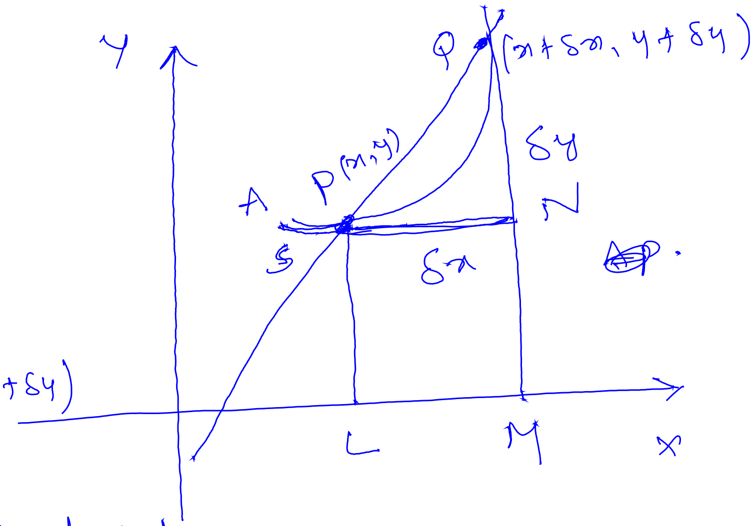
UNIT - I

Differential Calculus - II

Derivative of Arc length

Arc length $= \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Cartesian form



$y = f(x)$

Let A be the fixed point and $P(x, y)$ & $Q(x+\delta x, y+\delta y)$ be two neighbouring points

let $\widehat{AP} = s$ $\widehat{AQ} = s + \delta s$

PN ⊥r QM

$PN = LM = \delta x$

$QN = \delta y$

ΔPNQ is a right angled triangle.

$PQ^2 = PN^2 + NQ^2$

$PQ^2 = (\delta x)^2 + (\delta y)^2$

$$PQ^2 = (\delta x)^2 + (\delta y)^2$$

if $Q \to P$

$\widehat{AQ} = s + \delta s$

$$PQ = \delta s$$

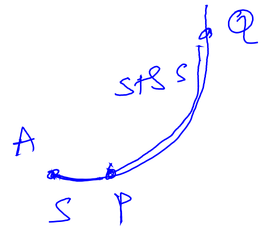
$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2 \to (1)$$

as $Q \to P$ $\delta x \to 0$. and $\delta y \to 0$

Divide (1) by $(\delta x)^2$

$\widehat{AP} = s$

$\widehat{AQ} = s + \delta s$



$PQ = \cancel{s} + \delta s - \cancel{s}$

$\underline{\underline{PQ = \delta s}}$

$$\lim_{\delta x \to 0} \frac{(\delta s)^2}{(\delta x)^2} = \lim_{\delta x \to 0} \frac{(\delta x)^2 + (\delta y)^2}{(\delta x)^2}$$

$$\lim_{\delta x \to 0} \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\boxed{\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$\frac{ds}{dx} = \sqrt{1 + y'^2}$$

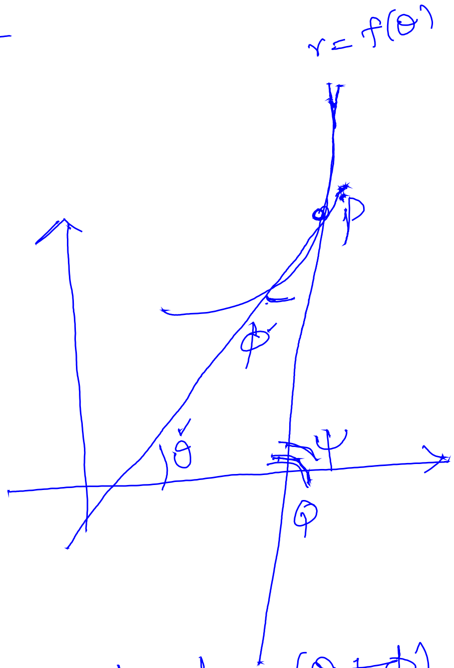
lly

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Derivative of arc length in parametric form

let
$x = x(t)$. $\quad y = y(t)$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$



Note :-

slope $= \dfrac{dy}{dx} = \tan\psi$

$\cot\psi = \dfrac{dx}{dy}$ ✓

$\tan\psi = \tan(\theta + \phi)$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{ds}{dx} = \sqrt{1 + \tan^2\psi}$$

$$\frac{ds}{dx} = \sqrt{\sec^2\psi} = \sec\psi$$

$$\boxed{\frac{ds}{dx} = \sec\psi}$$

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$\frac{ds}{dy} = \sqrt{1 + \cot^2\psi}$$

$$\frac{ds}{dy} = \sqrt{\operatorname{cosec}^2\psi}$$

$$\boxed{\frac{ds}{dy} = \operatorname{cosec}\psi}$$

$\dfrac{dx}{ds} = \cos\psi$ ✓

$\dfrac{dy}{ds} = \sin\psi$

Derivative of arc length in polar form.

