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ATME COLLEGE OF ENGINEERING

13th KM Stone, Bannur Road, Mysore - 560 028

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

(ACADEMIC YEAR 2020-21)

LESSON NOTES

SUBJECT: AUTOMATA THEORY AND COMPUTABILITY

SUB CODE: 18CS54

SEMESTER: V

INSTITUTIONAL MISSION AND VISION

Objectives

- To provide quality education and groom top-notch professionals, entrepreneurs and leaders for different fields of engineering, technology and management.
- To open a Training-R & D-Design-Consultancy cell in each department, gradually introduce doctoral and postdoctoral programs, encourage basic & applied research in areas of social relevance, and develop the institute as a center of excellence.
- To develop academic, professional and financial alliances with the industry as well as the academia at national and transnational levels.
- To develop academic, professional and financial alliances with the industry as well as the academia at national and transnational levels.
- To cultivate strong community relationships and involve the students and the staff in local community service.
- To constantly enhance the value of the educational inputs with the participation of students, faculty, parents and industry.

Vision

- Development of academically excellent, culturally vibrant, socially responsible and globally competent human resources.

Mission

- To keep pace with advancements in knowledge and make the students competitive and capable at the global level.
- To create an environment for the students to acquire the right physical, intellectual, emotional and moral foundations and shine as torch bearers of tomorrow’s society.
- To strive to attain ever-higher benchmarks of educational excellence.

Department of Computer Science & Engineering

Vision of the Department

- To develop highly talented individuals in Computer Science and Engineering to deal with real world challenges in industry, education, research and society.

Mission of the Department

- To inculcate professional behavior, strong ethical values, innovative research capabilities and leadership abilities in the young minds & to provide a teaching environment that emphasizes depth, originality and critical thinking.
- Motivate students to put their thoughts and ideas adoptable by industry or to pursue higher studies leading to research.

Program Educational Objectives (PEO'S):

1. Empower students with a strong basis in the mathematical, scientific and engineering fundamentals to solve computational problems and to prepare them for employment, higher learning and R&D.
2. Gain technical knowledge, skills and awareness of current technologies of computer science engineering and to develop an ability to design and provide novel engineering solutions for software/hardware problems through entrepreneurial skills.
3. Exposure to emerging technologies and work in teams on interdisciplinary projects with effective communication skills and leadership qualities.
4. Ability to function ethically and responsibly in a rapidly changing environment by applying innovative ideas in the latest technology, to become effective professionals in Computer Science to bear a life-long career in related areas.

Program Specific Outcomes (PSOs)

1. Demonstrate understanding of the principles and working of the hardware and software aspects of Embedded Systems.
2. Use professional Engineering practices, strategies and tactics for the development, implementation and maintenance of software.
3. Provide effective and efficient real time solutions using acquired knowledge in various domains.

Faculty Name/s :				Academic Year: 2019 - 2020			
Department: **COMPUTER SCIENCE & ENGINEERING**							
Course Code	Course Title	Core/Elective	Prerequisite	Contact Hours			Total Hrs/ Sessions
				L	T	P	
18CS54	**AUTOMATA THEORY AND COMPUTABILITY**	**Core**		**3**		**-**	**40**
Objectives	This course will enable students to • Introduce core concepts in Automata and Theory of Computation • Identify different Formal language Classes and their Relationships • Design Grammars and Recognizers for different formal languages • Prove or disprove theorems in automata theory using their properties • Determine the decidability and intractability of Computational problems						

Topics Covered as per Syllabus

Module – 1

Why study the Theory of Computation, Languages and Strings: Strings Languages. A Language Hierarchy, Computation, **Finite State Machines (FSM)**: Deterministic FSM, Regular languages, Designing FSM, Nondeterministic FSMs, From FSMs to Operational Systems, Simulators for FSMs, Minimizing FSMs, Canonical form of Regular languages, Finite State Transducers, Bidirectional Transducers. **Textbook 1: Ch 1,2, 3,4, 5.1 to 5.10**

Module – 2

Regular Expressions (RE): what is a RE? Kleene's theorem, Applications of REs, Manipulating and Simplifying REs. Regular Grammars: Definition, Regular Grammars and Regular languages. Regular Languages (RL) and Non- regular Languages: How many RLs, To show that a language is regular, Closure properties of RLs, to show some languages are not RLs.
Textbook 1: Ch 6, 7, 8: 6.1 to 6.4, 7.1, 7.2, 8.1 to 8.4

Module – 3

Context-Free Grammars(CFG): Introduction to Rewrite Systems and Grammars, CFGs and languages, designing CFGs, simplifying CFGs, proving that a Grammar is correct, Derivation and Parse trees, Ambiguity, Normal Forms. Pushdown Automata (PDA): Definition of non-deterministic PDA, Deterministic and Non-deterministic PDAs, Non-determinism and Halting, alternative equivalent definitions of a PDA, alternatives that are not equivalent to PDA.
Textbook 1: Ch 11, 12: 11.1 to 11.8, 12.1, 12.2, 12,4, 12.5, 12.6

Module – 4

Context-Free and Non-Context-Free Languages: Decidable questions, Un-decidable questions. Turing Machine: Turing machine model, Representation, Language acceptability by TM, design of TM, Techniques for TM construction.

Module – 5

Variants of Turing Machines (TM), The model of Linear Bounded automata: Decidability: Definition of an algorithm, decidability, decidable languages, Undecidable languages, halting problem of TM, Post correspondence problem. Complexity: Growth rate of functions, the classes of P and NP, Quantum Computation: quantum computers, Church-Turing thesis.

Textbook 2: Ch 9.7 to 9.8, 10.1 to 10.7, 12.1, 12.2, 12.8, 12.8.1, 12.8.2

List of Text Books

1. Elaine Rich, Automata, Computability and Complexity, 1st Edition, Pearson Education, 2012/2013
2. K L P Mishra, N Chandrasekaran , 3rd Edition, Theory of Computer Science, PhI, 2012.

List of Reference Books

1. John E Hopcroft, Rajeev Motwani, Jeffery D Ullman, Introduction to Automata Theory, Languages, and Computation, 3rd Edition, Pearson Education, 2013
2. Michael Sipser : Introduction to the Theory of Computation, 3rd edition, Cengage learning,2013
3. John C Martin, Introduction to Languages and The Theory of Computation, 3rd Edition, Tata McGraw –Hill Publishing Company Limited, 2013
4. Peter Linz, "An Introduction to Formal Languages an d Automata", 3rd Edition, Narosa Publishers, 1998
5. Basavaraj S. Anami, Karibasappa K G, Formal Languages and Automata theory, Wiley India, 2012
6. C K Nagpal, Formal Languages and Automata Theory, Oxford University press, 2012.

List of URLs, Text Books, Notes, Multimedia Content, etc

1.https://www.tutorialspoint.com/automata_theory/index.htm
2. https://www.geeksforgeeks.org/toc-finite-automata-introduction/

Course Outcomes	The students should be able to: • Acquire fundamental understanding of the core concepts in automata theory and Theory of Computation. • Learn how to translate between different models of Computation (e.g., Deterministic and Non-deterministic and Software models). • Design Grammars and Automata (recognizers) for different language classes and become knowledgeable about restricted models of Computation (Regular, Context Free) and their relative powers. • Develop skills in formal reasoning and reduction of a problem to a formal model, with an emphasis on semantic precision and conciseness. • Classify a problem with respect to different models of Computation.

MODULE I:

1. Introduction

The term "Automata" is derived from the Greek word "αὐτόματα" which means "self-acting". An **automaton** (Automata in plural) is an abstract self-propelled computing device which follows a predetermined sequence of operations automatically.

An automaton with a finite number of states is called a **Finite Automaton (FA)** or **Finite State Machine (FSM).**

2. Why to study Theory of Computation?

Theory of computation is mainly concerned with the study of how problems can be solved using algorithms. It is the study of mathematical properties both of problems and of algorithms for solving problems that depend on neither the details of today's technology nor the programming language.

It is still useful in two key ways:

- It provides a set of ***abstract structures*** that are useful for solving certain classes of problems. These abstract structures can he implemented on whatever hardware/software platform is available
- It defines provable limits to ***what can be computed*** regardless of processor speed or memory size. An understanding of these limits helps us to focus our design effort in areas in which it can pay off, rather than on the computing equivalent of the search for a perpetual motion machine.

The goal is to discover fundamental properties of the problems like:

- Is there any computational ***solution*** to the problem? 1f not. is there a restricted but useful variation of the problem for which a solution does exist?
- If a solution exists, can it be implemented using some ***fixed amount of memory***?
- If a solution exists. how ***efficient*** is it? More specifically. how do its time and space requirements grow as the size of the problem grows?
- Are there groups of problems that are ***equivalent*** in the sense that if there is an efficient

solution to one member of the group there is an efficient solution to all the others?

Applications of theory of computation:

- ***Development of Machine Languages***: Enables both machine-machine and person-machine communication. Without them, none of today's applications of computing could exist. Example: Network communication protocols, HTML etc.
- ***Development of modern programming languages:*** Both the design and the implementation of modern programming languages rely heavily on the theory of context-free languages. Context- free grammars are used to document the languages syntax and they form the basis for the parsing techniques that all compilers use.
- ***Natural language processing***: It is a field of computer science, artificial intelligence, and computational linguistics concerned with the interactions between computers and human (natural) languages.
- ***Automated hardware systems:*** Systems as diverse as parity checkers, vending machines, communication protocols, and building security devices can be straightforwardly described as finite state machines, which is a part of theory of computation.
- ***Video Games:*** Many interactive video games are use large nondeterministic finite state machines.
- ***Security*** is perhaps the most important property of many computer systems. The undecidability results of computation show that there cannot exist a general-purpose method for automatically verifying arbitrary security properties of programs.
- ***Artificial intelligence:*** Artificial intelligence programs solve problems in task domains ranging from medical diagnosis to factory scheduling. Various logical frameworks have been proposed for representing and reasoning with the knowledge that such programs exploit.
- ***Graph Algorithms:*** Many natural structures, including ones as different as organic molecules and computer networks can be modeled as graphs. The theory of complexity tells us that, is there exist efficient algorithms for answering some important questions about graphs. Some questions are "hard", in the sense that no efficient algorithm for them is known nor is one likely to be developed.

3. Strings

Alphabet

Definition: An **alphabet** is any finite set of symbols denoted by Σ (Sometimes also called as **characters** or **symbols**).

Example: Σ = {a, b, c, d} is an alphabet set where ‘a’, ‘b’, ‘c’, and ‘d’ are symbols.

String

Definition: A **string** is a finite sequence of symbols taken from Σ.

Example: 'cabcad' is a valid string on the alphabet set Σ = {a, b, c, d}

Alphabet name	*Alphabet symbols*	*Example strings*
The lower case English alphabet	{a, b, c, …, z}	ε, aabbcg, aaaaa
The binary alphabet	{0, 1}	ε, 0, 001100, 11
A star alphabet	{★, ✪, ☆, ✭, ✡, ✫}	ε, ✪✪, ✪✭✭✫★✫
A music alphabet	{𝅝, 𝅗𝅥, ♩, ♪, 𝅘𝅥𝅯, 𝅘𝅥𝅰, ●}	ε, ♩𝅝♪𝅘𝅥𝅯

3.1. Functions on Strings

Length of a String

Definition: It is the number of symbols present in a string. (Denoted by |.|).

Examples: If ***s*** ='cabcad', | ***s*** |= 6; Also |11001101| = 7

If | ***s*** |= 0, it is called an empty string, denoted by **ε.** |ε| = 0

Concatenation of strings: The *concatenation* of two strings *s* and *t*, written *s||t* or simply *st*, is the string formed by appending t to *s*. For example, if *x* = *good* and *y* = *bye*, then *xy* = *goodbye*. So |xy| = |x| + |y|.

The empty string, *e*, is the **identity** for concatenation of strings. *(xe = ex = x).* Concatenation, as a function defined on strings is associative. *(st)w = s (tw).*

String Replication

For each string *w* and each natural number *i*, the string w^i is defined as:

$$w^0 = \varepsilon$$
$$w^{i+1} = w^i w$$

Example: a^3 = aaa, $(bye)^2$ = byebye, a^0b^3 = bbb

String Reversal: For each string ***w***, the reverse of ***w***, written as w^R, is defined as:

If $|w| = 0$ then $w^R = w = \varepsilon$.
If $|w| \geq 1$ then $\exists a \in \Sigma\ (\exists u \in \Sigma^*\ (w = ua))$, (i.e., the last character of w is a.)
Then define $w^R = au^R$.

Theorem: If w and x are strings, then $(wx)^R = x^R w^R$.
For example, $(\text{nametag})^R = (\text{tag})^R(\text{name})^R = \text{gateman}$.

Proof: The proof is by induction on $|x|$:

Base case: $|x| = 0$. Then $x = \varepsilon$, and $(wx)^R = (w\varepsilon)^R = (w)^R = \varepsilon w^R = \varepsilon^R w^R = x^R w^R$.

Prove: $\forall n \geq 0\ (((|x| = n) \rightarrow ((wx)^R = x^R w^R)) \rightarrow ((|x| = n+1) \rightarrow ((wx)^R = x^R w^R)))$.

Consider any string x, where $|x| = n+1$. Then $x = ua$ for some character a and $|u| = n$. So:

$$\begin{aligned}
(wx)^R &= (w(ua))^R && \text{rewrite } x \text{ as } ua \\
&= ((wu)a)^R && \text{associativity of concatenation} \\
&= a(wu)^R && \text{definition of reversal} \\
&= a(u^R w^R) && \text{induction hypothesis} \\
&= (au^R)w^R && \text{associativity of concatenation} \\
&= (ua)^R w^R && \text{definition of reversal} \\
&= x^R w^R && \text{rewrite } ua \text{ as } x
\end{aligned}$$

3.2. Relations on strings

Substring: A string ***s*** is a substring of a string of t iff *s* occurs contiguously as part of t.

For example: aaa is a substring of aaabbbaaa, aaaaaa is not a substring of aaabbbaaa

Proper Substring: A string r is a proper substring of a string t, iff t is a substring of t and s ≠ t. Every string is a substring (although not a proper substring) of itself. The empty string. e. is a substring of every string.

Prefix: A string s is a prefix of t, iff $\exists x \in \sum^*(t = sx)$. A string s is a proper prefix of a string t iff s is a prefix of t and s≠t. Every string is a prefix (although not a proper prefix) of itself. The empty string ε, is a prefix of every string. For example. the prefixes of abba are: ε, a, ab, abb, abba.

Suffix: A string s is a suffix of t, iff $\exists x \in \sum^*(t = xs)$. A string s is a proper suffix of a string t iff s is a suffix of t and s≠t. Every string is a suffix (although not a proper suffix) of itself. The empty string ε, is a suffix of every string. For example. the prefixes of abba are: ε, a, ba, bba, abba.

4. Languages

A language is a (finite or infinite) set of strings over a finite alphabet ∑. When we are talking about more than one language, we will use the notation $\sum_L$, to mean the alphabet from which the strings in the language L are formed.

