UNIT-V

Periodic function :-

A function $f(t)$ is said to be a periodic function of period T if $f(t+nT) = f(t)$, where $n = 1, 2, 3, \ldots$

Ex :- 1) $\sin(2\pi + t) = \sin t$

$\sin t$ is a periodic function with period 2π

2) $\cos(2\pi + t) = \cos t$

period 2π

3) $\tan(\pi + t) = \tan t$, period π

If $f(t)$ is a periodic function with period T, i.e $f(t+nT) = f(t)$ then

$$L\{f(t)\} = \int_0^T \frac{e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

(or)

$$L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Proof :- We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^T e^{-st} f(t)\, dt + \int_T^{2T} e^{-st} f(t)\, dt + \int_{2T}^{3T} e^{-st} f(t)\, dt + \cdots + \int_{nT}^{\infty} e^{-st} f(t)\, dt$$

From second integral

put $t = u + T$

$dt = du$

if $t = T$, $u = 0$

$t = 2T$, $u = T$

third ....

$t = u + 2T$

$dt = du$

if $t = 2T$, $u = 0$

$t = 3T$, $u = T$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + \int_0^T e^{-s(u+T)} f(u+T)\, du + \int_0^T e^{-s(u+2T)} f(u+2T)\, du + \cdots$$

$$= \int_0^T e^{-st} f(t)\, dt + \int_0^T e^{-su} \cdot e^{-sT} f(u)\, du + \int_0^T e^{-su} \cdot e^{-2sT} f(u)\, du + \cdots$$

since T is period $f(u+T) = f(u+2T) = \cdots f(u)$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \int_0^T e^{-su} f(u)\,du + e^{-2sT} \int_0^T e^{-su} f(u)\,du + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \int_0^T e^{-st} f(t)\,dt + e^{-2sT} \int_0^T e^{-st} f(t)\,dt + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt \left[1 + e^{-sT} + e^{-2sT} + \cdots \right]$$

$$= \int_0^T e^{-st} f(t)\,dt \left[1 + e^{-sT} + \left(e^{-sT}\right)^2 + \cdots \right]$$

G.P

$a = 1$

$r = e^{-sT}$

$S_\infty = \frac{a}{1-r}$

$$= \int_0^T e^{-st} f(t)\,dt \left[\frac{1}{1 - e^{-sT}} \right]$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Problems

① If $f(t) = t^2,\ 0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$ find $L\{f(t)\}$

Sol:- $f(t)$ is a periodic function with period 2, i.e $T = 2$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 t^2 \cdot e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2s}} \left[\left(t^2 \cdot \frac{e^{-st}}{-s} \right)_0^2 - \left(2t\, \frac{e^{-st}}{(-s)^2} \right)_0^2 + 2 \cdot \left(\frac{e^{-st}}{-s^3} \right)_0^2 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\left(4\frac{e^{-2s}}{-s} - 0 \right) - \left(\frac{4}{s^2} e^{-2s} - 0 \right) + \left(-\frac{2}{s^3} e^{-2s} + \frac{2}{s^3} \right) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} e^{-2s} + \frac{2}{s^3} \right]$$

$$= \frac{2}{s^3(1-e^{-2s})} \left[-2s^2 e^{-2s} - 2s e^{-2s} - e^{-2s} + 1 \right]$$

(2) A periodic function of period $\frac{2\pi}{\omega}$ is (5) defined by

$$f(t) = \begin{cases} E \sin \omega t & \text{for } 0 \le t \le \pi/\omega \\ 0, & \text{for } \pi/\omega \le t \le 2\pi/\omega \end{cases}$$

where E and ω are positive constants

~~then~~ show that $L\{f(t)\} = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

Sol :- $T = \text{period} = \frac{2\pi}{\omega}$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{2\pi/\omega} e^{-st} f(t)\, dt \right]$$

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} E \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0 \, dt \right]$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt$$

