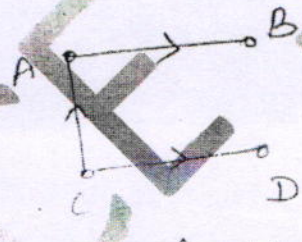
MODULE-5

DIRECTED GRAPHS AND GRAPHS

Definition of a Directed graph:

A directed graph (or a digraph) is a pair (V, E), where V is a nonempty set and E is a set of ordered pairs of elements taken from the set V.

For a directed graph (V, E), the elements of V are called vertices (points or nodes) and the elements of E are called directed edges. The set V is called the vertex set and the set E is called the directed edge set. vertex set is



$V = \{A, B, C, D\}$

and the edge set is, $E = \{AB, CD, CA\}$

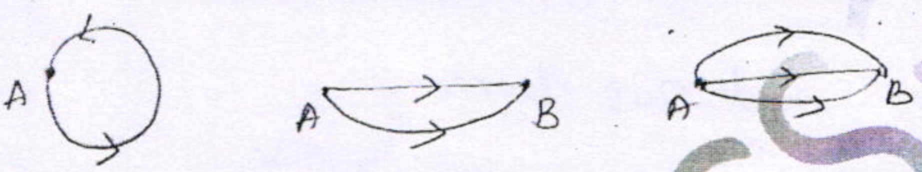
$= \{(A,B), (C,D), (C,A)\}$

Every directed edge of a digraph is determined by two vertices of the digraph - a vertex from which it begins and a vertex at which it ends.

If AB is a directed edge of a digraph D, then it is understood that this directed edge begins at the vertex A of D and terminates at the vertex B of D.

Here A is the initial vertex and B is the terminal vertex of AB.

AB is incident out of A and incident into B.



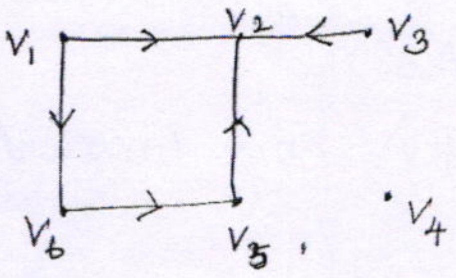
A directed loop which begins and ends at the vertex A.

Two directed edges having the same initial vertex and the same terminal vertex are called parallel directed edges.

Two or more directed edges having the same initial vertex and the same terminal vertex are called multiple directed edges.

A vertex of a digraph which is neither an initial vertex nor a terminal vertex of any directed edge is called an Isolated vertex of the digraph.

A non-isolated vertex which is not a terminal vertex for any directed edge is called a source and a non-isolated vertex which is not an initial vertex for any directed edge is called a sink.

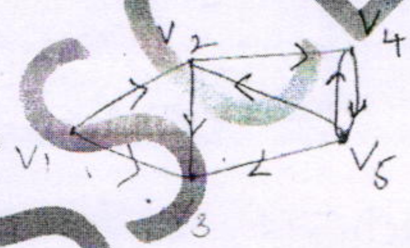


$v_4 \rightarrow$ Isolated vertex.
$v_1, v_3 \rightarrow$ Sources
$v_2 \rightarrow$ Sink

In-degree and Out degree:-

The indegree of a vertex v is a directed graph is the number of edges ending at it and is denoted as indeg(v)

Out degree: The outdegree of a vertex v is a directed graph is the number of edges begining from it and is denoted as outdeg(v)



Vertex:	v_1	v_2	v_3	v_4	v_5
Indegree:	0	2	3	2	1
Outdegree:	2	2	0	1	3

Note: In every digraph D, the sum of the out-degrees of all vertices is equal to the sum of the in-degrees of all vertices, each sum being equal to the number of edges in D.

Definition of a Graph:-

A graph written as $G = G(V, E)$ consists of two components

i) A finite set of vertices V, also called points or nodes.

ii) The finite set of (directed) edges E, also called lines or arcs connecting pair of vertices.

A graph/digraph containing no edges is called a null graph.

A null graph with only one vertex is called a trivial graph.

Adjacent vertices: Two vertices are said to be adjacent if they are connected by an edge.

Undirected graph: Graph with no direction ~~arrow~~ can be called undirected graph.

Simple Graph: A graph which has neither loops nor multiple edges is called a simple graph.

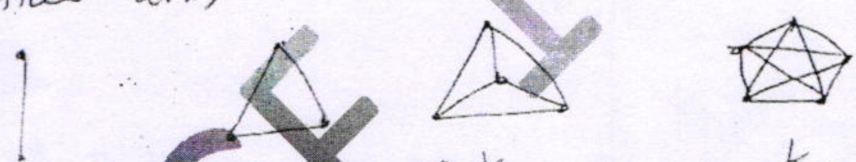
Multigraph: A graph which contain multiple edges But no loops called a multigraph.

A graph which contains multiple edges or loops (or both) is called a general graph.

Complete graph

A simple graph of order ≥ 2 in which there is an edge between every pair of vertices is called a complete graph (or a full graph).

Complete graphs with two, three, four and five vertices are,



K_2 K_3 K_4 K_5

A graph with five vertices, K_5 is called the Kuratowski's first graph.

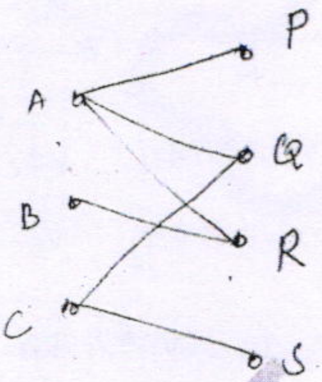
Bipartite graph

Suppose a simple graph G is such that its vertex set V is the union of two of its mutually disjoint nonempty subsets V_1 and V_2 which are such that each edge in G joins a vertex in V_1 and a vertex in V_2. The G is called a bipartite graph.

If E is the edge set of this graph, the is denoted by $G = (V_1, V_2 : E)$. (or) $G = G(V_1, V_2; E)$

The sets V_1 and V_2 are called bipartites (or partitions) of the vertex set V.

Ex:



It's a bipartite graph with $V_1 = \{A, B, C\}$ and $V_2 = \{P, Q, R, S\}$ as bipartities

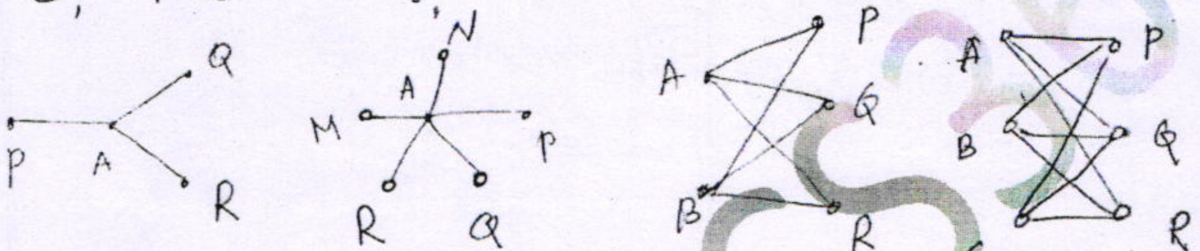
Complete Bipartite graph:

A bipartite graph $G = (V_1, V_2; E)$ is called a complete bipartite graph if there is an edge between every vertex in V_1 and every vertex in V_2.

