Numerical solutions of Simultaneous 1st order ODEs · NM-II

Picardé method:-

In this method we consider two eqns $\frac{dy}{dx} = f_1(x, y, z)$

$$\frac{dz}{dx} = f_2(x, y, z)$$

with given initial conditions $y(x_0) = y_0$ & $z(x_0) = z_0$

So by Picardé method, known to us we can write

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f_1(x, y, z)\, dx \quad \& \quad \int_{z_0}^{z} dz = \int_{x_0}^{x} f_2(x, y, z)\, dx$$

$$\Rightarrow y = y_0 + \int_{x_0}^{x} f_1(x, y, z)\, dx \quad \& \quad z = z_0 + \int_{x_0}^{x} f_2(x, y, z)\, dx.$$

using known values of y & z on RHS we can write

$$y = y_0 + \int_{x_0}^{x} f_1(x, y_0, z_0)\, dx \quad \& \quad z = z_0 + \int_{x_0}^{x} f_2(x, y_0, z_0)\, dx$$

& write them as y_1 & z_1 respectively

ie $y_1 = y_0 + \int_{x_0}^{x} f_1(x, y_0, z_0)\, dx$ & $z = z_0 + \int_{x_0}^{x} f_2(x, y_0, z_0)\, dx$

$||^{ly}$ we write

$$y_2 = y_0 + \int_{x_0}^{x} f_1(x, y_1, z_1)\, dx \quad \& \quad z = z_0 + \int_{x_0}^{x} f_2(x, y_1, z_1)\, dx$$

Further $y_3, y_4 \ldots$ are similarly obtaind

Problems:- Find second approximation to the solution of $x' = (y+1)/2$ & $y' = 2t/3x$; $x(1) = 2/3$ & $y(1) = 3$ using Picardé method

Soln:- Here $t_0 = 1$; $x_0 = 2/3$; $y_0 = 3$; $f(t, x, y) = \frac{y+1}{2}$; $g(t, x, y) = \frac{2t}{3x}$

So $x_1 = \frac{2}{3} + \frac{1}{2}\int_1^t (1 + y_0)\, dt$

$= \frac{2}{3} + \frac{1}{2}\int_1^t 4\, dt = 2t - \frac{4}{3}$

$y_1 = 3 + \int_1^t \frac{2t}{3 \cdot 2/3}\, dt$

$= 3 + \left[\frac{t^2}{2}\right]_1^t = \frac{5}{2} + \frac{t^2}{2}$

$$x_2 = \frac{2}{3} + \frac{1}{2}\int_1^t (1+y_1)\, dt \qquad y_2 = 3 + \int_1^t \frac{2t}{3x_1}\, dt$$

$$= \frac{2}{3} + \frac{1}{2}\int_1^t \left(1 + \frac{5}{2} + \frac{t^2}{2}\right) dt \qquad = 3 + \int_1^t \left(\frac{2t}{3\left(2t-\frac{4}{3}\right)}\right) dt$$

$$= \frac{2}{3} + \frac{1}{2}\left[\frac{7t}{2} + \frac{t^3}{6} - \frac{11}{3}\right] \qquad = 3 + \frac{1}{3}\int_1^t \left(1 + \frac{2}{3t-2}\right) dt$$

$$= 3 + \frac{1}{3}\left[t + \frac{2}{3}(\log(3t-2))\right]_1^t$$

$$= \frac{7t}{4} + \frac{t^3}{12} - \frac{7}{6} \qquad = \frac{8}{3} + \frac{t}{3} + \frac{2}{9}\log(3t-2)$$

$\therefore$ The second approx. are

$$x_2 = \frac{t^3}{12} + \frac{7t}{4} - \frac{7}{6} \quad \& \quad y_2 = \frac{t}{3} + \frac{2}{9}\log(3t-2) + \frac{8}{3}.$$

Problem 2 :- Using Picard's method of successive approximations find y & z at $x = 0.1$ given $\frac{dy}{dx} = z$ and $\frac{dz}{dx} = x^3(y+z)$, $y(0) = 1$ and $z(0) = 0.5$.

