UNIT-III HIGHER ORDER DIFFERENTIAL EQUATIONS

A differential equation of the form $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots a_n y = \phi(x)$ — ①

where $a_1, a_2, a_3, \dots, a_n$ are constant co-efficients is called a linear differential equation of order 'n'.

If $\phi(x) = 0$, the above DE is called homogeneous linear differential equation of higher order.

If $\phi(x) \neq 0$, the above DE is called non-homogeneous linear higher order differential equation

Note :- * The above differential equation can also be written as -

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots a_n y = \phi(x)$$

where, $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$, ...

* The general solution of non-homogeneous differential equation consist of two parts namely complementary function (CF) & particular integral (PI)

i.e, $y = y_c + y_p$

* If $\phi(x) = 0$, then the general solution contains only complementary function (CF), here PI = 0

* The given DE can also be written as -

$$f(D) y = \phi(x)$$

$$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots a_n$$

* The auxillary equation of ① is - $f(m) = 0$, i.e,

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

Rules to find complementary function CF

$\cdots -(1) \Rightarrow f(D)Y = \phi(x)$

The auxillary equation of above DE is - $m^2 + a_1 m + a_2 = 0$ - (2)

Clearly (2) has two roots, say m_1 & m_2.

Case 1 - If the roots are real & equal, i.e. $m_1 = m_2 = m$, i.e.

$$Y_c = (C_1 + C_2 x)e^{mx}$$

Case 2 - If the roots are real & distinct, i.e. $m_1 \neq m_2$.

$$Y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Case 3 - If the roots are complex, i.e. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

$$Y_c = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

Note :- For a 3rd order DE - $(D^3 + a_1 D^2 + a_2 D + a_3)Y = \phi(x)$

The auxillary equation is $m^3 + a_1 m^2 + a_2 m + a_3 = 0$ & the roots are m_1, m_2, m_3

Case 1 - If $m_1 = m_2 = m_3$ $\Rightarrow Y_c = (C_1 + C_2 x + C_3 x^2)e^{mx}$

Case 2 - If $m_1 \neq m_2 \neq m_3$ $\Rightarrow Y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$

Case 3 - If $m_1 = m_2 (= m)$ & m_3 $\Rightarrow Y_c = (C_1 + C_2 x)e^{mx} + C_3 e^{m_3 x}$

Case 4 - If $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, & m_3

$\Rightarrow Y_c = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3 e^{m_3 x}$

Solve the following DE

1. $(D^2 - 6D + 9)Y = 0$

Ans Auxillary equation is $f(m) = 0 \Rightarrow m^2 - 6m + 9 = 0$

$\Rightarrow m = 3, 3$

$$Y_c = (C_1 + C_2 x)e^{3x}$$

2. $(D^2 + 4)Y = 0$

Ans Auxillary equation is - $f(m) = 0 \Rightarrow m^2 + 4 = 0 \Rightarrow m = \pm 2i = 0 \pm 2i$

$Y_c = e^{(0)x}(C_1 \cos 2x + C_2 \sin 2x) = C_1 \cos 2x + C_2 \sin 2x$

3. $4y''' + 4y'' + y' = 0$

Ans $(4D^3 + 4D^2 + D)y = 0$

Auxillary equation $\Rightarrow$ $4m^3 + 4m^2 + m = 0$

$m(2m+1)^2 = 0$

$m = 0, -\frac{1}{2}, -\frac{1}{2}$

$y_c = (c_1 + c_2 x) e^{-\frac{1}{2}x} + c_3 e^{0(x)}$

$y_c = (c_1 + c_2 x) e^{-\frac{x}{2}} + c_3$

4. $D^3y + D^2y - Dy = 0$

Ans $(D^3 + D^2 - D) y = 0$

$\Rightarrow$ $m^3 + m^2 - m = 0$

$m(m^2 + m - 1) = 0$

$m(m^2 + m - 1) = 0$

$\Rightarrow$ $m = 0$, $m = \frac{-1 \pm \sqrt{5}}{2}$

$\therefore$ $y_c = c_1 e^{0x} + c_2 e^{\left(\frac{-1+\sqrt{5}}{2}\right)x} + c_3 e^{\left(\frac{-1-\sqrt{5}}{2}\right)x}$

Inverse differential operator

WKT, differential operator is denoted by $D \equiv \frac{d}{dx}$, then $\frac{1}{D}$ represents an inverse differential operator also called integral.

Methods to find the particular integral

Type 1 - Consider a differential equation of the form $f(D)y = \phi(x)$

If $\phi(x) = e^{ax}$, then

$$PI = \frac{1}{f(D)} \phi(x)$$

If $f($

If $f'(a$

Solve

1. $($

Ans

$$PI = \frac{1}{f(D)} e^{ax}$$

Replace D by a

$$PI = \frac{1}{f(a)} e^{ax} \quad , f(a) \neq 0$$

If $f(a) = 0$ $\Rightarrow PI = x \frac{1}{f'(a)} e^{ax}$, $f'(a) \neq 0$

If $f'(a) = 0$ $\Rightarrow PI = x^2 \frac{1}{f''(a)} e^{ax}$, $f''(a) \neq 0$

and so on...

Solve the following DE

1. $(D^3 + 3D^2 + 3D + 1) y = 5e^{2x} + 6e^{-x} + 7$

Ans $f(D) y = \phi(x)$

Auxillary equation is $f(m) = 0$

$\Rightarrow m^3 + 3m^2 + 3m + 1 = 0$

$(m+1)^3 = 0$

$\Rightarrow m = -1, -1, -1$

$\therefore y_c = (c_1 + c_2 x + c_3 x^2) e^{-1(x)}$

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{-x}$$

$$y_p = \frac{1}{f(D)} \phi(x)$$

$$y_p = \frac{1}{D^3 + 3D^2 + 3D + 1} (5e^{2x} + 6e^{-x} + 7e^{0x})$$

$$y_p = 5 \frac{1}{D^3 + 3D^2 + 3D + 1} e^{2x} + 6 \frac{1}{D^3 + 3D^2 + 3D + 1} e^{-x} + 7 \frac{1}{D^3 + 3D^2 + 3D + 1} e^{0x}$$

Put $D = 2$ $\quad D = -1$ $\quad D = 0$

$$y_p = \frac{5}{8 + 12 + 6 + 1} e^{2x} + \frac{6}{\cancel{-1} + \cancel{3} - \cancel{3} + 1} e^{-x} + \frac{7}{0 + 0 + 0 + 1} e^{0x}$$

we are getting zero in denominator

$\Rightarrow$ Now do $x \times$ (Differentiate denominator

$$\Rightarrow y_p = \frac{5}{27}e^{2x} + \frac{6x}{3D^2+6D+3}e^{-x} + 7$$

Put $D=-1$

$$y_p = \frac{5}{27}e^{2x} + \frac{6x}{\cancel{3-6+3}}e^{-x} + 7$$

$$\Rightarrow y_p = \frac{5}{27}e^{2x} + \frac{6x^2}{6D+6}e^{-x} + 7$$

Put $D=-1$

$$y_p = \frac{5e^{2x}}{27} + \frac{6x^2}{\cancel{-6+6}}e^{-x} + 7$$

$$\Rightarrow y_p = \frac{5e^{2x}}{27} + \frac{\cancel{6}x^3}{\cancel{6}}e^{-x} + 7$$

$$y_p = \frac{5}{27}e^{2x} + x^3e^{-x} + 7$$

General solution -

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = (c_1 + c_2x + c_3x^2)e^{-x} + \frac{5}{27}e^{2x} + x^3e^{-x} + 7}$$

2. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^x \cosh 2x + 2^x$

Ans $(D^2y - 4Dy + 13y) = e^x\left(\frac{e^{2x}+e^{-2x}}{2}\right) + e^{\log_e 2^x}$

$$\Rightarrow (D^2 - 4D + 13)y = \frac{e^{3x}}{2} + \frac{e^{-x}}{2} + e^{x(\log_e 2)}$$

Auxillary equation - $f(m) = 0$

$$\Rightarrow m^2 - 4m + 13 = 0$$

$$m = 2 \pm i3$$

$$\therefore y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2-4D+13}\left[\frac{e^{3x}}{2} + \frac{e^{-x}}{2} + e^{x\log_e 2}\right]$$

$$y_p = \frac{e^{3x}}{2}\left[\frac{1}{D^2-4D+13}\right]_{D=3} + \frac{e^{-x}}{2}\left[\frac{1}{D^2-4D+13}\right]_{D=-1} + e^{x\log_e 2}\left[\frac{1}{D^2-4D+13}\right]_{D=\log_e 2}$$

$$y_p = \frac{e^{3x}}{2}\left[\frac{1}{10}\right] + \frac{e^{-x}}{2}\left[\frac{1}{18}\right] + e^{x\log_e 2}\left[\frac{1}{(\log_e 2)^2-4(\log_e 2)+13}\right]$$

$$\Rightarrow \quad y_p = \frac{e^{3x}}{20} + \frac{e^{-x}}{36} + \frac{e^{x\log_e 2}}{10.707}$$

General solution -

$$y = y_c + y_p$$

$$\boxed{y = e^{2x}(c_1\cos 3x + c_2\sin 3x) + \frac{e^{3x}}{20} + \frac{e^{-x}}{36} + \frac{e^{x\log_e 2}}{10.707}}$$

3. $\frac{d^4y}{dx^4} - y = \cosh(1-x) + 3^x - 5$

Ans , $D^4y - y = \cosh(1-x) + 3^x - 5$

$$\Rightarrow (D^4-1)y = \frac{e^{1-x}+e^{-(1-x)}}{2} + e^{\log_e 3^x} - 5e^{0x}$$

