Linear combination of vectors :

Let $\{u_1, u_2, \dots u_n\}$ be n-vectors of vector space V and let $c_1, c_2, \dots c_n$ be n-scalars then the expression $c_1u_1 + c_2u_2 + \dots + c_nu_n = \sum_{i=1}^{n} c_i u_i$ is called the Linear combination of the vectors $\{u_1, u_2, \dots u_n\}$.

Span of set :

Let $\{u_1, u_2, \dots u_n\}$ be n-vectors in R^n, then the set of all linear combinations of $\{u_1, u_2, \dots u_n\}$ denoted by span $\{u_1, u_2, \dots u_n\}$ and is called the subset of R^n spanned (or generated) by $\{u_1, u_2, \dots u_n\}$.

i.e, Span $\{u_1, u_2, \dots u_n\}$ is the collection of all vectors that can be expressed as linear combinations of $\{u_1, u_2, \dots u_n\}$

i.e, If $b \in$ Span $\{u_1, u_2, \dots u_n\} \Rightarrow b = c_1u_1 + c_2u_2 + \dots + c_nu_n$ for some scalars $\{c_1, c_2, \dots c_n\}$.

The span of a subset S of a vector space V is denoted by $[S]$.

Linear Dependence :

If $\{u_1, u_2, \dots u_n\}$ are n-vectors of a vector space V, then $\{u_1, u_2, \dots u_n\}$ are said to be linearly dependent If $c_1u_1 + c_2u_2 + \dots + c_nu_n = 0 \Rightarrow$ atleast one of $c_i \neq 0$.

Linear Independence :

If $\{u_1, u_2, \ldots u_n\}$ are n-vectors of a vector space V, then $\{u_1, u_2, \ldots u_n\}$ are said to be linearly independent (LI) if $c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0 \Rightarrow$ all $c_i = 0$.

BASIS and DIMENSION :

1. A basis for a vector space is a set of vectors having two properties

   (i) It is Linearly independent

   and (ii) It spans the space

   (OR)

   A subset B of a vector space V is said to be a basis for V if

   (i) B is Linearly independent

   and (ii) $[B] = V$ (B generates V)

2. If a vectorspace V has a basis consisting of a ~~linearly~~ finite number of elements, the space is said to be finite-dimensional, The number of elements in a basis is called the dimension of the space and it is denoted by dim V

3. Standard basis :

   (i) $\{e_1 = (1,0),\ e_2 = (0,1)\}$ → standard basis for R^2

   (ii) $\{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$ → standard basis for $\mathbb{R}^3$

   (iii) $\{e_1 = (1,0,\ldots 0),\ e_2 = (0,1,\ldots 0),\ \ldots\ e_n = (0,0,\ldots 1)\}$ → standard basis for $\mathbb{R}^n$.

THEOREM :-

Prove that the set of all finite linear combinations of vectors of a vector space $V(F)$ is a subspace of (VF).

OR

Let S be a non-empty ~~subset~~ subset of a vector space V then the span of S ($[S]$) is a subspace of V.

Proof :- Let $[S] = \left\{ \sum_{i=1}^{n} c_i \alpha_i \,/\, c_i \in F,\ \alpha_i \in V \right\}$

$$\text{Let } \alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \cdots\cdots + c_n\alpha_n$$

$$\& \ \beta = c_1'\alpha_1 + c_2'\alpha_2 + c_3'\alpha_3 + \cdots\cdots + c_n'\alpha_n .$$

$$\text{then, } \alpha+\beta = (c+c_1')\alpha_1 + (c_2+c_2')\alpha_2 + \cdots\cdots + (c_n+c_n')\alpha_n .$$

$$\alpha+\beta = d_1\alpha_1 + d_2\alpha_2 + \cdots\cdots + d_n\alpha_n \in [S] \longrightarrow (1) .$$

$$\boxed{\text{where } d_i = c_i + c_i'} .$$

Again let $a \in F$ and $\alpha \in [S]$

$$\text{then } a\alpha = a[c_1\alpha_1 + c_2\alpha_2 + \cdots\cdots + c_n\alpha_n]$$

$$= (ac_1)\alpha_1 + (ac_2)\alpha_2 + \cdots\cdots + (ac_n)\alpha_n .$$

$$a\alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots\cdots + b_n\alpha_n \in [S] \longrightarrow (2) .$$

where $b_i = ac_i$

From (1) and (2), $[S]$ is closed under vector addition and scalar multiplication.

