1. Solve $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$

Sol:- Given $x^3y''' + 3x^2y'' + xy' + 8y = 65\cos(\log x)$

Put $\log x = t$

$x = e^t$

$x^3y''' = D(D-1)(D-2)y$

$x^2y'' = D(D-1)y$ where $D = \frac{d}{dt}$

$xy' = Dy$

then

$D(D-1)(D-2)y + 3\cdot D(D-1)y + Dy + 8y = 65\cos t$

$D(D^2-2D-D+2)y + 3(D^2-D)y + Dy + 8y = 65\cos t$

$(D^3 - 3\cancel{D^2} + 2\cancel{D} + 3\cancel{D^2} - 3\cancel{D} + \cancel{D} + 8)y = 65\cos t$

$(D^3+8)y = 65\cos t$

A.E is $m^3+8=0$ $\quad a^3+b^3=(a+b)(a^2-ab+b^2)$

$m^3+2^3=0$

$\Rightarrow (m+2)(m^2-2m+4)=0$

$m+2=0 \Rightarrow m=-2$, $m^2-2m+4=0$

$m = \frac{-(-2) \pm \sqrt{(-2)^2-4\cdot1\cdot4}}{2}$

$= \frac{2 \pm \sqrt{-12}}{2}$

$$m = \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm \sqrt{-4 \times 3}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}\, i}{2}$$

$$= 1 \pm \sqrt{3}\, i$$

Roots $= m = -2, \quad 1 \pm \sqrt{3}\, i$

$$CF = c_1 e^{-2t} + e^{t}\left(c_2 \cos\sqrt{3}\,t + c_2 \sin\sqrt{3}\,t\right)$$

$$P.I = \frac{1}{D^3 + 8} 65 \cos t$$

$a = 1$

Put $D^2 = -1^2$

$$= \frac{1}{D(-1) + 8} 65 \cos t$$

$$= \frac{1}{8 - D} 65 \cos t$$

$$= \frac{8 + D}{(8 - D)(8 + D)} 65 \cos t$$

$$= \frac{(8 + D)\, 65 \cos t}{64 - D^2}$$

Put $D^2 = -1^2$

$$= \frac{65\left(8 \cos t + D(\cos t)\right)}{64 - (-1)} = \frac{\cancel{65}\left[8 \cos t + (-\sin t)\right]}{\cancel{65}}$$

$P.I = 8\cos t - \sin t$

$y = CF + PI$

$y = c_1 e^{-2t} + e^{t}\left(c_2 \cos\sqrt{3}\,t + c_2 \sin\sqrt{3}\,t\right) +$
$8\cos t - \sin t$

Sol. is

$t = \log x$

$y = c_1\left(\frac{1}{x^2}\right) + x\left(c_2 \cos\sqrt{3}(\log x) + c_2 \sin\sqrt{3}(\log x)\right) +$

$8\cos(\log x) - \sin(\log x)$

$x = e^{t}$

$(e^{t})^2 = x^2$

$e^{-2t} = \frac{1}{x^2}$

(2) Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Sol:- $y'' + \frac{1}{x}y' = \frac{12\log x}{x^2}$

