Partial Differential Equations

Form the PDE by eliminating the arbitrary constants from

1) $z = ax + by + a^2 + b^2$ —— ①

Diff partially wrt x,

$\frac{\partial z}{\partial x} = p = a$ $\Rightarrow a = p$

Diff ① partially wrt y,

$\frac{\partial z}{\partial y} = b$ $\Rightarrow b = q$

Substituting these in ①

$z = px + qy + p^2 + q^2$

2) $(x-a)^2 + (y-b)^2 + z^2 = c^2$ —— ①

Diff ① partially wrt x,

$2(x-a) + 2zp = 0$ $\Rightarrow (x-a) = -zp$

Diff ① partially wrt y,

$2(y-b) + 2zq = 0$ $\Rightarrow (y-b) = -zq$

Substituting in ①,

$z^2p^2 + z^2q^2 + z^2 = c^2$

$\Rightarrow z^2(p^2+q^2+1) = c^2$

3) $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$ —— ①

Diff ① wrt x (partially)

$\cancel{2}(x-a) = \cancel{2} z p \cot^2 \alpha$

~~Let~~

Diff ① partially wrt y,

$\cancel{2}(y-b) = \cancel{2} z q \cot^2 \alpha$

$\therefore$ ① becomes,

$z^2 p^2 \cot^4 \alpha + z^2 q^2 \cot^4 \alpha = z^2 \cot^2 \alpha$

$\Rightarrow (p^2 + q^2) \cot^2 \alpha = 1$

$\Rightarrow \quad p^2 + q^2 = \tan^2 \alpha$

4) Find the differential equation of all spheres of fixed radius 3 having their centres in the xy-plane.

Eq^n of a sphere having centre in the xy-plane is given by $(x-a)^2 + (y-b)^2 + (z-0)^2 = 3^2$ where $(a, b, 0)$ is the centre and 3 is the radius.

$\Rightarrow (x-a)^2 + (y-b)^2 + z^2 = 9$ —— ①

Diff wrt x & y partially

$2(x-a) + 2zp = 0$ & $2(y-b) + 2zq = 0$

$\Rightarrow (x-a) = -zp \quad \& \quad (y-b) = -zq$

Substituting these in ①

$$(-zp)^2 + (-zq)^2 + z^2 = 9$$

$$\Rightarrow \quad z^2 (p^2+q^2+1) = 9$$

5. Find the differential equation of all spheres whose centres lie on the z-axis.

Let centre be $(0,0,c)$ $\because$ centre lie on z-axis

& r be the radius.

$\therefore$ Eqn of the sphere is $x^2+y^2+(z-c)^2 = r^2$ — ①

Diff ① partially wrt x,

$$2x + 2(z-c)p = 0$$

$$\Rightarrow z - c = -x/p \quad \text{— ②}$$

Diff ① partially wrt y.

$$2y + 2(z-c)q = 0$$

$$\Rightarrow z - c = -y/q \quad \text{— ③}$$

From ② & ③, $-x/p = -y/q$

$$\Rightarrow py - qx = 0$$

Form the partial equat DE by eliminating the arbitrary functions from:

1) $z = f(x^2 - y^2)$ —①

Diff partially wrt x,

$p = f'(x^2-y^2) \cdot 2x$ —②

Diff ① partially wrt y,

$q = f'(x^2-y^2) \cdot -2y$ —③

From ② & ③

$$p = \frac{q}{-2y} \cdot 2x \Rightarrow py + qx = 0$$

2) $z = y f(x) + x g(y)$ —①

Diff ① partially wrt x,

$p = y f'(x) + g(y)$ —②

Diff ① partially wrt y,

$q = f(x) + x g'(y)$ —③

Diff ② partially wrt y,

$$s = f'(x) + g'(y) = \frac{p - g(y)}{y} + \frac{q - f(x)}{x} \text{ using ② \& ③}$$

X'y throughout by xy.

$$\Rightarrow xys = px - x g(y) + q(y) - y f(x)$$

$$\Rightarrow xys = px + qy - z$$

3) $z = f(x^2+y^2) + x + y$

Diff wrt x & y partially-

$p = f'(x^2+y^2) \cdot 2x + 1$; $q = f'(x^2+y^2) \cdot 2y + 1$

Using one in the other

$$p = \left[\frac{q-1}{2y}\right] \cdot 2x + 1 \Rightarrow py - qx = y - x$$

4) $z = x^2 f(y) + y^2 g(x)$ — ①

$p = 2x f(y) + y^2 g'(x)$ — ② by diff ① partially wrt x.

