linear combination $y = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$ are defined by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \quad ; (j = 1, 2, \dots n)$$

Orthonormal vectors

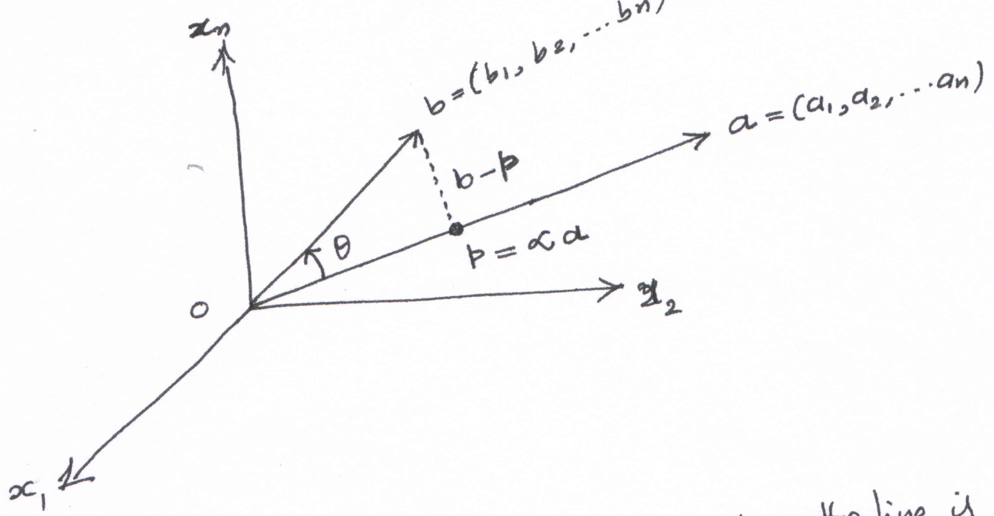
The non-zero vectors $\{ v_1, v_2, \dots v_K \}$ are said to be orthonormal if

(i) v_i are mutually orthogonal

(ii) $\|v_i\| = 1$, $\forall i$

(OR) $v_i^T v_j = \begin{cases} 0, & i \neq j \leftarrow \text{orthogonality} \\ 1, & i = j \leftarrow \text{normalization} \end{cases}$

Orthogonal projection



projection point $p = \alpha a$ [$\because$ every point on the line is a multiple of a]

Here $b - p$ is perpendicular to a

$$\Rightarrow (b - \alpha a) \cdot a = 0$$

(or)

$$a^T (b - \alpha a) = 0$$

$$\Rightarrow \boxed{\alpha = \frac{a^T b}{a^T a}}$$

$\therefore$ $\boxed{p = \alpha a = \frac{a^T b}{a^T a} a}$ is the projection of b onto the line through 0 and a

Orthonal Matrix

An orthogonal matrix is simply a square matrix with orthogonal columns and it is denoted by Q.

Also, $Q^T Q = I$ and $Q^T = Q^{-1}$ (The transpose is inverse)

Ex: Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$

Consider $U^T U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus, U is orthogonal matrix.

QR Factorization of Matrices :

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Note: Upper triangular matrix : $U = (u_{ij}) = 0$ for $i > j$

Ex: $U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 8 \\ 0 & 0 & 5 \end{bmatrix}$ $a_{21} = 0 = u_{31} = u_{32}$

Fundental theorem of orthogality

Given any matrix of order $m \times n$, Column subspace $C(A)$ and Left nullspace $N(A^T)$ are orthogonal in R^m and row space $C(A^T)$ and Null space $N(A)$ are orthogonal in R^n.

Proof: (i) To prove $C(A) \perp N(A^T)$

Let $x \in C(A)$ and $y \in N(A^T)$

$\Rightarrow Au = x$ and $A^T y = 0$ ← (null space)

Consider $x^T y = (Au)^T y$

$= (u^T A^T) y$

$= u^T (A^T y)$

$= u^T (0) = 0$

$\therefore$ $C(A)$ and $N(A^T)$ are orthogonal in R^m.