Let ∑ = {a, b}. ∑* = {ε, a, b, aa, ab , ba, bb, aaa, aab }.

Some examples of languages over ∑ are:

Φ, {ε}, {a, b}, {ε, a, aa, aaa, aaaa, aaaaa}, {ε, a, aa, aaa, aaaa, aaaaa, ...}

4.1. Techniques for Defining Languages

There are many ways. Since languages are sets. we can define them using any of the set-defining techniques

Ex-1: *All a's Precede All b's,*

L = {w ϵ {a,b}*: an a's precede all b's in w}. The strings ε, a, aa, aabbb, and bb are in *L* . The strings aba, ba, and abc are not in *L*.

Ex-2: ***Strings that end in 'a'***

L = {x : ∃yϵ {a, b}*, (*x* = ya)}. The strings a, aa, aaa, bbaa and ba are in *L*. The strings ε, bab, and bca are not in *L*. *L* consists of all strings that can be formed by taking some string in {a, b}* and concatenating a single a onto the end of it.

Ex-3: ***Empty language***

L = { } = Φ, the language that contains no strings. ***Note:*** L = { ε } the language that contains a single string, ε. Note that *L* is different from Φ.

Ex-4: *Strings of all 'a' s containing zero or more 'a's*

Let *L* = { a^n : $n \geq 0$}. *L* = (ε, a, aa, aaa, aaaa, aaaaa,.....)

Ex-5: We define the following languages in terms of the prefix relation on strings:

L1 = {wϵ{a, b}* : no prefix of *w* contains b}= { *e* , a, aa, aaa, aaaa, aaaaa, aaaaaa, } .

L2 ={wϵ {a, b}*: no prefix of *w* starts with b}={*w* ϵ{a,b}*: the first character of w is a }U{ε}.

L3= {wϵ {a, b}*; every prefix of w starts with b} =Φ. L3 is equal to Φ because ε is a prefix of every string. Since ε does not start with b, no strings meet L3 's requirement.

Languages are sets. So, a computational definition of a language can be given in two ways;

- a **language generator**, which enumerates (lists) the elements of the language, or
- a **language recognizer**, which decides whether or not a candidate string is in the language and returns *True* if it is and *False* if it isn't.

For example, the logical definition. $L = \{x: \exists y \in \{a, b\}^* (x = ya)\}$ can be turned into either a language generator (enumerator) or a language recognizer.

In some cases, when considering an enumerator for a language, we may care about the order in which the elements of *L* are generated. If there exists n total order *D* of the elements of $\sum_L$, then we can use *D* to define on *L* a total order called ***lexicographic order*** (written $<_L$.):

- Shorter strings precede longer ones: $\forall x (\forall y ((|x| < |y|) \textcircled{0} (x <_L y)))$ and
- Of strings that are the same length sort them in dictionary order using *D*.

Let L= {w ϵ {a, b}*; all a's precede all b's}. The lexicographic enumeration of Lis:

ε, a. b. aa. ab. bb. aaa. aab. abb. bbb. aaaa, aaab. aabb. abbb. bbbb. aaaaa ....

4.2. Cardinality of a Language

- Cardinality refers to the number of strings in the language.
- The smallest language over any alphabet is ϕ, whose cardinality is 0.
- The largest language over any alphabet ∑ is ∑*. Suppose that ∑ = Φ, then ∑* = {ε} and |∑*| = l. In general, |∑*| is infinite.

Theorem: If $\Sigma \neq \emptyset$ then Σ^* is countably infinite.

Proof: The elements of Σ^* can be lexicographically enumerated by a straightforward procedure that:

- Enumerates all strings of length 0, then length 1, then length 2, and so forth.
- Within the strings of a given length, enumerates them in dictionary order.

This enumeration is infinite since there is no longest string in Σ^*. By Theorem A.1, since there exists an infinite enumeration of Σ^*, it is countably infinite.

4.4. Functions on Languages

Since languages are sets. all of the standard set operations are well-defined on languages.

Union, **intersection**, **difference** and **complement** are quite useful

Let: $\Sigma = \{a, b\}$.
$L_1 = \{$strings with an even number of a's$\}$.
$L_2 = \{$strings with no b's$\} = \{\varepsilon, a, aa, aaa, aaaa, aaaaa, aaaaaa, \ldots\}$.

$L_1 \cup L_2 = \{$all strings of just a's plus strings that contain b's and an even number of a's$\}$.

$L_1 \cap L_2 = \{\varepsilon, aa, aaaa, aaaaaa, aaaaaaaa, \ldots\}$.

$L_2 - L_1 = \{a, aaa, aaaaa, aaaaaaa, \ldots\}$.

$\neg(L_2 - L_1) = \{$strings with at least one b$\} \cup \{$strings with an even number of a's$\}$.

Concatenation

Let L_1 and L_2 be two languages defined over some alphabet $\sum$. Then their concatenation. written L_1L_2 is:

$$L_1L_2 = \{w \in \Sigma^* : \exists s \in L_1 (\exists t \in L_2 (w = st))\}.$$

Example: Let: L_1 = {cat, dog, mouse, bird}. L_2 = { bone, food}.
L_1L_2 = { catbone, catfood, dogbane, dogfood, mousebone, mousefood, birdbone, birdfood}.

The language {ε} is the **identity for concatenation** of languages. So for all languages L, **L{ε} = {ε}L = L.**

The language Φ is a zero for concatenation of languages. So, for all languages *L*, $L\Phi = \Phi L = \Phi$. That Φ is a zero follows from the definition of the concatenation of two languages as the set consisting of all strings that can he formed by selecting some string 's' from the first language and some string 'l' from the second language and then concatenating them together. There are no ways to select a string from the empty set.

Concatenation on languages is **associative**. So, for all languages L_1L_2 and L_3:

$$((L_1L_2)L_3 = L_1 (L_2L_3)).$$

Reverse

Let L be a language defined over some alphabet $\sum$. Then the reverse of L , written L^R is:
$L^R = \{w \in \sum^* : w = x^R$ for some $x \in L\}$.
In other words, L^R is the set of strings that can be formed by taking some string in L and reversing it

Theorem: If L_1 and L_2 are languages, then $(L_1L_2)^R = L_2^R L_1^R$.

Proof: If x and y are strings, then $\forall x (\forall y ((xy)^R = y^R x^R))$ Theorem 2.1

$(L_1L_2)^R = \{(xy)^R : x \in L_1$ and $y \in L_2\}$	Definition of concatenation of languages
$= \{y^R x^R : x \in L_1$ and $y \in L_2\}$	Lines 1 and 2
$= L_2^R L_1^R$	Definition of concatenation of languages

Kleene Star

Definition: The **Kleene star** denoted by Σ^*, is a unary operator on a set of symbols or strings, Σ, that gives the infinite set of all possible strings of all possible lengths over Σ including ε.

Representation: $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots\ldots$ where Σ^p is the set of all possible strings of length p.

Example: If Σ = {a, b}, Σ^*= {λ, a, b, aa, ab, ba, bb, ...}

Let L = {dog, cat, fish}. Then:
L^* = {ε, dog, cat, fish, dogdog, dogcat, ...,
fishdog, ..., fishcatfish, fishdogfishcat, ...}.

Kleene Closure / Plus

Definition: The set Σ^+ is the infinite set of all possible strings of all possible lengths over Σ excluding ε. $\Sigma^+ = \Sigma^* - \{\varepsilon\}$ Representation:

$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots\ldots$

Example: If Σ = {a, b} , Σ^+ ={ a, b, aa, ab, ba, bb, ...}

Closure: A set S is closed under the operation @ if for every element x & y in S, x@y is also an element of S.

4.5. A Language Hierarchy

A Machine-Based Hierarchy of Language Classes are shown in the diagram.

We have four language classes:

1. **Regular languages**, which can be accepted by some finite state machine.
2. **Context-free languages**, which can be accepted by some pushdown automaton.
3. **Decidable** (or simply D) languages. which can decided by some Turing machine that always halts.
4. **Semi-decidable** (or SD) languages, which can be semi-decided by some Turing machine that halts on all strings in the language.

Each of these classes is a proper subset of the next class, as illustrated in the Figure. As we move outward in the language hierarchy, we have access to tools with greater and expressive power. We can define $A^nB^nC^n$ as a decidable language but not as a context-free or a regular one. These matters because expressiveness generally comes at a price. The price may be: Computational efficiency, decidability and clarity.

- *Computational efficiency:* Finite state machines run in time that is linear in the length of the input string. A general context-free parser based on the idea of a pushdown automaton requires time that grows as the cube of the length of the input string. A

Turing machine may require time that grows exponentially (or faster) with the length of the input string.

- *Decidability:* There exist procedures to answer many useful questions about finite state machines. For example, does an FSM accept some particular string? Is an FSM minimal? Are two FSMs identical? A subset of those questions can be answered for pushdown automata. None of them can be answered for Turing machines.
- *Clarity:* There exist tools that enable designers to draw and analyze finite state machines. Every regular language can also be described using the regular expression pattern language. Every context-free language, in addition to being recognizable by some pushdown automaton, can be described with a context-free grammar

5. Finite State Machines (FSM)

A ***finite state machines*** (or FSM) is a computational device whose input is a string and whose output is one of two values; Accept and Reject. FSMs are also sometimes called ***finite state automata*** or ***FSAs.***

5.1. Deterministic FSM

- We begin by defining the class of FSMs whose behavior is **deterministic**.
- These machines, makes exactly one move at each step
- The move is determined by the current state and the next input character.

Definition: Deterministic Finite State Machine (DFSM) is M: $M = (K, \Sigma, \delta, s, A)$, where:
K is a finite set of states
Σ is an alphabet
$s \in K$ is the initial state
$A \subseteq K$ is the set of accepting states, and
δ is the transition function from $(K \times \Sigma)$ to K

Configuration: A ***Configuration*** of a DFSM M is an element of $K \times \Sigma^*$. Configuration captures the two things that make a difference to M's future behavior: i) its current state, the input that remains to be read.

The ***Initial Configuration*** of a DFSM M, on input w, is (s_M, w) , where s_M is start state of M

The transition function δ defines the operation of a DFSM M one step at a time. δ is set of all pairs of states in M & characters in Σ. (Current State, Current Character) ⓪ New State

Relation 'yields': Yields-in-one-step relates configuration, to configuration-1to configuration- 2 iff M can move from canfiguration-1, to configuration-2 in one step. Let c be any element of Σ and let ***w*** be any element of Σ^*, then,

$$(q_1, cw) \vdash_M (q_2, w) \text{ iff } ((q_1, c), q_2) \in \delta$$

$|-_M^*$ is the reflexive, transitive closure of $|-_M$

Complete vs Incomplete FSM

Complete FSM: A transition is defined for every possible state and every possible character in the alphabet. Note: This can cause FSM to be larger than necessary, but ALWAYS processes the entire string

Incomplete FSM: One which defines a transition for every possible state & every possible character in the alphabet which can lead to an accepting state Note: If no transition is defined, the string is *Rejected*

Computation: A Computation by M is a finite sequence of configurations C_0, C_1, …, C_n for some $n \geq 0$ such that:

- C_0 is an initial configuration,
- C_n is of the form (q, ε), for some state $q \in K_M$
  - ε indicates empty string, entire string is processed & implies a complete DFSM
- $C_0 \vdash_M C_1 \vdash_M C_2 \vdash_M \ldots \vdash_M C_n$.

However, M **Halts** when the last character has to be processed or a next transition is not defined

Acceptance / Rejection

A DFSM M, **Accepts** a string w iff $(s, w) \vdash_M^* (q, \varepsilon)$, for some $q \in A_M$.

A DFSM M, **Rejects** a string w iff $(s, w) \vdash_M^* (q, \varepsilon)$, for some $q \notin A_M$.

The **language accepted by M**, denoted *L*(*M*), is the set of all strings accepted by *M*.

Regular languages

A language is regular iff it is accepted by some DFSM. Some examples are listed below.

- {w ∈{a, b}* | every a is immediately followed by b }.
- {w ∈ {a, b}* | every **a region** in w is of even length}
- binary strings with odd parity.

Designing Deterministic Finite State Machines

Given some language L. how should we go about designing a DFSM to accept L? In general. as in any design task. There is no magic bullet. But there are two related things that it is helpful to think about:

- Imagine any DFSM M that accepts L. As a string w is being read by M, what properties of the part of w that has been seen so far are going to have any bearing on the ultimate answer that M needs to produce? Those are the properties that M needs to record.
- If L is infinite but M has a finite number of states, strings must "cluster". In other words, multiple different strings will all drive M to the same state. Once they have done that, none of their differences matter anymore. If they've driven M to the same state, they share a fate. No matter what comes next, either all of them cause M to accept or all of them cause M to reject.

6. Finite State Transducers

So far, we have used finite state machines as language recognizers. All we have cared about, in analyzing a machine M, is whether or not M ends in an accepting state. But it is a simple matter to augment our finite state model to allow for output at each step of a machine's operation. Often, once we do that, we may cease to care about whether M actually accepts any strings. Many finite state transducers are loops that simply run forever, processing inputs.

An automaton that produces outputs based on current input and/or previous state is called a **transducer**. Transducers can be of two types:

- **Moore Machine** The output depends only on the current state.

Mealy Machine The output depends both on the current state and the current input.

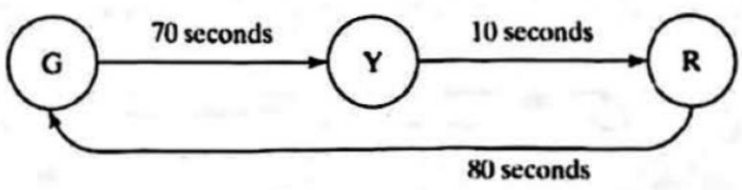
Moore Machine

One simple kind of finite state transducer associates an output with each state of a machine M. That output is generated whenever M enters the associated state. Deterministic finite state transducers of this sort are called Moore machines, after their inventor Edward Moore. A ***Moore machine*** M is a seven-tuple $(K, \Sigma, O, \delta, D, s, A)$, where:

- K is a finite set of states,
- Σ is an input alphabet,
- O is an output alphabet,
- $s \in K$ is the start state,
- $A \subseteq K$ is the set of accepting states (although for some applications this designation is not important),
- δ is the transition function. It is function from $(K \times \Sigma)$ to (K), and
- D is the display or output function. It is a function from (K) to (O^*).

A Moore machine M computes a function $f(w)$ iff, when it reads the input string w, its output sequence is $f(w)$.

