Method of Variation of Parameters

Consider a second order D.E of the form

$$y'' + a_1 y' + a_2 y = \phi(x)$$

where a_1, a_2 are functions of x (or) constants.

$$C.F = c_1 y_1 + c_2 y_2$$

where c_1, c_2 are arbitrary constants and y_1 and y_2 are functions of x.

$$P.I = A y_1 + B y_2$$

where $A = -\int \frac{y_2 \phi(x)}{W} dx$

$B = \int \frac{y_1 \phi(x)}{W} dx$

where $W = \text{Wronskian} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Sol is

$$y = CF + PI$$

Problems

1. Solve $y'' + a^2 y = \sec ax$ by the method of Variation of Parameters.

Sol:- $(D^2 + a^2) y = \sec ax$

A.E is $m^2 + a^2 = 0$

$m^2 = -a^2 \Rightarrow m = \pm ai$

$$CF = c_1 \underbrace{\cos ax}_{y_1} + c_2 \underbrace{\sin ax}_{y_2}$$

$y_1' = -\sin ax \cdot a$

$y_2' = \cos ax \cdot a$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$= a\cos^2 ax + a\sin^2 ax$

$= a(\cos^2 ax + \sin^2 ax)$

$= a(1)$

$$\boxed{W = a}$$

$$A = -\int \frac{y_2\,\phi(x)}{W}\,dx$$

$$A = -\int \frac{\sin ax \cdot \sec ax}{a}\,dx$$

$$= -\frac{1}{a}\int \frac{\sin ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \tan ax\,dx$$

$$= -\frac{1}{a}\,\frac{\log(\sec ax)}{a}$$

$$A = \frac{-1}{a^2}\log(\sec ax)$$

$$B = \int \frac{y_1\,\phi(x)}{W}\,dx$$

$$= \int \frac{\cos ax \cdot \sec ax}{a}\,dx$$

$$= \frac{1}{a}\int dx$$

$$B = \frac{x}{a}$$

$$P.I = Ay_1 + By_2$$

$$P.I = -\frac{1}{a^2}\log(\sec ax)\cdot\cos ax + \frac{x}{a}\cdot\sin ax$$

Sol $y = CF + PI$

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\log|\sec ax| \cos ax + \frac{x}{a}\sin ax$$

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② Solve $y'' + 2y' + 2y = e^{-x}\sec^3 x$ by the method of Variation of parameters

Sol :- $(D^2 + 2D + 2)y = e^{-x}\sec^3 x$

A.E is $m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2}$$

$$m = \frac{-2 \pm \sqrt{-4}}{2}$$

$$m = \frac{-2 \pm 2i}{2} \Rightarrow m = -1 \pm i$$

$$C.F = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$C.F = c_1 \underbrace{e^{-x}\cos x}_{y_1} + c_2 \underbrace{e^{-x}\sin x}_{y_2}$$

$$y_1' = e^{-x}(-\sin x) + \cos x(-e^{-x}) \Rightarrow y = -e^{-x}(\cos x + \sin x)$$

$$y_2' = e^{-x}\cos x + \sin x(-e^{-x}) \Rightarrow y = e^{-x}(\cos x - \sin x)$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}(\cos x + \sin x) & e^{-x}(\cos x - \sin x) \end{vmatrix}$$

$$W = y_1 y_2' - y_1' y_2$$

$$= e^{-x}\cos x\,\left(e^{-x}(\cos x - \sin x)\right) + e^{-x}\sin x\,\left(e^{-x}(\cos x + \sin x)\right)$$

$$= e^{-2x}\cos^2 x - \cancel{e^{-2x}\cos x\sin x} + \cancel{e^{-2x}\sin x\cos x} + e^{-2x}\sin^2 x$$

$$= e^{-2x}\cos^2 x + e^{-2x}\sin^2 x$$

$$= e^{-2x}(\cos^2 x + \sin^2 x)$$

$$\boxed{W = e^{-2x}}$$

$$A = -\int \frac{y_2\,\phi(x)}{W}\,dx$$

$$A = -\int \frac{e^{-x}\sin x \cdot e^{-x}\sec^3 x}{e^{-2x}}\,dx \Rightarrow$$

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$$A = -\int \frac{\cancel{e^{-2x}} \cdot \sin x \sec^3 x}{\cancel{e^{-2x}}} dx$$