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{ds}{dr} = \sqrt{1+\left(\frac{r\,d\theta}{dr}\right)^2}$$

Note :- we know that angle between radius vector and the tangent

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = r\cot\phi$$

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{r^2+(r\cot\phi)^2}$$

$$= \sqrt{r^2+r^2\cot^2\phi}$$

$$= \sqrt{r^2(1+\cot^2\phi)}$$

$$= \sqrt{r^2\operatorname{cosec}^2\phi}$$

$$\frac{ds}{d\theta} = r\operatorname{cosec}\phi$$

$$\operatorname{cosec}\phi = \frac{1}{r}\frac{ds}{d\theta} \Rightarrow \boxed{\sin\phi = r\frac{d\theta}{ds}}$$

$$\frac{ds}{dr} = \sqrt{1+\left(\frac{r\,d\theta}{dr}\right)^2}$$

$$\frac{ds}{dr} = \sqrt{1+(\tan\phi)^2}$$

$$= \sqrt{1+\tan^2\phi}$$

$$= \sqrt{\sec^2\phi}$$

$$\frac{ds}{dr} = \sec\phi$$

$$\boxed{\frac{dr}{ds} = \cos\phi}$$

Problems on Derivative of Arc Length

① Find $\frac{ds}{dx}$ for the Curve $ay^2 = x^3$

Sol:- $ay^2 = x^3 \Rightarrow y^2 = x^3/a$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Diff w r t 'x'

$$a \cdot 2y \cdot \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{3x^2}{2ay}\right)^2}$$

$$= \sqrt{1+\frac{9x^4}{4a^2y^2}}$$

$$= \sqrt{1+\frac{9x^{\cancel{4}}}{4a^{\cancel{2}} \cdot \cancel{x^3}/\cancel{a}}}$$

$$\frac{ds}{dx} = \sqrt{1+\frac{9x}{4a}}$$

② Find $\frac{ds}{dy}$ for $a^2y^2 = a^3 - x^3$ at $(a, 0)$

Sol:- $a^2y^2 = a^3 - x^3$

Diff w. r t 'x'

$$a^2 \cdot 2y \cdot \frac{dy}{dx} = 0 - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{2a^2y}$$

$$\frac{dx}{dy} = \frac{-2a^2y}{3x^2}$$

$$\frac{dx}{dy} = \frac{-2a^2y}{3x^2}$$

$$\frac{dx}{dy} \text{ at } (a,0) = \frac{-2a^2(0)}{3(a^2)}$$

$$= 0$$

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+0}$$

$$\frac{ds}{dy} = 1$$

(3) find $\frac{ds}{d\theta}$ for $r = a(1-\cos\theta)$

Sol:-

$$r = a(1-\cos\theta)$$

$$\frac{dr}{d\theta} = a(0-(-\sin\theta))$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{ds}{d\theta} = \sqrt{(a(1-\cos\theta))^2+(a\sin\theta)^2}$$

$$= \sqrt{a^2(1+\cos^2\theta-2\cos\theta)+a^2\sin^2\theta}$$

$$= \sqrt{a^2(1+\cos^2\theta-2\cos\theta+\sin^2\theta)}$$

$$= \sqrt{a^2(1+1-2\cos\theta)}$$

$$= \sqrt{a^2(2-2\cos\theta)} = \sqrt{2a^2(1-\cos\theta)}$$

$$\frac{ds}{d\theta} = \sqrt{a^2 \cdot 2(1-\cos\theta)}$$

$$= \cancel{\sqrt{2}a} \; a\sqrt{2(2\sin^2\theta/2)}$$

$1-\cos\theta = 2\sin^2\theta/2$

$$= a\, 2\sin\theta/2$$

$$\frac{ds}{d\theta} = 2a\sin\theta/2$$

(4) Show that for the Curve $r^2 = a^2\cos 2\theta$,

$$\frac{ds}{d\theta} = a^2/r$$

Sol:- $r^2 = a^2\cos 2\theta$

Diff w.r.t θ

$$\cancel{2}r\frac{dr}{d\theta} = a^2(-\sin 2\theta)\cdot\cancel{2}$$

$$r\frac{dr}{d\theta} = -a^2\sin 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2}{r}\sin 2\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{r^2 + \left(-a^2/r\,\sin 2\theta\right)^2}$$

