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MODULE - 1

NUMERICAL METHOD

Numerical solution of ordinary differential eq^{-ns} of first order and first degree

Numerical problem with initial condition.

consider a differential of eq^{n} of the form $\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0)=y_0$

This problem of finding y is called as initial value problem

1) Taylor's series

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

Examples on Taylor series

1. Use Taylor series method to find y at $x = 0.1, 0.2, 0.3$ considering terms upto 3^{rd} degree given that $\frac{dy}{dx} = x^2+y^2$ with the initial condition $y(0)=1$

A:- $$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) \longrightarrow (1)$$

$$\frac{dy}{dx} = x^2+y^2 \qquad x_0 = 0, \; y_0 = 1$$

$$y' = x^2 + y^2$$

$$y'' = 2x + 2yy'$$

$$y''' = 2 + 2[yy'' + [y']^2]$$

$$y''' = 2 + 2[\cancel{y}y'' + [y']^2]$$
$$= 2 + 2[2xy + 2y^2y' + [x^2+y^2]^2]$$
$$= 2+2[2xy + 2y^2[x^2+y^2] + x$$

$y(x_0) = 1 \qquad y(0) = 1$

$y' = x^2+y^2; \qquad y'(x_0) = 0^2+[y(0)]^2 = 0+1 = 1$

$y'' = 2x+2yy'; \qquad y''(x_0) = 2(0)+2(1)(1) = 2.$

$y''' = 2+2[yy''+(y')^2] \qquad y'''(x_0) = 2+2[1(2)+1]$

$= 8.$

eq^{-n} (1) becomes.

$$y(x) = 1 + x(1) + \frac{x^2}{2} \cdot 2 + \frac{x^3}{6} 8$$

$$\therefore y(0.1) = 1+0.1 + \frac{(0.1)^2}{2} 2 + \frac{(0.1)^3}{6} \times 8$$

$$y = 1.1113$$

$$y(0.2) = 1+0.2+\frac{(0.2)^2}{2} 2 + \frac{(0.2)^3}{6} \times 8$$

$$y = \cancel{0.3}\ 1.2506$$

$$y(0.3) = 1.426.$$

2. using the Taylors method Find, the third order approximate sol^{-n} at $x = 0.4$ of the problem $dy/dx = x^2y+1$ with $y(0) = 0$.

$dy/dx = x^2y+1 \qquad y(0) = 0$

$y' = 0+1 = 1$

$y'' = x^2 y' + y2x.$

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$y''(0) = 0 + 0$

$y''(0) = 0$

$y''' = x^2 y'' + y' 2x + 2[x y' + y]$

$= 0 + 0 + 2[0 + 0]$

$y''' = 0$

Taylors series

$y(x) = 0 + x \cdot (1) + 0$

$y(x) = x$

$y(0.4) = \underline{0.4}$

lly

at $y(0) = 1$

$y' = 1$

$y'' = 0$

$y''' = 2$

$y(x) = 1 + x + \frac{x^3}{3!} \times 2$

at $x = 0.4$

$y(0.4) = \cancel{1+0.4} \quad 1.05333$

3. using Taylor series method find an approximate sol^n corret to 4th decimal places for the following initial value problem at $\underline{x = 0.1}$ $dy/dx = x - y^2$ with $y(0) = 1$ [consider upto 4th degree]

$y' = x - y^2 \qquad y(0) = 1$

$y'(0) = -1$

$y'' = 1 - 2yy' \ ; \ y''(0) = 1 - 2(1)(-1)$

$y''(0) = 3$

$y''' = 0 - 2[yy'' + (y')^2]$

$y'''(0) = 0 - 2[1(3) + 1] =$

$0 - 2[4] = -8$.

$y'''' = -2[yy''' + y''y' + 2y'.y'']$.

$= -2[\cancel{1(8) - 3(-1) + 2(-1).(3)}]$.

$-2[\cancel{8 - (-3) + -6}]$

$-2[\cancel{8 + 3 - 6}]$

$-2[1(-8) + 3(-1) + 2(-1)(3)]$.

$= -2[-8 - 3 - 6]$

$= -2(-17)$

$= \underline{\underline{34}}$

$y(0.1) = 1 + 0.1(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{3!}(-8) + \frac{(0.1)^4}{4!}(34)$

$y(0.1) = 0.9L38$.

4. use taylor series method to obtain approximate value of Y at $x = 0.1$ and 0.2 for the differential eq^n $dy/dx = 2y + 3e^x$ $y(0) = 0$ considering upto 4th degree term.

$dy/dx = y' = 2y + 3e^x$.

$y(x) = y(x_0) + \frac{(x-x_0)}{1} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \frac{(x-x_0)^4}{4!} y^{iv}(x_0)$

$y' = 2y + 3e^x$

$y' = 3$

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$y'' = 2y' + 3e^{x}$
$= 2 \times 3 + 3 = 9.$

$y''' = 2y'' + 3e^{x}$
$= 2 \times 9 + 3 = 21.$

$y^{iv} = 2y''' + 3e^{x}$
$= 2 \times 21 + 3 = 45.$

$$y(x) = 0 + 3x + \frac{9x^2}{2} + \frac{21x^3}{3!} + \frac{45x^4}{4!}$$

at $x = 0.1$
$y = 0.3487$
at $x = 0.2$, $y = 0.811$

5. Use taylor series method to obtain approximate sol^{n} correct to 4 decimal places for the following initial value problem at $x = 0.1$

$\frac{dy}{dx} = x - y^2$; $y(0) = 1$ [4th degree] $x_0 = 0$, $y_0 = 1$

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \frac{(x - x_0)^4}{4!} y^{iv}(x_0)$$

$y(x_0) = 1$
$y' = x - y^2$
$y'(x_0) = -1.$

$y'' = 1 - 2yy'$; $y''(x_0) = 1 - 2(1)(-1) = 3$

$y''' = 1 - 2[yy'' + (y')^2]$; $y'''(x_0) = 1 - 2[(1)(3) + 1]$
$= -8.$

$y^{iv} = 1 - 2[yy''' + y''y' + 2y'y'']$; $y^{iv}(x_0) = \cancel{-2}$
$\cancel{-2[1(-8) + 3(-1) + 2(-1)]}$

$y^{iv}(0) = -2[-6 - 3 - 8]$
$= 34.$

$$y = 1 - 1x + \frac{3x^2}{2} - \frac{8x^3}{6} + \frac{34x^4}{24}$$

at $x = 0.1$

$y = 0.9138$

6. Use taylor series method to obtain a power series $(x-4)$ for the eq^n. $5xy_1 + y^2 - 2 = 0$; $x_0 = 4$ & $y_0 = 1$ and use it to find at $x = 4.1, 4.2$ & 4.3 consider upto 2nd degree.

$x_0 = 4.$

$$y = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)}{2!} y''(x_0) \quad \rightarrow (1)$$

$x_0 = 4$, $y(x_0) = 1$

$5xy' + y^2 - 2 = 0$

$5(4)y'(4) + (1)^2 - 2 = 0$

$20y'(4) + 1 - 2 = 0$

$y'(4) = 1/20 = 0.05$

$5xy'' + 5y' + 2yy' = 0$

put $x_0 = 4$

$5[4y''(4) + 0.05] + 2(1)(0.05) = 0$

$20y''(4) + 0.25 + 0.1 = 0$

$y''(4) = \frac{-0.35}{20} = -0.0175$

$$y(x) = 1 + (x-4)\,0.05 + \frac{(x-4)^2}{2}(-0.0175)$$

at $x = 4.1$

$y(4.1) = 1.0049$

at $x = 4.2$

$y(4.2) = 1.0096$

at $x = 4.3$

$y(4.3) = 1.0142$

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modified Euler's method

consider a D.E of the form $dy/dx = f(x, y)$ with the initial condition $y(x_0) = y_0$. we need to find y at $\boxed{x_1 = x_0 + h}$

ie. $y(x_1) = ?$

Eulers formula is given by

$$\boxed{y_1^{(0)} = y_0 + h f(x_0, y_0)}$$ where $y_1^{(0)}$ is initial approximation.

modified eulers formula is given by.

$$\boxed{y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]}$$

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

continuing the same process till we get a desired degree of accuracy.

Eulers formula and modified eulers formula jointly called as <u>Euler's predictor and Corrector formula</u>

1. given that $dy/dx = 1 + y/x$; $y = 2$ at $x = 1$ find the approximate value of y at $x = 1.2$ by taking step size $h = 0.2$ Apply modified Eulers method.

A: $x_1 = x_0 + h$ $\quad x_1 = 1.2 \quad x_0 = 1$

$x_0 = 1$, $y_0 = 2$, $h = 0.2$ $\quad x_1 = x_0 + h = 1.2$.

$f(x, y) = 1 + y/x$.

$\Rightarrow f(x_0, y_0) = 1 + \frac{2}{1} = 3$

Eulers formula is given by

$\Rightarrow y_1^{(0)} = y_0 + h f(x_0, y_0)$
$= 2 + 0.2 \times 3 = 2.6$

$\Rightarrow y_1^{(1)} = y_0 + h/2 [f(x_0, y_0) + f(x_1, y_1^{(0)})]$
$= 2 + \frac{0.2}{2}\left[3 + \cancel{1.2}\ \frac{2.6}{1.2}\right]$

$y_1^{(1)} = 2.616$

$\Rightarrow y_1^{(2)} = y_0 + h/2 [f(x_0, y_0) + f(x_1, y_1^{(1)})]$
$= 2 + \frac{0.2}{2}\left[3 + 1 + \frac{2.616}{1.2}\right]$

$y_1^{(2)} = 2.618$

$y_1^{(3)} = y_0 + h/2 [f(x_0, y_0) + f(x_1, y_1^{(2)})]$
$= 2 + \frac{0.2}{2}\left[3 + 1 + \frac{2.618}{1.2}\right]$

$y_1^{(3)} = \underline{2.618}$

(or)

Analytic solution

The given eq^{-n} can also be done in

$$\frac{dy}{dx} - y/x = 1 \Rightarrow \frac{dy}{dx} + Py = Q.$$

$P = \frac{-1}{x}$, $Q = 1$

$$I.F = e^{\int P dx}.$$

$$= e^{-\int \frac{1}{x} \cdot dx} = e^{-\log x} = e^{\log x^{-1}}$$

$$= 1/x.$$

$$y(I.F) = \int Q(I.F)\,dx + C.$$

$$y/x = \int 1/x \cdot dx + C.$$

$$= y/x = \log x + C. \rightarrow (1).$$

sub $x=1$, & $y=2$ in eq^{-n} (2).

$$\frac{2}{1} = \log 1 + C \Rightarrow C = 2.$$

$$y/x = \log x + 2.$$

$$y = x\log x + 2x.$$

~~y = 6~~ $y(1.2) = 2.618$

$\therefore$ numerical method is verified.

0.2 0 =

$$y_1^{(1)} = 1 + \frac{0.2}{2}\left[0 + e^{0.2} - 1\right]$$

$$y_1^{(1)} = 1.0221$$

$y_1^{(2)} = 1.0199.$

$y_1^{(3)} = 1.020$

$y_1^{(4)} = 1.020$ //

5. Using modified Eulers Formula Find $y(0.2)$ given $dy/dx = x+y$ $y(0)=1$, $h=0.1$ Perform 2 iterations in each step.

<u>step 1</u>

$y(0.1) = ?$

$f(x,y) = x+y$

$y_0 = 1$, $x_0 = 0$.

$f(x_0, y_0) = 1.$

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

$$= 1 + 0.1\,(1)$$

$$= 1.1$$

$$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[1 + 0.1 + 1.1\right]$$

$$y_1^{(1)} = 1.11$$

$$y_1^{(2)} = 1.1105$$ //

<u>step 2</u>

$$y(0.1) = 1.11.$$

$x_0 = 0.1$

$f(x_0, y_0) = 1.21$

$y_0 = 1.11$

$h = 0.1$

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

$$= 1.11 + 0.1\,(1.21) = 1.231$$

$y_1^{(1)} = y_0 + h/2 \left(f(x_0, y_0) + f(x_1, \ldots \right)$

$= 1.11 + \frac{0.1}{2} \left[1.244 + 0.2 + 1.231 \right]$

$y_1^{(1)} = \cancel{1.247}\ 1.242$

$\boxed{y_1^{(2)} = 1.242.}$

$y(0.2) = \cancel{1.11}$

$\underline{y(0.2) = 1.242}$

6. Using Modified eulers method find $y(0.1)$ with $dy/dx = x^2+y$ by taking $h = 0.05$ by taking 2 iterations in each step. $y(0)=1$

step 1 $y(0.05)$ $x_1 = 0.05$ $~~0.1~~ = 0+0.05$

$f(x,y) = x^2+y$ $x_0 = 0, y_0 = 1$

$f(x_0, y_0) = 1$

$y_1^{(0)} = y_0 + hf(x_0, y_0).$

$= 1 + 0.05(1) = 1.05.$

$y_1^{(1)} = y_0 + h/2 \left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$

$= 1 + \frac{0.05}{2} \left[1 + (0.05)^2 + 1.05 \right]$

$y_1^{(1)} = 1.051$

$y_1^{(2)} = 1.051.$

step 2

$y(0.05) = 1.051$ $x_1 = 0.1.$

$f(x_0, y_0) = (0.05)^2 + 1.051 = 1.0535$

$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1.051 + 0.05(1.0535)$

$= 1.10367$

$$y_1^{(1)} = 1 + \frac{0.2}{}[\ldots + f(x_1, y_1^{(1)})]$$

$$= 1.051 + \frac{0.05}{0.2}\left[1.0535 + (0.1)^2 + 1.1036\right]$$

$$y_1^{(1)} = 1.1051$$

$$y_1^{(2)} = 1.1052.$$

$$\therefore \boxed{y(0.1) = 1.105}$$

7. Using Eulers predictor and corrector formula compute $y(1.1)$ correct to 5 decimal places given that $dy/dx = y/x$

$$dy/dx + y/x = 1/x^2; \quad y(1) = 1$$

Ans:- $h = 0.1$, $x_0 = 1$, $y_0 = 1$, $x_1 = 1.1$

$$f(x,y) = \frac{1}{x^2} - y/x.$$

$$f(x_0, y_0) = 0$$

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$= 1 + 0.1 \times 0$$

$$= 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0 + \frac{1}{(1.1)^2} - \frac{1}{1.1}\right]$$

$$y_1^{(1)} = 0.99586.$$

$$y_1^{(2)} = 0.991529. \qquad \therefore y(1.1) = 0.99174$$

$$y_1^{(3)} = 0.99174$$

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R-K method.

3. Runge-kutta method of fourth order

$$y(x_0+h) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf\left[x_0 + h/2, y_0 + \frac{k_1}{2}\right]$

$k_3 = hf\left[x_0 + h/2, y_0 + \frac{k_2}{2}\right]$

$k_4 = hf\left[x_0 + h, y_0 + k_3\right]$

1. given $dy/dx = 3x + y/2$ with $y(0) = 1$. compute $y(0.2)$ by taking $h = 0.2$.

R.K

$x_0 = 0, \; y_0 = 1, \; h = 0.2$

$$y(x_0 + h) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$k_1 = hf(x_0, y_0)$

$= 0.2 \times \left[3.(0) + 1/2\right]$

$= 0.1$

$k_2 = \cancel{0.2}\, hf\left[x_0 + h/2, y_0 + k_1/2\right]$

$hf\left[0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right]$

$hf(0.1, 1.05)$

$0.2\left[3 \times 0.1 + \frac{1.05}{2}\right]$

$= 0.165$

$k_3 = \cancel{hf}\; \cancel{0.0765}\; 0.168$.

$k_4 = hf(0.2, 1.168)$

$= 0.2\left[3 \times 0.2 + \frac{\cancel{1.0765}\; 1.168}{2}\right]$

$k_4 = 0.2368$

$\therefore \; y(0.2) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$y(0.2) = 1 + \frac{1}{6}[0.1 + 2 \times 0.165 + 2 \times 0.168 + 0.2368]$

$y(0.2) = 1.1671$

2. solve $dy/dx = y^2 - x^2$ with $y(0) = 1$, y at $x = 0.2$ using R.K method of 4th order taking step length as $h = 0.2$ accurate upto 0.4 decimal places

given $x_0 = 0$, $y = 1$, $h = 0.2$.

$y(x_0 + h) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \longrightarrow (1)$

$k_1 = hf(x_0, y_0)$

$= 0.2\,[(1)^2 - 0]$

$k_1 = 0.2$

$k_2 = hf(x_0 + h/2,\; y_0 + \frac{k_1}{2})$

$= hf(0 + \frac{0.2}{2},\; 1 + \frac{0.2}{2})$

$= 0.2\,(0.1),\,(1.1)$

$= 0.24.$

$k_3 = hf(x_0 + h/2,\; y_0 + \frac{k_2}{2})$

$= 0.2488$

$k_4 = 0.3039.$

$k_1 = hf(x_0, y_0)$

$\therefore y(0.2) = 1 + \frac{1}{6}[0.2 + 2 \times 0.24 + 2 \times 0.2488 + 0.3039]$

$y(0.2) = 1.2468$

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3. solve $dy/dx = x+y$

$x=0,\ y=1$ & $y(0.2) = ?$ $h = 0.2$

$K_1 = hf(x_0, y_0)$

$= 0.2[1]$

$= 0.2$

$K_2 = hf(x_0 + h/2,\ y_0 + k_1/2]$

$= hf(0.1,\ 1 + \frac{0.2}{2})$

$= 0.2[0.1 + 1.1]$

$= 0.24$

$K_3 = 0.244$

$K_4 = 0.2888$

$y(0.2) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1.2428$

4. using R-K method of 4^{th} order, Find $y(0.2)$

$y' = y - \frac{2x}{y}$, $h = 0.2$; $y(0) = 1$

A:- $K_1 = hf(x_0, y_0)$

$= 0.2[1 - 0]$

$= 0.2$

$K_2 = hf(x_0 + h/2,\ y_0 + k_1/2)$

$= hf(0.1,\ 1 + 0.1)$

$= 0.2\left[1.1 - \frac{2 \times 0.1}{1.1}\right]$

$= \cancel{0.1836}$ 0.1836

~~$K_3 = 0.2[1.1836$~~

$$= 0.2[0.1, 1.0918]$$

$$= 0.2\left[1.0918 - \frac{2 \times 0.1}{1.0918}\right] = 0.1817$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.2(0.2, 1.1817)$$

$$= 0.2\left[1.1817 - \frac{2 \times 0.2}{1.1817}\right]$$

$$= 0.1686$$

$$y(0.2) = y_0 + \frac{1}{6}[0.2 + 2 \times 0.1836 + 2 \times 0.1817 + 0.1686]$$

$$y(0.2) = \underline{\underline{1.1832}}$$

5. using R-K method of 4^{th} order find $y(0.2)$ for $dy/dx = \frac{y-x}{y+x}$; $y(0) = 1$; $h = 0.1$

$y(x_0 + h)$. $h = 0.1$, $x = 0$, $y_0 = 1$

$$K_1 = hf(x_0, y_0)$$

$$= 0.1\left[\frac{1-0}{1+0}\right]$$

$$= 0.1$$

$$K_2 = hf[x_0 + h/2, y_0 + K_1/2]$$

$$= hf[0.05, 1.05]$$

$$= 0.1\left[\frac{1.05 - 0.05}{1.05 + 0.05}\right]$$

$$= 0.0909$$

$$K_3 = hf[x_0 + h/2, y_0 + K_2/2]$$

$$hf[0.05, 1.0454]$$

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$$k_3 = 0.1\left[\frac{1.0454 - 0.05}{1.0454 + 0.05}\right]$$

$$k_3 = 0.09087$$

$$k_4 = hf\left[x_0 + h, y_0 + k_3\right]$$

$$0.1\left[0.1, 1.0908\right]$$

$$= 0.1\left[\frac{1.0908 - 0.1}{1.0908 + 0.1}\right]$$

$$k_4 = 0.0832$$

$$y(0.1) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$y(0.1) = 1.0911$$

<u>step 2</u> $x_0 = 0.1$, $y_0 = 1.0911$, $h = 0.1$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1\left[\frac{1.0911 - 0.1}{1.0911 + 0.1}\right]$$

$$= 0.0832$$

$$k_2 = 0.1\left[0.15, 1.1327\right]$$

$$= 0.1\left[\frac{1.1327 - 0.15}{1.1327 + 0.15}\right]$$

$$= 0.0766.$$

$$k_3 = 0.1\left[0.15, 1.1294\right]$$

$$= 0.1\left[\frac{1.1294 - 0.15}{1.1294 + 0.15}\right]$$

$$k_3 = 0.07655$$

$$k_4 = 0.1\left[0.2, 1.16765\right] = 0.1\left[\frac{1.1676 - 0.2}{1.1676 + 0.2}\right]$$

~~y(0.1) = 0.0707~~ 0.0707

$$y(0.2) = 1.1678$$

b. Using 4th order R.K method solve $(x+y)\,dy/dx = 1$ with the initial condition $y(0.4)=1$ find y at $x = 0.5$ correct to 4 decimal places.

Ans: ~~Spt~~ step size is not mentioned

$$x_0 + h = 0.5$$

$$0.4 + h = 0.5 \qquad y' = \frac{1}{x+y}.$$

$$h = 0.1$$

$$K_1 = hf(x_0, y_0)$$

$$= 0.1\left[\frac{1}{0.4+1}\right] = 0.0714$$

$$K_2 = hf\left[0.4+0.05,\ 1.0357\right]$$

$$= 0.1\left[\frac{1}{0.45+1.0357}\right]$$

$$= 0.0673$$

$$K_3 = 0.0674.$$

$$K_4 = hf\left[0.4+0.1,\ 1+0.0674\right]$$

$$= 0.1\left[\frac{1}{0.5+1.0674}\right] = 0.06378$$

$$\therefore y(0.5) = 1 + \frac{1}{6}\left[0.0714 + 2\times 0.0673 + 2\times 0.0674 + 0.06378\right]$$

$$y(0.5) = 1.0674$$

4.

Predictor and correctors method

① Milne's method

② Adam - Bashforth method.

1. Milne's method

$$\boxed{y_4^{(p)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]}$$

Illy corrector formula is given by.

$$\boxed{y_4^{(c)} = y_2 + h/3\left[y_2' + 4y_3' + y_4'\right]}$$

2) Adam - Bashforth method

$$y_4^{p} = y_3 + h/24\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_4^{c} = y_3 + h/24\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

1. Given that $dy/dx = x - y^2$; $y(0) = 0$, $y(0.2) = 0.02$ $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. compute y at $x = 0.8$ by applying ⓐ milne's

ⓑ Adam - Bashfort

$h = 0.2$ { difference in x }

x	y	$y' = x - y^2$	
0	0	0	y_0
0.2	0.02	0.1996	y_1
0.4	0.0795	0.3936	y_2
0.6	0.1762	0.5689	y_3
0.8	0.305	0.7070	

a. milnes Predictor Formula

① $y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$

$= 0 + \frac{4\times(0.2)}{3}\left[2\cdot[0.1996] - 0.3936 + 2[0.5689]\right]$

$\boxed{y_4^P = 0.305}$

② $y_4^{(C)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$

$y_4^{(C)} = 0.0795 + \frac{0.2}{3}\left[0.3936 + 4[0.5689] + [0.7070]\right]$

$\boxed{y_4^{(C)} = 0.30458}$

b. Adam Bashforth.

① $y_4^P = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$

$= 0.1762 + \frac{0.2}{24}\left[55(0.5689) - 59(0.3936) + 37(0.1996) - 9(0)\right]$

$y_4^{(P)} = \cancel{0.1287}\ 0.3049.$

② $y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$

$= 0.1762 + \frac{0.2}{24}\left[9[0.7070] + 19(0.5689) - 5(0.3936) + 0.1996\right]$

$\boxed{y_4^{(c)} = 0.3045}$

2. given $y' = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$.

correct to 3 decimal places using milnes predictor formula

Ans Milne's method.

x	y	$y' = xy + y^2$	
0	1	1	y_0
0.1	1.1164	1.3591	y_1
0.2	1.2773	1.8869	y_2
0.3	1.5049	2.7161	y_3
0.4	1.8352	4.1016	y_4

$h = 0.1$

(a) Predictor

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4 \times 0.1}{3}\left[2 \times 1.3591 - 1.8869 + 2 \times 2.7161\right]$$

$$\boxed{y_4 = 1.8352}$$

$$\therefore \boxed{y_4' = 4.1016}$$

(b) $$y_4^{(c)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.2773 + \frac{0.1}{3}\left[1.8869 + 4(2.7161) + 4.1016\right]$$

$$y_4^{(C)} = 1.8390$$

3. $2\,dy/dx = (1+x^2)y^2$; $y(0) = 1$; $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$

Evaluate $y(0.4)$.

$$y' = \frac{(1+x^2)y^2}{2}$$

$h = 0.1$

x	y	y'
0	1	0.5
0.1	1.06	0.5674
0.2	1.12	0.6522
0.3	1.21	0.7979
0.4		0.9458

Milnes predictor formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4\times0.1}{3}\left[2(0.5674) - 0.6522 + 2(0.7979)\right]$$

$$\boxed{y_4^{(P)} = 1.2771}$$

$\cancel{y_4 = 1}$ 0.4 1.2771 $\underline{0.9459}$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.12 + \frac{0.1}{3}\left[0.6522 + 4(0.7979) + 0.9459\right]$$

$$\boxed{y_4^{(C)}(0.4) = 1.2796}$$

4. given that

$dy/dx = x^2(1+y)$; $y(1) = 1$, $y(1.1) = 1.2333$, $y(1.2) = 1.548$, $y(1.3) = 1.979$.

find y at $x = 1.4$ using Adam Bashforth.

Ans:-

x	y	y'
1	1	2
1.1	1.2333	2.7019
1.2	1.548	3.6691
1.3	1.979	5.0345
1.4	2.5722	7.0015

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 1.979 + \frac{0.1}{24}\left[55(5.0345) - 59(3.6691) + 37(2.7019) - 9(2)\right]$$

$$y_4^P = \underline{2.5722}$$

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$$y_4^{(c)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 1.979 + \frac{0.1}{24}\left[9(7.0015) + 19(5.03451) - 5(3.6691) + 2.7019\right]$$

$$\boxed{y(1.4) = 2.5749}$$

5. if $dy/dx = 2e^x - y$; $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.04$, $y(0.3) = 2.09$

find $y(0.4)$ by using ~~using~~ milnes predictor and corrector formula apply ~~predic~~ corrector twice.

x	y	y'
0	2	0
0.1	2.010	0.2003
0.2	2.04	0.4028
0.3	2.09	0.6097
0.4	2.1623	0.8213

// ~~lasi~~ by applying twice we get y_4' as 0.8216.

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 2 + \frac{4\times 0.1}{3}\left[2(0.2003) - (0.4028) + 2[0.6097]\right]$$

$$y_4^{(P)} = 2.1623$$

$$y_4^{(c)} = y_2 + h/3\left[y_2' + 4y_3' + y_4'\right]$$

$$= \cancel{0.4028}\ 2.04 + \frac{0.1}{3}\left[0.4028 + 4\times[0.6097) + \cancel{2x}\ 0.8213\right]$$

$$\boxed{y_4^{(c)} = 2.1620}$$

after applying twice, $y_4^{(c)} = \underline{2.1621}$

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6. using milnes predictor and corrector method find $y(0.3)$ correct to 3 decimal places

$x = -0.1, \quad 0 \quad 0.1 \quad 0.2.$

$y = 0.908783 \quad 1.000 \quad 1.11145 \quad 1.25253.$

$dy/dx = \log_e x + y$

x	y	y'
-0.1	0.908783	0.808783
0	1.00	1.000
0.1	1.11145	1.21145
0.2	1.2523	$1.4523.$
0.3	1.401203	$1.7012.$

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0.908783 + \frac{4(0.1)}{3}\left[2(1.00) - (1.21145) + 2(1.4523)\right]$$

$$\underline{y_4^{(P)} = 1.401203}$$

$$y_4^{(C)} = y_2 + h/3\left[y_2' + 4y_3' + y_4'\right].$$

$$= 1.11145 + \frac{0.1}{3}\left[1.21145 + 4(1.4523) + 1.7012\right]$$

$$\boxed{y_4^{(c)} = 1.4022.}$$

M-1

Numerical method

Numerical solution of 2^{nd} order ~~di~~ ordinary differential eq^{-n}

R-K method of 4^{th} order

consider a differential eq^{-n} of the form $\frac{d^2y}{dx^2} = g(x,y,y')$
with the initial condition $y(x_0)=y_0$: $y'(x_0)=y_0'$

Substitute $dy/dx = z \Rightarrow \frac{d^2y}{dx^2} = \frac{dz}{dx}$.

The given eq^{-n} becomes

$$\frac{dz}{dx} = g(x,y,z)$$

$$\therefore \frac{dy}{dx} = f(x,y,z) = z \; ; \; \frac{dz}{dx} = g(x,y,z)$$

with initial condition.

$$y(x_0) = y_0 \; \& \; z(x_0) = z_0$$

R-k method formula is given by

$$y(x_0+h) = y_0 + 1/6\,[k_1 + 2k_2 + 2k_3 + k_4].$$

where $k_1 = hf(x_0, y_0, z_0)$, $l_1 = hg[x_0, y_0, z_0]$

$$k_2 = hf\left[x_0 + h/2,\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right]$$

$$\cancel{k_0} = l_2 = hg\left[x_0 + h/2,\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_1}{2}\right]$$

$$k_3 = hf\left[x_0 + h/2,\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right]$$

$$l_3 = hg\left[x_0 + h/2,\ y_0 + \frac{k_3}{2},\ z_0 + \frac{l_2}{2}\right]$$

$$k_4 = hf[x_0 + h,\ y_0 + k_3,\ z_0 + l_3]$$

$$l_4 = hg[x_0 + h\ \ y_0 + k_3,\ z_0 + l_3]$$

$$z(x_0+h) = z_0 + 1/6\,[l_1 + 2l_2 + 2l_3 + l_4]$$

1. Compute $y(0.1)$ given $\frac{d^2y}{dx^2} = y^3$ and $y = 10$; $dy/dx = 5$ at $x = 0$.

given :- $h = 0.1$, $x_0 = 0$, $y_0 = 10$, $y_0' = 5$ $\quad y(0) = 10$, $y'(0) = 5$, $z_0 = 5$

$\frac{dy}{dx} = z = f(x, y, z)$.

$\frac{dz}{dx} = y^3 = g(x, y, z)$.

R-K method of 4th order is given by

$y(x_0 + h) = y_0 + 1/6 [k_1 + 2k_2 + 2k_3 + k_4] \rightarrow (1)$.

$k_1 = hf[x_0, y_0, z_0]$

$0.1 \times 5 = 0.5$

$l_1 = hg[x_0, y_0, z_0]$. $\quad 0.1 \times y_0^3$

$= 0.1 \times (10)^3 = 100$

$k_2 = hf[x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}]$.

$= 0.1[0 + \frac{0.1}{2}, 10 + \frac{0.5}{2}, 5 + \frac{100}{2}]$

$= 0.1[0.05, 10.25, 55]$.

$= 0.1 \times 55$

$= \boxed{k_2 = 5.5}$

$l_2 = hg[x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + \frac{l_1}{2}]$

$0.1\, g[0.05, 10.25, 55]$.

$= 0.1[10.25]^3$

$l_2 = 107.689$.

$k_3 = hf[x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}]$

$= 0.1 f[0 + \frac{0.1}{2}, 10 + \frac{5.5}{2}, 5 + \frac{107.689}{2}]$

$= 0.1 f[0.05, 12.75, 58.8445]$

$0.1[58.8445]$.

$k_3 = 5.8844$

$l_3 = (12.75)^3 \times 0.1 = \underline{207.2671}$

$k_4 = hf[x_0+h, y_0+k_3, z_0+l_3]$.

$0.1 f[0+0.1, 10+5.8844, 5+207.2671]$

$0.1 f[0.1, 15.8844, 212.2671]$.

$= 0.1[212.2671] = \underline{21.2267}$.

$l_4 = 400.7859$.

$y(x_0+h) = y_0 + 1/6 [k_1 + 2k_2 + 2k_3 + k_4]$.

$= 10 + 1/6 [0.5 + 2\times 5.5 + 2\times 5.8844 + 21.2267]$

$y(x_0+h) = 17.4159$.

$\boxed{y(0.1) = 17.4159}$

2. $y'' - xy' - y = 0$ $y(0) = 1$, $y'(0) = 0$. compute $y(0.2)$ & $y'(0.2)$.

given:- $h = 0.2$, $x_0 = 0$, $y_0 = 1$, $z_0 = 0$, $y_0' = 0$.

$\frac{dy}{dx} = z$, $\frac{dz}{dx} = xz + y$.

$f(x, y, z) = z$ & $g(x, y, z) = xz + y$.

R.K method of 4th order is given by.

$y(x_0+h) = y_0 + 1/6 [k_1 + 2k_2 + 2k_3 + k_4] \rightarrow (1)$.

$k_1 = hf[x_0, y_0, z_0]$

$0.1 \times 0 = 0$.

$l_1 = hg[x_0, y_0, z_0]$

$0.2[0 \times 0 + 1] = 0.2$.

$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$.

$0.2 = \left[0 + \frac{0.2}{2}, 1 + 0/2, 0 + \frac{0.2}{2}\right]$

$0.2[0.1, 1, 0.1]$

$k_2 = 0.2 \times 0.1 =$

$= \underline{0.02}$.

$l_2 = 0.2[(0.1) \times (0.1) + 1]$.

$\underline{l_2 = 0.202}$.

$k_3 = hf\left[x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$.

$= 0.2\left[0 + \frac{0.2}{2}, 1 + \overset{0.02}{0}, 0 + \frac{0.202}{2}\right]$

$= 0.2[0.1, 1.01, 0.101]$

$= 0.2 \times [0.101]$

$\underline{k_3 = 0.0202}$.

$l_3 = 0.2[(0.1 \times 0.101) + 1.01]$.

$l_3 = 0.204$.

$k_4 = hf[x_0 + h, y_0 + k_3, z_0 + l_3]$

$= 0.2[0 + 0.2, 1 + 0.0202, 0 + 0.204]$

$= 0.2[0.2, 1.0202, 0.204]$

$= 0.2[0.204] = \underline{0.0408}$

$l_4 = 0.2[(0.2)(0.204) + 1.0202]$

$= 0.2121$

$y(0.2) = \cancel{y_0}$

$= 1 + \frac{1}{6}\left[0 + 2\times0.02 + 2\times0.0202 + 0.0408\right]$.

$$\boxed{y(0.2) = \underline{\underline{1.0202}}}$$

$y'(0.2) = 0 + \frac{1}{6}\left[0.2 + 2\times0.202 + 2\times0.204 + 0.2121\right]$

$$\boxed{y'(0.2) = 0.2040}$$

3. $y'' = x(y')^2 - y^2$. for $x = 0.2$; $y = 1$ & $y' = 0$. at $x = 0$

$h = 0.2$.

$x_0 = 0$.

$y_0 = 1$.

$y'_0 = 0$. $\quad z_0 = 0$

$\frac{dy}{dx} = z$, $\frac{dz}{dx} = x(z)^2 - y^2$.

$f(x, y, z) = z$ and $g(x, y, z) = x(z)^2 - y^2$

$y(x_0 + h) = y_0 + 1/6\left[k_1 + 2k_2 + 2k_3 + k_4\right] \longrightarrow (1)$.

$k_1 = hf\left[x_0, y_0, z_0\right]$.

$0.2[0] = 0$. $\quad y' = z$

$\cancel{k_2}$

$l_1 = hg\left[x_0, y_0, z_0\right]$. $\quad hg(x_0, y_0, z_0)$

$= 0.2\left[0 \cdot (0)^2 - (1)^2\right]$. $\quad 0.2($

-0.2 .

$k_2 = hf\left[x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2\right]$.

$0.2\left[\cancel{0.0}\ 0.1, 1, -0.1\right]$.

$= 0.2\cancel{\left[(0.1)(0.1\right.}$ $\quad 0.2\left[-0.1\right] = -0.02$.

$l_2 = \cancel{hg}\left[\because 0.2\left[(0.1)(0.1)^2 - 1^2\right]\right.$.

$l_2 = -0.1992$

$k_3 = hf\left[x_0 + h/2 \;,\; y_0 + k_2/2 \;,\; z_0 + \frac{l_2}{2}\right]$ $\quad x = x_0 + h/2,\; y = 0.1$

$= 0.2\left[0 + \frac{0.2}{2} \;,\; 1 + \frac{-0.02}{2} \;;\; 0 + \frac{-0.1992}{2}\right]$

$= 0.2\left[0.1 \;,\; 0.99 \;,\; -0.09994\right]$

$k_3 = \cancel{0.2}\quad 0.2\left[-0.09994\right]$

$\boxed{k_3 = \cancel{0.14008}}\; -0.01992$

$l_3 = 0.2\left[0.1\left(\cancel{0.0004}\; -0.01992\right)^2 - 0.99^2\right]$

$l_3 = -\cancel{0.1998}\quad 0.1958$.

$k_4 = hf\left[x_0 + h \;,\; y_0 + k_3 \;,\; z_0 + l_3\right]$

$= 0.2\left[0.2 \;,\; 1.18008 \;,\; 0 + (-1.1958)\right]$

$= 0.2\left[-0.1958\right]$

$k_4 = -0.03916$

$y(0.2) = y_0 + 1/6\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= 1 + 1/6\left[0 + 2(-0.02) + 2(-0.01992) + (-0.03916)\right]$

$\boxed{y(0.2) = 0.9801}$

4 obtain the value of x and $\frac{dx}{dt}$ at $t = 0.1$

given that $\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$, $x = 3$, $\frac{dx}{dt} = 0$

when $t = 0$ $\qquad y = 3 \quad \frac{dy}{dx} = 0$

Put $t = x$

$dt = dx$.

