Extra Problems: Applications of Double and Triple Integrals, Beta-gamma Functions, Orthogonal Curvilinear co-ordinates*

Srikanth K S
Assistant Professor,
CMRIT

date: 16 January 2015

*problems borrowed from multiple sources. Thanks to original problem creators.

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example. Evaluate the integral $\int_0^1 \int_{y^2}^1 ye^{x^2}\, dx\, dy$. **Hint:** First reverse the order of integration.

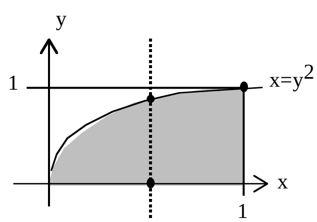
Solution

If we try to evaluate the integral as written above, then the first step is to compute the indefinite integral

$$\int e^{x^2}\, dx.$$

But e^{x^2} does not have an indefinite integral that can be written in terms of elementary functions. Then, we will fist reverse the order of integration. The region of integration is the Type II region

$$\begin{aligned} R: \quad & 0 \le y \le 1 \\ & y^2 \le x \le 1. \end{aligned}$$



Then, R can also be described as the Type I region

$$\begin{aligned} R: \quad & 0 \le x \le 1 \\ & 0 \le y \le \sqrt{x}. \end{aligned}$$

This gives

$$\begin{aligned} \int_0^1 \int_{y^2}^1 ye^{x^2}\, dx\, dy &= \int_0^1 \int_0^{\sqrt{x}} ye^{x^2}\, dy\, dx \\ &= \int_0^1 \left(\frac{y^2}{2} e^{x^2} \right) \Big|_0^{\sqrt{x}}\, dx \\ &= \frac{1}{2} \int_0^1 xe^{x^2}\, dx \\ &= \frac{1}{2} \left. \frac{e^{x^2}}{2} \right) \Big|_0^1 = \frac{1}{4}(e-1). \end{aligned}$$

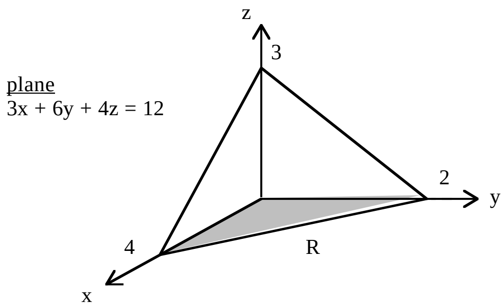
example. Find the volume of the tetrahedron bounded by the coordinate axes and the plane $3x + 6y + 4z = 12$.

Solution

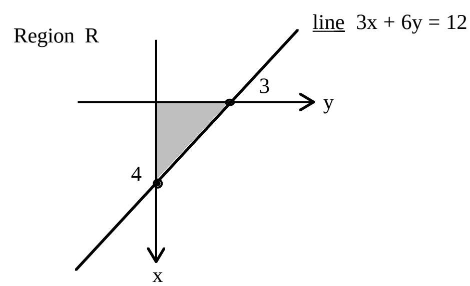
We have to find the volume of the tetrahedron S bounded by the plane

$$3x + 6y + 4z = 12 \iff z = \frac{12 - 3x - 6y}{4}$$

and the coordinate axes. This is the portion of the plane in the first octant, as one can see from Picture (1).



(1)



(2)

Then, we have

$$\text{volume}(S) = \int\int_R \frac{12 - 3x - 6y}{4} \, dA,$$

where R is the projection of the tetrahedron in the xy-plane. Then, R is the Type I region (see Picture (2))

$$R: \qquad \begin{aligned} &0 \le x \le 4 \\ &0 \le y \le \frac{12 - 3x}{6}. \end{aligned}$$

Finally, this gives

$$\begin{aligned}
\text{volume}(S) &= \frac{1}{4}\int_0^4 \int_0^{12-3x/6} (12 - 3x - 6y) \, dy \, dx \\
&= \frac{1}{4}\int_0^4 \left((12 - 3x)y - 3y^2\right) \Big|_0^{2-\frac{x}{2}} \, dx \\
&= \frac{1}{4}\int_0^4 (12 - 3x)\left(2 - \frac{x}{2}\right) - 3\left(2 - \frac{x}{2}\right)^2 \, dx \\
&= \frac{1}{4}\int_0^4 \frac{3}{4}x^2 - 6x + 12 \, dx \\
&= \frac{1}{4}\left(\frac{x^3}{4} - 3x^2 + 12x\right)\Big|_0^4 = 4.
\end{aligned}$$

example. Evaluate the integral

$$\int_0^1 \int_{\sqrt{3}\,y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\; dx\; dy.$$

Hint: Use polar coordinates.