Auxillary equation - $f(m)=0$

$$\Rightarrow \quad m^4 - 1 = 0 \quad \Rightarrow (m^2-1)(m^2+1) = 0$$

~~$m = 1,1,1,1$~~ $\quad m = \pm 1, \; m = \pm i = 0 \pm i$

~~$y_c = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^x$~~ $\quad \therefore y_c = c_1e^x + c_2e^{-x} + e^{0x}(c_3\cos x + c_4\sin x)$

$$\Rightarrow \boxed{y_c = c_1e^x + c_2e^{-x} + e^{0x}(c_3\cos x + c_4\sin x)}$$

$$y_p = \frac{1}{f(D)}\phi(x) \Rightarrow y_p = \frac{1}{D^4-1}\left[\frac{e^{1-x}}{2} + \frac{e^{x-1}}{2} + e^{x\log_e 3} - 5e^{0x}\right]$$

$$y_p = \frac{e^{1-x}}{2}\left[\frac{1}{D^4-1}\right]_{D=-1} + \frac{e^{x-1}}{2}\left[\frac{1}{D^4-1}\right]_{D=1} + e^{x\log_e 3}\left[\frac{1}{D^4-1}\right]_{D=\log_e 3} - 5e^{0x}\left[\frac{1}{D^4-1}\right]_{D=0}$$

$$y_p = \frac{e^{1-x}}{2}\left[\frac{1}{1-1}\right] + \frac{e^{x-1}}{2}\left[\frac{1}{1-1}\right] + e^{x\log_e 3}\left[\frac{1}{(\log_e 3)^4-1}\right] - \frac{5}{-1}$$

$$\Rightarrow \quad y_p = \frac{e^{1-x}}{2}\left[x \times \frac{1}{4D^3}\right]_{D=-1} + \frac{e^{x-1}}{2}\left[x \times \frac{1}{4D^3}\right]_{D=1} + e^{x\log_e 3}\left[\frac{1}{1.651}\right] + 5$$

$$y_p = \frac{e^{1-x}}{-8}x + \frac{e^{x-1}}{8}x + \frac{e^{x\log_e^3}}{1.651} + 5$$

General solution -

$$y = y_c + y_p$$

$$\boxed{y = c_1e^{x} + c_2e^{-x} + e^{0x}(c_3\cos x + c_4\sin x) + \frac{x\cdot e^{1-x}}{-8} + \frac{xe^{x-1}}{8} + \frac{e^{x\log_e^3}}{1.651} + 5}$$

4. $(D^3 - 3D^2 + 3D + 1)y = \sinh(x+2)$

Ans $(D^3-3D^2+3D-1)\,y = \dfrac{e^{x+2} - e^{-(x+2)}}{2}$

Auxillary equation -

$$f(m) = 0$$
$$m^3 - 3m^2 + 3m - 1 = 0$$
$$(m-1)^3 = 0$$
$$m = 1, 1, 1$$

$\Rightarrow y_c = (c_1 + c_2\cancel{x} + c_3x^2)e^{(x)}$ $\Rightarrow y_c = (c_1 + c_2x + c_3x^2)e^{x}$

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^3-3D^2+3D-1}\left[\frac{e^{x+2}}{2} - \frac{e^{-x-2}}{2}\right]$$

$$y_p = \frac{e^{x+2}}{2}\left[\frac{1}{D^3-3D^2+3D-1}\right]_{D=1} - \frac{e^{-x-2}}{2}\left[\frac{1}{D^3-3D^2+3D-1}\right]_{D=-1}$$

$$y_p = \frac{e^{x+2}}{2}\left[\frac{1}{\cancel{1}-\cancel{3}+\cancel{3}-1}\right] - \frac{e^{-x-2}}{2}\left[\frac{1}{-1-3-3-1}\right]$$

$$\Rightarrow y_p = \frac{e^{x+2}}{2}\left[x\times\frac{1}{3D^2-6D+3}\right] - \frac{e^{-x-2}}{2}\left[\frac{1}{-8}\right]$$

$$y_p = \frac{e^{x+2}}{2}\left[x\times\frac{1}{\cancel{3}-6+\cancel{3}}\right] + \frac{e^{-x-2}}{16}$$

$$\Rightarrow y_p = \frac{e^{x+2}}{2}\left[x^2\times\frac{1}{6D-6}\right] + \frac{e^{-x-2}}{16}$$

Type - 2

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If f(

If f

Solve

1. $D^3 y$

Ans

Auxil

$$y_p = \frac{e^{x+2}}{2}\left[\frac{x^2}{6\cdot 6}\right] + \frac{e^{-x-2}}{16}$$

$$\Rightarrow y_p = \frac{e^{x+2}}{2}\left[\frac{x^3}{6}\right] + \frac{e^{-x-2}}{16}$$

$$y_p = \frac{x^3}{12}e^{x+2} + \frac{e^{-x-2}}{16}$$

$\therefore\ y = y_c + y_p$

$$\boxed{y = (C_1 + C_2 x + C_3 x^2)e^{x} + \frac{x^3}{12}e^{x+2} + \frac{e^{-x-2}}{16}}$$

Type-2 - If $\phi(x) = \sin ax / \cos ax$, then particular integral,

$$PI = \frac{1}{f(D)}\phi(x) = \frac{1}{f(D)}(\sin ax)$$

Replace D^2 by $-a^2$

$$PI = \frac{1}{f(-a^2)}\sin ax \quad , f(-a^2) \neq 0.$$

If $f(-a^2) = 0$,

$$PI = x \times \frac{1}{f'(-a^2)}\sin ax \quad , f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$,

$$PI = x^2 \frac{1}{f''(-a^2)}\sin ax \quad , f''(-a^2) \neq 0$$

& so on:-

Solve the following DE

1. $D^3y + D^2y - Dy = 2\cos^2 x$

Ans
$$(D^3 + D^2 - D)y = 2\cos^2 x$$
$$(D^3 + D^2 - D)y = 1 + \cos 2x$$
$$(D^3 + D^2 - D)y = e^{0x} + \cos 2x$$

Auxillary equation -
$$f(m) = 0$$
$$m^3 + m^2 - m = 0$$
$$m(m^2 + m - 1) = 0$$
$$m = 0, \quad m = \frac{-1 \pm \sqrt{5}}{2}$$

$$\therefore y_c = C_1 e^{0x} + C_2 e^{\left(\frac{-1+\sqrt{5}}{2}\right)x} + C_3 e^{\left(\frac{-1-\sqrt{5}}{2}\right)x}$$

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^3+D^2-D}\left(e^{0x} + \cos 2x\right)$$

$$y_p = \frac{e^{0x}}{D^3+D^2-D} + \frac{1}{D^3+D^2-D}\cos 2x$$

$D = 0$ Replace D^2 by -2^2

$$y_p = \frac{e^{0x}}{\cancel{0+0-0}} + \frac{1}{(-2^2)D+(-2^2)-D}\cos 2x$$

$$\therefore y_p = \frac{x}{3D^2+2D-1}e^{0x} + \frac{1}{-5D-4}\cos 2x$$

$$y_p = \frac{x}{0+0-1}e^{0x} - \frac{1}{5D+4}\cos 2x$$

$$y_p = -x e^{0x} - \frac{1}{(5D+4)} \times \frac{(5D-4)}{(5D-4)}\cos 2x$$

$$y_p = -x - \frac{(5D-4)}{25D^2-16}\cos 2x$$

$D^2 = -2^2$

$$y_p = -x - \frac{(5D-4)}{-100-16}\cos 2x$$

$$y_p = -x + \frac{(5D-4)\cos 2x}{116}$$

$$y_p = -x + \frac{5D\cos 2x - 4\cos 2x}{116}$$

$$y_p = -x + \frac{5(-2\sin 2x) - 4\cos 2x}{116}$$

$$y_p = -x - \frac{(5\sin 2x + 2\cos 2x)}{58}$$

$$\therefore y = y_c + y_p$$

$$\boxed{y = C_1 e^{0x} + C_2 e^{\left(\frac{-1+\sqrt{5}}{2}\right)x} + C_3 e^{\left(\frac{-1-\sqrt{5}}{2}\right)x} - x - \frac{1}{58}(5\sin 2x + 2\cos 2x)}$$

2. (D^2-4D+3)

$(D^2-4D+$

Ans

Auxiliary e

$\therefore$ y

$y_p = $

$\therefore y = y_c$

2. $(D^2-4D+3)y = \sin 3x \cos 2x$

$(D^2-4D+3)y = \sin 3x \cos 2x$

Ans

Auxiliary eqn -

$$f(m) = 0$$
$$m^2 - 4m + 3 = 0$$
$$m^2 - 3m - m + 3 = 0$$
$$m(m-3) - 1(m-3) = 0$$
$$m = +1, \; m = 3$$

$$\therefore \; y_c = \underline{\underline{c_1 e^x + c_2 e^{3x}}}$$

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2-4D+3}(\sin 3x \cos 2x)$$

$$y_p = \frac{1}{D^2-4D+3}\left[\frac{\sin 5x + \sin x}{2}\right]$$

$$y_p = \frac{1}{2}\frac{\sin 5x}{D^2-4D+3} + \frac{1}{2}\frac{\sin x}{D^2-4D+3}$$
$$D^2 = -5^2 \qquad\qquad D^2 = -1^2$$

$$y_p = \frac{1}{2}\frac{\sin 5x}{-25-4D+3} + \frac{1}{2}\frac{\sin x}{-1-4D+3}$$

$$y_p = -\frac{1}{2}\frac{\sin 5x}{4D+22} + \frac{1}{2}\frac{\sin x}{2-4D}$$

$$y_p = -\frac{1}{2}\left[\frac{1}{4D+22} \times \frac{4D-22}{4D-22}\right]\sin 5x - \frac{1}{2}\sin x\left[\frac{1}{4D-2} \times \frac{4D+2}{4D+2}\right]$$

$$y_p = -\frac{1}{2}\left[\left[\frac{(4D-22)(\sin 5x)}{16D^2-22^2}\right] + \left[\frac{(4D+2)\sin x}{16D^2-4}\right]\right]$$
$$D^2 = -5^2 \qquad\qquad D^2 = -1^2$$

$$y_p = -\frac{1}{2}\left[\left[\frac{4D\sin 5x - 22\sin 5x}{16(-25)-484}\right] + \left[\frac{4D\sin x + 2\sin x}{-16-4}\right]\right]$$

$$y_p = -\frac{1}{2}\left[\frac{20\cos 5x - 22\sin 5x}{-884} + \frac{4\cos x + 2\sin x}{-20}\right]$$

$$y_p = \frac{10\cos 5x - 11\sin 5x}{884} + \frac{2\cos x + \sin x}{20}$$

$$\therefore \; y = y_c + y_p \Rightarrow \boxed{y = c_1 e^x + c_2 e^{3x} + \frac{10\cos 5x - 11\sin 5x}{884} + \frac{2\cos x + \sin x}{20}}$$

3. $(D^2+2D+1)y = e^{2x} - \cos^2 x$

Ans $(D^2+2D+1)y = e^{2x} - \cos^2 x = e^{2x} - \left(\frac{1+\cos 2x}{2}\right) = e^{2x} - \frac{e^{0x}}{2} - \frac{\cos 2x}{2}$

Auxillary equation.