$\Rightarrow y' = 0$

Put $x = y$

$dx = dy$ $\qquad x = 0$

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y and $\frac{dy}{dx}$ at $\cancel{x=0}$ $x_0 = 0$. $x = 0.1$, $h = 0.1$

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$z_0 = 0$

$$\frac{d^2y}{dx^2} = x \cdot \frac{dy}{dx} - 4y.$$

$h =$

$$\frac{dz}{dx} = xz - 4y.$$

$$\frac{dy}{dx} = z.$$

$$K_1 = hf[x_0, y_0, z_0]$$

$$0.1[0]$$

$$\underline{K_1 = 0}$$

$$l_1 = hg[x_0, y_0, z_0]$$

$$= 0.1[0 \times 0 - 4 \times 3]$$

$$= -1.2$$

$$K_2 = hf\left[x_0 + h/2,\ y_0 + \frac{K_1}{2},\ z_0 + \frac{l_1}{2}\right].$$

$$0.1\left[0 + 0.05,\ 3 + 0/2,\ 0 + -1.2/2\right]$$

$$0.1[0.05,\ 3,\ -0.6]$$

$$= [0.1][-0.6]$$

$$\underline{K_2 = -0.06}.$$

$$l_2 = \cancel{hg}\ 0.1[(0.05)(-0.6) - 4(3)].$$

$$\underline{l_2 = -1.203}$$

$$K_3 = hf\left[x_0 + h/2,\ y_0 + K_2/2,\ z_0 + l_2/2\right].$$

$$= 0.1\left[0.05,\ 3 + \frac{-0.06}{2},\ 0 + \frac{-1.203}{2}\right]$$

$$= 0.1[0.05,\ 2.97,\ -0.6015]$$

$$= 0.1[-0.6015]$$

$K_3 = -0.06015$

$$l_3 = 0.1[(0.05)[-0.6015] - 4[2.97]]$$
$$= -1.1910$$

$$k_4 = hf[x_0+h, y_0+k_3, z_0+l_3]$$
$$= 0.1[0+0.1, 3+-0.0601, 0+-1.1910]$$
$$= 0.1[0.1, 2.9399, -1.1910]$$
$$= 0.1[-1.1910]$$
$$= -0.1191$$

$$l_4 = [(0.1)(-1.1910) - 4(2.9399)]$$
$$\boxed{l_4 = -\cancel{1}.\cancel{3}878}$$

$$y(0.1) = y_0 + 1/6[k_1 + 2k_2 + 2k_3 + k_4]$$
$$3 + 1/6[0 + 2(-0.06) + 2(-0.06015) + (-0.1191)]$$
$$\boxed{y(0.1) = 2.94}$$

$$y'(0.1) = y_0' + 1/6[l_1 + 2l_2 + 2l_3 + l_4]$$
$$= 0 + 1/6[-1.2 + 2\times -1.203 + 2\times -1.1910 + -\cancel{1}.\cancel{3}875]$$
$$\boxed{y'(0.1) = \cancel{-1.0626}}$$
$$-1.1962$$

Milne's method :-

$$y_4^p = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$
$$z_4^p = z_0 + 4h/3[2z_1' - z_2' + 2z_3']$$

$y_4^c = y_2 + h/3 [y_2' + 4y_3' + y_4']$

$z_4^c = z_2 + h/3 [z_2' + 4z_3' + z_4']$.

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$y_4^p = y_0 + \frac{4h}{3} [2z_1 - z_2 + 2z_3]$

$y_4^c = y_2 + h/3 \cdot [z_2 + 4z_3 + z_4]$.

1. compute $y(0.8)$ given that $y'' = 1 - 2yy'$

and x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

x	y	y' [z]	$z' = y''$
0	0	0 z_0	1
0.2	0.02	0.1996 z_1	0.9920
0.4	0.0795	0.3937 z_2	0.9374
0.6	0.1996	0.5689 z_3	0.7995
0.8	0.3045	0.7054	0.5705

$h = 0.2$

$y_4^p = y_0 + 4h/3 [2z_1 - z_2 + 2z_3]$

$= 0 + \frac{4 \times 0.2}{3} [2(0.1996) - (0.3937) + 2[0.5689]]$

$= 0.3048$

$z_4^p = z_0 + 4h/3 [2z_1' - z_2' + 2z_3']$

$= 0 + \frac{4(0.2)}{3} [2(0.9920) - 0.9374 + 2[0.7995]]$

$= 0.7054$

$y_4^c = y_2$

Milnes corrector formula is given by

$y_4^c = y_2 + h/3 [z_2 + 4z_3 + z_4]$

0.0795

$= \cancel{1} + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.7054\right]$

$= \cancel{0.2249}$

$= \underline{0.3045}$

$Z_4' = 1 - 2y_4^c Z_4^p$

$= 1 - 2(0.3045)(0.7054)$

$Z_4^c = Z_2 + \frac{h}{3}\left[Z_2' + 4Z_3' + Z_4'\right]$

$= 0.3937 + \frac{0.2}{3}\left[0.9374 + 4[0.7995] + (0.5705)\right]$

$= \underline{0.7074}$

$\therefore \boxed{y(0.8) = 0.3045}$

2. using milne's method obtain an approximate sol^n at $x = 0.4$ by the eq^n. $y'' + 3xy' - 6y$

$y'' + 3xy' - 6y = 0$

$y(0) = 1, \quad y'(0) = 0.1$

$y'(0.1) = 0.6955$

$y(0.1) = 1.3995$

$y'(0.2) = 1.258$

$y(0.2) = 1.138036$

$y'(0.3) = 1.873$

$y(0.3) = 1.29865$

$\underline{y'' = 6y - 3xy'}$

x	y	Z	Z'
0	1	0.1	6
0.1	1.3995	0.6958	8.1883
0.2	1.138036	1.258	6.0734
0.3	1.29865	1.873	6.1062
0.4	~~1.7945~~ 1.53310	3.10208	~~7.04450~~ 5.476104

$h = 0.1$

Source diginotes.in

$$y_4^p = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.6955) - (1.258) + 2(1.5873)\right]$$

$$\boxed{y_4^p = 1.5172}$$

$$z_4^p = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$= 0.1 + \frac{4\times 0.1}{3}\left[2[8.1883] - 6.0734 + 2[6.1062]\right]$$

$$\boxed{z_4^p = 3.10208}$$

$$y_4^c = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + \overset{z_4^p}{z_4}\right]$$

$$= 1.3995 + \frac{0.1}{3}\left[1.258 + 4(1.5873) + 3.10208\right]$$

$$\cancel{\boxed{y_4^c = 1.7945}} \qquad \boxed{y_4^c = 1.53310}$$

$$z_4^c = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$= 1.258 + \frac{0.1}{3}\left[6.0734 + 4[6.1062] + \cancel{7.04450}\ 5.476104\right]$$

$$=$$

$$\cancel{\boxed{z_4^c = 2.5094}}$$

$$\cancel{\boxed{y(0.4) = 1.7945}} \qquad y(0.4) = 1.53310$$

$$\boxed{z_4' = 2.4571}$$

3. Apply milne's method to solve $y'' = 1 + y'$ find y at $x = 0.4$ i.e. $y(0.4)$

given $y(0) = 1$ $\quad y'(0) = 1$

$y(0.1) = 1.1103 \quad y'(0.1) = 1.2103$

$y(0.2) = 1.2427 \quad y'(0.2) = 1.4427$

$y(0.3) = 1.399 \quad y'(0.3) = 1.699$

x	y	$z = y'$	$z' = y''$
0	1	1	2
0.1	1.1103	1.2103	1.3103
0.2	1.2427	1.4427	1.6427
0.3	1.349	1.699	1.999
0.4	1.5727	1.6634	2.6634

mistake

$$y_4^P = y_0 + \frac{4h}{3}\left[2z_1 - z_2 + 2z_3\right]$$

$$= 1 + \frac{4\times 0.1}{3}\left[2(1.2103) - 1.4427 + 2\times 1.699\right]$$

$$\boxed{y_4^P = 1.5834}$$

$$z_4^P = z_0 + \frac{4h}{3}\left[2z_1' - z_2' + 2z_3'\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(1.3103) - 1.6427 + 2\times 1.999\right]$$

$$\boxed{z_4^P = 1.6634}$$

$$y_4^C = y_2 + \frac{h}{3}\left[z_2 + 4z_3 + z_4\right]$$

$$= 1.2427 + \frac{0.1}{3}\left[1.4427 + 4[1.699] + 1.6634\right]$$

$$\boxed{y_4^C = 1.5727}$$

$$z_4^C = z_2 + \frac{h}{3}\left[z_2' + 4z_3' + z_4'\right]$$

$$= 1.4427 + \frac{0.1}{3}\left[1.6427 + 4[1.999] + 2.6634\right]$$

$$\boxed{z_4^C = 1.8527}$$

$$y(0.4) = 1.5727$$

Simultaneous D.E

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Note :- Module 1

Simultaneous differential equations By Runge - Kutta method of 4^{th} order

1. Solve $\frac{dy}{dx} = 1+zx$ and $\frac{dz}{dx} + xy = 0$; $y(0)=0$, $z(0)=0$ at $x=0.3$ by using R.K method.

Sol^n $y(0.3) = ?$ $z(0.3) = ?$

$h = 0.3$, $x_0 = 0$, $y_0 = 0$, $z_0 = 0$.

$$\frac{dy}{dx} = \boxed{f(x,y,z) = 1+zx}$$

$$\frac{dz}{dx} = \boxed{g(x,y,z) = -xy}$$

$$y(x_0+h) = y_0 + 1/6\,[k_1 + 2k_2 + 2k_3 + k_4] \rightarrow 1$$

$k_1 = hf[x_0, y_0, z_0]$

$0.3[1+0] = 0.3$

$l_1 = hg[x_0, y_0, z_0] = 0$

$k_2 = hf[x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}]$

$= 0.3[0.3/2, 0 + 0.3/2, 0]$

$= 0.3[0.15, 0.15, 0]$

$= 0.3[1+0] = \underline{\underline{0.3}}$

$l_2 = hg[x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}]$

$= 0.3[0.3/2, 0.15, 0]$

$= 0.3[-(0.15)(0.15)]$

$= -6.75 \times 10^{-3}$

$k_3 = hf\left[x_0 + h/2,\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right]$

$= 0.3\left[0.15,\ 0.15,\ -3.375 \times 10^{-3}\right]$

$= 0.3\left[1 + (-3.375 \times 10^{-3})(0.15)\right]$

$k_3 = 0.2998$

$l_3 = hg\left[x_0 + h/2,\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right]$

$0.3\left[0.15,\ 0.15,\ -3.375 \times 10^{-3}\right]$

$0.3\left[-(0.15)(0.15)\right]$

$= -6.75 \times 10^{-3}$

$k_4 = hf\left[x_0 + h,\ y_0 + k_3,\ z_0 + l_3\right]$

$= 0.3\left[0.3,\ 0.2998,\ -6.75 \times 10^{-3}\right]$

$= 0.3\left[1 + (-6.75 \times 10^{-3})(0.3)\right]$

$k_4 = 0.2993$

$l_4 = hg\left[x_0 + h,\ y_0 + k_3,\ z_0 + l_3\right]$

$= 0.3\left[-(0.3)(0.2998)\right]$

$\underline{l_4 = -0.0269}$

$y(0.3) = y_0 + 1/6\left[0.3 + 2 \times 0.3 + 2 \times 0.2998 + 0.2993\right]$

$y(0.3) = 0.2998$

$z(0.3) = z_0 + 1/6\left[l_1 + 2l_2 + 2l_3 + l_4\right]$

$= 0 + 1/6\left[0 + 2(-6.75 \times 10^{-3}) + 2 \times (-6.75 \times 10^{-3})\right.$

$\left. + -0.0269\right]$

$= -8.9833 \times 10^{-3}$

Picard's method :-

consider a D.E of the form

$dy/dx = f(x, y)$; $y(x_0) = y_0$

$$dy = f(x, y)\,dx$$

Integrating B.S

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\cdot dx$$

$$y\Big]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)\cdot dx$$

$$= y - y_0 = \int_{x_0}^{x_0} f(x, y)\,dx$$

$$\boxed{y = y_0 + \int_{x_0}^{x} f(x, y)\,dx}$$

First approximation

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\cdot dx$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

To use picards method to obtain 4th approximation for ~~to~~ the eqn $dy/dx = x + y$ with the initial condition $y(0) = 1$ hence find ~~y at x = 0.1, 0.2~~

y at $x = 0.1, 0.2$

$f(x, y) = x + y$

$x_0 = 0,\ y_0 = 1$

From picards first approximation is given by

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0) \cdot dx \;.$$

$$= 1 + \int_{0}^{x} (x+1) \cdot dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + x + x^2/2$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x \cdot y_1) \cdot dx \;.$$

$$= y_0 + \int_{0}^{x} \left[x + 1 + x + x^2/2\right] \cdot dx \;.$$

$$= 1 + \frac{x^2}{2} + x + x^2/2 + x^3/3 \times 2$$

$$\cancel{y_2 = 2 + 2x + x^2/2}$$

$$1 + \frac{2x^2}{2} + x + \frac{x^3}{6}$$

$$y_2 = 1 + x^2 + x + x^3/6$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x \cdot y_2) \cdot dx$$

$$= y_0 + \int_{0}^{x} \left[x + 1 + x^2 + x + x^3/6\right] \cdot dx$$

$$= 1 + \int_{0}^{x} \left[\frac{x^2}{2} + x + x^3/3 + x^2/2 + \frac{x^4}{24}\right]_0$$

$$= 1 + x + x^2 + x^3/3 + x^4/24$$

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3) \cdot dx$$

$$= 1 + \int_0^x \left[x + 1 + x + x^2 + x^3/3 + \frac{x^4}{24}\right] dx$$

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$$= \left[1 + x^2 + x + x^3/3 + x^4/12 + \frac{x^5}{5 \times 24}\right]$$

$$y_4 = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^5}{120}$$

y at $x = 0.1$

$= \underline{1.1106}$.

y at $x = 0.2 \quad = \underline{1.2428}$

Q. Use picards method to obtain 3^{rd} approximation to the solution of $dy/dx + y = e^x$ with the initial condition $y(0) = 1$, find $y(0.2)$

Ans: $f(x, y) = e^x - y$.

$x_0 = 0, \quad y_0 = 1$

first approx

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y)\, dx.$$

$$= 1 + \int_0^x (e^x - 1)\, dx.$$

$$= 1 + \left[e^x - x\right]_0^x.$$

$$= 1 + \left[e^x - 1\right] - \left[x - 0\right].$$

$$1 + e^x - 1 - x$$

$$= e^x - x.$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx.$$

$$1 + \int_0^x \left[\cancel{e^x} - \cancel{e^x} + x\right].$$

$$= 1 + \int_0^x [2e^x - x]\, dx$$

$$= 1 + 2[e^x - 1] - [x - 0]$$

$$= 1 + 2e^x - 2 - x$$

$$= 2e^x - 1 - x$$ //

3rd $y_3 = y_0 + \int_0^x f(x, y_2)\, dx$

$$= y_0 + \int_0^x x +$$

$$1 + \int_0^x x = 1 + \frac{x^2}{2}$$

3rd $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$

$$= 1 + \int_{x_0}^{x} \left[e^x - 1 - \frac{x^2}{2}\right] dx$$

$$= \cancel{1 + \int_{x_0}^{x} \left[\frac{x^2}{2} - x - \frac{x^3}{6}\right]_0^x dx}$$

$$\cancel{1 + \frac{x^2}{2} - x - \frac{x^3}{6}}$$

$$1 + \int_0^x \left[e^x - 1 - x^2/2\right] dx$$

$$= 1 + e^x - x - \frac{x^3}{6} - 1$$

$$y_3 = y_3 = e^x - x - \frac{x^3}{6}$$

$y(0.2) = 1.02$ //

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3. upto 3rd approx. to the soln $\frac{dy}{dx} = 1+xy$ with initial condition. $y(0) = 2$, find $y(0.1)$, $y(0.2)$ & $y(0.3)$

$f(x, y) = 1+xy$.

$x = 0, \quad y_0 = 2$.

first approx

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

$$2 + \int_0^x (1+xy)\, dx$$

$$= 2 + \int_0^x [1+2x]\, dx$$

$$= 2 + [x + x^2]$$

$$= 2 + x + x^2$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 2 + \int_0^x 1 + x[2+x+x^2]\, dx$$

$$2 + \int_0^x (1 + 2x + x^2 + x^3)\, dx$$

$$= 2 + x + x^2 + x^3/3 + x^4/4$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$= 2 + \int_{x_0}^{x} 1 + x\left[2 + x + x^2 + x^3/3 + x^4/4\right] dx$$

$$= 2 + \int_0^x \left[1 + 2x + x^2 + x^3 + x^4/3 + x^5/4\right] dx$$

$y_3 = 2 + x + x^2 + x^3/3 + x^4/4 + \frac{x^5}{15} + \frac{x^6}{24}$

$y(0.1) = 2.1103$

$y(0.2) = 2.24$

$y(0.3) = 2.4012$

simultaneous differential eq^n by picards method

consider simultaneous differential eq^n of the form

$\frac{dy}{dx} = f(x,y,z)$ & $\frac{dz}{dx} = g(x,y,z)$

$y(x_0) = y_0$; $z(x_0) = z_0$

1st

$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0) \cdot dx$

$z_1 = z_0 + \int_{x_0}^{x} g(x, y_0, z_0) \cdot dx$

2nd

$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1) \cdot dx$

$z_2 = z_0 + \int_{x_0}^{x} g(x, y_1, z_1) \cdot dx$

$\vdots$

$y_n = y_0 + \int_{x_0}^{x} f(x, y_n, z_n) \cdot dx$

1. use picards method to solve the D.E

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$\frac{dy}{dx} = 1 + zx$, $\frac{dz}{dx} + xy = 0$.

$y(0) = 0$, $z(0) = 0$ at $x = 0.3$.

carry out 2 approximations .

$f(x, y, z) = 1 + zx$. $x_0 = 0$, $y_0 = z_0 = 0$.

$g(x, y, z) = -xy$.

$y_1 = y_0 + \int_0^x f(x, y, z) \cdot dx$.

$y_1 = 0 + \int_0^x (1 + zx) \cdot dx = 1 + 0 . = x$.

$z_1 = 0 + \int_0^x [-x y_0] \cdot dx$

$z_1 = 0$

$y_2 = y_0 + \int_0^x f(x, y_1, z_1)$.

$= 0 + \int_0^x (1 + x z_1) \cdot dx$

$= 0 + x = x$

$z_2 = z_0 + \int_0^x g(x, y_1, z_1) \cdot dx$

$= 0 + \int_0^x [-x^2] \cdot dx$

$z_2 = -x^3/3$

$\frac{dy}{dx} = -xy$
$= -x(x)$
$= -x^2$

$y(0.3) = 0.3$

$z(0.3) = -0.009$

2. use picards method to find $y(0.1)$ and $z(0.1)$ given that $dy/dx = x + z$. $\frac{dz}{dx} = x - y^2$.
$y(0) = 2$, $z(0) = 1$ upto 3rd approx

Sol

$f(x, y, z) = x + z$.

$g(x, y, z) = x - y^2$.

$x_0 = 0, \quad y_0 = 2, \quad z_0 = 1$.

* $y_1 = y_0 + \int_0^x f(x, y_0, z_0)$.

$= 2 + \int_0^x (x + 1) \cdot dx$.

$= \left[\frac{x^2}{2} + x\right]_0^x$

$2 + x^2/2 + x$.

$z_1 = z_0 + \int_0^x (x, y_0, z_0) \cdot dx$.

$= 1 + \int_0^x (x - y^2) \cdot dx$.

$= 1 + \int_0^x (x - 4) \cdot dx$.

$= 1 + \frac{x^2}{2} - 4x$.

$= 1 + x^2/2 - 4x$.

* $y_2 = y_0 + \int_0^x f(x, y_1, z_1)$

$= 2 + \int_0^x (x + 1 + x^2/2 - 4x) \cdot dx$.

$= 2 + \left[-\frac{3x^2}{2} + x + \frac{x^3}{6}\right]_0^x$.

$y_2 = 2 - \frac{3x^2}{2} + x + x^3/6$.

$$z_2 = z_0 + \int^x f(x, y_1, z_1)$$

$$= 1 + \int_0^x x - \left[2 + x^2/2 + x\right]^2 . dx$$

$$= 1 + \int_0^x x - \left[\frac{4 + x^2}{2} + x\right]^2 . dx$$

$$= \left[1 + \frac{x^2}{2} - \frac{\left[\frac{4 + x^2 + 2x}{2}\right]^3}{3}\right]_0^x$$

$$= \frac{2 + x^2}{2} - \frac{\frac{4 + x^2 + 2x}{2}}{3}$$

$$= \frac{3(2 + x^2) - \cancel{4 + x^2 + 2x}}{\frac{2}{3}}$$

$$= \frac{2 + x^2}{2} - \frac{12 + 3x^2 + 6x}{2}$$

$$z_2 = z_0 + \int_0^x f(x, y_1, z_1)$$

$$= 1 + \int_0^x x - \left[2 + x + \frac{x^2}{2}\right]\left[2 + x + x^2/2\right]$$

$$= 1 + \int_0^x x - \left[4 + 2x + x^2 + 2x + x^2 + x^3/2 + x^2 + x^3/2 + x^4/4\right]$$

$$1 + x - \left[4 + 4x + 3x^2 + x^3 + x^4/4\right]$$

$$1 + x - 4 - 4x - 3x^2 - x^3 - x^4/4$$

$$z_2 = 1 - 4x - \frac{3x^2}{2} - \frac{3x^3}{3} - \frac{x^4}{4} - x^5/20$$

Special function :-

1) Bessel's function

A D.E of the form.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

$\therefore$ bessel fn is given by ~~$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{x}{2}$~~

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

Note :- ① Gamma fn can be defined only for positive real no.

② $n! = \Gamma(n+1)$

③ $\Gamma(n) = (n-1)\Gamma(n-1)$

$n! = \Gamma(n+1)$

Properties of Bessel's function.

1. $J_{-n}(x) = (-1)^n J_n(x)$

By the defination of bessels fn

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)\, r!}$$

consider $\Gamma(-n+r+1)$

$$\Rightarrow \Gamma[r-(n-1)]$$

This fn is valid for $r \geq n$ ~~to~~

$$= \sum_{r=n}^{\infty} (-1)^r \left(x/2\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)\, r!} \quad \text{①}$$

Let $r - n = s \Rightarrow r = (n+s)$

for $r = n$, if $s = 0$.

$\therefore$ eq^n (1) $\Rightarrow$

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{n+s} \left(\frac{x}{2}\right)^{-n+2(n+s)} \frac{1}{\Gamma(-n+n+s+1)(n+s)!}$$

$$= \sum_{s=0}^{\infty} (-1)^n (-1)^s \left(\frac{x}{2}\right)^{-n+2n+2s} \frac{1}{\Gamma(s+1)\,(n+s)!}$$

$$\cancel{= \sum_{s=0}^{\infty} (-1)^n -1}$$

$$\left\{ \begin{array}{l} n! = \Gamma(n+1) \\ (n+s)! = \Gamma(n+s+1) \\ s! = \Gamma(s+1) \end{array} \right.$$

$$= (-1)^n \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \frac{1}{s!\,\Gamma(n+s+1)}$$

put $s = r$.

$$\therefore \boxed{J_{-n}(x) = (-1)^n J_n(x).}$$

2. $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$

Proof

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{-x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$(-1)^r(-1)^{2r} = (-1)^{3r}$

$$= \sum_{r=0}^{\infty} (-1)^r (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} \left[(-1)^3\right]^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_n(-x) = (-1)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$J_n(-x) = (-1)^n J_n(x) \longrightarrow$ ①

From the property ① we've.

$J_{-n}(x) = (-1)^n J_n(x)$.

Now eqⁿ ① becomes

$J_n(-x) = J_{-n}(x) = (-1)^n J_n(x)$

<u>Series solⁿ of Bessel differential eqⁿ leading to Bessel function</u>

<u>note</u>:- $n! = \Gamma(n+1)$

$\Gamma(n) = (n-1)\Gamma(n-1)$.

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n+2) = (n+1)\Gamma(n+1)$.

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$ $\quad \Gamma(7/2) = \frac{15}{8}\sqrt{\pi}$

$\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$

$\Gamma(1) = 1$

1. Series soⁿ

① $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Proof

Bessel D.E is given by.

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \longrightarrow (1)$$

Let us assume that the solⁿ of eqⁿ is of the form

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \longrightarrow (2)$$

Differentiating eqⁿ (2) w.r.t x twice.

$$\Rightarrow y' = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$\Rightarrow y'' = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

now eqⁿ (1) becomes

$$= \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} \cdot x^2 + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} \cdot x$$

$$+ \sum_{r=0}^{\infty} a_r x^{k+r} \cdot x^2 - n^2 \sum_{r=0}^{\infty} a_r x^{k+r} = 0.$$

$$\therefore \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r} +$$

$$\sum_{r=0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{k+r}.$$

$$= \sum_{r=0}^{\infty} a_r x^{k+r} \left[(k+r)(k+r-1) + (k+r) + x^2 - n^2 \right]$$

$$= \sum_{r=0}^{\infty} a_r x^{k+r} \left[k^2 + kr - k + kr + r^2 - r + k + r + x^2 - n^2 \right]$$

$$= \sum_{r=0}^{\infty} a_r x^{k+r} \left[(k+r)(k+r-\cancel{1}+\cancel{1}) - n^2 \right] + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$= \sum_{r=0}^{\infty} a_r x^{k+r}\left[(k+r)^2-n^2\right] + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0.$$

$$\left\{ \begin{aligned} & a_0 x^k[k^2-n^2] + a_1 x^{k+1}[(k+1)^2-n^2] + \cdots\cdots \\ & + [a_0 x^{k+2} + a_1 x^{k+3} + \cdots\cdots] = 0 \end{aligned} \right\}$$

Equate the co-efficient of lowest power of x to zero

$x^k : \; a_0[k^2-n^2] = 0$

$\Rightarrow a_0 \neq 0 \;\; \& \;\; k^2-n^2=0 \quad . \quad k = \pm n.$

The coefficient of x^{k+1}

$x^{k+1} : \; a_1[(k+1)^2-n^2] = 0$

$\Rightarrow a_1 = 0 \;\; \& \;\; (k+1)^2 \neq n^2 \quad \because$ we accepted already $k = \pm n$

Co-efficient of x^{k+r}.

$x^{k+r} ; \; a_r[(k+r)^2-n^2] + a_{r-2} = 0.$

$$\boxed{a_r = \frac{-a_{r-2}}{(k+r)^2-n^2}} \rightarrow (3) \qquad r \geq 2 .$$

eq^{-n} (3) is known as Recurrence Relation.

case 1 : Assume $k=n$. in eq^{-n} (3)

$$a_r = \frac{-a_{r-2}}{(n+r)^2-n^2}$$

$n^2 + r^2 + 2nr$, r^2

$\therefore r \geq 1^2$

$$a_r = \frac{-a_{r-2}}{2nr+r^2} \rightarrow (4) \qquad r \geq 2 .$$

Put $r = 2, 3, 4 \ldots$ in eq^{-n} (4)

$$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}$$

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$$a_3 = \frac{-a_1}{6n+9} = 0 \quad \because a_1 = 0$$

$$a_4 = \frac{-a_2}{8n+16} = \frac{-1}{8(n+2)}\left[\frac{-a_0}{4(n+1)}\right]$$

$$= \frac{a_0}{32(n+1)(n+2)} = \frac{a_0}{2^5(n+1)(n+2)}$$

$a_5 = 0$ and so on - - - -

Substitute the values a_0, a_2, a_3 - - - in expanded form of eq^n ②

$$= y_1 = x^k\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right]$$

$$= y_1 = x^n\left[a_0 + 0 - \frac{a_0 x^2}{2^2(n+1)} + 0 + \frac{a_0 x^4}{2^5(n+1)(n+2)} + \cdots\right]$$

$$= y_1 = x^n a_0\left[1 - \frac{1}{2^2(n+1)}x^2 + \frac{1}{2^5(n+1)(n+2)}x^4 + \cdots\right].$$

choose $a_0 = \dfrac{1}{2^n\Gamma(n+1)}$

$$y_1 = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \left(\frac{x}{2}\right)^2\frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4\frac{1}{(n+1)(n+2)2}\right]$$

$$= \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2\frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4\frac{1}{(n+1)(n+2)\Gamma(n+1)2}\cdots\right]$$

$$= \left(\frac{x}{2}\right)^n\left[\frac{(-1)^0}{\Gamma(n+1)0!}\left(\frac{x}{2}\right)^0 + \left(\frac{x}{2}\right)^2\frac{(-1)^1}{(n+1)\Gamma(n+1)1!} + \left(\frac{x}{2}\right)^4\frac{(-1)^2}{(n+1)(n+2)\Gamma(n+1)2!} + \cdots\right]$$

$$y_1 = \left(\frac{x}{2}\right)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma(n+r+1)r!}\left(\frac{x}{2}\right)^{n+2r} \longrightarrow ⑤$$

Eq^{-n} (5) is known as Bessel fn^{-n} of <u>first kind</u> of order n. denoted by $J_n(x)$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\, r!} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$\therefore$ complete sol^{-n} is given by

$$\boxed{y = A J_n(x) + B J_{-n}(x)}$$

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1. Prove that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

$$\left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right.$$
$$\left. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right]$$

from the defination of bessels fn

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r}.$$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(\frac{1}{2}+r+1)\, r!} \left(\frac{x}{2}\right)^{\frac{1}{2} + 2r}$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(3/2+r)\, r!} \left(\frac{x}{2}\right)^{2r}$$

$$= \sqrt{x/2} \cdot \left[\frac{1}{\Gamma(3/2)} + (-1)\left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(3/2+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(3/2+2)\,2} - \cdots \right]$$

$$= \sqrt{x/2} \left[\frac{1}{\Gamma(3/2)} - \frac{x^2}{4} \frac{1}{\Gamma(5/2)} + \frac{x^4}{16} \frac{1}{\Gamma(7/2)\,2} - \cdots \right]$$

$$= \sqrt{x/2}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{8}{15\sqrt{\pi}\,2}\right]$$

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$$= \sqrt{x/2}\;\frac{2}{\sqrt{\pi}}\left[1 - \frac{2x^2}{12} + \frac{4x^4}{16\times 30}\right]$$

$$\therefore\; J_{1/2}(x) = \sqrt{x/2}\cdot\frac{2}{\sqrt{\pi}}\left[1 - \frac{\cancel{2}x^2}{\cancel{12}\,6} + \frac{\cancel{4}x^4}{120} - \cdots\right]$$

~~x by x~~
x & ÷ by
x

$$= \sqrt{\frac{2x}{\pi}}\,\frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right].$$

$$\boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\;\sin x}$$

2. $J_{-1/2}(x) = \sqrt{2/\pi x}\cdot\cos x$.

from the defination of bessels fn.

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma(n+r+1)\,r!}\left(\frac{x}{2}\right)^{n+2r}.$$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma(-1/2+r+1)\,r!}\left(\frac{x}{2}\right)^{-1/2+2r}.$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2}\sum_{r=0}^{\infty}\frac{(-1)^r}{\Gamma(1/2+r)\,r!}\left(\frac{x}{2}\right)^{2r}.$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\Gamma(1/2)} + (-1)\left(\frac{x}{2}\right)^2\frac{1}{\Gamma(3/2)\,1!}\right.$$

$$\left. + \frac{(1)}{\Gamma(5/2)\,2!}\left(\frac{x}{2}\right)^4\right].$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4_{2}}\cdot\frac{2}{\sqrt{\pi}} + \frac{x^4}{16_{4}}\left[\frac{4}{3\sqrt{\pi}}\times\frac{\cancel{2}}{2}\right]\right]$$

$$= \sqrt{\frac{2}{x\pi}} \left[1 - \frac{x^2}{2} + \frac{x^4}{4!} \right]$$

$$\therefore \boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x}$$

Orthogonal property of bessel fn

If α and β are the two distinct root of the eqn $J_n(x) = 0$, then

$$\int_0^1 x \underbrace{J_n(\alpha x)}_{u} \underbrace{J_n(\beta x)}_{v} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2}[J_{n+1}(\alpha)]^2 & \text{if } \alpha = \beta \end{cases}$$

proof

Let us consider $J(n)(\lambda x)$ is the solⁿ of the eqn

$$x^2 y'' + xy' + (\lambda^2 x^2 - n^2) y = 0 \longrightarrow (1)$$

If $u = J_n(\alpha x)$ & $v = J(n)(\beta x)$.

The associated D.E is are given by

$$\cancel{x^2 y'' - xy' + (\alpha^2 x^2 - n^2) u = 0} \longrightarrow (2)$$
$$\cancel{x^2 y'' - xy' + (\beta^2 x^2 - n^2) v = 0} \longrightarrow (3)$$

$$\cancel{x^2 y''} \quad x^2 u'' + xu' + (\alpha^2 x^2 - n^2) u = 0 \longrightarrow (2)$$
$$x^2 v'' + xv' + (\beta^2 x^2 - n^2) v = 0 \longrightarrow (3)$$

Mly eqn (2) by v/x eqn by u/x.

$$(2) \Rightarrow x v u'' + v u' + \alpha^2 x u v - n^2 \frac{uv}{x} = 0$$

$$(3) \Rightarrow x u v'' + u v' + \beta^2 x u v - n^2 \frac{uv}{x} = 0$$

solving (subtracting) 2 eqn's

$$x[vu'' - uv''] + [vu' - uv'] + [\alpha^2 - \beta^2] xuv = 0$$

$$x[vu'' - uv''] + [uv' + vu'] = [\beta^2 - \alpha^2] xuv$$

$$\frac{d}{dx}[x(-uv' + vu')] = (\beta^2 - \alpha^2) x uv \longrightarrow (4)$$

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differentiating eq^{-n} (4) w.r.t x. b/w the limits 0 to 1.

$$= \left[x(-uv'+vu')\right]_{x=0}^{1} = (\beta^2-\alpha^2)\int_0^1 xuv\,dx.$$

$$= \left[x[-uv'+vu']\right]_{x=0} - 0 = (\beta^2-\alpha^2)\int_0^1 xuv\,dx.$$

$$= \int_0^1 xuv\,dx = \cancel{\frac{1}{\alpha^2-\beta^2}}\ \frac{1}{\beta^2-\alpha^2}\left[x(-uv'+vu')\right]_{x=1}$$

$\because$ we've considered $u = J_n(\alpha x)$ & $v = J_n(\beta x)$.

$u' = \alpha J_n'(\alpha x)$ $\quad v' = J_n'(\beta x)\beta$

$$= \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{\beta^2-\alpha^2}\left[-J_n(\alpha)\beta\, J_n'(\beta) + J_n(\beta)\alpha\, J_n'(\alpha)\right] \longrightarrow (6)$$

if α and β are the two distict root $J_n(x) = 0$

$\Rightarrow J_n(\alpha) = 0$ & $J_n(\beta) = 0$.

case i

$\alpha \neq \beta$

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0 \quad \text{if } \alpha \neq \beta.$$

case ii

$\alpha = \beta$.

$$\boxed{\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{2}\left[J_n'(\alpha)\right]^2 = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2}$$

Suppose $\alpha = \beta$

eq^{-n} (6) reduces to $0/0$ form.

we've to evaluate by taking limits on R.S as $\beta \rightarrow \alpha$ keeping α constant $\Rightarrow J_n(\alpha) = 0$

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By applying L-hospital rule.

$$\lim_{\beta \to \alpha} \int_0^1 x J_n(\alpha x) J_n(\beta x) \, dx = \lim_{\beta \to \alpha} \frac{1}{\beta^2 - \alpha^2} \left[\alpha J_n'(\alpha) J_n(\beta) \right]$$

$$= \lim_{\beta \to \alpha} \frac{\alpha J_n'(\alpha) J_n(\beta)}{2\beta}$$

$$\therefore \int_0^{1} x J_n(\alpha x) J_n(\alpha x) dx = \frac{\cancel{\alpha} J_n'(\alpha) J_n'(\alpha)}{2 \cancel{\alpha}}$$

$$= \frac{1}{2} [J_n'(\alpha)]^2 \longrightarrow (7)$$

J.

We know that recurrance relation is given by

$$\boxed{J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)}$$

Replace x as α.

$$J_n'(\alpha) = \frac{n}{\alpha} \cancel{J_n(\alpha)}^{\to 0} - J_{n+1}(\alpha)$$

$\because \alpha$ is constant $\alpha \to 0$.

$$\boxed{J_n'(\alpha) = -J_{n+1}(\alpha)}$$

$\therefore$ eq^{n} (7) becomes.

$$J_n'(\alpha) = \frac{1}{2} [J_{n+1}(\alpha)]^2$$

$$\int_0 x J_n(\alpha x) J_n(\alpha x) dx = \frac{1}{2} [J_n'(\alpha)]^2 = \frac{1}{2} [J_{n+1}(\alpha)]^2$$

It $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$x^2 y'' + xy' + (x^2 - n^2)y$

End

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Series Solution of Legendre diff eq^n

consider a D.E

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{—— (1)}$$

eq^n (1) is Lengendre's diff eq^n.

Let us assume that the sol^n of eq^n (1) is of the form

$$y = \sum_{r=0}^{\infty} a_r x^r \longrightarrow (2)$$

Differenciating eq^n (2) w.r.t x.

$$y' = \sum_{r=0}^{\infty} a_r r x^{r-1}$$

$$y'' = \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2}$$

∴ eq^n (1) becomes.

$$= (1-x^2)\sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - 2x \cdot \sum_{r=0}^{\infty} a_r r x^{r-1} + n(n+1)\cdot\sum_{r=0}^{\infty} a_r x^r = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} \cdot x^2 - 2\sum_{r=0}^{\infty} a_r r x^{r-1} \cdot x + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1) x^r - 2\sum_{r=0}^{\infty} a_r r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0,$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r x^r\left[r(r-1) + 2r - n(n+1)\right] = 0$$

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$$= \sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r x^r [r^2+r-n(n+1)] = 0$$

Equate the co-efficient of lowest power of x to 0.

$$[a_0(0)(-1)x^{-2} + a_1(1)(0)x^{-1} + \dots\dots]$$

x^{-2}: $a_0(0)(-1) = 0 \implies a_0 \neq 0$.

x^{-1}: $a_1(1)(0) = 0 \implies a_1 \neq 0$.

coefficient of x^r.

Replace r as $r+2$.

x^r: $a_{r+2}(r+2)(r+1) - a_r[r(r+1) - n(n+1)] = 0$

$$\implies a_{r+2} = \frac{[r(r+1) - n(n+1)]}{(r+1)(r+2)} a_r \rightarrow (3)$$

eq^n (3) is known as recurrence relation

Sub $r = 0, 1, 2 \dots$ in eq^n (3).

