MODULE – I.

MATHEMATICAL LOGIC – I.

Proposition: A proposition is a declarative sentence that is either true or false, but not both.

ex: 1. 2 is a prime number. (True)

2. All sides are equal in scalene (False)

3. $2+3=4$.

4. What is the time now.

5. Read this carefully.

From the above examples we note that 1. 2. 3 are propositions, where as 4 and 5 are not the propositions.

Connectives and Truth table

New propositions are obtained by starting with given propositions with the aid of words or phrases like 'not', 'and', 'if... then', and 'if and only if'. Such words or phrases are called Logical connectives

1. Negation: A proposition obtained by inserting the word 'not' at an appropriate place in the given proposition is called the negation of the given proposition

Truth Table.

p	$\neg p$
0	1
1	0

ex: p: 4 is an even number.
$\neg p$: 4 is not an even number.

Conjunction: A compound proposition obtained by combining two given propositions by inserting the word 'and' in between them is called the conjunction of the given proposition.

p: $\sqrt{2}$ is an irrational number.
q: 9 is a prime number.
$p \wedge q$: $\sqrt{2}$ is an irrational number and 9 is a prime number.

Truth Table for Conjunction.

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Disjunction: A compound proposition obtained by combining two given propositions by inserting the word 'or' in between them is called the disjunction of the given propositions.

p: All triangles are equilateral.
q: $2+5=7$
$p \vee q$: All triangles are equilateral or $2+5=7$

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive Disjuction: The compound proposition $p \underline{\vee} q$ (read as either p or q but not both) is called the exclusive disjunction of the propositions p & q. Its truth table is as given below.

Truth table.

p	q	$p \underline{\vee} q$.
0	0	0
0	1	1
1	0	1
1	1	0

ex: p: 9 is a prime number
q: All triangles are isosceles.

$p \underline{\vee} q$: Either 9 is a prime number or All triangles are isosceles but not both.

Conditional: A compound proposition obtained by combining two given propositions by using the words 'if' and 'then' at appropriate places is called a conditional.

ex: p: 3 is a prime number
q: 9 is a multiple of 6.

p	q	$p \to q$.
0	0	1
0	1	1
1	0	0
1	1	1

Biconditional: Let p and q be two simple propositions. Then the conjuction of the conditionals $p \to q$ and $q \to p$ is called the biconditional of p and q.

Truth table for Biconditional.

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

1. Construct the truth tables for the following compound propositions:
(i) $p \wedge (\neg q)$ (ii) $(\neg p) \vee q$ (iii) $p \to (\neg q)$ (iv) $(\neg p) \veebar (\neg q)$.

Sol: The desired truth tables are obtained by considering all possible combinations of the truth values of p and q. The combined form of the required truth tables is shown below.

p	q	$\neg p$	$\neg q$	$p \wedge (\neg q)$	$(\neg p) \vee q$	$p \to (\neg q)$	$(\neg p) \veebar (\neg q)$
0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	1	0	0	0	1	0	0

2. Let p, q and r be propositions having truth values 0, 0 and 1 respectively. Find the truth values of the following compound propositions:

(i) $(p \vee q) \vee r$ (ii) $(p \wedge q) \wedge r$ (iii) $(p \wedge q) \to r$

(iv) $p \to (q \wedge r)$ (v) $p \wedge (r \to q)$ (vi) $p \to (q \to \neg r)$

Sol:

(i). Since both p and q are false. Since r true, it follows that $(p \vee q) \vee r$ is true. Thus, the truth value of $(p \vee q) \vee r$ is 1.

(ii). Since both p and q are false, $p \wedge q$ is false. Since $p \wedge q$ is false and r is true $(p \wedge q) \wedge r$ is false. Thus, the truth value of $(p \wedge q) \wedge r$ is 0.

(iii) Since $p \wedge q$ is false and r is true. $(p \wedge q) \to r$ is true. Thus, the truth value of $(p \wedge q) \to r$ is 1.

(iv) Since q is false and r is true $q \wedge r$ is false. Also p is false. Therefore $p \to (q \wedge r)$ is true. Thus, the truth value of $p \to (q \wedge r)$ is 1.

(v). Since r is true and q is false $r \to q$ is false. Also, p is false. Hence, $p \wedge (r \to q)$ is false. Thus, the truth value of $p \wedge (r \to q)$ is 0.

(vi) Since r is true, $\neg r$ is false. Since q is false, $q \to (\neg r)$ is true. Also, p is false. Therefore, $p \to (q \to (\neg r))$ is true. Thus, the truth value of $p \to (q \to (\neg r))$ is 1.

3). Indicate how many rows are needed in the truth table for the compound proposition

$(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t\}$.

Find the truth value of this proposition if p and r are true and q, s, t are false.

Sol: The given compound proposition contains five primitives p, q, r, s, t. Therefore, the number of possible combinations of the truth values of these components which we have to consider is $2^5 = 32$. Hence 32 rows are needed in the truth table for the given compound proposition.

Next, suppose that p and r are true and q, s, t are false. Then $\neg q$ is true and $\neg r$ is false. Since p is true and $\neg q$ is true, $p \vee (\neg q)$ is true. On the other hand, since $\neg r$ is false and s is false, $\neg r \wedge s$ is false. Also, t is false. Hence $(\neg r \wedge s) \rightarrow t$ is true.

Since $p \vee (\neg q)$ is true and $(\neg r \wedge s) \rightarrow t$ is true, it follows that the truth value of the given proposition $(p \vee (\neg q)) \leftrightarrow \{(\neg r \wedge s) \rightarrow t\}$ is 1.

