SOP & POS Canonical Forms, Karnaugh Map Method

m.

terms where all the variables

complemented or uncomplemented

evaluates to 1. The product terms $a\bar{b}$

an assigned value of 1

The boolean expression written directly

truth table as $\boxed{f(a,b) = a\bar{b} + ab}$ is

Expressions constructed using such standard maxterms are said to be in Maxterm Canonical Forms or standard POS form.

While expressing functions in maxterm canonical form, we look out of rows for which the function evaluates to '0' rather than '1'.

Consider the same function truth table used in Minterm Canonical form. The maxterm corresponding to row '0' for which the function evaluates to '0' is $a+b$. Observe that $a=b=0$, $a+b$ evaluates to '0'.
Similarly, for row 2 the maxterm $a+\bar{b}$ evaluates to '0' for $a=0$ & $b=1$

Exercise 1

Write the Boolean expression in minterm canonical form for the function represented by the truth table below.

Row No.	a	b	c	f	terms
0	0	0	0	0	$\bar{a}\bar{b}\bar{c}$
1	0	0	1	1	$\bar{a}\bar{b}c$
2	0	1	0	0	$\bar{a}b\bar{c}$
3	0	1	1	0	$\bar{a}bc$
4	1	0	0	1	$a\bar{b}\bar{c}$
5	1	0	1	0	$a\bar{b}c$
6	1	1	0	1	$ab\bar{c}$
7	1	1	1	0	abc

Solution: The function can be represented by a boolean expression in Minterm Canonical form by collecting the terms corresponding to which the function evaluates to '1'

$\therefore \boxed{f(a,b,c) = \bar{a}\bar{b}c + a\bar{b}\bar{c} + ab\bar{c}}$

Exercise 2

Write the Boolean expression in maxterm canonical form for the function represented by the truth table below.

Row No.	a	b	c	f	terms
0	0	0	0	1	$a+b+c$
1	0	0	1	1	$a+b+\bar{c}$
2	0	1	0	0	$a+\bar{b}+c$
3	0	1	1	1	$a+\bar{b}+\bar{c}$
4	1	0	0	1	$\bar{a}+b+c$
5	1	0	1	0	$\bar{a}+b+\bar{c}$
6	1	1	0	0	$\bar{a}+\bar{b}+c$
7	1	1	1	1	$\bar{a}+\bar{b}+\bar{c}$

Solⁿ: The function can be represented by a boolean expression in Maxterm Canonical form by collecting the terms corresponding to which the function evaluates to '0'.

$\therefore \boxed{f(a,b,c) = (a+\bar{b}+c)(\bar{a}+b+\bar{c})(\bar{a}+\bar{b}+c)}$

m-Notation & M-Notation

The m-notation is used to simplify writing functions in minterms canonical forms. Minterms are represented as m_i, where i stands for the row number for which the function evaluates to '1'.

From exercise 1, $f(a,b,c) = \bar{a}\bar{b}c + a\bar{b}\bar{c} + ab\bar{c}$ can be represented in m-notation as

$\boxed{f = m_1 + m_4 + m_6}$ OR by using summation notation, $\boxed{f = \Sigma m(1, 4, 6)}$

The M-notation is used to simplify writing in maxterm canonical form. Maxterms are represented as M_i, where 'i' stands for the row no. for which the function evaluates to '0'

From exercise 2:

$$f(a,b,c) = (a+\bar{b}+c)(\bar{a}+b+\bar{c})(\bar{a}+\bar{b}+c)$$

can be represented in M-notation as

$$\boxed{f = M_2 \cdot M_5 \cdot M_6}$$

OR by using a product notation,

$$\boxed{f = \Pi_M(2,5,6)}$$

Exercise 3: Convert the following to its minterm canonical form. $f = \bar{a}(\bar{b}+c) + \bar{c}$

Soln:
$$f = \bar{a}(\bar{b}+c)+\bar{c}$$
$$= \bar{a}\bar{b} + \bar{a}c + \bar{c}$$
$$= \bar{a}\bar{b}(c+\bar{c}) + \bar{a}c(b+\bar{b}) + \bar{c}(a+\bar{a})(b+\bar{b})$$
$$= \bar{a}\bar{b}c + \bar{a}\bar{b}\bar{c} + \bar{a}bc + \bar{a}\bar{b}c + ab\bar{c} + a\bar{b}\bar{c} + \bar{a}b\bar{c} + \bar{a}\bar{b}\bar{c}$$
$$f = \bar{a}\bar{b}c + \bar{a}\bar{b}\bar{c} + \bar{a}bc + ab\bar{c} + a\bar{b}\bar{c} + \bar{a}b\bar{c}$$
$$= m_1 + m_0 + m_3 + m_6 + m_4 + m_2$$
$$\therefore \boxed{f(a,b,c) = \Sigma_m(0,1,2,3,4,6)}$$

Note: To make std. minterm, multiply $(\text{true}+\overline{\text{true}})$ of missing variable to the non-std term.

