PARSEVAL'S FORMULA

Prove that $\int_{-l}^{l}[f(x)]^2\,dx = l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right\}$,

provided the Fourier series for $f(x)$ converges uniformly in $(-l, l)$.

Proof : The Fourier series for $f(x)$ over $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right) \longrightarrow \boxed{1}$$

Multiplying bothsides of $\boxed{1}$ by $f(x)$ and integrating term by term from $-l$ to l, we get

$$\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0}{2}\int_{-l}^{l}f(x)\,dx + \sum_{n=1}^{\infty}a_n\int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \sum_{n=1}^{\infty}b_n\int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx \longrightarrow \boxed{2}$$

We know that

$$a_0 = \frac{1}{l}\int_{-l}^{l}f(x)\,dx \Rightarrow \int_{-l}^{l}f(x)\,dx = a_0 l$$

$$a_n = \frac{1}{l}\int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx \Rightarrow \int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx = a_n l$$

$$b_n = \frac{1}{l}\int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx \Rightarrow \int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx = b_n l$$

Using this, equation $\boxed{2}$ takes the form

$$\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2 l}{2} + \sum_{n=1}^{\infty}\left[l a_n^2 + l b_n^2\right]$$

(OR)

Case(i): If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$ in $(0, 2l)$

then $$\int_0^{2l} [f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

Case(ii): ~~If~~ In case of half-range Fourier cosine series of $f(x)$ over $(0, l)$

then $$\int_0^{l} [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \cdots\right]$$

Case(iii): In case of half-range Fourier sine series of $f(x)$ over $(0, l)$

then $$\int_0^{l} [f(x)]^2 dx = \frac{l}{2}\left[b_1^2 + b_2^2 + b_3^2 + \cdots\right]$$

NOTE:

Uniform convergence of series:

The series $\sum u_n(x)$ is said to be uniformly convergent to $S(x)$ (limit) in the interval (a, b), if for a given $\epsilon > 0$, a number N can be found independent of x such that for every x in the interval (a, b)

$$|S(x) - S_n(x)| < \epsilon \quad \text{for all } n > N$$

Problems

1 Using the Fourier series for $f(x) = x^2$ over $(-\pi, \pi)$ show that $\dfrac{\pi^4}{90} = \dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots$

Soln: Given $f(x) = x^2$, $l = \pi$

By Fourier series expansion we have

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4}{n^2}(-1)^n, \quad b_n = 0.$$

By Parseval's formula, we have

$$\int_{-\pi}^{\pi} [f(x)]^2 \, dx = l \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} x^4 \, dx = \pi \left[\frac{4\pi^4}{18} + \sum_{n=1}^{\infty} \frac{16}{n^4} (-1)^{2n} \right]$$

$$\Rightarrow \left. \frac{x^5}{5} \right]_{-\pi}^{\pi} = \pi \left[\frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \right] \quad \because (-1)^{2n} = 1$$

$$\Rightarrow \frac{1}{5}\left[\pi^5 - (-\pi)^5\right] = \pi \left[\frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \right]$$

$$\Rightarrow \frac{2\pi^5}{5} = \pi \left[\frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \right]$$

$$\Rightarrow \frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

2 Prove that $x = \dfrac{l}{2} - \dfrac{4l}{\pi^2} \left[\cos\left(\dfrac{\pi x}{l}\right) + \dfrac{1}{3^2} \cos\left(\dfrac{3\pi x}{l}\right) + \dfrac{1}{5^2} \cos\left(\dfrac{5\pi x}{l}\right) + \cdots \right]$

in $0 < x < l$ and hence deduce that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

3) Evaluate the value of $1^{-4} + 3^{-4} + 5^{-4} + \cdots$ from the half-range cosine series for the function.

$$f(x) = \begin{cases} \pi x & , 0 < x < 1 \\ \pi(2-x) & , 1 < x < 2 \end{cases} \quad \text{over } (0,2)$$

Solution: First we obtain the Half-range cosine series for the given function in $(0,2)$.

$(0, l) \downarrow (0,2)$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{\pi}{2}\left[1-0\right] + \pi\left[2(2-1) - \frac{1}{2}(2^2-1^2)\right] = \frac{\pi}{2} + \pi\left[2 - \frac{3}{2}\right]$$

$$a_0 = \pi$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi}{2}x\right)dx = \cancel{\int_0^1 \pi x \cos\left(\frac{n\pi}{2}x\right)dx}$$

$$= \int_0^1 \pi x \cos\left(\frac{n\pi}{2}x\right)dx + \int_1^2 \pi(2-x)\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \pi\left[x\,\frac{\sin\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}} - (1)\left(-\frac{\cos\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)^2}\right) + 0\right]_0^1$$

$$+\pi\left[(2-x)\,\frac{\sin\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}} - (0-1)\left(-\frac{\cos\left(\frac{n\pi}{2}x\right)}{\left(\frac{n\pi}{2}\right)^2}\right) + 0\right]_1^2$$

$$= \pi\left[\frac{2}{n\pi}\left(1\sin\left(\frac{n\pi}{2}\right) - 0\right) + \left(\frac{2}{n\pi}\right)^2\left(\cos\left(\frac{n\pi}{2}\right) - \cos 0\right)\right]$$

$$+\pi\left[\frac{2}{n\pi}\left\{(2-2)\sin\left(\frac{n\pi}{2}2\right) - (2-1)\sin\left(\frac{n\pi}{2}1\right)\right\} - \left(\frac{2}{n\pi}\right)^2\left\{\cos\left(\frac{n\pi}{2}2\right) - \cos\left(\frac{n\pi}{2}1\right)\right\}\right]$$

$$a_n = \pi\Big[\cancel{\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)} + \frac{4}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) - \cancel{\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$

$$- \frac{4}{n^2\pi^2}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right)\Big]$$

$$= \pi\left[\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}(-1)^n + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\right]$$

$$a_n = \pi\left[\frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left((-1)^n + 1\right)\right]$$

Next we use Parseval's Formula in the case of half-range Fourier cosine series of $f(x)$ over $(0,2)$ (in general $(0,2)$)

$$\int_0^2 [f(x)]^2\,dx = \frac{2}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \cdots\right]$$

$$\int_0^1 (\pi x)^2\,dx + \int_1^2 \pi^2 (2-x)^2\,dx = \frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \cdots$$

integrate and substitute the values of a_n.

~~This is left on~~

To arrive at the required answer.