4. Construct the truth tables for the following compound propositions:

(i) $(p \wedge q) \rightarrow (\neg r)$ (ii) $q \wedge ((\neg r) \rightarrow p)$

Sol: The required truth tables are shown below in a combined form.

p	q	r	$\neg r$	$p \wedge q$	$(p \wedge q) \to (\neg r)$	$(\neg r) \to p$	$q \wedge ((\neg r) \to p)$
0	0	0	1	0	1	0	0
0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	0	1	0	1	1

Tautology and Contradiction

A compound proposition which is true for all possible truth values of its components is called a tautology

A compound proposition which is false for all possible truth values of its components is called a contradiction or an absurdity

A compound proposition that can be true or false is called a contingency.

1. Show that; for any propositions p and q, the compound proposition $p \to (p \vee q)$ is a tautology and the compound proposition $p \wedge (\neg p \wedge q)$ is a contradiction

Sol: Let us first prepare the truth tables for $p \to (p \vee q)$ and $p \wedge (\neg p \wedge q)$. These truth tables are shown below in the combined form.

p	q	$p \vee q$	$p \to (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

From the above table we note that, for all possible truth values of p and q, the compound proposition $p \to (p \vee q)$ is true and the compound proposition $p \wedge (\neg p \wedge q)$ is false. Therefore $p \to (p \vee q)$ is a tautology and $p \wedge (\neg p \wedge q)$ is a contradiction.

2. Prove that, for any propositions p, q, r the compound proposition

$$[(p \to q) \wedge (q \to r)] \to (p \to r) \text{ is a tautology.}$$

Sol: The following truth table proves the required result.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

HW 1. Prove that for any propositions p, q, r, the compound proposition

$[(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}] \to r$ is a tautology.

2. Prove that for any propositions p, q, r the compound proposition

$[(p \to q) \vee (p \to r)] \Leftrightarrow [p \to (q \vee r)]$ is a tautology.

Logical Equivalence: The Laws of Logic.

Two statements s_1, s_2 are said to be logically equivalent, and we write $s_1 \Leftrightarrow s_2$, when the statement s_1 is true (respectively false) if and only if the statement s_2 is true (respectively, false).

OR The biconditional $s_1 \leftrightarrow s_2$ is a tautology.

1. Prove that for any two propositions p, q

$(p \rightarrow q) \Leftrightarrow (\neg p) \vee q$

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

2. For any two propositions p, q prove that

$(p \veebar q) \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$

p	q	$p \vee q$	$p \veebar q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	1	0	1	0	0

From columns 4 and 8 of the above truth table, we find that $p \veebar q$ and $(p \vee q) \wedge \neg(p \wedge q)$ have the same truth values of p and q.

Therefore $(p \veebar q) \Longleftrightarrow \{(p \vee q) \wedge (\neg p \vee \neg q)\}$.

3. Show that the compound propositions $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are not logically equivalent.

p	q	r	$\neg q$	$\neg r$	$\neg q \vee r$	$q \wedge \neg r$	$p \wedge (\neg q \vee r)$	$p \vee (q \wedge \neg r)$
0	0	0	1	1	1	0	0	0
0	0	1	1	0	1	0	0	0
0	1	0	0	1	0	1	0	1
0	1	1	0	0	1	0	0	0
1	0	0	1	1	1	0	1	1
1	0	1	1	0	1	0	1	1
1	1	0	0	1	0	1	0	1
1	1	1	0	0	1	0	1	1

From the last two rows we note that $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ do not have the same truth values in all possible situations. Therefore they are not logically equivalent.

Laws of Logic.

For any primitive statements p, q, r, any tautology T_0, and any contradiction F_0.

1. $\neg\neg p \Leftrightarrow p$ — Law of Double Negation.
2. $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
   $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ — DeMorgan's Laws.
3. $p \vee q \Leftrightarrow q \vee p$
   $p \wedge q \Leftrightarrow q \wedge p$ — Commutative laws.

4. $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ Associative Laws.
   $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

5. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ Distributive laws.
   $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

6. $p \vee p \Leftrightarrow p.$ Idempotent Laws.
   $p \wedge p \Leftrightarrow p.$

7. $p \vee F_0 \Leftrightarrow p$ Identity laws.
   $p \wedge T_0 \Leftrightarrow p$

8. $p \vee \neg p \Leftrightarrow T_0$ Inverse Laws.
   $p \wedge \neg p \Leftrightarrow F_0.$

9. $p \vee T_0 \Leftrightarrow T_0$ Domination Laws.
   $p \wedge F_0 \Leftrightarrow F_0.$

10. $p \vee (p \wedge q) \Leftrightarrow p$ Absorption Laws.
    $p \wedge (p \vee q) \Leftrightarrow p$

1) Prove Distributive law $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

From columns 5 and 8 of the above table, we find that $[p \vee (q \wedge r)]$ and $[(p \vee q) \wedge (p \vee r)]$ have the same truth values in all possible situations. Therefore,

$$[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)].$$

Law for the Negation of a conditional.

Given a conditional $p \to q$, its negation is obtained by using the following law.

$$\neg (p \to q) \Leftrightarrow [p \wedge (\neg q)].$$

Proof: The following table gives the truth values of $\neg(p \to q)$ and $(p \wedge \neg q)$ for all possible truth values of p and q.

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

We note that $\neg(p \to q)$ and $(p \wedge \neg q)$ have the same truth values in all possible situations.

Hence $\neg (p \to q) \Leftrightarrow p \wedge \neg q$.

