MODULE - 5 Laplace Transform

EXTRA Questions

Que Find the Laplace Transform of.

(i) $e^{-3t}\sin 5t \sin 3t$

Sol $L\{e^{-3t}\sin 5t \sin 3t\}$

$= L\{e^{-3t}\frac{1}{2}\, 2\sin 5t \sin 3t\}$

$= \frac{1}{2} L\{e^{-3t}(\cos 2t - \cos 8t)\}$

$= \frac{1}{2}\left[L\{e^{-3t}\cos 2t\} - L\{e^{-3t}\cos 8t\}\right]$

$= \frac{1}{2}\left[\frac{s+3}{(s+3)^2+2^2} - \frac{s+3}{(s+3)^2+8^2}\right]$

(ii) $f(t) = \begin{cases} \sin t & 0<t<\pi \\ 0 & t>\pi \end{cases}$

Sol $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$= \int_0^\pi e^{-st}\sin t\, dt + \int_\pi^\infty e^{-st}\cdot 0\, dt$

$$= \frac{e^{-st}}{(-s)^2+1^2}\left[-s\sin t - \cos t\right]_0^{\pi}$$

$$= \frac{e^{-\pi s}}{s^2+1}\left[-s\sin\pi - \cos\pi\right] - \frac{e^{0}}{s^2+1}\left[-s\sin 0 - \cos 0\right]$$

$$= \frac{e^{-\pi s}}{s^2+1} + \frac{e^{0}}{s^2+1} = \frac{1}{s^2+1}\left(e^{-\pi s}+1\right)$$

Que Find the L.T of square wave function of period $2a$ defined as $f(t) = \begin{cases} k, & 0<t<a \\ -k, & a<t\leq 2a \end{cases}$

Sol For any periodic function $f(t)$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a k e^{-st}\,dt + \int_a^{2a} (-k) e^{-st}\,dt\right]$$

$$= \frac{k}{1-e^{-2as}}\left[\left.\frac{e^{-st}}{-s}\right|_0^a - \left.\frac{e^{-st}}{-s}\right|_a^{2a}\right]$$

$$= \frac{k}{(1-e^{-2as})s}\left[-\left(e^{-st}\right)_0^a + \left(e^{-st}\right)_a^{2a}\right]$$

$$= \frac{k}{(1-e^{-2as})s}\left[-e^{-as}+e^{0}+e^{-2as}-e^{-as}\right]$$

$$= \frac{k\,(1-e^{-as})^2}{(1-e^{-2as})s}$$

$$= \frac{k\,(1-e^{-as})^{\cancel{2}}}{\cancel{(1-e^{-as})}(1+e^{-as})s} = \frac{k(1-e^{-as})}{(1+e^{-as})s}$$

$$= k\left(\frac{e^{+as/2}-e^{-as/2}}{s\,(e^{+as/2}+e^{-as/2})}\right) \quad \left[\text{Dividing num \& den. by } e^{as/4}\right]$$

$$L\{f(t)\} = \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

Que Find $L\left\{\int_0^t e^{-t}\cos t\,dt\right\}$

Sol We know $L\{\cos t\} = \frac{s}{s^2+1}$

$$L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}$$ (By Ist Shifting property)

$$L\left\{\int_0^t e^{-t}\cos t\,dt\right\} = \frac{1}{s}\frac{(s+1)}{\left((s+1)^2+1\right)}$$

Que Find $L\left\{\frac{1-\cos 3t}{t}\right\}$

Sol $L\{1-\cos 3t\} = L\{1\} - L\{\cos 3t\}$

$$= \frac{1}{s} - \frac{s}{s^2+9}$$

$$L\left\{\frac{1-\cos 3t}{t}\right\} = \int_s^\infty \frac{1}{s} - \frac{s}{s^2+9}\,ds$$

$$= \log s - \frac{1}{2}\log(s^2+9)\Big|_s^\infty$$

$$= \frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+9)$$

$$= \frac{1}{2}\log\frac{s^2}{s^2+9}\Big|_s^\infty$$

$$= \frac{1}{2} \log \frac{s^2+9}{s^2}.$$

Que Evaluate $\int_0^\infty t e^{-2t} \sin 3t \, dt$

Sol By def^n $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$ — (1)

Given $\int_0^\infty t e^{-2t} \sin 3t \, dt$ — (2)

Comp (1) & (2).