$q = x^2 f'(y) + 2y\, g(x)$ — ③ by diff ① partially wrt y.

Diff ② partially wrt y,

$$\frac{\partial^2 z}{\partial x \partial y} = s = 2x f'(y) + 2y g'(x)$$

$$\Rightarrow s = 2x\left[\frac{q - 2y g(x)}{x^2}\right] + 2y\left[\frac{p - 2x f(y)}{y^2}\right]$$

$$\Rightarrow sx^2y^2 = 2xy^2 q - 4xy^3 g(x) + 2yx^2 p - 4x^3 y f(y)$$

$$\Rightarrow sx^2y^2 = 2xy^2 q + 2yx^2 p - 4xy[z] \text{ using ①}$$

$$\Rightarrow xys = 2[px + qy - 2z]$$

5) $z = f(x) + e^y g(x)$ — ①

Diff wrt x and y respectively

$p = f'(x) + e^y g'(x)$ — ②

$q = g(x) e^y$ — ③

Diff ② wrt x & y and ③ wrt y.

$r = f''(x) + e^y g''(x)$ — ④

$s = g'(x) e^y$ — ⑤

$t = g(x) e^y$ — ⑥

From ③ & ⑥, $q = t$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2} \text{ is the required PDE.}$$

6) $z = x f_1(x+y) + f_2(x+y)$ — ①

~~Diff~~ From ①.

$p = f_1(x+y) + x f_1'(x+y) + f_2'(x+y)$

$q = x f_1'(x+y) + f_2'(x+y)$

& second order partial derivatives

$r = f_1'(x+y) + x f_1''(x+y) + f_1'(x+y) + f_2''(x+y)$

$s = f_1'(x+y) + x f_1''(x+y) + f_2''(x+y)$

$t = x f_1''(x+y) + f_2''(x+y)$

$\Rightarrow r = 2f_1'(x+y) + t$ — ②

& $s = f_1'(x+y) + t$ — ③

~~Substitute~~ $\Rightarrow f_1'(x+y) = s - t$

$\therefore$ ② becomes $r = 2[s-t] + t = 2s - t$

$\Rightarrow \quad r - 2s + t = 0$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

Solve the following equations:

1) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$

Let us solve this by direct integration:

Integrating with respect to x, treating y as a constant,

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

$\int^g$ wrt y, treating x as a constant.

$$z = \frac{x^2}{2} \log y + axy + F(y) + g(x)$$

2) $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

$\int^g$ wrt x, treating y as a constant,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x+3y)}{2} + f(y)$$

$\int^g$ wrt x, treating y as a const,

$$\frac{\partial z}{\partial y} = -\frac{\cos(2x+3y)}{4} + f(y).x + g(y)$$

$\int^g$ wrt y, treating x as a const,

$$z = -\frac{\sin(2x+3y)}{12} + F(y)x + G(y) + \phi(x)$$

3) Solve $\frac{\partial^2 z}{\partial y^2} = z$, gives that when $y=0$, $z=e^x$ & $\frac{\partial z}{\partial y} = e^{-x}$.

suppose z is a function of y only, then we have

$$\frac{d^2 z}{dy^2} = z \quad \Rightarrow (D^2-1)\, z = 0 \qquad D = \frac{d}{dy}$$

AE : $m^2 - 1 = 0$

$m = \pm 1$

$z = c_1 e^{y} + c_2 e^{-y}$

$\because$ given eq^n is a PDE,

Sol^n is $z = f(x) e^{y} + g(x) e^{-y}$ —①

Given when $y=0$, $z = e^x$

$e^x = f(x) + g(x)$ —②

$$\frac{\partial z}{\partial y} = f(x) e^{y} - g(x) e^{-y}$$

Given $\frac{\partial z}{\partial y} = e^{-x}$ when $y=0$

$\Rightarrow e^{-x} = f(x) - g(x)$ —③

From ② & ③,

$2f(x) = e^x + e^{-x}$

$\Rightarrow f(x) = \frac{e^x + e^{-x}}{2} = \cosh x$

$2g(x) = e^x - e^{-x} \Rightarrow g(x) = \sinh x$

$\therefore$ sol^n is : $z = \cosh x \, e^{y} + \sinh x \, e^{-y}$

4) Solve by the method of separation of variables

$py^3 + qx^2 = 0$.