$\therefore [S]$ is a subspace of $V(F)$.

THEOREM

Let V be a vector space over the Field F then

i) The set of vectors V containing the Null vector is Linearly dependent (L.D.)

ii) The set consisting of a single vector α of V is Linearly Independent Iff $\alpha \neq 0$.

iii) Any Superset of a Linearly dependent set is L.D.

iv) The set $\{v_1, v_2\}$ is Linearly dependent Iff one of them is a scalar multiple of other [$\{v_1, v_2\}$ are collinear]

v) The set $\{v_1, v_2, v_3\}$ is Linearly dependent Iff one of them is a Linear combination of other two [$\{v_1, v_2, v_3\}$ are Coplanar]

PROOF:

(i) Let $S = \{v_1, v_2, \ldots v_n\}$ be a set of vectors Containing the Null vector.

Let $v_1 = 0$, Then $1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n = 0$

$\therefore$ There exists a Linear combination of the form $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ in which $c_1 \neq 0$

$\therefore S = \{v_1, v_2, \ldots v_n\}$ is Linearly Dependent.

(ii) Let $\{\alpha\}$ is L.I., To prove that $\alpha \neq 0$

Suppose $\alpha = 0$, Then $\{\alpha\}$ is L.D. ($\because$ Set consists of Null vector is L.D.)

This is contradiction that $\{\alpha\}$ is L.I.

$\therefore \alpha \neq 0$.

Conversely, Let $\alpha \neq 0$, Then $c\alpha = 0$

$\Rightarrow c = 0$ (or) $\alpha = 0$

$\Rightarrow c = 0 \quad (\because \alpha \neq 0)$

$\therefore c\alpha = 0 \Rightarrow c = 0$

$\therefore \{\alpha\}$ is L. I.

(iii) Let $S = \{v_1, v_2, \dots v_n\}$ be a Linearly Dependent set

$\Rightarrow c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \Rightarrow$ atleast one of $c_i \neq 0$.

Consider $W = \{v_1, v_2, \dots v_n, v_m\}$ be a Superset of S

Then $c_1 v_1 + c_2 v_2 + \dots + c_n v_n + c_m v_m = 0 \Rightarrow$ atleast one of $c_i \neq 0$

$\therefore$ Any superset of a L. D. set is also L. D.

(iv) Let $\{v_1, v_2\}$ be a L. D. set

$\Rightarrow c_1 v_1 + c_2 v_2 = 0 \Rightarrow$ atleast one of $c_i \neq 0$.

$\Rightarrow v_1 = \left(-\frac{c_2}{c_1}\right) v_2$ in which $c_1 \neq 0$.

$\therefore v_1$ is a scalar multiple of ~~other~~ v_2.

Conversely Let $v_1 = a v_2$

$\Rightarrow 1 \cdot v_1 - a v_2 = 0$

$\Rightarrow 1 \cdot v_1 + (-a) v_2 = 0$ in which $c_1 = 1 \neq 0$

$\therefore \{v_1, v_2\}$ is L. D. set

(v) Let $\{v_1, v_2, v_3\}$ is L.D. set

Then $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow$ atleast one of $c_i \neq 0$.

$\therefore v_1 = \left(-\frac{c_2}{c_1}\right) v_2 + \left(\frac{-c_3}{c_1}\right) v_3 \quad ; \; c_1 \neq 0$

$\therefore v_1$ is a Linear Combination of other two vectors v_1, v_2.