A complete bipartite graph $G = (V_1, V_2; E)$ in which the bipartites V_1 and V_2 contain r and s vertices resp.

with $r \leq s$, is denoted by $K_{r,s}$. Each of r vertices in V_1 is joined to each of s vertices in V_2.

Thus, $K_{r,s}$ has $r+s$ vertices and rs edges; that is $K_{r,s}$ is of order $r+s$ and of size rs; it is therefore a $(r+s, rs)$ graph.



$V_1 = \{A\}$, $V_2 = \{P, Q, R\}$ | $V_1 = \{A\}$, $V_2 = \{P, Q, R, M, N\}$ | $V_1 = \{A, B\}$, $V_2 = \{P, Q, R\}$ | $V_1 = \{A, B, C\}$, $V_2 = \{P, Q, R\}$

* 1. Draw a diagram of the graph $G = (V, E)$ in each of the following cases:

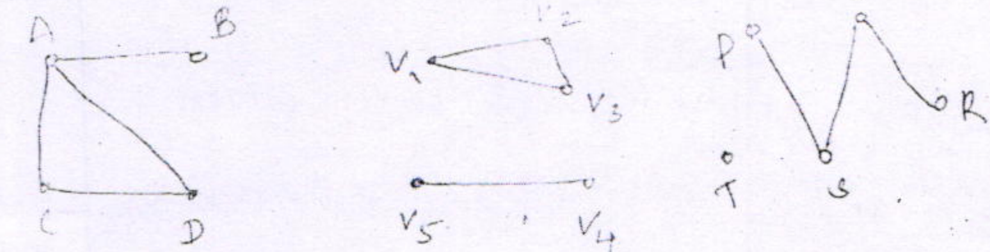
i) $V = \{A, B, C, D\}$, $E = \{AB, AC, AD, CD\}$

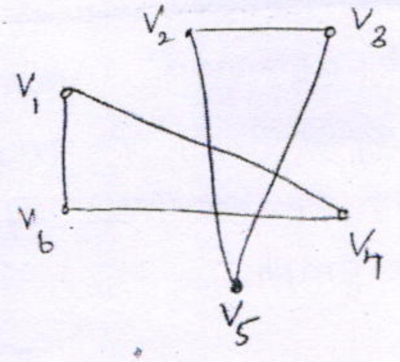
ii) $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{v_1v_2, v_1v_3, v_2v_3, v_4v_5\}$

iii) $V = \{P, Q, R, S, T\}$, $E = \{PS, QR, QS\}$

iv) $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $E = \{v_1v_4, v_1v_6, v_4v_6, v_3v_2, v_3v_5, v_2v_5\}$

$\Rightarrow$





Q. a) How many vertices and how many edges are there in the complete bipartite graphs $K_{4,7}$ and $K_{7,11}$?

b) If the graph $K_{r,12}$ has 72 edges, what is r?

$\Rightarrow$

The complete bipartite graph $K_{r,s}$ has $r+s$ vertices and rs edges.

a) $\therefore$ The graph $K_{4,7}$ has $4+7=11$ vertices and $4\times7=28$ edges, and the graph $K_{7,11}$ has 18 vertices and 77 edges.

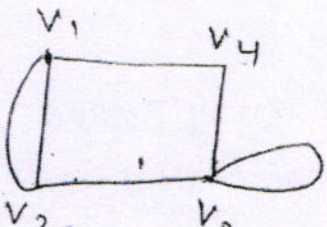
b) If the graph $K_{r,12}$ has 72 edges, solve $12r=72$ so that $r=6$

Vertex degree and Handshaking property

Let $G=(V,E)$ be a graph and v be a vertex of G. Then, the number of edges of G that are incident (That is, the number of edges that join v to other vertices of G) with the loops counted twice is called the degree of the vertex v and is denoted by $\deg(v)$, (or) $d(v)$.

The degree of all vertices of a graph arranged in non-decre-asing order is called the degree sequence of the graph. Also, the minimum of the degrees of vertices of a graph is called the degree of the graph.

Ex:



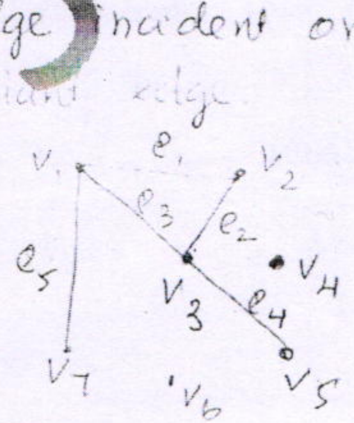
$d(v_1) = 3$
$d(v_2) = 4$
$d(v_3) = 4$
$d(v_4) = 3$

The degree sequence of this graph is 3, 3, 4, 4 and the degree of the graph is 3. The loop at v_3 is counted twice for determining the degree of v_3.

Isolated Vertex, Pendant Vertex

A vertex in a graph which is not an end vertex of any edge of the graph is called an isolated vertex. Obviously, a vertex is an isolated vertex if and only if its degree is zero.

A vertex of degree 1 is called a pendant vertex. An edge incident on a pendant vertex is called a pendant edge.



v_4 and v_6 are isolated vertices

v_5 & v_7 are pendant vertices and the e_4 and e_5 are pendant edges.

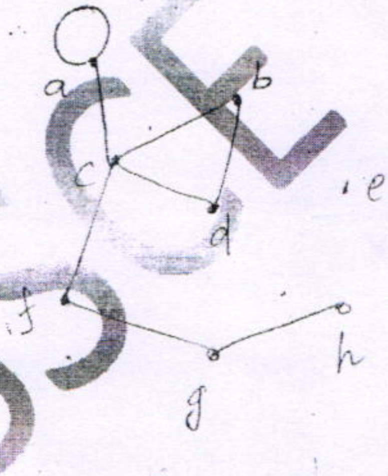
Regular graph: A graph in which all the vertices are of the same degree k is called a regular graph of degree k, or a k-regular graph.

In particular, a 3-regular graph is called a cubic-graph.

Property: The sum of the degrees of all the vertices in a graph is an even number and this number is equal to twice the number of edges in the graph.

Theorem: In every graph, the number of vertices of odd degrees is even.

(1). For the graph shown below, indicate the degree of each vertex and verify the handshaking property:



⇒ By examining the graph, we find that the degrees of its vertices are given below:

deg(a) = 3, deg(b) = 2, deg(c) = 4, deg(d) = 2

deg(e) = 0, deg(f) = 2, deg(g) = 2, deg(h) = 1

we note that e is an isolated vertex and h is

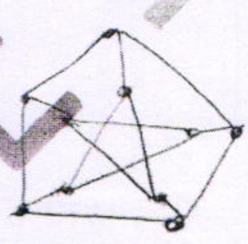
Pendant vertex.