$$= -\int \frac{\sin x}{\cos^3 x} dx$$

$$= -\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x} dx$$

$$= -\int \underline{\tan x} \cdot \sec^2 x \, dx$$

put $t = \tan x$
$dt = \sec^2 x \, dx$

$\int t \, dt$
$= t^2/2$

$$A = -\frac{\tan^2 x}{2}$$

$$B = \int \frac{y_1 \phi(x)}{W} dx$$

$$= \int \frac{e^{-x} \cos x \cdot e^{-x} \cdot \sec^3 x}{e^{-2x}} dx$$

$$= \int \frac{\cancel{e^{-2x}} \cdot \sec^2 x}{\cancel{e^{-2x}}} dx$$

$$B = \tan x$$

$$P.I = Ay_1 + By_2$$

$$= \left(-\frac{\tan^2 x}{2}\right) e^{-x} \cos x + (\tan x) e^{-x} \sin x$$

Sol is

$$y = CF + PI$$

$$y = e^{-x}(c_1\cos x + c_2\sin x) + \left(\frac{\tan^2 x}{2}\right)e^{-x}\cos x + (\tan x)\, e^{-x}\sin x$$

(3) Solve by the method of variation of parameters $y'' + y = \dfrac{1}{1+\sin x}$

Sol :- $(D^2+1)\,y = \dfrac{1}{1+\sin x}$

A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$C.F = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$$

$y_1' = -\sin x \qquad y_2' = \cos x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$W = \cos^2 x + \sin^2 x$$

$$\boxed{W = 1}$$

$$A = -\int \frac{y_2 \, \phi(x)}{W} \, dx$$

$$= -\int \frac{\sin x \cdot \frac{1}{1+\sin x}}{1} \, dx$$

$$= -\int \frac{\sin x}{1+\sin x} \, dx$$

$$= -\int \frac{\sin x \, (1-\sin x)}{(1+\sin x)(1-\sin x)} \, dx$$

$$= -\int \frac{\sin x - \sin^2 x}{1-\sin^2 x} \, dx$$

$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x} \, dx$$

$$= -\left[\int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right) dx\right]$$

$$= -\int \tan x \sec x \, dx + \int \tan^2 x \, dx$$

$$= -\int \tan x \sec x \, dx + \int (\sec^2 x - 1) \, dx$$

$$A = -\underline{\sec x} + \tan x - x$$

$$B = \int \frac{y_1 \phi(x)}{W} dx$$

$$= \int \frac{\cos x \cdot \frac{1}{1+\sin x}}{1} dx$$

$$= \int \frac{\cos x}{1+\sin x} dx$$

$$B = \log(1+\sin x)$$

$$P.I = Ay_1 + By_2$$

$$= (-\sec x + \tan x - x)\cos x + \log(1+\sin x) \cdot \sin x$$

Sol is

$$y = CF + PI$$

$$y = (c_1 \cos x + c_2 \sin x) + (-\sec x + \tan x - x)\cos x + \log(1+\sin x)\sin x$$

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4) Solve by the method of variation of parameters $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Sol :- $(D^2-1)y = \frac{2}{1+e^x}$

A.E is $m^2-1=0 \Rightarrow m=\pm 1$

$$Cf = \underset{\substack{\downarrow \\ y_1}}{c_1 e^x} + \underset{\substack{\downarrow \\ y_2}}{c_2 e^{-x}}$$

$y_1' = e^x \qquad y_2' = -e^{-x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -e^{x-x} - e^{-x+x}$$

$$= -1-1$$

$$\boxed{W = -2}$$

$$A = -\int \frac{y_2\, \phi(x)}{W}\, dx$$

$$= -\int \frac{e^{-x} \frac{\cancel{2}}{1+e^x}}{-\cancel{2}}\, dx = \int \frac{e^{-x}}{1+e^x}\, dx$$

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$$A = \int \frac{e^{-x}}{1+e^{x}}\,dx$$

$$= \int \frac{1}{e^{x}(1+e^{x})}\,dx$$

Partial fractions

let $e^{x} = t$

$$\frac{1}{t(1+t)} = \frac{A}{t} + \frac{B}{1+t}$$

$$\frac{1}{\cancel{t(1+t)}} = \frac{A(1+t)+Bt}{\cancel{t(1+t)}}$$

$$A(1+t) + Bt = 1$$

Put $t = -1$

$$0 + B(-1) = 1$$

$$\boxed{B = -1}$$

Put $t = 0$

$$A(1+0) + 0 = 1$$

$$\boxed{A = 1}$$

$$\frac{1}{t(1+t)} = \frac{1}{t} - \frac{1}{1+t}$$

$$\frac{1}{e^{x}(1+e^{x})} = \frac{1}{e^{x}} - \frac{1}{1+e^{x}}$$

$$A = \int\left(\frac{1}{e^{x}} - \frac{1}{1+e^{x}}\right)dx$$

$$A = \int e^{-x}\,dx - \int \frac{1}{1+e^{x}}\,dx$$

$$A = \frac{e^{-x}}{-1} - \int \frac{1}{e^{x}\left(1+\frac{1}{e^{x}}\right)}\,dx$$

