Partial Differential Equations

An equation involving a function of one or more independent variables and its partial derivatives is known as a partial differential equation.

Here usually, x and y will be taken as the independent variables and z as the dependent variable so that $z = f(x, y)$ and we use the notation

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

The order of a PDE is the order of the highest order derivative appearing in the equation. The degree of a PDE is the positive integral power to which the highest order derivative is raised.

1) $xy\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = xz$ is a PDE of first order first degree

2) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x} + 2x\frac{\partial z}{\partial y}$ is a second order first degree PDE.

3) $\left(\frac{\partial z}{\partial x}\right)^2 = 2z\frac{\partial z}{\partial y}$ is a first order second degree PDE.

Formation of PDE by eliminating arbitrary constants:

Let $f(x, y, z, a, b) = 0$ —— ① be a relation containing 2 independent variables x, y, dependent variable z and two arbitrary constants a and b.

Differentiating the relation ① partially with respect to x & y we get, respectively

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \quad ②$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \quad ③$$

Eliminating a and b from relation ①, ② & ③ we obtain an equation of the form $\phi\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$ which is a first order PDE.

1) Form a PDE by eliminating the arbitrary constants a & b from the relation $2z = (x+a)^{1/2} + (y-a)^{1/2} + b$

Differentiating the given relation partially wrt x & y

$$2\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x+a}}, \quad 2\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y-a}}$$

$$\Rightarrow x+a = \frac{1}{16\left(\partial z/\partial x\right)^2}, \quad y-a = \frac{1}{16\left(\partial z/\partial y\right)^2}$$

Adding these we get,

$$x+y = \frac{1}{16}\left\{\frac{1}{\left(\partial z/\partial x\right)^2} + \frac{1}{\left(\partial z/\partial y\right)^2}\right\} \Rightarrow \frac{1}{\left(\partial z/\partial x\right)^2} + \frac{1}{\left(\partial z/\partial y\right)^2} = 16(x+y)$$

2) Form a PDE by eliminating the arbitrary constants a and b from the eqⁿ $z = a \log\left\{\frac{b(y-1)}{1-x}\right\}$

$$\Rightarrow z = a\{\log b(y-1) - \log(1-x)\}$$

Diff this partially wrt x & y we get,

$$\frac{\partial z}{\partial x} = a\left\{-\frac{1}{1-x}(-1)\right\} \Rightarrow a = (1-x)\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y} = a\left\{\frac{b}{b(y-1)}\right\} \Rightarrow a = (y-1)\frac{\partial z}{\partial y}$$

$$\Rightarrow (1-x)\frac{\partial z}{\partial x} = (y-1)\frac{\partial z}{\partial y}$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

This is the required PDE.

Same as above

$\boxed{\text{3) H.W}}$ ~~Form a~~ $z = xy + y\sqrt{x^2-a^2} + b$

Ans : $\left(\frac{\partial z}{\partial x} - y\right)\left(\frac{\partial z}{\partial y} - x\right) = xy$

~~4)~~

3) Form a PDE by eliminating the arbitrary constants a, b, c from the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — ①

Diff ① partially wrt x,

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{c^2}{a^2}x = -z\frac{\partial z}{\partial x} \quad —②$$

Differentiating this partially wrt x.

$$\frac{c^2}{a^2} = -\left\{\left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2}\right\} \quad \text{— (3)}$$

Also from (2),

$$\frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x} \quad \text{— (4)}$$

From (3) & (4).

$$\frac{z}{x}\frac{\partial z}{\partial x} = \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2}$$

This is the required PDE.

1) Form a PDE by eliminating the arbitrary function f from the relation $z = f(\sin x + \cos y)$

Differentiating partially wrt x & y we get

$$\frac{\partial z}{\partial x} = f'(\sin x + \cos y) \cdot \cos x$$

$$\frac{\partial z}{\partial y} = f'(\sin x + \cos y) \cdot (-\sin y)$$

$$\Rightarrow f'(\sin x + \cos y) = \frac{1}{\cos x}\frac{\partial z}{\partial x} = \frac{-1}{\sin y}\frac{\partial z}{\partial y}$$

$$\Rightarrow (\sin y)\frac{\partial z}{\partial x} + (\cos x)\frac{\partial z}{\partial y} = 0$$

This is the required PDE.

2) $z = e^{my}\phi(x-y)$ —— ①

$\frac{\partial z}{\partial x} = e^{my}\phi'(x-y)\cdot 1$ —— ②

$\frac{\partial z}{\partial y} = e^{my}\phi'(x-y)(-1) + me^{my}\phi(x-y)$ —— ③

Using ① & ② in ③.

$\frac{\partial z}{\partial y} = -\frac{\partial z}{\partial x} + mz \implies \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = mz$

3) $z = f\left(\frac{xy}{z}\right)$

$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\left\{\frac{y}{z} - \frac{xy}{z^2}\frac{\partial z}{\partial x}\right\}$ —— ①

$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left\{\frac{x}{z} - \frac{xy}{z^2}\frac{\partial z}{\partial y}\right\}$ —— ②

From ① & ②

$$f'(xy/z) = \frac{\partial z/\partial x}{(y/z)\{1-(x/z)(\partial z/\partial x)\}} = \frac{\partial z/\partial y}{(x/z)\{1-(y/z)(\partial z/\partial y)\}}$$

$\implies x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$

This is the required PDE.

