5.2 Inverse Laplace Transforms

Inverse Laplace Transforms can as well be regarded as the reverse process of finding the Laplace transform of a given function. We also discuss *convolution theorem* which helps in finding the inverse Laplace transform. Finally we discuss solution of linear differential equations with a given set of initial conditions referred to as *initial value problems*. This method is highly useful in various branches of engineering.

We have made a mention of this while defining the Laplace transform of a function $f(t)$. If $L[f(t)] = \bar{f}(s)$ then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$ and is denoted by $L^{-1}[\bar{f}(s)]$.

Thus we can say that,

$$L[f(t)] = \bar{f}(s) \Leftrightarrow L^{-1}[\bar{f}(s)] = f(t)$$

Observe the following illustrations.

$$L(1) = \frac{1}{s} \Rightarrow L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L(\cos at) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

We revert the table of Laplace transforms of standard functions given earlier to present the basic table of inverse Laplace transforms.

	Function	Inverse Transform		Function	Inverse Transform
1	$\frac{1}{s}$	1	5	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
2	$\frac{1}{s-a}$	e^{at}	6	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
3	$\frac{s}{s^2+a^2}$	$\cos at$	7	$\frac{1}{s^{n+1}}$ $(n>-1)$	$\frac{t^n}{\Gamma(n+1)}$
4	$\frac{s}{s^2-a^2}$	$\cosh at$	8	$\frac{1}{s^{n+1}}$ $n=1,2,3..,$	$\frac{t^n}{n!}$

We present a few illustrative examples based on this table of inverse Laplace transforms.

1. $L^{-1}\left(\frac{1}{s-1}\right) = e^t$

2. $L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

3. $L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$

4. $L^{-1}\left(\frac{s}{s^2-16}\right) = \cosh 4t$

5. $L^{-1}\left(\frac{1}{s^2+5}\right) = \frac{1}{\sqrt{5}}\sin(\sqrt{5}\,t)$

6. $L^{-1}\left(\frac{1}{s^2-36}\right) = \frac{1}{6}\sinh 6t$

7. $L^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{3!}$

8. $L^{-1}\left(\frac{1}{s^{3/2}}\right) = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{t^{1/2}}{1/2\cdot\sqrt{\pi}} = 2\sqrt{t/\pi}$

Property : $L^{-1}[c_1\bar{f}(s)+c_2\bar{g}(s)] = c_1L^{-1}[\bar{f}(s)]+c_2L^{-1}[\bar{g}(s)]$

WORKED PROBLEMS
Set – 7

Find the inverse Laplace transform of the following functions

79. $\frac{1}{s+2}+\frac{3}{2s+5}-\frac{4}{3s-2}$

80. $\frac{s+2}{s^2+36}+\frac{4s-1}{s^2+25}$

81. $\frac{2s-5}{4s^2+25}+\frac{8-6s}{16s^2+9}$

82. $\frac{2s-5}{8s^2-50}+\frac{4s}{9-s^2}$

83. $\frac{(s+2)^3}{s^6}$

84. $\frac{3(s^2-1)^2}{2s^5}$

85. $\frac{1}{s\sqrt{s}}+\frac{3}{s^2\sqrt{s}}-\frac{8}{\sqrt{s}}$

86. $\frac{3s+5\sqrt{2}}{s^2+8}$

79. $L^{-1}\left[\frac{1}{s+2}\right]+\frac{3}{2}L^{-1}\left[\frac{1}{s+5/2}\right]-\frac{4}{3}L^{-1}\left[\frac{1}{s-2/3}\right]$

Thus the required inverse Laplace transform is

$$e^{-2t}+3/2\cdot e^{-5t/2}-4/3\cdot e^{2t/3}$$

80. $L^{-1}\left[\frac{s}{s^2+6^2}\right]+2L^{-1}\left[\frac{1}{s^2+6^2}\right]+4L^{-1}\left[\frac{s}{s^2+5^2}\right]-L^{-1}\left[\frac{1}{s^2+5^2}\right]$

Thus the required inverse Laplace transform is

$$\cos 6t+1/3\cdot\sin 6t+4\cos 5t-1/5\cdot\sin 5t$$

81. $2\,L^{-1}\left[\dfrac{s}{4s^2+25}\right]-5\,L^{-1}\left[\dfrac{1}{4s^2+25}\right]+8L^{-1}\left[\dfrac{1}{16s^2+9}\right]-6L^{-1}\left[\dfrac{s}{16s^2+9}\right]$

$$=\frac{1}{2}L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right]-\frac{5}{4}L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+(3/4)^2}\right]-\frac{3}{8}L^{-1}\left[\frac{s}{s^2+(3/4)^2}\right]$$

$$=\frac{1}{2}\cos(5t/2)-\frac{5}{4}\cdot\frac{2}{5}\sin(5t/2)+\frac{1}{2}\cdot\frac{4}{3}\sin(3t/4)-\frac{3}{8}\cos(3t/4)$$

Thus the required inverse Laplace transform is

$1/2\cdot\cos(5t/2)-1/2\cdot\sin(5t/2)+2/3\cdot\sin(3t/4)-3/8\cdot\cos(3t/4)$

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82. $L^{-1}\left[\dfrac{2s-5}{2(4s^2-25)}\right]-L^{-1}\left[\dfrac{4s}{s^2-9}\right]$

$$=\frac{1}{2}L^{-1}\left[\frac{1}{2s+5}\right]-4L^{-1}\left[\frac{s}{s^2-3^2}\right]=\frac{1}{2}\cdot\frac{1}{2}L^{-1}\left[\frac{1}{s+(5/2)}\right]-4L^{-1}\left[\frac{s}{s^2-3^2}\right]$$

Thus the required inverse Laplace transform is

$1/4\cdot e^{-5t/2}-4\cosh 3t$

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83. $L^{-1}\left[\dfrac{s^3+6s^2+12s+8}{s^6}\right]$

$$=L^{-1}\left(\frac{1}{s^3}\right)+6L^{-1}\left(\frac{1}{s^4}\right)+12L^{-1}\left(\frac{1}{s^5}\right)+8L^{-1}\left(\frac{1}{s^6}\right)$$

$$=\frac{t^2}{2!}+6\cdot\frac{t^3}{3!}+12\cdot\frac{t^4}{4!}+8\cdot\frac{t^5}{5!}$$

Thus the required inverse Laplace transform is

$$\frac{t^2}{2}+t^3+\frac{t^4}{2}+\frac{t^5}{15}$$

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84. $\dfrac{3}{2}L^{-1}\left[\dfrac{s^4-2s^2+1}{s^5}\right]$

$$=\frac{3}{2}\left[L^{-1}\left(\frac{1}{s}\right)-2L^{-1}\left(\frac{1}{s^3}\right)+L^{-1}\left(\frac{1}{s^5}\right)\right]$$

Thus the required inverse Laplace transform is

$$\frac{3}{2}\left[1-2\cdot\frac{t^2}{2!}+\frac{t^4}{4!}\right]=\frac{3}{2}\left[1-t^2+\frac{t^4}{24}\right]$$

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85. $L^{-1}\left(\dfrac{1}{s^{3/2}}\right)+3L^{-1}\left(\dfrac{1}{s^{5/2}}\right)-8L^{-1}\left(\dfrac{1}{s^{1/2}}\right)$

$$=\frac{t^{1/2}}{\Gamma(3/2)}+3\cdot\frac{t^{3/2}}{\Gamma(5/2)}-8\frac{t^{-1/2}}{\Gamma(1/2)}$$

$$=\frac{\sqrt{t}}{1/2\cdot\Gamma(1/2)}+3\frac{t\sqrt{t}}{3/2\cdot 1/2\cdot\Gamma(1/2)}-\frac{8}{\sqrt{t}\,\Gamma(1/2)}$$

$$=\frac{2\sqrt{t}}{\sqrt{\pi}}+4\frac{t\sqrt{t}}{\sqrt{\pi}}-\frac{8}{\sqrt{t}\sqrt{\pi}}$$

Thus the required inverse Laplace transform is

$$\frac{2}{\sqrt{\pi}}\left[\sqrt{t}+2t\sqrt{t}-\frac{4}{\sqrt{t}}\right]$$

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86. $3L^{-1}\left[\dfrac{s}{s^2+(\sqrt{8})^2}\right]+5\sqrt{2}\,L^{-1}\left[\dfrac{1}{s^2+(\sqrt{8})^2}\right]$

Thus the required inverse Laplace transform is

$3\cos(2\sqrt{2}\,t)+5/2\cdot\sin(2\sqrt{2}\,t)$

5.21 Computation of the inverse transform of $e^{-as}\,\bar{f}(s)$

We have proved that $L[f(t-a)\,u(t-a)]=e^{-as}\,\bar{f}(s)$

$$\therefore\quad L^{-1}[e^{-as}\,\bar{f}(s)]=f(t-a)\,u(t-a)$$

Working procedure for problems

Step-1 In the given function we should observe the presence of e^{-as} first and identify the remaining part of the function to be called as $\bar{f}(s)$.

Step-2 Taking the inverse of $\bar{f}(s)$ we obtain $f(t)$.

Step-3 The required inverse of $e^{-as}\,\bar{f}(s)$ is obtained by replacing t by $(t-a)$ in $f(t)$ to be multiplied by the unit step function $u(t-a)$

WORKED PROBLEMS
Set – 8

Find the inverse Laplace transforms of the following

87. $\dfrac{1+e^{-3s}}{s^2}$ 88. $\dfrac{3}{s^2}+\dfrac{2e^{-s}}{s^3}-\dfrac{3e^{-2s}}{s}$

89. $\dfrac{\cosh 2s}{e^{3s}s^2}$ 90. $\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}$

91. $\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{s\,e^{-2\pi s}}{s^2+4}$ 92. $\dfrac{s\,e^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

87. $L^{-1}\left(\dfrac{1}{s^2}\right)+L^{-1}\left(\dfrac{e^{-3s}}{s^2}\right)$. We have $L^{-1}(1/s^2)=t$

Thus $L^{-1}\left[\dfrac{1+e^{-3s}}{s^2}\right]=t+(t-3)\,u(t-3)$

---

88. $3L^{-1}\left(\dfrac{1}{s^2}\right)+2L^{-1}\left(\dfrac{e^{-s}}{s^3}\right)-3L^{-1}\left(\dfrac{e^{-2s}}{s}\right)$ ...(1)

We have $L^{-1}\left(\dfrac{1}{s^2}\right)=t,\ L^{-1}\left(\dfrac{1}{s^3}\right)=\dfrac{t^2}{2},\ L^{-1}\left(\dfrac{1}{s}\right)=1$

Hence (1) becomes,

$$3\cdot t+2\frac{(t-1)^2}{2}u(t-1)-3\cdot 1\cdot u(t-2)$$

Thus $L^{-1}\left[\dfrac{3}{s^2}+\dfrac{2e^{-s}}{s^3}-\dfrac{3e^{-2s}}{s}\right]=3t+(t-1)^2u(t-1)-3u(t-2)$

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89. $\dfrac{\cosh 2s}{e^{3s}\cdot s^2}=\dfrac{e^{-3s}}{s^2}\dfrac{(e^{2s}+e^{-2s})}{2}=\dfrac{1}{2}\left[\dfrac{e^{-s}}{s^2}+\dfrac{e^{-5s}}{s^2}\right]$

Now, $L^{-1}\left[\dfrac{\cosh 2s}{e^{3s}s^2}\right]=\dfrac{1}{2}\left\{L^{-1}\left(\dfrac{e^{-s}}{s^2}\right)+L^{-1}\left(\dfrac{e^{-5s}}{s^2}\right)\right\}$. But $L^{-1}(1/s^2)=t$

Thus $L^{-1}\left[\dfrac{\cosh 2s}{e^{3s}\cdot s^2}\right]=\dfrac{1}{2}\{(t-1)\,u(t-1)+(t-5)\,u(t-5)\}$

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90. $\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}=\dfrac{2-2e^{-s}-e^{-2s}+e^{-3s}}{s^3}$

Now, $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$

$$=2L^{-1}\left(\frac{1}{s^3}\right)-2L^{-1}\left(\frac{e^{-s}}{s^3}\right)-L^{-1}\left(\frac{e^{-2s}}{s^3}\right)+L^{-1}\left(\frac{e^{-3s}}{s^3}\right)$$

But, $L^{-1}(1/s^3)=t^2/2$

Thus $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right]$

$$=t^2-(t-1)^2u(t-1)-\frac{(t-2)^2u(t-2)}{2}+\frac{(t-3)^2u(t-3)}{2}$$

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91. $L^{-1}\left(\dfrac{e^{-\pi s}}{s^2+1}\right)+L^{-1}\left(e^{-2\pi s}\cdot\dfrac{s}{s^2+4}\right)$ ...(1)

We have, $L^{-1}\left(\dfrac{1}{s^2+1}\right)=\sin t,\ L^{-1}\left(\dfrac{s}{s^2+4}\right)=\cos 2t$

Hence (1) becomes,

$$\sin(t-\pi)\,u(t-\pi)+\cos 2(t-2\pi)\,u(t-2\pi)$$

Thus $L^{-1}\left[\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{s\,e^{-2\pi s}}{s^2+1}\right]=-\sin t\ u(t-\pi)+\cos 2t\ u(t-2\pi)$

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92. $L^{-1}\left(e^{-s/2}\cdot\dfrac{s}{s^2+\pi^2}\right)+L^{-1}\left(e^{-s}\cdot\dfrac{\pi}{s^2+\pi^2}\right)$ ...(1)

We have $L^{-1}\left(\dfrac{s}{s^2+\pi^2}\right)=\cos\pi t,\ L^{-1}\left(\dfrac{\pi}{s^2+\pi^2}\right)=\sin\pi t$

Hence (1) becomes,

$$\cos\pi(t-1/2)\,u(t-1/2)+\sin\pi(t-1)\,u(t-1)$$

$$=\sin\pi t\ u(t-1/2)-\sin\pi t\ u(t-1)$$

Thus $L^{-1}\left[\dfrac{s\,e^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right]=\sin\pi t\,[u(t-1/2)-u(t-1)]$

5.22 Inverse transform by completing the square

We have the property that if $L[f(t)] = \bar{f}(s)$ then $L[e^{at} f(t)] = \bar{f}(s-a)$.

This implies that

$$L^{-1}[\bar{f}(s)] = f(t) \qquad \ldots(1)$$

and $$L^{-1}[\bar{f}(s-a)] = e^{at} f(t) \qquad \ldots(2)$$

(2), as a consequence of (1) becomes

$$L^{-1}[\bar{f}(s-a)] = e^{at} L^{-1}[\bar{f}(s)] \qquad \ldots(3)$$

Working procedure for problems

Step-1 Given $\bar{f}(s) = \dfrac{\phi(s)}{(ps^2+qs+r)}$,

we first express (ps^2+qs+r) in the form $(s-a)^2 \pm b^2$ and later express $\phi(s)$ in terms of $(s-a)$ so that the given function of s reduces to a function of $(s-a)$.

Step-2 We then use (3) to obtain the result.

Step-3 However if $\bar{f}(s) = \phi(s)/\psi(s-a)$ we only need to express $\phi(s)$ in terms of $(s-a)$ to compute the inverse transform.

WORKED PROBLEMS
Set – 9

Find the inverse Laplace transform of the following functions

93. $\dfrac{s+5}{s^2-6s+13}$

94. $\dfrac{s^2}{(s+1)^3}$

95. $\dfrac{(s+2)e^{-s}}{(s+1)^4}$

96. $\dfrac{2s-1}{s^2+4s+29}$

97. $\dfrac{s+1}{s^2+6s+9}$

98. $\dfrac{e^{-4s}}{(s-4)^2}$

99. $\dfrac{s+1}{s^2+s+1}$

100. $\dfrac{2s+1}{s^2+3s+1}$

101. $\dfrac{7s+4}{4s^2+4s+9}$

102. $\dfrac{1}{(s+4)^{5/2}} + \dfrac{1}{\sqrt{2s+3}}$

93. $L^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right] = L^{-1}\left[\dfrac{s+5}{(s-3)^2+4}\right]$

ie., $= L^{-1}\left[\dfrac{s-3+3+5}{(s-3)^2+2^2}\right] = L^{-1}\left[\dfrac{(s-3)+8}{(s-3)^2+2^2}\right]$

Here, $a = 3$ and $(s-3)$ changes to s

ie., $= e^{3t} L^{-1}\left[\dfrac{s+8}{s^2+2^2}\right]$

$$= e^{3t}\left\{L^{-1}\left(\frac{s}{s^2+2^2}\right) + 8L^{-1}\left(\frac{1}{s^2+2^2}\right)\right\}$$

Thus $L^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right] = e^{3t}(\cos 2t + 4\sin 2t)$

94. Here we need to express s^2 in terms of $(s+1)$

$$s^2 = (s+1)^2 - 2s - 1 = (s+1)^2 - 2(s+1) + 2 - 1$$

ie., $s^2 = (s+1)^2 - 2(s+1) + 1.$

$\therefore$ $L^{-1}\left[\dfrac{s^2}{(s+1)^3}\right] = L^{-1}\left[\dfrac{(s+1)^2-2(s+1)+1}{(s+1)^3}\right]$

Here, $a = -1$ and $(s+1)$ changes to s. Hence R.H.S becomes

$$e^{-t} L^{-1}\left[\frac{s^2-2s+1}{s^3}\right] = e^{-t}\left\{L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^3}\right)\right\}$$

Thus $L^{-1}\left[\dfrac{s^2}{(s+1)^3}\right] = e^{-t}\left(1 - 2t + \dfrac{t^2}{2}\right)$

95. Let $\bar{f}(s) = \dfrac{s+2}{(s+1)^4}$

We shall first find $L^{-1}[\bar{f}(s)] = f(t)$

$$L^{-1}\left[\frac{s+2}{(s+1)^4}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^4}\right] = e^{-t} L^{-1}\left[\frac{s+1}{s^4}\right]$$

ie., $L^{-1}\left[\dfrac{s+2}{(s+1)^4}\right] = e^{-t}\left\{L^{-1}\left(\dfrac{1}{s^3}\right) + L^{-1}\left(\dfrac{1}{s^4}\right)\right\}$

Using $L^{-1}\left(\dfrac{1}{s^{n+1}}\right) = \dfrac{t^n}{n!}$ we get

$$f(t) = e^{-t}\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right) = e^{-t}\left(\frac{t^2}{2}+\frac{t^3}{6}\right)$$