$r=0$

$$a_2 = \left[\frac{-n(n+1)}{(1)(2)}\right] a_0$$

$$a_2 = \left[\frac{-n(n+1)}{2}\right] a_0$$

$r=1$

$$a_3 = \left[\frac{2-n(n+1)}{6}\right] a_1 .$$

$$a_3 = \left[\frac{-[(n-1)(n+2)]}{3!}\right] a_1 .$$

$r=2$

$$a_4 = \left[\frac{6-n(n+1)}{12}\right] . a_2 .$$

$$a_4 = \left[\frac{6-n(n+1)}{12}\right]\left[\frac{-n(n+1)}{2}\right] a_0 .$$

$$= +\frac{[n^2+n-6]}{12} \cdot \frac{n(n+1)}{2} a_0$$

$$a_4 = \frac{n(n+1)(n+3)(n-2)}{4!} \cdot a_0$$

$$a_5 = \left[\frac{12 - n(n+1)}{20}\right] \times \left[-\frac{[(n-1)(n+2)]}{3!}\right] a_1$$

$$\left[+\frac{[n^2+n-12] + [(n-1)(n+2)]}{120}\right] a_1.$$

$n^2 + 4n - 3n - 12$

$n(n+4)$

$$a_5 = \left[\frac{(n+4)(n-3)(n-1)(n+2)}{5!}\right] a_1$$

substitute These values in the expanded form of eq^{-n} (2)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + x a_1 - \frac{n(n+1)}{2!} a_0 x^2 - \left[\frac{(n-1)(n+2)}{3!}\right] a_1 x^3$$

$$+ \left[\frac{n(n+1)(n+3)(n-2)}{4!} a_0\right] x^4 +$$

$$\left[\frac{(n+4)(n-3)(n-1)(n+2)}{5!}\right] a_1 x^5 + \cdots$$

$$y = a_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+3)(n-2)}{4!} x^4 + \cdots\right]$$

$$+ a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n+4)(n-3)(n-1)(n+2)}{5!} x^5 + \cdots\right]$$

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Legendre Polynomials :-

$y = a_0 u(x) + a_1 v(x)$

if 'n' is positive even integer $a_0 u(x)$ reduce to polynomial of degree n.

if 'n' is +ve or. integer $a_1 v(x)$ reduce polynomial of degree 'n'

consider Legendre fn of second kind.

$$y = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \dots F(x) \rightarrow (1)$$

where

$$F(x) = \begin{cases} a_0 & \text{if } n \text{ is even} \\ a_1 & \text{if } n \text{ is odd} \end{cases}$$

we know that recurrence of legendre given by

$$a_{r+2} = -\frac{[n(n+1) - r(r+1)]}{(r+1)(r+2)} a_r \rightarrow (2)$$

using eq^n (2) we've to find the values of $a_{n-2}, a_{n-4} \dots$

Replace $r = n-2$ in eq^n (2)

$$a_{n-2+2} = -\left[\frac{[n(n+1) - (n-2)(n-2+1)]}{(n-2+1)(n-2+2)}\right] a_{n-2}$$

$$a_n = -\left[\frac{[n(n+1) - (n-2)(n-1)]}{n(n-1)}\right] a_{n-2}$$

$$\therefore a_n = -\left[\frac{[n^2 + n - n^2 + 3n - 2]}{n(n-1)}\right] a_{n-2}$$

$$a_n = \frac{-2[2n-1]}{n(n-1)} a_{n-2}.$$

$$\boxed{a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n}.$$

replace $r = n-4$ in eq^{-n} (2)

$$a_{n-2} = -\left[\frac{[n(n+1) - (n-4)(n-3)]}{(n-3)(n-2)}\right] a_{n-4}$$

$$= -\left[\frac{\cancel{n^2}+n-\cancel{n^2}+3n+4n-12}{(n-3)(n-2)}\right] a_{n-4}$$

$$a_{n-2} = \frac{-4[2n-3]}{(n-3)(n-2)} a_{n-4}.$$

$$\therefore \boxed{a_{n-4} = -\frac{(n-3)(n-2)}{4[2n-3]} \cdot a_{n-2}}$$

$$a_{n-4} = \frac{(n-3)(n-2)}{4[2n-3]} \cdot \frac{n(n-1)}{2[2n-1]} \cdot a_n.$$

now eq^{-n} (1) becomes.

$$y = a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2} + \frac{n(n-1)(n-2)(n-3) \cdot a_n}{8(2n-3)(2n-1)} x^{n-4}$$

where $G(x) = \begin{cases} a_0/a_n & \text{if } n \text{ even} \\ a_1/a_n & \text{if } n \text{ odd} \end{cases}$

$$= a_n\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-3)(2n-1)} x^{n-4} + \cdots\right]$$

constant a_n is so choosen such that $y = f(x)$ becomes 1 when $x = 1$,

The polynomial obtained are called Legendre polynomial denoted by $P_n(x)$

take

$$a_n = \frac{1, 3, 5 \ldots (2n-1)}{n!}$$

The above eq^n becomes

$$P_n(x) = \frac{1.3.5 \ldots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} + \ldots + G(x)\right] \rightarrow 3$$

sub $n=0$ in eq 3

$n=0$, $P_0(x) = 1$.

$n=1$, $P_1(x) = x$.

$n=2$, $P_2(x) = \cancel{x^2}\ \frac{1.3}{2}\left[x^2 - \frac{2(1)}{2\times 3} x\right]$

$= \frac{3}{2}\left[x^2 - \frac{1}{\cancel{6}3}\right]$

$= \frac{1}{2}\left[3x^2 - 1\right]$

$n=3$, $P_3(x) = \frac{15}{3!}\left[x^3 - \frac{3(2)}{2(6-1)} x^1\right]$

$= \frac{15}{6}\left[x^3 - \frac{6 \times 3x}{\cancel{10}\,5}\right]$

$= \frac{5}{2}\left[\frac{5x^3 - 3x}{5}\right]$

$P_3(x) = \frac{1}{2}\left[5x^3 - 3x\right].$

$$P_4(x) = \frac{1 \cdot 3 \cdot 5 \cdot 7}{4\times 3\times 2\times 1}\left[x^4 - \frac{\cancel{4}^{2}(3)}{\cancel{2}(8-1)} x^2 + \frac{\cancel{4}(3)(\cancel{2})(1)}{\cancel{8}(7)(5)}\right]$$

$$= \frac{35}{8}\left[x^4 - \frac{6}{7} x^2 + \frac{3}{35}\right]$$

$$p_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$$

It can be easily seen that all these expressions give 1 at x from the defination of Lengendre polynomial.

Rodrigue's formula

Derive Rodrigue's Formula for the legendre polynomial $P_n(x)$ in the form $\frac{1}{2^n n!}\frac{d^n}{dx^n}\{[x^2-1]^n\}$

Proof

Let $u = [x^2-1]^n \longrightarrow (1)$

Legendre diff eq^n is given by

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \longrightarrow (2)$$

diff eq^n (1) w.r.t x.

$$u_1 = n[x^2-1]^{n-1} 2x$$

$$u_1 = 2nx \frac{(x^2-1)^n}{(x^2-1)}$$

$$= (x^2-1)u_1 = 2nxu$$

diff above eq^n w.r.t x.

$$= (x^2-1)u_2 + u_1 2x = 2nu + 2nxu_1$$

$$= (x^2-1)u_2 + 2[1-n]xu_1 - 2nu = 0 \longrightarrow (3)$$

= Apply Leibnitz theorem to eq^n (3)

$$[uv]_n = uv_n + nu_1v_{n-1} + \frac{n(n-1)}{2!}u_2v_{n-2} + \dots u_nv$$

$$= \left[(x^2-1)u_{n+2} + n2xu_{n+1} + \frac{n(n-1)}{2}2u_n\right] -$$

$$2(n-1)\{xu_{n+1} + n(1)u_n\} - 2nu_n = 0.$$

Source diginotes.in

$= (x^2-1)u_{n+2} + [\cancel{n2x} - 2(n-1)x]u_{n+1}$

$+ [n(n-1) - 2n(n-1) - 2n]u_n = 0.$

$= (x^2-1)u_{n+2} + [+2x]u_{n+1} + [-n^2-n]u_n = 0$

$(x^2-1)u_{n+2} + 2xu_{n+1} - n(n+1)u_n = 0.$

$(1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0 \rightarrow (4)$

Compare eq^n (2) and (4) we conclude that U_n is a sol^n of legendres D.E also $P_n(x)$ satisfies the legandre D.E it also satisfied (4)

Hence U_n must be same as $P_n(x)$ but for some constant k written as $P_n(x) = kU_n \rightarrow (5)$

$$P_n(x) = k[(x^2-1)^n]_n.$$

$$= k[\underset{u}{(x-1)^n}\underset{v}{(x+1)^n}]_n.$$

$$= k[(x-1)^n\{(x+1)^n\}_n + n.n(x-1)^{n-1} \times \{(x+1)^n\}_{n-1} + \cdots$$

$$P_n(x) = k[(x-1)^n\{(x+1)^n\}_n + n^2(x-1)^{n-1}\{(x+1)^n\}_{n-1} + \cdots + n!(x+1)^n\}.$$

we need to find k by replacing $(x=1)$ in above eq^n.

$P_n(1) = kn!2^n$, $P_n(1) = 1$ from def^n LDE

$1 = kn!2^n$

$z = (x-1)^n$

$z_1 = n(x-1)^{n-1}$

$z_2 = n(n-1)(x-1)^{n-2}$

$\vdots$

$z_n = n(n-1)(n-2)\cdots 1(x-1)^{n-n} = n!$

$K = \frac{1}{2^n n!}$

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$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$ or

$\frac{1}{2^n n!}\left\{(x^2-1)^n\right\}_n$

Problems on Legendres polynomials.

$P_0(x) = 1$

$P_1(x) = x$.

$P_2(x) = 1/2\,[3x^2-1]$

$P_3(x) = 1/2\,[5x^3-3x]$

$P_4(x) = 1/8\,[35x^4-30x^2+3]$.

$2P_2(x) = 3x^2 - 1$

$P_0\ \frac{1}{3} + \frac{2P_2}{3}\ x \quad x^2$

$\frac{3x}{5} + \frac{2P_3(x)}{5}$,

$\cancel{P_4(x)}$

$8P_4(x) = 35x^4 - 30\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] + 3P_0(x)$.

$8P_4(x) = 35x^4 - 20P_2(x) - 7P_0(x)$.

$\Rightarrow x^4 = \frac{8}{35}P_4(x) + \frac{20}{35}P_2(x) + \frac{7}{35}P_0(x)$

$1 = P_0(x)$

$x = P_1(x)$

$x^2 = 2/3\,P_2(x) + 1/3\,P_0(x)$

$x^3 = 2/5\,P_3(x) + 3/5\,P_1(x)$.

$x^4 = 8/35\,P_4(x) + 4/7\,P_2(x) + 1/5\,P_0(x)$.

$\frac{8P_4(x)}{35} - \frac{3}{35} + \frac{30x^2}{35}$

1. Express $x^4+3x^3-x^2+5x-2$ in terms of Legendres polynomial.

$= x^4+3x^3-x^2+5x-2$.

$= \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x) + \frac{6}{5} P_3(x) + \frac{9}{5} P_1(x)$
$\quad - \frac{2}{3} P_2(x) - \frac{1}{3} P_0(x) + 5P_1(x) - 2P_0(x)$

$= \frac{8}{35} P_4(x) + \left(\frac{12-14}{21}\right) P_2(x) + \frac{6}{5} P_3(x) + \left(\frac{9+25}{5}\right) P_1(x)$

$+\left(\frac{3-5-30}{15}\right) P_0(x)$

$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \frac{32}{15} P_0(x)$

2. Express $2x^3-x^2-3x+2$.

$= \frac{4}{5} P_3(x) + \frac{6}{5} P_1(x) - \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x) - 3P_1(x)$
$\quad + 2P_0(x)$.

$= \frac{4}{5} P_3(x) - \frac{2}{3} P_2(x) + \left(\frac{6-15}{5}\right) P_1(x) - \frac{1}{3} P_0(x) + 2P_0(x)$

$= \frac{4}{5} P_3(x) - \frac{2}{3} P_2(x) - \frac{9}{5} P_1(x) + \frac{5}{3} P_0(x)$ //.

3. S.T.
$x^4-3x^2+x = \frac{8}{35} P_4(x) - \frac{10}{7} P_2(x) + P_1(x) - \frac{4}{5} P_0(x)$

$= \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x) - \frac{6}{3} P_2(x) + \frac{3}{3} P_0(x) + P_1(x)$.

$= \frac{8}{35} P_4(x) - \left(\frac{4-6}{7}\right) P_2(x) + P_1(x) - \left(\frac{1+3}{5}\right) P_2$

$= \frac{8}{35} P_4(x) - \frac{10}{7} P_2(x) + P_1(x) - \frac{4}{5} P_0(x)$

$\frac{24-1}{3}$ $\quad 1/3 - 8 \quad \frac{1-24}{3} \quad \cancel{8}\ 4/8 \quad \frac{24-1}{3} = \frac{23}{3}$

4. $4x^3 - x^2 - 3x + 8$

$= \frac{8}{5} P_3(x) + \frac{12}{5} P_1(x) - 2/3\, P_2(x) + 1/3\, P_0(x)$
$\quad - 3P_1(x) + 8P_0(x).$

$= \frac{8}{5} P_3(x) - \frac{2}{3} P_2(x) + \left(\frac{12-15}{5}\right) P_1(x) + \left(\frac{1-8}{3}\right) P_0(x)$

$= \frac{8}{5} P_3(x) - 2/3\, P_2(x) - 3/5\, P_1(x) + \left(\frac{23}{3}\right) P_0(x)$

5. $x^3 + 2x^2 - x + 1$

$= 2/5\, P_3(x) + 3/5\, P_1(x) + \frac{4}{3} P_2(x) + \frac{2}{3} P_0(x) - P_1(x) + P_0(x)$

$= 2/5\, P_3(x) + 4/3\, P_2(x) - \frac{2}{5} P_1(x) + 5/3\, P_0(x).$

$a = 8/3, \quad b = \frac{-2}{5}, \quad c = 4/3, \quad d = 2/5.$

6. $I = \int_{-1}^{1} x^2 P_4(x)\, dx.$

$= \int_{-1}^{1} x^2 \cdot \frac{1}{8}\left[35x^4 - 30x^2 + 3\right].$

$= \frac{1}{8}\int_{-1}^{1} (35x^6 - 30x^4 + 3x^2)\, dx$

$= \frac{2}{8}\int_{0}^{1} (35x^6 - 30x^4 + 3x^2)\, dx.$ $\Big\}$ even fn. so multiply by 2

$= \frac{2}{8}\left[35\frac{x^7}{7} - 30\frac{x^5}{5} + \frac{3x^3}{3}\right]_0^1$

$= \frac{1}{4}\left[\frac{35}{7} - \frac{30}{5} + 1\right] = \frac{1}{4}\left[5 - 6 + 1\right] = 0.$

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Module - III
Complex Variables

$i = \sqrt{-1}$

$i^2 = -1.$

$\frac{1}{i} = -i$

$z = x + iy.$

$\bar{z} = x - iy.$

$$\frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}.$$

$$= \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

$$\sin\theta = \frac{e^{\theta} - e^{-i\theta}}{2i}.$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}.$$

$$\sinh\theta = \frac{e^{\theta} + e^{-\theta}}{2}.$$

in polar form = $\boxed{z = re^{i\theta}}$

Series Solution.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

De-Moivre's Theorem:-

$$[\cos\theta + i\sin\theta]^n = \cos n\theta + i\sin n\theta.$$

$Z = x+iy$ & $z = re^{i\theta}$.

$x = r\cos\theta \quad y = r\sin\theta$

$\Rightarrow r = \sqrt{x^2+y^2} \longrightarrow$ modulus.

$\theta = \tan^{-1}(y/x) \longrightarrow$ amplitude.

properties associated with modulus + amplitude

① a. $|z_1 . z_2| = |z_1| . |z_2|$

b. $amp(z_1 . z_2) = amp\, z_1 + amp\, z_2$.

② a. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

b. $amp\left(\frac{z_1}{z_2}\right) = amp(z_1) - amp(z_2)$.

③ $|z_1 + z_2| \leq |z_1| + |z_2|$.

④ $|z_1 - z_2| \geq |z_1| - |z_2|$

Function of complex variable

$w = f(z) = u(x,y) + iv(x,y) \longrightarrow$ cartisian form.

$w = f(z) = u(r,\theta) + iv(r,\theta) \longrightarrow$ polar form.

Ex $f(z) = z^2$.

$u + iv = (x+iy)^2$

$u + iv = x^2 - y^2 + 2ixy$.

$u = (x^2 - y^2)$ & $v = 2xy$.

In polar.

$u + iv = (re^{i\theta})^2$

$= r^2 e^{2i\theta}$.

$= r^2(\cos 2\theta + i\sin 2\theta)$

$u = r^2\cos 2\theta$ & $v = r^2 \sin 2\theta$.

Analytic function :-

A complex value function $w = f(z)$ is said to be analytic at the point $z = z_0$ if $\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta(z)}$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta(z)}.$$

Should exist and unique at z_0 & neighbourhood of z_0

Analytic function is also known as
a Regular or holomorphic or Analytic.

Cauchy - Riemann eq^n's in Cartesian form.

Obtain the necessary condition in the cartision system for a fun^n $f(z)$ to be analytic in a Region R.

Statement:- The necessary condition that the function $w = f(z) = u(x,y) + iv(x,y)$ is said to be Analytic if $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$.

Provided

$\frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$ should exists.

Proof:- Let $w = f(z)$ is an analytic function

$\therefore f(z) = u + iv$

$\Rightarrow u + iv = f(x+iy) \longrightarrow (1)$

Diff eq^n (1) w.r.t x and w.r.t y seperately partially.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(x+iy)$$

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = if'(x+iy).$$

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]$$

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}.$$

comparing real and imaginary part:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \;\&\; \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\left\{ u_x = v_y \text{ or } v_x = -u_y \right.$

These are the necessary condition in the cartision for the complex value fn $f(z) = u + iv$ to be analytic

Cauchy - Riemann eq^n's in polar form.

Statement :-

The necessary condition for the fn $w = f(z) = u(r,\theta) + iv(r\theta)$, is said to be analytic if $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}}$

provided

$\frac{\partial u}{\partial r}, \frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial u}{\partial \theta}$ should exists,

proof :-

Let $w = f(z)$ is an analytic function.

$\therefore f(z) = u + iv$

$u + iv = f(re^{i\theta}) \rightarrow (1)$

Differentiating (1) w.r.t r and θ seperately

$$\frac{\partial u}{\partial r} + \frac{i\partial v}{\partial r} = f'(re^{i\theta}).\, e^{i\theta}$$

$$\frac{\partial u}{\partial \theta} + \frac{i\partial v}{\partial \theta} = f'(re^{i\theta})\, re^{i\theta} i$$

$$\frac{\partial u}{\partial \theta} + \frac{i\partial v}{\partial \theta} = \left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] i r$$

$$= ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}.$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \& \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{or} \quad \frac{1}{r}v_\theta$$

$$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta} \quad \text{or} \quad v_r = \frac{-1}{r}v_\theta$$

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properties of ~~A~~nalytic fn.

1. Harmonic property or harmonic fn.

A fn ϕ is said to be harmonic if it satisfies Laplacian eq^n.

$$\nabla^2\phi = 0.$$

$$\left\{ \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} i \right\}$$

In the cartisian form $\phi(x,y)$ is harmonic if

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

In polar form

$\phi(r,\theta)$ if $\boxed{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.}$

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properties

Show that Real and imaginary part of an analytic fn are Harmonic.

Proof:- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$

$f(z)$ is analytic.

C-R eq^n $\rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (1).$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \rightarrow (2).$

$v = x^2 y.$

$\frac{\partial u}{\partial x} = 2xy$ & $\frac{\partial v}{\partial y} = x^2.$

$\frac{\partial^2 u}{\partial y \partial x} = 2x$ & $\frac{\partial^2 u}{\partial x \partial y} = 2x$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}.$ Diff eq^n (1) w.r.t

$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$

In PDE $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}.$

always true

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}.$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Differentiate (1) w.r.t $y \rightarrow$

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}.$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}.$$

$\therefore v$ is the harmonic.

$$\boxed{\text{In PDE} \Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

<u>In polar form</u>

<u>Proof</u> :- Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic.

we've to show that.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial r^2} = 0.$$

Since $f(z)$ is analytic

C-R eq^{-n} is given by.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \Rightarrow r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \rightarrow (1)$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \Rightarrow r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} \rightarrow (2)$$

diff eq^{-n} (1) w.r.t r, (2) w.r.t θ.

$$\cancel{\frac{\partial^2 u}{\partial r^2} = \frac{-1}{r^2}\frac{\partial^2 v}{\partial r \partial \theta}}.$$

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$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{\partial^2 v}{\partial r \partial \theta}.$$

$$\frac{r\partial^2 v}{\partial \theta \partial r} = -\frac{\partial^2 u}{\partial \theta^2} \Rightarrow \frac{\partial^2 v}{\partial \theta \partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}.$$

In PDE $\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$ always true.

$$\Rightarrow \frac{r\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}.$$

$$\Rightarrow \frac{r\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

$\div$ by r.

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$\Rightarrow$ u is harmonic

Diff (1) w.r.t θ & (2) w.r.t r.

$$= \frac{\partial^2 v}{\partial \theta^2} = r\frac{\partial^2 u}{\partial r \partial \theta} \quad \& \quad r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} = -\frac{\partial^2 u}{\partial r \partial \theta}.$$

$$\therefore \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\partial v}{\partial r} - r\frac{\partial^2 v}{\partial r^2}.$$

$\times$ by $\frac{1}{r}$ on b.s.

$$\boxed{\frac{1}{r^2}\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial \theta^2} = 0}$$

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Show that the following functions are harmonic and hence find their harmonic conjugate

(i) $u = e^x \cos y + xy$ → (1)

diff (1) partially w.r.t x twice

$$\frac{\partial u}{\partial x} = \cos y\, e^x + y.$$

$$\frac{\partial^2 u}{\partial x^2} = \cos y\, e^x + 0 = e^x \cos y$$

diff (1) partially w.r.t y twice

$$\frac{\partial u}{\partial y} = -e^x \sin y + x.$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0.$$

$\Rightarrow u$ is harmonic

$\frac{\partial v}{\partial x} = \partial^2 u$

$\frac{\partial v}{\partial x}$

$u_x = v_y$

$v_x = -u_y$

from CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (2)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \rightarrow (3)$$

(2) $\Rightarrow \frac{\partial v}{\partial y} = e^x \cos y + y \rightarrow (4)$

(3) $\Rightarrow \frac{\partial v}{\partial x} = e^x \sin y - x \rightarrow (5)$

Integrate the eq^n w.r.t y partially.

(4) $\Rightarrow v = e^x \sin y + y^2/2 + f(x)$

w.r.t x.

(5) $\Rightarrow v = e^x \sin y - x^2/2 + g(y)$

Let $f(x) = -x^2/2$ & $g(y) = y^2/2$.

$$\Rightarrow \boxed{v = e^x \sin y - x^2/2 + y^2/2.}$$

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(2) $v = 2xy - 2x + 4y \rightarrow (1)$

diff (1) partially w.r.t x twice.

$$\frac{\partial v}{\partial x} = 2y - 2 + 0.$$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

diff (1) partially w.r.t y twice.

$$\frac{\partial v}{\partial y} = 2x - 0 + 4$$

$$\frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$0 + 0 = 0 //$$

$\Rightarrow v$ is harmonic

$+ f(x)$
$g(y)$

From CR - eq^{-n}

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (2)$$

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \rightarrow (3)$$

(2) $\Rightarrow \frac{\partial u}{\partial x} = 2x + 4 \rightarrow (4)$

(3) $-\frac{\partial u}{\partial y} = -2y + 2 \rightarrow (5)$

integrating (4) w.r.t x.

$$u = \frac{2x^2}{2} + 4x + f(y) \qquad x^2 + 4x$$

(5) w.r.t y.

$$u = -2y^2/2 + 2y + g(x)$$

$$= -2y^2/2 + 2y + g(x). \qquad -y^2 + 2y$$

Let $f(y) = -y^2 + 2y$

$g(x) = x^2 + 4x$

$$u = -\cancel{2}y^2/\cancel{2} + 2y + x^2 + 4x$$

$$u = x^2 + 4x - y^2 + 2y.$$

$$\boxed{u = x^2 + 4x - y^2 + 2y}$$

3. $u = e^x[x\cos y - y\sin y] \rightarrow ①$

diff ① w.r.t 'x' partially twice

$$\frac{\partial u}{\partial x} = e^x[x\cos y - y\sin y] + [\cos y - 0]e^x.$$

$$= e^x[x\cos y - y\sin y] + e^x\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x[x\cos y - y\sin y] + [\cos y - 0]e^x + e^x\cos y$$

$$= e^x[x\cos y - y\sin y] + 2e^x\cos y$$

diff ① w.r.t 'y' partially twice.

$$\frac{\partial u}{\partial y} = e^x[-x\sin y - [y\cos y + \sin y]]$$

$$= e^x[-x\sin y - y\cos y - \sin y]$$

$$\frac{\partial^2 u}{\partial y^2} = e^x[-x\cos y - [-y\sin y + \cos y] - \cos y]$$

$$= e^x[-x\cos y + y\sin y - \cos y - \cos y]$$

$$= e^x[-x\cos y + y\sin y - 2\cos y]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x[x\cos y - \cancel{y\sin y} + 2\cos y] + e^x[-x\cos y + \cancel{y\sin y} - 2\cos y]$$

$$= 0$$

$\Rightarrow$ u is harmonic

By CR eq^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow ②$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \rightarrow ③$$

(2) $\Rightarrow \frac{\partial v}{\partial y} = e^x(x\cos y - y\sin y + \cos y) \rightarrow (7)$

(3) $\Rightarrow \frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y + \sin y) \rightarrow (8)$

$$\frac{\partial v}{\partial y} = e^x[x\cos y - y\sin y + \cos y]$$

Integrate w.r.t y.

$\int y \sin y \, dy = y[-\cos y] - \int(-\cos y)dy.$

$= -y\cos y + \sin y.$

$$v = e^x[+x\sin y - [-y\cos y + \sin y] + \sin y]$$

$$= e^x[x\sin y + y\cos y - \cancel{\sin y} + \cancel{\sin y}]$$

$$= e^x[x\sin y + y\cos y] + f(x)$$

$v =$

$$\frac{\partial v}{\partial x} = e^x[x\sin y + y\cos y + \sin y]$$

Integrate w.r.t. x.

$$\frac{\partial v}{\partial x} = [x\sin y + y\cos y + \sin y]e^x - \int e^x \sin y\, dx + g(y).$$

$$e^x[x\sin y + y\cos y + \cancel{\sin y}] - e^x \cancel{\sin y} + g(y).$$

$$v = e^x[x\sin y + y\cos y] + g(y)$$

$$f(x) = g(y) = 0.$$

$$\boxed{v = e^x[x\sin y + y\cos y]}$$

4. $u = \frac{1}{r}\cos\theta \longrightarrow (1)$

diff eq^{-n} (1) w.r.t 'r' twice partially

$\frac{\partial u}{\partial r} = \frac{-1}{r^2}\cos\theta \qquad \frac{\partial u}{\partial \theta} = \frac{-1}{r}\sin\theta$.

$\frac{\partial^2 u}{\partial r^2} = \frac{2}{r^3}\cos\theta \qquad \frac{\partial^2 u}{\partial \theta^2} = \frac{-1}{r}\cos\theta$

$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$

$\cancel{\frac{2}{r^3}}\cos\theta + \cancel{\frac{(-1)}{r^3}}\cos\theta + \cancel{\frac{(-1)}{r^3}}\cos\theta =$

$\cancel{\frac{2}{r^3}}\cos\theta - \cancel{\frac{2}{r^3}}\cos\theta = 0$

u is harmonic.

By C-R eq^{-n}.

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \Rightarrow \frac{\partial v}{\partial \theta} = \frac{-1}{r}\cos\theta \longrightarrow (2)$

$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \Rightarrow \frac{\partial v}{\partial r} = \frac{1}{r^3}\sin\theta \rightarrow (3)$

Integrate (2) w.r.t θ.

$v = \frac{-1}{r}\sin\theta + f(r)$

r^{-3}

$(-3)r^{-3+1}$

$-3r^{-2}$

Integrate (3) w.r.t r.

$v = \frac{-1}{r}\sin\theta + g(\theta)$

$f(r) = g(\theta) = 0$

$\boxed{v = \frac{-1}{r}\sin\theta.}$

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Friend function:-

It is a non-member fn^{-n} that has special rights to access the private data member of any object of the class of whom it is a friend.

~~A Friend fn is a prototype within the~~ definition of the ~~class of whow it is a friend.~~ ×

---

5. $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 \rightarrow (1)$

show that u is harmonic & find its conjugate.

diff. eq^{-n} (1) twice w.r.t x.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \qquad \frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6 \qquad \frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$-\cancel{6x} + \cancel{6} + \cancel{6x} - \cancel{6} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x \rightarrow (2)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial v}{\partial x} = +\cancel{3x^2} + 6xy + 6y \rightarrow (3)$$

Integrate eq^{-n} (2) w.r.t y.

$$v = 3x^2y - y^3 + 6xy + f(x)$$

$$v = 3x^2y + 6xy + g(y)$$

$$f(x) = 0$$

$$g(y) = -y^3$$

$$\boxed{v = 3x^2y - y^3 + 6xy}$$

Construction of analytic fun^{-n} $f(z)$ given its real or imaginary ~~points~~ parts.

Cartisian

real

$$f'(z) = u_x + iv_x$$
$$f'(z) = u_x - iu_y$$
$$x = z, \; \& \; y = 0$$

Imaginary

$$f'(z) = v_y + iv_x$$

Polar form

real

$$f'(z) = e^{-i\theta}[u_r + iv_r]$$
$$= e^{-i\theta}\left[u_r - \frac{i}{r}u_\theta\right]$$

Imaginary

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}v_\theta + iv_r\right]$$

Substitute $x = z$ and $y = 0$ in cartisian form in polar form $r = z$ and $\theta = 0$ then $f'(z)$ should be a fn of z. This method is known as Milne - Thompson method

Milne - Thompson method

1. find the analytic fn whose real part is $u = y + e^x \cos y$

Ans $u = y + e^x \cos y \rightarrow (1)$

Diff eq^{-n} (1) partially w.r.t x and y.

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = 1 - e^x \sin y$$

$$f'(z) = e^x \cos y - i[1 - e^x \sin y] \rightarrow (2)$$

$$x = z, \quad y = 0.$$

$$f'(z) = e^z - i[1].$$

$$f(z) = e^z - iz + C.$$

2. $u = x \sin x \cosh y - y \cos x \sinh y \rightarrow (1)$

Diff eq^{-n} (1) partially w.r.t x and y.

$$\frac{\partial u}{\partial x} = [x \cdot \cos x + \sin x] \cdot \cosh y + y \sin x \sinh y.$$

$$\frac{\partial u}{\partial y} = x \sin x \sinh y - \cos x[y \cdot \cosh y + \sinh y]$$

$$f'(z) = [x\cos x + \sin x]\cosh y + y\sin x \sinh y$$
$$- i[x\sin x - \sinh y - \cos x[y\cdot\cosh y + \sinh y$$

$x = z, \quad y = 0.$

$$= z\cos z + \sin z + 0 - i[\cancel{z\sin z} - 0 - \cos z[0]]$$

$$f'(z) = z\cos z + \sin z - \cancel{iz\sin z}.$$

$$f(z) = z\sin z + \cos z - \cancel{\sin z}\cos z$$

$$\boxed{f(z) = z\sin z + c}$$

$z\sin z - \int \sin z \cdot \frac{dz}{dz}$

$z\sin z + \cos z.$

$e^{-i\theta}[u_r + iv_r]$

$u_x + iv_x$

$f'(z) = u_x + iv_x$

$f'(z) = u_x - iu_y$

$u_x + iv_y - \frac{1}{r}u_\theta$

$f'(z) = e^{-i\theta}[u_r + \frac{i}{r}u_\theta$

$u_r + \frac{i}{r}u_\theta$

$u_x + iv_x$

$e^{-i\theta}[u_r + iv_r]$

$u_x + iv_a$

$u_y + iv_y$

$u_x - iu_y$

$v_y + iv_y$

$f'(z)$

$e^{-i\theta}[u_r + \frac{\partial v}{r}u_\theta]$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$u_x + iv_x$

$u_x + iv_x$

$v_x = - u_x - u_y.$

$u_x + iv_y$

$u_y + iv_x$

J_{n} J_{n+1} $\frac{2n}{x}$

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3. $v = e^{x}[x \sin y + y \cos y] \rightarrow (1)$

Sol^n

diff eq^n (1) partially w.r.t. x and y.

$v_x = x e^{x} \sin y + e^{x} y \cos y \rightarrow (1)$.

$v_x = [x e^{x} + e^{x}] \sin y + e^{x} y \cos y$

$v_y = x e^{x} \cos y + e^{x}[y \sin y + \cos y]$

$f'(z) = v_y + i v_x$.

$= x e^{x} \cos y + e^{x}[-y \sin y + \cos y]$
$+ i[(x e^{x} + e^{x}) \sin y + e^{x} y \cos y] \rightarrow (2)$

substitute $x = z$ and $y = 0$. in eq^n (2)

$f'(z) = z e^{z} + e^{z}(+1) + i[0]$

$f'(z) = z e^{z} + e^{z}$

$f(z) = z e^{z} - \int e^{z} . \frac{dz}{dz} . dz + e^{z} + c$.

$z e^{z} - \cancel{e^{z}} + \cancel{e^{z}} + c$.

$\boxed{f(z) = z e^{z} + c}$

$\int uv = u v_1 - v_2 u' + v_3 u''$

4. $v = \cos x . \cosh y$.

$v_x = -\sin x . \cosh y$.

$v_y = \cos x . \sinh y$.

$f'(z) = v_y + i v_x$.

$= \cos x . \sinh y - i[\sin x . \cosh y]$.

$x = z, \ y = 0$.

$f'(z) = 0 - i[\sin z]$.

$f'(z) = -i \sin z$.

$\boxed{f(z) = i \cos z + c}$

5. polar form as a realpart.

$u = (r + \frac{1}{r})\cos\theta \cdot \rightarrow (1)$

diff eq^n (1) partially w.r.t. r. and θ

$V_r = \frac{}{}$

$u_r = [1 - \frac{1}{r^2}]\cos\theta$

$e^{i\theta}[u_r + iV_r]$

$u_\theta = -[r + \frac{1}{r}]\sin\theta$

$u_r - \frac{i}{r}$

$f'(z) = e^{-i\theta}[u_r - \frac{i}{r}u_\theta]$

$= e^{-i\theta}[(1 - \frac{1}{r^2})\cos\theta + \frac{i}{r}[r + \frac{1}{r}]\sin\theta] \rightarrow (2)$

Put $r = z$, $\theta = 0$ in eq^n (2)

$= [1[1 - \frac{1}{z^2}]1 + \frac{i}{z}[0]]$

$u_r =$

$= 1[\frac{z-1}{z^2}] + 0$

$\frac{1}{r}V_\theta - iV_r$

$f'(z) = 1 - \frac{1}{z^2}$

$\frac{z^{-2+1}}{-2+1}$

$\boxed{f(z) = z + \frac{1}{z} + c}$

$\frac{z^{-1}}{-1}$

6. $v = (r - \frac{1}{r})\sin\theta \cdot \rightarrow (1)$

$V_r = [1 + \frac{1}{r^2}]\sin\theta$

$V_\theta = [r - \frac{1}{r}]\cos\theta$

$f'(z) = e^{-i\theta}[\frac{1}{r}V_\theta + iV_r]$

$= e^{-i\theta}[\frac{1}{r}[r - \frac{1}{r}]\cos\theta + i[(1 + \frac{1}{r^2})\sin\theta]] \rightarrow (2)$

put $r = z$, $\theta = 0$ in eq^n (2)

$f'(z) = [\frac{1}{z}[z - \frac{1}{z}]]$

$f'(z) = [1 - \frac{1}{z^2}]$

$$\boxed{f(z) = z + \frac{1}{z} + C}.$$

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7. Find analytic f^n whose real part is $u = x^2 - y^2 + \frac{x}{x^2+y}$

$$u = x^2 - y^2 + \frac{x}{x^2+y^2} \rightarrow (1)$$

$$u_x = 2x + \left[\frac{(x^2+y^2) - x2x}{(x^2+y^2)^2}\right].$$

$$u_y = -2y + x\left[\frac{(x^2+y^2)\times 0 - 2y}{(x^2+y^2)^2}\right]$$

$$f'(z) = u_x - iu_y.$$

$$= 2x + \left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right] - i\left[-2y + x\left[\frac{-2y}{(x^2+y^2)^2}\right]\right]$$

$$x = z, \quad y = 0.$$

$$= 2z + \left[\frac{-z^2}{z^4}\right] - i\left[0\right]$$

$$= 2z - \frac{1}{z^2}$$

$$\boxed{f(z) = z^2 + \frac{1}{z} + C.}$$

8. Find the analytic f^n whose real part

given $u - v = e^x(\cos y - \sin y). \rightarrow (1)$.

differenciate eq^n (1) partially w.r.t x.

$$u_x - v_x = e^x[\cos y - \sin y]. \rightarrow (2).$$

$$u_y - v_y = e^x[-\sin y - \cos y] \rightarrow (3)$$

from CR equations.

we know that $u_x = v_y$.

$v_x = -u_y$.

$\therefore$ eq^n (3) becomes

$$-v_x - u_x = -e^x[\sin y + \cos y].$$

$$= u_x + v_x = e^x[\sin y + \cos y]. \rightarrow (4)$$

Subs eq^{-n} (2) and (4)

$$u_x + v_x = e^x[\sin y + \cos y]$$
$$u_x - v_x = e^x[\cos y - \sin y]$$

$$2u_x = 2e^x \cos y.$$

$$\therefore \; u_x = e^x \cos y \rightarrow (5)$$

substitute eq^{-n} (5) in (2).

$$e^x\cos y - v_x = e^x \cos y - e^x \sin y.$$

$$\boxed{v_x = e^x \sin y}$$

$$\therefore \; u_y = -e^x \sin y.$$

$$f'(z) = u_x - iu_y.$$
$$e^x\cos y + ie^x \sin y.$$
$$x = z, \quad y = 0.$$

$$f'(z) = e^z$$

$$\boxed{f(z) = e^z + c}$$

9. $u+v = x+y+e^x[\cos y + i\sin y] \rightarrow (1)$

$u_x + v_x = 1 + e^x[\cos y + i\sin y] \rightarrow (2)$

$u_y + v_y = 1 + e^x[-\sin y + i\cos y] \rightarrow (3)$

from C-R eq^{-n}

$u_y = v_x.$ $\quad v_x - u_x.$

$v_y = -u_x.$

$-u_x + v_x = 1 + e^x[-\sin y + i\cos y] \rightarrow (4)$

~~Substitute~~ subtract eq^{-n} (2) and (4)

e^{iy}

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$\cancel{iu_x + v_x = 1 + e^x[\cos y + i\sin y]}$

$\cancel{-iu_x + v_x = 1 + e^x[-\sin y + i\cos y]}$

$\cancel{2v_x = 2 +}$

$-(u_x - v_x) = \; i \cdot 1 + e^x - [\sin y$

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9. $u+v = x+y+e^x[\cos y + i\sin y]$

$u+v = x+y\; e^x e^{iy}$

$u_x + v_x = 1 + e^x e^{iy} \longrightarrow (1)$

$u_y + v_y = 1 + ie^x e^{iy} \longrightarrow (2)$

From CR equations.

$u_x = v_y$ & $v_x = -u_y$.

Now eq^n (2) becomes.