Solution

The region R of integration is the Type II region

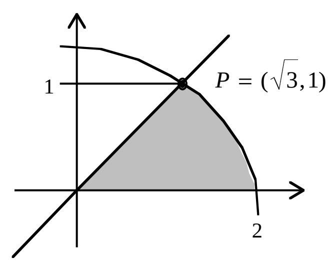
$$R: \qquad \begin{array}{l} 0 \le y \le 1 \\ \sqrt{3}\,y \le x \le \sqrt{4-y^2} \end{array}$$

We have

$$x = \sqrt{4-y^2} \quad \Rightarrow \quad x^2 = 4 - y^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4.$$

Then, x varies between the straight line $x = \sqrt{3}\,y$ and the circle $x^2 + y^2 = 4$.

The region R is



In polar coordinates, the region R is

$$R: \qquad \begin{array}{l} 0 \le r \le 2 \\ 0 \le \theta \le \alpha, \end{array}$$

where α is the angle made by the straight line $x = \sqrt{3}\,y$. The straight line and the cicle meet at the points

$$\left(\sqrt{3}y\right)^2 + y^2 = 4 \iff 4y^2 = 4 \iff y^2 = 1 \iff y = \pm 1.$$

The intersection point in the first quadrant is then $P = (\sqrt{3}, 1) = (2\cos\alpha, 2\sin\alpha)$. Then,

$$\alpha = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

Finally, the integrand $\sqrt{x^2+y^2}$ is r in polar coordinates. This gives

$$\begin{aligned} \int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\; dx\; dy &= \int_0^{\pi/6} \int_0^2 r\; r\; dr\; d\theta \\ &= \int_0^{\pi/6} \frac{r^3}{3} \Big|_0^2 \, d\theta \\ &= \frac{\pi}{6}\,\frac{8}{3} = \frac{4\pi}{9}. \end{aligned}$$

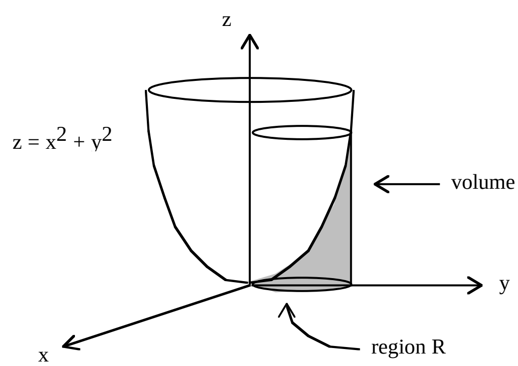
example. Find the volume below the surface $z = x^2 + y^2$, above the plane $z = 0$, and inside the cylinder $x^2 + y^2 = 2y$.

Solution

Completing the squares, we rewrite the equation of the cylinder as

$$x^2+y^2=2y \iff x^2+(y^2-2y)=0 \iff x^2+(y-1)^2=1.$$

The base of the cylinder is then the circle of radius 1 centered at $(0,1)$. Then, we have to find the volume of the 3D region:



From the picture above, we write

$$V=\int\int_R x^2+y^2\ dA,$$

where R is the projection of the 3D region in the plane, i.e. the circle $x^2+y^2=2y$. Using polar coordinates, this gives

$$V=\int\int_R r^2 r\ dr\ d\theta.$$

In polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$, the circle writes as

$$x^2+y^2=2y \iff r^2=2r\sin\theta \iff r=2\sin\theta,$$

and R is the region

$$\begin{aligned} R: \quad & 0\le\theta\le\pi \\ & 0\le r\le 2\sin\theta. \end{aligned}$$

Then,

$$\begin{aligned} V &= \int_0^\pi\int_0^{2\sin\theta} r^3\ dr\ d\theta=\int_0^\pi \frac{r^4}{4}\Big|_0^{2\sin\theta}\ d\theta \\ &= 4\int_0^\pi \sin^4\theta\ d\theta = 4\left(\frac{3\theta}{8}-\frac{1}{4}\sin 2\theta+\frac{1}{32}\sin 4\theta\Big|_0^\pi\right) \\ &= 4\left(\frac{3\pi}{8}\right)=\frac{3\pi}{2}. \end{aligned}$$