Next we have, $L^{-1}[e^{-s}\bar{f}(s)] = f(t-1)u(t-1)$

Thus $L^{-1}\left[e^{-s}\dfrac{s+2}{(s+1)^4}\right] = e^{-(t-1)}\left\{\dfrac{(t-1)^2}{2}+\dfrac{(t-1)^3}{6}\right\}u(t-1)$

96. $L^{-1}\left\{\dfrac{2s-1}{s^2+4s+29}\right\} = L^{-1}\left\{\dfrac{2s-1}{(s+2)^2+25}\right\} = L^{-1}\left\{\dfrac{2(s+2)-5}{(s+2)^2+25}\right\}$

ie., $= e^{-2t}L^{-1}\left\{\dfrac{2s-5}{s^2+5^2}\right\} = e^{-2t}\left\{2L^{-1}\left(\dfrac{s}{s^2+5^2}\right)-L^{-1}\left(\dfrac{5}{s^2+5^2}\right)\right\}$

Thus $L^{-1}\left[\dfrac{2s-1}{s^2+4s+29}\right] = e^{-2t}(2\cos 5t-\sin 5t)$

97. $L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\dfrac{(s+3)-2}{(s+3)^2}\right] = e^{-3t}L^{-1}\left[\dfrac{s-2}{s^2}\right]$

ie., $= e^{-3t}\left\{L^{-1}\left(\dfrac{1}{s}\right)-2L^{-1}\left(\dfrac{1}{s^2}\right)\right\}$

Thus $L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = e^{-3t}(1-2t)$

98. Let $\bar{f}(s) = \dfrac{1}{(s-4)^2}$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{(s-4)^2}\right] = e^{4t}L^{-1}\left(\frac{1}{s^2}\right) = e^{4t}\cdot t = f(t)$$

But, $L^{-1}[e^{-4s}\bar{f}(s)] = f(t-4)u(t-4)$

Thus $L^{-1}\left[\dfrac{e^{-4s}}{(s-4)^2}\right] = \left\{e^{4(t-4)}(t-4)\right\}u(t-4)$

99. $s^2+s+1 = (s+1/2)^2-1/4+1 = (s+1/2)^2+(\sqrt{3}/2)^2$

$$L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = L^{-1}\left[\frac{(s+1/2)+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] = e^{-t/2}L^{-1}\left[\frac{s+1/2}{s^2+(\sqrt{3}/2)^2}\right]$$

ie., $= e^{-t/2}\left\{L^{-1}\left[\dfrac{s}{s^2+(\sqrt{3}/2)^2}\right]+\dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+(\sqrt{3}/2)^2}\right]\right\}$

Thus $L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right] = e^{-t/2}\{\cos(\sqrt{3}\,t/2)+1/\sqrt{3}\cdot\sin(\sqrt{3}\,t/2)\}$

100. $s^2+3s+1 = (s+3/2)^2-9/4+1 = (s+3/2)^2-(\sqrt{5}/2)^2$

$$L^{-1}\left[\frac{2s+1}{s^2+3s+1}\right] = L^{-1}\left[\frac{2s+1}{(s+3/2)^2-(\sqrt{5}/2)^2}\right]$$

$$= L^{-1}\left[\frac{2(s+3/2)-2}{(s+3/2)^2-(\sqrt{5}/2)^2}\right]$$

$$= e^{-3t/2}\left\{2L^{-1}\left[\frac{s}{s^2-(\sqrt{5}/2)^2}\right]-2L^{-1}\left[\frac{1}{s^2-(\sqrt{5}/2)^2}\right]\right\}$$

Thus $L^{-1}\left[\dfrac{2s+1}{s^2+3s+1}\right] = e^{-3t/2}\{2\cosh(\sqrt{5}\,t/2)-4/\sqrt{5}\cdot\sinh(\sqrt{5}\,t/2)\}$

101. Consider $4s^2+4s+9$

$$= 4(s^2+s+9/4) = 4\{(s+1/2)^2+2\}$$

Hence, $7s+4 = 7(s+1/2)+1/2$

Now, $L^{-1}\left[\dfrac{7s+4}{4s^2+4s+9}\right] = \dfrac{1}{4}L^{-1}\left[\dfrac{7(s+1/2)+1/2}{(s+1/2)^2+2}\right]$

ie., $= \dfrac{e^{-t/2}}{4}\left\{7L^{-1}\left[\dfrac{s}{s^2+(\sqrt{2})^2}\right]+\dfrac{1}{2}L^{-1}\left[\dfrac{1}{s^2+(\sqrt{2})^2}\right]\right\}$

Thus $L^{-1}\left[\dfrac{7s+4}{4s^2+4s+9}\right] = \dfrac{e^{-t/2}}{4}\left\{7\cos(\sqrt{2}\,t)+\dfrac{1}{2\sqrt{2}}\sin(\sqrt{2}\,t)\right\}$

102. $L^{-1}\left[\frac{1}{(s+4)^{5/2}}\right]+L^{-1}\left[\frac{1}{\sqrt{2s+3}}\right]$

$$= e^{-4t}L^{-1}\left[\frac{1}{s^{5/2}}\right]+\frac{1}{\sqrt{2}}L^{-1}\left[\frac{1}{\sqrt{s+3/2}}\right]$$

$$= e^{-4t}\left[\frac{t^{3/2}}{\Gamma(5/2)}\right]+\frac{e^{-3t/2}}{\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right).$$

But $\Gamma(5/2)=\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}=\frac{3\sqrt{\pi}}{4}$

Thus we have $L^{-1}\left[\frac{1}{(s+4)^{5/2}}\right]+L^{-1}\left[\frac{1}{\sqrt{2s+3}}\right]$

$$=\frac{4}{3\sqrt{\pi}}e^{-4t}t^{3/2}+\frac{e^{-3t/2}}{\sqrt{2}}\cdot\frac{t^{-1/2}}{\Gamma(1/2)}$$ But $\Gamma(1/2)=\sqrt{\pi}$

$$=\frac{1}{\sqrt{\pi}}\left\{\frac{4}{3}e^{-4t}t^{3/2}+\frac{e^{-3t/2}}{\sqrt{2t}}\right\}$$

Find the inverse Laplace transform of the following functions.

103. $\frac{s}{s^4+4a^4}$ 104. $\frac{s^2}{s^4+4a^4}$ 105. $\frac{s}{s^4+s^2+1}$

Note : *In these problems we factorize the denominator and express the numerator interms of the factors of the denominator by simple adjustment.*

103. $s^4+4a^4=(s^2+2a^2)^2-4a^2s^2$ ...(1)

ie., $s^4+4a^4=(s^2+2a^2+2as)(s^2+2a^2-2as)$

Also, $4as=(s^2+2a^2+2as)-(s^2+2a^2-2as)$

Now, $\frac{s}{s^4+4a^4}=\frac{1}{4a}\left\{\frac{(s^2+2a^2+2as)-(s^2+2a^2-2as)}{(s^2+2a^2+2as)(s^2+2a^2-2as)}\right\}$

ie., $\frac{s}{s^4+4a^4}=\frac{1}{4a}\left\{\frac{1}{s^2+2a^2-2as}-\frac{1}{s^2+2a^2+2as}\right\}$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right]=\frac{1}{4a}\left\{L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]-L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right\}$$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right]=\frac{1}{4a}\left\{e^{at}L^{-1}\left[\frac{1}{s^2+a^2}\right]-e^{-at}L^{-1}\left[\frac{1}{s^2+a^2}\right]\right\}$$

$$=\frac{1}{4a}\left\{e^{at}\frac{\sin at}{a}-e^{-at}\frac{\sin at}{a}\right\}$$

Thus $L^{-1}\left[\frac{s}{s^4+4a^4}\right]=\frac{\sin at}{2a^2}\left\{\frac{e^{at}-e^{-at}}{2}\right\}=\frac{\sin at\sinh at}{2a^2}$

104. As in the previous problem, we have,

$$\frac{s}{s^4+4a^4}=\frac{1}{4a}\left\{\frac{1}{(s-a)^2+a^2}-\frac{1}{(s+a)^2+a^2}\right\}.$$

Multiplying by s we have

$$\frac{s^2}{s^4+4a^4}=\frac{1}{4a}\left\{\frac{s}{(s-a)^2+a^2}-\frac{s}{(s+a)^2+a^2}\right\}$$

or $\frac{s^2}{s^4+4a^4}=\frac{1}{4a}\left\{\frac{(s-a)+a}{(s-a)^2+a^2}-\frac{(s+a)-a}{(s+a)^2+a^2}\right\}$

$\therefore$ $L^{-1}\left[\frac{s^2}{s^4+4a^4}\right]=\frac{1}{4a}\left\{e^{at}L^{-1}\left[\frac{s+a}{s^2+a^2}\right]-e^{-at}L^{-1}\left[\frac{s-a}{s^2+a^2}\right]\right\}$

ie., $=\frac{1}{4a}\left\{e^{at}(\cos at+\sin at)-e^{-at}(\cos at-\sin at)\right\}$

$$=\frac{1}{2a}\left\{\frac{e^{at}-e^{-at}}{2}\cos at+\frac{e^{at}+e^{-at}}{2}\sin at\right\}$$

Thus $L^{-1}\left[\frac{s^2}{s^4+4a^4}\right]=\frac{1}{2a}(\sinh at\cos at+\cosh at\sin at)$

105. $s^4+s^2+1=(s^2+1)^2-s^2=(s^2+1-s)(s^2+1+s)$

Also, $2s=(s^2+1+s)-(s^2+1-s)$

$\therefore$ $\frac{s}{s^4+s^2+1}=\frac{1}{2}\left[\frac{(s^2+1+s)-(s^2+1-s)}{(s^2+1+s)(s^2+1-s)}\right]$

ie., $\frac{s}{s^4+s^2+1}=\frac{1}{2}\left[\frac{1}{s^2-s+1}-\frac{1}{s^2+s+1}\right]$

Now, $L^{-1}\left[\frac{s}{s^4+s^2+1}\right]=\frac{1}{2}\left\{L^{-1}\left[\frac{1}{s^2-s+1}\right]-L^{-1}\left[\frac{1}{s^2+s+1}\right]\right\}$

ie.,

$$= \frac{1}{2}\left\{L^{-1}\left[\frac{1}{(s-1/2)^2+(\sqrt{3}/2)^2}\right]-L^{-1}\left[\frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}\right]\right\}$$

$$= \frac{1}{2}\left\{e^{t/2}L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]-e^{-t/2}L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]\right\}$$

$$= \frac{1}{2}\left\{e^{t/2}\cdot\frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t/2)-e^{-t/2}\cdot\frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t/2)\right\}$$

$$= \frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t/2)\left\{\frac{e^{t/2}-e^{-t/2}}{2}\right\}$$

Thus $$L^{-1}\left[\frac{s}{s^4+s^2+1}\right] = \frac{2}{\sqrt{3}}\sin(\sqrt{3}\,t/2)\cdot\sinh(t/2)$$

5.23 Inverse transform by the method of partial fractions

We know that, the method of partial fractions is a technique of converting an algebraic fraction $\phi(s)/\psi(s)$ [*where degree of* $\phi(s)$ *is less than that of* $\psi(s)$] into a sum. Depending on the nature of terms in $\psi(s)$ we have to split into a sum of various terms with constants $A, B, C, D, \cdots$ which can be determined. Later the inverse is found term by term.

WORKED PROBLEMS
Set – 10

Find the inverse Laplace transform of the following functions.

106. $\dfrac{1}{s(s+1)(s+2)(s+3)}$

107. $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$

108. $\dfrac{3s+2}{s^2-s-2}$

109. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

110. $\dfrac{4s+5}{(s+1)^2(s+2)}$

111. $\dfrac{s+2}{s^2(s+3)}$

112. $\dfrac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}$

113. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

106. Let $$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{D}{s+3}$$

Multiplying by $s(s+1)(s+2)(s+3)$ we get,

$$1 = A(s+1)(s+2)(s+3)+Bs(s+2)(s+3)+Cs(s+1)(s+3)+Ds(s+1)(s+2)$$

Put $s=0$: $1=A(6)$ $\therefore$ $A=1/6$

Put $s=-1$: $1=B(-2)$ $\therefore$ $B=-1/2$

Put $s=-2$: $1=C(2)$ $\therefore$ $C=1/2$

Put $s=-3$: $1=D(-6)$ $\therefore$ $D=-1/6$

Now, $$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right]$$

$$= \frac{1}{6}L^{-1}\left[\frac{1}{s}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]-\frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

Thus $$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6}-\frac{1}{2}e^{-t}+\frac{1}{2}e^{-2t}-\frac{1}{6}e^{-3t}$$

107. $s^3+s^2-2s = s(s^2+s-2) = s(s-1)(s+2)$

Let $$\frac{2s^2+5s-4}{s(s-1)(s+2)} = \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}$$

Multiplying by $s(s-1)(s+2)$ we get,

$$2s^2+5s-4 = A(s-1)(s+2)+Bs(s+2)+Cs(s-1)$$

Put $s=0$: $-4=A(-2)$ $\therefore$ $A=2$

Put $s=1$: $3=B(3)$ $\therefore$ $B=1$

Put $s=-2$: $-6=C(6)$ $\therefore$ $C=-1$

Now, $$L^{-1}\left[\frac{2s^2+5s-4}{s(s-1)(s+2)}\right]$$

$$= 2L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{s+2}\right]$$

Thus $$L^{-1}\left[\frac{2s^2+5s-4}{s(s-1)(s+2)}\right] = 2+e^{t}-e^{-2t}$$

108. Note : *The problem can be done by completing the square. Since the quadratic is factorizable, the method of partial fractions is preferred.*

Let $\dfrac{3s+2}{(s-2)(s+1)} = \dfrac{A}{s-2}+\dfrac{B}{s+1}$

or $3s+2 = A(s+1)+B(s-2)$

Put $s = 2$: $8 = A(3)$ $\therefore$ $A = 8/3$

Put $s = -1$: $-1 = B(-3)$ $\therefore$ $B = 1/3$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3}L^{-1}\left[\frac{1}{s-2}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]$$

Thus $L^{-1}\left[\dfrac{3s+2}{(s-2)(s+1)}\right] = \dfrac{1}{3}(8e^{2t}+e^{-t})$

109. We have $\dfrac{s^2}{(s^2+1)(s^2+4)}$ and let $s^2 = t$ for convenience.

We now have $\dfrac{t}{(t+1)(t+4)} = \dfrac{A}{t+1}+\dfrac{B}{t+4}$ *(say)*

or $t = A(t+4)+B(t+1)$

Put $t = -1$: $-1 = A(3)$ $\therefore$ $A = -1/3$

Put $t = -4$: $-4 = B(-3)$ $\therefore$ $B = 4/3$

Hence $\dfrac{t}{(t+1)(t+4)} = \dfrac{-1}{3}\cdot\dfrac{1}{t+1}+\dfrac{4}{3}\cdot\dfrac{1}{t+4}$

Substituting $t = s^2$ and taking inverse we have,

$$L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{-1}{3}L^{-1}\left[\frac{1}{s^2+1}\right]+\frac{4}{3}L^{-1}\left[\frac{1}{s^2+4}\right]$$

Thus, $L^{-1}\left[\dfrac{s^2}{(s^2+1)(s^2+4)}\right] = \dfrac{1}{3}(2\sin 2t-\sin t)$

110. Let $\dfrac{4s+5}{(s+1)^2(s+2)} = \dfrac{A}{s+1}+\dfrac{B}{(s+1)^2}+\dfrac{C}{s+2}$

Multiplying by $(s+1)^2(s+2)$ we get,

$$4s+5 = A(s+1)(s+2)+B(s+2)+C(s+1)^2 \quad \ldots(1)$$

Put $s = -1$: $1 = B(1)$ $\therefore$ $B = 1$

Put $s = -2$: $-3 = C(1)$ $\therefore$ $C = -3$

Equating the coefficient of s^2 on both sides of (1) we get,

$0 = A+C$ $\therefore$ $A = 3$

Now, $L^{-1}\left[\dfrac{4s+5}{(s+1)^2(s+2)}\right]$

$$= 3L^{-1}\left[\frac{1}{s+1}\right]+L^{-1}\left[\frac{1}{(s+1)^2}\right]-3L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 3e^{-t}+e^{-t}L^{-1}[1/s^2]-3e^{-2t}$$

Thus, $L^{-1}\left[\dfrac{4s+5}{(s+1)^2(s+2)}\right] = 3e^{-t}+e^{-t}\cdot t-3e^{-2t}$

111. Let $\dfrac{s+2}{s^2(s+3)} = \dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{C}{s+3}$

Multiplying by $s^2(s+3)$ we get,

$$s+2 = As(s+3)+B(s+3)+Cs^2 \quad \ldots(1)$$

Put $s = 0$: $2 = B(3)$ $\therefore$ $B = 2/3$

Put $s = -3$: $-1 = C(9)$ $\therefore$ $C = -1/9$

Equating the coefficient of s^2 on both sides of (1) we get,

$0 = A+C$ $\therefore$ $A = -C = 1/9$

Now, $L^{-1}\left[\dfrac{s+2}{s^2(s+3)}\right] = \dfrac{1}{9}L^{-1}\left[\dfrac{1}{s}\right]+\dfrac{2}{3}L^{-1}\left[\dfrac{1}{s^2}\right]-\dfrac{1}{9}L^{-1}\left[\dfrac{1}{s+3}\right]$

Thus, $L^{-1}\left[\dfrac{s+2}{s^2(s+3)}\right] = \dfrac{1}{9}(1+6t-e^{-3t})$

112. Let $\dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1}$

or $\quad 3s+1 = A(s^2+1) + (Bs+C)(s-1) \qquad \ldots(1)$

Put $s = 1 \quad : 4 = A(2) \quad \therefore \; A = 2$

Put $s = 0 \quad : 1 = A - C \quad \therefore \; C = 1$

Equating the coefficient of s^2 on both sides of (1) we get,

$0 = A + B \quad \therefore \; B = -2.$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^t - 2\cos t + \sin t$$

Thus, $$L^{-1}\left[\frac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}\right] = [2e^{t-3} - 2\cos(t-3) + \sin(t-3)]\,u(t-3)$$

---

113. Let $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+2s+5}$

or $\quad 5s+3 = A(s^2+2s+5) + (Bs+C)(s-1) \qquad \ldots(1)$

Put $s = 1 \quad : 8 = A(8) \quad \therefore \; A = 1$

Put $s = 0 \quad : 3 = 5A - C \quad \therefore \; C = 2$

Equating the coefficient of s^2 on both sides of (1) we get

$0 = A + B \quad \therefore \; B = -1$

Now we have,

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{(s^2+2s+5)} = \frac{1}{s-1} + \frac{2-s}{(s+1)^2+4}$$

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{3-(s+1)}{(s+1)^2+4}$$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{3-(s+1)}{(s+1)^2+4)}\right]$$

$$= e^t + e^{-t}L^{-1}\left[\frac{3-s}{s^2+4}\right]$$

$$= e^t + e^{-t}\left\{3L^{-1}\left[\frac{1}{s^2+4}\right] - L^{-1}\left[\frac{s}{s^2+4}\right]\right\}$$

Thus $$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = e^t + e^{-t}\left[\frac{3}{2}\cdot\sin 2t - \cos 2t\right]$$

5.24 Inverse transform of logarithmic functions and inverse functions

Given $\bar{f}(s)$ we need to find $L^{-1}[\bar{f}(s)] = f(t)$

We have the property $: L[tf(t)] = -\bar{f}'(s)$

Equivalently, $L^{-1}[-\bar{f}'(s)] = tf(t) \qquad \ldots(1)$

Working procedure for problems

Step-1 In the case of logarithmic functions we apply the properties of logarithms and then differentiate w.r.t s to obtain $\bar{f}'(s)$.

