Solving non homogeneous L.D.E

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = \phi(x) \quad (\neq 0)$$

First find C.F

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$$

Particular Integral (P.I)

Given $f(D)\, y = \phi(x)$

$$P.I = \frac{1}{f(D)} \phi(x)$$

Ex

$(D^2 + D + 2)\, y = 2e^{2x}$

$f(D)\, y = e^{2x}$

$P.I = 2 \frac{1}{f(D)} e^{2x}$

$= 2 \frac{1}{D^2 + D + 2} e^{2x}$

Suppose

$$\phi(x) = a\, \phi_1(x) + b\, \phi_2(x)$$

$$P.I = a \frac{1}{f(D)} \phi_1(x) + b \cdot \frac{1}{f(D)} \phi_2(x)$$

$D = \frac{d}{dx}$

$D^2 = \frac{d^2}{dx^2}$

$D^n = \frac{d^n}{dx^n}$

why

$\frac{1}{D}, \frac{1}{D^2}, \frac{1}{D^3} \cdots$

Stands for Successive integrals

If $D = \frac{d}{dx}$ and $X = X(x)$ ✓ then prove that $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$.

Sol:- let $\frac{1}{D-a} X = y$

operating with $D-a$

$$D-a \cdot \frac{1}{D-a} X = (D-a) y$$

$$X = Dy - ay$$

$$X = \frac{dy}{dx} - ay$$

$$\frac{dy}{dx} - ay = X(x)$$

Linear in y

$$P = -a, \quad Q = X(x)$$

$$I.F = e^{\int p\,dx} = e^{\int -a\,dx} = e^{-\int a\,dx}$$

$$IF = e^{-ax}$$

Sol

$$y(IF) = \int Q\,(IF)\,dx + c$$

$$y e^{-ax} = \int X(x) e^{-ax} dx + c$$

$$y = e^{ax} \int X(x) e^{-ax} dx + c$$

If $D = \frac{d}{dx}$ and $X = X(x)$ then prove that $\frac{1}{D+a} X = e^{-ax} \int X e^{ax} dx$

Solution of non homogeneous L.D.E

Procedure

Let the given equation be $f(D)\,y = \phi(x)$

$(\phi(x) \neq 0)$

Let $f(D)y = 0$

Find complementary function (C.F)

To find $P.I = \frac{1}{f(D)}\phi(x)$.

General solution is

$$y = C.F + P.I$$

Type - I

P.I of the form $\frac{e^{ax}}{f(D)}$

For $P.I = \frac{e^{ax}}{f(D)}$ put $D = a$

when $f(a) \neq 0$.

If $f(a) = 0$

$P.I = x \cdot \frac{e^{ax}}{f'(a)}$ when $f'(a) \neq 0$

$P.I = x^2 \frac{e^{ax}}{f''(a)}$ when $f''(a) \neq 0$

Ex :- $(D^2+2D-2)y = e^{2x}$

$\underbrace{D^2+2D-2=0}$ ↓ CF

$P.I = \frac{1}{D^2+2D-2} e^{2x}$ put $D=2$

$= \frac{1}{2^2+2(2)-2} e^{2x}$

$= \frac{1}{6} e^{2x}$

Note:- This method can be applicable for

(i) e^{ax+b}

(ii) any constant $k = ke^{0\cdot x}$

(iii) $\sinh ax = \frac{e^{ax}-e^{-ax}}{2}$

(iv) $\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$

(v) $a^x = (e^{\log a})^x$ put $D = \log a$

Ex $2^x = (e^{\log 2})^x$

Problems

① Solve $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-x}$

Sol :- $(6D^2+17D+12)y = e^{-x}$

A.E is $6m^2+17m+12=0$

$6m^2+9m+8m+12=0 \Rightarrow 3m(2m+3)+4(2m+3)=0$

$(2m+3)(3m+4)=0$

$2m+3=0$ $\quad$ $3m+4=0$

$m=-3/2$ $\quad$ $m=-4/3$

Roots $= -3/2$ & $-4/3$

$C.F = C_1 e^{-3x/2} + c_2 e^{-4x/3}$

$P.I = \dfrac{1}{6D^2+17D+12} e^{-x}$

$a=-1$

Put $D=-1$

$= \dfrac{1}{6(-1)^2+17(-1)+12} e^{-x}$

$= \dfrac{1}{6-17+12} e^{-x}$

$= \dfrac{e^{-x}}{1}$

$P.I = e^{-x}$

General solution $y = C.F + P.I$

$y = c_1 e^{-3x/2} + c_2 e^{-4x/3} + e^{-x}$

2 Solve $(D^3 - 6D^2 + 11D - 6)\,y = e^{2x} + 2e^{-3x}$

Sol:- A.E is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$m = 1$$

$$(1)^3 - 6(1)^2 + 11(1) - 6 = 12 - 12 = 0$$

$m = 1$ satisfies

$$\begin{array}{c|cccc} m=1 & 1 & -6 & 11 & -6 \\ & 0 & \cdot & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$m^2 - 5m + 6 = 0$$