(ii) To prove $C(A^T)$ and $N(A)$ are Orthogonal

Let $x \in C(A^T)$ and $y \in N(A)$

$\Rightarrow A^T u = x$ and $Ay = 0$

Consider $x^T y = (A^T u)^T y$

$= u^T (Ay)$ $\quad (A^T)^T = A$

$= u^T (0)$

$= 0$

$\therefore$ $C(A^T) \perp N(A)$ in R^n

The Gram-Schmidt Process

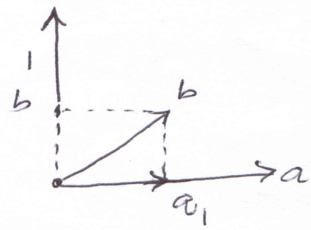
(orthogonalization of independent vectors)

Suppose a, b, c are three independent vectors, so we need to compute three orthonormal vectors q_1, q_2, q_3.

* Let $q_1 = a$ $\boxed{\text{or}}$ $a' = a$ (It can go in the direction of a),
  Divide by the length so that $\boxed{q_1 = \frac{a}{\|a\|}}$ $\boxed{\text{or}}$ $q_1 = \frac{a'}{\|a'\|}$.

* To find q_2 which has to be orthogonal to q_1
  Compute $b' = b - (q_1^T b) q_1$ [→ substract the component of b in the direction of q_1]
  which is orthogonal to q_1.

  $\therefore$ $\boxed{q_2 = \frac{b'}{\|b'\|}}$ which is orthoNormal



* Compute $c' = c - (q_1^T c) q_1 - (q_2^T c) q_2$ which is orthogonal to q_1 & q_2

  $\therefore$ $\boxed{q_3 = \frac{c'}{\|c'\|}}$ which is orthoNormal

Thus the whole idea in Gram-Schimdt process is to substract from every new vector its component in the directions that are ~~of~~ already settled.

Thus $a_j' = a_j - (q_1^T a_j) q_1 - \cdots - (q_{j-1}^T a_j) q_{j-1}$ and

$$q_j = \frac{a_j'}{\|a_j'\|}$$

where $a_1, a_2, \ldots, a_n$ are independent vectors & $q_1, q_2, \ldots q_n$ are orthonormal vectors.

The Factorization $A = QR$

$$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$\downarrow$ Columns of Independent vectors. $\downarrow$ Orthonormal columns $\downarrow$ R.

Where $R =$ upper Δlar Invertible Matrix with

Ex:

[1] Use Gram-Schmidt process to $a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $c = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

and factorise into $A = QR$.

Sol: Given $a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $c = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Let $q_1 = a \Rightarrow q_1 = \dfrac{a}{\|a\|} = \dfrac{1}{\sqrt{1^2+0^2+1^2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$$b' = b - (q_1^T b) q_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left\{ \left(\tfrac{1}{\sqrt{2}} \; 0 \; \tfrac{1}{\sqrt{2}} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$b' = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$\therefore q_2 = \frac{b'}{\|b'\|} = \frac{1}{\sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2}} \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}.$$

$$c' = c - (q_1^T c)q_1 - (q_2^T c)q_2$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left(\tfrac{1}{\sqrt{2}} \;\; 0 \;\; \tfrac{1}{\sqrt{2}}\right)\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \left(\tfrac{1}{\sqrt{2}} \;\; 0 \;\; -\tfrac{1}{\sqrt{2}}\right)\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$c' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \; q_3 = \frac{c'}{\|c'\|} = \frac{1}{\sqrt{0^2+1^2+0^2}}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \; \{q_1, q_2, q_3\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ forms a

orthonormal vectors

$$Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$ is the orthogonal matrix.

$$R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_3^T c \\ 0 & 0 & q_2^T c \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$\boxed{A = Q \quad R}$$

Q. What multiple of $a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be subtracted from $a_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to make the result orthogonal to a_1? Factor $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ into QR with orthonormal vectors in Q.

