UNIT - 3 DMS

Sum Rule :- If an event can occur in m ways and another event can occurs in n ways and if these two events cannot occur simult the one of the two events can occur in m+n ways.

product Rule :- If an event can occur in m ways and a second event can occur in n ways and if the number of ways the second event occurs does not depend upon how the first event occurs then the two events can occur simul in mn ways.

A bookshelf hold 6 different English books 8 different french books and 10 different German books

There are (6)(8)(10) = 480 ways of selecting ~~3bo~~ 1 in each language

ii) 6 + 8 + 10 = 24 ways of selecting 1 book in any one one of language.

) If each of the 8 question in a multiple choice examination has 3 answers ... ways of ... answering

Product Rule

~~Suppose~~ Suppose that 2 tasks T_1 & T_2 are to be performed one after the other. If T_1 can be performed in n_1 diff ways, & for each of these ways T_2 can be performed in n_2 diff ways, then both of the tasks can be performed in $n_1 n_2$ diff ways.

Examples

① Suppose a person has 3 shirts & 5 ties. Then he has $3 \times 5 = 15$ diff ways of choosing a shirt & a tie.

② Suppose we wish to construct sequence of 4 symbols in which the first 2 are English letters & the next 2 are single digit nos. If no letter or digit can be repeated, then the no of diff sequences that we can construct is

$$26 \times 25 \times 10 \times 9 = 58500.$$

If repetition of letters ~~..~~ & digits are allowed then, $26 \times 26 \times 10 \times 10 = 67600$.

(5) Suppose a restaurant sells 6 South Indian dishes, 4 north indian dishes, 3 hot beverages & 2 cold beverages. For breakfast, a student wishes to buy 1 south Indian dish & 1 hot beverage, or 1 north Indian dish & 1 cold beverage. Then he can have the first choice in $6 \times 3 = 18$ ways & second choice in $4 \times 2 = 8$ ways

The total no of ways he can buy his breakfast items is : $18 + 8 = 26$ (sum rule)

(4) There are 20 married couple in a party. Find the no of ways of choosing ~~a~~ one woman & one man from the party such that the 2 are not married to each other.

Sol A woman can be chosen in 20 ways, Among 20 men, one is her husband. out of 19, one can be chosen in 19 ways.

$20 \times 19 = 380.$

(5) A license plate consists of 2 English letters followed by 4 digits. If repetitions are allowed, how many of the plates have only vowels (A, E, I, O, U) & even digits?

soln

(0, 2, 4, 6, 8)

$$= (5\times5)\times(5\times5\times5\times5)$$

$$= 15,625$$

(6) There are 4 bus routes b/n the places A & B & 3 bus routes b/n the places B & C. Find the no of ways a person can make a round trip from A to A via B if he does ~~not~~ use a route more than once.

soln The person can travel from A to B in 4 ways & from B to C in 3 ways, but only in 2 ways from C to B & only in 3 ways from B to A if he does not use a route more than once.

$$\therefore 4\times3\times2\times3 = 72$$

(7) A bit is either 0 or 1. A byte is a sequence of 8 bits. Find

(a) the no of bytes

(b) the no of bytes that begin with 11 & end with 11

(c) the no of bytes that begin with 11 & do not end with 11,

(d) the no of bytes that begin with 11 or end with 11.

Sol:

(a) Since each byte contains 8 bits & each bit is a 1 or 0 (2 choices), the no bytes $2^8 = 256$.

(b) Beginning & ending with 11, so 4 open positions, $2^4 = 16$ ways.

(c) Six open positions for a byte beginning with 11, so $2^6 = 64$ ways, since there are 16 bytes that begin & end with 11 ($\because$ 4 open positions) $\therefore$ the no of bytes that begin with 11 but do not end with 11 is $64-16 = 48$.

(8) The no of bytes that end with 11 — 64

no of bytes begin & end with 11 — 16

no of bytes begin with 11 — 64

According $|A \cup B| = |A| + |B| - |A \cap B|$

$= 64 + 64 - 16$

$= 112$

(9) A telegraph ~~consists~~ can transmit 2

* diff signals, a dot & a dash; what length of these symbols is needed to encode 26 letter of the English alphabet & the ten digits 0, 1, -- 9.

sol:

(10) How many among the first 100,000 +ve integers contain exactly one 3, one 4 & one 5 in their decimal representation. ?

Soln The no. 100,000 does not contain 3 or 4 or 5 ∴ we have to consider all possible +ve integers with 5 places that meet the given conditions. In 5-place integers the digit 3 can be in any one of the 5 places,

4 can be any one of 4 — " —,

5 — " — 3 — " —

There are 2 places left & either of these may be filled by 7 digits (0, 1, 2, 6, 7, 8, 9).

$$\therefore \quad 5 \times 4 \times 3 \times 7 \times 7 = 2940$$

HM

(11) Find the number of

(a) 2-digit even no

(b) 2-digit odd no

(c) 2-digit odd not with distinct digits

(d) 2-digit even — " —

Sum Rule

(1) Ex - Suppose T_1 is the task of selecting a prime no less than 10 & T_2 is the task of selecting an even no less than 10. Then T_1 can be performed in 4 ways (by selecting 2 or 3 or 5 or 7) & T_2 can be performed in 4 ways (by selecting 2 or 4 or 6 or 8). But, since 2 is both a prime & an even no less than 10, the task T_1 or T_2 can be performed in $4+4-1 = 7$ ways.

Suppose 2 tasks T_1 & T_2 ~~can be~~ are to be performed. If the task T_1 can be performed in m diff ways & the task T_2 can be performed in n diff ways & if these 2 tasks cannot be performed simultaneously, then one of the 2 tasks can be performed in $m+n$ ways.

Consequently,

ABC BCA

$2 \times$ (No of arrangements of the letters BALL)

$=$ No of permutations of the symbols $B\,A\,L_1 L_2$

~~2 × 12 = perm of BAL1~~

$2 \times$ perm of BALL $= 4!$

perm BALL $= \dfrac{4!}{2}$

$= 12$

(2) What is the no of linear arrangements of 6 letters in PEPPER?

Sol:

PEPIER

$P_1 E P_2 P_3 E R$ $P_1 E P_3 P_2 E R$

$P_2 E P_1 P_3 E R$ $P_2 E P_3 P_1 E R$

$P_3 E P_1 P_2 E R$ $P_3 E P_2 P_1 E R$

There are ~~3!~~ $3! = 6$ arrangements with the P's distinguished for each arrangement in which the P's are not distinguished. Similarly for E,

$P_1 E P_2 P_3 E R$ — $P_1 E_1 P_2 P_3 E_2 R$ $P_1 E_2 P_2 P_3 E_1 R$

so

$(2!)(3!)$ No of arranges of PEPPER

= No of ~~arrange~~ perm ~~of~~ of symbols $P_1 E_1 P_2 P_3 E_2 R$

$$= \frac{6!}{2!\,3!}$$

perm of PEPPER = 60.

1) Find the no of permutations of the letters of the word MASSASAUGA. In how many of these, all 4 A's are together? How many of them begin with S?

Soln

The given word has 10 letters of which 4 are A, 3 are S & 1 each are M, U & G. ∴ the required no of permutations is,

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25{,}200$$

If, in a permutation, all A's are together, we treat all of A's as one single letter (AAAA), then the letters to be permuted read S, S, S, M, U, G & the no of permutations is,

$$\frac{7!}{1!\,3!\,1!\,1!\,1!} = 840$$

For permutations beginning with S, there occur 9 open positions to fill, where 2 are S, 4 are A, & one each are M, U, G. The no of such permutations is,

$$\frac{9!}{2!\,4!\,1!\,1!\,1!} = 7560$$

(2) It is required to seat 5 men & 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Soln The 5 men may be seated in odd places in 5! ways & the 4 women may be seated in even places in 4! ways, & corresponding to each arrangement of the men there is an arrangement of the women. ∴ the total no of arrangements $5! \times 4! = 2880$

(3) In how many ways can 6 men & 6 women be seated in a row

i) if any person may sit next to any other?

ii) if men & women must occupy alternate seats?

i) If any person may sit next to any other, no distinction need be made btn men & women in their seating. Accordingly, since there are 12 persons in all, the no of ways they can be seated is

$$12! = 479,001,600$$

ii) when men & women are to occupy alternate seats, the 6 men can be seated in 6! ways in odd places & the 6 women can be seated in 6! ways in even places; & corresponding to each arrangement of the men there is an arrangement of the women.