$x^2y'' + xy' = 12\log x$

put $\log x = t$

$x = e^{t}$

$x^2y'' = D(D-1)y$

$xy' = Dy$

$$D(D-1)y + Dy = 12t$$

$$(D^2 - \cancel{D} + \cancel{D})y = 12t$$

$$D^2 y = 12t$$

A.E is

$$m^2 = 0$$

$$m = 0, 0$$

$$CF = (c_1 + c_2 t)e^{0t}$$

$$P.I = \frac{1}{D^2} 12t$$

$$= \frac{1}{D}\left(\int 12t\,dt\right)$$

$$= 12 \cdot \frac{1}{D} \cdot \int t\,dt$$

$$= 12 \cdot \frac{1}{D}\left(\frac{t^2}{2}\right)$$

$$= 6 \cdot \int t^2\,dt$$

$$= 6 \cdot \frac{t^3}{3}$$

$$= 2t^3$$

$$y = CF + PI$$

$$y = (C_1 + c_2 t)e^{0t} + 2t^3$$

Sol :- $y = (c_1 + c_2 \cdot (\log x))\, e^{0t} + 2 \cdot (\log x)^3$

Solve

1. $x^2y'' - 4xy' + 6y = \cos(2\log x)$

2. $x^2y'' - 3xy' + 4y = 1 + x^2$

3. $x^3y''' + 2x^2y'' + 2y = 10(x + 1/x)$

4. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + y = (\log x)^2$

Problems on Legendre's linear equation

1. Solve $(x+1)^2y'' + 2(x+1)y' - y = 0$

$ax+b = x+1$

$a = 1$.

Put $t = \log(x+1)$

$e^t = x+1$

$x = e^t - 1$

Sol:-

$(x+1)^2y'' = 1^2 D(D-1)y$

$(x+1)y' = Dy$

where $D = \dfrac{d}{dt}$

$1\,D(D-1)y + 2Dy - y = 0$

$(D^2 - D + 2D - 1)y = 0$

$(D^2 + D - 1)y = 0$

A.E is $m^2 + m - 1 = 0$

$$m = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-1)}}{2}$$

$$m = \frac{-1 \pm \sqrt{5}}{2}$$

$$CF = c_1 e^{\left(\frac{-1+\sqrt{5}}{2}\right)t} + c_2 e^{\left(\frac{-1-\sqrt{5}}{2}\right)t}$$

Sol

$$y = c_1 (x+1)^{\left(\frac{-1+\sqrt{5}}{2}\right)} + c_2 (x+1)^{\left(\frac{-1-\sqrt{5}}{2}\right)}$$

$e^t = x+1$

$$e^{\left(\frac{-1+\sqrt{5}}{2}\right)t} = (x+1)^{\frac{-1+\sqrt{5}}{2}}$$

② Solve

$$(3x+2)^2 y'' + 3(3x+2)y' - 36y = 8x^2+4x+1$$

Sol:-

$ax+b = 3x+2$

$a = 3$

Put $t = \log(3x+2)$

$e^t = 3x+2$

$3x = e^t - 2$

$x = \frac{1}{3}(e^t - 2)$

$(3x+2)^2 y'' = 3^2 D(D-1)y$

$(3x+2)y' = 3Dy$

$$3^2 D(D-1)y + 3 \cdot 3Dy - 36y = 8\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1$$

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$$(9D(D-1)+9D-36)y = \frac{8}{9}(e^{t}-2)^{2} + \frac{4}{3}(e^{t}-2)+1$$

$$(9D^{2}-9\cancel{D}+9\cancel{D}-36)y = \frac{8}{9}(e^{2t}+4-4e^{t}) + \frac{4}{3}(e^{t}-2)+1$$

$$(9D^{2}-36)y = \frac{8}{9}e^{2t}+\frac{32}{9}-\frac{32}{9}e^{t}+\frac{4}{3}e^{t} - \frac{8}{3}+1$$

$$(9D^{2}-36)y = \frac{8}{9}e^{2t}+\underbrace{\frac{4}{3}e^{t}-\frac{32}{9}e^{t}}+\frac{32}{9}-\frac{8}{3}+1$$

$$(9D^{2}-36)y = \frac{8}{9}e^{2t}+\frac{12e^{t}-32e^{t}}{9}+\frac{32-24+9}{9}$$

$$(9D^{2}-36)y = \frac{8}{9}e^{2t} - \frac{20e^{t}}{9}+\frac{17}{9}$$

$$9(D^{2}-4)y = \frac{8}{9}e^{2t}-\frac{20}{9}e^{t}+\frac{17}{9}$$

$$(D^{2}-4)y = \frac{1}{9}\left(\frac{8}{9}e^{2t}-\frac{20}{9}e^{t}+\frac{17}{9}\right)$$

$$(D^{2}-4)y = \frac{8}{81}e^{2t}-\frac{20}{81}e^{t}+\frac{17}{81}$$

A.E is $m^{2}-4=0$

$m=\pm 2$

CF $= c_1e^{2t}+c_2e^{-2t}$.