$$\frac{\partial u}{\partial x} \cdot y^3 + \frac{\partial u}{\partial y} \cdot x^2 = 0$$

$$\frac{1}{x^2} \frac{\partial u}{\partial x} = -\frac{1}{y^3} \frac{\partial u}{\partial y}$$

Let

$$\frac{1}{x^2} \frac{\partial u}{\partial x} = k \quad ; \quad -\frac{1}{y^3} \frac{\partial u}{\partial y} = k$$

$$\frac{\partial u}{\partial x} = kx^2 \quad ; \quad \frac{\partial u}{\partial y} = -ky^3$$

Let $u = X(x)\, Y(y)$ be the required solution.

$$\frac{\partial}{\partial x}(XY)\, y^3 + \frac{\partial}{\partial y}(XY)\, x^2 = 0$$

$$Y \frac{dX}{dx} \cdot y^3 + X \frac{dY}{dy} \cdot x^2 = 0$$

$$Y \frac{dX}{dx} \cdot y^3 = -X \frac{dY}{dy} \cdot x^2$$

$$\frac{1}{Xx^2} \frac{dX}{dx} = \frac{1}{Yy^3} \frac{dY}{dy}$$

$$\frac{1}{Xx^2} \frac{dX}{dx} = k \quad ; \quad \frac{1}{Yy^3} \frac{dY}{dy} = k$$

$$\frac{1}{X} dX = kx^2 dx \quad ; \quad \frac{1}{Y} dY = ky^3 dy$$

$\int$ g both the sides.

$$\log X = \frac{kx^3}{3} + k_1 \quad ; \quad \log Y = \frac{ky^4}{4} + k_2$$

$$X = e^{\frac{kx^3}{3}} \cdot c_1 \quad ; \quad Y = e^{\frac{ky^4}{4} + k_2} = c_2 e^{\frac{ky^4}{4}}$$

$\therefore$ Required sol^n is

$$u = c_1 c_2 e^{\frac{kx^3}{3}} e^{\frac{ky^4}{4}} = c_1 c_2 e^{\frac{4kx^3 + 3ky^4}{12}}$$

Double and Triple Integrals

1) Evaluate the integral $\int_1^2 \int_1^3 xy^2 \, dy \, dx$.

$$\int_1^2 \int_1^3 xy^2 \, dy \, dx = \int_1^2 x \left[\frac{y^3}{3}\right]_1^3 dx = \int_1^2 \frac{x}{3}[27-1] \, dx$$

$$= \frac{26}{3}\left[\frac{x^2}{2}\right]_1^2 = \frac{26}{3}\left[\frac{4-1}{2}\right]$$

$$= \underline{\underline{13}}$$

Evaluate

2) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2) \, dy \, dx$.

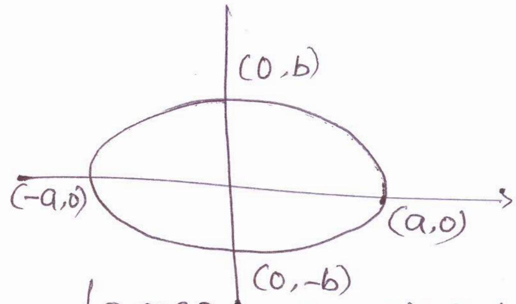
$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{\sqrt{x}} \cdot dx = \int_0^1 \left\{x^2[\sqrt{x} - x] + \frac{1}{3}(x\sqrt{x} - x^3)\right\} dx$$

$$= \int_0^1 \left[(x^{5/2} - x^3) + \frac{1}{3}(x^{3/2} - x^3)\right] \cdot dx = \left[\frac{x^{7/2}}{7/2} - \frac{x^4}{4} + \frac{1}{3}\left(\frac{x^{5/2}}{5/2} - \frac{x^4}{4}\right)\right]_0^1$$

$$= 2/7 - 1/4 + 2/15 - 1/12 = 0.0857$$

3) Evaluate $\iint (x+y)^2\, dx\, dy$ over the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Let $x = ar\cos\theta$, $y = br\sin\theta$

So, the ellipse transforms to

$r = 1$ with θ in $[0, 2\pi]$.

