① Evaluate $\int_0^\infty e^{-ax^{1/a}}\,dx$

Sol :- $\int_0^\infty e^{-ax^{1/a}}\,dx$

Put $ax^{1/a} = t$

$x^{1/a} = t/a$

$(x^{1/a})^a = (t/a)^a$

$x = \left(\frac{t}{a}\right)^a$

$dx = \frac{1}{a^a}\, a\, t^{a-1}\, dt$

Limits

if $x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$$\int_0 e^{-ax^{1/a}}\,dx = \int_0^\infty e^{-t}\,\frac{1}{a^a}\,a\,t^{a-1}\,dt$$

$$= \frac{a}{a^a}\int_0^\infty e^{-t}\,t^{a-1}\,dt$$

$$= \frac{1}{a^{a-1}}\int_0^\infty e^{-t}\,t^{a-1}\,dt \checkmark$$

$$= \frac{1}{a^{a-1}}\,\Gamma a.$$

$\because \Gamma n = \int_0^\infty e^{-x}\,x^{n-1}\,dx$

$$\int_0^\infty e^{-ax^{1/a}}\,dx = \frac{\Gamma a}{a^{a-1}}$$

② Prove that $\int_0^1 x^m \left(\log(1/x)\right)^n dx = \frac{\Gamma n+1}{(m+1)^{n+1}}$ ②

Sol :- LHS $\int_0^1 x^m \left(\log(1/x)\right)^n dx$

put $\log(1/x) = t$

$1/x = e^t$

$x = e^{-t}$

$dx = -e^{-t} dt$

Limits

If $x = 0$, $t = \infty$

$x = 1$, $t = 0$

$$\int_0^1 x^m (\log(1/x))^n dx = \int_\infty^0 (e^{-t})^m t^n (-e^{-t}) dt$$

$$= \int_0^\infty (e^{-t})^m t^n e^{-t} dt$$

$$= \int_0^\infty e^{-(m+1)t} t^n dt$$

put $(m+1)t = u$

$dt = \frac{du}{m+1}$

If $t = 0$, $u = 0$

$t = \infty$, $u = \infty$

$$= \int_0^\infty e^{-u} \left(\frac{u}{m+1}\right)^n \frac{du}{m+1}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{(m+1)^n} \frac{du}{(m+1)}$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-u} \cdot u^n \, du$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-u} u^{(n+1)-1} \, du$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$= \frac{1}{(m+1)^{n+1}} \Gamma n+1$$

$$= \frac{\Gamma n+1}{(m+1)^{n+1}}$$

(3) Evaluate $\int_0^1 x^a (\log x)^b \, dx$

Sol: -

Put $\log x = -t$

$x = e^{-t}$

$dx = -e^{-t} dt$

when $x=0$, $t = \infty$

$x=1$, $t=0$

4

$$\int_0^1 x^a (\log x)^b \, dx = \int_\infty^0 e^{-at} (-t)^b (-e^{-t}) dt$$

$$= \int_0^\infty e^{-at} (-t)^b e^{-t} dt$$

$$= \int_0^\infty e^{-(a+1)t} (-1)^b t^b \, dt$$

$$= (-1)^b \int_0^\infty e^{-(a+1)t} t^b \, dt$$

put $(a+1)t = u$

$t = \dfrac{u}{a+1}$

$dt = \dfrac{du}{a+1}$

if $t = 0, \ u = 0$

$t = \infty, \ u = \infty$

$$= (-1)^b \int_0^\infty e^{-u} \cdot \left(\frac{u}{a+1}\right)^b \cdot \frac{du}{a+1}$$

$$= (-1)^b \int_0^\infty e^{-u} \cdot \frac{u^b}{(a+1)^{b+1}} du$$

$$= \frac{(-1)^b}{(a+1)^{b+1}} \int_0^\infty e^{-u} \cdot u^{(b+1)-1} du$$

$$= \frac{(-1)^b}{(a+1)^{b+1}} \underline{\Gamma(b+1)}$$

4. If $p > -1$, $q > -1$ then prove that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$ and

hence evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$.