∴, the no of ways in which the men occupy the odd places & the women the even places is

$$6! \times 6! = 720 \times 720 = 518400$$

III^ly, the no of ways in which the women occupy the odd places & the men the even places, is 518400. ∴, the total no of ways is

$$518400 + 518400 = 1,036,800$$

(4) 4 diff maths books, 5 diff. CS books & 2 diff control theory books are to be arranged in a shelf. How many different arrangements are possible if

i) the books in each particular subjects must be together?

ii) only the maths books must be together

Soln

i) Maths books can be arranged among themselves in 4! differ

CS books ——— " — 5! ways

control theory ——— " — 2! ways

& the 3 groups in 3! ways. ∴ the no of possible arrangements is

$4! \times 5! \times 2! \times 3! = 34,560.$

ii) consider the 4 maths books as one single book. then we have 8 books which can be arranged in 8! ways. In all of these ways the maths books are together. But maths books can be arranged among themselves in 4! ways. Hence, the total arrangements

$$8! \times 4! = 967,680.$$

(5) Find the total no of +ve integers that can be formed from the digits 1, 2, 3, 4 if no digit is repeated in any one integer.

Sol: Note that no integer of the required type can contain more than 4 digits.

Since there are 4 digits, there are

i) 4 integers containing exactly 1 digit

ii) $4 \times 3 = 12$ — " — 2 — " —

iii) $4 \times 3 \times 2 = 24$ — " — 3 —

iv) $4 \times 3 \times 2 \times 1 = 24$ — " — 4 —

∴ required number is

$$4 + 12 + 24 + 24 = 64$$

(6) Find the value of n so that

$$2P(n,2)+50 = P(2n,2)$$

Soln

$$2P(n,2)+50 = P(2n,2)$$

$$2 \times \frac{n!}{(n-2)!} + 50 = \left(\frac{(2n)!}{(2n-2)!}\right)$$

$$= \frac{1}{2!}\left(\frac{(2n)!}{(n-1)!}\right)$$

$$2P(n,2)+50 = P(2n,2)$$

$$2 \times \frac{n!}{(n-2)!} + 50 = \frac{(2n)!}{(2n-2)!}$$

$$2 \times \frac{n \times n-1 \times \cancel{(n-2)!}}{\cancel{(n-2)!}} + 50 = \frac{2n \times (2n-1) \times \cancel{(2n-2)!}}{\cancel{(2n-2)!}}$$

$$2n(n-1)+50 = 2n(2n-1)$$

$$2n^2-2n+50 = 4n^2-2n$$

$$2n^2 = 50$$

$$n^2 = 25$$

$$\boxed{n = 5}$$

(7) P.T, for all integers $n, r \geq 0$, if $n+1 > r$

then, $p(n+1, r) = \left(\frac{(n+1)}{n+1-r}\right) p(n,r)$

Sol

WKT $p(n,r) = \frac{n!}{(n-r)!}$

$p(n+1, r) = \frac{(n+1)!}{(n+1-r)!}$

So $\frac{p(n+1, r)}{p(n,r)} = \frac{(n+1)!}{((n+1)-r)!} \cdot \frac{(n-r)!}{n!}$

$= \frac{(n+1)\, n!}{(n-(r-1))!} \cdot \frac{(n-r)!}{n!}$

$(n-r)\,(n-(r-1))!$

$= \frac{(n+1)}{(n-(r-1))!}$

so $$\frac{P(n+1, r)}{P(n, r)} = \frac{(n+1)!}{((n+1)-r)!} \cdot \frac{(n-r)!}{n!}$$

$$= \frac{(n+1)\cancel{n!}}{((n+1)-r)!} \times \frac{(n-r)!}{\cancel{n!}}$$

$$= \frac{(n+1)(n-r)!}{(n+1-r) \times ((n+1)-(r+1))!}$$

$$= \frac{(n+1)\cancel{(n-r)!}}{(n+1-r) \times \cancel{(n-r)!}}$$

$$\therefore \frac{P(n+1, r)}{P(n, r)} = \frac{(n+1)}{(n+1-r)}$$

Combinations

Suppose we are interested in selecting a set of r objects from a set of $n \geq r$ objects without regard to order (ie unordered set). The set of r objects being selected is called a combination of r objects

The total no of combinations of r different objects that can be selected from n different objects can be obtained by proceeding in the following way. Suppose this number say C, ie

there is a total of C no of combinations of r different objects chosen from n different objects. Take any one of these combinations. The r objects in this combination can be arranged in r! different ways.

Since there are C combinations, the total no of permutations is C . r! .

$$\therefore \quad P(n,r) = C \cdot r!$$

$$C = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

$c(n,n) = c(n,0) = 1$ &

$c(n,1) = c(n,n-1) = n$

problems

(1) 11 out of 20. ${}^{20}C_{11}$

(2) 3 Ques from first 5 & 4 from 5,

${}^{5}C_{3} \times {}^{5}C_{4}$

(3) ${}^{32}C_{20}$ i) ${}^{31}C_{20}$ iii) ${}^{31}C_{19}$

${}^{30}C_{18}$ ${}^{30}C_{19} + {}^{30}C_{19} + {}^{30}C_{18}$

$2 \times {}^{30}C_{19} = {}^{30}C_{19} + {}^{30}C_{19}$

${}^{30}C_{20} + {}^{30}C_{19} + {}^{30}C_{19}$

problems

(1) (1) A hostess is having a dinner party for some

(2) memembers of her charity committee. of

the size of her home, she can invite only 11 out of 20.

so she can invite "lucky 11" combinations she will get?
How many

$$C(20, 11) = \frac{20!}{11!\,9!} = 167,960 \text{ ways}$$

(2) A gym teacher must select 9 girls from the

(2) Junior & ~~senior classes for a volleyball team~~.

If there are 21 Juniors & 25 seniors, ~~she can~~ (a) How

many ways the selection can be done?

(b) If 2 Juniors & 1 seniors are the best spikers

& must be in the team.

(c) If team must contain 4 Juniors & 5 seniors.

Soln

(a) $\binom{53}{9} = \dfrac{53!}{44!\,9!} = 4,431,613,550$

ways of selection

(b) Rest of the team can be chosen in $\binom{50}{6} = 15,890,700$ ways

(c) She can select the 4 juniors in $\binom{28}{4} =$ ways

5 seniors in $\binom{25}{5} =$ ways

According to ROP, she can select her team in $\binom{21}{4} * \binom{25}{5} =$ ~~15,087,836,700~~ ways

(3) Gym teach must make up 4 volleyball teams of 9 girls each from the 36 ~~freshmen~~ girls ~~girls~~. In how many ways can she select these 4 teams? Call the teams A, B, C, D

sol:

To form team A, she can select any 9 girls from the 36 enrolled in

$\binom{36}{9}$ ways

For each team B $\binom{27}{9}$ ways

for team C $\binom{18}{9}$ ways

for team D $\binom{9}{9}$ ways

so according to ROP

$$= \binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}$$

$$= \frac{36!}{9!\,9!\,9!\,9!} = 2.145 \times 10^{19} \text{ ways}$$

wap (a)
(3)

The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831,600$$

How many of these arrangements have no adjacent A's ?

Ans

When we disregard A's, there are

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$$

ways to arrange the remaining letters one of these 5040 ways is shown below.

$$\uparrow E \uparrow E \uparrow S \uparrow T \uparrow L \uparrow L \uparrow S \uparrow H \uparrow$$

The arrows indicate 9 possible locations to the 3 A's. 3 of these locations can be selected in $\binom{9}{3} = 84$ ways, ∴ this is also possible for all

TLLHSSEE (AAA)

9!

the other 5039 arrangements . 84 the ROP .

$5040 \times 84 = 423,360$ arrangements .

(5) A string of length n is a sequence of the form $x_1 x_2 x_3 \cdots x_n$, where each x_i is a digit . The sum $x_1 + x_2 + \cdots + x_n$ is called the weight of the string . If each x_i can be one of $0, 1, 2$. find the no of strings of length $n = 10$. of these , find the no of strings whose weight is an even number.

00 10 20
01 11 21
02 12 22
9

soln Ex - strings of length 2 , $\left.\begin{matrix} 00, 01, 02 \\ 10, 11, 12 \\ 20, 21, 22 \end{matrix}\right\}$ 9

3^2

position 1 2

1 2
3 3

$3 \times 3 = 3^2$

3 choices x 3 choices = 3^2

length 3 : 1 2 3

3 choices x 3 choices x 3 choices = 3^3 = 27 .

and length n 1 2 3 - - - n

$3 \times 3 \times 3$ - - - n times

$= 3^n$ According to ROP

45360 -

In the problem's length is n,

so there 3^{10} strings, we have to determine even wt string,

A string has even wt when the no of 1's in the string is even. (0, 2, 4, 6, 8, 10)

there are 6 different cases to consider,

a) string x contains no 1's — then each of the 10 locations can be filled with 0 or 2 & by the ROP 2^{10} such strings.

b) x contains 2 1's —

the locations for these 2 1's can be selected in C(10,2) ways.

