UNIT - II

Linear differential equation :-

A D.E is said to be linear if the dependent variable and its derivative occur only in first degree and they are not multiplied ~~together~~.

Ex:- (1) $y'' + 2y' = e^{2x}$ — Linear DE
LDE

(2) $y''' + 2y'' + 3xy = x^2$ — LDE

(3) $(y'')^2 + 2yy' = 0$ — Not L.DE ×

Linear in y

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$ is a linear in y

$$I.F = e^{\int p\,dx}$$

Sol is $y\,(IF) = \int Q\,(IF)\,dx + c$

Linear in x

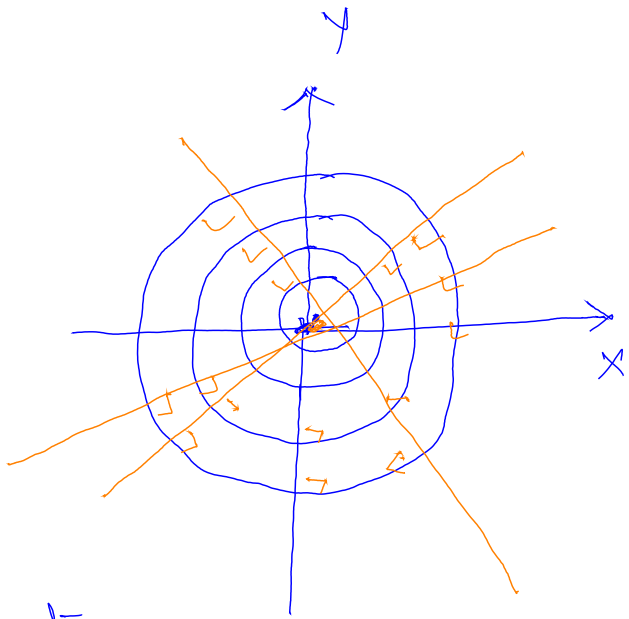
$$\frac{dx}{dy} + P(y) \cdot x = Q(y)$$ is a linear in x

$$I.F = e^{\int p\,dy}$$

Sol is $$x\,(IF) = \int Q\,(IF)\,dy + c$$

$$x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy} + c$$

Orthogonal Trajectories



Def

If two families of curves are such that every member of one family intersects every member of the other amily at right angles then they are said to be orthogonal trajectories of each other.

Ex:- The family of circles $x^2+y^2=a^2$ and the family of straight lines passing through the origin ($y=mx$) are orthogonal trajectories of each other.

Procedure to find O.T for Cartesian curve

(1) Given $f(x, y, c) = 0$, differentiate w.r.t 'x' and eliminate c

(2) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve the equation.

Note :- If the D.E of the given family remains unaltered after replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ then the given family of curves is said to be self orthogonal.

Procedure to find O.T for polar curves

(1) Given an equation in r and θ. Take log on both sides and differentiate w.r.t 'θ'

(2) After eliminating parameter replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ and solve the equation.

Problems

(1) Find the orthogonal trajectory of

$x^{2/3} + y^{2/3} = a^{2/3}$

Sol:- Given $x^{2/3} + y^{2/3} = a^{2/3}$

Diff w.r.t 'x'

$$\frac{2}{3} \cdot x^{2/3 - 1} + \frac{2}{3} \cdot y^{2/3-1} \cdot \frac{dy}{dx} = 0$$

Divide by $2/3$

$$x^{2/3 - 1} + y^{2/3-1} \frac{dy}{dx} = 0$$

$$x^{\frac{2-3}{3}} + y^{\frac{2-3}{3}} \frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3} \frac{dy}{dx} = 0$$

Put $\frac{dy}{dx} = -\frac{dx}{dy}$

$$x^{-1/3} + y^{-1/3} \left(-\frac{dx}{dy}\right) = 0$$

$$x^{-1/3} dy - y^{-1/3} dx = 0$$

$$x^{-1/3} dy = y^{-1/3} dx$$

$$\frac{1}{y^{-1/3}} dy = \frac{dx}{x^{-1/3}}$$

$$y^{1/3} dy = x^{1/3} dx$$

Integrating

$$\frac{y^{1/3+1}}{1/3+1} = \frac{x^{1/3+1}}{1/3+1} + C$$

$$\frac{y^{4/3}}{4/3} = \frac{x^{4/3}}{4/3} + c$$

$$\frac{y^{4/3} - x^{4/3}}{4/3} = c$$

$$y^{4/3} - x^{4/3} = \frac{4c}{3}$$

$$x^{4/3} - y^{4/3} = -\frac{4c}{3}$$

$$\boxed{x^{4/3} - y^{4/3} = c_1}$$

where $c_1 = -\frac{4c}{3}$

which is required O.T

(2) Show that the family of curves $x^3 - 3xy^2 = c_1$ and $y^3 - 3x^2y = c_2$ are O.T of each other