Soln :- Here $x_0 = 0$; $y_0 = 1$; $z_0 = 0.5$; $y' = f(x,y,z) = z$; $z' = g(x,y,z) = x^3(y+z)$

So we've $y_1 = y_0 + \int_{x_0}^x f(x, y_0, z_0)\, dx$ & $z_1 = z_0 + \int_{x_0}^x g(x, y_0, z_0)\, dx$

$$= 1 + \int_0^x \frac{1}{2}\, dx \qquad = \frac{1}{2} + \int_0^x \frac{3}{2}x^3\, dx$$

$$= 1 + \frac{x}{2} \qquad = \frac{1}{2} + \frac{3x^4}{8}$$

|||ly $y_2 = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8}\right) dx$ & $z_2 = \frac{1}{2} + \int_0^x x^3\left(\frac{3}{2} + \frac{x}{2} + \frac{3x^4}{8}\right) dx$

$$= 1 + \frac{x}{2} + \frac{3x^5}{40} \qquad z_2 = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}.$$

Taking $x = 0.1$ on the RHS of these two expressions for y_2 & z_2 we get $y(0.1) = 1.05$ & $z(0.1) = 0.50$.

Prob 3:- Solve $\frac{dy}{dx} = x + z$ & $\frac{dz}{dx} = x - y^2$ given that $y(0) = 2$ & $z(0) = 1$ using Picard's method of successive approximation of order 2.

Ans:- $y(0.1) = 2.0845$ & $z(0.1) = 0.5867$.

Prob 4:- Using Picard's method solve the system of equations $\frac{dy}{dx} = y + 2z$; $\frac{dz}{dx} = 3y + 2z$ given $y(0) = 6$; $z(0) = 4$ by finding three approximations & hence find y & z at $x = 0.02$

Ans:- $y_3 = 6 + 14x + 33x^2 + \frac{127}{3}x^3$; $z_3 = 4 + 26x + 47x^2 + \frac{193}{3}x^3$;

$y(0.2) = 6.2935$; $z(0.2) = 4.5393$.

Runge-Kutta 4th order method:- for this method also we consider the two simultaneous equations

$\frac{dy}{dx} = f_1(x, y, z)$ & $\frac{dz}{dx} = f_2(x, y, z)$ with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

We evaluate $y(x_1) = y(x_0 + h)$ & $z(x_1) = z(x_0 + h)$ using

$k_1 = h f_1(x_0, y_0, z_0)$ $\quad$ $l_1 = h f_2(x_0, y_0, z_0)$

$k_2 = h f_1(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$ $\quad$ $l_2 = h f_2(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$k_3 = h f_1(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$ $\quad$ $l_3 = h f_2(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

$k_4 = h f_1(x_0 + h, y_0 + k_3, z_0 + l_3)$ $\quad$ $l_4 = h f_2(x_0 + h, y_0 + k_3, z_0 + l_3)$

Then $y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ & $z(x_1) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

Similar procedure is adopted to find $y(x_2)$ & $z(x_2)$ using $y(x_1)$ & $z(x_1)$ etc.,

Problem1:- Apply 4^{th} order RK Method to solve $\frac{dy}{dx} = y+z$ and $\frac{dz}{dx} = -y+z$, given $y(0)=0$ & $z(0)=1$; at $x=0.1$. Take $h=0.1$

Soln:- Here $x_0=0$; $y_0=0$; $z_0=1$; $h=0.1$; $f(x,y,z) = y+z$

$g(x,y,z) = -y+z$

So $k_1 = h\,f(x_0, y_0, z_0) = 0.1$; $\quad l_1 = h\,g(x_0, y_0, z_0) = 0.1$

$k_2 = h\,f(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}) = 0.11$; $\quad l_2 = h\,g(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}) = 0.1$

