ATME COLLEGE OF ENGINEERING

13th KM Stone, Bannur Road, Mysore - 560 028

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

(ACADEMIC YEAR 2020-21)

NOTES OF LESSON

SUBJECT: DISCRETE MATHEMATICAL STRUCTURES

SUB CODE: 18CS36

SEMESTER: III

INSTITUTIONAL MISSION AND VISION

Objectives

- To provide quality education and groom top-notch professionals, entrepreneurs and leaders for different fields of engineering, technology and management.
- To open a Training-R & D-Design-Consultancy cell in each department, gradually introduce doctoral and postdoctoral programs, encourage basic & applied research in areas of social relevance, and develop the institute as a center ofexcellence.
- To develop academic, professional and financial alliances with the industry as well as the academia at national and transnationallevels.
- To develop academic, professional and financial alliances with the industry as well as the academia at national and transnationallevels.
- To cultivate strong community relationships and involve the students and the staff in local community service.
- To constantly enhance the value of the educational inputs with the participation of students, faculty, parents andindustry.

Vision

- Development of academically excellent, culturally vibrant, socially responsible and globally competent humanresources.

Mission

- To keep pace with advancements in knowledge and make the students competitive and capable at the globallevel.
- To create an environment for the students to acquire the right physical, intellectual, emotional and moral foundations and shine as torch bearers of tomorrow'ssociety.
- To strive to attain ever-higher benchmarks of educationalexcellence.

Department of Computer Science & Engineering

Vision of the Department

- To develop highly talented individuals in Computer Science and Engineering to deal with real world challenges in industry, education, research and society.

Mission of the Department

- To inculcate professional behavior, strong ethical values, innovative research capabilities and leadership abilities in the young minds & to provide a teaching environment that emphasizes depth, originality and criticalthinking.

- Motivate students to put their thoughts and ideas adoptable by industry or to pursue higher studies leading toresearch.

Program Educational Objectives (PEO'S):

Program Educational Objectives (PEO'S):

1. Empower students with a strong basis in the mathematical, scientific and engineering fundamentals to solve computational problems and to prepare them for employment, higher learning and R&D.

2. Gain technical knowledge, skills and awareness of current technologies of computer science engineering and to develop an ability to design and provide novel engineering solutions for software/hardware problems through entrepreneurial skills.

3. Exposure to emerging technologies and work in teams on interdisciplinary projects with effective communication skills and leadership qualities.

4. Ability to function ethically and responsibly in a rapidly changing environment by applying innovative ideas in the latest technology, to become effective professionals in Computer Science to bear a life-long career in related areas.

Program Specific Outcomes (PSOs)

1. Demonstrate understanding of the principles and working of the hardware and software aspects of Embedded Systems.
2. Use professional Engineering practices, strategies and tactics for the development, implementation and maintenance of software.
3. Provide effective and efficient real time solutions using acquired knowledge in various domains.

DISCRETE MATHEMATICAL STRUCTURES

[As per Choice Based Credit System (CBCS) scheme]
(Effective from the academic year 2018 -2019)
SEMESTER – III

Subject Code	**18CS36**	**IA Marks**	**40**
Number of Lecture Hours/Week	**04**	**Exam Marks**	**60**
Total Number of Lecture Hours	**50**	**Exam Hours**	**03**

CREDITS – 04

Course objectives: This course will enable students to

1. Provide theoretical foundations of computer science to perceive other courses in the programme.
2. Illustrate applications of discrete structures: logic, relations, functions, set theory and counting
3. Describe different mathematical proof techniques
4. Illustrate the use of graph theory in computer science

Module -1	**Teaching Hours**
Fundamentals of Logic: Basic Connectives and Truth Tables, Logic Equivalence – The Laws of Logic, Logical Implication – Rules of Inference. Fundamentals of Logic contd.: The Use of Quantifiers, Quantifiers, Definitions and the Proofs of Theorems.	**10Hours**
Module -2	
Properties of the Integers: Mathematical Induction, The Well Ordering Principle – Mathematical Induction. **Fundamental Principles of Counting:** The Rules of Sum and Product, Permutations, Combinations – The Binomial Theorem, Combinations with Repetition.	**10 Hours**
Module – 3	
Relations and Functions: Relations and Functions: Cartesian Products and Relations, Functions – Plain and One-to-One, Onto Functions. The Pigeon-hole Principle, Function Composition and Inverse Functions. Properties of Relations, Computer Recognition – Zero-One Matrices and Directed Graphs, Partial Orders – Hasse Diagrams, Equivalence Relations and Partitions.	**10 Hours**