Step-2 We then multiply by -1 and take inverse on both sides.

Step-3 LHS becomes $tf(t)$ by (1) and inverses are also found for the terms in RHS with the result we obtain the required $f(t)$.

Step-4 If logarithmic function persists in $\bar{f}'(s)$ we differentiate again w.r.t s to obtain $f''(s)$ and use the property that

$$L^{-1}[\bar{f}''(s)] = t^2 f(t) \text{ since } L[t^2 f(t)] = \bar{f}''(s)$$

Step-5 In the cases of inverse functions we simply differentiate the given $\bar{f}(s)$ and use the result (1) to obtain $f(t)$.

WORKED PROBLEMS

Set – 11

Find the inverse Laplace transform of the following functions

114. $\log\left(\dfrac{s+a}{s+b}\right)$

115. $\log\left(1-\dfrac{a^2}{s^2}\right)$

116. $\cot^{-1}(s/a)$

117. $\log\sqrt{(s^2+1)/(s^2+4)}$

118. $\log\left[\dfrac{s^2+4}{s(s+4)(s-4)}\right]$

119. $\tan^{-1}(2/s^2)$

120. $s\log\left(\dfrac{s+4}{s-4}\right)$

121. $\cot^{-1}\left(\dfrac{s+a}{b}\right)$

114. Let $\bar{f}(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$

$\therefore \quad -\bar{f}'(s) = -\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = \frac{1}{s+b} - \frac{1}{s+a}$

Now $\quad L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{1}{s+b}\right) - L^{-1}\left(\frac{1}{s+a}\right)$

ie., $\quad t f(t) = e^{-bt} - e^{-at}$

Thus $\quad f(t) = \frac{e^{-bt} - e^{-at}}{t}$

---

115. Let $\bar{f}(s) = \log\left(1 - \frac{a^2}{s^2}\right) = \log\left(\frac{s^2 - a^2}{s^2}\right)$

ie., $\quad \bar{f}(s) = \log(s^2 - a^2) - 2\log s$

$\therefore \quad -\bar{f}'(s) = -\left\{\frac{1}{s^2 - a^2} \cdot 2s - \frac{2}{s}\right\} = 2\left(\frac{1}{s} - \frac{s}{s^2 - a^2}\right)$

Now $\quad L^{-1}[-\bar{f}'(s)] = 2\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2 - a^2}\right)\right\}$

ie., $\quad t f(t) = 2(1 - \cos h\, at)$

Thus $\quad f(t) = \frac{2(1 - \cos h\, at)}{t}$

---

116. Let $\bar{f}(s) = \cot^{-1}(s/a)$.

Differentiate w.r.t s and multiply with -1

$\therefore \quad \bar{f}'(s) = \frac{-1}{1 + (s/a)^2} \cdot \frac{1}{a}$ and $-\bar{f}'(s) = \frac{a}{a^2 + s^2}$

Taking inverse, $\quad L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{a}{s^2 + a^2}\right)$

ie., $\quad t f(t) = \sin at$

Thus $\quad f(t) = \frac{\sin at}{t}$

---

117. Let $\bar{f}(s) = \log\sqrt{s^2 + 1/s^2 + 4}$

$= \frac{1}{2}\{\log(s^2 + 1) - \log(s^2 + 4)\}$

$\therefore \quad -\bar{f}'(s) = -\frac{1}{2}\left\{\frac{2s}{s^2 + 1} - \frac{2s}{s^2 + 4}\right\} = \frac{s}{s^2 + 4} - \frac{s}{s^2 + 1}$

Now $\quad L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{s}{s^2 + 2^2}\right) - L^{-1}\left(\frac{s}{s^2 + 1^2}\right)$

ie., $\quad t f(t) = \cos 2t - \cos t$

Thus $\quad f(t) = \frac{\cos 2t - \cos t}{t}$

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118. Let $\bar{f}(s) = \log\left[\frac{s^2 + 4}{s(s+4)(s-4)}\right]$

ie., $\quad \bar{f}(s) = \log(s^2 + 4) - \log s - \log(s+4) - \log(s-4)$

$\therefore \quad -\bar{f}'(s) = -\left\{\frac{2s}{s^2 + 4} - \frac{1}{s} - \frac{1}{s+4} - \frac{1}{s-4}\right\}$

Now $\quad L^{-1}[-\bar{f}'(s)] = L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+4}\right) + L^{-1}\left(\frac{1}{s-4}\right) - 2L^{-1}\left(\frac{s}{s^2 + 4}\right)$

ie., $\quad t f(t) = 1 + e^{-4t} + e^{4t} - 2\cos 2t = 1 + 2\cosh 4t - 2\cos 2t$

Thus $\quad f(t) = \frac{1 + 2(\cos h\, 4t - \cos 2t)}{t}$

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119. Let $\bar{f}(s) = \tan^{-1}(2/s^2)$

$\therefore \quad \bar{f}'(s) = \frac{1}{1 + (4/s^4)} \cdot \frac{-4}{s^3} = \frac{-4s}{s^4 + 4}$

Hence $\quad L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{4s}{s^4 + 4}\right]$

ie., $\quad t f(t) = L^{-1}\left[\frac{4s}{s^4 + 4}\right] \quad \ldots(1)$

Now $\quad s^4 + 4 = (s^2 + 2)^2 - 4s^2 = (s^2 + 2 + 2s)(s^2 + 2 - 2s)$

Also $\quad 4s = (s^2 + 2 + 2s) - (s^2 + 2 - 2s)$

Hence $\dfrac{4s}{s^4+4} = \dfrac{(s^2+2+2s)-(s^2+2-2s)}{(s^2+2+2s)(s^2+2-2s)}$

$$= \frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}$$

$\therefore \quad L^{-1}\left[\dfrac{4s}{s^4+4}\right] = L^{-1}\left[\dfrac{1}{s^2-2s+2}\right] - L^{-1}\left[\dfrac{1}{s^2+2s+2}\right]$

Using (1) in L.H.S we have,

$$tf(t) = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

ie., $\quad tf(t) = e^t L^{-1}\left[\dfrac{1}{s^2+1}\right] - e^{-t} L^{-1}\left[\dfrac{1}{s^2+1}\right]$

ie., $\quad tf(t) = e^t \sin t - e^{-t}\sin t = \sin t\,(e^t - e^{-t})$

ie., $\quad tf(t) = \sin t \cdot 2\sinh t$

Thus $\quad f(t) = \dfrac{2\sin t\,\sinh t}{t}$

120. Let $\bar{f}(s) = s\log\left(\dfrac{s+4}{s-4}\right)$

ie., $\quad \bar{f}(s) = s[\log(s+4) - \log(s-4)]$

$\therefore \quad \bar{f}'(s) = \dfrac{s}{s+4} - \dfrac{s}{s-4} + [\log(s+4) - \log(s-4)]$

We need to differentiate w.r.t s again,

$\therefore \quad \bar{f}''(s) = \dfrac{4}{(s+4)^2} - \dfrac{(-4)}{(s-4)^2} + \dfrac{1}{s+4} - \dfrac{1}{s-4}$

Now, $\quad L^{-1}[\bar{f}''(s)] = 4\left\{L^{-1}\left[\dfrac{1}{(s+4)^2}\right] + L^{-1}\left[\dfrac{1}{(s-4)^2}\right]\right\}$

$$+ L^{-1}\left[\frac{1}{s+4}\right] - L^{-1}\left[\frac{1}{s-4}\right]$$

ie., $\quad t^2 f(t) = 4\left\{e^{-4t} L^{-1}\left(\dfrac{1}{s^2}\right) + e^{4t} L^{-1}\left(\dfrac{1}{s^2}\right)\right\} + e^{-4t} - e^{4t}$

$$= 4(e^{-4t}t + e^{4t}\cdot t) - 2\sinh 4t$$

$$t^2 f(t) = 8t\cosh 4t - 2\sinh 4t$$

Thus $\quad f(t) = \dfrac{2(4t\cosh 4t - \sinh 4t)}{t^2}$

121. Let $\bar{f}(s) = \cot^{-1}\left(\dfrac{s+a}{b}\right)$

$\therefore \quad \bar{f}'(s) = \dfrac{-1}{1+\dfrac{(s+a)^2}{b^2}} \cdot \dfrac{1}{b} = \dfrac{-b}{b^2+(s+a)^2}$

Now, $\quad L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\dfrac{b}{(s+a)^2+b^2}\right] = e^{-at} L^{-1}\left[\dfrac{b}{s^2+b^2}\right]$

ie., $\quad tf(t) = e^{-at}\sin bt$

Thus $\quad f(t) = \dfrac{e^{-at}\sin bt}{t}$

122. *Find* (a) $L^{-1}\left[\dfrac{1}{s}\sin\left(\dfrac{1}{s}\right)\right]$ $\quad$ (b) $L^{-1}\left[\dfrac{1}{s}\cos\left(\dfrac{1}{s}\right)\right]$

>> Inverse transforms of $\sin(1/s)$ and $\cos(1/s)$ is not readily available. We consider the expansion of $\sin x$ and $\cos x$ in the neighbourhood of the origin (*ie.*, $x = 0$) which are given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \; ; \; \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Replacing x by $1/s$ where we shall assume that s is large in which case $x = 1/s \to 0$

$\therefore \quad \sin\left(\dfrac{1}{s}\right) = \dfrac{1}{s} - \dfrac{1}{3!}\dfrac{1}{s^3} + \dfrac{1}{5!}\dfrac{1}{s^5} - \cdots$ and $\cos\left(\dfrac{1}{s}\right) = 1 - \dfrac{1}{2!}\dfrac{1}{s^2} + \dfrac{1}{4!}\dfrac{1}{s^4} - \cdots$

Hence $\dfrac{1}{s}\sin\left(\dfrac{1}{s}\right) = \dfrac{1}{s^2} - \dfrac{1}{3!}\dfrac{1}{s^4} + \dfrac{1}{5!}\dfrac{1}{s^6} - \cdots$ and $\dfrac{1}{s}\cos\left(\dfrac{1}{s}\right) = \dfrac{1}{s} - \dfrac{1}{2!}\dfrac{1}{s^3} + \dfrac{1}{4!}\dfrac{1}{s^5} - \cdots$

$\therefore \quad L^{-1}\left[\dfrac{1}{s}\sin\left(\dfrac{1}{s}\right)\right] = \dfrac{t^1}{1!} - \dfrac{1}{3!}\dfrac{t^3}{3!} + \dfrac{1}{5!}\dfrac{t^5}{5!} - \cdots$

and $\quad L^{-1}\left[\dfrac{1}{s}\cos\left(\dfrac{1}{s}\right)\right] = 1 - \dfrac{1}{2!}\dfrac{t^2}{2!} + \dfrac{1}{4!}\dfrac{t^4}{4!} - \cdots$

Thus $\quad L^{-1}\left[\dfrac{1}{s}\sin\left(\dfrac{1}{s}\right)\right] = t - \dfrac{t^3}{(3!)^2} + \dfrac{t^5}{(5!)^2} - \cdots$

and $\quad L^{-1}\left[\dfrac{1}{s}\cos\left(\dfrac{1}{s}\right)\right] = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \cdots$

19. $3\cdot u(t-3)-(t-1)u(t-1)$

20. $5u(t-3)-u(t-1)$

21. $\dfrac{2(1-\cos at)}{t}$

22. $\dfrac{1+e^{-t}-2\cos t}{t}$

23. $\dfrac{2(e^{4t}-\cos 2t)}{t}$

24. $\dfrac{te^{-2t}\sin t}{2}$

25. $\dfrac{\sin at}{t}$

5.3 Convolution

Definition : The convolution of two functions $f(t)$ and $g(t)$ usually denoted by $f(t)*g(t)$ is defined in the form of an integral as follows:

$$f(t)*g(t)=\int_{u=0}^{t} f(u)g(t-u)\,du$$

Property : $f(t)*g(t)=g(t)*f(t)$

*That is to say that the convolution operation * is commutative.*

Proof : We have from the definition of convolution

$$f(t)*g(t)=\int_{u=0}^{t} f(u)g(t-u)\,du \qquad \ldots(1)$$

Put, $t-u=v$ in (1) $\quad\therefore\ -du=dv$ or $du=-dv$

If $\quad u=0,\ v=t$; If $u=t,\ v=0$. Also $t-v=u$

Hence, $f(t)*g(t)=\int_{v=t}^{0} f(t-v)g(v)(-dv)$

ie., $\quad f(t)*g(t)=\int_{v=0}^{t} f(t-v)g(v)\,dv$ or $\int_{v=0}^{t} g(v)f(t-v)\,dv$

Comparing the R.H.S with (1) we have,

$$f(t)*g(t)=g(t)*f(t)$$

This proves that the operation * is commutative.

5.31 Convolution theorem

If $L^{-1}[\bar{f}(s)]=f(t)$ *and* $L^{-1}[\bar{g}(s)]=g(t)$ *then*

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)]=\int_{u=0}^{t} f(u)g(t-u)\,du$$

Proof : We shall show that

$$L\left[\int_{u=0}^{t} f(u)g(t-u)\,du\right]=\bar{f}(s)\cdot\bar{g}(s)$$

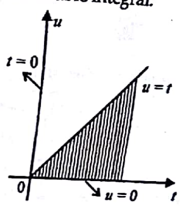
We have LHS by the definition

$$\text{LHS}=\int_{t=0}^{\infty} e^{-st}\left[\int_{u=0}^{t} f(u)g(t-u)\,du\right]dt$$

$$\text{LHS}=\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st} f(u)g(t-u)\,du\,dt$$

We shall change the order of integration in respect of this double integral.

Existing region :



$t=0$ to ∞ (Horizontal strip)

$u=0$ to t (Vertical strip)

On changing the order :

$u=0$ to ∞ (Vertical strip)

$t=u$ to ∞ (Horizontal strip)

On changing the order of integration, (1) becomes

$$\text{LHS}=\int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)g(t-u)\,dt\,du$$

Now, let us put $t-u=v$ where u is fixed $\quad\therefore\ dt=dv$

If $t=u,\quad v=0$; If $t=\infty\ \ v=\infty$ and hence (2) becomes

$$\text{LHS}=\int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-s(u+v)} f(u)g(v)\,dv\,du$$

$$\text{LHS} = \left[\int_{u=0}^{\infty} e^{-su} f(u)\,du\right]\cdot\left[\int_{v=0}^{\infty} e^{-sv} g(v)\,dv\right]$$

$$\text{LHS} = \bar{f}(s)\cdot\bar{g}(s) = \text{RHS}$$

Hence we have proved that

$$L\left[\int_{u=0}^{t} f(u)g(t-u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s)$$

Thus $L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$

This proves the convolution theorem.

Remarks :

1. *The integral in RHS is the convolution of* $f(t)$ *and* $g(t)$ *denoted by* $f(t)*g(t)$

Thus the convolution theorem can be put in the form

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t)*g(t)$$

2. *Since we have proved that* $f(t)*g(t) = g(t)*f(t)$,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{u=0}^{t} f(t-u)g(u)\,du$$

The integral in either of the forms is called as the convolution integral.

WORKED PROBLEMS

Set – 12

Type-1 ***Verification of the convolution theorem -***

Working procedure for problems

We need to verify the theorem in respect of the two given functions $f(t)$ and $g(t)$

Step-1 We find $\bar{f}(s) = L[f(t)]$ and $\bar{g}(s) = L[g(t)]$

Step-2 We evaluate $f(t)*g(t) = \int_0^t f(u)g(t-u)\,du$

Step-3 We find $L[f(t)*g(t)]$.

Step-4 If $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$, the theorem is verified.

Verify convolution theorem for the following pair of functions

123. $f(t) = t$ *and* $g(t) = \cos t$

124. $f(t) = \sin t$ *and* $g(t) = e^{-t}$

125. $f(t) = \cos at$ *and* $g(t) = \cos bt$

126. $f(t) = t$ *and* $g(t) = te^{-t}$

123. $\bar{f}(s) = L[f(t)] = L(t) = \dfrac{1}{s^2}$

$$\bar{g}(s) = L[g(t)] = L(\cos t) = \frac{s}{s^2+1}$$

$$f(t)*g(t) = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_0^t u\cos(t-u)\,du$$

Applying Bernoulli's rule to the integral we get,

$$f(t)*g(t) = [u\cdot -\sin(t-u)]_{u=0}^{t} - [1\cdot -\cos(t-u)]_{u=0}^{t}$$

$$f(t)*g(t) = -(0-0)+(1-\cos t) = 1-\cos t$$

$\therefore$ $L[f(t)*g(t)] = L(1) - L(\cos t) = \dfrac{1}{s} - \dfrac{s}{s^2+1} = \dfrac{1}{s(s^2+1)}$

Also $\bar{f}(s)\cdot\bar{g}(s) = \dfrac{1}{s^2}\cdot\dfrac{s}{s^2+1} = \dfrac{1}{s(s^2+1)}$

Thus $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$. The theorem is verified.