$$= \sqrt{r^2 + \frac{a^4}{r^2}\sin^2 2\theta}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \frac{a^4}{r^2}\sin^2 2\theta}$$

$$= \sqrt{\frac{r^4 + a^4\sin^2 2\theta}{r^2}}$$

$$= \sqrt{\frac{(a^2\cos 2\theta)^2 + a^4\sin^2 2\theta}{r^2}}$$

$$= \sqrt{\frac{a^4\cos^2 2\theta + a^4\sin^2 2\theta}{r^2}}$$

$$= \sqrt{\frac{a^4(\cos^2 2\theta + \sin^2 2\theta)}{r^2}}$$

$$= \sqrt{\frac{a^4(1)}{r^2}}$$

$$\boxed{\frac{ds}{d\theta} = a^2/r}$$

5 Show that for the Curve $2r\cos^2\theta = 1$

$$\frac{ds}{d\theta} = r\sqrt{8r-3}$$

Sol:- $2r\cos^2\theta = 1$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$r = \frac{1}{2\cos^2\theta}$$

$$r = \frac{1}{2}\sec^2\theta$$

$$\frac{dr}{d\theta} = \frac{1}{\cancel{2}} \cdot \cancel{2}\sec\theta \cdot \sec\theta \cdot \tan\theta$$

$$\frac{dr}{d\theta} = \sec^2\theta \tan\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + (\sec^2\theta\tan\theta)^2}$$

$$= \sqrt{\left(\tfrac{1}{2}\sec^2\theta\right)^2 + (\sec^2\theta\tan\theta)^2}$$

$$\frac{ds}{d\theta} = \sqrt{\left(\tfrac{1}{2}\sec^2\theta\right)^2 + \left(\sec^2\theta\tan\theta\right)^2}$$

$$= \sqrt{\tfrac{1}{4}\sec^4\theta + \sec^4\theta\tan^2\theta}$$

$$= \sqrt{\frac{\sec^4\theta + 4\sec^4\theta\tan^2\theta}{4}}$$

$$= \frac{1}{2}\sqrt{\sec^4\theta\left(1+4\tan^2\theta\right)}$$

$$= \frac{1}{2}\cdot\sec^2\theta\sqrt{1+4\tan^2\theta}$$

$$= \frac{1}{2}\sec^2\theta\sqrt{1+4\left(\sec^2\theta-1\right)}$$

$$= \frac{\sec^2\theta}{2}\sqrt{1+4(2r-1)}$$

$\because r = \frac{1}{2}\sec^2\theta$

$\sec^2\theta = 2r$

$$= \frac{\cancel{2}r}{\cancel{2}}\sqrt{1+8r-4}$$

$$= r\sqrt{8r-3}$$

⑥ Find $\frac{ds}{dt}$ for the Curve $x = e^t \sin t$

$y = e^t \cos t$

Sol :-

$x = e^t \sin t$

$\frac{dx}{dt} = e^t . \cos t + \sin t . e^t$

$\frac{dx}{dt} = e^t (\cos t + \sin t)$

$y = e^t \cos t$

$\frac{dy}{dt} = e^t (-\sin t) + \cos t . e^t$

$\frac{dy}{dt} = - e^t \sin t + e^t \cos t$

$\frac{dy}{dt} = e^t (\cos t - \sin t)$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{\left(e^t(\cos t + \sin t)\right)^2 + \left(e^t(\cos t - \sin t)\right)^2}$

$= \sqrt{e^{2t}(\cos t + \sin t)^2 + e^{2t}(\cos t - \sin t)^2}$

$\sqrt{e^{2t}\left(\cos^2 t + \sin^2 t + \cancel{2\cos t \sin t} + \cos^2 t + \sin^2 t - \cancel{2\cos t \sin t}\right)}$

$= \sqrt{e^{2t}(2\cos^2 t + 2\sin^2 t)}$

11

$$= \sqrt{e^{2t}(2\cos^2 t + 2\sin^2 t)}$$

$$= \sqrt{e^{2t} \cdot 2(\underline{\cos^2 t + \sin^2 t})}$$

$$= \sqrt{2e^{2t}}$$

$$\frac{ds}{dt} = \underline{\sqrt{2}\, e^{t}}$$

(7) Find ψ, $\frac{ds}{dt}$, $\frac{ds}{dx}$, and $\frac{ds}{dy}$ for the Curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