Example: Traffic light



Mealy Machine

A different definition for a deterministic finite state transducer permits each machine to output any finite sequence of symbols as it makes each transition (in other words, as it reads each symbol of its input). FSMs that associate outputs with transitions

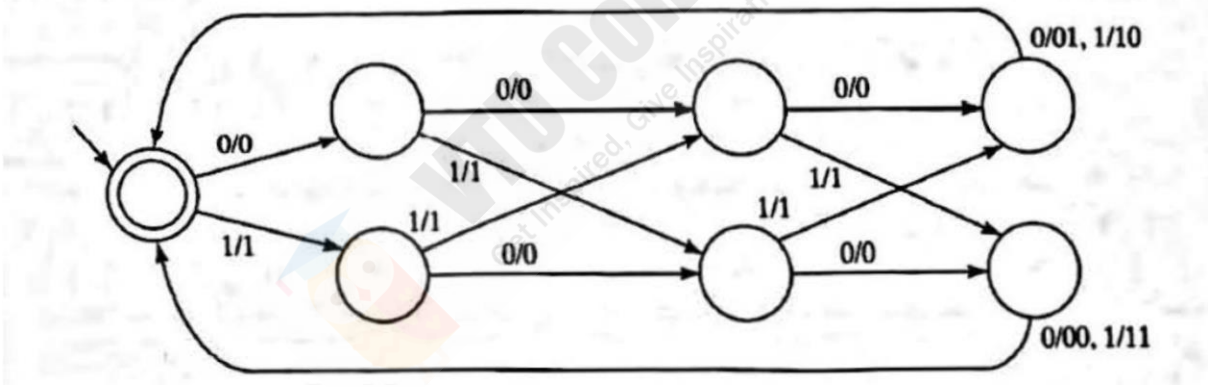
are called Mealy machines, after their inventor George Mealy. A ***Mealy machine*** M is a six-tuple $(K, \Sigma, O, \delta, s, A)$, where:

- K is a finite set of states,
- Σ is an input alphabet,
- O is an output alphabet,
- $s \in K$ is the start state,
- $A \subseteq$ is the set of accepting states, and
- δ is the transition function. It is a function from $(K \times \Sigma)$ to $(K \times O^*)$.

A Mealy machine M computes a function $f(w)$ iff, when it reads the input string w, its output sequence is $f(w)$.

Example: Generating Parity bits

The following Mealy machine adds an odd parity bit after every four binary digits that it reads. We will use the notation a/b on an arc to mean that the transition may be followed if the input character is a. If it is followed, then the string b will be generated.



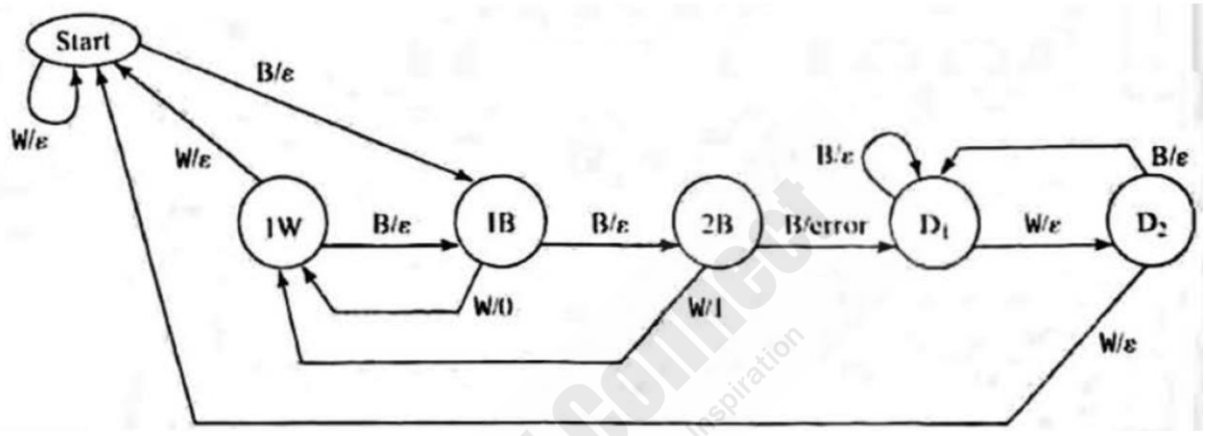
Example: Bar Code reader

Bar codes are ubiquitous. We consider here a Simplification: a bar code system that encodes just binary numbers. Imagine a bar code such as:

It is composed of columns, each of the same width. A column can he either white or black. If two black columns occur next to each other, it will look to us like a single-wide-black column, but the reader will see two adjacent black columns of the standard width. The job of the white columns is to delimit the black ones. A single black column encodes 0. A double black column encodes 1.

We can build a finite state transducer to read such a bar code and output a string of binary digits. We will represent a black bar with the symbol B and a white bar with the symbol W. The input to the transducer will be a sequence of those symbols corresponding to reading the bar code left to right. We'll assume that every correct bar code starts with a black column, so

white space ahead of the first black column is ignored. We will also assume that after every complete bar code there are at least two white columns. So, the reader should, at that point, reset to be ready to read the next code. If the reader sees three or more black columns in a row, it must indicate an error and stay in its error state until it is reset by seeing two while columns.



Interpreters for finite stale transducers can be built using techniques similar to the ones that we used to interpreters for finite state machines.

7. Bidirectional Transducers

A process that reads an input string and constructs a corresponding output string can be described in a variety of different ways. Why should we choose the finite state transducer model? One reason is that it provides a declarative, rather than a procedural, way to describe the relationship between inputs and outputs, such a declarative model can then be run in two directions.

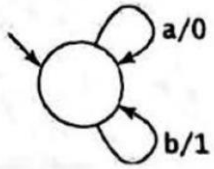
For example, to read an English text requires transforming a word like "liberties" into the root word "liberty" and the affix PLURAL. To generate an English text requires transforming a root word like "liberty" and the semantic marker “PLURAL” into the surface word "liberties". If we could specify, in a single declarative model, the relationship between surface words (the ones we see in text) and underlying root words and affixes, we could use it for either application.

The facts about English spelling rules and morphological analysis can be described with a bidirectional finite state transducer.

If we expand the definition of a Mealy machine to allow non-determinism, then any of these bidirectional

processes can be represented. A nondeterministic Mealy machine can be thought of as defining a relation between one set of strings (for example, English surface words) and a second set of strings (for example. English underlying root words. along with affixes). It is possible that we will need a machine that is nondeterministic in one or both directions because the relationship between the two sets may not be able to be described as a function.

Example: When we define a regular language, it doesn't matter what alphabet we use. Anything that is true of a language L defined over the alphabet {a,b} will also be true of the language L' that contains exactly the strings in L except that every a has been replaced by a 0 and every b has been replaced by a L We can build a simple bidirectional transducer that can convert strings in L to strings in L' and vice-versa.



Of course. the real power of bidirectional finite state transducers comes from their ability to model more complex processes.

MODULE II:

1. **Regular Expression**
2. **Equivalence of Re and NFA**
3. **DFA**
4. **Pumping lemma for regular languages**
5. **Grammars**
6. **Parse Trees**

1. Regular Expressions

- A regular expression is used to specify a language, and it does so precisely.

- Regular expressions are very intuitive.
- Regular expressions are very useful in a variety of contexts.
- Given a regular expression, an NFA-ε can be constructed from it automatically.
- Thus, so can an NFA, a DFA, and a corresponding program, all automatically!

Definition:

- Let Σ be an alphabet. The regular expressions over Σ are:

– Ø Represents the empty set { }
– ε Represents the set {ε}
– a Represents the set {a}, for any symbol a in Σ

Let r and s be regular expressions that represent the sets R and S, respectively.

– r+s Represents the set R U S (precedence 3)
– rs Represents the set RS (precedence 2)
– r^* Represents the set R* (highest precedence)
– (r) Represents the set R (not an op, provides precedence)

- If r is a regular expression, then L(r) is used to denote the corresponding language.

- **Examples:** Let Σ = {0, 1}

(0 + 1)*	All strings of 0's and 1's
0(0 + 1)*	All strings of 0's and 1's, beginning with a 0
(0 + 1)*1	All strings of 0's and 1's, ending with a 1
(0 + 1)*0(0 + 1)*	All strings of 0's and 1's containing at least one 0
(0 + 1)*0(0 + 1)*0(0 + 1)*	All strings of 0's and 1's containing at least two 0's
(0 + 1)*01*01*	All strings of 0's and 1's containing at least two 0's
(1 + 01*0)*	All strings of 0's and 1's containing an even number of 0's

1*(01*01*)*	All strings of 0's and 1's containing an even number of 0's
(1*01*0)*1*	All strings of 0's and 1's containing an even number of 0's

Identities:

Øu = uØ = Ø Multiply by 0

- u + Ø = u
- u + u = u

8. u* = (u*)*

9. u(v+w) = uv+uw
10. (u+v)w = uw+vw
11. (uv)*u = u(vu)*
12. (u+v)* = (u*+v)*
    = u*(u+v)*
    = (u+vu*)*
    = (u*v*)*
    = u*(vu*)*
    = (u*v)*u*

2. Equivalence of Regular Expressions and NFA-ε

- **Note:** Throughout the following, keep in mind that a string is accepted by an NFA-ε if there exists a path from the start state to a final state.
- **Lemma 1:** Let r be a regular expression. Then there exists an NFA-ε M such that L(M) = L(r). Furthermore, M has exactly one final state with no transitions out of it.
- **Proof:** (by induction on the number of operators, denoted by OP(r), in r).
- **Basis:** OP(r) = 0

Then r is either Ø, ε, or **a**, for some symbol **a** in Σ

- **Inductive Hypothesis:** Suppose there exists a k ≥ 0 such that for any regular expression r where 0 ≤ OP(r) ≤ k, there exists an NFA-ε such that L(M) = L(r). Furthermore, suppose that M has exactly one final state.

- **Inductive Step:** Let r be a regular expression with k + 1 operators (OP(r) = k + 1), where k + 1 >= 1.

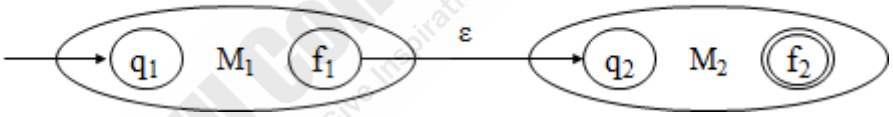
Case 1) r = r_1 + r_2

Since OP(r) = k +1, it follows that 0<= OP(r_1), OP(r_2) <= k. By the inductive hypothesis there exist NFA-ε machines M_1 and M_2 such that L(M_1) = L(r_1) and L(M_2) = L(r_2). Furthermore, both M_1 and M_2 have exactly one final state.

Case 2) r = r_1r_2

Since OP(r) = k+1, it follows that 0<= OP(r_1), OP(r_2) <= k. By the inductive hypothesis there exist NFA-ε machines M_1 and M_2 such that L(M_1) = L(r_1) and L(M_2) = L(r_2). Furthermore, both M_1 and M_2 have exactly one final state.

Construct M as:



Case 3) r = r_1*

Since OP(r) = k+1, it follows that 0<= OP(r_1) <= k. By the inductive hypothesis there exists an NFA-ε machine M_1 such that L(M_1) = L(r_1). Furthermore, M_1 has exactly one final state.

- **Example:**

Problem: Construct FA equivalent to RE, r = 0(0+1)*

Solution:
r = r1r2
r1 = 0
r2 = (0+1)*
r2 = r3*
r3 = 0+1
r3 = r4 + r5
r4 = 0
r5 = 1

Transition graph:

3. Definitions Required to Convert a DFA to a Regular Expression

- Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA with state set $Q = \{q_1, q_2, \ldots, q_n\}$, and define:
  $R_{i,j} = \{ x \mid x \text{ is in } \Sigma^* \text{ and } \delta(q_i,x) = q_j\}$
  $R_{i,j}$ is the set of all strings that define a path in M from q_i to q_j.

- **Lemma 2:** Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA. Then there exists a regular expression r such that $L(M) = L(r)$.

if i=j.Case 1) No transitions from q_i to q_j and i != j $r^0_{i,j} = \emptyset$

Case 2) At least one ($m \geq 1$) transition from q_i to q_j and i != j

$$r^0_{i,j} = a_1 + a_2 + a_3 + \ldots + a_m \quad \text{where } \delta(q_i, a_p) = q_j, \text{ for all } 1 \leq p \leq m$$

Case 3) No transitions from q_i to q_j and i = j

$$r^0_{i,j} = \varepsilon$$

Case 4) At least one ($m \geq 1$) transition from q_i to q_j and i = j

$$r^0_{i,j} = a_1 + a_2 + a_3 + \ldots + a_m + \varepsilon \quad \text{where } \delta(q_i, a_p) = q_j \text{ for all } 1 \leq p \leq m$$

- **Inductive Hypothesis:**
  Suppose that $R^{k-1}_{i,j}$ can be represented by the regular expression $r^{k-1}_{i,j}$ for all $1 \leq i,j \leq n$, and some $k \geq 1$.

- **Inductive Step:**
  Consider $R^k_{i,j} = R^{k-1}_{i,k} (R^{k-1}_{k,k})^* R^{k-1}_{k,j} \cup R^{k-1}_{i,j}$. By the inductive hypothesis there exist regular expressions $r^{k-1}_{i,k}$, $r^{k-1}_{k,k}$, $r^{k-1}_{k,j}$, and $r^{k-1}_{i,j}$ generating $R^{k-1}_{i,k}$, $R^{k-1}_{k,k}$, $R^{k-1}_{k,j}$, and $R^{k-1}_{i,j}$, respectively. Thus, if we let

$$r^k_{i,j} = r^{k-1}_{k,k}(r^{k-1}_{k,k})^* r^{k-1}_{k,j} + r^{k-1}_{i,j}$$

  then $r^k_{i,j}$ is a regular expression generating $R^k_{i,j}$,i.e., $L(r^k_{i,j}) = R^k_{i,j}$.

- Finally, if $F = \{q_{j1}, q_{j2}, \ldots, q_{jr}\}$, then
  $r^n_{1,j1} + r^n_{1,j2} + \ldots + r^n_{1,jr}$
  is a regular expression generating L(M).•

4. Pumping Lemma for Regular Languages

- Pumping Lemma relates the size of string accepted with the number of states in a DFA
- What is the largest string accepted by a DFA with *n* states?
- Suppose there is no loop?
  Now, if there is a loop, what type of strings are accepted *via* the loop(s)?
- **Lemma:** (the pumping lemma)

  Let M be a DFA with |Q| = n states. If there exists a string x in L(M), such that $|x| \geq n$, then there exists a way to write it as x = uvw, where u,v, and w are all in Σ^* and:

  - $1 \leq |uv| \leq n$
  - $|v| \geq 1$
  - such that, the strings uv^iw are also in L(M), for all $i \geq 0$

- Let:
  - $u = a_1 \ldots a_s$
  - $v = a_{s+1} \ldots a_t$
- Since $0 \leq s < t \leq n$ and $uv = a_1 \ldots a_t$ it follows that:
  - $1 \leq |v|$ and therefore $1 \leq |uv|$
  - $|uv| \leq n$ and therefore $1 \leq |uv| \leq n$
- In addition, let:
  - $w = a_{t+1} \ldots a_m$
- It follows that $uv^iw = a_1 \ldots a_s (a_{s+1} \ldots a_t)^i a_{t+1} \ldots a_m$ is in L(M), for all $i \geq 0$.

