Lagrange's Linear PDE

The equation of the form $Pp+Qq=R$ where $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$

P, Q, R functions

Form A.E $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \rightarrow (1)$

We can consider suitable pairs which can be put in the forms like

$f(x)\,dx = g(y)\,dy$, $g(y)\,dy = h(z)\,dz$ &

$f(x)\,dx = h(z)\,dz$ (Separation Variables)

Integrate and get $u(x,y,z)=c_1$, $v(x,y,z)=c_2$

General solution is $\phi(u,v)=0$

Method of Multipliers

We have a property in ratio and proportion

$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$ this equal to

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\frac{k_1a_1+k_2a_2+k_3a_3}{k_1b_1+k_2b_2+k_3b_3}$$

w.r.t. (1) we get

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{K_1 dx + K_2 dy + K_3 dz}{K_1 P + K_2 Q + K_3 R}$$

choose K_1, K_2, K_3 s.t

$$K_1 P + K_2 Q + K_3 R = 0$$

then $K_1 dx + K_2 dy + \underline{K_3 dz} = 0$

then $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$

General sol is

$$\phi(u,v) = 0$$

Problems

(1) Solve $xp + yq = z$ By Lagrange's method

Sol:-

$$Pp + Qq = R$$

$$P = x, \quad Q = y, \quad R = z$$

A.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log (x/y) = \log c_1$$

$$x/y = c_1 \longrightarrow (1)$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z} \Longrightarrow \log y = \log z + \log c_2$$

$$\log y - \log z = \log c_2$$

$$\log (y/z) = \log c_2$$

$$y/z = c_2 \longrightarrow (2)$$

General sol

$$\phi(x/y, \ y/z) = 0 \quad (\text{from } (1) \ \& \ (2))$$

(2) Solve $pyz + qzx = xy$

Sol:- $Pp + Qq = R$

$P = yz, \quad Q = zx \quad R = xy$

A.E $\quad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

$$\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x} \qquad\qquad \frac{dy}{\cancel{z}\cancel{x}} = \frac{dz}{\cancel{x}y}$$

$$x\,dx = y\,dy \qquad\qquad y\,dy = z\,dz$$

$$\int x\,dx = \int y\,dy \qquad\qquad \int y\,dy = \int z\,dz$$

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \qquad\qquad \frac{y^2}{2} = \frac{z^2}{2} + c_2$$

$$\frac{x^2}{2} - \frac{y^2}{2} = c_1 \qquad\qquad \frac{y^2}{2} - \frac{z^2}{2} = c_2$$

$$\underbrace{x^2 - y^2}_{u} = 2c_1 \qquad\qquad \underbrace{y^2 - z^2}_{v} = 2c_2$$

General sol is $\phi(u, v) = 0$

$$\phi(x^2 - y^2,\ y^2 - z^2) = 0$$

③ Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ by Lagrange's method

Sol :-

$$Pp + Qq = R$$

$P = \sqrt{x}$, $Q = \sqrt{y}$, $R = \sqrt{z}$

A.E is $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} \qquad\qquad \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}} \qquad\qquad \int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$2\sqrt{x} = 2\sqrt{y} + c_1 \qquad\qquad 2\sqrt{y} = 2\sqrt{z} + c_2$$

$$\boxed{\int x^{-1/2}\,dx = \frac{x^{-1/2+1}}{-1/2+1} = \frac{x^{1/2}}{1/2} = 2\sqrt{x}}$$

$$2\sqrt{y} - 2\sqrt{z} = c_2$$

$$\sqrt{y} - \sqrt{z} = \frac{c_2}{2}$$

$$2\sqrt{x} - 2\sqrt{y} = c_1$$

$$\sqrt{x} - \sqrt{y} = \frac{c_1}{2}$$

General sol

$$\phi(u, v) = 0$$

$$\phi\left(\sqrt{x} - \sqrt{y},\ \sqrt{y} - \sqrt{z}\right) = 0$$

④ Solve $p \tan x + q \tan y = \tan z$.