Conversely Let $v_1 = a v_2 + b v_3$

$\Rightarrow 1 \cdot v_1 + (-a) v_2 + (-b) v_3 = 0$ in which $c_1 = 1 \neq 0$

$\therefore \{v_1, v_2, v_3\}$ is L.D. set

THEOREM :

If W is a proper subspace of a finite-dimensional vector space V, then W is finite-dimensional and $\dim W < \dim V$.

PROOF: Let $\alpha \in W$ such that $\alpha \neq 0$.

Then there is a Basis of W containing α which contains NO more than $\dim V$ elements

Hence W is finite-dimensional and $\dim W \leq \dim V$.

Since W is proper subspace of V, There is a vector $\beta \in V$ and $\beta \notin W$.

$\therefore \boxed{\dim W < \dim V.}$

* If $\{v_1, v_2, \ldots v_K\}$ is a basis for a vector space V, then prove that every vector space in V can be written as a Linear combination of the Basis in one & only one way.

PROOF: Let $\{v_1, v_2, \ldots v_K\}$ is a basis for vector space V

Then any vector in V can be expressed as Linear combination of $\{v_1, v_2, \ldots v_K\}$

i.e. $v = c_1 v_1 + c_2 v_2 + \cdots + c_K v_K \longrightarrow \boxed{1}$

Also $v = c_1' v_1 + c_2' v_2 + \cdots + c_K' v_K \longrightarrow \boxed{2}$

$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_K v_K = c_1' v_1 + c_2' v_2 + \cdots + c_K' v_K$

$\Rightarrow (c_1 - c_1') v_1 + (c_2 - c_2') v_2 + \cdots + (c_K - c_K') v_K = 0$

$\Rightarrow c_1 - c_1' = 0, \quad c_2 - c_2' = 0, \ldots \; c_K - c_K' = 0$

$\left(\because \{v_1, v_2, \ldots v_K\} \text{ are L.I.} \right)$

$\Rightarrow c_1 = c_1', \quad c_2 = c_2', \ldots, \; c_K = c_K'$

$\Rightarrow c_i = c_i', \; \forall i$

$\therefore$ The Expression $\boxed{1}$ is UNIQUE

* If $\{v_1, v_2, \cdots v_m\}$ and $\{w_1, w_2, \cdots w_n\}$ are both bases for the same vector space then prove that $m = n$.

(OR)

Any Two bases of a Finite dimension vector space V have the Same finite number of elements.

PROOF: Let V be a finite dimensional vector space.

Let $S = \{v_1, v_2, \cdots v_m\}$ and $T = \{w_1, w_2, \cdots w_n\}$ be two bases for V

Since S is a Basis, S spans V

Since T is a Basis, T is Linearly Independent

$\therefore$ we have $m \leq n \longrightarrow \boxed{1}$ (If $\{v_1, v_2, \cdots v_m\}$ spans V & $\{w_1, w_2, \cdots w_n\}$ is L.I. then $m \leq n$)

Similarly

Since T is a Basis, T spans V

Since S is a Basis, S is Linearly Independent

$\therefore$ we have $n \leq m \longrightarrow \boxed{2}$

From $\boxed{1}$ & $\boxed{2}$ $\boxed{m = n}$

Hence the Theorem

THEOREM: If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is a finite dimensional and $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

PROOF: $W_1 \cap W_2$ has a finite Basis $\{\alpha_1, \alpha_2, \dots \alpha_K\}$ which is a part of a Basis $\{\alpha_1, \alpha_2, \dots \alpha_K, \beta_1, \beta_2, \dots \beta_m\}$ for W_1 and a part of a Basis $\{\alpha_1, \alpha_2, \dots \alpha_K, \gamma_1, \gamma_2, \dots \gamma_n\}$ for W_2

$\therefore$ $W_1 + W_2$ is spanned by the vectors $\{\alpha_1, \alpha_2, \dots \alpha_K, \beta_1, \dots \beta_m, \gamma_1, \dots \gamma_n\}$ and these vectors are linearly Independent.