  In other words, when processing the accepted string x, the loop was traversed once, but could have been traversed as many times as desired, and the resulting string would still be accepted.

4.1 Closure Properties of Regular Languages

1. Closure Under Union
- If L and M are regular languages, so is L ∪ M.
- Proof: Let L and M be the languages of regular expressions R and S, respectively.
- Then R+S is a regular expression whose language is L ∪ M.

2. Closure Under Concatenation and Kleene Closure
- RS is a regular expression whose language is LM.
- R* is a regular expression whose language is L*.

3. Closure Under Intersection
- If L and M are regular languages, then so is L ∩ M.
- Proof: Let A and B be DFA's whose languages are L and M, respectively.

4. Closure Under Difference
- If L and M are regular languages, then so is L – M = strings in L but not M.
- Proof: Let A and B be DFA's whose languages are L and M, respectively.

5. Closure Under Complementation
- The complement of language L (w.r.t. an alphabet Σ such that Σ* contains L) is Σ* – L.
- Since Σ* is surely regular, the complement of a regular language is always regular.

6. Closure Under Homomorphism
- If L is a regular language, and h is a homomorphism on its alphabet, then h(L) = {h(w) | w is in L} is also a regular language.

5. Grammar

- **Definition:** A grammar G is defined as a 4-tuple, G = (V, T, S, P)
  Where,
  - V is a finite set of objects called variables,
  - T is a finite set of objects called terminal symbols,
  - S ∈ V is a special symbol called start variable,
  - P is a finite set of productions.

  Assume that V and T are non-empty and disjoint.

- **Example:**
  Consider the grammar G = ({S}, {a, b}, S, P) with P given by
  S → aSb, S →ε_.
  For instance, we have S ⇒ aSb ⇒ aaSbb ⇒ aabb.
  It is not hard to conjecture that L(G) = $\{a^n b^n \mid n \geq 0\}$.

5.1 Right, Left-Linear Grammar

- **Right-linear Grammar:** A grammar G = (V, T, S, P) is said to be right-linear if all productions are of the form:
  A → xB,
  A → x,
  Where A, B ∈ V and x ∈ T*.

  - **Example#1:**

    S* → *abS* | *a is an example of a right-linear grammar.

    - Can you figure out what language it generates?
    - ***L* = {*w* ∈{*a,b*}* | *w***
      Contains alternating ***a***'s and ***b***'s , begins with an ***a***, and ends with a ***b}***

      ∪ {*a*}

    - ***L*((*ab*)**a*)**

- **Left-linear Grammar:** A grammar G = (V, T, S, P) is said to be left-linear if all productions are of the form:
  A → Bx,
  A → x,
  Where A, B ∈ V and x ∈ T*.
  - **Example#2:**

    S* → *Aab
    A* → *Aab* | *aB
    B* → *a
    is an example of a left-linear grammar.

    - Can you figure out what language it generates?
    - ***L* = {*w* Î {*a,b*}* | *w*** is ***aa*** followed by at least one set of alternating ***ab***'s**}**

    - ***L*(*aaab*(*ab*)*)**

  - **Example#3:**

    Consider the grammar
    S* → *A
    ***A* → *aB* | λ**
    B* → *Ab
    This grammar is NOT regular.

- No "mixing and matching" left- and right-recursive productions.

5.2 Regular Grammar

- A linear grammar is a grammar in which at most one variable can occur on the right side of any production without restriction on the position of this variable.

- An example of linear grammar is G = ({S, S1, S2}, {a, b}, S, P) with
  S → S1ab,
  S1 → S1ab | S2,
  S2 → a.

- A **regular grammar** is one that is either right-linear or left-liner.

5.3 Testing Equivalence of Regular Languages

- Let L and M be reg langs (each given in some form).

  To test if L = M

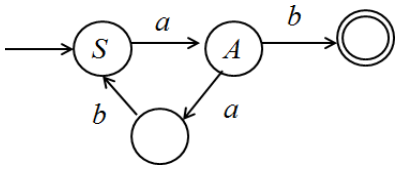
  1. Convert both L and M to DFA's.
  2. Imagine the DFA that is the union of the two DFA's (never mind there are two start states)
  3. If TF-algo says that the two start states are distinguishable, then L 6= M, otherwise, L = M.

  We can “see" that both DFA accept L(ε+(0+1)*0). The result of the TF-algo is

  Therefore the two automata are equivalent.

5.4 Regular Grammars and NFA's

- It's not hard to show that regular grammars generate and nfa's accept the same class of languages: the regular languages!
- It's a long proof, where we must show that
  - Any finite automaton has a corresponding left- or right-linear grammar,
  - And any regular grammar has a corresponding nfa.
- **Example:**
  - We get a feel for this by example.



Let $S \rightarrow aA \quad A \rightarrow abS \mid b$CONTEXT FREE-GRAMMAR

- **Definition:** Context-Free Grammar (CFG) has 4-tuple: G = (V, T, P, S)

  Where,

V	-	A finite set of variables or *non-terminals*
T	-	A finite set of *terminals* (V and T do not intersect)
P	-	A finite set of *productions*, each of the form A –> α, Where A is in V and α is in (V U T)* Note: that α may be ε.
S	-	A starting non-terminal (S is in V)

- **Example#1 CFG:**

  G = ({S}, {0, 1}, P, S)
  P:

  (1) S –> 0S1 or just simply S –> 0S1 | ε
  (2) S –> ε

- **Example Derivations:**

S	=> 0S1	(1)
S	=> ε	(2)
	=> 01	(2)
S	=> 0S1	(1)
	=> 00S11	(1)
	=> 000S111	(1)
	=> 000111	(2)

- Note that G "generates" the language $\{0^k1^k | k>=0\}$

6. Derivation (or Parse) Tree

- **Definition:** Let G = (V, T, P, S) be a CFG. A tree is a derivation (or parse) tree if:
  - Every vertex has a label from V U T U {ε}
  - The label of the root is S
  - If a vertex with label A has children with labels $X_1, X_2,\ldots, X_n$, from left to right, then

    $$A \rightarrow X_1, X_2,\ldots, X_n$$

    must be a production in P
  - If a vertex has label ε, then that vertex is a leaf and the only child of its' parent

- More Generally, a derivation tree can be defined with any non-terminal as the root.

Definition: A derivation is *leftmost (rightmost)* if at each step in the derivation a production is applied to the leftmost (rightmost) non-terminal in the sentential form.

- The first derivation above is **leftmost**, second is **rightmost** and the third is neither.

MODULE III:

1. **Context Free Grammar**
2. **Minimization of Context Free Grammar**
3. **Chomsky Normal Form**
4. **Pumping Lemma for Context-Free Languages**
5. **Pushdown Automata (PDA)**

1. Ambiguity in Context Free Grammar

- **Definition:** Let G be a CFG. Then G is said to be ambiguous if there exists an x in L(G) with >1 leftmost derivations. Equivalently, G is said to be ambiguous if there exists an x in L(G) with >1 parse trees, or >1 rightmost derivations.
- Note: Given a CFL L, there may be more than one CFG G with L = L(G). Some ambiguous and some not.
- Definition: Let L be a CFL. If every CFG G with L = L(G) is ambiguous, then L is inherently ambiguous.
- **Example:** Consider the string aaab and the preceding grammar.
- The string has two left-most derivations, and therefore has two distinct parse trees and is ambiguous .

1.1 Eliminations of Useless Symbols

- **Definition:**
  Let $G = (V, T, S, P)$ be a context-free grammar. A variable $A \in V$ is said to be useful if and only if there is at least one $w \in L(G)$ such that

  $$S \Rightarrow^* xAy \Rightarrow^* w$$
  with $x, y \in (V \cup T)^*$.

  In words, a variable is useful if and only if it occurs in at least on derivation. A variable that is not useful is called useless. A production is useless if it involves any useless variable
- For a grammar with productions

$S \rightarrow aSb \mid \lambda \mid A$
$A \rightarrow aA$

A is useless variable and the production $S \rightarrow A$ plays no role since A cannot be eventually transformed into a terminal string; while A can appear in a sentential form derived from S, this sentential form can never lead to sentence!

Hence, removing $S \rightarrow A$ (and $A \rightarrow aA$) does not change the language, but does simplify the grammar.

- For a grammar with productions

$S \rightarrow A$
$A \rightarrow aA \mid \lambda$
$B \rightarrow bA$

B is useless so is the production $B \rightarrow bA$! Observe that, even though a terminal string can be derived from B, there is no way to get to B from S, i.e. cannot achieve $S \Rightarrow^* xBy$.

- **Example:**
Eliminate useless symbols and productions from $G = (V, T, S, P)$, where $V = \{S, A, B, C\}$, $T = \{a, b\}$ and P consists of

$S \rightarrow aS \mid A \mid C$
$A \rightarrow a$
$B \rightarrow aa$
$C \rightarrow aCb$

First, note that the variable C cannot lead to any terminal string, we can then remove C and its associated productions, we get G_1 with $V_1 = \{S, A, B\}$, $T_1 = \{a\}$ and P_1 consisting of

$S \rightarrow aS \mid A$
$A \rightarrow a$
$B \rightarrow aa$

Next, we identify variables that cannot be reached from the start variable. We can create a dependency graph for V_1. For a context-free grammar, a dependency graph has its vertices labeled with variables with an edge between any two vertices I and J if there is a production of the form

$I \rightarrow xJy$

Consequently, the variable B is shown to be useless and can be removed together with its associated production.

The resulting grammar $G' = (V', T', S, P')$ is with $V' = \{S, A\}$, $T' = \{a\}$ and P' consisting of

$$S \rightarrow aS \mid A$$
$$A \rightarrow a$$

1.2 Eliminations of λ-Production

- **Definition :**
  a) Any production of a context-free grammar of the form
  $$A \rightarrow \lambda$$
  is called a λ-production.

  b) Any variable A for which the derivation
  $$A \Rightarrow^* \lambda$$
  is possible is called nullable.

- If a grammar contains some λ-productions or nullable variables but does not generate the language that contains an empty string, the λ-productions can be removed!

- **Example:**
  Consider the grammar, G with productions
  $$S \rightarrow aS_1b$$
  $$S_1 \rightarrow aS_1b \mid \lambda$$
  $L(G) = \{a^n b^n \mid n \geq 1\}$ which is a λ-free language. The λ-production can be removed after adding new productions obtained by substituting λ for S_1 on the right hand side.

  We get an equivalent G' with productions
  $$S \rightarrow aS_1b \mid ab$$
  $$S_1 \rightarrow aS_1b \mid ab$$

- **Theorem:**
  Let G be any context-free grammar with $\lambda \notin L(G)$. There exists an equivalent grammar G' without λ-productions.

  Proof :
  Find the set V_N of all nullable variables of G
  1. For all productions $A \rightarrow \lambda$, put A in V_N
  2. Repeat the following step until no further variables are added to V_N:
  For all productions
  $$B \rightarrow A_1A_2 \ldots A_n$$

  where $A_1, A_2, \ldots, A_n$ are in V_N, put B in V_N.

  With the resulting V_N, P' can be constructed by looking at all productions in P of the form
  $$A \rightarrow x_1x_2 \ldots x_m,\ m \geq 1$$

where each $x_i \in V \cup T$.

For each such production of P, we put in P' the production plus all productions generated by replacing nullable variables with λ in all possible combinations. However, if all x_i are nullable, the resulting production $A \rightarrow \lambda$ is not put in P'.

- **Example:**

For the grammar G with

$S \rightarrow ABaC$
$A \rightarrow BC$
$B \rightarrow b \mid \lambda$
$C \rightarrow D \mid \lambda$
$D \rightarrow d$

the nullable variables are A, B, and C.

The equivalent grammar G' without λ-productions has P' containing

$S \rightarrow ABaC \mid BaC \mid AaC \mid ABa \mid aC \mid Ba \mid Aa \mid a$
$A \rightarrow BC \mid C \mid B$
$B \rightarrow b$
$C \rightarrow D$
$D \rightarrow d$

1.3 Eliminations of MODULE-Production

- **Definition:**
Any production of a context-free grammar of the form

$A \rightarrow B$

where $A, B \in V$ is called a MODULE-production.

- **Theorem:**
Let $G = (V, T, S, P)$ be any context-free grammar without λ-productions. There exists a context-free grammar $G' = (V', T', S, P')$ that does not have any MODULE-productions and that is equivalent to G.

Proof:
First of all, Any MODULE-production of the form $A \rightarrow A$ can be removed without any effect. We then need to consider productions of the form $A \rightarrow B$ where A and B are different variables.

Straightforward replacement of B (with $x_1 = x_2 = \lambda$) runs into a problem when we have

$A \rightarrow B$
$B \rightarrow A$

We need to find for each A, all variables B such that

$A \Rightarrow^* B$

This can be done via a dependency graph with an edge (I, J) whenever the grammar G has a MODULE-production $I \to J$; $A \Rightarrow^* B$ whenever there is a walk from A to B in the graph.

The new grammar G' is generated by first putting in P' all non-MODULE-productions of P. Then, for all A and B with $A \Rightarrow^* B$, we add to P'

$$A \to y_1 \mid y_2 \mid \ldots \mid y_n$$

where $B \to y_1 \mid y_2 \mid \ldots \mid y_n$ is the set of all rules in P' with B on the left. Not that the rules are taken from P', therefore, none of y_i can be a single variable! Consequently, no MODULE- productions are created by this step.

- **Example:**

Consider a grammar G with

$$S \to Aa \mid B$$
$$A \to a \mid bc \mid B$$
$$B \to A \mid bb$$

We have $S \Rightarrow^* A$, $S \Rightarrow^* B$, $A \Rightarrow^* B$ and $B \Rightarrow^* A$.

First, for the set of original non-MODULE-productions, we have

$$S \to Aa$$
$$A \to a \mid bc$$
$$B \to bb$$

We then add the new rules

$$S \to a \mid bc \mid bb$$
$$A \to bb$$
$$B \to a \mid bc$$

We finally obtain the equivalent grammar G' with P' consisting of

$$S \to Aa \mid a \mid bc \mid bb$$
$$A \to a \mid bc \mid bb$$
$$B \to bb \mid a \mid bc$$

Notice that B and its associate production become useless.

2 Minimization of Context Free Grammar

- **Theorem:**

Let L be a context-free language that does not contain λ. There exists a context-free grammar that generates L and that does not have any useless productions, λ-productions or MODULE-productions.

Proof:
We need to remove the undesirable productions using the following sequence of steps.

1. Remove λ-productions
2. Remove MODULE-productions
3. Remove useless productions

3. Chomsky Normal Form

- **Definition:**
A context-free grammar is in Chomsky normal form if all productions are of the form

$$A \rightarrow BC$$

or

$$A \rightarrow a$$

where $A, B, C \in V$, and $a \in T$.

Note: that the number of symbols on the right side of productions is strictly limited; not more than two symbols.

