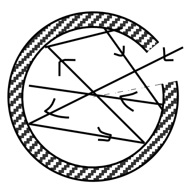
Modern Physics

Syllabus*Black body radiation spectrum, Assumptions of quantum theory of radiation, Plank's law, Weins law and Rayleigh Jeans law, for shorter and longer wavelength limits.Wave Particle dualism,deBroglie hypothesis. Compton Effect and its Physical significance. Matter waves and theirCharacteristic properties, Phase velocity and group velocity. Relation between phase velocity andgroup velocity, Relation between group velocity and particle velocity*

Black body radiation spectrum:

The radiation emitted by a body as a result of its temperature is called thermal radiation. When radiation incident on material object either it is absorbed, reflected or transmitted. These processes are dependent on the radiation and the object involved. A body that is capable of absorbing all radiation incident on it is called a perfectly black body. It also emits all radiations when maintained at a constant temperature. Practically we can't have a perfectly black body but can have objects that are only close to a black body.



In the laboratory a black body can be approximated by a hollow object with a very small hole leading to its interior. Any radiation striking the hole enters the cavity where it is completely trapped by reflections back and forth until it is absorbed. Object that absorb a particular wavelength of radiation are also found to be a good emitter of radiation of that particular wavelength. Hence, a black body is also a good emitter of all radiation it has absorbed. Radiation emitted by the black body is called blackbody radiation. Black body radiation depends on the temperature of the body.

Distribution of energy in the spectrum of a black body radiation:

Wien studied the spectrum of radiation by a black body by heating it to a higher and higher temperature by maintaining the entire black body at the same temperature (i.e. at thermal equilibrium). The distribution of radiant energy density among the different wavelength varies at constant temperature as shown in fig. It is called black body spectrum. The important features of these distribution curves may be summarized as follows;

- The energy is not uniformly distributed in the radiation spectrum of black body at a given temperature.

- At a given temperature, the energy density initially increases with increase in wavelength and becomes maximum at λ_m and with further increase in wavelength the intensity of radiation decreases.

- An increase in temperature results in an increase in the total energy emitted and also the energy emitted at all wavelengths.

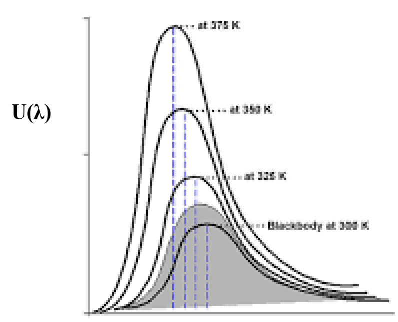


figure:black body spectrum

- As the temperature of the black body increases, λ_m shifts towards shorter wavelength region. Here, λ_m is the wavelength at which maximum emission of energy takes place.

i.e. $\lambda_m \alpha 1/T$

$\lambda_m T$ = constant=2.898 X10^{-3}mK

This result is known as **Wien's displacement law**.

The term displacement refers to the way the peak is moved or displaced as the temperature is varied.

- The area under the curve is a measure of the total energy of radiation at that temperature and is proportional to the fourth power of the absolute temperature of black body.

Laws of radiation:

Kirchoffs Law

The law states that the ratio of emissive power to the absorptive power is same for all surfaces at the same temperature and is equal to emissive power of a perfectly black body at that temperature.

Stefan's law

Stefan's law states that "the total energy radiated per unit time per unit surface area of a black body is proportional to the fourth power of the absolute temperature of the body expressed in Kelvin" i.e. $E = \sigma T^4$ where σ is constant of proportionality known as Stefan's constant, having value $5.67 \times 10^{-8} Wm^{-2} K^{-4}$

Wien's distribution law:

Wien deduced the relation between the wavelength of emission and temperature of the black body based on classical physics as,

$$U(\lambda)\, d\lambda = \frac{C_1}{\lambda^5} \frac{1}{e^{C_2/\lambda T}} d\lambda$$

Where C_1 and C_2 are constants and $U_\lambda d\lambda$ is the energy density of black body for wavelength in the range λ and $\lambda+d\lambda$.