124. $\bar{f}(s) = L(\sin t) = \dfrac{1}{s^2+1}$, $\bar{g}(s) = L(e^{-t}) = \dfrac{1}{s+1}$

$$f(t)*g(t) = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{u=0}^{t} \sin u\cdot e^{-(t-u)}\,du$$

? $= e^{-t}\int_{u=0}^{t} e^u \sin u\,du$

$$= e^{-t}\left[\frac{e^u}{1+1}(\sin u - \cos u)\right]_{u=0}^{t}$$

$$= \frac{e^{-t}}{2}[e^t(\sin t - \cos t)+1] = \frac{\sin t - \cos t}{2} + \frac{e^{-t}}{2}$$

$$\therefore \quad L[f(t)*g(t)] = \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{1}{s+1}\right]$$

$$= \frac{1}{2}\left[\frac{(s+1)-s(s+1)+(s^2+1)}{(s+1)(s^2+1)}\right] = \frac{1}{(s+1)(s^2+1)}$$

Also $\bar{f}(s)\cdot\bar{g}(s) = \dfrac{1}{(s^2+1)(s+1)}$

Thus $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$. The theorem is verified.

---

125. $\bar{f}(s) = L(\cos at) = \dfrac{s}{s^2+a^2}$; $\bar{g}(s) = L(\cos bt) = \dfrac{s}{s^2+b^2}$

$$f(t)*g(t) = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{u=0}^{t} \cos au \cdot \cos(bt-bu)\,du.$$

$$f(t)*g(t) = \frac{1}{2}\int_{u=0}^{t}[\cos(au+bt-bu)+\cos(au-bt+bu)]\,du$$

$$= \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b} + \frac{\sin(au-bt+bu)}{a+b}\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left[\frac{1}{a-b}\{\sin at - \sin bt\} + \frac{1}{a+b}\{\sin at + \sin bt\}\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b}+\frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b}-\frac{1}{a-b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\cdot\frac{2a}{a^2-b^2} + \sin bt\cdot\frac{-2b}{a^2-b^2}\right]$$

$$= \frac{1}{a^2-b^2}(a\sin at - b\sin bt),\ a \neq b$$

$$\therefore \quad L[f(t)*g(t)] = \frac{1}{a^2-b^2}\left[a\cdot\frac{a}{s^2+a^2} - b\cdot\frac{b}{s^2+b^2}\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{a^2(s^2+b^2)-b^2(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}\right]$$

ie., $$L[f(t)*g(t)] = \frac{1}{a^2-b^2}\cdot\frac{s^2(a^2-b^2)}{(s^2+a^2)(s^2+b^2)} = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Also $\bar{f}(s)\cdot\bar{g}(s) = \dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Thus $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$. The theorem is verified.

---

126. $\bar{f}(s) = L(t) = \dfrac{1}{s^2}$, $\bar{g}(s) = L(e^{-t}\cdot t) = \dfrac{1}{(s+1)^2}$

$$f(t)*g(t) = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{u=0}^{t} u\,e^{-(t-u)}(t-u)\,du$$

$$= e^{-t}\int_{u=0}^{t}(tu-u^2)e^u\,du$$

$$= e^{-t}[(tu-u^2)e^u - (t-2u)e^u + (-2)e^u]_{u=0}^{t}$$

$$= e^{-t}[(0-0)-(-te^t-t)-2(e^t-1)]$$

$$= t + te^{-t} - 2 + 2e^{-t}$$

$$L[f(t)*g(t)] = \frac{1}{s^2} + \frac{1}{(s+1)^2} - \frac{2}{s} + \frac{2}{s+1}$$

$$= \frac{(s+1)^2 + s^2 - 2s(s+1)^2 + 2s^2(s+1)}{s^2(s+1)^2} = \frac{1}{s^2(s+1)^2}$$

Thus $L[f(t)*g(t)] = \bar{f}(s)\cdot\bar{g}(s)$. The theorem is verified.

Type-2 ***Computation of the inverse transform by using convolution theorem***

Working procedure for problems.

Step-1 The given function is expressed as the product of two functions say $\bar{f}(s)$ and $\bar{g}(s)$

Step-2 We find $L^{-1}[\bar{f}(s)] = f(t)$ and $L^{-1}[\bar{g}(s)] = g(t)$

Step-3 We apply the convolution theorem in one of the form :

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

Step-4 We evaluate the convolution integral to obtain the required inverse.

Using convolution theorem obtain the inverse Laplace transform of the following functions

127. $\frac{1}{s(s^2+a^2)}$

128. $\frac{s}{(s^2+a^2)^2}$

129. $\frac{s^2}{(s^2+a^2)^2}$

130. $\frac{1}{(s^2+a^2)^2}$

131. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

132. $\frac{1}{(s-1)(s^2+1)}$

133. $\frac{1}{s^2(s+1)^2}$

134. $\frac{s+2}{(s^2+4s+5)^2}$

135. $\frac{1}{(s^2+4s+13)^2}$

136. $\frac{4s+5}{(s-1)^2(s+2)}$

127. Let $\bar{f}(s) = \frac{1}{s}$; $\bar{g}(s) = \frac{1}{s^2+a^2}$

Taking inverse,

$$f(t) = L^{-1}\left[\frac{1}{s}\right] = 1 \; ; \; g(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$$

We have convolution theorem,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

$$\therefore \quad L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_{u=0}^{t} 1\cdot\frac{\sin(at-au)}{a}\,du$$

$$= \left[\frac{\cos(at-au)}{a^2}\right]_{u=0}^{t} = \frac{1}{a^2}(1-\cos at)$$

Thus $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}(1-\cos at)$

128. Let $\bar{f}(s) = \frac{1}{s^2+a^2}$; $\bar{g}(s) = \frac{s}{s^2+a^2}$

$$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \frac{\sin at}{a}; \; g(t) = L^{-1}[\bar{g}(s)] = \cos at$$

We have convolution theorem,

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

$$\therefore \quad L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{t}\frac{\sin au}{a}\cdot\cos(at-au)\,du$$

$$= \frac{1}{2a}\int_{u=0}^{t}[\sin(au+at-au)+\sin(au-at+au)]\,du$$

$$= \frac{1}{2a}\int_{u=0}^{t}[\sin at+\sin(2au-at)]\,du$$

$$= \frac{1}{2a}\left\{\sin at\,[u]_{u=0}^{t} - \left[\frac{\cos(2au-at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2a}\left\{\sin at\,(t-0) - \frac{1}{2a}(\cos at-\cos at)\right\} = \frac{t\sin at}{2a}$$

Thus $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$

129. Let $\bar{f}(s) = \frac{s}{s^2+a^2} = \bar{g}(s)$

$$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \cos at = g(t)$$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_{u=0}^{t}\cos au\,\cos(at-au)\,du$$

$$= \frac{1}{2}\int_{u=0}^{t}[\cos(au+at-au)+\cos(au-at+au)]\,du$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\int_{u=0}^{t}[\cos at + \cos(2au - at)]\,du$$

$$= \frac{1}{2}\left\{\cos at\,[u]_0^t + \left[\frac{\sin(2au - at)}{2a}\right]_0^t\right\}$$

$$= \frac{1}{2}\left\{\cos at\,(t-0) + \frac{1}{2a}(\sin at + \sin at)\right\} = \frac{1}{2}\left\{t\cos at + \frac{\sin at}{a}\right\}$$

Thus $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}(at\cos at + \sin at)$

130. Let $\bar{f}(s) = \frac{1}{s^2+a^2} = \bar{g}(s)$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \frac{\sin at}{a} = g(t)$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \int_{u=0}^{t}\frac{\sin au}{a}\cdot\frac{\sin(at-au)}{a}\,du$$

$$= \frac{1}{2a^2}\int_{u=0}^{t}[\cos(au - at + au) - \cos(au + at - au)]\,du$$

$$= \frac{1}{2a^2}\int_{u=0}^{t}[\cos(2au - at) - \cos at]\,du$$

$$= \frac{1}{2a^2}\left\{\left[\frac{\sin(2au - at)}{2a}\right]_0^t - \cos at\,[u]_0^t\right\}$$

$$= \frac{1}{2a^2}\left\{\frac{1}{2a}(\sin at + \sin at) - \cos at\cdot t\right\}$$

Thus $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

131. Let $\bar{f}(s) = \frac{s}{s^2+a^2}$; $\bar{g}(s) = \frac{s}{s^2+b^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = \cos at$; $g(t) = L^{-1}[\bar{g}(s)] = \cos bt$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_{u=0}^{t}\cos au\cdot\cos(bt - bu)\,du$$

[Refer problem-125 for the integration process]

Thus $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2},\ a\neq b$

132. Let $\bar{f}(s) = \frac{1}{s-1}$; $\bar{g}(s) = \frac{1}{s^2+1}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = e^t$; $g(t) = L^{-1}[\bar{g}(s)] = \sin t$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_{u=0}^{t}e^u\cdot\sin(t-u)\,du$$

But $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \left[\frac{e^u}{1+1}\{\sin(t-u) + \cos(t-u)\}\right]_0^t$$

$$= \frac{1}{2}\{e^t(0+1) - 1(\sin t + \cos t)\}$$

Thus $L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{1}{2}(e^t - \sin t - \cos t)$

133. Let $\bar{f}(s) = \frac{1}{s^2}$; $\bar{g}(s) = \frac{1}{(s+1)^2}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = t$; $g(t) = L^{-1}[\bar{g}(s)] = e^{-t}t.$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = \int_{u=0}^{t} u\,e^{-(t-u)}(t-u)\,du = e^{-t}\int_{u=0}^{t}(tu-u^2)\,e^{u}\,du$$

[Refer Problem-54 for the integration process]

Thus $L^{-1}\left[\dfrac{1}{s^2(s+1)^2}\right] = t+te^{-t}-2+2e^{-t} = 2(e^{-t}-1)+t(1+e^{-t})$

134. Let $\bar{f}(s) = \dfrac{s+2}{s^2+4s+5}$; $\bar{g}(s) = \dfrac{1}{s^2+4s+5}$

$\Rightarrow$ $f(t) = L^{-1}\left[\dfrac{s+2}{(s+2)^2+1}\right]$; $g(t) = L^{-1}\left[\dfrac{1}{(s+2)^2+1}\right]$

$f(t) = e^{-2t}L^{-1}\left[\dfrac{s}{s^2+1}\right]$; $g(t) = e^{-2t}L^{-1}\left[\dfrac{1}{s^2+1}\right]$

$\therefore$ $f(t) = e^{-2t}\cos t$; $g(t) = e^{-2t}\sin t$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \int_{u=0}^{t} e^{-2u}\cos u\, e^{-2(t-u)}\sin(t-u)\,du$$

$$= e^{-2t}\int_{u=0}^{t}\sin(t-u)\cos u\,du$$

$$= \frac{e^{-2t}}{2}\int_{u=0}^{t}[\sin(t-u+u)+\sin(t-u-u)]\,du$$

$$= \frac{e^{-2t}}{2}\int_{u=0}^{t}[\sin t+\sin(t-2u)]\,du$$

$$= \frac{e^{-2t}}{2}\left\{\sin t\,[u]_0^t + \frac{\cos(t-2u)}{2}\Big|_{u=0}^{t}\right\}$$

$$= \frac{e^{-2t}}{2}\left\{t\sin t+\frac{1}{2}(\cos t-\cos t)\right\}$$

Thus $L^{-1}\left[\dfrac{s+2}{(s^2+4s+5)^2}\right] = \dfrac{e^{-2t}\,t\sin t}{2}$

135. Let $\bar{f}(s) = \dfrac{1}{s^2+4s+13} = \bar{g}(s)$

$\Rightarrow$ $f(t) = L^{-1}\left[\dfrac{1}{(s+2)^2+3^2}\right] = g(t)$

ie., $f(t) = e^{-2t}L^{-1}\left[\dfrac{1}{s^2+3^2}\right] = \dfrac{e^{-2t}\sin 3t}{3} = g(t)$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \int_{u=0}^{t}\frac{e^{-2u}\sin 3u}{3}\cdot\frac{e^{-2(t-u)}\sin(3t-3u)}{3}\,du$$

$$= \frac{e^{-2t}}{9}\int_{u=0}^{t}\sin 3u\cdot\sin(3t-3u)\,du$$

$$= \frac{e^{-2t}}{18}\int_{u=0}^{t}[\cos(3u-3t+3u)-\cos(3u+3t-3u)]\,du$$

$$= \frac{e^{-2t}}{18}\int_{u=0}^{t}[\cos(6u-3t)-\cos 3t]\,du$$

$$= \frac{e^{-2t}}{18}\left\{\left[\frac{\sin(6u-3t)}{6}\right]_0^t - \cos 3t\,[u]_0^t\right\}$$

$$= \frac{e^{-2t}}{18}\left\{\frac{\sin 3t+\sin 3t}{6} - \cos 3t\cdot t\right\}$$

Thus $L^{-1}\left[\dfrac{1}{(s^2+4s+13)^2}\right] = \dfrac{e^{-2t}}{54}(\sin 3t-3t\cos 3t)$

136. Let $\bar{f}(s) = \dfrac{1}{s+2}$; $\bar{g}(s) = \dfrac{4s+5}{(s-1)^2}$

$\Rightarrow$ $f(t) = L^{-1}[\bar{f}(s)] = e^{-2t}$

Also, $g(t) = L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{4s+5}{(s-1)^2}\right] = L^{-1}\left[\dfrac{4(s-1)+9}{(s-1)^2}\right]$

$$g(t) = e^t L^{-1}\left[\frac{4s+9}{s^2}\right] = e^t(4+9t)$$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{s+2}\cdot\frac{4s+5}{(s-1)^2}\right] = \int_{u=0}^{t} e^{-2u}\,.\,e^{(t-u)}[4+9(t-u)]\,du$$

$$= e^t\int_{u=0}^{t} e^{-3u}(4+9t-9u)\,du$$

$$= e^t\int_{u=0}^{t} (4+9t-9u)\,e^{-3u}\,du$$

Integrating R.H.S by parts we get,

$$L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = e^t\left\{(4+9t-9u)\frac{e^{-3u}}{-3}-(-9)\frac{e^{-3u}}{9}\right\}_{u=0}^{t}$$

$$= e^t\left\{4\frac{e^{-3t}}{-3}-\frac{(4+9t)}{-3}+e^{-3t}-1\right\}$$

$$= e^t\left\{\frac{1}{3}-\frac{1}{3}e^{-3t}+3t\right\}$$

Thus $L^{-1}\left[\dfrac{4s+5}{(s-1)^2(s+2)}\right] = \dfrac{1}{3}e^t-\dfrac{1}{3}e^{-2t}+3\,t\,e^t$

Remark.

It is advisable to remember the following three inverse Laplace transforms.

(i) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right] = \dfrac{t\sin at}{2a}$

(ii) $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right] = \dfrac{1}{2a}(\sin at+at\cos at)$

(iii) $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = \dfrac{1}{2a^3}(\sin at-at\cos at)$

Type-3 ***Laplace transform of the convolution integral and solution of integral equations***

Working procedure for problems

Step-1 We can find the Laplace transform of the convolution integral by using the result

$$L\left[\int_0^t f(u)\,g(t-u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s) = L\left[\int_0^t f(t-u)\,g(u)\,du\right]$$

in the appropriate form.

Step-2 Given an equation for $f(t)$ involving the convolution integral we first take Laplace transform on both sides.

Step-3 We evaluate the convolution integral and simplify to obtain $L[f(t)] = \bar{f}(s)$ as a function of s.

Step-4 Taking inverse we obtain $f(t)$

Find the Laplace transform of the following convolution integrals.

