Vector Space

A non empty set V is called a real vector space If the following axioms are satisfied.

i) For u, v & $w \in V$ and c & $c' \in F$

i) $u+v \in V$ — closed under addition

ii) $cu \in V$ — closed under scalar multiplication

iii) $u+v = v+u$ - commutative law under $+$

iv) $u+(v+w) = (u+v)+w$ — Associative law

v) There is a unique '0' such that $u+0 = u = 0+u$, $\forall u$

vi) For each u there is a unique $-u$ such that $u+(-u) = 0 = (-u)+u$.

vii) $c(u+v) = cu + cv$

viii) $(c+c')u = cu + c'u$

(ix) $c(c'u) = (cc')u$

(x) $1 \cdot u = u$. , where 1 is the unit element.

The elements of the vector space V are called vectors.

EX:

1 The set M of all $m \times n$ matrices is a vector space under the matrix addition and scalar multiplication.

2 The set $V = \{a + b\sqrt{2} \mid a, b \in Q\}$ where Q be the set of all rational numbers is a vector space over the field Q.

3. Prove that the set V_n of all ordered n-tuples of real Numbers is a Vector Space.

PR

Sol: Let $u = (x_1, x_2, \dots x_n)$, $v = (y_1, y_2, \dots y_n)$ and $W = (z_1, z_2, \dots z_n) \in V$

(i) $u + v = (\quad) + (\quad) = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in V$

(ii) $cu = c(x_1, x_2, \dots x_n) = (cx_1, cx_2, \dots cx_n) \in V$

(iii) $u+v = (x_1+y_1, x_2+y_2, \dots x_n+y_n) = v+u$

(iv) $u+(v+w) = (x_1+y_1+z_1, x_2+y_2+z_2, \dots x_n+y_n+z_n) = (u+v)+w$

(v) There exists $0 = (0, 0, \dots 0)$ such that

$u + 0 = u$

(vi) For $u = (x_1, x_2, \dots x_n)$ there exists $-u = (-x_1, -x_2, \dots -x_n)$ such that $u + (-u) = 0$

(vii) $c(u+v) = c(x_1+y_1, x_2+y_2, \dots, x_n+y_n)$

$= (cx_1+cy_1, cx_2+cy_2, \dots cx_n+cy_n)$

$= (cx_1, cx_2, \dots cx_n) + (cy_1, cy_2, \dots cy_n)$

$= c(x_1, x_2, \dots x_n) + c(y_1, y_2, \dots y_n)$

$= cu + cv.$

(viii) $(c+c')u = (c+c')(x_1, x_2, \dots x_n)$

$= ((c+c')x_1, (c+c')x_2, \dots (c+c')x_n)$

$= (cx_1, cx_2, \dots cx_n) + (c'x_1, c'x_2, \dots c'x_n)$

$= c(x_1, x_2, \dots x_n) + c'(x_1, x_2, \dots x_n) = cu + c'u.$

(ix) $c(c'x) = c(c'x_1, c'x_2, \ldots c'x_n)$

$= ((cc')x_1, (cc')x_2, \ldots, (cc')x_n)$

$= (cc')(x_1, x_2, \ldots x_n)$

$= (cc')x$

(x) $1 \cdot x = 1 \cdot (x_1, x_2, \ldots x_n)$

$= (1 \cdot x_1, 1 \cdot x_2, \ldots 1 \cdot x_n)$

$= (x_1, x_2, \ldots x_n) = x.$

$\therefore$ The V_n of n-tuples is a vector space

$\boxed{4}$ Show that the set $V = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x, y \in C \right\}$, where C is the set of complex numbers is a vector space over the field C.

$\boxed{5}$ Show that the set $V = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in R \right\}$ is a vector space over the field R of real numbers.

$\boxed{6}$ For $n \geq 0$, the set P_n of polynomials of degree atmost n consists of all polynomials of the form $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$ is a vector space.

