* Obtain $Z_T[a^n \cos n\theta]$ and $Z_T[a^n \sin n\theta]$

Damping rule

App: $Z_T[k^n\, n] = \left[Z_T[n]\right]_{z \to z/k}$

Actual rule: $Z_T[k^n u_n] = u(z/k)$

$k = a$

$$Z_T[a^n \cos n\theta] = \left[Z_T[\cos n\theta]\right]_{z \to z/a}$$

$$= \left\{ \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} \right\}_{z \to z/a}$$

$$= \frac{z/a\,(z/a - \cos\theta)}{(z/a)^2 - 2(z/a)\cos\theta + 1}$$

$Z_T[k^n u_n]$

$Z_T[u_n]_{z \to z/k}$

$$Z_T[a^n \cos n\theta] = \frac{z(z - a\cos\theta)}{z^2 - 2za\cos\theta + a^2}$$

$$Z_T[a^n \sin n\theta] = \left[Z_T[\sin n\theta]\right]_{z \to z/a}$$

$$= \left[\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right]_{z \to z/a}$$

$$= \frac{z/a\,\sin\theta}{z^2/a^2 - 2z/a\cos\theta + 1} = \frac{za\sin\theta}{z^2 - 2za\cos\theta + a^2}$$

Note : $Z_T[e^{-an}\cos n\theta] = Z_T[(e^{-a})^n \cos n\theta]$

$= Z_T[\cos n\theta]_{z \to z/e^{-a}}$

$$= \frac{z(z - e^{-a}\cos\theta)}{z^2 - 2e^{-a}z\cos\theta + e^{-2a}}$$

HW : Find $Z_T[e^{-an}\sin n\theta] = ?$

* obtain $Z_T[\cosh n\theta]$ and $Z_T[\sinh n\theta]$

Sol : $Z_T[\cosh n\theta] = Z_T\left[\frac{e^{n\theta} + e^{-n\theta}}{2}\right]$

$= \frac{1}{2}\left\{ Z_T[(e^{\theta})^n] + Z_T[(e^{-\theta})^n]\right\}$

$$= \frac{1}{2}\left[\frac{z}{z - e^{\theta}} + \frac{z}{z - e^{-\theta}}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-\theta} + z^2 - ze^{\theta}}{(z - e^{\theta})(z - e^{-\theta})}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{\theta} + e^{-\theta})}{z^2 - ze^{-\theta} - ze^{\theta} + e^{\theta-\theta}}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{\theta} + e^{-\theta})}{z^2 - z(e^{-\theta} + e^{\theta}) + 1}\right] = \frac{1}{2}\left[\frac{2z^2 - 2z\cosh\theta}{z^2 - 2z\cosh\theta + 1}\right]$$

$\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$

$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

$Z_T[k^n] = \frac{z}{z - k}$

$$= \frac{z(z - \cosh\theta)}{z^2 - 2z\cosh\theta + 1}$$

$$Z_T[k^n] = \frac{z}{z-k}$$

$$Z_T[\sinh n\theta] = Z_T\left[\frac{e^{n\theta} - e^{-n\theta}}{2}\right] = Z_T\left[\frac{(e^{\theta})^n - (e^{-\theta})^n}{2}\right]$$

$$= \frac{1}{2}\left[Z_T[(e^{\theta})^n] - Z_T[(e^{-\theta})^n]\right]$$

$$= \frac{1}{2}\left[\frac{z}{(z - e^{\theta})} - \frac{z}{(z - e^{-\theta})}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{z^2} - ze^{-\theta} - \cancel{z^2} + ze^{+\theta}}{z^2 \underbrace{- ze^{-\theta} - ze^{\theta}} + e^{\theta - \theta}}\right]$$

$$= \frac{1}{2}\left[\frac{z(e^{\theta} - e^{-\theta})}{z^2 + 1 - z(e^{-\theta} + e^{\theta})}\right]$$