137. $\int_0^t (t-u)\sin 2u\,du$

138. $\int_0^t e^{-u}\sin(t-u)\,du$

139. $\int_0^t \sin a(t-u)\cos au\,du$

140. $\int_0^t (t-u)\,u\,e^{-2u}\,du$

137. To find $L\left[\int_0^t (t-u)\sin 2u\,du\right]$ we use the result

$$L\left[\int_0^t f(t-u)\,g(u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots(1)$$

By comparing, we have $f(t-u) = (t-u)$ and $g(u) = \sin 2u$

Equivalently, $f(t) = t$ and $g(t) = \sin 2t$

$\therefore\quad \bar{f}(s) = L(t) = \dfrac{1}{s^2}$ and $\bar{g}(s) = L(\sin 2t) = \dfrac{2}{s^2+4}$

Thus from (1) we get,,

$$L\left[\int_0^t (t-u)\sin 2u\,du\right] = \frac{1}{s^2}\cdot\frac{2}{s^2+4} = \frac{2}{s^2(s^2+4)}$$

138. *To find* $L\left[\int_0^t e^{-u}\sin(t-u)\,du\right]$ *we use the result*

$$L\left[\int_0^t f(u)\,g(t-u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots(1)$$

By comparing, we have, $f(u) = e^{-u}$ and $g(t-u) = \sin(t-u)$

Equivalently, $f(t) = e^{-t}$ and $g(t) = \sin t$

$\therefore \quad \bar{f}(s) = L(e^{-t}) = \dfrac{1}{s+1}$ and $\bar{g}(s) = L(\sin t) = \dfrac{1}{s^2+1}$

Thus from (1) we get,

$$L\left[\int_0^t e^{-u}\sin(t-u)\,du\right] = \frac{1}{(s+1)(s^2+1)}$$

---

139. *To find* $L\left[\int_0^t \sin a(t-u)\cos au\,du\right]$ *we use the result*

$$L\left[\int_0^t f(t-u)\,g(u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots(1)$$

By comparing we have, $f(t-u) = \sin a(t-u)$ and $g(u) = \cos au$

Equivalently, $f(t) = \sin at$ and $g(t) = \cos at$

$\therefore \quad \bar{f}(s) = L(\sin at) = \dfrac{a}{s^2+a^2}$ and $\bar{g}(s) = L(\cos at) = \dfrac{s}{s^2+a^2}$

Thus from (1) we get,

$$L\left[\int_0^t \sin a(t-u)\cos au\,du\right] = \frac{as}{(s^2+a^2)^2}$$

---

140. *To find* $L\left[\int_0^t (t-u)\,u\,e^{-2u}\,du\right]$ *we use the result*

$$L\left[\int_0^t f(t-u)\,g(u)\,du\right] = \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots(1)$$

By comparing we have, $f(t-u) = (t-u)$ and $g(u) = u\,e^{-2u}$

Equivalently, $f(t) = t$ and $g(t) = t\,e^{-2t}$

$\therefore \quad \bar{f}(s) = L(t) = \dfrac{1}{s^2}$ and $\bar{g}(s) = L(e^{-2t}\cdot t) = \dfrac{1}{(s+2)^2}$

Thus from (1) we get,

$$L\left[\int_0^t (t-u)\,u\,e^{-2u}\,du\right] = \frac{1}{s^2(s+2)^2}$$

---

141. *Find* $f(t)$ *from the equation* $f(t) = 1 + 2\int_0^t f(t-u)\,e^{-2u}\,du$

>> Taking Laplace transform on both sides we have,

$$L[f(t)] = L(1) + 2L\left[\int_0^t f(t-u)\,e^{-2u}\,du\right]$$

ie., $\quad \bar{f}(s) = \dfrac{1}{s} + 2L\left[\int_0^t f(t-u)\,e^{-2u}\,du\right]$

ie., $\quad \bar{f}(s) = \dfrac{1}{s} + 2\bar{f}(s)\cdot\bar{g}(s) \qquad \ldots(1)$

where $\bar{g}(s) = L[g(u)] = L(e^{-2u}) = \dfrac{1}{s+2}$

Hence (1) becomes

$$\bar{f}(s) = \frac{1}{s} + 2\bar{f}(s)\cdot\frac{1}{s+2}$$

or $\quad \bar{f}(s)\left[1 - \dfrac{2}{s+2}\right] = \dfrac{1}{s}$ or $\bar{f}(s)\left[\dfrac{s}{s+2}\right] = \dfrac{1}{s}$

$\therefore \quad \bar{f}(s) = \frac{s+2}{s^2} = \frac{1}{s} + \frac{2}{s^2}$

By taking inverse we have,

$$L^{-1}[\bar{f}(s)] = L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right)$$

Thus $f(t) = 1 + 2t$

142. *Solve the integral equation* : $f(t) = 1 + \int_0^t f(u)\sin(t-u)\,du$

>> Taking Laplace transform on bothsides we have,

$$L[f(t)] = L(1) + L\left[\int_0^t f(u)\sin(t-u)\,du\right]$$

ie., $\quad \bar{f}(s) = \frac{1}{s} + \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots(1)$

where $g(t-u) = \sin(t-u)$ or $g(t) = \sin t \quad \therefore \quad \bar{g}(s) = \frac{1}{s^2+1}$

Hence (1) becomes,

$$\bar{f}(s) = \frac{1}{s} + \bar{f}(s)\cdot\frac{1}{s^2+1} \quad \text{or} \quad \bar{f}(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s}$$

ie., $\quad \bar{f}(s)\cdot\frac{s^2}{s^2+1} = \frac{1}{s} \quad$ or $\quad \bar{f}(s) = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}$

Now $\quad L^{-1}[\bar{f}(s)] = L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^3}\right)$

Thus $\quad f(t) = 1 + \frac{t^2}{2}$

143. *Find* $f(t)$ *from the equation* $f(t) = \frac{t^2}{2} - \int_0^t f(u)(t-u)\,du$

>> Taking Laplace transform on both sides we have,

$$L[f(t)] = L\left(\frac{t^2}{2}\right) - L\left[\int_0^t f(u)\,(t-u)\,du\right]$$

ie., $\quad \bar{f}(s) = \frac{1}{s^3} - \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots(1)$

where, $g(t-u) = (t-u)$ or $g(t) = t \quad \therefore \quad \bar{g}(s) = \frac{1}{s^2}$

Hence (1) becomes,

$$\bar{f}(s) = \frac{1}{s^3} - \frac{\bar{f}(s)}{s^2} \quad \text{or} \quad \bar{f}(s)\left[1 + \frac{1}{s^2}\right] = \frac{1}{s^3}$$

ie., $\quad \bar{f}(s)\cdot\frac{s^2+1}{s^2} = \frac{1}{s^3} \quad$ or $\quad \bar{f}(s) = \frac{1}{s(s^2+1)}$

Now $\quad L^{-1}[\bar{f}(s)] = f(t) = L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

But $\quad \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$ by partial fractions.

$$\therefore \quad L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+1}\right)$$

Thus $\quad f(t) = 1 - \cos t = 2\sin^2(t/2)$

144. *Solve the integral equation* : $f(t) = 4t^2 - \int_0^t f(t-u)e^{-u}\,du$

>> Taking Laplace transform on both sides we have,

$$L[f(t)] = 4L(t^2) - L\left[\int_0^t f(t-u)e^{-u}\,du\right]$$

ie., $\quad \bar{f}(s) = \frac{8}{s^3} - \bar{f}(s)\cdot\bar{g}(s) \qquad \ldots(1)$

where $g(u) = e^{-u}$ and hence $\bar{g}(s) = L(e^{-u}) = \frac{1}{s+1}$

Now (1) becomes,

$$\bar{f}(s) = \frac{8}{s^3} - \frac{\bar{f}(s)}{s+1} \quad \text{or} \quad \bar{f}(s)\left[1 + \frac{1}{s+1}\right] = \frac{8}{s^3}$$

ie., $\quad \bar{f}(s)\cdot\frac{s+2}{s+1} = \frac{8}{s^3} \quad$ or $\quad \bar{f}(s) = \frac{8(s+1)}{s^3(s+2)}$

Further, $L^{-1}[\bar{f}(s)] = f(t) = L^{-1}\left[\frac{8(s+1)}{s^3(s+2)}\right] \qquad \ldots(2)$

Let $\dfrac{8s+8}{s^3(s+2)} = \dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{C}{s^3}+\dfrac{D}{s+2}$

or $8s+8 = As^2(s+2)+Bs(s+2)+C(s+2)+Ds^3$ ...(3)

Put $s=0$: $8 = C(2)$ $\therefore$ $C = 4$

Put $s=-2$: $-8 = D(-8)$ $\therefore$ $D = 1$

Equating the coefficients of s^3, s^2 separately on both sides of (3) we get,

$0 = A+D$ and $0 = 2A+B$. $\therefore$ $A = -1$ and $B = 2$

Now $L^{-1}\left[\dfrac{8s+8}{s^3(s+2)}\right] = -L^{-1}\left(\dfrac{1}{s}\right)+2L^{-1}\left(\dfrac{1}{s^2}\right)+4L^{-1}\left(\dfrac{1}{s^3}\right)+L^{-1}\left(\dfrac{1}{s+2}\right)$

Thus $f(t) = -1+2t+2t^2+e^{-2t}$.

5.4 Solution of linear differential equations using Laplace transforms (Initial value problems)

Laplace transform of the derivatives

We derive an expression for $L[y'(t)]$ and hence deduce the expressions for $L[y''(t)]$, $L[y'''(t)]$...

Further we use the principle of mathematical induction to establish the result for $L[y^{(n)}(t)]$

$$L[y'(t)] = \int_0^\infty e^{-st}y'(t)\,dt$$

Integrating by parts we have,

$$L[y'(t)] = \left[e^{-st}y(t)\right]_{t=0}^{\infty} - \int_0^\infty y(t)\cdot e^{-st}(-s)\,dt$$

$$= [0-1\cdot y(0)]+s\int_0^\infty e^{-st}y(t)\,dt$$

$$= -y(0)+sL[y(t)]$$

Thus $L[y'(t)] = sL[y(t)]-y(0)$ ...(1)

Now, $L[y''(t)] = L\left[\{y'(t)\}'\right]$ and applying (1) we have,

$= sL[y'(t)]-y'(0)$

ie., $= s\{sL[y(t)]-y(0)\}-y'(0)$ by using (1).

$\therefore$ $L[y''(t)] = s^2L[y(t)]-sy(0)-y'(0)$ ...(2)

Also, $L[y'''(t)] = s^3L[y(t)]-s^2y(0)-sy'(0)-y''(0)$ ...(3)

In general we shall show that

$$L[y^{(n)}(t)] = s^nL[y(t)]-s^{n-1}y(0)-s^{n-2}y'(0)-\cdots-y^{(n-1)}(0)$$

The result is established by induction.

When $n=1$ we have

$$L[y'(t)] = s^1L[y(t)]-s^0y(0) = sL[y(t)]-y(0) \quad \text{...(i)}$$

Comparing with (1) we conclude that the result is true for $n=1$. Let us assume the result to be true for $n=k$.

$$L[y^{(k)}(t)] = s^kL[y(t)]-s^{k-1}y(0)-s^{k-2}y'(0)-\cdots-y^{(k-1)}(0) \quad \text{...(ii)}$$

$L[y^{(k+1)}(t)] = L\left[\{y^{(k)}(t)\}'\right] = sL[y^{(k)}(t)]-y^{(k)}(0)$ by (i).

Using (ii) in the RHS for $L[y^{(k)}(t)]$ we have,

$$L[y^{(k+1)}(t)] = s\left\{s^kL[y(t)]-s^{k-1}y(0)-s^{k-2}y'(0) -\cdots-y^{(k-1)}(0)\right\}-y^{(k)}(0)$$

ie., $= s^{k+1}L[y(t)]-s^ky(0)-s^{k-1}y'(0)-\ldots-sy^{(k-1)}(0)-y^{(k)}(0)$

ie., $L[y^{(k+1)}(t)] = s^{k+1}L[y(t)]-s^{(k+1)-1}y(0)-s^{(k+1)-2}y'(0)-\ldots-y^{(k+1)-1}(0)$ ... (iii)

Comparing (ii) and (iii) we conclude that the result is true for $n=k+1$. Thus by the principle of induction the result is true for all positive integral values of n.

We have already said that a differential equation with a set of initial conditions is called an initial value problem. However if the boundary conditions are given the problem is called a boundary value problem. Laplace transform serves as a useful tool in solving such problems.

Working procedure for problems

Step-1 The given differential equation is expressed in the notation :

$y'(t), y''(t), y'''(t)$ ... for the derivatives.

Step-2 We take Laplace transform on both sides of the given equation.

Step-3 We use the expressions for $L[y'(t)], L[y''(t)]$ ...

Step-4 We substitute the given initial conditions and simplify to obtain $L[y(t)]$ as a function of s.

Step-5 We find the inverse to obtain $y(t)$.

WORKED PROBLEMS
Set – 13

145. *Solve by using Laplace transforms* : $\dfrac{d^2y}{dt^2}+k^2y=0$ *given that* $y(0)=2,\ y'(0)=0$

\>> The given equation is $y''(t)+k^2y(t)=0$

Taking Laplace transform on both sides we have,

$$L[y''(t)]+k^2L[y(t)]=L(0)$$

ie., $$\{s^2L[y(t)]-sy(0)-y'(0)\}+k^2L[y(t)]=0$$

Using the given initial conditions we obtain,

$$(s^2+k^2)\,L[y(t)]-2s=0 \quad\text{or}\quad L[y(t)]=\frac{2s}{s^2+k^2}$$

$\therefore$ $$y(t)=2L^{-1}\left[\frac{s}{s^2+k^2}\right]=2\cos kt$$

Thus $\boldsymbol{y(t)=2\cos kt}$

146. *Solve* $y'''+2y''-y'-2y=0$ *given* $y(0)=y'(0)=0$ *and* $y''(0)=6$ *by using Laplace transform method.*

\>> Taking Laplace transform on both sides of the given equation,

$$L[y'''(t)]+2L[y''(t)]-L[y'(t)]-2L[y(t)]=L(0)$$

ie., $$\{s^3L[y(t)]-s^2y(0)-sy'(0)-y''(0)\}+2\{s^2L[y(t)]-sy(0)-y'(0)\}$$
$$-\{sL[y(t)]-y(0)\}-2L[y(t)]=0$$

Using the given initial conditions we obtain,

$$L[y(t)]\{s^3+2s^2-s-2\}-6=0$$

ie., $$L[y(t)]\{s^2(s+2)-1(s+2)\}=6$$

or $$L[y(t)]\{(s+2)(s^2-1)\}=6$$

or $$L[y(t)]=\frac{6}{(s+2)(s-1)(s+1)}$$

$\therefore$ $$y(t)=L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\}$$

Let $$\frac{6}{(s+2)(s-1)(s+1)}=\frac{A}{s+2}+\frac{B}{s-1}+\frac{C}{s+1}$$

or $$6=A(s-1)(s+1)+B(s+2)(s+1)+C(s+2)(s-1)$$

Put $s=-2$: $6=A(-3)(-1)$ $\therefore$ $A=2$

Put $s=1$: $6=B(3)(2)$ $\therefore$ $B=1$

Put $s=-1$: $6=C(1)(-2)$ $\therefore$ $C=-3$

Hence, $$\frac{6}{(s+2)(s-1)(s+1)}=\frac{2}{s+2}+\frac{1}{s-1}+\frac{-3}{s+1}$$

$\therefore$ $$L^{-1}\left\{\frac{6}{(s+2)(s-1)(s+1)}\right\}=2L^{-1}\left(\frac{1}{s+2}\right)+L^{-1}\left(\frac{1}{s-1}\right)-3L^{-1}\left(\frac{1}{s+1}\right)$$

Thus $\boldsymbol{y(t)=2e^{-2t}+e^{t}-3e^{-t}}$

76. *Solve* $\dfrac{d^4y}{dt^4}-k^4y=0$ *given* $y(0)=1$ *and* $y'(0)=0=y''(0)=y'''(0)$

\>> The given equation is $y^{(4)}(t)-k^4y(t)=0$

Taking Laplace transform on both sides we have,

$$L\left[y^{(4)}(t)\right]-k^4L\left[y(t)\right]=L(0)$$

ie., $$\{s^4L[y(t)]-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)\}-k^4L[y(t)]=0$$

Using the given initial conditions we obtain,

$$L[y(t)]\,[s^4-k^4]-s^3\cdot 1=0 \quad\text{or}\quad L[y(t)]=\frac{s^3}{s^4-k^4}$$

$\therefore$ $$y(t)=L^{-1}\left[\frac{s^3}{s^4-k^4}\right]$$

Now, $s^4-k^4 = (s^2-k^2)(s^2+k^2) = (s-k)(s+k)(s^2+k^2)$

Let $\dfrac{s^3}{(s-k)(s+k)(s^2+k^2)} = \dfrac{A}{(s-k)}+\dfrac{B}{(s+k)}+\dfrac{Cs+D}{s^2+k^2}$

ie., $s^3 = A(s+k)(s^2+k^2)+B(s-k)(s^2+k^2)+(Cs+D)(s-k)(s+k)\ldots(1)$

Put $s=k$: $k^3 = A(2k)(2k^2)$ $\therefore$ $A = 1/4$

Put $s=-k$: $-k^3 = B(-2k)(2k^2)$ $\therefore$ $B = 1/4$

Put $s=0$: $0 = (1/4)(k)(k^2)+(1/4)(-k)(k^2)+D(-k^2)$

ie., $0 = (k^3/4)-(k^3/4)+D(-k^2)$ $\therefore$ $D = 0$

Comparing the coefficient of s^3 on both sides of (1) we have

$1 = A+B+C$ *ie.,* $1 = 1/4+1/4+C$ $\therefore$ $C = 1/2$

Hence $\dfrac{s^3}{(s-k)(s+k)(s^2+k^2)} = \dfrac{1}{4}\dfrac{1}{s-k}+\dfrac{1}{4}\dfrac{1}{s+k}+\dfrac{1}{2}\dfrac{s}{s^2+k^2}$

$\therefore$ $L^{-1}\left[\dfrac{s^3}{s^4-k^4}\right] = \dfrac{1}{4}L^{-1}\left[\dfrac{1}{s-k}\right]+\dfrac{1}{4}L^{-1}\left[\dfrac{1}{s+k}\right]+\dfrac{1}{2}L^{-1}\left[\dfrac{s}{s^2+k^2}\right]$

ie., $y(t) = \dfrac{1}{4}e^{kt}+\dfrac{1}{4}e^{-kt}+\dfrac{1}{2}\cos kt = \dfrac{1}{4}(e^{kt}+e^{-kt})+\dfrac{1}{2}\cos kt$

ie., $y(t) = \dfrac{1}{4}(2\cos h\, kt)+\dfrac{1}{2}\cos kt.$

Thus $y(t) = \dfrac{1}{2}(\cosh kt+\cos kt)$

148. *Solve the following initial value problem by using Laplace transforms :*

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y = e^{-t};\ y(0) = 0,\ y'(0) = 0$$

>> The given equation is $y''(t)+4y'(t)+4y(t) = e^{-t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)]+4L[y'(t)]+4L[y(t)] = L(e^{-t})$$

ie., $\{s^2L[y(t)]-sy(0)-y'(0)\}+4\{sL[y(t)]-y(0)\}+4L[y(t)] = \dfrac{1}{s+1}$

Using the given initial conditions we obtain,

$L[y(t)]\{s^2+4s+4\} = \dfrac{1}{s+1}$ or $Ly(t) = \dfrac{1}{(s+1)(s+2)^2}$

$\therefore$ $y(t) = L^{-1}\left[\dfrac{1}{(s+1)(s+2)^2}\right]$

Let $\dfrac{1}{(s+1)(s+2)^2} = \dfrac{A}{s+1}+\dfrac{B}{s+2}+\dfrac{C}{(s+2)^2}$

Multiplying with $(s+1)(s+2)^2$ we obtain

$$1 = A(s+2)^2+B(s+1)(s+2)+C(s+1)$$

Putting $s=-1$ we get $A = 1$

Putting $s=-2$ we get $C = -1$

Putting $s=0$ we have $1 = 1(4)+B(2)-1(1)$ $\therefore$ $B = -1$

Hence $\dfrac{1}{(s+1)(s+2)^2} = \dfrac{1}{s+1}+\dfrac{-1}{s+2}+\dfrac{-1}{(s+2)^2}$

$\therefore$ $L^{-1}\left[\dfrac{1}{(s+1)(s+2)^2}\right] = L^{-1}\left[\dfrac{1}{s+1}\right]-L^{-1}\left[\dfrac{1}{s+2}\right]-L^{-1}\left[\dfrac{1}{(s+2)^2}\right]$

ie., $y(t) = e^{-t}-e^{-2t}-e^{-2t}L^{-1}\left(\dfrac{1}{s^2}\right)$

Thus $y(t) = e^{-t}-e^{-2t}-e^{-2t}t = e^{-t}-(1+t)e^{-2t}$

149. *Employ Laplace transform to solve the equation :* $y''+5y'+6y = 5e^{2x}$, $y(0) = 2,\ y'(0) = 1$

