(1) find $L\left\{\frac{\sin t \sin 3t}{t}\right\}$

Sol:- $f(t) = \sin t \sin 3t$

$$\sin A \sin B = \tfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin t \sin 3t = \frac{1}{2}\left[\cos(t-3t) - \cos(t+3t)\right]$$

$$= \frac{1}{2}\left[\cos 2t - \cos 4t\right]$$

$$L\{\sin t \sin 3t\} = \frac{1}{2}\left[L\{\cos 2t\} - L\{\cos 4t\}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+2^2} - \frac{s}{s^2+4^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+16}\right] = F(s)$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+16}\right) ds$$

$$= \frac{1}{2}\left[\frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+4} - \frac{2s}{s^2+16}\right) ds\right]$$

$$= \frac{1}{4}\left[\log(s^2+4) - \log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2+4}{s^2+16}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\lim_{s\to\infty} \log\left(\frac{s^2+4}{s^2+16}\right) - \log\left(\frac{s^2+4}{s^2+16}\right)\right]$$

$$= \frac{1}{4}\left[\lim_{s\to\infty} \log\left(\frac{\cancel{s^2}(1+4/s^2)}{\cancel{s^2}(1+16/s^2)}\right) - \log\left(\frac{s^2+4}{s^2+16}\right)\right]$$

$$= 1/4\left[\log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s^2+14}{s^2+16}\right)\right]$$

$$= \frac{1}{4}\left[\log 1 - \log\left(\frac{s^2+4}{s^2+16}\right)\right]$$

$$= 1/4\left[0 - \log\left(\frac{s^2+4}{s^2+16}\right)\right]$$

$$= -1/4 \log\left(\frac{s^2+4}{s^2+16}\right)$$

$$= \log\left(\frac{s^2+4}{s^2+16}\right)^{-1/4}$$

$$= \log\left(\frac{s^2+16}{s^2+4}\right)^{1/4}$$

2. find $L\left\{\frac{\sin at}{t}\right\}$

Sol:- $f(t) = \sin at$

$F(s) = L\{\sin at\}$

$= \frac{a}{s^2+a^2}$

$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\, ds$

$= a \cdot \left[\frac{1}{a}\tan^{-1}(s/a)\right]_s^\infty$

$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$

$= \pi/2 - \tan^{-1}(s/a)$

$= \cot^{-1}(s/a)$

3. find $L\left\{\frac{1-\cos t}{t}\right\}$

Sol:- $f(t) = 1-\cos t$

$F(s) = L\{1-\cos t\}$

$= L\{1\} - L\{\cos t\}$

$F(s) = 1/s - \frac{s}{s^2+1}$

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \int_s^\infty \frac{1}{s}\, ds - \int_s^\infty \frac{s}{s^2+1}\, ds$$

$$= (\log s)_s^\infty - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+1}\, ds$$

$$= \left[\log s - \frac{1}{2}\cdot \log(s^2+1)\right]_s^\infty$$

$$= \left[\frac{1}{2}(\log s)2 - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[2\log s - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{s^2}{s^2+1}\right) - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{\cancel{s^2}}{\cancel{s^2}(1+1/s^2)}\right) - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)$$

$$= \log\left(\frac{s^2}{s^2+1}\right)^{-1/2}$$

$$= \log\left(\frac{s^2+1}{s^2}\right)^{1/2}$$

$$= \log\sqrt{\frac{s^2+1}{s^2}}$$

1. find $L\left\{\frac{\sin 2t \sin 4t}{t}\right\}$

2. find $L\left\{\frac{\sin 2t}{t}\right\}$

3. find $L\left\{\frac{e^{2t} - e^{3t}}{t}\right\}$

4. Find $L\left\{\frac{\cos 4t - \cos 2t}{t}\right\}$

1. Evaluate $L\left\{\int_0^t e^{-t}\cos t\,dt\right\}$

Sol :-

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$$

$$f(t) = \cos t$$

$$F(s) = L\{\cos t\}$$

$$= \frac{s}{s^2+1^2}$$

$$L\{e^{-t}\cos t\} = \frac{s}{s^2+1}$$

$s \to s+1$ (shifting rule)

$$= \frac{s+1}{(s+1)^2+1}$$

$$L\{e^{-t}\cos t\} = \frac{s+1}{s^2+2s+2} \quad F(s)$$

$$L\left\{\int_0^t e^{-t}\cos t\right\} = \frac{1}{s}\cdot\frac{s+1}{s^2+2s+2}$$

2. Evaluate $L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\}$

Sol:-

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left(\tan^{-1}s\right)_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \pi/2 - \tan^{-1}s$$

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$$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1} s = F(s)$$

$$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cdot\cot^{-1} s$$

③ Evaluate $L\left\{\int_0^t e^{t}\left(\frac{\sin t}{t}\right)dt\right\}$

Sol:- $L\{\sin t\} = \frac{1}{s^2+1}$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left(\tan^{-1} s\right)_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \pi/2 - \tan^{-1}(s)$$

$$= \cot^{-1}(s)$$

$$L\left\{e^{t}\frac{\sin t}{t}\right\} = \cot^{-1}(s) \quad s \to s-1$$

$$= \cot^{-1}(s-1)$$

$$L\left\{\int_0^t e^{t}\left(\frac{\sin t}{t}\right)dt\right\} = \frac{1}{s}\cdot\cot^{-1}(s-1)$$

4) Evaluate $L\left\{\int_0^t t\cdot\cos at\,dt\right\}$

Sol :-

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{t\cos at\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\left[\frac{(s^2+a^2)\frac{d}{ds}(s) - s\cdot\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$L\{t\cdot\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$L\left\{\int_0^t t\cdot\cos at\,dt\right\} = \frac{1}{s}\cdot\frac{s^2-a^2}{(s^2+a^2)^2}$$

5. Evaluate $\int_0^\infty e^{-t} \frac{\sin t}{t} \, dt$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

Sol :-

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1} \, ds$$

$$= \left(\tan^{-1} s\right)_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \pi/2 - \tan^{-1}(s)$$

$$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1} s$$

$$\int_0^\infty e^{-t} \frac{\sin t}{t} \, dt = \cot^{-1}(s)$$

put $s = 1$

$$= \cot^{-1}(1)$$

$$= \pi/4$$