$k_3 = h\,f(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}) = 0.1105$; $\quad l_3 = h\,g(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}) = 0.0995$

$k_4 = h\,f(x_0+h, y_0+k_3, z_0+l_3) = 0.121$; $\quad l_4 = h\,g(x_0+h, y_0+k_3, z_0+l_3) = 0.0989$

$$\therefore y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 0 + \frac{1}{6}\left[0.1+2(0.11+0.1105)+0.121\right] = 0.11033$$

$$z_1 = z(0.1) = z_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

$$= 1 + \frac{1}{6}\left[0.1+2(0.1+0.0995)+0.0989\right] = 1.09965$$

Problem2:- Using 4^{th} order RK Method, solve the eqns $\frac{dx}{dt} = 2x+y$; $\frac{dy}{dt} = x-3y$ where $x_0=0$; $y_0=0.5$ at $t_0=0$. Find x & y at $t=0.2$

Soln:- Here $x_0=0$; $y_0=0.5$; $t_0=0$; $f(t,x,y) = 2x+y$; $g(t,x,y) = x-3y$

So $k_1 = h\,f(t_0, x_0, y_0) = 0.1$; $\quad l_1 = h\,g(t_0, x_0, y_0) = -0.3$

$k_2 = h\,f(t_0+\frac{h}{2}, x_0+\frac{k_1}{2}, y_0+\frac{l_1}{2}) = 0.09$; $\quad l_2 = h\,g(t_0+\frac{h}{2}, x_0+\frac{k_1}{2}, y_0+\frac{l_1}{2}) = -0.2$

$k_3 = h\,f(t_0+\frac{h}{2}, x_0+\frac{k_2}{2}, y_0+\frac{l_2}{2}) = 0.098$; $\quad l_3 = h\,g(t_0+\frac{h}{2}, x_0+\frac{k_2}{2}, y_0+\frac{l_2}{2}) = -0.231$

$k_4 = h\,f(t_0+h, x_0+k_3, y_0+l_3) = 0.093$; $\quad l_4 = h\,g(t_0+h, x_0+k_3, y_0+l_3) = -0.142$

$\therefore\ x_1 = x(t_0+h) = x(0.2) = x_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.0948$

$y_1 = y(t_0+h) = y(0.2) = y_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4) = 0.2827.$

Prob 3:- Solve the DEs $\frac{dv}{du} = uvw$, $\frac{dw}{du} = \frac{uv}{w}$ given that $v(1) = 1/3$; $w(1) = 1$ using RK Method of fourth order at $u = 1.1$

Soln:- Here $u_0 = 1$; $v_0 = \frac{1}{3}$; $w_0 = 1$; $h = 0.1$; $u_1 = 1.1$

$f(u,v,w) = uvw$; $g(u,v,w) = uv/w$;

So $k_1 = hf(u_0,v_0,w_0) = 0.0333$; $l_1 = hg(u_0,v_0,w_0) = 0.0333$

$k_2 = hf(u_0+\frac{h}{2}, v_0+\frac{k_1}{2}, w_0+\frac{l_1}{2}) = 0.03736$; $l_2 = hg(u_0+\frac{h}{2}, v_0+\frac{k_1}{2}, w_0+\frac{l_1}{2}) = 0.03615$

$k_3 = hf(u_0+\frac{h}{2}, v_0+\frac{k_2}{2}, w_0+\frac{l_2}{2}) = 0.03763$; $l_3 = hg(u_0+\frac{h}{2}, v_0+\frac{k_2}{2}, w_0+\frac{l_2}{2}) = 0.0363$

$k_4 = hf(u_0+h, v_0+k_3, w_0+l_3) = 0.0423$; $l_4 = hg(u_0+h, v_0+k_3, w_0+l_3) = 0.03938$

So $v_1 = v(u_0+h) = v(1.1) = v_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.3709$