Sol:- $x = a(\cos t + \underline{t\sin t})$

$$\frac{dx}{dt} = a(-\sin t + t \cdot \cos t + \cancel{\sin t \cdot 1})$$

$$\frac{dx}{dt} = a(t\cos t)$$

$$y = a(\sin t - \underline{t\cos t})$$

$$\frac{dy}{dt} = a(\cos t - (t(-\sin t) + \cos t \cdot 1))$$

$$= a(\cancel{\cos t} + t\sin t - \cancel{\cos t})$$

$$\frac{dy}{dt} = a t \sin t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(at\cos t)^2 + (at\sin t)^2}$$

$$= \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t}$$

$$= \sqrt{a^2t^2(\cos^2 t + \sin^2 t)}$$

$$\frac{ds}{dt} = at$$

$$\tan\psi = \frac{dy}{dx}$$

$$= \frac{dy/dt}{dx/dt} = \frac{\cancel{at}\sin t}{\cancel{at}\cos t} = \tan t$$

$$\tan\psi = \tan t$$

$$\boxed{\psi = t}$$

$$\frac{dx}{dy} = \cot t$$

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad (\text{or}) \quad \frac{ds}{dx} = \sec\psi$$

$$\frac{ds}{dx} = \sec t$$

$$= \sqrt{1+\tan^2 t}$$

$$= \sqrt{\sec^2 t}$$

$$= \sec t$$

$$\frac{ds}{dy} = \operatorname{cosec}\psi = \operatorname{cosec} t$$

(8) Find $\frac{ds}{dr}$ for the Curve $r\theta = a$

Sol:-

$$r\theta = a$$

$$\theta = a/r$$

$$\frac{d\theta}{dr} = a\cdot\left(\frac{-1}{r^2}\right)$$

$$\frac{d\theta}{dr} = -\frac{a}{r^2}$$

$$\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

(or)

$$\frac{ds}{dr} = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}$$

$$\frac{ds}{dr} = \sqrt{1+\left(\cancel{r}\cdot\left(-a/\cancel{r^2}\right)\right)^2}$$

$$= \sqrt{1+\frac{a^2}{r^2}}$$

$$= \sqrt{\frac{r^2+a^2}{r^2}}$$

(or) $\frac{ds}{dr} = \sqrt{1+(a/r)^2}$

$$= \sqrt{1+\theta^2}$$

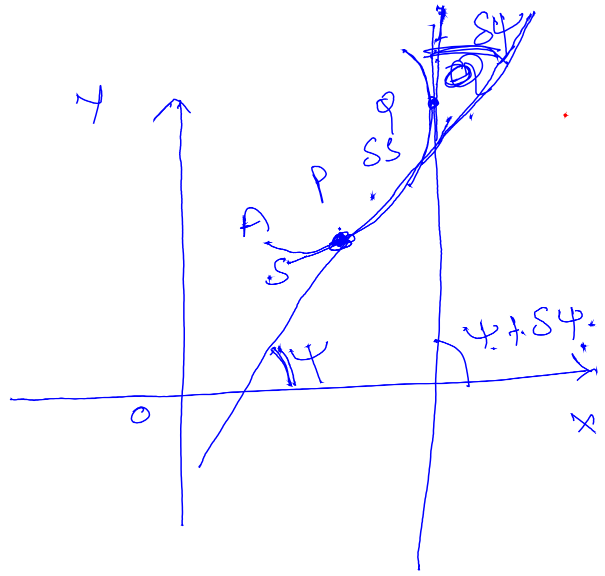
(1) find $\frac{ds}{dx}$ for $y = \ln\left(\frac{e^x-1}{e^x+1}\right)$

(2) If $x = a\cos^3 t$ $y = a\sin^3 t$ find $\frac{ds}{dt}$

(3) Find $\frac{ds}{dx}$, $\frac{ds}{dy}$, $\frac{ds}{d\theta}$ for $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

(4) find $\frac{ds}{dx}$ for $x^{2/3}+y^{2/3} = a^{2/3}$

Curvature



Consider a Curve in xy-plane. Let A be a fixed point on it. Let P & Q be two neighbouring points on the Curve.

$\overset{\frown}{AP} = s$ $\quad \overset{\frown}{AQ} = s+\delta s$, $\quad \overset{\frown}{PQ} = \delta s$

Let ψ and $\psi+\delta\psi$ respectively be the angles made by the tangents at P and Q with the x-axis.

The angle $\delta\psi$ between the tangents is called bending of the curve.

[The amount of bending of the curve at P is called the Curvature at P and defined as

$$\lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$$

denoted by K $\quad \therefore$ Curvature $K = \frac{d\psi}{ds}$]

Radius of Curvature

The reciprocal of the curvature is called the radius of curvature and is denoted by ρ.

$\therefore$ Radius of curvature $= \rho = \frac{1}{K} = \frac{ds}{d\psi}$