$f(m) = 0$

$(m^2+2m+1) = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$y_c = (C_1 + C_2 x)e^{-x}$

$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{(D^2+2D+1)}\left[e^{2x} - \frac{e^{0x}}{2} - \frac{\cos 2x}{2}\right]$

$y_p = \frac{1}{D^2+2D+1}e^{2x} - \frac{1}{2(D^2+2D+1)}e^{0x} - \frac{1}{2}\frac{\cos 2x}{D^2+2D+1}$

$D=2 \qquad D=0 \qquad D^2=-2^2$

$y_p = \frac{1}{9}e^{2x} - \frac{1}{2}e^{0x} - \frac{1}{2}\frac{\cos 2x}{2D-3}$

$y_p = \frac{1}{9}e^{2x} - \frac{1}{2} - \frac{1}{2}\left[\frac{1}{2D-3} \times \frac{2D+3}{2D+3}\right]\cos 2x$

$y_p = \frac{e^{2x}}{9} - \frac{1}{2} - \frac{1}{2}\left[\frac{2D\cos 2x + 3\cos 2x}{4D^2-9}\right]$

$D^2 = -2^2$

$y_p = \frac{e^{2x}}{9} - \frac{1}{2} - \frac{1}{2}\left[\frac{-4\sin 2x + 3\cos 2x}{-25}\right]$

$y_p = \frac{e^{2x}}{9} - \frac{1}{2} + \frac{1}{2}\left[\frac{3\cos 2x - 4\sin 2x}{25}\right]$

$\Rightarrow y = y_c + y_p$

$$\boxed{y = (C_1 + C_2 x)e^{-x} + \frac{e^{2x}}{9} - \frac{1}{2} + \frac{1}{2}\left[\frac{3\cos 2x - 4\sin 2x}{25}\right]}$$

Type - 3 - If $\phi(x)$ is a polynomial in 'x', in this case the particular integral can be obtained by writing $\phi(x)$ in descending powers of 'x' and $f(D)$ as ascending powers of 'D', then divide $\phi(x)$ with $f(D)$, the quotient so obtained is the required particular integral (PI) without any remainder

Solve the follow

1. $\frac{D^2y}{dx^2} + \frac{dy}{dx}$

Ans $(D^2+D)y =$

Auxillary equ

PI = y_p

2. $(2D^2+2D+$

Ans $(2$

Auxillary e

$y_p =$

Solve the following DE

1. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2+2x+4$

Ans $(D^2+D)y = x^2+2x+4$

Auxillary equation - $f(m)=0$

$m^2+m=0$

$m(m+1)=0$

$\Rightarrow m=0,-1$

$y_c = c_1 e^{0x} + c_2 e^{-x}$

$PI = y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2+D}(x^2+2x+4)$

$D(q) = x^2$

$\Rightarrow q = \int x^2$

$D^2\left(\frac{x^3}{3}\right) = 2x$

$\Rightarrow D+D^2)\, x^2+2x+4\,\left(\frac{x^3}{3}+4x\right.$

x^2+2x

4

$4+0$

0

$\Rightarrow y_p = \frac{x^3}{3}+4x$

$y = y_c + y_p \Rightarrow \boxed{y = c_1 e^{0x} + c_2 e^{-x} + \frac{x^3}{3} + 4x}$

2. $(2D^2+2D+3)y = x^2+2x-1$

Ans $(2D^2+2D+3)y = x^2+2x-1$

Auxillary equation - $f(m)=0$

$2m^2+2m+3=0$

$m = \frac{-2\pm\sqrt{-20}}{4} = \frac{-2\pm 2i\sqrt{5}}{4}$

$m = \frac{-1\pm\sqrt{5}\,i}{2}$

$y_c = e^{-\frac{1}{2}x}\left(c_1\cos\left(\frac{\sqrt{5}}{2}x\right) + c_2\sin\left(\frac{\sqrt{5}x}{2}\right)\right)$

$y_p = \frac{1}{f(D)}\phi(x) \Rightarrow y_p = \frac{1}{2D^2+2D+3}(x^2+2x-1)$

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$3+2D+2D^2) \; x^2+2x-1 \; \left(\frac{x^2}{3}+\frac{2x}{9}-\frac{25}{27}\right.$

$3(y)=\frac{2x}{3}$ || $3(y)=\frac{-25}{3}$

$y=\frac{2x}{9}$

$\cancel{x^2}+\frac{4x}{3}+\frac{4}{3}$

$(-)\;(-)\;(-)$

$\frac{2x}{3}-\frac{7}{3}$

$\cancel{\frac{2x}{3}}+\frac{4}{9}+0$

$(-)\;(-)$

$\frac{-25}{9}$

$\cancel{\frac{-25}{9}}$

$(+)$

0

$$y_p=\frac{x^2}{3}+\frac{2x}{9}-\frac{25}{27} \Rightarrow y=y_c+y_p$$

$$\boxed{y=e^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt5}{2}x\right)+c_2\sin\left(\frac{\sqrt5}{2}x\right)\right)+\frac{x^2}{3}+\frac{2x}{9}-\frac{25}{27}}$$

3. $D^3y+3D^2y+2Dy-1=x^2+2x$

Ans $(D^3+3D^2+2D)y=x^2+2x+1$

$f(m)=0$

$m^3+3m^2+2m=0$

$m(m^2+3m+2)=0$

$m(m+1)(m+2)=0$

$m=0,-1,-2$

$$y_c=c_1e^{0x}+c_2e^{-x}+c_3e^{-2x}$$

$2D(y)=x^2$ $\quad$ $2D(y)=-x$

$y=\int\frac{x^2}{2}$ $\quad$ $y=\int\frac{-x}{2}$

$$y_p=\frac{1}{D^3+3D^2+2D}(x^2+2x+1)$$

$2D(y)=\frac{3}{2}$

$y=\int\frac{3}{4}$

$2D+3D^2+D^3) \; x^2+2x+1 \; \left(\frac{x^3}{6}-\frac{x^2}{4}+\frac{3}{4}x\right.$

$\cancel{x^2}+3x+1$

$(-)\;(-)\;(-)$

$-\cancel{x}$

$\cancel{-x}-\frac{3}{2}$

$(+)\;(+)$

$\frac{3}{2}$

$\cancel{\frac{3}{2}}$

$(-)$

0

$$y_p=\frac{x^3}{6}-\frac{x^2}{4}+\frac{3}{4}x \Rightarrow y=y_c+y_p$$

$$y=c_1e^{0x}+c_2e^{-x}+c_3e^{-2x}+\frac{x^3}{6}-\frac{x^2}{4}+\frac{3}{4}x$$

the PI is

by PI

solve the

1. $(D^2+4$

Ans

$y_p=$

Type - 4 — If $\phi(x) = e^{ax} V$. where 'V' is any function of 'x', then the PI is obtained by replacing 'D' by 'D+a' in $f(D)$ & it is given by $PI = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} e^{ax} V$ 'V' is sine or cosine or any polynomial of x

$$PI = e^{ax}\left[\frac{1}{f(D+a)} V\right]$$

solve the following DE

1. $(D^2+4D+5)y = e^{2x} x^2$

Ans $(D^2+4D+5)y = e^{2x} x^2$

$\Rightarrow f(m) = 0$

$m^2+4m+5 = 0$

$m = -2 \pm i$

$y_c = e^{-2x}(C_1 \cos x + C_2 \sin x)$

$y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{D^2+4D+5}(e^{2x} x^2)$

Replace 'D' by 'D+2'

$\Rightarrow y_p = e^{2x}\left[\frac{1}{(D+2)^2+4(D+2)+5}(x^2)\right]$

$y_p = e^{2x}\left[\frac{1}{D^2+12+8D}(x^2)\right]$

$y_p = e^{2x}\left[\frac{1}{D^2+8D+17}(x^2)\right]$

$\Rightarrow \quad 17+8D+D^2)\, x^2 \left(\frac{x^2}{17} - \frac{16x}{289} + \frac{94}{4913}\right.$

$x^2 + \frac{16x}{17} + \frac{2}{17}$
$(-)\ (-)\ (-)$

$-\frac{16x}{17} - \frac{2}{17}$

$-\frac{16x}{17} - \frac{128}{289}$
$(+)\ (+)$

$\frac{94}{289}$

$\frac{94}{289}$
$(-)$

0

$y_p = \left(\frac{x^2}{17} - \frac{16x}{289} + \frac{94}{4913}\right)e^{2x} \Rightarrow \left(y = e^{-2x}(C_1\cos x + C_2 \sin x) + \left(\frac{x^2}{17} - \frac{16x}{289} + \frac{94}{4913}\right)e^{2x}\right)$

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2. $(D^2-2D+4)y = e^x \cos x$

Ans

$f(m) = 0$

$m^2 - 2m + 4 = 0$

$m^2 - 2m + 4 = 0$

$m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2i\sqrt{3}}{2}$

$m = 1 \pm i\sqrt{3}$

$y_c = e^x (c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

$y_p = \frac{1}{(D^2-2D+4)} e^x \cos x$

Replace 'D' by 'D+1'