$w_1 = w(u_0+h) = w(1.1) = w_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4) = 1.03626.$

Prob 4:- Apply 4th order RK Method to solve $\frac{dy}{dx} = 1+zx$; $\frac{dz}{dx} + xy = 0$ given $y(0) = 0$; $z(0) = 1$ at $x = 0.3$

Soln:- Here $x_0 = 0$; $y_0 = 0$; $z_0 = 1$; $h = 0.3$; $f(x,y,z) = 1+zx$; $g(x,y,z) = -xy$

$k_1 = 0.3$; $k_2 = 0.345$; $k_3 = 0.34485$; $k_4 = 0.3893$

$l_1 = 0$; $l_2 = -0.00675$; $l_3 = -0.0077625$; $l_4 = -0.03104$

$y(0.3) = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.34483$

$z(0.3) = z_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4) = 0.98999.$

Prob 5:- Use RK Method of order 4 to solve $\frac{dx}{dt} = y-t$; $\frac{dy}{dt} = x+t$; $x(0) = 1$; $y(0) = 1$; Find $x(0.1)$ & $y(0.1)$

Soln:- $k_1 = 0.1$; $k_2 = 0.1$; $k_3 = 0.1005$; $k_4 = 0.101$; $l_1 = 0.1$; $l_2 = 0.11$; $l_3 = 0.11$; $l_4 = 0.12$
$x(0.1) = 1.1$ & $y(0.1) = 1.11$

Numerical solutions of 2nd order DEs :-

Picard's Method :- In this method, the given 2nd order DE is reduced to two first order simultaneous differential equations and then apply Picard's method to solve simultaneous DEs.

Problem :- Solve the 2nd order DE $y'' + y^2 y' = x^3$ given that $y(1) = 1$ & $y'(1) = 1$ at $x = 1.1$

Soln :- Take $y' = \frac{dy}{dx} = z$ then we get $y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}$;

So the given 2nd order DE becomes $\frac{dz}{dx} + y^2 z = x^3$; $y(1) = 1$, $z(1) = 1$

Thus the two 1st order simultaneous eqns are

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = x^3 - y^2 z; \quad y(1) = 1; \quad z(1) = 1;$$

$x_0 = 1$; $y_0 = 1$; $z_0 = 1$;

$$\Rightarrow \int_1^y dy = \int_1^x z\, dx$$

$$y_1 = 1 + \int_1^x z\, dx$$

$$= 1 + x - 1 = x.$$

$$\Rightarrow y_1 = x$$

$$\int_1^z dz = \int_1^x (x^3 - y^2 z)\, dx$$

$$z = 1 + \int_1^x (x^3 - y^2 z)\, dx$$

$$= 1 + \int_1^x (x^3 - 1)\, dx$$

$$= 1 + \left(-x + \frac{x^4}{4}\right) - \left(-1 + \frac{1}{4}\right)$$

$$z_1 = \frac{7}{4} - x + \frac{x^4}{4}$$

$$y_2 = 1 + \int_1^x z_1\, dx$$

$$= 1 + \int_1^x \left(\frac{7}{4} - x + \frac{x^4}{4}\right) dx$$

$$= 1 + \left(\frac{7x}{4} - \frac{x^2}{2} + \frac{x^5}{20}\right) - \left(\frac{7}{4} - \frac{1}{2} + \frac{1}{20}\right)$$

$$y_2 = \frac{-3}{10} + \frac{7x}{4} - \frac{x^2}{2} + \frac{x^5}{20}$$

z_2 is not required as only y value is required

Prob 2:- When a pendulum swings in a resisting medium, its equation of motion is of the form $\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} + b\sin\theta = 0$ where a & b are constants. Let $a = 0.2$; $b = 10$. Find the second approximation to the solution of the equation with $\frac{d\theta}{dt} = 0$ & $\theta = 0.3$ when $t = 0$, using Picard's method. Find θ at $t = 0.01$, $t = 0.02$ & $t = 0.03$;

Soln:- Given $t_0 = 0$; $\theta(t_0) = 0.3$ & $\left.\frac{d\theta}{dt}\right|_{t_0} = 0$; $\frac{d^2\theta}{dt^2} + 0.2\frac{d\theta}{dt} + 10\sin\theta = 0$;

put $\phi = \frac{d\theta}{dt} \Rightarrow \frac{d\phi}{dt} = \frac{d^2\theta}{dt^2}$; So the 2nd order DE becomes.

