Transformation.

For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. we call this transformation or mapping of z plane into w-plane. if a point z_0 maps into the point w_0, w_0 is also known as the image of z_0.

Example: Transform the rectangular region ABCD in z plane bounded by $x = 1$, $x = 3$, $y = 0$ and $y = 3$. Under the transformation

$$w = z + (2 + i)$$

Soln:

$$w = z + (2 + i)$$

$$u + iv = x + iy + (2 + i)$$

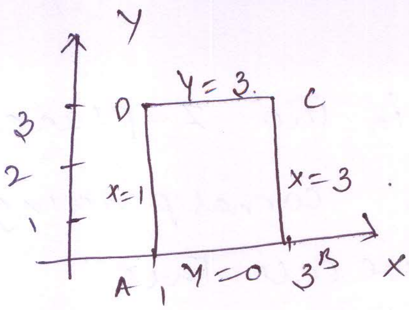
$$= (x + 2) + i(y + 1)$$

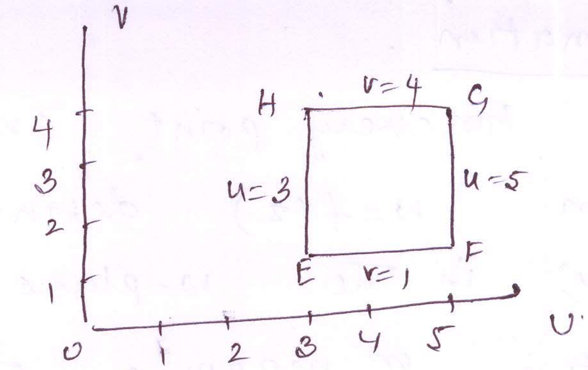
By equating real and imaginary quantities we have $u = x + 2$, $v = y + 1$.

z plane	w-plane	z plane	w plane.
x	$u = x + 2$	y	$v = y + 1$
1	$= 3$	0	$= 1$
3	$= 5$	3	$= 4$.

Here the lines $x = 1$, $x = 3$, $y = 0$ and $y = 3$ in the z plane are transformed onto the line

$u=3, u=5$ $\quad v=1$ and $v=4$. in The w plane.





confolmal Transformation

let two curves C, C_1 in the z-plane intersect at the point P and the corresponding curve C', C_1' in the w-plane intersect at P'. if the angle of intersection of the curves at P in z-plane is the same as the angle of intersection of the curves of w plane at P' in magnitude and direction, then the transfolmation is called conformal.

conditions
(i) $f(z)$ is analytic
(ii) $f(z) \neq 0$.

Bilinear Transfolmation (Mobius transformation)

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

is known as bilinear transfomation.

If $ad-bc \neq 0$ then $\frac{dw}{dz} \neq 0$ i.e transformation is conformal.

$$z = \frac{-dw+b}{cw-a}$$

This is also bilinear except $w = \frac{a}{c}$.

Invariant points of bilinear transformation

we know that $w = \frac{az+b}{cz+d}$ — ① if z maps into itself, then $w = z$.

Then $z = \frac{az+b}{cz+d}$ — ②

roots of (2) are the invariants or fixed points of the bilinear transformation. if the roots are equal the bilinear transformation is said to be parabolic.

<u>cross ratio</u> :

if there are four points z_1, z_2, z_3, z_4 taken in order, then the ratio $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of z_1, z_2, z_3, z_4

A bilinear transformation preserves cross-ratio of four points.

Proof: we know that $w = \frac{az+b}{cz+d}$.

As w_1, w_2, w_3, w_4 are images of z_1, z_2, z_3, z_4 respectively. so

$$w_1 = \frac{az_1+b}{cz_1+d} \qquad w_2 = \frac{az_2+b}{cz_2+d}.$$

$$w_1 - w_2 = \frac{ad-bc}{(cz_1+d)(cz_2+d)} (z_1 - z_2)$$

$$w_2 - w_3 = \frac{ad-bc}{(cz_2+d)(cz_3+d)}(z_2-z_3)$$

$$w_3 - w_4 = \frac{ad-bc}{(cz_3+d)(cz_4+d)}(z_3-z_4)$$

$$w_4 - w_1 = \frac{ad-bc}{(cz_4+d)(cz_1+d)}(z_4-z_1)$$

From the above equations

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

$$\Rightarrow (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4).$$

Problems: (1) Find a bilinear transformation which maps the points $i, -i, 1$ of the z plane into $0, 1, \infty$ of the w-plane respectively.

By cross ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$w_1=0$, $w_2=1$, $w_3=\infty$ and $z_1=i$ $z_2=-i$ $z_3=1$.