$$P.I = \frac{1}{D^2-4}\left(\frac{8}{81}e^{2t} - \frac{20}{81}e^{t} + \frac{17}{81}e^{0t}\right)$$ 8

$$= \frac{8}{81}\frac{1}{D^2-4}e^{2t} - \frac{20}{81}\frac{1}{D^2-4}e^{t} + \frac{17}{81}\frac{1}{D^2-4}e^{0t}$$

Put $D=2$ Put $D=1$ Put $D=0$

$$= \frac{8}{81}\cdot\frac{1}{\underline{2^2-4}}e^{2t} - \frac{20}{81}\cdot\frac{1}{1^2-4}e^{t} + \frac{17}{81}\cdot\frac{1}{0-4}e^{0t}$$

$$= \frac{8}{81}t\cdot\frac{1}{2D}e^{2t} - \frac{20}{81}\frac{1}{(-3)}e^{t} + \frac{17}{81}\cdot\frac{1}{(-4)}\cdot 1$$

Put $D=2$

$$= \frac{8}{81}\cdot t\cdot\frac{1}{2(2)}e^{2t} + \frac{20}{243}e^{t} - \frac{17}{324}$$

$$P.I = \frac{8t\,e^{2t}}{324} + \frac{20}{243}e^{t} - \frac{17}{324}$$

$$y = CF + PI$$

$$y = \left(c_1 e^{2t} + c_2 e^{-2t}\right) + \frac{8te^{2t}}{324} + \frac{20}{243}e^{t} - \frac{17}{324}$$

Sol is

$$y = c_1(3x+2)^2 + c_2\frac{1}{(3x+2)^2} + \frac{8}{324}\log(3x+2)(3x+2)^2 + \frac{20}{243}(3x+2) - \frac{17}{324}$$

3 Solve $(2x+1)^2 y'' - 2(2x+1)y' - 12y = 6x+5$

Sol:-

$ax+b = 2x+1$

$a = 2$

Put $t = \log(2x+1)$

$e^t = 2x+1$

$2x = e^t - 1$

$x = \frac{e^t - 1}{2}$

$(2x+1)^2 y'' = 2^2 D(D-1)y$

$(2x+1)\, y' = 2Dy$

where $D = \frac{d}{dt}$

$$2^2 D(D-1)y - 2 \cdot 2Dy - 12y = \cancel{6}^{\,3}\left(\frac{e^t-1}{\cancel{2}}\right) + 5$$

$$(4D^2 - 4D - 4D - 12)y = 3e^t - 3 + 5$$

$$(4D^2 - 8D - 12)\, y = 3e^t + 2$$

A.E is

$$4m^2 - 8m - 12 = 0$$

$$4m^2 - 12m + 4m - 12 = 0$$

$$4m(m-3) + 4(m-3) = 0$$

$$(m-3)(4m+4) = 0$$

$$m = 3, \quad m = -1$$

$$CF = c_1 e^{3t} + c_2 e^{-t}$$

$$P.I = \frac{1}{4D^2 - 8D - 12}(3e^t + 2)$$

$$= 3 \cdot \frac{1}{4D^2 - 8D - 12} e^t + 2 \cdot \frac{1}{4D^2 - 8D - 12} e^{0t}$$

Put $D = 1$ Put $D = 0$

$$= 3 \cdot \frac{1}{4(1) - 8(1) - 12} e^t + 2 \cdot \frac{1}{0 - 0 - 12} e^{0t}$$

$$= 3 \cdot \frac{1}{-16} e^t - \frac{2}{12}$$

$$= -\frac{3}{16} e^t - \frac{1}{6}$$

$$y = CF + PI$$

$$y = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{16} e^t - \frac{1}{6}$$