$y_p = e^x \left[\frac{\cos x}{D^2 + \cancel{2D} + 1 - \cancel{2D} - 2 + 4} \right]$

$y_p = e^x \left[\frac{1}{D^2+3} \cos x \right]$

Replace D^2 by -1^2

$y_p = e^x \left[\frac{1}{-1+3} \cos x \right]$

$y_p = \frac{e^x \cos x}{2}$

$y = y_c + y_p \Rightarrow \boxed{y = e^x (c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + \frac{e^x \cos x}{2}}$

Type-5 - If $\phi(x) = xV$ or $x^2 V$ or $x^n V$ where 'V' is any function of x

Note:- w.k.t,

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\cos\theta \rightarrow$ Real part of $e^{i\theta}$

$\sin\theta \rightarrow$ Imaginary part of $e^{i\theta}$

Solve the foll

1. $(D^2+1)y$

Ans

$\Rightarrow y$

$y_p =$

Solve the following DE

1. $(D^2+1)y = x\cos x$

Ans
$$f(m) = 0$$
$$m^2+1=0$$
$$m = \pm i$$
$$\Rightarrow y_c = e^{0x}(c_1\cos x + c_2\sin x)$$

$$y_p = \frac{1}{\phi(D)}\phi(x) = \frac{1}{D^2+1}(x\cos x)$$

$e^{ix} = \cos x + i\sin x$
$x e^{ix} = x\cos x + i\sin x(x)$
$\Rightarrow$ R.P of $xe^{ix} = x\cos x$

$$y_p = \frac{1}{D^2+1} \text{ Real part of } (xe^{ix})$$

$$y_p = \text{R.P of } \left[\frac{1}{D^2+1}(xe^{ix})\right]$$

Replace 'D' by 'D+i'

$$y_p = \text{R.P of } e^{ix}\left[\frac{1}{D^2+\cancel{i^2}+2iD+\cancel{1}}\, x\right]$$

$$y_p = \text{R.P of } e^{ix}\left[\frac{1}{D^2+2iD}\, x\right]$$

$2iD+D^2)\ x\ \left(\frac{x^2}{4i}+\frac{x}{4}\right.$

$\ \underline{\cancel{x}+\frac{1}{2i}}$

$\ -\frac{1}{2i}$

$\ \underline{\cancel{-\frac{1}{2i}}\ +0}$

$\ 0$

$2iD(q) = x$
$D(q) = \frac{x}{2i}$
$q = \int\frac{x}{2i} = \frac{x^2}{4i}$

$2iD(q) = -\frac{1}{2i}$
$D(q) = \frac{1}{4}$
$q = \frac{x}{4}$

$$\Rightarrow y_p = \text{R.P of } e^{ix}\left[\frac{x^2}{4i}+\frac{x}{4}\right]$$

But $\frac{1}{i} = -i$

$$y_p = \text{R.P of } e^{ix}\left[-\frac{x^2}{4}i+\frac{x}{4}\right]$$

$$y_p = \text{R.P of } (\cos x + i\sin x)\left(-\frac{x^2}{4}i+\frac{x}{4}\right)$$

we will get real part

$$y_p = \frac{x\cos x}{4} + \frac{x^2\sin x}{4}$$

$$\Rightarrow \boxed{y = e^{0x}(c_1\cos x + c_2\sin x) + \frac{x\cos x + x^2\sin x}{4}}$$

2. $(D^2 - 2D + 1)y = xe^x \sin x$

Ans

$f(m) = 0$

$m^2 - 2m + 1 = 0$

$m = \frac{2 \pm \sqrt{4 - 4(1)}}{2}$

$(m-1)^2 = 0$

$m - 1 = 0$

$m = 1, 1$

$y_c = (C_1 + C_2 x)e^x$

$y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{D^2 - 2D + 1} xe^x \sin x$

Imaginary part of $xe^{ix} = x \sin x$

$y_p = e^x \left[\frac{1}{D^2 - 2D + 1} (\text{Imaginary part of } xe^{ix}) \right]$

Replace 'D' by 'D+1'

$y_p = e^x \left[\frac{1}{D^2 + 1 + \cancel{2D} - \cancel{2D} + 1} \text{ I.P of } xe^{ix} \right]$

$y_p = e^x \left[\frac{1}{D^2 + 2} \text{ I.P of } xe^{ix} \right]$

Replace 'D' by 'D+i'

~~$y_p = e^x \cdot e^{ix}$~~ $y_p = e^x \cdot \text{I.P of } e^{ix} \left[\frac{1}{D^2 + i^2 + 2iD + 2} x \right]$

$y_p = e^x \cdot \text{I.P of } e^{ix} \left(\frac{1}{D^2 + 2iD + 1} x \right)$

$-1 + 2iD + D^2) \; x \; (-x - 2i$
$\quad x + 2i$
$\quad \overline{\quad -2i}$
$\quad -2i$
$\quad \overline{\quad 0}$

$\Rightarrow y_p = e^x \text{ I.P of } e^{ix} (-x - 2i)$

$y_p = e^x \text{ I.P } (\cos x + i \sin x)(-x - 2i)$

$y_p = e^x (-x \sin x - 2 \cos x)$

$$\boxed{y = (c_1 + c_2 x)e^x + e^x(-x\sin x - 2\cos x)}$$

Note :- $\int uv\,dx = u\int v\,dx - u'\iint v\,dx\,dx + u''\iiint v\,dx\,dx\,dx - u'''\iiiint v\,dx\,dx\,dx\,dx$

Eg :- $\int x^3 \sin x\,dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x(\cos x) - 6(\sin x)$

3. $(D^2 - 4D + 4)y = 8x^2 e^{2x}\sin 2x$

ANS

$f(m) = 0$

$\Rightarrow m^2 - 4m + 4 = 0$

$m^2 - 2m - 2m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$

$y_c = (c_1 + c_2 x)e^{2x}$

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2 - 4D + 4} e^{2x}(8x^2 \sin 2x)$$

Replace 'D' by 'D+2'

$$y_p = e^{2x}\left(\frac{1}{D^2 + \cancel{4} + \cancel{4D} - \cancel{4D} - \cancel{8} + \cancel{4}}(8x^2\sin 2x)\right)$$

$$y_p = e^{2x}\frac{1}{D^2}(8x^2 \sin 2x)$$

$$y_p = e^{2x}\left[\frac{1}{D^2}(8x^2\sin 2x)\right]$$

$$y_p = e^{2x}\left[\frac{1}{D}\left[\int 8x^2 \sin 2x\,dx\right]\right]$$

$$y_p = e^{2x}\left[\frac{1}{D}\left(8x^2\left(-\frac{\cos 2x}{2}\right) - 16x\left(-\frac{\sin 2x}{4}\right) + 16\left(\frac{\cos 2x}{8}\right)\right)\right.$$

$$y_p = e^{2x}\left[\int(-4x^2\cos 2x + 4x\sin 2x + 2\cos 2x)\,dx\right]$$

$$y_p = e^{2x}\left[(-4x^2)\left(\frac{\sin 2x}{2}\right) - (-8x)\left(-\frac{\cos 2x}{4}\right) + (-8)\left(-\frac{\sin 2x}{8}\right)\right.$$
$$\left. + 4x\left(-\frac{\cos 2x}{2}\right) - 4\left(-\frac{\sin 2x}{4}\right) + 2\frac{\sin 2x}{2}\right]$$

$$y_p = e^{2x}\left[-2x^2\sin 2x - 2x\cos 2x + \sin 2x - 2x\cos 2x + \sin 2x + \sin 2x\right]$$

$$y_p = e^{2x}\left[3\sin 2x - 4x\cos 2x - 2x^2\sin 2x\right]$$

$$\therefore \boxed{y = (C_1 + C_2x)e^{2x} + e^{2x}\left[3\sin 2x - 4x\cos 2x - 2x^2\sin 2x\right]}$$

Differential equations with variable co-efficients

Legendre's differential equation -

A differential equation of the form -

$$a_0(ax+b)^2\frac{d^2y}{dx^2} + a_1(ax+b)\frac{dy}{dx} + a_2y = \phi(x) \quad -(1)$$ is called legendre's linear DE of 2nd order with variable co-efficients

The above DE (1) can be reduced into a differential equation with constant co-efficients using suitable substitution,

Put $\boxed{\log_e(ax+b) = t} \Rightarrow \boxed{ax+b = e^t} \Rightarrow x = \frac{e^t - b}{a}$

where

$$\left.\begin{aligned}(ax+b)\frac{dy}{dx} &= aD_1y\\ (ax+b)^2\frac{d^2y}{dx^2} &= a^2(D_1)(D_1-1)y\\ (ax+b)^3\frac{d^3y}{dx^3} &= a^3(D_1)(D_1-1)(D_1-2)y\end{aligned}\right\} \text{ where } D_1 \equiv \frac{d}{dt}$$

Cauchy's Differential equation

If $a = 1$, $b = 0$ then legendre's differential equation becomes -

$$a_0x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = \phi(x) \quad -(2)$$ which is known as Cauchy's DE of 2nd order with variable co-efficients. It can be reduced to DE with constant co-efficients by putting

$\boxed{\log_e x = t} \Rightarrow \boxed{x = e^t}$

where

$$\left.\begin{aligned}x\frac{dy}{dx} &= D_1y\\ x^2\frac{d^2y}{dx^2} &= D_1(D_1-1)y\end{aligned}\right\} \quad D_1 \equiv \frac{d}{dt}$$

Solve the following D

1\. $x^4y''' + 2x^3y'' -$

Ans $x^4y''' + 2x^3$

Divide (1) by

$\Rightarrow x^3y''' +$

I

Put $\log_e x$

$xy' =$

$x^2y'' =$

$x^3y''' =$

$\Rightarrow$ (2) becomes

$D_1(D_1 -$

$D_1(D_1^2$

$\Rightarrow (D_1^3$

$(D$

Auxillary

$\therefore$

$y_p =$

$\Rightarrow y_p =$

Solve the following DE

$x^4 y''' + 2x^3 y'' - x^2 y' + xy = \sin(\log_e x)$

1. $x^4 y''' + 2x^3 y'' - x^2 y' + xy = \sin(\log_e x)$ — ①

Ans

Divide ① by x.