Exercise 4. Convert the following to its maxterm canonical form $f(a,b,c) = (b+\bar{c})(a\bar{b}+c)$

Soln:
$$f = (b+\bar{c})(a\bar{b}+c)$$

Applying distributive law for second term, we get

$$f = (b+\bar{c})(a+c)(\bar{b}+c)$$
$$= (b+\bar{c}+a\bar{a})(a+c+b\bar{b})(\bar{b}+c+a\bar{a})$$
$$= (a+b+\bar{c})(\bar{a}+b+\bar{c})(a+b+c)(a+\bar{b}+c)(a+\bar{b}+c)(\bar{a}+\bar{b}+c)$$
$$f = (a+b+\bar{c})(\bar{a}+b+\bar{c})(a+b+c)(a+\bar{b}+c)(\bar{a}+\bar{b}+c)$$
$$= M_1 \cdot M_5 \cdot M_0 \cdot M_2 \cdot M_6$$
$$\therefore \boxed{f(a,b,c) = \Pi_M(0,1,2,5,6)}$$

Note: To make std. Maxterm, add $(\text{true}\cdot\overline{\text{true}})$ of missing variable to the non-std. term.

Minterms & Maxterms for Three Variables

Row No.	A	B	C	Minterms	Maxterms
0	0	0	0	$\bar{A}\bar{B}\bar{C} = m_0$	$A+B+C = M_0$
1	0	0	1	$\bar{A}\bar{B}C = m_1$	$A+B+\bar{C} = M_1$
2	0	1	0	$\bar{A}B\bar{C} = m_2$	$A+\bar{B}+C = M_2$
3	0	1	1	$\bar{A}BC = m_3$	$A+\bar{B}+\bar{C} = M_3$
4	1	0	0	$A\bar{B}\bar{C} = m_4$	$\bar{A}+B+C = M_4$
5	1	0	1	$A\bar{B}C = m_5$	$\bar{A}+B+\bar{C} = M_5$
6	1	1	0	$AB\bar{C} = m_6$	$\bar{A}+\bar{B}+C = M_6$
7	1	1	1	$ABC = m_7$	$\bar{A}+\bar{B}+\bar{C} = M_7$

Complements of Canonical forms

Looking at the truth table of a given function, we can immediately say that the complement of the function will have 1's where the original functional values were '0' and vice versa

Let $f(a,b,c) = \Sigma m(0,2,5)$

Term.	a	b	c	f	$\overline{f}$
m_0	0	0	0	1	0
m_1	0	0	1	0	1
m_2	0	1	0	1	0
m_3	0	1	1	0	1
m_4	1	0	0	0	1
m_5	1	0	1	1	0
m_6	1	1	0	0	1
m_7	1	1	1	0	1

$\therefore \boxed{\overline{f}(a,b,c) = \Sigma m(1,3,4,6,7)}$

in SOP form.

Now, consider a function $f = a\overline{b}c = m_5$

Computing $\overline{f}$ using DeMorgan's Theorem gives.

$$\overline{f} = \overline{a\overline{b}c} = \overline{a}+\overline{\overline{b}}+\overline{c} = \overline{a}+b+\overline{c} = M_5$$

Thus, a function can be complemented by simply changing the notation from m to M.

Consider $f(a,b,c) = m_0 + m_1 + m_5$

$$\therefore \overline{f}(a,b,c) = \overline{m_0+m_1+m_5}$$

$$= \overline{m_0}\cdot\overline{m_1}\cdot\overline{m_5}$$

$\therefore \boxed{f(a,b,c) = M_0 \cdot M_1 \cdot M_5}$ in POS form.

Exercise 5: Find the complement of the following in SOP and POS form.

(a) $f(a\,b\,c) = \Sigma m(0,3,7)$

(b) $f(a\,b\,c\,d) = \Sigma m(4,6,7,9,11,15)$

Soln: (a) For obtaining the complement in SOP form, consider minterms not listed in 'f'

$\therefore \boxed{\overline{f}(a\,b\,c) = \Sigma m(1,2,4,5,6)}$ SOP form

For obtaining the complement in POS form, complement 'f' and apply DeMorgan's thm.

$$\therefore f(a\,b\,c) = \overline{m_0+m_3+m_7}$$

$$= \overline{m_0}\cdot\overline{m_3}\cdot\overline{m_7}$$

$$= M_0 \cdot M_3 \cdot M_7$$

$\boxed{f(a\,b\,c) = \Pi M(0,3,7)}$ POS form.

(b)

$\overline{f}(a\,b\,c\,d) = \Sigma m(0,1,2,3,5,8,10,12,13,14)$.

SOP form.

$f(a\,b\,c\,d) = \Pi_M(4,6,7,9,11,15)$ POS form.

