Solution of Separation of variables

let $z = XY$ be the solution where $X = X(x)$, $Y = Y(y)$

Sub $z = XY$ in the given equation.

Re write the equation in such a way that the left hand side involves x & related terms, and RHS involves y & related terms.

Equate LHS & RHS separately to constant resulting eq's for x and y

Sub resulting expressions in $z = XY$,

write the general solution.

① Solve by the method of separation of variables

$$py^3 + qx^2 = 0$$

let $z = XY$

where $X = X(x)$ & $Y = Y(y)$

$$\frac{\partial(XY)}{\partial x} \cdot y^3 + \frac{\partial(XY)}{\partial y} \cdot x^2 = 0$$

$$Y \frac{dX}{dx} \cdot y^3 + \frac{dY}{dy} X x^2 = 0$$

$$Y \frac{dX}{dx} y^3 = -\frac{dY}{dy} X x^2$$

$$\frac{1}{X} \frac{dX}{dx} \cdot \frac{1}{x^2} = -\frac{dY}{Y y^3}$$

Let

$$\frac{1}{x}\frac{dx}{dx}\cdot\frac{1}{x^2} = c$$

$$-\frac{1}{y}\cdot\frac{dy}{dy}\frac{1}{y^3} = c$$

$$\frac{1}{x}\frac{dx}{dx} = c_1 x^2$$

$$\frac{1}{y}\frac{dy}{dy} = -c\,y^3$$

$$\frac{dx}{dx} = c x^2\,dx$$

$$\frac{dy}{dy} = -c y^3\,dy$$

Int.

$$\log x = c\,x^3/3 + \log c_1$$

$$\log y = -c\frac{y^4}{4} + \log c_2$$

$$\cancel{x = c\,e^{x^3/3}}$$

$$y = c_2 e^{-cy^4/4}$$

$$x = e^{c x^3/3}\cdot c_1$$

Sol

$$z = xy$$

$$z = c_1 e^{cx^3/3}\cdot c_2 e^{-cy^4/4}$$

$$z = k\cdot e^{cx^3/3 - cy^4/4}$$

(2) Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ by separation of Variables.

Sol :- let $z = XY$ be the sol

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$Y \cdot \frac{d^2 X}{dx^2} + X \cdot \frac{d^2 Y}{dy^2} = 0$$

$$Y \cdot \frac{d^2 X}{dx^2} = -X \frac{d^2 Y}{dy^2}$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2}$$

let $\frac{1}{X} \frac{d^2 X}{dx^2} = K$

$$\frac{d^2 X}{dx^2} = KX$$

$$\frac{d^2 X}{dx^2} - KX = 0$$

$$(D^2 - K) X = 0$$

$$m^2 - K = 0$$

$$m = \pm\sqrt{K}$$

$$X = c_1 e^{\sqrt{K} \cdot x} + c_2 e^{-\sqrt{K} x}$$

let $-\frac{1}{Y} \frac{d^2 Y}{dy^2} = K$

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = -K$$

$$\frac{d^2 Y}{dy^2} + KY = 0$$

$$(D^2 + K) Y = 0$$

$$m^2 + K = 0$$

$$m = \pm\sqrt{K}\, i$$

$$Y = C_3 \cos\sqrt{K} y + C_4 \sin\sqrt{K} y$$

General sol

$$z = XY$$

$$z = \left(C_1 e^{\sqrt{K} \cdot x} + C_2 e^{-\sqrt{K} x}\right)\left(C_3 \cos\sqrt{K} y + C_4 \sin\sqrt{K} y\right)$$

(3) Solve by the method of Separation of Variables

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0$$

Sol :- Let $u = XY$ be the sol

$$x^2 \frac{\partial^2 (XY)}{\partial x^2} + x \cdot \frac{\partial (XY)}{\partial x} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$x^2 Y \cdot \frac{d^2 X}{dx^2} + x Y \frac{dX}{dx} + X \frac{d^2 Y}{dy^2} = 0$$

Dividing by XY

$$x^2 Y \cdot \frac{d^2 X}{dx^2} \frac{1}{XY} + x Y \frac{dX}{dx} \frac{1}{XY} + X \frac{d^2 Y}{dy^2} \cdot \frac{1}{XY} = 0$$

$$\frac{x^2}{X} \frac{d^2 X}{dx^2} + x \cdot \frac{dX}{X\, dx} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

$$\frac{x^2}{X} \frac{d^2 X}{dx^2} + \frac{x}{X} \frac{dX}{dx} = \frac{-1}{Y} \frac{d^2 Y}{dy^2}$$

$$\frac{x^2}{X} \frac{d^2 X}{dx^2} + \frac{x}{X} \frac{dX}{dx} = \frac{-1}{Y} \frac{d^2 Y}{dy^2}$$

