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Unit - IV
Infinite Series

* Examples:

$A = (1, -1, \frac{1}{2}, \frac{7}{4}, \sqrt{2})$ — finite
$B = (1, -1, 1, -1 \ldots \ldots \infty)$ — infinite
$C = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16})$ — finite
$D = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \ldots \ldots \infty)$ — infinite

* Definition: A sequence is defined to be a mapping x from the set to all natural number N to a set of real number R.

$x : N \rightarrow R$, Thus $\{x_n\} = \{x_1, x_2 \ldots \ldots x_n\}$

Eg:- $f : N \rightarrow R$

$f = \{\frac{1}{2^n}\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots \ldots \infty\}$

x_n is called as n^{th} term of sequence $\{x_n\}$

More examples:- $\{\frac{1}{2^n}\}$, $\{n\}$, $\{\frac{1}{n}\}$, $\{(-1)^n\}$, $\{2n\}$

→ Limit of sequence

$\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3} \ldots \frac{1}{n} \ldots\}$

when $n \rightarrow \infty$ then $\{\frac{1}{n}\} \rightarrow 0$

$\{n\} = \{1, 2, 3 \ldots n, \ldots\}$

$\lim_{n \rightarrow \infty} n = \infty$

$\{(-1)^n\} = \{-1, 1, -1, 1, -1 \ldots \ldots\}$

when $n \rightarrow$ odd $\{(-1)^n\} = -1$

when $n \rightarrow$ even $\{(-1)^n\} = 1$

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* Definition: Limit of a sequence is the value that the terms of a sequence "tend to" ∞. (OR)
A real number 'l' is said to be limit of sequence $\{x_n\}$ as n tends to '∞' if for ~~every~~ query $\epsilon > 0$, $\exists$ a +ve integer m, such that
$$|x_n - l| < \epsilon \quad \forall\ n > m.$$

Symbolically
$$\lim_{n \to \infty} \text{ or } lt\ x_n = l$$

$$|x_n - l| < \epsilon$$

* If 'l' is finite and unique then the sequence $\{x_n\}$ is said to be convergent
* If 'l' is infinite & unique then the sequence $\{x_n\}$ is said to be divergent.
* If 'l' is neither finite nor infinite, then such sequence $\{x_n\}$ is said to be oscillatory

① $\lim_{n \to \infty} 1/n = 0 \Rightarrow |1/n - 0| < \epsilon \quad \forall\ n > m$

② $\lim_{n \to \infty} n = \infty$ Hence the seqn: divergent

→ Infinite Series :- Consider an infinite series seq $\{x_n\}$, then the sum of $x_1 + x_2 + x_3 + \dots x_n + x_{n+1} \dots$

$$\sum_{i=\phi}^{\infty} x_i = x_1 + x_2 + x_3 \dots x_n + \dots$$

$$\{1/n\} = \{1, 1/2, 1/3, 1/4 \dots\}$$

$$\text{sum} = \sum_{n=1}^{\infty} 1/n$$

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* A series Σx_n is said to be a series of +ve terms if $x_n > 0 \ \forall \ n \in \mathbb{N}$. Eg:- $\Sigma n^2 = 1^2 + 2^2 + 3^2 \ldots$

* A series Σx_n is said to be an alternating series if the terms are positive & negative alternatively

Eg:- $\sum_{n=1}^{\infty} (-1)^n = -1+1-1+1 \ldots$

$$\Sigma (-1)^n \cdot \frac{1}{n(n+1)} = \frac{-1}{1\cdot 2} + \frac{1}{2\cdot 3} - \frac{1}{3\cdot 4} + \ldots$$

→ Let Σx_n be a series in general

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \ldots \ x_n + \ldots$$

n^{th} partial sum of Σx_n ← $S_n = x_1 + x_2 + \ldots + x_n$ (sum till n^{th} term)

$S_1 = (x_1)$ → first partial sum of series Σx_n

$S_3 = x_1 + x_2 + x_3$ → third partial sum of Σx_n

Definition :- Let Σx_n be a given series Σx_n and $S_1, S_2, S_3 \ldots \ldots S_n$ are different partial sum of Σx_n. Then the $\{S_n\}$ of this partial sum is called "sequence of partial sum of the series Σx_n.

Nature of the Series :- Convergent, Divergent and Oscillatory

Let Σx_n be a series & $\{S_n\}$ be the corresponding sequence of partial sum then

(i) The series is Σx_n is convergent if the $\{S_n\}$ is convergent i.e. $\lim_{n\to\infty} S_n = l$ (finite & unique)

(ii) The series is Σx_n is divergent if the $\{S_n\}$ is divergent i.e. $\lim_{n\to\infty} S_n = \infty$ or $-\infty$

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(iii) (a) The series $\sum x_n$ is said to be oscillatory finitely if the $\{S_n\}$ oscillate finitely.

Eg:- $\sum (-1)^n$.

(b) The series $\sum x_n$ is said to be oscillate infinitely, if the $\{S_n\}$ oscillate infinitely.

Eg:- $\sum (-1)^n . n$.

11/11/20 Nature of an Infinite series:-

1. Convergent $\lim_{n \to \infty} S_n = l$ (finite & unique)

2. Divergent

$\lim_{n \to \infty} S_n = +\infty$ or $-\infty$ $\{S_n\}$ is divergent
$\Rightarrow \sum X_n$ is divergent

3. (i) Oscillates finitely $\Rightarrow \{S_n\}$ oscillates finitely
(ii) Oscillates infinitely

Q. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots \frac{1}{n(n+1)} + \dots \infty$

$$\sum u_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \cancel{\frac{n}{n} - \frac{n}{n+1}} \quad \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \dots \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \cancel{1 - \frac{n}{n+1}} \qquad S_n = 1 - \frac{1}{n+1}$$

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$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \underline{\underline{1}}$$

$\{S_n\}$ is convergent

Hence $\sum u_n$ is convergent to 1.

(2) $\sum n^2 = 1^2 + 2^2 + 3^2 \ldots + n^2 + \ldots$

$\sum u_n = \sum n^2$

n^{th} term $u_n = n^2$

n^{th} partial series of $\sum u_n$ is

$S_n = 1^2 + 2^2 + 3^2 \ldots + n^2$

$$S_n = \frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n^3 (1+1/n)(2+1/n)}{6}$$

$$= +\infty$$

$\{S_n\}$ diverges to infinity

therefore $\sum u_n$ also diverges to infinity.

① Geometric Series:-

$1 + x + x^2 + x^3 + \ldots x^n + \ldots \infty$

$\sum u_n = \sum x^n$, n^{th} term of $\sum u_n$ is x^n

(i) Convergant if $-1 < x < 1$ $\quad |x| < 1$

(ii) divergent if $x \geq 1$

(iii) Oscillates finitely if $x = -1$.

Oscillates infinitely if $x < -1$.

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(2) Comparison Test :-

(i) If Σu_n & Σv_n are series of +ve term & Σv_n is convergent & there is the constant 'k' such that $u_n \le k v_n$, then Σu_n is also convergent

(ii) If Σu_n & Σv_n are series of +ve term & Σv_n is divergent & there is a constant 'k' such that $u_n > k v_n$.

Note :- A positive term series either converge or diverge to $+\infty$.

3) P- series or Harmonic Series:-

Let the series $\Sigma \frac{1}{n^p}$ is called P-series, then the series is convergent if $P > 1$ & divergent $P \le 1$

4) Limit form of test / Quotient test

Let Σu_n & Σv_n be two series of +ve term &
$\lim_{n \to \infty} \frac{u_n}{v_n} = l \neq 0$, then both the series behave like.

Q. Discuss the comparison of series

$$1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} \dots$$

$$= 1 + \frac{1}{(2)^{2\times 2/3}} + \frac{1}{(3)^{2\times 2/3}} + \frac{1}{(4^2)^{2/3}}$$

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$$= \frac{1}{1^{4/3}} + \frac{1}{2^{4/3}} + \frac{1}{3^{4/3}} + \frac{1}{4^{4/3}} \dots$$

$$\Sigma u_n = \Sigma \frac{1}{n^{4/3}}$$

Here $p = 4/3$ & $p > 1$

Hence it is convergent in nature.