From (1)

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)\left(\frac{1}{\infty}-1\right)}{\left(\frac{w}{\infty}-1\right)(1-0)} = \frac{(z-i)(-i-1)}{(z-1)(-i-1)}$$

$$w = \frac{(z-i)(i+1)}{(z-1)(2i)} = \frac{(z-i)\,(-1+i)}{(z-1)(-2)}$$

$$= \frac{(i-1)\,z + (i+1)}{-2z+2}.$$

② Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$. Also find the image of the unit circle $|z|=1$.

Soln: On putting $z=0,-1,\ i$ into $w = i, 0, \infty$ respectively in

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(-1)}{(-1)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \frac{w-i}{-i} = \frac{z\,(1+i)}{z-i}$$

$$\Rightarrow w-i = \frac{(-i+1)\,z}{z-i}$$

$$\Rightarrow w = \frac{(1-i)z}{z-i} + i = \frac{(1-i)\,z + iz + 1}{z-i}$$

$$w = \frac{z+1}{z-i}$$

$$z = \frac{iw+1}{w-1}$$

and $|z| = 1$.

$$\left|\frac{iw+1}{w-1}\right| = 1 \Rightarrow |1+iw| = |w-1|$$

$$\Rightarrow |1+i(u+iv)| = |u+iv-1|$$

$$\Rightarrow |1-v+iu| = |u-1+iv|$$

$$(1-v)^2 + u^2 = (u-1)^2 + v^2$$

$$\Rightarrow 1+v^2-2v+u^2 = u^2+1-2u+v^2$$

$$\Rightarrow u - v = 0$$

$$\Rightarrow v = u.$$

(3) Show that the transformation $w = i\frac{1-z}{1+z}$ transforms the circle $|z|=1$ onto the real axis of the w peane and the interior of the circle into the upper half of the w-plane.

Soln: $w = i\left(\frac{1-z}{1+z}\right)$

$$u+iv = i\left(\frac{1-(x+iy)}{1+(x+iy)}\right) = \frac{i-ix+y}{[1+(x+iy)]}\frac{[(1+x)-iy]}{[(1+x)-iy]}$$

$$= \frac{i+ix+y-ix-ix^2-xy+y+xy-iy^2}{(1+x)^2+y^2}$$

$$= \frac{y - xy + y + xy + i + ix - ix - ix^2 - iy^2}{(1+x)^2 + y^2}$$

$$= \frac{2y + i(1 - x^2 - y^2)}{(1+x)^2 + y^2}$$

Equating the real and imaginary parts we get

$$u = \frac{2y}{(1+x)^2 + y^2} \quad -① \qquad v = \frac{1-(x^2+y^2)}{(1+x)^2+y^2} \quad -②$$

when $x^2+y^2 = 1$, then $v = \frac{1-1}{(1+x)^2+y^2} = 0$.

$v = 0$, is the equation of the real axis in the w-plane.

Now the equation of the interior circle is $x^2+y^2 < 1$

dividing (1) by (2)

$$\frac{u}{v} = \frac{2y}{1-(x^2+y^2)} \qquad u - u(x^2+y^2) = 2vy.$$

$$u(x^2+y^2) = u.$$

$$x^2+y^2 = 1 - \frac{2vy}{u} \qquad 1 - \frac{2vy}{u} < 1$$

$$-\frac{2vy}{u} < 0, \qquad 2vy > 0$$

$$v > 0$$

$\Rightarrow$ $v > 0$ is the equation of the upper half of w-plane.

④ Find Bilinear transformation which send the points $z = 0, 1, \infty$ into the points $w = -5, -1, 3$ respectively. What are the invariant points in this transformation ?

$w = \dfrac{az+b}{cz+d}$ be the required BLT.

when $z = 0$, $w = -5$ we get $-5 = b/d$.

$\Rightarrow b + 5d = 0$.

when $z = 1$, $w = -1$ we get $-1 = \dfrac{a+b}{c+d}$

$\Rightarrow a + b + c + d = 0$.

$$w = \frac{az+b}{cz+d} = \frac{z(a+b/z)}{z(c+\frac{d}{z})} = \frac{a+b/z}{c+d/z}$$

$z = \infty$, $w = 3$ we get

$3 = \dfrac{a}{c} \Rightarrow a = 3c$.

$\therefore$ $b + 4c + d = 0$ $\qquad \dfrac{b}{0-20} = \dfrac{-c}{1-5} = \dfrac{d}{4}$

$b + 5d = 0$

$$\frac{b}{-20} = \frac{c}{4} = \frac{d}{4}$$

$a = 3$, $b = -5$, $c = 1$, $d = 1$.

$\therefore$ $w = \dfrac{az+b}{cz+d}$ reduces to $\dfrac{3z-5}{z+1}$.

The invariant points are obtained by taking $w = z$

$\therefore z = \dfrac{3z-5}{z+1} \Rightarrow z^2+z-3z+5=0$

$z^2 - 2z + 5 = 0$.

$z = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm 4i}{2}$.

$\therefore z = 1 \pm 2i$ are invariant points.