$$m^2 - 3m - 2m + 6 = 0$$

$$m(m-3) - 2(m-3) = 0$$

$$(m-3)(m-2) = 0 \Rightarrow m = 2, 3$$

Roots $1, 2, 3$

$$cf = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

$$P.I = \frac{1}{D^3 - 6D^2 + 11D - 6}\left(e^{2x} + 2e^{-3x}\right)$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6}\, e^{2x} + 2 \cdot \frac{1}{D^3 - 6D^2 + 11D - 6}\, e^{-3x}$$

put $D = 2$ put $D = -3$

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$$P.I = \frac{1}{\underbrace{2^3-6(2)^2+11(2)-6}_{0}} e^{2x} + 2\cdot\frac{1}{(-3)^3-6(-3)^2+11(-3)-6} e^{-3x}$$

$$= x\cdot\frac{1}{3D^2-12D+11} e^{2x} + 2\cdot\frac{1}{\underbrace{-27-54-33-6}} e^{-3x}$$

put $D=2$

$$= x\cdot\frac{1}{3(2)^2-12(2)+11} e^{2x} + 2\cdot\frac{1}{(-120)} e^{-3x}$$

$$= x\cdot\frac{1}{(-1)} e^{2x} - \frac{2}{120} e^{-3x}$$

$$= -x e^{2x} - \frac{1}{60} e^{-3x}$$

$$P.I = -\left(x e^{2x} + \frac{1}{60} e^{-3x}\right)$$

General Solution is

$$y = CF + PI$$

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \left(x e^{2x} + \frac{1}{60} e^{-3x}\right)$$

(3) Solve $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} + 14y = (1+e^{-x})^2$

Sol:- $(D^2+9D+14)y = (1+e^{-x})^2$

$(D^2+9D+14)y = 1+(e^{-x})^2+2e^{-x}\cdot 1$

$(D^2+9D+14)y = 1+e^{-2x}+2e^{-x}$

A.E is $m^2+9m+14=0$

$m^2+7m+2m+14=0$

$m(m+7)+2(m+7)=0$

$(m+2)(m+7)=0$

$m=-2, \quad m=-7$

Roots $= -2, -7$

C.F $= c_1e^{-2x}+c_2e^{-7x}$

$$P.I = \frac{1}{D^2+9D+14}(1+e^{-2x}+2e^{-x})$$

$$P.I = \frac{1}{D^2+9D+14}1 + \frac{1}{D^2+9D+14}e^{-2x} + \frac{1}{D^2+9D+14}2e^{-x}$$

$$= \frac{1}{D^2+9D+14}e^{0x} + \frac{1}{D^2+9D+14}e^{-2x} + 2\cdot\frac{1}{D^2+9D+14}e^{-x}$$

Put $D=0$, Put $D=-2$ Put $D=-1$

$$P.I = \frac{1}{0+0+14}e^{0x} + \underbrace{\frac{1}{(-2)^2+9(-2)+14}}e^{-2x} + 2.\frac{1}{(-1)^2+9(-1)+14}e^{-x}$$

$$= \frac{1}{14}e^{0x} + \frac{1}{\underset{\sim}{18-18}}e^{-2x} + 2.\frac{1}{6_3}e^{-x}$$

$$= \frac{1}{14} + x.\frac{1}{2D+9}e^{-2x} + \frac{1}{3}e^{-x}$$

Put $D = -2$

$$= \frac{1}{14} + x.\frac{1}{2(-2)+9}e^{-2x} + \frac{e^{-x}}{3}$$

$$P.I = \frac{1}{14} + x.\frac{1}{5}e^{-2x} + \frac{e^{-x}}{3}$$

General solution is

$y = CF + PI$

$$y = c_1e^{-2x} + c_2e^{-7x} + \frac{1}{14} + \frac{x}{5}e^{-2x} + \frac{e^{-x}}{3}$$

(4) Solve $\frac{d^3y}{dx^3} - 3.\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = \sinh x + 2$