example. Evaluate

$$\int_{x=0}^1\int_{y=0}^1\int_{z=\sqrt{x^2+y^2}}^2 xyz\ dz\ dy\ dx.$$

Solution

$$
\begin{aligned}
\int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^2 xyz\,dz\,dy\,dx &= \int_0^1\int_0^1 \frac{xyz^2}{2}\Big|_{z=\sqrt{x^2+y^2}}^2\,dy\,dx\\
&= \int_0^1\int_0^1 2xy-\frac{xy(x^2+y^2)}{2}\,dy\,dx\\
&= \int_0^1\int_0^1 2xy-\frac{x^3y}{2}-\frac{y^3x}{2}\,dy\,dx\\
&= \int_0^1\left(xy^2-\frac{x^3y^2}{4}-\frac{y^4x}{8}\right)\Big|_0^1\,dx\\
&= \int_0^1 x-\frac{x^3}{4}-\frac{x}{8}\,dx\\
&= \int_0^1 \frac{7x}{8}-\frac{x^3}{4}\,dx\\
&= \left(\frac{7x^2}{16}-\frac{x^4}{16}\right)\Big|_0^1\\
&= \frac{7}{16}-\frac{1}{16}=\frac{3}{8}.
\end{aligned}
$$

example. Find the mass of the 3D region B given by

$$x^2+y^2+z^2\le 4,\ x\ge 0,\ y\ge 0,\ z\ge 0,$$

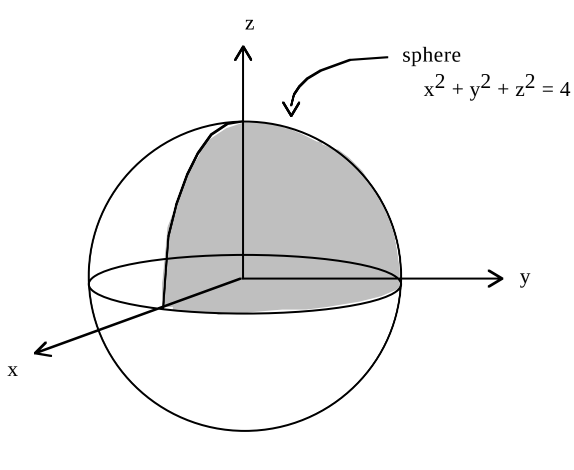
if the density is equal to xyz.

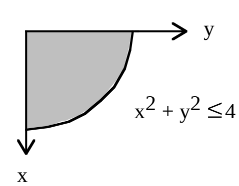
Solution

We have

$$\text{mass}(B)=\int\int\int_B xyz\,dV,$$

and the region B is the portion of the sphere of radius 2 in the first octant.





3D region B 2D region R

Then, B can be described as

$$0 \le z \le \sqrt{4 - x^2 - y^2}, \quad \text{for all } (x, y) \in R,$$

where R is the projection of B in the xy-plane. Describing R as a Type I region, this gives

$$B: \quad \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4 - x^2} \\ 0 \le z \le \sqrt{4 - x^2 - y^2}. \end{array}$$

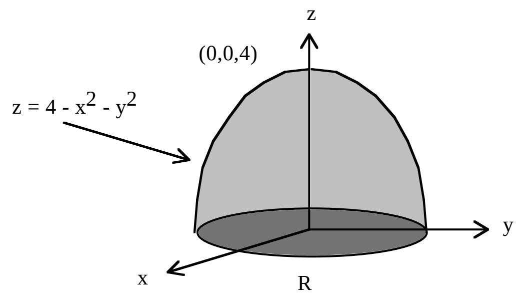
Then,

$$\begin{aligned}
\text{mass}(B) &= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz \; dz \; dy \; dx \\
&= \int_0^2 \int_0^{\sqrt{4-x^2}} \left(xy\frac{z^2}{2} \right) \Big|_0^{\sqrt{4-x^2-y^2}} dy \; dx \\
&= \frac{1}{2} \int_0^2 \int_0^{\sqrt{4-x^2}} xy(4 - x^2 - y^2) \; dy \; dx \\
&= \frac{1}{2} \int_0^2 \int_0^{\sqrt{4-x^2}} 4xy - x^3y - y^3x \; dy \; dx \\
&= \frac{1}{2} \int_0^2 \left(2xy^2 - \frac{x^3y^2}{2} - \frac{y^4x}{4} \right) \Big|_0^{\sqrt{4-x^2}} dx \\
&= \frac{1}{2} \int_0^2 2x(4 - x^2) - \frac{x^3(4 - x^2)}{2} - \frac{(4 - x^2)^2 x}{4} \; dx \\
&= \frac{1}{2} \int_0^2 \frac{x^5}{4} - 2x^3 + 4x \; dx \\
&= \frac{1}{2} \left(\frac{x^6}{24} - \frac{x^4}{2} + \frac{4x^2}{2} \right) \Big|_0^2 \\
&= \frac{1}{2} \left(\frac{64}{24} - 8 + 8 \right) = \frac{4}{3}.
\end{aligned}$$

example. Find the volume of the region B bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

Solution

We have volume$(B) = \int\int\int_B 1\ dV$.