$\Rightarrow \quad x^3 y''' + 2x^2 y'' - x y' + y = \dfrac{\sin(\log_e x)}{x}$ — ②

It is in cauchy's form

Put $\log_e x = t \Rightarrow x = e^t$

$$\left.\begin{aligned} xy' &= D_1 y \\ x^2 y'' &= D_1(D_1 - 1)y \\ x^3 y''' &= D_1(D_1-1)(D_1-2)y \end{aligned}\right\} \quad D_1 \equiv \frac{d}{dt}$$

$\Rightarrow$ ② becomes,

$D_1(D_1-1)(D_1-2)y + 2(D_1(D_1-1)y) - D_1 y + y = \dfrac{\sin(t)}{e^t}$

$D_1(D_1^2 - 3D_1 + 2)y + 2(D_1^2 - D_1)y - D_1 y + y = \dfrac{\sin t}{e^t}$

$\Rightarrow (D_1^3 - 3D_1^2 + 2D_1 + 2D_1^2 - 2D_1 - D_1 + 1)y = e^{-t}\sin t$

$(D_1^3 - D_1^2 - D_1 + 1)y = e^{-t}\sin t$

Auxillary equation is .

$f(m) = 0$

$m^3 - m^2 - m + 1 = 0$

$m^2(m-1) - 1(m-1) = 0$

$(m-1)(m+1)(m-1) = 0$

$\Rightarrow m = 1, 1, -1$

$\therefore \quad y_c = (c_1 + c_2 t)e^{t} + c_3 e^{-t}$

$y_p = \dfrac{1}{f(D)}\phi(t) = \dfrac{1}{D_1^3 - D_1^2 - D_1 + 1}(e^{-t}\sin t)$

Replace 'D_1' by '$D_1 - 1$'

$\Rightarrow y_p = e^{-t}\left[\dfrac{1}{(D_1-1)^3 - (D_1-1)^2 - (D_1-1) + 1}\sin t\right]$

$$y_p = e^{-t}\left[\frac{1}{D_1^3 + \cancel{1} - 3D_1^2 - 3D_1 - D_1^2 - \cancel{1} + 2D_1 - D_1 - 1 + 1}\sin t\right]$$

$$y_p = e^{-t}\left[\frac{1}{D_1^3 - 4D_1^2 - 2D_1}\sin t\right]$$

Put $D_1^2 = -1^2$

$$y_p = e^{-t}\left[\frac{1}{(-1^2)D_1 - 4(-1^2) - 2D_1}\sin t\right]$$

$$y_p = e^{-t}\left[\frac{1}{-D_1 + 4 - 2D_1}\sin t\right]$$

$$y_p = e^{-t}\left[\sin t\left(\frac{1}{4 - 3D_1} \times \frac{4 + 3D_1}{4 + 3D_1}\right)\right]$$

$$y_p = e^{-t}\left[\frac{\sin t\,(4 + 3D_1)}{16 - 9D_1^2}\right]$$

Put $D_1^2 = -1^2$

$$y_p = e^{-t}\left[\frac{4\sin t + 3\cos t}{16 - 3(-1^2)}\right]$$

$$y_p = e^{-t}\left[\frac{4\sin t + 3\cos t}{25}\right]$$

$\therefore\ y = y_c + y_p \Rightarrow \boxed{y = (c_1 + c_2\log x)x + \frac{c_3}{x} + \frac{1}{25x}[4\sin(\log x) + 3\cos(\log x)]}$

2. $(2x+1)^2 y'' - 2(2x+1)y' - 12y = 6x + 5$

Ans Put $\log(2x+1) = t \Rightarrow 2x + 1 = e^t$

$\Rightarrow x = \frac{e^t - 1}{2}$

$(2x+1)\frac{dy}{dx} = 2D_1 y$

$(2x+1)^2\frac{d^2y}{dx^2} = 2^2 D_1(D_1 - 1)y$

$\therefore\ 4D_1(D_1 - 1)y - 2(2D_1 y) - 12y = 6\left(\frac{e^t - 1}{2}\right) + 5$

$4(D_1^2 - D_1)y - 4D_1 y - 12y = 3e^t - 3 + 5$

$(4D_1^2 - 4D_1 - 4D_1 - 12)y = 3e^t + 2$

$(4D_1^2$

$f(m) = 0$

$\Rightarrow 4$

$\therefore\ y_c =$

$y_p = \frac{1}{f(D)}$

$\therefore\ y =$

3. $(3x+2)^2 y'' +$

$f(m) = 0$

$\Rightarrow \quad 4m^2 - 8m - 12 = 0$

$m^2 - 2m - 3 = 0$

$m^2 - 3m + m - 3 = 0$

$m(m-3) + 1(m-3) = 0$

$m = -1, 3$

$\therefore \quad Y_c = c_1 e^{-t} + c_2 e^{3t}$

$\Rightarrow \quad Y_c = c_1 e^{-\log(2x+1)} + c_2 e^{3\log(2x+1)}$

$Y_c = c_1 e^{\log(2x+1)^{-1}} + c_2 e^{\log(2x+1)^3}$

$\Rightarrow \quad Y_c = \frac{c_1}{2x+1} + c_2 (2x+1)^3$

$Y_p = \frac{1}{f(D)} \phi(t) = \frac{1}{4(D_1^2 - 2D_1 - 3)} 3e^t + 2$

$Y_p = \frac{1}{4}\left[\frac{3e^t}{D_1^2 - 2D_1 - 3} + \frac{2e^{0t}}{D_1^2 - 2D_1 - 3}\right]$

Put $D = 1$ $\quad D = 0$

$Y_p = \frac{1}{4}\left[\frac{3e^t}{-4} + \frac{2e^{0t}}{-3}\right]$

$Y_p = -\frac{1}{4}\left[\frac{3e^t}{4} + \frac{2}{3}\right]$

$Y_p' = -\frac{1}{4}\left[\frac{3}{4} e^{\log(2x+1)} + \frac{2}{3}\right]$

$Y_p = -\frac{1}{4}\left[\frac{3}{4}(2x+1) + \frac{2}{3}\right]$

$\therefore \quad Y = Y_c + Y_p \Rightarrow \boxed{Y = \frac{c_1}{2x+1} + c_2(2x+1)^3 - \frac{1}{4}\left[\frac{3}{4}(2x+1) + \frac{2}{3}\right]}$

3. $(3x+2)^2 y'' + 3(3x+2)y' - 36y = 8x^2 + 4x + 1$

Ans

Put $\log_e(3x+2) = t \Rightarrow e^t = 3x+2 \Rightarrow x = \frac{e^t-2}{3}$

$(3x+2)y' = 3D_1 y$

$(3x+2)^2 y'' = 3^2 D_1(D_1-1)y$

$\therefore \quad 9D_1(D_1-1)y + 3(3D_1 y) - 36y = 8\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1$

$(9D_1^2 - \cancel{9D_1} + \cancel{9D_1} - 36)y = 8\left(\frac{e^{2t}+4-4e^t}{9}\right) + \frac{4e^t-8}{3} + 1$

$(9D_1^2 - 36)y = 8\left(\frac{e^{2t}+4-4e^t}{9}\right) + \frac{4e^t-8}{3} + 1$

Auxillary equation -

$f(m) = 0$

$9m^2 - 36 = 0$

$m^2 - 4 = 0$

$m = \pm 2$

$y_c = c_1 e^{-2t} + c_2 e^{2t}$

$y_c = c_1 e^{-2\log(3x+2)} + c_2 e^{2\log(3x+2)}$

$y_c = \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$

$y_p = \frac{1}{f(D)}\phi(t) = \frac{1}{9D_1^2-36}\left[8\left(\frac{e^{2t}+4-4e^t}{9}\right) + 4\left(\frac{e^t-2}{3}\right) + 1\right]$

$y_p = \frac{1}{D_1^2-4}\left[\frac{8e^{2t}+32-32e^t}{81} + \frac{4e^t-8}{27} + \frac{1}{9}\right]$

$y_p = \frac{1}{81}\left[\frac{e^{2t}}{D_1^2-4} + \frac{4e^{0t}}{D_1^2-4} - \frac{4e^t}{D_1^2-4}\right] + \frac{1}{27}\left[\frac{4e^t}{D_1^2-4} - \frac{8e^{0t}}{D_1^2-4}\right] + \frac{1}{9}\frac{e^{0t}}{D_1^2-4}$

$D_1 = 2 \qquad D_1 = 0 \qquad D_1 = 1 \qquad D_1 = 1 \qquad D_1 = 0 \qquad D_1 = 0$

$y_p = \frac{1}{81}\left[\frac{e^{2t}}{2^2-4} + \frac{4(1)}{-4} - \frac{4e^t}{1-4}\right] + \frac{1}{27}\left[\frac{4e^t}{1-4} - \frac{8(1)}{0-4}\right] + \frac{1(1)}{9\,(0-4)}$

$\Rightarrow \quad y_p = \frac{1}{81}\left[t\frac{e^{2t}}{2D_1} - 1 + \frac{4e^t}{3}\right] + \frac{1}{27}\left[-\frac{4e^t}{3} + 2\right] - \frac{1}{36}$

$D_1 = 2$

$y_p = \frac{1}{81}\left[\frac{e^{2t}t}{4} - 1 + \frac{4e^t}{3}\right] + \frac{1}{27}\left[2 - \frac{4e^t}{3}\right] - \frac{1}{36}$ //

$y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 e^{-2t} + c_2 e^{2t} + \frac{1}{81}\left[\frac{e^{2t}t}{4} - 1 + \frac{4e^t}{3}\right] + \frac{1}{27}\left[7 - \frac{4e^t}{3}\right] - \frac{1}{36}}$$

where, $e^t = (3x+2)$

$\log_e(3x+2) = t$

4. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

Ans $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$ $\quad \times$ by x^2