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

(10 location

Once these 2 locations have been specif

there are 2^8 ways to place either 0 or 2 in the other eight positions. Hence there are

$\binom{10}{2} 2^8$ strings of even wt that contains 2 1's

(c) string of length 4 -

locations for 4 1's $c(10,4)$

other locations to fill by 0's & 2's $= 2^6$

$\therefore$ $c(10,4) \times 2^6$ strings have 4 1's.

d) $c(10,6) \times 2^4$ strings have 6 1's

e) $c(10,8) \times 2^2$ " " 8 1's

f) $c(10,10) \times 2^0$ " " 10 1's

According to rule of sum (ROS),

the strings of length 10 that have even wt is

$$= 2^{10} + \binom{10}{2}2^8 + \binom{10}{4}2^6 + \binom{10}{6}2^4 + \binom{10}{8}2^2 + \binom{10}{10}$$

$$= \sum_{n=0}^{5} \binom{10}{2n} 2^{10-2n}$$

⑥ A party is attended by n persons. If each person shakes hands with all others in the party, find the no of handshakes.

Sol: Each handshake is determined by exactly 2 persons.

$$c(n,2) = \frac{n!}{(n-2)!\,2!} = \frac{1}{2}n(n-1)$$

⑦ Prove that ~~foll identities~~ -

$$c(n+1, r) = c(n, r-1) + c(n, r)$$

~~b) $c(m+n, 2) - c(m,2) - c(n,2) = mn$~~

Sol:

a) $LHS = c(n, r-1) + c(n, r)$

$$= \frac{n!}{(n-r+1)!\,(r-1)!} + \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n!}{(r-1)!\,(n-r+1) \times (n-r)!} + \frac{n!}{r!\,(n-r)!}$$

$$\left[\because (n-r+1)! = (n-r+1) \times (n-r) \times (n-r-1) \times \cdots 3 \times 2 \times 1 \right]$$

$$\left\{ r! = r \times (r-1) \times (r-2) \times \cdots 3 \times 2 \times 1 \right.$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left[\frac{1}{(n-r+1)} + \frac{1}{r} \right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left[\frac{\cancel{r} + n - \cancel{r} + 1}{r(n-r+1)} \right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left[\frac{n+1}{r(n-r+1)} \right]$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!}$$

$$= c(n+1, r)$$

$$= \text{LHS}$$

b) $\Rightarrow C(m+n,2) - C(m,2) - C(n,2) = mn$

~~so $C(m+n,2) = mn + C(m,2) + C(n,2)$~~

$$= \frac{m!}{(m-2)!\,2!} + \frac{n!}{(n-2)!\,2!} + mn$$

$$= \frac{m \times (m-1) \times \cancel{(m-2)!}}{\cancel{(m-2)!}\,2!} + \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2!} + mn$$

$$= \frac{1}{2}\left[m(m-1) + n(n-1)\right] + mn$$

=

(7) P.T

$$C(n,r)\cdot C(r,k) = C(n,k)\cdot C(n-k, r-k)$$

for $n \geq r \geq k$.

~~Deduce that, if n is a prime number,~~

~~then $C(n,r)$ is divisible by n.~~

Sol

$$C(n,r)\cdot C(r,k) = \frac{n!}{\cancel{r!}(n-r)!}\cdot\frac{\cancel{r!}}{k!(r-k)!}$$

~~Divide~~ & multiply by $(n-k)!$

$$= \frac{n!}{(n-k)!\,k!}\cdot\frac{(n-k)!}{(n-r)!\,(r-k)!}$$

$$= \frac{n!}{(n-k)!\,k!}\cdot\frac{(n-k)!}{[(n-k)-(r-k)]!\,(r-k)!}$$

$[\because n-r = n-k-r+k]$, Add & sub k

$$= C(n,k)\cdot C(n-k, r-k)$$

$= LHS$

~~FACKLst~~

(8) ~~A gym teacher must make up 4 volleyball teams of 9 girls ~~from~~ ~~such from~~ from 36. In how many ways can she select these 4 teams~~

(4) vap

(9) A woman has 11 close relatives & she wishes to invite 5 of them to dinner. In how many ways can she invite them in the foll situations

a) There is no restriction on the choice

b) 2 particular persons will not attend separately

c) 2 particular persons will not attend together.

Sol

a) Since there is no ~~choice~~ restriction on the choice of invitees, 5 out of 11 can be invited in

$$C(11,5) = \frac{11!}{6!\,5!} = 462 \text{ ways}$$

b) Since 2 particular persons will not attend separately, they should both be invited or not invited.

i) If both of them are invited, then only 3 ~~more~~ invitees are to be selected from the remaining 9.

$$C(9,3) = \frac{9!}{6!\,3!} = 84 \text{ ways}$$

ii) If both of them are not invited, then 5 invitees are to be selected from 9,

$$C(9,5) = \frac{9!}{5!\,4!} = 126 \text{ ways}$$

∴ Total no of ways in which the invitees can be selected is

$$84 + 126 = 210.$$

c) Since 2 (A & B) particular persons will not attend together, only one of them or none of them can be invited.

i) Inviting only one A,

$$C(9,4) = \frac{9!}{5!\,4!} = 126$$

ii) Inviting only B,

$C(9,4) = 126$

iii) ~~Inviting~~ Not inviting both

$$C(9,5) = 126.$$

∴ Total no of ways the invitees can be selected is $126 + 126 + 126 = 378$

combinations with repetition

Ex. How many ways are there to select 4 pieces of fruit from a bowl containing apples, oranges & pears if the order in which the pieces are selected does not matter.

soln

4 apples	4 oranges	4 pears
3 A, 1 Or	3 A, 1 P	3 Or, 1 A
3 Or, 1 P	3 P, 1 A	3 P, 1 Or
2 A, 2 Or	2 A, 2 P	2 Or, 2 P
2A, 1 Or, 1P	2 Or, 1A, 1P	2P, 1A, 1 Or

The soln is the number of 4-combinations from a 3 element set {A, Or, P} with repetition allowed.

We need a general method for counting the r-combinations of n-element set.

$$\frac{12C_4}{4!\,4!\,4!}$$

Ex - How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills & \$100 bills? Assume the order in which the bills are chosen does not matter.

sol - This problem involves counting 5-combinations with repetition allowed from a set with 7 elements. Listing all possibilities would be tedious, since there are a large no of solutions. Instead we use a technique for counting combinations with repetitions allowed.

Suppose that a cash box has 7 compartments, one to hold each type of bills, (fig). They are separated by 6 dividers (fig) The choice of 5 bills corresponds to placing 5 markers in the compts holding diff types of bills.

Fig a illustrates this correspondence for 3 diff ~~ways~~ ways to select 5 bills, where the

6 dividers are represented by bars & the 5 bills by stars.