1. Simplify the following compound propositions using the laws of logic.

(i) $(p \vee q) \wedge [\neg\{(\neg p) \wedge q\}]$ (ii) $(p \vee q) \wedge [\neg\{(\neg p) \vee q\}]$

(iii) $\neg[\neg\{(p \vee q) \wedge r\} \vee \neg q]$.

Sol: (i). $(p \vee q) \wedge [\neg\{(\neg p) \wedge q\}]$

$\equiv (p \vee q) \wedge \{(\neg\neg p) \vee (\neg q)\}$ by D'Morgans law.

$\equiv (p \vee q) \wedge \{p \vee (\neg q)\}$ by law of double negation.

$\equiv p \vee \{q \wedge (\neg q)\}$ by distributive law.

$\equiv p \vee F_0$ by inverse law.

$\equiv p$ by identity law.

(ii) $(p \vee q) \wedge [\neg\{(\neg p) \vee q\}]$

$\equiv (p \vee q) \wedge (p \wedge \neg q)$

$\equiv \{(p \vee q) \wedge p\} \wedge (\neg q)$ using Associative law.

$\equiv \{p \wedge (p \vee q)\} \wedge (\neg q)$, using commutative law.

$\equiv p \wedge (\neg q)$ using Absorption Law.

(iii) $\neg[\neg\{(p \vee q) \wedge r\} \vee \neg q\}$

$\equiv \neg[\neg\{((p \vee q) \wedge r) \wedge q\}]$ using Demorgan law.

$\equiv ((p \vee q) \wedge r) \wedge q$ Law of Double negation.

$\equiv (p \vee q) \wedge (q \wedge r)$ Using Associative and Commutative Law.

$\equiv \{(p \vee q) \wedge q\} \wedge r$, using Associative law.

$\equiv q \wedge r$ using Absorption law.

2. Prove the following logical equivalence without using truth tables.

(i) $[p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$.

(ii) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$.

(iii) $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$.

Sol: (i). We have

$\neg p \wedge \neg q \wedge r \Leftrightarrow (\neg p \wedge \neg q) \wedge r$ by Associative law.

$\Leftrightarrow \neg(p \vee q) \wedge r$ by Demorgan law.

Therefore,

$p \vee q \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow (p \vee q) \vee [\neg(p \vee q) \wedge r]$

$\Leftrightarrow [(p \vee q) \vee \neg(p \vee q)] \wedge [(p \vee q) \vee r]$ - by Distribution law.

$\Leftrightarrow T_0 \wedge (p \vee q \vee r)$ by Inverse and Associative Law.

$\Leftrightarrow p \vee q \vee r$ by Commutative and Identiti law.

(ii) We have

$(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r) \Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$ because $u \rightarrow v \Leftrightarrow (\neg u \vee v)$

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$. by De Morgan law and Associative Law.

$\Leftrightarrow p \wedge q$ by Absorption law.

(iii) We have

$p \rightarrow (q \rightarrow r) \Leftrightarrow \neg p \vee (\neg q \vee r)$ $\quad u \rightarrow v \Leftrightarrow (\neg u \vee v)$

$\Leftrightarrow (\neg p \vee \neg q) \vee r$ Associative law

$\Leftrightarrow \neg(p \wedge q) \vee r$ DeMorgan law.

$\Leftrightarrow (p \wedge q) \rightarrow r$. $\quad u \rightarrow v \Leftrightarrow \neg(u \vee v)$.

Duality: Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, the the dual of s denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$ respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

ex: Given the primitive statements p, q, r and the compound statement

$s: (p \wedge \neg q) \vee (r \wedge T_0)$

$s^d: (p \vee \neg q) \wedge (r \vee F_0)$

Principle of Duality: Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$. If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

1) Write duals of the following propositions.

(i) $p \to q$ (ii) $(p \to q) \to r$ (iii) $p \to (q \to r)$

Sol: We recall that $(u \to v) \Leftrightarrow (\neg u \vee v)$

Therefore by the principle of duality, we find that

(i) $(p \to q)^d \Leftrightarrow (\neg p \vee q)^d \equiv \neg p \wedge q$.

(ii) $[(p \to q) \to r]^d \Leftrightarrow [\neg(p \to q) \vee r]^d$

$\Leftrightarrow [(p \wedge \neg q) \vee r]^d$.

$\Leftrightarrow (p \vee \neg q) \wedge r$.

iii) $[p \to (q \to r)]^d \Leftrightarrow [\neg p \vee (q \to r)]^d$

$\Leftrightarrow [\neg p \vee (\neg q \vee r)]^d \Leftrightarrow \neg p \wedge (\neg q \wedge r)$.

NAND and NOR.

The compound proposition $\neg(p \wedge q)$ is read as "Not p and q" and is also denoted by $(p \uparrow q)$

The symbol $\uparrow$ is called the NAND connective.

The compound proposition $\neg(p \vee q)$ is read as "Not (p or q)" and is also denoted by $(p \downarrow q)$.

The symbol $\downarrow$ is called the NOR connective.

Truth table

p	q	$p \uparrow q$	$p \downarrow q$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

$p \uparrow q = \neg(p \wedge q)$
$\Leftrightarrow \neg p \vee \neg q.$

$p \downarrow q = \neg(p \vee q)$
$\Leftrightarrow \neg p \wedge \neg q.$

1. For any propositions p, q prove the following.

(i) $\neg(p \downarrow q) \Leftrightarrow (\neg p \uparrow \neg q)$ (ii) $\neg(p \uparrow q) \Leftrightarrow \neg p \downarrow \neg q$.