$x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 12\log x$

let, $\log_e x = t \Rightarrow e^t = x$

$x\,y' = (1)D_1 y$

$x^2 y'' = (1)(D_1)(D_1 - 1)y$

$\therefore \quad D_1(D_1-1)y + D_1 y = 12t$

$(D_1^2 - \cancel{D_1} + \cancel{D_1})y = 12t$

$D_1^2 y = 12t$

$f(m) = 0$

$\Rightarrow m^2 = 0$

$m = 0, 0$

$y_c = (c_1 + c_2 t)e^{0x} = \underline{\underline{c_1 + c_2 t}}$

$y_p = \dfrac{1}{f(D)}\phi(t) = \dfrac{1}{D_1^2} 12t$

$y_p = \dfrac{1}{D_1}\left(\int 12t\right)$

$y_p = \dfrac{1}{D_1}\left(\dfrac{12t^2}{2}\right)$

$y_p = \int 6t^2$

$y_p = \dfrac{6t^3}{3}$

$y_p = \underline{\underline{2t^3}}$

$\therefore \; y = y_c + y_p \Rightarrow \boxed{y = c_1 + c_2 t + 2t^3}$

where, $\log_e x = t$

Solution of differential equations by the method of variation of parameters

* The given DE of 2nd order is of the form $f(D)Y = \phi(x)$, we write the complementary function as $Y_c = C_1 Y_1(x) + C_2 Y_2(x)$

* Let us assume that the general solution is of the form $Y = AY_1 + BY_2$ to be the complete solution of given DE where A & B are functions of x.

* Obtain wronskian constant $W = Y_1 Y_2' - Y_2 Y_1'$

* Find the values of A & B using,

$$A = -\int \frac{y_2(x)\,\phi(x)}{W}\,dx + K_1$$

$$B = \int \frac{y_1(x)\,\phi(x)}{W}\,dx + K_2$$

* Substitute A & B in $Y = AY_1 + BY_2$, we get the complete solution.

Solve the following DE using method of variation of parameters

1. $Y'' + 3Y' + 2Y = e^{e^x}$

Ans

$$(D^2 + 3D + 2)Y = e^{e^x}$$

Auxillary equation -

$$f(m) = 0$$

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2$$

$$\Rightarrow Y_c = C_1 e^{-x} + C_2 e^{-2x}$$

$$Y_c = C_1 Y_1(x) + C_2 Y_2(x)$$

$$\therefore Y_1 = e^{-x} \ \& \ Y_2 = e^{-2x}$$

$\therefore \quad W = y_1 y_2' - y_2 y_1'$

$$W = e^{-x}(-2e^{-2x}) - e^{-2x}(-e^{-x})$$

$$W = -2e^{-3x} + e^{-3x}$$

$$W = -e^{-3x}$$

$\therefore \quad A = -\int \frac{y_2\,\phi(x)}{W}\,dx + K_1 \qquad B = \int \frac{y_1\,\phi(x)}{W}\,dx + K_2$

$$A = -\int \frac{e^{-2x}\,e^{e^x}}{-e^{-3x}}\,dx + K_1 \qquad B = \int \frac{e^{-x} e^{e^x}}{-e^{-3x}}\,dx + K_2$$

$$A = \int e^{x} e^{e^x}\,dx + K_1 \qquad B = \int e^{2x} e^{e^x}\,dx + K_2$$

$A = \int e^t\,dt + K_1$ | Put, $e^x = t$, $\Rightarrow e^x dx = dt$

$A = e^t + K_1$

$$\underline{\underline{A = e^{e^x} + K_1}}$$

$B = \int e^x e^{e^x} e^x\,dx + K_2$

$B = \int t\,e^t\,dt + K_2$ | Put, $e^x = t$, $e^x dx = dt$

$B = t e^t - (1)e^t + K_2$

$B = e^t(t-1) + K_2$

$$\underline{\underline{B = e^{e^x}(e^x - 1) + K_2}}$$

$\Rightarrow \quad Y = Ay_1 + By_2$

$$Y = (e^{e^x} + K_1)e^{-x} + \left(e^{e^x}(e^x - 1) + K_2\right)e^{-2x}$$

2. $y'' - 2y' + 2y = e^x \tan x$

Ans $\qquad (D^2 - 2D + 2)Y = e^x \tan x$

Auxillary equation -

$$f(m) = 0$$

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \frac{2 \pm i2}{2}$$

$$m = 1 \pm i$$

$$y_c = e^x(C_1 \cos x + C_2 \sin x)$$

$$y_c = \underline{\underline{C_1 e^x \cos x + C_2 e^x \sin x}}$$

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

$$y_1 = e^x \cos x \qquad y_2 = e^x \sin x$$

$$\therefore W = y_1 y_2' - y_2 y_1'$$

$$W = e^x \cos x \left(e^x \cos x + \sin x\, e^x \right) - e^x \sin x \left(e^x(-\sin x) + e^x \cos x \right)$$

$$W = e^{2x} \cos^2 x + \cancel{e^{2x} \sin x \cos x} + e^{2x} \sin^2 x - \cancel{e^{2x} \sin x \cos x}$$

$$W = e^{2x}$$

$$A = -\int \frac{y_2\, \phi(x)}{W}\, dx + K_1 \qquad B = \int \frac{y_1\, \phi(x)}{W}\, dx + K_2$$

$$= -\int \frac{\cancel{e^x} \sin x\, \cancel{e^x} \tan x}{\cancel{e^{2x}}}\, dx + K_1 \qquad = \int \frac{\cancel{e^x} \cos x\, \cancel{e^x} \tan x}{\cancel{e^{2x}}} + K_2$$

$$= -\int \sin x \frac{\sin x}{\cos x}\, dx + K_1 \qquad = \int \cancel{\cos x} \frac{\sin x}{\cancel{\cos x}}\, dx + K_2$$

$$= -\int \frac{1 - \cos^2 x}{\cos x}\, dx + K_1 \qquad B = -\cos x + K_2$$

$$= -\int \sec x - \cos x\, dx + K_1$$

$$A = -\log(\sec x + \tan x) + \sin x + K_1$$

$$\therefore \quad y = A y_1 + B y_2$$

$$\boxed{y = \left(-\log(\sec x + \tan x) + \sin x + K_1\right) e^x \cos x + \left(-\cos x + K_2\right) e^x \sin x}$$

3. $x^2 y'' + x y' - y = x^2 e^x$

Ans Put, $t = \log_e x \Rightarrow x = e^t$ $\cancel{\Rightarrow dx = e^t dt}$

$$x y' = D_1 y$$

$$x^2 y'' = D_1(D_1 - 1) y$$

$$\Rightarrow D_1(D_1 - 1) y + D_1 y - y = e^{2t} e^{e^t}$$

$$(D_1^2 - \cancel{D_1} + \cancel{D_1} - 1)y = e^{2t} e^{e^t}$$

$$(D_1^2 - 1)y = e^{2t} e^{e^t}$$

Auxiliary equation - $f(m) = 0$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$\underline{y_c = c_1 e^t + c_2 e^{-t}}$$

$$\therefore y_c = c_1 y_1(x) + c_2 y_2(x)$$

$$y_1 = e^t \qquad y_2 = e^{-t}$$

$$W = y_1 y_2' - y_2 y_1'$$

$$= e^t(-e^{-t}) - e^{-t}(e^t)$$

$$= -1 - 1$$

$$= -2$$

$$A = -\int \frac{y_2 \phi(t)}{W} dt + K_1$$

$$\cancel{= -\int \frac{e^{-t} e^{2t} e^{e^t}}{-2} dt + K_1}$$

$$\cancel{\frac{1}{2}\int e^{2t} e^{e^t} dt + K_1}$$

$$A = -\int \frac{e^{-t} e^{2t} e^{e^t}}{-2} dt + K_1$$

$$A = +\frac{1}{2}\int e^t e^{e^t} dt + K_1$$

$$e^t = P$$

$$e^t dt = dP$$

$$A = \frac{1}{2}\int e^P dP + K_1$$

$$A = \frac{e^{e^t}}{2} + K_1$$

$$B = \int \frac{y_1 \phi(t) dt}{W} + K_2$$

$$= \int \frac{e^t e^{2t} e^{e^t}}{-2} dt + K_2$$

$$= -\frac{1}{2}\int e^{3t} e^{e^t} dt + K_2$$

$$e^t = P$$

$$e^t dt = dP$$

$$= -\frac{1}{2}\int e^{2t} e^{e^t} e^t dt + K_2$$

$$= -\frac{1}{2}\int P^2 e^P dP + K_2$$

$$= -\frac{1}{2}\left[P^2 e^P - 2Pe^P + 2e^P\right] + K_2$$

$$= -\frac{P^2 e^P}{2} + Pe^P - e^P + K_2$$

$$= -\frac{e^{2t} e^{e^t}}{2} + e^t e^{e^t} - e^{e^t} + K_2$$

$$\therefore y = Ay_1 + By_2$$

$$\boxed{y = \left(\frac{e^{e^t}}{2} + K_1\right)e^t + \left(-\frac{e^{2t} e^{e^t}}{2} + e^{e^t} e^t - e^{e^t} + K_2\right)e^{-t}}$$

$$y = \left(\frac{e^x}{2} + K_1\right) x + \left(-\frac{x^2 e^x}{2} + e^x x - e^x + K_2\right)\left(\frac{1}{x}\right)$$

$$\boxed{y = \frac{xe^x}{2} + K_1 x - \frac{x}{2} e^x + e^x - \frac{e^x}{x} + \frac{K_2}{x}}$$

$$\therefore\ y = \underbrace{c_1 e^t + c_2 e^{-t}}_{y_c} + \underbrace{\frac{e^{e^t} e^t}{2} + e^{-t} e^{e^t}\left(e^t - 1 - \frac{e^{2t}}{2}\right)}_{y_p} \quad \text{where } e^t = x$$

