(2) $Z_T[k^n n^2] = \dfrac{kz^2+k^2z}{(z-k)^3}$

LHS : $Z_T[k^n n^2]$

$$= \left\{\frac{z^2+z}{(z-1)^3}\right\}_{z \to z/k}$$

$$= \frac{(z/k)^2 + z/k}{(z/k-1)^3}$$

$$= \frac{z^2}{k^2} + z/k \Big/ \frac{(z-k)^3}{k^3}$$

$$= \frac{\frac{z^2+zk}{\cancel{k^2}}}{\cancel{(z-k)^3}} \cdot \frac{k^3}{(z-k)^3}$$

$$\boxed{Z_T[k^n n^2] = \frac{kz^2+k^2z}{(z-k)^3}}$$

$Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$ ✓

$Z_T(k^n u_n) = \underline{\underline{U(z/k)}}$

Write $Z_T(n^2)$ Replace $z \to z/k$

(3) HW $\boxed{Z_T[k^n n^3] = \dfrac{kz^3+4k^2z^2+k^3z}{(z-k)^4}}$

$Z_T(n^3) = \dfrac{z^3+4z^2+z}{(z-1)^4}$. Replace $z \to z/k$

* Shifting rule

i) Right shifting rule

If $Z_T[U_n] = U(z)$ then $Z_T[U_{n-k}] = z^{-k}U(z)$ $\forall$ $k>0$

proof : $Z_T[U_{n-k}] = \sum_{n=0}^{\infty} U_{n-k} z^{-n}$

$$\cancel{\left[= \sum_{n=k}^{\infty} U_{n-k} z^{-n} \right.}$$ $n = k, k+1, k+2 \ldots$

$$\cancel{= U_{k-k} z^{-k} + U_{k-k+1} z^{-(k+1)} + U_{k-k+2} z^{-(k+2)} + \ldots}$$

$$\cancel{= \sum_{k=0}^{\infty}}$$

$$Z_T[U_{n-k}] = \sum_{n=0}^{\infty} U_{n-k} z^{-n} \quad \text{—— (1)}$$

$U_n = 0$ for $n<0$ in general context

(Def z-transform) —— (2)

from (2) $\boxed{U_{n-k} = 0}$ for $n-k<0$ $\Rightarrow$ $n = 0, 1, 2 \ldots (k-1)$

ppt is valid only for $k>0$

$$\therefore \quad Z_T(U_{n-k}) = \sum_{n=k}^{\infty} U_{n-k} z^{-n}$$

$$= U_0 z^{-k} + U_1 z^{-(k+1)} + U_2 z^{-(k+2)} + \ldots$$

$$= z^{-k}(U_0 + U_1 z^{-1} + U_2 z^{-2} + \ldots)$$

$$= z^{-k}(u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots)$$

$$= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k} U(z)$$

$$\boxed{Z_T(u_{n-k}) = z^{-k} U(z)}$$

② Left shifting rule

$$Z_T(u_{n+k}) = z^k \left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)} \right] \quad \text{OR}$$

$$Z_T(u_{n+k}) = z^k \left[U(z) - \sum_{r=0}^{k-1} u_r z^{-r} \right]$$

proof : Consider $Z_T[u_{n+k}] = \sum_{n=0}^{\infty} u_{n+k} z^{-n}$

$$= \sum_{n=0}^{\infty} u_{n+k} z^{-n-k+k} = z^k \sum_{0}^{\infty} u_{n+k} z^{-(n+k)}$$

$$= z^k \left[u_k z^{-k} + u_{k+1} z^{-(k+1)} + u_{k+2} z^{-(k+2)} + \dots \right]$$

[RHS $= U(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots)$
$\sum u_n z^{-n} = U(z)$]

Add & sub $u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots u_{k-1} z^{-(k-1)}$

$$= z^k \Big[\underbrace{(u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_{k-1} z^{-(k-1)}) + u_k z^{-k} + u_{k+1} z^{-(k+1)} + u_{k+2} z^{-(k+2)} + \dots}_{\sum_{n=0}^{\infty} u_n z^{-n}} - (u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}) \Big]$$

$$Z_T[U_{n+k}] = z^k \Big[\underbrace{\sum_{n=0}^{\infty} u_n z^{-n}}_{U(z)} - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)} \Big]$$

$$\boxed{\begin{aligned} Z_T(U_{n+k}) &= z^k \left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} \cdots - u_{k-1} z^{-(k-1)} \right] \\ Z_T(U_{n+k}) &= z^k \left[U(z) - \sum_{r=0}^{k-1} u_r z^{-r} \right] \end{aligned}}$$