Result: In any vector space V,

(i) $c\,0 = 0$ for every scalar c

(ii) $0u = 0$ for every vector $u \in V$

(iii) $(-1)u = -u$ for every vector $u \in V$

Proof: (i) We have $0 = 0 + 0$

$$\Rightarrow \quad c\cdot 0 = c(0+0)$$
$$= c\cdot 0 + c\cdot 0$$

Adding $-(c\cdot 0)$ to both sides, we get

$$(c\cdot 0) - (c\cdot 0) = \underbrace{-(c\cdot 0) + (c\cdot 0)} + (c\cdot 0)$$

$$0 = 0 + c\cdot 0$$

$$0 = c\cdot 0$$

$$\therefore \quad \boxed{c\cdot 0 = 0}$$

(ii)
$$0u = (0+0)u$$
$$= 0u + 0u$$

Adding $-(0u)$ to both sides, we get

$$-(0u) + 0u = \underbrace{-(0u) + 0u} + 0u$$

$$\Rightarrow \quad 0 = 0 + 0u$$

$$0 = 0u$$

$$\therefore \quad \boxed{0u = 0}$$

(iii)
$$(-1)u + u = (-1)u + 1\cdot u$$
$$= (-1+1)u$$

$$(-1)u + u = 0u = 0.$$

$\Rightarrow$ $(-1)u$ is the negative of u. $\Rightarrow \boxed{(-1)u = -u}$

SUBSPACE

Let S be a non-empty subset of a vector space V, then S is said to be a subspace of V if S is a vector space under vector addition and scalar multiplication as in V.

The set $\{0\}$ consisting of zero vectors of V and the whole set $\{V\}$ of V are called trivial (or) improper subspaces of V, all other subspaces of V are called the non-trivial subspaces of V.

Ex 1. The $V_2(R)$ of all vectors lying on a plane passing through the origin is a subspace of the vector space $V_3(R)$.

Ex 2. Construct a subset of xy-plane in R^2 that is
(i) closed under vector addition ~~but~~ not scalar multiplication
(ii) closed under scalar multiplication but not vector addition.

Sol: (i) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \,/\, x, y \in \underbrace{Z}_{\text{integers}} \right\}$

For $\alpha = \begin{bmatrix} x \\ y \end{bmatrix}$, $\beta = \begin{bmatrix} x' \\ y' \end{bmatrix} \in R^2 \Rightarrow \alpha + \beta = \begin{bmatrix} x+x' \\ y+y' \end{bmatrix} \in S$ → closed

For $c = \sqrt{2}$, $c\alpha = \sqrt{2}\begin{bmatrix} x \\ y \end{bmatrix} \notin S$ ← not closed

(ii) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \,\middle|\, x = 0 \text{ or } y = 0 \right\}$

For $\alpha = \begin{bmatrix} x \\ 0 \end{bmatrix}$, $\beta = \begin{bmatrix} 0 \\ y \end{bmatrix} \in S$

$\alpha + \beta = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \notin S$ ← not closed

THEOREM : A non-empty subset S of a vector space $V(F)$ is a subspace of $V(F)$ if and only if the following conditions are satisfied
i) If $\alpha, \beta \in S$, then $\alpha+\beta \in S$
ii) If $\alpha \in S$, $c \in F$, then $c\alpha \in S$

(OR)

A subset S of a vector space $V(F)$ is a subspace of $V(F)$ if and only if it is closed under vector addition and scalar multiplication of vectors defined in V.

Proof : If part : Given S is a ~~vector~~ subspace of $V(F)$

$\Rightarrow S$ itself is a vector space

$\therefore$ (i) For $\alpha, \beta \in S$, $\alpha+\beta \in S$

and (ii) For $\alpha \in S$, $c \in F$, $c\alpha \in S$

only if part : Given (i) For $\alpha, \beta \in S$, $\alpha+\beta \in S$
(ii) For $\alpha \in S$, $c \in F$, $c\alpha \in S$

From (i) S is closed under vector addition
From (ii) S is closed under scalar multiplication

Put $c=0$ in (ii), $0\alpha = 0 \in S$

$\therefore$ '0' is the additive identity of S. $[0+\alpha=\alpha \in S]$

Put $c=-1$ in (ii), $(-1)\alpha = -\alpha \in S$

$\therefore$ '$-\alpha$' is the additive inverse of S $[\alpha+(-\alpha)=0 \in S]$

Since the vector addition is associative, commutative in V, so is in S also.