$-v_x + u_x = 1 + ie^{x+iy}$.

$$u_x + v_x = 1 + e^{x+iy}$$
$$u_x - v_x = 1 + ie^{x+iy} \quad \text{(add)}$$
$$2u_x = 2 + (1+i)e^{x+iy}$$
$$u_x = 1 + \frac{(1+i)}{2} e^{x+iy}$$

$$u_x + v_x = 1 + e^{x+iy}$$
$$u_x - v_x = 1 + ie^{x+iy} \quad \text{(sub)}$$
$$2v_x = (1-i)e^{x+iy}$$
$$v_x = \frac{(1-i)}{2} \cdot e^{x+iy}$$

$f'(z) = u_x + iv_x$

$f'(z) = 1 + \frac{(1+i)}{2} e^{x+iy} + i\left\{\frac{(1-i)}{2} e^{x+iy}\right\}$

$$f'(x) = 1 + \frac{(1+i)}{2} e^z + i\frac{(1-i)}{2} e^z$$

$$= 1 + \frac{e^z}{2} + \frac{i}{2}e^z + \frac{i}{2}e^z + \frac{i}{2}\cdot e^z$$

$$= 1 + e^z + ie^z$$

$$f'(z) = 1 + (1+i)e^z$$

$$\boxed{f(z) = z + (1+i)e^z + C}$$

10. If $f(z)$ is a regular function of z show that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |f(z)|^2 = 4|f'(z)|^2$$

proof: Let $f(z) = u + iv$ be an analytic or regular or holomorphic function, then

$$|f(z)| = \sqrt{u^2+v^2}.$$

$$|f(z)|^2 = u^2+v^2. = \phi \text{ (say)}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi = 4|f'(z)|^2.$$

$$= \phi_{xx} + \phi_{yy} = 4|f'(z)|^2$$

$$= \cancel{\phi_{xx} + \phi_{yy}} =$$

$$\Rightarrow \phi = u^2+v^2 \rightarrow (1)$$

$\cancel{|f'(z)| = \sqrt{u_x^2+i}}$

$|f'(z)| = \sqrt{u_x^2+v_x^2}$

diff eq^n (1) w.r.t x and w.r.t y partially

$$\phi_x = 2u\cdot u_x + 2vv_x.$$

$$\phi_{xx} = 2[uu_{xx} + (u_x)^2] + 2[vv_{xx} + (v_x)^2]$$

lly

$$\phi_{yy} = 2[uu_{yy} + (u_y)^2] + 2[vv_{yy} + (v_y)^2].$$

$$\Rightarrow \phi_{xx} + \phi_{yy}$$

$$= 2u[u_{xx} + u_{yy}] + 2u_x^2 + 2v[v_{xx} + v_{yy}] + 2v_x^2 + 2u_y^2 + 2v_y^2$$

$\because f(z) = u + iv$ is analytic

u and v are harmonic.

$\Rightarrow u_{xx} + u_{yy} = 0.$

$v_{xx} + v_{yy} = 0.$

from C-R eq^n.

$u_x = v_y$ & $v_x = -u_y$.

Now eq^n (5) becomes.

$$\phi_{xx} + \phi_{yy} = 2u(0) + 2v(0) + 2u_x^2 + 2v_x^2 + 2v_x^2 + 2u_x^2$$

$$= 4(u_x^2 + v_x^2)$$

$$\phi_{xx} + \phi_{yy} = 4|f'(z)|^2.$$

11. If $f(z)$ is a regular function of z

Show that $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$.

Proof: Let $f(z) = u + iv$ be an analytic or regular or holomorphic function, then.

$$|f(z)| = \sqrt{u^2+v^2} = \phi \text{ (say)}.$$

$$\phi^2 = u^2 + v^2.$$

To prove that: $\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = |f'(z)|^2.$

$$\Rightarrow (\phi_x)^2 + (\phi_y)^2 = |f'(z)|^2.$$

$$\Rightarrow \phi^2 = u^2 + v^2 \rightarrow (1).$$

$$\Rightarrow \cancel{2}\phi\phi_x = \cancel{2}uu_x + \cancel{2}vv_x.$$

lly. $\cancel{2}\phi\phi_y = \cancel{2}uu_y + \cancel{2}vv_y$

$$\phi^2\phi_x^2 + \phi^2\phi_y^2 = (uu_x + vv_x)^2 + (uu_y + vv_y)^2.$$

$$= \phi^2[\phi_x^2 + \phi_y^2] = u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x +$$

$$u^2u_y^2 + v^2v_y^2 + 2uv\,u_yv_y.$$

$$= u^2[u_x^2 + u_y^2] + v^2[v_x^2 + v_y^2] + 2uv[u_x v_x + u_y v_y] \rightarrow (2)$$

$\because f(z)$ is analytic,

u and v are harmonic.

$\therefore$ from C-R equations $u_x = v_y$ & $v_x = -u_y$.

$\therefore$ eq^{-n} (2) $\Rightarrow$

$$= u^2[u_x^2 + v_x^2] + v^2[v_x^2 + u_x^2] + 2uv[\cancel{u_x v_x} - v_x u_x]$$

$$\phi^2[\phi_x^2 + \phi_y^2] = [u_x^2 + v_x^2][u^2 + v^2]$$

$$\Rightarrow (\phi_x)^2 + (\phi_y)^2 = u_x^2 + v_x^2.$$

$$\Rightarrow \boxed{(\phi_x)^2 + (\phi_y)^2 = |f'(z)|^2}$$

12. $u = e^{-x}[x\cos y + y\sin y] \rightarrow (1)$

$f'(z) = u_x - iu_y$ $\qquad v_x + iv_y$ $\qquad u_x - iu_y$

Diff eq^{-n} (1) w.r.t x and y.

$u_x = -xe^{-x}\cos y + e^{-x}y\sin y$. $\quad u_x - iu_y$

$u_x = [-xe^{-x} + e^{-x}]\cos y - e^{-x}y\sin y$.

$u_y = -xe^{-x}\sin y + e^{-x}[y\cos y + \sin y]$

$x = z, \; y = 0.$

$$f'(z) = [-ze^{-z} + e^{-z}]$$

$$f(z) = -z\left[\frac{e^{-z}}{-1}\right] - \int\left[\frac{e^{-z}}{-1}\right]dz - \frac{e^{-z}}{-1} + c.$$

$$= ze^{-z} + \cancel{e^{-z}} - \cancel{e^{-z}} + c.$$

$$\boxed{f(z) = ze^{-z} + c.}$$

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complex Integration.

$z \to (0,1)$ to $3+i$ $(3,1)$

$\int_{(0,0)} \to \int_{(0,0)}$

Line Integral.

The complex line Integral along the path C denoted by

$$\int_C f(z).dz.$$

if $z = x+iy$, $\Rightarrow dz = dx + idy$.

$f(z) = u+iv$.

if 'c' is simple closed curve.

then $\oint_c f(z)\,dz$.

properties of complex integral.

(1) $$\int_{-c} f(z).dz = -\int_c f(z)\,dz.$$

(2) $$\int_c f(z).dz = \int_{c_1} f(z).dz + \int_{c_2} f(z).dz + \dots\dots$$

(3) $$\int_c [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\,dz = \lambda_1 \int_c f_1(z)\,dz \pm \lambda_2 \int_c f_2(z)\,dz$$

Line integral of complex valued function.

Let $f(z) = u(x,y)+iv(x,y)$ be a complex valued function defined over a region 'R' and 'c' be the curve in the region.

$$\int_c f(z).dz = \int_c (u+iv)(dx+idy)$$

$$= \int_c [udx + iudy + ivdx - vdy]$$

$$= \int_c [udx - vdy] + i\int_c [udy + vdx]$$

Source diginotes.in

This shows that the evaluation of line integral of a complex value function is nothing but the evaluation of two integrals of real value function.

Example

1. Evaluate $\int_C z^2 dz$

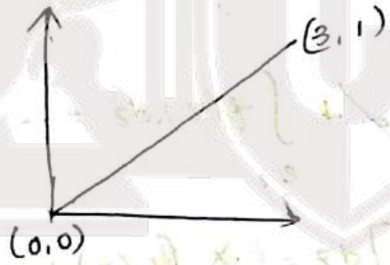
a. along the straight line from $z=0$ to $z = 3+i$

b. along the curve made up of two line segments one from $z=0$ to $z=3$ & another from $z=3$ to $z=3+i$

Sol^n

$$z^2 dz = (x+iy)^2 (dx+idy)$$
$$= (x^2-y^2+i2xy)(dx+idy).$$

ⓐ



since $z \to 0$ to $3+i$

$\Rightarrow (x,y) \to (0,0)$ to $(3,1)$

eq^n of line joining two points is given by

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\Rightarrow \frac{y}{x} = 1/3$$

$$\Rightarrow y = x/3 \text{ or } x = 3y.$$

$$\int_C z^2 dz = \int_{(0,0)}^{(3,1)} \{x^2-y^2+i2xy)(dx+idy).$$

$$= \int_{(0,0)}^{(3,1)} \left[(x^2-y^2)dx - 2xy\,dy\right] + i\left[2xy\,dx + (x^2-y^2)dy\right] \rightarrow (1).$$

Let $x = 3y \Rightarrow dx = 3dy \Rightarrow y \rightarrow 0$ to 1

$$I = \int_0^1 \{(9y^2-y^2)3\,dy - 2\cdot 3y(y)dy\} + i\left[2(3y)y\,3dy + (9y^2-y^2)dy\right]$$

$$= \int_0^1 \left[24y^2dy - 6y^2dy\right] + i\left[18y^2dy + 8y^2dy\right].$$

$$= \int_0^1 \left[18y^2dy + i26\cdot y^2dy\right]$$

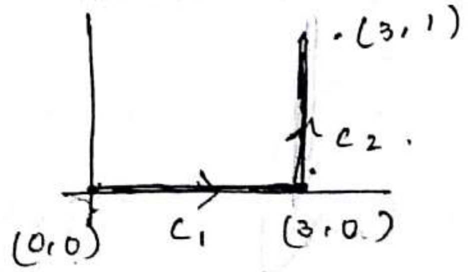
$$= \int_0^1 18y^2dy + i\int_0^1 26y^2\,dy$$

$$I = 18\left[\frac{y^3}{3}\right]_0^1 + i26\left[\frac{y^3}{3}\right]_0^1$$

$$= 18\times \frac{1}{3} + i26\,\frac{1}{3}$$

$I = 6 + \frac{26i}{3}$. along the path.

(b)



$$\int_c z^2dz = \int_{c_1} z^2dz + \int_{c_2} z^2dz$$

along c_1 $y=0$, $dy=0 \Rightarrow x \to 0$ to 3.

along c_2 $x=3$ & $dx=0 \Rightarrow y \to 0$ to 1

$$\therefore \int_C z^2 dy = \int_{x=0}^{3} x^2\,dx + \int_{y=0}^{1} \cancel{(3+iy)}\,(9-y^2+i6y)\,i\,dy$$

$$\left[\frac{x^3}{3}\right]_0^3 + i\int_0^1 \left[9y - \frac{y^3}{3} + i\frac{6y^2}{2}\right]$$

$$\frac{27}{3} + i\left[9 - \frac{1}{3} + 3i\right]$$

$$9 + \frac{26}{3}i - 3$$

$$= 6 + \frac{26}{3}i$$

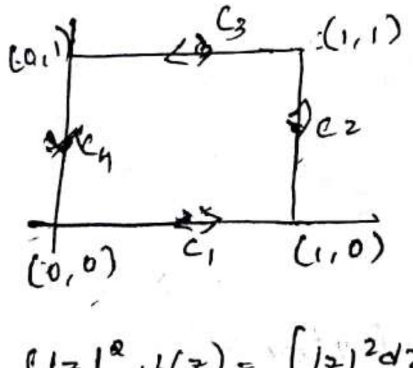
2. Evaluate $\int_C |z|^2 dz$, where c is a square with the vertices $(0,0)$ $(1,0)$ $(1,1)$ $(0,1)$

Sol^n:- $z = x+iy$ $\quad dz = dx + i\,dy$.

$$|z| = \sqrt{x^2+y^2}$$

$$|z|^2 = x^2+y^2$$

$$|z|^2 dz = (x^2+y^2)(dx+i\,dy)$$



$$\int_C |z|^2 f(z) = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz \quad \to (1)$$

along C_1 :- $y=0$ & $dy=0$ $\int$ w.r.t x $[0 \le x \le 1]$

$c_2: x=1, dx=0.$ $\int$ w.r.t y $[0 \le y \le 1]$ $y \to 0$ to 1

$c_3: y=1 . dy=0$ $\int$ w.r.t x $[1 \le x \le 0]$ $x \in$

$c_4: x=0$, ∴ $dx=0$ $\int$ w.r.t $y[1 \le y \le 0]$ $y \in (1, 0)$

<u>along c_1</u>

$$\int_C |z|^2 dz = \int_{x=0}^{1} (x^2)dx + \int_{y=0}^{1} (1+y^2)\, i\, dy$$

$$+ \int_{x=1}^{0} (x^2+1)dx + \int_{y=1}^{0} y^2 i\, dy \ldots$$

$$\frac{x^3}{3} + i\left(y + \frac{y^3}{3}\right) + x^2 + y + \frac{y^3}{3} i$$

~~along~~ c_2 $\cancel{\frac{1}{3} + i\left[1 + \frac{1}{3}\right]} +$

$$\boxed{I = i-1}.$$

3. Integrate $\int_0^{2+i} (\bar{z})^2 dz$ along

(a) the line $x = 2y$.

(b) the real axis upto 2 and then vertically to $2+i$.

<u>sol^n</u>

$\cancel{(x+iy)}$. $\bar{z} = x - iy$.

$(\bar{z})^2 = (x^2 - y^2 - i2xy)$.

$\therefore (\bar{z})^2 dz = (x^2 - y^2 - i2xy)(dx + idy)$.

(a) $x = 2y \Rightarrow dx = 2dy \Rightarrow \int$ w.r.t y (y 0 to 1)

$$I = \int_{y=0}^{1} [4y^2 - y^2 - i4y^2](2dy + idy)$$

$$= I = \int_0^1 [3y^2 - i4y^2][2+i]dy$$

$$= (2+i)\left[\frac{3y^3}{3} - i\frac{4y^3}{3}\right]_0^1$$

$= (2+i)\left(1 - \frac{i4}{3}\right)$

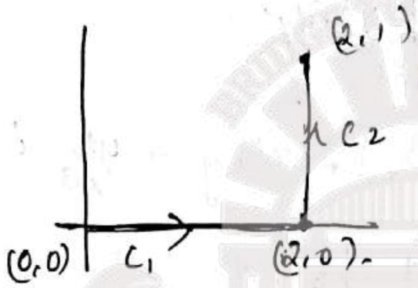
$= 2 - i\,8/3 + i + 4/3.$

$= \cancel{2/3} - \cancel{i5/3}.$

$\frac{10}{3} - 5/3\, i$

$5/3\,[2 - i].$

(10)



$$\int_0^{2+i} (\bar{z})^2\,dz = \int_{C_1} (\bar{z})^2\,dz + \int_{C_2} (\bar{z})^2\,dz \rightarrow (1)$$

along C_1 $y = 0$ $dy = 0$, $x \rightarrow (0, 2)$.

along C_2 $x = 2$, $dx = 0$, $y = (0, 1)$

$$\int_{x=0}^{2} x^2\,dx + \int_0^1 (4 - y^2 - i4y)\, i\,dy$$

$$\left[\frac{x^3}{3}\right]_0^2 + i\left[4y - y^3/3 - i\,\frac{4y^2}{2}\right]_0^1$$

$= 8/3 + i\left[4 - 1/3 - 2i\right].$

$= 8/3 + \frac{11}{3} i + 2.$

$= \frac{14}{3} + \frac{11}{3} i$

$= 1/3\,[14 + 11 i]$

RIP logic

$|z| = r = 1$

$z = re^{i\theta}$

$x = r\cos\theta$

$y = r\sin$

$\bar{z} = e^{-i\theta}$ $\quad ie^{i\theta}\,d\theta$

$\bar{z}$

A. Evaluate ~~Integ~~ $\int_c \bar{z}\, dz$ where

c represents the following path.

(a) the straigh line from $-i$ to i

(b) the right half of the unit circle $|z|=1$ from $-i$ to i

sol^n $\bar{z}\, dz = (x-iy)(dx+idy)$.

The points $(0,-1)$ to $(0,1)$.

$x=0$ & $dx=0$. $\int$ w.r.t y $y \to -1$ to 1

$$\int_{y=-1}^{1} (-iy)(idy) = \int_{-1}^{1} y\,dy = 0.$$

b. $|z|=1 \Rightarrow r=1$

$z = re^{i\theta} = e^{i\theta}$, $dz = ie^{i\theta} \cdot d\theta$

$\bar{z} = e^{-i\theta}$

$(\bar{z}) \cdot dz = e^{-i\theta} \cdot ie^{i\theta} \cdot d\theta = i\,d\theta$

$x = r\cos\theta$. $y = r\sin\theta$.

$y = \sin\theta$.

$y=-1 \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\pi/2$.

$y=1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$.

$$\bar{z}\, dz = \int_{\theta=-\pi/2}^{\pi/2} i\,d\theta = i\theta \Big|_{-\pi/2}^{\pi/2}$$

$= i\pi$.

$|z|=1$

$r=1$

$\bar{z}\, dz$

$\bar{z}$ $(0,-1)$ $(0,1)$

$dz = e^{i\theta}$

$e^{i\theta} \cdot i\,d\theta$

$e^{-i\theta}$

$e^{-i\theta}$

$ie^{i\theta} \cdot d\theta$

$i\,d\theta$ $i\,d\theta$

$\theta = \pi$

Cauchy's theorem

If $f(z)$ is analytic at all the points and on a simple closed curve c then $\int_c f(z)dz = 0$

proof

$$f(z) = u+iv, \quad z = x+iy$$

$$dz = dx+idy$$

$$\int_c f(z)\cdot dz = \int_c (u+iv)(dx+idy)$$

$$\int_c f(z)dz = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy) \quad \to (1)$$

From greens theorem $M = u, N = -v$

$$\int_c (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Now eq^n (1) becomes

$$\int_c f(z)dz = \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \to (2)$$

since $f(z)$ is analytic from C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Now eq^n (2) becomes

$$\int_c f(z)dz = \iint_R \left(-\cancel{\frac{\partial v}{\partial x}}^{\,\partial u/\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint \left(\cancel{\frac{\partial u}{\partial x}}^{\,\partial v/\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$\int_c f(z)dz = 0.$$

$$f(a) = \frac{1}{2\pi i}$$

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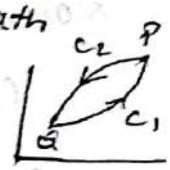
Consequences of cauchy's theorem

1) If $f(z)$ is an analytic function ~~and~~ in a region R & P & Q are any two points in it then

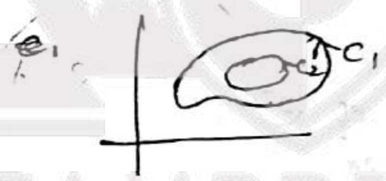
$$\int_P^Q f(z)\,dz$$ is independent of the ~~join~~ path joining P & Q.



ie $$\int_P^Q f(z)\,dz$$ if $f(z)$ is analytic

(2) If C_1, C_2 are two simple closed curve such that C_2 lies entirely within C_1 if $f(z)$ is analytic on C_1 and C_2 & in the region bounded by C_1, C_2 [known as annular region].

ie $$\int_{C} f(z)\,dz = \int_{C} f(z)\,dz$$



Cauchy's Integral theorem

Statement :- If $f(z)$ is analytic inside and on a simple closed curve 'c' and if 'a' is any point within 'c' then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.}$$

Proof :-

since 'a' is a point within 'c' we shall enclose it by a circle C_1 with $z = a$ as the center

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and 'r' is the radius such that C_1 lies entirely within 'c'

the f^n $\frac{f(z)}{(z-a)}$ is analytic inside and on the boundary of annular region b/w c and C_1.

Now, From the consequence of Cauchy's Theorem

$$\int_c \frac{f(z)}{z-a} dz = \int_{c_1} \frac{f(z)}{z-a} dz \rightarrow (1).$$

The eq^n of C_1 [eq^n of a circle with the center a and radius r]

$\Rightarrow |z-a| = r \Rightarrow (z-a) = re^{i\theta}$.

$\Rightarrow z = a + re^{i\theta}$

$dz = ire^{i\theta} . d\theta$

$$(1) \Rightarrow \int_c \frac{f(z)}{z-a} dz = \int_c \frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}} i\cancel{re^{i\theta}}$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\, d\theta \rightarrow (2)$$

As $r \rightarrow 0$, eq (2)

$$\int \frac{f(z)}{z-a} dz = i\int_0^{2\pi} f(a)\, d\theta.$$

$$= f(a)\, i[\theta]_0^{2\pi}$$

$$= 2\pi i f(a)$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz.$$

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Generalized Cauchy's integral theorem

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$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} \, dz .$$

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \, dz .$$

1. $\int_C \frac{e^{3z}}{(z+2)} \, dz . \qquad a = -2, \quad f(z) = e^{3z}$

Note :- ① → If $z = a$ is inside 'c' curve, then we have to evaluate the cauchys Integral theorem of the form.

$$\int_C \frac{f(z)}{z-a} \, dz = 2\pi i f(a).$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}} \, dz = \frac{2\pi i}{n!} f^n(a) .$$

② → If $z = a$ is outside c, then the Integral value is zero.

1. evaluate $\int_C \frac{e^z}{(z+i\pi)} \, dz$ over each of the following contours c.

(a) $|z| = 2\pi$

(b) $|z| = \pi/2$.

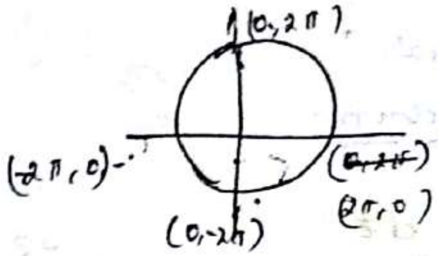
(c) $|z-1| = 1$

a. given that $\int_C \frac{e^z}{(z+i\pi)} \, dz = \int_C \frac{f(z)}{z-a} \, dz$.

$$f(z) = e^z , \qquad a = -i\pi \Rightarrow (0, -\pi).$$

a. $|z| = 2\pi$

eq^n of a circle of radius $(0, 2\pi)$,



$z = -i\pi$ is lies inside C, therefore from cauchys integral theorem

$$= \int_c \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

$$= \int_c \frac{e^z}{z+i\pi}\, dz = 2\pi i f(-i\pi) \qquad f(z) = e^z$$

$$f(-i\pi) = e^{-i\pi}$$

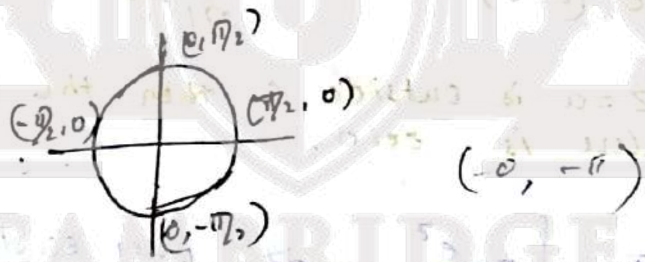
$$= (\cos\pi - i\sin\pi)$$

$$= -1$$

$$\int_c \frac{e^z}{z+i\pi}\, dz = 2\pi i e^{-i\pi}$$

$$= -2\pi i$$

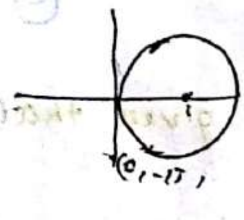
b. eq^{-n} of a circle with radius $\pi/2$. ie $(0, \pi/2)$



$z = -i\pi$ is lies outside the circle C, therefore integral value is zero.

c. $|z-1| = 1$, eq^{-n} of a circle with centre 1 ie, $(1, 0)$ and radius $= 1$

$z = -i\pi$ lies outside the circle c, therefore integral value is zero.



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2. $\int_C \frac{e^z}{z - i\pi}\, dz$.

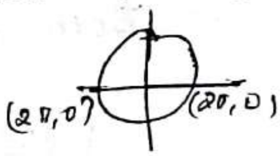
(a). $|z| = 2\pi$

(b) $|z| = \pi/2$.

given that $\int_C \frac{e^z}{(z - i\pi)}\, dz = \int \frac{f(z)}{z-a}\, dz$.

$f(z) = e^z \cdot \quad a = i\pi$

a) . $|z| = 2\pi$ is eq^{-n} of circle with radius 2π and center 0 .



. $z = +i\pi$ lies inside c.

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

$$= -2\pi i$$

$f(z) = e^z$.

$f(i\pi) = e^{i\pi}$

$\cos\pi + i\sin\pi$

-1

(b) $|z| = \pi/2$

. centre $= 0$, radius $= \pi/2$.

$\therefore$ $z = i\pi$ lise outside c, therefore integral value is zero

3. $\int_C \frac{e^{2z}}{(z+1)(z-2)} \cdot dz \quad ; \quad c : |z| = 3$.

consider $\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$

$$1 = A(z-2) + B(z+1)$$

$$z = 2$$

$$B = 1/3$$

$$z = -1$$

$$A = -1/3$$

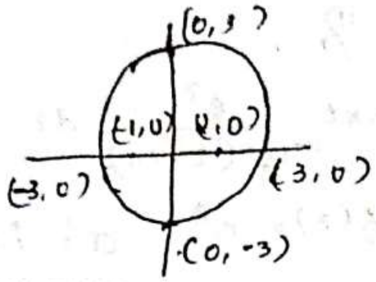
$$\therefore \frac{1}{(z+1)(z-2)} = \frac{-1}{3(z+1)} + \frac{1}{3(z-2)}$$

$$= \frac{1}{3}\int \frac{e^{2z}}{z-2}\,dz - \frac{1}{3}\int_c \frac{e^{2z}}{z+1}\,dz.$$

9.2

$f(z) = e^{2z}$, $a = 2$, & $a = -1$.

$C: |z| = 3$ $\quad a = (2,0)$ and $a = (-1,0)$



$\therefore$ Both $z = 2$ and $z = -1$ lies inside C.

$$\therefore \int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a).$$

$\cancel{f(z) = e^{2z}}$

$$\frac{1}{(z+1)(z-2)} = \frac{-1}{3}\left[2\pi i e^{-2}\right] + \frac{1}{3}\left[2\pi i f(a)\right]$$

$$= -\frac{1}{3}\left[2\pi i e^{-2}\right] + \frac{1}{3}\left[2\pi i e^{4}\right]$$

$$= \frac{2\pi i}{3}\left[e^{4} - \frac{1}{e^{2}}\right]$$

4.

$$4 \int \frac{dz}{(z^2-4)}$$

C. (a) $|z| = 1$. (b) $|z| = 3$.

(c) $|z+2| = 1$.

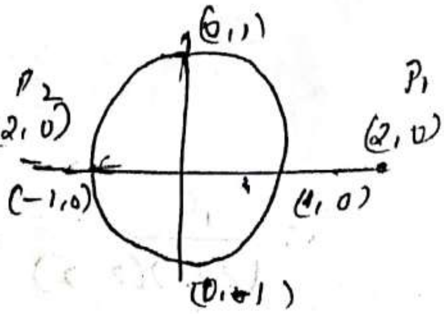
$f(z) = 1$.

$$\frac{1}{(z^2-4)} = \frac{1}{(z+2)(z-2)} = \frac{A}{z+2} + \frac{B}{z-2}.$$

$$1 = A(z-2) + B(z+2)$$

$A = -1/4$

$B = 1/4$.



$$\frac{1}{(z^2-4)} = \frac{-1}{4(z+2)} + \frac{1}{4(z-2)}$$

$$= \frac{1}{4}\int \frac{1}{z-2}\,dz - \frac{1}{4}\int \frac{1}{z+2}$$

$f(z) = 1, \quad a = 2, \quad a = -2.$

P_1 & P_2 lies outside the c.

$\therefore$ integral value is zero.

b. $|z| = 3$.

eq^{-n} of the circle with radius 3 and centre = 0.

. $z = -2, 2$. lises inside c.

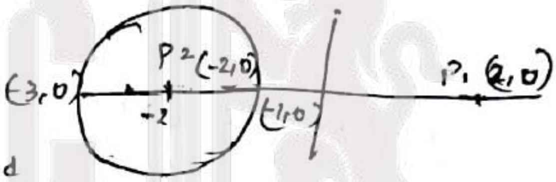
$$\therefore \frac{1}{(z^2-4)} = \frac{1}{4}\left[2\pi i f(a)\right] - \frac{1}{4}\left[2\pi i f(a)\right]$$

$$= \frac{1}{4}\left[\cancel{4\pi i}\right] - \frac{1}{4}\left[2\pi i \cancel{f}(-2)\right].$$

$\because f(z) = 1$

$= 0$.

c. $|z+2| = 1$



One is inside and the other is outside.

4

$$0 - \frac{1}{4} \cdot 2\pi i f(-2).$$

$$= -\frac{\pi i}{2}.$$

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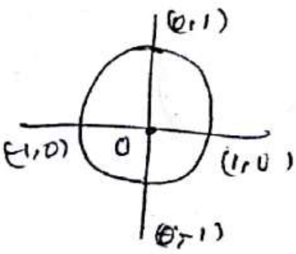
5. evaluate $\int_c \frac{e^{3z}}{z^2} dz : |z| = 1 \rightarrow$ radius.

$$\frac{e^{3z}}{(z-0)^2} \Rightarrow a = 0. \quad f(z) = e^{3z}$$

$$f'(z) = 3e^{3z}$$

$$f'(0) = 3.$$

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a).$$



$$= \int_c \frac{e^{3z}}{(z-0)^2} \cdot dz = \frac{2\pi i}{1!} f'(0)$$

$$= 6\pi i$$

6. $\int \frac{e^{\pi z}}{(2z-i)^3}\, dz \quad : |z| = 1$.

$f(z) = e^{\pi z} \qquad a = i/2$

$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^n(a)$

$= \frac{2\pi i}{2} f^{11}(a)$

$= \pi^3 i = e^{i\pi/2}$.

$\int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz = \frac{2\pi i}{n!} f^n(a)$.

$= \frac{e^{\pi z}}{2^3 \left[z - i/2\right]^3}$

$f(z) = e^{\pi z}$

$f'(z) = \pi e^{\pi z}$

$f''(z) = \pi^2 e^{\pi z}$.

$f''(i/2) = \pi^2 e^{i\pi/2}$

$= \pi^2 \left[\cos \pi/2 + i \sin \pi/2\right]$

$= i\pi^2$

$= \frac{1}{8} \frac{2\pi i}{2} f''(a)$

$= \frac{1}{8} \frac{2\pi i}{2}\, i\pi^2$

$= \frac{-\pi^3}{8}$.

7. $\int_C \frac{(z^2+z+1)}{(z-2)^3}\, dz : |z| = 3$.

$\int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz = \frac{2\pi i}{n!} f^n(a)$

$f(z) = z^2 + z + 1$

$a = 2$,

$\oint_C = \frac{2\pi i}{2!} f''(2)$

$f(z) = z^2+z+1$

$2z+1$

$f''(z)\ 2$

$\frac{2\pi i}{2}\, 2$.

$= 2\pi i$

8. Evaluate $\int_C \frac{e^{2z}}{(z+1)^2(z-2)}$: $|z|=3$

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we can solve by partial fraction.

$$\frac{1}{(z-2)(z+1)^2} = \frac{A}{(z-2)} + \frac{B}{(z+1)} + \frac{C}{(z+1)^2}$$

$$1 = A(z+1)^2 + B(z-2)(z+1) + C(z-2).$$

$z=-1$

$C = -1/3$

$z=+2.$

$A = 1/9.$

$z=0.$

$$1 = A(1)^2 + B(-2)(1) + C(-2)$$

$$1 = 1/9 + -B + 2/3.$$

$$\cancel{\frac{1+3(2)}{9}}\ 1 = 7/9 - B(-2)$$

$$1 - 7/9 = -B$$

$$\frac{9-7}{9} = \frac{2}{9}$$

$$= \frac{2}{9} = -2B.$$

$$\boxed{B = -1/9}$$

$$= \int_C \frac{1}{9}\frac{e^{2z}}{(z-2)} - \int_C \frac{1}{9}\frac{e^{2z}}{(z+1)} - \int_C \frac{1}{3}\frac{e^{2z}}{(z+1)^2}$$

$f(z) = e^{2z}$ $\quad$ $f(z) = e^{2z}$ $\quad$ $f(z) = e^{2z}$

$a=2$ $\quad$ $a=-1$ $\quad$ $a=-1$

$f(2) = e^4$ $\quad$ $f(-1) = e^{-2}$ $\quad$ $f'(z) = 2e^{2z}$

$f'(a) = 2e^{-2}.$

$$= \frac{1}{9}\left[2\pi i f(2)\right] - \frac{1}{9}\left[2\pi i f(-1)\right]$$

$$- 1/3\left[2\pi i f(-1)\right]$$

$$= \frac{1}{9}\left[2\pi i e^{4}\right] - \frac{1}{9}\left[2\pi i e^{-2}\right] - \frac{1}{3}\left[2\pi i \cdot 2e^{-2}\right]$$

$$= \frac{2\pi i}{9}\left[e^{4} - \frac{7}{e^{2}}\right]$$

9. Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} \cdot dz \quad |z| = 3.$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \qquad B = 1, \quad A = -1.$$

$$\int \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} = \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)} dz - \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)} \cdot dz.$$

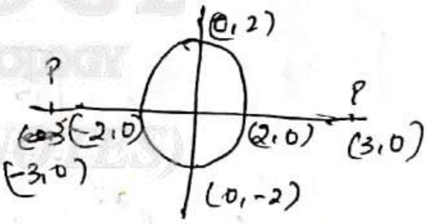
$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 4\pi i$$

10. $\int_C \frac{z}{(z^2+1)(z^2-9)} \cdot dz \quad \therefore |z| = 2.$

$\underline{Sol^n}$: $(z^2 - 9) = (z+3)(z-3)$

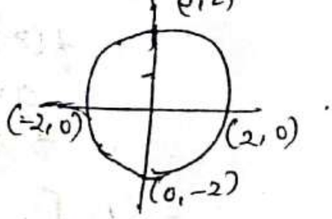
But $|z| = 2$.



$\therefore$ P lies outside.
hence Integral value will be zero.

$$\therefore \int \frac{z}{(z^2+1)} \cdot dz = \int \frac{z}{(z+i)(z-i)}$$

$(0,-1) \quad (0,1)$



$$1 = \frac{A}{(z+i)} + \frac{B}{(z-i)}$$

$$1 = A(z-i) + B(z+i)$$

$z = i$

$1 = 2iB \qquad B = 1/2i$

$A = -\frac{1}{2i}$

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$$\int \frac{z}{(z^2+1)} = \int \frac{1}{2i} \frac{z}{(z-i)} dz - \int \frac{1}{2i} \frac{z}{(z+i)} \cdot dz$$

$$= \frac{1}{2i}\left[\left\{ 2\pi i f(i) - 2\pi i f(-i) \right] \right.$$

$f(z) = z$.
$f(i) = i$
$f(-i) = -i$

$$= \frac{1}{2i}\left[2\pi i^2 - 2\pi i(-i)\right]$$

$$= \frac{1}{\cancel{2}i}\left[-\cancel{2}\pi - 2\pi\right]$$

$$-\frac{2\pi}{i} \qquad i = \frac{-1}{i}$$

$$= 2\pi i$$

6/4/17

Pole and Residue

$$* \int_C \frac{f(z)}{(z-a)^1} dz.$$

Pole of $z = a$ of order (m) is 1.

$$\int \frac{f(z)}{(z^2+1)(z-2)}$$

Coeff $= 1/z$

$$R[m, a] = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

Cauchy Residue Theorem

If $f(z)$ is analytic fn inside and one the bondry of simple closed curve c, then the in $\int_C f(z)$

then $\boxed{\int_C f(z) \cdot dz = 2\pi i \sum R.}$

where $\sum R$ is sum of the Residues.

1. Find the residue of the fn $f(z) = \frac{z}{(z+1)(z-2)^2}$

at $z=-1$ & $z=2$.

Sol^n The pole of $z=-1$ of order 1

The pole of $z=2$ of order 2.

Residual theorem

$$R[m,a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

case i) $z=-1$, $a=-1$, $m=1$

$$= \frac{1}{0!} \lim_{z\to -1} \frac{d^0}{dz^0} \left[(z+1)^1 \frac{z}{(z+1)(z-2)^2}\right]$$

$$= \lim_{z\to -1} \left[\cancel{(z+1)} \frac{z}{\cancel{(z+1)}(z-2)^2}\right]$$

$$= \lim_{z\to -1} \left[\frac{z}{(z-2)^2}\right]$$

$$\frac{-1}{(-1-2)^2} = -1/9$$

$$R[1,-1] = -1/9.$$

case ii) $z=2$, $a=+2$, $m=2$.

$$R[m,a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

$$= \frac{1}{1!} \lim_{z\to 2} \frac{d^1}{dz^1} \left[\cancel{(z-2)^2} \frac{z}{(z+1)\cancel{(z-2)^2}}\right]$$

$$= \lim_{z\to 2} \frac{d}{dz}\left[\frac{z}{(z+1)}\right]$$

$$= \lim_{z\to 2} \left[\frac{(\cancel{z}+1) - \cancel{z}}{(z+1)^2}\right]$$

$$= \lim_{z\to 2} \frac{1}{(z+1)^2}$$

$$R[2,2] = 1/9.$$

Note :- If we have to find cauchy Residue theorem, then we have to consider the above ~~Bottom~~ example is of the form

$$\int_C f(z).dz = 2\pi i \sum R.$$

$$= 2\pi i\left[R[1,-1] + R[2,2]\right]$$

$$= 2\pi i\left[\cancel{-1/a} + \cancel{1/a}\right]$$

$$= 0$$

2. Find the residue of the function $f(z) = \dfrac{2z+1}{z^2-z-2}$.

$$f(z) = \frac{2z+1}{(z-2)(z+1)}$$

case i $z=+2$, $a=2$, $m=1$

$$R[m,a] = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$= \frac{1}{0!}\lim_{z\to 2}\frac{d^0}{dz}\left[\cancel{(z-2)}\frac{2z+1}{\cancel{(z-2)}(z+1)}\right]$$

$$\lim_{z\to 2}\left[\frac{2z+1}{z+1}\right]$$

$$R[1,2] = 5/3$$

case ii) $z=-1$, $a=-1$, $m=1$.

$$R[m,a] = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$= \lim_{z\to -1}\left[\cancel{(z+1)}\frac{2z+1}{(z-2)\cancel{(z+1)}}\right]$$

$$\lim_{z\to -1}\left[\frac{2z+1}{z-2}\right] = 1/3$$

3. $f(z) = \dfrac{\sin z}{(2z-\pi)^2}$

~~$z = \pi, a = \pi, m = 2$.~~

$$f(z) = \frac{\sin z}{4[z - \pi/2]^2}$$

$z = \pi/2, a = \pi/2, m = 2$.

$$R[2, \pi/2] = \frac{1}{1} \lim_{z \to \pi/2} \frac{d}{dz}\left[\cancel{(z-\pi/2)^2} \frac{\sin z}{4\cancel{[z-\pi/2]^2}}\right]$$

$$= \lim_{z \to \pi/2} \frac{d}{dz}\left[\frac{\sin z}{4}\right]$$

$$= \lim_{z \to \pi/2} \frac{1}{4} \cos \pi/2$$

$= 0$.