H.W

4) Form a PDE by eliminating the arb func^n f from the relation $x+y+z = f(x^2+y^2+z^2)$

Ans: $(y-z)\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} = x-y$

5) $z = x\phi(y) + y\psi(x)$ —— ①

$\Rightarrow \frac{\partial z}{\partial x} = \phi(y) + y\psi'(x)$; $\frac{\partial z}{\partial y} = x\phi'(y) + \psi(x)$ —— ②

$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \phi'(y) + \psi'(x)$ —— ③

Substituting for $\phi'(y)$ & $\psi'(x)$ in ③,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x}\left\{\frac{\partial z}{\partial y} - \psi(x)\right\} + \frac{1}{y}\left\{\frac{\partial z}{\partial x} - \phi(y)\right\}$$

$$\Rightarrow xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - [x\phi(y) + y\psi(x)]$$

Using ①,

$$xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$$

6) Form a PDE by ---- $z = \phi(x+at) + \psi(x-at)$

Let $x + at = u$, $x - at = v$ —— ①

$\Rightarrow z = \phi(u) + \psi(v)$ —— ②

$$\frac{\partial z}{\partial x} = \phi'(u)\frac{\partial u}{\partial x} + \psi'(v)\frac{\partial v}{\partial x} = \phi'(u) + \psi'(v)$$

$$\frac{\partial^2 z}{\partial x^2} = \phi''(u)\frac{\partial u}{\partial x} + \psi''(v)\frac{\partial v}{\partial x} = \phi''(u) + \psi''(v) \quad —— ③$$

III^{ly} $\frac{\partial z}{\partial t} = a[\phi'(u) - \psi'(v)]$

$$\frac{\partial^2 z}{\partial t^2} = a^2[\phi''(u) + \psi''(v)] \quad —— ④$$

From ③ & ④ $a^2\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$.

Solution of PDE by direct integration

Solve $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 y$ subject to the conditions $z(x,0) = x^2$ & $z(1,y) = \cos y$.

Given $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = x^2 y$

On integrating wrt x, we obtain

$$\frac{\partial z}{\partial y} = \frac{x^3}{3} y + f(y)$$

Integrating wrt y, we obtain

$$z = \frac{x^3}{3} \cdot \frac{y^2}{2} + \int f(y)\,dy + g(x)$$

$$\Rightarrow z = \frac{x^3 y^2}{6} + F(y) + g(x) \quad \text{--- (1)}$$

Given $z = x^2$ at $y = 0$

$$\therefore x^2 = 0 + F(0) + g(x)$$

$$\Rightarrow g(x) = x^2 - F(0)$$

Put it in (1),

$$z = \frac{x^3 y^2}{6} + F(y) + x^2 - F(0) \quad \text{--- (2)}$$

Given $z(1,y) = \cos y$

$$\cos y = \frac{y^2}{6} + F(y) + 1 - F(0)$$

Putting this in (2), we obtain

$$z = \frac{x^3 y^2}{6} + \cos y - \frac{y^2}{6} - 1 + F(0) + x^2 - F(0)$$

$$\Rightarrow z = \frac{x^3 y^2}{6} + \cos y - \frac{y^2}{6} - 1 + x^2$$

2) Solve $\frac{\partial^2 z}{\partial y^2} = z$, if $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$

If z is a function of y alone, then

Already there in other notes.

2) Solve $xys = 1$

$$\Rightarrow xy \frac{\partial^2 z}{\partial x \partial y} = 1$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{xy}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{1}{y} \log x + f(y)$$

$\int^g$ wrt y

$$z = (\log y)(\log x) + \int f(y)\, dy + g(x)$$

$$= (\log x)(\log y) + F(y) + g(x)$$

Solution of PDE involving only one ~~arbitrary~~ derivatives wrt variable.

1) $x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} = 9x^2 y^3$.

Treating z as a funcn of only x, above PDE can be written as $x \frac{d^2 z}{dx^2} + \frac{dz}{dx} = 9x^2 y^3$

1) Solve the eqn $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + 2z = 0$ given that $z = e^y$ & $\frac{\partial z}{\partial x} = 0$ when $x = 0$

Given eq^n may be put in the form

$(D^2-2D+2)z=0$ where $D=\frac{\partial}{\partial z}$

AE: $m^2-2m+2=0$

$$m=\frac{2\pm\sqrt{4-8}}{2}=1\pm i$$

$$\Rightarrow z=e^x(A\cos x+B\sin x) \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial x}=e^x(A\cos x+B\sin x)+e^x(-A\sin x+B\cos x)$$

$$=e^x\{(B+A)\cos x+(B-A)\sin x\} \quad \text{--- (2)}$$

Using given conditions $A=e^y$ & $B=-A=-e^y$

$$\therefore z=e^x(e^y\cos x-e^y\sin x)=(\cos x-\sin x)e^{x+y}$$

2) Solve the eq^n $\frac{\partial^3 u}{\partial y^3}+4\frac{\partial u}{\partial y}=0$, given that

$u=0$, $\frac{\partial u}{\partial y}=x$, $\frac{\partial^2 u}{\partial y^2}=x^2-1$ when $y=0$

Setting $\frac{\partial u}{\partial y}=v$, we get $\frac{\partial^2 v}{\partial y^2}+4v=0$ --- (1)

$\Rightarrow (D^2+4)v=0$ where $D=\frac{\partial}{\partial y}$

AE: $m^2+4=0$

$$\Rightarrow v=A\cos 2y+B\sin 2y \quad \text{--- (2)}$$

$\int^g$ wrt y we get $u=\frac{A}{2}\sin 2y-\frac{B}{2}\cos 2y+C$ --- (3)

Here A, B, C are arbitrary funcns of x.

From (2), $\frac{\partial v}{\partial y} = -2A \sin 2y + 2B \cos 2y$ — (4)

Given $u=0$, $\frac{\partial u}{\partial y} = x$ & $\frac{\partial^2 u}{\partial y^2} = x^2-1$ when $y=0$

Using these conditions,

$$0 = -\frac{B}{2} + C, \quad x = A, \quad x^2-1 = 2B$$

$A = x$, $B = \frac{1}{2}(x^2-1)$, $C = \frac{1}{4}(x^2-1)$.