$f(t) = t \sin 3t$, $\boxed{s = 2}$

Now $L\{f(t)\} = L\{t \sin 3t\}$

$L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{t \sin 3t\} = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$= (-1)\, 3 \cdot (-1) \frac{2s}{(s^2+9)^2}$

$L\{t \sin 3t\} = \frac{6s}{(s^2+9)^2}$

Now $\int_0^\infty e^{-2t}\, t\sin 3t \, dt = L\{t\sin 3t\}\Big|_{s=2}$

$$= \frac{6s}{(s^2+9)^2}\Bigg|_{s=2} = \frac{12}{(13)^2}$$

$$= \frac{12}{169}$$

<u>Que</u> Find the inverse laplace transform

$$\frac{3s+2}{s^2-s-2}$$

<u>Sol</u> $L^{-1}\left\{\frac{3s+2}{s^2-s-2}\right\} = L^{-1}\left\{\frac{3s+2}{(s-2)(s+1)}\right\}$

$$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$3s+2 = A(s+1) + B(s-2)$$

$s=-1$ | $s=2$

$B = 1/3$ | $A = 8/3$

$$\therefore L^{-1}\left\{\frac{3s+2}{(s-2)(s+1)}\right\} = L^{-1}\left\{\frac{1/3}{s+1} + \frac{8/3}{s-2}\right\}$$

$$= \frac{1}{3}e^{-t} + \frac{8}{3}e^{2t}$$

Que Find the Inverse L.T of $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ (4)

Sol

$$L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{(s+1)^2+1+1}{((s^2+2s+1)+1)(s^2+2s+1+4)}\right\}$$

$$= L^{-1}\left\{\frac{\cancel{(s+1)^2+1}}{\cancel{((s+1)^2+1)}((s+1)^2+2^2)}\right\} + L^{-1}\left\{\frac{1}{((s+1)^2+1)((s+1)^2+2^2)}\right\}$$

$$= e^{-t}\frac{1}{2}\sin 2t + e^{-t}\left[L^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}\right] \quad \text{—(1)}$$

[$\because$ By Ist shifting property]

Now $$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \int_0^t \sin u \, \frac{1}{2}\sin 2(t-u)\,du.$$

[By convolution thm $L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du$]

$$= \frac{1}{4}\int_0^t 2\sin(2t-2u)\sin u\,du$$

$$= \frac{1}{4}\int_0^t \cos(2t-3u) - \cos(2t-u)\,du$$

$$= \frac{1}{4}\left[\frac{\sin(2t-3u)}{-3} - \frac{\sin(2t-u)}{-1}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{\sin(2t-3u)}{-3} + \frac{\sin(2t-u)}{}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{4}{3}\sin t - \frac{2}{3}\sin 2t\right]$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t.$$

Now from (1)

$$L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = e^{-t}\frac{1}{2}\sin 2t + e^{-t}\left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right)$$

$$= \frac{e^{-t}}{3}\left[\sin t + \sin 2t\right] \text{ Ans.}$$

Que Find $L^{-1}\left\{\log\left(\frac{s^2+1}{(s-1)^2}\right)\right\}$

Sol Let $L^{-1}\left\{\log(s^2+1) - 2\log(s-1)\right\} = f(t)$
$= L^{-1}\{\bar{f}(s)\}$

Now $L^{-1}\left\{\frac{d}{ds}\log(s^2+1) - 2\frac{d}{ds}\log(s-1)\right\} = (-1)^1 t^1 f(t)$

where

$$L^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s-1}\right\} = -t\, f(t)$$

$$\Rightarrow 2L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2L^{-1}\left\{\frac{1}{s-1}\right\} = -t\,f(t)$$

$$\Rightarrow \quad 2\cos t - 2e^{t} = -t\,f(t)$$

$$f(t) = \frac{2(e^{t} - \cos t)}{t}$$

<u>Que</u> Solve $\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^{t}$,

where $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2$.

