Radius of Curvature in Cartesian form

Let $y = f(x)$ be the Cartesian Curve.

we know that $y' = \frac{dy}{dx} = \tan\psi$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\psi)$$

$$= \sec^2\psi \cdot \frac{d\psi}{dx}$$

$$= (1+\tan^2\psi)\, \underline{\frac{d\psi}{ds} \cdot \frac{ds}{dx}}$$

$$y'' = (1+\tan^2\psi)\, \frac{1}{\rho} \cdot \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$y'' = \left[1+\left(\frac{dy}{dx}\right)^2\right] \cdot \frac{1}{\rho} \cdot \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right] \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}{y''}$$

$$\boxed{\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}}$$

Note :- Radius of Curvature is independent of axes of reference.

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2}$$

Radius of Curvature in Parametric form

For $y = f(x)$

$$x = f(t), \quad y = g(t)$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \longrightarrow (1)$$

We express y' & y'' in terms of parameter t

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'} \longrightarrow (2) \text{ where } \left.\begin{aligned} y' &= \frac{dy}{dt} \\ x' &= \frac{dx}{dt} \end{aligned}\right\}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left[\frac{y'}{x'}\right]$$

$$= \left(\frac{x'y'' - y'x''}{(x')^2}\right) \cdot \frac{dt}{dx}$$

$$= \frac{x'y'' - y'x''}{(x')^2} \cdot \frac{1}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{(x')^3} \longrightarrow (3), \text{ where } y'' = \frac{d^2y}{dt^2}, \quad x'' = \frac{d^2x}{dt^2}$$

Substituting ② & ③ in ①. we get. ③

$$\rho = \frac{\left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2}}{\frac{x'y''-y'x''}{(x')^3}}$$

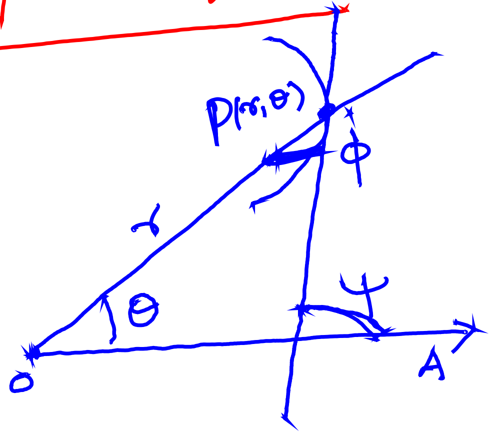
$$\rho = \frac{\left[\frac{(x')^2+(y')^2}{(x')^2}\right]^{3/2}}{\frac{x'y''-y'x''}{(x')^3}}$$

$$\rho = \frac{\left((x')^2+(y')^2\right)^{3/2}}{\left((x')^{\cancel{2}}\right)^{3/\cancel{2}}} \times \frac{(x')^3}{x'y''-y'x''}$$

$$= \frac{\left[(x')^2+(y')^2\right]^{3/2}}{\cancel{(x')^3}} \times \frac{\cancel{(x')^3}}{x'y''-y'x''}$$

$$\rho = \frac{\left[(x')^2+(y')^2\right]^{3/2}}{x'y''-y'x''}$$

Radius of Curvature in polar form :-



Let OA be the initial line. $OP = r$, and ϕ be the angle between radius vector and the tangent

& ψ be the angle made by the tangent at P with the initial line.

$$\psi = \theta + \phi$$

Diff w.r.t 's'

$$\frac{d\psi}{ds} = \frac{d}{ds}(\theta + \phi)$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds}$$

But $\frac{d\psi}{ds} = \frac{1}{\rho}$

$$\frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds} \Rightarrow \frac{1}{\rho} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$\frac{1}{\rho} = \frac{d\theta}{ds}\left(1+\frac{d\phi}{d\theta}\right)$$

$$\rho = \frac{\frac{ds}{d\theta}}{1+\frac{d\phi}{d\theta}} \longrightarrow (1)$$

But $\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \longrightarrow (2)$

we know that $\tan\phi = \frac{r\,d\theta}{dr}$

$$\tan\phi = \frac{r}{\frac{dr}{d\theta}}$$

$$\tan\phi = \frac{r}{r_1}$$ where $r_1 = \frac{dr}{d\theta}$

Diff w.r.t 'θ'

$$\sec^2\phi \cdot \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - r \cdot r_2}{r_1^2}$$

