1) Consider the basis $B = \{(1,2), (3,-1)\}$ & $B' = \{(3,1), (5,2)\}$ of $\mathbb{R}^2$

Find ~~B' to B'~~ transition matrix from B to B' if

U is a vector $U_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find U_B'.

Such that

→

$u_1 = (1,2) \in B$

$u_1' = (3,1) \quad u_2' = (5,2) \in B'$

$u_1 = c_1 u_1' + c_2 u_2'$

$(1,2) = c_1(3,1) + c_2(5,2)$

$3c_1 + 5c_2 = 1$

$c_1 + 2c_2 = 2$

$(U_1)_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$

$u_2 = (3,-1) \in B$

$u_1' = (3,1), \quad u_2 = (5,2)$

$u_2 = c_1 u_1' + c_2 u_2'$

$(3,-1) = c_1(3,1) + c_2(5,2)$

$3c_1 + 5c_2 = 3$

$c_1 + 2c_2 = -1$

$(U_2)_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 11 \\ -6 \end{bmatrix}$

$P = [(U_1)_{B'} \quad (U_2)_{B'}]$

$P = \begin{bmatrix} -8 & 11 \\ 5 & -6 \end{bmatrix}$

$U_{B'} = P U_B$

$U_{B'} = \begin{bmatrix} -8 & 11 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 4 \end{bmatrix}$

2) Find the transition matrix P from the basis B to the standard ~~matrix~~ basis B' of $\mathbb{R}^2$. Use this matrix to find the coordinate vectors u, v & w relative to B'.

$$B = \{(2,3)\ (1,2)\} \ \& \ B' = \{(1,0)\ (0,1)\}$$

$$u_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad v_B = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad w_B = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$\rightarrow$ $P = ?$, $u_{B'} = ?$, $v_{B'} = ?$ $w_{B'} = ?$

$$u_{B'} = P u_B$$

$$v_{B'} = P v_B$$

$$w_{B'} = P w_B$$

$$u_1 = (2,3)$$

$$u_1' = (1,0), \quad u_2' = (0,1)$$

$$u_1 = c_1 u_1' + c_2 u_2'$$

$$(2,3) = c_1(1,0) + c_2(0,1)$$

$$c_1 = 2$$

$$c_2 = 3$$

$$(u_1)_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$u_2 = (1,2)$$

$$u_1' = (1,0), \quad u_2' = (0,1)$$

$$u_2 = c_1 u_1' + c_2 u_2'$$

$$(1,2) = c_1(1,0) + c_2(0,1)$$

$$c_1 = 1$$

$$c_2 = 2$$

$$(u_2)_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P = [(u_1)_{B'}\ (u_2)_{B'}]$$

$$= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$U_{B'} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$V_{B'} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

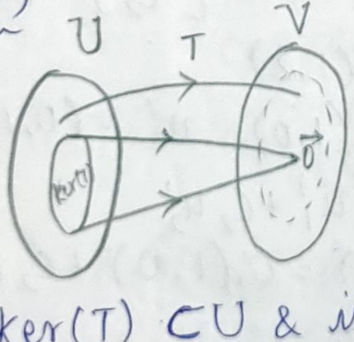
$$W_{B'} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

Kernel and range of linear transformation:-

Let $T: U \to V$ be a linear transformation.

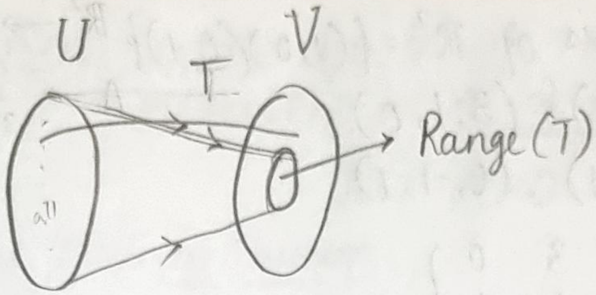
Then,

* The set of vectors in U that are mapped into the zero vector(s) of V is called the Kernel of the linear transformation T & is denoted by $ker(T)$



NOTE: $ker(T) \subset U$ & is called Null space of the linear transformation T.

* The dimension Kernal - T is denoted by $dim(Ker(T))$ is called the Nullity of the linear transformation T.