Q. $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \dots$

$$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n(2-1/n)}{n^3(1+1/n)(1+2/n)}$$

$$= \frac{1(2-1/n)}{n^2(1+1/n)(1+2/n)}$$

Let us choose $\Sigma v_n = \Sigma \frac{1}{n^2}$

$p=2$, so it is convergent

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\frac{1(2-1/n)}{\cancel{n^2}(1+1/n)(1+2/n)}}{1/\cancel{n^2}}$$

$= \frac{2}{(1)(1)} = 2$, Both will behave same

hence even u_n is also convergent

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Q. $\sqrt{1/4} + \sqrt{2/6} + \sqrt{3/8} + \dots \sqrt{\frac{n}{2(n+1)}} + \dots$

$$u_n = \sqrt{\frac{n}{2(n+1)}}$$

$$= \sqrt{\frac{n}{2n(1+1/n)}} = \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}}\sqrt{\frac{1}{2(1+1/n)}}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{1}{2(1+1/n)}}$$

$$= \sqrt{1/2} \neq 0$$

$\Rightarrow$ Σu_n does not converge

(Note :- If Σu_n is converging then $\lim_{n\to\infty} u_n = 0$)

$\because$ The given series is of +ve terms, i.e. the series should be convergent or divergent.
Since Σu_n is not convergent, then it must be divergent.

This is for applying limit to entire infinite series (u_n) not partial (S_n).

Q. $\Sigma \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$

$$u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^p} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{(n+1) - n}{n^p(\sqrt{n+1} + \sqrt{n})} = \frac{1}{n^{p+1/2}(\sqrt{1+1/n} + 1)}$$

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Consider $V_n = \frac{1}{n^{p+1/2}}$

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1. D' Alembert's Ratio Test :-

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$$

If $l < 1$, then Σu_n is convergent
If $l > 1$ then Σu_n is divergent

Note : If $l = 1$, the test fails

Hint:

If $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l$

$l > 1$, convergent
$l < 1$, divergent
(Reverse the condition)

* If condition fails, applying Raabe's test will be easy.

2. Raabe's Test :-

$$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$$

If $l > 1$, then Σu_n is convergent
$l < 1$, then Σu_n is divergent

Note: ① If $l = 1$ then Raabe's test fails
② Raabe's test is used when D' Alembert's test fail & when $\frac{u_{n+1}}{u_n}$ does not involve the number 'e'.

3. Cauchy's Root test :-

$$\lim_{n \to \infty} u_n^{1/n} = l$$

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If $l < 1$, Σu_n is convergent
$l > 1$, Σu_n is divergent

If $l = 1$, Again the test fails

Imp limit condition:

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

Problems:

(1) $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$

Soln:

$$u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$$

$$u_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} \cdot x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} x^{n+1} \cdot \frac{\sqrt{n^2+1}}{\sqrt{n} \cdot x^n}$$

$$= \frac{\sqrt{n+1} \cdot \sqrt{n^2+1}}{\sqrt{n} \cdot \sqrt{(n+1)^2+1}} \cdot x$$

$$= \frac{\cancel{\sqrt{n}} \cdot \cancel{n} \sqrt{1+1/n} \cdot \sqrt{1+1/n^2}}{\cancel{\sqrt{n}} \cdot \cancel{n} \sqrt{(1+1/n)^2 + 1/n^2}} \cdot x$$

$$= \frac{\sqrt{1+1/n} \cdot \sqrt{1+1/n^2}}{\sqrt{(1+1/n)^2 + 1/n^2}} \cdot x$$

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Applying limits

$$= \lim_{n\to\infty} \frac{\sqrt{1+1/n} \cdot \sqrt{1+1/n^2}}{\sqrt{(1+1/n)^2 + 1/n^2}} \cdot x$$

$$= \frac{\sqrt{1+0} \cdot \sqrt{1+0}}{\sqrt{(1+0)^2+0}} \cdot x$$

$$= \frac{1 \cdot 1}{1} \cdot x$$

$$= x$$

By D' Alembert's ratio test

If $x < 1$, Σu_n is convergent

$x > 1$, Σu_n is divergent

$x = 1$, the test fails

Put $x=1$, $u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} = \frac{\sqrt{n}}{n\sqrt{1+1/n^2}} = \frac{1}{\sqrt{n}\sqrt{1+1/n^2}}$

Comparison test let $v_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

By P series $p = 1/2 < 1 \Rightarrow \Sigma v_n$ is divergent

By limit of Test

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\cancel{\frac{1}{\sqrt{n}}} \frac{1}{\sqrt{1+1/n^2}}}{\cancel{\frac{1}{\sqrt{n}}}} = \frac{1}{\sqrt{1+0}} = 1$$

Both series behave like

Hence, $\Sum u_n$ is divergent

So

$\Sum u_n$ is convergent $x<1$

$\Sum u_n$ is divergent $x>=1$.

(2)

H.W

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} \cdots$$

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

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(3) $1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} \ldots + \infty$

$$U_n = \frac{(\alpha+1)(2\alpha+1)\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots(n\beta+1)}$$

$$U_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots(n\alpha+1)((n+1)\alpha+1)}{(\beta+1)(2\beta+1)\ldots(n\beta+1)((n+1)\beta+1)}$$

$$\frac{U_{n+1}}{U_n} = \frac{((n+1)\alpha+1)}{((n+1)\beta+1)}$$

$$= \frac{\cancel{n}((1+1/n)\alpha + 1/n)}{\cancel{n}((1+1/n)\beta + 1/n)}$$

Applying limits

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(1+1/n)\alpha + 1/n}{(1+1/n)\beta + 1/n}$$

$$= \frac{\alpha}{\beta}$$

If $\alpha/\beta < 1 \longrightarrow$ convergent

$\alpha/\beta > 1 \longrightarrow$ divergent

If $\alpha/\beta = 1$ i.e $\alpha = \beta \Rightarrow \lim_{n\to\infty} U_n = 1 \neq 0$, then ΣU_n does not converge, being a +ve series it must be divergent.

Raabe's Test

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(1) $\frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots$

$$u_n = \frac{1.3.5\dots(2n-1)}{2.4.6\dots(2n)} \cdot \frac{x^{2n+1}}{(2n+1)}$$

$$u_{n+1} = \frac{1.3.5\dots(2n-1)(2n+1)}{2.4.6\dots(2n)(2n+2)} \frac{x^{2n+3}}{(2n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)}{(2n+1)} \cdot \frac{(x^{2n+1})}{(2n+1)} \frac{(2n+3)}{(x^{2n+3})}$$

$$= \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} \cdot \frac{1}{x^2}$$

$$= \lim_{n \to \infty} \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} \cdot \frac{1}{x^2}$$

$$= \lim_{n \to \infty} \frac{\cancel{n^2}(2+2/n)(2+3/n)}{\cancel{n^2}(2+1/n)(2+1/n)} \frac{1}{x^2}$$

$$= \frac{2.2}{2.2} \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

$\frac{1}{x^2} > 1 \longrightarrow$ convergent

$\frac{1}{x^2} < 1 \longrightarrow$ divergent

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If $\frac{1}{x^2} = 1$, test fails

$\Rightarrow x^2 = 1$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1} - 1\right)$$

let $\Sigma V_n = n$, $p = -1$

$$= 3/2 > 1$$

Hence Σu_n is convergent

Q.
H.W

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - 3/2\right)^{-2} + \left(\frac{4^4}{3^4} - 4/3\right)^{-3} \ldots + \infty$$

Hint $$u_n = \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-n}$$

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Fourier Series :

→ Fourier series is a representation of period function is a series of simple periodic function such as sine and cosine

→ Fourier series representation of a function is more general than taylor series because many periodic function with finite no. of discontinuous can be expanded in fourier series, but they donot have taylor series representation.

