(1) According to Newton's law of cooling the rate of which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. of the temperature of the air is 30°c and the substance cools from 100°c to 70°c in 15 min. find time when the temperature will be 40°c.

Sol:- By Newton's law of cooling

$$\frac{d\theta}{dt} \propto (\theta-\theta_0)$$

θ = Substance temp
t = time
θ_0 = Temp of air

$$\frac{d\theta}{dt} = -K(\theta-\theta_0)$$

Given

$$\frac{dT}{dt} = -K(T-30)$$

$$\frac{dT}{T-30} = -K\,dt$$

Integrating

$$\int \frac{dT}{T-30} = -K\int dt$$

$$\log(T-30) = -Kt + c \rightarrow (1)$$

at $t=0$, $T=100°c$

$$\log(100-30) = -K(0) + c$$

$$\boxed{\log 70 = c}$$

Sub c in eqn (1)

2

$$\log(T-30) = -kt + \log 70$$

$$kt = \log 70 - \log(T-30) \longrightarrow (2)$$

at $t = 15$, $T = 70$

Sub t & T in (2)

$$15k = \log 70 - \log(70-30)$$

$$15k = \log 70 - \log 40 \longrightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{kt}{15k} = \frac{\log 70 - \log(T-30)}{\log 70 - \log 40}$$

when $T = 40$

$$\frac{t}{15} = \frac{\log 70 - \log(40-30)}{\log 70 - \log 40}$$

$$\frac{t}{15} = \frac{\log 70 - \log 10}{\log 70 - \log 40}$$

$$\frac{t}{15} = \frac{\log_e(70/10)}{\log(70/40)}$$

$$\frac{t}{15} = \frac{\log_e 7}{\log_e(7/4)}$$

$$t = \frac{\log_e 7}{\log_e(7/4)} \times 15$$

$$t = \underline{52.20}$$

Hence the temp will be $40°c$ after 52.20 min.

2) The temperature of a cup of Coffee is 92°c, when freshly poured the temperature being 24°c. In one minute it was cooled to 80°c. How long or period must elapse before the temperature of cup becomes 65°c.

Sol:- Given

$$\frac{dT}{dt} = -K(T - T_0)$$

$$\frac{dT}{dt} = -K(T - 24)$$

$$\frac{dT}{T-24} = -K\,dt$$

Integrating

$$\int \frac{dT}{T-24} = -K\int dt$$

$$\log(T-24) = -Kt + c \longrightarrow (1)$$

at $t = 0$, $T = 92$

$$\log(92 - 24) = -K(0) + c$$

$$\boxed{c = \log 68}$$

Sub c in (1)

$$\log(T-24) = -Kt + \log 68$$

$$Kt = \log 68 - \log(T-24) \longrightarrow (2)$$

at $t = 1$, $T = 80$

$$K\cdot(1) = \log 68 - \log(80-24)$$

$$K = \log 68 - \log 56 \rightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{Kt}{K} = \frac{\log 68 - \log(T-24)}{\log 68 - \log 56}$$

$$t = \frac{\log 68 - \log(T-24)}{\log 68 - \log 56}$$

when $T=65$

$$t = \frac{\log 68 - \log(65-24)}{\log 68 - \log 56}$$

$$t = 2.608 \text{ min}$$

The temperature of the coffee will be $65^\circ c$, after 2.608 minutes

Solve

(1) Water at temperature $100^\circ c$ cools in 10 minutes to $88^\circ c$ in a room temperature $25^\circ c$. Find the temperature of water after 20 min.

Ans: $77.9^\circ c$

(2) A resistance of 100 ohms, an inductance of 0.5 Henry is connected in series with a battery of 20 volts. Find the current in a circuit as a function of t?

Linear D.E of Second and higher Order with Constant Coefficients

A D.E of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = \phi(x) \rightarrow (1)$$

is called n^{th} order Linear Differential Equation with Constant Coefficients. where $a_0, a_1, \ldots a_n$ are Constants