$\frac{1}{4} \cos \pi/2$

$= 0$ //

7/4/17

4. Find Residue at the pole for $f(z) = \dfrac{z}{(z+1)^2(z^2+4)}$

sol

$$\frac{z}{(z+1)^2(z+2i)(z-2i)}$$

case 1

$z = -1, a = -1, m = 2$

$$R[2, -1] = \frac{1}{1} \times \lim_{z \to -1} \frac{d}{dz}\left[\cancel{(z+1)^2} \times \frac{z}{\cancel{(z+1)^2}(z+2i)(z-2i)}\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{z}{(z+2i)(z-2i)}\right]$$

$$= \lim_{z \to -1} d/dz\left[\frac{z}{z^2+4}\right]$$

$$= \frac{(z^2+4)\,1 - z(2z)}{(z^2+4)^2}$$

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$$\lim_{z \to -1} \frac{4-2z^2}{(z^2+4)^2}$$

$$= \frac{2}{25}$$

Case 2

$z = -2i$, $a = -2i$, $M = 1$.

$$R[m, a] = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a) f(z) \right]$$

$$= \lim_{z \to -2i} \left[\cancel{(z+2i)} \frac{z}{(z+1)^2 \cancel{(z+2i)}(z-2i)} \right]$$

$$= \lim_{z \to -2i} \left[\frac{z}{(z+1)^2 (z-2i)} \right]$$

$$= \frac{-2i}{(-2i+1)^2 (-2i-2i)}$$

$$= \frac{\cancel{-2i}}{(-4+1-4i)(\cancel{-4i})}$$

$$= \frac{1}{(-3-4i)(2)}$$

$$-6i - 8 \qquad \frac{1}{6i-8}$$

$$= \frac{-1}{-8i-6}$$

$$= -\frac{1}{2} \left[\frac{1}{3+4i} \times \frac{3-4i}{3-4i} \right] = -\frac{1}{2} \left[\frac{3-4i}{(9+16)} \right]$$

$$= -\frac{1}{50} [3-4i]$$

case iii:

$a = 2i \quad m=1, \; z = 2i.$

$$R[m,a] = \lim_{z\to 2i}\left[\cancel{(z-2i)}\; \frac{z}{(z+1)^2(z+2i)\cancel{(z-2i)}}\right]$$

$$= \frac{\cancel{2i}}{(2i+1)^2(\cancel{4}_2 i)}$$

$$= \frac{1}{(-4+1+4i)}$$

$$= \frac{1}{8i-6}$$

$$= \frac{1}{2}\left[\frac{1}{-3+4i} \times \frac{-3-4i}{-3-4i}\right]$$

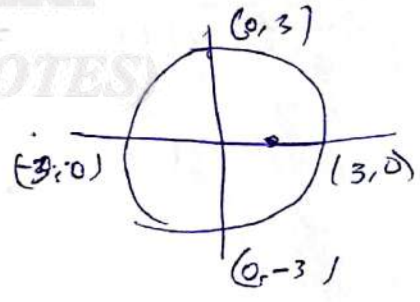
$$\frac{-3-4i}{9+16}$$

$$= \frac{1}{2}\left[\frac{-3-4i}{25}\right]$$

$$= \frac{1}{50}\left[-3-4i\right]$$

5. Cauchy's Residue method.

$$\int_c \frac{e^{2z}}{(z+1)(z-2)}\,dz \quad c: |z| = 3$$



Sol^n:- $\quad f(z) = \dfrac{e^{2z}}{(z+1)(z-2)}$

(i) $\quad a = -1, \; z = -1, \; m = 1$

$$R[m,a] = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$= \lim_{z\to -1}\left[\cancel{(z+1)}\,\frac{e^{2z}}{\cancel{(z+1)}(z-2)}\right]$$

$$\lim_{z \to -1} \left[\frac{e^{2z}}{z-2} \right]$$

$$= \frac{e^{-2}}{-3} \;//$$

<u>case ii</u>

$a = 2, \quad m = 1, \quad z = 2.$

$$R[m,a] = \frac{1}{1} \lim_{z \to 2} \frac{e^{2z}}{3}.$$

$$R[m,a] = \frac{e^{\cancel{2z}\,4}}{3}.$$

By Cauchy's Residue theorem

$$\int f(z).dz = 2\pi i \sum R.$$

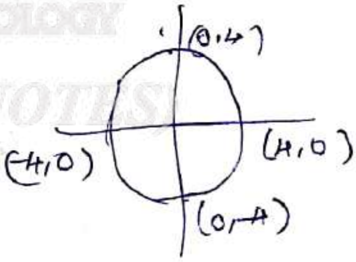
$$= 2 \times \frac{2\pi i}{3} \left[\frac{e^{4} - e^{-2}}{2} \right] \quad \cancel{\times \;\&\; \div \text{ by } 2}.$$

$$= \cancel{\frac{4\pi i}{3} \sinh 2z}.$$

$$= \frac{2\pi i}{3} \left[e^{4} - e^{-2} \right]$$

$$= \frac{2\pi i}{3} \left[e^{4} - \frac{1}{e^{2}} \right] \;//$$

6. $\displaystyle\int_C \frac{(z^2+5)}{(z-2)(z-3)}$ $\quad c : |z| = 4.$



<u>sol^n</u> $\quad f(z) = \dfrac{z^2+5}{(z-2)(z-3)}$

<u>case i</u>

$z = 2, \; a = 2, \; m = 1$

$$R[m,a] = 1 \times \lim_{z \to 2} \left[(z-2) \times \frac{z^2+5}{(z-2)(z-3)} \right]$$

$$= \lim_{z \to 2} \left[\frac{z^2+5}{z-3} \right] = -9$$

Case ii) $z = 3,\ a = 3\ ,\ m = 1$

$$R[m,a] = \lim_{z\to 3}\left[\frac{z^2-5}{z-2}\right]$$

$$= \underline{\underline{14}}$$

$$\int f(z)\,dz = 2\pi i \sum R.$$

$$= 2\pi i\,(-9+14)$$

$$= 10\pi i$$

$$= \underline{\underline{31.42\,i}}$$

7. $\displaystyle\int \frac{dz}{z^3(z-1)}$: $C\,|z| = 2$.

Case i $z = 0,\ a = 0\ ,\ m = 3$ - pole of order

$$R[m,a] = \frac{1}{2}\times \lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{1}{z-1}\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\cdot\frac{1}{(z-1)}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[-(z-1)^{-2}\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\frac{+2}{(z-1)^3}\right] = -1$$

$= -1$.

Case ii) $a = 1,\ z = 1\ ,\ m = 1$

$$R[m,a] = \lim_{z\to 1}\left[\frac{+1}{z^3}\right] = 1$$

$$\int f(z)\,dz = 2\pi i \sum R$$

$$= 2\pi i\,[\cancel{-1+1}]^{\,0} = 0$$

(3). $\int_c \frac{3z^3+2}{(z-1)(z^2+9)} \, dz$: $|z|=4$.

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$$f(z) = \frac{3z^3+2}{(z-1)(z+3i)(z-3i)}$$

case i

$a=1, \ z=1, \ m=1$

$$R[m,a] = \frac{1}{1} \cdot \lim_{z\to 1} \left[\frac{3z^3+2}{(z+3i)(z-3i)}\right]$$

$$= \cancel{\lim_{z}} \left[\frac{3+2}{(1+3i)(1-3i)}\right]$$

$$\frac{5}{1+9} = 1/2.$$

ii)

$z=-3i, \ a=-3i, \ m=1$

$$R[m,a] = 1 \times \lim_{z\to -3i} \left[\frac{3z^3+2}{(z-1)(z-3i)}\right]$$

$$= \frac{3(-3i)^2+2}{(-3i-1)(-3i-3i)}$$

$$\frac{-27+2}{(-3i-1)(-6i)}.$$

$$= \frac{+25}{-18+6i}$$

$$= \frac{25}{-18+6i} \times \frac{-18-6i}{-18-6i}$$

$$= \frac{-450-150i}{324+36}$$

$$= \frac{\cancel{-600}\,-450-150i}{360}$$

ii) $\lim_{z \to -3i} \frac{3z^3 + 2}{(z-1)(z-3i)}$

$= \frac{3(-27)(-i)+2}{(-3i-1)(-3i-3i)} = \frac{81i + 2}{6i(3i+1)}$

$= \frac{81i+2}{-18-6i}$

$= \frac{81i+2}{6[i-3]} \times \frac{i+3}{i+3}$

$= \frac{-81+i243+i2+6}{6[-1-9]}$

$= \frac{-1}{60}[-75+i245]$

$= \frac{-5}{60}[49i-15]$

$= \frac{1}{12}[15-i49]$

iii) $a = 3i$, $z = 3i$; $m = 1$

$R[m,a] = \lim_{z \to 3i} \frac{3z^3 + 2}{(z-1)(z+3i)}$

$= \frac{3(3i)^3 + 2}{(3i-1)(3i+3i)}$

$\frac{3 \cdot 9i + 2}{(3i-1) \; 6i}$

$18i^2$

9. Evaluate $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2(z-2)} dz$. $|z| = 3$

i) $a = 1, z = 1, m = 2$

$$R[m, a] = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z)\right]$$

$$= \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left[\frac{\cancel{(z-1)^2}\, \sin(\pi z^2) + \cos(\pi z^2)}{\cancel{(z-1)^2}(z-2)}\right]$$

$$\lim_{z \to 1} \frac{d}{dz} \left[\frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-2)}\right]$$

$$\frac{(z-2)\left[\cos(\pi z^2) - \sin(\pi z^2)\right] 2z\pi -}{(z-2)^2}$$

$$= \lim_{z \to 1} \left[\frac{(z-2)\left[\cos(\pi z^2) - \sin(\pi z^2)\right] 2\pi z - \left(\sin(\pi z^2) + \cos \pi z^2\right)}{(z-2)^2}\right]$$

$$= \frac{-1\left[\cos\pi - \sin\pi\right] 2\pi - \left[\sin(\pi) + \cos\pi\right]}{1}$$

$$-1[-1 - 0]\, 2\pi - [0 - 1]$$

$$(1)\, 2\pi + 1$$

$$= 2\pi + 1$$

case ii

$z = 2, \quad a = 2, \quad m = 1$

$$R[m, a] = \lim_{z \to 2} \left[\frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2}\right]$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1} \qquad (-1)^2$$

$$= 1$$

Cauchy Residue theorem $\int_C f(z) \cdot dz = 2\pi i \sum R$

$$= 2\pi i\left[2\pi + 1 + 1\right]$$

$$= \oint_C = 2\pi i\left[2\pi + 2\right]$$

$$4\pi i\left[\pi + 1\right]$$

10. Evaluate $\int_C \frac{2z^2+1}{(z+1)^2(z-2)} dz \quad |z| = 3$.

$z = -1, \quad a = -1, \quad m = 2$.

case i) $R[2,-1] = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz}\left[\frac{2z^2+1}{z-2}\right]$.

$= \lim_{z \to -1} \left[\frac{(z-2)4z - (1)(2z^2+1)}{(z-2)^2}\right]$

$= \lim_{z \to -1} \left[\frac{4z^2 - 8z - 2z^2 - 1}{(z-2)^2}\right]$

$= \lim_{z \to -1} \left[\frac{2z^2 - 8z - 1}{(z-2)^2}\right]$

$= \frac{2+8-1}{9} = 1$

case ii $R[1,2] = \lim_{z \to 2} \left[\frac{2z^2+1}{(z+1)^2}\right]$

$\lim_{z \to 2} \left[\frac{2z^2+1}{(z+1)^2}\right]$

$= 9/9 = 1$

$\int_C f(z) dz = 2\pi i [R_1 + R_2]$.

$2\pi i [1 + 1]$

$= 4\pi i$

11. Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)} \quad |z| = 5/2$.

case i $z = 1, a = 1, m = 2$

$R[m,a] = 1 \lim_{z \to 1} \left\{\frac{d}{dz}\left[\frac{z^2}{z+2}\right]\right.$

$= \lim_{z \to 1} \left[\frac{(z+2)2z - z^2 \cdot 1}{(z+2)^2}\right] = \lim_{z \to 1} \left[\frac{z^2+4z}{(z+2)^2}\right]$

$= 5/9$

case ii $R(m,a) \quad a = -2, \ z = -2, \ m = 1$

$= \lim_{z \to -2} \left[\frac{z^2}{(z-1)^2}\right] = \frac{4}{9}$

$\int_C f(z) \cdot dz = 2\pi i \, \Sigma R$

$= 2\pi i [5/9 + 4/9] = 2\pi i$

$y_0 + \frac{h}{24}\left[y_3' \quad y_0 + \right.$

2

y_4 $\quad + \quad h/24$

$9y_4' \quad 19 \quad -5$

$+ 55y_3' \; 9y_2' \; 37 \; 9y$

12. $\int_C \frac{ze^z}{(z^2-1)}\,dz \quad |z| = 2$

$= \int_C \frac{ze^z}{(z+1)(z-1)}\,dz$

~~z = -1 is the pole~~

case(i) $z = -1,\ a = -1,\ m = 1$

$\lim_{z \to -1} \frac{ze^z}{(z-1)}$

$\frac{-1\,e^{-1}}{-2} \quad \frac{1}{2e}$

case(ii) $\lim_{z \to 1} \frac{ze^z}{(z+1)} = \frac{e}{2}$

$\int_C f(z)\,dz = 2\pi i\left[\frac{1}{2e} + \frac{e}{2}\right]$

$\cancel{2\pi i\left[\frac{2+e}{4e}\right]} = \cancel{\pi i\left[\frac{2+2e^2}{e}\right]}$

$= \cancel{\frac{\pi i}{e}\left[2+e\right]}$

$= \pi i\left(\frac{1+e^2}{e}\right)$

$\pi i\left(\frac{1+e^2}{e}\right)$

Conformal transformation

1. $w = z^2$, $w = e^z$ & $w = z + \frac{1}{z}$.

$w = z^2$.

Sol^n $u + iv = (x+iy)^2$.

$u + iv = x^2 - y^2 + i2xy$.

$u = x^2 - y^2 \rightarrow (1)$

$v = 2xy \rightarrow (2)$

Case i $x = c_1$ (constant) replace in eq^n (1) & (2)

$\therefore u = c_1^2 - y^2$

$v = 2c_1 y$.

$\Rightarrow y = \frac{v}{2c_1}$

$\Rightarrow u = c_1^2 - \frac{v^2}{4c_1^2}$.

$v^2 = -4c_1^2[u - c_1^2]$ $[\because y = 4a(x-a)]$

$\rightarrow (3)$

eq^n (3) represent the eq^n of the parabola which is symmetrical about real-axis & focus at the origin.

Case ii $y = c_1$ in eq^n (1) & (2).

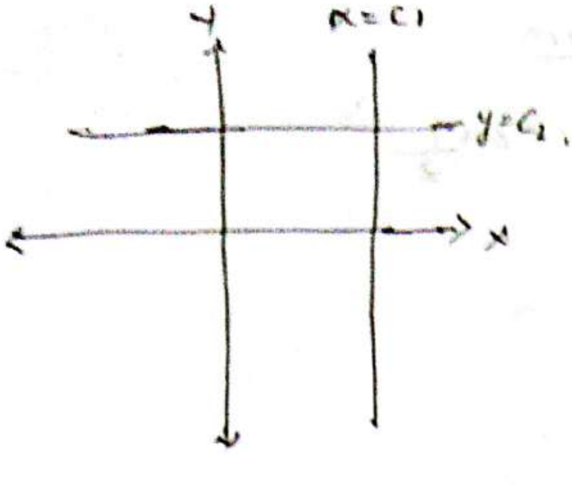
$u = x^2 - c_1^2$ & $v = 2c_1 x$.

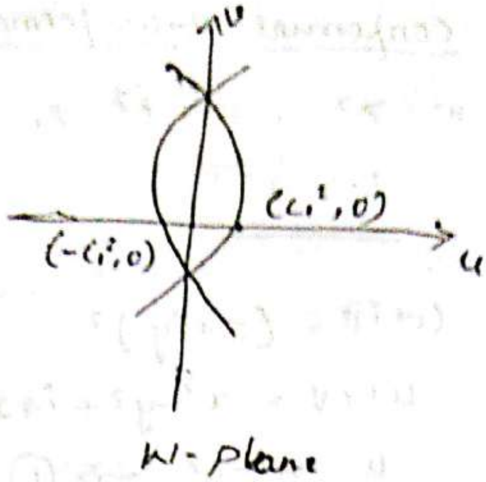
$\Rightarrow x = \frac{v}{2c_1}$

$u = \frac{v}{4c_1} - c_1^2$.

$v^2 = 4c_1^2[u + c_1^2] \longrightarrow (4)$

so eq^n (4) represent eq^n of the parabola which is symmetrical about the real axis & focus at the origin





Conclusion

Hence we can conclude that, the line which is parallel to co-ordinate axis in z-plane which maps onto eq-n of the parabola in w-plane.

2.

$W = e^z$

$u + iv = e^{x+iy}$.

$\Rightarrow u + iv = e^x e^{iy}$.

$u + iv = e^x[\cos y + i\sin y]$.

$u = e^x \cos y$ & $v = e^x \sin y$.

$u^2 + v^2 = e^{2x} \rightarrow ①$ & $\frac{v}{u} = \frac{\sin y}{\cos y} = \tan y \rightarrow ②$

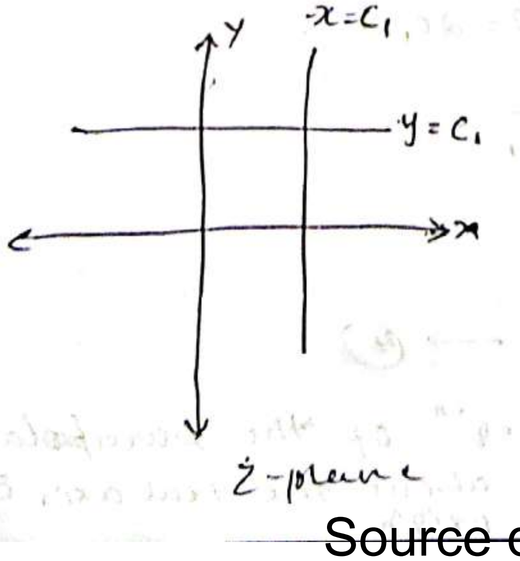
$x = c_1$ in eq-n ①

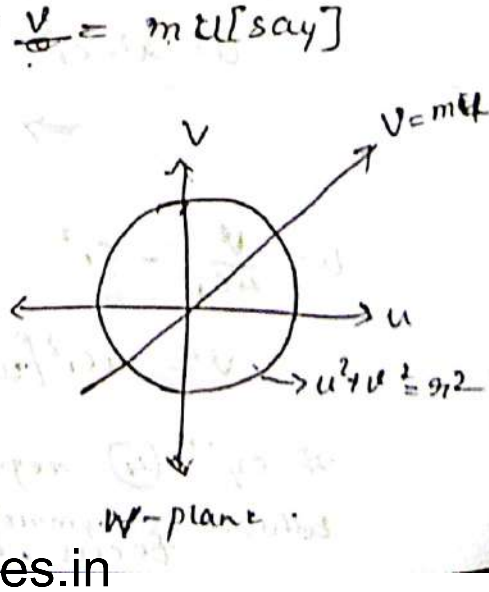
$u^2 + v^2 = e^{2c_1} = r^2$ [say]

$y = c_1$ in eq-n ②

$\frac{v}{u} = \tan c_1$

$\Rightarrow v = mu$ [say]





conclusion

The line which is parallel to y-axis $[x=c_1]$ in the z-plane maps on to eq^n of the circle $[u^2+v^2=r^2]$ in w-plane

lly the line parallel to x-axis $[y=c_1]$ in the z-plane maps on to the eq^n of straight line $[v=mu]$. in w-plane.

3. $w = z + \frac{1}{z}$.

~~we~~ if we do in cartesian form ~~we~~ it will be complicated. so we solve this by polar form

$w = z + \frac{1}{z}$ $\quad z \to re^{i\theta}$

$w = re^{i\theta} + \frac{1}{re^{i\theta}}$.

$\cancel{w = r[\cos\theta + i\sin\theta] + \frac{1}{r[\cos\theta + i\sin\theta]}}$

$u+iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$.

$= r[\cos\theta + i\sin\theta] + \frac{1}{r}[\cos\theta - i\sin\theta]$

$= \left[r+\frac{1}{r}\right]\cos\theta + i\left[r-\frac{1}{r}\right]\sin\theta$.

$\Rightarrow u = \left(r+\frac{1}{r}\right)\cos\theta$ & $v = \left(r-\frac{1}{r}\right)\sin\theta$

$\frac{u}{\cos\theta} = r+\frac{1}{r}$. & $\frac{v}{\sin\theta} = r-\frac{1}{r}$

$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$.

$\frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \to (1)$,

$\frac{u}{r+\frac{1}{r}} = \cos\theta$ & $\frac{v}{r-\frac{1}{r}} = \sin\theta$.

$\cancel{\cos^2}$

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1 \rightarrow (2)$$

$z = re^{i\theta}$ $\quad r = \sqrt{x^2+y^2}$

$\Rightarrow r^2 = x^2+y^2$

$\theta = \tan^{-1}(y/x) \Rightarrow y/x = \tan\theta$

$y = x\tan\theta$

case i

suppose $r = c_1$ or $x^2+y^2 = c_1^2$ [eq^n of circle]

in eq^n (2)

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \quad \text{wher } A^2 = \left[c_1+\frac{1}{c_1}\right]^2$$

$\rightarrow (3)$ $\quad B^2 = \left[c_1 - \frac{1}{c_1}\right]^2$

eq^n (3) represent ~~rep~~ eq^n of the ellipse with the foci $\left[\pm\sqrt{A^2-B^2}, 0\right] = (\pm 2a, 0)$ $\quad \therefore 2a = \left(\frac{2}{c_1}\right)^2$

$2a = \left(\frac{2}{c_1}\right)^2$

case ii

suppose $\theta = c_1$ or $y = x\tan c_1 = mx$ [say]

eq^n (1) becomes.

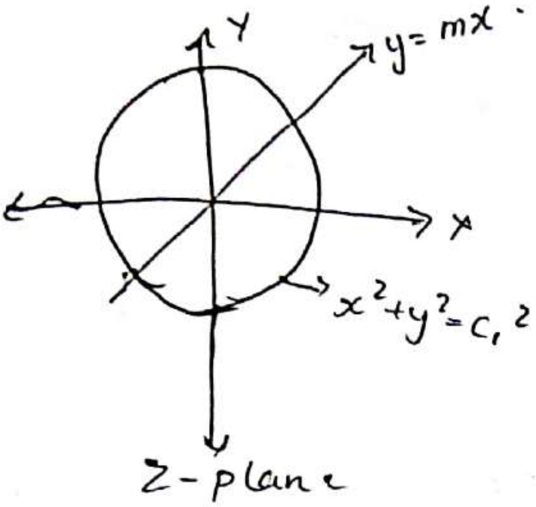
$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \longrightarrow (4)$$

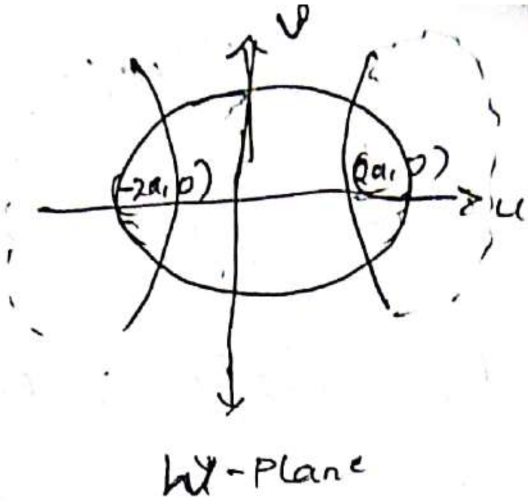
$(2\cos c_1)^2 = A^2$

$(2\sin c_1) = B^2$

eq^n (4) represent eq^n of the ellipse with the ~~foci~~

foci $\left[\pm\sqrt{A^2+B^2}, 0\right] = [\pm 2a, 0]$





Conclusion

- The eq^n of circle in z-plane $[x^2+y^2=c_1^2]$ in z-plane maps onto the eq^n of the ellipse in $\left[\frac{u^2}{A^2}+\frac{v^2}{B^2}=1\right]$ in w-plane.

Illy

The eq^n of straight line in z plane $[y=mx]$ maps onto the eq^n of the ellipse $\left[\frac{u^2}{A^2}-\frac{v^2}{B^2}=1\right]$ in w-plate.

$\frac{u^2}{(2\cos\theta)^2}+\frac{v^2}{(2\sin\theta)^2}=1$

$z=re^{i\theta}$

u^2

$z=re^{i\theta}$

$r^2=x^2+y^2$

$\boxed{r=\frac{1}{r}}$

$r^2=x^2+y^2 \quad \theta=t$

$\theta=\tan^{-1}y/x \quad y/t$

$r^2=x^2+y^2$

12|A|17

$w = i, 1, 0 \quad z = 1, -1\infty$

$w = \dfrac{az+b}{cz+d}$

Bilinear Transformation (BT)

The transformation $w = \dfrac{az+b}{cz+d}$ where a, b, c, d are real / complex constant such that $ad - bc = 0$ is called bilinear Transformation (BT)

Invariant Points :-

If the pt z maps itself ie $w = z$ under the Bilinear transformation, then the point is called as invariant point or Fixed point.

Example 1.-

To find bilinear transformation, we've

$$w = \frac{az+b}{cz+d} \quad \text{OR} \quad \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} =$$

$$= \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

① Find bilinear transformation which maps the point $z = 1, i, -1$ into $w = i, 0, -i$

Ans: $w = \dfrac{az+b}{cz+d}$ $\quad w_1 = i, w_2 = 0, w_3 = -i$

$z_1 = 1, z_2 = i, z_3 = -1$

$$w_1 = \frac{az_1+b}{cz+d}$$

$$i = \frac{a(1)+b}{c(1)+d} \Rightarrow a+b-ic-id = 0 \rightarrow ①$$

$$0 = \frac{a(i)+b}{c(i)+d} \Rightarrow ai + b = 0 \rightarrow ②$$

$$-i = \frac{a(-1)+b}{c(-1)+d} \Rightarrow -a+b-ic+id = 0 \rightarrow ③$$

eq^n ① + eq^n ③ $\Rightarrow 2b - 2ic = 0$

$\Rightarrow b - ic = 0.$

$$ai + b + 0c = 0$$
$$0a + b - ic = 0$$

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}} = \frac{c}{\begin{vmatrix} i & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{a}{-i} = \frac{-b}{-(i)^2} = \frac{c}{i}$$

$$= \frac{a}{-i} = \frac{-b}{1} = \frac{c}{i}$$

$a = -i$, $b = -1$, $c = i$

substitute in eq^{-n} ①.

$$-i - 1 - (i)^2 - id = 0$$
$$-i - \cancel{1} + \cancel{1} - id = 0$$
$$-i - id = 0$$
$$\cancel{d(i+i) = 0}$$
$$\cancel{d(2i) = 0}$$
$$-\cancel{i}d = \cancel{i}$$
$$\boxed{d = -1}$$

$$w = \frac{-iz - 1}{zi - 1} = \frac{1 + iz}{1 - iz} \quad \text{Or}$$

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\Rightarrow \frac{(w - i)(0 + i)}{(i - 0)(-i - w)} = \frac{(z - 1)(i + 1)}{(1 - i)(-1 - z)}$$

$$\Rightarrow \frac{\cancel{i}(w - i)}{-\cancel{i}(i + w)} = \frac{(z - 1)(i + 1)}{-(1 - i)(1 + z)}$$

$$(w - i)[1 + z - i - iz] = (i + w)[zi + z - i - 1]$$

$$w + \cancel{wz} - i\cancel{w} - izw - \cancel{i} - iz + i^2 + i^2 z$$
$$= i^2 z + iz - i^2 - \cancel{i} + izw + z\cancel{w} - i\cancel{w} - w.$$

$w - izw - iz - 1 = iz + 1 + izw - w$.

$\Rightarrow 2w - 2izw - 2iz - 2 = 0$.

$\Rightarrow w(1-iz) = iz + 1$

$$\boxed{w = \frac{1+iz}{1-iz}}$$

2. Find B.T which maps the point $z = 1, i, -1$ into $w = 2, i, -2$ also find invariant points or fixed point of the transformation.

Ans

$$w = \frac{az+b}{cz+d}$$

$$w_1 = \frac{az_1+b}{cz+d}$$

$$2 = \frac{a+b}{c+d}$$

$2c + 2d = a+b$.

$a+b-2c-2d = 0 \rightarrow (1)$

$i(ci+d) = ai+b$.

$ai+b+c-di = 0 \rightarrow (2)$

$$-2 = \frac{-a+b}{-c+d}$$

$2c-2d = -a+b$

$a-b+2c-2d = 0 \rightarrow (3)$

Substitute eq^n (1) and (3).

$\cancel{a}+\cancel{b} - \cancel{2c} - 2\cancel{d} = 0$

$a - \cancel{b} + \cancel{2c} - \cancel{2}d = 0$.

$2a - 4d = 0 \rightarrow (4)$

sub eq^n (2) and (4),

$ai + b + c - id = 0$

$2a + 0b + 0c - 4d = 0$,

Source diginotes.in

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$= \frac{(w-2)(i+2)}{(2-i)(-2-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$= \frac{(w-2)(i+2)}{-(2-i)(2+w)} = \frac{(z-1)(i+1)}{-(1-i)(1+z)}$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{z-1}{z+1}\left\{\frac{(i+1)(2-i)}{(1-i)(i+2)}\right\}$$

$$\frac{(w-2)}{w+2} = \frac{(z-1)}{(z+1)}\left[\frac{2i-i^2-2-i}{i+2-i^2-2i}\right]$$

$$= \frac{(z-1)}{(z+1)}\frac{(i+3)}{(-i+3)}$$

$$\Rightarrow \frac{(w-2)(-i+3)}{(w+2)(i+3)} = \frac{z-1}{z+1}$$

$$\Rightarrow (w-2)(-i+3)(z+1) = (z-1)(w+2)(i+3)$$

$$\Rightarrow (w-2)[-iz-i+3z+3] = (w+2)[zi+3z-i-3]$$

$$\Rightarrow -iwz-iw+3wz+3w+2zi+2i-6z-6$$

$$= iwz+3wz-iw-3w+2iz+6z-2i-6$$

$$\Rightarrow -iwz+3w+2i-6z = iwz-3w-2i+6z = 0.$$

$$\Rightarrow -2iwz+6w+4i-12z = 0$$

$$iwz-3w-2i+6z = 0$$

$$w(iz-3) = 2i-6z.$$

$$w = \frac{2i-6z}{iz-3}.$$

for invariant pt

$$z = \frac{2i-6z}{iz-3},$$

$$z(iz-3) = 2i-6z$$

$$iz^2-3z = 2i-6z$$

$iz^2+3z-2i=0$.

$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. $\quad b=3,\ a=i,\ c=-2i$.

$z = \frac{-3 \pm \sqrt{9-4(i)(-2i)}}{2i}$

$z = \frac{-3 \pm \sqrt{9-8}}{2i}$

$z = \frac{-3 \pm 1}{2i}$

$z = \frac{-3+1}{2i}$ or $z = \frac{-3-1}{2i}$

$= \frac{-2}{2i}$ $\quad = \frac{-4}{2i}$

$z = \frac{-1}{i}$ $\quad z = \frac{-2}{i}$

3. Find the BT which maps $z=0, -i, -1$, & $w = i, 1, 0$

Ans

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$= \frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

$$= \frac{(w-i)(1)}{(i-1)(-w)} = \frac{z(-i+1)}{(i)(-1-z)}$$

$$\frac{(w-i)}{(-w)(i-1)} = \frac{z(i-1)}{i(z+1)}$$

$i(z+1)(w-i) = (-w)(i-1)(z)(i-1)$

$(zi+i)(w-i) = (-wi+w)(zi-z)$

$\cancel{wzi} + iw + z + 1 = wz + \cancel{zwi} + wzi - zw$.

$iw+z+1 = wz[1+i-1]$

$iw+z+1 = wzi$

$= wi - iwz + 1 + z = 0.$

$\Rightarrow wi(1-z) = -(1+z).$

$$w = \frac{-(1+z)}{i(1-z)}$$

$$w = \frac{i(1+z)}{1-z} \qquad \boxed{\frac{1}{i} = -i}$$

4. Find B.T which maps $z = 0, i, \infty$. $w = 1, -i, -1$ and also find the invariant point. $\to 1$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}.$$

$$\left\{ \frac{\cancel{z_3}\left[\frac{z_2}{z_3} - 1\right]}{\cancel{z_3}\left[1 - \frac{z}{z_3}\right]} = \frac{\frac{1}{\infty} - 1}{1 - \frac{1}{\infty}} = \frac{0-1}{1-0} = -1 \right.$$

$$= \frac{(w-1)(-i+1)}{(1+i)(-1-w)} = -1\left[\frac{z-0}{0-i}\right].$$

$$\frac{(w-1)(1-i)}{-(1+i)(w+1)} = \frac{z}{i}$$

$$\frac{(w-1)(1-i)}{-(1+i)(w+1)} = -zi$$

$= (w-1)(1-i) = zi(1+i)(w+1).$

$= w - iw - 1 + i = (zi - z)(w+1).$

$= w - iw - 1 + i = zwi + zi - zw - z.$

$\cancel{zw(i-1) + z}$

$= -w + 1 - iw + i = z + iz + wz + iwz.$

$= -w - iw - wz = z + iz - 1 - i.$

$w[-1-i-z-iz] = z + iz - 1 - i$

$-w(1+i+z+iz) = (z-1)(z+1).$

$= -w(1+i+z+iz) = (z+1)(1+i)$

$= -w\left[(1 \cancel{+} i)(z \cancel{+} i)\right] = (z-1)(1 \cancel{+} i)\right].$

$$w = \frac{(1-z)}{(1+z)} \Rightarrow z = \frac{1-z}{1+z}.$$

$$z + z^2 - 1 + z = 0.$$

$$z^2 + 2z - 1 = 0.$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{4+4}}{2}.$$

$$= z = \frac{\cancel{-2}\left[1 \pm \sqrt{2}\right]}{\cancel{2}}$$

$$z = -1+\sqrt{2}, \ -1-\sqrt{2}.$$

H.W

5. $z = -1, i, 1, \quad w = 1, i, -1$
6. $z = 0, -i, -1, \quad w = i, 1, 0.$ (Repeated).
7. $z = 0, i, \infty \quad w = 1, -i, -1$
8. $z = i, 1, -1 \quad w = 1, 0, \infty$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-1)(-1)}{[1-0]} = \frac{(z-i)(1+1)}{(i-1)(-1-z)}$$

$$\cancel{-}\frac{(w-1)}{1} = \cancel{-}\frac{2(z-i)}{(i-1)(z+1)}.$$

$\cancel{(w-1)(i-1)(z+1) = 2z-2i}$

$\cancel{(wi-w-i+1)(z+1) = 2z-2i}$

$\cancel{zwi-zw-iz+z+wi-w-i+1 = 2z-2i}$

$$w = \frac{2(z-i)}{(i-1)(z+1)} + 1$$

$$w = \frac{2(z-i)+(i-1)(z+1)}{(i-1)(z+1)}$$

$$w = \frac{iz - zi + i - 1 + iz - z}{(1+z)(i-1)}$$

$$w = \frac{z - i - 1 + iz}{(1+z)(i-1)}$$

$$= \frac{z(1+i) - (1+i)}{(1+z)(i-1)}$$

$$w = \frac{(z-1)(1+i)}{(z+1)(i-1)}$$

9. Find B.T $z = \infty, i, 0$ $w = -1, -i, 1$ and also find the fixed point or invariant point.

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w+1)(-i-1)}{(-1+i)(1-w)} = \frac{(-1)(i)}{-z}$$

$$\frac{w+1}{1-w} = \frac{i(-1+i)}{-z(i+1)} = \frac{-i+i^2}{-z(i+1)} = \frac{\cancel{-}(1+i)}{\cancel{-}z(i+1)}$$

$$\frac{w+1}{1-w} = \frac{1}{z}$$

$$(w+1)z = 1 - w$$

$$wz + z - 1 + w = 0$$

$$w(1+z) = 1 - z$$

$$\boxed{w = \frac{1-z}{1+z}}$$

invariant point. $z = \dfrac{-2[1 \pm \sqrt{2}]}{-2}$

$$z = -1 + \sqrt{2}, \quad -1 - \sqrt{2}$$

10. $z = \infty, i, 0 \quad , \quad w = 0, i, \infty$

$$\frac{\cancel{(w-w_1)}^{-1}\,(w_2-w_3)}{\cancel{(w_1-w_2)}\,(w_3-w)} = \frac{(z-z_1)\,\cancel{(z_2-z_3)}^{-1}}{(z_1-z_2)\,\cancel{(z_3-z)}}$$

$$\frac{(i-0)}{(0-w)} = \frac{(z-0)}{(0-i)}$$

$$\frac{i}{-w} = \frac{z}{-i}$$

$$\boxed{w = \frac{-1}{z}}$$

$$z = \frac{-1}{z}$$

$$z^2 = -1$$

$$z = \pm i$$

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MODULE – IV

Probability Distribution & Joint Probability :-

Defination :- The probability of any Event can be defined as ratio of

$$P(E) = \frac{\text{No of favourable cases.}}{\text{No of possible cases}} = \frac{m}{n}$$

⟨i⟩ $P(E) + P(\bar{E}) = 1$.

$p + q = 1$
$p \rightarrow$ success
$q \rightarrow$ failure.

⟨ii⟩ $P(E) \leq 1$

Random Experiment :- The Random Experient when we perform repeatedly giving different results are called Random expriment.

Random variable :-
A variable whose value is determined by the outcome of Random experiment is called Random variable. It is also known as

Random variable (RV)

- discrete RV
  - Binomial Distribution
  - Poisson D
- continous RV
  - Normal - D
  - Exponential - D

D – Distribution.

Discrete RV :- If a RV takes finite no of values then it is called as discrete RV.

Continous RV:- If a RV takes continous no of values then it is called as continous R.V.

Note :-

Mutually Exclusive Event	Not mutually Exclusive Event
→ It is also called as dependent.	→ It is Independent.
① → Two or more event are said to be mutually Exclusive, if the happening of the one event prevent the simultaneous happening of other event.	→ do-not prevent the simultaneous happening of other event.
→ $P(A \cup B) = P(A) + P(B)$. [Addition]	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
② → $P(A \cap B) = P(A) \cdot P(B/A)$ [multiplication Rule]	$P(A \cap B) = P(A) \cdot P(B)$.

Discrete Probability Distribution :-

If for each value of x_i of a discrete Random variable 'X' we assign a real no $P(x_i)$ such that.

⟨i⟩ $P(x_i) \geq 0$

⟨ii⟩ $\sum P(x_i) = 1$

then the function $P(x)$ is called as probability fn or probability density fn or probability mass fn.

The Distribution fn $f(x)$ define as

$$f(x) = P(X \leq x) = \sum_{i=1}^{x} P(x_i) \longrightarrow \text{Cumulative fn.}$$

x

f

In distribution table, we are finding mean

$$\text{Mean}(\mu) = \sum x_i P(x_i)$$

$$\text{Variance } (v) = \sum (x_i - \mu)^2 P(x_i)$$

(or)

$$\sum x_i^2 P(x_i) - \mu^2.$$

(σ) Standard deviation $= \sqrt{V}$

1. show that the following distribution represent a discrete probability, find mean & variance.

Ans Condition

x	10	20	30	40
$P(x)$	1/8	3/8	3/8	1/8

(i) $P(x_i) \geq 0$.