Putting these in (3),

$$u = \frac{1}{2} x \sin 2y + \frac{1}{4}(x^2-1)(1-\cos 2y)$$

is the required soln.

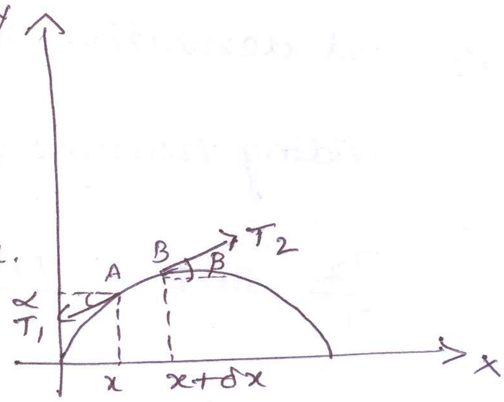
Derivation of one dimensional wave eq^n

Consider a flexible string tightly stretched b/w two fixed points at a distance l apart. Let ρ be the mass per unit length of the string. Let us assume the following:

(i) The tension T of the string is same throughout.

(ii) The effect of gravity can be ignored due to large tension T.

(iii) The motion of the string is in small transverse vibrations.



Let us consider the forces acting on a small element AB of length δx.

Let T_1 & T_2 be the tensions at the points A & B.

Since there is no motion in the horizonal direction, the horizontal components T_1 & T_2 must cancel eachother.

$$\therefore T_1 \cos\alpha = T_2 \cos\beta = T \quad \text{—— (1)}$$

where α & β are the angles made by T_1 & T_2 with the horizontal.

vertical components of tension are $-T_1 \sin\alpha$ & $T_2 \sin\beta$, where the $-$ve sign is used because T_1 is directed downwards. Hence the resultant force acting vertically upwards is $T_2 \sin\beta - T_1 \sin\alpha$

Applying Newton's second law of motion,

Force = mass × acceleration

$$\Rightarrow \quad T_2 \sin\beta - T_1 \sin\alpha = (\rho\, \delta x) \frac{\partial^2 u}{\partial t^2}$$

$\rho \delta x$ is the mass of the element portion AB & second derivative w.r.t 't' represents acceleration.

Dividing throughout by T we have,

$$\frac{T_2}{T} \sin\beta - \frac{T_1}{T} \sin\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

But from ①, $\dfrac{T_1}{T} = \dfrac{1}{\cos\alpha}$; $\dfrac{T_2}{T} = \dfrac{1}{\cos\beta}$

$$\therefore \quad \frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \quad \tan\beta - \tan\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2} \quad \text{--- ②}$$

But $\tan\beta$ & $\tan\alpha$ represent the slopes at $B(x+\delta x)$ & $A(x)$ respectively

$\therefore \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ & $\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x$

Now ② becomes

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2}$$

Dividing δx & taking limit as $\delta x \to 0$ we have

$$\lim_{\delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

But the LHS is nothing but the derivative of $\frac{\partial u}{\partial x}$ wrt x treating t as constant.

ie $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ or $\frac{\partial^2 u}{\partial x^2}$

Hence $\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$ or $\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$

Denoting T/ρ by c^2 we get

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad u_{tt} = c^2 u_{xx}$$

This is the wave eq^n in one dimension.

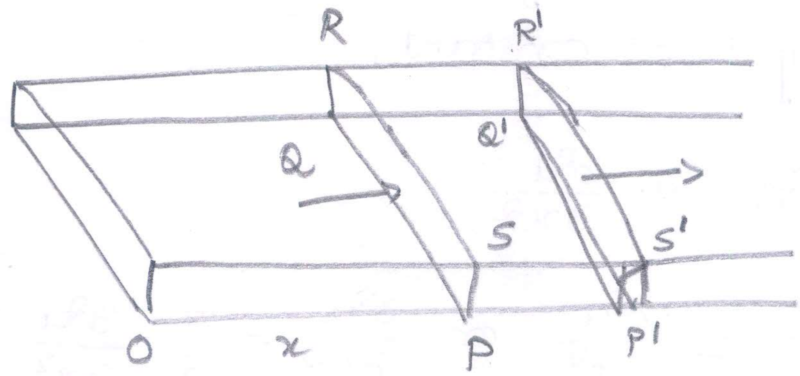
Derivation of one-dimensional heat eqn.

We have the following laws in respect of heat flow,

(1) Heat flows from a higher temp to a lower temp.

(2) The amount of heat in a body is proportional to its mass & temp.

(3) The rate of heat flow accross an area is proportional to the area & to the temp gradient normal to the area where the constant of proportionality (K) is called the thermal conductivity.



Consider a homogeneous bar of const cross-sectional area A. Let ρ be the density, s be the specific heat and K be the thermal conductivity of the material. Let the sides be insulated so that the stream lines of heat flow are parallel and perpendicular to the area A.

Let one end of the bar be taken as the origin O & the direction of heat flow be the positive x-axis.

Let $u = u(x,t)$ be the temp of the slab at a distance x from the origin

Consider an element of bar b/w the planes PQRS & P'Q'R'S' at a distance x & $x+\delta x$ from the end O.

Let δu be the change in temp in a slab of thickness δx of the bar.