Exercise 6: Find the complement of the following function in SOP & POS form. $f(a\,b\,c) = \Pi M(1,4,6,7)$

Soln:

POS form $\boxed{\overline{f}(a,b,c) = \Pi M(0,2,3,5)}$

SOP form $f(a,b,c) = \overline{M_1 \cdot M_4 \cdot M_6 \cdot M_7}$

$$= \overline{M_1}+\overline{M_4}+\overline{M_6}+\overline{M_7}$$

Incomplete Boolean Functions & Don't Care Conditions

Consider the following design specification:

→ Design a system which accepts 3 i/ps a, b, & c and generates an output '0', when the no. of 1's in the input is one and generates an o/p '1' when the no. of 1's in the i/p is two.

Soln

Row No.	a	b	c	f
0	0	0	0	X
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	0
5	1	0	1	1
6	1	1	0	1
7	1	1	1	X

The o/p is not specified for the first and last row of table. This table describes an incomplete Boolean function.

Truth tables whose o/p for all i/p combinations is either a '1' or a '0' describes a complete boolean funcn.

The below table shows the complement of the incomplete boolean function of above table.

Row No.	a	b	c	$\bar{f}$
0	0	0	0	X
1	0	0	1	1
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	0
6	1	1	0	0
7	1	1	1	X

Observe that undefined o/ps continue to remain undefined when an incomplete boolean function is complemented. All defined o/ps are however complemented.

The unspecified o/ps in the truth table of an incomplete boolean function is called "DON'T-CARE" conditions. Minterms or Maxterms are written for the specified o/ps and the unspecified o/ps are grouped together as dont-care terms.

The dont-care terms in above table's are written as $\boxed{dc(0,7)}$.

The boolean function in minterm canonical form for the above function can be written as

$$\boxed{f(a,b,c) = \Sigma m(3,5,6) + dc(0,7)}$$

On the same lines, the same boolean function can be expressed in maxterm canonical form as

$$\boxed{f(a,b,c) = \Pi M(1,2,4) + dc(0,7)}$$

In practical implementation, these don't cares may be taken either as '1' or '0'

Don't-Care Conditions in Logic Design.

Dont-Care conditions could arise in two scenarios.

1) The first scenario is the example what we saw above. In that i/p condition for don't cares do occur, but o/ps are not defined.

2) Second scenario is that i/p conditions for dont care o/ps never occurs eg: BCD number as it ranges only from 0 to 9 taking 4 bits to represent, after 9 rest 6 o/ps are dont cares

Simplification of Boolean Expression Using Karnaugh Maps.

The complexity of the digital logic gates that implement a boolean function is directly related to the algebraic expression from which the function is implemented.

Boolean expressions may be simplified by algebraic manipulation using set of rules & theorems. However this procedure of simplification is awkward, because it lacks specific rules to predict each succeeding step in the manipulative process and it is difficult to determine whether the simplest expression has been acheived.

By contrast, the map method provides a straightforward procedure for simplifying boolean functions of upto four variables.

The map is also known as "KARNAUGH Map or K-Map".

Boolean functions can be diagrammatically represented by means of Karnaugh Maps.

A 3 variable function has $2^3 = 8$ rows in the truth table.
A 4 variable function has $2^4 = 16$ rows in the truth table.
Similarly a 'n' variable function would have '2^n' rows in the truth table.

A Karnaugh map is so drawn such that each row of the truth table corresponds to cell (square) in the map.
A '1' is placed in the cell corresponding to a row for which the function evaluates to '1' and '0' is placed in the cells corresponding to a row for which the function evaluates '0'.

The cells (squares) are so arranged that physically adjacent cells are also logically adjacent.

Two terms are logically adjacent if they differ only in one literal (variable).

eg: $a\bar{b}c$ is adjacent to abc, $\bar{a}\bar{b}c$ and $a\bar{b}\bar{c}$.
$\bar{a}b$ is adjacent to ab and $\bar{a}\bar{b}$
$abcd$ is adjacent to $\bar{a}bcd$, $a\bar{b}cd$, $ab\bar{c}d$ & $abc\bar{d}$

Observe that a two variable term is logically adjacent to 2 terms, a three variable term is logically adjacent to 3 terms and an 'n' variable term is logically adjacent to 'n' terms.

On the map, a square (cell) corresponding to a 'n' variable term would be logically adjacent to 'n' other cells.

Two-Variable Map.

There are 4 minterms for a boolean function with two variables. Hence the two-variable map consists of 4 squares., one of each minterm as shown below.

m_0	m_1
m_2	m_3

fig. (a)

A \ B	0	1
0	$\bar{A}\bar{B}$ [0]	$\bar{A}B$ [1]
1	$A\bar{B}$ [2]	AB [3]

fig. (b)

	A	B	Terms
0	0	0	$\bar{A}\ \bar{B}$
1	0	1	$\bar{A}\ B$
2	1	0	$A\ \bar{B}$
3	1	1	$A\ B$

In fig (b), map is redrawn to show the relationship between the two squares and two variable A & B.