(6)

$$= \frac{E}{1 - e^{-\frac{2\pi s}{\omega}}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt$$

$$\boxed{\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)}$$

$$= \frac{E}{1 - e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{(-s)^2+\omega^2} \left(-s \cdot \sin \omega t - \omega \cdot \cos \omega t \right) \right]_0^{\pi/\omega}$$

$$= \frac{E}{1 - e^{-2\pi s/\omega}} \left[\frac{e^{-\frac{\pi s}{\omega}}}{s^2+\omega^2} \left(-s \cdot \sin (\cancel{\omega} \cdot \pi/\cancel{\omega}) - \omega \cdot \cos(\cancel{\omega} \cdot \pi/\cancel{\omega}) \right) - \frac{e^0}{s^2+\omega^2} \left(-s \cdot \sin 0 - \omega \cdot \cos 0 \right) \right]$$

$$= \frac{E}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-\pi s/\omega}}{s^2+\omega^2}^{*} \left(-s \cdot \sin \pi - \omega \cdot \cos \pi \right) - \frac{1}{s^2+\omega^2} \left(0 - \omega \right) \right]$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-\pi s/\omega}}{s^2+\omega^2}\left(-\omega(-1)\right)+\frac{\omega}{s^2+\omega^2}\right]$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{\omega}{s^2+\omega^2}\cdot e^{-\pi s/\omega}+\frac{\omega}{s^2+\omega^2}\right]$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\cdot\frac{\omega}{s^2+\omega^2}\left(e^{-\pi s}+1\right)$$

$$= \frac{E\omega}{1^2-\left(e^{-\pi s/\omega}\right)^2}\cdot\frac{1}{s^2+\omega^2}\left(1+e^{-\pi s/\omega}\right)$$

$$= \frac{E\omega}{\cancel{\left(1+e^{-\pi s/\omega}\right)}\left(1-e^{-\pi s/\omega}\right)}\cdot\frac{1}{s^2+\omega^2}\cancel{\left(1+e^{-\pi s/\omega}\right)}$$

$$= \frac{E\omega}{\left(s^2+\omega^2\right)\left(1-e^{-\pi s/\omega}\right)}$$

$$L\{f(t)\} = \frac{E\omega}{\left(s^2+\omega^2\right)\left(1-e^{-\pi s/\omega}\right)}$$

③ Find the Laplace transform of periodic function $f(t) = \frac{Kt}{T}$, $0 < t < T$

Sol:- Period $= T$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot \frac{Kt}{T}\, dt$$

$$= \frac{1}{1-e^{-sT}} \cdot \frac{K}{T} \int_0^T e^{-st} \cdot t\, dt$$

$$= \frac{K}{T(1-e^{-sT})} \left[\left(t \cdot \frac{e^{-st}}{-s} \right) - 1 \cdot \frac{e^{-st}}{s^2} \right]_0^T$$

$$= \frac{K}{T(1-e^{-sT})} \left[\left(T \cdot \frac{e^{-sT}}{-s} - \frac{e^{-sT}}{s^2} \right) - \left(0 - \frac{e^{-s(0)}}{s^2} \right) \right]$$

$$= \frac{K}{T(1-e^{-sT})} \left[\frac{Te^{-sT}}{-s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{K}{T(1-e^{-sT})} \left[\frac{-Te^{-sT}}{s} + \frac{1}{s^2} \left(1 - e^{-sT} \right) \right]$$

4. A periodic function $f(t)$ of period $2a$, $a>0$. is defined by

$$f(t) = \begin{cases} E, & \text{for } 0 \le t \le a \\ -E & \text{for } a \le t \le 2a \end{cases}$$

then show that $L\{f(t)\} = \frac{E}{s} \tanh\left(\frac{as}{2}\right)$

Sol :- $T =$ Period $= 2a$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} E\, dt + \int_a^{2a} e^{-st} (-E)\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[E \int_0^a e^{-st}\, dt - E \int_a^{2a} e^{-st}\, dt \right]$$

$$= \frac{E}{1-e^{-2as}} \left[\int_0^a e^{-st}\, dt - \int_a^{2a} e^{-st}\, dt \right]$$

$$= \frac{E}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^a - \left(\frac{e^{-st}}{-s} \right)_a^{2a} \right]$$

$$= \frac{E}{1-e^{-2as}}\left[\left(\frac{e^{-as}}{-s}-\frac{e^{0}}{-s}\right)-\left(\frac{e^{-2as}}{-s}-\frac{e^{-as}}{-s}\right)\right]$$

$$= \frac{E}{1-e^{-2as}}\left[-\frac{1}{s}\left(e^{-as}-1\right)+\frac{1}{s}\left(e^{-2as}-e^{-as}\right)\right]$$

$$= \frac{E}{s\left(1-e^{-2as}\right)}\left[-e^{-as}+1+e^{-2as}-e^{-as}\right]$$

$$= \frac{E}{s\left(1-e^{-2as}\right)}\left[e^{-2as}-2e^{-as}+1\right]$$

$$= \frac{E}{s\left(1-e^{-2as}\right)}\left[\left(e^{-as}\right)^{2}-2\cdot e^{-as}\cdot 1+1^{2}\right]$$