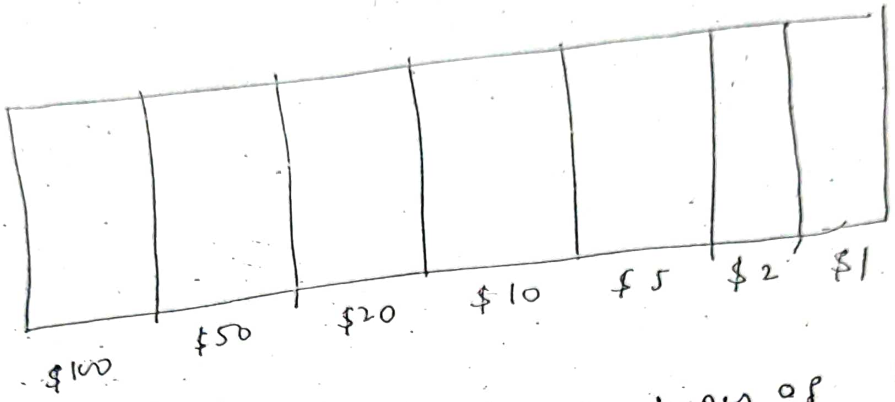


Fig 1 - cash box with 7 types of bills

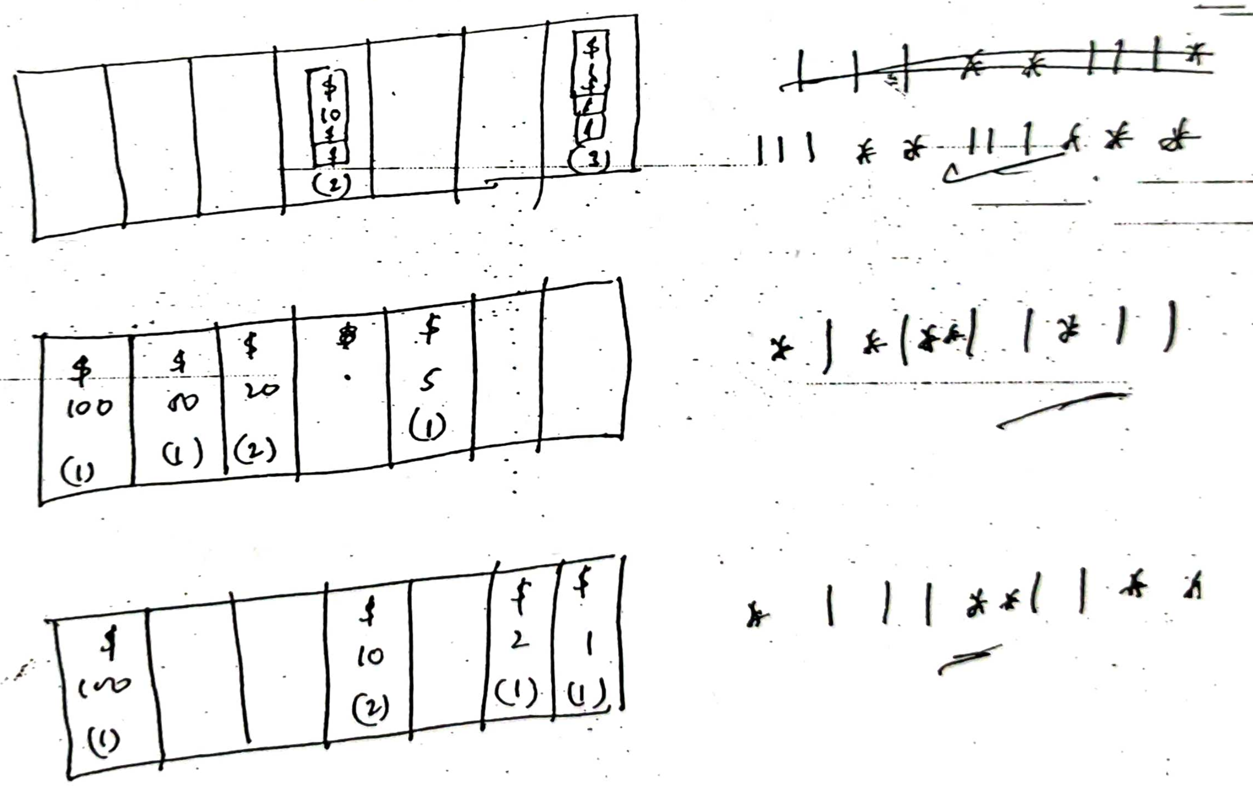


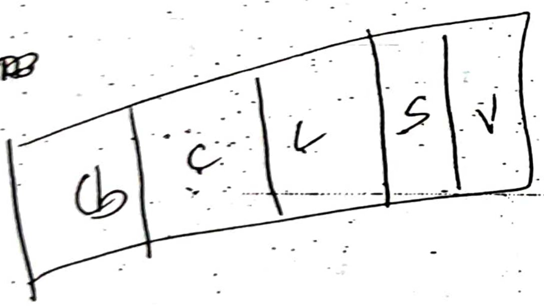
Fig (2)

the no of ways to select 5 bells corresponds to the no of ways of to arrange 6 bars & 5 stars.

consequently, the no of ways to select the 5 bells is the no of ways to select the positions of the 5 stars, from 11 possible ~~selections~~ positions. This corresponds to the no of unordered selections of 5 objects from a set of 11 objects which can be done in

$C(11, 5)$ ways

$$= \frac{11!}{5!\,6!} = 462$$



→ 0 0 0 → → → →

ccc → 0 0 0 → → → 0

bcv 0 → ~~0~~ 0 → → 0 0

bvv 0 → → → → 0 0

b

$x(n-1)$

Theorem

There are $C(n+r-1, r)$ r-combination from a set with n elements when repetition of elements is allowed.

proof -

Each r-combinations of a set with n elements when repetitions is allowed can be represented by $(n-1)$ bars & r stars since each list corresponds ~~to~~ a choice of r positions to place r stars from the $(n-1+r)$ positions, so the no of such lists is

$$c(n+r-1, r)$$

problems

(1) ~~suppose~~ ~~the~~ A cookie shop has 4 diff kinds of cookies. How many diff ways can 6 cookies be chosen? Assume that only the type of cookie, & not the individual cookies or the order in which they are chosen, matters.

Sol: The no of ways to ~~select~~ choose 6 cookies is the no of 6-combinations of a set with 4 elements.

$n = 4$

$r = 6$

$$C(n-1+r, r) = \frac{(n-1+r)!}{(n-1)!\,r!}$$

$$= \frac{(4-1+6)!}{3!\,6!}$$

$$= \frac{9!}{3!\,6!}$$

$$= 84$$

② In how many ways ~~we~~ we can distribute 7 apples & 6 oranges among 4 children so that each child receives at least one appl

Sol:

giving each child an apple, we have

1111 $n=4$ $r=3$.

$n=4$
$r=3$

$C(4+3-1, 3) = 20$ ways to distribute the other 3 apples

$n=4$
$r=6$

$C(4+6-1, 6) = 84$ ways to distribute 6 oranges.

$n=4$
$r=6$

so by ROP,

$20 \times 84 = 1680$ ways

③ A message is made up of 12 diff symbols & is to be transmitted through a communicat channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces b/n the symbols, with at least 3 spaces b/n each pair of consecutive symbo

In how many ways can the transmitter send such a message?

Soln: There are 12! ways to arrange the 12 different symbols.

~~For~~ each of these arrangements there are 11 positions b/n 12 symbols. ∵ there must be atleast 3 spaces b/n successive symbols, we use upto 33 of the 45 spaces & must now locate the remaining 12 spaces.

This is now a selection, with repetition of size 12 (spaces) from a collection of size 11 (the locations) & this can be accomplished in

$r = 12$
$n = 11$

$11 \times 3 = 33$ 12 blanks

|·|||..| |||| ||| | ||| | |||·| ||·| ||||||||||||

$$C(11+12-1, 12) = 646,646 \text{ ways.}$$

By the ROP, the transmitter can send messages ~~with~~ in

$$(12!)\binom{22}{12} = 3.097 \times 10^{14} \text{ ways}$$

Combination with repetition

⑥ In how many ways can we distribute 7 apple & 6 oranges among 4 children so that each child receives at least one apple?

Soln Giving each child an apple, we have $c(4+3-1, 3) = 20$ ways to distribute the other 3 apples & $c(4+6-1, 6) = 84$ ways to distribute the 6 oranges. So by product rule,

$$20 \times 84 = 1680 \text{ ways}$$

⑦ In how many ways can one distribute 10 (identical) marbles among 6 distinct container

Soln

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$$

$n = 6$

$r = 10$

$$= c(n+r-1, r)$$

$$= c(15, 10) = 3003$$

(8) Find the no of non-negative integer solns of the inequality

$$x_1+x_2+x_3+ \cdots +x_6 < 10$$

Soln: we have to find the no of non-negative integer solns of the eqn,

$$x_1+x_2+ \cdots +x_6 = 9-x_7$$

where $9-x_7 \le 9$ so that x_7 is a non-negative integer

$$x_1+x_2+ \cdots +x_6+x_7 = 9$$

$n=7$
$r=9$ $\quad C(15,9) = 5005$

(9) A total amount of Rs. 1500 is to be distributed to 3 poor students A, B, C of a class. In how many ways the distribution can be made in multiples of Rs 100

i) if everyone of these must get at least Rs. 300;

ii) if A must get at least Rs. 500 & B & C must get at least Rs 400 each?

consider Rs. 100 as 1

$A+B+C = 1500$ (Total 500)

i) $A \le 300$, $B \le 300$, $C \le 300$

$A+B+C = 900$

$n=3$, $r=6$

$n = 3$ $\quad C(11, 9)$

$r = 9$ $\quad C(3+6-1, 6)$. $\quad C(9,6)$.

800, 500, 1300

ii) $A \le 5$, $B \le 4$, $C \le 4$ $\quad$ 500 400 400.

$r = 200$.

$A+B+C = 2$ $\quad n=3$ $\quad n=3$, $r=2$

$r=2$

$n = 3$ $\quad C(4, 2)$

$r = 2$ $\quad C(n+r-1, r)$

$C(4, 2)$

---

(10) Find the no of ways of giving 10 identical gift boxes to 6 persons A, B, C, D, E, F in such a way that the total no of boxes given to A & B together does not exceed 4.