Sol:- Given $x^3 - 3xy^2 = c_1$

Diff w.r.t 'x'

$$3x^2 - 3\left(x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1\right) = 0$$

$\div 3$

$$x^2 - \left(x \cdot 2y\frac{dy}{dx} + y^2\right) = 0$$

$$x^2 - 2xy\frac{dy}{dx} - y^2 = 0$$

$$(x^2 - y^2) - 2xy\frac{dy}{dx} = 0$$

$$x^2-y^2 = 2xy\frac{dy}{dx}$$

Put $\frac{dy}{dx} = -\frac{dx}{dy}$

$$x^2-y^2 = 2xy\left(-\frac{dx}{dy}\right)$$

$$(x^2-y^2)\,dy = -2xy\,dx$$

$$2xy\,dx + (x^2-y^2)\,dy = 0$$

$M = 2xy$ $\qquad N = x^2-y^2$

$$\frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = 2x,$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ This is an exact D.E

Exact D.E

A D.E of the form $M\,dx + N\,dy = 0$ is said to be exact if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Sol

$$\int\limits_{y\ \text{constant}} M\,dx + \int\limits_{\text{Not } x} N\,dy = c$$

Sol:- $\displaystyle\int\limits_{y\ \text{constant}} M\,dx + \int\limits_{\text{Not } x} N\,dy = c$

$$\int\limits_{y=\text{constant}} 2xy\,dx + \int\limits_{\text{Not } x} (x^2-y^2)\,dy = c$$

$$2y\int x\,dx + \int 0 - y^2\,dy = c$$

$$\cancel{2}y\cdot x^2/\cancel{2} - y^3/3 = c$$

$$x^2y - \frac{y^3}{3} = c$$

$$\frac{3x^2y - y^3}{3} = c$$

$$3x^2y - y^3 = 3c$$

$$y^3 - 3x^2y = -3.c \qquad \text{let } -3c = c_2$$

$$\boxed{y^3 - 3x^2y = c_2}$$

(3) Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter

Sol :- Given $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$

Diff w.r.t 'x'

$$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\cdot\frac{dy}{dx} = 0$$

$\div 2$

$$\frac{x}{a^2} + \frac{y}{b^2+\lambda}\cdot\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} + \frac{y y_1}{b^2+\lambda} = 0$$

$$\frac{x}{a^2} = \frac{-y y_1}{b^2+\lambda} \rightarrow 1$$

Given $$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$$

$$\frac{x^2}{a^2} - 1 = \frac{-y^2}{b^2+\lambda}$$

$$\frac{x^2-a^2}{a^2} = \frac{-y^2}{b^2+\lambda} \rightarrow 2$$

$$\frac{1}{2} \Rightarrow \frac{x/\cancel{a^2}}{\frac{x^2-a^2}{\cancel{a^2}}} = \frac{\frac{-y y_1}{\cancel{b^2+\lambda}}}{\frac{-y^2}{\cancel{b^2+\lambda}}}$$

$$\frac{x}{x^2-a^2} = \frac{y_1}{y}$$

$$\frac{x}{x^2-a^2} = \frac{1}{y} \cdot \frac{dy}{dx}$$

Put $$\frac{dy}{dx} = -\frac{dx}{dy}$$

$$\frac{x}{x^2-a^2} = \frac{1}{y}\left(-\frac{dx}{dy}\right)$$

$$y\,dy \cdot x = -(x^2-a^2)\,dx$$

$$\Rightarrow y\,dy = -\frac{(x^2-a^2)}{x}\,dx$$

$$y\,dy = -\left(\frac{x^2}{x} - \frac{a^2}{x}\right)$$

$$y\,dy = -(x - a^2/x)$$

Integrating

$$\int y\,dy = -\int (x - a^2/x)\,dx$$

$$\frac{y^2}{2} = -\left(\frac{x^2}{2} - a^2\log x\right) + c$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + a^2\log x + c$$

$$\frac{x^2}{2} + \frac{y^2}{2} - a^2\log x - c = 0$$

$$\frac{x^2 + y^2 - 2a^2\log x - 2c}{2} = 0$$

$$\underline{x^2 + y^2 - 2a^2\log x - 2c = 0}$$

4) Find the O.T of the family of coaxial circles $x^2+y^2+2gx+c=0$ where g is a parameter