Then, B can be described as

$$0 \le z \le 4 - x^2 - y^2, \quad \text{for all } (x, y) \in R,$$

where R is the projection of B in the xy-plane. Then, R is the interior of the circle $x^2 + y^2 = 4$. In polar coordinates, the region R is

$$R: \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 2, \end{array}$$

and in cylindrical coordinates, the region B is

$$B: \qquad \begin{array}{l} 0 \le z \le (4 - r^2) \\ 0 \le \theta \le 2\pi \\ 0 \le r \le 2. \end{array}$$

Then,

$$\begin{aligned} \text{volume}(B) &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\ dz\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^2 (4 - r^2) r\ dr\ d\theta \\ &= 2\pi \int_0^2 4r - r^3\ dr \\ &= 2\pi \left(2r^2 - \frac{r^4}{4} \Big|_0^2 \right) = 2\pi(4) = 8\pi. \end{aligned}$$

example. Find the center of gravity of the region in **12.**, assuming constant density σ.

Solution

By symmetry, $\overline{x} = \overline{y} = 0$. Also, as the density is $d(x, y, z) = \sigma$,

$$\begin{aligned} \overline{z} &= \frac{\int\int\int_B d(x,y,z) z\ dV}{\int\int\int_B d(x,y,z)\ dV} = \frac{\sigma \int\int\int_B z\ dV}{\sigma \int\int\int_B 1\ dV} \\ &= \frac{\int\int\int_B z\ dV}{\text{volume}(B)} = \frac{1}{8\pi}\int\int\int_B z\ dV. \end{aligned}$$

Using the description of the region B in cylindrical coordinates of **12.**, we get

$$\begin{aligned}
\int\int\int_B z\,dV &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} z\ r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 \frac{(4-r^2)^2}{2} r\ dr\ d\theta \\
&= \frac{2\pi}{2}\int_0^2 16r - 8r^3 + r^5\ dr \\
&= \pi\left(8r^2 - 2r^4 + \frac{r^6}{6}\Big|_0^2\right) = \frac{32}{3}\pi.
\end{aligned}$$

Then,

$$\overline{z} = \frac{1}{8\pi}\left(\frac{32}{3}\pi\right) = \frac{4}{3}.$$

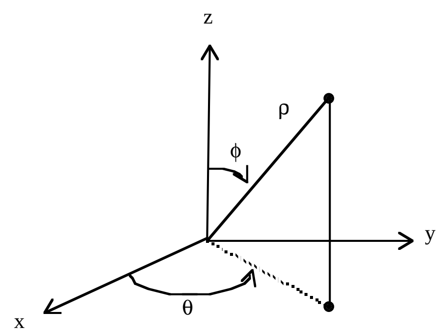
example . Evaluate

$$\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV,$$

where B is the region bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2+y^2}$.

<u>Solution</u>

We will use the spherical coordinates



to describe the region B. In those coordinates,

$$\begin{aligned}
x &= \rho\ \sin\phi\ \cos\theta \\
y &= \rho\ \sin\phi\ \sin\theta \\
z &= \rho\ \cos\phi.
\end{aligned}$$

Then, the cone $z = \sqrt{x^2+y^2}$ writes as

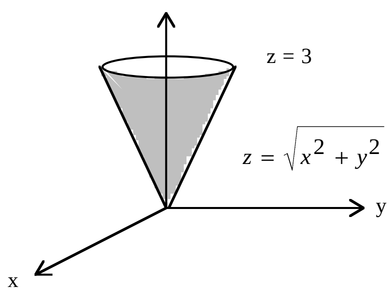
$$\rho\cos\phi = \rho\sin\phi \iff \frac{\sin\phi}{\cos\phi} = \tan\phi = 1 \iff \phi = \frac{\pi}{4},$$

the plane $z = 3$ as

$$3 = \rho\cos\phi \iff \rho = \frac{3}{\cos\phi},$$

and the region B can be described as

$$B: \quad \begin{array}{l} 0 \le \phi \le \dfrac{\pi}{4} \\ 0 \le \phi \le 2\pi \\ 0 \le \rho \le \dfrac{3}{\cos\phi}. \end{array}$$