Let $\frac{x^2}{X} \frac{d^2 X}{dx^2} + \frac{x}{X} \frac{dX}{dx} = K_1$

$$x^2 \cdot \frac{d^2 X}{dx^2} + x \frac{dX}{dx} = K_1 X$$

$$x^2 \frac{d^2 X}{dx^2} + x \cdot \frac{dX}{dx} - K_1 X = 0$$

Cauchy's LDE

$$x^2\frac{d^2X}{dx^2} = D(D-1)X \qquad \text{put } \log x = t$$
$$x = e^t$$

$$x\frac{dX}{dx} = DX \qquad \text{where } D = \frac{d}{dt}$$

$$D(D-1)X + DX - K_1X = 0$$

$$(D^2 - D + D - K_1)X = 0$$

$$(D^2 - K_1)X = 0$$

$$m^2 - K_1 = 0$$

$$m = \pm\sqrt{K_1}$$

$$X = c_1 e^{\sqrt{K_1}t} + c_2 e^{-\sqrt{K_1}t}$$

$$X = c_1(e^t)^{\sqrt{K_1}} + c_2(e^t)^{-\sqrt{K_1}}$$

$$X = c_1 \cdot x^{\sqrt{K_1}} + c_2 x^{-\sqrt{K_1}}$$

let $\quad -\frac{1}{Y}\frac{d^2Y}{dy^2} = K_1$

$$\frac{d^2Y}{dy^2} = -K_1Y$$

$$\frac{d^2Y}{dy^2} + K_1Y = 0$$

$$(D^2 + K_1)Y = 0$$

$$m^2 + K_1 = 0 \Rightarrow m^2 = -K_1$$

$$m = \pm\sqrt{K_1}\, i$$

$$Y = c_3 \cdot \cos\sqrt{K_1}\, y + c_4 \sin\sqrt{K_1}\, y$$

General sol is

$$u = XY$$

$$u = \left(c_1 x^{\sqrt{K_1}} + c_2 x^{-\sqrt{K_1}}\right)\left(c_3 \cos\sqrt{K_1}\, y + c_4 \sin\sqrt{K_1}\, y\right)$$

(4) Solve by the method of separation of Variables

$$\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + z, \qquad z(x,0) = 6e^{-3x}$$

Sol:- Let $z = XY$ be the sol

$$\frac{\partial(XY)}{\partial x} = 2\frac{\partial(XY)}{\partial y} + XY$$

$$Y \cdot \frac{dX}{dx} = 2X \frac{dY}{dy} + XY$$

Dividing by XY

$$\frac{1}{X}\frac{dX}{dx} = \frac{2}{Y}\frac{dY}{dy} + 1$$

$$\frac{1}{X}\frac{dX}{dx} = K$$

$$\frac{1}{X}\, dX = K\, dx$$

$$\int \frac{1}{X}\, dX = K\int dx \Rightarrow \log_e X = Kx + c_1$$

$$X = e^{Kx + c_1}$$

$$\frac{2}{y}\frac{dy}{dy}+1=k$$

$$\frac{2}{y}\frac{dy}{dy}=k-1$$

$$\frac{1}{y}dy=\frac{k-1}{2}dy$$

$$\log y=\left(\frac{k-1}{2}\right)\cdot y+c_2$$

$$y=e^{\left(\frac{k-1}{2}\right)y+c_2}$$

General sol

$$z=xy$$

$$z=e^{kx+c_1}\cdot e^{\left(\frac{k-1}{2}\right)y+c_2}$$

$$z=e^{kx}\cdot e^{c_1}\cdot e^{\left(\frac{k-1}{2}\right)y}\cdot e^{c_2}$$

$$z=e^{c_1}\cdot e^{c_2}\, e^{kx+\left(\frac{k-1}{2}\right)y}$$

$$z=A\,e^{kx+\left(\frac{k-1}{2}\right)y}\rightarrow (1)$$

Given $z(x,0)=6e^{-3x}$ when $y=0$, $z=6e^{-3x}$

$$6e^{-3x}=A\cdot e^{kx+\left(\frac{k-1}{2}\right)0}$$

$$6e^{-3x}=Ae^{kx}$$

Comparing $A=6,\quad k=-3.$

Sub A&k in (1)

$$z=6e^{-3x+\left(\frac{k-1}{2}\right)y}$$

$$z=6e^{-3x+\left(\frac{-4}{2}\right)y}$$

$$z=6e^{-3x-2y}$$

5) $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u.$ $\quad u(0,y) = 3e^{-y} - e^{-5y}$

$$4\frac{d(XY)}{dx} + \frac{d(XY)}{dy} = 3(XY)$$

$$\frac{4}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 3$$

$$\frac{4}{X}\frac{dX}{dx} = 3 - \frac{1}{Y}\frac{dY}{dy}$$

$$\frac{4}{X}\frac{dX}{dx} = K. \qquad 3 - \frac{1}{Y}\frac{dY}{dy} = K$$

$$\frac{dX}{X} = \frac{K}{4}dx \qquad \frac{dY}{Y} = (3-K)\,dy$$

$$\log X = \frac{K}{4}x + \log c_1 \qquad \log Y = 3y - Ky$$

$$X = c_1 e^{(K/4)x} \qquad Y = c_2 e^{(3-K)y}$$

$u = c_1 c_2 \, e^{(k/4)x} \, e^{(3-k)y}$

$= A \, e^{(k/4)x} \, e^{(3-k)y}$

Case(i) $\quad u(0,y) = 3e^{-y}$

$u = A \, e^{0} \, e^{(3-k)y}$

$3e^{-y} = A \, e^{(3-k)y}$

$A = 3, \quad k = 4$

$u = 3 \, e^{x} \, e^{-y}$

$= 3e^{x-y}$

case(ii) $\quad u(0,y) = e^{-5y}$

$e^{-5y} = A \, e^{0} \, e^{(3-k)y}$

$A = 1, \quad k = 8.$

$u = e^{2x} \, e^{-5y}$

$= e^{2x-5y}$

Case (iii) $\quad u(0,y) = 3e^{-y} - e^{-5y}$

$u = u_1 - u_2$

$u = u_1(0,y) - u_2(0,y)$

$= 3e^{-y} - e^{-5y}$

$u = 3e^{x-y} - e^{2x-5y}.$