Sol :- We know that

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

put $2m-1 = p$, $2n-1 = q$

$2m = p+1$ $\quad$ $2n = q+1$

$m = \frac{p+1}{2}$. $\quad$ $n = \frac{q+1}{2}$

$$\beta(m,n) = \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\boxed{\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta}$$ ✓

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta$$ ✓

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$$

$p = 1/2$, $q = -1/2$

$=$

$$= \frac{1}{2} \beta \left(\frac{p+1}{2}, \frac{q+1}{2} \right)$$

$$= \frac{1}{2} \beta \left(\frac{1/2+1}{2}, \frac{-1/2+1}{2} \right)$$

$$= \frac{1}{2} \beta \left(\frac{3/2}{2}, \frac{1/2}{2} \right)$$

$$= 1/2 \, \beta \left(3/4, 1/4 \right)$$

$$= 1/2 \cdot \frac{\Gamma 3/4 \; \Gamma 1/4}{\Gamma 3/4 + 1/4}$$

$$\beta(m, n) = \frac{\Gamma m \; \Gamma n}{\Gamma m+n}$$

$$= 1/2 \cdot \frac{\Gamma 3/4 \cdot \Gamma 1/4}{\Gamma 1}$$

$$\Gamma 1 = 1$$

$$= 1/2 \cdot \Gamma 3/4 \cdot \Gamma 1/4$$

Note :-

✓ $$\Gamma m \cdot \Gamma 1-m = \frac{\pi}{\sin m\pi}$$

$$= \frac{1}{2} \cdot \Gamma 1/4 \cdot \Gamma 1-1/4$$

$$= 1/2 \cdot \frac{\pi}{\sin 1/4 (\pi)} = \frac{1}{2} \frac{\pi}{\sin \pi/4} = \frac{1}{2} \cdot \frac{\pi}{1/\sqrt{2}}$$

$$= \pi/\sqrt{2}$$

⑤ Show that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \frac{\pi}{\,}$

Sol :- LHS $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$

let $I_1 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

$= \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta \quad \cos^0\theta\, d\theta$

$p = 1/2 \quad q = 0$

$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$= \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right)$

$= \frac{1}{2}\beta\left(\frac{3/2}{2}, 1/2\right)$

$= \frac{1}{2}\beta\left(3/4, 1/2\right)$

$I_1 = \frac{1}{2} \cdot \frac{\Gamma 3/4\ \Gamma 1/2}{\Gamma(3/4 + 1/2)}$ $\qquad \beta(m,n) = \frac{\Gamma m\ \Gamma n}{\Gamma m+n}$

$I_1 = \frac{1}{2} \frac{\Gamma 3/4\ \Gamma 1/2}{\Gamma 5/4}$

let $I_2 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$

$= \int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta \quad \cos^0\theta \, d\theta$

$p = -1/2, \quad q = 0$

$= \frac{1}{2} \beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$

$= 1/2\, \beta\left(\frac{1/2}{2}, 1/2\right)$

$= 1/2\, \beta(1/4, 1/2)$

$I_2 = 1/2 \cdot \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma(1/4+1/2)}$

$I_2 = 1/2 \cdot \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma 3/4}$

LHS $I_1 \times I_2 = 1/2 \frac{\cancel{\Gamma 3/4} \cdot \Gamma 1/2}{\Gamma 5/4} \times \frac{1}{2} \frac{\Gamma 1/4 \, \Gamma 1/2}{\cancel{\Gamma 3/4}}$

$= \frac{1}{4} \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 5/4} \cdot \Gamma 1/4$

But $\Gamma 1/2 = \sqrt{\pi}$

$$= \frac{1}{4} \cdot \frac{\Gamma(1/4)\,\sqrt{\pi}\,\sqrt{\pi}}{\Gamma(5/4)}$$

But

$$\Gamma(5/4) = \Gamma(1+1/4)$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(1+1/4) = 1/4\,\Gamma(1/4)$$

$$= \frac{1}{\cancel{4}} \cdot \frac{\cancel{\Gamma(1/4)} \cdot \pi}{\cancel{1/4} \cdot \cancel{\Gamma(1/4)}}$$

$$= \pi$$

<u>RHS</u>

$$\therefore \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \pi$$

⑥ Show that $\displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx \times \int_0^\infty x^2 e^{-x^4}\,dx = \frac{\pi}{4\sqrt{2}}$