Finally, in spherical coordinates, $\sqrt{x^2+y^2+z^2} = \rho$.
Then,

$$\begin{aligned} \int\int\int_B \sqrt{x^2+y^2+z^2}\, dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3/\cos\phi} \rho\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= 2\pi \int_0^{\pi/4} \sin\phi \int_0^{3/\cos\phi} \rho^3\, d\rho\, d\phi \\ &= 2\pi \int_0^{\pi/4} \sin\phi \left(\frac{\rho^4}{4} \Big|_0^{3/\cos\phi} \right) d\phi \\ &= 2\pi \frac{3^4}{4} \int_0^{\pi/4} \frac{\sin\phi}{\cos^4\phi}\, d\phi \\ &= \frac{81\pi}{2} \left(\frac{(\cos\phi)^{-3}}{3} \Big|_0^{\pi/4} \right) \\ &= \frac{81\pi}{6} \left(\left(\frac{\sqrt{2}}{2} \right)^{-3} - 1 \right) = \frac{27\pi}{2} \left(2\sqrt{2} - 1 \right). \end{aligned}$$

Extra Problems: Gamma & Beta Functions

Srikanth K S
Asst Professor,
CMRIT

Date: 16th January 2015

I. Gamma Function

Definition

$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \;;\; n > 0$

& $\Gamma(n) = \Gamma(n+1) / n \;;\; n \in R - Z^{\leq 0}$

Results:

(1) $\Gamma(n+1) = n\,\Gamma(n) \;;\; n > 0$, where $\Gamma(1) = 1$

(2) $\Gamma(n+1) = n! \;;\; n \in \mathbf{N}$ (convention: $0! = 1$)

(3) $\Gamma(n)\,\Gamma(1-n) = \pi/\sin(n\pi) \;;\; 0 < n < 1$

In Particular;

$\Gamma(1/2) = \sqrt{\pi}$

Examples:

Example(1)

Evaluate $\int_0^\infty x^4 e^{-x}\,dx$

Solution

$\int_0^\infty x^4 e^{-x}\,dx = \int_0^\infty x^{5-1} e^{-x}\,dx = \Gamma(5)$

$\Gamma(5) = \Gamma(4+1) = 4! = 4(3)(2)(1) = 24$

Exercise

Evaluate $\int_0^\infty x^5 e^{-x}\,dx$

Example(2)

Evaluate $\int_0^\infty x^{1/2} e^{-x}\,dx$

$\int_0^\infty x^{1/2} e^{-x}\,dx = \int_0^\infty x^{3/2-1} e^{-x}\,dx = \Gamma(3/2)$

$3/2 = \frac{1}{2} + 1$

$\Gamma(3/2) = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\,\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

Exercise

Evaluate $_0\int^{\infty} x^{3/2} e^{-x} dx$

Example(3)

Evaluate $_0\int^{\infty} x^{3/2} e^{-x} dx$

$_0\int^{\infty} x^{3/2} e^{-x} dx = {_0\int^{\infty}} x^{5/2-1} e^{-x} dx = \Gamma(5/2)$

$5/2 = 3/2 + 1$

$\Gamma(5/2) = \Gamma(3/2+1) = 3/2\ \Gamma(3/2) = 3/2 . ½\ \Gamma(½) = 3/2 . ½ . \sqrt{\pi} = ¾ \sqrt{\pi}$

Exercise

Evaluate $_0\int^{\infty} x^{5/2} e^{-x} dx$

Example(4)

Find $\Gamma(-½)$

$(-½) + 1 = ½$

$\Gamma(-1/2) = \Gamma(-½ + 1) / (-½) = -2\ \Gamma(1/2) = -2\sqrt{\pi}$

Example(5)

Find $\Gamma(-3/2)$

$(-3/2) + 1 = -\frac{1}{2}$

$\Gamma(-3/2) = \Gamma(-3/2 + 1) / (-3/2) = \Gamma(-1/2) / (-2/3) = (-2\sqrt{\pi}) / (-2/3) = 4\sqrt{\pi} /3$

Exercise

Evaluate $\Gamma(-5/2)$

II. Beta Function

Definition

$B(m,n) = \ {}_0\!\int^1 \ x^{m-1} \ (1 - x)^{n-1} \, dx \ ; \quad m > 0 \ \& \ n > 0$

Results:

(1) $B(m,n) = \Gamma(m)\ \Gamma(n) \ / \ \Gamma(m+n)$

(2) $B(m,n) = B(n,m)$

(3) ${}_0\!\int^{\pi/2} \sin^{2m-1}x \ . \ \cos^{2n-1}x \ dx = \Gamma(m)\ \Gamma(n) \ / \ 2\,\Gamma(m+n) \quad ; \ m>0 \ \& \ n>0$

(4) ${}_0\!\int^{\infty} x^{q-1} \ / \ (1+x) \ . dx = \Gamma q)\ \Gamma(1-q) = \Pi / \sin(q\pi) \quad ; \ 0<q<1$

Examples:

Example(1)