The mass of the element $= A\rho\,\delta x$

The quantity of heat stored in this slab element

$$= A\rho s\,\delta x . \delta u$$

Hence the rate of increase of heat in this slab element is

$$R = (A\rho s\,\delta x)\frac{\partial u}{\partial t} \qquad \text{①}$$

If R_I is the rate of inflow of heat & R_O is the rate of out flow of heat we have

$$R_I = -KA\left[\frac{\partial u}{\partial x}\right]_x \quad \& \quad R_O = -KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} \qquad \text{②}$$

where the $-$ve sign is due to empirical law ①

Hence we have from ① & ②.

$$R = R_I - R_O$$

$$\Rightarrow \quad A\rho s\,\delta x\frac{\partial u}{\partial t} = KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - KA\left[\frac{\partial u}{\partial x}\right]_x$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{K}{\rho s}\left\{\frac{\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \left[\frac{\partial u}{\partial x}\right]_{x}}{\delta x}\right\} \quad \text{--- (3)}$$

Taking limit as $\delta x \to 0$, RHS of (3) is equal to

$$\frac{K}{\rho s}\frac{\partial^2 u}{\partial x^2}$$

Further denoting $c^2 = K/\rho s$ which is called the diffusity of the substance, (3) becomes

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \Rightarrow u_t = c^2 u_{xx}$$

This is one-dimensional heat eq^n.

Various possible solutions of the one-dimensional wave eq^n $u_{tt} = c^2 u_{xx}$ by the method of separation of variables.

Consider $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

Let $u = XT$; $X = X(x)$, $Y = Y(y)$ be the sol^n of the PDE.

Hence the PDE becomes

$$\frac{\partial^2 (XT)}{\partial t^2} = c^2\frac{\partial^2 (XT)}{\partial x^2} \Rightarrow X\frac{d^2 T}{dt^2} = c^2 T\frac{d^2 X}{dx^2}$$

Dividing by c^2XT

$$\Rightarrow \frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$

Equating both sides to a common constant k we have,

$$\frac{1}{X}\frac{d^2X}{dx^2} = k \quad \& \quad \frac{1}{c^2T}\frac{d^2T}{dt^2} = k$$

$$(D^2-k)X = 0 \qquad (D^2-c^2k)T = 0$$

$$D = \frac{d}{dx} \qquad D = \frac{d}{dt}$$

Case (i): Let $k=0$

$$\Rightarrow D^2X = 0 \quad \& \quad D^2T = 0$$

In both the equations AE is $m^2 = 0$ $\therefore m = 0, 0$

Solutions are given by

$$X = (c_1+c_2x)e^{0x} \quad \& \quad T = (c_3+c_4x)e^{0t}$$

ie $\therefore u = XT = (c_1+c_2x)(c_3+c_4t)$

Case (ii): Let k be +ve say, $k = +p^2$

The eqns become

$$(D^2-p^2)X = 0 \quad \& \quad (D^2-c^2p^2)T = 0$$

$$\Rightarrow m^2 = p^2 \quad \& \quad m^2 = c^2p^2$$

$$X = (c_1e^{px} + c_2e^{-px}) \qquad T = (c_3e^{cpt} + c_4e^{-cpt})$$

Hence the solution of the PDE is given by

$$u = XT = (c_1' e^{px} + c_2' e^{-px})(c_3' e^{cpt} + c_4' e^{-cpt})$$

Case (iii) : Let k be $-$ve, say $k = -p^2$

$$\Rightarrow (D^2 + p^2) X = 0 \quad \& \quad (D^2 + c^2p^2) T = 0$$

AE : $m^2 + p^2 = 0$

$$m = \pm ip \quad \& \quad m = \pm icp$$

Solutions are :

$$X = (c_1'' \cos px + c_2'' \sin px) \; ; \; T = (c_3'' \cos cpt + c_4'' \sin cpt)$$

Note: Solution obtained in the case (iii) is ~~obtained~~ considered as the befitting / suitable solⁿ to solve a B.V.P connected with one dimensional wave eqⁿ as the solⁿ involves periodic funcⁿs.

The befitting solⁿ of the one-d wave eqⁿ for solving B.V.Ps is :

$$u(x,t) = (A \cos px + B \sin px)(C \cos cpt + D \sin cpt)$$

Heat Eqⁿ $\longrightarrow$ same as that of wave :

case (i) $u(x,t) = Ax + B$

case (ii) $u(x,t) = e^{c^2p^2t}(A' e^{px} + B' e^{-px})$

case (iii) $u(x,t) = e^{-c^2p^2t}(A'' \cos px + B'' \sin px)$

Double & Triple Integrals.

Consider a funcn $f(x,y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \ldots \delta A_n$. Let (x_r, y_r) be any point within the r^{th} elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \cdots + f(x_n, y_n)\,\delta A_n$$

$$\text{ie } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum if it exists, as the number of subdivisions increases indefinitely & area of each sub division decreases to zero, is defined as the double integral of $f(x,y)$ over the region R and is defined as

$$\iint_R f(x,y)\, dA = \lim_{\substack{n \to \infty \\ \delta A \to 0}} f(x_r, y_r)\,\delta A_r$$

(i) When y_1, y_2 are funcns of x & x_1, x_2 are constants, $f(x,y)$ is first integrated wrt y keeping x fixed b/w limits y_1, y_2 & then resulting expression is integrated wrt x within the limits x_1, x_2

$$\text{ie} \quad I = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)\, dy} \, . \, dx$$ & vice versa

(ii) When both the pairs of limits are constants, the region of integration is the rectangle.

For constant limits, it hardly matters whether we first integrate wrt x & then wrt y or vice versa.

1) Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

$$= \int_{x=0}^{1} \left\{ \int_{y=0}^{x^2} e^{y/x}\, dy \right\} dx = \int_0^1 x(e^x - 1)\, dx$$

$$= \left[(x-1)e^x - \frac{1}{2}x^2 \right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

2) $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$

$$= \int_{x=0}^{1} \left\{ \int_{y=0}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-y^2}}\, dy \right\} dx$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}} \{ \sin^{-1} 1 - \sin^{-1} 0 \}\, dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0 \right) . dx$$

$$= \frac{\pi}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} (\sin^{-1} 1 - \sin^{-1} 0) = \frac{\pi}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$

Double integral by changing into polar form

We can obtain the relation connecting a double integral in cartesian form and the corresponding double integral in polar form.

Let (r, θ) be the polar coordinates of a point (x, y)

Then $x = r\cos\theta$, $y = r\sin\theta$ so that

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

Hence, $$\iint_R f(x,y)\,dx\,dy = \iint_{\bar{R}} \phi(r,\theta)\, r\,dr\,d\theta$$

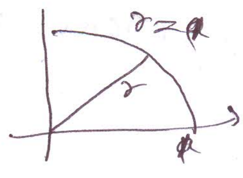
Here $\bar{R}$ is the region in which (r, θ) vary as (x, y) vary in R.

1) Evaluate the integral $I = \iint \left(\frac{1-x^2y^2}{1+x^2+y^2}\right)^{1/2} dx\,dy$ over the +ve quadrant bounded by the circle $x^2+y^2=1$

~~Sol~~

~~Let~~ $r : 0 \longrightarrow 1$

$\theta : 0 \longrightarrow \pi/2$



On changing to polar coordinates,

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \left(\frac{1-r^2}{1+r^2}\right)^{1/2} r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 \frac{1-r^2}{\sqrt{1-r^4}}\, r\,dr\,d\theta$$

On multiplying & dividing by $\sqrt{1-r^2}$

Change of order of integration

1) By changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy$, show that $\int_0^\infty \frac{\sin px}{x} \cdot dx = \pi/2$

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy = \int_0^\infty \left\{ \int_0^\infty e^{-xy} \sin px \, dx \right\} dy$$

$$= \int_0^\infty \left[\frac{e^{-xy}}{p^2+y^2} (p\cos px + y \sin px) \right]_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2} \cdot dy = \left[\tan^{-1} \frac{y}{p} \right]_0^\infty = \pi/2 \quad \text{---} (1)$$

On changing the order of $\int^n$.

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy = \int_0^\infty \sin px \left\{ \int_0^\infty e^{-xy} dy \right\} dx$$

$$= \int_0^\infty \sin px \left| \frac{e^{-xy}}{-x} \right|_0^\infty \cdot dx = \int_0^\infty \frac{\sin px}{x} \cdot dx \quad \text{---} (2)$$

From (1) & (2),

$$\int_0^\infty \frac{\sin px}{x} \cdot dx = \pi/2 .$$

$$= \int_0^{\pi/2} \left\{ \int_0^1 \frac{r}{\sqrt{1-r^4}}\, dr - \int_0^1 \frac{r^3}{\sqrt{1-r^4}}\, dr \right\} d\theta .$$

Setting $r^2 = t$ in the I integral & $r^4 = s$ in the second one,

$$I = \int_0^{\pi/2} \left\{ \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} - \frac{1}{4}\int_0^1 \frac{ds}{\sqrt{1-s}} \right\} d\theta$$

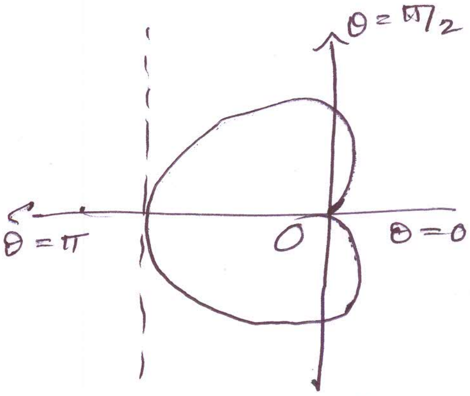
$$= \int_0^{\pi/2} \left\{ \frac{1}{2} \sin^{-1} t \Big|_0^1 + \frac{1}{2} \left(\sqrt{1-s}\right) \Big|_0^1 \right\} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\frac{\pi}{2} - 1\right) d\theta = \frac{1}{2}\left(\frac{\pi}{2}-1\right)\frac{\pi}{2} = \frac{\pi}{8}(\pi - 2)$$

2) Evaluate $\iint_R r \sin\theta \, dr\, d\theta$, where R is the region bounded by the cardioid $r = a(1-\cos\theta)$ above the initial line.

$\theta : 0 \rightarrow \pi$

$r : 0 \rightarrow a(1-\cos\theta)$



$$\iint_R r \sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1-\cos\theta)} r \sin\theta \, dr\, d\theta$$

$[t = 1-\cos\theta]$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} \sin\theta = \frac{a^2}{2}\int_0^{\pi} (1-\cos\theta)^2 \sin\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^2 t^2 \, dt = \frac{4a^2}{3}$$

Some more problems ~~[illegible]~~ →

Partial Differential Equations

Form the PDE by eliminating the arbitrary constants from

1) $z = ax + by + a^2 + b^2$ —①

Diff partially wrt x,

$\frac{\partial z}{\partial x} = p = a$ $\Rightarrow a = p$

Diff ① partially wrt y.