The '0' and '1' marked on the left side and the top of map designate the values of the variables.

A function of two variables can be represented in a map by marking the squares that correspond to the minterms of the function. as shown below.

eg: $f(a,b) = \bar{a}b + a\bar{b}$

row/square no.	a	b	f
0	0	0	0
1	0	1	1
2	1	0	1
3	1	1	0

a \ b	0	1
0	0^{0}	1^{1}
1	1^{2}	0^{3}

Note: To represent any function into the K-map, the function should be a standard function i-e; all its minterms must contain all the variables either in complemented or uncomplemented form. If not, convert the non-standard function to standard function and then map it into K-Map.

eg: $f(a,b) = a + b\bar{a}$ ← non-std. function.

$= a(b+\bar{b}) + \bar{a}b$

$\boxed{f(a,b) = ab + a\bar{b} + \bar{a}b}$ ← std. function.

a \ b	0	1
0	0^{0}	1^{1}
1	1^{2}	1^{3}

$\because f(a,b) = \Sigma m(\underbrace{1, 2, 3}_{\text{Square no.s}})$

corresponding to minterms of function.

Three-Variable Map.

There are eight minterms for 3 binary variables. Therefore, a three variable map consists of eight squares shown below, each square corresponding to particular i/p variable combinations.

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6

fig (a)

A \ BC	00	01	11	10
0	$\bar{A}\bar{B}\bar{C}^{\,0}$	$\bar{A}\bar{B}C^{\,1}$	$\bar{A}BC^{\,3}$	$\bar{A}B\bar{C}^{\,2}$
1	$A\bar{B}\bar{C}^{\,4}$	$A\bar{B}C^{\,5}$	$ABC^{\,7}$	$AB\bar{C}^{\,6}$

fig (b).

The map drawn in fig (b) is marked with binary no.s for each row and each column to shown the binary values of the minterms.

Note that the numbers along the columns do not follow the binary count sequence.

The characteristic of the listed sequence is that, only one bit changes in value from one adjacent column to the next.

The function of three variables can be represented in a map by marking the squares that corresponds to the minterms of the function.

eg: $f(abc) = \bar{a}\bar{b}c + \bar{a}\bar{b}\bar{c} + \bar{a}bc + ab\bar{c} + a\bar{b}\bar{c} + \bar{a}b\bar{c}$

$= m_1 + m_0 + m_3 + m_6 + m_4 + m_2$

a \ bc	00	01	11	10
0	1^{0}	1^{1}	1^{3}	1^{2}
1	1^{4}	0^{5}	0^{7}	1^{6}

Four-Variable Map

There are 16 minterms for four binary variables, and therefore, a four-variable map consists of 16 squares. Shown below

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6
m_{12}	m_{13}	m_{15}	m_{14}
m_8	m_9	m_{11}	m_{10}

AB \ CD	00	01	11	10
00	0 $\bar{A}\bar{B}\bar{C}\bar{D}$	1 $\bar{A}\bar{B}\bar{C}D$	3 $\bar{A}\bar{B}CD$	2 $\bar{A}\bar{B}C\bar{D}$
01	4 $\bar{A}B\bar{C}\bar{D}$	5 $\bar{A}B\bar{C}D$	7 $\bar{A}BCD$	6 $\bar{A}BC\bar{D}$
11	12 $AB\bar{C}\bar{D}$	13 $AB\bar{C}D$	15 $ABCD$	14 $ABC\bar{D}$
10	8 $A\bar{B}\bar{C}\bar{D}$	9 $A\bar{B}\bar{C}D$	11 $A\bar{B}CD$	10 $A\bar{B}C\bar{D}$

The rows and columns are numbered so that only one bit of the binary number changes in value between any two adjacent columns or rows.

The minterms corresponding to each square can be optained by combining the row number with the column number.

for $f(abcd) = \Sigma m(1, 3, 4, 5, 10, 12, 13)$

ab \ cd	00	01	11	10
00	0^{0}	1^{1}	1^{3}	0^{2}
01	1^{4}	1^{5}	0^{7}	0^{6}
11	1^{12}	1^{13}	0^{15}	0^{14}
10	0^{8}	0^{9}	0^{11}	1^{10}

Exercise: Write the boolean algebraic form of the function from the K-map given below.

a \ bc	00	01	11	10
0	1^{0}	0^{1}	0^{3}	1^{2}
1	1^{4}	1^{5}	0^{7}	1^{6}

Sol^n.

$$f = \Sigma m(0, 2, 4, 5, 6)$$

$$\therefore f(a,b,c) = \bar{a}\bar{b}\bar{c} + \bar{a}b\bar{c} + a\bar{b}\bar{c} + a\bar{b}c + ab\bar{c}$$

The truth table is as follows.