Evaluate $_0\int^1 x^4 \; (1 - x)^3 \, dx$

Solution

$_0\int^1 \; x^4 \; (1 - x)^3 \, dx \; = \; x^{5-1} \; (1 - x)^{4-1} \, dx$

$= B(5,4) = \Gamma(5)\ \Gamma(4) \ / \ \Gamma(9) \ = 4! \,.\, 3! / 8! \ = 3!/(8.7.6.5) = 1/ (8.7.5) = 1/280$

Exercise

Evaluate $_0\int^1 x^2 \; (1 - x)^6 \, dx$

Example(2)

Evaluate $I \; = \; _0\int^1 \; \left[\; 1 \; / \; {}^3\sqrt{[x^2 (1 - x)]} \; \right] dx$

Solution

$I \; = \; _0\int^1 \; x^{-2/3} \; (1 - x)^{-1/3} \; dx \; = \; _0\int^1 \; x^{1/3 - 1} \; (1 - x)^{2/3 - 1} \, dx$

$= B(1/3,2/3) = \; \Gamma(1/3)\ \Gamma(2/3) \ / \ \Gamma(1)$

$\Gamma(1/3)\ \Gamma(2/3) = \Gamma(1/3)\ \Gamma(1-1/3) = \pi/\sin(\pi/3) = \pi/(\sqrt{3}/2) = 2\pi/\sqrt{3}$

Exercise

Evaluate $I = {}_0\int^1 \left[\, 1 / \sqrt[4]{[x^3(1-x)]}\, \right] dx$

Example(3)

Evaluate $I = {}_0\int^1 \sqrt{x}\ .\ (1-x)\ dx$

Solution

$I = {}_0\int^1 x^{1/2}\ (1-x)\ dx = {}_0\int^1 x^{3/2-1}\ (1-x)^{2-1}\ dx$

$= B(3/2, 2) = \Gamma(3/2)\ \Gamma(2) / \Gamma(7/2)$

$\boldsymbol{\Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}}$

$\Gamma(5/2) = \Gamma(3/2+1) = (3/2)\ \Gamma(3/2) = (3/2)\ .\ \tfrac{1}{2}\sqrt{\pi} = 3\sqrt{\pi}/4$

$\boldsymbol{\Gamma(7/2) = \Gamma(5/2+1) = (5/2)\ \Gamma(5/2) = (5/2)\ .\ (3\sqrt{\pi}/4) = 15\sqrt{\pi}/8}$

Thus,

$I = (\tfrac{1}{2}\sqrt{\pi})\ .\ 1!\ /\ (\mathbf{15\sqrt{\pi}/8}) = 4/15$

Exercise

Evaluate $I = {}_0\int^1 \sqrt{x^5}\ .\ (1-x)\ dx$

II. Using Gamma Function to Evaluate Integrals

Example(1)

Evaluate: $I = {}_0\int^{\infty} x^{6} \; e^{-2x} \, dx$

Solution:

Letting $y = 2x$, we get

$I = (1/128) \, {}_0\int^{\infty} y^{6} \; e^{-y} \, dy = (1/128) \;\; \Gamma(7) = (1/128) \; 6! \; = 45/8$

Example(2)

Evaluate: $I = {}_0\int^{\infty} \sqrt{x} \;\; e^{-x\text{^}3} \, dx$

Solution:

Letting $y = x^3$, we get

$I = (1/3) \, {}_0\int^{\infty} y^{-1/2} \;\; e^{-y} \, dy = (1/3) \;\; \Gamma(1/2) = \sqrt{\pi} \, / \, 3$

Example(3)

Evaluate: $I = \int_0^{\infty} x^m \; e^{-k x^n} dx$

Solution:

Letting $y = k x^n$, we get

$$I = [1 / (n . k^{(m+1)/n})] \int_0^{\infty} y^{[(m+1)/n - 1]} \; e^{-y} dy = [1 / (n . k^{(m+1)/n})] \; \Gamma[(m+1)/n]$$

II. Using Beta Function to Evaluate Integrals

Formulas

(1) $\int_0^1 x^{m-1} (1 - x)^{n-1} dx = B(m,n) = \Gamma(m) \Gamma(n) / 2 \Gamma(m+n)$; $m > 0$ & $n > 0$

(3) $\int_0^{\pi/2} \sin^{2m-1}x \,.\, \cos^{2n-1}x \, dx = (1/2) B(m,n)$; $m>0$ & $n>0$

(4) $\int_0^{\infty} x^{q-1} / (1+x) \,.\, dx = \Gamma(q) \Gamma(1-q) = \Pi / \sin(q\pi)$; $0 < q < 1$

Using Formula (1)