$r_2 = \frac{d^2r}{d\theta^2}$

$$\sec^2\phi \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2}$$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2 \sec^2\phi}$$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2(1+\tan^2\phi)}$$

$\sec^2\phi = 1+\tan^2\phi$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2\left(1 + \frac{r^2}{r_1^2}\right)}$$

$$= \frac{r_1^2 - r r_2}{\cancel{r_1^2}\left(\frac{r_1^2 + r^2}{\cancel{r_1^2}}\right)}$$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r^2 + r_1^2}$$

$$1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2 - r r_2}{r^2 + r_1^2}$$

$$= \frac{r^2 + r_1^2 + r_1^2 - r r_2}{r^2 + r_1^2}$$

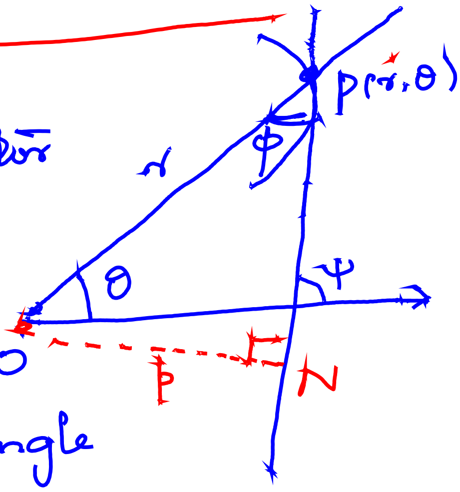
$$1 + \frac{d\phi}{d\theta} = \frac{r^2 + 2r_1^2 - r r_2}{r^2 + r_1^2} \longrightarrow (3)$$

Sub (2) & (3) in (1)

$$\rho = \frac{\sqrt{r^2 + r_1^2}}{\frac{r^2 + 2r_1^2 - r r_2}{r^2 + r_1^2}} = \frac{\sqrt{r^2 + r_1^2}\,(r^2 + r_1^2)}{r^2 + 2r_1^2 - r r_2}$$

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}}$$

Radius of Curvature for pedal Curve :-



Let $OP = r$ be the radius vector

let ϕ be the angle made by the radius vector with the tangent. let ψ be the angle made by the tangent with x-axis

Draw $ON = p$. which is perpendicular from pole to the tangent

From ONP, $\sin\phi = \frac{p}{r}$

$$p = r\sin\phi \longrightarrow (1)$$

Diff (1) w.r.t 'r'

$$\frac{dp}{dr} = r \cdot \cos\phi \cdot \frac{d\phi}{dr} + \sin\phi \cdot 1$$

$$\left.\begin{aligned} \sin\phi &= r\frac{d\theta}{ds} \\ \cos\phi &= \frac{dr}{ds} \end{aligned}\right\}$$

$$\frac{dp}{dr} = r \cdot \frac{dr}{ds} \cdot \frac{d\phi}{dr} + r\frac{d\theta}{ds}$$

$$\frac{dp}{dr} = r \cdot \frac{d\phi}{dr} \cdot \frac{dr}{ds} + r\frac{d\theta}{ds}$$

$$\frac{dp}{dr} = r\cdot\frac{d\phi}{ds} + r\cdot\frac{d\theta}{ds}$$

$$\frac{dp}{dr} = r\left[\frac{d\phi}{ds} + \frac{d\theta}{ds}\right]$$

$$\frac{dp}{dr} = r\cdot\frac{d}{ds}(\theta+\phi)$$ But $\theta+\phi=\psi$

$$\frac{dp}{dr} = r\cdot\frac{d\psi}{ds}$$

$$\frac{d\psi}{ds} = \frac{1}{r}\cdot\frac{dp}{dr}$$ But $\frac{d\psi}{ds} = 1/e$

$$\frac{1}{e} = \frac{1}{r}\frac{dp}{dr}$$

$$\boxed{\rho = r\frac{dr}{dp}}$$

Problems on radius of curvature

① Show that the radius of curvature of the curve $y = 4\sin x - \sin 2x$ at $x = \pi/2$ is $\frac{5\sqrt{5}}{4}$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