4. $y'' - 3y' + 2y = \frac{e^x}{1+e^x}$

Ans

$$f(m) = 0$$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$$y_c = c_1 e^x + c_2 e^{2x}$$

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

$$y_1 = e^x \qquad y_2 = e^{2x}$$

$$\therefore\ W = y_1 y_2' - y_2 y_1'$$

$$W = e^x(2e^{2x}) - e^{2x}(e^x)$$

$$W = 2e^{3x} - e^{3x}$$

$$W = e^{3x}$$

$$A = -\int \frac{y_2(x)\,\phi(x)}{W}\,dx \qquad B = \int \frac{y_1(x)\,\phi(x)}{W}\,dx$$

$$A = -\int \frac{\cancel{e^{2x}}\, e^x}{\cancel{(e^{3x})}\, 1+e^x}\,dx \qquad B = \int \frac{\cancel{e^x} \cdot e^x}{(e^{3x})(1+e^x)}\,dx$$

$$A = -\int \frac{1}{1+e^x}\,dx \qquad B = \int \frac{e^{-x}}{1+e^x}\,dx$$

$$A = -\int \frac{1}{e^x\left(\frac{1}{e^x}+1\right)}\,dx \qquad B = \int \frac{e^{-x}\ \text{[redacted]}}{1+e^x}\,dx$$

$$A = -\int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$$A = -\left(-\log(e^{-x}+1)\right) + K_1$$

$$A = \log(e^{-x}+1) + K_1$$

$$B = \int \frac{1}{e^x(1+e^x)}\,dx$$

$$B = \int \frac{1}{e^x} - \frac{1}{1+e^x}\,dx$$

$$B = \int e^{-x}\,dx - \int \frac{1}{1+e^x}\,dx$$

$$B = \frac{e^{-x}}{-1} - \left(-\log(e^{-x}+1)\right)$$

$$B = -e^{-x} + \log(e^{-x}+1) + K_2$$

$$\therefore\ y = Ay_1 + By_2$$

$$y = \log(e^{-x}+1) + K_1 - e^{-x} + \log(e^{-x}+1) + K_2$$

5. If $D \equiv \frac{d}{dx}$ & $X = X(x)$, then P.T $\frac{1}{(D-a)}X = e^{ax}\int X e^{-ax}\,dx$

a → constant

Ans let,

$$\frac{1}{(D-a)}X = y$$

$$X(x) = (D-a)y$$

$$X(x) = \frac{dy}{dx} - ay$$

$$\frac{dy}{dx} - ay = X(x)$$

$$\Rightarrow\ I.F = e^{\int P\,dx} = e^{\int -a\,dx} = e^{-ax}$$

$$\therefore\ y(e^{-ax}) = \int X(x)e^{-ax}\,dx$$

$$y e^{-ax} = \int X(x) e^{-ax}\,dx$$

$$y = e^{ax}\int X e^{-ax}\,dx$$

Hence proved

6. If $D = \frac{d}{dx}$ & $X = X(x)$ then PT. $\frac{1}{D+a}X = e^{-ax}\int X e^{ax}\,dx$

a → constant

Ans let,

$$\frac{1}{(D+a)}X = y$$

$$X(x) = (D+a)y$$

$$X(x) = \frac{dy}{dx} + ay$$

$$\frac{dy}{dx} + ay = X(x)$$

$\Rightarrow$ I.F $= e^{\int a\,dx} = e$

$$ye^{ax} = \int X(x)\, e^{ax}\, dx$$

$$y = e^{-ax} \int X e^{ax} dx$$

Hence proved

Initial & boundary value problems

Solve the following

1. $y'' - y = 0 \quad ; \quad y(0) = 1 \ , \ y(1) = 2 - e$

Ans $(D^2 - 1)\, y = 0$

$\Rightarrow \quad m^2 - 1 = 0$

$m = \pm 1$

$y_c = c_1 e^{x} + c_2 e^{-x}$

Given, when $x = 0$, $y = 1$ & $x = 1$, $y = 2 - e$

$\therefore \quad 1 = c_1(1) + c_2(1) \qquad 2 - e = c_1 e^{1} + c_2 e^{-1}$

$1 = c_1 + c_2 \qquad 2 - e = e c_1 + \frac{c_2}{e}$

$\therefore \quad c_1 + c_2 = 1 \quad \times e$

$c_1 e + \frac{c_2}{e} = 2 - e \quad \times 1$

$\cancel{c_1 e} + c_2 e = e$

$\cancel{c_1 e} + \frac{c_2}{e} = 2 - e$

$(-) \quad (-) \quad (-)$

$$c_2 e - \frac{c_2}{e} = e - 2 + e$$

$$\frac{c_2 e^2 - c_2}{e} = 2e - 2$$

$$c_2 e^2 - c_2 = 2e^2 - 2e$$

$$c_2(e^2 - 1) = 2e(e - 1)$$

$$c_2 = \frac{2e(e-1)}{e^2 - 1} \Rightarrow \boxed{c_2 = \frac{2e}{e+1}}$$

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$$c_1 = 1 - \frac{2e}{e+1}$$

$$c_1 = \frac{e+1-2e}{e+1}$$

$$\boxed{c_1 = \frac{1-e}{e+1}}$$

$$\therefore \quad y_c = \left(\frac{1-e}{e+1}\right)e^{x} + \left(\frac{2e}{e+1}\right)e^{-x}$$

Boundary value problem -

The DE in which conditions are specified for a given set of 'n' values of the independent value of x is called a boundary value problem.

If $y = y(x)$ is the general solution of given DE, then the corresponding boundary conditions will be of the form $y_1 = y(x_1)$, $y_2 = y(x_2) \ldots\ldots y_n = y(x_n)$.

Solve the following BVP

1. $y'' + y = 0$, $y(0) = 2$, $y\left(\frac{\pi}{2}\right) = -2$, ~~$(D^2+1)y = 0$~~

Ans $(D^2+1)y = 0$

Auxillary equation -

$$f(m) = 0$$
$$m^2 + 1 = 0$$
$$m^2 = -1$$
$$m = \pm i$$

$$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$$

$$y = y_c + y_p \Rightarrow y = y_c$$

$$y = c_1 \cos x + c_2 \sin x$$

when, $x = 0$, $y = 2$ $\qquad$ $x = \frac{\pi}{2}$, $y = -2$

$\Rightarrow 2 = c_1$ $\qquad$ $\Rightarrow -2 = c_2$

$$\therefore \quad y = 2\cos x - 2\sin x$$

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2. $x^2y'' - 2y + x = 0, \quad y(2) = y(3) = 0$

Ans

$x^2y'' - 2y = -x$

Put $\log x = t \Rightarrow e^t = x$

$xy' = D_1 y$

$x^2y'' = D_1(D_1-1)y$

$(D_1^2 - D_1)y - 2y = -x$

$(D_1^2 - D_1 - 2)y = -x \Rightarrow (D_1^2 - D_1 - 2)y = -e^t$

Auxillary equation -

$f(m) = 0$

$m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$m = 2, -1$

$y_c = c_1 e^{2t} + c_2 e^{-t}$

$y_p = \frac{1}{f(D)} \phi(t) = \frac{(-e^t)}{D_1^2 - D_1 - 2}$

$y_p = \frac{-e^t}{D_1^2 - D_1 - 2}$

$D_1 = 1$

$\Rightarrow y_p = \frac{-e^t}{1-1-2}$

$y_p = \frac{e^t}{2}$

$\therefore \quad y = y_c + y_p$

$y = c_1 e^{2t} + c_2 e^{-t} + \frac{e^t}{2}$

$y = c_1 x^2 + \frac{c_2}{x} + \frac{x}{2}$

when, $x = 2, y = 0$ $\qquad$ $x = 3, y = 0$

$\Rightarrow 0 = c_1(4) + \frac{c_2}{2} + \frac{2}{2}$ $\qquad$ $0 = 9c_1 + \frac{c_2}{3} + \frac{3}{2}$

$8c_1 + c_2 = -2$ $\qquad$ $-\frac{3}{2} = \frac{27c_1 + c_2}{3}$

$\Rightarrow 27c_1 + c_2 = -\frac{9}{2}$

Initial value

The DE

of the i

problem.

If

boundary c

$y''(x_0) =$

Solve the

1. $y'' -$

Ans

when

$$\therefore -2-8C_1 = \frac{-9}{2} - 27C_1$$

$$19C_1 = \frac{-9}{2}+2$$

$$19C_1 = \frac{-5}{2}$$

$$\boxed{C_1 = \frac{-5}{38}}$$

$$\therefore C_2 = -2 - \cancel{8}^{4}\left(\frac{-5}{\cancel{38}_{19}}\right)$$

$$C_2 = -2+\frac{20}{19}$$

$$\boxed{C_2 = \frac{-18}{19}}$$

$$\therefore y = \frac{-5}{38}x^2 + \left(\frac{-18}{19}\right)\left(\frac{1}{x}\right)+\frac{x}{2}$$

Initial value problem -

The DE in which the conditions are specified at a single value of the independent variable say $x = x_0$ is called an initial value problem.

If $y = y(x)$ is the solution of given DE and the corresponding boundary condition will be of the form, $y(x_0) = y_0$, $y'(x_0) = y_1$, $y''(x_0) = y_2$, ...