$\frac{d\phi}{dt} + 0.2\phi + 10\sin\theta = 0$ where $\phi(t_0) = 0$; $\theta(t_0) = 0.3$; $t_0 = 0$;

So we've the system of equations

$\frac{d\theta}{dt} = \phi$ & $\frac{d\phi}{dt} = -0.2\phi + 10\sin\theta$;

So we get

$$\int_{0.3}^{\theta} d\theta = \int_0^t \phi\, dt$$

$$\Rightarrow \theta = 0.3 + \int_0^t 0 \cdot dt$$

$$\Rightarrow \theta_1 = 0.3$$

$$\int_0^{\phi} d\phi = -\int_0^t (0.2\phi + 10\sin\theta)\,dt.$$

$$\Rightarrow \phi = -\int_0^t (0.2\phi + 10\sin\theta)\,dt$$

$$= -\int_0^t 10\sin(0.3)\,dt = -2.9552t.$$

$$\Rightarrow \phi_1 = -2.9552t.$$

ϕ_2 is not required as we need only θ value

$$\theta_2 = 0.3 + \int_0^t \phi_1\, dt$$

$$= 0.3 - \int_0^t 2.9552\, dt$$

$$= 0.3 - 2.9552\frac{t^2}{2}.$$

$$\therefore \theta_2 = 0.3 - 1.4776t^2$$

$\therefore \theta(0.01) = 0.29985$; $\theta(0.02) = 0.29941$ & $\theta(0.03) = 0.29867$

Prob3:- By 4^{th} order RK Method, find $y(0.1)$ given that

$$\frac{d^2y}{dx^2} = y^3 \text{ and } y = 10, \ \frac{dy}{dx} = 5 \text{ at } x = 0$$

Soln:- Put $\frac{dy}{dx} = z \Rightarrow \frac{d^2y}{dx^2} = \frac{dz}{dx}$. So that we've

$\frac{dy}{dx} = z$ & $\frac{dz}{dx} = y^3$ where $x_0 = 0$; $y(x_0) = 10$; $z(x_0) = 5$

$f(x, y, z) = z$; $g(x, y, z) = y^3$; $h = 0.1$;

$k_1 = h f(x_0\ y_0\ z_0) = 0.5$ $\quad$ $l_1 = h\, g(x_0, y_0, z_0) = 100$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = 5.5$; $l_2 = h\, g(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = 107.7$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 5.885$; $l_3 = h\, g(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 207.27$

$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3) = 21.227$; $l_4 = h\, g(x_0 + h, y_0 + k_3, z_0 + l_3) = 400.83$

$\Rightarrow y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 10 + \frac{1}{6}[0.5 + 2(5.5) + 2(5.885) + 21.227]$

$$y_1 = y(0.1) = 17.4162$$

Prob4:- Solve $y'' - xy' - y = 0$, $y(0) = 1$; $y'(0) = 0$. Find $y(0.2)$ & $y'(0.2)$ using RK 4^{th} order method.

Soln:- Take $y' = \frac{dy}{dx} = z$; $\Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{dz}{dx}$.