$\therefore (S, +)$ is an abelian group

The other axioms of the vector space hold in S also,

since they hold in the whole space V also.

$\therefore$ S is a vector space and $S \subset V$

Thus, S is a subspace of $V(F)$.

<u>THEOREM</u>: Prove that a non-empty subset S of a vector space $V(F)$ is a subspace of $V(F)$ If and only if for $a, b \in F$ and $\alpha, \beta \in S$, $a\alpha + b\beta \in S$.

<u>Proof:</u> <u>If part</u>: Given S is a subspace of $V(F)$

$\Rightarrow$ S itself a vector space

For $a \in F$, $\alpha \in S$ $\Rightarrow$ $a\alpha \in S$

For $b \in F$, $\beta \in S$ $\Rightarrow$ $b\beta \in S$

$\therefore$ For $a\alpha \in S$ and $b\beta \in S$, $a\alpha + b\beta \in S$.

<u>only If part</u>: Given $a\alpha + b\beta \in S \rightarrow \boxed{1}$, for $a, b \in F$, $\alpha, \beta \in S$

By putting $a = 1$ and $b = 1$ in $\boxed{1}$, we get $\alpha + \beta \in S \rightarrow \boxed{2}$

By putting $a = c$ and $b = 0$ in $\boxed{1}$, we get $c\alpha \in S \rightarrow \boxed{3}$

From $\boxed{2}$ and $\boxed{3}$, S is closed under vector addition and scalar multiplication

$\therefore$ S is a subspace of $V(F)$

THEOREM : Prove that the intersection of two subspaces of a vector space is a subspace of V. Is the result is true for union also? Justify.

Proof : Let S and T be two subspaces of V

Since S and T are subspaces, $0 \in S$ and $0 \in T$

$\Rightarrow 0 \in S \cap T$ [Zero vector]

$\Rightarrow S \cap T \neq \phi$

For $u, v \in S \cap T \Rightarrow u, v \in S$ and $u, v \in T$

$\Rightarrow u + v \in S$ and $u + v \in T$ ($\because$ S & T are subspaces)

$\Rightarrow u + v \in S \cap T \rightarrow \boxed{1}$

For $c \in F$, $cu \in S$ and $cu \in T$

$\Rightarrow cu \in S \cap T$ ($\because$ S & T are subspaces) $\rightarrow \boxed{2}$

From $\boxed{1}$ & $\boxed{2}$, $S \cap T$ is closed under vector addition and scalar multiplication

$\therefore S \cap T$ is a subspace of V.

Union of two subspaces need not be a subspace of V.

Ex. Consider $S = \{(a, 0) \mid a \in R\}$ and $T = \{(0, b) \mid b \in R\}$ are two subspaces of $V_2(R)$.

$$S \cup T = \{(a, 0), (0, b)\}$$

Consider $(a, 0) + (0, b) = (a, b) \notin S \cap T$ ← not closed under vector addition

$\therefore S \cup T$ is NOT a subspace of V.

NOTE: Union ($S \cup T$) is a subspace of V provided S (or T) is a subset of T (or S). [S = plane $z = 0$ in R^3 & T = x-axis in R^3, $S \cup T$ is subspace of R^3]

Problems on subspace

1. Let $S = (x, 2x, -3x, x)$ in V_4, Show that S is a subspace of V_4.

Sol: Let $u = (x, 2x, -3x, x)$ and $v = (y, 2y, -3y, y) \in S$

consider (i) $u + v = (x+y,\ 2(x+y),\ -3(x+y),\ x+y) \in S$

(ii) $cu = (cx,\ 2(cx),\ -3(cx),\ (cx)) = c(x, 2x, -3x, x) \in S$

Thus, S is closed under vector addition & scalar multiplication

$\therefore$ $S = \{(x, 2x, -3x, x) / x \in R\}$ is a subspace of V_4.