Limitation:Wien's law holds good only for shorter wavelengths region (i.e. $\lambda < \lambda_m$) and high temperature value of the source. It failed to explain gradual drop in the intensity for radiations whose wavelength are longer than the one corresponding to peak values.

Rayleigh -Jean's law:

In a different approach, Rayleigh and Jean treated the radiations inside a cavity as standing electromagnetic waves and obtained the following formula for energy density.

The number of modes of vibration/unit volume whose wavelength in the range λ and $\lambda+d\lambda$.is given by

$$8\pi\lambda^{-4} d\lambda$$

And the numbers of standing waves, formed in the cavity of block body possess average thermal energy by law of equipartition energy per degree of freedom is given by

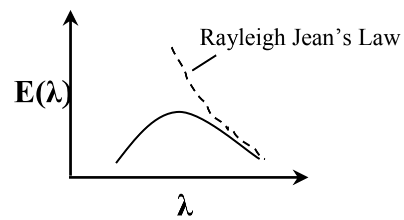
$$\bar{E} = kT$$

Then, The Energy emitted per unit volume per unit second of a black body in unit frequency is given by

$U_\lambda d\lambda =$ (Number of standing waves formed in the cavity) x

(Average thermal energy)

$$\boxed{U(\lambda)\, d\lambda = \frac{8\pi kT}{\lambda^4}\, d\lambda}$$



According to this law the energy radiated by the black body should rapidly decrease with increase in wavelength. Thus, Rayleigh -Jean's law correctly predicts the fall of intensity of the radiation towards the longer wavelength side (i.e. $\lambda > \lambda_m$)

For very short wavelength i.e, $\lambda \rightarrow 0$, $U_\lambda d\lambda \rightarrow \infty$

i.e., according to the Rayleigh and Jean's law, radiant energy increases enormously with decreasing wavelength and approaches infinity for short wavelengths.

<u>Limitation of Rayleigh -Jean's law (Ultraviolet Catastrophe)</u>

According to Rayleigh -Jean's law, the radiant energy increases enormously with the decreasing wavelength i.e. the black body must radiate practically all the energy at very short wave length side. This is not at all in agreement with the experimental observation. A black body radiates mainly not in the very short wavelength range but in the infra-red or visible region of the em spectrum. Hence, Rayleigh -Jean's law fail to explain the lower wavelength side of the black body radiation spectrum. Thus the failure of Rayleigh Jean's law to explain the aspect of very little emission of radiation beyond the violet region towards the lower wavelength side of the spectrum is referred to as **"Ultraviolet Catastrophe"**.

Wien's formula agreed with the experimental curves for shorter wavelengths while Rayleigh formula agrees for longer wavelengths. Thus both the laws of classical physics failed to explain the entire spectrum of the black body radiation.

Planck's radiation Law and assumptions of quantum theory

Max Planck derived an equation which successfully explained the entire spectrum of blackbody radiation. This law is based on quantum theory. He proposed that atoms or molecules absorb or emit radiation in quanta or small energy packets called photons. Energy of each photon can be expressed as,

$E = h\nu$, where, ν is the frequency of the radiation corresponding to the energy E, h is the Planck's constant. ($h=6.63 \times 10^{-34}$Js)

The assumption in the derivation of Planck's law is that the wall of the experimental blackbody consists of very large number of electrical oscillators (i.e. atoms and molecules), with each oscillator vibrating with frequency of its own. Planck postulated the following assumptions in his theory:

1. The atoms in the walls of black body behaves like simple harmonic oscillators.
2. The oscillators can vibrate with all possible frequencies.
3. The frequency of radiation emitted by an oscillator is same as that of frequency of its vibration.
4. An oscillator does not radiate or absorb energy continuously as suggested in classical physics.
5. An oscillator emits or absorbs the energy in discrete packets of energy i.e. '**quanta**' or photon of size '**hυ**'i.e., E**=**E = nhν ,where n=1,2,3,…..