$\therefore$ we have $\frac{dy}{dx} = z$; $\frac{dz}{dx} = xz + y$; $x_0 = 0$; $y_0 = 1$; $z_0 = 0$; $h = 0.2$

$k_1 = h f(x_0, y_0, z_0) = 0$; $l_1 = h\, g(x_0, y_0, z_0) = 0.2$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = 0.02$; $l_2 = h\, g(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = 0.202$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 0.0202$; $l_3 = h\, g(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 0.204$

$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.0408$; $l_4 = h\, g(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2122$

$y_1 = y(x_0 + h) = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + 0.0202 = 1.0202$

$z_1 = z(x_0 + h) = z(0.2) = y'(0.2) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 0 + 0.204 = 0.204$.

Prob 5:- Using 4^{th} order RK Method find the value of x and $\frac{dx}{dt}$ at $t=0.1$ given that $\frac{d^2x}{dt^2} - t\frac{dx}{dt} + 4x = 0$ & $x(0)=3$; $\frac{dx}{dt}=0$ at $x=0$;

Soln:- $\frac{dx}{dt} = y$; $\frac{dy}{dt} = ty - 4x$; $t_0 = 0$, $x_0 = 3$; $y_0 = 0$; $h = 0.1$

$k_1 = 0$; $k_2 = -0.06$; $k_3 = -0.06015$; $k_4 = -0.1191$; $l_1 = -1.2$; $l_2 = -1.203$

$l_3 = -1.191$; $l_4 = -1.18785$; $x(t_0+h) = 2.9401$; $y(t_0+h) = \frac{dx}{dt} = -1.196$.

Prob 6:- Solve $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$ using Picard's method given $x=0$ & $y=1$, $\frac{dy}{dx} = 0.1$;

Soln:- $\frac{dy}{dx} = z$; $\frac{dz}{dx} + 3xz - 6y = 0$; $x_0 = 0$; $y_0 = 1$; $z_0 = 0.1$ $h = 0.1$

$y_1 = 1 + \frac{x}{10}$; $y_2 = 1 + \frac{x}{10} + 3x^2 - \frac{x^3}{20}$; $y_3 = 1 + \frac{x}{10} + \frac{x^3}{20} - \frac{3x^4}{2} + \frac{9x^5}{400}$

$z_1 = \frac{1}{10} + 6x - \frac{3x^2}{20}$; $z_2 = \frac{1}{10} + 6x + \frac{3x^2}{20} - 6x^3 + \frac{9x^4}{80}$; z_3 is not required.

Milne's method to solve 2nd order ODE :-

Q1: Find the value of $y(0.8)$ given $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} - 1 = 0$ and $y = y' = 0$ at $x=0$; $y(0.2) = 0.02$; $y'(0.2) = 0.1996$

$y(0.4) = 0.0795$; $y'(0.4) = 0.3937$; $y(0.6) = 0.1762$; $y'(0.6) = 0.5689$

Use the corrector formula twice to obtain the y value of $x=0.8$

Soln:- Here $\frac{dy}{dx} = z$; $\frac{dz}{dx} = 1 - 2yz$;

when $x_0 = 0$; $y_0 = 0$ $z_0 = 0$ & $z_0' = 1$

when $x_1 = 0.2$ $y_1 = 0.02$ $z_1 = 0.1996$ $z_1' = 0.992$

when $x_2 = 0.4$ $y_2 = 0.0795$ $z_2 = 0.3937$; $z_2' = 0.9374$

when $x = 0.6$ $y_3 = 0.1762$ $z_3 = 0.5689$; $z_3' = 0.7995$

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3); \quad z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

$$y_4^{(P)} = 0.3049 \quad ; \quad z_4^{(P)} = 0.7055$$

Using the corrector $y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$

$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

we've $z_4' = 1 - 2y_4^{(P)} z_4^{(P)} = 1 - 2(0.3049)(0.7055) = 0.5698$

$$y_4^{(c)} = 0.3045; \quad z_4^{(c)} = 0.7074$$

again using corrector formula

$$y_4^{(c)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7074] = 0.3046$$

i.e. $y(0.8) = 0.3046$; $z(0.8)$ is not required.