<u>Sol</u>:- Let $\displaystyle I_1 = \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$

Put $x^2 = t \rightarrow x = \sqrt{t}$

$\sqrt{x} = t^{1/4}$

$2x\,dx = dt$

$dx = \dfrac{dt}{2x}$

$dx = \dfrac{dt}{2\sqrt{t}}$

If $x = 0,\ t = 0$

$x = \infty,\ t = \infty$

$$I_1 = \int_0^\infty \frac{e^{-t}}{t^{1/4}}\,\frac{dt}{2\sqrt{t}}$$

$$I_1 = \frac{1}{2}\int_0^\infty \frac{e^{-t}}{t^{1/4} \cdot t^{1/2}}\, dt$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-t}}{t^{3/4}}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-3/4}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{1/4 - 1}\, dt \qquad \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$I_1 = \frac{1}{2} \cdot \Gamma \tfrac{1}{4}$$

Let $I_2 = \int_0^\infty x^2 e^{-x^4} dx$

Put $x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2x}$

$$I_2 = \int_0^\infty e^{-t^2}\, t \, \frac{dt}{2\sqrt{t}} \qquad dx = \frac{dt}{2\sqrt{t}}$$

$x = 0,\ t = 0$

$x = \infty,\ t = \infty$

$$= \frac{1}{2}\int_0^\infty e^{-t^2} \sqrt{t}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t^2} \cdot t^{1/2}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t^2} \cdot t^{2(3/4)-1}\, dt$$

$\Gamma n = 2\int_0^\infty e^{-t^2} t^{2n-1} dt$

(alternate form)

$I_2 = \frac{1}{2} \cdot \frac{1}{2} \, \Gamma\!\left(\frac{3}{4}\right)$

$\underline{\text{LHS}}$ $\quad I_1 \times I_2 = \frac{1}{2}\,\Gamma\!\left(\frac{1}{4}\right) \times \frac{1}{2} \cdot \frac{1}{2} \cdot \Gamma\!\left(\frac{3}{4}\right)$

$= \frac{1}{8} \cdot \Gamma\!\left(\frac{1}{4}\right) \cdot \Gamma\!\left(\frac{3}{4}\right)$

$= \frac{1}{8} \cdot \Gamma\!\left(\frac{1}{4}\right) \cdot \Gamma\!\left(1 - \frac{1}{4}\right)$

$$\boxed{\Gamma(m) \cdot \Gamma(1-m) = \frac{\pi}{\sin m\pi}}$$

$= \frac{1}{8} \cdot \frac{\pi}{\sin \frac{1}{4} \cdot \pi}$

$= \frac{1}{8} \cdot \frac{\pi}{\sin \pi/4}$

$= \frac{1}{8} \cdot \frac{\pi}{1/\sqrt{2}}$

$= \underline{\underline{\frac{\pi}{4\sqrt{2}}}}$ $\quad \underline{\text{RHS}}$

(7) Show that $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

$\underline{\text{Sol}}$:— let $\displaystyle I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$

Put $x^2 = \sin\theta$

$\Rightarrow x = \sqrt{\sin\theta}$

$2x\,dx = \cos\theta\, d\theta$

$dx = \dfrac{\cos\theta\, d\theta}{2x}$

$dx = \dfrac{\cos\theta\, d\theta}{2\sqrt{\sin\theta}}$

If $x = 0,\ \theta = 0$

$x = 1,\ \theta = \pi/2$

$$I_1 = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \frac{\cos\theta\, d\theta}{2\sqrt{\sin\theta}}$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cancel{\cos\theta}} \frac{\cancel{\cos\theta}\, d\theta}{2\sqrt{\sin\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\, d\theta$$

$$= \frac{1}{2} \cdot \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta\, d\theta$$