   In other words it states that exchange of energy between the radiation and matter can not takes place continuously. The energies of the atoms are said to be **quantized** and the allowed energy levels are called as **quantum levels**.

The number of oscillators per unit volume in a cavity having the wavelength range λ and λ+dλ is given by $8\pi\lambda^{-4}d\lambda$

Average energy of an oscillator is given by

$$\bar{E} = \frac{h\upsilon}{e^{\frac{h\upsilon}{kT}} - 1}$$

Then, the total Energy emitted per unit volume of a black body having the wavelength range and λ+dλ is given by

$U_\lambda d\lambda =$ (Number of oscillators per unit volume) x (Average thermal energy)

$$U_\lambda d\lambda = \frac{8\pi d\lambda}{\lambda^4}\frac{h\upsilon}{e^{\frac{h\upsilon}{kT}} - 1}$$

The empirical relation given by the Planck to explain the energy distribution of black body radiation spectrum is

$$\boxed{U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{\left[e^{hc/\lambda kT} - 1\right]}d\lambda} \qquad (1)$$

This relation is known as **Planck's radiation law**. This law does not suffer from an ultraviolet catastrophe and agrees well with the experimental data.

Reductions of Planck's radiation law to Wien's law and Rayleigh Jean's law:

Case i: For shorterwavelength regions $e^{\frac{hc}{\lambda kT}}$ is very large

i.e $e^{\frac{hc}{\lambda kT}} >> 1$

Therefore in eqn. (1) denominator is $\left(e^{\frac{hc}{\lambda kT}} - 1\right) \approx e^{\frac{hc}{\lambda kT}}$

Substituting the above quantity in Planck's radiation law

$$U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{\left[e^{hc/\lambda kT} - 1\right]}d\lambda$$

$$U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{\left[e^{hc/\lambda kT}\right]}d\lambda$$

$$U_\lambda d\lambda = \frac{C_1}{\lambda^5}\frac{1}{\left[e^{C_2/\lambda T}\right]}d\lambda$$ where $C_1 = 8\pi hc$ and $C_2 = hc/k$

This is the Wien's distribution law.

Case ii: for longer wavelength

The quantity $\frac{hc}{\lambda kT}$ is small. Hence $e^{\frac{hc}{\lambda kT}}$ is also small.

Expanding $e^{\frac{hc}{\lambda kT}}$ by exponential series

$$e^{\frac{hc}{\lambda kT}} = 1 + \frac{hc}{\lambda kT} + \frac{(hc/\lambda kT)^2}{2!} + \frac{(hc/\lambda kT)^3}{3!} + \dots\dots\dots\dots\dots\dots\dots$$

Since $\frac{hc}{\lambda kT}$ is very small neglecting the higher order terms.

$$e^{\frac{hc}{\lambda kT}} = 1 + \frac{hc}{\lambda kT} \text{ or } e^{\frac{hc}{\lambda kT}} - 1 = \frac{hc}{\lambda kT}$$

Substituting above quantity in the Planck's radiation law, we get

$$U(\lambda)\,d\lambda = \frac{8\pi kT}{\lambda^4}\,d\lambda$$

Thus Wien's law and Rayleigh-Jeans law come out as special cases showing the general form of Planck's law of radiation.

Compton Effect:

When a beam of monochromatic X-rays is scattered by the electrons present in the suitable target, the scattered radiations consists of two components, one of the same wavelength (coherent) and another of longer wavelength (incoherent) than that of incident wavelength. The change in wavelength of scattered beam is independent of the target material but depends on scattering angle. The phenomenon in which wavelength of X-rays scattered by an element, is greater than that of incident rays is called Compton effect and the change in wavelength is called Compton shift. The scattering of photon by an electron is called Compton scattering.

Explanation: Compton succeeded in accounting for the change in wavelength of the scattering of X-rays by making use of quantum theory of radiations. The incident beam of X-rays is assumed to be consisting of a stream of photons. He treated incidence of photon is equal to incidence of a particle of energy E given by $E=hc/\lambda$ where h is Planck's constant, c is velocity of light and λ is wavelength of the incident X-rays.