(ii) $\sum P(x_i) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

mean $= \sum P(x_i) x_i$

$= \frac{10}{8} + \frac{60}{8} + \frac{90}{8} + \frac{40}{8}$

$\frac{200}{8} = 25$

$V = x_i^2 P(x_i) - \mu^2$

$= \left[\left(100 \times \frac{1}{8}\right) + \frac{400 \times 3}{8} + \frac{900 \times 3}{8} + \frac{1600}{8}\right] - 625$

$= 75$ $[700 - (25)^2]$

$S.D = \sqrt{V}$

$S.D = \sqrt{75} = 5\sqrt{3}$

2. Find the value of 'k' such that the following represent the finite probability distribution. Hence find mean and S.D and also find $P(x \leq 1)$, $P(x > 1)$ & $P(-1 \leq x \leq 2)$

x	-3	-2	-1	0	1	2	3
$P(x)$	k	2k	3k	4k	3k	2k	k.

given that $P(x)$ is discrete probability distribution.

ie $P(x_i) \geq 0$

$\sum P(x_i) = 1$

$\Rightarrow k + 2k + 3k + 4k + 3k + 2k + k = 1$

$k = 1/16$

$\mu = \sum P(x_i)\, x_i$

$= \cancel{-3}\left[\frac{1}{16} \times (-3)\right] + \left[\frac{2}{16} \times (-2)\right] + \left[\frac{3}{16}(-1)\right] + \left[0 \times \frac{4}{16}\right]$

$+ \left[\frac{3}{16} \times 1\right] + \left[\frac{2}{16} \times 2\right] + \left[\frac{1}{16} \times 3\right]$

$= \underline{0}$

$\text{Variance} = 9\left(\frac{1}{16}\right) + 4\left(\frac{2}{16}\right) + \frac{3}{16} + \frac{3}{16} + 4\left(\frac{2}{16}\right)$
$+ \left(\frac{9}{16}\right)$

$= 2.5$

$S.D = \sqrt{2.5} = 1.58$

$P(x \le 1) = 1 - P(x > 1)$

$= 1 - \{P(x=2) + P(x=3)\}$

$1 - \left[\frac{2}{16}\right] + \left[\frac{1}{16}\right]$

$\cancel{1 - \left[\frac{1}{4} + \frac{3}{16}\right]} = 13/16$

$P(x > 1) = P(x=2) + P(x=3)$

$= 2K + K = \frac{3}{16}$

$P(-1 \le x \le 2) = P(x=0) + P(x=1) + P(x=-1) +$
$P(x=2)$

$= 3K + 4K + 3K + 2K$

$= 12K = 12/16 = 3/4$

3. A Random variable x has a density fn

$$P(x) = \begin{cases} kx^2 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Evaluate K and find $P(x \le 1)$, $P(1 \le x \le 2)$, $P(x \le 2)$, $P(x > 1)$, $P(x > 2)$

sol) $\because P(x)$ is a density fn)

x	0	1	2	3
$P(x)$	0	k	$4k$	$9k$

$P(x) \geq 0$ & $\sum P(x) = 1$.

$k + 4k + 9k = 1$

$k = 1/14$

$\mu = \sum P(x_i) x_i$

$P(x \leq 1) = \frac{1}{0} + \frac{1}{k} = \frac{1}{14}$.

$P(1 \leq x \leq 2) = \frac{\cdot 1}{k} + \frac{\cdot 2}{4k}$ ~~$= \frac{1}{14} + \frac{0 \cdot 2 \times 4}{14 \cdot 7}$~~

~~$\frac{2+1}{28} = 3/28$.~~

$k + 4k$

$= 5k$.

~~$\frac{1 + 4(2)}{14} = \frac{9}{14}$.~~

$= \frac{5}{14}$

$P(x \leq 2) \quad k + 4k = 5k = \frac{9}{14}$.

$P(x > 1) = 4k + 9k = 13k = \frac{13}{14}$.

$P(x > 2) = 9k = \frac{9}{14}$.

Binomial Distribution

If p is the probability of success and 'q' is the probability of failure, then the probability of x successes out of n trials is given by

$$\boxed{P(x) = {}^nC_x \cdot p^x q^{n-x}}$$

x	0	1	2	3	...	n
$P(x)$	q^n	${}^nC_1 p q^{n-1}$	-	-	-	p^n

$p(x) \geq 0$ & $\sum p(x) = q^n + {}^nC_1\, p q^{n-1} + \dots + p^n$

$= (q+p)^n = 1^n = 1.$

$\because \{p+q=1\}$

$$\sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} = 1$$

|||ly

$$\sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1}\, q^{(n-1)-(x-1)} = 1.$$

$$\sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2}\, q^{[(n-2)-(x-2)]} = 1$$

Obtain the mean and S.D of binomial distribution.

$$\text{Mean}(\mu) = \sum_{x=0}^{n} x_i\, P(x_i)$$

$$= \sum_{x=0}^{n} x\, {}^nC_x\, p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{(n-x)!\,x!}\, p^x q^{n-x}.$$

$\frac{n!}{(n)!\,(-1)!}$

$$= \sum_{x=0}^{n} \cancel{x}\, \frac{n!}{(n-x)!\,\cancel{x}(x-1)!}\, p^x q^{n-x}.$$

$$= \sum_{x=\phi}^{n} \frac{n(n-1)!}{[(n-1)-(x-1)]!\,(x-1)!}\, p^{x-1}\, p\, q^{[(n-1)-(x-1)]}$$

$$= np \sum_{x=1}^{n} {}^{(n-1)}C_{x-1}\, p^{x-1}\, q^{[(n-1)-(x-1)]}$$

mean $= np(1)$

$$\text{Variance} = \sum_{x=0}^{n} x^2 P(x) - \mu^2$$

$x^2 = x(x-1) + x$

$$= \sum_{x=0}^{n} [x(x-1)+x]\, P(x) - \mu^2$$

$$= \sum_{x=0}^{n} x(x-1)\,p(x) + \underbrace{\sum_{x=0}^{n} x\,p(x)}_{\mu} - \mu^2 .$$

$$= \sum_{x=0}^{n} x(x-1)\left[{}^{n}C_x\, p^x q^{n-x}\right] + \mu - \mu^2 .$$

$$= \cancel{\sum_{x=0}^{n} x(x-1) + \mu - \mu^2} .$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\,x!} p^x q^{n-x} + \mu - \mu^2 .$$

$$= \sum_{x=2}^{n} \frac{\cancel{x(x-1)}\; n!}{(n-x)!\;\cancel{x(x-1)}(x-2)!} p^x \cdot q^{n-x} + \mu - \mu^2$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{[(n-2)-(x-2)]!\,(x-2)!} p^{x-2} \cdot p^2\, q^{[(n-2)-(x-2)]} + \mu - \mu^2$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{[(n-2)-(x-2)]!\,(x-2)!} p^{x-2} q^{[(n-2)-(x-2)]} + \mu - \mu^2$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{[(n-2)-(x-2)]} + \mu - \mu^2$$

$$= np^2(n-1)(1) + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2 \qquad S.D = \sqrt{npq}$$

$$= np(1-p) = npq .$$

1. When a coin is tossed 4 times find the probability of getting.

(i) Exactly one head

(ii) atmost 3 head.

(iii) atleast 2 heads.

Sol^n given $n=4$, $p = 1/2$, $q = 1/2$.

$x \to$ denotes no of heads in a coin

$\therefore P(x) = {}^{n}C_x\, p^x \cdot q^{n-x}$.

$= {}^4C_x (1/2)^x (1/2)^{4-x}$

$$\boxed{P(x) = \frac{1}{16}\, {}^4C_x}$$

$\frac{0}{4}$ = $\frac{0}{4}$

(i) Exactly one head:

$P(x=1) = \frac{1}{16}\, {}^4C_1 = \frac{1}{16}(4) = \frac{1}{4}$.

(ii) at most 3 head $= P(x \le 3) = 1 - P(x > 3)$

$= \frac{1}{16} +$

$ {}^4C_0 \times \frac{1}{16} + {}^4C_1 \frac{1}{16} + {}^4C_2 \frac{1}{16} + {}^4C_3 \frac{1}{16}$.

$= \frac{1}{16} + \frac{1}{4} + \frac{6}{16} + 4 \times \frac{1}{16}$

$\frac{2}{4} + \frac{7}{16} \quad \frac{2(4)+7}{16} = \frac{15}{16}$

(iii) atleast 2 heads : $P(x \ge 2) =$

$P(2) + P(3) + P(4)$

$\frac{4!}{2!\,2!} = 6$

$\frac{1}{16}\, {}^4C_2 + \frac{1}{16}\, {}^4C_3 + \frac{1}{16}\, {}^4C_4$

$\frac{4!}{0!\,4!}$

$\frac{1}{16} \times 6 + \frac{1}{16} \times 4 + \frac{1}{16}$

$= \frac{11}{16}$.

24/4/17

3. In a consignment of electric lamps 5% are defective, if a Random sample of a class are inspected, what is the probability that one or more lamp are defective in 8 lamps

$p = \frac{5}{100} = 0.05,$ $p+q=1$

$q = 0.95$

$n=8.$

Let x Denote the no of lamps which are defective.

$P(x \geq 1) = 1 - P(x<1)$

$1 - P(x=0)$

$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x} = {}^{8}C_{x} (0.05)^{x} (0.95)^{8-x}$

$P(x=0) = {}^{8}C_{0} (0.05)^{0} (0.95)^{8-0}$

$= 1 \times 1 \times 0.66342$

$= 0.6634.$

$P(x) = 1 - P(x=0)$

$1 - 0.6634.$

$= 0.3366$

4. The probability that 60 years will live upto 70 years is 0.65. what is the probability that out of 10 persons age 60 atleast 7 of them will live 70 years.

$P=0.65$

$n=10.$

$q = 0.35$

$P(x \geq 7) = ?$

$= 1 - P(x<7).$

or

$P(x=7) + P(x=8) + P(x=9) + P(x=10)$

$n=10$

$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$P(x \geq 7)$

$n=10$

${}^{n}C_{x}\, p^{x} q$

$x=7$.

$$P(x=7) = {}^{10}C_7 P^7 q^{10-7} = {}^{10}C_7 (0.65)^7 (0.35)^3$$

$$= 0.252188.$$

$$P(x=8) = {}^{10}C_8 (0.65)^8 \times (0.35)^2$$

$$= 0.17565$$

$$P(x=9) = {}^{10}C_9 (0.65)^9 \times (0.35)^1$$

$$= 0.07249$$

$$P(x=10) = {}^{10}C_{10} (0.65)^{10} \times (0.35)^0$$

$$= 0.0134627.$$

$$P(x \geq 7) = 0.252188 + 0.17565 + 0.07249 + 0.0134627$$

$$P(x \geq 7) = 0.51379.$$

5. In a Quiz Contest of the answering yes or no, what is the probability of atleast 6 answers out of 10 question asked also find the probability of the same if there are 4 options for the correct option

1st option

$P = 1/2 = q$, $n=10$, $P(x \geq 6)$

$$P(x \geq 6) = 1 - P(x < 6)$$

$$= 1 - [P(x=0) + \dots P(x=5)]$$

or

$$P(x=7) + P(x=8) + P(x=9) + P(x=10)$$

$$P(x) = {}^nC_x p^x q^{n-x}.$$

$${}^{10}C_x (0.5)^x (0.5)^{10-x}.$$

$$= {}^{10}C_x (1/2)^x (1/2)^{10-x}$$

$${}^{10}C_x (1/2)^{10}.$$

$$P(x=7) = {}^{10}C_7 (0.5)^7 (0.5)^3$$

$$= 0.1171875$$

$$P(x=9) = 0.009765$$

$$P(x=8) = 0.043945$$

$$P(x=10) = 9.7656 \times 10^{-7}.$$

$$P(x \geq 6) = 0.376$$

ii) 4 options for the correct answer

$P = 1/4$, $q = 3/4$, $n = 10$

$$P(x \geq 6) = P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)$$

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x=6) = {}^{10}C_6 (0.25)^6 (0.75)^4$$

6. If the mean and S.D of no. of correctly answer question given to 4096 student are 2.5 and $\sqrt{1.875}$ find the Estimate no of ~~condition~~ candidate answering correctly.

1. 8 or more.
2) 2 or less.
3) 5

$$\sqrt{npq}$$

$$\left\{ \begin{matrix} p = 1 - q \\ p + q = 1 \end{matrix} \right\}$$

$$mean(\mu) = np = 2.5$$

$$S.D = (\sigma) = \sqrt{npq} \Rightarrow npq = 1.875$$

$$q = \frac{1.875}{2.5} = 0.75 \qquad p = 0.25$$

$$np = 2.5 \Rightarrow n = \frac{2.5}{0.25} = 10$$

$$P(x) = {}^{10}C_x (0.25)^x (0.75)^{10-x}$$

$$P(x \geq 8) = P(x=8) + P(x=9) + P(x=10)$$

$$P(8) = 3.8623 \times 10^{-4}$$

$$P(9) = 2.8610 \times 10^{-5}$$

$$P(10) = 9.5367 \times 10^{-7}$$

$$P(x \geq 8) = \boxed{4.1579 \times 10^{-4}}$$

$$\text{Estimate value } f(x) = 4096\, P(x)$$

$$= \boxed{1.7030} \approx 2.$$

(ii) $P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$

$$P(0) = 0.05631$$

$$P(1) = 0.1877$$

$$P(2) = 0.5255$$

$$\text{Estimate value} = 4096\,(0.5255) = 2152.448$$

(iii) $P(x=5) = {}^{10}C_5 (0.25)^5 (0.75)^5$

$$= \boxed{0.05839}$$

$$\text{Est value} = 239.165$$

7. In 800 families with 5 children each, How many families would be expected to have

(i) 3B (ii) 5G.

(iii) Either 2 or 3B.

(iv) atmost 2G.

by assuming that probability of Boys and girls are equal.

Sol^n

$P = 1/2 = q$. , $n = 5$.

Let x = No of boys.

$P(x) = {}^nC_x \, p^x q^{n-x}$.

$= {}^5C_x (0.5)^x (0.5)^{5-x}$.

$= \frac{1}{2^5} \cdot {}^5C_x$.

(i) $P(x=3) \cdot \frac{1}{2^5} \cdot {}^5C_3$.

$= 0.3125$

$0.3125 \times 800 = 250$.

ii) $P = 1/2 = q$, $n = 5$.

$x = 0$ [as ~~we~~ x denotes boys].

$P(0) = \frac{1}{2^5} \, {}^5C_x$.

$= 0.03125$

No of $f(x) = 25$

iii) Either 2 or 3 B.

$P(x=2) + P(x=3)$

$= \frac{1}{2^5} \, {}^5C_2 + \frac{1}{2^5} \, {}^5C_3$

$= 0.625$

$f(x) =$ $= 0.625 \times 800 = \underline{500}$

(iv) ~~P~~(. atmost 2 Girls.

$= 5B\ \&\ 0G + 4B\ \&\ 1G + 3B\ \&\ 2G.$

$= P(x=5) + P(x=4) + P(x=3).$

$= \frac{1}{2^5}\,{}^5C_5 + \frac{1}{2^5}\,{}^5C_4 + \frac{1}{2^5}\,{}^5C_3.$

$= 0.5$

$\underline{f(x) = 400}$

25/4/18

Poisson Distribution.

Poisson Distribution is Regarded as limiting form of Binomial distribution where n is very large ($n \to \infty$) and p is small $p \to 0$ so that np tends to finite constant say m is given by

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$n \to \infty.$

$P \to 0.$

$np = m.$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ series sol

$\lim_{x \to 0} (1+x)^{1/x} = e.$

From binomial distribution.

$$P(x) = {}^nC_x\, p^x q^{n-x}.$$

$$= \frac{n!}{(n-x)!\,x!}\, p^x q^{n-x}.$$

$$p(x) = \frac{n(n-1)(n-2)\cdots[n-(x-1)]\cancel{(n-x)!}}{\cancel{(n-x)!}\; x!} \times \frac{p^x q^n}{q^x}$$

$$= \frac{n.\ n[1-1/n]\ n[1-2/n]\ \cdots\ n\left[1-\frac{(x-1)}{n}\right]}{x!}\ p^x \cdot \frac{q^n}{q^x}$$

$$= \frac{n^x[1-1/n][1-2/n]\cdots\left[1-\frac{(x-1)}{n}\right]}{x!}\ p^x \frac{q^n}{q^x}$$

$$P(x) = \frac{np^x}{x!}\left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\cdots\left[1-\frac{(x-1)}{n}\right]\frac{q^n}{q^x} \rightarrow (1)$$

$$np = m \implies p = \frac{m}{n}.$$

$$q^n = [1-p]^n$$
$$= \left[1-\tfrac{m}{n}\right]^n$$
$$= \left[\left(1-\frac{m}{n}\right)^{-n/m}\right]^{-m}, \quad \text{take } k = \frac{-m}{n}.$$

$$q^n = \left[(1+k)^{1/k}\right]^{-m} \quad \text{as } n \to \infty, \; k \to 0.$$

$$q^n = e^{-m}$$

$$\boxed{(1+k)^{1/k} = e} \quad \text{as } k \to 0.$$

$$q^x = (1-p)^x = 1 \quad \text{as } p \to 0.$$

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left[1-\frac{(x-1)}{n}\right] = 1 \quad \text{as } n \to \infty.$$

eq^n (1) becomes

$$\boxed{P(x) = \frac{m^x e^{-m}}{x!}}$$

Obtain the mean and S.D of Poisson Distribution

$$\text{Mean } (\mu) = \sum_{x=0}^{n} x\,P(x)$$

$$= \sum_{x=0}^{n} x\left[\frac{m^x e^{-m}}{x!}\right]$$

$$\sum_{x=0}^{n} \cancel{x}\left[\frac{m^x e^{-m}}{\cancel{x}(x-1)!}\right]$$

$$e^{-m}\sum_{x=1}^{n}\left[\frac{m}{1} + \frac{m^2}{1!} + \frac{m^3}{2!} + \cdots\right].$$

$$= me^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots\right]$$

$= me^{-m}e^{m}$ ...

$$\boxed{\mu = m = np}$$

Variance

$$V = \sum x_i^2 P(x_i) - \mu^2$$

$$= \sum_{x=0}^{n} x^2 P(x) - \mu^2$$

$$= \sum_{x=0}^{n} x^2 \left[\frac{m^x e^{-m}}{x!}\right] - m^2$$

$$= \sum_{x=0}^{n} \frac{x\left[m^x e^{-m}\right]}{(x-1)!} - m^2$$

$$= \sum_{x=0}^{n} \left[x(x-1) + x\right] P(x) - \mu^2$$

$$= \sum_{x=0}^{n} x(x-1) P(x) + \sum_{x=0}^{n} x P(x) - \mu^2$$

$$= \sum_{x=0}^{n} x(x-1) \frac{m^x e^{-m}}{x!} + \mu - \mu^2$$

$$= \sum_{x=2}^{n} x\cancel{(x-1)} \frac{m^x e^{-m}}{x\cancel{(x-1)}(x-2)!} + \mu - \mu^2$$

$$= e^{-m}\left[\frac{m^2}{1} + \frac{m^3}{1!} + \frac{m^4}{2!} + \cdots\right] + \mu - \mu^2$$

$$V = e^{-m} m^2 \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots\right] + m - m^2$$

$$\cancel{e^{-m}} \cdot \cancel{m^2} \cancel{e^{m}} + m - \cancel{m^2}$$

$$= m$$

$$\boxed{V = m}$$

$\therefore$ in poisson Distribution $\boxed{V = \mu = m}$

$$S.D = \sqrt{V} = \sqrt{m}.$$

Note in binomial distribution n is small

p is large.

ex:- $n = 10, 15, 20, 25, 30$.

$p = 0.1, 0.2, 0.3, 0.4, 0.5$

in poisson distribution n is large.

p is small.

$n = 35, 40, 40, \ldots 100$

$p = 0.01, 0.02, 0.03, \ldots$

Examples on poission distribution

① The no of accident per day is recorded in a textile industry over a period of 400 days is given fit a poisson distribution for the data and calculate the theoretical frequency.

(x) 400 days

$$m = \frac{\Sigma f x}{\Sigma f}$$

x	0	1	2	3	4	5
f	173	168	37	18	3	1

$$\text{mean} = m = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

$$= \frac{0\times173 + 1\times168 + 2\times37 + 3\times18 + 4\times3 + 5\times1}{173 + 168 + 37 + 18 + 3 + 1}.$$

$$m = 0.7825$$

$$P(x) = \frac{m^x e^{-m}}{x!} = \frac{(0.7825)^x e^{-0.7825}}{x!}$$

theoretical frequency is given by.

$$f(x) = 400 \times P(x)$$

$$f(x) = \frac{400(0.7825)^x e^{-0.7825}}{x!}$$

$\therefore f(0) = 182.90 \approx 183$

$f(1) = 143.122 \approx 143$

$f(2) = 55.9968 \approx 56$ 400

$f(3) = 14.6058 \approx 15$ $f(x)\ 400 P(x)$

$f(4) = 2.8572 \approx 3$

$f(5) = 0.44 \approx 0$.

Q. 2% of the fuses manufactured by a firm are found to be defective. Find the probability that a box containing 200 fuses

(i) no defect.

(ii) 3 or more

$m = np$

$P = 2\% \Rightarrow 0.02$ $2/100$ $m = np$

$n = 200$.

$m = np$

$m = 200 \times 0.02 = 4$.

$$P(x) = \frac{4^x e^{-4}}{x!}$$

Let x denote the no of fuses to be defective.

(i) $\therefore x = 0$.

$$\therefore P(0) = \frac{4^0 e^{-4}}{0!} =$$

$$= 0.0183$$

ii) 3 or more.

$p + q = 1$

$p = (1 - q$

$P(x \geq 3) = 1 - P(x < 3)$

$= 1 - \{P(x=0) + P(x=1) + P(x=2)\}$.

$$1 - \left[0.0183 + \frac{4e^{-4}}{1!} + \frac{16 \cdot e^{-4}}{2}\right]$$

$$= \underline{0.76191}$$

3. if the probability of a bad reaction from a certain injection is 0.001 determine the chance out of 200 individuals more than 2 will get bad reaction

$p = 0.001$. $\underline{0.001}$ $m = np$

$n = 200$

$m = np = 200 \times 0.001 = 0.2$.

$$P(x) = \frac{m^x e^{-m}}{x!}$$

Let x be the chance for bad reaction.

$$P(x>2) = 1 - P(x \leq 2)$$
$$= 1 - [P(x=0) + P(x=1) + P(x=2)]$$
$$= 1 - \left[\frac{(0.2)^0 e^{-0.2}}{0!} + \frac{(0.2)^1 e^{-0.2}}{1!} + \frac{(0.2)^2 e^{-0.2}}{2!}\right]$$

$= \cancel{1 - 0.03887}$

$= 0.06387$.

$= 1 - [0.8187 + 0.1637 + 0.01637]$

$= 0.016$

4. The no of accidents in a year to taxi driver in a city follows a poisson distribution with mean 3. Out of thousand taxi drivers find approximately the no of the drivers with

i) no accident

ii) more than 3 accident in a year

$m = 3$, out of 1000

$$P(x) = \frac{m^x e^{-m}}{x!}$$

x is the no of accidents in a year

$$\boxed{P(x) = \frac{3^x e^{-3}}{x!}}$$

$m = 3$

Source diginotes.in

i) $P(x=0) = \frac{e^{-3}}{0} = e^{-3} \times 1000 = 49.787 \approx 50$

ii) $P(x>3) = 1-P(x\leq 3)$

$= 1-[P(x=0)+P(x=1)+P(x=2)+P(x=3)]$

$= 1-\{e^{-3}+3e^{-3}+\frac{9e^{-3}}{2}+\frac{27e^{-3}}{6}]$

$= 1-\{13 \cdot e^{-3})$

$= 0.3527 \times 1000$

$= 352.768 \approx 353$

5. The probability that the new reader commits no mistake is $1/e^3$ find the probability that on a particular news board case he commits

i) only 2 mistakes.

ii) more than 3 mistakes.

iii) atmost 8 mistakes

sol^n $= \frac{1}{e^3} = e^{-3} = e^{-m}$

$\boxed{m=3}$

$P = \frac{1}{e^3}$

e^{-3}

e^{-m}

$m:3$

$P(x) = \frac{3^x e^{-3}}{x!}$

i) $P(x=2)$

$= \frac{9e^{-3}}{2}$

$= 0.2240$

ii) $P(x>3)$

$= 1-[P(x\leq 3)]$

$= 1-[P(x=0)+P(x=1)+P(x=2)+P(x=3)]$

$= 1-\{0.0497+0.1493+0.2240+0.2240]$

$= 1-0.64701$

$= 0.35299$

6. The probability of x (poisson variate) taking the values 3 and 4 are equal, calculate the probability of variate taking the values 0 and 1

P.T.O

Given, $P(3) = P(4)$, $m = ?$

$$\frac{m^3 e^{-m}}{3!} = \frac{m^4 e^{-m}}{4!}$$

$$\frac{4!}{3!} = m$$

$$\boxed{m=4}$$

$$P(x) = \frac{4^x e^{-4}}{x!}$$

$P(0) = e^{-4}$ & $P(1) = 4e^{-4}$.

7. If x follow poisson low such that $P(x=2) = 2/3\,P(x=1)$ find $P(x=0)$ & $P(x=3)$

$$\frac{m^2 e^{-m}}{2!} = \frac{2}{3}\left[\frac{m^1 e^{-m}}{1!}\right]$$

$$\frac{m}{2!} = \frac{2}{3}[1]$$

$$\boxed{m = \frac{4}{3}}$$

$$P(x=0) = \frac{1 e^{-4/3}}{0!} = 0.26359$$

$$P(x=3) = \frac{\left(\frac{4}{3}\right)^3 e^{-4/3}}{3!} = 0.10413$$

8. Compute the mean and variance in poisson.

$$P(x=2) = 9P(x=4) + 90P(x=6)$$

$$\frac{m^2 e^{-m}}{2!} = 9\left[\frac{m^4 e^{-m}}{4!}\right] + 90\left[\frac{m^6 e^{-m}}{6!}\right] \quad \because P(x) = \frac{m^x e^{-m}}{x!}$$

$$= \frac{3}{8} m^4 e^{-m} + \frac{1}{8} m^6 e^{-m}$$

$$\frac{m^2 e^{-m}}{2!} = \frac{1}{8}[3m^4 e^{-m} + m^6 e^{-m}]$$

$$4m^2 e^{-m} = 3m^4 e^{-m} + m^6 e^{-m}$$

$$4m^2 e^{-m} = m^4 e^{-m}[3 + m^2]$$

$4 = m^2[3+m^2]$

$4 = m^2[3+m^2]$

$4 = 3m^2+m^4$

$= m^4+3m^2-4=0 \Rightarrow m^4+4m^2-m^2-4=0.$

$= m^2[m^2+4]-1[m^2+4]=0$

$= m^2-1=0\ ,\ m^2+4=0$

$\Rightarrow m=\pm 1\ ,\ m=\pm 2i$

$\boxed{m=\pm 1} \Rightarrow m=+1$

$\boxed{\text{mean} = \text{variance} = 1}$

9. 10% of the tools produced during certain manufacturing process turn out to be defective find the probability that in a sample of 10 tools choose at random exactly 2 will be defective by using poisson distribution.

$n=10\ ,\ P=10\% = 0.1$

$\text{mean} = m = np = 10\times 0.1$

$m=1.$

$$p(x) = \frac{1^x e^{-1}}{x!}$$

$$p(x=2) = \frac{1^2 e^{-1}}{2!} = 0.1839$$

10. 5% of the items produced in a certain manufacturing company turn out to be defective if the manufacturer sells the items in a box of 100. and guarantees that not more than 5 items will be defective what is the probability that the box will fail to meet the guaranteed quality.

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Continous probability Distribution

If for every x belonging to the range of constant random variable, $x \in X$, we assign a real no $f(x)$ satisfying condition $f(x) \geq 0$ & integral $\int_{-\infty}^{\infty} f(x)dx = 0$, then $f(x)$ is called as continous probability fn or probability density fn.

$x \in X$

$f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x).dx = 1$

$$\text{if } P.(a \leq x \leq b) = \int_a^b f(x).dx.$$

Continous distribution function [CDF].

1. If x is constant random variable with probability density fn then $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x).dx$.

$$\Rightarrow f(x) = \frac{d}{dx}[f(x)].$$

Note

1. If r is any real no, then the probability of

$$P(x \geq r) = \int_r^{\infty} f(x)\,dx.$$

$$P(x < r) = \int_{-\infty}^{r} f(x).dx.$$

(or)

$$1 - P(x \geq r)$$

$$= 1 - \int_r^{\infty} f(x).dx.$$

Mean & variance

$$\text{Mean}(\mu) = \int_{-\infty}^{\infty} x\, f(x).dx.$$

$$\text{variance} = \int_{-\infty}^{\infty} x^2 f(x).dx - \mu^2 \text{ or } \int_{-\infty}^{\infty} [x-\mu]^2 f(x).d$$

$S.D = \sqrt{V}.$

α [alpha] should be positive real no. $\alpha > 0$

Exponential distribution

The continous probablity distribution with probablity density fn $f(x)$ is defined by $f(x)$ =

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0, & \text{other wise.} \end{cases}$$

$f(x)$ is known as exponential distribution.

(i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x).dx = 1$

$$\int_{-\infty}^{\infty} f(x).dx = \cancel{\int_{-\infty}^{0} f(x).dx} + \int_{0}^{\infty} f(x).dx$$

$$= \int_{0}^{\infty} \alpha e^{-\alpha x}.dx$$

$$= \left[\frac{\alpha e^{-\alpha x}}{-\alpha}\right]_0^{\infty}$$

$$= -[e^{-\infty} - 1]$$

$$= -[0 - 1] = 1$$ Hence proved

1. Obtain mean and S.D of exponential distribution.

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \alpha > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} x f(x).dx$$

$$= \cancel{\int_{-\infty}^{0} x f(x).dx} + \int_{0}^{\infty} f(x).dx$$

$$= \int_{0}^{\infty} x.\alpha e^{-\alpha x}.dx$$

$$= \alpha \int_0^\infty x e^{-\alpha x} dx = \alpha \left[x \left[\cancel{\frac{e^{-\alpha x}}{-\alpha}}\right] - \left[\frac{e^{-\alpha x}}{\alpha^2}\right](1) \right]_0^\infty$$

$$= \alpha \left\{ \frac{-1}{\alpha}\left[e^{-\infty} - 0\right] - \frac{1}{\alpha^2}\left[e^{-\infty} - 1\right] \right\}$$

$$\text{mean } \boxed{= \alpha\left[\frac{1}{\alpha^2}\right] = \frac{1}{\alpha}}$$

$uv_1 - u'v_2 + u''v_3$

Variance

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$\left[\cancel{\int_{-\infty}^{0} x^2 f(x)\, dx} + \int_0^\infty x^2 f(x)\, dx \right] - \mu^2$$

$$= \int_0^\infty x^2 \alpha e^{-\alpha x}\, dx - \mu^2$$

$$= \alpha \int_0^\infty \underset{u}{x^2} \underset{v}{e^{-\alpha x}}\, dx - \mu^2$$

$$= \cancel{\alpha} \left[x^2 \cancel{\left[\frac{e^{-\alpha x}}{-\alpha}\right]_0} - (2x)\cancel{\left[\frac{e^{-\alpha x}}{\alpha^2}\right]_0^\infty} + 2\left[\frac{e^{-\alpha x}}{-\alpha^3}\right]_0^\infty \right]$$

$$\cancel{= \left[\frac{-x^2 e^{-\alpha x}}{}\right]_0^\infty + (2x)}$$

$$= \alpha \left[\frac{-2}{\alpha^3}\left[e^{-\infty} - 1\right] \right] - \frac{1}{\alpha^2}$$

$$= \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}.$$

$e^{-\alpha x}\Big]_0^\infty = e^{-\infty} - e^0$
$= 0 - 1$

Since we can conclude that in exponential distribution. mean = standard deviation

1. Find the value of 'C' such that

$$f(x) = \begin{cases} \frac{x}{6} + c & 0 \le x \le 5 \\ 0 & \text{elsewhere.} \end{cases}$$

$f(x) \rightarrow$ pdf also find $P(1 \le x \le 2)$

Sol

Since $f(x)$ is pdf,

i.e $f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$= \int_{-\infty}^{\infty} f(x) \cdot dx = 1$$

$$= \int_0^3 \left(\frac{x}{6} + c\right) \cdot dx \Rightarrow \left[\frac{x^2}{12} + cx\right]_0^5 = 1$$

$$\Rightarrow \frac{3^3}{12} + c3 = 1$$

$$\Rightarrow 3c = 1 - \frac{3}{4}$$

$$\therefore c = \frac{1}{3}\left[1 - \frac{3}{4}\right]$$

$$\boxed{c = \frac{1}{12}}$$

Subs c in $f(x)$

$$f(x) = \begin{cases} \frac{x}{6} + \frac{1}{12} \; : \; 0 \le x \le 3. \end{cases}$$

$$P(1 \le x \le 2) = \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx.$$

$$= \left.\frac{x^2}{12} + \frac{x}{12}\right]_1^2$$

$$= \frac{1}{12}\left[4 + 2 - 2\right]$$

$$= \frac{1}{3}$$

2. Find A constant k such that

$$f(x) = \begin{cases} kx^2 & 0 < x < 3 \\ 0 & \text{Otherwise} \end{cases}$$ is pdf

also compute

i) $P(1 < x < 2)$

ii) $P(x \le 1)$

iii) $P(x > 1)$

iv) mean (v) variance

Sol) $f(x)$ is PDF

$$\int_0^3 kx^2\,dx = 1$$

$$\frac{k}{3}\left[x^3\right]_0^3 = 1$$

$$\frac{k}{\cancel{3}} \cdot \cancel{27}^{\,9} = 1$$

$$\boxed{k = \frac{1}{3}} \implies f(x) = \frac{x^2}{9}.$$

i) $$P(1 < x < 2) = \int_1^2 \frac{x^2}{9}\,dx$$

$$= \frac{1}{9 \times 3}\left[x^3\right]_1^2$$

$$= \frac{1}{9 \times 3}[8 - 1] = \frac{7}{27}.$$

ii) $$P(x \le 1) = \int_0^1 \frac{x^2}{9}\,dx$$

$$= \frac{1}{9 \times 3}\left[x^3\right]_0^1 = \frac{1}{27}.$$

iii) $$P(x > 1) = 1 - P(x \le 1)$$

$$1 - \left[\frac{1}{27}\right] = \frac{26}{27}.$$

iv) $$\text{Mean}(\mu) = \int_{-\infty}^{\infty} x f(x)\cdot dx.$$

$$= \int_0^3 x \cdot \frac{x^2}{9} \cdot dx$$

$$= \frac{1}{9} \int_0^3 x^3 dx$$

$$= \frac{1}{9 \times 4} \cdot [x^4]_0^3 = \frac{1}{9 \times 4} [9 \times 9] = \frac{9}{4}.$$

(v) <u>variance</u>

$$V = \int_{-\infty}^{\infty} x^2 f(x) \cdot dx - \mu^2$$

$$= \int_0^3 x^2 \cdot \frac{x^2 dx}{9} - \frac{81}{16}.$$

$$= \frac{1}{9} \int_0^3 x^4 - \frac{81}{16}$$

$$= \frac{1}{9 \times 5} [x^5]_0^3 - \frac{81}{16}$$

$$= \frac{27}{5} - \frac{81}{16}$$

$$= \frac{432 - 405}{80}$$

$$= \frac{27}{80}$$

3 Is the following fn ~~a~~ is density fn.

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$ (If it is PDF)

if so determine the probability if ~~it is PDF~~ $P(1 < x < 2)$.

For PDF

$f(x) \geq 0$ true.

and $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

$$= \int_{-\infty}^{\infty} e^{-x} \cdot dx.$$

$$= -e^{-x}\Big]_0^{\infty}$$

$$= -[0-1] = 1$$

$\therefore$ $P(1 < x < 2) = \int_1^2 f(x) \cdot dx$

$$= \int_1^2 e^{-x} \, dx$$

$$= -e^{-x}\Big]_1^2$$

$$= -[e^{-2} - e^{-1}]$$

$$= \underline{0.2325}$$

A. A Random variable x has following density f^n.

$$f(x) = \begin{cases} Kx^2 & -3 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

Evaluate find $P(1 \leq x \leq 2)$

iii) $P(x \leq 2)$ ii) $P(x > 1)$

The ~~gi~~ given f^n is PDF ie. $f(x) \geq 0$.

i) $\int_{-3}^{3} f(x) = 1$.

$$\int_{-3}^{3} kx^2 = 1$$

$$\frac{k}{1}\left[\frac{x^3}{3}\right]_{-3}^{3}$$

$$\frac{K}{\cancel{3}}[\cancel{27}^{9} + \cancel{27}^{9}] = 1$$

$$18K = 1$$

$$\boxed{K = 1/18}$$

$$\therefore f(x) = \frac{x^2}{18}$$

i) $P(1 \leq x \leq 2)$

$$\int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{x^2}{18}\,dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_1^2$$

$$\cancel{\frac{1}{18\times 3}[8-1]} = \frac{1}{18\times 3}[8-1]$$

$$= \frac{7}{18\times 3} = \frac{7}{54}$$

ii) $p(x \le 2)$.

$$= \int_{-3}^{2} f(x)\,dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{2}$$

$$= \frac{1}{18\times 3}[8+27]$$

$$= \frac{35}{18\times 3} = \frac{35}{54}$$

iii) $p(x>1)$.

$$= \int_1^3 f(x)\,dx$$

$$= \frac{1}{18}\int_1^3 x^3$$

$$= \frac{1}{18\times 3}\left[x^3\right]_1^3$$

$$\frac{[27-1]}{18\times 3} = \frac{26}{18\times 3} = \cancel{\frac{26}{54}} = \frac{1}{2}$$

$$= \frac{26}{54}.$$

5. Find CDF for the following PDF

(i) $f(x) = \begin{cases} 6x - 6x^2 & 0 \le x \le \infty. \\ 0 & \text{Otherwise.} \end{cases}$

(ii) $f(x) = \begin{cases} \frac{x}{4} e^{-x/2} & 0 < x < \infty. \\ 0 & \text{otherwise.} \end{cases}$

iii) Exponential distribution.

CDF

$$F(x) = \int_{-\infty}^{x} f(x)\,.dx$$

~~i) $f(x) = \int_{-\infty}^{x}$~~

i) $F(x) = \cancel{\int_{-\infty}^{0} f(x)\,dx} + \int_{0}^{x} f(x)\,.dx.$

$$= \int_{0}^{x} (6x - 6x^2)\,dx = \left.\frac{6x^2}{2} - \frac{6x^3}{3}\right]_0^x$$

$$= 3x^2 - 2x^3$$

$F(x) = 3x^2 - 2x^2 \quad 0 \le x \le \infty.$

ii) $F(x) = \int_{0}^{x} \frac{x}{4} e^{-x/2}\,.dx.$

$$= \frac{1}{4}\left[x\left(\frac{e^{-x/2}}{-1/2}\right) - (1)\left(\frac{e^{-x/2}}{1/4}\right)\right]_0^x$$

$$= \frac{1}{4}\left[-2xe^{-x/2} - 4\left[e^{-x/2} - 1\right]\right].$$

$$= \frac{1}{4}\left[-2e^{-x/2}(x+2) + 4\right] \quad . \; 0 < x < \infty.$$

iii) <u>Exponential distribution</u>

$f(x) = \begin{cases} \alpha e^{-\alpha x} & x \ge 0 \\ 0 & \text{Otherwis} \end{cases} \qquad x > 0.$

$$F(x) = \int_{0}^{x} \alpha e^{-\alpha x}\,.dx.$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right]_0^x$$

$$= -1\left[e^{-\alpha x} - 1 \right]$$

$$= \underline{1 - e^{-\alpha x}} \qquad x \geq 0$$

5 continous random variable as distribution fn

$$\cancel{f(x)} = F(x) = \begin{cases} 0 & x \leq 1 \\ c(x-1)^4 & 1 \leq x \leq 3 \\ 1 & x > 3. \end{cases}$$

Find c and also the pdf.