2. prove that the set of all solutions (a, b, c) of the equation $a + b + 2c = 0$ is a subspace of R^3.

Sol: Let $S = \{(a, b, c) \mid a + b + 2c = 0\}$

Let $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2) \in S$

then $a_1 + b_1 + 2c_1 = 0$ and $a_2 + b_2 + 2c_2 = 0$

consider (i) $u + v = (a_1, b_1, c_1) + (a_2, b_2, c_2)$

$= (a_1 + b_1 + 2c_1) + a_2 + b_2 + 2c_2$

$= 0 + 0 = 0 \in S$

(ii) $cu = (ca_1, cb_1, cc_1)$

$= c(a_1 + b_1 + 2c_1)$

$= c(0)$

$= 0 \in S$

$\therefore$ $S = \{(a, b, c) \mid a + b + 2c = 0\}$ is a subspace of R^3

3 prove that the set of vectors of the form $(a,0,0)$ is a subspace of R^3.

field of real numbers.

Sol: let $S = \{ (a,0,0) \in R^3 / a \in R \}$

suppose $u = (a,0,0)$ and $v = (b,0,0) \in S$

Consider $u+v = (a+b, 0, 0) \in S$

and $cu = (ca, 0, 0) = c\,(a,0,0) = cu \in S$

$\therefore$ S is a subspace of R^3.

4 Is the set $\{(x,y,z) / x^2+y^2+z^2 \le 1\}$ is a subspace of R^3 or not?

Sol: let $S = \{ (x,y,z) \in R^3 / x^2+y^2+z^2 \le 1$ and $x,y,z \in R \}$

suppose $u = (1,0,0)$ and $v = (0,1,0) \in S$

Consider $u+v = (1,0,0) + (0,1,0) = (1,1,0) = 1^2+1^2+0^2 = 2 \not\le 1$

$\Rightarrow u+v \notin S$ ← not closed vector addition.

Hence, S is not a subspace of R^3.

5 Determine which of the following are subspaces of M_{22}, set of all 2×2 matrices.

(i) The set of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in Z$ set of integers.

(ii) The set of 2×2 matrices such that $A = A^T$, where T denotes the transpose of matrix

(iii) The set of 2×2 matrices such that $|A| = 0$.

Soln: (i) Let $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \mid a, b, c, d \in Z \right\}$

suppose $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in S$, then $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \notin S$

Hence, S is not a subspace of M_{22}.

(ii) Let $S = \{ A \in M_{22} \mid A = A^T \}$

suppose $A, B \in S$ and $c \in R$

Consider $(A+B)^T = A^T + B^T$

$= A + B \in S \quad [\because A, B \in S]$

and $(cA)^T = c A^T$

$= cA \in S$

Hence, S is a subspace of M_{22}.

(iii) Let $S = \{ A \in M_{22} \mid |A| = 0 \}$

suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S \; (|A| = 0 = |B|)$

Consider $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |A+B| = 1 \notin S$

$\Rightarrow S$ is not closed under matix Addition

Hence, S is not a subspace of M_{22}.

$\boxed{6}$ show that the set S of the elements of the vector space $V_3(R)$ of the form $(x+2y, y, -x+3y)$, where $x, y \in R$ is a sub space of $V_3(R)$

7 Determine which of the following are subspaces of V, the vector space of all real valued functions defined on $\mathbb{R}$.

(i) The family of all f's such that $f(0)=0$
(ii) The family of all f's such that $f(0)=1$
(iii) The family of all continuous functions f
(iv) The family of all differentiable functions f

Sol: (i) Let $S = \{ f \in V \mid f(0)=0 \}$

suppose $f, g \in S$ and $c \in \mathbb{R}$

consider $(f+g)(0) = f(0)+g(0) = 0 \in S$

and $(cf)(0) = c\,f(0) = c(0) = 0 \in S$

Hence, S is a subspace of V

(ii) Let $S = \{ f \in V \mid f(0) = 1 \}$

Suppose $f(x) = 1+x$ and $g(x) = 1+x^2 \in S$
($\because f(0)=1$ & $g(0)=1$)

consider $(f+g)(0) = f(0)+g(0)$
$= 1+1 = 2 \notin S$

Hence, S is not a subspace of S.

