* Employing Parseval's identity to the fn

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases} \quad \text{S.T} \quad \int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{\pi a}{2}$$

Sol :

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-a}^{a} 1\, e^{iux}\, dx = \left.\frac{e^{iux}}{iu}\right]_{-a}^{a}$$

$$= \frac{1}{iu}\left[e^{iua} - e^{-iua} \right] \qquad \Bigg| \; \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \frac{1}{iu}\,(2i \sin au)$$

$$F(u) = \frac{2\sin au}{u}$$

Using Parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(u)|^2\, du = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{2\sin au}{u}\right|^2 du = \int_{-a}^{a} 1\, dx$$

$$\cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty} 4\,\frac{\sin^2 au}{u^2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 au}{u^2}\, du$$

$$= \left. x \right]_{-a}^{a}$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 au}{u^2}\, du = a + a$$

$$\int_{\infty}^{\infty} \frac{\sin^2 au}{u^2}\, du = \frac{\pi(2a)}{2}$$

$$2\int_0^{\infty} \frac{\sin^2 au}{u^2}\, du = \pi a$$

$$\int_0^{\infty} \frac{\sin^2 au}{u^2}\, du = \frac{\pi a}{2} \qquad \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$u \to x$

$$\int_0^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \frac{\pi a}{2}$$

(2) Employing Parseval's identity to the fn

$$f(x) = \begin{cases} a^2 - x^2 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases} \qquad a > 0$$

S.T $\displaystyle\int_0^{\infty} \left(\frac{\sin x - x\cos x}{x^6}\right)^2 dx = \pi/15$

Sol : $\displaystyle F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx = \int_{-a}^{a} (a^2 - x^2)\, e^{iux}\, dx$

$$\Rightarrow F(u) = \frac{4}{u^3}\left[-au\cos au + \sin au\right]$$

Using Parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(u)|^2\, du = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left| 4\left(\frac{\sin au - au\cos au}{u^3}\right)\right|^2 du = \int_{-a}^{a} |a^2 - x^2|^2\, dx$$

$$\frac{16}{2\pi}\int_{-\infty}^{\infty}\frac{(\sin au - au\cos au)^2}{u^6}\,du = \int_{-a}^{a}(a^2-x^2)^2\,dx$$

$$\frac{8}{\pi}\,2\int_{0}^{\infty}\frac{(\sin au - au\cos au)^2}{u^6}\,du = \int_{-a}^{a}(a^4+x^4-2a^2x^2)\,dx$$

$$\frac{16}{\pi}\int_{0}^{\infty}\frac{(\sin au - au\cos au)^2}{u^6}\,du = \left[a^4x+\frac{x^5}{5}-\frac{2a^2x^3}{3}\right]_{-a}^{a}$$

$$\text{—॥—} = \left[a^5+\frac{a^5}{5}-\frac{2a^5}{3}-\left(-a^5-\frac{a^5}{5}+\frac{2a^2a^3}{3}\right)\right]$$

$$\text{—॥—} = a^5+\frac{a^5}{5}-\frac{2a^5}{3}+a^5+\frac{a^5}{5}-\frac{2a^5}{5}$$

$$\frac{\cancel{16}}{\pi}\int_{0}^{\infty}\frac{(\sin au - au\cos au)^2}{u^6}\,du = \frac{\cancel{16}}{15}a^5$$

$$\int_{0}^{\infty}\frac{(\sin au - au\cos au)^2}{u^6}\,du = \frac{\pi}{15}a^5$$

put $x = au$

$dx = a\,du$

$$\int_{0}^{\infty}\frac{(\sin x - x\cos x)^2}{(x/a)^6}\left(\frac{dx}{a}\right) = \frac{\pi}{15}a^5$$

$$\int_{0}^{\infty}\frac{(\sin x - x\cos x)^2(\cancel{a^6})}{x^6}\frac{dx}{\cancel{a}} = \pi/15\,\cancel{a^5}$$

$$\int_{0}^{\infty}\left(\frac{\sin x - x\cos x}{x^6}\right)^2 dx = \pi/15$$

other way

$a = 1$

$u \to x$

$$\int_{0}^{\infty}\frac{(\sin x - x\cos x)^2}{x^6}\,dx = \frac{\pi}{15}$$

* Find the Fourier Cosine Transform for the function $f(x) = e^{-ax}$ & $g(x) = e^{-bx}$ and then employ Parseval's identity to show that $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}$

Sol:

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$= \int_0^\infty e^{-ax} \cos ux \, dx$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$= \left[\frac{e^{-ax}}{a^2+u^2}\left[-a \cos ux + u \sin ux\right]\right]_0^\infty$$

$$= \frac{1}{a^2+u^2}\left[0-(1)\left[-a(1)+u(0)\right]\right] \qquad e^{-\infty} = 0$$

$$F_c(u) = \frac{a}{a^2+u^2} \qquad \{ f(x) = e^{-ax} \}$$

$G_c(u)$

$$G_c(u) = \int_0^\infty g(x) \cos ux \, dx$$

$$= \int_0^\infty e^{-bx} \cos ux \, dx$$

$$G_c(u) = \frac{b}{b^2+u^2}$$

Using Parseval's identity for FCT

$$\frac{2}{\pi}\int_0^\infty F_c(u)\, G_c(u)\, du = \int_0^\infty f(x) g(x)\, dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{a}{a^2+u^2} \cdot \frac{b}{b^2+u^2}\, du = \int_0^\infty e^{-ax}\, e^{-bx}\, dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{du}{(a^2+u^2)(b^2+u^2)}\, du = \int_0^\infty e^{-x(a+b)}\, dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= 0 + \frac{1}{a+b}$$

$$\int_0^\infty \frac{du}{(a^2+u^2)(b^2+u^2)} = \frac{\pi}{2ab(a+b)}$$

Replace $u \rightarrow x$

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)} \quad //$$

→ Def FT, IFT, Problems

FCT IFCT, FST, IFST, Problems

Convolution theorem, $f(x) * g(x) = g(x) * f(x)$,

Parseval's identity FT, FCT, FST

— Problems

Laplace Transform & F.T → ode, pde

✓

Z- Transforms

Laplace Transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

Solve ODE

Fourier Transform

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

2 parameter

Solve Pde

Z - Transform

1 — parameter

Solve difference eq

Difference equation : It is a relationship b/w the differences of an unknown fn (dependent variable y) at several values of the independent variable (argument x).

They are called recurrence relation

$$\begin{cases} a_r y_{n+r} + a_{r-1} y_{n+r-1} + a_{r-2} y_{n+r-2} + \cdots + \\ a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = \phi(n) \quad n = 0, 1, 2 \ldots \\ a_r, a_{r-1}, \ldots a_0 \text{ — constants} \end{cases}$$

linear difference eq of order n

Ex - 1

Definition of Z-Transform

Z-transforms operates on a sequence u_n of discrete integer value arguments ~~i.e n=0,±1,~~ ~~±2~~, ±3, ... $n = 0, 1, 2, 3 \ldots$

Z-transforms are useful to solve difference equation which represents a discrete system.

Def: If $u_n = f(n)$ (sequence) defined for all $n = 0, 1, 2 \ldots$ & $u_n = 0$ for $n < 0$ then the Z-transform of u_n is defined by

$$\boxed{Z_T(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}}$$

whenever the series in RHS converges to fn of z

Inverse Z-Transform

It is denoted by $\boxed{Z^{-1}[U(z)] = u_n}$, determines the sequence u_n which generates the given Z-transform

Property : $\boxed{Z_T(n^k) = -z \frac{d}{dz} Z_T(n^{k-1})}$

Sol : WKT $\boxed{Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}}$ — defn of Z-Trans

$$Z_T(n^k) = \sum_{n=0}^{\infty} n^k z^{-n} \quad \text{--- (1)}$$

Consider RHS of the ppt

$$-z \frac{d}{dz} Z_T(n^{k-1}) = -z \frac{d}{dz} \sum_{n=0}^{\infty} n^{k-1} z^{-n}$$

(from (2))

[WKT $\boxed{Z_T(n^{k-1}) = \sum_{n=0}^{\infty} n^{k-1} z^{-n}}$ — (2)

$$= (-z) \sum_{n=0}^{\infty} n^{k-1} (-n) z^{-n-1}$$

$$= \sum_{n=0}^{\infty} n^{k-\cancel{1}+\cancel{1}} z^{-n-\cancel{1}+\cancel{1}}$$

$$= \sum_{n=0}^{\infty} n^k z^{-n} = Z_T(n^k)$$

RHS = LHS

$$\boxed{Z_T(n^k) = -z \frac{d}{dz} Z_T(n^{k-1})}$$

Z- transform of some standard functions

1)

① $Z_T[k^n]$

Def of z - transform

$$Z_T[u_n] = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$Z_T[k^n] = \sum_{n=0}^{\infty} k^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{k}{z}\right)^n$$

$$= \left(\frac{k}{z}\right)^0 + \left(\frac{k}{z}\right)^1 + \left(\frac{k}{z}\right)^2 + \cdots$$

$$= 1 + \frac{k}{z} + \frac{k^2}{z^2} + \frac{k^3}{z^3} + \cdots$$

The above series is an geometric series

$a = 1 \quad r = k/z \quad S_n = \frac{a}{1-r}$

$$Z_T[k^n] = \frac{1}{1 - k/z}$$

$$\boxed{Z_T[k^n] = \frac{z}{z-k}}$$

② $Z_T[k^{-n}]$

$$Z_T[k^{-n}] = \sum_{n=0}^{\infty} k^{-n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{kz}\right)^n$$

$$= \left(\frac{1}{kz}\right)^0 + \left(\frac{1}{kz}\right)^1 + \left(\frac{1}{kz}\right)^2 + \cdots$$

$$= 1 + 1/kz + 1/(kz)^2 + \cdots$$

$$Z_T[k^{-n}] = \frac{1}{1-(1/kz)} = \frac{kz}{kz-1}$$

$$\boxed{Z_T[k^{-n}] = \frac{kz}{kz-1}}$$

* $Z_T[1]$

$$Z_T[1] = \sum_{n=0}^{\infty} 1\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= \left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \cdots$$