Suppose $x_1\alpha_1 + x_2\alpha_2 + \dots + x_K\alpha_K + y_1\beta_1 + y_2\beta_2 + \dots + y_m\beta_m + z_1\gamma_1 + z_2\gamma_2 + \dots + z_n\gamma_n = 0$

(OR)

$$\sum x_i\alpha_i + \sum y_j\beta_j + \sum z_r\gamma_r = 0$$

Then
$$-\sum z_r\gamma_r = \sum x_i\alpha_i + \sum y_j\beta_j \longrightarrow (*)$$

$\Rightarrow$ $\sum z_r\gamma_r \in W_1$, also $\sum z_r\gamma_r \in W_2$.

$\therefore$ $\sum z_r\gamma_r = \sum c_i\alpha_i$, for some scalars $c_1, c_2, \dots c_K$.

But the set $\{\alpha_1, \dots \alpha_K, \gamma_1, \dots \gamma_n\}$ is independent, so each $z_r = 0$.

$\therefore$ From (*) $\sum x_i\alpha_i + \sum y_j\beta_j = 0$.

Since $\{\alpha_1, \dots \alpha_K, \beta_1 \dots \beta_m\}$ is Independent, so each $x_i = y_j = 0$.

$\therefore$ ~~From (*)~~ $\{\alpha_1, \dots \alpha_K, \beta_1, \dots \beta_m, \gamma_1 \dots \gamma_n\}$ is a Basis for $W_1 + W_2$

Finally $\dim W_1 + \dim W_2 = (K+m) + (K+n)$
$= K + (m + K + n)$
$= \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

Hence the proof.

Problems

1. Show that $(3,7)$ belongs to $[(1,2),(0,1)]$ in $V_2(R)$.

Sol: Let $(3,7) = c_1(1,2) + c_2(0,1)$

$\Rightarrow (3,7) = (c_1, 2c_1 + c_2)$

$\Rightarrow \boxed{c_1 = 3}, \quad 2c_1 + c_2 = 7 \Rightarrow \boxed{c_2 = 1}$

$\therefore \quad (3,7) = 3(1,2) + 1(0,1)$ is the required L.C.

2. Determine whether the vector $(8,0,5)$ is a linear combination of $\{(1,2,3), (0,1,4), (2,-1,1)\}$.

Sol: Let $(8,0,5) = c_1(1,2,3) + c_2(0,1,4) + c_3(2,-1,1)$

$\Rightarrow (8,0,5) = (c_1 + 2c_3, \ 2c_1 + c_2 - c_3, \ 3c_1 + 4c_2 + c_3)$

$$\Rightarrow \begin{aligned} c_1 + 2c_3 &= 8 \\ 2c_1 + c_2 - c_3 &= 0 \\ 3c_1 + 4c_2 + c_3 &= 5 \end{aligned}$$

solving, we get $c_1 = 2, \ c_2 = -1, \ c_3 = 3$

$\therefore \quad (8,0,5) = 2(1,2,3) + (-1)(0,1,4) + 3(2,-1,1)$ is the required L.C.

3. Show that $h = 4x^2 + 3x - 7$ lies in span $\{f, g\}$ where $f = 2x^2 - 5$ and $g = x + 1$.

Soln: Let $h = c_1 f + c_2 g$

$\Rightarrow 4x^2 + 3x - 7 = 2c_1 x^2 - 5c_1 + c_2 x + c_2$

comparing the coefficients on both sides, & solving, we get

$\boxed{c_1 = 2} \quad \boxed{c_2 = 3} \quad \therefore 4x^2 + 3x - 7 = 2(2x^2 - 5) + 3(x+1)$

4. Show that $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

Sol: let $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} c_1 + 2c_2 &= -1 \\ -3c_2 + c_3 &= 7 \\ 2c_1 + 2c_3 &= 8 \\ c_1 + 2c_2 &= -1 \end{aligned}\right\} \text{solving we get } c_1 = 3,\ c_2 = -2,\ \underline{c_3 = 1}$$

$\therefore \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + (-2) \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + (1) \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ is the required L.C.