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$= \frac{1}{2}\left[\frac{z(2\sinh\theta)}{z^2 + 1 - z(2\cosh\theta)}\right]$$

$$= \frac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1} \quad //$$

* Initial Value theorem

If $Z_T[u_n] = U(z)$ then $u_0 = \lim_{z\to\infty} U(z)$

proof :

Def Z - transform

$$Z_T[u_n] = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$= u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots \quad \text{--- (1)}$$

$$\boxed{\lim_{z\to\infty} U(z) = u_0}$$

~~III'ly u_1 can be obtained as~~

~~wkt $u_{n+1} = z[U(z) - u_0]$~~

~~put $n = 0$~~

~~$u_1 = [U(z) - u_0]$~~

III'ly u_1 can be obtained as

(1) $\Rightarrow$ $U(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots$

$$z(U(z) - u_0) = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots$$

$$\boxed{\lim_{z\to\infty} \underbrace{z[U(z) - u_0]}_{(u_{n+1})} = u_1}$$

III'ly

$$\boxed{u_2 = \lim_{z\to\infty} z^2\underbrace{\left[u(z) - u_0 - \frac{u_1}{z}\right]}_{}}$$

$$= \lim_{z\to\infty} u_{n+2}$$

$$\boxed{u_3 = \lim_{z\to\infty} z^3\underbrace{\left[u(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]}_{}}$$

$$\lim_{z\to\infty} u_{n+3}$$

① If $u(z) = \dfrac{2z^2+3z+12}{(z-1)^4}$, Find the value u_0, u_1, u_2, u_3.

Sol :

$$u_0 = \lim_{z\to\infty} u(z)$$

$$= \lim_{z\to\infty} \frac{2z^2+3z+12}{(z-1)^4}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^2}\left(2+3/z+12/z^2\right)}{z^2\cancel{z^4}\left(1-1/z\right)^4}$$

$$= \frac{2+0+0}{\infty(1-0)} = 0 \qquad \Big|\ 1/\infty = 0$$

$$u_0 = 0$$

$$u_1 = \lim_{z\to\infty} z\left[u(z) - u_0\right]$$

$$= \lim_{z\to\infty} z\left[\frac{2z^2+3z+12}{(z-1)^4} - 0\right]$$

$$u_1 = 0$$

$$u_2 = \lim_{z\to\infty} z^2\left[u(z) - u_0 - \frac{u_1}{z}\right]$$

$$= \lim_{z\to\infty} z^2\left[\frac{2z^2+3z+12}{(z-1)^4} - 0 - 0\right]$$

$$= \lim_{z\to\infty} \frac{\cancel{z^2}\left(\cancel{z^2}\left(2+3/z+\frac{12}{z^2}\right)\right)}{\cancel{z^4}(1-1/z)^4}$$

$$u_2 = 2$$

$$u_3 = \lim_{z\to\infty} z^3\left[u(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^2+3z+12}{(z-1)^4} - 0 - 0 - \frac{2}{z^2}\right]$$

$(z-1)^2(z-1)^2$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^4+3z^3+12z^2-2(z-1)^4}{z^2(z-1)^4}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^4+3z^3+12z^2-2(z^4-4z^3+6z^2-4z+1)}{z^2(z-1)^4}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2\cancel{z^4}+3z^3+12\cancel{z^2}-2\cancel{z^4}+8z^3-12\cancel{z^2}+8z-2}{z^2(z-1)^4}\right]$$

$$= \lim_{z\to\infty} \frac{z^3(11z^3+8z-2)}{z^2(z-1)^4}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^3}\cdot\cancel{z^3}\left(11+8/z^2-2/z^3\right)}{\cancel{z^2}\cdot\cancel{z^4}(1-1/z)^4}$$

$$\boxed{u_3 = 11}$$

* If $Z_T[u_n] = \frac{z}{z-1} - \frac{z}{z^2+1}$ then find

$Z_T[u_{n+2}]$

Sol:

$Z_T[u_{n+2}] = ? =$

$Z_T[u_{n+2}] = z^2\left[u(z) - u_0 - \frac{u_1}{z}\right]$

$u_0 = ?$ $u_1 = ?$

$$u_0 = \lim_{z\to\infty} u(z)$$

$$= \lim_{z\to\infty}\left[\frac{z}{z-1} - \frac{z}{z^2+1}\right] = \lim_{z\to\infty}\left[\frac{\cancel{z}}{\cancel{z}(1-1/z)} - \frac{\cancel{z}}{z^{\cancel{2}}(1+1/z^2)}\right] = \left[\frac{1}{1} - 0\right]$$

$u_0 = 1$

$$U_1 = \lim_{z\to\infty} z\left[U(z) - U_0\right]$$

$$= \lim_{z\to\infty} z\left[\frac{z}{z-1} - \frac{z}{z^2+1} - 1\right]$$

$$= \lim_{z\to\infty} z\left[\frac{z(z^2+1) - z(z-1) - 1(z-1)(z^2+1)}{(z-1)(z^2+1)}\right]$$

$$= \lim_{z\to\infty} z\left[\frac{\cancel{z^3}+z-\cancel{z^2}+\cancel{z}-\cancel{z^3}-\cancel{z}+\cancel{z^2}+1}{(z-1)(z^2+1)}\right]$$

$$= \lim_{z\to\infty} z\left(\frac{z+1}{(z-1)(z^2+1)}\right) = \lim_{z\to\infty}\frac{\cancel{z}\,\cancel{z}(1+1/z)}{z^3(1-1/z)(1+1/z^2)}$$

$$U_1 = 0$$

$$Z_T\left[U_{n+2}\right] = z^2\left[U(z) - U_0 - \frac{U_1}{z}\right]$$

$$= z^2\left[\frac{z}{z-1} - \frac{z}{z^2+1} - 1 - 0\right]$$

$$= z^2\left[\frac{z(z^2+1) - z(z-1)}{(z-1)(z^2+1)} - 1\right] = \cancel{z}$$

$$= z^2\left[\frac{z^3+z-z^2+z - (z-1)(z^2+1)}{(z-1)(z^2+1)}\right]$$

$$= z^2\left[\frac{\cancel{z^3}+z-\cancel{z^2}+\cancel{z}-\cancel{z^3}-\cancel{z}+\cancel{z^2}-1}{(z-1)(z^2+1)}\right]$$

$$= \frac{z^2(z+1)}{(z-1)(z^2+1)}$$

$$Z_T(u_{n+2}) = \frac{z^3+z^2}{(z-1)(z^2+1)}$$

* Final value theorem

If $Z_T[u_n] = U(z)$ then

$$\boxed{\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)\,U(z)}$$

Sol :

Linearity ppt

Consider $Z_T[u_{n+1} - u_n] = \underbrace{Z_T[u_{n+1}]}_{} - \underbrace{Z_T[u_n]}_{}$

$$= z[U(z) - u_0] - \underbrace{U(z)}_{\text{Defn } Z_T}$$

$$\underbrace{Z_T(u_{n+1}-u_n)}_{(\text{Dfn } Z-T)} = U(z)(z-1) - z u_0$$

$$\sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n} = U(z)(z-1) - z u_0$$

$\lim_{z\to 1}$ on B.S

$$\lim_{z\to 1} \sum_{n=0}^{\infty} (u_{n+1} - u_n) \underbrace{z^{-n}}_{(1)^{-n}} = \lim_{z\to 1} \left(U(z)(z-1) - z u_0\right)$$

Splited

$$\sum_{n=0}^{\infty} (u_{n+1} - u_n) = \lim_{z\to 1} U(z)(z-1) - \lim_{z\to 1} z u_0$$

$$\sum_{n=0}^{\infty} u_{n+1} - u_n = \lim_{z\to 1} U(z)(z-1) - u_0$$

$$\lim_{m\to\infty} \sum_{n=0}^{m} (u_{n+1} - u_n) = \lim_{z\to 1} U(z)(z-1) - u_0$$

$$\lim_{m\to\infty} \left[(\cancel{u_1} - u_0) + (\cancel{u_2} - \cancel{u_1}) + (\cancel{u_3} - \cancel{u_2}) + \cancel{\;} \dots + (u_{m+1} - \cancel{u_m}) \right] = RHS$$