Solve the following IVP

1. $y''-y=0$, $y(0)=3$; $y'(0)=-3$

Ans Auxillary equation -

$$f(m)=0$$

$$(m^2-1)=0$$

$$m=\pm 1$$

$$y_c = C_1e^x + C_2e^{-x}$$

$$y_p = 0$$

$$\Rightarrow y = C_1e^x + C_2e^{-x}$$

$$y' = C_1e^x - C_2e^{-x}$$

when $x=0$, $y=3$

$$\Rightarrow 3 = C_1+C_2$$

when $x=0$, $y'=-3$

$$\Rightarrow -3 = C_1-C_2$$

$c_1+c_2=3$
$c_1-c_2=-3$
$2c_1=0$
$c_1=0$

$\Rightarrow c_2=3$

$\therefore \boxed{y=3e^{-x}}$

2. $y'''+2y''-y'-2y=0$, $y(0)=0$, $y'(0)=0$, $y''(0)=6$

Ans
$$D^3y+2D^2y-Dy-2y=0$$
$$(D^3+2D^2-D-2)y=0$$

Auxiliary equation -
$$m^3+2m^2-m-2=0$$
$$m^2(m+2)-1(m+2)=0$$
$$m^2=+1, \ m=-2$$
$$m=\pm1, -2$$
$$\therefore \ y_c=c_1e^{x}+c_2e^{-x}+c_3e^{-2x}$$
$$y_p=0$$
$$\Rightarrow y=c_1e^{x}+c_2e^{-x}+c_3e^{-2x}$$
$$y'=c_1e^{x}-c_2e^{-x}-2c_3e^{-2x}$$
$$y''=c_1e^{x}+c_2e^{-x}+4c_3e^{-2x}$$

when $x=0$, $y=0$ $\qquad$ $x=0$, $y'=0$ $\qquad$ $x=0$, $y''=6$

$0=c_1+c_2+c_3$ $\qquad$ $0=c_1-c_2-2c_3$ $\qquad$ $6=c_1+c_2+4c_3$

$\Rightarrow 6=-c_3+4c_3$

$c_3=2$

$c_1+c_2=-2$
$c_1-c_2=4$
$2c_1=2$
$c_1=1$

$\Rightarrow 1+c_2+2=0$
$c_2=-3$

$\therefore \ y=e^{x}-3e^{-x}+2e^{-2x}$

3. $(D^2+D)y=2+2x+x^2$, $y(0)=8$, $y'(0)=-1$

Ans
$$m^2+m=0$$
$$m(m+1)=0$$
$$m=0, -1$$

$y_p=$

when

4. A

are

$i=$

Ans

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$$y_c = c_1 e^{0x} + c_2 e^{-x}$$

$$y_c = c_1 + c_2 e^{-x}$$

$$y_p = \frac{1}{f(D)} \phi(x) = \frac{2+2x+x^2}{D^2+D}$$

$$D+D^2 \big) \; x^2+2x+2 \; \big(\frac{x^3}{3} + 2x$$

$$\frac{x^2 + 2x}{(-)\quad(-)}$$

$$2$$

$$\frac{2+0}{(-)}$$

$$0$$

$$\Rightarrow \quad y_p = \frac{x^3}{3} + 2x$$

$$\therefore \; y = y_c + y_p$$

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 2x$$

$$y' = -c_2 e^{-x} + x^2 + 2$$

when, $x = 0$, $y = 8$ $\qquad$ $x = 0$, $y' = -1$

$$8 = c_1 + c_2 \qquad -1 = -c_2 + 2$$

$$c_2 = 3 \Rightarrow c_1 = 5$$

$$\therefore \; y = 5 + 3e^{-x} + \frac{x^3}{3} + 2x$$

4. A resistor $R = 10\Omega$, an inductor of $L = 2H$ & a battery of 'E' V are connected in series. At $t = 0$, the switch is closed, current $i = 0$. Find the current i for $t > 0$ where $E = 40V$.

Ans Differential equation for R-L circuit -

$$iR + L\frac{di}{dt} = E$$

$$2\frac{di}{dt} + 10i = 40$$

$$\frac{di}{dt} + 5i = 20$$

$$\Rightarrow (D+5)i = 20$$

Auxillary equation -

$$f(m) = 0$$

$$m + 5 = 0$$

$$m = -5$$

$i_c = C_1 e^{-5t}$

$i_P = \frac{1}{f(D)} \phi(t) = \frac{(20)}{D+5}$

$i_P = 20 \frac{e^{0t}}{D+5}$

$D=0$

$\Rightarrow i_P = \frac{20 e^{0t}}{5}$

$i_P = 4$

$\therefore i = i_c + i_P$

$i = C_1 e^{-5t} + 4$

At $t=0$, $i=0$

$\Rightarrow 0 = C_1 e^{0} + 4$

$C_1 = -4$

$\Rightarrow \boxed{i = -4e^{-5t} + 4}$

5. A condenser a capacity 'C' discharge through an inductance 'L' and resistance 'R' in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$. Given $L = 0.25H$, $R = 250\Omega$, $C = 2\times10^{-6}F$ and when $t=0$, $q = 0.002C$, and the current $i = \frac{dq}{dt} = 0$. Obtain the value of q in terms of t

Ans

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$0.25\frac{d^2q}{dt^2} + 250\frac{dq}{dt} + \frac{q}{2\times10^{-6}} = 0$$

$$0.25\frac{d^2q}{dt^2} + 250\frac{dq}{dt} + 5\times10^{5}q = 0$$

$$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + 2\times10^{6}q = 0$$

$$\Rightarrow (D^2 + 1000D + 2\times10^6)q = 0$$

Auxillary equation -

$f(m) = 0$

$\Rightarrow m^2 + 1000m + 2\times10^6 = 0$

$$m = -500 \pm i\, 1322.9$$

$$\Rightarrow m = -500 \pm 1323 i$$

$$q_c = e^{-500t}(c_1 \cos 1323t + c_2 \sin 1323t)$$

$q_p = 0$

$$\Rightarrow q = q_c + q_p$$

$$q = e^{-500t}(c_1 \cos 1323t + c_2 \sin 1323t)$$

when: $t = 0$, $q = 0.002$.

$$0.002 = e^0(c_1 + 0)$$

$$c_1 = 0.002$$

$$\frac{dq}{dt} = e^{-500t}\left(c_1(1323)(-\sin 1323t) + c_2(1323)\cos 1323t\right)$$
$$+ (c_1 \cos 1323t + c_2 \sin 1323t)(-500 e^{-500t})$$

when $t = 0$, $\frac{dq}{dt} = 0$

$$\Rightarrow 0 = 1(1323 c_2) + (c_1)(-500)$$

$$c_2 = \frac{500 c_1}{1323}$$

$$c_2 = 7.5585 \times 10^{-4}$$

$$\therefore q = e^{-500t}(0.002 \cos 1323t + 7.5585 \times 10^{-4} \sin 1323t)$$

6. The rotating displacement 'u' in a rotating disc has a distance 'r' from the axis is given by- $r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u + Kr^3 = 0$

$K \to$ Constant. Solve the equation under the condition $u(0) = u(a) = 0$

Ans
$$r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -Kr^3$$

Put $\log_e r = t \Rightarrow e^t = r$

$$r\frac{du}{dr} = D_1 u$$

$$r^2 \frac{d^2u}{dr^2} = D_1(D_1 - 1)u$$

$$\Rightarrow (D_1^2 - D_1 + D_1 - 1)u = -K e^{3t}$$

$(D_1^2-1)u = -Ke^{3t}$

$\Rightarrow \quad f(m) = 0$

$m^2 - 1 = 0$

$m = \pm 1$

$U_c = C_1 e^{t} + C_2 e^{-t}$

$U_p = \frac{1}{f(D)} \phi(t) = \frac{1}{D_1^2-1}(-Ke^{3t})$

$U_p = -K \frac{e^{3t}}{D_1^2-1}$

$D_1 = 3$

$U_p = \frac{-Ke^{3t}}{8}$

$\therefore \quad U = U_c + U_p$

$U = C_1 e^{t} + C_2 e^{-t} - \frac{Ke^{3t}}{8}$

when $t=0, \ U=0$ $\qquad$ $t=a, \ U=0$

$\Rightarrow 0 = C_1 + C_2 - \frac{K}{8}$ $\qquad$ $0 = C_1 e^{a} + C_2 e^{-a} - \frac{Ke^{3a}}{8}$

$C_1 + C_2 = \frac{K}{8}$ $\qquad$ $C_1 e^{a} + C_2 e^{-a} = \frac{Ke^{3a}}{8}$

$U = C_1 r + \frac{C_2}{r} - \frac{Kr^3}{8}$

$\Rightarrow \ Ur = C_1 r^2 + C_2 - \frac{Kr^4}{8}$

when $r=0, \ U=0$ $\qquad$ when $r=a, \ U=0$

$0 = C_2$ $\qquad$ $0 = C_1 a^2 + 0 - \frac{Ka^4}{8}$

$\boxed{C_1 = \frac{Ka^2}{8}}$

$\Rightarrow \quad U = \frac{+Ka^2}{8} r + 0 - \frac{Kr^4}{8}$

$$\boxed{U = \frac{Ka^2}{8} r - \frac{Kr^4}{8}}$$

7. A particle

$\frac{d^2x}{dt^2} + 6$

with init

Ans

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7. A particle moves along the x-axis according to the law - $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$. If the particle started at $x=0$ ($t=0$) with initial velocity $\frac{dx}{dt} = 12 ft/s$ leftwards. Determine 'x' in terms of 't'

Ans
$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$$

$$(D^2+6D+25)x = 0$$

$$\Rightarrow \phi(m) = 0$$

$$m^2+6m+25 = 0$$

$$m = \frac{-6 \pm \sqrt{36-100}}{2} = \frac{-6 \pm i8}{2} = -3 \pm i4$$

$$x_c = e^{-3t}(c_1 \cos 4t + c_2 \sin 4t)$$

$$x_p = \frac{1}{\phi(D)} \phi(t) = 0 \qquad \Rightarrow x = x_c + x_p$$

$$x = e^{-3t}(c_1 \cos 4t + c_2 \sin 4t)$$

when $t=0$, $x=0$

$$\Rightarrow 0 = (c_1 + 0)$$

$$c_1 = 0$$

$$\frac{dx}{dt} = e^{-3t}(-4c_1 \sin 4t + 4c_2 \cos 4t) + (c_1 \cos 4t + c_2 \sin 4t)(-3e^{-3t})$$

At $t=0$, $\frac{dx}{dt} = -12$

$$-12 = 1(0+4c_2) + (0+0)(-3)$$

$$-12 = 4c_2$$

$$\boxed{c_2 = -3}$$

$$\therefore \boxed{x = e^{-3t}(-3\sin 4t)}$$