$\therefore$ Jacobian $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix}$$

$$= abr$$

Hence $\iint (x+y)^2\, dx\, dy$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (ar\cos\theta + br\sin\theta)^2 \cdot abr\, dr\, d\theta$$

$$= ab \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^3 (a\cos\theta + b\sin\theta)^2\, dr\, d\theta$$

$$= ab \int_{\theta=0}^{2\pi} (a\cos\theta + b\sin\theta)^2 d\theta \cdot \int_{r=0}^{1} r^3\, dr$$

$$= ab \int_{\theta=0}^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta)\, d\theta \cdot \left[\frac{r^4}{4}\right]_0^1$$

$$= \frac{ab}{4} \int_{\theta=0}^{2\pi} \left[\frac{a^2}{2}(1+\cos 2\theta) + \frac{b^2}{2}(1-\cos 2\theta) + ab\sin 2\theta\right] d\theta$$

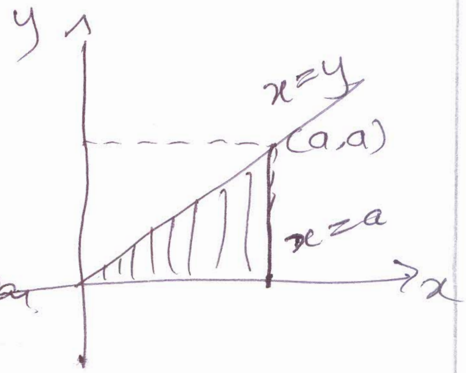
$$= \frac{ab}{8} \left[a^2\Big(\theta + \cancelto{0}{\frac{\sin 2\theta}{2}}\Big) + b^2\Big(\theta - \cancelto{0}{\frac{\sin 2\theta}{2}}\Big) - \cancel{2}ab\frac{\cos 2\theta}{\cancel{2}}\right]_0^{2\pi}$$

$$= \frac{ab}{8}\left[a^2 \cdot 2\pi + 2\pi b^2 - (\cancelto{0}{ab - ab})\right] = \frac{\pi ab}{4}(a^2+b^2)$$

4) Evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ by changing the order of $\int^n$.

Here, $y : 0 \to a$

$x : y \to a$



By changing the order of integration

we can see that ~~...~~ in the given area,

$x : 0 \to a$

& $y : 0 \to x$.

$\therefore$ Given integral can be written as

$$\int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_{x=0}^{a} \left[\tan^{-1} y/x\right]_0^x \cdot dx \quad = \int_0^a \frac{\pi}{4} \cdot dx = \frac{\pi a}{4}$$

5. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \cdot dz \cdot dx\,dy$

$$= \int_1^e \int_1^{\log y} \left[\log z \cdot z - \int \cancel{z} \cdot \frac{1}{\cancel{z}} \cdot dz\right]_1^{e^x} \cdot dx \cdot dy$$

$$= \int_1^e \int_1^{\log y} \left[z \log z - z\right]_1^{e^x} \cdot dx\,dy$$

$$= \int_1^e \int_1^{\log y} \left[e^x \cdot x - e^x + 1\right] \cdot dx\,dy$$

$$= \int_1^e \cancel{\int} \left[x e^x - e^x - e^x + x\right]_1^{\log y} \cdot dy$$

$$= \int_1^e (y \log y - y - y + \log y - \cancel{e} + \cancel{e} + e - 1) \cdot dy$$

$$= \int_1^e (y \log y - 2y + \log y + e - 1) \cdot dy$$

$$= \left[\log y \cdot \frac{y^2}{2} - \int \frac{y^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{y}} \cdot dy - y^2 + y \log y - y + ey - y\right]_1^e$$

$$= \frac{e^2}{2} - \frac{1}{4}\left[e^2 - 1\right] - \left[e^2 - 1\right] + e - (e-1) + e(e-1) - (e-1)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} - \cancel{e^2} + 1 + \cancel{e} - \cancel{e} + 1 + \cancel{e^2} - e - e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{1}{4} + 3 = \frac{1}{4}\left[e^2 - 8e + \cancel{13}\, 12\right]$$

$$= \frac{1}{4}\left[e^2 - 8e + 13\right]$$

6. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} r\left[z\right]_0^{\frac{a^2-r^2}{a}} dr\cdot d\theta$$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} r\left[\frac{a^2-r^2}{a}\right]\cdot dr\cdot d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta}\cdot d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left[\frac{a^4}{2}\sin^2\theta - \frac{a^4\sin^4\theta}{4}\right]\cdot d\theta$$

$$= \frac{1}{a}\left[\frac{a^4}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{a^4}{4}\cdot\frac{3}{4\cdot 2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{a^3\pi}{2}\left[\frac{1}{4} - \frac{3}{32}\right] = \frac{5\pi a^3}{64}$$