<u>Prob 2</u>:- Find a solution of $y'' + xy' + y = 0$, $y(0) = 1$; $y'(0) = 0$. First apply Picard's method to find the 3rd approx. and then Milne's method to find $y(0.4)$.

<u>Soln</u>:- Take $\frac{dy}{dx} = z \Rightarrow \frac{dz}{dx} = -xz - y$; $x_0 = 0$; $y_0 = 1$; $z_0 = 0$

$$\int_1^y dy = \int_0^x z\,dx. \qquad \int_0^z dx = -\int_0^x (\cancel{xz} + y)\,dx.$$

$$\Rightarrow y_1 = 1 + \int_0^x 0\,dx = 1 \qquad z_1 = -\int_0^x 1\,dx = -x;$$

$$z_2 = -\int_0^x (xz_1 + y_1)\,dx = -\int_0^x (-x^2 + 1)\,dx$$

$$y_2 = 1 + \int_0^x z_1\,dx$$

$$= 1 + \int_0^x -x\,dx = 1 - \frac{x^2}{2} \qquad \Rightarrow z_2 = -x + \frac{x^3}{3}$$

$$z_3 = -\int_0^x (xz_2 + y_2)\,dx = -\int_0^x \left(-x^2 + \frac{x^4}{3} + 1 - \frac{x^2}{2}\right)dx$$

$$y_3 = 1 + \int_0^x \left(-x + \frac{x^3}{3}\right)dx \qquad = -x + \frac{x^3}{2} - \frac{x^5}{15}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{12}$$

Taking $x = 0.1, 0.2$ & 0.3 we get

$y(0.1) = 0.995$; $y(0.2) = 0.9801$; $y(0.3) = 0.956$

$z(0.1) = -0.0995$; $z(0.2) = -0.196$; $z(0.3) = -0.2867$.

also $z'(0) = -(0+1) = -1$; $z'(0.1) = -[(0.1)(-0.0995) + 0.995] = -0.985$

$z'(0.2) = -[(0.2)(-0.196) + 0.980] = -0.941$; $z'(0.3) = -[(0.3)(-0.2867) + 0.956]$
$= -0.87$.

Now

when $x_0 = 0$; $y_0 = 1$; $z_0 = 1$; $z_0' = -1$

when $x_1 = 0.1$ $y_1 = 0.995$ $z_1 = -0.0995$; $z_1' = -0.985$

when $x_2 = 0.2$ $y_2 = 0.9801$ $z_2 = -0.196$; $z_2' = -0.941$

when $x_3 = 0.3$ $y_3 = 0.956$; $z_3 = -0.2867$; $z_3' = -0.87$.

So by Milne's Predictor, $y_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right] = 0.9231$

$z_4^{(P)} = y_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right] = -0.3692$

$\Rightarrow z_4' = -(x_4 z_4^{(P)} + y_4^{(P)}) = -0.7754$

Now using the correctors we get

$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4) = 0.9230$

$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4') = -0.3692$

$\therefore$ $y_4 = 0.9230$

Prob3 :- Solve the DE $y'' = 2yy'$ at $x = 0.8$ using Milne's Predictor-corrector formula given

$y(0) = 0$; $y(0.2) = 0.2027$; $y(0.4) = 0.4228$ $y(0.6) = 0.6841$

$y'(0) = 1$; $y'(0.2) = 1.041$; $y'(0.4) = 1.179$ $y'(0.6) = 1.468$.

Use corrector formula trice

Soln - we've $\frac{dy}{dx} = z$; $\frac{dz}{dx} = 2yz$ where

$z'(0) = 0$; $z'(0.2) = 0.422$; $z'(0.4) = 0.997$; $z'(0.6) = 2.009$;

$y_4^{(P)} = 1.0237$; $z_4^{(P)} = 2.0307$; $y_4^{(C)} = 1.0282$; $z_4^{(C)} = 2.0584$

$z_4' = 4.1157$. $y_4^{(C)} = 1.03009$.