(5) Show that the transformation $w = \dfrac{i-z}{i+z}$ maps the x-axis of the z plane onto a circle $|w| = 1$ and the points in the half plane $y > 0$ onto the points $|w| < 1$

Given $w = \dfrac{i-z}{i+z}$

$\Rightarrow w(i+z) = i - z$

$\Rightarrow z(w+1) = i - wi$

$z = \dfrac{i(1-w)}{(1+w)}$

$\Rightarrow x+iy = \dfrac{i[1-(u+iv)]}{[1+(u+iv)]} = \dfrac{u+i(1-u)}{(1+u)+iv}$

$\Rightarrow x+iy = \dfrac{v+i(1-u)}{(1+u)+iv} - \dfrac{(1+u)-iv}{(1+u)-iv}$

$$= \frac{v(1+u) - iv^2 + i(1+u)(1-u) + v(1-u)}{(1+u)^2 - i^2v^2}$$

$$= \frac{2v + i\,[1-u^2-v^2]}{(1+u)^2+v^2}.$$

$$x+iy = \frac{2v}{(1+u)^2+v^2} + \frac{i\,(1-u^2-v^2)}{(1+u)^2+v^2}$$

Equating real and imaginary parts

$$x = \frac{2v}{(1+u)^2+v^2} \qquad y = \frac{1-u^2-v^2}{(1+u)^2+v^2}$$

since $y=0$ is the equation of x-axis

$\therefore$ we have $\dfrac{1-u^2-v^2}{(1+u)^2+v^2} = 0$

$\Rightarrow 1-u^2-v^2 = 0$

$\therefore u^2+v^2 = 1$ which is a circle

$|w|^2 = 1$ or $|w| = 1$

Further $y>0$

$\Rightarrow \dfrac{1-u^2-v^2}{(1+u)^2+v^2} > 0 \Rightarrow 1-u^2-v^2 > 0$

$\Rightarrow u^2+v^2 < 1$ or $|w| < 1$.

6. Show that the transformation $w = \dfrac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ onto the straight line $4u+3=0$.

Solution: we have $w = \frac{2z+3}{z-4}$

The inverse transformation is $z = \frac{4w+3}{w-2}$.

Now the circle $x^2+y^2-4x=0$ can be written as

$z\bar{z} - 2(z+\bar{z}) = 0$

substituting for z and $\bar{z}$ we get -

$$\frac{4w+3}{w-2} \quad \frac{4\bar{w}+3}{\bar{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

$$16w\bar{w} + 12w + 12\bar{w} + 9 - 2(4w\bar{w} + 3\bar{w} - 8w - 6 + 4w\bar{w} + 3w - 8\bar{w} - 6) = 0.$$

$22(w+\bar{w}) + 33 = 0 \quad \Rightarrow 22(2u) + 33 = 0$

$\Rightarrow 4u + 3 = 0.$

Thus the circle is transformed into a straight line.

1. Transformation of $w = e^z$.

let $w = f(z) = e^z$

$\Rightarrow f'(z) = e^z$

clearly $f'(z) \neq 0 \quad \forall z$

$\therefore$ $w = f(z) = e^z$ is confolmal at everypoint.

let $z = x+iy$ and $w = u+iv$

$u+iv = e^{x+iy}$

$= e^x e^{iy}$

$= e^x (\cos y + i \sin y)$

$u = e^x \cos y \qquad v = e^x \sin y$

case (i) consider a straight line parallel to y axis. equation of a line parallel to y axis is $x = a$.

$u = e^a \cos y \qquad v = e^a \sin y$.

$\Rightarrow \quad u^2 + v^2 = e^{2a} = (e^a)^2$

which is the equation of the circle with centre at the origin. and radius e^a.

$\therefore$ $w = e^z$ transforms a st line parallel to y axis in z plane to a circle with centre at the origin in w plane.

case (ii): consider a st line parallel to x axis

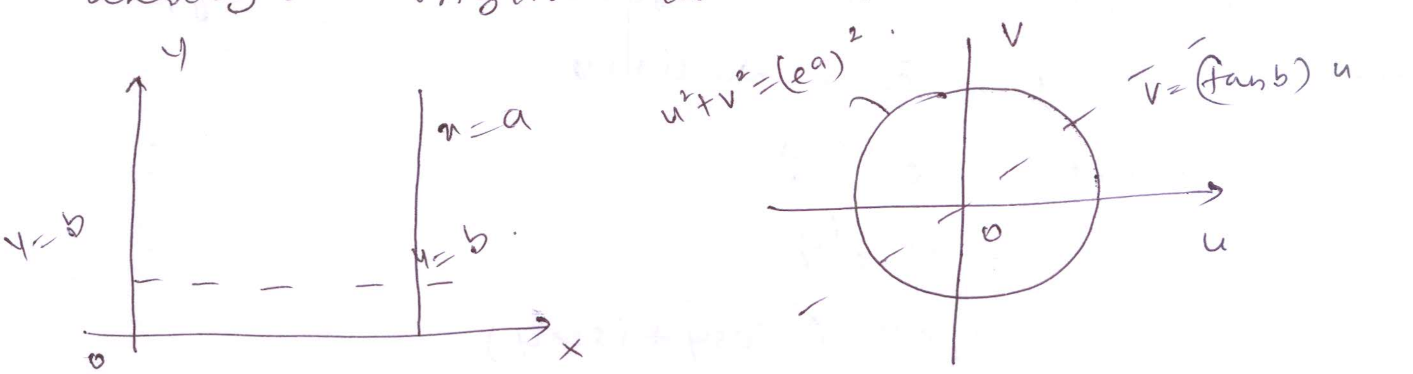
$y = b$.