5. Let $S = \{(1,2,1), (1,1,-1), (4,5,-2)\}$. Determine which of the following vectors are in $[S]$

(i) $(0,0,0)$ ✓

(ii) $(1,1,0)$

(iii) $(2,-1,-8)$ ✓

(iv) $\left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$ ✓

(v) $(1,0,1)$

(vi) $(1,-3,5)$

6. Verify whether $(3,7) \in [(1,2),(2,4)]$

Sol: Let $(3,7) = c_1(1,2) + c_2(2,4)$

$$\Rightarrow \left.\begin{aligned} c_1 + 2c_2 &= 3 \\ 2c_1 + 4c_2 &= 7 \end{aligned}\right\} \Rightarrow c_1 + 2c_2 = 3 \quad (\because \text{ one is Scalar Multiple of other})$$

$\Downarrow$

which is inconsistent.

$\therefore (3,7) \notin [(1,2),(2,4)]$

Note : Linearly Independent vectors → trivial linear combination ($|A| \neq 0$)

Linearly dependent vectors → non trivial linear combination ($|A| = 0$)

Ex: Decide ~~whether~~ the dependence (or) independence of the following :

(i) $\{(3,0,0), (4,1,0), (2,5,2)\}$

(ii) $\{(1,1,2), (1,2,1), (3,1,1)\}$

(iii) $\{(1,1,0), (0,1,1), (1,0,-1)\}$

(iv) $\{(2,1,0), (1,-1,2), (1,2,1)\}$

Sol: (i) Consider $\begin{vmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 5 & 2 \end{vmatrix} = 3(2-0) - 0(\,"\,) + 0(\,"\,)$

$= 6 \neq 0$

$\therefore \{(3,0,0), (4,1,0), (2,5,2)\}$ are L.I.

Similarly others.

problems on basis and dimension

III show that $S = \{(1,1,2), (1,2,1), (3,1,1)\}$ forms a basis for $V_3(R)$.

Sol: Given $S = \{(1,1,2), (1,2,1), (3,1,1)\}$

(i) Consider $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = -7 \neq 0$

$\therefore$ S is L.I.

(ii) Let $(x,y,z) \in V_3(R)$ then

$(x,y,z) = c_1(1,1,2) + c_2(1,2,1) + c_3(3,1,1)$

$\Rightarrow$ ~~~~

$$c_1 + c_2 + 3c_3 = x \rightarrow \boxed{1}$$
$$c_1 + 2c_2 + c_3 = y \rightarrow \boxed{2}$$
$$2c_1 + c_2 + c_3 = z \rightarrow \boxed{3}$$

solving, we get $c_1 = \frac{1}{7}(5z - x - 2y)$

$c_2 = \frac{1}{7}(5y - x - 2z)$

$c_3 = \frac{1}{7}(3x - y - z)$

$\therefore (x,y,z) = \frac{1}{7}(5z-x-2y)(1,1,2) + \frac{1}{7}(5y-x-2z)(1,2,1) + \frac{1}{7}(3x-y-z)(3,1,1)$

$\therefore$ S spans $V_3(R)$

From (i) and (ii) S forms a basis for $V_3(R)$.

$\boxed{2}$ Find the basis and dimension of the subspace spanned by the vectors $\{(1,1,2,4)\ (2,-1,-5,2),\ (1,-1,-4,0),$ ~~and~~ $(2,1,1,6)\}$.

Sol: Consider

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix} \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \\ R_4 = R_4 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix} \quad R_2 \leftrightarrow -R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & -3 & -9 & -6 \end{bmatrix} \quad \begin{array}{l} R_3 = R_3 + 2R_2 \\ R_4 = R_4 + 3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{Echelon form}$$

In this there are 2 non zero rows (2 independent rows)

$\therefore$ Basis for $A = \{(1,1,2,4),\ (0,1,3,2)\}$

dimension $= \boxed{2}$