Module-4	
The Principle of Inclusion and Exclusion: The Principle of Inclusion and Exclusion, Generalizations of the Principle, Derangements – Nothing is in its Right Place, Rook Polynomials. **Recurrence Relations:** First Order Linear Recurrence Relation, The Second Order Linear Homogeneous Recurrence Relation with Constant Coefficients.	**10 Hours**
Module-5	
Introduction to Graph Theory: Definitions and Examples, Sub graphs, Complements, and Graph Isomorphism, Vertex Degree, Euler Trails and Circuits , Trees: Definitions, Properties, and Examples, Routed Trees, Trees and Sorting, Weighted Trees and Prefix Codes	**10 Hours**
Course outcomes:	
After studying this course, students will be able to: 1. Use propositional and predicate logic in knowledge representation and truth verification. 2. Demonstrate the application of discrete structures in different fields of computer science. 3. Solve problems using recurrence relations and generating functions. 4. Application of different mathematical proofs techniques in proving theorems in the courses. 5. Compare graphs, trees and their applications.	

Text Books: Grimaldi: Discrete and Combinatorial Mathematics, , 5th Edition, Pearson Education. 2004.
Refernce Books:
1. Basavaraj S Anami and Venakanna S Madalli: Discrete Mathematics – A Concept based approach, Universities Press,2016 2. Kenneth H. Rosen: Discrete Mathematics and its Applications, 6th Edition, McGrawHill,2007. 3. JayantGanguly: A Treatise on Discrete Mathematical Structures, Sanguine-Pearson,2010. 4. D.S. Malik and M.K. Sen: Discrete Mathematical Structures: Theory and Applications, Thomson, 2004. 5. Thomas Koshy: Discrete Mathematics with Applications, Elsevier, 2005, Reprint2008.

MODULE 1

Contents

Module 1:

1.1. Introduction

This module consists of basic conncetvities where we used to write the truth tables. Further this module consists of laws of logic where it consists of laws using that we prove the logical equivalence. It also consists of logical implications; similar to the laws in logics we have inference rules. By using this we solve the inference problems.

Quantifiers are the statement which consists of for all and for some values. By using quantifiers we will solve the truthness of the compound statement. Finally we conclude with the proofs of theorems with direct proof, indirect proof and proof by contradiction.

Objective

- Understand the concept of basic connectvities and truthtable.
- Understand the concept of application of laws for solving aproblem.

Basic connectivities and TruthTables

Introduction: Logic is being used as a tool in a number of situations for a variety of reasons. Logic is found to be extremely useful in decision taking problems. In computer science and engineering, knowledge of logic is essential in in the following field:

- Analysis of algorithms andimplementation
- Development of algorithms into a structured program in a programminglanguage.
- A material of logic theory forms a basis in theoretical computer science such as Artificial Intelligence, Fuzzy logic, functioning of expert systemsetc.

Many times while working on a project after the beginning steps, we always have a doubt regarding whether the direction followed is right or wrong? Are we doing the job correctly? Whether the decisions taken are correct or incorrect etc? It is here plays an important role;
Using which one can solve a problem with lot of confidence and satisfaction. With these few remarks, the next sections introduce formal symbolic logic. .

A proposition is denoted. by using lowercase letters such as p, q, r, s etc. Every proposition has exactly one of the two truth values; either true or false. When it is true, the same **Basic Terminologies of logic:**

A Proposition is a declarative sentence written in some language, usually English, which is either true or false, but not both (i.e. true and false) at the same time. It is also referred as a simple statement or primitive proposition or an atomic statement.

is assigned a symbol **"T"or"1".** If the primitive statement is false, it is assigned the symbol **–F" or "0‖**. In view of this, a proposition may be defined as a pair either **(p, T) or (p, F).**

This situation may be compared to what we discussed earlier in set theory. It is seen that with respect to a set A and an element x, there are only two options; either x is an element of A or x need not be an element of A. There too we used the numbers 1 and 0 to describe the situation.