Sol. Using definition, we find that

(i) $\neg(p \downarrow q) \Leftrightarrow \neg\{\neg(p \vee q)\}$.
$\Leftrightarrow \neg(\neg p \wedge \neg q)$
$\Leftrightarrow (\neg p) \uparrow (\neg q)$.

(ii) $\neg(p \uparrow q) \Leftrightarrow \neg\{\neg(p \wedge q)\}$
$\Leftrightarrow \neg(\neg p \vee \neg q)$
$\Leftrightarrow (\neg p) \downarrow (\neg q)$.

Converse, Inverse and Contrapositive.

Consider a conditional $p \to q$ Then:

1. $q \to p$ is called the converse of $p \to q$
2. $\neg p \to \neg q$ is called the inverse of $p \to q$
3. $\neg q \to \neg p$ is called the contrapositive of $p \to q$

Truth table for converse, Inverse and Contrapositive.

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

Note: 1. A conditional and its contrapositive are logically equivalent. i.e, $(p \to q) \Leftrightarrow (\neg q) \to (\neg p)$.

2. A converse and the inverse of a conditional are logically equivalent

$$(q \to p) \Leftrightarrow (\neg p) \to (\neg q).$$

Logical Implication – Rules of Inference.

Let us consider the implication

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q.$$

Here n is a positive integer, the statements $p_1, p_2 \ldots p_n$ are called the premises of the argument

1. State the converse inverse and contrapositive of
i) If a triangle is not isosceles, then it is not equivalateral
ii). If a real number x^2 is greater than zero, then x is not equal to zero.

Sol: let p: Triangle is ~~not~~ isosceles.
(i) q: Triangle is not equilateral.

Implication: $p \to q$. If a tringle is not isosceles then it is not equilateral.

Converse: $q \to p$. If a triangle is not equilateral then it is not isosceles.

Inverse : $\neg p \to \neg q$ If a triangle is isosceles then it is not equilateral.

Contrapositive : $\neg q \to \neg p$. If a tringle is equilateral then it is isoscelesn.

ii). p: A real number x^2 is greater than zero.
q: x is not equal to zero.

Implication $p \to q$ If a real number x^2 is greater than zero, then x is not equal to zero.

Converse: $q \to p$ If a real number x^2 is not equal to zero, then x^2 is greater than zero.

Inverse: If a real number x^2 is not greater than zero then x is equal to zero. $\neg p \to \neg q$.

Contrapositive: If a real number x is equal to zero then x^2 is not greater than zero.

Name of Rule and Rule of Inference.

Rule of Inference		Name of Rule.
1.	p $p \to q$ $\therefore q$ p antecedent, q consequent	Rule of Detatchment (Modus Ponens)
2.	$p \to q$ $q \to r$ $\therefore p \to r$	Law of Syllogism.
3.	$p \to q$ $\neg q$ $\therefore \neg p$	Modus Tollens
4.	p q $\therefore p \wedge q$	Rule of conjunction
5.	$p \vee q$ $\neg p$ $\therefore q$	Rule of Disjunctive Syllogism.
6.	$\neg p \to F_0$ $\therefore p$	Rule of contradiction
7.	$p \wedge q$ $\therefore p$	Rule of Conjunctive Simplification.
8.	p $\therefore p \vee q$	Rule of Disjunctive Amplification

1. Test whether the following is a valid argument.

If Sachin hits a century, he gets a free car.
Sachin gets a free car.
∴ Sachin has hit a century.

Sol: let p: Sachin hits a century
q: Sachin gets a free car.

the given statement reads
$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$
Modus pones.

We note that if $p \to q$ and q is true it is valid there is no rule which asserts that p must be true. Indeed p can be false when $p \to q$ and q are true

p	q	$p \to q$	$(p \to q) \wedge q$
0	1	1	1

Thus, $[(p \to q) \wedge q] \to p$ is not a tautology. ∴ the given argument is not a valid one.

2. Test the validity of the following argument.

If Ravi goes out with friends, he will not study.
If Ravi does not study, his father becomes angry.
His father is not angry
∴ Ravi has not gone out with friends.

Sol: let p: Ravi goes out with friends.
q: Ravi does not study

r: His father gets angry

Then the given argument reads.

$$\begin{array}{c} p \to q \\ q \to r \\ \neg r \\ \hline \therefore \neg p. \end{array}$$

this argument is logically equivalent to.

$$\begin{array}{c} p \to r \\ \neg r. \\ \hline \therefore \neg p \end{array}$$ using Rule of syllogism.

In view of Modus Tollens rule, this is a valid argument.

3. Test the validity of the following argument

If I study, I will not fail in the examination

If I do not watch TV in the evenings. I will study

I failed in the examination.

∴ I must have watched TV in the evenings.

Sol: let p: I study . q: I fail in the examination

r: I watch TV in the evenings,

Then the given argument reads.

$$\begin{array}{c} p \to \neg q \\ \neg r \to p. \\ q \\ \hline \therefore r \end{array}$$

This argument is logically equivalent to

$$q \to \neg p \quad (\text{because } (p \to \neg q) \Leftrightarrow (\neg\neg q \to \neg p))$$
$$\neg p \to r \quad (\text{because } (\neg r \to p) \Leftrightarrow (\neg p \to r)).$$
$$\frac{q}{\therefore r.}$$

This is equivalent to.

$$q \to r \quad (\text{using Rule of Syllogism})$$
$$\frac{q.}{\therefore r.}$$

This argument is valid, by the modus Pones Rule.