Further, we observe that the sum of the degrees of vertices is equal to 16. Also, the graph has 8 edges. Thus, the sum of the degrees of vertices is equal to twice the number of edges. This verifies the handshaking property for the given graph.

(2) Can there be a graph consisting of the vertices A, B, C, D with deg(A) = 2, deg(B) = 3, deg(C) = 2, deg(D) = 2?

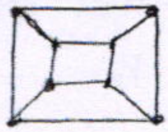
⇒ In every graph, the sum of the degrees of the vertices has to be an even number. Here this sum is 9 which is not even. Therefore, there does not exist a graph of the given kind.

Petersen Graph:



The graph is a 3-regular graph (cubic graph). This particular cubic graph which contains 10 vertices and 15 edges is called the Petersen graph.

Three-dimensional hypercube:



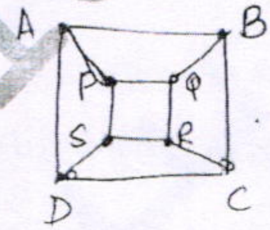
The graph is a cubic graph with $8 = 2^3$ vertices. This particular graph is called the three-dimensional hypercube and is denoted by Q_3.

In general, for any positive integer k, a loop-free k-regular graph with 2^k vertices is called the k-dimensional hypercube (or k-cube) and is denoted by Q_k.

(1) Show that the hypercube Q_3 is a bipartite graph which is not a complete bipartite graph.

$\Rightarrow$

The hypercube Q_3 is, with the vertices labeled as A, B, C, D, P, Q, R, S so that,



for this graph, the vertex set is $V = \{A, B, C, D, P, Q, R, S\}$

let,

$V_1 = \{A, C, Q, S\}$ and $V_2 = \{B, D, P, R\}$

[illegible] $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \phi$ and that every edge of the graph has one end vertex in the set V_1 and the other end vertex in the set V_2; further, no edge of the graph has both of its end vertices in V_1 or V_2. Hence, this graph is a bipartite graph.

We observe that this graph is not a complete bipartite graph. Because, there is no edge between every vertex in V_1 and every vertex in V_2.

(2) a) What is the dimension of the hypercube with 524288 edges?

b) How many vertices are there in a hypercube with 4980736 edges?

$\Rightarrow$ For the k-dimensional hypercube Q_k, the number of vertices is 2^k and the number of edges is $k2^{k-1}$.

a) If Q_k has 524288 edges, we've $k2^{k-1} = 524288$.

We check that, $524288 = 2^{19} = 2^4 \times 2^{15}$
$= 16 \times 2^{15}$

Accordingly, $k2^{k-1} = 524288$ holds if $k=16$. Thus, the dimension of the hypercube with 524288 edges is $k=16$.

b) We check that $4980736 = 19 \times 2^{18}$, which indicates that Q_k has 4980736 edges when $k=19$.

The number of vertices in this hypercube is $2^k = 2^{19} = 524288$

(3) Show that there is no graph with 28 edges and 12 vertices in the following cases:

i) The degree of a vertex is either 3 or 4

ii) The degree of a vertex is either 3 or 6

=>

Suppose there is a graph with 28 edges and 12 vertices of which k vertices are of degree 3 (each). Then:

i) If all of the remaining $(12-k)$ vertices have degree 4, then we should have (by the handshaking property) $3k+4(12-k) = 2\times28 = 56$ or $k=-8$ which is not possible (because, k has to be nonnegative)

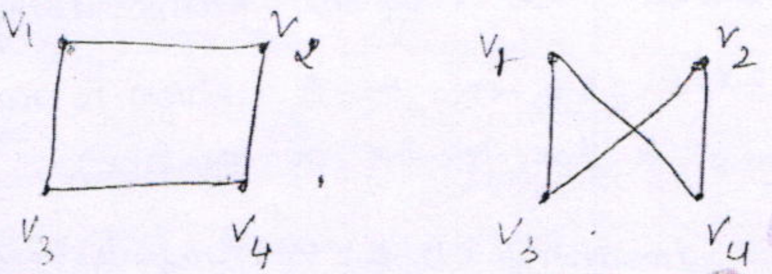
ii) If all of the remaining $(12-k)$ vertices have degree 6, the we should have $3k+6(12-k)=56$ (or) $k=16/3$. This is not possible (because k has to be a nonnegative integer)

Isomorphism:

Consider two graphs $G=(V,E)$ and $G'=(V',E')$. Suppose there exists a function $f: V \to V'$ such that
i) f is a one-to-one correspondence and
ii) for all vertices A, B of G, {A, B} is an edge of G if and only if {f(A), f(B)} is an edge of G'.
Then f is called an isomorphism between G and G', and we say that G and G' are isomorphic graphs.

Isomorphism of Digraphs: Two digraphs D_1 and D_2 are said to be isomorphic if there is a one-to-one correspondence between their vertices and between their edges such that adjacency of vertices along with direction is preserved.

① Prove that two graphs shown below are Isomorphic



⇒

We first observe that both graphs have four vertices and four edges. Consider the following one-to-one correspondence between the vertices of the graphs: $u_1 \leftrightarrow v_1$, $u_2 \leftrightarrow v_4$, $u_3 \leftrightarrow v_3$, $u_4 \leftrightarrow v_2$

This correspondence gives the following correspondence between the edges

$\{u_1, u_2\} \leftrightarrow \{v_1, v_4\}$, $\{u_1, u_3\} \leftrightarrow \{v_1, v_3\}$,

$\{u_2, u_4\} \leftrightarrow \{v_4, v_2\}$, $\{u_3, u_4\} \leftrightarrow \{v_3, v_2\}$

These represent one-to-one correspondence between the edges of the two graphs under which the adjacent vertices in the first graph correspond to adjacent vertices in the second graph and vice-versa.

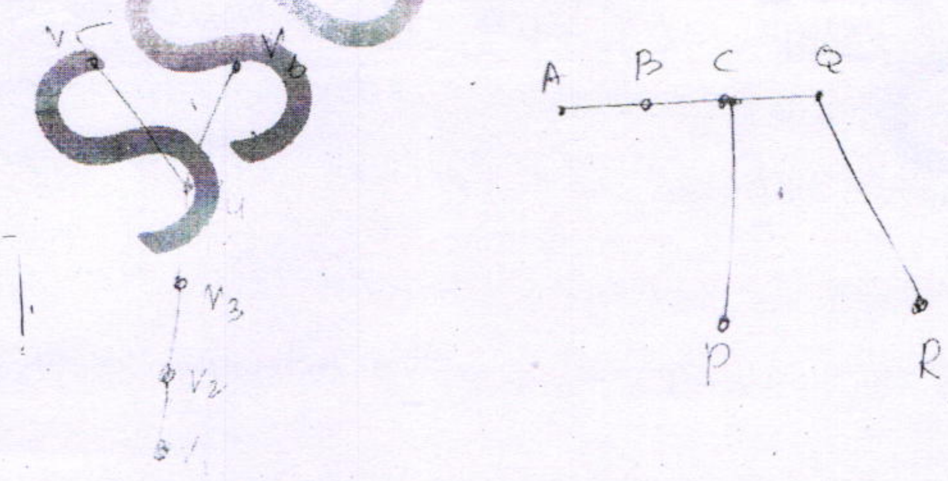
Accordingly, the two graphs are Isomorphic.