- **Example:**
The following grammar is in Chomsky normal form.

$$S \rightarrow AS \mid a$$
$$A \rightarrow SA \mid b$$

On the other hand, the grammar below is not.

$$S \rightarrow AS \mid AAS$$
$$A \rightarrow SA \mid aa$$

- **Theorem:**
Any context-free grammar $G = (V, T, S, P)$ with $\lambda \notin L(G)$ has an equivalent grammar $G' = (V', T', S, P')$ in Chomsky normal form.

Proof:
First we assume (based on previous Theorem) without loss of generality that G has no λ-productions and no MODULE-productions. Then, we show how to construct G' in two steps.

Step 1:

Construct a grammar $G_1 = (V_1, T, S, P_1)$ from G by considering all productions in P of the form

$$A \rightarrow x_1x_2\ldots x_n$$

Where each x_i is a symbol either in V or in T.

Note that if $n = 1$, x_1 must be a terminal because there is no MODULE-productions in G. In this case, put the production into P_1.

If $n \geq 2$, introduce new variables B_a for each $a \in T$. Then, for each production of the form $A \rightarrow x_1x_2\ldots x_n$, we shall remove all terminals from productions whose right side has length greater than one

This is done by putting into P_1 a production

$$A \rightarrow C_1C_2\ldots C_n$$

Where

$$C_i = x_i \text{ if } x_i \in V$$

And

$$C_i = B_a \text{ if } x_i = a$$

And, for every B_a, we also put into P_1 a production

$$B_a \rightarrow a$$

As a consequence of Theorem 6.1, it can be claimed that

$$L(G_1) = L(G)$$

Step 2:

The length of right side of productions is reduced by means of additional variables wherever necessary. First of all, all productions with a single terminal or two variables ($n = 2$) are put into P'. Then, for any production with $n > 2$, new variables D_1, D_2, … are introduced and the following productions are put into P'.

$$A \rightarrow C_1D_1$$
$$D_1 \rightarrow C_2D_2$$
$$\ldots$$
$$D_{n-2} \rightarrow C_{n-1}C_n$$

G' is clearly in Chomsky normal form.

- **Example:**

Convert to Chomsky normal form the following grammar G with productions.

$$S \rightarrow ABa$$
$$A \rightarrow aab$$
$$B \rightarrow Ac$$

Solution:

Step 1:

New variables B_a, B_b, B_c are introduced and a new grammar G_1 is obtained.

$$S \rightarrow ABB_a$$
$$A \rightarrow B_aB_aB_b$$
$$B \rightarrow AB_c$$
$$B_a \rightarrow a$$
$$B_b \rightarrow b$$
$$B_c \rightarrow c$$

Step 2:

Additional variables are introduced to reduce the length of the first two productions making them into the normal form, we finally obtain G'.

$S \rightarrow AD_1$
$D_1 \rightarrow BB_a$
$A \rightarrow B_aD_2$
$D_2 \rightarrow B_aB_b$
$B \rightarrow AB_c$
$B_a \rightarrow a$
$B_b \rightarrow b$
$B_c \rightarrow c$

Greibach normal form

- **Definition:**
  A context-free grammar is said to be in Greibach normal form if all productions have the form

  $$A \rightarrow ax$$

  where $a \in T$ and $x \in V^*$

  Note that the restriction here is not on the number of symbols on the right side, but rather on the positions of the terminals and variables.

- **Example:**
  The following grammar is not in Greibach normal form.

  $S \rightarrow AB$
  $A \rightarrow aA \mid bB \mid b$
  $B \rightarrow b$

  It can, however, be converted to the following equivalent grammar in Greibach normal form.

  $S \rightarrow aAB \mid bBB \mid bB$
  $A \rightarrow aA \mid bB \mid b$
  $B \rightarrow b$

- **Theorem:**
  For every context-free grammar G with $\lambda \notin L(G)$, there exists an equivalent grammar G' in Greibach normal form.

Conversion

- Convert from Chomsky to Greibach in two steps:
  1. From Chomsky to intermediate grammar
     a) Eliminate direct left recursion
     b) Use $A \rightarrow uBv$ rules transformations to improve references (explained later)

2. From intermediate grammar into Greibach

1.a) Eliminate direct left recursion

Step1:

- Before
  $A \rightarrow A\underline{a} \mid \boldsymbol{b}$
- After
  $A \rightarrow \boldsymbol{b}Z \mid \boldsymbol{b}$
  $Z \rightarrow \underline{a}Z \mid \underline{a}$

- Remove the rule with direct left recursion, and create a new one with recursion on the right

Step2:

- Before
  $A \rightarrow A\underline{a} \mid A\underline{b} \mid \boldsymbol{b} \mid \boldsymbol{c}$
- After
  $A \rightarrow \boldsymbol{b}\underline{Z} \mid \boldsymbol{c}\underline{Z} \mid \boldsymbol{b} \mid \boldsymbol{c}$
  $Z \rightarrow \underline{a}Z \mid \underline{b}Z \mid \underline{a} \mid \underline{b}$
- Remove the rules with direct left recursion, and create new ones with recursion on the right

Step3:

- Before
  $A \rightarrow A\underline{B} \mid \boldsymbol{BA} \mid \boldsymbol{a}$
  $B \rightarrow b \mid c$
- After
  $A \rightarrow \boldsymbol{BA}\underline{Z} \mid \boldsymbol{a}\underline{Z} \mid \boldsymbol{BA} \mid \boldsymbol{a}$
  $Z \rightarrow \underline{B}Z \mid \underline{B}$
  $B \rightarrow b \mid c$

Transform $A \rightarrow uBv$ rules

- Before
  $A \rightarrow uBb$
  $B \rightarrow w_1 \mid w_1 \mid \ldots \mid w_n$
- After
  Add $A \rightarrow uw_1b \mid uw_1b \mid \ldots \mid uw_nb$
  Delete $A \rightarrow uBb$

Background Information for the Pumping Lemma for Context-Free Languages

- **Definition:** Let G = (V, T, P, S) be a CFL. If every production in P is of the form

  $A \rightarrow BC$

  or $A \rightarrow a$

  where A, B and C are all in V and a is in T, then G is in Chomsky Normal Form (CNF).

- **Example:**

  $S \rightarrow AB \mid BA \mid aSb$
  $A \rightarrow a$
  $B \rightarrow b$

- **Theorem:** Let L be a CFL. Then L – {ε} is a CFL.

- **Theorem:** Let L be a CFL not containing {ε}. Then there exists a CNF grammar G such that L = L(G).

- **Definition:** Let T be a tree. Then the height of T, denoted h(T), is defined as follows:
  - If T consists of a single vertex then h(T) = 0
  - If T consists of a root r and subtrees T_1, T_2, … T_k, then h(T) = $\max_i\{h(T_i)\} + 1$

- **Lemma:** Let G be a CFG in CNF. In addition, let w be a string of terminals where A=>*w and w has a derivation tree T. If T has height h(T)≥1, then $|w| \leq 2^{h(T)-1}$.

- **Proof:** By induction on h(T) (exercise).

- **Corollary:** Let G be a CFG in CNF, and let w be a string in L(G). If $|w| \geq 2^k$, where k ≥ 0, then any derivation tree for w using G has height at least k+1.

- **Proof:** Follows from the lemma.

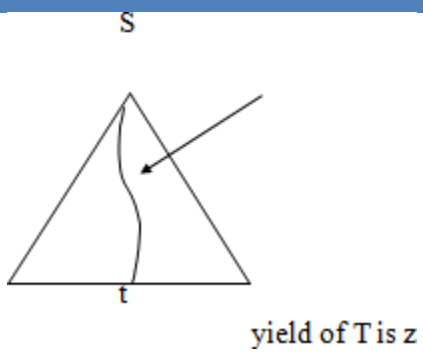
4. Pumping Lemma for Context-Free Languages

- **Lemma:**
  Let G = (V, T, P, S) be a CFG in CNF, and let $n = 2^{|V|}$. If z is a string in L(G) and |z| ≥ n, then there exist strings u, v, w, x and y in T* such that z=uvwxy and:
  - |vx| ≥ 1 (i.e., |v| + |x| ≥ 1)
  - |vwx| ≤ n
  - uv^iwx^iy is in L(G), for all i ≥ 0

- **Proof:**
  Since $|z| \geq n = 2^k$, where k = |V|, it follows from the corollary that any derivation tree for z has height at least k+1.

  By definition such a tree contains a path of length at least k+1.

  Consider the longest such path in the tree:



yield of T is z

Such a path has:
- Length ≥ k+1 (i.e., number of edges in the path is ≥ k+1)
- At least k+2 nodes
- 1 terminal

At least k+1 non-terminals

- Since there are only k non-terminals in the grammar, and since k+1 appear on this long path, it follows that some non-terminal (and perhaps many) appears at least twice on this path.
- Consider the first non-terminal that is repeated, when traversing the path from the leaf to the root.

  This path, and the non-terminal A will be used to break up the string z.

- **In addition, (2) also tells us:**

  S =>* uAy (1)
  =>* uvAxy (2)

  =>* uv^2Ax^2y (2)
  =>* uv^2wx^2y (3)

- **More generally:**

  S =>* uv^iwx^iy for all i>=1

- **And also:**

  S =>* uAy (1)
  =>* uwy (3)

- **Hence:**

  S =>* uv^iwx^iy for all i>=0

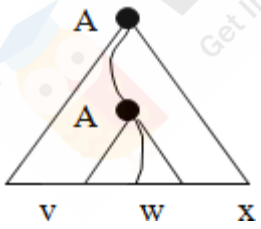
- **Consider the statement of the Pumping Lemma:**

  – *What is n?*
  n = 2^k, where k is the number of non-terminals in the grammar.

  – *Why is $|v| + |x| \geq 1$?*

  Since the height of this subtree is ≥ 2, the first production is A->V_1V_2. Since no non-terminal derives the empty string (in CNF), either V_1 or V_2 must derive a non-empty v or x. More specifically, if w is generated by V_1, then x contains at least one symbol, and if w is generated by V_2, then v contains at least one symbol.

  – *Why is $|vwx| \leq n$?*
  Observations:
  - The repeated variable was the first repeated variable on the path from the bottom, and therefore (by the pigeon-hole principle) the path from the leaf to the second occurrence of the non-terminal has length at most k+1.
  - Since the path was the largest in the entire tree, this path is the longest in the subtree rooted at the second occurrence of the non-terminal. Therefore the subtree has height ≤k+1. From the lemma, the yield of the subtree has length ≤ 2^k=n.



CFL Closure Properties

- **Theorem#1:**
  The context-free languages are closed under concatenation, union, and Kleene closure.

- **Proof:**
  Start with 2 CFL *L*(*H1*) and *L*(*H2*) generated by *H1* = (*N1*,*T1*,*R1*,*s1*) and *H2* = (*N2*,*T2*,*R2*,*s2*).
  Assume that the alphabets and rules are disjoint.

  Concatenation:
  Formed by *L*(*H1*)·*L*(*H2*) or a string in *L*(*H1*) followed by a string in *L*(*H2*) which can be

generated by *L*(*H3*) generated by *H3* = (*N3*,*T3*,*R3*,*s3*). *N3* = *N1* ⋃ *N2*, *T3* = *T1* ⋃ *T2*, *R3* = *R1* ⋃ *R2* ⋃ {*s3* -->*s1s2*} where *s3* →*s1s2* is a new rule introduced. The new rule generates a string of *L*(H1) then a string of *L*(*H2*). Then *L*(*H1*) ·*L*(*H2*) is context-free.

Union:
Formed by *L*(*H1*) ⋃ *L*(*H2*) or a string in *L*(*H1*) or a string in *L*(*H2*). It is generated by *L*(*H3*) generated by *H4* = (*N4*,*T4*,*R4*,*s4*) where *N4* = *N1* ⋃ *N2*, *T4* = *T1* ⋃ *T2*, and *R4* = *R1* ⋃ *R2* ⋃ {*s4*-->*s1*, *s4* → *s2*}, the new rules added will create a string of *L*(*H1*) or *L*(*H2*). Then *L*(*H1*) ⋃ *L*(*H2*) is context-free.

Kleene:
Formed by *L*(*H1*)* is generated by the grammar *L*(*H5*) generated by *H5* = (*N1*,*T1*,*R5*,*s1*) with *R5* = *R1* ⋃ {*s1*→*e*, *s1*→*s1s1*}. *L*(*H5*) includes *e*, every string in *L*(*H1*), and through *i*-1 applications of *s1*→*s1s1*, every string in *L*(*H1*)*i*. Then *L*(*H1*)* is generated by *H5* and is context-free.

- **Theorem#2:**
The set of context-free languages is not closed under complementation or intersection.

- **Proof:**
Intersections of two languages *L1* □ □*L2* can be defined in terms of the Complement and Union operations as follows:
L1 □□*L2* = □* - (□* - *L1*) □□(□* - *L2*)

Therefore if CFL are closed under intersection then it is closed under compliment and if closed under compliment then it is closed under intersection.

The proof is just showing two context-free languages that their intersection is not a context-free language.

Choose *L1* = {*anbncm* | *m*,*n* □ □0} is generated by grammar *H1* = {*N1*,*T1*,*R1*,*s1*}, where
N1 = {*s*, *A*, *B*}
T1 = {*a*, *b*, *c*}
R1 = {*s*□*AB*,
A *aAb*,
A *e*,
B *Bc*,
B *e*}.

Choose *L2* = {*ambncn* | *m*,*n* *H2* = {*N2*,*T2*,*R2*,*s2*}, where
N1 = {*s*, *A*, *B*}
T1 = {*a*, *b*, *c*}
R2 = {*s* *AB*,
A *aA*,
A *e*,
B *bBc*,
B *e*}.

Thus *L1* and *L2* are both context-free.

The intersection of the two languages is *L3* = {*anbncn* | *n*
already been proven earlier in this paper to be not context-free. Therefore CFL are not closed under intersections, which also means that it is not closed under complementation.

5. Pushdown Automata (PDA)

- **Informally:**

– A PDA is an NFA-ε with a stack.

–Transitions are modified to accommodate stack operations.

- **Questions:**

–What is a stack?

–How does a stack help?

- A DFA can "remember" only a finite amount of information, whereas a PDA can "remember" an infinite amount of (certain types of) information.

- **Example:**

$\{0^n1^n \mid 0=<n\}$	Is *not* regular.
$\{0^n1^n \mid 0\leq n\leq k$, for some fixed k$\}$	Is regular, for any fixed k.

- **For k=3:**

$L = \{\varepsilon, 01, 0011, 000111\}$

- In a DFA, each state remembers a finite amount of information.

- To get $\{0^n1^n \mid 0 \leq n\}$ with a DFA would require an infinite number of states using the preceding technique.