Example(1)

Evaluate: $I = \int_0^2 x^2 / \sqrt{(2 - x)} \,.\, dx$

Solution:

Letting $x = 2y$, we get

$I = (8/\sqrt{2}) \int_0^1 y^2 (1 - y)^{-1/2} dy = (8/\sqrt{2}) \,.\, B(3, 1/2) = 64\sqrt{2}/15$

Example(2)

Evaluate: $I = {}_0\int^a x^4 \sqrt{(a^2 - x^2)} \;.\, dx$

Solution:

Letting $x^2 = a^2 y$, we get

$I = (a^6 / 2)\, {}_0\int^1 y^{3/2} (1 - y)^{1/2} dy = (a^6 / 2) \,.\, B(5/2, 3/2) = a^6 /3\,2$

Exercise

Evaluate: $I = {}_0\int^2 x \sqrt{(8 - x^3)} \;.\, dx$

Hint

Lett $x^3 = 8y$

Answer

$I = (8/3)\, {}_0\int^1 y^{-1/3} (1 - y)^{1/3} \;.\, dy = (8/3)\, B(2/3, 4/3) = 16\,\pi / (9 \sqrt{3})$

Using Formula (3)

Example(3)

Evaluate: $I = {}_0\int^{\infty} dx \ / \ (1+x^4)$

Solution:

Letting $x^4 = y$, we get

$I = (1 / 4)\, {}_0\int^{\infty} y^{-3/4} \ dy \ / \ (1 + y) = (1 / 4) \,.\, \Gamma (1/4) \,.\, \Gamma (1 - 1/4)$

$= (1/4) \,.\, [\pi \ / \ \sin (¼ \,.\, \pi)] = \pi \sqrt{2} \ / \ 4$

Using Formula (2)

Example(4)

a. Evaluate: $I = {}_0\int^{\pi/2} \sin^{3} \ . \ \cos^{2}x \ dx$

b. Evaluate: $I = {}_0\int^{\pi/2} \sin^{4} \ . \ \cos^{5}x \ dx$

Solution:

a. Notice that: $2m - 1 = 3 \rightarrow m = 2$ & $2n - 1 = 2 \rightarrow m = 3/2$

$I = (1/2)\ B(2, 3/2) = 8/15$

b. $I = (1/2)\ B(5/2, 3) = 8/315$

Example(5)

a. Evaluate: $I = {}_0\int^{\pi/2} \sin^6\ dx$

b. Evaluate: $I = {}_0\int^{\pi/2} \cos^6 x\ dx$

Solution:

a. Notice that: $2m - 1 = 6 \rightarrow m = 7/2$ & $2n - 1 = 0 \rightarrow m = 1/2$

$I = (1/2)\ B(7/2, 1/2) = 5\pi/32$

b. $I = (1/2)\ B(1/2, 7/2) = 5\pi/32$

Example(6)

a. Evaluate: $I = {}_0\int^{\pi} \cos^4 x\ dx$

b. Evaluate: $I = {}_0\int^{2\pi} \sin^8\ dx$

Solution:

a. $I = {}_0\int^{\pi} \cos^4 x = 2\ {}_0\int^{\pi/2} \cos^4 x = 2\,(1/2)\ B(1/2, 5/2) = 3\pi/8$

b. $I = I = {}_0\int^{\pi} \sin^8 x = 4\ {}_0\int^{\pi/2} \sin^8 x = 4\,(1/2)\ B(9/2, 1/2) = 35\pi/64$

Details

I.

Example(1)

Evaluate: $I = {}_0\int^{\infty} x^6 \; e^{-2x} \, dx$

$x = y/2$

$x^6 = y^6/64$

$dx = (1/2)dy$

$x^6 \; e^{-2x} \, dx = y^6/64 \; e^{-y} \,.\, (1/2)dy$

Example(2)

$I = {}_0\int^{\infty} \sqrt{x} \; e^{-x^3} \, dx \quad x=y^{1/3}$

$\sqrt{x}= y^{1/6}$

$dx=(1/3)y^{-2/3} \, dy$

$\sqrt{x} \; e^{-x^3} \, dx = y^{1/6} \, e^{-y} \,.\, (1/3)y^{-2/3} \, dy$

Example(3)

Evaluate: $I = {}_0\int^{\infty} x^m \; e^{-k x^n} \, dx$

$y = k x^n$

$x = y^{1/n} / k^{1/n}$

$x^m = y^{m/n} / k^{m/n}$

$dx = (1/n) \, y^{(1/n-1)} / k^{1/n} \quad dy$

$x^m \; e^{-k x^n} \, dx = (y^{m/n} / k^{m/n}) \,.\, e^{-y} \,.\, (1/n) \, y^{(1/n-1)} / k^{1/n} \quad dy$

