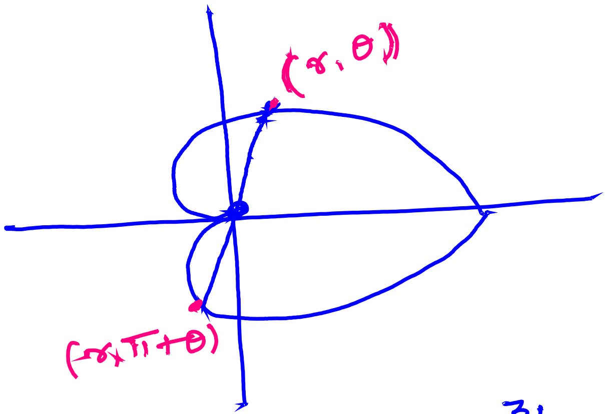
① If ρ_1 and ρ_2 be the radii of Curvature at the extremities of any chord of the Cardiod $r = a(1+\cos\theta)$, which passes through the pole, show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

Sol:-



$r = a(1+\cos\theta)$

$r_1 = \frac{dr}{d\theta} = a(0-\sin\theta)$

$= -a\sin\theta$

$r_2 = \frac{d^2r}{d\theta^2} = -a\cdot\cos\theta$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$r^2+r_1^2 = [a(1+\cos\theta)]^2 + (-a\sin\theta)^2$

$= a^2(1+\cos^2\theta+2\cos\theta) + a^2\sin^2\theta$

$= a^2[1+\cos^2\theta+2\cos\theta+\sin^2\theta]$

$= a^2[1+\underbrace{\cos^2\theta+\sin^2\theta}+2\cos\theta]$

$= a^2[1+1+2\cos\theta]$

$= a^2(2+2\cos\theta) \Rightarrow 2a^2(1+\cos\theta)$

$$r^2 + 2r_1^2 - r r_2 = (a(1+\cos\theta))^2 + 2(-a\sin\theta)^2 - a(1+\cos\theta)(-a\cos\theta)$$

$$= a^2(1+\cos^2\theta + 2\cos\theta) + 2a^2\sin^2\theta + a^2(\cos\theta + \cos^2\theta)$$

$$= a^2\left[1+\cos^2\theta + 2\cos\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta\right]$$

$$= a^2\left[1+2\cos^2\theta + 2\sin^2\theta + 3\cos\theta\right]$$

$$= a^2\left[1+2(1) + 3\cos\theta\right]$$

$$= a^2\left[3+3\cos\theta\right]$$

$$= \underline{3a^2(1+\cos\theta)}$$

Radius of Curvature $= \rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$$\rho = \frac{\left[2a^2(1+\cos\theta)\right]^{3/2}}{3a^2(1+\cos\theta)}$$

$$\rho = \frac{2^{3/2}(a^2)^{3/2}(1+\cos\theta)^{3/2}}{3a^2(1+\cos\theta)}$$

$$\rho = \frac{2^{3/2}}{3} a (1+\cos\theta)^{3/2-1}$$

$$\rho = \frac{2\sqrt{2}}{3} a (1+\cos\theta)^{1/2}$$

at (r, θ) $\rho_1 = \frac{2\sqrt{2}}{3} a (1+\cos\theta)^{1/2}$

A pair of extremities (end points) of a chord (given cardiod) are (r, θ) and $(r, \pi+\theta)$ at these points

ρ_1 = Radius of curvature at (r, θ)

$$\rho_1 = \frac{2\sqrt{2}}{3} a (1+\cos\theta)^{1/2} \longrightarrow (1)$$

ρ_2 = Radius of curvature at $(r, \pi+\theta)$

$$= \frac{2\sqrt{2}}{3} a (1+\cos(\pi+\theta))^{1/2}$$

$$= \frac{2\sqrt{2}}{3} a (1+(-\cos\theta))^{1/2}$$

$$\rho_2 = \frac{2\sqrt{2}}{3} a (1-\cos\theta)^{1/2} \longrightarrow (2)$$

<u>LHS</u>

$$\rho_1^2 + \rho_2^2 = \left[\frac{2\sqrt{2}}{3} a (1+\cos\theta)^{1/2}\right]^2 + \left[\frac{2\sqrt{2}a}{3} (1-\cos\theta)^{1/2}\right]^2$$

$$= \frac{8a^2}{9}(1+\cos\theta) + \frac{8a^2}{9}(1-\cos\theta)$$

$$= \frac{8a^2}{9}(1+\cancel{\cos\theta}+1-\cancel{\cos\theta}) = \frac{16a^2}{9}$$ <u>RHS</u>

(2) Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$, varies inversely as r^{n-1} (4)

Sol :- $r^n = a^n \cos n\theta$

$\log r^n = \log(a^n \cos n\theta)$

$\log r^n = \log a^n + \log(\cos n\theta)$

$n \log r = n \log a + \log(\cos n\theta)$

Diff w.r.t. θ

$n \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} \cdot (-\sin n\theta) \cdot n$

$\cancel{n} \cdot \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta \cdot \cancel{n}$

$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$

$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2$

$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}(-\tan n\theta)^2$

$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\tan^2 n\theta$

$\frac{1}{p^2} = \frac{1}{r^2}(1 + \tan^2 n\theta)$

$\frac{1}{p^2} = \frac{1}{r^2}(\sec^2 n\theta)$

$p^2 = r^2 \cos^2 n\theta \implies p = r \cdot \cos n\theta$

$p = r \cos n\theta$

But given $r^n = a^n \cos n\theta$

$\cos n\theta = \frac{r^n}{a^n}$

$p = r \cdot \frac{r^n}{a^n}$

$p = \frac{r^{n+1}}{a^n}$

Diff w.r.t 'r'