Cell no.	a	b	c	f
0	0	0	0	1
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	0
7	1	1	1	1

Exercise: Construct the K-map for the following function $f(abcd) = \bar{a}\bar{b}c\bar{d} + a\bar{b}\bar{c}\bar{d} + \bar{a}b\bar{c}d + abc\bar{d} + ab\bar{c}d + ab\bar{c}\bar{d}$

Sol^n: Let us locate the terms in truth table.

P.T.O

Cell no.	a	b	c	d	minterms	f
0	0	0	0	0	$\bar{a}\bar{b}\bar{c}\bar{d}$	0
1	0	0	0	1	$\bar{a}\bar{b}\bar{c}d$	0
2	0	0	1	0	$\bar{a}\bar{b}c\bar{d}$	1
3	0	0	1	1	$\bar{a}\bar{b}cd$	0
4	0	1	0	0	$\bar{a}b\bar{c}\bar{d}$	0
5	0	1	0	1	$\bar{a}b\bar{c}d$	1
6	0	1	1	0	$\bar{a}bc\bar{d}$	0
7	0	1	1	1	$\bar{a}bcd$	0
8	1	0	0	0	$a\bar{b}\bar{c}\bar{d}$	1
9	1	0	0	1	$a\bar{b}\bar{c}d$	0
10	1	0	1	0	$a\bar{b}c\bar{d}$	0
11	1	0	1	1	$a\bar{b}cd$	0
12	1	1	0	0	$ab\bar{c}\bar{d}$	1
13	1	1	0	1	$ab\bar{c}d$	1
14	1	1	1	0	$abc\bar{d}$	1
15	1	1	1	1	$abcd$	0

$\therefore f = \Sigma m(2, 5, 8, 12, 13, 14)$

The Karnaugh Map can be drawn as follows.

ab \ cd	00	01	11	10
00	$0^{(0)}$	$0^{(1)}$	$0^{(3)}$	$1^{(2)}$
01	$0^{(4)}$	$1^{(5)}$	$0^{(7)}$	$0^{(6)}$
11	$1^{(12)}$	$1^{(13)}$	$1^{(15)}$	$0^{(14)}$
10	$1^{(8)}$	$0^{(9)}$	$0^{(11)}$	$0^{(10)}$

<u>Exercise</u> : Write the Karnaugh maps for the following functions.

(a) $f(abcd) = \Sigma m(1, 6, 7, 8, 10, 12, 14)$

(b) $f(abcd) = \Pi M(0, 3, 4, 7, 9, 13, 14)$.

<u>Soln</u>: Place a '1' corresponding to each minterm in (a) and fill the remaining cells with 0's.

Place a '0' corresponding to each maxterm in (b) and fill the remaining cells with 1's.

ab \ cd	00	01	11	10
00	$0^{(0)}$	$1^{(1)}$	$0^{(3)}$	$0^{(2)}$
01	$0^{(4)}$	$0^{(5)}$	$1^{(7)}$	$1^{(6)}$
11	$1^{(12)}$	$0^{(13)}$	$0^{(15)}$	$1^{(14)}$
10	$1^{(8)}$	$0^{(9)}$	$0^{(11)}$	$1^{(10)}$

(a)

ab \ cd	00	01	11	10
00	$0^{(0)}$	$1^{(1)}$	$0^{(3)}$	$1^{(2)}$
01	$0^{(4)}$	$1^{(5)}$	$0^{(7)}$	$1^{(6)}$
11	$1^{(12)}$	$0^{(13)}$	$1^{(15)}$	$0^{(14)}$
10	$1^{(8)}$	$0^{(9)}$	$1^{(11)}$	$1^{(10)}$

(b).

<u>Exercise</u>: Write the K-Map for the function

$f(abc) = a\bar{b} + \bar{a}b + \bar{b}c$

<u>Soln</u>. $f(abc) = a\bar{b} + \bar{a}b + \bar{b}c$

as the given function is not in standard SOP form $\therefore$ minterms are not having all the variables of function. So first we have to convert it to std. SOP.

$\therefore f(abc) = a\bar{b}(c+\bar{c}) + \bar{a}b(c+\bar{c}) + \bar{b}c(a+\bar{a})$

$= a\bar{b}c + a\bar{b}\bar{c} + \bar{a}bc + \bar{a}b\bar{c} + a\bar{b}c + \bar{a}\bar{b}c$

$\therefore f(abc) = a\bar{b}c + a\bar{b}\bar{c} + \bar{a}bc + \bar{a}b\bar{c} + \bar{a}\bar{b}c$

P.T.O

Now once we get the std SOP form. Let us write truth table and map for the std. SOP function.