(iii) Let $S = \{ f \in V \mid f$ is continuous $\}$
suppose $f, g \in S$ and $c \in \mathbb{R}$
~~consider~~ Then, $f+g$ and cf are continuous
$\Rightarrow f+g \in S$ and $cf \in S$

Hence, S is a subspace of V

(iv) similler to (iii)

NULL SPACES

The null space of an $m \times n$ matrix A, is the set of all solutions of the homogeneous equation $AX = 0$. It is denoted by Null A. or N(A)

i.e., $\text{Null } A = \{ x \mid x \in R^n \text{ and } Ax = 0 \}$

Ex: Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$

then $AX = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad X \in \text{Null } A.$

COLUMN SPACES

The column space of an $m \times n$ matrix, is the set of all linear combinations of the columns of A. It is denoted by Col A, or C(A).

i.e. $\text{Col } A = \{ \text{All linear combinations of columns of } A \}$

$\text{Col } A = \{ b \mid b = Ax \text{ for some } x \in R^n \}$ $\boxed{n}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$ then

$$\text{Col } A = \left\{ c_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \middle/ c_i \in R \right\}$$

ROW SPACES

The row space of an $m \times n$ matrix, is the set of all linear combinations of the rows of A. It is denoted by row A or $R(A)$ or $C(A^T)$

[i.e. set of all linear combinations of the columns of A^T]

i.e, Row A = { All linear combinations of rows of A }

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ then

$$\text{Row } A = C(A^T) = \{ c_1(a_{11}, a_{12}, \ldots a_{1n}) + c_2(a_{21}, a_{22}, \ldots a_{2n}) + \cdots + c_n(a_{m1}, a_{m2}, \ldots a_{mn}) \mid c_i \in R \}$$

LEFT NULL SPACE

The Left null space of an $m \times n$ matrix, is the set of all solutions of the homogeneous equations $A^T x = 0$. It is denoted by $N(A^T)$.

i.e, $N(A^T) = \{ x \mid x \in R^n \text{ and } A^T x = 0 \}$

THEOREM: The null space of an $m\times n$ matrix is a subspace of R^n.

Proof: $Nul\,A = \{x / x\in R^n \text{ and } Ax=0\}$

clearly $0\in Nul\,A \Rightarrow Nul\,A \neq \phi$

Let u and $v \in Nul\,A$ then $Au=0$ and $Av=0$

consider $A(u+v) = Au+Av = 0+0 = 0 \in Nul\,A$

and $A(cu) = c(Au) = c(0) = 0 \in Nul\,A$

Thus, $Nul\,A$ is a subspace of R^n.

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THEOREM: The column space of $m\times n$ matrix A is a subspace of R^m.

proof: $col\,A = \{b / b=Ax, \text{ for some } x\in R^n\}$

clearly $0\in col\,A$ $(\because A0=0) \Rightarrow C(A)\neq\phi$

Let $b, b' \in C(A)$ then $b=Ax$ and $b'=Ax'$

consider $b+b' = Ax+Ax'$

$= A(x+x')$

$\Rightarrow b+b' \in C(A)$.

Similarly, $cb = c(Ax) = A(cx)$, $c\in R$

$\Rightarrow cb \in C(A)$

Thus, $col\,A$ is a subspace of R^m.

Elementary transformations of a matrix

(i) The interchange of any two rows (columns)

i.e $R_i \leftrightarrow R_j$ (or) $C_i \leftrightarrow C_j$

(ii) The multiplication of any row (column) by a non-zero number

i.e. $R_i = K R_i$ (or) $C_i = K C_i$

(iii) The addition of a scalar multiple of the elements of any row (column) to the corresponding elements of any other row (column)

i.e, $R_i = R_i \pm K R_j$ (or) $C_i = C_i \pm K C_j$

Echelon form of a Matrix

A non-zero matrix A of order $m \times n$ is said to be in the echelon form, if the following conditions hold.

i) The leading entry of each row is unity (or non-zero)

First ENTRY

ii) All the entries below this leading entry is a zero.

iii) The number of zeros appearing before the leading entry in each row must be greater than that appear in its previous row.

iv) The zero rows (rows with every entry is zero) must appear below the non zero rows.