Quantifiers

Open statement: A declarative sentence is an open statement if
i) It contains one or more variables
ii) it is not a statement, but
iii) it becomes a statement when the varidbles in it are replaced by certain allowable choices.

ex: "The number $x+2$ is an even integer" is denoted by $P(x)$ then $\neg P(x)$ may be read as "The number $x+2$ is not an even integer"

Quantifiers: The words "all", "every", "some", "there exists" are associated with the idea of a quantity, such words are called quantifiers.

1. Universal Quantifiers: The symbol $\forall$ has been used to denote the phrases "for all" and "for every" in logic. "for each" and "for any" are also taken up to be equivalent to these. These equivalent phrases is called the Universal quantifiers.

ex: All squares are rectangles ie, $\forall x \in S, P(x)$.

2. Existential quantifiers: The symbol $\exists$ has been used to denote the phrases "there exists" "for some" "for atleast one". Each of these equivalent phrases is called the Existential Quantifiers.

ex: for every integer x, x^2 is a non-negative integer $\exists x \in Z$, $P(x)$.

ex: 1. for the universe of all integers, let

$p(x): x>0$, $q(x): x$ is even

$r(x): x$ is a perfect square.

$s(x): x$ is divisible by 3.

$t(x): x$ is divisible by 7.

Write down the following quantified statements in symbolic form:

i) At least one integer is even.

ii) There exists a positive integer that is even.

iii) Some integers are divisible by 3.

iv) Every integer is either even or odd.

v) If x is even and a perfect square, then x is not divisible by 3.

vi) If x is odd or is not divisible by 7, then x is divisible by 3.

Sol: Using the definition of quantifiers, we find that the given statements read as follows in symbolic form

i) $\exists x,\ q(x)$ ii) $\exists x,\ [p(x) \wedge q(x)]$

iii) $\exists x,\ [q(x) \wedge s(x)]$ iv) $\forall x,\ [q(x) \underline{\vee} \neg q(x)]$

v) $\forall x\ [\{q(x) \wedge r(x)\} \rightarrow \neg s(x)]$.

vi) $\forall x,\ [\{\neg q(x) \vee \neg t(x)\} \rightarrow s(x)]$.

Rules employed for determining truth value.

Rule 1: The statement "$\forall x \in S,\ P(x)$" is true only when $P(x)$ is true for each $x \in S$.

Rule 2: The statement "$\exists x \in S,\ P(x)$" is false only when $P(x)$ is false for every $x \in S$.

Rules of inference

Rule 3: If an open statement $P(x)$ is known to be true for all x in a universe S and if $a \in S$ then $p(a)$ is true

(This is Known as the Rule of Universal Specification)

Rule 4: If an open statement $P(x)$ is proved to be true for any (arbitrary) x chosen from a set S then the quantified statement $\forall x \in S, P(x)$ is true.

(This is known as the Rule of Universal Generalization).

Logical Equivalence: Two quantified statements are said to be logically equivalent whenever they have the same truth values in all possible situations

The following results are easy to prove.

i) $\forall x\,[P(x) \wedge q(x)] \iff (\forall x\, P(x)) \wedge (\forall x,\, q(x))$

ii) $\exists x\,[P(x) \vee q(x)] \iff (\exists x, P(x)) \vee (\exists x, q(x))$

iii) $\exists x, [P(x) \rightarrow q(x)] \iff \exists x, [\neg P(x) \vee q(x)]$.

Rule for Negation of a Quantified Statement

Rule 5: To construct the negation of a quantified statement change the quantifier from universal to existential and vice versa.

i.e, $\neg[\forall x, P(x)] \equiv \exists x, [\neg P(x)]$

$\neg[\exists x, P(x)] \equiv \forall x\ [\neg P(x)]$

1. Consider the open statements with the set of real numbers as the universe.

$P(x): |x| > 3$, $q(x): x > 3$

Find the truth value of the statement

$\forall x, [P(x) \rightarrow q(x)]$ — (i).

Also, write down the converse, inverse and the contrapositive of this statement and find their truth values

Sol: We readily note that

$p(-4) \equiv |-4| > 3$ is true and $q(-4) \equiv -4 > 3$ is false

Thus, $p(x) \to q(x)$ is false for $x = -4$. Accordingly the given statement (i) is false.

The converse of the statement (i) is

$\forall x, [q(x) \to p(x)]$ (ii)

In words, this reads

"For every real number x, if $x > 3$ then $|x| > 3$" or equivalently,

"Every real number greater than 3 has its absolute value (magnitude) greater than 3"

This is a true statement.

Next, the inverse of the statement (i) is

$\forall x, [\neg p(x) \to \neg q(x)]$ (iii)

In words this reads

"For every real number x, if $|x| \not> 3$ then $x \leq 3$" or equivalently

"If the magnitude of a real number x is less than or equal to 3, then the number x is less than or equal to 3.

Since the converse and inverse of a conditional are logically equivalent the statements (ii) and (iii)

(iii) have the same truth values. Thus iii) is a true statements,

Contra positive

$\forall x, [\neg q(x) \rightarrow \neg p(x)]$ — iv

In words

"Every real number which is less than or equal to 3 has its magnitude less than or equal to 3."

2). Let- $p(x): x^2-7x+10=0$, $q(x): x^2-2x-3=0$ $r(x): x<0$ Determine the truth or falsity of the following statements. when the universe U contains only the integers 2 and 5. If a statement is false. provide a counter example or explanation.

(i) $\forall x, p(x) \rightarrow \neg r(x)$, (ii) $\forall x, q(x) \rightarrow r(x)$

(iii) $\exists x, q(x) \rightarrow r(x)$ (iv) $\exists x, p(x) \rightarrow r(x)$.