$\frac{\partial z}{\partial y} = b$ $\Rightarrow b = q$

Substituting these in ①

$z = px + qy + p^2 + q^2$

2) $(x-a)^2 + (y-b)^2 + z^2 = c^2$ —①

Diff ① partially wrt x,

$2(x-a) + 2zp = 0$ $\Rightarrow (x-a) = -zp$

Diff ① partially wrt y,

$2(y-b) + 2zq = 0$ $\Rightarrow (y-b) = -zq$

Substituting in ①,

$z^2p^2 + z^2q^2 + z^2 = c^2$

$\Rightarrow z^2(p^2+q^2+1) = c^2$

3) $(x-a)^2+(y-b)^2 = z^2\cot^2\alpha$ —— ①

Diff ① wrt x (partially)

$\cancel{2}(x-a) = \cancel{2}zp\cot^2\alpha$

~~Let~~

Diff ① partially wrt y,

$\cancel{2}(y-b) = \cancel{2}zq\cot^2\alpha$

$\therefore$ ① becomes,

$z^2p^2\cot^4\alpha + z^2q^2\cot^4\alpha = z^2\cot^2\alpha$

$\Rightarrow (p^2+q^2)\cot^2\alpha = 1$

$\Rightarrow \quad p^2+q^2 = \tan^2\alpha$

4) Find the differential equation of all spheres of fixed radius 3 having their centres in the xy-plane.

Eq^n of a sphere having centre in the xy-plane is given by $(x-a)^2+(y-b)^2+(z-0)^2 = 3^2$ where $(a,b,0)$ is the centre and 3 is the radius.

$\Rightarrow (x-a)^2+(y-b)^2+z^2 = 9$ —— ①

Diff wrt x & y partially

$2(x-a)+2zp=0$ & $2(y-b)+2zq=0$

$\Rightarrow (x-a) = -zp$ & $(y-b) = -zq$

Substituting these in ①

$$(-zp)^2 + (-zq)^2 + z^2 = 9$$

$$\Rightarrow \quad z^2 (p^2+q^2+1) = 9$$

5. Find the differential equation of all spheres whose centres lie on the z-axis.

Let centre be $(0,0,c)$ ∵ centre lie on z-axis

& r be the radius.

∴ Eqⁿ of the sphere is $x^2+y^2+(z-c)^2 = r^2$ —— ①

Diff ① partially wrt x,

$$2x + 2(z-c)p = 0$$

$$\Rightarrow z-c = -x/p \quad \text{—— ②}$$

Diff ① partially wrt y.

$$2y + 2(z-c)q = 0$$

$$\Rightarrow z-c = -y/q \quad \text{—— ③}$$

From ② & ③, $-x/p = -y/q$

$$\Rightarrow py - qx = 0$$

Form the partial equat DE by eliminating the arbitrary functions from:

1) $z = f(x^2-y^2)$ — ①

Diff partially wrt x, $p = f'(x^2-y^2) \cdot 2x$ — ②

Diff ① partially wrt y, $q = f'(x^2-y^2) \cdot -2y$ — ③

From ② & ③

$$p = \frac{q}{-2y} \cdot 2x \quad \Rightarrow \quad py + qx = 0$$

2) $z = y f(x) + x g(y)$ — ①

Diff ① partially wrt x,

$p = y f'(x) + g(y)$ — ②

Diff ① partially wrt y,

$q = f(x) + x g'(y)$ — ③

Diff ② partially wrt y,

$$s = f'(x) + g'(y) = \frac{p - g(y)}{y} + \frac{q - f(x)}{x} \text{ using ② \& ③}$$

X'ly throughout by xy.

$\Rightarrow xy\,s = px - x\,g(y) + q\,(y) - y\,f(x)$

$\Rightarrow xy\,s = px + qy - z$

3) $z = f(x^2+y^2) + x + y$

Diff wrt x & y partially -

$p = f'(x^2+y^2) \cdot 2x + 1$; $q = f'(x^2+y^2) \cdot 2y + 1$

Using one in the other

$$p = \left[\frac{q-1}{2y}\right] \cdot 2x + 1 \quad \Rightarrow \quad py - qx = y - x$$

4) $z = x^2 f(y) + y^2 g(x)$ —— ①

$p = 2x f(y) + y^2 g'(x)$ —② by diff ① partially wrt x.

$q = x^2 f'(y) + 2y\, g(x)$ —③ by diff ① partially wrt y.

Diff ② partially wrt y,

$$\frac{\partial^2 z}{\partial x \partial y} = s = 2x f'(y) + 2y g'(x)$$

$$\Rightarrow s = 2x\left[\frac{q - 2y g(x)}{x^2}\right] + 2y\left[\frac{p - 2x f(y)}{y^2}\right]$$

$$\Rightarrow s x^2 y^2 = 2xy^2 q - 4xy^3 g(x) + 2yx^2 p - 4x^3 y f(y)$$

$$\Rightarrow s x^2 y^2 = 2xy^2 q + 2yx^2 p - 4xy[z] \text{ using ①}$$

$$\Rightarrow xys = 2[px + qy - 2z]$$

5) $z = f(x) + e^y g(x)$ —— ①

Diff wrt x and y respectively

$p = f'(x) + e^y g'(x)$ —— ②

$q = g(x) e^y$ —— ③

Diff ② wrt x & y and ③ wrt y.

$r = f''(x) + e^y g''(x)$ —— ④

$s = g'(x) e^y$ —— ⑤

$t = g(x) e^y$ —— ⑥

From ③ & ⑥. $q = t$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2} \text{ is the required PDE.}$$

6) $z = x f_1(x+y) + f_2(x+y)$ — ①

From ①.

$p = f_1(x+y) + x f_1'(x+y) + f_2'(x+y)$

$q = x f_1'(x+y) + f_2'(x+y)$

& second order partial derivatives

$r = f_1'(x+y) + x f_1''(x+y) + f_1'(x+y) + f_2''(x+y)$

$s = f_1'(x+y) + x f_1''(x+y) + f_2''(x+y)$

$t = x f_1''(x+y) + f_2''(x+y)$

$\Rightarrow r = 2f_1'(x+y) + t$ — ②

& $s = f_1'(x+y) + t$ — ③

~~Substitute~~ $\Rightarrow f_1'(x+y) = s - t$

$\therefore$ ② becomes $r = 2[s-t] + t = 2s - t$

$\Rightarrow \quad r - 2s + t = 0$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

Solve the following equations:

1) $$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{x}{y} + a$$

Let us solve this by direct integration:

Integrating with respect to x, treating y as a constant.