Sol: Given $a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Consider $a_2 - c a_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} c \\ c \end{bmatrix} = \begin{bmatrix} 4-c \\ -c \end{bmatrix}$

given $(a_2 - c a_1) \cdot a_1 = 0$

$$\Rightarrow \begin{bmatrix} 4-c \\ -c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow (4-c)(1) + (-c)(1) = 0$$

$$\Rightarrow 2c = 4 \Rightarrow \boxed{c = 2}$$

$\therefore$ '2' must be multiplied to a_1.

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{1^2+1^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$a_2' = a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \left(\tfrac{1}{\sqrt{2}} \ \tfrac{1}{\sqrt{2}}\right) \begin{bmatrix} 4 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$q_2 = \frac{a_2'}{\|a_2'\|} = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} \end{bmatrix} \rightarrow \text{orthogonal matrix}$$

$$R = \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 4 \end{bmatrix} \quad \therefore \boxed{A = QR}$$

3 If u is a unit ~~matrix~~ vector, show that $Q = I - 2uu^T$ is an orthogonal Matrix. Compute Q when

$$u^T = \left[\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\right]$$

Sol: Consider $Q^TQ = (I - 2uu^T)^T(I - 2uu^T)$

$$= (I - 2uu^T)(I - 2uu^T)$$

$$= I - \cancel{2uu^T} - \cancel{2uu^T} + \cancel{4uu^Tuu^T} = I$$

$\therefore$ $Q = I - 2uu^T$ is orthogonal Matrix

Now $$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\underbrace{\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}}_{4\times 1}\underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{1\times 4}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & +1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{bmatrix}$$

$$Q = \begin{bmatrix} \quad \end{bmatrix}$$

4 Let the vector space P_2 have the inner product $\langle p(x), q(x) \rangle$
$= \int_0^1 p(x) q(x)\, dx$. Apply the Gram-Schmidt procedure to transform the standard Basis $1, x, x^2$ to an orthoNormal Basis.

Sol: Given $u_1 = 1$, $u_2 = x$, $u_3 = x^2$

Let $u_1' = u_1 = 1$

$$\therefore q_1 = \frac{u_1'}{\|u_1'\|} = \frac{1}{\left\{\int_0^1 1^2 dx\right\}^{1/2}} = \boxed{1}$$

$\int_0^1 1^2 dx = x]_0^1 = 1$. $(1^{1/2} = 1)$

$[\frac{x^2}{2}]_0^1 = \frac{1}{2}$

Let $u_2' = u_2 - \langle u_2, q_1 \rangle q_1 = x - \left(\int_0^1 x\, dx\right) . 1 = x - \frac{1}{2}$

$$\therefore q_2 = \frac{u_2'}{\|u_2'\|} = \frac{x - 1/2}{\left\{\int_0^1 (x-\frac{1}{2})^2 dx\right\}^{1/2}} = \frac{x-1/2}{\left\{\frac{1}{3}(x-\frac{1}{2})^3\right\}^{1/2}} = \frac{x - 1/2}{(1/12)^{1/2}}$$

$$\therefore \boxed{q_2 = \sqrt{12}\left(x - \frac{1}{2}\right)}$$

$$u_3' = u_3 - \langle u_3, q_1 \rangle q_1 - \langle u_3, q_2 \rangle q_2$$
$$= x^2 - \left(\int_0^1 x^2 dx\right) . 1 - \sqrt{12}\left(\int_0^1 x^2 (x - \tfrac{1}{2})\, dx\right)(x - \tfrac{1}{2})$$
$$= x^2 - \frac{1}{3} - \sqrt{12}\left[\frac{x^2}{4} - \frac{x^3}{6}\right]_0^1 (x - 1/2)$$
$$= x^2 - \frac{1}{3} - \sqrt{12} \times \left(\frac{2}{24}\right)(x - 1/2)$$

$$\therefore q_3 = \frac{u_3'}{\|u_3'\|} = \frac{(x^2 - 1/3) - \frac{1}{\sqrt{12}}(x - 1/2)}{\sqrt{\int_0^1 \left(x^2 - 1/3 - 1/\sqrt{12}\,(x - 1/2)\right) dx}}$$