(2) Show that the following graphs are not isomorphic

⇒ we note that the first graph has a pair of vertices of degree 4 which are not adjacent whereas the second graph has a pair of vertices of degree 4 which are adjacent. Therefore the two graphs are not isomorphic.

(3) Show that two graphs need not be isomorphic even if they have the same number of vertices, the same number of edges and equal number of vertices with the same degree.

⇒ consider the two graph



we observe that both set graphs have same (6) number of vertices and the same (5) number of edges. Further in each of them there are 3 vertices of degree 1 (namely v_1, v_5, v_6 in the 1st graph and A, P, R in the second graph), there are 2 vertices of degree 2 (namely v_2, v_3 in the 1st graph and B, Q in the second graph), and there is 1 vertex of degree 3 (namely v_4 in the first graph and C in the second graph). Thus the two graphs have equal number of vertices with the same degree.

But the two graphs are not isomorphic. Because there are 2 pendant vertices adjacent to the vertex v_4 (which is of degree 3) in the first graph but there is only one pendant vertex adjacent to the vertex C (which is of degree 3) in the second graph. As such, the adjacency of the vertices cannot be preserved under a one-to-one correspondence between the vertices of the graphs.

Subgraph: Given two graphs G and G_1, we say that G_1 is a subgraph of G if the following conditions hold

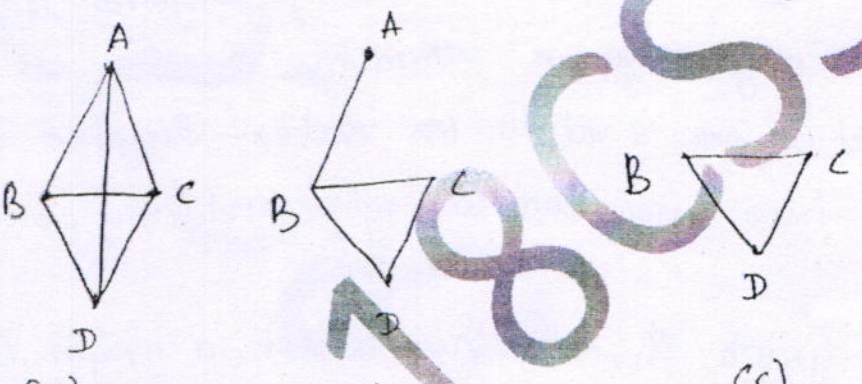
i) All the vertices and all the edges of G_1 are in G

ii) Each edge of G_1 has the same end vertices in G as in G_1.

Spanning Subgraph.

Given a graph $G=(V,E)$ if there is a subgraph, $G_1=(V_1,E_1)$ of G such that $V_1=V$, then G_1 is called a spanning subgraph of G.

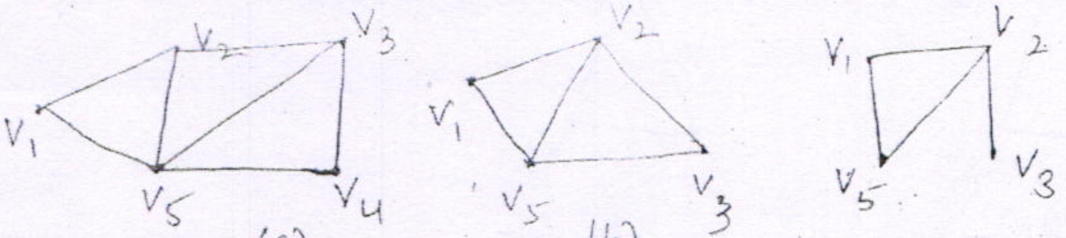
For example, for the graph shown below, (a) the graph (b) is a spanning subgraph whereas the graph (c) is a subgraph but not a spanning Subgraph



(a) (b) (c)

Induced subgraph.

Given a graph $G=(V,E)$, suppose there is a subgraph $G_1=(V_1,E_1)$ of G such that every edge $\{A,B\}$ of G, where $A,B \in V_1$ is an edge of G_1 also. Then G_1 is called an induced subgraph of G (induced by V_1) and is denoted by $\langle V_1 \rangle$



(a) (b) (c).

For the graph (a), the graph is a induced subgraph - induced by the set of vertices $V_1 = \{v_1, v_2, v_3, v_5\}$ whereas the graph is (c) is not an induced subgraph

Edge-disjoint and vertex-disjoint subgraphs:

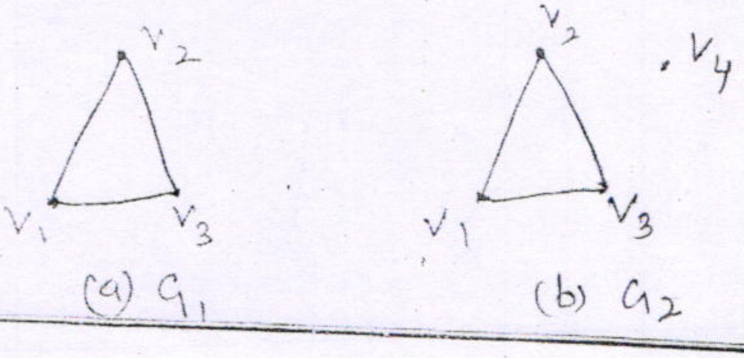
Let G be a graph and G_1 and G_2 be two subgraphs of G. Then.

1) G_1 and G_2 are said to be edge-disjoint if they do not have any edge in common.

2) G_1 and G_2 are said to be vertex-disjoint if they do not have any common edge and any common vertex

① Give a graph G_1, can there exist a graph G_2 such that G_1 is a subgraph of G_2 but not a spanning subgraph of G_2 and yet G_1 and G_2 have the same size?

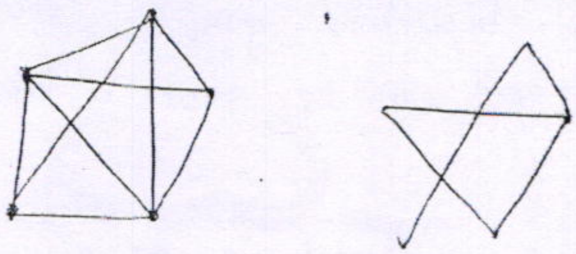
⇒ Yes, consider a graph G_1 which contains all the vertices and all the edges of G and at least [illegible] isolated vertex.



(a) G_1 (b) G_2

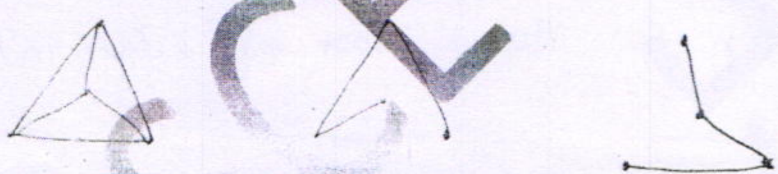
Complement of a Subgraph

Given a graph G and a subgraph G_1 of G, the subgraph of G obtained by deleting from G all the edges that belong to G_1 is called the complement of G_1 in G; it is denoted by $G - G_1$ or $\bar{G_1}$.