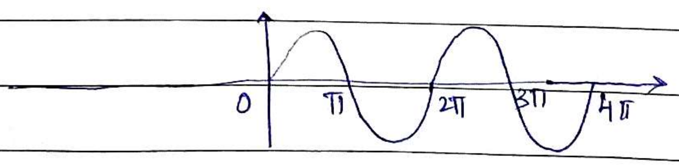
→ Fourier series are often used to represent the respone of system to a periodic input. This response depends directly on frequency content of input.

Periodic functions:

→ A function $f(x)$ is said to be a periodic if
$f(x+T) = f(x) \quad \forall\, x$, where 'T' is a +ve no.

* The no. 'T' is called period of $f(x)$
* It means that the value of $f(x)$ is repeated at interval 'T'.

Ex :- $\sin x \Rightarrow \sin(x+2\pi) = \sin x$, Hence 2π is the period of $\sin x$.



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$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \sin(x+6\pi) \ldots$

Hence $\sin x$ has period $2\pi, 4\pi, 6\pi \ldots 2n\pi$

Since 2π is the smallest period, we say that the period of $\sin x$ is 2π.

Fourier Series definition:

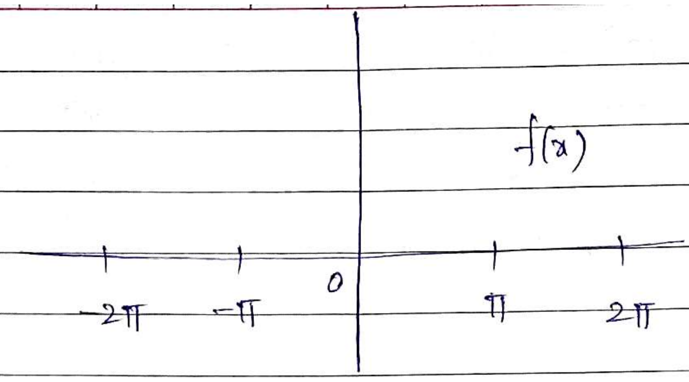
Fourier series for a function $f(x)$ in the interval where x is defined as $a < x < a+2\pi$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are constant and are called as Fourier co-efficients of $f(x)$.

($\frac{1}{2} a_0$ is the constant co-efficient for $n=0$ i.e $\cos(0x)=1$ but if $\sin(0x)=0$, $b_0=0$)

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_a^{a+2\pi} f(x) \cdot dx \\ a_n &= \frac{1}{\pi}\int_a^{a+2\pi} f(x) \cos nx \cdot dx \\ b_n &= \frac{1}{\pi}\int_a^{a+2\pi} f(x) \sin nx \cdot dx \end{aligned}\right\} \text{Euler's formula}$$



(i) $(0, 2\pi)$

(ii) $(-\pi, \pi)$

If $a = 0$, then the interval $(0, 2\pi)$

i.e $0 < x < 2\pi$, then the euler's formula will be

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \cdot dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \cdot dx$$

If $a = -\pi$, interval $(-\pi, \pi)$

i.e. $-\pi < x < \pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \cdot dx$$

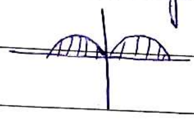
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \cdot dx$$

Even & odd function:

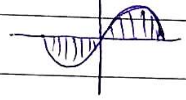
→ A function $f(x)$ is said to be an even function if $f(-x) = f(x)$

→ A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$

⇒ If $f(x)$ is even function, then the graph is symmetrical about y-axis.



⇒ If $f(x)$ is odd function, then the graph is symmetrical about origin



formula

① $$\int_{-a}^{a} f(x) \cdot dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_{0}^{a} f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$$

→ If $f(x)$ is an odd function then,

$$a_0 = a_n = 0, \quad b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \cdot dx$$

→ If $f(x)$ is an even function then,

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx \cdot dx$$

$$b_n = 0$$

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* **Important formula**

① $$\int_a^{a+2\pi} \sin nx \cdot dx = \left[\frac{-\cos nx}{n}\right]_a^{a+2\pi} = 0$$

② $$\int_a^{a+2\pi} \cos nx \cdot dx = \left[\frac{\sin nx}{n}\right]_a^{a+2\pi} = 0$$

③ $$\int_a^{a+2\pi} \sin mx \cdot \cos nx \cdot dx = \frac{1}{2}\int_a^{a+2\pi}\left\{\sin(m+n)x + \sin(m-n)x\right\} dx$$

$$= 0 \quad , \text{if } (m \neq n)$$

④ $$\int_a^{a+2\pi} \cos mx \cos nx \cdot dx = \int_a^{a+2\pi} \sin mx \sin nx \cdot dx = 0 \quad , \text{if } (m \neq n)$$

If $m = n$

⑤ $$\int_a^{a+2\pi} \cos^2 nx \cdot dx = \int_a^{a+2\pi} \sin^2 nx \cdot dx = \pi$$

⑥ $$\int_a^{a+2\pi} \sin nx \cos nx \cdot dx = 0$$

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⑦ $$\int_{-a}^{a} f(x)\cdot dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(x) \text{ is even} \\ 0, & f(x) \text{ is odd} \end{cases}$$

⑧ $$\int e^{ax} \sin bx \cdot dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx) + c$$

$$\int e^{ax} \cos bx \cdot dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$$

⑨ $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ & f(x) \text{ is even} \\ 0 & \text{if } f(2a-x) = -f(x) \\ & f(x) \text{ is odd.} \end{cases}$$

⑩ To integrate the product of two function, we apply the rule of integrating by parts.

$$\int u\cdot v\,dx = u\int v\cdot dx - \int u'\int v\,dx\,dx$$

$$= uv_1 - u'v_2 + u''v_3 - u'''v_4 \ldots$$

(dash $\rightarrow$ derivatives and suffix $\rightarrow$ integration)

* $\sin n\pi = 0$ $\qquad$ $\sin(n+\frac{1}{2})\pi = (-1)^n$

$\cos n\pi = (-1)^n$ $\qquad$ $\cos(n+\frac{1}{2})\pi = 0$

where 'n' is an integer.

Problems:-

1. Obtain the fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

(i) $$a_0 = \int_0^{2\pi} f(x) \cdot dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cdot dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{-1} \right]_0^{2\pi}$$

$$= \frac{-e^{-2\pi} + 1}{\pi}$$

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(ii) $$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx \cdot dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}(-\cos 2n\pi + 0) - \frac{1}{1+n^2}(-1+0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[-e^{-2\pi}+1\right]$$

(iii) $$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(0 - n\cos 2n\pi) - (0 - n\cos 0)\right]$$

$$= \frac{n}{\pi(1+n^2)}\left[-e^{-2\pi}+1\right]$$

$$f(x) = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx$$

$$f(x) = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x \ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x \ldots\right)\right]$$

2. Find fourier series for $f(x) = e^{ax}$ in the interval $-\pi < x < \pi$

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

① $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cdot dx$$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a} \right]_{-\pi}^{\pi} = \frac{1}{a\pi} \left[e^{a\pi} - e^{-a\pi} \right] = \frac{2 \sinh a\pi}{a\pi}$$

② $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos nx \cdot dx$$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a^2+n^2} (a \cos nx + n \sin nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{a\pi}}{a^2+n^2} (a \cos n\pi + n \sin n\pi) - \frac{e^{-a\pi}}{a^2+n^2} (a \cos n\pi - n \sin n\pi) \right]$$

$$= \frac{a}{\pi(a^2+n^2)} \left[e^{a\pi} ((-1)^n) - e^{-a\pi} ((-1)^n) \right]$$

$$= \frac{a(-1)^n}{\pi(a^2+n^2)} \left[e^{a\pi} - e^{-a\pi} \right]$$

$$= \frac{2(-1)^n a \sinh a\pi}{\pi(a^2+n^2)}$$

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(3) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} \sin nx \cdot dx$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\sin n\pi - n\cos n\pi) - \frac{e^{-a\pi}}{a^2+n^2}(-a\sin n\pi - n\cos n\pi)\right]$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{a\pi}(-n(-1)^n) - e^{-a\pi}(-n(-1)^n)\right]$$

$$= \frac{-n\cdot(-1)^n}{\pi(a^2+n^2)} \cdot 2\sinh a\pi$$

$$= \frac{-2n\sinh a\pi\,(-1)^n}{\pi(a^2+n^2)}$$

$$f(x) = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}\cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{-2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

$$= \cancel{\frac{\sinh a\pi}{\pi}}\left[\cancel{1/a +}\right.$$

3 Express $f(x) = |x|$ in $-\pi < x < \pi$ as Fourier series.

w.k.t (for $-\pi < x < \pi$ $\quad f(-x) = f(x) \rightarrow$ even
$f(-x) = -f(x) \rightarrow$ odd)