Sol

$$y = c_1 (2x+1)^3 + c_2 \cdot \frac{1}{(2x+1)} - \frac{3}{16}(2x+1) - \frac{1}{6}$$

④ Solve $(2x+1)^2 y'' - 6(2x+1)y' + 16y = 8(2x+1)^2$ (11)

Sol:-

$ax+b = 2x+1$

$a = 2$

Put $t = \log(2x+1)$

$2x+1 = e^t$

$2x = e^t - 1$

$x = \frac{e^t - 1}{2}$

$(2x+1)^2 y'' = 2^2 D(D-1)y$

$(2x+1)y' = 2Dy$

where $D = \frac{d}{dt}$

$2^2 D(D-1)y - 6 \cdot 2Dy + 16y = 8(e^t)^2$

$(4D^2 - 4D - 12D + 16)y = 8e^{2t}$

$(4D^2 - 16D + 16)y = 8e^{2t}$

A.E is $4m^2 - 16m + 16 = 0$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$cf = (c_1 + c_2 t)e^{2t}$

$P.I = \frac{1}{4D^2 - 16D + 16} 8e^{2t}$

$$PI = 8 \cdot \frac{1}{4D^2-16D+16} e^{2t}$$

Put $D=2$

$$= 8 \cdot \frac{1}{4(2)^2-16\cdot 2+16} e^{2t}$$

$$= 8 \cdot \frac{1}{16-32+16} e^{2t}$$

$$= 8 \cdot t \cdot \frac{1}{8D-16} e^{2t}$$

put $D=2$

$$= 8t \frac{1}{8(2)-16} e^{2t}$$

$$= \cancel{8} t^2 \cdot \frac{1}{\cancel{8}} e^{2t}$$

$$P.I = \underline{\underline{t^2 e^{2t}}}$$

$$y = CF + PI$$

$$y = (c_1+c_2 t) e^{2t} + t^2 e^{2t}$$

Sol

$$y = \left(c_1 + c_2 \log(2x+1)\right)(2x+1)^2 + \left(\log(2x+1)\right)^2 \cdot (2x+1)^2$$