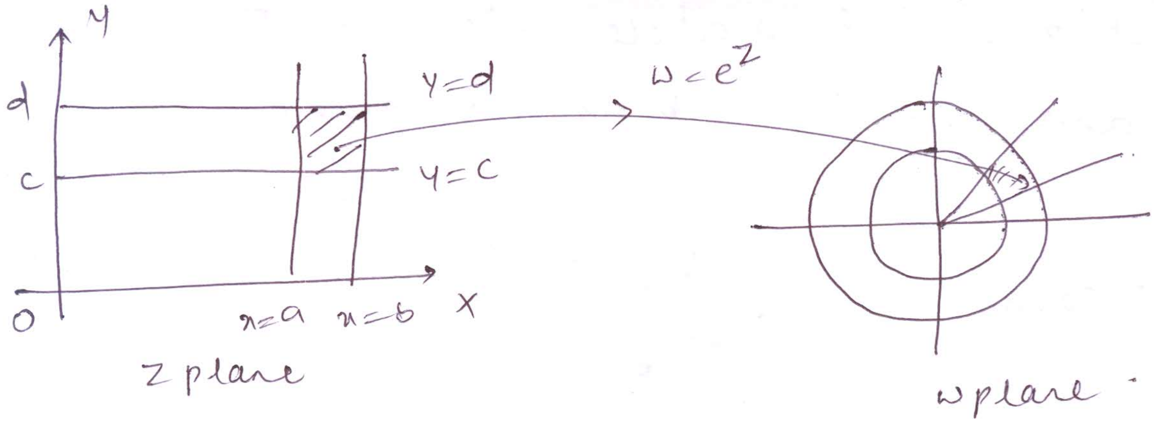
$u = e^x \cos b \qquad v = e^x \sin b$.

$\frac{u}{v} = \tan b$

$v = (\tan b) u$. which is the equation of a st. line passing through the origin and having slope $(\tan b)$ in w plane.

$\therefore$ $w = e^z$, Transforms the st. line parallel to x-axis in z plane to a st. line passing through origin in w-plane.





(2) Show that the transformation $w = z + \frac{1}{z}$ transforms circles from z-plane into family of ellipse and radial lines into family of hyperbolae.

Soln: Let $w = f(z)$

$$= z + \frac{1}{z}, \quad z \neq 0. \quad -(1)$$

$$f'(z) = 1 - \frac{1}{z^2}$$

clearly $f(z)$ is conformal for all values of z except $x = \pm 1$.

let $z = re^{i\theta}$

eqn (1) reduces to

$$w = u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

equating real and imaginary parts we get

$$u = \left(r + \frac{1}{r}\right)\cos\theta$$

$$v = \left(r - \frac{1}{r}\right)\sin\theta \quad -(2)$$

origin and radius b in z-plane.

from (2)

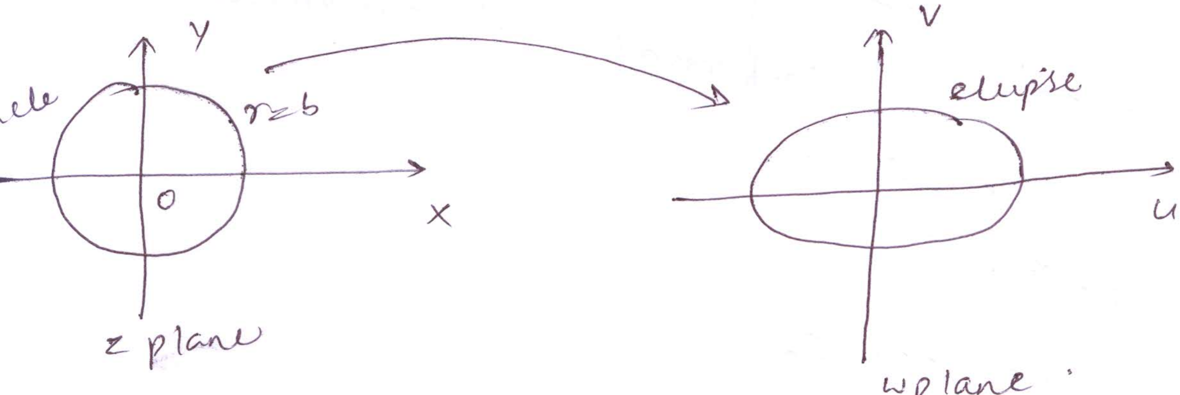
$$\frac{u}{\left(b+\frac{1}{b}\right)} = \cos\theta \qquad \frac{v}{\left(b-\frac{1}{b}\right)} = \sin\theta$$