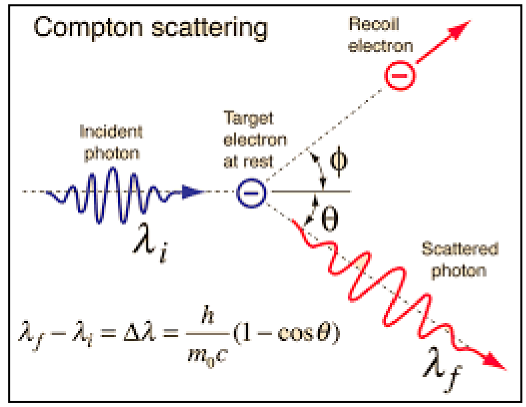
The scattering process is analyzed as a collision between two particles, the incident photon of energy $E=hc/\lambda$ and the electron of the target of rest energy m_0c^2. As a result the photon is scattered at an angle of θ to the incident direction and its energy reduces from $E=hc/\lambda$ to $E'=hc/\lambda'$. In order to account for the change in energy of the photon, the electron is assumed to recoil at an angle ϕ with the incident direction of photon.

Applying law of conservation of energy and momentum, $E_{initial}=E_{final}$

i.e. $$\frac{hc}{\lambda}+m_0c^2=\frac{hc}{\lambda'}+mc^2 \qquad \text{where } m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Here m_0 is rest mass of electron and m is relativistic mass of electron.

$$\frac{h}{\lambda}-\frac{h}{\lambda'}+m_0c=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$



Let us take β=v/c, then we can write above equation as

$$\left(\frac{h}{\lambda}-\frac{h}{\lambda'}\right)+m_0c=\frac{m_0}{\sqrt{1-\beta^2}}$$

Squaring on both sides

$$\left(\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}-\frac{2h^2}{\lambda\lambda'}\right)+m_0^2c^2+2m_0ch\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right)=\frac{m_0^2c^2}{1-\beta^2} \qquad (1)$$

Now using law of conservation of momentum along X-direction

$$(P_x)_{initial}=(P_x)_{final}$$

$$\frac{h}{\lambda}+0=mv\ \cos\phi+\frac{h}{\lambda'}\cos\theta$$

$$\frac{h}{\lambda}-\frac{h}{\lambda'}\cos\theta=\frac{m_0v\ \cos\phi}{\sqrt{1-\beta^2}} \qquad (2)$$

Similarly using momentum conservation along Y-direction,

$$(P_y)_{initial}=(P_y)_{final}$$

$$0=mv\ \sin\phi-\frac{h}{\lambda'}\sin\theta$$

$$\frac{h}{\lambda'}\sin\theta=\frac{m_0v\ \sin\phi}{\sqrt{1-\beta^2}} \qquad (3)$$

Squaring and adding equations (2) and (3), we get

$$\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}\cos^2\theta-\frac{2h^2}{\lambda\lambda'}\cos\theta+\frac{h^2}{\lambda'^2}\sin^2\theta=\frac{m_0^2v^2}{1-\beta^2}$$

$$\frac{h^2}{\lambda^2}+\frac{h^2}{\lambda'^2}-\frac{2h^2}{\lambda\lambda'}\cos\theta=\frac{m_0^2\beta^2c^2}{1-\beta^2} \qquad \left(\Theta\ \beta=\frac{v}{c}\right) \qquad (4)$$

Subtracting equation (4) from(2)

$$-\frac{2h^2}{\lambda\lambda'}+m_0^2c^2+2m_0ch\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right)+\frac{2h^2}{\lambda\lambda'}\cos\theta=\frac{m_0^2c^2}{1-\beta^2}-\frac{m_0^2\beta^2c^2}{1-\beta^2}$$

$$-\frac{2h^2}{\lambda\lambda'}+2m_0ch\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right)+\frac{2h^2}{\lambda\lambda'}\cos\theta=0$$

$$\frac{m_0c}{h}\left(\lambda'-\lambda\right)=1-\cos\theta$$

$$\left(\lambda' - \lambda\right) = \frac{h}{m_0 c}\left(1 - \cos\theta\right)$$

This change in wavelength of scattered X-rays is called Compton shift ($\Delta\lambda = \lambda' - \lambda$). Above equation is known as **Compton equation**. The Compton equation shows that the Compton shift depends only on the scattering angle 'θ' and is independent of the wavelength of the incident X-rays and the nature of the target material. This is experimentally verified.