$$A = -e^{-x} - \int \frac{e^{-x}}{1+e^{-x}}\,dx$$

$$A = -e^{-x} + \log(1+e^{-x})$$

$$B = \int \frac{y_1\,\phi(x)}{W}\,dx = \int \frac{e^{x}\cdot\frac{\cancel{2}}{1+e^{x}}}{-\cancel{2}}\,dx$$

$$B = -\int \frac{e^{x}}{1+e^{x}}\,dx$$

$$B = -\log(1+e^{x})$$

$$P.I = Ay_1 + By_2$$

$$= \left(-e^{-x} + \log(1+e^{-x})\right)e^{x} - \log(1+e^{x})\,e^{-x}$$

Sol is

$$y = CF + PI$$

$$y = c_1 e^{x} + c_2 e^{-x} + \left(-e^{-x} + \log(1+e^{-x})\right)e^{x} - \log(1+e^{x})\cdot e^{-x}$$

5 Solve $y'' - 3y' + 2y = \cos(e^{-x})$ by the method of variation of parameters

Sol:- $(D^2 - 3D + 2)y = \cos(e^{-x})$

A.E is $m^2 - 3m + 2 = 0$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$cf = c_1 e^{x} + c_2 e^{2x}$$

$y_1 = e^{x}$ $\quad$ $y_2 = e^{2x}$

$y_1' = e^{x}$ $\quad$ $y_2' = 2e^{2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x}$$

$$= e^{3x}$$

$$\boxed{W = e^{3x}}$$

$$A = -\int \frac{y_2 \phi(x)}{W} dx$$

$$= -\int \frac{e^{2x}\cos(e^{-x})}{e^{3x}} dx$$

$$= -\int \frac{\cos(e^{-x})}{e^{x}} dx$$

$$= -\int e^{-x}\cos(e^{-x}) dx$$

Put $e^{-x} = t$

$-e^{-x}dx = dt$

$$= \int \cos t \, dt$$

$$= \sin t \implies A = \sin(e^{-x})$$

$$B = \int \frac{y_1 \phi(x)}{W} dx$$

$$B = \int \frac{e^{x} \cdot \cos(e^{-x})}{e^{3x}} dx$$

$$B = \int \frac{\cos(e^{-x})}{e^{2x}} dx$$

$$B = \int e^{-2x} \cos(e^{-x}) dx$$

put $e^{-x} = t$

$-e^{-x} dx = dt$

$e^{-x} dx = -dt$

$$B = \int e^{-x} \cdot e^{-x} \cos(e^{-x}) dx$$

$$B = \int t \cdot \cos t \, (-dt)$$

$$B = -\int t \cdot \cos t \, dt$$

$$B = -\left[t \cdot \sin t - 1 \cdot (-\cos t) \right]$$

$$B = -t \sin t - \cos t$$

$$B = -e^{-x} \sin(e^{-x}) - \cos(e^{-x})$$

$$P.I = Ay_1 + By_2$$

$$= \sin(e^{-x}) \cdot e^{x} + (-e^{-x} \sin(e^{-x}) - \cos(e^{-x})) \cdot e^{2x}$$

Sol

$$y = CF + PI$$

$$y = c_1 e^{x} + c_2 e^{2x} + \sin(e^{-x}) e^{x} - (e^{-x} \sin(e^{-x}) + \cos(e^{-x})) e^{2x}$$

6 Solve $y'' + 3y' + 2y = e^{e^x}$

Sol :- $(D^2 + 3D + 2)y = e^{e^x}$

A.E is $m^2 + 3m + 2 = 0$

$m^2 + 2m + m + 2 = 0$

$m(m+2) + 1(m+2) = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$$CF = \underbrace{c_1 e^{-x}}_{y_1} + \underbrace{c_2 e^{-2x}}_{y_2}$$

$y_1' = -e^{-x}$ $\qquad$ $y_2' = -2e^{-2x}$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$= -2e^{-3x} + e^{-3x}$