Squaring and adding we get.

$$\frac{u^2}{\left(b+\frac{1}{b}\right)^2} + \frac{v^2}{\left(b-\frac{1}{b}\right)^2} = 1.$$

which is the equation of ellipse with centre at the origin w-plane.

$w = z + \frac{1}{z}$ transforms a circle in z-plane to an ellipse in w-plane.



case (ii) Let $\theta = c$ be any line in z-plane.

$$u = \left(r+\frac{1}{r}\right)\cos c \quad , \quad v = \left(r-\frac{1}{r}\right)\sin c$$

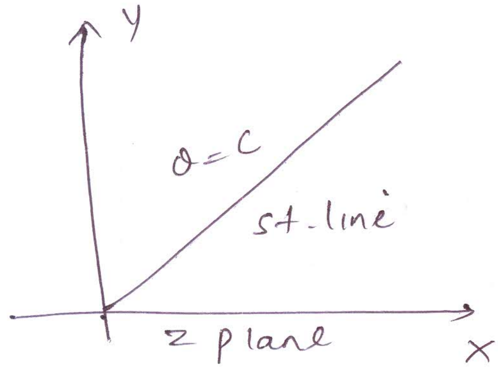
$$\Rightarrow \frac{u^2}{\cos^2 c} = \left(r+\frac{1}{r}\right)^2 \qquad \frac{v^2}{\sin^2 c} = \left(r-\frac{1}{r}\right)^2$$

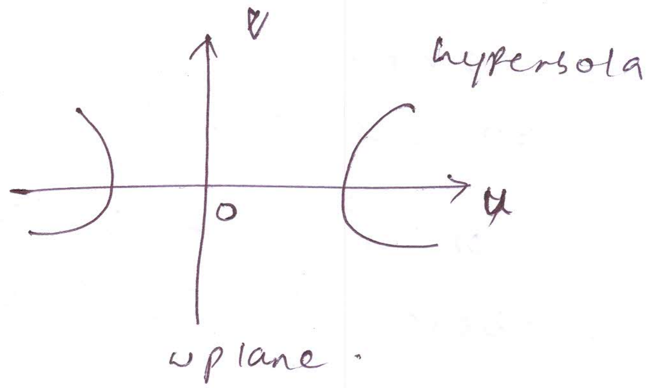
$$\Rightarrow \frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = 4$$

$$\frac{u^2}{(2\cos c)^2} - \frac{v^2}{(2\sin c)^2} = 1.$$

which is the equation of hyperbola is w-plane

$\therefore$ $w = z + \frac{1}{z}$ transforms a st-line passing through origin in z-plane. To a hyperbola in w plane.





(3) Find the image of $x - y = 1$ and $x^2 - y^2 = 1$ under the transformation $w = z^2$.

consider $w = z^2$

$z = x + iy$ $w = u + iv$

$u + iv = (x+iy)^2 = (x^2 - y^2) + i(2xy)$

equating real and imaginary parts we get

$u = x^2 - y^2$ $v = 2xy$.

(i) consider $x - y = 1$, or $y = x - 1$

we have $u = x^2 - y^2 = x^2 - (x-1)^2$

$= 2x - 1$

$\Rightarrow$ $2x = u + 1$.

also $v = 2xy$.

$= 2x(x-1)$

since $2x = u+1$ wehave

$V = \cancel{2xy}\ (u+1)\left\{\frac{u+1}{2} - 1\right\}$

$$V = \frac{u^2-1}{2}$$

$u^2 - 1 = 2V$

$u^2 = 2V + 1$

$\Rightarrow u^2 = 2\left(v + \frac{1}{2}\right)$

or $u^2 = 2\left\{v - \left(\frac{-1}{2}\right)\right\}$ which is the equation of parabola in the w plane with vertex $\left(0, -\frac{1}{2}\right)$

(ii) consider $x^2 - y^2 = 1$

$\Rightarrow u = 1$

$\because$ from (1) $u = x^2 - y^2$

which is a st-line parallel to the v axis in the w plane.

Complex Integration

In case of a real variable, the path of integration of $\int_a^b f(x)dx$ is always along the x-axis from $x=a$ to $x=b$. But in case of a complex function $f(z)$ the path of the definite integral $\int_a^b f(z)dz$ can be along any curve from $z=a$ to $z=b$.

$$z = x+iy \Rightarrow dz = dx + i dy \quad -(1)$$

$$dz = dx \text{ if } y=0 \quad -(2)$$

$$dz = i dy \text{ if } x=0 \quad -(3)$$

In (1), (2), (3) the direction of dz are different. its value depends upon the path of integration. But the value of integral from a to b remains the same along any regular curve from a to b.