$$\lim_{m\to\infty} (\underset{c}{-u_0} + u_{m+1}) = RHS$$

$$\cancel{-u_0} + \lim_{m\to\infty} u_{m+1} = \lim_{z\to 1} U(z)(z-1) \cancel{-u_0}$$

$$\lim_{z\to 1} U(z)(z-1) = \lim_{m\to\infty} u_{m+1}$$

If $m\to\infty$ $\quad m+1 \to \infty$

$$LHS = \lim_{m+1\to\infty} u_{m+1}$$

put $m+1$ as n

$$\boxed{\lim_{z\to 1} U(z)(z-1) = \lim_{n\to\infty} u_n}$$

1. Find the final value of the sequence with the z-transform

a) $U(z) = \dfrac{10z^2 + 2z}{(z-1)(5z-1)^2}$

Sol:

$$\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)\,U(z)$$

$$= \lim_{z\to 1} \cancel{(z-1)} \frac{10z^2+2z}{\cancel{(z-1)}(5z-1)^2}$$

$$= \frac{10+2}{(5-1)^2} = 12/16 = 3/4$$

b) $U(z) = \dfrac{4z^2 - z}{2z^2 - 3z + 1}$

$$\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)\,U(z) = \lim_{z\to 1} \frac{(z-1)(4z^2-z)}{(2z^2-3z+1)}$$

$$\lim_{z\to 1} \frac{\cancel{(z-1)}(4z^2-z)}{\cancel{(z-1)}(2z-1)} = \frac{4-1}{2-1} = 3$$

Inverse z- transform

$$Z_T^{-1}[U(z)] = u_n$$

List of standard inverse z-transform

① $Z_T^{-1}\left[\frac{z}{z-1}\right] = 1$

② $Z_T^{-1}\left[\frac{z}{(z-1)^2}\right] = n$

③ $Z_T^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = n^2$

④ $Z_T^{-1}\left[\frac{z^3+4z^2+z}{(z-1)^4}\right] = n^3$

⑤ $Z_T^{-1}\left[\frac{z}{z-k}\right] = k^n$

⑥ $Z_T^{-1}\left[\frac{kz}{(z-k)^2}\right] = k^n \cdot n$

⑦ $Z_T^{-1}\left[\frac{kz^2+k^2z}{(z-k)^3}\right] = k^n \cdot n^2$

⑧ $Z_T^{-1}\left[\frac{kz^3+4k^2z^2+k^3z}{(z-k)^4}\right]$

$= k^n \cdot n^3$

⑨ $Z_T^{-1}\left[\frac{z}{z^2+1}\right]$

$= \sin(n\pi/2)$

⑩ $Z_T^{-1}\left(\frac{z^2}{z^2+1}\right)$

$= \cos(n\pi/2)$

Working procedure

① $U(z) = \frac{f(z)}{g(z)}$ given. We have to express $g(z)$ in terms of non repeated linear factors only

② Consider $\frac{U(z)}{z}$ in the form of proper fraction & resolve into partial fractions

③ X'ly by z to have $U(z)$ involving various terms of form $c \cdot z/z-k$ $c \to$ constant

④ Finally comput $IZ-T$

Note : If $g(z)$ involves repeated linear factors of form $(z-k)^2$ $(z-k)^3$ $(z-k)^4$ we need to take it into acc the corresponding terms in numerator : kz, kz^2+k^2z, $kz^3+4k^2z^2+k^3z$ respectively & express $U(z)$ suitably ~~by mult~~ with terms multiplied by A, B, C

(1) Find inverse z-transform for the foll

a) $u(z) = \frac{z}{(z-1)(z-2)}$

Let $u(z) = \frac{z}{(z-1)(z-2)}$

$\frac{u(z)}{z} = \frac{1}{(z-1)(z-2)}$ —— (1)

RHS : $\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$ —— (2)