Ex :- $3\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = 2e^x$

If $\phi(x) = 0$, then the above D.E is called Homogeneous D.E

If $\phi(x) \neq 0$ then it is called non homogeneous D.E

Note :- $D = \frac{d}{dx}$

$D^2 = \frac{d^2}{dx^2}$

$D^3 = \frac{d^3}{dx^3}$

$\vdots$

$D^n = \frac{d^n}{dx^n}$

(1) $\Rightarrow$ $a_0 D^n y + a_1 D^{n-1} y + \cdots a_n y = \phi(x)$

If $D = \frac{d}{dx}$ and $X = X(x)$ then P.T

$$\frac{1}{D} X = \int X(x)\,dx$$

Sol:- let $\frac{1}{D} \cancel{X} = y$

operating with D

$$D \cdot \frac{1}{D} X = Dy$$

$$X = Dy$$

$$X = \frac{dy}{dx} \Rightarrow X\,dx = dy$$

Integrating

$$\int X\,dx = \int dy$$

$$y = \int X\,dx$$

$$\boxed{\frac{1}{D} X = \int X(x)\,dx}$$

Solution of Homogeneous Linear D.E

Consider the equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

where a_0, a_1, a_2 are Constants

$$a_0 D^2 y + a_1 Dy + a_2 y = 0$$

$$(a_0 D^2 + a_1 D + a_2) y = 0$$

Auxiliary equation

$a_0 m^2 + a_1 m + a_2 = 0$ we can find roots

Case(i) Suppose the roots are m_1 and m_2 real and distinct $(m_1 \neq m_2)$.

Sol complementary function

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Note:- $m_1, m_2, m_3 \ldots m_n$ $(m_1 \neq m_2 \neq m_3 = \ldots)$

$$C.F = y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

Case(ii):- Suppose the roots are m_1 & m_2 real and equal $(m_1 = m_2 = m)$

$$CF = y = (c_1 + c_2 x) e^{mx}$$

Note:- $m_1 = m_2 = m_3 = \ldots = m$

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}$$

Case(iii):- Suppose the roots are complex which occurs in pairs $p \pm iq$

then $y = CF = e^{px}(c_1 \cos qx + c_2 \sin qx)$

If the roots are purely imaginary

$0 \pm qi$

$y = c_1 \cos qx + c_2 \sin qx$

Note:- If the complex roots pairs $p \pm iq$ repeated n times

$$y = e^{px}\left[\left(c_1 + c_2 x + c_3 x^2 , \ldots + c_{n-1} x^{n-1}\right)\cos qx + \left(k_1 + k_2 x + k_3 x^2 + \ldots + k_{n-1} x^{n-1}\right)\sin qx\right]$$

Problems

1. Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$

Sol:- $(D^2 - D - 12)y = 0$

(A.E is Auxilary equation)

A.E is $m^2 - m - 12 = 0$

$m^2 - 4m + 3m - 12 = 0$

$m(m-4) + 3(m-4) = 0$

$(m+3)(m-4) = 0$

$m = -3, \quad m = 4$

C.F = (Complementary function)