Since $f(x) = |x|$ is an even function, then F.S in $|x|$ is

$$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

① $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cdot dx = \frac{2}{\pi}\int_0^{\pi} x \cdot dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \underline{\underline{\pi}}$$

② $$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \cdot dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi(0) + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \cos nx$$

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\left(-2\cos x - \frac{2\cos 3x}{9} - \frac{2\cos 5x}{25} - \dots\right)$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$$

Put $x = 0$

$$0 = \pi/2 - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots\right)$$

$$\frac{\pi}{2} \times \frac{\pi}{4} = \left(\frac{1}{1^2} + \frac{1}{3^2} \dots\right)$$

$$\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \dots\right) = \frac{\pi^2}{8}$$

So, $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

(4) Obtain FS for $f(x) = x^2$ in $-\pi < x < \pi$ and also deduce

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots = \frac{\pi^2}{8}$

since $f(x)$ is even

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 . dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx . dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + 2x\frac{\cos nx}{n^2} + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2(0) + 2\pi\frac{\cos n\pi}{n^2} - 2(0) - 0\right]$$

$$= \frac{2}{\pi}\left[\cancel{\pi^2} \cancel{+} \frac{2\pi}{n^2}\cos n\pi\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n\right]$$

$$= \frac{4(-1)^n}{n^2}$$

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$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx \, (-1)^n$$

$$x^2 = \frac{\pi^2}{3} + \frac{4}{\cancel{1}}\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} \cdots\right]$$

→ when $x=0$

$$-\frac{\pi^2}{3} = 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} \cdots\right]$$

$$\frac{\pi^2}{12} = \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots\right)$$

Hence (ii) proved.

→ when $x=\pi$

$$\pi^2 - \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\right)$$

$$\frac{2\pi^2}{12} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\right)$$

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\right) = \frac{\pi^2}{6}$$

Hence (i) proved.

→ ~~when~~ for proving (iii) add (i) & (ii)

$$\left(\frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} \cdots\right) = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

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$$\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\right) = \frac{3\pi^2}{12\times 2}$$

$$\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\right) = \frac{\pi^2}{8}$$

Hence (iii) is proved

5. Obtain F.S for $f(x) = x \qquad -\pi < x < \pi$

odd function.

$a_0 = a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \cdot dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n} + \frac{\overset{0}{\sin n\pi}}{n^2} + 0 - 0\right]$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= -2\frac{(-1)^n}{n}$$

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$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx.$$

6. (i) Obtain F.S for $f(x) = x\sin x$ in the interval $-\pi < x < \pi$

even ~~odd~~ function.

Deduce that $\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \frac{1}{7\cdot9} + \dots = \frac{\pi-2}{4}$

(ii) Same ques $0 < x < 2\pi$ $\longrightarrow$ neither even nor odd

Soln: (i) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cdot dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - (-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-x\cos x + \sin x\right)_0^{\pi}$$

$$= \frac{2(-\pi\cos\pi)}{\pi}$$

$$= 2\pi/\pi = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cdot \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{2\pi} x\left[\frac{1}{2}\left(\sin(1+n)x - \sin(n-1)x\right]dx\right.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin(n+1)x\,dx - \int_0^{\pi} x \sin(n-1)x \cdot dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} - \frac{1}{\pi}\left[x\left(\frac{-\cos(n-1)x}{(n-1)}\right) + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi \cdot \left(\frac{-\cos(n+1)\pi}{(n+1)}\right) - \pi\left(\frac{-\cos(n-1)\pi}{(n-1)}\right)\right]$$

$$a_n = \left(\frac{-\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)}\right) \quad n \neq 1$$

when $n = \text{odd} \Rightarrow (n-1)$ & $(n+1)$ are even

$$a_n = \frac{-1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}$$

when $n = \text{even}$

$$a_n = \frac{1}{n+1} - \frac{1}{n-1} = \frac{-2}{n^2-1}$$

when $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \frac{\sin 2x}{2}\, dx$$

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$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\right]$$

$$a_1 = -1/2$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$x \sin x = 1 - 1/2 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

so, $$x \sin x = 1 - 1/2 \cos x \cancel{+} \overline{\ } \frac{2\cos 2x}{2^2-1} + \frac{2\cos 3x}{3^2-1} - \cdots$$

$$x \sin x = 1 - \frac{\cos x}{2} - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \cdots\right)$$

when $x = \pi/2$

$$\frac{\pi}{2} = 1 - 2\left(\frac{-1}{3} + 0 + \frac{1}{15} + 0 \cdots\right)$$

$$\frac{\pi}{2} - 1 = +2\left(\frac{1}{3} \cancel{+} \overline{\ } \frac{1}{15} + \frac{1}{35} \cdots\right)$$

$$\frac{\pi - 2}{4} = \left(\frac{1}{1\cdot 3} \cancel{+} \overline{\ } \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \right) //$$ Hence proved.

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(ii) $0 < x < 2\pi \longrightarrow$ Neither odd nor even.

$$f(x) = x\sin x$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \, dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - (-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[2\pi(-1)\Big]$$

$$= \underline{\underline{-2}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cdot \cos nx \, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \left(x\, \tfrac{1}{2}\Big[\sin(1+n)x - \sin(n-1)x\Big]\right) dx.$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos(n+1)x}{n+1}\right) + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi}$$

$$- \frac{1}{\pi}\left[x\left(-\frac{\cos(n-1)x}{(n-1)}\right) + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos(n+1)2\pi}{n+1}\right) + \frac{\sin(n+1)2\pi}{(n+1)^2}\right] - \frac{1}{\pi}\left[2\pi\left(-\frac{\cos(n-1)2\pi}{(n-1)}\right) + \frac{\sin(n-1)2\pi}{(n-1)^2}\right]$$

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$$\frac{1}{\pi} \cdot \cancel{2}\left[\frac{-\cos(n+1)2\pi}{n+1} + \frac{\cos(n-1)2\pi}{n-1}\right]$$

$$= \cancel{2}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{2}{n^2-1} \quad \{ n \neq 1$$

a_1 when $n=1$

$$a_1 = \cancel{\frac{\cos}{2}} \quad \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\cancel{\pi}}\left[2\cancel{\pi}\left(-\frac{1}{2}\right)\right] = \underline{\underline{-1/2}}$$

$$b_n = \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n+1)^2}} - \cancel{\frac{1}{(n-1)^2}} + \cancel{\frac{1}{(n+1)^2}}\right]$$

$$= 0 \quad \text{for } n \neq 1$$

$b_1 = ?$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin nx \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \{x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (x - x\cos 2x) \, dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\Big|_0^{2\pi} - \left(x \frac{\sin 2x}{2} - \left(-\frac{\cos 2x}{4}\right)\Big|_0^{2\pi}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - \left(2\pi \times 0 + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2}\right)$$

$$= \pi$$

$$f(x) = -1 \cancel{+} -\frac{1}{2} + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx + \pi$$

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Q. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the range 0 to 2π, then show that $f(x) = \pi^2/12 + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

$\Rightarrow$ $f(2\pi - x) = f(x) \longrightarrow$ even

$$f(2\pi - x) = \left(\frac{\pi - (2\pi - x)}{2}\right)^2$$

$$= \left(\frac{-\pi + x}{2}\right)^2$$

$$= (-1)^2 \left(\frac{\pi - x}{2}\right)^2$$

$$= \left(\frac{\pi - x}{2}\right)^2$$

$$= f(x)$$

Hence it is a even function.

$$b_n = 0$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi - x}{2}\right)^2 \cdot dx$$

$$= \frac{2}{\pi} \left[\left(\frac{\pi - x}{2}\right)^3 \times -\frac{2}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \times \frac{2}{3} \times \frac{\pi^3}{8} = \frac{\pi^2}{6}$$

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$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nx \cdot dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi-x}{2}\right)^2 \frac{\sin nx}{n} - \left[-\frac{(\pi-x)}{2} \times -\frac{\cos nx}{n^2}\right] + \left[\frac{1}{2}\times -\frac{\sin nx}{n^3}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - [0] + 0 - \left[\frac{\pi^2}{4}\times 0 - \left(\frac{-\pi}{2} \times \frac{-1}{n^2}\right) + 0\right]\right.$$

$$= \frac{2}{\pi} \times \frac{\pi}{2n^2} = \frac{1}{n^2}$$

$$f(x) = \frac{\pi^2}{6\times 2} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

Dirichlet's conditions:-

* The sufficient conditions for the uniform convergence of a fourier series are called Dirichlet's condition.