$1 = A(z-2) + B(z-1)$

put $z=2$ $\quad 1 = B$ $\quad$ put $z=1$ $\quad 1 = -A$

$\boxed{A=-1}$ $\quad \boxed{B=1}$ $\quad$ ~~(2) in (1)~~

(1) ⇒ $\cancel{\frac{u(z)}{z} = \frac{-1}{}}$

(2) ⇒ $\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$ —— (3)

(3) in (1)

$\frac{u(z)}{z} = \frac{-1}{(z-1)} + \frac{1}{z-2}$

$u(z) = -\frac{z}{(z-1)} + \frac{z}{(z-2)}$

$= z/z-2 - z/z-1$

Inverse Z- transform on B.S

$$Z_T^{-1}\left[U(z)\right] = Z_T^{-1}\left[\frac{z}{z-2}\right] - Z_T^{-1}\left[\frac{z}{z-1}\right]$$

$$\boxed{Z_T^{-1}\left[U(z)\right] = 2^n - 1^n}$$

Formula

$$Z_T^{-1}\left[\frac{z}{z-k}\right] = k^n$$

(B) ~~$u(z) = 3z^2$~~

$$u(z) = \frac{3z^2+2z}{(5z-1)(5z+2)}$$

$$u(z) = \frac{z(3z+2)}{(5z-1)(5z+2)}$$

$$\frac{u(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)} \quad \text{——— (1)}$$

Let $$\frac{3z+2}{(5z-1)(5z+2)} = \frac{A}{(5z-1)} + \frac{B}{(5z+2)}$$

$$3z+2 = A(5z+2) + B(5z-1)$$

Put $z = 1/5$: $3(1/5)+2 = A(5(1/5)+2)$

$A = 13/15$

$z = -2/5$: $B = -4/15$

$$\frac{3z+2}{(5z-1)(5z+2)} = \frac{13}{15}\frac{1}{(5z-1)} - \frac{4}{15}\frac{1}{5z+2} \quad \boxed{2}$$

② in ①

$$\frac{U(z)}{z} = \frac{13}{15 \cdot 5}\frac{1}{(z-1/5)} - \frac{4}{15 \cdot 5}\left(\frac{1}{z+2/5}\right)$$

$$U(z) = \frac{13}{75}\frac{z}{(z-1/5)} - \frac{4}{75}\left(\frac{z}{z+2/5}\right)$$

IZT

$$Z_T^{-1}[U(z)] = \frac{13}{75} Z_T^{-1}\left(\frac{z}{(z-1/5)}\right) - \frac{4}{75} Z_T^{-1}\left(\frac{z}{\underbrace{z+2/5}_{z-(-2/5)}}\right)$$

$$Z_T^{-1}\left(\frac{z}{z-k}\right) = k^n$$

$$Z_T^{-1}[U(z)] = \frac{13}{75}(1/5)^n - \frac{4}{75}(-2/5)^n$$

c) $$U(z) = \frac{8z - z^3}{(4-z)^3}$$

imp: Note that denominator has repeated linear factor

Note

$$\left\{\begin{array}{l} \dfrac{1}{(z-4)^3} = \dfrac{A}{(z-4)} + \dfrac{B}{(z-4)^2} + \dfrac{C}{(z-4)^3} \\[2ex] Z_T^{-1}\left[\dfrac{z}{z-4}\right] = 4^n \qquad Z_T^{-1}\left[\dfrac{4z^2+4^2z}{(z-4)^3}\right] = 4^n n^2 \\[2ex] Z_T^{-1}\left[\dfrac{4z}{(z-\underset{K}{4})^2}\right] = 4^n \cdot n \end{array}\right.$$