$\boxed{W = -e^{-3x}}$

$A = -\int \frac{y_2 \phi(x)}{W} dx$

$A = -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}} dx$

$A = \int \frac{e^{e^x}}{e^{-x}} dx$

$A = \int e^x e^{e^x} dx$

$$A = \int e^{x} e^{e^{x}}\, dx$$

Put $e^{x} = t$

$e^{x}\, dx = dt$

$$A = \int e^{t}\, dt$$

$$A = e^{t}$$

$$A = e^{e^{x}}$$

$$B = \int \frac{y_1\, \phi(x)}{W}\, dx$$

$$B = \int \frac{e^{-x} \cdot e^{e^{x}}}{-e^{-3x}}\, dx$$

$$B = -\int \frac{e^{e^{x}}}{e^{-2x}}\, dx$$

$$B = -\int e^{2x} e^{e^{x}}\, dx$$

Put $e^{x} = t$

$e^{x}\, dx = dt$

$$B = -\int e^{x} e^{x} e^{e^{x}}\, dx$$

$$= -\int t\, e^{t}\, dt$$

$$= -\left[t \cdot e^{t} - 1 \cdot e^{t}\right]$$

$$B = -t e^{t} + e^{t}$$

$$B = -e^{x} e^{e^{x}} + e^{e^{x}}$$

$$B = e^{e^{x}}(1 - e^{x})$$

$P.I = Ay_1 + By_2$

$P.I = e^{e^x} \cdot e^{-x} + e^{e^x}(1-e^x)e^{-2x}$

Sol is

$y = CF + PI$

$y = c_1 e^{-x} + c_2 e^{-2x} + e^{e^x} \cdot e^{-x} + e^{e^x}(1-e^x)e^{-2x}$

⑦ Solve $x^2y'' + xy' - y = x^2 e^x$ by the method of variation of parameters

Sol: -

Put $t = \log x$
$x = e^t$

$x^2y'' = D(D-1)y$ where $D = \frac{d}{dt}$

$xy' = Dy$

$D(D-1)y + Dy - y = (e^t)^2 \cdot e^{e^t}$

$(D^2 - \cancel{D} + \cancel{D} - 1)y = e^{2t} e^{e^t}$

$(D^2 - 1)y = e^{2t} e^{e^t}$.

A.E is $m^2 - 1 = 0$

$m = \pm 1$

$CF = c_1 \underset{y_1}{e^t} + c_2 \underset{y_2}{e^{-t}}$

$y_1' = e^t$

$y_2' = -e^{-t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix}$$

$$W = -e^{t}\cdot e^{-t} - e^{t}\cdot e^{-t}$$

$$\boxed{W = -2}$$

$$A = -\int \frac{y_2\,\phi(t)}{W}\,dt$$

$$= -\int \frac{e^{-t}\,e^{2t}\cdot e^{e^{t}}}{-2}\,dt$$

$$= \frac{1}{2}\int e^{t}\,e^{e^{t}}\,dt$$

put $e^{t} = p$

$e^{t}\,dt = dp$

$$= \frac{1}{2}\int e^{p}\,dp$$

$$= \frac{1}{2}\cdot e^{p}$$

$$= \frac{1}{2}\cdot e^{e^{t}}$$

$$= \frac{1}{2}\cdot e^{x}$$

$\boxed{\because e^{t} = x}$

$$B = \int \frac{y_1 \phi(t)}{W} dt$$

$$B = \int \frac{e^{t} \cdot e^{2t} e^{e^t}}{-2} dt$$

$$= -\frac{1}{2} \int e^{t} \cdot e^{2t} e^{e^t} dt$$

Put $e^t = K$
$e^t dt = dK$

$$= -\frac{1}{2} \int K^2 e^K dK$$

$$= -\frac{1}{2} \left[K^2 \cdot e^K - 2K \cdot e^K + 2 \cdot e^K \right]$$

$$= -\frac{1}{2} \left[K^2 - 2K + 2 \right] e^K$$

$$B = -\frac{1}{2} \left[(e^t)^2 - 2e^t + 2 \right] e^{e^t}$$

$$B = -\frac{1}{2} \left[x^2 - 2x + 2 \right] e^x$$

$$P.I = Ay_1 + By_2$$

$$= \frac{e^x}{2} \cdot x + \left(-\frac{e^x}{2} (x^2 - 2x + 2) \right) \frac{1}{x}$$

$y_1 = e^t$
$y_1 = x$
$y_2 = e^{-t}$
$= 1/x$

Sol

$$y = CF + PI$$

$$y = c_1 x + c_2 \cdot \frac{1}{x} + \frac{x e^x}{2} - \frac{e^x}{2x}(x^2 - 2x + 2)$$