Complement of a simple graph:

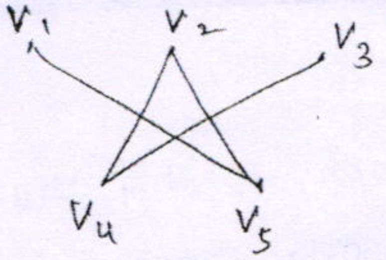
The complement $\bar{G}$ of a simple graph G with n vertices is that graph which is obtained by deleting those edges in K_n which belong to G. Thus, $\bar{G} = K_n - G$
$= K_n \Delta G$.



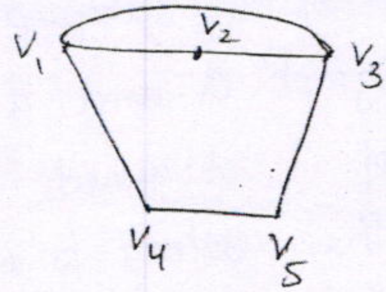
(a) K_4. b) G. (c) $\bar{G}$

① Show that the complement of bipartite graph need not be a bipartite graph.

⇒ Consider the bipartite graph of order 5. The complement of this graph is shown,



(a)



(b)

The graph (a) is a bipartite graph. The complement of this graph (b) is not a bipartite graph.

2. Let G be a simple graph of order n. If the size of G is 56 and the size of $\bar{G}$ is 80, what is n?

$\Rightarrow$

w.k.T, $\bar{G} = K_n - G$.

$\therefore$ Size of $\bar{G}$ = (size of K_n) − (size of G),

Since size of K_n (ie) the number of edges in K_n is $\frac{1}{2} n(n-1)$,

$$80 = \frac{1}{2} n(n-1) - 56$$

$$n(n-1) = 160 + 112 = 272$$

$$n(n-1) = 17 \times 16$$

$$\boxed{n = 17}$$

That is G is of order 17.

Walks and their classification

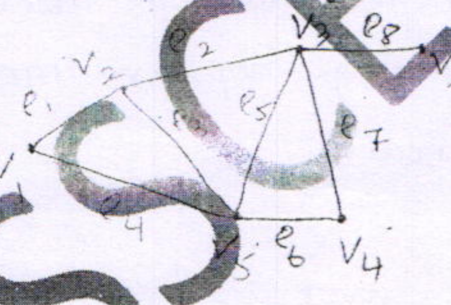
Walk

let G be a graph having at least one edge In G, consider a finite, alternating sequence of vertices and edges of the form,

$$v_i e_j, v_{i+1} e_{j+1} v_{i+2} \dots e_k v_m$$

which begins and ends with vertices and which is such that each edge in the sequence is incident on the vertices preceding and following it in the sequence. Such a sequence is called a walk in G, In a walk, a vertex or an edge (or both) can appear more than once.

The number of edges present in a walk is called its length.



In this graph

i) The sequence $v_1 e_1 v_2 e_2 v_3 e_8 v_6$ is a walk of length 3 (because, this walk contains 3 edges e_1, e_2, e_8).

(ii) The sequence $v_1 e_4 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4$ is a walk of length 5. In this walk, the vertex v_5 is repeated, but no edge is repeated.

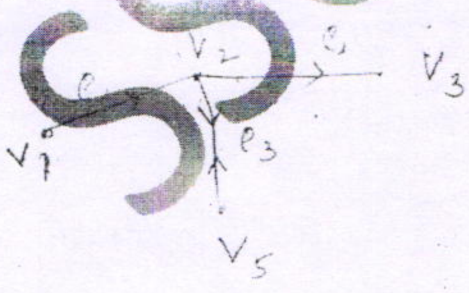
(ii) The sequence $v_1 e_1 v_2 e_3 v_5 e_3 v_2 e_2 v_3$ is a walk of length 4. In this walk, the edge e_3 is repeated and the vertex v_2 is repeated.

The vertex with which a walk begins is called the initial vertex (or the origin) of the walk and the vertex with which a walk ends is called the final vertex (or the terminus) of the walk.

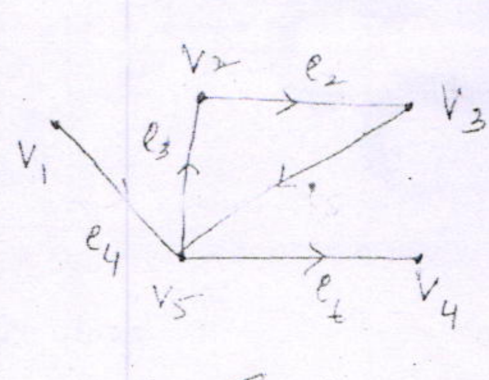
In the graph, $v_1 e_1 v_2 e_3 v_5 e_4 v_1$ is a closed walk and $v_1 e_1 v_2 e_2 v_3 e_5 v_5$ is an open walk.

<u>Trail and Circuit:</u>

In a walk, vertices and/or edges may appear more than once. If in an open walk no edge appears more than once, then the walk is called a trail. A closed walk in which no edge appears more than once is called a circuit.



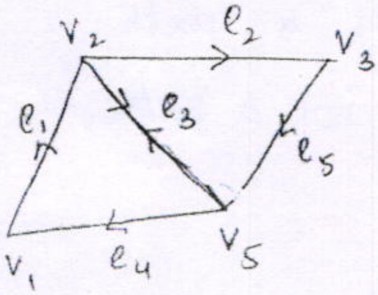
a) Not a trail



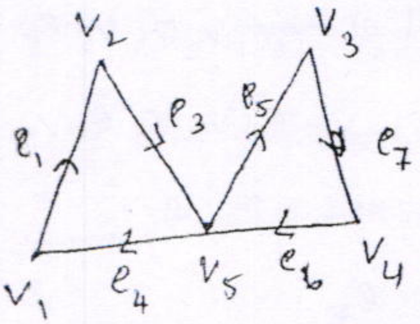
b) Trail

The graph (a) is an open walk $v_1 e_1 v_2 e_3 v_5 e_3 v_2 e_2 v_3$ but not a trail, because in this walk, the edge e_3 is repeated.

The grap (b) the walk $v_1 e_4 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4$ is an open walk which is trail.



a) Not a circuit

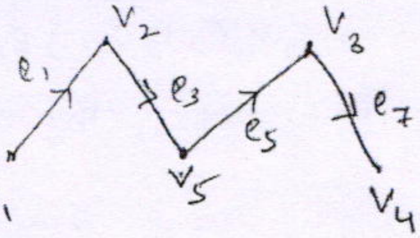


b) circuit

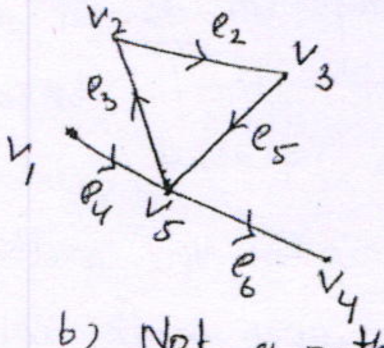
(a) The walk $v_1 e_1 v_2 e_3 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_4 v_1$ is a closed walk but not a circuit (because e_3 is rep.) where as (b) $v_1 e_1 v_2 e_3 v_5 e_5 v_3 e_7 v_4 e_6 v_5 e_4 v_1$ is a closed walk which is a circuit.