$= \sqrt{180}\,(x^2 - x + \frac{1}{6})$

Thus $\{q_1, q_2, q_3\} = \{1, \sqrt{12}(x - \frac{1}{2}), \sqrt{180}(x^2 - x + \frac{1}{6})\}$. forms an orthonormal Basis for P_2.

*

1 Let V be an inner product space and v_1, v_2, v_3 be vectors in V with $\langle v_1, v_2\rangle = 3$, $\langle v_2, v_3\rangle = -2$, $\langle v_1, v_3\rangle = 1$ and $\langle v_1, v_1\rangle = 1$ calculate

(i) $\langle v_1, 2v_2 + 3v_3\rangle$ (ii) $\langle 2v_1 - v_2, v_1 + v_3\rangle$

(iii) $\|v_2\|$ If $\langle v_2, v_1 + v_2\rangle = 13$.

Sol: Given $\langle v_1, v_2\rangle = 3$, $\langle v_2, v_3\rangle = -2$, $\langle v_1, v_3\rangle = 1$, $\langle v_1, v_1\rangle = 1$

(i) $\langle v_1, 2v_2 + 3v_3\rangle = \langle v_1, 2v_2\rangle + \langle v_1, 3v_3\rangle = 2\langle v_1, v_2\rangle + 3\langle v_1, v_3\rangle = 6+3 = 9$

(ii) $\langle 2v_1 - v_2, v_1 + v_3\rangle = 2\langle v_1, v_1\rangle + 2\langle v_1, v_3\rangle - \langle v_2, v_1\rangle - \langle v_2, v_3\rangle$

$= 2\langle v_1, v_1\rangle + 2\langle v_1, v_3\rangle - \langle v_1, v_2\rangle - \langle v_2, v_3\rangle$

$= 2 + 2 - 3 + 2 = \underline{3}$

(iii) $\langle v_2, v_1 + v_2\rangle = \langle v_2, v_1\rangle + \langle v_2, v_2\rangle = 13$

$\langle v_2, v_1\rangle + \langle v_2, v_2\rangle = 13$

$3 + \|v_2\|^2 = 13 \Rightarrow \|v_2\|^2 = 10$

$\|v_2\| = \sqrt{10}$

2 Let V be an inner product space, $u, v \in V$, 0, the zero vector in V and $\alpha \in \mathbb{R}$ prove that

(i) $\langle 0, u\rangle = 0$ (ii) $\|\alpha u\| = |\alpha|\,\|u\|$

(iii) $\|u\| = \|-u\|$ (iv) $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

Proof: (i) $\langle 0, u\rangle = \langle 0.0, u\rangle = 0\langle 0, u\rangle = 0$.

(ii) $\|\alpha u\|^2 = \langle \alpha u, \alpha u\rangle = \alpha^2\langle u, u\rangle = \alpha^2\|u\|^2$

$\therefore \|\alpha u\| = \sqrt{\alpha^2}\,\|u\| = |\alpha|\,\|u\|$

(iii) $\|-u\| = \|(-1)u\| = |-1|\,\|u\| = \|u\|$

(iv) $\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$

$$= \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle + \langle u,u\rangle - 2\langle u,v\rangle + \langle v,v\rangle$$

$$= 2\left[\langle u,u\rangle + \langle v,v\rangle\right]$$

$$= 2\left[\|u\|^2 + \|v\|^2\right]$$

3. Express $(1,2,3)$ as a Linear Combination of the vectors in the orthogonal basis $\{(1,-2,1)\ (2,1,0)\ (-1,2,5)\}$

Sol: Let $y = (1,2,3)$

$$u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad u_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

$$\therefore\ y = c_1 u_1 + c_2 u_2 + c_3 u_3 \longrightarrow \boxed{1}$$

Where $c_1 = \dfrac{y \cdot u_1}{u_1 \cdot u_1} = \dfrac{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}} = \dfrac{0}{6} = 0$