- An infinite stack solves the problem for $\{0^n1^n \mid 0 \leq n\}$ as follows:
  - Read all 0's and place them on a stack
  - Read all 1's and match with the corresponding 0's on the stack

- Only need two states to do this in a PDA

- Similarly for $\{0^n1^m0^{n+m} \mid n,m \geq 0\}$

Formal Definition of a PDA

- A pushdown automaton (PDA) is a seven-tuple:

$M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$

Q	A finite set of states
Σ	A finite input alphabet
Γ	A finite stack alphabet
q_0	The initial/starting state, q_0 is in Q
z_0	A starting stack symbol, is in Γ
F	A set of final/accepting states, which is a subset of Q
δ	A transition function, where

$\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow$ finite subsets of $Q \times \Gamma^*$

- Consider the various parts of δ:

$Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow$ finite subsets of $Q \times \Gamma^*$

- Q on the LHS means that at each step in a computation, a PDA must consider its' current state.
- Γ on the LHS means that at each step in a computation, a PDA must consider the symbol on top of its' stack.
- $\Sigma \cup \{\varepsilon\}$ on the LHS means that at each step in a computation, a PDA may or may not consider the current input symbol, i.e., it may have epsilon transitions.

- "Finite subsets" on the RHS means that at each step in a computation, a PDA will have several

options.

–Q on the RHS means that each option specifies a new state.

–Γ* on the RHS means that each option specifies zero or more stack symbols that will replace the top stack symbol.

- **Two types of PDA transitions #1:**

$$\delta(q, a, z) = \{(p_1,\gamma_1), (p_2,\gamma_2),\ldots, (p_m,\gamma_m)\}$$

–Current state is q

–Current input symbol is a

–Symbol currently on top of the stack z

–Move to state p_i from q

–Replace z with γ_i on the stack (leftmost symbol on top)

–Move the input head to the next input symbol

- **Two types of PDA transitions #2:**

$$\delta(q, \varepsilon, z) = \{(p_1,\gamma_1), (p_2,\gamma_2),\ldots, (p_m,\gamma_m)\}$$

–Current state is q

–Current input symbol is not considered

–Symbol currently on top of the stack z

–Move to state p_i from q

–Replace z with γ_i on the stack (leftmost symbol on top)

–**No input symbol is read**

- **Example:** (balanced parentheses)

$M = (\{q_1\}, \{\text{“(“, “)”}\}, \{L, \#\}, \delta, q_1, \#, Ø)$

δ:

(1)	$\delta(q_1, (, \#) = \{(q_1, L\#)\}$
(2)	$\delta(q_1,), \#) = Ø$
(3)	$\delta(q_1, (, L) = \{(q_1, LL)\}$
(4)	$\delta(q_1,), L) = \{(q_1, \varepsilon)\}$
(5)	$\delta(q_1, \varepsilon, \#) = \{(q_1, \varepsilon)\}$
(6)	$\delta(q_1, \varepsilon, L) = Ø$

- **Goal:** (acceptance)

–Terminate in a non-null state

–Read the entire input string

–Terminate with an empty stack

•Informally, a string is accepted if there exists a computation that uses up all the input and leaves the stack empty.

•Example Computation:

Current Input	Stack	Transition	
(())	#		
())	L#	(1)	- Could have applied rule
))	LL#	(3)	(5), but it would have
)	L#	(4)	done no good
ε	#	(4)	
ε	-	(5)	

•**Example PDA #1:** For the language $\{x \mid x = wcw^r \text{ and } w \text{ in } \{0,1\}^*\}$

$M = (\{q_1, q_2\}, \{0, 1, c\}, \{R, B, G\}, \delta, q_1, R, \emptyset)$

δ:

(1)	$\delta(q_1, 0, R) = \{(q_1, BR)\}$	(9)	$\delta(q_1, 1, R) = \{(q_1, GR)\}$
(2)	$\delta(q_1, 0, B) = \{(q_1, BB)\}$	(10)	$\delta(q_1, 1, B) = \{(q_1, GB)\}$
(3)	$\delta(q_1, 0, G) = \{(q_1, BG)\}$	(11)	$\delta(q_1, 1, G) = \{(q_1, GG)\}$
(4)	$\delta(q_1, c, R) = \{(q_2, R)\}$		
(5)	$\delta(q_1, c, B) = \{(q_2, B)\}$		
(6)	$\delta(q_1, c, G) = \{(q_2, G)\}$		
(7)	$\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$	(12)	$\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$
(8)	$\delta(q_2, \varepsilon, R) = \{(q_2, \varepsilon)\}$		

• **Notes:**

–Only rule #8 is non-deterministic.

–Rule #8 is used to pop the final stack symbol off at the end of a computation.

- **Example Computation:**

(1) $\delta(q_1, 0, R) = \{(q_1, BR)\}$
(2) $\delta(q_1, 0, B) = \{(q_1, BB)\}$
(3) $\delta(q_1, 0, G) = \{(q_1, BG)\}$
(4) $\delta(q_1, c, R) = \{(q_2, R)\}$
(5) $\delta(q_1, c, B) = \{(q_2, B)\}$
(6) $\delta(q_1, c, G) = \{(q_2, G)\}$
(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$
(8) $\delta(q_2, \varepsilon, R) = \{(q_2, \varepsilon)\}$
(9) $\delta(q_1, 1, R) = \{(q_1, GR)\}$
(10) $\delta(q_1, 1, B) = \{(q_1, GB)\}$
(11) $\delta(q_1, 1, G) = \{(q_1, GG)\}$
(12) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

State	Input	Stack	Rule Applied	Rules Applicable
q_1	**0**1c10	R	-	(1)
q_1	**1**c10	BR	(1)	(10)
q_1	**c**10	GBR	(10)	(6)
q_2	**1**0	GBR	(6)	(12)
q_2	**0**	BR	(12)	(7)
q_2	**ε**	R	(7)	(8)
q_2	ε	ε	(8)	-

- **Example Computation:**

(1) $\delta(q_1, 0, R) = \{(q_1, BR)\}$
(2) $\delta(q_1, 0, B) = \{(q_1, BB)\}$
(3) $\delta(q_1, 0, G) = \{(q_1, BG)\}$
(4) $\delta(q_1, c, R) = \{(q_2, R)\}$
(5) $\delta(q_1, c, B) = \{(q_2, B)\}$
(6) $\delta(q_1, c, G) = \{(q_2, G)\}$
(7) $\delta(q_2, 0, B) = \{(q_2, \varepsilon)\}$
(8) $\delta(q_2, \varepsilon, R) = \{(q_2, \varepsilon)\}$
(9) $\delta(q_1, 1, R) = \{(q_1, GR)\}$
(10) $\delta(q_1, 1, B) = \{(q_1, GB)\}$
(11) $\delta(q_1, 1, G) = \{(q_1, GG)\}$
(12) $\delta(q_2, 1, G) = \{(q_2, \varepsilon)\}$

State	Input	Stack	Rule Applied
q_1	**1**c1	R	
q_1	**c**1	GR	(9)
q_2	**1**	GR	(6)
q_2	**ε**	R	(12)
q_2	ε	ε	(8)

- **Definition:** |—* is the reflexive and transitive closure of |—.

–I |—* I for each instantaneous description I

–If I |— J and J |—* K then I |—* K

•Intuitively, if I and J are instantaneous descriptions, then I |—* J means that J follows from I by zero or more transitions.

•**Definition:** Let M = (Q, Σ, Γ, δ, q_0, z_0, F) be a PDA. The *language accepted by empty stack*, denoted $L_E(M)$, is the set

$$\{w \mid (q_0, w, z_0) \vdash^* (p, \varepsilon, \varepsilon) \text{ for some } p \text{ in } Q\}$$

•**Definition:** Let M = (Q, Σ, Γ, δ, q_0, z_0, F) be a PDA. The *language accepted by final state*, denoted $L_F(M)$, is the set

$$\{w \mid (q_0, w, z_0) \vdash^* (p, \varepsilon, \gamma) \text{ for some } p \text{ in } F \text{ and } \gamma \text{ in } \Gamma^*\}$$

•**Definition:** Let M = (Q, Σ, Γ, δ, q_0, z_0, F) be a PDA. The *language accepted by empty stack and final state*, denoted L(M), is the set

$$\{w \mid (q_0, w, z_0) \vdash^* (p, \varepsilon, \varepsilon) \text{ for some } p \text{ in } F\}$$

•**Lemma 1:** Let $L = L_E(M_1)$ for some PDA M_1. Then there exits a PDA M_2 such that $L = L_F(M_2)$.

•**Lemma 2:** Let $L = L_F(M_1)$ for some PDA M_1. Then there exits a PDA M_2 such that $L = L_E(M_2)$.

•**Theorem:** Let L be a language. Then there exits a PDA M_1 such that $L = L_F(M_1)$ if and only if there exists a PDA M_2 such that $L = L_E(M_2)$.

•**Corollary:** The PDAs that accept by empty stack and the PDAs that accept by final state define the same class of languages.

•**Note:** Similar lemmas and theorems could be stated for PDAs that accept by both final state and empty stack.

Greibach Normal Form (GNF)

•**Definition:** Let G = (V, T, P, S) be a CFL. If every production in P is of the form

$$A \rightarrow a\alpha$$

Where A is in V, a is in T, and α is in V*, then G is said to be in Greibach Normal Form (GNF).

•**Example:**

$S \rightarrow aAB \mid bB$
$A \rightarrow aA \mid a$

B –> bB | c

•**Theorem:** Let L be a CFL. Then L – {ε} is a CFL.

•**Theorem:** Let L be a CFL not containing {ε}. Then there exists a GNF grammar G such that L = L(G).

•**Lemma 1:** Let L be a CFL. Then there exists a PDA M such that L = L_E(M).

•**Proof:** Assume without loss of generality that ε is not in L. The construction can be modified to include ε later.

Let G = (V, T, P, S) be a CFG, and assume without loss of generality that G is in GNF. Construct M = (Q, Σ, Γ, δ, q, z, Ø) where:

Q = {q}
Σ = T
Γ = V
z = S

δ: for all a in Σ and A in Γ, δ(q, a, A) contains (q, γ) if A –> aγ is in P or rather:
δ(q, a, A) = {(q, γ) | A –> aγ is in P and γ is in Γ*}, for all a in Σ and A in Γ

•For a given string x in Σ* , M will attempt to simulate a leftmost derivation of x with G.

•**Example #1:** Consider the following CFG in GNF.

S → aS
S → a

G is in GNF
L(G) = a+

Construct M as:
Q = {q}
Σ = T = {a}
Γ = V = {S}
z = S

δ(q, a, S) = {(q, S), (q, ε)}
δ(q, ε, S) = Ø

•**Example #2:** Consider the following CFG in GNF.

(1) S –> aA
(2) S –> aB
(3) A –> aA
(4) A –> aB

G is in GNF
L(G) = a+b+

(5) B –> bB
(6) B –> b

Construct M as:

Q = {q}
Σ = T = {a, b}
Γ = V = {S, A, B}
z = S

(1)δ(q, a, S) = {(q, A), (q, B)} From productions #1 and 2, S->aA, S->aB
(2)δ(q, a, A) = {(q, A), (q, B)} From productions #3 and 4, A->aA, A->aB
(3) δ(q, a, B) = Ø
(4) δ(q, b, S) = Ø
(5) δ(q, b, A) = Ø
(6)δ(q, b, B) = {(q, B), (q, ε)} From productions #5 and 6, B->bB, B->b
(7) δ(q, ε, S) = Ø
(8) δ(q, ε, A) = Ø
(9) δ(q, ε, B) = Ø Recall δ: Q x (Σ U {ε}) x Γ –> finite subsets of Q x Γ*

•For a string w in L(G) the PDA M will simulate a leftmost derivation of w.

–If w is in L(G) then $(q, w, z_0) \vdash^* (q, \varepsilon, \varepsilon)$

–If $(q, w, z_0) \vdash^* (q, \varepsilon, \varepsilon)$ then w is in L(G)

•Consider generating a string using G. Since G is in GNF, each sentential form in a *leftmost* derivation has form:

•And each step in the derivation (i.e., each application of a production) adds a terminal and some non-terminals.

$$A_1 \rightarrow t_{i+1}\alpha$$

$$\Rightarrow t_1t_2\ldots t_i\, t_{i+1}\, \alpha A_1A_2\ldots A_m$$

•Each transition of the PDA simulates one derivation step. Thus, the i^{th} step of the PDAs' computation corresponds to the i^{th} step in a corresponding leftmost derivation.

•After the i^{th} step of the computation of the PDA, $t_1t_2\ldots t_{i+1}$ are the symbols that have already

been read by the PDA and $\alpha A_1A_2\ldots A_m$ are the stack contents.

- For each leftmost derivation of a string generated by the grammar, there is an equivalent accepting computation of that string by the PDA.

- Each sentential form in the leftmost derivation corresponds to an instantaneous description in the PDA's corresponding computation.

- For example, the PDA instantaneous description corresponding to the sentential form:

$$=> t_1t_2\ldots t_i\, A_1A_2\ldots A_m$$

would be: $(q, t_{i+1}t_{i+2}\ldots t_n, A_1A_2\ldots A_m)$

- **Example:** Using the grammar from example #2:

S => aA	(1)
=> aaA	(3)
=> aaaA	(3)
=> aaaaB	(4)
=> aaaabB	(5)
=> aaaabb	(6)

- The corresponding computation of the PDA:

•(q, aaaabb, S)	|— (q, aaabb, A)	(1)/1
	|— (q, aabb, A)	(2)/1
	|— (q, abb, A)	(2)/1
	|— (q, bb, B)	(2)/2
	|— (q, b, B)	(6)/1
	|— (q, ε, ε)	(6)/2

–String is read
–Stack is emptied
–Therefore the string is accepted by the PDA

- **Example #3:** Consider the following CFG in GNF.

(1)	S –> aABC	
(2)	A –> a	G is in GNF
(3)	B –> b	
(4)	C –> cAB	
(5)	C –> cC	

Construct M as:

Q = {q}
Σ = T = {a, b, c}
Γ = V = {S, A, B, C}
z = S

(1)	δ(q, a, S) = {(q, ABC)}	S->aABC	(9)	δ(q, c, S) = Ø
(2)	δ(q, a, A) = {(q, ε)}	A->a	(10)	δ(q, c, A) = Ø
(3)	δ(q, a, B) = Ø		(11)	δ(q, c, B) = Ø
(4)	δ(q, a, C) = Ø	C->cAB|cC	(12)	δ(q, c, C) = {(q, AB), (q, C))
(5)	δ(q, b, S) = Ø		(13)	δ(q, ε, S) = Ø
(6)	δ(q, b, A) = Ø		(14)	δ(q, ε, A) = Ø
(7)	δ(q, b, B) = {(q, ε)}	B->b	(15)	δ(q, ε, B) = Ø
(8)	δ(q, b, C) = Ø		(16)	δ(q, ε, C) = Ø

•Notes:

–Recall that the grammar G was required to be in GNF before the construction could be applied.

–As a result, it was assumed at the start that ε was not in the context-free language L.

•Suppose ε is in L:

1) First, let L' = L – {ε}

Fact: If L is a CFL, then L' = L – {ε} is a CFL.

By an earlier theorem, there is GNF grammar G such that L' = L(G).

2) Construct a PDA M such that L' = L_E(M)

How do we modify M to accept ε?

Add δ(q, ε, S) = {(q, ε)}? No!