The following are examples ofpropositions:

p: Dr. Manmohan Singh is the president of India.

q: Mumbai is the financial capital of India

r: Bangalore is the silicon valley of India.

s:3+3=5

t: Dr. Abdul Kalam was awarded BharathRathna

- x+3 is aninteger.
- Please comein!
- Are u alright? Complete work todayitself
- What are youdoing?
- What a beautifulevening!

These are not considered as propositions. Because, x + 3 is an integer cannot be a proposition, as x is not specified. A statement of this kind is called an open statement. the statement gets a meaning only when x is assigned a value chosen from the universe of discourse of the problem considered. Others are either commands, or enquiries, or exclamatory sentences. These are not referred to as non-propositions.

Discussion of various logical connectives

Negation: Let ***p*** be a proposition. It is not the case that ***p*** is called as negation of the ***p***.

Simply, it is NOT ***p*** . This is denoted by the symbol Ø***p***. This situation can well be explained by using a truth table which is shown below:

P	Ø***p***	***p***	Ø***p***
T	F	1	0
F	T	0	1

Disjunction or logical OR:Let ***p*** **a**nd ***q*** be simple propositions. A proposition obtainedby combining***p*** and ***q*** using logical OR is called ***p***disjunction ***q*** . It is denoted by ***p*** U***q***. The truthvalue of ***p***U***q*** is false onlywhenboth ***p*** and ***q*** are false, otherwise, ***p*** U***q*** is a true proposition. This is explained in the truth table givenbelow.

p	***q***	***p*** U***q***
T	T	T
T	F	T
F	T	T
F	F	F

Note: Logical operator is similar to the union operator in the case of set theory.

Conjunction

Let ***p*** and***q*** besimple propositions. A proposition obtained by combining***p*** and ***q*** using logical AND is called ***p***conjunction ***q*** . It is denoted by ***p*** ∩***q*** . The truth value of ***p*** ∩ ***q*** is true only when both ***p*** and***q*** are true otherwise ***p***∩***q*** is a falseproposition. The same is explained in the truth table given below.

p	*q*	*p* ∩*q*
T	T	T
T	F	F
F	T	F
F	F	F

Note: Logical operator is similar to the intersection operator in the case of set theory. Also, logical OR and logical AND have dual characteristics. Thus, conjunction and disjunction are examples of dual operators.

<u>One sided implication or logical IMPLICATION or IF THEN</u>:

Let ***p*** and***q*** be simple propositions. A proposition obtained by combining ***p*** and ***q*** using logical connective, namely, one sided IMPLICATION Called ***p***implies ***q*** . It is denoted by ***p→q*** . The truth value of ***p →q*** is false only when ***p*** is true but ***q*** is a false proposition. In all other instances, ***p →q*** is a true statement. The truth table is given below.

P	q	*p* → *q*
0	0	1
0	1	1
1	0	0
1	1	1

Few examples:

- If 3+3=7, then Sunday is a Christmas is aholiday.
- If 3+2 = 5, then sun raises in theeast.

<u>Bi-conditional or two sided implication OR IF AND ONLY IF or IFF</u>:

Let ***p*** and ***q*** be two simple propositions. ***p***Bi-conditional ***q*** is a compound proposition denoted by the symbol ***p q*** . he truth value of ***p q*** is true only when both ***p*** and ***q*** are assigned the same truth value; otherwise, its truth value is false.

p	***q***	***P<->q***
T	T	T
T	F	F
F	T	T
F	F	F

•

Note: Here, p is necessary and also sufficient for q. p bi-conditional q is to be read as p if and only if q.

<u>Exclusive logical OR:</u> Consider two propositions ,say, ***p*** and ***q***.Then ***p*** exclusive or ***q*** isa compound proposition whose true value is false when both ***p*** and ***q*** have the same truth values. Otherwise, the truth value is true. It is denoted by ***p***$\underline{\cup}$q. The truth table is given below.

p	***q***	***p***$\underline{\cup}$***q***
T	T	F
T	F	T
F	T	T
F	F	F

<u>Note:</u> This logical operator is similar to the symmetric difference operation for sets.