Sol:- $(D^3 - 3D^2 + 3D - 1)y = \sinh x + 2$

A.E is

$m^3 - 3m^2 + 3m - 1 = 0$ Put $m = 1$

$(1)^3 - 3(1)^2 + 3(1) - 1 = 0$

$$\begin{array}{c|cccc} m=1 & 1 & -3 & 3 & -1 \\ & 0 & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

$m^2 - 2m + 1 = 0 \implies m^2 - m - m + 1 = 0$

$m(m-1) - 1(m-1) = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

Roots are $1, 1, 1$

$$C.F = (c_1 + c_2 x + c_3 x^2) e^x$$

$$P.I = \frac{1}{D^3 - 3D^2 + 3D - 1} (\sinh x + 2)$$

$$= \frac{1}{D^3 - 3D^2 + 3D - 1} \left[\left(\frac{e^x - e^{-x}}{2} \right) + 2e^{0x} \right]$$

$$P.I = \frac{1}{2} \left[\frac{1}{D^3 - 3D^2 + 3D - 1} e^x - \frac{1}{D^3 - 3D^2 + 3D - 1} e^{-x} \right] + 2 \cdot \frac{1}{D^3 - 3D^2 + 3D - 1} e^{0x}$$

put $D=1$ put $D=-1$ put $D=0$

$$= \frac{1}{2} \left[\frac{1}{(1)^3 - 3(1)^2 + 3(1) - 1} e^x - \frac{1}{(-1)^3 - 3(-1)^2 + 3(-1) - 1} e^{-x} \right] + 2 \cdot \frac{1}{0 - 0 + 0 - 1} e^{0x}$$

$$= \frac{1}{2} \left[\frac{1}{\underbrace{1 - 3 + 3 - 1}} e^x - \frac{1}{(-8)} e^{-x} \right] + 2 \cdot \frac{1}{(-1)} e^{0x}$$

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$$P.I = \frac{1}{2}\left[x\frac{1}{3D^2-6D+3}e^{x} - \frac{1}{(-8)}e^{-x}\right] + \frac{2}{-1}e^{0x}$$

Put $D=1$

$$= \frac{1}{2}\left[x\underbrace{\frac{1}{3(1)^2-6(1)+3}}\,e^{x} + \frac{1}{8}e^{-x}\right] - 2$$

$$= \frac{1}{2}\left[x^2\cdot\frac{1}{6D-6}e^{x} + \frac{1}{8}e^{-x}\right] - 2$$

Put $D=1$

$$= \frac{1}{2}\left[x^2\cdot\underbrace{\frac{1}{6(1)-6}}\,e^{x} + \frac{1}{8}e^{-x}\right] - 2$$

$$= \frac{1}{2}\left[x^3\cdot\frac{1}{6}e^{x} + \frac{1}{8}e^{-x}\right] - 2$$

$$P.I = \frac{1}{12}\underline{x^3 e^{x}} + \frac{1}{16}e^{-x} - 2$$

General solution is

$$y = CF + PI$$

$$y = (c_1 + c_2 x + c_3 x^2)e^{x} + \frac{e^{x}x^3}{12} + \frac{e^{-x}}{16} - 2$$

5) Solve $\frac{d^4y}{dx^4} - y = \cosh(x-1) + 3^x - 5$

<u>Sol:-</u> $(D^4-1)y = \cosh(x-1) + 3^x - 5$

A.E is

$m^4 - 1 = 0$

$m = -1$ satisfies $(-1)^4 - 1 = 0$

$m = 1$ satisfies

$$\begin{array}{r|rrrrr} m=1 & 1 & 0 & 0 & 0 & -1 \\ & 0 & 1 & 1 & 1 & 1 \\ \hline m=-1 & 1 & 1 & 1 & 1 & \boxed{0} \\ & 0 & -1 & 0 & -1 & \\ \hline & 1 & 0 & 1 & \boxed{0} & \end{array}$$

$m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm i$

Roots $= 1, -1, +i, -i$

(Or)