a	b	c	minterm	f
0	0	0	$\bar{a}\bar{b}\bar{c}$	0
0	0	1	$\bar{a}\bar{b}c$	1
0	1	0	$\bar{a}b\bar{c}$	1
0	1	1	$\bar{a}bc$	1
1	0	0	$a\bar{b}\bar{c}$	1
1	0	1	$a\bar{b}c$	1
1	1	0	$ab\bar{c}$	0
1	1	1	abc	0

a \ bc	00	01	11	10
0	0 [0]	1 [1]	1 [3]	1 [2]
1	1 [4]	1 [5]	0 [7]	0 [6]

Exercise: Write the K-map for the function.

$$f(a\,b\,c) = (a+c)(b+c)(\bar{b}+\bar{c})$$

Sol^n: $f(a\,b\,c) = (a+c)(b+c)(\bar{b}+\bar{c})$

As the above eq^n is not in standard POS. So we have convert it to std. POS.

$$f(a\,b\,c) = (a+b\bar{b}+c)(a\bar{a}+b+c)(a\bar{a}+\bar{b}+\bar{c})$$

$$= (a+b+c)(a+\bar{b}+c)(a+b+c)(\bar{a}+b+c)$$
$$(a+\bar{b}+\bar{c})(\bar{a}+\bar{b}+\bar{c})$$

$$f(a\,b\,c) = (a+b+c)(a+\bar{b}+c)(\bar{a}+b+c)(a+\bar{b}+\bar{c})(\bar{a}+\bar{b}+\bar{c})$$

a	b	c	maxterm	f
0	0	0	$a+b+c$	0
0	0	1	$a+b+\bar{c}$	1
0	1	0	$a+\bar{b}+c$	0
0	1	1	$a+\bar{b}+\bar{c}$	0
1	0	0	$\bar{a}+b+c$	0
1	0	1	$\bar{a}+b+\bar{c}$	1
1	1	0	$\bar{a}+\bar{b}+c$	1
1	1	1	$\bar{a}+\bar{b}+\bar{c}$	0

a \ bc	00	01	11	10
0	0 [0]	1 [1]	0 [3]	0 [2]
1	0 [4]	1 [5]	0 [7]	1 [6]

Use of Karnaugh Maps to Simplify/Minimize Boolean Expressions

The simplified expressions produced by the map are always in SOP or POS form.

The simplest algebraic expression is one with a minimum no. of terms and with the fewest possible no. of literals (variables) in each term.

This produces a two-level implementation having a logic circuit diagram with a minimum no. of gates and the minimum no. of i/ps to the gates.

Rules for Grouping together adjacent cells/squares containing 1's.

1) Groups must contain 1, 2, 4, 8, 16 .... 2^n squares.
2) Groups must contain only '1' (and 'X' if dont-care is allowed).
3) Groups may be horizontal or vertical, but not diagonal.
4) Groups should be as large as possible
5) Each square/cell containing a '1' must be in atleast one group.
6) Groups may overlap.
7) Groups may wrap around the table. The leftmost cell in a row may be grouped with the rightmost cell and top cell in the column may be grouped with the bottom cell.
8) There should be as few groups as possible.

Note: Grouping '1' in map leads to minimal SOP form.
Grouping '0' in the map leads to minimal POS form.

Rules for grouping 0's in the map is same as for 1's explained above. Only difference is groups must contain only '0' (and 'x' if dont-care is allowed)

Example: Find the minimal sums for the following boolean functions

(a) $f(abc) = \Sigma m(1,3,4,5,6,7)$.

(b) $f(abc) = \Sigma m(2,3,4,5,7)$.

Solⁿ

(a) $f(abc) = \Sigma m(1,3,4,5,6,7)$

a \ bc	00	01	11	10	
0	0	1	1	0	→ c
1	1	1	1	1	→ a

∴ minimal sum of $f(abc) = \Sigma m(1,3,4,5,6,7)$ is

$$\boxed{f = a + c}$$

(b) $f(abc) = \Sigma m(2,3,4,5,7)$

a \ bc	00	01	11	10	
0	0	0	1	1	→ $\bar{a}b$
1	1	1	1	0	
	↓ $a\bar{b}$		↓ ac		

$$\therefore \boxed{f = a\bar{b} + ac + \bar{a}b}$$

The above example can also be solved in other way as shown below.

a \ bc	00	01	11	10	
0	0	0	1	1	→ $\bar{a}b$
1	1	1	1	0	
	↓ $a\bar{b}$		↓ bc		

$$\therefore \boxed{f = a\bar{b} + bc + \bar{a}b}$$

a \ bc	00	01	11	10
0	0	0	1	1
1	1	1	1	0

"Redundant grouping."

∵ same '1' is grouped in other groups though it is already grouped.
By this we get more terms & that will not said to be minimal sum expression. (OR) efficient expression.

Note: (i) Try to make as less as possible no. of groups to get an efficient minimal sum expression.

(ii) Avoid more overlappings of the groups as it leads to Redundant Grouping as shown above.