* All functions that normally arise in engineering problem satisfy these conditions & hence they can be expressed as fourier series

Any function $f(x)$ can be expressed as a Fourier series provided

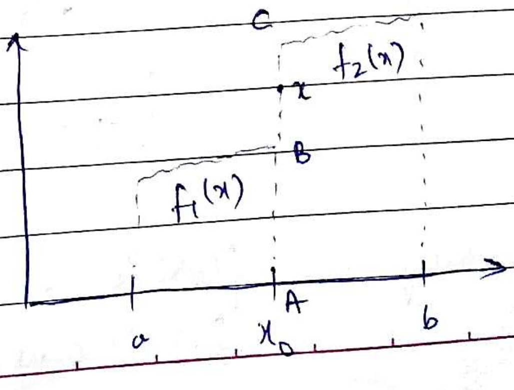
(i) $f(x)$ is periodic, single valued & finite
(ii) $f(x)$ has a finite number of finite discontinuous in any one period.
(iii) $f(x)$ has a finite number of maxima and minima

When these conditions are satisfied the Fourier series converges to $f(x)$ at every point of continuity.

At a point of discontinuity, the sum of the series is equal to the mean of the limit on the right & left side

i.e. $\frac{1}{2}\left[f(x-0)+f(x+0)\right]$

Fourier series for Discontinuous function:



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If $f(x)$ has a finitely many points of finite discontinuity, even then it can be expressed as a Fourier series.

The integral for a_0, a_n, b_n are to be evaluated by breaking up the range of integration

Let $f(x)$ be defined as

$$f(x) = f_1(x) \qquad a < x < x_0$$
$$f(x) = f_2(x) \qquad x_0 < x < b \,/\, (a+2\pi)$$

where x_0 is the point of discontinuity in the interval.

The value of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_a^{x_0} f_1(x)\cdot dx + \int_{x_0}^{a+2\pi} f_2(x)\cdot dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_a^{x_0} f_1(x)\cos nx\cdot dx + \int_{x_0}^{a+2\pi} f_2(x)\cos nx\, dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_a^{x_0} f_1(x)\sin nx\cdot dx + \int_{x_0}^{a+2\pi} f_2(x)\sin nx\cdot dx\right]$$

Note 1 : If $f(x) = \begin{cases} \phi(x) & \text{in } -\pi < x < 0 \\ \psi(x) & \text{in } 0 < x < \pi \end{cases}$

We say that $f(x)$ is even if $\phi(-x) = \psi(x)$

$f(x)$ is odd if $\phi(-x) = -\psi(x)$

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Fourier Series for Discontinuous functions:

(1) $f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Soln: $\phi(2\pi - x) = \psi(x)$

Hence it is even function

so $b_n = 0$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} 2\pi - x \cdot dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[2\pi^2 - \pi^2\right] = \underline{\underline{\pi}}$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} x \cos nx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} + \left[\frac{2\pi \sin nx}{n}\right]_{\pi}^{2\pi} - \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - 2\left(\frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \pi/2 + \sum_{n=1}^{\infty} \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2} \cos nx$$

$$f(x) = \pi/2 - \frac{2}{\pi}\left[\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} \cdots\right]$$

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put $x = 0$

at $x = 0$, since $f(x) = x$

$f(0) = 0$

$$0 = \pi/2 - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right)$$

$$\frac{\pi}{2} \times \frac{\pi}{4} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots$$

$$= \frac{\pi^2}{8}$$ Hence proved

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Q. $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$

P.T $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$

& hence S.T

(i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \ldots = 1/2$

(ii) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\ldots = \frac{\pi - 2}{4}$

Soln: $a_0 = \frac{1}{\pi}\left[\int_0^{\pi} \sin x \cdot dx\right]$

$= \frac{1}{\pi}\left(-\cos x\right)_0^{\pi}$

$= \frac{1}{\pi}(1+1)$

$= \frac{2}{\pi}$

$a_n = \frac{1}{\pi}\left[\int_0^{\pi} \sin x \cos nx \cdot dx\right]$

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$$a_n = \frac{1}{2\pi} \int_0^{\pi} \left(\sin(1+n)x - \sin(n-1)x \right) dx$$

$$= \frac{1}{2\pi} \left[-\frac{\cos(1+n)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)} \right]_0^{\pi}$$

$$= \frac{1}{2\pi} \left[-\frac{\cos(1+n)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right]$$

when n = odd

$$= \frac{1}{2\pi} \left[\frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= 0$$

when n = even

$$= \frac{1}{2\pi} \left[\frac{2}{n+1} - \frac{2}{n-1} \right]$$

$$= \frac{1}{2\pi} \frac{2(n-1) - 2(n+1)}{(n^2-1)}$$

$$= \frac{-2}{\pi(n^2-1)}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\cos(1-n)x - \cos(1+n)x] \, dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)x}{(1-n)} - \frac{\sin(1+n)x}{(1+n)}\right]_0^{\pi}$$

$$\underline{\underline{= 0}} \quad \text{for } n \neq 1$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \cdot dx$$

$$\frac{1}{2\pi}\int_0^{\pi} 1 - \cos 2x \, dx$$

$$\frac{1}{2\pi} \times x - \frac{\sin 2x}{2}\Big|_0^{\pi}$$

$$= \frac{1}{2\pi}(\pi)$$

$$= \underline{\frac{1}{2}}$$

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so, $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=1}^{\infty} \frac{-2}{\pi(n^2-1)} \cos nx$ for n = even

when n is even

$n \to 2n$. (starting from 1)

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$$

Hence proved

(i) put $x = 0$

$f(x) = \sin x$

$= \sin(0) = 0$

$$0 = \frac{1}{\pi} + \frac{1}{2} \times 0 - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} \cdots\right)$$

$$-\frac{1}{\pi} = -\frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} \cdots\right.$$

$$\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} \cdots$$

(ii) put $x = \pi/2$

$f(x) = \sin \pi/2 = 1$

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\left(\frac{-1}{3} + \frac{1}{15} \cancel{+} \frac{1}{35} \cdots\right)$$

$$1 - \frac{1}{2} = \frac{1}{\pi} + \frac{2}{\pi}\left(\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right)$$

$$\frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\left(\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right)$$

$$\frac{\pi - 2}{2\pi} = \frac{2}{\pi}\left(\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right)$$

$$= \frac{\pi - 2}{4}$$

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Change of interval

Let $f(x)$ be a periodic function with period $2l$ in the interval $(a, a+2l)$. Then the Fourier series of function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This interval must be transferred into an interval of length 2π.

we put $\frac{x}{l} = \frac{z}{\pi}$ or $z = \frac{\pi x}{l}$

$\Rightarrow$ at $x = a$, $z = \frac{\pi a}{l} = c$

at $x = a+2l$, $z = \frac{\pi a}{l} + 2\pi = c + 2\pi$

Thus, the function $f(x)$ of period $2l$ in interval $(a, a+2l)$ is transferred into the function $f(z)$ of period 2π in a limit $(c, c+2\pi)$.