Sol: Here, the universe is $U = \{2, 5\}$.

We note that $x^2-7x+10 = (x-5)(x-2)$, Therefore $p(x)$ is true for $x=5$ and 2. That is $p(x)$ is true for all $x \in U$.

Further $x^2-2x-3 = (x-3)(x+1)$. Therefore, $q(x)$ is true only for $x=3$ and $x=-1$. Since $x=3$ and $x=-1$ are not in the universe, $q(x)$ is false for all $x \in U$.

obviously $r(x)$ is false for all $x \in U$.

Accordingly:

(i) Since $p(x)$ is true for all $x \in U$ and $\neg r(x)$ is true for all $x \in U$, the statement $\forall x, p(x) \rightarrow \neg r(x)$ is true.

(ii) Since $q(x)$ is false for all $x \in U$ and $r(x)$ is false for all $x \in U$, the statement $\forall x, q(x) \rightarrow r(x)$ is true.

(iii) Since $q(x)$ and $r(x)$ are false for $x = 2$, the statement $\exists x, q(x) \rightarrow r(x)$ is true.

(iv). Since $p(x)$ is true for all $x \in U$ but $r(x)$ is false for all $x \in U$. the statement $p(x) \rightarrow r(x)$ is false for every $x \in U$. Consequently, $\exists x, p(x) \rightarrow r(x)$ is false.

3) Negate and simplify each of the following.

(i) $\exists x\, [p(x) \vee q(x)]$ (ii) $\forall x, [p(x) \wedge \neg q(x)]$

(iii) $\forall x, [p(x) \rightarrow q(x)]$ (iv) $\exists x, [p(x) \vee q(x)] \rightarrow r(x)$.

Sol: By using the rule of negation for quantified statements and the laws of logic, we find that

(i) $\neg [\exists x, \{p(x) \vee q(x)\}] \equiv \forall x\, [\neg \{p(x) \vee q(x)\}]$
$\equiv \forall x\, [\neg p(x) \wedge \neg q(x)]$

(ii) $\neg [\forall x, [p(x) \wedge \neg q(x)]] \equiv \exists x\, [\neg \{p(x) \wedge \neg q(x)\}]$
$\equiv \exists x\, [\neg p(x) \vee q(x)]$.

(iii) $\neg [\forall x, \{p(x) \rightarrow q(x)\}] \equiv \exists x.\, [\neg \{p(x) \rightarrow q(x)\}]$
$\equiv \exists x.\, [p(x) \wedge \neg q(x)]$.

(iv) $\neg[\exists x, \{p(x) \vee q(x)\} \rightarrow r(x)] \equiv \forall x [\neg\{p(x) \vee q(x)\} \rightarrow r(x)]$
$\equiv \forall x [\{p(x) \vee q(x)\} \wedge \neg r(x)]$

4. Write down the following proposition in symbolic form, and find its negation:
"If all triangles are right-angled, then no triangle is equiangular".

Sol: Let T denote the set of all triangles. Also, let
$p(x)$: x is right-angled, $q(x)$: x is equiangular
Then, in symbolic form, the given proposition reads
$\{\forall x \in T, p(x)\} \rightarrow \{\forall x \in T, \neg q(x)\}$
The negation of this is
$\{\forall x \in T, p(x)\} \wedge \{\exists x \in T\ q(x)\}$
In words, this reads "All triangles are right angled and some triangles are equiangular".

Logical Implication involving Quantifiers.

1. Prove that
$\exists x, [p(x) \wedge q(x)] \Rightarrow \exists x, p(x) \wedge \exists x, q(x)$.
Is the converse true

Sol: Let S denote the universe. We find that
$\exists x, [p(x) \wedge q(x)] \Rightarrow p(a) \wedge q(a)$ for some $a \in S$
$\Rightarrow p(a)$, for $a \in S$ and $q(a)$ for some $a \in S$
$\Rightarrow \exists x, p(x) \wedge \exists x, q(x)$.

This proves the required implication.

Next, we observe that $\exists x, P(x) \Rightarrow P(a)$ for some $a \in S$ and $\exists x, q(x) \Rightarrow q(b)$ for some $b \in S$.

Therefore,

$$\exists x, P(x) \land \exists x, q(x) \Rightarrow P(a) \land q(b)$$

$$\not\Leftrightarrow P(a) \land q(a) \text{ because } b \text{ need not be } a.$$

Thus,

$\exists x, [P(x) \land q(x)]$ need not be true when $\exists x\, P(x) \land \exists x, q(x)$ is true, That is

$$\exists x, P(x) \land \exists x, q(x) \not\Rightarrow \exists x, [P(x) \land q(x)]$$

Accordingly, the converse of the given implication is not necessarily true.

2. Find whether the following argument is valid:

No engineering student of first or second semester studies logic

Anil is an Engineering student who studies logic.

∴ Anil is not in Second Semester.

Sol: Let us take the universe to be the set of all engineering students.

$P(x)$: x is in first semester $q(x)$: x is in Second semester

$r(x)$: x studies logic. a: Anil.

Then the given argument reads

$$\forall x, [\{P(x) \lor q(x)\} \to \neg r(x)]$$

$$\frac{r(a)}{\therefore \neg q(a).}$$

We note that

$\forall x, [\{p(x) \vee q(x)\} \to \neg r(x)] \Rightarrow \{p(a) \vee q(a)\} \to \neg r(a)$

by rule of Universal specification.

Therefore.