EXS.

$$\begin{pmatrix} 1 & -3 & 2 & 4 & 5 & 1 \\ 0 & 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 4 & 2 & 1 \end{pmatrix}_{4\times 5}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{3\times 4}$$

$$\begin{pmatrix} 4 & 3 & -2 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4}, \quad \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 4 & 2 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{44}$$

Rank of a matrix

The rank of a matrix A of order $m \times n$ is the dimension of Row space of A (or) column space of A

[Number of independent (non zero) rows or Number of pivot columns]

The rank of the matrix A is denoted by $\rho(A)$ (rho) or rank A

NOTE:

1. The rank of a null matrix is zero
2. The rank of a singular matrix $(|A| = 0)$ of order $n \times n$ is less than n.
3. If A is the matrix of order $m \times n$, then $\rho(A) \leq \min(m, n)$

   i.e $\rho(A)$ cannot exceed m (or) n.
4. The rank of unit matrix of order n, is $\underline{n}$ $[\rho(I_n) = n]$
5. The rank of A and its transpose have the same rank
6. The rank of a matrix A is not altered by elementary transformations applied to the matrix.
7. $\rho(A) = \rho(AB)$; If B is non-Singular

The rank theorem

The dimensions of the column space and the row space of an $m \times n$ matrix are equal (rank A) and satisfies the equation

nullity

$$\boxed{\text{rank } A + \dim \text{Nul } A = n} \rightarrow \text{Number of columns.}$$

rank A: Number of non-zero rows (or) Number of pivot Columns

dim Nul A: Number of non-pivot columns (or) dimensions of Basis of N(A).

Note: Nullity is called the dimensions of Null space of A.

Note: The variables corresponds to the pivot columns are called basic variables and the variables corresponds to the non basic variables are called free variables.

Ex:

$$\begin{bmatrix} 1 & 0 & -5 & 1 & 5 \\ 0 & 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Here $x_1, x_2, x_5 \rightarrow$ basic variables (pivot columns)

$x_3, x_4 \rightarrow$ free variables (non pivot columns)

Problems on R(A), C(A), N(A) and $N(A^T)$

1 Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}_{3\times 4}$, then

(i) If the C(A) is a subspace of R^K, then what is K?

(ii) If the N(A) is a subspace of R^K, then what is K?

Soln: (i) Since C(A) is a subspace of R^m

$\Rightarrow m = 3 \Rightarrow \boxed{K=3}$

(ii) Since N(A) is a subspace of R^n

$\Rightarrow n = 4 \Rightarrow \boxed{K=4}$

2 With $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

(i) Determine if u is in N(A). Could u be in C(A)?

(ii) Determine if v is in C(A). Could v be in N(A)?

Soln: (i) consider $Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore u \notin N(A)$

Also, u has 4-entries and C(A) is a subspace of R^3

$\therefore u \notin C(A)$

(ii) Consider $(A ; v) = \begin{bmatrix} 2 & 4 & -2 & 1 & ; & 3 \\ -2 & -5 & 7 & 3 & ; & -1 \\ 3 & 7 & -8 & 6 & ; & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & ; & 3 \\ 0 & -1 & 5 & 4 & ; & 2 \\ 0 & 0 & 0 & 17 & ; & 0 \end{bmatrix}$

which is consistent (No. of Eqns = No. of unknowns)

$\therefore v \in C(A)$

Also, v has 3 entries and N(A) subspace of R^4

$\therefore v \notin N(A)$

$\boxed{3}$ Find the basis for $C(A)$, $C(A^T)$, $N(A)$ of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$ Also verify rank theorem.