w.k.t

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(z)\, dz$$

$$z = \frac{\pi x}{l}$$

$$dz = \frac{\pi}{l} dx$$

$z = c \qquad x = a$

$z = c+2\pi \qquad x = a+2l$

$$a_0 = \frac{1}{\pi} \int_{a}^{a+2l} f(x) \frac{\pi}{l} dx$$

so $$a_0 = \frac{1}{l} \int_{a}^{a+2l} f(x)\, dx$$

|||ly, $$a_n = \frac{1}{l} \int_{a}^{a+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_{a}^{a+2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

case 1: $a=0$, the interval is $0 < x < 2l$

$$a_0 = \frac{1}{l} \int_{0}^{2l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} \, dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} \, dx$$

case 2 : If $a = -l \longrightarrow (-l, l)$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) \, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx$$

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① Find fourier expansion for the function

$$f(x) = x - x^2 \quad \text{in } -1 < x < 1$$

Soln: Here $a = -1$

$a + 2l = 1$

$2l = 2$

so $l = 1$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$= \int_{-1}^{1} x - x^2 \cdot dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-1}^{1}$$

$$= -2/3$$

$$a_n = \int_{-1}^{1} (x - x^2) \cos \frac{n\pi x}{1}\, dx$$

$$= \int_{-1}^{1} x \cos n\pi x\, dx - \int_{-1}^{1} x^2 \cos n\pi x\, dx$$

$$a_n = \left(\frac{x \cancel{\sin n\pi x}}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2} \right)_{-1}^{1} - \left(\frac{x^2 \cancel{\sin n\pi x}}{n\pi} \right.$$

$$\left. + \frac{2x \cos n\pi x}{(n\pi)^2} - \frac{2 \cancel{\sin n\pi x}}{(n\pi)^3} \right)_{-1}^{1}$$

$$a_n = \left(\cancel{\frac{(-1)^n}{(n\pi)^2}} - \cancel{\frac{(-1)^n}{(n\pi)^2}} \right) - \left(\frac{2(-1)^n}{(n\pi)^2} - \frac{(-2)(-1)^n}{(n\pi)^2} \right)$$

$$= \frac{-4(-1)^n}{(n\pi)^2}$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin n\pi x \, dx = \int_{-1}^{1} x \sin n\pi x \, dx - \int_{-1}^{1} x^2 \sin n\pi x$$

$$= \left(\frac{-x \cos n\pi x}{n\pi} + \frac{\cancel{\sin n\pi x}}{(n\pi)^2} \right)_{-1}^{1}$$

$$- \left(\frac{-x^2 \cos n\pi x}{n\pi} + \frac{2x \cancel{\sin n\pi x}}{(n\pi)^2} + \frac{2 \cos n\pi x}{(n\pi)^3} \right)_{-1}^{1}$$

$$b_n = \left(-\frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}\right) - \left(-\cancel{\frac{\cos n\pi}{n\pi}} + \cancel{\frac{2\cos n\pi}{(n\pi)^3}} + \cancel{\frac{\cos n\pi}{n\pi}} - \cancel{\frac{2\cos n\pi}{(n\pi)^3}}\right)$$

$$= \frac{-2(-1)^n}{n\pi}$$

$$f(x) = \frac{-1}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{(n\pi)^2} \cos n\pi x - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin n\pi x$$

(or)

$$= \frac{-1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{(n\pi)^2} \cos n\pi x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin n\pi x$$

(2) Find the Fourier series $f(x) = x^2 - 2$ when $-2 \le x \le 2$

Soln: Here $a = -2$

$a + 2l = 2$

so, $l = 2$

$f(x) = x^2 - 2$

$f(-x) = x^2 - 2$

Hence the function is even.

so, $b_n = 0$

$$a_0 = \frac{1}{2}\int_{-2}^{2} (x^2 - 2)\, dx$$

$$a_0 = \frac{1}{2}\left[\frac{x^3}{3} - 2x\right]_{-2}^{2}$$

$$= \frac{1}{2}\left[\frac{8}{3} - 4 + \frac{8}{3} - 4\right]$$

$$a_0 = -4/3$$

$$a_n = \frac{1}{2}\int_{-2}^{2}(x^2 - 2)\cos\frac{n\pi x}{2}$$

$$= \frac{1}{2}\left[\int_{-2}^{2} x^2 \cos\frac{n\pi x}{2}\,dx - 2\int_{-2}^{2} \cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\cancelto{0}{x^2\frac{\sin\frac{n\pi x}{2}}{n\pi/2}} - 2x\frac{\left(-\cos\frac{n\pi x}{2}\right)}{(n\pi/2)^2} + 2\cancelto{0}{\frac{\left(-\sin\frac{n\pi x}{2}\right)}{(n\pi/2)^3}}\right]_{-2}^{2}$$

$$- \left[\cancelto{0}{\frac{\sin\frac{n\pi x}{2}}{n\pi/2}}\right]_{-2}^{2}$$

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$$= \frac{1}{2}\left[\frac{2x \cos\frac{n\pi x}{2}}{(n\pi/2)^2}\right]_{-2}^{2}$$

$$= \frac{2\cos n\pi \times 4}{(n\pi)^2} + \frac{2\cos n\pi \times 4}{(n\pi)^2}$$

$$= \frac{16(-1)^n}{(n\pi)^2}$$

$$f(x) = \frac{-2}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{(n\pi)^2} \cos\frac{n\pi x}{2}$$

(3) $f(x) = \begin{cases} 1+\frac{4x}{3} & -3/2 \leq x \leq 0 \\ 1-\frac{4x}{3} & 0 \leq x < 3/2 \end{cases}$

Soln: $\phi(-x) = \phi(x)$ Hence the function is even

$b_n = 0$.

Here $l = 3/2$

so, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

Sulekha

$$a_0 = \frac{2}{3}\left[\int_{-3/2}^{0} 1+\frac{4x}{3}\,dx + \int_{0}^{3/2} 1-\frac{4x}{3}\,dx\right]$$

$$= \frac{2}{3}\left[x + \frac{2x^2}{3}\Big|_{-3/2}^{0} + x - \frac{2x^2}{3}\Big|_{0}^{3/2}\right]$$

$$= \frac{2}{3}\left[\frac{+3}{2} \cancel{+} - \frac{2}{3}\times\frac{9}{4} + \frac{3}{2} - \frac{2}{3}\times\frac{9}{4}\right]$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{18}{12}\right]$$

$$= \frac{4}{3}\left[\frac{18-18}{12}\right] = \underline{\underline{0}}$$

$$a_n = \frac{4}{3}\int_{0}^{3/2}\left(1 - \frac{4x}{3}\right)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{4}{3}\left[\int_{0}^{3/2}\cos\frac{2n\pi x}{3}\,dx - \int_{0}^{3/2}\frac{4x}{3}\cos\frac{2n\pi x}{3}\,dx\right]$$

$$= \frac{4}{3}\left[\frac{\sin\frac{2n\pi x}{3}}{\cancel{2n\pi/3}}\right]_{0}^{3/2} - \frac{4}{3}\left[\frac{x\sin\frac{2n\pi x}{3}}{\cancel{2n\pi/3}} + \frac{\cos\frac{2n\pi x}{3}}{(2n\pi/3)^2}\right]_{0}^{3/2}$$

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$$= \frac{4}{3} \times \frac{-4}{3}\left[\frac{(-1)^n \times 9 - 1}{4n^2\pi^2}\right]$$

$$= \cancel{\frac{-4(-1)^n}{n^2\pi^2}} \quad \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

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(28) Find F.S for $f(t)$ $-1<t<1$

$$f(t) = \begin{cases} 1 & -1<t<0 \\ \cos(\pi t) & 0<t<1 \end{cases}$$

To what value does this series converge when $t=1$?