Note this step

$$u(z) = \frac{8z-z^3}{(4-z)^3} = \frac{z^3-8z}{(z-4)^3}$$

Let $u(z) = \dfrac{z^3-8z}{(z-4)^3} = A\cdot\dfrac{z}{(z-4)} +$

$$B\cdot\frac{4z}{(z-4)^2} + C\cdot\frac{(4z^2+16z)}{(z-4)^3} \quad \text{—— (1)}$$

$$z^3-8z = Az(z-4)^2 + B(4z)(z-4) + C(4z^2+16z)$$

$$z^2-8 = A(z-4)^2 + 4B(z-4) + 4C(z+4)$$

$$z^2-8 = Az^2 - 8Az + 16A + 4Bz - 16B + 4Cz + 16C$$

put $z = 4$

$\cancel{16 = 0 + 0 + 4C(8)}$

$\cancel{C =}$ $4^2 - 8 = 0 + 0 + 4C(8)$

$\cancel{8} = 4C\cancel{8}$

$\boxed{C = 1/4}$

Coeff of z^2 on B.S

$\boxed{1 = A}$

$A(z-4)^2$
$A(z^2+16-8z)$

Coeff of z on B.S

$0 = -8A + 4B + 4C$

$-8(1) + 4B + 4(1/4) = 0$

$4B = 8 - 1 = 7$

$\boxed{B = 7/4}$

$$\frac{z^3 - 8z}{(z-4)^3} = \frac{z}{z-4} + \frac{7}{4}\frac{4z}{(z-4)^2} + \frac{1}{4}\frac{4z^2+16z}{(z-4)^3}$$

Apply Inverse Z-transform

$$Z_T^{-1}\left[\frac{z^3 - 8z}{(z-4)^3}\right] = Z_T^{-1}\left(\frac{z}{z-4}\right) + 7/4 \, Z_T^{-1}\left(\frac{4z}{(z-4)^2}\right) + 1/4 \, Z_T^{-1}\left((4z^2+162)/(z-4)^3\right)$$

$$Z_T^{-1}(U(z)) = 4^n + \frac{7}{4}4^n \cdot n + \frac{1}{4}4^n n^2$$

$$U_n = \frac{4^n}{4}(4 + 7n + n^2)$$

d) $$U(z) = \frac{z^3 - 20z}{(z-2)^3(z-4)}$$

$$\frac{z^3 - 20z}{(z-2)^3(z-4)} = A\frac{z}{(z-2)} + B\frac{2z}{(z-2)^2} +$$

$$C\frac{(2z^2+4z)}{(z-2)^3} + \frac{Dz}{(z-4)}$$

$$z^3 - 20z = Az(z-2)^2(z-4) + B2z(z-2)(z-4)$$

$$+ C(2z^2+4z)(z-4) + Dz(z-2)^3$$

$$z^2 - 20 = A(z-2)^2(z-4) + 2B(z-2)(z-4)$$

$$+ C(2z+4)(z-4) + D(z-2)^3$$

$A = 1/2$ $\quad B = 0$ $\quad C = 1$ $\quad D = -1/2$

$$U_n = 2^{n-1} + 2^n n^2 - 2^{2n-1}$$

Solution of Difference equation using Z-transform

① Solve by using Z-transforms:

$y_{n+2} + 2y_{n+1} + y_n = n$ with $y_0 = 0 = y_1$

Sol: $y_{n+2} + 2y_{n+1} + y_n = n$

put Z_T on B.S

$Z_T[y_{n+2}] + 2Z_T[y_{n+1}] + Z_T[y_n] = Z_T[n]$ — ①

$Z_T[y_{n+2}] = z^2\left[y(z) - y_0 - \frac{y_1}{z}\right]$ | $u_{n+2} = y_{n+2}$

$Z_T[y_{n+1}] = z[y(z) - y_0]$ — ②

$Z_T[y_n] = y(z)$

$Z_T(n) = z/(z-1)^2$

② in ①

$z^2\left[y(z) - y_0 - \frac{y_1}{z}\right] + 2[z(y(z) - y_0)] + y(z)$

$= z/(z-1)^2$

$z^2y(z) - z^2 \cancel{y_0}^{0} - z\cancel{y_1}^{0} + 2zy(z) - 2z\cancel{y_0}^{0} +$