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

$\int^g$ wrt y, treating x as a constant.

$$z = \frac{x^2}{2} \log y + axy + F(y) + g(x)$$

2) $$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$$

$\int^g$ wrt x, treating y as a constant,

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\sin(2x+3y)}{2} + f(y)$$

$\int^g$ wrt x, treating y as a const,

$$\frac{\partial z}{\partial y} = -\frac{\cos(2x+3y)}{4} + f(y) \cdot x + g(y)$$

$\int^g$ wrt y, treating x as a const,

$$z = -\frac{\sin(2x+3y)}{12} + F(y)x + G(y) + \phi(x)$$

3) Solve $\frac{\partial^2 z}{\partial y^2} = z$, gives that when $y=0$, $z = e^x$ & $\frac{\partial z}{\partial y} = e^{-x}$.

Suppose z is a function of y only, then we have

$$\frac{d^2 z}{dy^2} = z \quad \Rightarrow (D^2-1)\, z = 0 \qquad D = \frac{d}{dy}$$

AE : $m^2 - 1 = 0$

$m = \pm 1$

$z = c_1 e^{y} + c_2 e^{-y}$

$\because$ given eq^n is a PDE,

Sol^n is $\quad z = f(x) e^{y} + g(x) e^{-y}$ —— ①

Given when $y=0$, $z = e^x$

$e^x = f(x) + g(x)$ —— ②

$$\frac{\partial z}{\partial y} = f(x) e^{y} - g(x) e^{-y}$$

Given $\frac{\partial z}{\partial y} = e^{-x}$ when $y = 0$

$\Rightarrow e^{-x} = f(x) - g(x)$ —— ③

From ② & ③.

$2 f(x) = e^{x} + e^{-x}$

$\Rightarrow f(x) = \frac{e^{x} + e^{-x}}{2} = \cosh x$

$2 g(x) = e^{x} - e^{-x} \quad \Rightarrow g(x) = \sinh x$

$\therefore$ sol^n is : $\Rightarrow$

$$z = \cosh x\, e^{y} + \sinh x\, e^{-y}$$

4) Solve by the method of separation of variables

$py^3 + qx^2 = 0.$

$$\frac{\partial u}{\partial x} \cdot y^3 + \frac{\partial u}{\partial y} \cdot x^2 = 0$$

$$\frac{1}{x^2} \frac{\partial u}{\partial x} = -\frac{1}{y^3} \frac{\partial u}{\partial y}$$

Let

$$\frac{1}{x^2} \frac{\partial u}{\partial x} = k \quad ; \quad -\frac{1}{y^3} \frac{\partial u}{\partial y} = k$$

$$\frac{\partial u}{\partial x} = kx^2 \quad ; \quad \frac{\partial u}{\partial y} = -ky^3$$

Let $u = X(x)\, Y(y)$ be the required solution.

$$\frac{\partial}{\partial x}(XY)\, y^3 + \frac{\partial}{\partial y}(XY)\, x^2 = 0$$

$$Y \frac{dX}{dx} \cdot y^3 + X \frac{dY}{dy} \cdot x^2 = 0$$

$$Y \frac{dX}{dx} \cdot y^3 = -X \frac{dY}{dy} \cdot x^2 \cancel{=0}$$

$$\frac{1}{Xx^2} \frac{dX}{dx} = \frac{1}{Yy^3} \frac{dY}{dy}$$

$$\frac{1}{Xx^2} \frac{dX}{dx} = k \quad ; \quad \frac{1}{Yy^3} \frac{dY}{dy} = k$$

$$\frac{1}{X} dX = kx^2 dx \quad ; \quad \frac{1}{Y} dY = ky^3 dy$$

$\int$g both the sides.

$\log X = \frac{kx^3}{3} + k_1$; $\log Y = \frac{ky^4}{4} + k_2$

$X = e^{\frac{kx^3}{3}} \cdot c_1$; $Y = e^{\frac{ky^4}{4} + k_2} = c_2 e^{\frac{ky^4}{4}}$

$\therefore$ Required solⁿ is

$$u = c_1 c_2 e^{\frac{kx^3}{3}} e^{\frac{ky^4}{4}} = c_1 c_2 e^{\frac{4kx^3+3ky^4}{12}}$$

Double and Triple Integrals

1) Evaluate the integral $\int_1^2 \int_1^3 xy^2\,dy\,dx$.

$$\int_1^2 \int_1^3 xy^2\,dy\,dx = \int_1^2 x\left[\frac{y^3}{3}\right]_1^3 dx = \int_1^2 \frac{x}{3}[27-1]\,dx$$

$$= \frac{26}{3}\left[\frac{x^2}{2}\right]_1^2 = \frac{26}{3}\left[\frac{4-1}{2}\right]$$

$$= \underline{\underline{13}}$$

Evaluate

2) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$.