Soln: Here $l=1$

$$a_0 = 1\left[\int_{-1}^{0} 1\, dt + \int_{0}^{1} \cos(\pi t)\, dt\right]$$

$$a_0 = x\Big|_{-1}^{0} + \cancel{\frac{\sin \pi t}{\pi}}\Big|_{0}^{1}$$

$$a_0 = 1$$

Sulekha

$$f(t) = \begin{cases} 1 & -1 < t < 0 \\ \cos\pi t & 0 < t < 1 \end{cases}$$

$$a_0 = \int_{-1}^{0} 1\, dt + \int_0^1 \cos\pi t\, dt = 1$$

$$a_n = \int_{-1}^{0} \cos n\pi t\, dt + \int_0^1 \cos\pi t \cos n\pi t\, dt$$

$$a_n = 0 + \int_0^1 \cos(n+1)\pi t + \cos(n-1)\pi t\, dt$$

$$a_n = \left[\frac{\sin(n+1)\pi t}{(n+1)\pi} + \frac{\sin(n-1)\pi t}{(n-1)\pi}\right]_0^1 = 0$$

$$b_n = \int_{-1}^{0} \sin n\pi t\, dt + \int_0^1 \cos\pi t \sin n\pi t\, dt$$

$$b_n = \left(-\frac{\cos n\pi t}{n\pi}\right)_{-1}^{0} + \frac{1}{2}\int_0^1 \sin(n+1)\pi t + \sin(n-1)\pi t\, dt$$

$$b_n = \frac{-1 + (-1)^n}{n\pi} + \frac{1}{2}\left[-\frac{\cos(n+1)\pi t}{(n+1)\pi} - \frac{\cos(n-1)\pi t}{(n-1)\pi}\right]_0^1$$

$$b_n = \frac{-1 + (-1)^n}{n\pi} + \frac{1}{2}\left[-\frac{(-1)^{n+1}}{(n+1)\pi} - \frac{(-1)^{n-1}}{(n-1)\pi} + \frac{1}{(n+1)\pi} + \frac{1}{(n-1)\pi}\right]$$

n - odd

$$b_n = \frac{-2}{n\pi} + \frac{2}{2\pi}\left[\frac{2n}{n^2-1}\right] =$$

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Half-Range Series:-

Sometimes it is required to expand f(x) in the range $(0, \pi)$ in a Fourier series of period 2π or in the range $(0, l)$ in a F S of period $2l$.

If required to expand f(x) in the interval $(0, l)$ then it is immaterial what the function may be outside the range $(0, l)$.

We are free to choose it arbitarily in the interval $(-l, 0)$

If we extend the function f(x) by reflecting it in the y-axis i.e $f(-x) = f(x)$ then extend. function is even. The F.S of f(x) will only contain cosine terms

If we extend the function f(x) by reflecting it in the origin i.e $f(-x) = -f(x)$ then extended function is odd. The F.S of f(x) will contain only sine terms.

Ex: Q. Find F.S half range for $f(x)=x$, $0<x<\pi$

Hence the function $f(x)$ defined over the interval $0<x<l$: capable of two distinct half range series

1. Half-range cosine series
2. Half-range sine series

① Half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

② Half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

① Find Half range sine series

$$f(x) = \pi x - x^2 \quad \text{in } (0,\pi)$$

$\Rightarrow$ HSS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\left[\pi\int_0^{\pi} x\sin nx\, dx - \int_0^{\pi} x^2 \sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\pi\left(-\frac{x\cos nx}{n} + \cancel{\frac{\sin nx}{n^2}}^{\,0}\right)_0^{\pi} - \left(-\frac{x^2\cos nx}{n} + \cancel{\frac{2x\sin nx}{n^2}}^{\,0} + \frac{2\cos nx}{n^3}\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\cancel{-\pi^2\frac{(-1)^n}{n}} + \cancel{\pi^2\frac{(-1)^n}{n}} - \frac{2(-1)^n}{n^3} + \frac{2}{n^3}\right]$$

$$b_n = \frac{4}{\pi n^3}\left[1 - (-1)^n\right] \quad //$$

Hence

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left[1-(-1)^n\right] \sin nx$$

2. Find H CS

$$f(x) = (x-1)^2 \quad \text{in } (0,1)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{1}$$

$$a_0 = \frac{2}{l}\left[\int_0^l f(x)\,dx\right] = 2\left[\int_0^1 (x-1)^2\,dx\right]$$

$$\cancel{a_0 = \frac{2}{1}} \qquad a_0 = 2\int_0^1 x^2 - 2x + 1\,dx$$

$$= 2\left[\frac{x^3}{3} - x^2 + x\right]_0^1$$

$$= 2 \times \left(\frac{1}{3} - 1 + 1\right)$$

$$a_0 = 2/3$$

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$$a_n = 2\left[\int_0^1 (x-1)^2 \cos n\pi x \, dx\right]$$

$$= 2\left[(x-1)^2 \cancel{\frac{\sin n\pi x}{n\pi}}^{0} + 2(x-1)\frac{\cos n\pi x}{(n\pi)^2} + 2\left(\cancel{\frac{-\sin n\pi x}{(n\pi)^3}}^{0}\right)\right]_0^1$$

$$= 4 \left.\frac{(x-1)\cos n\pi x}{(n\pi)^2}\right|_0^1$$

$$a_n = \frac{4}{(n\pi)^2}$$

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x$$

(8) Obtain F.S for

$$f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$$

deduce (i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} \cdots\right]$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \cdots\right]$

(i) Half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\Big|_0^{\frac{\pi}{2}} + \frac{-\pi\cos nx}{n}\Big|_{\pi/2}^{\pi}\right.$$

$$\left. - \left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right)\Bigg|_0^{\pi/2} - \pi\frac{\cos nx}{n}\Bigg|_{\pi/2}^{\pi} + \left(\frac{x\cos nx}{n} - \frac{\sin nx}{n^2}\right)\Bigg|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\cancelto{0}{\frac{\cos n\pi/2}{n}} + \frac{\sin n\pi/2}{n^2} - \cancel{\pi\frac{\cos n\pi}{n}} + \cancelto{0}{\pi\frac{\cos n\pi/2}{n}} + \cancel{\pi\frac{\cos n\pi}{n}} - \cancelto{0}{\frac{\sin n\pi}{n^2}} - \frac{\pi}{2}\cancelto{0}{\frac{\cos n\pi/2}{n}} + \frac{\sin n\pi/2}{n^2}\right]$$

$$= \frac{2}{\pi}\left[2\,\frac{\sin n\pi/2}{n^2}\right] = \frac{4}{\pi n^2}\left(\sin n\pi/2\right)$$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \sin\frac{n\pi}{2}\,\sin nx$$

$$= \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} \dots\right]$$

(ii) Half Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} \pi - x\,dx\right]$$

$$a_0 = \frac{2}{\pi}\left[\frac{\pi^2}{8} + \cancel{\pi^2} - \frac{\cancel{\pi^2}}{2} - \frac{\cancel{\pi^2}}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \cos nx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\right]$$

$$= \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi/2}$$

$$+ \left.\frac{\pi \sin nx}{n}\right|_{\pi/2}^{\pi} - \left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right)_{\pi/2}^{\pi}$$

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$$= \frac{2}{\pi}\left[\frac{\pi}{2}\frac{\sin n\pi/2}{n} - \frac{1}{n^2} - \pi\frac{\sin n\pi/2}{n}\right.$$

$$\left. - \left(\frac{(-1)^n}{n^2} - \frac{\pi}{2}\frac{\sin n\pi/2}{n}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\cancel{\frac{\sin n\pi/2}{n}} + \frac{\pi}{2}\cancel{\frac{\sin n\pi/2}{n}} - \pi\cancel{\frac{\sin n\pi/2}{n}}\right.$$

$$\left. - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$a_n = \frac{2}{\pi n^2}\left[-1 - (-1)^n\right]$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[1 + (-1)^n\right]}{n^2}\cos nx$$

Practical Harmonic Analysis

When the function $f(x)$ is not given by an analytical expression rather by its graph or a table of corresponding values at a no. of equi spaced points, we cannot evaluate the integral for fourier coefficients. However using the rule of approximate integration, we can find approximate value of the first few terms of Fourier series.