$[\forall x, [p(x) \vee q(x)] \to \neg r(x)] \wedge r(a)$

$\Rightarrow [\{p(a) \vee q(a)\} \to \neg r(a)] \wedge r(a)$

$\Rightarrow r(a) \wedge [r(a) \to \neg [p(a) \vee q(a)]]$, using commutative law and contrapositive

$\Rightarrow \neg [p(a) \vee q(a)]$ by the Modus Pones Rule.

$\Rightarrow \neg p(a) \wedge \neg q(a)$, by Demorgan law.

$\Rightarrow \neg q(a)$ by the Rule of Conjunctive specification.

This proves that the given argument is valid.

3. Find whether the following argument is valid.

If a triangle has two equal sides, then it is isosceles

If a triangle is isosceles, then it has two equal angles

A certain triangle ABC does not have two equal angles

$\therefore$ The triangle ABC does not have two equal sides

Sol: Let the universe be the set of all triangles and let

$p(x)$: x has equal sides

$q(x)$: x is isosceles.

$r(x)$: x has two equal angles.

Also let c denote the triangle ABC.

Then, in symbols, the given argument reads as follows:

$$\begin{array}{l} \forall x, [p(x) \to q(x)] \\ \forall x, [q(x) \to r(x)] \\ \underline{\neg r(c)} \\ \therefore \neg p(c) \end{array}$$

We note that

$\forall x, [p(x) \to q(x)] \land \forall x [q(x) \to r(x)] \land \neg r(c)$

$\Rightarrow \{\forall x, [p(x) \to r(x)] \land \neg r(c)$ Rule of syllogism.

$\Rightarrow [p(c) \to r(c)] \land \neg r(c)$, by rule of Universal specification.

$\Rightarrow \neg p(c)$ by Modus Tollens Rule.

This proves that the given argument is valid.

4. Prove that the following argument is valid

$$\begin{array}{l} \forall x, [p(x) \lor q(x)] \\ \exists x, \neg p(x) \\ \forall x, [\neg q(x) \lor r(x)] \\ \underline{\forall x, [s(x) \to \neg r(x)]} \\ \therefore \exists x, \neg s(x). \end{array}$$

Sol: We note that

$\{\forall x [p(x) \lor q(x)]\} \land \{\exists x, \neg p(x)\}$.

$\Rightarrow \{p(a) \lor q(a)\} \land \neg p(a)$. for some a in the universe.

$\Rightarrow q(a)$ by disjunctive syllogism.

Therefore $\{\forall x [p(x) \lor q(x)]\} \land \{\exists x, \neg p(x)\} \land \{\forall x, [\neg q(x) \lor r(x)]\}$

$\Rightarrow q(a) \land \{\neg q(a) \lor r(a)\}$

$\Rightarrow r(a)$ by rule of disjunctive Syllogism.

Consequently,

$\{\forall x, [p(x) \vee q(x)]\} \wedge \{\exists x, \neg p(x)\} \wedge \{\forall x [\neg q(x) \vee r(x)]\} \wedge \{\forall x, [s(x) \rightarrow \neg r(x)]\}$

$\Rightarrow r(a) \wedge \{s(a) \rightarrow \neg r(a)\}$.

$\Rightarrow \neg s(a)$ by Modus Tollens rule

$\Rightarrow \exists x \neg s(x)$.

This proves the given argument is valid

Quantified Statements with more than one variable

1. Determine the truth value of each of the following quantified statements, the universe being the set of all non-zero integers.

(i) $\exists x, \exists y [xy = 1]$ (ii) $\exists x \forall y [xy = 1]$ (iii) $\forall x \exists y [xy = 1]$

(iv) $\exists x, \exists y, [(2x + y = 5) \wedge (x - 3y = -8)]$

(v) $\exists x, \exists y [(3x - y = 17) \wedge (2x + 4y = 3)]$

Sol: (i) True (Take $x = 1$, $y = 1$.

(ii) False (For a specified x, $xy = 1$ for every y is not true.

(iii) False [For $x = 2$, there is no integer y such that $xy = 1$)

(iv). True (Take $x = 1$, $y = 3$).

(v) False (Equations $3x - y = 7$ and $2x + 4y = 3$ do not have a common integer solution)

Methods of Proof and Methods of Disproof.

Direct proof:

1. Hypothesis first - assume that p is true
2. Analysis: Starting with the hypothesis and employing the rules/laws of logic and other known facts infer that q is true.
3. Conclusion: $p \to q$ is true.

Indirect Proof: A conditional $p \to q$ and its contrapositive $\neg q \to \neg p$ are logically equivalent. In some situations given a conditional $p \to q$, a direct proof of the contrapositive $\neg q \to \neg p$ is easier. On the basis of this proof we infer that the conditional $p \to q$ is true. This method of proving a conditional is called an indirect method of proof.

Proof by Contradiction:

1. Hypothesis: Assume that $p \to q$ is false. That is assume that p is true and q is false.
2. Analysis: Starting with the hypothesis that q is false and employing the rules of logic and other known facts, infer that p is false. This contradicts the assumption that p is true.
3. Conclusion: Because of the contradiction arrived

in the analysis, we infer that $p \to q$ is true.

1. Prove that for all integers k and l, if k and l are both odd then k+l is even and kl is odd.

Sol: Take any two integers k and l, and assume that both of these are odd (hypothesis).

Then $k = 2m+1$, $l = 2n+1$ for some integers m and n. Therefore,

$$k + l = (2m+1) + (2n+1) = 2(m+n+1).$$
$$kl = (2m+1)(2n+1) = 4mn + 2(m+n)+1$$

We observe that k+l is divisible by 2 and kl is not divisible by 2. Therefore k+l is an even integer and kl is an odd integer.