Sol:- $P = \tan x$, $Q = \tan y$ $\quad R = \tan z$

$$Pp + Qq = R$$

A.E is $\quad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\cot x \, dx = \cot y \, dy = \cot z \, dz$$

$$\cot x \, dx = \cot y \, dy$$

$$\int \cot x \, dx = \int \cot y \, dy$$

$$\log(\sin x) = \log(\sin y) + \log c_1$$

$$\log(\sin x) - \log(\sin y) = \log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$$

$$\Rightarrow \quad \frac{\sin x}{\sin y} = c_1$$

$$\cot y \, dy = \cot z \, dz$$

$$\int \cot y \, dy = \int \cot z \, dz$$

$$\log(\sin y) = \log(\sin z) + \log c_2$$

$$\log(\sin y) - \log(\sin z) = \log c_2$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log c_2$$

$$\Rightarrow \quad \frac{\sin y}{\sin z} = c_2$$

General sol

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

⑤ Solve $(y^2+z^2)p + x(yq - z) = 0$

Sol:-

$$(y^2+z^2)p + xyq - xz = 0$$

$$(y^2+z^2)p + xyq = xz$$

$$Pp + Qq = R$$

$$P = y^2+z^2, \quad Q = xy, \quad R = xz$$

A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2+z^2} = \underbrace{\frac{dy}{xy} = \frac{dz}{xz}}$$

$$\frac{dy}{xy} = \frac{dz}{xz}$$

$$\int \frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$$

$$\log y - \log z = \log c_2$$

$$\log(y/z) = \log c_2$$

$$\underline{\underline{y/z = c_2}}$$

using the multipliers $x, -y, -z$

$$\frac{dx}{y^2+z^2} = \frac{dy}{xy} = \frac{dz}{xz} = \frac{x\,dx - y\,dy - z\,dz}{x(y^2+z^2) - y(xy) - z(xz)}$$

$$\frac{dx}{y^2+z^2} = \frac{dy}{xy} = \frac{dz}{xz} = \frac{x\,dx - y\,dy - z\,dy}{0}$$

$$x\,dx - y\,dy - z\,dz = 0$$

$$\int (x\,dx - y\,dy - z\,dz) = 0$$

$$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = c$$

$$x^2 - y^2 - z^2 = 2c$$

General sol is $\phi(u,v) = 0$

$$\phi\left(y/z,\ x^2-y^2-z^2\right) = 0$$

⑥ Solve $(x^2-y^2-z^2)\,p + 2xy\,q = 2xz$

Sol:- $P = x^2-y^2-z^2$, $Q = 2xy$, $R = 2xz$

A.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z} \Rightarrow \log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1 \Rightarrow y/z = c_1$$

using multipliers x, y, z

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx+y\,dy+z\,dz}{x(x^2-y^2-z^2)+y(2xy)+z(2xz)}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{x^3+xy^2+xz^2}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{dz}{2\cancel{x}z} = \frac{x\,dx+y\,dy+z\,dz}{\cancel{x}(x^2+y^2+z^2)}$$

$$\frac{dz}{2z} = \frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2}$$

$$\int\frac{dz}{z} = \int\frac{2(x\,dx+y\,dy+z\,dz)}{x^2+y^2+z^2} \qquad \left\{ \int\frac{f'(x)}{f(x)}dx = \log(f(x)) \right.$$

$$\log z = \log(x^2+y^2+z^2) + \log c_2$$

$$\log z - \log(x^2+y^2+z^2) = \log c_2$$

$$\log\left(\frac{z}{x^2+y^2+z^2}\right) = \log c_2$$

$$\frac{z}{x^2+y^2+z^2} = c_2$$

General sol is $\phi\left(y/z, \frac{z}{x^2+y^2+z^2}\right) = 0$

8 Solve $(y+z)p - (x+z)q = x-y$ by Lagrange's method

Sol :- $P = y+z$, $Q = -(x+z)$, $R = x-y$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y}$$

using multipliers 1, 1, 1

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{dx+dy+dz}{y+z-x-z+x-y}$$

$$= \frac{dx+dy+dz}{0}$$

$$dx+dy+dz = 0$$

$$\int dx+dy+dz = 0 \Rightarrow x+y+z = c_1$$

Using multipliers $x, y, -z$

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy - z\,dz}{x(y+z) - (x+z)y + z(x-y)}$$

$$= \frac{x\,dx + y\,dy - z\,dz}{0}$$

$$x\,dx + y\,dy - z\,dz = 0$$

$$\Rightarrow \int x\,dx + y\,dy - z\,dz = c_2$$

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c_2$$

$$\Rightarrow x^2 + y^2 - z^2 = 2c_2$$

General sol is $\phi(x+y+z,\ x^2+y^2-z^2) = 0$

(9) Solve $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

(Or) $x(z^2-y^2)\frac{\partial z}{\partial x} + y(x^2-z^2)\frac{\partial z}{\partial y} = z(y^2-x^2)$

by Lagrange's method.