$y(z) = z/(z-1)^2$

$(z^2 + 2z + 1)\, y(z) = z/(z-1)^2$

$$y(z) = \frac{z}{(z-1)^2(z^2+2z+1)}$$

$$y(z) = \frac{z}{(z-1)^2(z+1)^2} \quad \text{—— ①}$$

$$\frac{z}{(z-1)^2(z+1)^2} = A\frac{z}{(z-1)} + B\frac{(1)z}{(z-1)^2} +$$

$$\frac{cz}{(z+1)} + D\frac{(-1)z}{(z+1)^2}$$

$z-(-1)$ $\quad$ $(z-(-1))^2$

~~$z = Az + Bz +$~~

$$z = Az(z-1)(z+1)^2 + Bz(z+1)^2 +$$
$$cz(z+1)(z-1)^2 - Dz(z-1)^2$$

$$1 = A(z-1)(z+1)^2 + B(z+1)^2 + c(z+1)(z-1)^2$$
$$- D(z-1)^2$$

Put $z=1$ $\qquad$ $z=-1$

$1 = 0 + 4B$ $\qquad$ $1 = 0 + 0 + 0 - D(4)$

$B = 1/4$ $\qquad$ $D = -1/4$

put $z = 0$

$1 = -A + B + c - D$ —— ⊛

Equating coeff z^3

$A + c = 0$

$\boxed{A = -c}$

$$1 = -A + B - A - D$$

$$1 = -2A + 1/4 + 1/4$$

$$1 - 2/4 = -2A$$

$$2/4 = -2A \qquad A = -1/4$$

$$C = 1/4$$

$$u(z) = \frac{z}{(z-1)^2 (z+1)^2} = \frac{-1}{4} \frac{z}{(z-1)} + \frac{1}{4} \frac{z}{(z-1)^2}$$

$$+ \frac{1}{4} \frac{z}{(z-(-1))} - \frac{1}{4} \frac{(-z)}{[z-(-1)]^2}$$

Apply IZT

$$Z_T^{-1}(y(z))$$

$$\cancel{Z_T^{-1}(u(z))} = \frac{-1}{4} Z_T^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{4} Z_T^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$+ \frac{1}{4} Z_T^{-1}\left[\frac{z}{z-(-1)}\right] - \frac{1}{4} Z_T^{-1}\left[\frac{(-1)\, z}{(z-(-1))^2}\right]$$

$$y_n = \frac{-1}{4}(1) + \frac{1}{4} n + \frac{1}{4}(-1)^n - \frac{1}{4}(-1)^n n$$

$$Z_T^{-1}\left[\frac{z}{(z-(-k))^2}\right] = (-k)^n n \qquad Z_T^{-1}\left[\frac{z}{z-1}\right] = 1$$

for

$$Z_T^{-1}\left[\frac{z}{z-(-k)}\right] = (-k)^n$$

(2) Solve difference eq using z-transforms:

$$U_{n+2} - 5U_{n+1} - 6U_n = 2^n$$

Take z-transform on B.S

$$Z_T[U_{n+2}] - 5Z_T[U_{n+1}] - 6Z_T[U_n] = Z_T[2^n]$$

$$z^2\left[U(z) - U_0 - \frac{U_1}{z}\right] - 5z[U(z) - U_0]$$

$$-6U(z) = \frac{z}{z-2}$$

$$(z^2 - 5z - 6)U(z) - U_0(z^2 - 5z) - U_1 z = z/z-2$$

$$(z-6)(z+1)U(z) = U_0(z^2-5z) + U_1 z + z/z-2$$

$$U(z) = U_0 \frac{z^2 - 5z}{(z-6)(z+1)} + U_1 \frac{z}{(z-6)(z+1)} + \frac{z}{(z-2)(z-6)(z+1)}$$