When $\theta = 90^o$, $\Delta\lambda = \frac{h}{m_0 c} = 2.42X10^{-12}$ m this constant is called Compton wavelength.
when $\theta = 0^o$, $\Delta\lambda$ is maximum and will be twice the Compton wavelength.

Physical significance of Compton effect:

In the Compton Effect, the elastic collisions between photon and electron takes place. Collision and momentum are characteristics of particle behavior. Thus Compton effect confirms the particle behavior of radiation.

Wave particle dualism - de-Broglie's hypothesis:

Matter and electromagnetic radiation are both forms of energy and are inconvertible, as established by Einstein's mass energy equivalence relationship, $E=mc^2$. Electromagnetic radiations commonly exhibits wavelike properties (interference, diffraction and polarization) and in certain cases (black body radiation, photoelectric effect and Compton scattering)behave like a stream of photons.

Louis de-Broglie extended the wave particle duality of light to all the fundamental entities of physics such as electrons, protons neutrons etc. de-Broglie put a bold suggestion that the wave particle dualism need not be special feature of light alone but material particles must also exhibit dual behavior. His suggestion was based on the fact that *"**nature loves symmetry**"*, if radiation can behave as particle under certain circumstances then, one can even expect that entities which ordinarily behaves as particles to exhibit wave properties under appropriate circumstances.

According to de-Broglie's hypothesis, every moving material particle is associated with a wave whose wavelength is given by $\lambda = \frac{h}{mv} = \frac{h}{p}$, where p is the momentum of the particle. The wave associated with moving material particle is called **matter waves** or de-Broglie waves. λ is called de-Broglie wavelength.

The expression of the wavelength associated with a material particle can be deduced for photons and in analogy, assumed for matter waves as follows:
Considering the Planck's theory of radiation, the energy of a photon (quanta) is given by

$$E = \frac{hc}{\lambda}$$

And according to Einstein's mass energy equivalence relation, we have

$$E = mc^2 = pc$$

Equating above two equations, we get

$$\frac{hc}{\lambda} = pc \quad or \quad \lambda = \frac{h}{p}$$

Where c is the velocity of light and $h= 6.62x10^{-34}$Js is Planck's constant.

Similarly, for matter waves, the de Broglie wavelength, $\lambda = \frac{h}{p} = \frac{h}{mv}$
where m is mass and v is velocity of particle under consideration.

Note:1) For ordinary objects, the wave like behavior cannot be observed as Planck's constant is very small($\approx 10^{-34}$Js). In subatomic scale (microscopic), the momentum can be sufficiently small to bring the de-Broglie wavelength into observable range.
Note:2)
If E_k is the kinetic energy of the particle, then

$$E_k = \frac{1}{2} mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{1}{2}\frac{p^2}{m}$$

or p= $\sqrt{2mE_K}$

$\therefore$ De-Broglie wavelength $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_K}}$

Note:3) Also if charge 'q' is accelerated through a potential difference of 'V' volts, then K.E.=E_K=qV
Hence de-Broglie wavelength of an accelerated electron λ = $\lambda = \frac{h}{\sqrt{2mqV}} = \frac{1.226}{\sqrt{V}}$ nm

Distinction between the expression for energy of a photon and a particle

Photon is mass less particle and for photon E=mc^2=pc (mc=p as the velocity is equal to c)
E=pc=hc/λ for photon
For a matter particle, the velocity is always less than the velocity of light and as the velocity approaches the velocity of light, mass of the particle tends to ∞ as seen from the relation,