In case the initial point and final point coincide so that c is a closed curve, then this integral is called contour integral and is denoted by $\oint_c f(z)dz$.

if $f(z) = u(x,y) + i v(x,y)$ then since $dz = dx + i dy$ we have

$$\oint_c f(z)dz = \int_c (u+iv)(dx+idy)$$

$$= \int_c (udx - vdy) + i\int_c (vdx + udy)$$

which shows that the evaluation of a line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

1. Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$ along the two paths.

(i) $x = t+1$, $y = 2t^2-1$

(ii) The straight line joining $1-i$ and $2+i$.

(i) $x = t+1 \Rightarrow dx = dt$

$y = 2t^2-1 \Rightarrow dy = 4t\,dt$

$dz = dx + i\,dy$

$= dt + i\,4t\,dt = (1+i4t)\,dt$

The limits of the integral are $(1,-1)$ and $(2,1)$. Corresponding to these points, the limits of t are 0 and 1.

$$\int_{1-i}^{2+i} (2x+iy+1)\,dz$$

$$\int_0^1 [2(t+1) + i(2t^2-1) + 1]\,(1+4it)\,dt$$

$$\int_0^1 [\{2(t+1) + i(2t^2-1) + 1\} + \{8it(t+1) - 4t(2t^2-1) + 4it\}]\,dt$$

$$\int_0^1 [2(t+1) + 1 - 4t(2t^2-1) + i\{2t^2-1 + 8t(t+1) + 4t\}]\,dt$$

$$\int_0^1 \{(2t+2+1 - 8t^3+4t) + i(2t^2-1+8t^2+8t+4t)\}\,dt$$

$$\int_0^1 \{(-8t^3+6t+3) + i(10t^2+12t-1)\}\,dt$$

$$\left[-2t^4+3t^2+3t+i\left(\frac{10}{3}t^3+6t^2-t\right)\right]_0^1$$

$$= -2+3+3+i\left(\frac{10}{3}+6-1\right)-0 = 4+\frac{25}{3}i$$

(ii) $\int_{1-i}^{2+i} (2x+iy+1)\,dz$

The equation of the straight line joining $(1,-1)$ and $(2,1)$ is

$$y+1 = \frac{1+1}{2-1}(x-1) \Rightarrow y+1 = 2x-2$$
$$\Rightarrow y = 2x-3$$
$$\Rightarrow dy = 2dx.$$

$$\int_{1,-1}^{(2,1)} (2x+iy+1)(dx+idy) = \int_1^2 (2x+2ix-3i+1)(dx+2i\,dx)$$

$$= \int_1^2 (2x+2ix-3i+1)(1+2i)\,dx$$

$$= \int_1^2 (2x+2ix-3i+1+4ix-4x+6+2i)\,dx$$

$$= \int_1^2 (-2x+6ix-i+7)\,dx$$

$$= \left(-x^2+3ix^2-ix+7x\right)_1^2 = 4+8i$$

(2) Evaluate $\int_0^{1+i} (x^2-iy)\,dz$, along the path

(a) $y=x$ (b) $y=x^2$.

(a) Along the line $y=x$.

$dy = dx$ so that $dz = dx + idy$

$$\Rightarrow dz = dx + idx$$
$$= (1+i)\,dx$$

$\int_0^{1+i} (x^2 - iy)\, dz$

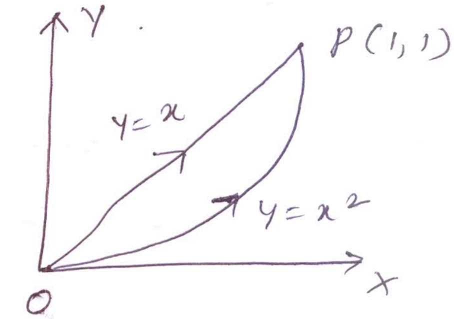
[on putting $y = x$ and $dz = (1+i)dx$]

$= \int_0^1 (x^2 - ix)(1+i)\, dx$

$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$

$= (1+i)\left(\frac{1}{3} - \frac{1}{2}i\right)$

$= \frac{(1+i)(2-3i)}{6} = \frac{5}{6} - \frac{1}{6}i$



which is the required value of the given integral.

(b) Along the parabola $y = x^2$, $dy = 2x\,dx$

$dz = dx + i\,dy$

$\Rightarrow dz = dx + 2i\,x\,dx = (1 + 2ix)\,dx$

and x varies from 0 to 1.