Sol $f(x) = \frac{d}{dx}\left[F(x) \right]$

$$\Rightarrow f(x) = \begin{cases} 0 & x \leq 1 \\ 4c\,(x-1)^3 & 1 \leq x \leq 3 \\ 0 & x > 3 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_1^3 4c(x-1)^3 \cdot dx$$

$$= \cancel{4}c \int_1^3 \frac{(x-1)^4}{\cancel{4}} \cdot dx$$

$$= c \left[(x-1)^4 \right]_1^3$$

$$= c\left[16\right] = 1$$

$$\boxed{c = \frac{1}{16}}$$

$\therefore$ probability density fn is given by

$$f(x) = 4 \times \frac{1}{16} (x-1)^3 \qquad 1 \leq x \leq 3.$$

$$= \frac{1}{4} (x-1)^3 \qquad 1 \leq x \leq 3.$$

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Exponential Distribution.

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0,\ \alpha > 0 \\ 0 & \text{otherwise} \end{cases}$$

7. $f(x) = \dfrac{K}{1+x^2}, \quad -\infty < x < \infty$.

(i) $P(x \geq 0)$ (ii) $P(0 < x < 1)$.

Sol $f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\left\{ \int \frac{1}{1+x^2}\,dx = \tan^{-1} x \right.$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{K}{1+x^2}\,dx = 1 \longrightarrow (1)$$

$f(x) = \dfrac{K}{1+x^2}$ $\quad \tan^{-1} x \Big]_0^{\infty}$

$\Rightarrow f(x) = \dfrac{K}{1+x^2}$. $\quad K = 1/\pi \quad \pi/2$

$\Rightarrow f(x) = f(-x)$

$f(x)$ is even.

Now eq^n (1) becomes.

$$2\int_{0}^{\infty} \frac{K}{1+x^2}\,dx = 1. \quad \left\{ \text{since the } f(x) \text{ is even} \right.$$

$$2K\left[\tan^{-1}(x)\right]_0^{\infty} = 1$$

$$2K\left[\pi/2 - 0\right] = 1 \Rightarrow K = 1/\pi$$

$$f(x) = \frac{1}{\pi}\,\frac{1}{1+x^2}.$$

(i) $P(x \geq 0) = \int_0^{\infty} f(x).dx$

$= \int_0^{\infty} \frac{1}{\pi} \left(\cdot \tan^{-1} x\right]_0^{\infty} = \frac{1}{\pi}\left[\pi/2 - 0\right]$

ii) $P(0 < x < 1) = \int_0^1 f(x).dx = \frac{1}{\pi}\int_0^1 \frac{1}{1+x^2}.dx$

$= \frac{1}{\pi}\left[\tan^{-1} x\right]_0^1$

$= \frac{1}{\pi}\left[\pi/4 - 0\right] = 1/4$

8. if x is exponential ~~variant~~ variate with the mean = 3 find (i) $P(x>1)$ (ii) $P(x<3)$

mean $= \frac{1}{\alpha} = 3 \quad \therefore \alpha = \frac{1}{3}$.

sol^n $f(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x>0 \\ 0 & \text{otherwise.} \end{cases}$

$\left\{ f(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0,\ \alpha > 0 \\ 0 & \text{otherwise.} \end{cases} \right.$

i) $P(x>1) = \int_1^{\infty} f(x).dx = \int_1^{\infty} \frac{1}{3} e^{-x/3}.dx$.

$= \frac{1}{\cancel{3}}\left[\frac{e^{-x/3}}{-1/\cancel{3}}\right]_1^{\infty}$

$= -\frac{1}{3}\left[e^{-\infty/3} - e^{-1/3}\right]$

$= -\left[e^{-\infty/3} - e^{-1/3}\right]$

$= e^{-1/3} = 0.7165$

ii). $P(x < 3) = \int_0^3 f(x) \cdot dx$.

$$= \int_0^3 \frac{1}{3} e^{-x/3} \cdot dx .$$

$$= \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3}\right]_0^3$$

$$= -\left[e^{-1} - 1\right]$$

$$= 1 - e^{-1}$$

$$= 1 - \frac{1}{e} = \frac{e-1}{e}.$$

$\frac{1}{\alpha}$ $\frac{1}{\alpha}$ $\frac{1}{\alpha^2}$ $\frac{1}{\alpha^2}$

9. If x is a exponential variant with the mean as 5

evaluate (i) $P(0 < x < 1)$.

(ii) $P(-\infty < x < 10)$

(iii) $P(x \le 0 \text{ or } x \ge 1)$.

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0, \ \alpha > 0 \\ 0 & \text{otherwise.} \end{cases}$$

sinc mean $= 5$, $\alpha = \frac{1}{5}$.

$$\therefore f(x) = \begin{cases} \frac{1}{5} e^{-x/5} & . x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) $P(0 < x \le 1)$.

$$= \int_0^1 f(x) \cdot dx = \frac{1}{5} \int_0^1 e^{-x/5} \cdot dx .$$

$$= \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5}\right]_0^1$$

$$= -\left[e^{-1/5} - 1\right] = 1 - e^{-1/5}$$

$$= 0.1812$$

ii) . $P(-\infty < x < 10)$. $= \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{10} f(x)\cdot dx$.

$\int_{0}^{10} f(x)\cdot dx$.

$= \frac{1}{5}\int_{0}^{10} e^{-x/5}\cdot dx$.

$= -\left[e^{-10/5} - 1\right]$

$= -\left[e^{-2} - 1\right]$

$= \left[1 - e^{-2}\right]$

$= 0.8646$.

$1 - \frac{1}{e^2} = \frac{e^2 - 1}{e^2}$

iii) $.P(x \le 0 \text{ or } x \geq 1)$.

$\int_{-\infty}^{0} f(x)\cdot dx + \int_{1}^{\infty} f(x)\cdot dx$

$= \frac{1}{5}\int_{1}^{\infty} e^{-x/5}\cdot dx$.

$= \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_1^{\infty}$

$= -1\left[e^{-\infty} - e^{-1/5}\right]$

$-1\left[0 - e^{-1/5}\right]$

$e^{-1/5}$

$= \underline{0.8187}$

10. In a certain town the duration of shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for

i) 10 m or more. $= P(x \geq 10)$

ii) less than 10 m $= P(x < 10)$

iii) .10 & 12 m $= P(10 < x < 12)$

$$f(x) = \begin{cases} \frac{1}{5}e^{-x/5} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\text{mean} = \frac{1}{\alpha}.$

(i) $P(x \geq 10)$

$$= \int_{10}^{\infty} f(x)\,dx = \frac{1}{5}\int_{10}^{\infty} e^{-x/5}\,dx$$

$$= \left[e^{-x/5}\right]_{10}^{\infty}$$

$$= -\left[0 - e^{-10/5}\right]$$

$$= e^{-2} = \frac{1}{e^2}$$

$$= 0.1353$$

ii) $P(x < 10)$

$$\int_{-\infty}^{10} f(x)\,dx = \cancel{\int_{-\infty}^{0} f(x)\,dx} + \int_{0}^{10} f(x)\,dx$$

$$= -\left[e^{-x/5}\right]_0^{10} = -\left[e^{-2} - 1\right].$$

$$\left[1 - e^{-2}\right]$$

$$= 1 - \frac{1}{e^2} = \frac{e^2 - 1}{e^2}$$

$$= 0.8646.$$

iii) $P(10 < x < 12)$

$$= \int_{10}^{12} f(x)\,dx = -\left[e^{-x/5}\right]_{10}^{12}$$

$$= -\left[e^{-12/5} - e^{-2}\right].$$

$$\left[e^{-2} - e^{-12/5}\right]$$

$$= \cancel{10.88}$$

$$0.04461$$

Normal Distribution :-

The continous probability distribution having probability density f^{n}

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$$
$$-\infty < \mu < \infty.$$

is called normal distribution.

sol

given $f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x).dx = 1.$

W.K.t in 3rd sem

$$\int_0^{\infty} e^{-t^2}.dt = \sqrt{\pi/2}$$

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}.dx.$$

consider $t = \frac{(x-\mu)}{\sqrt{2}\sigma}.$

$$dt = \frac{1}{\sqrt{2}\sigma} \cancel{(x-\mu)}.dx.$$

t varies from $-\infty$ to $+\infty$

$$\Rightarrow \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}.\sqrt{2}\cancel{\sigma}.dt.$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}.dt = \frac{2}{\sqrt{\pi}}.\int_0^{\infty} e^{-t}.dt = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{\cancel{2}} = 1$$

Hence proved

$\boxed{\text{mean} = \mu}$

$\boxed{\text{standard deviation} = \sigma}$

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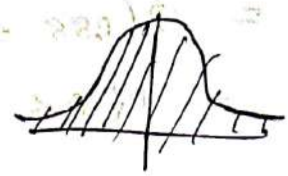
⑥ $P(-z_1 < z < z_1) = 2\phi(z_1)$



Note :-

① $P(0 < z < z_1) = \phi(z_1)$

② $P(-\infty < z < \infty) = 1$

③ $P(-\infty < z < 0) = 0.5 = (P\ 0 < z < \infty)$

④ $P(z < z_1) = 0.5 + \phi(z_1)$

⑤ $P(z > z_1) = 0.5 - \phi(z_1)$

The Random variable x say to have normal distribution if it has probability distribution that is symmetric and bell shaped

Standard Normal Variate

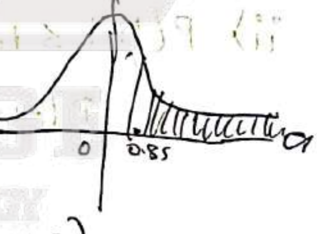
$$z = \frac{x - \mu}{\sigma} \quad \text{where} \quad \mu \to \text{mean}, \quad \sigma \to S.D$$

$0 < z < \infty$

ex

1- Find the Probability of $P(z \geq 0.85)$

$\therefore P(z \geq 0.85) = P(0 < z < \infty) - P(0 < z < 0.85)$



$= 0.5 - P(0 < z < 0.85)$

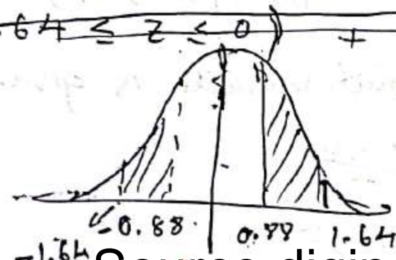
$= 0.5 - \phi(0.85)$

$= 0.5 - \cancel{0.3203}\ \cancel{0.3132}\ 0.3023$

$= \cancel{0.1867} = 0.19766$

2- $P(-1.64 \leq z \leq -0.88)$

$= \cancel{P(-1.64 \leq z \leq 0) + P(0 \leq z \leq -0.88)}$



Source diginotes.in

$= P(0.88 \le z < 1.64)$

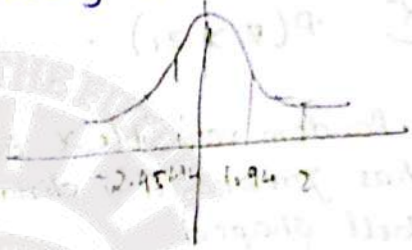
$= P(0 < z < 1.64) - P(0 < z < 0.88)$

$= \phi(1.64) - \phi(0.88)$

$= 0.4495 - 0.31057$

$= 0.1389$

3. Evaluate the following probality with the help of probability table.



i) $P(z \le -2.43)$

ii) $P(|z| \le 1.94)$

i) $P(z \le -2.43)$

by applying symmetric rule.

$P(z \ge 2.43)$

$= P(0 < z < \infty) - P(0 < z < 2.43)$

$= 0.5 - \phi(2.43)$

$= 0.5 - 0.49254$

$= 0.0075$

ii) $P(|z| \le 1.94)$

$= P(-1.94 \le z \le 1.94)$

$= 2\phi(z_1)$

$= 2\phi(1.94)$

$= 2 \times 0.47381$

$= \underline{0.94762}$.

4. If x is the Normal variate with mean 30 and S.D 5 Find the probability that

i) $P(26 \le x \le 40)$ ii) $P(x \ge 45)$

w.k.T standard normal variate is given by

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{x - 30}{5}$$

i) $P(26 \le z \le 40)$

$\Rightarrow$ at $x = 26$, $z = \frac{26-30}{5} = -0.8$.

$\Rightarrow$ at $x = 40$, $z = \frac{40-30}{5} = 2$.

$P(26 \le x \le 40) = P(0.8 \le z \le 2)$

$= P(0.8 \le z < 0) + P(0 \le z \le 2)$

$= P(0 \le z < 0.8) + P(0 \le z \le 2)$

$= \phi(0.8) + \phi(2) = 0.76539$

ii) $P(x \geq 45)$

$\Rightarrow$ at $x = 45$

$z = \frac{45-30}{5} = 15/5 = 3$.

$\therefore P(x \geq 40) \Rightarrow P(z \geq 3)$

$P(z \geq 45) = P(0 < z < \infty) - P(0 < z < 3)$.

$0.5 - \phi(3)$

$= 0.5 - 0.49865$

$= 1.35 \times 10^{-3}$.

5. x, $\mu = 12$, $\sigma = 4$

i) $P(x \geq 20)$

ii) $P(x \le 20)$

$z = \frac{x-\mu}{\sigma} = \frac{x-12}{4}$.

i) $P(x \geq 20)$

$\Rightarrow$ at $x = 20$

$z = \frac{20-12}{4}$

$z = 2$.

$P(x \geq 20) \Rightarrow P(z \geq 2)$

$\Rightarrow 0.5 - \phi(2)$

$= 0.02275$

$z = \frac{x-\mu}{\sigma}$

$z = \frac{x-\mu}{\sigma}$

Source diginotes.in

ii) $P(x \leq 20)$

$\Rightarrow$ at $x = 20$

$z = \frac{20-12}{4}$

$z = 2$

$P(x \leq 20) \Rightarrow P(z \leq 2)$

$= 0.5 + \phi(2)$

$= \underline{0.97725}$

6. The marks of 1000 students in an examiner follows Normal distribution with $\mu = 70$, & S.D $= 5$, find no of students whose marks will be

i) less than 65 $P(x < 65)$

ii) more than ~~75~~ $P(x > 75)$

iii) b/w 65 & 77 $P(65 \leq x \leq 75)$

$\mu = 70, \quad \sigma = 5$

$z = \frac{x-70}{5}$

i) $P(x < 65)$

$z = \frac{65-70}{5} = -1$

$P(x < 65) = P(z < -1) \Rightarrow P(z > 1) = 0.5 - \phi(1)$

$P(z > 1) = 0.15866$

$0.15860 \times 1000 = 158.66$

$\approx$ 159 students

ii) $P(x > 75)$

$x = \frac{75-70}{5}$

$z = 1$

$P(z > 1) = 0.5 - \phi(1)$

$\approx$ 159 students.

iii) $P(65 \leq x \leq 75)$

$z = \frac{65-70}{5} = -1 \qquad z = \frac{75-70}{5} = 1$

$P(65 \leq x \leq 75) = P(-1 \leq z \leq 1) = 2\phi(0 < z < 1) = 2\phi(1)$

$= 0.68268 \times 1000$

≈ 683 Students

7. In a test on electric bulbs, It was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and SD of 60 hours. If a firm purchase 2500 bulbs, find the no of bulbs that are likely to last for

i) more than 2100 hours
ii) Less than 1950 hrs
iii) b/w 1900 to 2100 hrs.

$np = 70$

$\sqrt{npq} = \frac{525}{70}$

$SD = 0.35$

$p = 0.65$

q

$n = 108$

$n = \frac{70}{0.6}$

$108\, C_x\, (0.65)^x (0.35)^{108-x}$

5/5/17

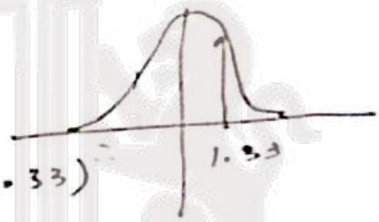
8. If the heights of 300 students are normally distributed with the mean $\mu = 68.0$ inches and S.D $\sigma = 3.0$ inches. How many students have (i) greater than 72 inches
(ii) less than or equal to 64 inches.

Sol^n: $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-68}{3}$.

(i) $P(x > 72) = \cancel{\dfrac{72-68}{3}} = 4/3 = 1.333$

at $x = 72$ $z = \dfrac{72-68}{3} = 4/3 = 1.33$

$P(z > 1.33) =$



$(1.33 < z < \infty)$.

$P(0 < z < \infty) - P(0 < z < 1.33)$

$= 0.5 - \phi(1.33)$

$= 0.5 - 0.40825$

$= \underline{0.09176}$ ∴ no of students greater than 72 inches is 0.09176×300
$= 28$

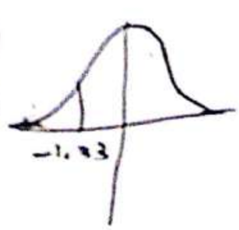
~~(ii) $P(z \leq -1.33)$~~ $=$ ~~$P(z \geq 1.33)$~~

(ii) ~~$z = 64 - 6$~~ at $x = 64$

$z = \dfrac{x-\mu}{\sigma} = \dfrac{64-68}{3} = -4/3 = -1.33$

∴ $P(z \leq -1.33) = P(z \geq 1.33)$

$= 0.5 - \phi(1.33)$



$= 0.5 - 0.40825$

$= 0.09176$

$= 0.09176 \times 300$

$= 28$

9. 15,000 students appear for BCA examination of Bangalore university, the mean marks were 49 ($\mu = 49$) & the standard ~~mat~~ deviation of marks is 6 ($\sigma = 6$) assuming the marks to be normally distributed, find the no. of students scored more than 55 marks.

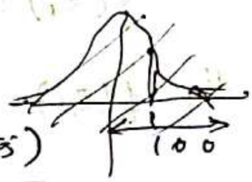
Sol^{+n}

$$\frac{100-49}{6}$$

$$z = \frac{x-\mu}{\sigma}$$

at $x = 55$, $\frac{55-49}{6} = 1$

$P(x > 55) = P(z > 1)$



$\cancel{P(0 < z < 100) - P(0 < z < 55)}$

$\cancel{\phi(100)} - \phi$

$= 0.5 - \phi(1)$

$= 0.5 - 0.34134$

$= 0.15866$

$\therefore$ no of students scored less than 55 is $0.15866 \times 15,000 = 2379.9$

10. In a normal dist, 31% of the items are under 45 & 8% of items are over 64, find the mean & the S.D. of the distribution.

$[\phi(0.5) = 0.19$ & $\phi(1.4) = 0.419]$

Given

31% are under 45

& 8% of over 64

$P(x \leq 45) = 0.31$

$P(x \geq 64) = 0.08$.

$P(z \leq z_1) = 0.5$

$\phi(z < z_1)$

$0.5 + \phi(z_1)$

$$z = \frac{x-\mu}{\sigma}$$

$$z_1 = \frac{45-\mu}{\sigma}, \quad z_2 = \frac{64-\mu}{\sigma}$$

$\therefore P(z \leq z_1)$, $P(z \geq z_2)$

with the help of basic property.

$P(z \le z_1) = 0.31$

$0.5 + \phi(z_1) = 0.31 \Rightarrow \phi(z_1) = -0.19.$

lly

$P(z \ge z_1) = 0.08.$

$0.5 - \phi(z_2) = 0.08 \Rightarrow \phi(z_2) = 0.42$

From the given table.

$\phi(z_1) = -\phi(0.5)$

lly $\phi(z_2) = \phi(1.4)$

$\therefore \; z_1 = -0.5$

$z_2 = 1.4$

Substitute z_1 & z_2 in previous eqn.

$\frac{45-\mu}{\sigma} = -0.5$ & $\frac{64-\mu}{\sigma} = 1.4$

$\Rightarrow 45 - \mu = -0.5\sigma$

$64 - \mu = 1.4\sigma$

$-\mu + 0.5\sigma = -45$

$-\mu - 1.4\sigma = -64.$

$\mu = \cancel{49.87} \simeq \cancel{49.87}\; 50$

$\sigma = \cancel{9.74} \simeq 10.$

11 In an Exam 7% of the student score less than 35 marks and 89% of the student score less than 60. Find the mean and S.D if marks are normally distributed, it is given that

$$p(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2} \cdot dz.$$

given

$\phi(1.2263) = 0.39$

$\phi(1.4757) = 0.43$

$P(x<35) = 0.07$

$P(x<60) = 0.89$

$$z = \frac{x-\mu}{\sigma}$$

$$z_1 = \frac{35-\mu}{\sigma} \qquad z_2 = \frac{60-\mu}{\sigma}$$

$$0.5+\phi(z_1) = 0.07 \qquad 0.5+\phi(z_2) = 0.89$$

$$\phi(z_1) = 0.07-0.5 \qquad \phi(z_2) = 0.89-0.5$$

$$\phi(z_1) = -0.43 \qquad \phi(z_2) = 0.39$$

$$\Rightarrow \phi(1.4757) = 0.43 \qquad \phi(1.2263) = 0.39$$

$$\Rightarrow z_1 = -1.457 \qquad z_2 = 1.2263$$

Substitute in previous eq^n

$$-1.457 = \frac{35-\mu}{\sigma} \qquad 1.2263 = \frac{60-\mu}{\sigma}$$

$$= -\mu + 1.457\sigma = -35 \qquad -\mu - 1.2263\sigma = -60$$

$$\mu = 48.57 \approx 49$$

$$S.D = 9.31 \approx 9$$

12. For the following normal dist find c and also the mean and ~~also~~ $\mu - 0$ S.D of the frequency distribution $f(x) = c e^{-1/24(x^2-6x+4)}$

Sol

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$x^2-6x+4$$

$$x^2-6x+9-9+4$$

$$= (x-3)^2-5$$

$$\therefore f(x) = c e^{-1/24[(x-3)^2-5]}$$

$$= c e^{5/24} e^{\frac{-(x-3)^2}{24}}$$

$$= c e^{5/24} e^{\frac{-(x-3)^2}{2\times(\sqrt{12})^2}}$$

compare with standard eq^n.

$$\therefore \mu = 3, \quad \sigma = \sqrt{12}$$

$\therefore x = \frac{1}{\sqrt{\;}}$

$c = \frac{1}{\sigma\sqrt{2\pi}}$

$\therefore c e^{5/24} = \cdot c = \frac{e^{-5/24}}{\sqrt{12}\sqrt{2\pi}}$

$c = \underline{0.0935}$

9/5/17

Joint probability

x

$P(x)$

$x \backslash y$	0	1	2	
0	3	4	5	11
1	2	0	5	17
	5	4	10	

$E(x) = \mu_x = \sum x\, P(x) \;\; \& \;\; \mu_y = \sum y\, P(y).$

$V_x = \sum x^2 P(x) - \mu_x^2$

$\sigma = \sqrt{V}$

Marginal probability distribution

The sum of the rows and column of X and Y table is called as Marginal Proba. distribution

Independent Random variable

$P(X = x, Y = y) = P(X = x) \cdot P(Y = y).$

Expectation, variance & covariance.

$E(x) = \text{mean}\,(\mu) = \sum x\, P(x)$ or $\sum y\, p(y)$

$V_x = \sum x^2 P(x) - \mu_x^2$ (or) $V_y\; \sum y^2 P(y) - \mu_y^2$

co-variance

$$\sigma_x = \sqrt{V_x} \;\&\; \sigma_y = \sqrt{V_y}.$$

$$cov(x, y) = E(xy) - \mu_x \mu_y.$$

<u>co-relation co-efficient</u>

$$r = \frac{cov(x, y)}{\sigma_x \sigma_y}.$$

1. The joint probability of 2 random variables X & Y given by the following table.
   (i) Marginal distribution of X & Y.
   (ii) E(x), E(y), E(x, y).
   (iii) Verify that X & Y are independent.

x \ y	2	3	4	sum
1	0.06	0.15	0.09	0.3 → P(x)
2	0.14	0.35	0.21	0.70
sum	0.2	0.5	0.3	

P(y) ↓

(i) Marginal distribution.

$-E(y) = \sum y\, P(y)$

$\cancel{P(x)}$ - x : 1 2

P(x) : 0.3 0.7

y : 2 3 4

P(y) : 0.2 0.5 0.3.

ii) $E(x) = \mu_x = \sum x\, P(x)$

$= 1(0.3) + 2(0.7) = 1.7.$

$E(y) = \mu_y = \sum y\, P(y)$

$= 2(0.2) + 3(0.5) + 4(0.3) = 3.1$

$E(x, y) =$

P.T.O.

$E(x,y) = 1\times 0.06\times 2 + 1\times 0.15\times 3 + 1\times 4\times 0.09 +$

$2\times 2\times 0.14 + 2\times 3\times 0.35 + 2\times 4\times 0.21$

$= 5.27$

(iii) $P(X=x, Y=y) = P(X=x), P(Y=y)$

$P(X=1, Y=2) = 0.06$

RHS

$P(X=1), P(Y=2) = 0.3\times 0.2 = 0.06.$

$\therefore$ X & Y are independent.

Q The J.P of 2 random variable X & Y given by the following table.

(i) Marginal distribution of X, Y.

ii) Find cov(X, Y)

(iii) verify that X & Y are independent.

x \ y	-3	2	4	sum
1	0.1	0.2	0.2	0.5
3	0.3	0.1	0.1	0.5
sum	0.4	0.3	0.3	

(i)

x : 1 3.

$P(x)$: 0.5 0.5

y ; -3 2 4

$P(y)$: 0.4 0.3 0.3.

(ii) $\text{cov}(x, y) = E(x, y) - \mu_x \mu_y$

$E(x) = \mu_x = \Sigma x P(x) = 1(0.5) + 3(0.5) = 2.$

$E(y) = \mu_y = \Sigma y P(y) = -1.2 + 0.6 + 1.2 = 0.6$

$E(x,y) = 1 \times 0.1 \times (-3) + 1 \times 0.2 \times 2 + 1 \times 0.2 \times 4$
$+ 3 \times 0.3 \times (-3) + 0.1 \times 3 \times 2 + 3 \times 0.1 \times 4$

$= 0.$

$\therefore \text{cov}(x,y) = 0 - 2(0.6)$

$\text{cov}(x,y) = -1.2.$

iii) X, Y are independent.

$P(X = 1, Y = -3) = 0.1$

$P(X=1) = 0.5$

$P(Y=-3) = 0.4$

$P(X=1) \times P(Y=-3) = 0.2.$

$\therefore$ X & Y are not independent.

3.

i) Marginal distribution of x, y.

ii) Find cov(x, y)

iii) Verify that X & Y are independent.

iv) Find correlation co-eff of X & Y.

$x \backslash y$	1	3	6
1	1/9	1/6	1/18
3	1/6	1/4	1/12
6	1/18	1/12	1/36

$\frac{C_{xy}}{\sigma_x \sigma_y}$

$\sqrt{V_x}\sqrt{V_y}$

x : 1 3 6, y : 1 3 6

~~$P(x)$: 1/9 + 1/6 + 1/18~~ $P(y)$:

$xP(x)$: $yP(y)$:

$x^2P(x)$: $y^2P(y)$:

P.T.O.

$\cancel{P(x) : 1/9 + 1/6 + 1/18}$

x :	1	3	6	y :	1	3	6
$P(x)$:	0.33	0.5	0.166	$P(y)$:	0.33	0.5	0.166
$xP(x)$:	0.33	0.15	0.996	$yP(y)$:	0.33	0.15	0.996
$x^2P(x)$:	0.33	4.5	5.976	$y^2P(y^2)$:	0.33	4.5	5.976

$\mu_x = \sum x\, P(x)$, $\mu_y = \sum y\, P(y)$

$= \cancel{1(0.33) + 3}(0.5) + 6(0.166)$

$\mu x \cdot 0.33 + 0.15 + 0.996 = 1.476$

$\mu y = 0.33 + 0.15 + 0.996 = 1.476$

$V_x = \sum x^2 P(x) - \mu^2 x$

$= 0.33 + 4.5 + 5.976$

$= 10.806$

$\mu_y \; V_y = 10.806$

$\sigma_x = \sqrt{V_x} = \sqrt{10.806} = 3.287$

$\sigma_y = \sqrt{V_y} = \sqrt{10.806} = 3.287$

ii) $\text{cov}(x, y) = E(xy) - \mu_x \mu_y$

$= 1 \times 1/3 \times 1 + 1 \times 1/6 \times 3 + 1 \times 1/18 \times 6 + 3 \times 1/6 \times 1$

$+ 3 \times 1/4 \times 3 + 3 \times 1/12 \times 6 + 6 \times 1/18 \times 1 +$

$6 \times 1/12 \times 3 + 6 \times 1/36 \times 6$

$= 8.03$

11. A fair coin is tossed 3-times, Let X denote zero or one (0 or 1) according to head or tail occurs on the first toss. Let Y denotes the no of heads which occurs

i) Find distribution of x & y.

ii) $\mathrm{cov}(x, y)$.

iii) ρ.

Sol possibilities

$S = \{HHH, HHT, HTH, HTT, THT, THT, TTH, TTT\}$

First toss Tail = 1

x	0	0	0	0	1	1	1	1
y	3	2	2	1	2	1	1	0

$\Rightarrow$

x :	0	1
$P(x)$:	$4/8$	$4/8$
	$= 1/2$	$1/2$

y :	0	1	2	3
$P(y)$:	$1/8$	$3/8$	$3/8$	$1/8$

~~mean~~

Joint probability of x and y is given by

$x \backslash y$	0	1	2	3	
0	0	$1/8$	$\cancel{2}/8$	$1/8$	$\frac{4}{8} = 1/2$
1	$1/8$	$\cancel{2}/8$	$1/8$	0	$= 4/8 = 1/2$
	$1/8$	$\cancel{8}/8$ $= 3/8$	$3/8$	$1/8$.	

$E(x) = \mu_x = \sum x\, P(x)$

$0 \times 1/2 + 1/2 \times 1 = 1/2$

$E(y) = \mu_y = \sum y\, P(y) = 3/2$

$= 0 + 3/8 + 6/8 + 3/8$

$\cancel{E(x,y)}$

$E(xy) = 0 + 1/4 + 1/4 = 2/4 = 1/2$.

ii) $cov(x,y) = E(xy) - \mu_x \mu_y$.

$= 1/2 - 1/2 \times 3/2$

$1/2 - 3/4$

$= \frac{-1}{4}$

iii) $r = \frac{cov(x,y)}{\sigma_x \sigma_y}$.

$\sigma_x = \sqrt{V_x}$.

$V_x = \sum x^2 p(x) - \mu_x^2$

$V_x = 1/2 - 1/4$

$\cancel{\sigma_x = \frac{1}{\sqrt{2}}}$ $V_x = \frac{1}{4}$.

$\sigma_x = \sqrt{\frac{1}{4}}$

$\sigma_x = 1/2$.

$\sigma_y = \sqrt{V_y}$.

$V_y = \sum y^2 p(y) - \mu_y^2$.

$= \left[3/8 + 12/8 + 9/8\right] - 9/4$

$= \frac{24}{8} - 9/4$

$V_y = 3 \quad 6/8 = 3/4$.

$\sigma_y = \frac{\sqrt{3}}{2}$.

$r = \frac{cov(x,y)}{\sigma_x \sigma_y}$.

$= \frac{-1/4}{1/2 \times \sqrt{3}/2}$

$= \frac{-1/4}{\sqrt{3}/4}$

$= -1/4 \times \frac{4}{\sqrt{3}}$

$$\boxed{r = -1/\sqrt{3}}$$

5. The J.P distribution of two discrete Random variable X & Y is given by

$$f(x,y) = k(2x+y), \quad 0 \le x \le 2, \quad 0 \le y \le 3$$

i) Find constant k

ii) Find marginal distribution of X & Y.

iii) S.T X & Y are dependent Random Variable.

Sol

x \ y	0	1	2	3	
0	0	K	2K	3K.	6K.
1	2K.	3K	4K	5K.	14K.
2	4K	5K	6K	7K	22K
	6K	9K	12K	15K	

X : 0 1 2

P(x) 6K 14K 22K

Y: 0 1 2 3

P(y) = 6K 9K 12K 15K.

W.K.T

$\sum P(x) = 1$ & $\sum P(y) = 1$

$42K = 1$

$\therefore \boxed{K = 1/42}$

ii) Marginal distribution of X & Y.

X : 0 1 2

P(x): $\frac{6}{42}$ $14/42$ $22/42$

Y : 0 1 2 3

P(Y) : $6/42$ $9/42$ $12/14$ $15/42$

iii) X & Y are dependent

$cov(XY) = E(XY) - \mu_x \mu_y$.

$$E(XY) = 0 + 1\times\frac{3}{42}\times 1 + 1\times\frac{4k}{42}\times 2 + 1\times\frac{5}{42}\times 3$$

$$+ 2\times\frac{5}{42}\times 1 + 2\times\frac{6}{42}\times 2 + 2\times\frac{7}{42}\times 3.$$

$$= \cancel{102}\ \cancel{\frac{99}{48}}$$

$$= 3k + 8k + 15k + 10k + 24k + 42k$$

$$= 102k = \frac{102}{42} = \underline{\underline{2.43}}$$

$$\mu_x = \frac{14}{42} + \frac{44}{42} = \frac{58}{42}$$

$$\mu_y = \frac{9}{42} + \frac{24}{42} + \frac{45}{42} = \frac{78}{42}.$$

$$cov(x.y) = E(XY) - \mu_x\mu_y$$

$$2.43 - \frac{58}{42}\times\frac{78}{42}$$

$$= -0.1246 \neq 0$$

$\therefore$ They are not independent, they are dependent

6. $\underline{\underline{HW}}$

The J.P of two Random variable is given by

$f(x,y) = Cxy$ for x & $y = 1, 2, 3$

i) find C

ii) $P(X=2, Y=3)$

iii) $P[1 \le x \le 2, y \le 2]$.

iv) $P(x \geq 2) = P(x=2) + P(x=3)$

x \ y	1	2	3
1	C	2C	3C
2	2C	4C	6C
3	3C	6C	9C

7. X & Y are independent Random variable, X takes the values 2, 5, 7 with probability 1/2, 1/4, 1/4 & Y takes the values 3, 4, 5 with probability 1/3, 1/3, 1/3

i) find the joint probability distribution.

ii) show that cov(XY) = 0

iii) find the probability dist $z = x+y$

A:-

Given that

X:	2	5	7	Y:	3	4	5
P(X):	1/2	1/4	1/4	P(Y):	1/3	1/3	1/3

i)

X \ Y	3	4	5	
2	1/6	1/6	1/6	3/6 = 1/2
5	1/12	1/12	1/12	3/12 = 1/4
7	1/12	1/12	1/12	3/12 = 1/4
	4/12 = 1/3	4/12 = 1/3	4/12 = 1/3	

$P(X=2, Y=3) = P(X=2) \times P(Y=3)$
$= 1/2 \times 1/3$

$P(X=2, Y=4) = P(X=2) \times P(Y=4)$
$= 1/2 \times 1/3$

ii) $cov(XY) = E(XY) - \mu_x \mu_y$

$\mu_x = 1 + 5/4 + 7/4 = 4.$

$\mu_y = \sum y\,p(y)$
$= 1 + 4/3 + 5/3 = 12/3 = 4.$

$E(X,Y) = 6/6 + 8/6 + 10/6 + 15/12 + 20/12 + 25/12 + 21/12 + \frac{28}{12} + 35/12$

$= \frac{\cancel{24}^{4}}{\cancel{6}} + \frac{\cancel{144}^{12}}{\cancel{12}}$

$E(XY) = 16.$

$cov(XY) = E(XY) - \mu_x \mu_y.$
$= 16 - 4 \times 4 = 0$

iii)

$Z = X+Y$.	5	6	7	8	9	10	11	12
$P(Z)$	$1/6$	$1/6$	$1/6$	$1/12$	$1/12$	$1/6$	$1/12$	$1/10$.

for $10 = 1/12 + 1/12 = 1/6$.

HW

8. Two marbles are drawn from box containing 3 blue, 2 red & 3 green marbles. If X is the number of blue marbles and Y the number of red marbles.

i) Form the joint distribution of X & Y $\quad X, Y$ $\boxed{x \backslash y}$

ii) Find the marginal distribution. $\Sigma x P(x)$

iii) Find the expectations of X, Y, XY

$\mu_x \quad \mu_y \quad E(XY)$

$S =$	RR	BB	GG	RB	RG	BG.
$X =$	0	2	0	1	0	1
$Y =$	2	0	0	1	1	0

ii)

X: 0 1 2

$P(x)$: $3/6 \quad 2/6 \quad 1/6 = 1/2, 1/3, 1/6$

Y: 0 1 2

$P(y)$: $3/6 \quad 2/6 \quad 1/6 = 1/2, 1/3, 1/6$.

$x \backslash y$	0	1	2	
0	$1/6$	$1/6$	$1/6$	$= 1/2$.
1	$1/6$.	$1/6$	0	
2	$1/6$	0	0	

$E(XY) = 1/6$.

$\mu = E(X) = \sum x P(x)$

$= 1/3 + 1/3 = 2/3$

$\mu_y \quad \mu_y = 2/3$

Sampling Theory & Stochastic process

Stochastic process

Probability vector

A vector whose component are non-negative and their sums are equal to 1 $v_i \geq 0$ & $\sum_{i=1}^{n} v_i = 1$

Stochastic Matrix

A square matrix P is called stochastic matrix if all the entries of the P are non-negative & sum of the entries of any row is 1

Ex :- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1/3 & 2/3 \\ 0 & 1 \end{bmatrix}$

Ex:- $[1/4, 1/4, 1/2]$.

Fixed vector or fixed point :-

A vector 'V' is said to be fixed vector or a fixed point of a matrix 'A' if

$$\boxed{VA = V}$$

Regular Stochastic matrix

A stochastic matrix P said to be regular if all the entries of sum of P^m are +ve [positive]

$P \Rightarrow$ sum of P^m are +ve.

$A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$ $A^2 = A \cdot A = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

Note 1. Let 'P' be a regular stocastic matrix

i) P has a unique fixed ~~vector~~ probability vector.

ii) P is a stocastic matrix

2. A stochastic matrix is not regular if 1 occurs in the principle main diagonal.

$$Ex: \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \text{ not regular.}$$

Examples

1. Find unique Fixed probability vector of Regular stochastic matrix $P = \begin{bmatrix} 1/4 & 3/4 \\ 1/4 & 1/2 \end{bmatrix}$

Sol

let $v = (x, y)$ be a unique fixed probability vector associated with P

To Prove $\boxed{VA = V}$ ie $x + y = 1$

W.K.T by the property of regular stochastic matrix

ie $VA = V$

$$\Rightarrow (x, y) \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix} = (x, y).$$

$$= \left[\frac{x}{4} + y/2 \; , \; 3x/4 + y/2\right] = (x, y).$$

$\cancel{\frac{x+2y}{4}}$

$$\Rightarrow \frac{x}{4} + y/2 = x \quad \& \quad \frac{3x}{4} + y/2 = y.$$

$$\Rightarrow y/2 = x - x/4 \quad \& \Rightarrow y = 3x/2.$$

$$y/2 = \frac{3x}{4}.$$

$\therefore$ substitute $y = 3x/2$ to get x

$\cancel{\frac{3x}{4}} + \cancel{y/2}$

$$\cancel{\frac{3x}{4} + \frac{3x}{4} = 3x/2}.$$

$$\cancel{\frac{3x}{2} = 3x/2}.$$

$x+y=1 \Rightarrow x+3x/2 = 1$;

$\Rightarrow \frac{5x}{2} = 1 \Rightarrow x = 2/5$.