$p = 1/2,\ q = 0$

$$= \frac{1}{2} \cdot \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{4}\beta\left(3/4, 1/2\right)$$

$\beta(m,n) = \dfrac{\Gamma m\, \Gamma n}{\Gamma m+n}$

$$= \frac{1}{4} \frac{\Gamma 3/4\ \Gamma 1/2}{\Gamma 3/4 + 1/2}$$

$$I_1 = \frac{1}{4}\,\frac{\Gamma 3/4\ \Gamma 1/2}{\Gamma 5/4}$$

$$= \frac{1}{4}\,\frac{\Gamma 3/4 \cdot \Gamma 1/2}{\Gamma 1+1/4}$$

$\Gamma n+1 = n\,\Gamma n$

$\Gamma 1/2 = \sqrt{\pi}$

$$= \cancel{\frac{1}{4}}\,\frac{\Gamma 3/4\ \sqrt{\pi}}{\cancel{\frac{1}{4}}\ \Gamma 1/4}$$

$$I_1 = \frac{\Gamma 3/4\ \sqrt{\pi}}{\Gamma 1/4}$$

let $\quad I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

put $\quad x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\,d\theta$

$x = \sqrt{\tan\theta}$

$$dx = \frac{\sec^2\theta\,d\theta}{2x}$$

If $x=0, \quad \theta = 0$

$x = 1, \quad \theta = \pi/4$

$$dx = \frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}}$$

$$I_2 = \int_0^{\pi/4} \frac{\sec^2\theta\ d\theta}{\sqrt{1+\tan^2\theta}\ \ 2\sqrt{\tan\theta}}$$

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^{\cancel{2}}\theta}{\cancel{\sec\theta}\cdot\sqrt{\tan\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\cos\theta}\cdot\frac{1}{\frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}}\,d\theta$$

$$= \frac{1}{2}\cdot\int_0^{\pi/4} \frac{1}{\sqrt{\cos\theta}}\cdot\frac{1}{\sqrt{\sin\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\sin\theta}\sqrt{\cos\theta}}\,d\theta$$

$$= \frac{1}{2}\cdot\int_0^{\pi/4} \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{\sin\theta}\sqrt{\cos\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sqrt{2}}{\sqrt{2\sin\theta\cdot\cos\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\cdot\int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}}\,d\theta$$

Put $2\theta = \phi$

$\theta = \phi/2$

$d\theta = \frac{d\phi}{2}$

if $\theta = 0, \quad \phi = 0$

$\theta = \pi/4, \quad \phi = \pi/2$

$$I_2 = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{d\phi}{2\sqrt{\sin\phi}}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{\sin\phi}}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\phi \, d\phi$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\phi \cdot \cos^0\phi \, d\phi$$

$p = -1/2, \quad q = 0$

$$= \frac{1}{2\sqrt{2}} \, \frac{1}{2} \, \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$$

$$I_2 = \frac{1}{4\sqrt{2}} \beta\left(\tfrac{1}{4}, \tfrac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma\tfrac{1}{4}\,\Gamma\tfrac{1}{2}}{\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}\right)}$$

$$I_2 = \frac{1}{4\sqrt{2}} \frac{\Gamma\tfrac{1}{4}\,\Gamma\tfrac{1}{2}}{\Gamma\tfrac{3}{4}}$$

LHS

$$I_1 \times I_2 = \frac{\cancel{\Gamma\tfrac{3}{4}} \times \sqrt{\pi}}{\cancel{\Gamma\tfrac{1}{4}}} \times \frac{1}{4\sqrt{2}} \cdot \frac{\cancel{\Gamma\tfrac{1}{4}}\,\Gamma\tfrac{1}{2}}{\cancel{\Gamma\tfrac{3}{4}}}$$

$$= \frac{\sqrt{\pi} \cdot \sqrt{\pi}}{4\sqrt{2}} \qquad \because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$= \frac{\pi}{4\sqrt{2}}$$

RHS