Sol:- $y = 4\sin x - \sin 2x$

$$\frac{dy}{dx} = 4\cdot\cos x - \cos 2x \times 2$$

$$\frac{d^2y}{dx^2} = -4\sin x - 2\cdot(-\sin 2x)\cdot 2$$

$$\frac{d^2y}{dx^2} = -4\sin x + 4\sin 2x$$

$$\frac{dy}{dx} \text{ at } x = \pi/2 = (4\cos x - 2\cos 2x) \text{ at } x = \pi/2$$

$$= 4\cos\pi/2 - 2\cdot\cos(\cancel{2}\cdot\pi/\cancel{2})$$

$$= 4\cdot 0 - 2\cdot(-1)$$

$$= \underline{2}$$

$$\frac{d^2y}{dx^2} \text{ at } x = \pi/2$$

$$\frac{d^2y}{dx^2} = (-4\sin x + 4\sin 2x) \text{ at } x = \pi/2$$

$$= -4\sin\pi/2 + 4(\sin\cancel{2}\cdot\pi/\cancel{2})$$

$$= -4\cdot 1 + 4(0)$$

$$= \underline{-4}$$

Radius of curvature $= \rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$= \frac{[1+2^2)^{3/2}}{-4}$$

$$= \frac{5^{3/2}}{-4} = -\frac{5\cdot 5^{1/2}}{4}$$

$$\rho = \frac{5\sqrt{5}}{4} \qquad = \frac{5\sqrt{5}}{4}$$

② Find the radius of curvature for the Curve $x^3+y^3=3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Sol:- Given $x^3+y^3=3axy$

Diff w.r.t 'x'

$$3x^2+3y^2\cdot\frac{dy}{dx} = 3a\left[x\cdot\frac{dy}{dx}+y\cdot 1\right]$$

$$\cancel{3}\left[x^2+y^2\frac{dy}{dx}\right] = \cancel{3}a\left[x\cdot\frac{dy}{dx}+y\right]$$

$$x^2+y^2\frac{dy}{dx} - ax\frac{dy}{dx} - ay = 0$$

$$(y^2-ax)\frac{dy}{dx} = ay-x^2$$

$$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax} \rightarrow ①$$

$\frac{dy}{dx}$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$\frac{dy}{dx} = \frac{a\cdot\frac{3a}{2} - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\cdot\frac{3a}{2}}$$

$$= \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$$

Diff ② w.r.t. 'x'

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \leftarrow ③$$

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\frac{d}{dx}(ay-x^2) - (ay-x^2)\frac{d}{dx}(y^2-ax)}{(y^2-ax)^2}$$

$$= \frac{(y^2-ax)\left(a\cdot\frac{dy}{dx} - 2x\right) - (ay-x^2)\left(2y\cdot\frac{dy}{dx} - a\right)}{(y^2-ax)^2}$$

at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$\frac{d^2y}{dx^2} = \frac{\left(\left(\frac{3a}{2}\right)^2 - a\cdot\frac{3a}{2}\right)\left(a\cdot(-1) - \cancel{2}\cdot\frac{3a}{\cancel{2}}\right) - \left(a\cdot\frac{3a}{2} - \left(\frac{3a}{2}\right)^2\right)\left(2\left(\frac{3a}{2}\right)(-1) - a\right)}{\left(\left(\frac{3a}{2}\right)^2 - a\cdot\frac{3a}{2}\right)^2}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-a - 3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-4a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-32}{3a}$$

Radius of Curvature $= \rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$\rho = \frac{\left[1+(-1)^2\right]^{3/2}}{-32/3a}$$

$$\rho = \frac{2^{3/2}}{-32/3a} \Rightarrow \rho = \frac{2\sqrt{2}\cdot 3a}{\cancel{-}32_{16}}$$

$$\rho = \frac{3\sqrt{2}\,a}{16}$$

$$\rho = \frac{3\cancel{\sqrt{2}}\,a}{8\cdot\cancel{\sqrt{2}}\cdot\sqrt{2}}$$

$$\rho = \frac{3a}{8\sqrt{2}}$$

(3) Find the radius of Curvature of

$x = a\left(\cos t + \log\left(\tan(t/2)\right)\right)$, $y = a \sin t$

$$x = a\left(\cos t + \log\left(\tan \tfrac{t}{2}\right)\right)$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\cdot \sec^2 t/2 \cdot \tfrac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{\frac{\sin t/2}{\cancel{\cos t/2}}}\cdot \frac{1}{\cos^{\cancel{2}} t/2}\cdot \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{2\sin t/2 \cos t/2}\right)$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$= a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$\frac{dx}{dt} = a\,\frac{\cos^2 t}{\sin t}$$