Relativistic mass, $$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (1)$$

where m_0 is the rest mass and m is mass corresponding to velocity v.
Squaring the equation (1) and rearranging the terms, we obtain

$$m^2\left(1-\frac{v^2}{c^2}\right) = m_0^2c^2 \qquad (2)$$

Multiplying on both sides with c^2 we get,

$$m^2c^4 - m^2c^2v^2 = m_0^2c^4$$

$$E^2 = E_0^2 + p^2c^2$$

The energy of a relativistic matter particle,

$$E = \sqrt{E_0^2 + p^2c^2}$$

For a particle whose velocity, v≪ c, (non-relativistic particle), m=m_0
and $E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

Properties of matter waves

Following are properties of matter waves:

1. Lighter is the particle, longer will be the wavelength of matter waves associated with it, velocity being constant.
2. Smaller the velocity of the particle, longer will be the wavelength of the matter waves associated with it.
3. When the particle is at rest, the wavelength associated with it becomes infinite.

i.e, when v=0 then $\lambda=\infty$ i.e, wave becomes indeterminate. This shows that only the moving particle produces the matter waves.

4. Matter waves are produced, whether the particles are charged or uncharged. But e. m. waves are produced by the motion of charged particle. This fact reveals that, matter waves are not electromagnetic waves.
5. The velocity of matter waves depends on velocity of material particle and it is not constant, while velocity of e. m. waves is constant.
6. Matter waves can travel even in vacuum, hence they are not mechanical waves.
7. Matter waves are represented by a wave packet made up of a group of waves of slightly different wavelengths. Hence, we talk of group velocity of matter waves rather than the phase velocity. The group velocity can be shown to be equal to the particle velocity.
8. Matter waves show properties similar to other waves such as interference and diffraction.

Phase velocity and group velocity of matter waves:

Definition: Phase velocity is the velocity with which a phase point located on a progressive wave is transported. Phase velocity is same as wave velocity. $v_p=\nu\lambda$ (ν is the frequency)

Let P be a phase point located on a travelling wave represented by the equation

y = A sin (kx-ωt)

where y is the displacement at any instant t, A is the amplitude of vibration, w is the angular frequency (= 2πν) and k is the wave vector/angular wave number (= 2π/λ). The phase velocity of such a wave is the velocity with which a particular phase point of the wave travels. This corresponds to the phase being constant. The velocity of the phase point will be the same as wave velocity and is given by

$$v_p = \nu\lambda = \frac{\omega}{2\pi}.\frac{2\pi}{k} = \frac{\omega}{k} \qquad \textbf{(i)}$$

$$\textbf{Also } \nu = \frac{E}{h} \textbf{ and } E = mc^2 \qquad \textbf{(ii)}$$

$$\lambda = \frac{h}{p}. \textbf{ and } p = m\mathbf{v} \qquad \textbf{(iii)}$$

Substituting (ii) and (iii) in (i), we obtain $v_p = \frac{E}{p} = \frac{c^2}{v}$

Here ,v is the velocity of the particle associated with the wave.

Note: (i) For photons (light quanta), v=c in vacuum and hence phase velocity is equal to velocity of light.

(ii) Material particles like electrons travel with velocities less than c and so phase velocity of matter waves becomes greater than speed of light ($v_p>c$) which has no physical significance. Hence for a physical representation of matter waves, we require to consider a wave packet to be associated with the moving particle.

Group velocity:

According to Schrodinger a moving is not equivalent to single wave, but it can be represented by **wave packet**. A wave packet is formed due to superposition of two or more individual waves each with slightly different wavelengths moving in same direction whose interference with one another results in variation in amplitude, such variation represents the wave group as a whole, and is called the wave packet.