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(x)]+5L[y'(x)]+6L[y(x)] = 5L(e^{2x})$$

ie., $\{s^2L[y(x)]-sy(0)-y'(0)\}+5\{sL[y(x)]-y(0)\}+6L[y(x)] = \dfrac{5}{s-2}$

Using the given initial conditions we obtain,

$$(s^2+5s+6)\,L[y(x)]-2s-1-10 = \frac{5}{s-2}$$

ie., $(s^2+5s+6)\,L[y(x)] = (2s+11)+\dfrac{5}{s-2}$

$$L[y(x)] = \frac{(2s+11)(s-2)+5}{(s-2)(s^2+5s+6)} = \frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}$$

$\therefore \quad y(x) = L^{-1}\left[\dfrac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right]$

Let $\quad \dfrac{2s^2+7s-17}{(s-2)(s+2)(s+3)} = \dfrac{A}{s-2}+\dfrac{B}{s+2}+\dfrac{C}{s+3}$

or $\quad 2s^2+7s-17 = A(s+2)(s+3)+B(s-2)(s+3)+C(s-2)(s+2)$

Put $s=2$: $5 = A(4)(5)$ $\quad \therefore \quad A = 1/4$

Put $s=-2$: $-23 = B(-4)(1)$ $\quad \therefore \quad B = 23/4$

Put $s=-3$: $-20 = C(-5)(-1)$ $\quad \therefore \quad C = -4$

Hence $\quad L^{-1}\left[\dfrac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right]$

$$= \frac{1}{4}L^{-1}\left[\frac{1}{s-2}\right]+\frac{23}{4}L^{-1}\left[\frac{1}{s+2}\right]-4L^{-1}\left[\frac{1}{s+3}\right]$$

Thus $\quad y(x) = \dfrac{1}{4}e^{2x}+\dfrac{23}{4}e^{-2x}-4e^{-3x}$

---

150. *Using Laplace transform technique solve* $x''-2x'+x = e^{2t}$ *with* $x(0)=0,\ x'(0)=-1$

>> Taking Laplace transform on both sides of the given equation we have,

$$L[x''(t)]-2L[x'(t)]+L[x(t)] = L(e^{2t})$$

ie., $\quad \left\{s^2L[x(t)]-sx(0)-x'(0)\right\}-2\left\{sL[x(t)]-x(0)\right\}+L[x(t)] = \dfrac{1}{s-2}$

Using the given initial conditions we obtain,

$$\left\{s^2-2s+1\right\}L[x(t)]+1 = \frac{1}{s-2}$$

ie., $\quad (s-1)^2L[x(t)] = \dfrac{1}{s-2}-1 = \dfrac{3-s}{(s-2)}$

or $\quad L[x(t)] = \dfrac{3-s}{(s-1)^2(s-2)}$

$\therefore \quad x(t) = L^{-1}\left[\dfrac{3-s}{(s-1)^2(s-2)}\right]$

Let $\quad \dfrac{3-s}{(s-1)^2(s-2)} = \dfrac{A}{s-1}+\dfrac{B}{(s-1)^2}+\dfrac{C}{s-2}$

Multiplying by $(s-1)^2(s-2)$ we get,

$$3-s = A(s-1)(s-2)+B(s-2)+C(s-1)^2 \qquad \ldots(1)$$

Put $s=1$: $2 = B(-1)$ $\quad \therefore \quad B=-2$

Put $s=2$: $1 = C(1)$ $\quad \therefore \quad C=1$

Equating the coefficient of s^2 on both sides of (1) we get,

$0 = A+C.\ \therefore\ A=-1$

Hence $L^{-1}\left[\dfrac{3-s}{(s-1)^2(s-2)}\right] = -L^{-1}\left[\dfrac{1}{s-1}\right]-2L^{-1}\left[\dfrac{1}{(s-1)^2}\right]+L^{-1}\left[\dfrac{1}{s-2}\right]$

Thus $\quad x(t) = -e^t-2e^t\cdot t+e^{2t} = e^{2t}-(1+2t)e^t$

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151. *Solve the DE* $y''+4y'+3y = e^{-t}$ *with* $y(0)=1=y'(0)$ *using Laplace transforms.*

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)]+4L[y'(t)]+3L[y(t)] = L(e^{-t})$$

ie., $\quad \left\{s^2L[y(t)]-sy(0)-y'(0)\right\}+4\left\{sL[y(t)]-y(0)\right\}+3L[y(t)] = \dfrac{1}{s+1}$

Using the given initial conditions we obtain,

$$(s^2+4s+3)L[y(t)]-s-1-4 = \frac{1}{s+1}$$

ie., $\quad (s^2+4s+3)L[y(t)] = (s+5)+\dfrac{1}{(s+1)}$

ie., $\quad (s+1)(s+3)L[y(t)] = \dfrac{s^2+6s+6}{s+1}$

or $\quad L[y(t)] = \dfrac{s^2+6s+6}{(s+1)^2(s+3)}$

$\therefore \quad y(t) = L^{-1}\left[\dfrac{s^2+6s+6}{(s+1)^2(s+3)}\right]$

Let $\quad \dfrac{s^2+6s+6}{(s+1)^2(s+3)} = \dfrac{A}{s+1}+\dfrac{B}{(s+1)^2}+\dfrac{C}{s+3}$

Multiplying by $(s+1)^2(s+3)$ we get,

$$s^2+6s+6 = A(s+1)(s+3)+B(s+3)+C(s+1)^2 \quad \cdots(1)$$

Put $s=-1$: $1 = B(2)$ $\quad \therefore\ B = 1/2$

Put $s=-3$: $-3 = C(4)$ $\quad \therefore\ C = -3/4$

Equating the coefficient of s^2 on both sides of (1) we get,

$$1 = A+C \quad \therefore\ A = 7/4$$

Hence $L^{-1}\left[\dfrac{s^2+6s+6}{(s+1)^2(s+3)}\right] = \dfrac{7}{4}L^{-1}\left[\dfrac{1}{s+1}\right]+\dfrac{1}{2}L^{-1}\left[\dfrac{1}{(s+1)^2}\right]-\dfrac{3}{4}L^{-1}\left[\dfrac{1}{s+3}\right]$

Thus $\quad y(t) = \dfrac{7}{4}e^{-t}+\dfrac{1}{2}e^{-t}\cdot t-\dfrac{3}{4}e^{-3t}$

152. *Solve by using Laplace transforms* $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+x = 3t\,e^{-t}$ *given that* $x = 4,\ \dfrac{dx}{dt} = 2$ *when* $t = 0$.

>> The given equation is $x''(t)+2x'(t)+x(t) = 3t\,e^{-t}$.

Initial conditions are $x(0) = 4,\ x'(0) = 2$

Taking Laplace transform on both sides of the equation we have,

$$L[x''(t)]+2L[x'(t)]+L[x(t)] = 3L(e^{-t}\cdot t)$$

ie., $\quad \left\{s^2L[x(t)]-s\,x(0)-x'(0)\right\}+2\left\{s\,L[x(t)]-x(0)\right\}+L[x(t)] = \dfrac{3}{(s+1)^2}$

Using the given initial conditions we obtain,

$$(s^2+2s+1)\,L[x(t)]-4s-2-8 = \frac{3}{(s+1)^2}$$

ie., $\quad (s+1)^2\,L[x(t)] = (4s+10)+\dfrac{3}{(s+1)^2}$

or $\quad L[x(t)] = \dfrac{4s+10}{(s+1)^2}+\dfrac{3}{(s+1)^4}$

$\therefore \quad x(t) = L^{-1}\left[\dfrac{4(s+1)+6}{(s+1)^2}\right]+L^{-1}\left[\dfrac{3}{(s+1)^4}\right]$

$$= e^{-t}L^{-1}\left[\frac{4s+6}{s^2}\right]+3\,e^{-t}L^{-1}\left[\frac{1}{s^4}\right]$$

$$x(t) = e^{-t}\left\{4L^{-1}\left(\frac{1}{s}\right)+6L^{-1}\left(\frac{1}{s^2}\right)+3L^{-1}\left(\frac{1}{s^4}\right)\right\}$$

$$= e^{-t}(4+6t+3\cdot t^3/6)$$

Thus $\quad x(t) = e^{-t}(4+6t+t^3/2)$

153. *Solve by using Laplace transforms* $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+2y = 5\sin t$ *given that* $y(0) = 0 = y'(0)$

>> The given equation is $y''(t)+2y'(t)+2y(t) = 5\sin t$

Taking Laplace transform on both sides we have,

$$L[y''(t)]+2L[y'(t)]+2L[y(t)] = 5L(\sin t).$$

ie., $\quad \left\{s^2L[y(t)]-s\,y(0)-y'(0)\right\}+2\left\{sL[y(t)]-y(0)\right\}+2L[y(t)] = \dfrac{5}{s^2+1}$

Using the given initial conditions we obtain,

$$L[y(t)]\left\{s^2+2s+2\right\} = \frac{5}{s^2+1} \quad \text{or} \quad L[y(t)] = \frac{5}{(s^2+1)(s^2+2s+2)}$$

$\therefore \quad y(t) = L^{-1}\left[\dfrac{5}{(s^2+1)(s^2+2s+2)}\right]$

Let $\quad \dfrac{5}{(s^2+1)(s^2+2s+2)} = \dfrac{As+B}{s^2+1}+\dfrac{Cs+D}{s^2+2s+2}$

ie., $\quad 5 = (As+B)(s^2+2s+2)+(Cs+D)(s^2+1)$

ie., $\quad 5 = (A+C)s^3+(2A+B+D)s^2+(2A+2B+C)s+(2B+D)$

Comparing the coefficients on both sides, we get

$$A+C = 0\ ;\ 2A+B+D = 0\ ;\ 2A+2B+C = 0\ ;\ 2B+D = 5$$

Solving these simultaneously we obtain

$$A = -2,\ B = 1,\ C = 2,\ D = 3$$

Hence $\quad \dfrac{5}{(s^2+1)(s^2+2s+2)} = \dfrac{-2s+1}{s^2+1}+\dfrac{2s+3}{s^2+2s+2}$

$\therefore \quad L^{-1}\left[\dfrac{5}{(s^2+1)(s^2+2s+2)}\right]$

$$y(t) = -2L^{-1}\left(\frac{s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s^2+1}\right)+L^{-1}\left(\frac{2s+3}{s^2+2s+2}\right)$$

ie., $$y(t) = -2\cos t+\sin t+L^{-1}\left\{\frac{2(s+1)+1}{(s+1)^2+1}\right\}$$

$$= -2\cos t+\sin t+e^{-t}L^{-1}\left\{\frac{2s+1}{s^2+1}\right\}$$

$$= -2\cos t+\sin t+e^{-t}\left[2L^{-1}\left(\frac{s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

Thus $\boldsymbol{y(t) = -2\cos t+\sin t+e^{-t}(2\cos t+\sin t)}$

154. *Solve* $y''+6y'+9y=12t^2e^{-3t}$ *subject to the conditions,* $y(0)=0=y'(0)$ *by using Laplace transforms.*

>> The given equation is $y''(t)+6y'(t)+9y(t) = 12t^2e^{-3t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)]+6L[y'(t)]+9L[y(t)] = 12L[e^{-3t}t^2]$$

ie., $$\{s^2L[y(t)]-sy(0)-y'(0)\}+6\{sL[y(t)]-y(0)\}+9L[y(t)] = \frac{12\cdot 2!}{(s+3)^3}$$

Using the given initial conditions we obtain,

$$(s^2+6s+9)\,L[y(t)] = \frac{24}{(s+3)^3} \quad\text{or}\quad L[y(t)] = \frac{24}{(s+3)^5}$$

$\therefore$ $$y(t) = L^{-1}\left[\frac{24}{(s+3)^5}\right]$$

$$y(t) = 24e^{-3t}L^{-1}\left(\frac{1}{s^5}\right) = 24e^{-3t}\frac{t^4}{4!}$$

Thus $\boldsymbol{y(t) = e^{-3t}t^4}$

155. *Solve by using Laplace transform method* $y''(t)+y(t) = H(t-1)$ *given* $y(0) = 0$ *and* $y'(0) = 1$

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)]+L[y(t)] = L[H(t-1)]$$

ie., $$\{s^2L[y(t)]-sy(0)-y'(0)\}+L[y(t)] = \frac{e^{-s}}{s}$$

Using the given initial conditions we obtain,

$$(s^2+1)\,L[y(t)]-1 = \frac{e^{-s}}{s} \quad\text{or}\quad (s^2+1)\,L[y(t)] = 1+\frac{e^{-s}}{s}$$

$\therefore$ $$L[y(t)] = \frac{1}{s^2+1}+\frac{e^{-s}}{s(s^2+1)}$$

$\Rightarrow$ $$y(t) = L^{-1}\left[\frac{1}{s^2+1}\right]+L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right]$$

ie., $$y(t) = \sin t+L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] \qquad \ldots(1)$$

In respect of the second term, let $\bar{f}(s) = \dfrac{1}{s(s^2+1)}$

let $\bar{f}(s) = \dfrac{1}{s(s^2+1)} = \dfrac{1}{s}-\dfrac{s}{s^2+1}$ by partial fractions.

Now, $L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s}\right]-L^{-1}\left[\dfrac{s}{s^2+1}\right]$

ie., $f(t) = 1-\cos t$

We also have, $L^{-1}[e^{-s}\bar{f}(s)] = f(t-1)H(t-1)$

ie., $$L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right] = [1-\cos(t-1)]H(t-1) \qquad \ldots(2)$$

We shall use (2) in (1),

Thus $\boldsymbol{y(t) = \sin t+[1-\cos(t-1)]H(t-1)}$

156. *Solve by using Laplace transforms* $y''(t)+y(t) = F(t)$ *where* $F(t) = \begin{cases} 4, & 0<t<1 \\ 3, & t>1 \end{cases}$ *subject to the conditions,* $y(0) = 0$ *and* $y'(0) = 1$

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)]+L[y(t)] = L[F(t)]$$

ie., $$\{s^2L[y(t)]-sy(0)-y'(0)\}+L[y(t)] = L[F(t)]$$

Using the given initial conditions we obtain,

$$(s^2+1)\,L[y(t)]-1 = L[F(t)] \qquad \ldots(1)$$

Now $L[F(t)] = \int_0^{\infty} e^{-st} F(t)\,dt = \int_0^1 e^{-st} F(t)\,dt + \int_1^{\infty} e^{-st} F(t)\,dt$

ie., $L[F(t)] = \int_0^1 4e^{-st}\,dt + \int_1^{\infty} 3e^{-st}\,dt$

$$= 4\left[\frac{e^{-st}}{-s}\right]_0^1 + 3\left[\frac{e^{-st}}{-s}\right]_1^{\infty} = \frac{-4}{s}(e^{-s}-1) - \frac{3}{s}(0-e^{-s})$$

Hence, $L[F(t)] = \frac{4}{s} - \frac{1}{s}e^{-s}$ ...(2)

Using (2) in the RHS of (1) we get,

$$(s^2+1)L[y(t)] = 1 + \frac{4}{s} - \frac{e^{-s}}{s}$$

or $L[y(t)] = \frac{1}{s^2+1} + \frac{4}{s(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}$

$\therefore$ $y(t) = L^{-1}\left[\frac{1}{s^2+1}\right] + 4L^{-1}\left[\frac{1}{s(s^2+1)}\right] - L^{-1}\left[\frac{e^{-s}}{s(s^2+1)}\right]$

[Refer Problem-84 for the inverse of the second and third terms]

Thus $\mathbf{y(t) = \sin t + 4(1-\cos t) - [1-\cos(t-1)]H(t-1)}$

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157. *Solve the following boundary value problem by using Laplace transforms*
$y''(t) + y(t) = 0$; $y(0) = 2$, $y(\pi/2) = 1$

>> Taking Laplace transform on both sides of the given equation we have,

$$L[y''(t)] + Ly(t) = L(0)$$

ie., $\left\{s^2 Ly(t) - s\,y(0) - y'(0)\right\} + Ly(t) = 0$ ...(1)

Let us assume $y'(0) = a$ where a is a constant to be found later and we have $y(0) = 2$ by data.