•Counter Example:

Consider L = {ε, b, ab, aab, aaab, …}
Then L' = {b, ab, aab, aaab, …}

•**The GNF CFG for L':**

(1) S –> aS
(2) S –> b

•**The PDA M Accepting L':**

Q = {q}
Σ = T = {a, b}
Γ = V = {S}
z = S

δ(q, a, S) = {(q, S)}
δ(q, b, S) = {(q, ε)}
δ(q, ε, S) = Ø

•If δ(q, ε, S) = {(q, ε)} is added then:

L(M) = {ε, a, aa, aaa, …, b, ab, aab, aaab, …}

3) Instead, add a new *start* state q' with transitions:

δ(q', ε, S) = {(q', ε), (q, S)}

•**Lemma 1:** Let L be a CFL. Then there exists a PDA M such that L = L_E(M).

•**Lemma 2:** Let M be a PDA. Then there exists a CFG grammar G such that L_E(M) = L(G).

•**Theorem:** Let L be a language. Then there exists a CFG G such that L = L(G) iff there exists a PDA M such that L = L_E(M).

•**Corollary:** The PDAs define the CFLs.

Equivalence of CFG to PDAs

- **Example:** Consider the grammar for arithmetic expressions we introduced earlier. It is reproduced below for convenience. G = ({E, T, F}, {n, v, +, *, (,)}, P, E), where

E = { 1: E → E + T,
2: E → T,
3: T → T* F,
4: T → F,
5: F → n,
6: F → v,
7: F → (E),
}

Suppose the input to our parser is the expression, n*(v+n*v). Since G is unambiguous this expression has only one leftmost derivation, p = 2345712463456. We describe the behavior of the PDA in general, and then step through its moves using this derivation to guide the computation.

- **PDA Simulator:**
  - Step 1: Initialize the stack with the start symbol (E in this case). The start symbol will serve as the bottom of stack marker (Z_0).
  - Step 2: Ignoring the input, check the top symbol of the stack.
    - Case (a) Top of stack is a nonterminal, "X": non-deterministically decide which
      X-rule to use as the next step of the derivation. After selecting a rule, replace X in the stack with the rightpart of that rule. If the stack is non-empty, repeat step 2. Otherwise, halt (input may or may not be empty.)
    - Case(b) Top of stack is a terminal, "a": Read the next input. If the input matches a, then pop the stack and repeat step 2.
      Otherwise, halt (without popping "a" from the stack.)
  - This parsing algorithm by showing the sequence of configurations the parser would assume in an accepting computation for the input, n*(v+n*v).
    Assume "q0" is the one and only state of this PDA.
  - p (leftmost derivation in G) = 2345712463456

    (q0, n*(v+n*v), E)

    2⇒M (q0, n*(v+n*v), T)

    3⇒M (q0, n*(v+n*v), T*F)

    4⇒M (q0, n*(v+n*v), F*F)

5⇒M (q0, n*(v+n*v), n*F)	read⇒M (q0, *(v+n*v), *F)
	read⇒M (q0, (v+n*v), F)
7⇒M (q0, (v+n*v), (E))	read⇒M (q0, v+n*v), E))
1⇒M (q0, v+n*v),E+T))	
2⇒M (q0, v+n*v), T+T))	
4⇒M (q0, v+n*v), F+T))	
6⇒M (q0, v+n*v), v+T))	read⇒M (q0, +n*v), +T))
	read⇒M (q0, n*v), T))
3⇒M (q0, n*v), T*F))	
4⇒M (q0, n*v), F*F))	
5⇒M (q0, n*v), n*F))	read⇒M (q0, *v), *F))
	read⇒M (q0, v), F))
6⇒M (q0, v), v))	read⇒M (q0,),))
	read⇒M (q0, l, l) accept!

Deterministic PDAs and DCFLs

- **Definition:** A *Deterministic Pushdown Automaton* (DPDA) is a 7-tuple,

  $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, A)$,
  where
  Q = finite set of states,
  Σ = input alphabet,
  Γ = stack alphabet,

  $q_0 \in Q$ = the initial state,
  $Z_0 \in \Gamma$ = bottom of stack marker (or initial stack symbol), and

  $\delta: Q \times (\Sigma \cup \{L\}) \times \Gamma \rightarrow Q \times \Gamma^*$ = the transition function (not necessarily total).
  Specifically,

  [1] if $d(q, a, Z)$ is defined for some $a \in \Sigma$ and $Z \in \Gamma$, then $d(q, L, Z) = \Phi$ and $|d(q, a, Z)| = 1$.

[2] Conversely, if $d(q, L, Z) \neq \Phi$, for some Z, then $d(q, a, Z) = \Phi$, for all $a \in \Sigma$, and $|d(q, L, Z)| = 1$.

- **NOTE:** DPDAs can accept their input either by final state or by empty stack – just as for the non-deterministic model. We therefore define D_{stk} and D_{ste}, respectively, as the corresponding families of Deterministic Context-free Languages accepted by a DPDA by empty stack and final state.

MODULE IV:

1. **Turing Machines**
2. **The Halting Problem**
3. **The Universal language**
4. **A Church- Turing thesis**
5. **Linear Bounded Automata.**

1. Turing Machines (TM)

- **Generalize the class of CFLs:**

- Recursively enumerable languages are also known as *type 0* languages.
- Context-sensitive languages are also known as *type 1* languages.
- Context-free languages are also known as *type 2* languages.
- Regular languages are also known as *type 3* languages.

- TMs model the computing capability of a general purpose computer, which informally can be described as:
  - Effective procedure
    - Finitely describable
    - Well defined, discrete, "mechanical" steps
    - Always terminates
  - Computable function
    - A function computable by an effective procedure

- TMs formalize the above notion.

1.1 Deterministic Turing Machine (DTM)

- Two-way, infinite tape, broken into cells, each containing one symbol.
- Two-way, read/write tape head.
- Finite control, i.e., a program, containing the position of the read head, current symbol being scanned, and the current state.
- An input string is placed on the tape, padded to the left and right infinitely with blanks, read/write head is positioned at the left end of input string.
- In one move, depending on the current state and the current symbol being scanned, the TM 1) changes state, 2) prints a symbol over the cell being scanned, and 3) moves its' tape head one

cell left or right.

- Many modifications possible.

1.2 Formal Definition of a DTM

– A DTM is a seven-tuple:

$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$

Q	A finite set of states
Γ	A finite tape alphabet
B	A distinguished blank symbol, which is in Γ
Σ	A finite input alphabet, which is a subset of Γ– {B}
q_0	The initial/starting state, q_0 is in Q
F	A set of final/accepting states, which is a subset of Q
δ	A next-move function, which is a *mapping* from Q x Γ –> Q x Γ x {L,R}

Intuitively, δ(q,s) specifies the next state, symbol to be written and the direction of tape head movement by M after reading symbol s while in state q.

- **Example #1:** $\{0^n1^n \mid n >= 1\}$

	0	1	X	Y	B
q_0	(q_1, X, R)	-	-	(q_3, Y, R)	-
q_1	$(q_1, 0, R)$	(q_2, Y, L)	-	(q_1, Y, R)	-
q_2	$(q_2, 0, L)$	-	(q_0, X, R)	(q_2, Y, L)	-
q_3	-	-	-	(q_3, Y, R)	(q_4, B, R)
q_4	-	-	-	-	-

- **Example #1:** $\{0^n1^n \mid n >= 1\}$

	0	1	X	Y	B
q_0	(q_1, X, R)	-	-	(q_3, Y, R)	-
q_1	$(q_1, 0, R)$	(q_2, Y, L)	-	(q_1, Y, R)	-
q_2	$(q_2, 0, L)$	-	(q_0, X, R)	(q_2, Y, L)	-
q_3	-	-	-	(q_3, Y, R)	(q_4, B, R)
q_4	-	-	-	-	-

  - The TM basically matches up 0's and 1's
  - q_1 is the "scan right" state
  - q_2 is the "scan left" state
  - q_4 is the final state

- **Example #2:** $\{w \mid w$ is in $\{0,1\}^*$ and w ends with a $0\}$

0
00
10
10110
Not ε

$Q = \{q_0, q_1, q_2\}$
$\Gamma = \{0, 1, B\}$
$\Sigma = \{0, 1\}$

$F = \{q_2\}$

	0	1	B
q_0	$(q_0, 0, R)$	$(q_0, 1, R)$	(q_1, B, L)
q_1	$(q_2, 0, R)$	-	-
q_2	-	-	-

  - q_0 is the "scan right" state
  - q_1 is the verify 0 state

– **Definition:** Let M = (Q, Σ, Γ, δ, q_0, B, F) be a TM, and let w be a string in Σ*. Then w is *accepted* by M iff

$$q_0w \vdash^* \alpha_1 p \alpha_2$$

Where p is in F and α_1 and α_2 are in Γ*

– **Definition:** Let M = (Q, Σ, Γ, δ, q_0, B, F) be a TM. The *language accepted by M*, denoted L(M), is the set

L={w | w is in Σ* and w is accepted by M}

In contrast to FA and PDAs, if a TM simply *passes through* a final state then the string is accepted.

- Given the above definition, no final state of an TM need have any exiting transitions. *Henceforth, this is our assumption*.
- **If x is not in L(M) then M may enter an infinite loop, or halt in a non-final state**.
- Some TMs halt on all inputs, while others may not. In either case the language defined by TM is still well defined.

– **Definition:** Let L be a language. Then L is *recursively enumerable* if there exists a TM M such that L = L(M).

– If L is r.e. then L = L(M) for some TM M, and
  •If x is in L then M halts in a final (accepting) state.
  •If x is not in L then M may halt in a non-final (non-accepting) state, or loop forever.

– **Definition:** Let L be a language. Then L is *recursive* if there exists a TM M such that L = L(M) and M halts on all inputs.

– If L is recursive then L = L(M) for some TM M, and
  •If x is in L then M halts in a final (accepting) state.
  •If x is not in L then M halts a non-final (non-accepting) state.

– The set of all recursive languages is a subset of the set of all recursively enumerable languages

– Terminology is easy to confuse: A *TM* is not recursive or recursively enumerable, rather a *language* is recursive or recursively enumerable.

– **Observation:** Let L be an r.e. language. Then there is an infinite list M_0, M_1, … of TMs such that L = L(M_i).

– **Question:** Let L be a recursive language, and M_0, M_1, … a list of all TMs such that L = L(M_i), and choose any i>=0. Does M_i always halt?

Answer: Maybe, maybe not, but *at least one in the list does*.

– **Question:** Let L be a recursive enumerable language, and M_0, M_1, … a list of all TMs such that L = L(M_i), and choose any i>=0. Does M_i always halt?

Answer: Maybe, maybe not. Depending on L, none might halt or some may halt.

– If L is also recursive then L is recursively enumerable.

Question: Let L be a recursive enumerable language that is not recursive (L is in r.e. – r), and M_0, M_1, … a list of all TMs such that L = L(M_i), and choose any i>=0. Does M_i always halt?
Answer: No! If it did, then L would not be in r.e. – r, it would be recursive.

- **Let M be a TM.**
  - Question: Is L(M) r.e.?
    Answer: Yes! By definition it is!
  - Question: Is L(M) recursive?
    Answer: Don't know, we don't have enough information.
  - Question: Is L(M) in r.e – r?
    Answer: Don't know, we don't have enough information.
- **Let M be a TM that halts on all inputs:**
  - Question: Is L(M) recursively enumerable?
    Answer: Yes! By definition it is!
  - Question: Is L(M) recursive?
    Answer: Yes! By definition it is!
  - Question: Is L(M) in r.e – r?
    Answer: No! It can't be. Since M always halts, L(M) is recursive.
- **Let M be a TM.**
  - As noted previously, L(M) is recursively enumerable, but may or may not be recursive.
  - Question: Suppose that L(M) is recursive. Does that mean that M always halts?
    Answer: Not necessarily. However, some TM M' must exist such that L(M') = L(M) and M' always halts.
  - Question: Suppose that L(M) is in r.e. – r. Does M always halt?
    Answer: No! If it did then L(M) would be recursive and therefore not in r.e. – r.
- **Let M be a TM, and suppose that M loops forever on some string x.**

- Question: Is L(M) recursively enumerable?
  Answer: Yes! By definition it is.

- Question: Is L(M) recursive?
  Answer: Don't know. Although M doesn't always halt, some other TM M' may exist such that L(M') = L(M) and M' always halts.

- Question: Is L(M) in r.e. – r?
  Answer: Don't know.

Closure Properties for Recursive and Recursively Enumerable Languages

- **TMs Model General Purpose Computers:**
  - If a TM can do it, so can a GP computer
  - If a GP computer can do it, then so can a TM

  If you want to know if a TM can do X, then some equivalent question are:
  - *Can a general purpose computer do X?*
  - *Can a C/C++/Java/etc. program be written to do X?*

  For example, is a language L recursive?
  - *Can a C/C++/Java/etc. program be written that always halts and accepts L?*

- **TM Block Diagrams:**
  - If L is a recursive language, then a TM M that accepts L and always halts can be pictorially represented by a "chip" that has one input and two outputs.

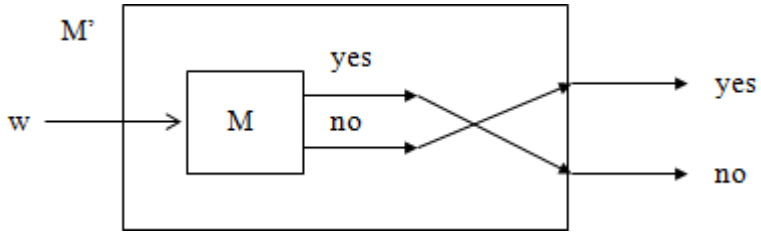
  - If L is a recursively enumerable language, then a TM M that accepts L can be pictorially represented by a "chip" that has one output.

  - Conceivably, M could be provided with an output for "no," but this output cannot be counted on. Consequently, we simply ignore it.

– **Theorem:** The recursive languages are closed with respect to complementation, i.e., if L is a recursive language, then so is

Proof: Let M be a TM such that L = L(M) and M always halts. Construct TM M' as

follows:



- **Note That:**
  - M' accepts iff M does not
  - M' always halts since M always halts

From this it follows that the complement of L is recursive. •

- **Theorem:** The recursive languages are closed with respect to union, i.e., if L_1 and L_2 are recursive languages, then so is

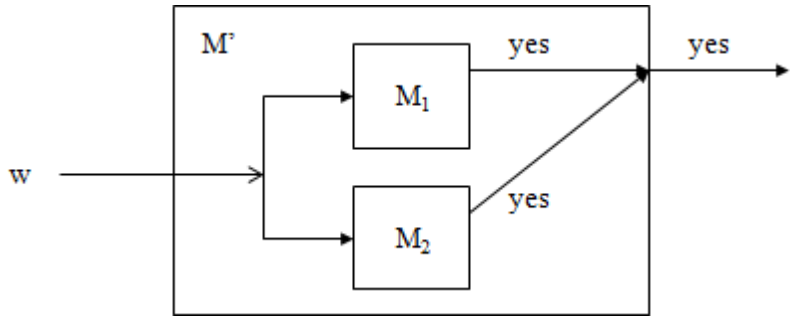
Proof: Let M_1 and M_2 be TMs such that $L_1 = L(M_1)$ and $L_2 = L(M_2)$ and M_1 and M_2 always halts. Construct TM M' as follows:

- **Note That:**
  - $L(M') = L(M_1) \cup L(M_2)$
    - $L(M')$ is a subset of $L(M_1) \cup L(M_2)$
    - $L(M_1) \cup L(M_2)$ is a subset of $L(M')$
  - M' always halts since M_1 and M_2 always halt

It follows from this that $L_3 = L_1 \cup L_2$ is recursive.