Group velocity is defined as the rate at which the amplitude is modulated in resultant pattern or the rate at which energy is transported by group of waves or the velocity of motion of wave packet.

$$v_{group} = \frac{d\omega}{dk} \qquad (2)$$

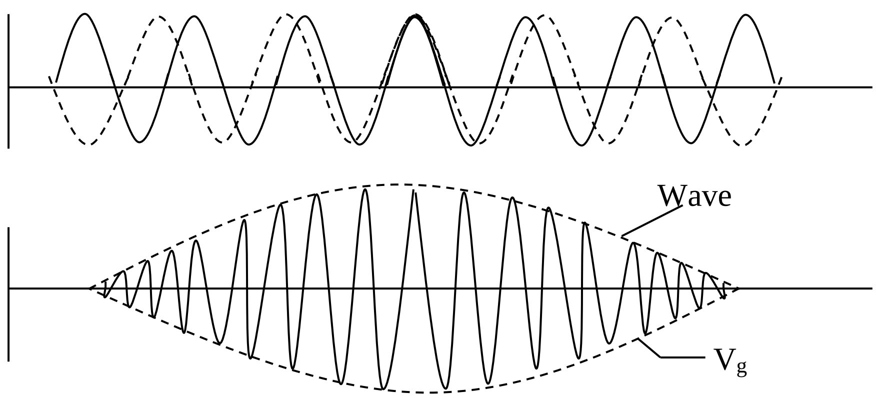


Figure : The super position of phase waves of nearly equal wavelengths to give the wave packet.

<u>Expression for Group velocity:</u>

Consider two travelling waves of same amplitude 'A', but of slightly different wave numbers and angular frequencies. The two waves can be represented by the following two equations,

$$y_1 = A \sin (kx-\omega t) \qquad (1)$$

$$y_2 = A \sin [(k+\Delta k)x- (\omega+\Delta\omega)t] \qquad (2)$$

Where, y_1 and y_2 are the displacements in the directions normal to the direction of propagation, at any instant t,
The resultant displacement y due to the superposition of two waves is given by,

$$y = y_1 + y_2$$

Therefore $\quad y = A \sin (kx-\omega t) + A \sin [(k+\Delta k)x- (\omega+\Delta\omega)t] \qquad (3)$
Using the relation,

Sin A + Sin B = 2 cos $\left(\frac{A-B}{2}\right)$ sin $\left(\frac{A+B}{2}\right)$

$$y = 2A\cos\left[\left(\frac{\Delta k}{2}\right)x - \left(\frac{\Delta\omega}{2}\right)t\right]\sin\left[\left(\frac{2k+\Delta k}{2}\right)x - \left(\frac{2\omega+\Delta\omega}{2}\right)t\right] \qquad (4)$$

since Δk and Δω are very small, we can take 2k+Δk≈2k and 2ω+Δω≈2ω

$$y = 2A\cos\left[\left(\frac{\Delta k}{2}\right)x - \left(\frac{\Delta\omega}{2}\right)t\right]\sin(kx-\omega t) \qquad (5)$$

Comparing equations (1) and (5), it is possible to treat the coefficient of sin $\{wt - kx\}$ in both the equations to be the amplitude of the representative waves. In this sense, in eq. (1), the amplitude will be A, which is constant. But in eq. (5) the amplitude becomes,

2 A cos $[\left(\frac{\Delta\omega}{2}\right) t - \left(\frac{\Delta k}{2}\right) x]$

This is not a constant, but varies as a wave. Thus above equation (5) represents a sine wave of angular frequency ω and wave number k whose amplitude is modulated with the angular frequency Δω/2 and wave number Δk/2.

The velocity with which the variation in amplitude is transmitted in the resultant wave is the group velocity, and from equation (5) rate of modulation of amplitude is

$$v_{group} = \frac{\frac{\Delta\omega}{2}}{\frac{\Delta k}{2}} = \frac{\Delta\omega}{\Delta k}$$

In the limit, $\left(\frac{\Delta\omega}{\Delta k}\right) \rightarrow \left(\frac{d\omega}{dk}\right)$

Therefore, $\boxed{v_{group} = \frac{d\omega}{dk}}$

The wave packet consists of regions of constructive and destructive interference. The probability of finding the particle in a given region depends on the amplitude of the wave group in that region. The wave group representation enables the localization of the particle. **It is the motion of wave group**, not the motion of individual waves that makes up the wave group, **corresponds to the motion of the particle.**

Relation between Group velocity (V_{group}) and Particle velocity ($V_{Particle}$)