$m^4 - 1 = 0 \Longrightarrow (m^2)^2 - 1^2 = 0$

$(m^2+1)(m^2-1) = 0$

$m^2 - 1 = 0$ $m^2 + 1 = 0$

$m = \pm 1$ $m^2 = -1$

$m = \pm i$

$$CF = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$P.I = \frac{1}{D^4-1} \cosh(x-1) + 3^x - 5$$

$$= \frac{1}{D^4-1}\cosh(x-1) + \frac{1}{D^4-1} 3^x - \frac{1}{D^4-1} 5$$

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$$P.I = \frac{1}{D^4-1}\frac{1}{2}\left(e^{x-1}+e^{-(x-1)}\right)+\frac{1}{D^4-1}\left(e^{\log 3}\right)^x - 5\frac{1}{D^4-1}e^{0x}$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{D^4-1}e^{x-1}}_{\text{put } D=1} + \underbrace{\frac{1}{D^4-1}e^{(1-x)}}_{\text{put } D=-1}\right] + \frac{1}{D^4-1}e^{(\log 3)x}$$

put $D = \log 3$

$$5\underbrace{\frac{1}{D^4-1}e^{0x}}_{\text{put } D=0}$$

$$= \frac{1}{2}\left[\frac{1}{(1)^4-1}e^{x-1} + \frac{1}{(-1)^4-1}e^{(1-x)}\right] + \frac{1}{(\log 3)^4-1}e^{(\log 3)x} - 5\cdot\frac{1}{0-1}e^{0x}$$

$$= \frac{1}{2}\left[\underbrace{x\cdot\frac{1}{4D^3}e^{x-1}}_{\text{put } D=1.} + \underbrace{x\cdot\frac{1}{4D^3}e^{(1-x)}}_{\text{put } D=-1}\right] + \frac{1}{(\log 3)^4-1}e^{(\log 3)x} - \frac{5}{(-1)}$$

$$= \frac{1}{2}\left[x\cdot\frac{1}{4(1)^3}e^{x-1} + x\cdot\frac{1}{4(-1)^3}e^{(1-x)}\right] + \frac{1}{(\log 3)^4-1}e^{(\log 3)x} + 5$$

$$3^x = \left(e^{\log_e 3}\right)^x = e^{(\log 3)x}$$

$$= \frac{1}{2}\left[\frac{x}{4}e^{x-1} - \frac{x}{4}e^{1-x}\right] + \frac{1}{(\log 3)^4-1}3^x + 5$$

General solution is

$$y = CF + PI$$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{1}{2}\left[\frac{x}{4} e^{x-1} - \frac{x}{4} e^{1-x}\right] + \frac{1}{(\log 3)^4 - 1} 3^{x+5}$$

Solve

1. $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 3e^{-x}$
2. $(D^3 + 3D^2 + 3D + 1)y = 5e^{2x} + 6e^{-x} + 7$
3. $y'' - 4y' + 13y = \underline{e^{x}\cosh 2x} + 2^{x}$

$$e^{x}\left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

$$= \frac{1}{2}\left[e^{3x} + e^{-x}\right]$$

4. $(D^3 + 2D^2 - D - 2)y = 2\cosh x$
5. $(D^3 - 3D^2 + 3D - 1)y = \sinh(x+2)$

Type - II P.I of the form $\frac{\sin ax}{f(D)}$ (or) $\frac{\cos ax}{f(D)}$

Put $D^2 = -a^2$

$\frac{\sin ax}{f(-a^2)}$ (or) $\frac{\cos ax}{f(-a^2)}$ when $f(-a^2) \neq 0$

if $f(-a^2) = 0$,

$x \cdot \frac{\sin ax}{f'(-a^2)}$ (or) $\frac{x \cdot \cos ax}{f'(-a^2)}$ when $f'(-a^2) \neq 0$

if $f'(-a^2) = 0$

$x^2 \cdot \frac{\sin ax}{f''(-a^2)}$ (or) $\frac{x^2 \cdot \cos ax}{f''(-a^2)}$ $f''(-a^2) \neq 0$

Ex :-

$\frac{1}{D^2-1} \sin(x)$ $a=1$

Put $D^2 = -1$

$= \frac{1}{-1-1} \sin x$

$= \frac{1}{-2} \sin x$

$\frac{1}{D^2+D-1} \sin x$

Put $D^2 = -1$

$= \frac{1}{-1+D-1} \sin x$

$= \frac{1}{D-2} \sin x$

$= \frac{D+2}{(D-2)(D+2)} \sin x$

$= \frac{(D+2)\sin x}{D^2-4}$ $D^2 = -1$

$\frac{F}{D}$ $= \frac{D(\sin x) + 2\sin x}{-1-4}$

Note

This method can be applicable for

(1) $\sin(ax+b)$ and $\cos(ax+b)$

(2) $\sin ax \cdot \cos bx$, $\sin ax \cdot \sin bx$, $\cos ax \cdot \cos bx$, $\cos ax \cdot \sin bx$

(3) $\sin^2 x = \frac{1-\cos 2x}{2}$ & $\cos^2 x = \frac{1+\cos 2x}{2}$