Note :

when $k=1$

$$Z_T(U_{n+1}) = z[U(z) - u_0]$$

$k=2$

$$Z_T(U_{n+2}) = z^2 [U(z) - u_0 - u_1 z^{-1}]$$

$k=3$

$$Z_T(U_{n+3}) = z^3 [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$$

Z- Transforms of Standard fn

	u_n	$Z_T[u_n] = U(z)$
①	1	$z/z-1$
②	n^k	$-z \frac{d}{dz} U(n^{k-1})$
③	k^n	$z/z-k$
④	k	$kz/z-1$
⑤	n	$z/(z-1)^2$
⑥	$(-1)^n$	$\frac{z}{z+1}$
⑦	$\frac{1}{n!}$	$e^{1/z}$
⑧	$k^n n$	$\frac{zk}{(z-1)^2}$
⑨	$k^n n^2$	$\frac{kz^2+k^2z}{(z-k)^3}$

① Obtain Z- transform of $\cos n\theta$ & $\sin n\theta$

Sol: WKT $Z_T[k^n] = \frac{z}{z-k}$ ✓

$$Z_T[e^{in\theta}] = Z_T[(e^{i\theta})^n]$$

$$\boxed{Z_T[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}}}$$

$$Z_T[\cos n\theta + i\sin n\theta] = \frac{z}{z-e^{i\theta}}$$

$e^{in\theta} = \cos n\theta + i\sin n\theta$

$$= \frac{z}{z-e^{i\theta}} \times \frac{z-e^{-i\theta}}{z-e^{-i\theta}}$$

$$= \frac{z^2 - ze^{-i\theta}}{z^2 - ze^{-i\theta} - ze^{i\theta} + e^{0}}$$

$e^{i\theta}+e^{-i\theta} = 2\cos\theta$

$$= \frac{z^2 - z(\cos\theta - i\sin\theta)}{z^2 - z(e^{-i\theta}+e^{i\theta})+1}$$

$$= \frac{z^2 - z\cos\theta + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z_T[\cos n\theta] + i Z_T[\sin n\theta] = \frac{z^2 - z\cos\theta}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

(using linearity ppt)

$$\boxed{Z_T[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}}$$

$$\boxed{Z_T[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}}$$

(2) obtain Z-transform of $\sin(3n+5)$

$\sin(3n+5) = \sin 3n \cos 5 + \cos 3n \sin 5$

$Z_T[\sin(3n+5)] = Z_T[\sin 3n \cos 5 + \cos 3n \sin 5]$

$= \cos 5 \, Z_T(\sin 3n] + \sin 5 \, Z_T(\cos 3n)$

$= \cos 5 \left[\frac{z\sin 3}{z^2 - 2z\cos 3 + 1}\right] + \sin 5 \left[\frac{z^2 - z\cos 3}{z^2 - 2z\cos 3 + 1}\right]$

$$Z_T[\sin(3n+5)] = \cos 5 \left[\frac{z\sin 3}{z^2 - 2z\cos 3 + 1}\right] + \sin 5 \left[\frac{z^2 - z\cos 3}{z^2 - 2z\cos 3 + 1}\right]$$

(3) obtain the $Z_T[(n+1)^2]$

$Z_T[(n+1)^2] = Z_T[n^2 + 1 + 2n]$

$= Z_T[n^2] + Z_T[1] + 2 Z_T[n]$

$$= \frac{z^2+z}{(z-1)^3} + \frac{2z}{(z-1)^2} + \frac{z}{(z-1)}$$ //

* S.T $Z_T(1/n!) = e^{1/z}$ & hence evaluate

$Z_T(U_{n+1}) = ?$ $\qquad$ $Z_T(U_{n+2}) = ?$

$Z_T\left(\frac{1}{(n+1)!}\right)$ and $Z_T\left(\frac{1}{(n+2)!}\right)$

Sol: (First part is proved.)

WKT $Z_T(1/n!) = e^{1/z} = U(z)$ $\qquad$ $U_0 = \frac{1}{0!}$

$Z_T(U_n) = \sum U_n z^{-n}$ $\qquad$ $U_n = 1/n!$ $\qquad$ $U_1 = 1/1!$ $\qquad$ $U_1 = 1/1!$

$U_{n+1} = \frac{1}{(n+1)!}$ $\qquad$ $U_{n+2} = \frac{1}{(n+2)!}$

$$Z_T\left(\frac{1}{(n+1)!}\right) = Z_T(U_{n+1})$$
$$= z^1[U(z) - U_0]$$
$$= z[e^{1/z} - 1]$$

$$\boxed{Z_T\left(\frac{1}{(n+1)!}\right) = z(e^{1/z}-1)}$$

$Z_T(U_{n+1})$
$= z[U(z)-U_0]$

$Z_T(U_{n+2})$
$= z^2(U(z)-U_0 - U_1/z)$

$$Z_T\left(\frac{1}{(n+2)!}\right) = Z_T(U_{n+2})$$
$$= z^2\left[U(z) - U_0 - \frac{U_1}{z}\right]$$

$$\boxed{Z_T\left(\frac{1}{(n+2)!}\right) = z^2\left[e^{1/z} - 1 - \frac{1}{z}\right]}$$ //