$$U(z) = U_0 P(z) + U_1 q(z) + r(z) \quad (1)$$

$$p(z) = \frac{z^2 - 5z}{(z-6)(z+1)} \quad \text{--- } ②$$

$$q(z) = \frac{z}{(z-6)(z+1)} \quad \text{--- } ③$$

$$r(z) = \frac{z}{(z-2)(z-6)(z+1)} \quad \text{--- } ④$$

② $\Rightarrow$

$$\frac{z^2 - 5z}{(z-6)(z+1)} = A\frac{z}{(z-6)} + B\frac{z}{(z+1)}$$

$(z-(-1))$

$$z^2 - 5z = Az(z+1) + Bz(z-6)$$

$$z - 5 = A(z+1) + B(z-6)$$

$$B = 6/7$$

$$A = 1/7$$

$$p(z) = \frac{1}{7}\frac{z}{z-6} + \frac{6}{7}\frac{z}{z+1}$$

IZT

$$Z_T^{-1}[p(z)] = \frac{1}{7} Z_T^{-1}\left(\frac{z}{z-6}\right) + \frac{6}{7} Z_T^{-1}\left[\frac{z}{z+1}\right]$$

$$Z_T^{-1}[p(z)] = \frac{1}{7} 6^n + \frac{6}{7}(-1)^n \quad \text{--- } ⑤$$

$$q(z) = E \frac{z}{(z-6)(z+1)} - \frac{1}{12} \qquad F = 1/28 \qquad G = 1/21$$

$$\frac{\cancel{20}(z)}{(z-6)(z-1)} = -\frac{1}{12}\cancel{A}\frac{z}{(z^2 6)^2} + \frac{1}{28}B\frac{z}{\cancel{z-6}\,(z+1)} +$$

$$z = \frac{1}{2}Az(z+1) + Bz(z-6)$$

$$1 = A(z+1) + B(z-6)$$

Solving $\qquad C = 1/7 \quad D = -1/7$

$$q(z) = \frac{1}{7}\frac{z}{z-6} - \frac{1}{7}\frac{z}{(z-(-1))}$$

IZT

$$Z_T^{-1}[q(z)] = \frac{1}{7} Z_T^{-1}\left[\frac{z}{z-6}\right] - \frac{1}{7} Z_T^{-1}\left[\frac{z}{z-(-1)}\right]$$

$$Z_T^{-1}[q(z)] = \frac{1}{7}6^n - \frac{1}{7}(-1)^n \text{ ——— } (6)$$

$$r(s) = \frac{z}{(z-2)(z-6)(z+1)}$$

$$\frac{z}{(z-2)(z-6)(z+1)} = \frac{\overset{E}{A}z}{(z-2)} + \frac{\overset{F}{B}z}{(z-6)} + \frac{\overset{G}{C}z}{(z+1)}$$

$\cancel{E} = -1/12 \quad F = 1/28 \quad G = 1/21$

$$r(z) = \frac{-1}{12} \frac{z}{z-2} + \frac{1}{28} \frac{z}{z-6} +$$

$$\frac{1}{21} \frac{z}{z+1}$$

$$Z_T^{-1}[r(z)] = \frac{-1}{12} Z_T^{-1}\left(\frac{z}{z-2}\right) + \frac{1}{28}$$

$$Z_T^{-1}\left(\frac{z}{z-6}\right) + \frac{1}{21}\left(Z_T\left(\frac{z}{(z-(-1))}\right)\right)$$

$$= -\frac{1}{12} 2^n + \frac{1}{28} 6^n + \frac{1}{21}(-1)^n$$

$\longrightarrow$ (7)

$$Z_T^{-1}[U(z)] = u_0 Z_T^{-1}[p(z)] + u_1 Z_T^{-1}[q(z)]$$

$$+ Z_T^{-1}(r(s))$$

$$u_n = u_0\left[\frac{1}{7} 6^n + \frac{6}{7}(-1)^n\right] +$$

$$u_1\left[\frac{1}{7} 6^n - \frac{1}{7}(-1)^n\right] +$$

$$\frac{1}{28} 6^n + \frac{1}{21}(-1)^n - \frac{1}{12} 2^n$$

$$U_n = \underbrace{\left[\frac{U_0}{7} + \frac{U_1}{7} + \frac{1}{28}\right]}_{c_1} 6^n + \underbrace{\left[\frac{6U_0}{7} - \frac{U_1}{7} + \frac{1}{21}\right]}_{c_2} (-1)^n$$

$$- \frac{2n}{12}$$

$$U_n = c_1 6^n + c_2 (-1)^n - 2^n/12$$