We have the equation for group velocity as,

$$v_{group} = \frac{d\omega}{dk} \qquad (1)$$

But, w= 2πν = 2π $\frac{E}{h}$ (∵ E = hν)

$$\therefore\ dw = \left(\frac{2\pi}{h}\right) dE \qquad (2)$$

Also, we have, k = $\frac{2\pi}{\lambda}$ = 2π $\frac{p}{h}$ (∵λ = $\frac{h}{p}$)

$\therefore\ dk = \left(\frac{2\pi}{h}\right) dp$ (3)

Substituting eq. (2) and (3) in eq. (1) we get,

$v_{group} = \frac{d\omega}{dk} = \frac{dE}{dp}$ (4)

But we know that $E = \frac{p^2}{2m}$ (5)
Where, p is the momentum of the particle.

$\therefore dE = \frac{2p\,dp}{2m}$ OR

$\frac{dE}{dp} = \frac{2p}{2m} = \frac{p}{m} = \frac{m v_{particle}}{m} = v_{particle}$ (6)

Using the eq. (6) in eq. (4) we get,

$v_{group} = \frac{dE}{dp} = v_{particle}$

$$\boxed{v_{group} = v_{particle}}$$

That is, velocity of a particle is always equal to the group velocity of the corresponding wave packet. In other words the particle and the associated wave packet move together.

Relation between Phase velocity (V_{phase}) and Group velocity (V_{group})

Depending on how the phase velocity varies with wavelength, the group velocity may be less or greater than the phase velocities of member waves. If the phase velocity is independent of wavelength, it is a non dispersive medium. Variation of phase velocity with wavelength is called dispersion.
We obtain the relationship between group and phase velocities in a dispersive medium in the following way:

We know that Phase velocity is given by,

$v_{phase} = \frac{\omega}{k} \Rightarrow \quad \omega = v_{phase} k$ (1)

and group velocity

$v_{group} = \frac{d\omega}{dk}$ (2)

from Equation (1)

$$v_{group} = \frac{d}{dk}(v_{phase} k)$$

$$= v_{Phase} + k\frac{d(v_{phase})}{dk}$$

$$= v_{Phase} + k\frac{dv_{Phase}}{d\lambda} \times \frac{d\lambda}{dk} \quad (3)$$

We have a relation,

$$k = \frac{2\pi}{\lambda}$$

$$dk = -\frac{2\pi}{\lambda^2} d\lambda$$

$$\therefore \frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi} \quad (4)$$

$$v_{group} = v_{phase} + k.\frac{d(v_{phase})}{d\lambda}\left(\frac{-\lambda^2}{2\pi}\right)$$

$$v_{group} = v_{phase} + \frac{2\pi}{\lambda}.\frac{d(v_{ph})}{d\lambda}\left(\frac{-\lambda^2}{2\pi}\right)$$

$$\boxed{\therefore v_{group} = v_{phase} - \lambda \cdot \frac{dv_{phase}}{d\lambda}} \quad (5)$$

This is the relation between phase velocity and group velocity in a dispersive medium.

<u>In a non dispersive medium</u>

$\frac{dv_{phase}}{d\lambda}$ =0. Hence $v_g = v_{phase}$

<u>Relation between Phase velocity, Group velocity and velocity of light</u>

We have Phase velocity

$$\therefore v_{phase} = \frac{\omega}{k} \quad \text{-----------(1)}$$

Consider Angular velocity and Wave vector

$$\omega = 2\pi\gamma \quad k = \frac{2\pi}{\lambda}$$

We have, $E = h\gamma \quad \lambda = \frac{h}{p}$

$$\gamma = \frac{E}{h} \quad \frac{1}{\lambda} = \frac{p}{h}$$

$$\omega = \frac{2\pi E}{h} \quad k = \frac{2\pi p}{h}$$

On substituting in the above equation (1), we get

$$\therefore v_{phase} = \frac{E}{p}$$

$$v_{phase} = \frac{mc^2}{mv_{Particle}}$$

$$\boxed{v_{phase}.v_{group} = c^2}$$