C.F $\quad y = c_1 e^{-3x} + c_2 e^{4x}$

2) Solve $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$

Sol:- $(D^3 - 2D^2 + 4D - 8)y = 0$

A.E is $m^3 - 2m^2 + 4m - 8 = 0$

$m^2(m-2) + 4(m-2) = 0$

$(m-2)(m^2+4) = 0$

$m - 2 = 0$ $m^2 + 4 = 0$

$m = 2$ $m^2 = -4$

$m = \pm 2i$

Roots = $2, \pm 2i$

CF
Sol $y = c_1 e^{2x} + (c_2 \cos 2x + c_3 \sin 2x)$

3) Solve $4y''' + 4y'' + y' = 0$

Sol:- $(4D^3 + 4D^2 + D)y = 0$

A.E is $4m^3 + 4m^2 + m = 0$

$m(4m^2 + 4m + 1) = 0$

$m = 0,$ $4m^2 + 4m + 1 = 0$

$(2m)^2 + 2 \cdot 2m \cdot 1 + 1 = 0$

$(2m+1)^2 = 0 \Rightarrow 2m + 1 = 0$

$m = -1/2, -1/2$

Roots
$m = 0, -1/2, -1/2$

CF

$y = c_1 e^{0x} + (c_2 + c_3 x) e^{-\frac{1}{2}x}$

④ Solve $(D^2 - 6D + 9) y = 0$

Sol:- A.E is $m^2 - 6m + 9 = 0$

$m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3) = 0$

$(m-3)(m-3) = 0$

$m = 3, 3$

CF

$y = (c_1 + c_2 x) e^{3x}$

⑤ Solve $(D^3 - 9D^2 + 23D - 15) y = 0$

Sol:- A.E is

$m^3 - 9m^2 + 23m - 15 = 0$

$m = 1$ satisfies

$m = 1$	1	-9	23	-15
	0	1	-8	15
	1	-8	15	0

$m^2 - 8m + 15 = 0$

$m^2 - 5m - 3m + 15 = 0$

$m(m-5) - 3(m-5) = 0$

$(m-3)(m-5) = 0 \Rightarrow m = 3, m = 5$

Roots $m = 1, 3, 5$

$$y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}$$

$m = 1$

$m^3 - 9m^2 + 23m - 15 = 0$

$(1)^3 - 9(1)^2 + 23(1) - 15 = 0$

$1 - 9 + 23 - 15 = 0$

$m = 1$ $\quad$ $\neq 0$ $\quad$ $m = -1$ $\neq 0$

$m = 1$

1	−9	23	−15
0	1	−8	15
1	−8	15	0

Solve

(6) $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

Sol:- $(D^3 + 6D^2 + 11D + 6)y = 0$

$m = 1,$ $\quad 1^3 + 6(1)^2 + 11(1) + 6 \neq 0$

$m = -1$ $\quad (-1)^3 + 6(-1)^2 + 11(-1) + 6 =$

$12 - 12 = 0$

$m = -1$

1	6	11	6
0	−1	−5	−6
1	5	6	0

$m^2 + 5m + 6 = 0$

$m^2 + 3m + 2m + 6 = 0$

$m(m+3) + 2(m+3) = 0$

$\Rightarrow (m+2)(m+3) = 0$

$m = -2, -3$

Roots

$m = -1, -2, -3$

CF

$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

7 Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$

Sol:- $(D^2 - D + 1) y = 0$

A.E is $m^2 - m + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot (1)(1)}}{2(1)}$

$m = \frac{1 \pm \sqrt{1-4}}{2}$

$m = \frac{1 \pm \sqrt{-3}}{2}$

$p \pm iq$

$m = \frac{1 \pm \sqrt{3}\, i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

c.f

$y = e^{x/2} \left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right]$

$p = \frac{1}{2}, \quad q = \frac{\sqrt{3}}{2}$

$\underline{p \pm iq}$

$y = e^{px} (c_1 \cos qx + c_2 \sin qx)$

8) Solve $(D^3-5D^2+8D-4)y=0$

Sol:- A.E is

$$m^3-5m^2+8m-4=0$$

$m=1,\quad (1)^3-5(1)^2+8(1)-4=$

$9-9=0$

$$m=1\;\begin{array}{|cccc}1 & -5 & 8 & -4\\ 0 & 1 & 4 & 4\\ \hline 1 & -4 & 4 & \lfloor 0\end{array}$$

$$m^2-4m+4=0$$

$$m^2-2m-2m+4=0$$

$$(m-2)(m-2)=0$$

$$m=2,2$$

Roots $=1,\ 2,\ 2$

$$y_{cf}=c_1e^{x}+(c_2+c_3x)e^{2x}$$

9) Solve $\dfrac{d^4y}{dx^4}+4y=0$

Sol:- $(D^4+4)y=0$

A.E is $m^4+4=0$

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$m^4+4=0$

$(m^2)^2 \pm 2^2 = 0$ ×

$m^4+4m^2-4m^2+4=0$

$(m^4+4m^2+4)-4m^2=0$

$(m^4+4m^2+4)-(2m)^2=0$

$(m^2+2 \cdot m^2 \cdot 2+2^2)-(2m)^2=0$

$(m^2+2)^2-(2m)^2=0$

$(m^2+2+2m)\quad(m^2+2-2m)=0$

$m^2+2m+2=0,\qquad m^2-2m+2=0$

$m=\dfrac{-2\pm\sqrt{2^2-4\cdot1\cdot2}}{2}\qquad m=\dfrac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot2}}{2}$

$m=\dfrac{-2\pm\sqrt{4-8}}{2}\qquad m=\dfrac{2\pm\sqrt{4-8}}{2}$

$m=\dfrac{-2\pm\sqrt{-4}}{2}\qquad =\dfrac{2\pm\sqrt{-4}}{2}$

$=\dfrac{-2\pm2i}{2}\qquad =\dfrac{2\pm2i}{2}$

$=\dfrac{\cancel{2}(-1\pm i)}{\cancel{2}}\qquad =\dfrac{\cancel{2}(1\pm i)}{\cancel{2}}$

$m=-1\pm i\qquad m=1\pm i$

$m = -1 \pm i$, $\quad m = 1 \pm i$

cf

$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$

Solve

(1) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4 = 0$

(2) $2y'' - 2y' - y = 0$

(3) $(D^4 - 5D^2 + 4)\, y = 0$

(4) $\frac{d^3y}{dx^3} + y = 0$