Sol:- Given $x^2+y^2+2gx+c=0 \rightarrow (1)$

Diff w.r.t 'x'

$$2x + 2y\cdot y_1 + 2g = 0$$

$\div 2$ $\quad x + yy_1 + g = 0 \Rightarrow g = -(x+yy_1) \rightarrow (2)$

Sub (2) in (1)

$$x^2+y^2+2(-(x+yy_1))x+c=0$$

$$x^2+y^2-2x^2-2xyy_1+c=0$$

$$y^2-x^2-2xyy_1+c=0$$

$$-2xyy_1=x^2-y^2-c$$

$$-y_1=\frac{x^2-y^2-c}{2xy}$$

$$-\frac{dy}{dx}=\frac{x^2-y^2-c}{2xy}$$

Put $\frac{dy}{dx}=-\frac{dx}{dy}$

$$-\left(-\frac{dx}{dy}\right)=\frac{x^2-y^2-c}{2xy}$$

$$\frac{dx}{dy}=\frac{x^2-y^2-c}{2xy}$$

$$2x\frac{dx}{dy}=\frac{x^2-y^2-c}{y}$$

$$2x\frac{dx}{dy}=\frac{x^2}{y}-y-c/y$$

$$2x\frac{dx}{dy}-x^2/y=-(y+c/y)$$

Put $x^2 = t$ (Diff w.r.t y)

$$2x \frac{dx}{dy} = \frac{dt}{dy}$$

(Then above ③ eq Becomes)

$$\frac{dt}{dy} - t/y = -(y + c/y)$$

linear in t

$$P = -1/y, \quad Q = -(y + c/y)$$

$$I.F = e^{\int p\,dy}$$

$$= e^{\int -1/y\,dy}$$

$$= e^{-\int 1/y\,dy}$$

$$= e^{-\log y}$$

$$= e^{\log y^{-1}}$$

$$= y^{-1}$$

$$\boxed{I.F = 1/y}$$

Sol is

$$t(IF) = \int Q\,(IF)\,dy + c_1$$

$$t \cdot 1/y = \int -(y + c/y) \cdot 1/y\,dy + c_1$$

$$t/y = \int -(1 + c/y^2)\,dy + c_1$$

$$t/y = -\int dy + c\int -1/y^2\,dy + c_1$$

$t/y = -y + c \cdot 1/y + c_1$

$$\frac{x^2}{y} = -y + c/y + c_1$$

$$\frac{x^2}{\cancel{y}} = \frac{-y^2 + c + c_1 y}{\cancel{y}}$$

$x^2 = -y^2 + c + c_1 y$

$x^2 + y^2 - c_1 y - c = 0$ ✓

$x^2 + y^2 - c_1 y + c_2 = 0$ ✓ where $c_2 = -c$

(5) Show ~~that the~~ family of parabolas $y^2 = 4a(x+a)$ is self orthogonal

Sol :- Given $y^2 = 4a(x+a) \rightarrow (1)$

Diff w.r.t 'x'