Sol: Of the 10 boxes, suppose r boxes are given to A & B together, then $0 \le r \le 4$. The no of ways of giving r boxes to A & B is

$$\frac{(r+1)!}{r!}$$

$$C(2+r-1, r) = C(r+1, r)$$

$$\frac{(2+r-1)!}{r!(2+r-1-1)!} \quad \frac{(r+1)!}{r!(r+1-1)!} = \boxed{r+1}$$

$$\frac{(r+1)!}{r! \times r!} \quad (r+1). \qquad \frac{(r+1)!}{r!\,r!} = \frac{(r+1) \times r!}{r!\,r!}$$

$$\frac{(r+1)!}{r!\,(r+1-1)!} \qquad \frac{(r+1)\,r!}{r!\,r!}$$

The no. of ways in which the remaining (10−r) boxes can be given to C, D, E, F is

$$C(4+(10-r)-1, (10-r))$$

$$= C(13-r, 10-r)$$

$10-r$, $C(13-r, 10-r)$, $13-r$

The no. of ways in which r boxes may be given to A & B, (10−r) boxes to C, D, E, F is,

$$\sum_{r=0}^{4} (r+1) \times C(13-r, 10-r)$$

---

(11) How many ~~distinct~~ terms are there in the expansion of $(w+x+y+z)^{10}$?

Soln: According to multinomial thm,

$$n_1+n_2+n_3+n_4 = 10$$

$n=4$
$r=10$

$$= C(n+r-1, r)$$

$$= C(13, 10) = 286$$

(4) How many solns does the eqn

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2 & x_3 are non-negative integers?

soln

To count the number of solns, we note that a soln corresponds to a way of selecting 11 items from a set with 3 elements, so that x_1 items of type one, x_2 items of type two & x_3 items of type three are chosen.

Hence the no of solns is equal to the number of 11-combinations with repetition allowed from a set with 3 elements.

$n = 3$

$r = 11$

$$C(3+11-1, 11) = \frac{13!}{2! \cdot 11!}$$

$$= 78.$$

(4)(b) Let say the constraints are

$x_1 \geq 1, \quad x_2 \geq 2 \quad \& \quad x_3 \geq 3$

That is ~~atmost~~ at least 1 item of type one, 2 items of type two, 3 items of type three

Left out 5 items $(11-(1+2+3))$

$n = 3$

$r = 5$

$$C(3+5-1, 5) = \frac{7!}{2!\,5!}$$

$= 21$

Binomial Theorem

Binomial co-efficients –

The number of r-combinations from a set with n elements is often denoted by $\binom{n}{r}$ $[{}^nC_r]$ this number is also called BC ∴ these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(x+y)^n$.

The expansion of $(x+y)^3$ can be found using combinatorial reasoning instead of multiplying the 3 terms. when $(x+y)^3 = (x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, & a term in the third sum are added.

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

$$= \cancel{x^3 + 3x}\ (xx + xy + yx + yy)(x+y)$$

$$= xxx + xxy + xyx + xyy + yxx$$
$$+ yxy + yyx + yyy$$

$= x^3 + 3x^2y + 3xy^2 + y^3$

To obtain a term of the form x^3, an x must be chosen in each of the sums & this can be done in only one way. Thus the x^3 term is ~~the~~ has a coefficient 1.

To obtain a term of the form x^2y, an x must be chosen in 2 of the 3 sums (and consequently a y in the other sum). Hence, the no. of such terms is the number of 2-combination of 3 objects, ${}^3C_2$.

III^ly the xy^2, the no of ways to pick one of the 3 sums to obtain x (& consequently take a y from each of the other 2 sums). this can be done in ~~3~~ ${}^3C_1$ ways.

Finally, only one way to obtain a y^3 term, ~~3~~

$\therefore (x+y)^3 =$

Binomial Theorem

Let x & y be variables, & let n be a non-negative integer. Then

$$\binom{n}{n-j} x^{j} y^{n-j}$$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^{j} \quad \text{or} \quad \sum_{j=0}^{n} \binom{n}{j} x^{j} y^{n-j}$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Proof - A combinatorial proof of the theorem will be given. The terms in the product when it is expanded are of the form

$$x^{n-j} y^{j} \text{ for } j = 0, 1, 2 \ldots n.$$

To count the no of terms of the form $x^{n-j}y^{j}$, note that to obtain such a term it is necessary to chose $(n-j)$ xs from the n sums (so that the other j terms in the product are ys).

$\therefore$ the coefficient of

$x^{n-j} y^j$ is nC_{n-j} or $\binom{n}{n-j}$

which is equal to $\binom{n}{j}$

$$\left[\binom{n}{n-j} = \frac{n!}{(n-\cancel{n}+j)!\,(n-j)!} = \frac{n!}{(n-j)!\,j!} \right.$$

$$\left. \therefore \binom{n}{j} = \frac{n!}{(n-j)!\,j!} \right]$$

problems

① what is the expansion of $(x+y)^4$?

sol:- From BT,

$$= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

(4) P.T the foll for all of positive integer n.

a) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+ --- +\binom{n}{n} = 2^n$

b) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+ --- (-1)^n\binom{n}{n} = 0$

From BT,

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r} x^{n-r} y^r$$

when $x=y=1$,

$$2^n = \sum_{r=0}^{n}\binom{n}{r} 1^r . 1^{n-r}$$

$$= \sum_{r=0}^{n}\binom{n}{r} = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+ --- +\binom{n}{n}$$

(b) ~~when $x=y=$~~

when $x=+1$, $y=-1$

$$0 = \sum_{r=0}^{n}\binom{n}{r}(-1)^r = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}+ --- (-1)^n\binom{n}{n}$$

(2) what is the co-efficient of $x^{12}y^{13}$ in the expn of $(x+y)^{25}$? (3)

sol: From BT, this co-efficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5,200,300$$

(3) what is the co-efficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

sol: Expression according to BT $(2x+(-3y))^{25}$

$$(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} 2x^{25-j} (-3y)^{j} \quad \text{--- (1)}$$

The coefficient of $x^{12}y^{13}$ in the expr is obtained

when $j = 13$,

$$\cancel{\binom{25}{13}2}$$

$$(1) \Rightarrow \quad = \binom{25}{j} 2^{25-j} (-3)^{j}$$

$$= \binom{25}{13} 2^{12} (-3)^{13} = \frac{25!}{12!\,13!} 2^{12} (-3)^{13}$$

(4) Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^{k} \binom{n}{k} = 3^{n}$$

Sol: We recognise that the LHS of this formula is the expansion of $(1+2)^n$ according to B.T.

$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^{k} = \sum_{k=0}^{n} \binom{n}{k} 2^{k}$$

Hence

$$\sum_{k=0}^{n} 2^{k} \binom{n}{k} = 3^{n}$$

Multinomial Theorem

For +ve integers n, t, the co-efficients of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ in the expansion of $(x_1+x_2+\cdots+x_n)^n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_t!}$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$ & $n_1+n_2+n_3+\cdots+n_t = n$.

sol: As in the proof of the binomial theorem, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $(n-n_1)$ remaining factors, x_3 from n_3 of the $(n-n_1-n_2)$ remaining factors & so on & x_t from n_t of the last $(n-n_1-n_2-n_3 - \cdots - n_{t-1})$ remaining factors. So

$\cancel{{}^{n}C_{n_1} \times (n-n_1)}$

$$ {}^{n}C_{n_1} \times {}^{n-n_1}C_{n_2} \times {}^{n-n_1-n_2}C_{n_3} \cdots $$

$$ {}^{n-n_1-n_2-n_3 \cdots -n_{t-1}}C_{n_t} \quad \text{ways} $$

$$ = \frac{n!}{\cancel{(n-n_1)!}\, n_1!} \times \frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\, n_2!} \times \frac{\cancel{(n-n_1-n_2)!}}{(n-n_1-n_2-n_3)!\, n_3!} $$

$$ \cdots \frac{\cancel{(n-n_1-n_2-n_3 \cdots n_{t-1})!}}{(n-n_1-n_2 \cdots n_{t-1}-n_t)!\, n_t!} = $$

$$ \cancel{= \frac{n(n-1)!}{(n-n_1)!\, n_1!} \times \frac{(n-n_1)(n-n_2)!}{(n-n_1-n_2)!}} $$

$$ = \frac{n\cancel{(n-n_1)!}}{\cancel{(n-n_1)!}\, n_1!} \times \frac{(n-n_1)\cancel{(n-n_1-n_2)!}}{\cancel{(n-n_1-n_2)!}\, n_2!} \times \cdots $$

$$ \cdots \frac{(n-n_1-n_2 \cdots n_{t-1})(n-n_1 \cdots n_{t-2})}{(n-n_1-n_2 \cdots n_{t-1}-n_t)!\, n_t!} $$

$$ = \frac{n \times (n-n_1)(n-n_2) \cdots (n-n_1-n_2 \cdots n_{t-1})}{n_1!\, n_2!\, n_3! \cdots n_t!} $$

$$\left(\because n_1+n_2+ \cdots + n_t = 1 + n_t = n,\quad n_t = n-n_1-n_2-\cdots-n_{t-1} \right)$$

$$= \frac{n!}{n_1!\, n_2! \cdots (n-n_1-n_2-\cdots-n_{t-1}-n_t)!\, n_t!}$$

is called multinomial co-efficient.

$$= \frac{n!}{n_1!\, n_2! \cdots (n_t/n_t)!\, n_t!} = \frac{n!}{n_1!\, n_2! \cdots n_t!}$$

problems

(1) ~~Find the term which contains x^{11} & y^4 in the expansion of $(2x^3-3xy^2+z^2)^6$~~

(1) Find the co-efficient of $x^2y^2z^3$, xyz^5 & x^3z^4 in the expansion of $(x+y+z)^7$

Sol:

From multinomial theorem

i) co-efficient of $x^2y^2z^3$ is

$$\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = 210$$

ii) co-efficient of xyz^5 $\binom{7}{1,1,5} = \frac{7!}{1!\,1!\,5!}$

$= 42$

iii) $x^3 z^4$ $\quad \binom{7}{3,0,4} = \dfrac{7!}{3!\,0!\,4!} = 35$

(2) Find the co-efficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.