$\int_0^{1+i} (x^2 - iy)\, dx = \int_0^1 (x^2 - ix^2)(1 + 2ix)\, dx$

$= \int_0^1 x^2 (1-i)(1+2i\,x)\, dx$

$= (1-i)\int_0^1 x^2 (1 + 2ix)\, dx = (1-i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1$

$= (1-i)\left[\frac{1}{3} + \frac{1}{2}i\right] = \frac{(1-i)(2+3i)}{6}$

$= \frac{5}{6} + \frac{1}{6}i$

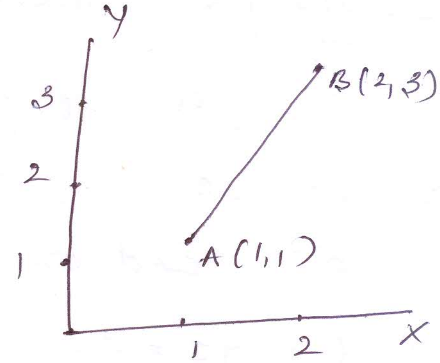
(3) Evaluate $\int_C (12z^2 - 4iz)\, dz$ along the curve C joining the points $(1,1)$ and $(2,3)$

$$\int_C (12z^2 - 4iz)\, dz$$

$$= \int_C \left[12(x+iy)^2 - 4i(x+iy)\right](dx+idy)$$

$$= \int_C \left[12(x^2-y^2+2ixy) - 4ix + 4y\right](dx+idy)$$

$$= \int_C (12x^2 - 12y^2 + 24ixy - 4ix + 4y)(dx+idy) \quad - (1)$$



Equation of the line AB passing through (1,1) and (2,3) is

$$y-1 = \frac{3-1}{2-1}(x-1)$$

$$(y-1) = 2(x-1) \Rightarrow y = 2x-1 \Rightarrow dy = 2dx.$$

Putting the values of y and dy in (1)

$$= \int_1^2 \left[12x^2 - 12(2x-1)^2 + 24ix(2x-1) - 4ix + 4(2x-1)\right][dx + i2dx]$$

$$= \int_1^2 \left[12x^2 - 48x^2 + 48x - 12 + 48ix^2 - 24ix - 4ix + 8x - 4\right](1+2i)\,dx$$

$$= (1+2i)\int_1^2 \left[(-36+48i)x^2 + (56-28i)x - 16\right] dx$$

$$= (1+2i)\left[(-36+48i)\frac{x^3}{3} + (56-28i)\frac{x^2}{2} - 16x\right]_1^2$$

$$= -156 + 38i$$

Cauchy's integral theorem.

If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve C, then $\int_C f(z)\,dz = 0$.

Proof: Let the region enclosed by the curve C be R and let $f(z) = u+iv$, $z = x+iy$, $dz = dx+idy$.

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx-vdy) + i\int_C (vdx+udy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_C \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Replacing $-\frac{\partial v}{\partial x}$ by $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ we get.

$$\int_C f(z)dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$$= 0.$$

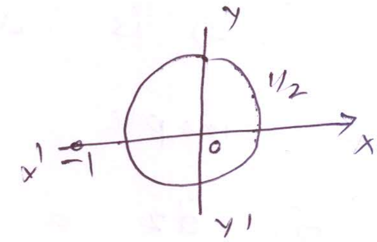
$$\Rightarrow \int_C f(z)dz = 0.$$

① Find the integral $\int_C \frac{3z^2+7z+1}{z+1}\,dz$, where C is the circle $|z| = \frac{1}{2}$.

Poles of the integrand are given by putting the denominator equal to zero.

$$z+1 = 0 \quad \Rightarrow \quad z = -1.$$

The given circle $|z| = \frac{1}{2}$ with centre at $z=0$. and radius $\frac{1}{2}$. doesnot enclose any singularity of the given function.



(2) Find the value of $\int_C \frac{z+4}{z^2+2z+5}\, dz$ if c is the circle $|z+1| = 1$.

Soln:

$$z^2+2z+5 = 0$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

The given circle $|z+1|=1$ with centre at $z=-1$ and radius unity doesnot enclose any singularity of the function $\frac{z+4}{z^2+2z+5}$.

$$\therefore \int_C \frac{z+4}{z^2+2z+5}\, dz = 0.$$

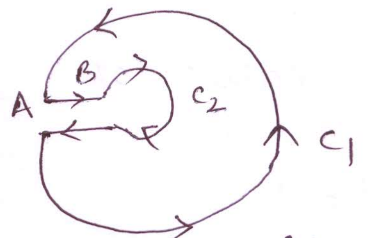
Assignment.