Example: Find minimal sum for the following function

$f(abc) = \Pi M(2,4,7)$.

Solⁿ:

a \ bc	00	01	11	10	
0	1	1	1	0	
1	0	1	0	1	→ $ab\bar{c}$
	↓ $\bar{a}\bar{b}$	↓ $\bar{b}c$	↓ $\bar{a}c$		

$$\therefore \boxed{f = \bar{a}\bar{b} + \bar{b}c + \bar{a}c + ab\bar{c}}$$

Here we can observe that '1' present in square of abc = 001 is getting grouped thrice just to make bigger group.
It is not called Redundant Grouping.

In the above eg the given function is POS function, but asked to get minimal sum i.e, in terms of SOP form.

So, what we did is we mapped those numbers of ΠM in K-map by 0's and rest all squares by 1's.

To get minimal sum, we grouped 1's instead of 0's.

If they asked to get minimal product expression for the same function, we would have grouped 0's of the numbers of ΠM and get minimal product expression i.e, POS form.

Exercise: Find the minimal products of boolean function (a) $f = \Sigma m(1, 3, 4, 5, 6, 7)$.

Sol^n

(a) $f = \Sigma m(1, 3, 4, 5, 6, 7)$.

a \ bc	00	01	11	10
0	0	1	1	0
1	1	1	1	1

$\rightarrow a+c$

$\therefore \boxed{f = a+c}$

(b) $f(abc) = \Pi M(1, 2, 5, 6, 7)$

a \ bc	00	01	11	10
0	1	0	1	0
1	1	0	0	0

$\rightarrow \bar{b}+c$

$b+\bar{c}$ $\quad \bar{a}+\bar{c}$

$\therefore \boxed{f = (b+\bar{c})(\bar{a}+\bar{c})(\bar{b}+c)}$

Here each group terms are written in the form of sums by taking variable value '0' as true & variable value '1' as complement & finally all sums are ANDed to get POS form.

Example: Write all the minimal sums and minimal products of the following Boolean functions.

(a) $f(abcd) = \bar{a}\bar{b}d + bcd + a\bar{b}d + b\bar{c}\bar{d}$

(b) $f(abcd) = (a+\bar{b})(a+c+d)(\bar{a}+\bar{b}+\bar{d})(a+\bar{c}+d)$.

Sol^n. (a) $f(abcd) = \bar{a}\bar{b}d + bcd + a\bar{b}d + b\bar{c}\bar{d}$

as the given function is not in standard form. i.e; all terms are not containing all the i/p variables.

So we have to convert it to std. form.

$\therefore f(abcd) = \bar{a}\bar{b}d(c+\bar{c}) + bcd(a+\bar{a}) + a\bar{b}d(c+\bar{c}) + b\bar{c}\bar{d}(a+\bar{a})$.

$= \bar{a}\bar{b}dc + \bar{a}\bar{b}d\bar{c} + abcd + \bar{a}bcd + a\bar{b}cd + a\bar{b}\bar{c}d + ab\bar{c}\bar{d} + \bar{a}b\bar{c}\bar{d}$.

ab \ cd	00	01	11	10
00	0	1	1	0
01	1	0	1	0
11	1	0	1	0
10	0	1	1	0

$b\bar{c}\bar{d}$, $\rightarrow cd$, $\rightarrow \bar{b}d$

$\therefore$ Minimal sum is $\boxed{f(abc) = b\bar{c}\bar{d} + \bar{b}d + cd}$

ab \ cd	00	01	11	10
00	0	1	1	0
01	1	0	1	0
11	1	0	1	0
10	0	1	1	0

$\rightarrow b+d$, $\rightarrow \bar{c}+d$, $\bar{b}+c+\bar{d}$

$\therefore$ Minimal product is

$\boxed{f(abc) = (\bar{b}+c+\bar{d})(\bar{c}+d)(b+d)}$

(b) $f(abcd) = (a+\bar{b})(a+c+d)(\bar{a}+\bar{b}+\bar{d})(a+\bar{c}+d)$

Its not in standard POS, so we have to convert it to std expression.

$= (a+\bar{b}+c\bar{c}+d\bar{d})(a+b\bar{b}+c+d)(\bar{a}+\bar{b}+c\bar{c}+\bar{d})(a+b\bar{b}+\bar{c}+d)$

$= (a+\bar{b}+c+d)(a+\bar{b}+c+\bar{d})(a+\bar{b}+\bar{c}+d)(a+\bar{b}+\bar{c}+\bar{d})$
$(a+b+c+d)\cancel{(a+\bar{b}+c+d)}(\bar{a}+\bar{b}+c+\bar{d})(\bar{a}+\bar{b}+\bar{c}+\bar{d})$
$(a+b+\bar{c}+d)\cancel{(a+\bar{b}+\bar{c}+d)}$.