$\frac{dp}{dr} = \frac{1}{a^n} \cdot (n+1)\, r^{n+1-1}$

$\frac{dp}{dr} = \frac{1}{a^n} (n+1)\, r^n$

Radius of Curvature in pedal form $= \rho = \frac{r}{dp/dr}$

$r \frac{dr}{dp}$

(or) $\rho = \frac{r}{dp/dr}$

$\rho = \frac{r}{\frac{(n+1)\, r^n}{a^n}}$

$\rho = \frac{a^n}{n+1} \cdot \frac{1}{r^{n-1}}$

The radius of curvature varies inversely as r^{n-1}

3. If $r(1-\cos\theta) = 2a$, then show that p^2 varies as r^3.

Sol:- $r(1-\cos\theta) = 2a$

$\log r + \log(1-\cos\theta) = \log 2a$

Diff w.r.t 'θ'

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{1-\cos\theta}\cdot(0-(-\sin\theta)) = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\cancel{2}\,\cancel{\sin\theta/2}\,\cos\theta/2}{\cancel{2}\,\sin^{\cancel{2}}\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\cot\theta/2$$

$$\cot\phi = -\cot\theta/2$$

$$\cot\phi = \cot(\pi-\theta/2)$$

$$\phi = \pi-\theta/2$$

$$p = r\sin\phi$$

$$p = r\sin(\pi-\theta/2)$$

$$p = r\sin\theta/2 \rightarrow (1)$$

Given

$r(1-\cos\theta) = 2a$

$r(\cancel{2}\sin^2\theta/2) = \cancel{2}a$

$\sin^2\theta/2 = a/r$

$\sin\theta/2 = \sqrt{\frac{a}{r}} \rightarrow (2)$

Sub. ② in ①

$$p = r \cdot \sqrt{\frac{a}{r}}$$

$$p = r \cdot \frac{\sqrt{a}}{\sqrt{r}}$$

$$p = \sqrt{r} \cdot \sqrt{a}$$

$$p = \sqrt{ar} = \sqrt{a} \cdot r^{1/2}$$

Diff w.r.t 'r'.

$$\frac{dp}{dr} = \sqrt{a} \cdot \frac{1}{2} r^{\frac{1}{2}-1}$$

$$\frac{dp}{dr} = \frac{\sqrt{a}}{2} \cdot r^{-1/2}$$

Radius of Curvature $= \rho = \dfrac{r}{\frac{dp}{dr}} = \dfrac{r}{\frac{\sqrt{a}}{2} r^{-1/2}}$

$$\rho = \frac{2r}{\sqrt{a}\, r^{-1/2}}$$

$$\rho = \frac{2}{\sqrt{a}} \cdot r \cdot r^{1/2}$$

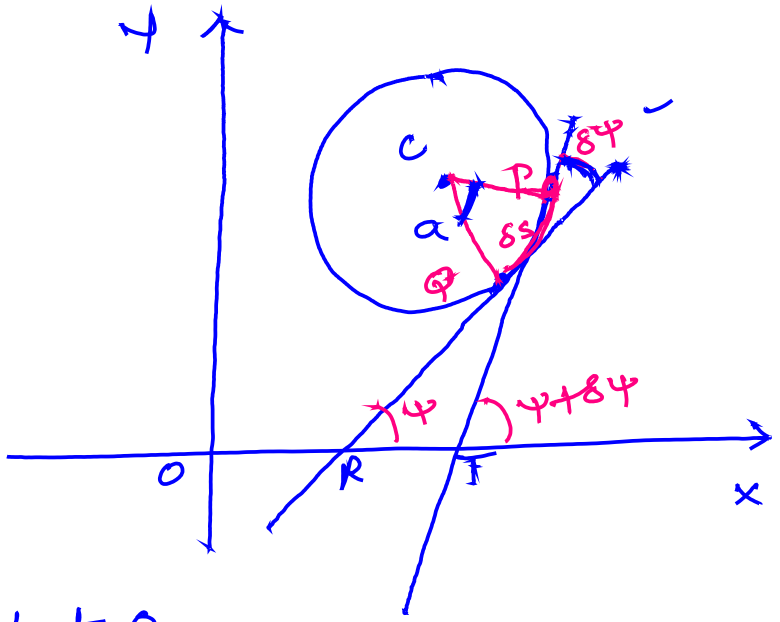
$$\rho = \frac{2}{\sqrt{a}} r^{3/2}$$

$$\rho^2 = \left(\frac{2}{\sqrt{a}} r^{3/2}\right)^2$$

$$\rho^2 = \frac{4}{a} \cdot \left(r^{3/2}\right)^2$$

$$\rho^2 = \frac{4}{a} r^3 \qquad \therefore \rho^2 \text{ varies as } r^3$$

Proove that Curvature of a Circle is constant (8)



Let us Consider a Circle with the radius $\underline{a}$ units. Let P & Q be two points on the circle.

As we move from P to Q

Then $Q \to P$ the tangent rotates then the angle through $\delta\psi$

$$\underline{s = r\theta}$$

$$\underline{\delta s = a \cdot \delta\psi}$$

From, the above fig

$$\underline{\delta s} = r\,\underline{\delta\psi} = a\,\delta\psi$$

$$K = \frac{d\psi}{ds}$$

$$\text{Curvature at } P = \lim_{Q \to P} \frac{\cancel{\delta\psi}}{a\,\cancel{\delta\psi}}$$

$$= \frac{1}{a}$$

$$= \text{constant}$$

$\therefore$ Curvature of a circle is a constant

Solve

1. Find the radius of curvature for the curve $r^n = a^n \sin n\theta$

2. Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $4a\cos\theta/2$