(3) Evaluate $\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)}\, dz$ where c is the square with the vertices at $1+i,\ 2+i,\ 2+2i,\ 1+2i$

Extension of cauchy's theorem to multiple connected Region

if $f(z)$ is analytic in the region R between two simple closed curves c_1 and c_2 then

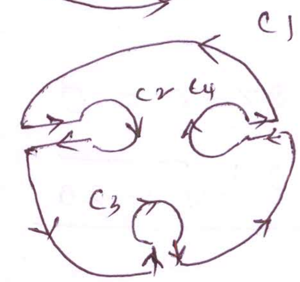
$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz.$$



Proof :

$\int f(z)\,dz = 0$.

where the path of integration is along AB, and curves C_2 in clock wise direction and along BA and along C_1 in anticlock wise direction.



$$\int_{AB} f(z)\,dz - \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

$$\Rightarrow \; -\int_{C_2} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

as $\int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Cauchy Integral formula :

If $f(z)$ is analytic with in and on a closed curve C, and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\,dz.$$

Proof: consider the function $\frac{f(z)}{z-a}$, which is analytic at all points within C, except $z=a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1. Hence by Cauchy's integral theorem for multiple connected region, we have

$$\int_C \frac{f(z)\,dz}{z-a} = \int_{C_1} \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)-f(a)+f(a)}{z-a}\,dz$$

$$= \int_{C_1} \frac{f(z)-f(a)}{z-a}\,dz + f(a)\int_{C_1} \frac{dz}{z-a} \quad \text{--- (1)}$$

For any point on C_1

Now $$\int_{C_1} \frac{f(z)-f(a)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})-f(a)}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$= \int_0^{2\pi} [f(a+re^{i\theta})-f(a)]\,i\,d\theta = 0.$$

$$\int_{C_1} \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{re^{i\theta}} = \int_0^{2\pi} i\,d\theta = i\,[\theta]_0^{2\pi}$$

$$= 2\pi i$$

Putting the values of the integrals in R.H.S of (1) we have

$$\int \frac{f(z)\,dz}{z-a} = 0 + f(a)\,2\pi i \Rightarrow f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\,dz$$

Cauchy integral formula for the derivative of an analytic function.

If a function $f(z)$ is analytic in a region R, then its derivative at any point $z = a$ of R is also analytic in R and is given by

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$ where c is any closed curve in R surrounding the point $z = a$.

Proof: We know Cauchy's Integral formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz \quad - (1)$$

Differentiating (1) w.r.t 'a' we get

$$f'(a) = \frac{1}{2\pi i} \int_C f(z) \frac{\partial}{\partial a}\left(\frac{1}{z-a}\right) dz$$

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^3}$$

$$\boxed{f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{n+1}}}$$

① Evaluate $\int_C \frac{e^{2z}}{(z+1)(z+2)} dz$ where C is the circle $|z|=3$.

Solution : The points $z=a=-1$, $z=a=-2$ being $(-1,0)$ $(-2,0)$ lies inside the circle $|z|=3$.

Resolve $\frac{1}{(z+1)(z+2)}$ into partial fraction

$$\frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$1 = A(z+2) + B(z+1)$$

When $z=-2$, we set $1=-B \Rightarrow B=-1$

$z=-1$ $A=1$.

$$\frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$\int_C \frac{e^{2z}}{(z+1)(z+2)} dz = \int_C \frac{e^{2z}}{z+1} dz - \int \frac{e^{2z}}{(z+2)} dz$$

By Cauchy integral formula we have

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Taking $f(z) = e^{2z}$, $a=-1,-2$ respectively.

$$\int \frac{e^{2z}}{(z+1)(z+2)} dz = 2\pi i f(-1) - 2\pi i f(-2)$$

$$= 2\pi i \{e^{-2} - e^{-4}\}$$

$$= \frac{2\pi i}{?} \left\{ \frac{1}{e^2} - \frac{1}{e^4} \right\}.$$

(2) Evaluate $\int_C \frac{2z+1}{z^2+z}\,dz$ where $C: |z| = \frac{1}{2}$

Soln : $\int_C \frac{2z+1}{z^2+z}\,dz = \int_C \frac{2z+1}{z(z+1)}\,dz$ which is of the form $\int_C \frac{f(z)}{z-a}\,dz$, where $f(z) = \frac{2z+1}{z+1}$ and $a=0$.

It is evident that $z=0$ is inside the given circle $|z| = \frac{1}{2}$.

$\therefore$ $f(z)$ is analytic on and within C.

$\therefore$ $\int_C \frac{2z+1}{z^2+1}\,dz = \int_C \frac{2z+1}{(z-0)(z+1)}\,dz = \int \frac{f(z)}{z-0}\,dz$

$= 2\pi i\, f(a) = 2\pi i\, f(0)$

$= 2\pi i \cdot 1$

$\left(\because f(z) = \frac{2z+1}{z+1} \Rightarrow f(0) = 1\right)$

$\therefore$ $\int_C \frac{2z+1}{z^2+1}\,dz = 2\pi i$

(3) Evaluate $\int_C \frac{z^3-2z+1}{(z-i)^2}\,dz$ where C is $|z| = 2$

Soln :