Sol: Consider

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \downarrow R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \downarrow \begin{array}{l} R_2 = R_2 + 3R_1 \\ R_3 = R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \downarrow R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 10 & -10 \end{bmatrix} \downarrow R_3 = R_3 - 5R_2$$

$$\sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$ which is echelon form

In this echelon form, there are 2-independent rows (or) 2-pivot columns

$\therefore$ Basis for $C(A^T) = R(A) = \{(-3, 6, -1, 1, -7), (1, -2, 2, 3, -1)\}$

(or)

$\{(1, -2, 2, 3, -1), (0, 0, 1, 2, -2)\}$

rank $A = \boxed{2}$

Basis for $C(A) = \left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ pivot columns.

<u>To find Basis for $N(A)$;</u>

consider $Ax = 0$

$$\Updownarrow$$

$\Rightarrow \quad Ux = 0$ Echelon form

$$\Rightarrow \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here x_1, x_3 are basic and x_2, x_4, x_5 are free

Express <u>basic</u> variables in terms of $\{x_2, x_4, x_5\}$

ie. $x_1 - 2x_2 + 2x_3 + 3x_4 - x_5 = 0 \rightarrow \boxed{1}$

$x_3 + 2x_4 - 2x_5 = 0 \rightarrow \boxed{2}$

from $\boxed{2}$, $\quad x_3 = -2x_4 + 2x_5$

From $\boxed{1}$, $\quad x_1 - 2x_2 + 2(-2x_4 + 2x_5) + 3x_4 - x_5 = 0$

$$\Rightarrow x_1 = 2x_2 + x_4 - 3x_5$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Basis for $N(A) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\because$ Any linear combinations of these three vectors satisfies $Ax = 0$ (ie, $\in N(A)$)

$\Rightarrow \dim N(A) = \boxed{3}$

$$\underset{\dim R(A)}{\text{rank } A} + \underset{\dim N(A)}{\text{Nullity } A} = n \longrightarrow \text{columns}$$

$$2 + 3 = 5$$

Hence, rank theorem is satisfied (verified)

$\boxed{4}$ Find the basis for $C(A)$, $C(A^T)$, $N(A)$ and $N(A^T)$ for the matrix $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 6 \\ 5 & 15 & 25 \\ -3 & -2 & -6 \end{bmatrix}$

Soln: Consider $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 6 \\ 5 & 15 & 25 \\ -3 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -9 \\ 0 & 0 & 0 \\ 0 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Echelon form

Basis for $C(A^T) = R(A) = \{(1, 3, 5), (0, -7, -9)\}$

(independent rows)

rank $A = \dim R(A) = \boxed{2}$

Bases for $C(A) = \text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 15 \\ -2 \end{bmatrix} \right\}$ pivot columns

Null Space:

Consider $Ax = 0$

$\Updownarrow$

Echelon form $\leftarrow Ux = 0$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here x_1, x_2 are basic, x_3 is free

$$\Rightarrow \quad x_1 + 3x_2 + 5x_3 = 0 \rightarrow \boxed{1}$$
$$-7x_2 - 9x_3 = 0 \rightarrow \boxed{2}$$

From $\boxed{2}$, $\quad x_2 = -\frac{9}{7}x_3$

From $\boxed{1}$, $\quad x_1 = -3\left(-\frac{9}{7}x_3\right) - 5x_3 = -\frac{8}{7}x_3$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8/7\, x_3 \\ -9/7\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -8/7 \\ -9/7 \\ 1 \end{bmatrix}$$

Basis for $N(A) = \left\{ \begin{bmatrix} -8/7 \\ -9/7 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim N(A) = 1.$

Left Null space:

consider $A^T y = 0 \Rightarrow \begin{bmatrix} 1 & 3 & 5 & -3 \\ 3 & 2 & 15 & -2 \\ 5 & 6 & 25 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\sim \begin{array}{c} \begin{matrix} \checkmark & \checkmark & \times & \times \end{matrix} \\ \begin{bmatrix} 1 & 3 & 5 & -3 \\ 0 & -7 & 0 & +7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $y_1, y_2, \rightarrow$ basic, $y_3, y_4 \rightarrow$ free

$$\Rightarrow \quad y_1 + 3y_2 + 5y_3 - 3y_4 = 0 \rightarrow (1)$$
$$-7y_2 + 7y_4 = 0 \rightarrow (2)$$