Hence (1) becomes,

$$(s^2+1)\,L[y(t)] - 2s - a = 0$$

ie., $(s^2+1)\,L[y(t)] = 2s + a$ or $L[y(t)] = \frac{2s+a}{s^2+1}$

$\therefore$ $y(t) = 2L^{-1}\left[\frac{s}{s^2+1}\right] + a\,L^{-1}\left[\frac{1}{s^2+1}\right]$

ie., $y(t) = 2\cos t + a\sin t$ ...(2)

Now we shall use the condition $y(\pi/2) = 1$

Hence (2) becomes $y(\pi/2) = 2\cos(\pi/2) + a\sin(\pi/2)$

ie., $1 = 0 + a$ $\therefore$ $a = 1$

Thus $\mathbf{y(t) = 2\cos t + \sin t}$

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158. *Solve by using Laplace transforms :*
$\frac{d^2x}{dt^2} + \omega^2 x = a\sin(\omega t + \alpha)$, $x(0) = 0 = x'(0)$

>> The given equation is $x''(t) + \omega^2 x(t) = a\sin(\omega t + \alpha)$

ie., $x''(t) + \omega^2 x(t) = a\left\{\sin\omega t\cos\alpha + \cos\omega t\sin\alpha\right\}$

Taking Laplace transform on both sides we have,

$$L[x''(t)] + \omega^2 Lx(t) = a\cos\alpha\, L(\sin\omega t) + a\sin\alpha\, L(\cos\omega t)$$

ie., $\left\{s^2 Lx(t) - sx(0) - x'(0)\right\} + \omega^2 Lx(t) = a\cos\alpha\cdot\frac{\omega}{s^2+\omega^2} + a\sin\alpha\cdot\frac{s}{s^2+\omega^2}$

ie., $(s^2+\omega^2)Lx(t) = \frac{a\cos\alpha\cdot\omega}{s^2+\omega^2} + \frac{a\sin\alpha\cdot s}{s^2+\omega^2}$

or $Lx(t) = \frac{a\cos\alpha\cdot\omega}{(s^2+\omega^2)^2} + a\sin\alpha\cdot\frac{s}{(s^2+\omega^2)^2}$

$\therefore$ $x(t) = \omega a\cos\alpha\, L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right] + a\sin\alpha\, L^{-1}\left[\frac{s}{(s^2+\omega^2)^2}\right]$

Recollecting the standard inverse Laplace transforms we have

$$x(t) = \omega a\cos\alpha\cdot\frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t) + a\sin\alpha\cdot\frac{t\sin\omega t}{2\omega}$$

$$= \frac{a\cos\alpha}{2\omega^2}\sin\omega t - \frac{a\cos\alpha}{2\omega}t\cos\omega t + \frac{a\sin\alpha}{2\omega}t\sin\omega t$$

$$= \frac{a\cos\alpha}{2\omega^2}\sin\omega t - \frac{at}{2\omega}(\cos\omega t\cos\alpha - \sin\omega t\sin\alpha)$$

$$= \frac{a\cos\alpha}{2\omega^2}\sin\omega t - \frac{at}{2\omega}\cos(\omega t + \alpha)$$

Thus $\mathbf{x(t) = \frac{a}{2\omega^2}\left\{\cos\alpha\sin\omega t - \omega t\cos(\omega t + \alpha)\right\}}$

159. *Using Laplace transforms solve* $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y = \sin t$, *given* $y=0$, $\frac{dy}{dt}=0$ *when* $t=0$

>> The given equation is $y''(t)+2y'(t)-3y(t)=\sin t$ with the initial conditions $y(0)=0$ and $y'(0)=0$

Taking Laplace transforms as both sides of the given equation we have,

$$L[y''(t)]+2L[y'(t)]-3L[y(t)] = L(\sin t)$$

ie., $$\{s^2L[y(t)]-sy(0)-y'(0)\}+2\{sL[y(t)]-y(0)\}-3L[y(t)] = \frac{1}{s^2+1}$$

Using the given initial conditions wehave,

$$(s^2+2s-3)L[y(t)] = \frac{1}{s^2+1}$$

or $$L[y(t)] = \frac{1}{(s^2+1)(s^2+2s-3)} = \frac{1}{(s^2+1)(s-1)(s+3)}$$

$\therefore$ $$y(t) = L^{-1}\left[\frac{1}{(s^2+1)(s-1)(s+3)}\right]$$

Let $$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{As+B}{s^2+1}+\frac{C}{s-1}+\frac{D}{s+3}$$

or $$1 = (As+B)(s-1)(s+3)+C(s^2+1)(s+3)+D(s^2+1)(s-1)$$

Put $s=1$: $1 = C(8)$ or $C = 1/8$

Put $s=-3$: $1 = D(-40)$ or $D = -1/40$

Put $s=0$: $1 = B(-3)+C(3)+D(-1)$

ie., $1 = -3B+3/8+1/40$ or $B=-1/5$

Equating the coefficients of s^3 on both sides we get

$0 = A+C+D$ or $A = -1/10$

$\therefore$ $$L^{-1}\left[\frac{1}{(s^2+1)(s-1)(s+3)}\right]$$

$$= \frac{-1}{10}L^{-1}\left[\frac{s}{s^2+1}\right]-\frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]+\frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right]-\frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right]$$

Thus $$y(t) = \frac{-1}{10}\cos t-\frac{1}{5}\sin t+\frac{1}{8}e^t-\frac{1}{40}e^{-3t}$$

160. *Using Laplace transform method solve,* $\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+3\frac{dy}{dt}-y = t^2e^t$ *given* $y(0)=1,\ y'(0)=0,\ y''(0)=-2$

>> The given equation is

$$y'''(t)-3y''(t)+3y'(t)-y(t) = t^2e^t$$

Taking Laplace transform on both sides we have,

$$L[y'''(t)]-3L[y''(t)]+3L[y'(t)]-Ly(t) = L(e^tt^2)$$

ie., $$\{s^3L[y(t)]-s^2y(0)-sy'(0)-y''(0)\}-3\{s^2L[y(t)]-sy(0)-y'(0)\}$$
$$+3\{sL[y(t)]-y(0)\}-L[y(t)] = \frac{2}{(s-1)^3}$$

Using the given initial conditions we have,

ie., $$(s^3-3s^2+3s-1)L[y(t)]-s^2+2+3s-3 = \frac{2}{(s-1)^3}$$

ie., $$(s-1)^3L[y(t)] = (s^2-3s+1)+\frac{2}{(s-1)^3}$$

or $$L[y(t)] = \frac{s^2-3s+1}{(s-1)^3}+\frac{2}{(s-1)^6}$$

$\therefore$ $$y(t) = L^{-1}\left[\frac{s^2-3s+1}{(s-1)^3}\right]+2L^{-1}\left[\frac{1}{(s-1)^6}\right] \quad \ldots(i)$$

Now, $$L^{-1}\left[\frac{s^2-3s+1}{(s-1)^3}\right] = L^{-1}\left[\frac{\{(s-1)^2+2s-1\}-3s+1}{(s-1)^3}\right]$$

ie., $$= L^{-1}\left[\frac{(s-1)^2-s}{(s-1)^3}\right]$$

$$= L^{-1}\left[\frac{(s-1)^2-(s-1)-1}{(s-1)^3}\right]$$

$$= e^tL^{-1}\left[\frac{s^2-s-1}{s^3}\right]$$

$$= e^t\left\{L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s^2}\right]-L^{-1}\left[\frac{1}{s^3}\right]\right\}$$

$$\therefore \quad L^{-1}\left[\frac{s^2-3s+1}{(s-1)^3}\right] = e^t\left(1-t-\frac{t^2}{2}\right)$$

$$\text{Also} \quad L^{-1}\left[\frac{2}{(s-1)^6}\right] = 2e^t L^{-1}\left[\frac{1}{s^6}\right] = 2e^t\frac{t^5}{5!} = \frac{e^t t^5}{60}$$

Thus by using these results in the R.H.S of (i) we have,

$$y(t) = e^t\left\{1-t-\frac{t^2}{2}+\frac{t^5}{60}\right\}$$

Solution of a few simultaneous differential equaitons and Application problems using Laplace Transforms is provided for the benefit of readers.

Solution of simultaneous differential equations

1. *Solve by using Laplace transforms* $\frac{dx}{dt}-2y = \cos 2t,\ \frac{dy}{dt}+2x = \sin 2t$;

$x = 1,\ y = 0\ at\ t = 0$

>> The given system of equations are,

$$x'(t)-2y(t) = \cos 2t \qquad \ldots(1)$$

$$2x(t)+y'(t) = \sin 2t \qquad \ldots(2)$$

with the initial conditions $x(0) = 1$ and $y(0) = 0$

Taking Laplace transform on both sides of (1) and (2) we have

$$L[x'(t)]-2L[y(t)] = L(\cos 2t)$$

$$2L[x(t)]+L[y'(t)] = L(\sin 2t)$$

ie., $$\{sL[x(t)]-x(0)\}-2L[y(t)] = s/s^2+4$$

$$2L[x(t)]+\{sL[y(t)]-y(0)\} = 2/s^2+4$$

Using the given initial conditions we have,

$$sL[x(t)]-2L[y(t)] = 1+(s/s^2+4)$$

$$2L[x(t)]+sL[y(t)] = 2/s^2+4$$

Let us multiply (3) by s and (4) by 2

$$s^2L[x(t)]-2sL[y(t)] = s+(s^2/s^2+4)$$

$$4L[x(t)]+2sL[y(t)] = 4/s^2+4$$

Adding we get, $(s^2+4)L[x(t)] = s+\frac{s^2}{s^2+4}+\frac{4}{s^2+4}$

ie., $$(s^2+4)L[x(t)] = s+1$$

or $$L[x(t)] = \frac{s+1}{s^2+4}$$

$$\therefore \quad x(t) = L^{-1}\left(\frac{s}{s^2+2^2}\right)+L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

Thus $\mathbf{x(t) = \cos 2t + \frac{1}{2}\sin 2t}$ $\ldots(5)$

To find $y(t)$, let us consider $\frac{dx}{dt}-2y = \cos 2t$

$$\therefore \quad y = \frac{1}{2}\left[\frac{dx}{dt}-\cos 2t\right] = \frac{1}{2}\left[\frac{d}{dt}\left(\cos 2t+\frac{1}{2}\sin 2t\right)-\cos 2t\right]$$

ie., $y(t) = \frac{1}{2}[-2\sin 2t+\cos 2t-\cos 2t] = -\sin 2t$

Thus $\mathbf{y(t) = -\sin 2t}$ $\ldots(6)$

(5) and (6) represents the required solution.

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2. *Solve by using Laplace transforms* $\frac{dx}{dt} = 2x-3y,\ \frac{dy}{dt} = y-2x$ *given that* $x(0) = 8$ *and* $y(0) = 3$

>> The given equations are, $x'(t)-2x(t)+3y(t) = 0$

$$2x(t)+y'(t)-y(t) = 0$$

Taking Laplace transform on both sides of these equations we have,

$$L[x'(t)]-2L[x(t)]+3L[y(t)] = 0$$

$$2L[x(t)]+L[y'(t)]-L[y(t)] = 0$$

ie., $$sL[x(t)]-x(0)-2L[x(t)]+3L[y(t)] = 0$$

$$2L[x(t)]+sL[y(t)]-y(0)-L[y(t)] = 0$$

Using the given initial conditions we obtain,

$$(s-2)L[x(t)]+3L[y(t)] = 8$$

$$2L[x(t)]+(s-1)L[y(t)] = 3$$

or $$(s-1)(s-2)L[x(t)]+3(s-1)L[y(t)] = 8s-8$$

$$6L[x(t)]+3(s-1)L[y(t)] = 9$$

Subtracting we get, $(s^2-3s-4)L[x(t)] = 8s-17$

$\therefore \quad x(t) = L^{-1}\left[\dfrac{8s-17}{s^2-3s-4}\right] = L^{-1}\left[\dfrac{8s-17}{(s-4)(s+1)}\right]$

Let $\dfrac{8s-17}{(s-4)(s+1)} = \dfrac{A}{s-4}+\dfrac{B}{s+1}$

or $8s-17 = A(s+1)+B(s-4)$

Put $s=4$: $15 = 5A$ $\therefore$ $A=3$

Put $s=-1$: $-25=-5B$ $\therefore$ $B=5$

Now, $x(t) = 3L^{-1}\left[\dfrac{1}{s-4}\right]+5L^{-1}\left[\dfrac{1}{s+1}\right]$

Thus $x(t) = 3e^{4t}+5e^{-t}$ ...(1)

Consider $\dfrac{dx}{dt} = 2x-3y$ $\therefore$ $y = \dfrac{1}{3}\left[2x-\dfrac{dx}{dt}\right]$

ie., $y(t) = \dfrac{1}{3}\left[2(3e^{4t}+5e^{-t})-(12e^{4t}-5e^{-t})\right] = \dfrac{1}{3}(-6e^{4t}+15e^{-t})$

Thus $y(t) = 5e^{-t}-2e^{4t}$ ...(2)

(1) and (2) represents the solution of the given equations.

3. *Solve the following system of equations by using Laplace transforms.*
$\dfrac{dx}{dt}+\dfrac{dy}{dt} = 2z$, $\dfrac{dy}{dt}+\dfrac{dz}{dt} = 2x$, $\dfrac{dz}{dt}+\dfrac{dx}{dt} = 2y$ *given that* $x=y=z=1$ *when* $t=0$.

>> The given equaitons are,

$x'(t)+y'(t) = 2z(t)$

$y'(t)+z'(t) = 2x(t)$ $\quad x(0)=1=y(0)=z(0)$ are the initial conditions.

$z'(t)+x'(t) = 2y(t)$

Taking Laplace transform on bothsides of these equations we have,

$sL[x(t)]-x(0)+sL[y(t)]-y(0) = 2L[z(t)]$

$sL[y(t)]-y(0)+sL[z(t)]-z(0) = 2L[x(t)]$

$sL[z(t)]-z(0)+sL[x(t)]-x(0) = 2L[y(t)]$

Using the given initial conditions we obtain,

$sL[x(t)]+sL[y(t)]-2L[z(t)] = 2$ ...(1)

$-2L[x(t)]+sL[y(t)]+sL[z(t)] = 2$ ...(2)

$sL[x(t)]-2L[y(t)]+sL[z(t)] = 2$ ...(3)

$(1)\times 2+(2)\times s$: $(s^2+2s)L[y(t)]+(s^2-4)L[z(t)] = 2s+4$ ...(4)

$(1)-(3)$: $(s+2)L[y(t)]-(s+2)L[z(t)] = 0$ ...(5)

Now $(4)+(s-2)\times(5)$ will give us

$(s^2+2s+s^2-4)L[y(t)] = 2s+4$

ie., $(2s^2+2s-4)L[y(t)] = 2(s+2)$

ie., $2(s^2+s-2)L[y(t)] = 2(s+2)$

ie., $(s+2)(s-1)L[y(t)] = (s+2)$

or $L[y(t)] = \dfrac{1}{s-1} \Rightarrow y(t) = L^{-1}\left[\dfrac{1}{s-1}\right] = e^t$

Now (5) becomes $(s+2)\dfrac{1}{s-1}-(s+2)L[z(t)] = 0$

$\therefore$ $L[z(t)] = \dfrac{1}{s-1} \Rightarrow z(t) = e^t$

But $y'(t)+z'(t) = 2x(t)$

ie., $e^t+e^t = 2x(t)$ or $2x(t) = 2e^t$ $\therefore$ $x(t) = e^t$

Thus $x(t) = e^t = y(t) = z(t)$ represents the solution of the given system of equations.

Application of Laplace transforms

Vibrations of string

It is obvious that a spring vibrates when it is put into motion. Let us suppose that an elastic spring is suspended downwards and is appended with a body of mass m to its lower end. When the string carrying the mass vibrates freely there is a resistance due to the medium opposing the movement resulting in *damped vibrations*.

If y is the downward displacement of the body from the equilibrium position at time t then $\dfrac{dy}{dt}$ is the velocity, $\dfrac{d^2y}{dt^2}$ is the acceleration of the body at time t. y satisfies the differential equation

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky = 0 \qquad \ldots(1)$$

where c is the damping coefficient and k is the spring modulus.

If the medium does not cause any resistance to the vibrations, then we have *undamped vibrations*.

In such a case $c = 0$ and (1) becomes

$$m\frac{d^2y}{dt^2}+ky = 0 \quad \ldots(2)$$

If $c>0$ then we have *resonance damped vibrations*

If an extra time dependent force function $f(t)$ is appended onto the body (1) and (2) assumes the form

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky = f(t) \quad \ldots(3)$$

and
$$m\frac{d^2y}{dt^2}+ky = f(t) \quad \ldots(4)$$

Deflection of beams

Consider a uniform beam supported at both the ends is set to deflect from a horizontal position. Suppose it is subjected to a vertical load $w(x)$ then the deflection $y(x)$ at a distance x from one end of the beam satisfies the differential equation

$$EI\frac{d^4y}{dx^4} = w(x) \quad \ldots(5)$$

where E is the modulus of elasticity and I is the moment of inertia. EI is a constant and is called flexural rigidity of the beam.

L-R-C Circuits

An electric circuit consisting of an inductance of L henrys, capacitance of C farads and resistance of R ohms connected in series is called an L-R-C circuit. If E volts is the e.m.f applied to an L-R-C circuit then the current i measured in amperes in the circuit at time t is given by the differential equation

$$L\frac{di}{dt}+Ri+\frac{q}{C} = E(t) \quad \ldots(6)$$

Here q is the charge measured in coulombs. This is connected with the current i by the relation $i = \frac{dq}{dt}$

Thus (6) can also be put in the form

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = E(t) \quad \ldots(7)$$

so that q can be found. Differentiating (6) we have

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C} = E'(t) \quad \ldots(8)$$

so that i can be found.

The differential equation in an L-R circuit is given by

$$L\frac{di}{dt}+Ri = E(t) \quad \ldots(9)$$

ILLUSTRATIVE PROBLEMS

1. *A particle moves along the x-axis according to the law* $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+25x = 0$. *If the particle is started at* $x = 0$ *with an initial velocity of 12ft/sec to the left, determine* x *in terms of* t *using Laplace transform method.*

\>> The given equation is

$$x''(t)+6x'(t)+25x(t) = 0 \quad \ldots(1)$$

and $x = 0$ at $t = 0$, $\frac{dx}{dt} = -12$ at $t = 0$, by data.

ie., $x(0) = 0,\ x'(0) = -12$ are the initial conditions.

Now taking Laplace transform on both sides of (1) we have

$$L[x''(t)]+6L[x'(t)]+25L[x(t)] = L(0)$$

ie., $\{s^2L[x(t)]-sx(0)-x'(0)\}+6\{sL[x(t)]-x(0)\}+25L[x(t)] = 0$

Using the initial conditions we obtain,

$$(s^2+6s+25)\,L[x(t)] = -12 \quad \text{or} \quad L[x(t)] = \frac{-12}{s^2+6s+25}$$

$$\therefore \quad x(t) = -12L^{-1}\left[\frac{1}{s^2+6s+25}\right]$$

$$= -12L^{-1}\left[\frac{1}{(s+3)^2+4^2}\right] = -12e^{-3t}L^{-1}\left(\frac{1}{s^2+4^2}\right)$$

ie., $x(t) = -12e^{-3t}\frac{\sin 4t}{4}$

Thus $x(t) = -3e^{-3t}\sin 4t$

2. *Using Laplace transforms method solve the problem of resonance damped vibration of a spring.*

>> The governing d.e is given by $m\dfrac{d^2y}{dt^2}+c\dfrac{dy}{dt}+ky=0\,;\ c>0$

or $$\frac{d^2y}{dt^2}+\frac{c}{m}\frac{dy}{dt}+\frac{k}{m}y=0$$

Let us denote $c/m=2\lambda$ and $k/m=\mu^2$ for convenience so that the d.e assumes the form

$$y''(t)+2\lambda y'(t)+\mu^2 y(t)=0 \qquad \ldots(1)$$

Let $y(0)=y_0$ and $y'(0)=y_1$ be the initial conditions.