* Cancel the repeated terms

$= (a+\bar{b}+c+d)(a+\bar{b}+c+\bar{d})(a+\bar{b}+\bar{c}+d)(a+\bar{b}+\bar{c}+\bar{d})$
$(a+b+c+d)(\bar{a}+\bar{b}+c+\bar{d})(\bar{a}+\bar{b}+\bar{c}+\bar{d})(a+b+\bar{c}+d)$.

ab \ cd	00	01	11	10
00	0	1	1	0
01	0	0	0	0
11	1	0	0	1
10	1	1	1	1

$\rightarrow a\bar{d}$

$\rightarrow \bar{b}d$

$\therefore \boxed{f(abcd) = a\bar{d} + \bar{b}d}$

Minimal Sum.

ab \ cd	00	01	11	10
00	0	1	1	0
01	0	0	0	0
11	1	0	0	1
10	1	1	1	1

$\bar{b}+\bar{d}$ $a+d$

$\therefore \boxed{f(abcd) = (\bar{b}+\bar{d})(a+d)}$

Minimal Product.

Note: To get minimal sum, group 1's.
To get minimal product, group 0's.
} From any form of expressions

Minimal Expressions of Incomplete Boolean functions.

Incomplete boolean expressions are those which evaluates to a don't care condition for some i/p combinations.
It is represented as 'X' or '–' in the truth table.

Karnaugh maps of incomplete boolean expressions would have cells with 'X' entries besides cells with 1's and 0's. Such 'X' cells are called don't care cells.

Minimal sum expression is obtained by grouping 1's along with 'X' cells by considering 'X' as 1.

Similarly Minimal product expression is obtained by grouping 0's along with 'X' cells by considering 'X' as 0.

Note: Rules for grouping is as same as what we followed earlier.

Exercise: Find minimal sum of the following incomplete boolean functions using K-maps.

(a) $f(abcd) = \Sigma m(0,2,5,7,8,10,13,15) + dc(1,4,11,14)$

(b) $f(abcd) = \Pi M(0,1,4,5,8,9,11) + dc(2,10)$

Sol^n.

(a) $f(abcd) = \Sigma m(0,2,5,7,8,10,13,15) + dc(1,4,11,14)$

ab \ cd	00	01	11	10
00	1	X	0	1
01	X	1	1	0
11	0	1	1	X
10	1	0	X	1

$\rightarrow \bar{b}\bar{d}$

$\rightarrow bd$

$\rightarrow ac$

$\bar{a}\bar{c}$

$\therefore$ minimal sum is $\boxed{f(abcd) = \bar{a}\bar{c} + \bar{b}\bar{d} + bd + ac}$

(b) $f(a\,b\,c\,d) = \Pi M(0,1,4,5,8,9,11) + dc(2,10)$.

ab \ cd	00	01	11	10	
00	0	0	1	X	$\to \bar{a}c$
01	0	0	1	1	$\to bc$
11	1	1	1	1	$\to ab$
10	0	0	0	X	$\to c\bar{d}$

$\therefore$ minimal sum is $\boxed{f(a\,b\,c\,d) = \bar{a}c + bc + ab + c\bar{d}}$

Here the given expression is POS form, so for the numbers of ΠM we substituted '0' in K map & dont cares as 'x'. Remaining cells will be filled by '1's. So to get minimal sum, group 1's along with x's.

Exercise: Find the minimal products of the following incomplete boolean functions using K-maps.

(a) $f(a\,b\,c\,d) = \Sigma m(7,9,11,12,13,14) + dc(3,5,6,15)$.

(b) $f(a\,b\,c\,d) = \Pi M(2,8,11,15) + dc(3,12,14)$.

Soln. $f(a\,b\,c\,d) = \Sigma m(7,9,11,12,13,14) + dc(3,5,6,15)$

ab \ cd	00	01	11	10
00	0	0	X	0
01	0	X	1	X
11	1	1	X	1
10	0	1	1	0

$a+c$ $\quad$ $b+d$

$\therefore \boxed{f(a\,b\,c\,d) = (a+c)(b+d)}$

Minimal product.

We can observe from the above example, that all 'x's are not grouped. It is not necessary, because once all 0's are grouped with x's by considering those grouped x's as 0's its done. So no need to consider ungrouped x's as 0's and grouping. Because it violates the logic of given function.

i(b) $f(a\,b\,c\,d) = \Pi M(2,8,11,15) + dc(3,12,14)$

ab \ cd	00	01	11	10	
00	1	1	X	0	$\to a+b+\bar{c}$
01	1	1	1	1	
11	X	1	0	X	
10	0	1	0	1	

$\bar{a}+c+d$ $\quad$ $\bar{a}+\bar{c}+\bar{d}$

$\therefore$ Minimal product is

$$\boxed{f(a\,b\,c\,d) = (\bar{a}+c+d)(\bar{a}+\bar{c}+\bar{d})(a+b+\bar{c})}$$