Sols: Replace a by v

$2b$ by w

$-3c$ by x

$2d$ by y

5 by z

According to MT,

$(v+w+x+y+z)^{16}$, co-efficient of $a^2b^3c^2d^5$

is $\binom{16}{2,3,2,5} = \dfrac{16!}{2!\,3!\,2!\,5!} = 302,702,400$

(2) Find the term which contains x^{11} & y^4 in the expansion of $(2x^3 - 3xy^2 + z^2)^6$

Soln

By the MT, the general term in the given expansion is

$$= \binom{6}{n_1\ n_2\ n_3} (2x^3)^{n_1} (-3xy^2)^{n_2} (z^2)^{n_3}$$

$$= \binom{6}{n_1\ n_2\ n_3} 2^{n_1} (-3)^{n_2} \cdot x^{3n_1+2n_2} \cdot y^{2n_2} z^{2n_3}$$

Then, for the term containing x^{11} & y^4

we should have $3n_1 + n_2 = 11$ & $2n_2 = 4$,

so that $n_1 = 3$ & $n_2 = 2$.

Since $n_1 + n_2 + n_3 = 6$, then $n_3 = 1$

According the term containing x^{11} & y^4 is

$$= \binom{6}{3\ 2\ 1} 2^3 (-3)^2 x^{11} y^4 z^2$$

$$= \left\{ \frac{6!}{3!\,2!\,1!} \times 8 \times 9 \right\} x^{11} y^4 z^2$$

$$= 4320\, x^{11} y^4 z^2 \qquad \boxed{4320}$$

③ Determine the co-efficient of

i) xyz^2 in the expansion of $(2x-y-z)^4$

ii) $a^2b^3c^2d^5$ in " " $(a+2b-3c+2d+5)^{16}$

Sol

i) The general term of in the expansion of $(2x-y-z)^4$ is

$$\binom{4}{n_1\ n_2\ n_3} (2x)^{n_1} (-y)^{n_2} (-z)^{n_3}$$

(xyz^2) $n_1 = 1,\ n_2 = 1,\ n_3 = 2$

Then $\binom{4}{1,1,2}(2x)(-4)(-z)^2$

$= \binom{4}{1,1,2} \times 2 \times (-1) \times (-1)^2 \, x y z^2$

$= -2 \times \frac{4!}{1!\,1!\,2!} \, x y z^2$

$= -24 \, x y z^2$ $\qquad$ $\boxed{-24}$

ii) The genl term in the expansion of

$(a+2b-3c+2d+5)^{16}$ is.

$\binom{16}{n_1, n_2, n_3, n_4, n_5} a^{n_1} (2b)^{n_2} (-3c)^{n_3} (2d)^{n_4} (5)^{n_5}$

$(a^2 b^3 c^2 d^5)$; $n_1 = 2$, $n_2 = 3$, $n_3 = 2$, $n_4 = 5$, $n_5 = ?$

$n_1 + n_2 + n_3 + n_4 + n_5 = 16$

$\therefore n_5 = 4$

$$= \binom{16}{2,3,2,5,4} a^2(2b)^3(-3c)^2(2d)^5\, 5^4$$

$$= \binom{16}{2,3,2,5,4} \times 2^3 \times (-3)^2 \times 2^5 \times 5^4 \times a^2b^3c^2d^5$$

$$= 2^8 \times 3^2 \times 5^4 \times \frac{16!}{2!\,3!\,2!\,5!\,4!}\, a^2b^3c^2d^5$$

$$= 3 \times 2^5 \times 5^3 \times \frac{16!}{(4!)^2}\, a^2b^3c^2d^5$$

$$= \boxed{\frac{16! \times 2^5 \times 5^3 \times 3}{(4!)^2}}$$

Principles of Inclusion & Exclusion

Theorem -

Consider a set S, with $|S| = N$, & conditions c_i, $1 \le i \le t$, satisfied by some of the elements of S. The no of elements of S that satisfy none of the conditions c_i, $1 \le i \le t$ is denoted by $\overline{N} = N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \ldots \bar{c}_t)$ where

$$\overline{N} = N - [N(c_1) + N(c_2) + \cdots + N(c_t)]$$
$$+ [N(c_1 c_2) + N(c_1 c_3) + \cdots + N(c_1 c_t) + N(c_2 c_3) + \cdots + N(c_{t-1} c_t)] -$$
$$[N(c_1 c_2 c_3) + N(c_1 c_2 c_4) + \cdots + N(c_1 c_2 c_t) + N(c_1 c_3 c_4) + \cdots + N(c_1 c_3 c_t) + \cdots$$
$$\cdots + N(c_{t-2} c_{t-1} c_t)] + \cdots$$
$$\cdots (-1)^t N(c_1 c_2 \cdots c_t),$$

(or)

$$\bar{N} = N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \quad (2)$$

$$\sum_{1 \le i < j < k \le t} N(c_i c_j c_k) + \cdots + (-1)^t N(c_1 c_2 \cdots c_t)$$

Proof - By induction on t.

For each $x \in S$ we show that x contributes the same count, either 0 or 1, to each side of eqn (2).

If x satisfies none of the conditions, then x is counted once in $\bar{N}$ & once in N, but not in any of the other terms in eqn (2). Consequently, x contributes a count of 1 to each side of the eqn.

The other possibility is that x satisfies exactly r of the conditions where $1 \le r \le t$. In this case x contributes nothing to $\bar{N}$. But on the right hand side of eqn (2), x is counted

1) One time in N

2) r times in $\sum_{1 \le i \le t} N(c_i)$ (once for each of the r conditions)

3) $\binom{r}{2}$ times in $\sum_{1 \le i < j \le t} N(c_i c_j)$

(once for each pair of conditions selected from the r conditions it satisfies)

(4) $\binom{r}{3}$ times in $\sum_{1 \le i < j \le k \le t} N(C_i C_j C_k)$

⋮

(r+1) $\binom{r}{r} = 1$ times in $\sum N(C_{i_1} C_{i_2} \cdots C_{i_r})$, where summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand ~~side~~ of eqn (2), x is counted

$$= 1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r}$$

$$= [1 + (-1)]^r \quad \text{[By Bionomical thm]}$$

$$= 0^r$$

$$= 0 \text{ times.}$$

∴, the two sides of eqn (2) count the same elements of S, & the equality is verified.