* 

* The set of vectors in V that are the images of all vectors in U is called the Range of the linear transformation 'T' & is denoted by Range (T).

NOTE: Range (T) $\subset$ V and the dimension of Range (T), denoted by dim(Range(T)) = Rank of its matrix of the linear transformation T.

$T: U \to V$

$T(x,y) = (x, x+y, z)$

$A = [T(e_1)\ T(e_2)]$

Rank - Nullity theorem:
Let $T: U \to V$ be a linear transformation, then dimension of domain (T) = dim(Ker(T)) + dim(Range(T))

$$\dim(\text{domain}(T)) = \dim(\text{Ker}(T)) + \dim(\text{Range}(T))$$

$$\boxed{\dim(U) = \text{Nullity} + \text{Rank}}$$

Problems:

1) Determine the kernel, Range of L.T for the following.

a) $T: \mathbb{R}^2 \to \mathbb{R}^3$ where $T(x,y) = (3x, x-y, y)$.
Also, verify Rank - Nullity theorem.

$\rightarrow$ Ker(T) = ? , Range (T) = ?

dim(U) = ?
Nullity = ?
Rank = ?

Standard basis of $\mathbb{R}^2 = \{(1,0)(0,1)\}$ $\quad \mathbb{R}^2 \xrightarrow{2\times3} \mathbb{R}^3$

$A \to 3\times2$

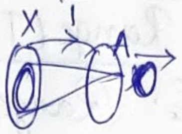
$T(1,0) = (3,1,0)$

$T(0,1) = (0,-1,1)$

$$A = \begin{bmatrix} 3 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}_{3\times2}$$

$\dim(\text{dom}(T)) = \dim(U) =$ no. of columns in A $= 2$

$$A^T = \begin{bmatrix} 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$



$T(X) = AX$

fig-a

in Echelon form

$$= \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$\rho(A^T) = 2$

$\rho(A^T) = \rho(A) = 2$.

$\because$ W.K.T rank of matrix and its transpose are same $\rho(A) = 2$

$\dim(\text{range}(T)) = \text{rank} = \underline{\underline{2}}$. of $A^T \to$ Echelon form

Range $= \text{Span}\{c_1(1, 1/3, 0) + c_2(0,1,-1)\}$

$= \{c_1, \frac{c_1}{3} + c_2, -c_2\}$

Ker(T)

$TX = AX = \vec{0}$ (fig-a)

$AX = 0 \to V$

$$\begin{bmatrix} 3 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x = 0 \qquad x + y = 0$

$x = 0 \qquad y = 0$

$\therefore$ $Ker(T) = \{0, 0\}$

$dim(Ker(T))$ = no. of free parameters $= 0$

Nullity = 0

<u>Rank Nullity theorem:</u>

$$dim(u) = Rank + Nullity$$

$$\underline{\underline{2}} = \underline{\underline{2}} + 0$$

b) $T: R^3 \to R^2$ where $T(x, y, z) = (x+y, z)$
& verify Rank Nullity theorem.

→ Standard basis of $\mathbb{R}^3 = \{(1,0,0)\ (0,1,0)(0,0,1)\}$

$T(1,0,0) = (1,0)$

$T(0,1,0) = (1,0)$

$T(0,0,1) = (0,1)$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2\times 3} \leftarrow \underset{(3\times 2)}{\text{ulta } \&}$$

$\boxed{dim(u) = \text{no. of columns} = 3}$

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \to \text{Echelon form}$$

$\rho(A^T) = 2$

Rank → $\boxed{\rho(A^T) = \rho(A) = 2}$ ($\because$ W.K.T A & A' have same rank)

Range = Span $\{c_1(1,0) + c_2(0,1) + c_3(0,0)\}$

$= \{c_1, c_2\}$

$Ker(T)$: $\vec{A}\vec{X} = \vec{0}$ ($\vec{U}$, V)

$$\cancel{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \quad \overset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \overset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\left.\begin{array}{l} x+y=0 \\ z=0 \end{array}\right\}$$ 2 independent equations

F.P = No. of unknowns − No. of independent equations

$= 3-2$

$= \underline{\underline{1}}$

$\because z=0, \quad x=k, y=-k.$

$Ker(T) = \{k, -k, 0\}$

$\dim(Ker(T)) = \text{Nullity} = \underline{\underline{1}}.$

<u>Rank-Nullity theorem</u>:

$\dim(U) = \text{Nullity} + \text{Rank}$

$3 = 1 + 2$

$\underline{\underline{3=3}}$

c) $T: \mathbb{R}^3 \rightarrow \mathbb{R}, \quad T(x,y,z) = x+y+z.$

→ Standard basis of $\mathbb{R}^3 = \{(1,0,0), (0,1,0)(0,0,1)\}$

$T(1,0,0) = 1$

$T(0,1,0) = 1$

$T(0,0,1) = 1$

$A = \cancel{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{3\times 1}} \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}_{1\times 3}$

No. of columns in A $= \dim(U) = \underline{\underline{3}}.$

$A^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\rho(A^T) = \rho(A) = \underline{\underline{1}}.$

Range $=$ ~~Span$(c_1(1) + c_2(0) + c_3$~~

$=$ Span $\{1,0,0\}$

$= c_1(1,0,0) = c_1$

Ker(T) :

$AX = 0$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x + y + z = 0.$

no. of IE = 1

no. of unknowns = 3

FP = no. of U.K − No. of IE

= 3 − 1

= $\underline{\underline{2}}$

Let $x = k_1$, $y = k_2$

$z = -(k_1 + k_2)$.

dim(Ker(T)) = nullity = 2.

$ker(T) = \{k_1, k_2, -(k_1 + k_2)\}$

Rank-Nullity theorem:

dim (U) = Nullity + Rank

3 = 2 + 1

$\underline{3 = 3}$

2) Determine the Kernel & Range of Transformation T defined by each of the following matrices & hence verify Rank-Nullity theorem:

$T : R^4 \to R^3$

a) $$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 1 & -1 & 8 & 7 \\ 2 & -4 & 6 & 10 \end{bmatrix}_{3\times 4}$$

→ $$A = \begin{bmatrix} 1 & -2 & 3 & 5 \\ 1 & -1 & 8 & 7 \\ 2 & -4 & 6 & 10 \end{bmatrix}$$

dim(u) = No. of columns = 4.

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -4 \\ 3 & 8 & 6 \\ 5 & 7 & 10 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 5R_1 \end{array}$$

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} R_3 \to \frac{R_3}{8} - 5R_2 \\ R_4 \to R_4 - 2R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A^T) = \rho(A) = \underline{\underline{2}}$$

Range $= \text{Span}\{(1,1,2)\ (0,1,0)\ (0,0,0)\ (0,0,0)\}$

$= \{c_1(1,1,2) + c_2(0,1,0)\}$

Range $= \{c_1,\ c_1 + c_2,\ 2c_1\}$

<u>Ker(T)</u> :

$$AX = 0$$

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ 1 & -1 & 8 & 7 \\ 2 & -4 & 6 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - 2y + 3z + 5w = 0$

$x - y + 8z + 7w = 0$

same eq^{n}s $2x - 4y + 6z + 10w = 0$

No. of independent eq^{n}s $= 2$

No. of unknowns $= 4$

Free parameters $= 4 - 2$

$= \underline{\underline{2}}$

Nullity $= \underline{\underline{2}}$.

Rank-Nullity Theorem:

$$\dim(V) = \text{Rank} + \text{Nullity}$$

$$4 = 2 + 2$$

$$4 = \underline{\underline{4}}$$

$$[A:B] = \left[\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 1 & -1 & 8 & 7 & 0 \\ 2 & -4 & 6 & 10 & 0 \end{array}\right] \qquad R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 1 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 \to R_3 - 2R_1$$

$\rho(A) = 2 = \rho(A:B) = 2.$

Infinitely many solutions,

$$x - 2y + 3z + 5w = 0$$

$$y + 5z + 2w = 0$$

$$z\,\cancel{y} = k_1, \quad w\,\cancel{z} = k_2$$

$$y = -(5k_1 + 2k_2)$$

$$x + 10k_1 + 4k_2 + 3k_1 + 5k_2 = 0$$

$$x = -13k_1 - 9k_2$$

$$\text{Ker}(T) = \{-(13k_1 + 9k_2), -(5k_1 + 2k_2), k_1, k_2\}$$

b) $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$