$$\cancel{2} y \frac{dy}{dx} = \cancel{4}^{2} a(1+0)$$

$$y \frac{dy}{dx} = 2a$$

$$a = \frac{y \frac{dy}{dx}}{2} \Rightarrow a = \frac{y y_1}{2}$$

Sub a in (1), we get

$$y^2 = 4 \frac{y y_1}{2} \left(x + \frac{y y_1}{2} \right)$$

$$\cancel{y^2} = 2\cancel{y}y_1\left(x+\frac{yy_1}{2}\right)$$

$$y = 2y_1\left(x+\frac{yy_1}{2}\right)$$

$$y = \cancel{2}y_1\left(\frac{2x+yy_1}{\cancel{2}}\right)$$

$$y_1 = \frac{dy}{dx}$$

$$-1/y_1 = \frac{-1}{dy/dx}$$

$$= -dx/dy$$

$$y = y_1(2x+yy_1)$$

$$y = 2xy_1 + yy_1^2 \rightarrow (2).$$

Put $\boxed{y_1 = -1/y_1}$ (or) $\dfrac{dy}{dx} = -\dfrac{dx}{dy}$

$$y = 2x(-1/y_1) + y(-1/y_1)^2$$

$$y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$$

$$y = \frac{-2xy_1 + y}{y_1^2}$$

$$yy_1^2 = y - 2xy_1$$

$$\Rightarrow y = 2xy_1 + yy_1^2 \rightarrow (3)$$

(2) = (3)

$\therefore$ $y^2 = 4a(x+a)$ is self orthogonal

6) Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ where λ is a parameter

(or)

Show that the family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, (where λ is a parameter) is self orthogonal

Sol:- Given $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$

Diff w.r.t 'x'

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\cdot y_1=0$$

$\div 2$

$$\frac{x}{a^2+\lambda}+\frac{yy_1}{b^2+\lambda}=0$$

$$\frac{x(b^2+\lambda)+yy_1(a^2+\lambda)}{(a^2+\lambda)(b^2+\lambda)}=0$$

$$x(b^2+\lambda)+yy_1(a^2+\lambda)=0$$

$$b^2x+\lambda x+yy_1a^2+yy_1\lambda=0$$

$$\lambda x+yy_1\lambda+(b^2x+yy_1a^2)=0$$

$$\lambda x+yy_1\lambda=-(b^2x+yy_1a^2)$$

$$\lambda(x+yy_1)=-(b^2x+yy_1a^2)$$

$$\lambda = \frac{-(b^2x + yy_1a^2)}{x+yy_1}$$

$$a^2+\lambda = a^2 - \frac{(b^2x+yy_1a^2)}{x+yy_1}$$

$$\frac{a^2(x+yy_1) - (b^2x+yy_1a^2)}{x+yy_1}$$

$$= \frac{a^2x+\cancel{a^2yy_1} - b^2x - \cancel{yy_1a^2}}{x+yy_1}$$

$$a^2+\lambda = \frac{(a^2-b^2)x}{x+yy_1} \longrightarrow 1$$

$$b^2+\lambda = b^2 - \left(\frac{b^2x+yy_1a^2}{x+yy_1}\right)$$

$$= \frac{b^2(x+yy_1) - (b^2x+yy_1a^2)}{x+yy_1}$$

$$= \frac{\cancel{b^2x}+b^2yy_1 - \cancel{b^2x} - yy_1a^2}{x+yy_1}$$

$$b^2+\lambda = \frac{(b^2-a^2)yy_1}{x+yy_1} \longrightarrow 2$$

Sub ① & ② in the given eq

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

$$\frac{x^{\cancel{2}}}{\frac{(a^2-b^2)\cancel{x}}{x+yy_1}} + \frac{y^{\cancel{2}}}{\frac{\cancel{y}y_1(b^2-a^2)}{x+yy_1}} = 1$$

$$\frac{x(x+yy_1)}{a^2-b^2} + \frac{y(x+yy_1)}{y_1(b^2-a^2)} = 1$$

$$\frac{x(x+yy_1)}{a^2-b^2} - \frac{y(x+yy_1)}{y_1(a^2-b^2)} = 1$$

$$\frac{x+yy_1}{a^2-b^2}\left[x - \frac{y}{y_1}\right] = 1$$

$$x+yy_1\,(x - y/y_1) = a^2-b^2 \rightarrow ③$$

Put $y_1 = -1/y_1$ (or) $\frac{dy}{dx} = -\frac{dx}{dy}$

$$x+y(-1/y_1)\left(x - \frac{y}{(-1/y_1)}\right) = a^2-b^2$$

$$(x - y/y_1)(x+yy_1) = a^2-b^2 \rightarrow ④$$

③ = ④

$\therefore \dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$ is self orthogonal

Solve

① find O.T of family of curves $x^2 - y^2 = c$

② find O.T of $xy = k$

③ If $\dfrac{xy}{a^2-x^2} + \dfrac{dy}{dx} = 0$ is the differential equation of the family of curves $f(x,y,c)=0$ then find its orthogonal trajectory.

④ find O.T of $x^2+y^2=a^2$