Path and cycle:

- A trail in which no vertex appears more than once is called a path.
- A circuit in which the terminal vertex does not appear as an internal vertex and no vertex is repreated is called a cycle.



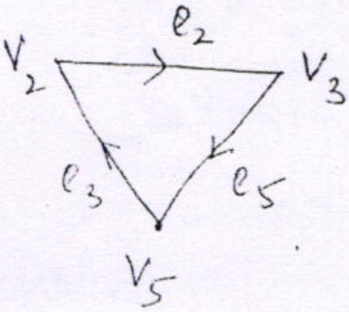
a) path



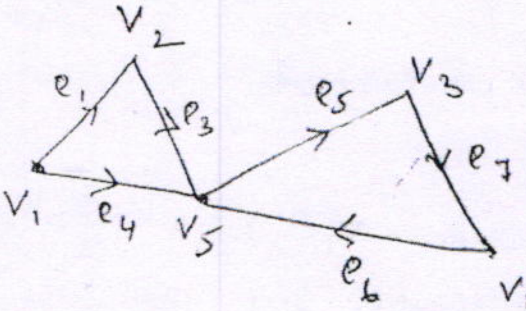
b) Not a path

The trail, a) $v_1 e_1 v_2 e_3 v_5 e_5 v_3 e_7 v_5$ is a path.

b) $v_1 e_4 v_5 e_3 v_2 v_3 e_5 v_5 e_6 v_4$ is not a path, v_5 appears twice

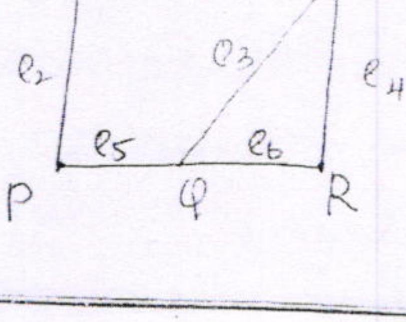


a) cycle



b) Not a cycle.

1. consider the graph shown below. Find all paths from vertex A to vertex R. Also indicate their lengths



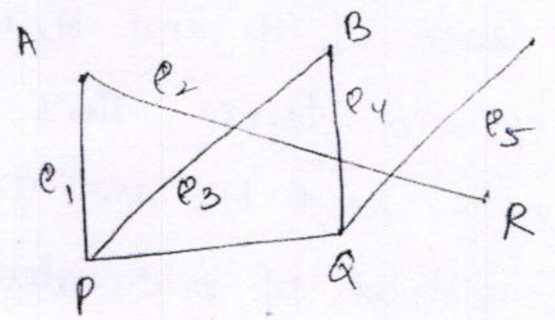
There are four paths from A to R. These are

Ae_1Be_4R, $Ae_1Be_3Qe_6R$, $Ae_2Pe_5Qe_6R$,

$Ae_2Pe_5Qe_3Be_4R$.

These paths contain, respectively, two, three, three and four edges. Their lengths are therefore two, three, three and four resp.

2. Find all the cycles present in the graph



⇒ In the given graph, there are no cycles beginining and ending with vertices B, P, Q are $Qe_4Be_3Pe_6Q$.

$Be_3Pe_6Qe_4B$, $Pe_6Qe_4Be_3P$,

Connected and Disconnected Graphs

The graph G is a connected graph if every pair of distinct vertices in G are connected. otherwise, G is called a disconnected graph.

(1) Theorem: If a graph has exactly two vertices of odd degree, then there must be a path connecting these vertices.

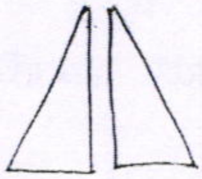
Proof: Denote the two vertices of odd degree by v_1 and v_2. Suppose there is no path connecting these. Then the graph is disconnected, and v_1 and v_2 belong to two different components, say H_1 and H_2. consequently each of H_1 and H_2 contains only one vertex of odd degree. This is not possible, because H_1 and H_2 are graphs and the number of vertices of odd degree is always even. Hence there must be a path connecting v_1 and v_2

(2) Theorem: A connected graph with n vertices has at least n-1 edges.

(3) A graph G is disconnected if and only if its vertex set V can be partitioned in two non-empty disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and other is in V_2

4) Let G be a graph with n vertices where n is even and > 2. If the degree of every vertex in G is $\frac{1}{2}(n-2)$, disprove that G is connected.

=) consider the disconnected graph.



In this graph the number of vertices is n=6, which is even and greater than 2, and the degree of every vertex is

$$2 = (6-2)/2 = (n-2)/2$$

Thus, the graph consider meets the given conditions and is disconnected. This counter example disproves that G is connected.

Euler circuits and Euler trails.

Consider a connected graph G. If there is a circuit in G that contains all the edges of G, then that circuit is called an Euler circuit in G.

If there is a trail in G that contains all the edges of G, then that trail is called an Euler trail.

Theorem: 1 A connected graph G has an Euler circuit if and only if all vertices of G are of even dgree.

Theorem: 2 A connected graph G has an Euler circuit if and only if G can be decomposed into edge - disjoint cycles.

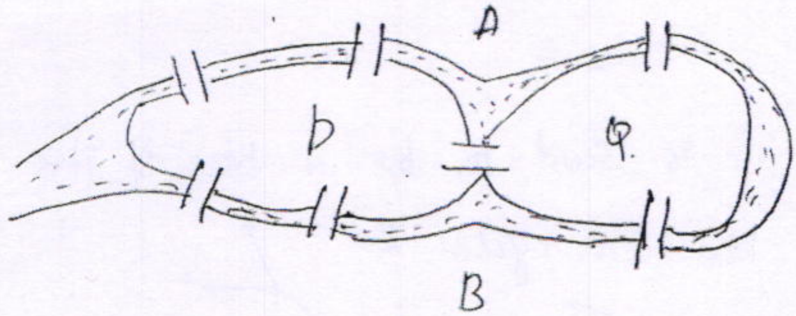
The Konigsberg Bridge Problem:

This problem remained unsolved for several years. In the year 1736, Euler analyzed the problem with the help of a graph and gave the solution. This was indeed the starting point for the development of graph theory.

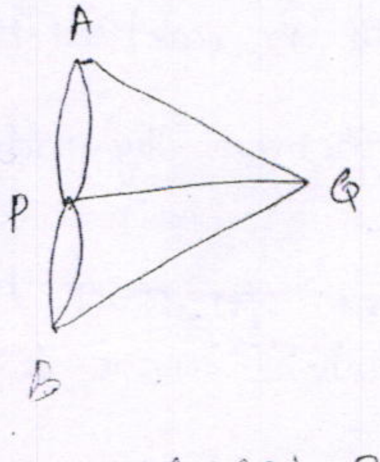
Let us see what the solution is.