$$= \frac{E}{s\left(1-e^{-2as}\right)}\left[\left(1-e^{-as}\right)^{2}\right]$$

$$= \frac{E\left(1-e^{-as}\right)^{2}}{s\left(1^{2}-\left(e^{-as}\right)^{2}\right)}$$

$$= \frac{E}{s\left((1+e^{-as})\cancel{(1-e^{-as})}\right)}(1-e^{-as})^{\cancel{2}}$$

$$= \frac{E}{s} \cdot \left(\frac{1-e^{-as}}{1+e^{-as}}\right)$$

$$= \frac{E}{s} \cdot \frac{e^{as/2}}{e^{as/2}}\left(\frac{1-e^{-as}}{1+e^{-as}}\right)$$

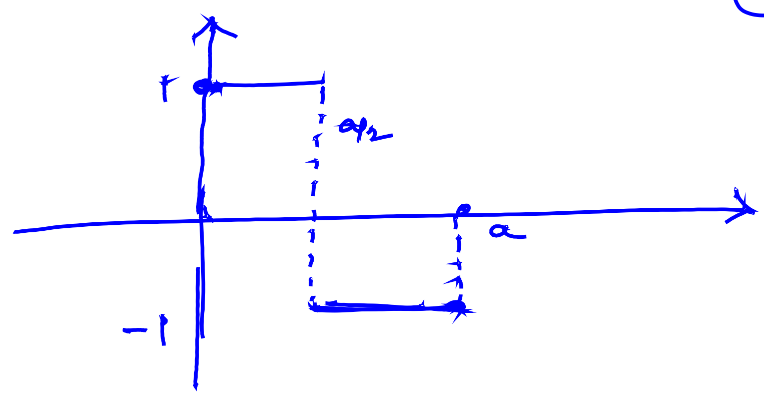
$$= \frac{E}{s} \cdot \left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right)$$

$$= \frac{E}{s} \cdot \tanh\left(\frac{as}{2}\right)$$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

5 Find the Laplace transform of square wave function of period a, defined by

$$f(t) = \begin{cases} 1, & \text{for } 0 < t < a/2 \\ -1, & \text{for } a/2 < t < a \end{cases}$$



Sol:- Period $= T = a$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} f(t)\, dt + \int_{a/2}^{a} e^{-st} f(t)\, dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} \cdot 1\, dt + \int_{a/2}^{a} e^{-st} \cdot (-1)\, dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{a/2} - \left(\frac{e^{-st}}{-s} \right)_{a/2}^{a} \right]$$

$$= \frac{1}{1-e^{-as}} \left[-\frac{1}{s} \left(e^{-as/2} - e^{0} \right) + \frac{1}{s} \left(e^{-as} - e^{-as/2} \right) \right]$$

(13)

$$= \frac{1}{1-e^{-as}}\left[-\frac{1}{s}\left(e^{-as/2}-1\right)+\frac{1}{s}\left(e^{-as}-e^{-as/2}\right)\right]$$

$$= \frac{1}{1-e^{-as}}\cdot\frac{1}{s}\left[1-e^{-as/2}+e^{-as}-e^{-as/2}\right]$$

$$= \frac{1}{s\left(1-e^{-as}\right)}\left[1-2e^{-as/2}+e^{-as}\right]$$

$$= \frac{1}{s\left(1-e^{-as}\right)}\left[1^2-2\cdot 1\cdot e^{-as/2}+\left(e^{-as/2}\right)^2\right]$$

$$= \frac{1}{s\left(1-e^{-as}\right)}\left(1-e^{-as/2}\right)^2$$

$$= \frac{1}{s\left(1^2-\left(e^{-as/2}\right)^2\right)}\left(1-e^{-as/2}\right)^2$$

$$= \frac{1}{s\left(1+e^{-as/2}\right)\cancel{\left(1-e^{-as/2}\right)}}\left(1-e^{-as/2}\right)^{\cancel{2}}$$

$$= \frac{\left(1-e^{-as/2}\right)}{s\left(1+e^{-as/2}\right)}$$

$$= \frac{e^{as/4}\left(1 - e^{-as/2}\right)}{s\, e^{as/4}\left(1 + e^{-as/2}\right)}$$

$$= \frac{e^{as/4} - e^{-as/4}}{s\left(e^{as/4} + e^{-as/4}\right)}$$

$$\tanh ax = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$$

$$= \frac{1}{s} \cdot \tanh\left(\frac{as}{4}\right)$$