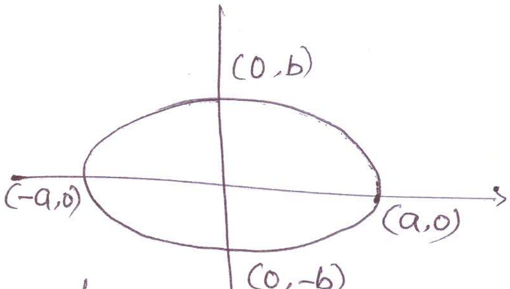
$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{\sqrt{x}} \cdot dx = \int_0^1 \left\{x^2[\sqrt{x}-x] + \frac{1}{3}(x\sqrt{x}-x^3)\right\} dx$$

$$= \int_0^1 \left[(x^{5/2}-x^3) + \frac{1}{3}(x^{3/2}-x^3)\right] \cdot dx = \left[\frac{x^{7/2}}{7/2} - \frac{x^4}{4} + \frac{1}{3}\left(\frac{x^{5/2}}{5/2} - \frac{x^4}{4}\right)\right]_0^1$$

$$= 2/7 - 1/4 + 2/15 - 1/12 = 0.0857$$

3) Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Let $x = ar\cos\theta$, $y = br\sin\theta$

So, the ellipse transforms to

$r = 1$ with θ in $[0, 2\pi]$.

$\therefore$ Jacobian $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix}$

$= abr$

Hence $\iint (x+y)^2\,dx\,dy$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (ar\cos\theta + br\sin\theta)^2 \cdot abr\,dr\,d\theta$$

$$= ab\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^3(a\cos\theta + b\sin\theta)^2\,dr\,d\theta$$

$$= ab\int_{\theta=0}^{2\pi}(a\cos\theta + b\sin\theta)^2\,d\theta \cdot \int_{r=0}^{1} r^3\,dr$$

$$= ab\int_{\theta=0}^{2\pi}(a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta)\,d\theta \cdot \left[\frac{r^4}{4}\right]_0^1$$

$$= \frac{ab}{4}\int_{\theta=0}^{2\pi}\left[\frac{a^2}{2}(1+\cos 2\theta) + \frac{b^2}{2}(1-\cos 2\theta) + ab\sin 2\theta\right]d\theta$$

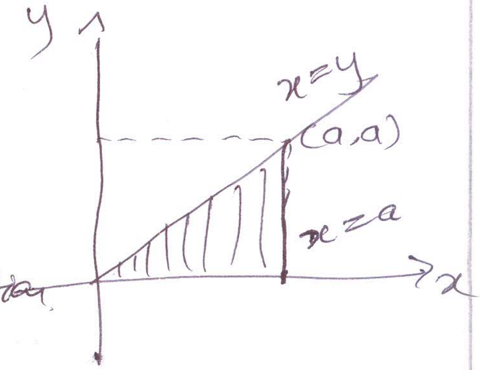
$$= \frac{ab}{8}\left[a^2\left(\theta + \cancelto{0}{\frac{\sin 2\theta}{2}}\right) + b^2\left(\theta - \cancelto{0}{\frac{\sin 2\theta}{2}}\right) - 2ab\frac{\cos 2\theta}{\cancel{2}}\right]_0^{2\pi}$$

$$= \frac{ab}{8}\left[a^2\cdot 2\pi + 2\pi b^2 - (\cancelto{0}{ab - ab})\right] = \frac{\pi ab}{4}(a^2+b^2)$$

4) Evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ by changing the order of $\int^n$.

Here, $y : 0 \to a$

$x : y \to a$



By changing the order of integration

we can see that in the given area,

$x : 0 \to a$

& $y : 0 \to x$.

$\therefore$ Given integral can be written as

$$\int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} \left[\tan^{-1} y/x\right]_0^x . dx = \int_0^a \frac{\pi}{4} . dx = \frac{\pi a}{4}$$

5. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z . dz . dx\, dy$

$$= \int_1^e \int_1^{\log y} \left[\log z . z - \int z . \frac{1}{z} . dz\right]_1^{e^x} . dx . dy$$

$$= \int_1^e \int_1^{\log y} \left[z \log z - z\right]_1^{e^x} . dx\, dy$$

$$= \int_1^e \int_1^{\log y} \left[e^x . x - e^x + 1\right] . dx\, dy$$

$$= \int_1^e \int_1^{\log y} [x e^x - e^x - e^x + x]_1^{\log y} \cdot dy$$

$$= \int_1^e (y \log y - y - y + \log y - \cancel{e} + \cancel{e} + e - 1) \cdot dy$$

$$= \int_1^e (y \log y - 2y + \log y + e - 1) \cdot dy$$

$$= \left[\log y \cdot \frac{y^2}{2} - \int \frac{y^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{y}} \cdot dy - y^2 + y \log y - y + ey - y\right]_1^e$$

$$= \frac{e^2}{2} - \frac{1}{4}[e^2 - 1] - [e^2 - 1] + e - (e - 1)$$
$$+ e(e-1) - (e-1)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} - \cancel{e^2} + 1 + \cancel{e} - \cancel{e} + 1 + \cancel{e^2} - e - e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{1}{4} + 3 = \frac{1}{4}[e^2 - 8e + 12]$$

$$= \frac{1}{4}[e^2 - 8e + 13]$$

6. Evaluate $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\, dz\, dr\, d\theta$

$$= \int_0^{\pi/2} \int_0^{a\sin\theta} r\,[z]_0^{\frac{a^2-r^2}{a}}\, dr \cdot d\theta$$

$$= \int_0^{\pi/2} \int_0^{a\sin\theta} r\left[\frac{a^2-r^2}{a}\right] \cdot dr \cdot d\theta$$

$$= \frac{1}{a} \int_0^{\pi/2} \left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} \cdot d\theta$$

$$= \frac{1}{a} \int_0^{\pi/2} \left[\frac{a^4}{2}\sin^2\theta - \frac{a^4\sin^4\theta}{4}\right] \cdot d\theta$$

$$= \frac{1}{a}\left[\frac{a^4}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{a^4}{4} \cdot \frac{3}{4 \cdot 2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{a^3\pi}{2}\left[\frac{1}{4} - \frac{3}{32}\right] = \frac{5\pi a^3}{64}$$