Let the F.S for $y = f(x)$ in $(0, 2\pi)$

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \dots b_1 \sin x + b_2 \sin 2x \dots$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} y \, dx$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} y \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} y \sin nx \, dx$$

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Let the range $(0, 2\pi)$ be divided into m equal parts i.e $x = x_0, x_1, x_2 \ldots x_m$ so that each sub interval is of length $\frac{2\pi}{m}$. Let the ordinates at these points be denoted by $y_0, y_1, y_2 \ldots y_m$

x	y
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_m	y_m

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} y \, dx$$

By numerical integration (Trapezoidal rule)

If x is divided into 'm' equal parts, y values corresponding to each x is integrated by

$$I = \frac{h}{2}\left[(y_0 + y_m) + 2(y_1 + y_2 + y_3 \cdots y_{m-1})\right]$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} y\,dx = 2\left[\text{mean value of } y \text{ in } (0,2\pi)\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y \cos nx\,dx = 2\left[\text{mean value of } y\cos nx \text{ in } (0,2\pi)\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} y \sin nx\,dx = 2\left[\text{mean value of } y\sin nx \text{ in } (0,2\pi)\right]$$

This process of finding the fourier series for a function given by numerical values is known as Harmonic analysis.

P: (1) Analyse harmonically the data given below & express 'y' in fourier series upto third harmonic

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y :	1	1.4	1.9	1.7	1.5	1.2	1

Since last value of y is a repetition of the first, only the first six values will be considered

x	y	$\cos x$	$\cos 2x$	$\cos 3x$	$\sin x$	$\sin 2x$	$\sin 3x$
0	1	1	1	1	0	0	0
$\pi/3$	1.4	0.5	-0.5	-1	0.866	0.866	0
$2\pi/3$	1.9	-0.5	-0.5	1	0.866	-0.866	0
π	1.7	-1	1	-1	0	0	0
$4\pi/3$	1.5	-0.5	-0.5	1	-0.866	0.866	0
$5\pi/3$	1.2	0.5	-0.5	-1	-0.866	-0.866	0

$$\Sigma y = 1 + 1.4 + 1.9 + 1.7 + 1.5 + 1.2$$
$$= \underline{\underline{8.7}}$$

$$\Sigma y \cos x = 1 + (1.4)(0.5) + (1.9)(-0.5) + (1.7)(-1) + (1.5)(-0.5) + (1.2)(0.5) = \underline{\underline{-1.1}}$$

$$\Sigma y \cos 2x = \underline{\underline{-0.3}}$$

$$\Sigma y \cos 3x = 0.1$$

$$a_0 = \frac{2\,\Sigma y}{m} = \frac{2}{6} \times 8.7 = \underline{\underline{2.9}}$$

$$a_1 = \frac{2\sum y \cos x}{m}$$

$$= \frac{2}{6} \times -1.1 = -0.37$$

$a_2 = -0.1$

$a_3 = 0.03$

$\sum y \sin x = 0.5196$

$\sum y \sin 2x = -0.1732$

$\sum y \sin 3x = 0$

$$b_1 = \frac{2\sum y \sin x}{m}$$

$$= 0.17$$

$b_2 = -0.06$

$b_3 = 0$

$$\therefore y = \frac{2.9}{2} + \left(-0.37\cos x - 0.1\cos 2x + 0.03\cos 3x \ldots\right)$$

$$+ \left(0.17 \sin x - 0.06 \sin 2x \ldots\right)$$

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H.W

(2) Find the F.S from the data, calculate only 1st harmonic

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Here $N=6$ $\quad 0<x<6$

$(0,2l) = 2l=6$ so $l=3$

*imp

$$\left(\theta = \frac{\pi x}{l}\right)$$

θ	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	9	1	0	9	0
$\pi/3$	18	0.5	0.866	9	15.588
$2\pi/3$	24	-0.5	0.866	-12	20.784
π	28	-1	0	-28	0
$4\pi/3$	26	-0.5	-0.866	-13	-22.516
$5\pi/3$	20	0.5	-0.866	10	-17.32

$\Sigma y = 125$

$\Sigma y\cos\theta = -25$ $\quad \Sigma y\sin\theta = -3.464$

$$a_0 = \frac{2\Sigma y}{m} = \frac{2\times 125}{6} = 41.66$$

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$$a_1 = \frac{2\Sigma y\cos x}{m} = \frac{2\times -25}{6}$$

$$= -8.33$$

$$b_1 = \frac{2\Sigma y\sin x}{m} = \frac{2\times -3.464}{6}$$

$$= -1.1546$$

$$y = \frac{41.66}{2} - 8.33\cos\frac{\pi x}{3}$$

$$-1.1546\sin\frac{\pi x}{3}$$

qus changed 1.5 → 1.3

Q. The following table gives variation of periodic current over a period of 2π. Show by numerical analysis that there is a direct current part of 0.75(amps) in the variable current. & obtain amplitude to first harmonic

t sec	0	T/6	T/3	T/2	2T/3	5T/6	T
A amp	1.98	1.5	1.05	1.3	-0.88	-0.25	1.98

Soln:

$$\theta = \frac{2\pi}{T}$$

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Amplitude of first harmonic $= \sqrt{a_1^2 + b_1^2}$

$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$

$\frac{a_0}{2} \rightarrow$ direct current

t	A	θ	cos θ	sin θ	y cos θ	y sin θ
0	1.98	0	1	0	1.98	0
T/6	1.5	π/3	0.5	0.866	0.75	1.299
T/3	1.05	2π/3	−0.5	0.866	−0.525	0.9093
T/2	1.3	π	−1	0	−1.3	0
2T/3	−0.88	4π/3	−0.5	−0.866	0.44	0.762
5T/6	−0.25	5π/3	0.5	−0.866	−0.125	0.2165
T	1.98					

repeat, so ignore

$\Sigma y = 4.7$

$\Sigma y \cos\theta = 1.22$

$\Sigma y \sin\theta = 3.1868$

$a_0 = \frac{2 \Sigma y}{m} = \frac{2}{6} \times 4.7 = \cancel{1.57}\ 1.57$

$a_1 = \frac{2 \Sigma y \cos x}{m} = \frac{2}{6} \times 1.22 = 0.406$

$b_1 = \frac{2 \Sigma y \sin x}{m} = \frac{2}{6} \times 3.1868 = 1.0623$

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direct current $= a_0/2 = \frac{1.57}{2} = \underline{\underline{0.785}}$

$$A = \sqrt{a_1^2 + b_1^2} = \sqrt{(0.406)^2 + (1.0623)^2}$$

$$= \cancel{1.053}\ \underline{\underline{1.1372}}$$

Note:- (The direct current part of the variable current is the constant term in the f.s i.e $a_0/2$).

Q. The turning moment 'T' units of the crank shaft of a steam engine is given for a series of values of the crank shaft angle in degree

θ	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

Find the first four terms in a series of sines to represent T. Also calculate T when $\theta = 75°$

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soln: Let the half range sine series of T be

$$T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta$$

θ	T	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$
0	0	0	0	0	0
30	5224	0.5	0.866	1	0.866
60	8097	0.866	0.866	0	-0.866
90	7850	1	0	-1	0
120	5499	0.866	-0.866	0	0.866
150	2626	0.5	-0.866	1	-0.866
180	0				

repeated

$\Sigma T = 29596$

$\Sigma T \sin\theta = 23549.136$

$\Sigma T \sin 2\theta = 4499.736$

$\Sigma T \sin 3\theta = 0$

$\Sigma T \sin 4\theta = 0$

$$b_1 = \frac{2 \Sigma T \sin\theta}{m} = \frac{2}{6} \times 23549.136$$

$$= 7849.712$$

$$b_2 = \frac{2 \Sigma T \sin 2\theta}{m} = \frac{2}{6} \times 4499.736 = 1499.912$$

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$b_3 = 0$

$b_4 = 0$

$T = 7849.714 \sin\theta + 1499.912 \sin 2\theta$