——— X ———

simplification of eqn (1) or (2)

$$S_0 = N$$

$$S_1 = [N(C_1) + N(C_2) + \cdots + N(C_t)]$$

$$S_2 = [N(C_1C_2) + N(C_1C_3) + \cdots + N(C_1C_t) + N(C_2C_3) + \cdots + N(C_{t-1}C_t)]$$

In general,

$$S_k = \sum N(C_{i_1} C_{i_2} \cdots C_{i_k}), \quad 1 \le k \le t.$$

Note

we can represent as sets,

$$N = |S|, \quad N(C_i) = |A_i|, \quad N(\overline{C_i}) = |\overline{A_i}|$$

$$N(C_iC_j) = |(A_i \cap A_j)|, \quad N(\overline{C_i}\,\overline{C_j}) = |\overline{A_i} \cap \overline{A_j}|$$

$$N(C_iC_jC_k) = |(A_i \cap A_j \cap A_k)|, \quad N(\overline{C_i}\,\overline{C_j}\,\overline{C_k}) = |\overline{A_i} \cap \overline{A_j} \cap \overline{A_k}$$

$$\vdots$$

$$N(C_1C_2C_3 \cdots C_n) = |A_1 \cap A_2 \cap A_3 \cdots \cap A_n|$$

$$\overline{N} = N(\overline{C_1}\,\overline{C_2} \cdots \overline{C_n}) = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cdots \cap \overline{A_n}|$$

& $N(c_i \text{ or } c_j) = |A_i \cup A_j|$

$N(c_i \text{ or } c_j \text{ or } c_k) = |A_i \cup A_j \cup A_k|$

$\vdots$

$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } \cdots \text{ or } c_n) = |A_1 \cup A_2 \cup A_3 \cdots A_n|$

$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } \cdots \text{ or } c_n) = |S| - \cancel{\overline{A \cap B}}$ — ③

$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \cdots \bar{c}_n)$

$= N - \bar{N}$

$\because |A \cup B| = |A| + |B| - |A \cap B|$

$|\overline{A \cup B}| = |S| - |A \cup B|$

$\therefore |\bar{A} \cap \bar{B}| = S - |A \cup B|$

$|A \cup B| = S - |\bar{A} \cap \bar{B}|$

③ $\Rightarrow$ $N(c_1 \text{ or } c_2 \text{ or } c_3 \cdots \text{ or } c_n) = \cancel{|S|} - \cancel{|S|}$

$+ S_1 - S_2 + S_3 - \cdots - (-1)^{n-1} S_n.$

(1) Among the students in a hostel, 12 students study maths (A), 20 study physics (B), 20 chemistry (C) & 8 Biology (D). There are 5 students in A & B, 7 in A & C, 4 in A & D, 16 in B & C, 4 in B & D, & 3 in C & D. There are 3 students in A, B & C, 2 in A, B & D, 2 in B, C & D, 3 in A, C & D. Finally, there are 2 who study all of these subjects. Furthermore, there are 71 students who do not study any of these subjects. Find the total no of students in the hostel

Soln

$|A| = 12, \ |B| = 20, \ |C| = 20, \ |D| = 8$

$|A \cap B| = 5, \ |A \cap C| = 7, \ |A \cap D| = 4, \ |B \cap C| = 16$

$|B \cap D| = 4, \ |C \cap D| = 3$

$|A \cap B \cap C| = 3, \ |A \cap B \cap D| = 2, \ |B \cap C \cap D| = 2$

$|A \cap C \cap D| = 3, \ |A \cap B \cap C \cap D| = 2, \ |\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = 71$

$|S| = ?$

$$71 = |S| - (12 + 20 + 20 + 8) + (5 + 7 + 4 + 16 + 4 + 3) - (3 + 2 + 2 + 3) + 2$$

$\therefore 100$

pigeonhole principle

If m pigeons occupy n pigeonholes & $m > n$, then at least one pigeonhole must contain 2 or more pigeons in it

Generalized pigeon-hole principle

If m pigeons occupy n pigeonholes then at least one pigeonhole must contain ~~(p+1) pigeons~~ $(p+1)$ or more pigeons $(\geq (p+1))$,

~~where~~ $p = \lfloor m-1/n \rfloor$

proof - By method of contradiction

Assume that the conclusion part is not true. Then, no pigeonhole contains $(p+1)$ or more pigeons. This means every pigeonhole contains p or less number of pigeons

then,

Total no of pigeons $\leq np$

$$\leq n \lfloor m-1/n \rfloor$$

$$\leq n\left(\frac{m-1}{n}\right)$$

$$\leq (m-1)$$

But total no of pigeons is m. Hence our assumption is wrong, hence the proof

problems

① P.T if 30 dictionaries in a library contain a total of 61,327 pages, then at least one of the dictionaries must have at least 2045 pages

sol: pages as pigeons

Dictionaries - pigeonholes

According to generalized princ.

at least one of the dictionaries must contain $(p+1)$ or more pages i.e

Pigeonhole Principle

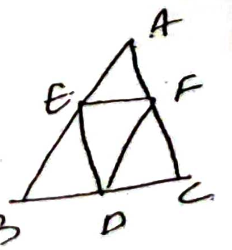
If m pigeons occupy n pigenholes $m > n$ then at least one pigeonhole must contain two or more pigeons in it.

Generalization

The ~~follo~~wing is an extension/generalization of the pigeonhole principle

If m pigeons occupy n pigeonholes then at least one pigeonhole must contain $(p+1)$ or more pigeons, where $p = \lfloor (m-1)/n \rfloor$

1) ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that at least two of these points are such that the distance b/w them is less than $\frac{1}{2}$cm.



A bag contains 12 pairs of socks (each pair in different colour). If a person draws the socks one by one at random, determine at most how many draws are required to get at least one pair of matched socks.

Let n denote the number of the draw. For $n \le 12$, it is possible that socks drawn are of different colours, b/coz there are 12 colours. For $n = 13$ all socks cannot have different colours, at least two must have the same colour (Treat 13 as the number of pigeons and 12 colour as 12 pigeonholes). Thus at most 13 draws are required to have at least one pair of socks of the same colour.

Prove that in any set of 29 persons at least five person must have been born on the same day of the week.

$\left\lfloor \frac{29-1}{7} \right\rfloor + 1 = 5$ ans

$\left\lceil \frac{29}{7} \right\rceil = 4.1 = 5$

Problems on inclusion & exclusion

(2) How many integers b/n 1 & 300 (inclusive).

i) divisible by at least one of 5, 6, 8 ?

ii) divisible by none of 5, 6, 8 ?

Sol: $|S| = 300$.

Let A_1, A_2, A_3 be subsets of S divisible by 5, 6 & 8.

i) Atleast one of 5, 6 & 8 is $|A_1 \cup A_2 \cup A_3|$

$$|A_1 \cup A_2 \cup A_3| = S_1 - S_2 + S_3$$

$$= |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$$

$|A_1| = \lfloor 300/5 \rfloor = 60.$

$|A_2| = \lfloor 300/6 \rfloor = 50.$

$|A_3| = \lfloor 300/8 \rfloor = 37.$

$|A_1 \cap A_2| = \lfloor 300/30 \rfloor = 10.$ [LCM of 5, 6 is 30]

$|A_1 \cap A_3| = \lfloor 300/40 \rfloor = 7$ [— 5, 8 is 40]

$|A_2 \cap A_3| = \lfloor 300/24 \rfloor = 12$ [— 6, 8 is 24]

$|A_1 \cap A_2 \cap A_3| = \lfloor 300/120 \rfloor = 2$ [— 5, 6, 8 is 120]

$$|A_1 \cup A_2 \cup A_3| = (60+50+37) - (10+7+12) + 2$$

$$= 120.$$

ii) Divisible by none of 5, 6, 8 is,

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |S| - |A_1 \cup A_2 \cup A_3|$$

$$= 300 - 120$$

$$= 180.$$

problems

① Determine the no of +ve integers n where $1 \le n \le 100$ & n is not divisible by 2, 3, 5

Sol: Condition C_1 is — is divisible by 2

C_2 — — " — 3

C_3 — — " — 5

$N(C_1) = \lfloor 100/2 \rfloor = 50$

$N(C_2) = \lfloor 100/3 \rfloor = 33$

$N(C_3) = \lfloor 100/5 \rfloor = 20$

$N(C_1 C_2) = \lfloor 100/2.3 \rfloor = 16$ (LCM of 2, 3 = 6

$N(C_1 C_3) = \lfloor 100/2.5 \rfloor = 10$ (LCM of 2, 5 = 10

$N(C_2 C_3) = \lfloor 100/3.5 \rfloor = 6$ (LCM of 3, 5 = 15

$N(C_1 C_2 C_3) = \lfloor 100/30 \rfloor = 3$ (LCM of 2, 3, 5 = 30

$N(\bar{C_1} \bar{C_2} \bar{C_3}) = S_0 - S_1 + S_2 - S_3$

$= N - [N(C_1) + N(C_2) + N(C_3)]$

$+ [N(C_1 C_2) + N(C_1 C_3) + N(C_2 C_3)]$

$- N(C_1 C_2 C_3) \quad 26$

$N(C_1 \cup C_2 \cup C_3)$

(2) No of non-negative integer solns to the eqn

$x_1 + x_2 + x_3 + x_4 = 18$, $x_i \le 7$ for all $1 \le i \le 4$

soln

~~S is the set of solns of $x_1+x_2+x_3+x_4=18$ with $x_i \ge 0$~~

~~$|S| = N = c(n+r-1, r)$~~

Let S denote the set of all non-negative integer solns.