Sol:- $P = x(y^2-z^2)$ $\quad Q = y(z^2-x^2)$ $\quad R = z(x^2-y^2)$

A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

Using multipliers x, y, z

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx+y\,dy+z\,dz}{x(x(y^2-z^2))+y(y(z^2-x^2))+z.z(x^2-y^2)}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{x^2y^2-x^2z^2+y^2z^2-y^2x^2+z^2x^2-z^2y^2}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{0}$$

$$x\,dx+y\,dy+z\,dz=0$$

$$\int x\,dx+y\,dy+z\,dz = c_1$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = c_1 \Rightarrow \underline{\underline{x^2+y^2+z^2=2c_1}}$$

$$\frac{\frac{dx}{x}}{y^2-z^2} = \frac{\frac{dy}{y}}{z^2-x^2} = \frac{\frac{dz}{z}}{x^2-y^2}$$

using $1, 1, 1$

$$\frac{\frac{dx}{x}}{y^2-z^2} = \frac{dy/y}{z^2-x^2} = \frac{dz/z}{x^2-y^2} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\int \left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\right) = \log c_2$$

$$\log x + \log y + \log z = \log c_2$$

$$\log (xyz) = \log c_2$$

$$\underline{\underline{xyz = c_2}}$$

General Sol is $\underline{\underline{\phi(x^2+y^2+z^2,\ xyz) = 0}}$

(10) Solve $(mz-ny)p + (nx-lz)q = ly-mx$ by Lagrange's method

Sol:- $P = mz-ny$, $Q = nx-lz$,

$R = ly-mx$

A.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

using multipliers l, m, n

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{l\,dx + m\,dy + n\,dz}{\underbrace{l(mz-ny) + m(nx-lz) + n(ly-mx)}}$$

$$l\,dx + m\,dy + n\,dz = 0$$

$l\,dx + m\,dy + n\,dz = 0$

$\int l\,dx + m\,dy + n\,dz = c_1$

$lx + my + nz = c_1$

Using multipliers x, y, z

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{x\,dx + y\,dy + z\,dz}{x(mz-ny) + y(nx-lz) + z(ly-mx)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$x\,dx + y\,dy + z\,dz = 0$

$\int x\,dx + y\,dy + z\,dz = c_2$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$

$x^2 + y^2 + z^2 = 2c_2$

General sol is $\phi(u, v) = 0$

$\phi(lx + my + nz,\; x^2 + y^2 + z^2) = 0$

(11) Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$ by Lagrange's method.

Sol :- $P = x^2(y-z) \quad Q = y^2(z-x) \quad R = z^2(x-y)$

A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

using multipliers $1/x^2$, $1/y^2$, $1/z^2$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{1/x^2\,dx + 1/y^2\,dy + 1/z^2\,dz}{1/x^2 \cdot x^2(y-z) + 1/y^2 \cdot y^2(z-x) + 1/z^2 \cdot z^2(x-y)}$$

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\Rightarrow \frac{1}{x^2}dx + 1/y^2\,dy + 1/z^2\,dz = 0$$

$$\int \frac{1}{x^2}dx + 1/y^2\,dy + 1/z^2\,dz = c_1$$

$$-1/x - 1/y - 1/z = c_1 \Rightarrow 1/x + 1/y + 1/z = -c_1$$

Using multipliers $1/x, 1/y, 1/z$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{1/x \cdot x^2(y-z) + 1/y\, y^2(z-x) + 1/z \cdot z^2(x-y)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \quad 1/x\,dx + 1/y\,dy + 1/z\,dz = 0$$

$$\int \frac{1}{x}dx + 1/y\,dy + 1/z\,dz = \log c_2$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

General sol is

$$\phi\left(1/x + 1/y + 1/z,\ xyz\right) = 0$$

Solve by Lagrange's method

1. $z(xp - yq) = y^2 - x^2$

2. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

3. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Hint

$\frac{dx - dy}{} = \frac{dy - dz}{} = \frac{dz - dx}{}$

4. $(y - z)p + (z - x)q = x - y$