Taking Laplace transform on both sides of (1) we have,

$$L[y''(t)]+2\lambda L[y'(t)]+\mu^2 L[y(t)]=0$$

ie., $$\left\{s^2L[y(t)]-sy(0)-y'(0)\right\}+2\lambda\left\{sL[y(t)]-y(0)\right\}+\mu^2L[y(t)]=0$$

Using the initial conditions we obtain,

$$(s^2+2\lambda s+\mu^2)L[y(t)]-y_0s-y_1-2\lambda y_0=0$$

ie., $$L[y(t)]=\frac{y_0s}{s^2+2\lambda s+\mu^2}+\frac{(y_1+2\lambda y_0)}{(s^2+2\lambda s+\mu^2)}$$

$$y(t)=y_0L^{-1}\left[\frac{s}{s^2+2\lambda s+\mu^2}\right]+L^{-1}\left[\frac{y_1+2\lambda y_0}{s^2+2\lambda s+\mu^2}\right]$$

$$=y_0L^{-1}\left[\frac{(s+\lambda)-\lambda}{(s+\lambda)^2+(\mu^2-\lambda^2)}\right]+(y_1+2\lambda y_0)L^{-1}\left[\frac{1}{(s+\lambda)^2+(\mu^2-\lambda^2)}\right]$$

Denoting $\mu^2-\lambda^2=v^2$ we have,

$$y(t)=y_0e^{-\lambda t}L^{-1}\left[\frac{s-\lambda}{s^2+v^2}\right]+(y_1+2\lambda y_0)e^{-\lambda t}L^{-1}\left[\frac{1}{s^2+v^2}\right]$$

$$y(t)=y_0e^{-\lambda t}[\cos vt-(\lambda/v)\sin vt]+(y_1+2\lambda y_0)e^{-\lambda t}\sin vt/v$$

$$y(t)=e^{-\lambda t}[y_0\cos vt+(\lambda y_0/v)\sin vt+y_1\sin vt]$$

Thus $$y(t)=e^{-\lambda t}\left[y_0\cos vt+\left\{y_1+(\lambda y_0/v)\right\}\sin vt\right]$$

where $\lambda=c/2m$, $\mu=\sqrt{k/m}$ and $v=\sqrt{\mu^2-\lambda^2}$

3. *By using Laplace transforms, solve the problem of undamped forced vibrations of a spring in the case where the forcing function is $f(t)=A\sin\omega t$*

>> The differential equation associated with the problem is

$$m\frac{d^2y}{dt^2}+ky=A\sin\omega t \quad\text{or}\quad \frac{d^2y}{dt^2}+\frac{k}{m}y=\frac{A}{m}\sin\omega t.$$

Let $\lambda^2=k/m$ and $\mu=A/m$

The d.e assumes the form

$$y''(t)+\lambda^2y(t)=\mu\sin\omega t \qquad \ldots(1)$$

Let $y(0)=y_0$ and $y'(0)=y_1$ be the initial conditions.

Taking Laplace transform on both sides of (1) we have,

$$L[y''(t)]+\lambda^2L[y(t)]=\mu L(\sin\omega t)$$

ie., $$\left\{s^2L[y(t)]-sy(0)-y'(0)\right\}+\lambda^2L[y(t)]=\frac{\mu\omega}{s^2+\omega^2}$$

Using the initial conditions we obtain,

ie., $$(s^2+\lambda^2)\,L[y(t)]-sy_0-y_1=\frac{\mu\omega}{s^2+\omega^2}$$

ie., $$L[y(t)]=\frac{sy_0+y_1}{(s^2+\lambda^2)}+\frac{\mu\omega}{(s^2+\lambda^2)(s^2+\omega^2)}$$

$\therefore$ $$y(t)=y_0L^{-1}\left[\frac{s}{s^2+\lambda^2}\right]+y_1L^{-1}\left[\frac{1}{s^2+\lambda^2}\right]+\mu\omega L^{-1}\left[\frac{1}{(s^2+\lambda^2)(s^2+\omega^2)}\right]$$

$$y(t)=y_0\cos\lambda t+\frac{y_1}{\lambda}\sin\lambda t+\mu\omega L^{-1}\left[\frac{1}{(s^2+\lambda^2)(s^2+\omega^2)}\right] \qquad \ldots(2)$$

In respect of the last term, let $s^2=t$ for convenience and we resolve into partial fractions.

Let $$\frac{1}{(t+\lambda^2)(t+\omega^2)}=\frac{a}{t+\lambda^2}+\frac{b}{t+\omega^2} \quad\text{or}\quad 1=a(t+\omega^2)+b(t+\lambda^2)$$

Put $t=-\omega^2$: $1=b(\lambda^2-\omega^2)$ $\therefore$ $b=1/\lambda^2-\omega^2$

Put $t=-\lambda^2$: $1=a(\omega^2-\lambda^2)$ $\therefore$ $a=-1/\lambda^2-\omega^2$

$$L^{-1}\left[\frac{1}{(s^2+\lambda^2)(s^2+\omega^2)}\right]=\frac{1}{\lambda^2-\omega^2}\left\{L^{-1}\left(\frac{1}{s^2+\omega^2}\right)-L^{-1}\left(\frac{1}{s^2+\lambda^2}\right)\right\}$$

ie., $$L^{-1}\left[\frac{1}{(s^2+\lambda^2)(s^2+\omega^2)}\right] = \frac{1}{\lambda^2-\omega^2}\left(\frac{\sin\omega t}{\omega}-\frac{\sin\lambda t}{\lambda}\right) \quad \ldots(3)$$

Thus by using (3) in the RHS of (2) we get,

$$y(t) = y_0\cos\lambda t+\frac{y_1}{\lambda}\sin\lambda t+\frac{\mu}{\lambda^2-\omega^2}\left[\sin\omega t-\frac{\omega}{\lambda}\sin\lambda t\right]$$

4. *A particle undergoes forced vibrations according to the law $x''(t)+25x(t) = 21\cos 2t$. If the particle starts from rest at $t = 0$ find the displacement at any time $t>0$ using Laplace transforms.*

>> We have $x''(t)+25x(t) = 21\cos 2t$; $x(0) = 0,\ x'(0) = 0$ from the given data.

Taking Laplace transform on both sides of the equation we have,

$$L[x''(t)]+25L[x(t)] = 21L(\cos 2t)$$

ie., $$\{s^2L[x(t)]-s\,x(0)-x'(0)\}+25L[x(t)] = \frac{21s}{s^2+4}$$

Using the initial conditions we obtain,

$$(s^2+25)\,L[x(t)] = \frac{21s}{s^2+4}$$

$$\therefore \quad x(t) = L^{-1}\left[\frac{21s}{(s^2+4)(s^2+25)}\right]$$

Let $$\frac{21s}{(s^2+4)(s^2+25)} = \frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+25}$$

or $$21s = (As+B)(s^2+25)+(Cs+D)(s^2+4)$$

Comparing the coefficients of s^3, s^2, s and constant on both sides we get,

$$A+C = 0 \quad ; \quad B+D = 0$$

$$25A+4C = 21 \quad ; \quad 25B+4D = 0$$

By solving these we get, $A = 1,\ B = 0,\ C = -1$ and $D = 0$

Hence $$L^{-1}\left[\frac{21s}{(s^2+4)(s^2+25)}\right] = L^{-1}\left[\frac{s}{s^2+4}\right]-L^{-1}\left[\frac{s}{s^2+25}\right]$$

Thus $\boldsymbol{x(t) = \cos 2t-\cos 5t}$

5. *A particle is moving with damped motion according to the law $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+25x = 0$. If the initial position of the particle is at $x = 20$ and the initial speed is 10, find the displacement of the particle at any time t using Laplace transforms.*

>> The given equation is $x''(t)+6x'(t)+25x(t) = 0$

Initial conditions are $x(0) = 20,\ x'(0) = 10$

Taking Laplace transform on both sides of the equation we have,

$$L[x''(t)]+6L[x'(t)]+25L[x(t)] = 0$$

ie., $$\{s^2L[x(t)]-sx(0)-x'(0)\}+6\{sL[x(t)]-x(0)\}+25L[x(t)] = 0$$

Using the initial conditions we obtain,

$$(s^2+6s+25)\,L[x(t)]-20s-10-120 = 0$$

ie., $$L[x(t)] = \frac{20s+130}{s^2+6s+25}$$

$$\therefore \quad x(t) = L^{-1}\left[\frac{20s+130}{(s+3)^2+16}\right]$$

ie., $$x(t) = L^{-1}\left[\frac{20(s+3)+70}{(s+3)^2+4^2}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{20s+70}{s^2+4^2}\right] = e^{-3t}\left(20\cos 4t+\frac{70}{4}\sin 4t\right)$$

Thus $\boldsymbol{x(t) = 10e^{-3t}(2\cos 4t+7/4\cdot\sin 4t)}$

6. *A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L resistance R. Show that the current at any time t is $\frac{E}{R-aL}\{e^{-at}-e^{-Rt/L}\}$*

>> The differential equation in respect of the L-R circuit is

$$L\frac{di}{dt}+Ri = E(t) \text{ where } E(t) = Ee^{-at} \text{ by data.}$$

Now the equation is put in the form

$$Li'(t)+Ri(t) = Ee^{-at} \ ; \ i(0) = 0$$

Taking Laplace transform (L_T) on both sides we get,

$$LL_T[i'(t)]+RL_T[i(t)] = EL_T(e^{-at})$$

ie., $L\left\{sL_T[i(t)]-i(0)\right\}+RL_T[i(t)] = \dfrac{E}{s+a}$

ie., $L_T[i(t)]\,(Ls+R) = \dfrac{E}{s+a}$ or $L_T[i(t)] = \dfrac{E}{(s+a)(Ls+R)}$

$\therefore$ $i(t) = L_T^{-1}\left[\dfrac{E}{(s+a)(Ls+R)}\right]$

Now, let $\dfrac{E}{(s+a)(Ls+R)} = \dfrac{A}{s+a}+\dfrac{B}{Ls+R}$

or $E = A(Ls+R)+B(s+a)$ ...(1)

Put $s=-a$: $E = A(-aL+R)$ $\therefore$ $A = \dfrac{E}{R-aL}$

Set $Ls+R=0$ or $s=-R/L$ and from (1)

$EL = B(-R+aL)$ $\therefore$ $B = \dfrac{-EL}{R-aL}$

Now $L_T^{-1}\left[\dfrac{E}{(s+a)(Ls+R)}\right] = \dfrac{E}{R-aL}\left\{L_T^{-1}\left[\dfrac{1}{s+a}\right] - L\,L_T^{-1}\left[\dfrac{1}{L(s+R/L)}\right]\right\}$

Thus $i(t) = \dfrac{E}{R-aL}\left\{e^{-at}-e^{-Rt/L}\right\}$

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7. *The current i and charge q in a series circuit containing an inductance L, capacitance C, e.m.f E satisfy the D.E.* $L\dfrac{di}{dt}+\dfrac{q}{C} = E$; $i = \dfrac{dq}{dt}$ *Express i and q in terms of t given that L,C,E are constants and the value of* i, q *are both zero initially.*

>> Since $i = \dfrac{dq}{dt}$ the D.E becomes

$$L\frac{d^2q}{dt^2}+\frac{q}{C} = E \text{ or } \frac{d^2q}{dt^2}+\frac{q}{LC} = \frac{E}{L}$$

ie., $q''(t)+\lambda^2 q(t) = \mu$, where $\lambda^2 = 1/LC$ and $\mu = E/L$

Taking Laplace transform (L_T) on both sides we have,

$$L_T[q''(t)]+\lambda^2 L_T[q(t)] = L_T(\mu)$$

ie., $\left\{s^2L_T[q(t)]-sq(0)-q'(0)\right\}+\lambda^2 L_T[q(t)] = \dfrac{\mu}{s}$

But $i=0,\ q=0$ at $t=0$ by data.

That is $q(0)=0,\ q'(0)=0$

Hence $(s^2+\lambda^2)L_T[q(t)] = \dfrac{\mu}{s}$ or $L_T[q(t)] = \dfrac{\mu}{s(s^2+\lambda^2)}$ ...(1)

$\therefore$ $q(t) = L_T^{-1}\left[\dfrac{\mu}{s(s^2+\lambda^2)}\right]$

Now $\dfrac{1}{s(s^2+\lambda^2)} = \dfrac{1}{\lambda^2}\left(\dfrac{1}{s}-\dfrac{s}{s^2+\lambda^2}\right)$ by partial fractions.

$\therefore$ $L_T^{-1}\left[\dfrac{\mu}{s(s^2+\lambda^2))}\right] = \dfrac{\mu}{\lambda^2}L_T^{-1}\left[\dfrac{1}{s}-\dfrac{s}{s^2+\lambda^2}\right]$

ie., $q(t) = \dfrac{\mu}{\lambda^2}(1-\cos\lambda t)$ where $\lambda^2 = 1/LC$ and $\mu = E/L$

Thus $q(t) = EC\left\{1-\cos(\sqrt{1/LC}\,t)\right\}$

---

8. *A resistance R in series with inductance L is connected with e.m.f* $E(t)$. *The current i is given by* $L\dfrac{di}{dt}+Ri = E(t)$. *If the switch is connected at* $t=0$ *and disconnected at* $t=a$ *find the current i in terms of t.*

>> We have by data $i=0$ at $t=0$ *ie.,* $i(0)=0$ and

$$E(t) = \begin{cases} E & \text{in } 0<t<a \\ 0 & \text{if } t \ge a \end{cases}$$

Also $Li'(t)+Ri(t) = E(t)$ is the given equation,

Taking Laplace transform (L_T) on both sides we have

$$L\,L_T[i'(t)]+RL_T[i(t)] = L_T[E(t)]$$

ie., $L\left\{sL_T[i(t)]-i(0)\right\}+RL_T[i(t)] = L_T[E(t)]$

ie., $L_T[i(t)]\,(Ls+R) = L_T[E(t)]$ ...(1)

Now to find $L_T[E(t)]$ we have by the definition,

$$L_T[E(t)] = \int_0^\infty e^{-st} E(t)\,dt = \int_0^a e^{-st}\cdot E\,dt + \int_a^\infty e^{-st}\cdot 0\,dt$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^a = \frac{-E}{s}(e^{-as}-1) = \frac{E}{s}(1-e^{-as})$$

Using this in the RHS of (1) we now have,

$$L_T[i(t)]\,(Ls+R) = \frac{E}{s}(1-e^{-as})$$

$$L_T[i(t)] = \frac{E(1-e^{-as})}{s(Ls+R)} = \frac{E}{s(Ls+R)} - \frac{Ee^{-as}}{s(Ls+R)}$$

$$\therefore \quad i(t) = L_T^{-1}\left[\frac{E}{s(Ls+R)}\right] - L_T^{-1}\left[\frac{Ee^{-as}}{s(Ls+R)}\right] \qquad \ldots(2)$$

Now, let $\dfrac{E}{s(Ls+R)} = \dfrac{A}{s} + \dfrac{B}{Ls+R}$ by resolving into partial fractions.

or $\quad E = A(Ls+R) + Bs$

Put $s = 0 : A = \dfrac{E}{R}$; Put $s = \dfrac{-R}{L} : E = B\left(\dfrac{-R}{L}\right) \quad \therefore \quad B = -\dfrac{EL}{R}$

Hence $\dfrac{E}{s(Ls+R)} = \dfrac{E}{R}\cdot\dfrac{1}{s} - \dfrac{EL}{R}\cdot\dfrac{1}{Ls+R}$

$$\therefore \quad L_T^{-1}\left[\frac{E}{s(Ls+R)}\right] = \frac{E}{R}L_T^{-1}\left(\frac{1}{s}\right) - \frac{E}{R}L_T^{-1}\left(\frac{1}{s+R/L}\right)$$

$$\text{ie.,} \quad L_T^{-1}\left[\frac{E}{s(Ls+R)}\right] = \frac{E}{R}(1-e^{-Rt/L}) \qquad \ldots(3)$$

Further we have the property of the unit step function,

$$L_T[f(t-a)u(t-a)] = e^{-as}\bar{f}(s) \text{ where } \bar{f}(s) = L_T[f(t)]$$

Taking $\bar{f}(s) = \dfrac{E}{s(Ls+R)}$ then $L_T^{-1}[\bar{f}(s)] = L_T^{-1}\left[\dfrac{E}{s(Ls+R)}\right]$

ie., $\quad f(t) = \dfrac{E}{R}(1-e^{-Rt/L})$ by (3).

Also $\quad L_T^{-1}[e^{-as}\bar{f}(s)] = f(t-a)u(t-a)$

$$\text{ie.,} \quad L_T^{-1}\left[e^{-as}\frac{E}{s(Ls+R)}\right] = \frac{E}{R}\left[1-e^{-R(t-a)/L}\right]u(t-a)$$

But $\quad u(t-a) = \begin{cases} 0 \text{ in } 0<t<a \\ 1 \text{ if } t \geq a \end{cases}$

$$\therefore \quad L_T^{-1}\left[e^{-as}\frac{E}{s(Ls+R)}\right] = \frac{E}{R}\left[1-e^{-R(t-a)/L}\right] \text{ when } t \geq a$$

$$= 0 \text{ in } 0<t<a \qquad \ldots(4)$$

Using the results (3) and (4) in (2) we get

$$i(t) = \frac{E}{R}\left[1-e^{-Rt/L}\right] \text{ in } 0<t<a$$

Also, $\quad i(t) = \dfrac{E}{R}\left[1-e^{-Rt/L}\right] - \dfrac{E}{R}\left[1-e^{-R(t-a)/L}\right]$ when $t \geq a$ $\qquad \ldots(5)$

ie., $\quad i(t) = \dfrac{E}{R}\left[e^{-R(t-a)/L} - e^{-Rt/L}\right]$ when $t \geq a$ $\qquad \ldots(6)$

Thus (5) and (6) represents the required $i(t)$ in terms of t.

Remark : *Many of these application problems has also been solved in differential equations method (Refer Module-1)*

EXERCISES

Verify convolution theorem for the following functions. **[1 to 4]**

1. $f(t) = 1,\ g(t) = \cos t$
2. $f(t) = \sin at,\ g(t) = \cos bt$
3. $f(t) = t^2,\ g(t) = te^{-2t}$
4. $f(t) = e^t,\quad g(t) = \cos t$

Applying convolution theorem find the inverse Laplace transform of the following functions. **[5 to 10]**

5. $\dfrac{s}{(s^2+4)^2}$
6. $\dfrac{s^2}{(s^2+9)^2}$