$m/n + 1/n - 1 = (m+1)/n - 1$

$-m/n - 1/n = -(m+1)/n$

$I = [1 / (n \,.\, k^{(m+1)/n})] \; {}_0\int^{\infty} y^{[(m+1)/n - 1]} \; e^{-y} \, dy$

II.
Example(1)

Example(1)

$I = {}_0\int^2 x^2 / \sqrt{(2 - x)} \ . dx$

$x = 2y$

$dx=2dy$

$x^2 = 4 y^2$

$\sqrt{(2 - x)} = \sqrt{(2 - 2y)} = \sqrt{2} \quad \sqrt{(1 - y)}$

$x^2 / \sqrt{(2 - x)} \ . dx = 4 y^2 / \sqrt{2} \quad \sqrt{(1 - y)} \quad 2dy$

y=0 when x=0

y=1 when x=2

Example(2)

Evaluate: $I = {}_0\int^a x^4 \ \sqrt{(a^2 - x^2)} \ . dx$

$x^2 = a^2 y$, we get

$x^4 = a^4 y^2$

$x= a y^{1/2}$

$dx= (1/2)a y^{-1/2} dy$

$\sqrt{(a^2 - x^2)} = \sqrt{(a^2 - a^2 y)} = a (1 - y)^{1/2}$

$x^4 \sqrt{(a^2 - x^2)} \ . dx = a^4 y^2 \ a (1 - y)^{1/2} \ (1/2) a y^{-1/2} dy$

y=0 when x=0
y=1 when x=a

Example(3)

$I = {}_0\int^{\infty} dx \ / \ (1+x^4)$
$x^4 = y$

$x=y^{1/4}$
$dy= (1/4) y^{-3/4} \ dy$
$dx \ / \ (1+x^4) = (1/4) \ y^{-3/4} \ dy \ / \ (1+y)$

Proofs of formulas (2) & (3)

Formula (2)

We have,

$B(m,n) = \ {}_0\int^1 \ x^{m-1} \ (1-x)^{n-1} dx$

Let $x = \sin^2 y$
Then $dy = 2 \sin x \ \cos x \ dx$

&

$x^{m-1} \ (1-x)^{n-1} dx \ = (\sin^2 y)^{m-1} \ (\cos^2 y)^{n-1} \ (dy \ / \ 2 \sin x \ \cos x)$

$= \ 2 \sin^{2m-1} y \ . \ \cos^{2n-1} y \ dy$

When x=0 , we have $y = 0$
When x=1, we hae $y = \pi/2$

Thus,

$I = 2 \ {}_0\int^{\pi/2} \ \sin^{2m-1} y \ . \ \cos^{2n-1} y \ dy$

$I \ = \ {}_0\int^{\pi/2} \ \sin^{2m-1} y \ . \ \cos^{2n-1} y \ dy \ = \ B(m,n) \ / \ 2$

Formula (3)

We have,

$$I = {}_0\int^{\infty} x^{q-1} \;/\; (1+x)\; dx$$

Let

$y = x / (1+x)$

Hence, $x = y / 1-y$

, $1 + x = 1 + (y / 1-y) = 1/(1-y)$

& $dx = -[(1-y) - y(-1)] \;/\; (1-y)^2 \;.\, dy = 1 / (1-y)^2 \;.\, dy$

whn $x = 0$, we have $y = 0$

when $x \to \infty$, we have $y = \lim_{x\to\infty} x / (1+x) = 1$

Thus,

$$I = {}_0\int^{\infty} [\, x^{q-1} \;/\; (1+x)\,]\, dx = {}_0\int^{\infty} [\, (\, y / 1-y\,)^{q-1} \;/\; (1/(1-y))\,] \;.\; 1 / (1-y)^2 \;.\, dy$$

$$= {}_0\int^{1} [\, y^{q-1} \;/\; (1-y)^{-q}\,]\; dy$$

$$= B(q\,,\, 1-q) = \Gamma(q)\;\Gamma(1-q)$$

Proving that $\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(1/2) = {}_0\int^{\infty} x^{1/2-1} e^{-x} dx = {}_0\int^{\infty} x^{-1/2} e^{-x} dx$

Let $y = x^{½}$

$x = y^2$

$dx = 2y\, dy$

$\Gamma(1/2) = {}_0\int^{\infty} y^{-1} e^{-y^2} 2y\, dy$

$= 2\ {}_0\int^{\infty} e^{-y^2} dy$

$=2 (\sqrt{\pi} / 2) = \sqrt{\pi}$