$y = 3x/2 \Rightarrow y = 3/2 \cdot 2/5 = 3/5$.

$V(x,y) = V(2/5, 3/5)$.

Hence unique probability vector is $2/5$ & $3/5$

2. Find the unique fixed probability vector of Regular Stochastic matrix

$$A = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Let $v = (x,y,z)$ be unique fixed prob. vector with A

A. ie $x+y+z=1$.

W.K.T by the prob of regular stochastic matrix

ie $VA = V$.

$$(x,y,z)\begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (x,y,z).$$

$$\left[1/2 y, \ 3/4 x + 1/6 y + z, \ 1/4 x\right] = (x,y,z)$$

$\Rightarrow x = y/2, \quad y = 3/4 x + 1/6 y + z, \quad z = 1/4 x$

$x+y+z = 1$

$x + 2x + x/4 = 1$

$\frac{13}{4} x = 1$

$\therefore x = 4/13$.

$\therefore y = 2x = 8/13$.

$||^{ly} \quad z = 1/4 \times 4/13 = 1/13$

$\Rightarrow V = [4/13, \ 8/13, \ 1/13]$

H.W

3. $A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$

4 $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.3 & 0.5 & 0 \end{bmatrix}$

$V = (x, y, z)$

$x + y + z = 1$

To prove.

$VA = V$

$(x, y, z) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix} = (x, y, z).$

$[z(0.5), \ x + z(0.5), \ y] = (x, y, z)$

$z(0.5) = x, \quad x + 0.5z = y, \quad y = z.$

$x + y + z = 1$

$\frac{1}{2}z + z + z = 1 \qquad = \frac{5}{2}z = 1$

$z = 2/5, \quad y = 2/5, \quad x = \frac{2}{5} \times \frac{1}{2} = 1/5.$

$V = [1/5, \ 2/5, \ 2/5].$

$$A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$V = (x, y, z)$

To prove $VA = V$.

$$(x, y, z) \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = (x, y, z).$$

$$\left[x/2 + y/2,\ x/4 + z,\ x/4 + y/2\right] = (x, y, z).$$

$$x/2 + y/2 = x, \quad x/4 + z = y, \quad x/4 + y/2 = z.$$

$$x + y + z = 1.$$

$$1/2\, y = 1/2\, x \qquad 1/4\, x + 1/2\, y = z.$$

$$3/4\, x = z.$$

$$x + y + z = 1$$

$$x + x + 3/4\, x = 1$$

$$2x + 3/4\, x = 1$$

$$11/4\, x = 1$$

$$x = 4/11$$

Transition probability matrix or stochastic matrix

The probabilities of moving from one state to another state or ~~the~~ remaining in the same state are called transition probability ~~matrix~~.

Transition probability from a square matrix is called ∴ Transition probability matrix or stochastic matrix.

~~Markov chain~~

$$P = [P_{ij}] = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

Markov chain

The stochastic process which is such that the generation of p the probability distribution depends only on the present state is called Markov chain.

Then entry P_{ij} in the transition probability matrix P of the markov chain is the probability that the system changes from the state a_i to a_j in a single step.

$$p^{(1)} = p^{(0)}, p^{(1)}$$
$$p^{(2)} = p^{(0)} \; p^{(2)}$$

Note

Markov chain is irrecible if the associated transition probability matrix is regular

explain the following

1. Absorbing state :- In a Markov Chain, the process Reaches to a certain state after which it continous to remain in the same state such a state is called as Absorbing state.

Ex P.T.O

Ex:- $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & \phi & 0 \\ 1/2 & 1/2 & 1/2 \end{bmatrix}$

2. Transient State :-
A state i said to be transiend iff if there is a positive prob that the process will not return to this state

Ex If we model a program as a Markov chain, then all except final step of the program are transient state.

3. Recurrent State :-
A state i is said to be Recurrent state iff starting from the state i, the process eventually return to state i with the probability 1

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The system remains in the same state after two steps.

4. Periodic state :- The recurrent state i is said to be periodic state if GCD $(d_i > 1)$

ex Identity matrix

Problems

1. Define regular stochastic matrix & show that

$A = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible [Regular] matrix

Ans A stochastic matrix P said to be regular if all the entries of sum of P^m are +ve

$P \Rightarrow$ sum of P^m are +ve.

$$A = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0.5 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

$\therefore$ It is Regular.

2. Define a stochastic matrix and show that

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$ is a regular stochastic matrix

Ans. ~~st~~ ~~st~~

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

Defination is written Backside.

$$P^2 = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 1/4 & 5/8 & 1/8 \\ 1/4 & 9/16 & 3/16 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/4 & 5/8 & 1/8 \\ 3/8 & 9/32 & 11/32 \\ 3/8 & 19/64 & 21/64 \end{bmatrix}$$

3. A student's study habits are as follows
If he studies One night, he is 70% sure Not ~~to~~ study the next night, One the other hand if he does not study One night, he is 60% sure not to study the next night.
In the Long Run how often does he study?

Sol

Let $S = \{$ Studying(A) , not studying (B) $\}$

$$\Rightarrow \quad P = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$

we have to find unique fixed probability vector

Let 'v' be a unique fixed prob vector $v = (x, y)$

$\Rightarrow \quad x + y = 1 \longrightarrow (1)$

∴ from stochastic matrix we've

$$vP = v$$

$$(x, y)\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = (x, y)$$

$$[0.3x + 0.4y \, , \; 0.7x + 0.6y] = [x, y]$$

$$0.3x + 0.4y = x \quad , \quad 0.7x + 0.6y = y.$$

$$3x + 4y = 10x.$$

$$4y = 7x.$$

$$\boxed{y = \frac{7}{4}x}$$

Sub $y = \frac{7x}{4}$

$$x + y = 1$$

$$x + \frac{7x}{4} = 1$$

$$11x = 4$$

$$x = 4/11 \Rightarrow y = \frac{7x}{4} \Rightarrow 7/11$$

$$x = 4/11 \, , \quad y = 7/11$$

Hence require fixed probability is

$$v = [4/11 \, , \; 7/11]$$

Oftenly he does study 4/11 or 36.36 %.

Q. A man's smoking habits are as follows.
If he smokes filter cigarettes one week he switches to non-filter cigerates the next week with probability 0.2. If he smokes ~~non~~ non-filter cigarettes one week, there is a probability of 0.7 that he will smoke non-filter cigarette the next week as well. In the long run how often does he smoke filter cigarette?

Let $S = \{\text{filter cig }(A), \text{ Non-filter cig }(B)\}$

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

$$x + y = 1 \longrightarrow (1)$$

$$VP = V$$

$$(x, y)\begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = (x, y)$$

$$[0.8x + 0.3y, \; 0.2x + 0.7y] = (x, y)$$

$$0.8x + 0.3y = x, \quad 0.2x + 0.7y = y$$

$$8x + 3y = 10x$$

$$3y = 2x$$

$$y = \frac{2x}{3}$$

$$x + y = 1 \qquad x + y =$$

$$x + \frac{2x}{3} = 1$$

$$\frac{3x + 2x}{3} = 1$$

$$3x + 2x = 3 \implies y = \frac{2 \times 3}{3 \cdot 5}$$

$$5x = 3 \qquad = 2/5$$

$$\underline{x = 3/5}$$

$$V = [3/5, \; 2/5]$$

15/6/17

5. A habitant gambler is a member of two clubs A & B. He visits either of the clubs everyday for playing cards. He never visits club A on two consecutive days. But if he visits club B on a particular day, then the next day, he is as likely to visit club B or club A

i) After long run how often does he visit club A

ii) If the person has visited club B on monday, find the probability that he visit club A on tuesday.

SOL^n

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

i) $x + y = 1$

$VP = V$

$\Rightarrow [x, y] \cdot \begin{bmatrix} & \\ & \end{bmatrix} = [x, y]$

ii) If the person visited club B on Monday, i.e. Day 1 the he visit club A on thursday day 3

$$P^3 = \begin{bmatrix} 0.25 & 0.75 \\ 0.375 & 0.675 \end{bmatrix}$$

$= 0.375$

6) Every year a man trades his car for a new car. If he has Maruthi he trades it for an Ambassdor. If he has an Ambassador he trades it for Santro. However if he has a Santro he is just as likely to trade it for a new santro as to trade it for maruthi or an Ambassador. In the year 2000, he bought his first car which was a santro. find the Probability that he has

i) 2002 maruthi

ii) 2003 Santro

$\underline{\text{Sol}^n}$

$$P = \begin{array}{c} \\ m \\ n \\ s \end{array} \begin{array}{c} \begin{array}{ccc} M & A & J \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

After 2 years.

i) $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.33 & 0.33 & 0.33 \\ 0.11 & 0.44 & 0.44 \end{bmatrix}$

$A_{31} = 0.11$

ii) $P^3 = \begin{bmatrix} 0.33 & 0.33 & 0.33 \\ 0.11 & 0.44 & 0.44 \\ 0.148 & 0.259 & 0.592 \end{bmatrix}$

$A_{33} = 0.592.$

7. 3 boys A, B, C are throwing the ball to each other. A always throws a ball to B & B always throws ball to C. But C is just as likely to throw a ball to B as to A. If C was the 1st person to throw the ball, find the ~~to~~ probability that:

i) A has a ball

ii) b has a ball

iii) C has a ball for the 4th throw

$\underline{\text{Sol}^n}$

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

$P^3 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$

i) $0.25 = 1/4$

ii) $0.25 = 1/4$

iii) $0.5 = 1/2$

8. 2 Boys B_1, B_2 & 2 girls G_1, G_2 are throwing the ball from one to another. Each boy throws the ball to other boy with the probability $1/2$ & to each girl with the probability $1/4$ on the other hand throws a ball to each boy with probability $1/2$ & never to the other girl. In the long run how often does each receive the ball?

$$P = \begin{array}{c} \\ B_1 \\ B_2 \\ G_1 \\ G_2 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & G_1 & G_2 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$

To find the long run we have to use UFP vector

Let $V = (a, b, c, d)$ - are U.F.P

$$\Rightarrow a+b+c+d = 1 \longrightarrow (1)$$

From stochastic matrix we have

$$VP = V$$

$$[a\ b\ c\ d] = \begin{bmatrix} 0 & 1/2 & 1/4 & 1/2 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [a\ b\ c\ d]$$

$a+b+c+d = $
$1-a$

$$\Rightarrow \frac{b}{2} + c/2 + d/2 = a.$$

$$a/2 + c/2 + d/2 = b$$

$$a/4 + b/4 = c$$

$$a/4 + b/4 = d$$

$$\Rightarrow c = d.$$

$$\Rightarrow b/2 + c/2 + c/2 = a.$$

$$\Rightarrow a/2 + c/2 + c/2 = b$$

$$1/2[b+c+d] = a$$

$$1/2[1-a] = a$$

$$1/2 - a/2 = a$$

$$1/2 = a + a/2$$

$$1/2 = 3a/2$$

$$1/3 = a$$

$\Rightarrow \frac{b}{2} + c = a$

$\frac{a}{2} + c = b$

$\overline{(-)\ \ (-)\ \ (-)}$

$\frac{b}{2} - \frac{a}{2} = a - b$

$\frac{b}{2} - \frac{a}{2} = a - b$

$\frac{b}{2} + b = a + \frac{a}{2}$

$\frac{3b}{\cancel{2}} = \frac{3}{\cancel{2}}a$

$a = b$

$a = 1/3, \ b = 1/3$

$c = 1/6, \ d = 1/6$

SAMPLING THEORY

A large collection of data of individuals or attributes is known as population or universe

A small part of the population is called a ~~samp~~ sample. The process of selecting a sample from the population is called sampling.

There are 2-types of sampling.

1) Random sampling with replacement

Here the items are drawn 1 by 1 and are put back to the population before the next draw. In this method mean $= \mu$ & variance $= \frac{\sigma^2}{n}$.

2) Random sampling without Replacement

Here the items are drawn 1 by 1 but are not put back to the population befor the next draw. In this method

$$\text{Variance} = \left(\frac{N-n}{N-1}\right)\frac{\sigma^2}{n}.$$

16/5/17

Testing of Hyphothesis

Hypothesis is a decision making statement which is true or false. There are 2 types of hypothesis

- Null Hypothesis
- Alternate Hypothesis

Null Hypothesis (H_0)

The hypotheses formulated for the purpose of rejection is called Null Hypothesis
denoted by H_0

Alternate Hypothesis (H_1)

Any hypothesis which is not Null or exceptance is called alternate hypothesis denoted by H_1.

Confidence interval :- that acts as good estimates of the unknown population parameter

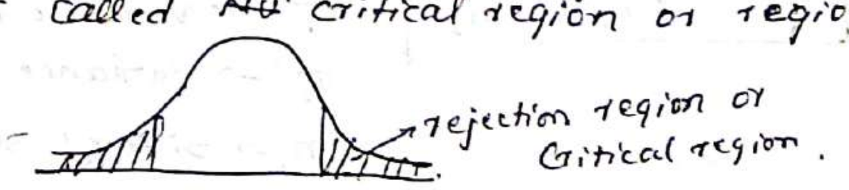
ERRORS

There are 2 types of Errors of testing hypothesis.

1. **Type-I error** :- rejecting the null hypothesis H_0 and accepting the alternate hypothesis H_1 when actually H_0 is true is called type-I error.
2. **Type-I error** :- accepting H_0 and rejecting H_1 actually H_1 is true.

Critical Region.

A region which amounts to the rejection of Null Hypothesis is called the critical region or region of rejection



Signifance level or Level of significance [LOS]

The probability level below which leads to regection is called significance Level.

→ generally 1% or 5% is the significance level

if we're (5% or 1%). (5% or 1%)

(-1.96 or -2.58) (1.96 or 2.58).

Note :- critical value of z (two-tailed test)

5% LOS	1% LOS
-1.96 & 1.96	-2.58 & 2.58.

Testing of population mean (μ) is given by

$$\boxed{z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}}$$

where $\bar{x}$ → The sample mean
μ → population mean
σ → S.D.

Testing of propotion (p)

$$\boxed{z = \frac{x - np}{\sqrt{npq}}}$$

np → Expected no of success
x → observed no of succees
p → probability of success

$q = 1 - p$

Testing of mean

$$\boxed{z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}}$$

$\bar{x}_1$ → mean of sample 1
$\bar{x}_2$ → ——— " ——— 2
σ_1^2 → variance of sample 1
σ_2^2 → variance of sample 2.
n_1 → size of sample 1
n_2 ——— " ——— 2

Testing of significance for the difference of properties of two samples is given by

$$Z = \frac{P_1 - P_2}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}.$$

$$P = \frac{n_1P_1 + n_2P_2}{n_1+n_2}.$$

5. confidence intervals for μ.

95% C.I $\bar{x} \pm 1.96\left(\frac{\sigma}{\sqrt{n}}\right)$ confidence interval.

99% C.I $\bar{x} \pm 2.58\left(\frac{\sigma}{\sqrt{n}}\right)$.

6. confidence intervals for (p)

95% CI $p \pm 1.96\left(\sqrt{\frac{pq}{n}}\right)$.

99% CI $p \pm 2.58\sqrt{\frac{pq}{n}}$.

7. C.I for difference of two means.

95% C.I $(\bar{x}_1 - \bar{x}_2) \pm 1.96\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$.

99% CI $(\bar{x}_1 - \bar{x}_2) \pm 2.58\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$.

C

Example

1. The mean life time of the sample ~~100~~ of 100 flurosecent light bulbs produced by a company is computed to be 1570 hours with SD = 120 hours. If μ is the life time of all bulb's produced by the company, thest the hypothesis $\mu = 1600$ hours against the alternate hypothesis $\mu \neq 1600$ hous. using LOS as 0.01

Ans

LOS = 0.01
= 1% = 2.58.

given $n = 100$, $\bar{x} = 1570$, $\mu = 1600$.

$\sigma = 120$

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{1570 - 1600}{\frac{120}{\sqrt{100}}} = -2.5.$$

$$|z|_{cal} = 2.5$$

level of significance

$$Z_{0.01} = 2.58.$$

$$|z|_{cal} < Z_{0.01}$$

$\therefore$ H_0 is accepted

2. ~~Ad~~ A dice was thrown 9000 times and a throw of 5 or 6 was obtained 3240 times. On the assumption of random throwing, Do the data indicate that the data is unbiased.?

Ans $n = 9000$, $x = 3240$

$$p = \frac{2}{6} = 1/3$$

$$q = 1 - p = 1 - 1/3 = \frac{2}{3}$$

$$\therefore Z = \frac{x - np}{\sqrt{npq}} = \frac{3240 - 9000 \times 1/3}{\sqrt{9000 \times 1/3 \times 2/3}}$$

$$= 5.37$$

$$|Z|_{cal} = 5.37$$

1% LOS $Z_{tab} = 2.58$.

$Z_{cal} > Z_{tab} \Rightarrow H_0$ is rejected.

$\therefore$ the data is unbiased.

3. A dice was thrown ~~4000~~ 1200 time & the no 6 is obtained 236 times, can the dice be fair at 0.01 LOS.

Ans $n = 1200$, $x = 236$.

$p = 1/6$

$q = 1 - 1/6 = 5/6$.

$$Z = \frac{x - np}{\sqrt{npq}} = 2$$

$$\frac{236 - 1200 \times 1/6}{\sqrt{1200 \times 1/6 \times 5/6}} = 2.8$$

$|Z|_{cal} = 2.8$

$|Z|_{tab} = 2.58$.

$|Z|_{cal} > |Z|_{tab} \Rightarrow H_0$ is unfair [rejected]

4. A coin is tossed 400 times and the head turn up 216 times, test the hypothesis at 5% LOS. that the coin is unbiased.

A: $n = 400$

$p = 1/2$.

$q = 1 - 1/2 = 1/2$.

5. A Dice is tossed 960 times, it falls with 5 upwards 184 times, it the die biased at 5% of LOS ?

6. T

6. To know the mean weights of all 10 year boys in delhi, a sample of 225 was taken, the mean weight of the sample was found to be 67 pounds with the S.D 12 pounds. What can we infer about the mean weight of population

A: $n = 225$, $\bar{x} = 67$, $\sigma = 12$.

We've solve this by confidence interval for μ.

$\rightarrow$ 99% of CI $\quad \bar{x} \pm 2.58\left(\frac{\sigma}{\sqrt{n}}\right)$.

$$= 67 \pm 2.58\left(\frac{12}{\sqrt{225}}\right).$$

$\mu = 69.064,\ 64.936$.

$\therefore$ mean weight lies b/w $64.936 \leq \mu \leq 69.064$.

$\rightarrow$ 95% of CI $\quad \bar{x} \pm 1.96\left(\frac{\sigma}{\sqrt{n}}\right)$

$$= 67 \pm 1.96\left(\frac{12}{\sqrt{225}}\right).$$

$= 68.56 \quad , \quad 65.432$.

~~$68.56 < \mu < 65.432$~~ .

$\therefore$ The mean weigh lies b/w .

$65.432 < \mu < 68.56$.

~~The mean weight lies b/w~~ .

7. The mean & S.D of maximum load supported by 60 cables are 01.09 tonnes & 0.37 tonnes respectively . Find 99%. confidence limit for the mean of the maximum loads of all the cables produced by a company.

A: $n = 60$, $\bar{x} = 01.09$.

$\sigma = 0.73$.

~~$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$~~

~~$= \bar{x}\ 1.09$~~

95% of CI $= \bar{x} \pm 1.96 \left(\frac{\sigma}{\sqrt{n}}\right)$

$= 1.09 \pm 1.96 \cdot \left(\frac{0.73}{\sqrt{60}}\right)$.

$= 1.2747 , \quad 0.9052$.

99% of CI $= \bar{x} \pm 2.58 \left(\frac{\sigma}{\sqrt{n}}\right)$

$= 1.333, \quad 0.8468$.

18/5/17

Note suppose in the question , if infer confidence limit or probable limit is mentioned, then we've to solve by confidence method .

8. A sample of 900 days was taken in a coastal town & it was found that the 100 days the weather was very ~~high~~ hot , ~~optain~~ obtain the probable limit of 1%. of very hot weather .

Sol

$n = 900$

$P = \frac{100}{900} = 1/9$

$q = 1 - P$

$q = 8/9$

$\therefore$ 99% CI for proportion

$$99\% \text{ of } CI(P) : P \pm 2.58\sqrt{\frac{Pq}{n}}$$

$$= 1/9 \pm 2.58\sqrt{\frac{1/9 \times 8/9}{900}}$$

$$= 0.084$$

$$\frac{1}{9} - 2.58\sqrt{\frac{1/9 \times 8/9}{900}}$$

$$= 0.13$$

$$\therefore 0.084 \leq P \leq 0.13$$

9. In a sample of 500 men, it was found that 60% of them had overweight. What can we infer about the proportion of people having overweight in the population.

$n = 500$

$P = 0.6 \Rightarrow q = 0.4$.

use 99% ~~of~~ of Level of significance.

$$99\% \text{ of } CI(P) = P \pm 2.58\sqrt{\frac{Pq}{n}}$$

$$= 0.6 \pm 2.58\sqrt{\frac{(0.6)(0.4)}{500}}$$

$$= 0.6565 \text{ or } 0.5434$$

~~95% of CI(P) = P ±~~

10. A sample of 100 bulbs produced by a company. Showed a mean life of 1190 hours. & SD of 19 hrs. and also a sample of 75 bulbs produced by company B showed a mean life of 1230 hrs. & S.D is 120 hrs. Is there a difference b/w the mean life of the bulb produced by 2 companies at 5% LOS. 1% LOS

Sol:

Company - A.

$n_1 = 100$

$\bar{x}_1 = 1190$

$\sigma_1 = 19.$

Company - B

$n_2 = 75$

$\bar{x}_2 = 1230$

$\sigma_2 = 120.$

H_0 = there is no significant difference b/w mean life of A & B companies.

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$Z = \frac{1190 - 1230}{\sqrt{\frac{(19)^2}{100} + \frac{(120)^2}{75}}} \implies Z = -2.85$$

$\therefore |Z| = 2.85.$

$$Z_{tab} = \begin{cases} 2.58 \text{ at } 1\% \text{ LOS} \\ 1.96 \text{ at } 5\% \text{ LOS} \end{cases}$$

$\therefore |Z|_{cal} > Z_{tab}$

$\implies$ Ho is rejected.

11. In an elementary school examination, the mean rate of 32 boys was 72. with $\sigma_1 = 8$ and while the mean rate of 36 girls was 75 with $\sigma_2 = 6$. Test the hypothesis that the performance of the girls is better than the boys.

A:-

$n_1 = 32$ $\quad n_2 = 36$

$\bar{x}_1 = 72$ $\quad \bar{x}_2 = 75$

$\sigma_1 = 8$ $\quad \sigma_2 = 6$.

Ho : there is no significant diffrencees b/w the performance of girls and boys

$$\therefore Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{\bar{x}_1} + \frac{\sigma_2^2}{\bar{x}_2}}}$$

$$Z = \frac{72 - 75}{\sqrt{\frac{(8)^2}{72} - \frac{(6)^2}{75}}}$$

$$= \frac{-3}{\sqrt{\frac{64}{72} - \frac{36}{75}}} = 1.73$$

1.96 5%

2.58 1%

$Z_{cal} = 1.73$.

$$Z_{tab} = \begin{cases} 2.58 & \text{at } 1\% \text{ LOS} \\ 1.96 & \text{at } 5\% \text{ LOS} \end{cases}$$

$Z_{cal} < Z_{tab}$

$\therefore$ Ho is accepted.

12. In a city A, 20% of rand sample of 900 Sheboo school boys has physical defects. In another city B 18.5% of random sample of 1600 boys has the same defect.. Is the diffreence b/w the proportion is significant. [use 1% of level of significance].

Ans 20% 900

18.5% 1600

$n_1 = 900$ $\quad n_2 = 1600$

$P_1 = 0.2$ $\quad P_2 = 0.185$

$$Z = \frac{P_1 - P_2}{\sqrt{PQ\left[\frac{1}{n_1}+\frac{1}{n_2}\right]}} \quad . \quad P_0 = \frac{n_1 P_1 + n_2 P_2}{n_1 + n_2}.$$

Ho: There is no significant difference b/w the proportion.

$Z_{cal} =$

$$P = \frac{900(0.2) + 1600(0.185)}{900 + 1600}$$

$p = 0.1904.$

$q = 1 - p = 1 - 0.1904 = 0.8096.$

$\therefore Z = 0.916.$

$Z_{cal} = 0.916. \quad Z_{tab}$ at $1\% = 2.58.$

$\Rightarrow Z_{cal} < Z_{tab}$

$\Rightarrow$ Ho is accepted.

13. A Random sample of 1000 Engg students from a city A and 800 from city B were taken. It was found that 400 students in each of the sample were from payment quota. ~~Then~~ Does the data reveal a significant diffence b/w two cities w.r.t payment ~~sto~~ quota students.

Ans

A: $n_1 = 1000$ B: $n_1 = 800$

400 - payment seats

$P_1 = \frac{400}{1000} = 2/5 = 0.4$, $P_2 = \frac{400}{800} = 1/2 = 0.5$.

Ho: there is no significant diff b/w two cities

P.T.O.

$$Z_1 = \frac{P_1 - P_2}{\sqrt{PQ\left(1/n_1 + 1/n_2\right)}}$$

$$P = \frac{n_1 P_1 + n_2 P_2}{n_1 + n_2} = \frac{1000(0.4) + 800(0.5)}{1000 + 800}$$

$$P = 4/9.$$

$$\Rightarrow Q = 5/9.$$

$$\therefore Z = \frac{1000 - 800}{\sqrt{4/9 \times 5/9 \left(\frac{1}{1000} + \frac{1}{800}\right)}}$$

$$Z_{cal} = 4.243$$

$$\therefore Z_{tab} = \begin{cases} 2.58 \text{ at } 1\% \text{ LOS} \\ 1.96 \text{ at } 5\% \text{ LOS} \end{cases}$$

$$\sum (x - \bar{x})^2$$

$Z_{cal} > Z_{tab}$

$\Rightarrow$ Ho is rejected.

14. A Random Sample of 1000 workers in a company has mean wage of 50 Rs/day and S.D. is 15. Another sample of 1500 workers from the anoth Company has mean wage of 45 Rs/day & SD is 20 Rs. Find the 95% of confidence limit for the difference of the mean wages of population of 2-companies.

Sol

A

$n_1 = 1000$

$\bar{x}_1 = 50$

$\sigma_1 = 15$

B

$n_2 = 1500$

$\bar{x}_2 = 45$

$\sigma_2 = 20$

$$(\bar{x}_1 - \bar{x}_2) \pm 1.96 \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$(50 - 45) \pm 1.96 \sqrt{\frac{(15)^2}{1000} + \frac{(20)^2}{1500}}$$

$$5 \pm 1.96 \sqrt{0.225 + 0.2666}$$

$$= 3.62 \text{ \& } 6.3$$

19/5/17

t - distribution & chi - Squar test

student t distribution [t - test]

(1) $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ $\quad s - S.D$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

(2) $t = \dfrac{\bar{x_1} - \bar{x_2}}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ $\quad S^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$.

s_1 is the S.D of I^{st} sample

s_2 —— " —— 2^{nd} sample.

$\bar{x}_1$ is mean of 1^{st} sample

$\bar{x}_2$ —— " 2^{nd} samle

n_1 - size of 1^{st} sample.

n_2 size of 2^{nd} sample.

(3) 95% Confidence limits for (μ):

$$\bar{x} \pm t_{0.05}\left[\frac{s}{\sqrt{n}}\right]$$

Note :- Degree of freedom. [df]

Degree of freedom is nothing but no of values — 1

$$[df] = n - 1$$

Problems

1. 10 individuals are choosen at random from the population & their height in inches are found to be 63, 63, 66, 67, 68, 69, 70, 70, 71, 71

Test the hypothesis that the mean height of the universe is 66 inches. at $[t_{0.05} = 2.262$ for 9df$]$

Ans

Given $n = 10$

~~sol~~ :- $\mu = 66$ inches.

$$\bar{x} = \frac{\sum x}{n} = \frac{63+63+66+67+68+69+70+70+71+71}{10}$$

$$= 67.8.$$

x	$(x-\bar{x})^2$
63	23.04
63	23.04.
66	3.24.
67	0.64
68	0.04.
69	1.44
70	4.84
70	4.84
71	10.24
71	10.24

$\sum(x-\bar{x})^2$
$= 81.6$

$$S^2 = \frac{1}{n-1}\sum(x-\bar{x})^2$$

$$= \frac{1}{9}(81.5). = 9.067$$

$$\therefore \boxed{S = 3.011}$$

$$S^2 = \frac{1\sum(x-\bar{x})}{n-1} \qquad \bar{x} = \frac{\sum x}{n}$$

$$t = \frac{\bar{x}-\mu}{S/\sqrt{n}} = \frac{67.8-66}{3.011/\sqrt{10}} = 1.89$$

$t_{cal} < t_{tab} \Rightarrow H_0$ is accepted.

2. A certain stimulus administered to each of the 12 pateints resulted in the following change in BP
5 2 8 -1 3 0 6 -2 1 5 0 4
can it be concluded that this stimulus will increase the B.P. [$t_{0.05} = 2.201$ for 11 df]

Given $n = 12$, $\mu = 0$.

$$\bar{x} = \frac{\sum x}{n} = \frac{31}{12} = 2.58$$

x	$(x-\bar{x})^2$
5	5.8564
2	0.3364
8	29.3764
-1	12.816
3	0.1764
0	6.6564
6	11.6964
-2	20.9764
1	2.4964
5	5.5864
0	6.6564
4	2.0164
	104.908

$$s^2 = \frac{1}{n-1} \sum (x-\bar{x})^2$$

$$= \frac{1}{11} 104.908$$

$$\boxed{s^2 = 9.5370}$$

$\Rightarrow s = 3.088$

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{2.58 - 0}{\frac{3.088}{\sqrt{12}}}$$

$$= 2.894$$

$t_{cal} > t_{tab} \Rightarrow H_0$ is not accepted.

$\bar{x} = \frac{\sum x}{n}$

3. A sample of 10 measurements of the diameter of a sphere gave a mean 12 cm & S.D is 0.15 cm find 95% of confidence limit for the actual diameter [$t_{0.05} = 2.201$ for 9 df]

$n = 10$ $\mu = 12$
σ 0.15

Ans

$\bar{x} = 12$ cm & $\sigma = 0.15$ cm, $n = 10$

95% of confidence limits

$$\bar{x} \pm t_{0.05} \left[s/\sqrt{n} \right]$$

$$12 \pm 2.20 \times \frac{0.15}{\sqrt{10}}$$

$\therefore$ μ lies b/w 12.104 or 11.8951

$$\therefore \boxed{11.895 < \mu < 12.104}.$$

4. A machine is expected to produce nails of length 3 inches. A random sample of 25 nails gave an average length of 3.1 inch with S.D 0.3. Can it be said that the machine is producing as per specification? [$t_{0.05} = 2.201$ for 24 df].

Ans Sol^n

$n = 25$

$\mu = 3$ $\qquad t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$. $\qquad S^2 = \frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + ...}$

$\bar{x} = 3.1$

$S = 0.3$.

$$S^2 = \frac{\sum (x - \bar{x})^2}{(n-1)}$$

Q5. A sample of 11 rates from a central population had an average blood viscosity of 3.92 with a S.D of 0.61 on the basis of this sample, establish 95% confidence limits for μ, the mean viscosity of the central population [~~to~~ $t_{0.05} = 2.228$ for 10 df].

Sol^n $n = 11$

$\bar{x} = 3.92$.

$S = 0.61$

$\bar{x} \pm t_{0.05}\left[S/\sqrt{n}\right] = \qquad \le \mu \le$

6. A group of boys and girls where given an intelligence test. The mean score, SD score and number in each group are as follows

	BOYS	Girls		
mean	74	70		
S.D	8	10		
n	12	10		

Is the difference b/w the means of the two groups significant at 5% LOS ? [$t_{0.05} = 2.086$ for 20df].

sol

~~$t = \bar{x}_1 - \bar{x}_2$~~

Ho: there is no significant difference b/w girls and boys.

$\bar{x}_1 = 74 \quad \bar{x}_2 = 70$

$s_1 = 8 \quad s_2 = 10$

$n_1 = 12 \quad n_2 = 10.$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{1/n_1 + 1/n_2}}$$

$$s^2 = \frac{12\times(8)^2 + 10(10)^2}{12+10.}$$

$= 80.36.$

$s = 8.964$

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}.$$

$t_{cal} =$ $\quad t_{tab} = 2.086$

$$t = \frac{74 - 70}{8.964\sqrt{1/12 + 1/10}} = 1.044$$

$t_{cal} = 1.044 \quad t_{tab} = 2.086$

20/5/17

Chi - square test

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

where 'O' is observed frequency

E is expected frequency χ - (chi)

$\sum O = N$ $F(x) = N.P(x)$

$n = df$

1. Fit a poisson distribution for the following data and test the goodness of fit given that

$\chi^2_{0.05} = 7.815$ for 3 df

x	0	1	2	3	4
$O = f$	122	60	15	2	1
E	121.306	60.65	15.16	2.52	0.31

Sol

$E = F$?

$N = \sum f = 200$

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$$m = \frac{\sum f \times x}{\sum f} = \frac{60+30+6+4}{200} = \frac{100}{200} = 1/2$$

$$P(x) = \frac{(0.5)^x e^{-0.5}}{x!}$$

Expected frequency $F(x) = N P(x) = \frac{200}{x!} (0.5)^x e^{-0.5}$

~~at x=0~~ $E \to F = \frac{200}{x!} (0.5)^x e^{-0.5}$

at $x = 0$.

$$= \frac{200}{1!} (0.5)^0 e^{-0.5}$$

$$= 121.306$$

at $x = 1$

$$\frac{200}{1} (0.5) \cdot e^{-0.5} = 60.65$$

$$\frac{(O-E)^2}{E} = \frac{(122 - 121.13)^2}{121.3} = 6.2399 \times 10^{-3}$$

$$\frac{(O-E)^2}{E} = \frac{(60 - 60.65)^2}{60.65} = 6.966 \times 10^{-3}$$

$$\frac{(O-E)^2}{E} = \frac{(15 - 15.16)^2}{15.16} = 1.688 \times 10^{-3}$$

$$\frac{(2 - 2.25)^2}{2.25} = 0.0277$$

$$\frac{(1 - 0.31)^2}{0.31} = 1.5358$$

$$\sum \frac{(O-E)^2}{E} = 6.2399 \times 10^{-3} + 6 - - -$$

$$= 1.578$$

$$\psi^2_{cal} < \psi^2_{tab}$$

$\therefore$ Hypothesis is accepted.

2. Fit the poisson distribution for the following data and test for the goodness of fit given that

$$\psi^2_{0.05} = 9.49 \quad \text{fo} \quad 4 \text{ df}.$$

x	0	1	2	3	4
$O \to f$	419	352	154	56	19
E	404.94	366.071	165.46	49.85	11.268
$\frac{(O-E)^2}{E}$	0.4881	0.540	0.7937	0.758	5.3058

$$\sum \frac{(O-E)^2}{E} = 7.88$$

$N = \Sigma f = 1000$

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$$m = \frac{\Sigma fx}{\Sigma f} = \frac{904}{1000} = 0.904$$

$$P(x) = \frac{(0.904)^x e^{-0.904}}{x!}$$

$$F(x) = N\,P(x) = \frac{1000}{x!}(0.904)^x e^{-0.904}$$

at $x=0$. $1000\,(1) \times e^{-0.904} = 404.94$

$x=1$ $1000\,(0.904)\,e^{-0.904} = 366.071$

$x=2$ $\frac{1000}{2!}(0.904)^2 e^{-0.904} = 165.46$

$x=3$ $\frac{1000}{3!}(0.904)^3 e^{-0.904} = 49.8599$

$x=4$ 11.2683

$$\Sigma \frac{(O-E)^2}{E} = 7.885$$

$\psi_{cal} < \psi_{tab}$

$\therefore$ Hypothesis is accepted.

3. Four coin were tossed 160 times and the follo result were obtained. Test the goodness of fit of the binomial distribution given that. $[\psi^2_{0.05} = 9.49 \text{ for } 4 df]$

No of heads	0	1	2	3	4
frequency	17	52	54	31	6

$n = 4$.

$N = 160$.

$p = 1/2$, $q = 1/2$ {Probability of tossing a coin.

Binomial Distribution.

$P(x) = {}^nC_x \cdot p^x q^{n-x}$.

$P(x) = 4C_x (1/2)^x (1/2)^{4-x} = \frac{1}{2^4}\, {}^4C_x$.

$F = NP(x) = \frac{160}{16} \cdot {}^4C_x = 10\, {}^4C_x$.

x =	0	1	2	3	4
O = ~~freq~~	17	52	54	31	6
E → F =	10	40	60	40	10
$\frac{(O-E)^2}{E}$ =	~~2/5~~ 4.9	18/5 3.6	3/5	81/40	8/5

$\sum \frac{(O-E)^2}{E} = \frac{\cancel{329}}{\cancel{40}} = \cancel{8.225}$ 12.0725

$\chi^2_{cal} > \chi^2_{tab}$

⟹ Hypothesis is ~~accepted~~ rejected.

5 diece were thrown 96 times and the numbers 1 or 2 or 3 appering on the face of the diece follows the frequency distribution as follows

[~~$\chi^2_{0.05} = 9.49$ for 4 df~~].

No of the diece showing 1 or 2 or 3	5	4	3	2	1	0
frequency	7	19	35	24	8	3

Test the hypothesis that the data follows a binomial distribution $[\chi^2_{0.05} = 9.49$ for 5 df$]$.

$n = 5 \qquad N = 96$

$p = 3/6 = 1/2$ {out of 6 we need only 3 sides

$\Rightarrow q = 3/6 = 1/2$.

Binomial Distribution

$$P(x) = {}^{n}C_x \cdot p^x \cdot q^{n-x}$$

$$= {}^{5}C_x (1/2)^x (1/2)^{5-x}$$

$$= 1/32 \; {}^{5}C_x$$

$$F = N\,P(x) = \frac{96}{32}\, {}^{5}C_x = 3({}^{5}C_x)$$

x	5	4	3	2	1	0
O	7	19	35	24	8	3
F = E	3	15	30	30	15	3
$\frac{(O-E)^2}{E}$	5.333	1.066	0.833	1.2	3.26	0

$$\sum \frac{(O-E)^2}{E} = 11.692$$

$\psi_{cal} > \psi_{tab} \Rightarrow$ hypothesis is rejected.