⑤ Solve $(x+2)^2 y'' - (x+2)y' + y = 3x+4$

Sol:-

$ax+b = x+2$

$a = 1$

$(x+2)^2 y'' = 1^2 D(D-1)y$

$(x+2)y' = Dy$

Put $t = \log(x+2)$

$x+2 = e^t$

$x = e^t - 2$

$$D(D-1)y - Dy + y = 3(e^t - 2) + 4$$

$$(D^2 - D - D + 1)y = 3e^t - 6 + 4$$

$$(D^2 - 2D + 1)y = 3e^t - 2$$

A.E is

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1, 1$

$CF = (c_1 + c_2 t) e^t$

$$P.I = \frac{1}{D^2 - 2D + 1}(3e^t - 2)$$

$$= 3 \cdot \frac{1}{D^2 - 2D + 1} e^t - \frac{2}{D^2 - 2D + 1} e^{0t}$$

put $D = 1$, Put $D = 0$

$$= 3 \cdot \frac{1}{1 - 2 + 1} e^t - \frac{2}{0 - 0 + 1} e^{0t}$$

$P.I = 3 \cdot t \frac{1}{2D-2} e^{t} - 2$

put $D=1$

$= 3t \cdot \frac{1}{2(1)-2} e^{t} - 2$

$= 3t^2 \cdot \frac{1}{2} e^{t} - 2$

$P.I = \frac{3t^2e^t}{2} - 2$

$y = CF + PI$

$y = (c_1 + c_2 t) e^{t} + \frac{3t^2e^t}{2} - 2$

Sol is

$y = (c_1 + c_2 \log(x+2))(x+2) + \frac{3}{2}(\log(x+2))^2(x+2) - 2$

⑥ Solve $(2x-1)^2 y'' + (2x-1)y' - 2y = 8x^2 - 2x + 3$

Sol :-

$ax + b = 2x - 1$

$a = 2$

Put $\log(2x-1) = t$

$e^t = 2x - 1$

$2x = e^t + 1$

$x = \frac{e^t+1}{2}$

$$(2x-1)^2y'' = 2^2D(D-1)y$$

$$(2x-1)y' = 2Dy \qquad \text{where } D = \frac{d}{dt}$$

$$2^2D(D-1)y + 2Dy - 2y = 8\left(\frac{e^t+1}{2}\right)^2 - \cancel{2}\left(\frac{e^t+1}{\cancel{2}}\right) + 3$$

$$(4D^2-4D+2D-2)y = \cancel{8}^{\,2}\frac{(e^t+1)^2}{\cancel{4}} - \cancel{2}(e^t+1) + 3$$

$$(4D^2-2D-2)y = 2(e^{2t}+1+2e^t) - e^t - 1 + 3$$

$$(4D^2-2D-2)y = 2e^{2t} + 2 + 4e^t - e^t - 1 + 3$$

$$(4D^2-2D-2)y = 2e^{2t} + 3e^t + 4$$

A.E is

$$4m^2 - 2m - 2 = 0$$

$$2m^2 - m - 1 = 0$$

$$2m^2 - 2m + m - 1 = 0$$

$$2m(m-1) + 1(m-1) = 0$$

$$(m-1)(2m+1) = 0$$

$$m = 1, \quad m = -1/2$$

$$CF = c_1e^t + c_2e^{-1/2\,t}$$

$$P.I = \frac{1}{4D^2 - 2D - 2}\, 2e^{2t} + 3e^{t} + 4$$

$$= 2 \cdot \frac{1}{4D^2 - 2D - 2} e^{2t} + 3 \cdot \frac{1}{4D^2 - 2D - 2} e^{t} +$$

Put $D = 2$ Put $D = 1$

$$4 \cdot \frac{1}{4D^2 - 2D - 2} e^{0t}$$

Put $D = 0$

$$= 2 \cdot \frac{1}{4(2)^2 - 2(2) - 2} e^{2t} + 3 \cdot \frac{1}{4(1)^2 - 2(1) - 2} e^{t}$$

$$+ 4 \cdot \frac{1}{4(0) - 2(0) - 2} e^{0t}$$

$$= \cancel{2} \cdot \frac{1}{\cancel{10}_{5}} e^{2t} + 3 \cdot \frac{1}{\underbrace{4-4}} e^{t} +$$

$$\frac{4}{(-2)} e^{0t}$$

$$= \frac{e^{2t}}{5} + 3t \cdot \frac{1}{8D - 2} e^{t} - 2$$

Put $D = 1$

$$= \frac{e^{2t}}{5} + 3t \cdot \frac{1}{8(1) - 2} e^{t} - 2$$

$$= \frac{e^{2t}}{5} + 3t\, \frac{1}{6} e^{t} - 2$$

$$y = CF + PI$$

$$y = c_1 e^{t} + c_2 e^{-\frac{1}{2}t} + \frac{e^{2t}}{5} + \frac{\cancel{3}\, t e^{t}}{\cancel{6}_2} - 2$$

$$y = c_1 e^{t} + c_2 e^{-t/2} + \frac{e^{2t}}{5} + \frac{t e^{t}}{2} - 2$$

Sol is

$$y = c_1 (2x-1) + c_2 \cdot \frac{1}{(2x-1)^{1/2}} + \frac{1}{5}(2x-1)^2$$

$$+ \frac{\log(2x-1) \cdot (2x-1)}{2} - 2$$

Solve

① $(3x+2)^2 y'' + 3(3x+2) y' - 36y = 3x^2+4x+1$

② $(x+1)^2 y'' + (x+1) y' + y = 4 \cos \log(x+1)$

③ $(1-2x)^2 y'' - 2(2x-1) y' = 0$

④ $(2x+3)^2 y'' + 2(2x+3) y' - 12y = 0$