$c_2 = \dfrac{y \cdot u_2}{u_2 \cdot u_2} = \dfrac{4}{5}$

$c_3 = \dfrac{y \cdot u_3}{u_3 \cdot u_3} = \dfrac{3}{5}$

$$\therefore\ (1,2,3) = 0(1,-2,1) + \frac{4}{5}(2,1,0) + \frac{3}{5}(-1,2,5)$$

4 Let $V = (1, -2, 2, 0)$. Find a unit vector u in the same direction ~~as~~ V.

Sol: $\|V\| = \sqrt{9} = 3$

$$\text{unit vector} = u = \frac{V}{\|V\|} = \frac{1}{3}\begin{bmatrix}1\\-2\\2\\0\end{bmatrix}$$ //.

5 Show that $\{u_1, u_2, u_3\}$ is an orthogonal set. where

$$u_1 = \begin{bmatrix}3\\1\\1\end{bmatrix}, \quad u_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix}, \quad u_3 = \begin{bmatrix}-1/2\\-2\\7/2\end{bmatrix}$$

consider $u_1 \cdot u_2 = \begin{bmatrix}3\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\2\\1\end{bmatrix} = -3 + 2 + 1 = 0$

$u_1 \cdot u_3 = \qquad = 0$

$u_2 \cdot u_3 = \qquad = 0$

$\therefore \{u_1, u_2, u_3\}$ is an orthogonal set

~o~

6 Find all the vectors in $\mathbb{R}^3$ that are orthogonal to $(1, 1, 1)$ and $(1, -1, 0)$. produce an ~~orthog~~ orthonormal bases from these vectors.

Sol: Let $A = \begin{bmatrix}1 & 1 & 1\\1 & -1 & 0\end{bmatrix} \downarrow R_2 = R_2 - R_1$

$$\sim \begin{bmatrix}1 & 1 & 1\\0 & -2 & -1\end{bmatrix} \downarrow R_1 = R_1 + \tfrac{1}{2}R_2$$

$$\sim \begin{bmatrix}1 & 0 & 1/2\\0 & -2 & -1\end{bmatrix} \downarrow R_2 = -\tfrac{1}{2}R_2$$

$$\sim \begin{bmatrix}1 & 0 & 1/2\\0 & 1 & 1/2\end{bmatrix} = R$$

Consider $Rx = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x + \frac{1}{2}z = 0 \qquad \text{Let } z = k$$

$$y + \frac{1}{2}z = 0 \qquad x = -\frac{1}{2}k \ \& \ y = -\frac{1}{2}k$$

$$\therefore \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \text{ (or) } \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore$ The set of vectors orthogonal to given vectors $= (-1, -1, 2)$.

Verify
$x_1y_1 + x_2y_2 + x_3y_3 = 0$

Orthonormal Bases are given by

$$\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) \left(\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) \right\}$$

7 Given the three vectors $v_1 = (2, 0, -1)$ $v_2 = (0, -1, 0)$, $v_3 = (2, 0, 4)$ in $\mathbb{R}^3$. Verify that $\{v_1, v_2, v_3\}$ form an orthonormal set.

Sol: $\langle v_1, v_2 \rangle = \qquad = 0$

$\langle v_1, v_3 \rangle = \qquad = 0$

$\langle v_2, v_3 \rangle = \qquad = 0$

$\|v_1\| = \qquad = \sqrt{5}$

$\|v_2\| = \qquad = 1$

$\|v_3\| = \qquad = 2\sqrt{5}$

Since $\|v_2\| = 1$ & $\|v_1\| \neq 1$ & $\|v_3\| \neq 1$

$\therefore$ $\{v_1, v_2, v_3\}$ is NOT an orthonormal set