Since k and l are arbitrary integers, the proof of the given statement is complete.

2. For each of the following statements, provide an indirect proof by stating and proving the contrapositive of the given statement

(i) For all integers k and l, if kl is odd then both k and l are odd.

(ii) For all integers k and l if k+l is even, then k and l are both even or both odd.

Sol: The contrapositive of the given statement is "For all integers k and l, if k is even or l is even then kl is even.

We now prove this contrapositive.

For any integers k and l, assume that k is even. Then $k = 2m$ for some integer m, and $kl = (2m)l = 2(ml)$ which is evidently even. Similarly, if l is even, then $kl = k(2n) = 2kn$ for some integer n so that kl is even. This proves the contrapositive.

This proof of the contrapositive serves as an indirect proof of the given statement.

(ii). Here, the contrapositive of the given statement is "for all integers k and l, if one of k and l is odd and the other is even, then $k+l$ is odd"

We now prove this contrapositive.

For any odd integers k and l, assume that, one of k and l is odd and the other is even. Suppose k is odd and l is even. Then $k = 2m+1$ and $l = 2n$ for some integers m and n. Consequently, $k+l = (2m+1) + 2n = 2(m+n)+1$ which is evidently odd. Similarly, if k is even and l is odd, we find that $k+l$ is odd. This proves the contrapositive.

This proof of the contrapositive serves as an indirect proof of the given statement.

3. Give (i) a direct proof (ii) an indirect proof and (iii) Proof by contradiction for the following statement.

"If n is an odd integer, then $n+9$ is an even integer"

Sol: (i) Direct proof: Assume that n is an odd integer. Then $n = 2k+1$ for some integer k. This gives $n+9 = (2k+1)+9 = 2(k+5)$ from which it is evident that $n+9$ is even. This establishes the truth of the given statement by a direct proof.

(ii) Indirect proof: Assume that $n+9$ is not an even integer. Then $n+9 = 2k+1$ for some integer k. This gives $n = (2k+1) - 9 = 2(k-4)$, which shows that n is even. Thus, if $n+9$ is not even, then n is not odd. This proves the contrapositive of the given statement. This proof of the contrapositive serves as an indirect proof of the given statement.

(iii) Proof by contradiction: Assume that the given statement is false. That is, assume that n is odd and $n+9$ is odd. Since $n+9$ is odd, $n+9 = 2k+1$ for some integer k so that $n = (2k+1) - 9 = 2(k-4)$ which shows that n is even. This contradicts the assumption that n is odd. Hence the given statement must be true.

Proof by Exhaustion:

Recall that a proposition of the form "$\forall x \in S$, $P(x)$" is true if $P(x)$ is true for every (each) x in S. If S consists of only a limited number of elements, we can prove that the statement "$\forall x \in S, P(x)$" is true by considering $p(a)$ for each a in S and verifying that $p(a)$ is true (in each case). Such a method of proof is called the method of exhaustion.

1. Prove that every even integer n with $2 \le n \le 26$ can be written as a sum of at most three perfect squares.

Sol: Let $S = \{2, 4, 6, \ldots 24, 26\}$. We have to prove that the statement: "$\forall x \in S$, $P(x)$" is true, where $p(x)$: x is a sum of at most three perfect squares.

We observe that

$2 = 1^2 + 1^2$
$4 = 2^2$
$6 = 2^2 + 1^2 + 1^2$
$8 = 2^2 + 2^2$
$10 = 3^2 + 1^2$
$12 = 2^2 + 2^2 + 2^2$
$14 = 3^2 + 2^2 + 1^2$

$16 = 4^2$
$18 = 4^2 + 1^2 + 1^2$
$20 = 3^2 + 3^2 + 2^2$
$22 = 3^2 + 3^2 + 2^2$
$24 = 4^2 + 2^2 + 2^2$
$26 = 5^2 + 1^2$

The above facts verify that each x in S is a sum of at most three perfect squares,

Disproof by counter example.

The best way of disproving a proposition involving the universal quantifier is to exhibit just one case where the proposition is false. This method of disproof is called Disproof by Counterexample.

1. Prove or disprove that the sum of squares of any four non-zero integers is an even integer.

Sol: Here the proposition is

"For any four non-zero integers a, b, c, d. $a^2+b^2+c^2+d^2$ is an even integer.

We check that for $a=1, b=1, c=1, d=2$, the proposition is false. Thus, the given proposition is not a true proposition. The proposition is disproved through the counterexample $a=b=c=1$ and $d=2$.

2. Consider the following statement for the universe of integers. If n is an integer then $n^2=n$ or $\forall n \{n^2=n\}$.

Sol: Now for $n=0$ it is true that $n^2=0^2=0=n$. And if $n=1$ it is also true that $n^2=1^2=1=n$. However we cannot conclude that $n^2=n$ for every integer n. The rule of universal generalization does not apply here, for we cannot consider the choice of 0 (or 1)

as an arbitrarily chosen integer. If $n=2$, we have $n^2=4 \neq 2=n$, and this one counterexample is enough to tell us that the given statement is false.

However, either replacement - namely $n=0$ or $n=1$ is not enough to establish the truth of the statement.

For some integer n, $n^2=n$ or $\exists n \{n^2=n\}$.

3. For all positive integers x and y. if the product xy exceeds 25, then $x>5$ or $y>5$.

Proof: Consider the negation of the conclusion that is suppose that $0<x\leq 5$ and $0<y\leq 5$. Under these circumstances we find that $0<x \cdot y \leq 5 \cdot 5 = 25$. So the product xy does not exceed 25.