From $\boxed{2}$, $\quad y_2 = y_4$

From $\boxed{1}$, $\quad y_1 = -3y_4 - 5y_3 + 3y_4 = -5y_3$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -5y_3 \\ y_4 \\ y_3 \\ y_4 \end{bmatrix} = y_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $N(A^T) = \left\{ \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim N(A^T) = 2$

$\boxed{5}$ Find the bases for the row space, the column space and the null space of the matrix. Also verify rank theorem

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Sol: Given $A =$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \begin{matrix} R_2 = R_2 + 2R_1 \\ R_3 = R_3 - 3R_1 \\ R_4 = R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \quad \begin{matrix} R_3 = R_3 - 2R_2 \\ R_4 = R_4 - 4R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\begin{array}{c} \begin{matrix} \checkmark & \checkmark & \times & \checkmark & \times \end{matrix} \\ \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

which is the Echelon form of A

Here there are 3 non zero rows (independent rows) (or)

3 pivot columns.

$\therefore$ Basis for Row $A = \{ (1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20) \}$

dim Row $A = \boxed{3} =$ rank.

Basis for Col A = pivot columns (1^{st}, 2^{nd}, 4^{th} columns)

$$= \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

To find basis for N(A) :

Consider $Ax = 0$

$\Leftrightarrow$ $Ux = 0$

Echelon form

$$\Rightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $x_1, x_2, x_4 \to$ basic variables, x_3, x_5 are free variables.

$$\Rightarrow x_1 + 3x_2 - 5x_3 + x_4 + 5x_5 = 0 \longrightarrow \boxed{1}$$

$$x_2 - 2x_3 + 2x_4 - 7x_5 = 0 \longrightarrow \boxed{2}$$

$$-4x_4 + 20x_5 = 0 \longrightarrow \boxed{3}$$

From $\boxed{3}$, $x_4 = 5x_5$

From $\boxed{2}$, $x_2 = 2x_3 - 3x_5$ (on simplification)

From $\boxed{1}$, $x_1 = -x_3 - x_5$ (on simplification)

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{u} + x_5 \underbrace{\begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}}_{v}$$

Any linear combinations of u & v forms a basis for N(A).

$$\therefore \text{Basis for } N(A) = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim N(A) = \boxed{2}$$

rank Theorem : $\text{rank} A + \dim N(A) = n$ → number of columns

$$3 + 2 = 5$$

NOTE: (Points to remember)

1. Given $n \times n$ system $Ax = b$, The system has UNIQUE soln if

i) $|A| \neq 0$

ii) $Ax = 0$ has only Trivial soln when $|A| \neq 0$

iii) $\rho(A) = \rho(A;b) = n$

iv) All pivots are non-zero

v) All the n-planes intersects at a UNIQUE point [Row picture]

vi) RHS vector can be uniquely Expressed as a Linear combination of the column vectors [Column picture]

vii) RHS vector is a vertex of the completed parallelopiped in n-dimension.

2. The system has Infinitely Many solution If

i) $|A| = 0$

ii) $Ax = 0$ has infinitely many solution

iii) $\rho(A) = \rho(A;b) < n$

iv) SOME pivots are zero

v) planes do not intersect at a UNIQUE point

vi) RHS vector can be expressed as a Linear combination of vectors

vii) RHS is NOT a unique vertex of the completed parallelo piped in n-dimension.

3 The system has NO Solution, If

(i) $|A| = 0$

(ii) $Ax = 0$ has Infinetely many solution

(iii) $\rho(A) \neq \rho(A;B)$

(iv) Some pivots are zero.

(v) planes do not intersects at all

(vi) RHS vector cannot be expressed as a Linear Combination of ~~vectors~~ column vectors.

(vii) RHS vector is a vertex of the completed Parallelopiped in n-dimension.