Sol $\quad (y''' - 3y'' + 3y' - y) = t^2e^{2t}$

Taking L.T. on b.s.

$$L\{y'''(t)\} - 3L\{y''(t)\} + 3L\{y'(t)\} - L\{y(t)\} = L\{t^2e^{t}\}$$

$$\Rightarrow s^3L\{y(t)\} - s^2y(0) - sy'(0) - y''(0) - 3[s^2L\{y(t)\} - sy(0) - y'(0)]$$
$$+3\{sL\{y(t)\} - y(0)\} - L\{y(t)\} = \frac{2}{(s-1)^3}.$$

$$\Rightarrow (s^3 - 3s^2 + 3s - 1)L\{y(t)\} - (s^2 - 3s + 1) = \frac{2}{(s-1)^3}$$

$$\Rightarrow (s-1)^3 L\{y(t)\} = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

$$\Rightarrow L\{y(t)\} = \frac{s^2-3s+1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$L\{y(t)\} = \frac{(s-1)^2-(s-1)-1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$L\{y(t)\} = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$y(t) = L^{-1}\left\{\frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}\right\}$$

$$= e^t\left\{1 - t - \frac{t^2}{2} + \frac{t^5}{30}\right\}$$

__Que__ Solve. $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$

given that $x(0) = 1$, $y(0) = 0$

__Sol__ $Dx - y = e^t$, $Dy + x = \sin t$

Taking L.T on b.s

$L\{Dx\} - L\{y\} = L\{e^t\}$, $L\{Dy\} + L\{x\} = L\{\sin t\}$

$s L\{x(t)\} - x(0) - L\{y\} = \frac{1}{s-1}$, $s L\{y(t)\} - y(0) + L\{x(t)\} = \frac{1}{s^2+1}$

$\Rightarrow$ $s L\{x(t)\} - L\{y\} = \frac{1}{s-1} + 1$ — (1)

$+L\{x(t)\} + s L\{y(t)\} = \frac{1}{s^2+1}$ — (2)

Multiplying eqⁿ ② by s and adding (ii) to it.

$$(s^2+1)\,L\{x(t)\} = \frac{s}{s-1} + s + \frac{1}{s^2+1}.$$

$$(s^2+1)\,L\{x(t)\} = \frac{s^2}{s-1} + \frac{1}{s^2+1}$$

$$L\{x(t)\} = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}$$

$$= \frac{s^2+1-1}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + \frac{1}{(s^2+1)^2}$$

$$x(t) = \frac{1}{2} L^{-1}\left\{\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right\} + L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}\{e^t + \cos t + \sin t\} + \frac{1}{2}[\sin t - t\cos t]$$

$$\boxed{x(t) = \frac{1}{2}e^t + \frac{1}{2}(1-t)\cos t + \sin t}$$

Que Express in term of unit step function and hence evaluate L.T. of $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi < t < 2\pi \\ \cos 3t & t > 2\pi \end{cases}$

Sol

$$f(t) = \cos t + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)$$

$$= \cos t + \left[\cos(2t - 2\pi) + \cos(t-\pi)\right] u(t-\pi) + (\cos 3(t-2\pi) - \cos 2(t-2\pi))\, u(t-2\pi)$$

$$L\{f(t)\} = L\{\cos t\} + L\{\cos 2(t-\pi)\, u(t-\pi)\} + L\{\cos(t-\pi)\, u(t-\pi)\} + L\{\cos 3(t-2\pi)\, u(t-2\pi)\} - L\{\cos 2(t-2\pi)\, u(t-2\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\frac{s}{s^2+4} + e^{-\pi s}\frac{s}{s^2+1} + e^{-2\pi s}\left\{\frac{s}{s^2+9} - \frac{s}{s^2+4}\right\}$$

Que find $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

sol. $L^{-1}\left\{e^{-\pi s}\frac{1}{s^2+1}\right\} = L^{-1}\left\{e^{-as} F(s)\right\}$

$$= f(t-a)\, u(t-a)$$

$= \cos(t-\pi)\, u(t-\pi)$

Que Find the L.T. of (a) $\frac{1}{t}\,\delta(t-a)$

(b) $\sin at\; \delta(t-b)$

Sol $L\left\{\frac{\delta(t-a)}{t}\right\}$

Now $L\{\delta(t-a)\} = e^{-as}$

$$L\left\{\frac{\delta(t-a)}{t}\right\} = \int_s^\infty e^{-as}\, ds$$

$$= \left.\frac{e^{-as}}{-a}\right|_s^\infty$$

$$= \frac{e^{-\infty}}{-a} + \frac{e^{-as}}{a}$$

$$= \frac{e^{-as}}{a} \quad \text{Ans}$$

(b) By def^n $L\{\sin at\; \delta(t-b)\} = \int_0^\infty e^{-st} \sin at\; \delta(t-b)$

Comparing with $\int_0^\infty f(t)\, \delta(t-a)\, dt = f(a)$

$\therefore \int_0^\infty e^{-st} \sin at\; \delta(t-b) = \boxed{e^{-bs} \sin ab.}$ Ans