$$= 1 + 1/z + 1/z^2 + \cdots$$

$$= \frac{1}{1 - 1/z} = \frac{z}{z-1}$$

$$\boxed{Z_T[1] = \frac{z}{z-1}}$$

* $Z_T[k]$

$$Z_T[k] = \sum_{n=0}^{\infty} k\, z^{-n} = k \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= k\left(1 + 1/z + 1/z^2 + \cdots\right)$$

$$= k\left(\frac{z}{z-1}\right)$$

$$\boxed{Z_T[k] = \frac{kz}{z-1}}$$

* Let us find z- transform the foll :

1) $Z_T(n)$ 2) $Z_T(n^2)$ 3) $Z_T(n^3)$

1. $Z_T(n)$

$$\boxed{Z_T(n^k) = -z\frac{d}{dz} Z_T(n^{k-1})} \quad \text{--- (1)}$$

put $k=1$ in (1)

$$Z_T(n) = -z\frac{d}{dz} Z_T(n^0)$$

$$= -z\frac{d}{dz} Z_T(1)$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z\left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]$$

$$= -z\left[\frac{\cancel{z}-1-\cancel{z}}{(z-1)^2}\right]$$

$Z[1] = \frac{z}{z-1}$

$$\boxed{Z_T(n) = \frac{z}{(z-1)^2}}$$

2. put $k=2$ in (1)

$$Z_T(n^2) = -z\frac{d}{dz} Z_T(n^{2-1})$$

$$= -z\frac{d}{dz} Z_T(n)$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z(2(z-1))}{(z-1)^4}\right]$$

$Z_T(n) = \frac{z}{(z-1)^2}$

put $K = 3$ in (1)

HW

$Z_T(n^3) =$

$$= -z\left[\frac{z^2+1-\cancel{2z}-2z^2+\cancel{2z}}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2+1}{(z-1)^4}\right]$$

$$Z_T(n^2) = \frac{z^3-z}{(z-1)^4} = \frac{z(z^2-1)}{(z-1)^4}$$

$$= \frac{z\cancel{(z-1)}(z+1)}{(z-1)^{\cancel{4}\,3}}$$

$$\boxed{Z_T(n^2) = \frac{z^2+z}{(z-1)^3}}$$

put $K=3$ in (1)

HW

$$Z_T(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$$

* Find z- transform of unit step sequence defined by

$$U_n = \begin{cases} 0 & n<0 \\ 1 & n\geq 0 \end{cases}$$

$$Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} = \sum_{n=0}^{\infty} (1)\, z^{-n}$$

$$= \sum (1/z)^n = 1 + 1/z + 1/z^2 + \ldots$$

$$\boxed{Z_T[u_n] = \frac{z}{z-1}}$$

* Find ZT of unit impulse function

$Z_T(\delta(n))$ defined by

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$Z_T(\delta(n)) = \sum_{n=0}^{\infty} \delta(n)\, z^{-n}$$

$$= \delta(0)\, z^{0} + \delta(1)\, z^{-1} + \delta(2)\, z^{-2} + \cdots$$

$$= 1(1) + 0 + 0 + 0 \cdots$$

$$\boxed{Z_T(\delta(n)) = 1}$$

* $Z_T\left[\frac{1}{n!}\right]$

$$Z_T[u_n] = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$Z_T\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^{n}$$

$$= \frac{1}{0!} + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^{2} + \cdots$$

$$= 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \frac{1/z^3}{3!} + \cdots$$

$$\boxed{Z_T\left[\frac{1}{n!}\right] = e^{1/z}}$$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

* $Z_T(-1)^n$

$$Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$= \sum (-1)^n z^{-n} = \sum \left(\frac{-1}{z}\right)^n$$

$$= 1 + \left(\frac{-1}{z}\right) + \left(\frac{-1}{z}\right)^2 + \left(\frac{-1}{z}\right)^3 + \cdots$$

$$= 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots$$

$r = -1/z$

$$= \frac{1}{1-(-1/z)}$$

$$\boxed{Z_T(-1)^n = \frac{z}{z+1}}$$

other properties of z-transform

① Linearity property

If u_n & v_n are any two discrete valued functions then

$$Z_T[c_1 u_n + c_2 v_n] = c_1 Z_T(u_n) + c_2 Z_T(v_n)$$

where c_1 & c_2 are constant

Consider LHS : $Z_T(c_1 u_n + c_2 v_n]$

$$= \sum_{n=0}^{\infty} (c_1 u_n + c_2 v_n) z^{-n}$$

$$= \sum_{n=0}^{\infty} c_1 u_n z^{-n} + \sum_{n=0}^{\infty} c_2 v_n z^{-n}$$

$$= c_1 \sum_0^{\infty} u_n z^{-n} + c_2 \sum_0^{\infty} v_n z^{-n} = c_1 Z_T(u_n) + c_2 Z_T(v_n)$$

RHS