$= c(4+18-1, 18) =$

$|S| = c(21, 18)$

~~Let A_1 denote the~~

Let A_1 be the subset of S that contains the non-negative integer solns of the given eqn under the conditions $x_1 > 7$, $x_2 \ge 0$, $x_3 \ge 0$, $x_4 \ge 0$

$A_1 = \{(x_1, x_2, x_3, x_4) \in S \mid x_1 > 7\}$

$A_2 = \{(x_1, x_2, x_3, x_4) \in S \mid x_2 > 7\}$

$A_3 = \{(x_1, x_2, x_3, x_4) \in S \mid x_3 > 7\}$

$A_4 = \{(x_1, x_2, x_3, x_4) \in S \mid x_4 > 7\}$

then the required no of soln $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4|$

$x_1 > 7$ i.e $x_1 \geq 8$, minimum 8

$x_1 + x_2 + x_3 + x_4 = 10 \quad (\because 18 - 8)$

i) $|A_1| = |A_2| = |A_3| = |A_4| = C(4+10-1, 10)$

$= C(13, 10)$

ii) $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4|$

$x_1 + x_2 + x_3 + x_4 = 2 \quad (18 - 16)$

$C(4+2-1, 2) = C(5, 2)$

iii) $|A_1 \cap A_2 \cap A_3|$, $x_1 + x_2 + x_3 + x_4 = 0$

$|A_1 \cap A_2 \cap A_4|$

$|A_2 \cap A_3 \cap A_4|$

$|A_1 \cap A_3 \cap A_4|$

iv) $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= C(21,18) - \binom{4}{1} \times C(13,10) + \binom{4}{2} \times C(5,2) - 0 + 0$$

22 6 9 8 3

9 2 4 3 7 5 5 1 2 1

$= 366$

3) In how many can the 26 letters of the alphabet be permuted so that none of the patterns car, dog, pun, or byte occurs

Soln

Let S denote the set of all permutations of the 26 letters, $|S| = 26!$.

A permutation in S is said to satisfy condition C_i if the permutation contains car, dog, pun or byte resp.

$N(C_1) = N(C_2) = N(C_3) = 24!$

$N(C_4) = 23!$

$N(C_1C_3) = N(C_2C_3) = 22!$

$N(C_iC_4) = 21!$

$N(C_1C_2C_3) = 20!$

$N(C_iC_jC_4) = 19!$

$N(C_1C_2C_3C_4) = 17!$

Generalization of the principle.

Consider a set S with $|S| = N$, & conditions $c_1, c_2, \ldots, c_t$ satisfied by some of the elements of S. The inclusion - exclusion provides a way to determine $N(\bar{c}_1 \bar{c}_2 \ldots \bar{c}_t)$, the no of elements in S that satisfy none of the t condition.

Let say $1 \le m \le t$, we want to determine E_m which denotes the no of elements in S that satisfy exactly m of the t conditions.

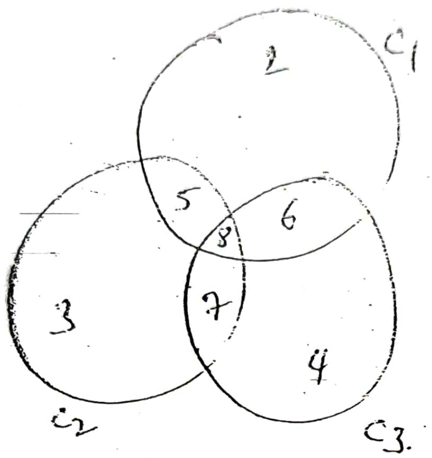
$$E_1 = N(c_1 \bar{c}_2 \bar{c}_3 \ldots \bar{c}_t) + N(\bar{c}_1 c_2 \bar{c}_3 \ldots \bar{c}_t) + \ldots + N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \ldots \bar{c}_{t-1} c_t)$$

&

$$E_2 = N(c_1 c_2 \bar{c}_3 \bar{c}_4 \ldots \bar{c}_t) + N(c_1 \bar{c}_2 c_3 \bar{c}_4 \ldots \bar{c}_t) + \ldots + N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \ldots c_t c_t)$$

By taking an ex, we see, ~~let take t~~

i) $t = 3$



E_1 equals the no of elements in regions 2, 3 & 4.

$$E_1 = N(C_1) + N(C_2) + N(C_3) - 2[N(C_1C_2) + N(C_1C_3) + N(C_2C_3)] + 3N(C_1C_2C_3)$$

In $[N(C_1) + N(C_2) + N(C_3)]$ we count the elements in regions 5, 6 & 7 twice & those in region 8 three times.

In the second term, the elements in regions 5, 6 & 7 are deleted twice. we remove the elements in region 8 six times, ~~...~~

so we then add the term $3N(C_1C_2C_3)$ &

end up not counting the elements in region 8

at all.

Hence $E_1 = S_1 - 2S_2 + 3S_3$

$E_1 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3$

For E_2, we have to count the element

of S in regions 5, 6 & 7.

$E_2 = N(C_1C_2) + N(C_2C_3) + N(C_1C_3) - 8 \cdot N(C_1C_2C_3)$

$= S_2 - 3S_3$

$E_2 = S_2 - \binom{3}{1} S_3$

ii) $t = 4$

$$E_1 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3 - \binom{4}{3} S_4$$

$$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$$

$$\cancel{E_3 = S_3 - \binom{4}{1} S_4}$$

from this we can generalize

m conditions ; $1 \leq m \leq t$

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - - - - (-1)^{t-m} \binom{t}{t-m} S_t$$

De arrangements

A permutation of n distinct objects in which none of the objects is in its natural place is called a derangement

$$d_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \frac{(-1)^n}{n!}\right\}$$

Find the number of derangements of 1, 2, 3, 4

$$d_4 = 4! \times \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\}$$

$$= 24 \times \left\{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right\}$$

$$= 12 - 4 + 1 = 9$$

2143	2341	2413
3142	3412	3421
4123	4312	4321

$d_5 = 5!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}$

$= (120)\left\{\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right\} = 44$

$d_6 = (6!)\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right\}$

$= 265$

$d_7 = 1845$ $\qquad d_8 = 148.33$

Rook Polynomials

$1 + nx + r_2 x^2 + \dots + r_n x^n$

$r(C, x) = 1 + r_1 x + r_2 x^2 + \dots + r_n x^n$

Consider the board shown

1	2
3	4

$r_1 = 4 \qquad (1,4)\ (2,3) \qquad r_2 = 2$

$r(C,x) = 1 + 4x + 2x^2$

(1 5 7) 4, 5 (6)

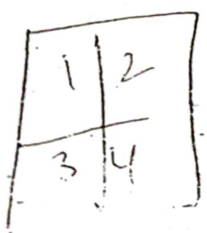
2 3 4.

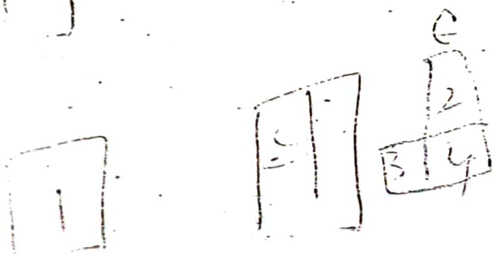
$r(C,x) = 1+8x+14x^2+4x^3$

Expansion Formula

In a given board C, suppose we choose a particular square and mark it is *. let D be the board obtained from C by deleting the row and the colomn containing the square (*) and Let E be the board obtained from C by deleting only the square *

$$r(C,x) = x\,r(D,x) + r(E,x)$$

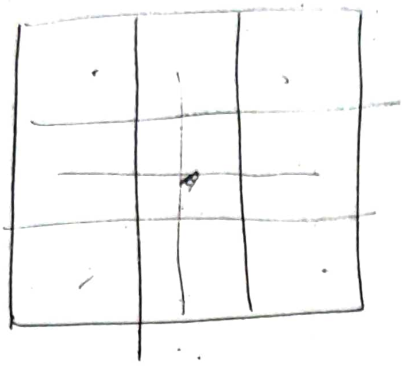


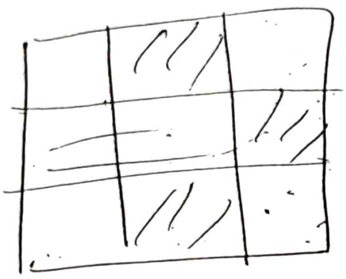


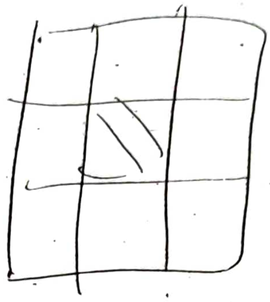
D

$r(D,x)$ 1 + 2

$1+3x+x^2$







D E

$r_1 = 4 \quad r_2 = 2 \quad r_3 = r_4 = 0$

$r(D, x) = 1 + 4x + 2x^2$

$r(E, x) = 1 + 8x + 14x^2 + 4x^3$

$x\, r\, D(x) + r(E, x)$

$x(1 + 4x + 2x^2 + (1 + 8x + 14x^2 + 4x^3$

$= 1 + 9x + 18x^2 + 6x^3$