- **Theorem:** The recursive enumerable languages are closed with respect to union, i.e., if L_1 and L_2 are recursively enumerable languages, then so is $L_3 = L_1 \cup L_2$

Proof: Let M_1 and M_2 be TMs such that $L_1 = L(M_1)$ and $L_2 = L(M_2)$. Construct M' as follows:



- **Note That:**
  - $L(M') = L(M_1) \cup L(M_2)$
    - $L(M')$ is a subset of $L(M_1) \cup L(M_2)$
    - $L(M_1) \cup L(M_2)$ is a subset of $L(M')$
  - M' halts and accepts iff M_1 or M_2 halts and accepts

It follows from this that is recursively enumerable.

2. The Halting Problem – Background

- **Definition:** A decision problem is a problem having a yes/no answer (that one presumably wants to solve with a computer). Typically, there is a list of parameters on which the problem is based.
  - Given a list of numbers, is that list sorted?
  - Given a number x, is x even?
  - Given a C program, does that C program contain any syntax errors?
  - Given a TM (or C program), does that TM contain an infinite loop?

From a practical perspective, many decision problems do not seem all that interesting. However, from a theoretical perspective they are for the following two reasons:

  - Decision problems are more convenient/easier to work with when proving complexity results.
  - Non-decision counter-parts are typically at least as difficult to solve.

- **Notes:**
  - The following terms and phrases are analogous:

Algorithm	-	A halting TM program
Decision Problem	-	A language
(un)Decidable	-	(non)Recursive

Statement of the Halting Problem

- **Practical Form:** (P1)
  Input: Program P and input I.
  Question: Does P terminate on input I?

- **Theoretical Form:** (P2)
  Input: Turing machine M with input alphabet Σ and string w in Σ*.
  Question: Does M halt on w?

- **A Related Problem We Will Consider First:** (P3)
  Input: Turing machine M with input alphabet Σ and one final state, and string w in Σ*.
  Question: Is w in L(M)?

- **Analogy:**
  Input: DFA M with input alphabet Σ and string w in Σ*.
  Question: Is w in L(M)?

  Is this problem decidable? Yes!

- **Over-All Approach:**
  - We will show that a language L_d is not recursively enumerable
  - From this it will follow that is not recursive
  - Using this we will show that a language L_u is not recursive
  - From this it will follow that the halting problem is undecidable.

3. The Universal Language

- Define the language L_u as follows:

  L_u = {x | x is in {0, 1}* and x = <M,w> where M is a TM encoding and w is in L(M)}

- Let x be in {0, 1}*. Then either:

1. x doesn't have a TM prefix, in which case x is **not** in L_u
2. x has a TM prefix, i.e., x = <M,w> and either:
   a) w is not in L(M), in which case x is **not** in L_u
   b) w is in L(M), in which case x is in L_u

- **Compare P3 and L_u:**

  (P3):
  Input: Turing machine M with input alphabet Σ and one final state, and string w in Σ*.

- **Notes:**
  - L_u is P3 expressed as a language
  - Asking if L_u is recursive is the same as asking if P3 is decidable.
  - We will show that L_u is not recursive, and from this it will follow that P3 is un-decidable.
  - From this we can further show that the halting problem is un-decidable.
  - Note that L_u is recursive if M is a DFA.

4. Church-Turing Thesis

- There is an effective procedure for solving a problem if and only if there is a TM that halts for all inputs and solves the problem.

- There are many other computing models, but all are equivalent to or subsumed by TMs. *There is no more powerful machine* (Technically cannot be proved).

- DFAs and PDAs do not model all effective procedures or computable functions, but only a subset.

- If something can be "computed" it can be computed by a Turing machine.

- Note that this is called a ***Thesis***, not a theorem.

- It can't be proved, because the term "can be computed" is too vague.

- But it is universally accepted as a true statement.

- Given the ***Church-Turing Thesis***:
  - What does this say about "computability"?
  - Are there things even a Turing machine can't do?
  - If there are, then there are things that simply can't be "computed."
    - Not with a Turing machine

- Not with your laptop
- Not with a supercomputer

- There ARE things that a Turing machine can't do!!!

- The ***Church-Turing Thesis:***
  - In other words, there is no problem for which we can describe an algorithm that can't be done by a Turing machine.

The Universal Turing machine

- If Tm's are so damned powerful, can't we build one that simulates the behavior of any Tm on any tape that it is given?
- Yes. This machine is called the ***Universal Turing machine***.
- How would we build a Universal Turing machine?
  - We place an encoding of any Turing machine on the input tape of the Universal Tm.
  - The tape consists entirely of zeros and ones (and, of course, blanks)
  - Any Tm is represented by zeros and ones, using unary notation for elements and zeros as separators.
- Every Tm instruction consists of four parts, each a represented as a series of **1**'s and separated by **0**'s.
- Instructions are separated by **00**.
- We use unary notation to represent components of an instruction, with
  - 0 = **1**,
  - 1 = **11**,
  - 2 = **111**,

- 3 = **1111**,
- ***n*** = **111...111** (***n***+1 1's).

- We encode q_n as $n + 1$ 1's
- We encode symbol a_n as $n + 1$ 1's
- We encode move left as 1, and move right as 11

11110111011111011101001011011011011100

q_3, a_2, q_4, a_2, L q_0, a_1, q_1, a_1, R

- Any Turing machine can be encoded as a unique long string of zeros and ones, beginning with a **1**.
- Let T_n be the Turing machine whose encoding is the number ***n***.

5. Linear Bounded Automata

- A Turing machine that has the length of its tape limited to the length of the input string is called a linear-bounded automaton (LBA).
- A linear bounded automaton is a 7-tuple *nondeterministic* Turing machine M = (Q, S, G, d, q_0,q_{accept}, q_{reject}) except that:

a. There are two extra tape symbols < and >, which are not elements of G.

b. The TM begins in the configuration (q_0≤*x*>), with its tape head scanning the symbol < in cell 0. The > symbol is in the cell immediately to the right of the input string *x*.

c. The TM cannot replace < or > with anything else, nor move the tape head left of < or right of >.

Context-Sensitivity

- *Context-sensitive production* any production □ □ □ satisfying |□≤ | □|.
- *Context-sensitive grammar* any generative grammar *G* = ⟨□ □, □, Π, □ ⟩ such that every production in Π context-sensitive.
- No empty productions.

Context-Sensitive Language

- Language *L context-sensitive* if there exists context-sensitive grammar *G* such that either *L* = *L*(*G*) or *L* = *L*(*G*) ∪ { }.

- **Example**:
  The language L = $\{a^n b^n c^n : n \geq 1\}$ is a C.S.L. the grammar is

  S → abc/ aAbc,

  Ab → bA,

  AC → Bbcc,

  bB → Bb,

  aB → aa/ aaA

  The derivation tree of $a^3b^3c^3$ is looking to be as following

  S ⇒ aAbc

  ⇒ abAc

  ⇒ abBbcc

  ⇒ aBbbcc ⇒ aaAbbcc

  ⇒ aabAbcc

  ⇒ aabbAcc ⇒ aabbBbccc

  ⇒ aabBbbccc

  ⇒ aaBbbbccc

  ⇒ aaabbbccc

CSG = LBA

- A language is accepted by an LBA iff it is generated by a CSG.
- Just like equivalence between CFG and PDA
- Given an x ∈ CSG G, you can intuitively see that and LBA can start with S, and nondeterministically choose all derivations from S and see if they are equal to the input string x. Because CSL's are non-contracting, the LBA only needs to generate derivations of length ≤ |x|. This is because if it generates a derivation longer than |x|, it will never be able to shrink to the size of |x|.

MODULE V

1. **Chomsky Hierarchy Languages**
2. **Turing Reducibility**
3. **The Class P**

1. Chomsky Hierarchy of Languages

- A containment hierarchy (strictly nested sets) of classes of formal grammars

The Hierarchy

Class	Grammars	Languages	Automaton
Type-0	Unrestricted	Recursively enumerable (Turing-recognizable)	Turing machine
	none	Recursive (Turing-decidable)	Decider
Type-1	Context-sensitive	Context-sensitive	Linear-bounded
Type-2	Context-free	Context-free	Pushdown
Type-3	Regular	Regular	Finite

Type 0 Unrestricted:

Languages defined by Type-0 grammars are accepted by Turing machines .

Rules are of the form: $\alpha \rightarrow \beta$, where α and β are arbitrary strings over a vocabulary V and $\alpha \neq \varepsilon$

Type 1 Context-sensitive:

Languages defined by Type-1 grammars are accepted by linear-bounded automata.

Syntax of some natural languages (Germanic)

Rules are of the form:

$$\alpha A \beta \rightarrow \alpha B \beta$$

$$S \rightarrow \varepsilon$$

where

Type 2 Context-free:

Languages defined by Type-2 grammars are accepted by push-down automata.

Natural language is almost entirely definable by type-2 tree structures

Rules are of the form:

$A \rightarrow \alpha$

Where

$A \in N$

$\alpha \in (N \cup \Sigma)*$

Type 3 Regular:

Languages defined by Type-3 grammars are accepted by finite state automata

Most syntax of some informal spoken dialog

Rules are of the form:

$A \rightarrow \varepsilon$

$A \rightarrow \alpha$

$A \rightarrow \alpha B$

where

$A, B \in N$ and $\alpha \in \Sigma$

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- Yes. This machine is called the ***Universal Turing machine***.
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q_3, a_2, q_4, a_2, L q_0, a_1, q_1, a_1, R

- Any Turing machine can be encoded as a unique long string of zeros and ones, beginning with a **1**.
- Let T_n be the Turing machine whose encoding is the number ***n***.

2. Turing Reducibility

- A language A is Turing reducible to a language B, written $A \leq_T B$, if A is decidable relative to B
- Below it is shown that E_{TM} is Turing reducible to EQ_{TM}
- Whenever A is mapping reducible to B, then A is Turing reducible to B
  - The function in the mapping reducibility could be replaced by an oracle
- An oracle Turing machine with an oracle for EQ_{TM} can decide E_{TM}

$T^{EQ\text{-}TM}$ = "On input <M>

1. Create TM M_1 such that $L(M_1) = \varnothing$

   M_1 has a transition from start state to reject state for every element of Σ

1. Call the EQ_{TM} oracle on input <M,M_2>
2. If it accepts, accept; if it rejects, reject"

- $T^{EQ\text{-}TM}$ decides E_{TM}
- E_{TM} is decidable relative to EQ_{TM}
- **Applications**
  - If $A \leq_T B$ and B is decidable, then A is decidable
  - If $A \leq_T B$ and A is undecidable, then B is undecidable
  - If $A \leq_T B$ and B is Turing-recognizable, then A is Turing-recognizable
  - If $A \leq_T B$ and A is non-Turing-recognizable, then B is non-Turing-recognizable

3. The class P

A decision problem *D* is *solvable in polynomial time* or *in the class P*, if there exists an algorithm *A* such that

- *A Takes instances* of *D* as inputs.
- *A* always outputs the correct answer "Yes" or "No".
- There exists a polynomial *p* such that the execution of *A* on inputs of size *n* always terminates in *p(n)* or fewer steps.
- **EXAMPLE**: The Minimum Spanning Tree Problem is in the class P.

The class *P* is often considered as synonymous with the class of computationally feasible problems, although in practice this is somewhat unrealistic.

The class NP

A decision problem is *nondeterministically polynomial-time solvable* or *in the class NP* if there exists an algorithm *A* such that

- *A* takes as inputs potential witnesses for "yes" answers to problem *D*.
- *A* correctly distinguishes true witnesses from false witnesses.

- There exists a polynomial *p* such that for each potential witnesses of each instance of size *n* of *D*, the execution of the algorithm *A* takes at most *p(n)* steps.
- Think of a non-deterministic computer as a computer that magically "guesses" a solution, then has to verify that it is correct
    - If a solution exists, computer always guesses it
    - One way to imagine it: a parallel computer that can freely spawn an infinite number of processes
        - Have one processor work on each possible solution
        - All processors attempt to verify that their solution works
        - If a processor finds it has a working solution
    - So: **NP** = problems *verifiable* in polynomial time
    - Unknown whether **P = NP** (most suspect not)

NP-Complete Problems

- We will see that NP-Complete problems are the "hardest" problems in NP:
    - If any *one* NP-Complete problem can be solved in polynomial time.
    - Then *every* NP-Complete problem can be solved in polynomial time.
    - And in fact *every* problem in **NP** can be solved in polynomial time (which would show **P = NP**)
    - Thus: solve hamiltonian-cycle in $O(n^{100})$ time, you've proved that **P = NP**. Retire rich & famous.
- The crux of NP-Completeness is *reducibility*
    - Informally, a problem P can be reduced to another problem Q if *any* instance of P can be "easily rephrased" as an instance of Q, the solution to which provides a solution to the instance of P
        - *What do you suppose "easily" means?*
        - This rephrasing is called *transformation*
    - Intuitively: If P reduces to Q, P is "no harder to solve" than Q
- An example:
    - P: Given a set of Booleans, is at least one TRUE?
    - Q: Given a set of integers, is their sum positive?

- Transformation: $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ where $y_i = 1$ if x_i = TRUE, $y_i = 0$ if x_i = FALSE

- Another example:
  - Solving linear equations is reducible to solving quadratic equations
    - *How can we easily use a quadratic-equation solver to solve linear equations?*

- Given one NP-Complete problem, we can prove many interesting problems NP-Complete
  - Graph coloring (= register allocation)
  - Hamiltonian cycle
  - Hamiltonian path
  - Knapsack problem
  - Traveling salesman
  - Job scheduling with penalties, etc.

NP Hard

- **Definition:** Optimization problems whose decision versions are NP- complete are called *NP-hard.*

- **Theorem:** If there exists a polynomial-time algorithm for finding the optimum in any *NP*-hard problem, then *P = NP.*
  Proof: Let *E* be an *NP*-hard optimization (let us say minimization) problem, and let *A* be a polynomial-time algorithm for solving it. Now an instance *J* of the corresponding decision problem *D* is of the form *(I, c),* where *I* is an instance of *E,* and *c* is a number. Then the answer to *D* for instance *J* can be obtained by running *A* on *I* and checking whether the cost of the optimal solution exceeds *c.* Thus there exists a polynomial-time algorithm for *D,* and *NP*-completeness of the latter implies *P= NP.*