$$y = a \sin t$$

$$\frac{dy}{dt} = a \cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cancel{a}\,\cancel{\cos t}}{\cancel{a}\,\frac{\cos^{\cancel{2}} t}{\sin t}}$$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t} = \tan t$$

$$\frac{dy}{dx} = \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx}$$

But $\frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$

$$= \sec^2 t \cdot \frac{\sin t}{a\cos^2 t}$$

$$= \frac{1}{a} \cdot \sec^2 t \cdot \sin t \cdot \sec^2 t$$

$$\frac{d^2y}{dx^2} = 1/a \sec^4 t \sin t$$

Radius of Curvature $= \rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$\rho = \frac{\left[1+(\tan t)^2\right]^{3/2}}{1/a \sec^4 t \sin t}$$

$$\rho = \frac{a \cdot (1+\tan^2 t)^{3/2}}{\sec^4 t \sin t}$$

$$\rho = \frac{a \cdot (\sec^{\cancel{2}} t)^{3/\cancel{2}}}{\sec^4 t \sin t}$$

$$\rho = \frac{a \cdot \cancel{\sec^3 t}}{\cancel{\sec^4} t \sin t}$$

$$\rho = \frac{a \cdot \cos t}{\sin t}$$

$$\rho = \underline{a \cot t}$$

(4) Find the radius of curvature for $r^3 = a^2 p$

Sol:- $r^3 = a^2 p$

Diff w r t p

$$3r^2 \cdot \frac{dr}{dp} = a^2 (1)$$

$$\rho = r\frac{dr}{dp}$$

$$\frac{dr}{dp} = \frac{a^2}{3r^2}$$

Radius of Curvature $= \rho = r\frac{dr}{dp}$

$$\rho = \cancel{r} \cdot \frac{a^2}{3r^{\cancel{2}}}$$

$$\rho = \frac{a^2}{3r}$$

(5) Show that for the curve $r = a e^{\theta \cot \alpha}$ where a and α are constants ρ/r is a constant

Sol:- $r = a e^{\theta \cot \alpha}$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

$$\log r = \log(a e^{\theta \cot \alpha})$$

$$\log r = \log a + \log_e e^{\theta \cot \alpha}$$

$$\log r = \log a + \theta \cot \alpha$$

$\log r = \log a + \theta \cot\alpha$

D.r w.r.t 'θ'

$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1 \cdot \cot\alpha$

$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha$

$\frac{dr}{d\theta} = r\cot\alpha \Rightarrow r_1 = r\cot\alpha \rightarrow (1)$

Diff (1) w.r.t 'θ'

$r_2 = r_1 \cot\alpha$

$r_2 = (r\cot\alpha)\cot\alpha$

$r_2 = r\cot^2\alpha$

Radius of Curvature $= \rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$= \frac{[r^2+(r\cot\alpha)^2]^{3/2}}{r^2+2\cdot(r\cot\alpha)^2 - r\cdot r\cot^2\alpha}$

$= \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha - r^2\cot^2\alpha}$

$= \frac{(r^2(1+\cot^2\alpha))^{3/2}}{r^2+r^2\cot^2\alpha}$

$$\rho = \frac{(r^2)^{3/2}(\operatorname{cosec}^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)}$$

$$\rho = \frac{r^3 \operatorname{cosec}^3\alpha}{\cancel{r^2}\cdot \cancel{\operatorname{cosec}^2\alpha}}$$

$$\rho = r\operatorname{cosec}\alpha$$

$$\frac{\rho}{r} = \operatorname{cosec}\alpha \implies \frac{\rho}{r} = \text{constant}$$

Since α is constant

Problems

1. Show that the radius of curvature for the curve $x^4+y^4=2$ at $(1,1)$ is $\frac{\sqrt{2}}{3}$.

2. Find the radius of curvature for $xy^3 = a^4$ at (a,a)

3. Find the radius of curvature for $y = ax^2+bx+c$ at $x = \frac{1}{2a}\left[\sqrt{a^2-1} - b\right]$

4. $x = a\cos\theta,\ y = a\sin\theta$ at $(a/\sqrt{2}, a/\sqrt{2})$