Denote the land areas of the city by

A, B, P, Q. where A, B are the banks of



the river and P, Q are islands construct a graph by treating the four land areas as four vertices and the seven bridges connecting them as seven edges. The graph is as,

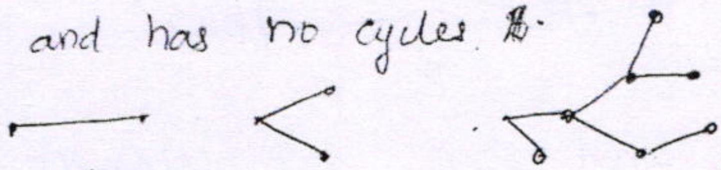


$deg(A) = deg(B) = deg(Q) = 3$, , $deg(P) = 5$

which are not even, Therefore the graph does not have an Euler circuit. This means that there does not exist a closed walk the contains all the edges exactly once. This amounts to saying that it is not possible to walk over each of of the seven bridges exactly once and return to the starting point.

Trees:

A graph G is said to be a tree if it is connected and has no cycles.



(a) (b) (c)

Each of these possesses at least two pendant vertices. A pendant vertex of a tree is also called a leaf.

(1) Theorem. A tree with n vertices has $n-1$ edges.

Proof: We prove the theorem by induction on n. The theorem is obvious for $n=1, n=2$ and $n=3$ (fig a, b, c). Assume that the theorem ~~for $n=k$~~ holds for all trees with n vertices where $n \le k$, for a specified positive integer k.

Consider a tree T with $k+1$ vertices. In T, let e be an edge with end vertices u and v. Since T is a tree, it has no cycles and therefore there exists no other edge or path between u and v. Hence deletion of e from T will disconnect graph and $T-e$ consists of exactly two components say T_1 and T_2. Since T does not contain any cycle the components T_1 and T_2 too do not contain any cycles. Hence T_1 and T_2 are trees in their own right. Both of these trees have less than $k+1$ vertices each, and therefore, according to the assumption made, the theorem holds for these trees; that is, each of T_1 and T_2 contains one less edge than the number of vertices in it. Therefore, since the total number of vertices in T_1 and T_2 (taken together) is $k+1$, the total number of edges in T_1 and T_2 (taken together) is $(k+1)-2 = k-1$. But T_1 and T_2 taken together is $T-e$. Thus, $T-e$ contains $k-1$ edges consequently T has exactly k edges.

Thus, if the therorem is true for a tree with $n \le k$ vertices, it is true with $n=k+1$ vertices. Hence, by induction, the theorem is true for all positive integers n.

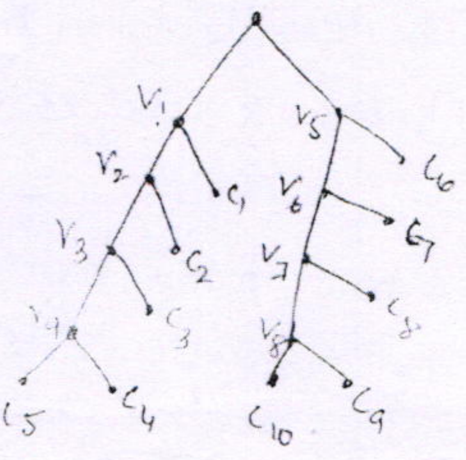
(2) Let $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ be two trees. If $|E_1| = 19$ and $|V_2| = 3|V_1|$. determine, $|V_1|, |V_2|$ and $|E_2|$.

$\Rightarrow$ It is given that $|E_1| = 19$. Therefore,

$$|V_1| = |E_1| + 1 = 19 + 1 = 20$$

The wall socket may be regarded as root of a complete binary tree having the computers as its leaves and the internal vertices, other than the root, as extension cords.

Then the number of leaves in the tree is $p = 10$. Therefore, the number of internal vertices in the tree is $q = p - 1 = 10 - 1 = 9$. Accordingly the number of extension cords needed (namely the number of internal vertices minus the root) is $q - 1 = 8$.



<u>Sorting</u> (rearrange/reorganize)

Using the merge-sort method, sort the list

7, 3, 8, 4, 5, 10, 6, 2, 9.

=> First, we recursively split the given list and all subsequent lists in half or as close as possible to half until each sublist contains a single element.

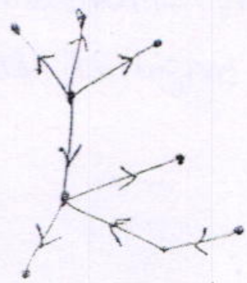
Since $|V_2| = 3|V_1|$ as given, we get,

$$|V_2| = 3|V_1| = 3 \times 20 = 60$$

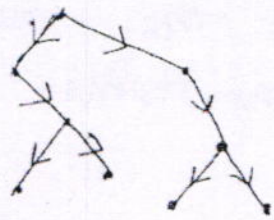
$$|E_2| = |V_2| - 1 = 60 - 1 = 59$$

Rooted Trees:

A directed tree T is called a rooted tree if (i) T contains a unique vertex, called the root, whose in-degree is equal to 0 and (ii) the in-degrees of all other vertices of T are equal to 1.



Not rooted Tree

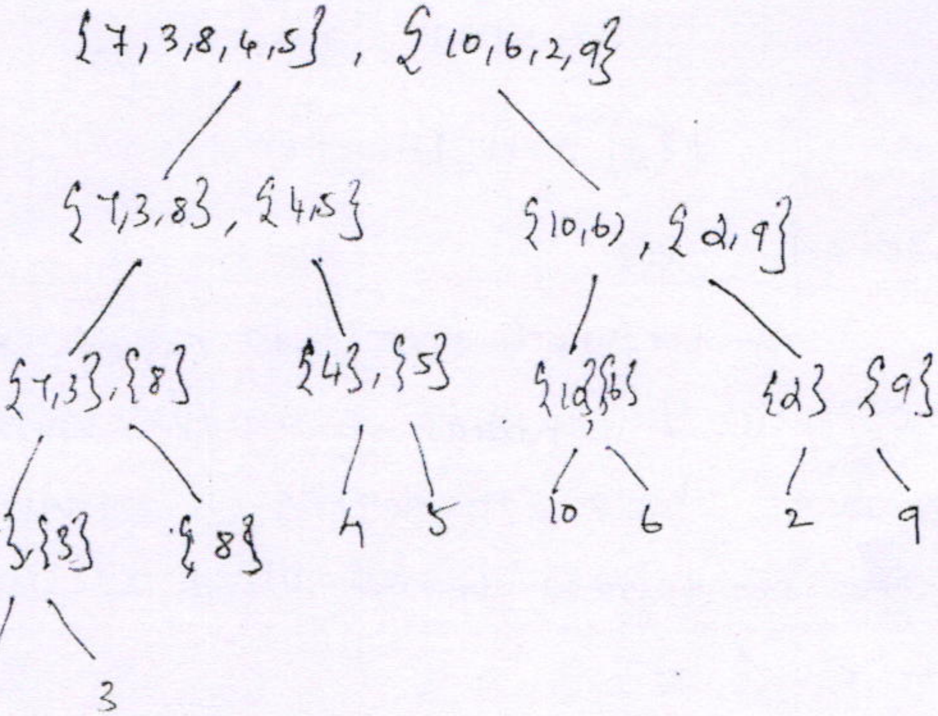


Rooted Tree

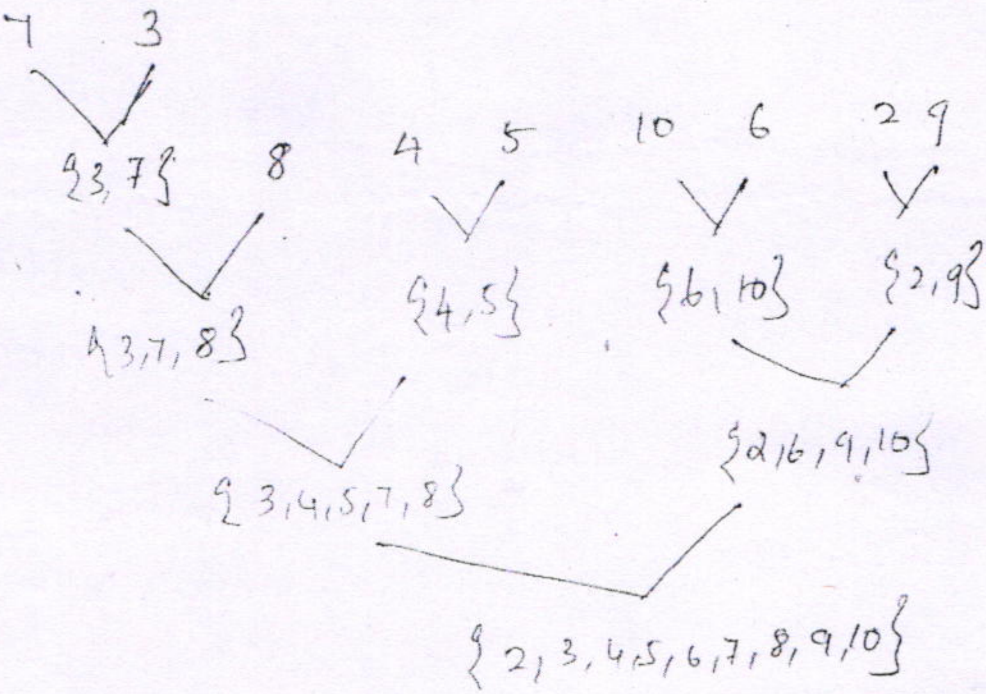
(1) The computer laboratory of a school has 10 computers that are to be connected to a wall socket that has 2 outlets. Connections are made by using extension cords that have 2 outlets each. Find the least number of cords needed to get these computers set up for use

=)

The splitting process is represented in the tree



Now merge the sublists in nondecreasing order until the items in the original list have been sorted.



Prefix codes and weighted trees.

A sequence consisting of only 0 and 1 is called a binary sequence or a binary string.

01, 001, 101, 11001, 1000100

are all binary sequence with length 2, 3, 3, 5, 7 resp.

(1) Consider the prefix code:

a) 111, b: 0 c: 1100, d: 1101, e: 10

Using this code, decode the following sequence.

i) 1001111101, (ii) 1011110011000110 1 (iii) 110111111 00 10

=> (i) splitting: 10|0|111|1101 — ebad

(ii) 10|111|10|0|1100|0|1101 — eaebcbd

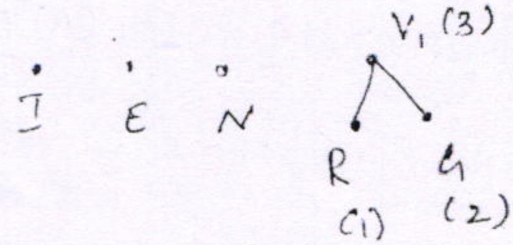
iii) 1101|111|1100|10 — d a c e.

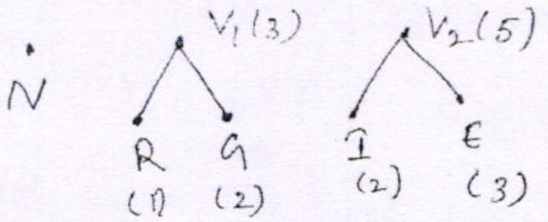
2. Construct an optimal prefix code for the letters of the word ENGINEERING. Hence deduce the code for this word.

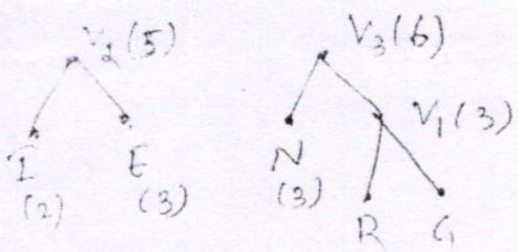
=> The given word consists of the letters E, N, G, I, R with frequencies 3, 3, 2, 2, 1 respectively. First, we arrange these letters in the non-decreasing order of their frequencies (which may be regarded as weights). Their representation as isolated vertices is shown below

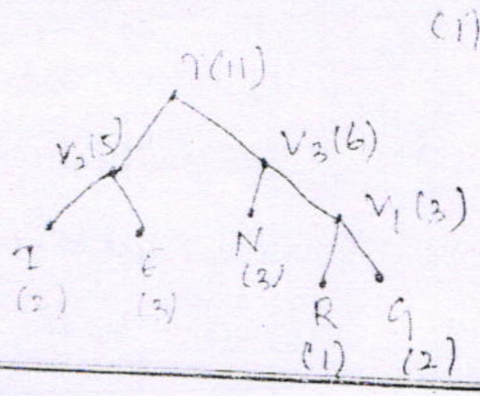
R	G	I	E	N
(1)	(2)	(2)	(3)	(3)

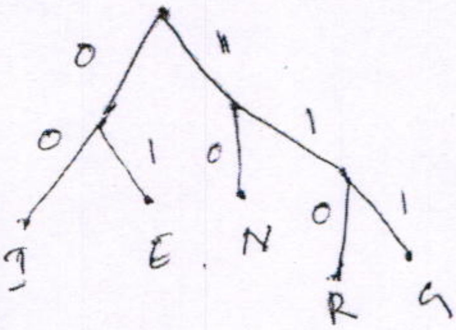
we now construct an optimal tree having these letters as leaves by using Huffman's procedure. the graph obtained in successive steps of the procedure are shown below in figures, in the order of their occurrence. the labeled version of the final tree is shown below,











from this tree, we obtain the optimal prefix codes shown below for the letters with which we started

R : 110 G ; 111 I : 00 E ; 01 N : 10

Accordingly the code for the given word ENGINEERING is

0110111001001011101011