A proposition obtained by combining many logical variables(i.e.propositions when not specified) using a number of logical connectives, and, containing proper parentheses is called a statement formula or simply a statement. For example, the following are statementformulas:

- ***p*** U[(***q*** →***r***)UØ***q***]
- (***p*** U***q***) →[(***q*** →***r***) U(Ø***q***)

Hence, the truth table of the formula willcontain 2^n rows.

What is meant by Tautology? Consider a statement,say

A : ***A*** (***p***$_1$,***p***$_2$,***p***$_3$, . . . ***p***$_k$, ,U,Ú, ® , , Ø) . If for each of the 2^koptions, truth value of **A** turns out to be true, then **A** is called a **tautology**, or as universally accepted formula, or as universally valid formula, Thus, if a statement formula **A** is always true, then it said to be a tautology. Suppose, the truth value of **A** is always false, then it is called an **absurdity, or contradiction**, or universally invalid formula. On the other hand, if the truth value of **A** is some times true and at others false, then it is called**contingency**.

Illustrative examples: By constructing the truth table of the following, determine which one is a tautology? Which one is a contradiction? Which is one a contingency?

1. (***p*** →***q***)U(Ø***p***Ú***q***)
2. (***p*** →***q***)U(Ø***q***→Ø***p***)

Solution to problem number 1 (***p***→***q***)U(Ø***p***U***q***)

p	***q***	***A*** : ***p*** →***q***	Ø***p***	***B*** : Ø***p***U***q***	***AB***
T	T	**T**	F	**T**	**T**
T	F	**F**	F	**F**	**T**
F	T	**T**	T	**T**	**T**
F	F	**T**	T	**T**	**T**

Therefore, (***p***→***q***)U(Ø***p***Ú***q***) is a Tautology

Solution to problemnumber:2. $(p \to q) U (Øq \to Øp)$

p	q	$A : p \to q$	$Øp$	$Øq$	$B : Øq \to Øp$	AB
T	T	**T**	F	F	**T**	**T**
T	F	**F**	F	T	**F**	**T**
F	T	**T**	T	F	**T**	**T**
F	F	**T**	T	T	**T**	**T**

Tutorial on fundamentals of logic

Let ***p*** , ***q*** , ***r*** , ***s*** denote the following statements

p : I finish writing my computer program before lunch
q : I shall play tennis this afternoon
r : The sun is shining
s : The humidity is low

1. Determine the truth value of each of thefollowing:

(i) If 3 + 4 = 12, then 3 + 2 = 6.(True)

(ii) If 3 + 3 = 6, then 3 + 4 = 9.(False)

(iii) If George Bush was the third president of USA, then 2 + 3 = 5.(True)

2. Rewrite the following statements as an implication in the form if . . . Then. .form.

(a) For practicing her serve daily is a sufficient condition for Ms. SaniaMirzato have a good chance of winning the Australian Open tennistournament.

(b) Fix my air-conditioner or I won't paytherent.

(c) Manavi will be allowed to sit on Mohan's motor bike only if she wears herhelmet.

Solution :(a) If Ms. SaniaMirza practices her serve daily, then she will have a good chance of winning the tennis tournament.

(b) If you do not fix my air-conditioner, then I shall not pay the rent.

(c) If Manaviis to be allowed on Mohan‘s motor bike, then she must wear her helmet as true, and s Ú t as false. Thus, truth values of these primitive statements are:

The implication is given to be fse;we must have (pUq) Ur as true and sÚ$t$asfalse.

Thus, truth values of these primitive statements are:

Primitive statements	p	Q	r	s	t
Truth values	**T**	**T**	**T**	**T**	**T**
	T	**T**	**T**	**F**	**F**

Harold finishes his data structures project but fails to graduate at the end of semester.

<u>Concept of converse, inverse and contra positive statements:</u>

Write the converse, inverse and contra -positive of the statement “If today is a labor day, then tomorrow is Tuesday”.

Solution:

- The converse statement is –If tomorrow is Tuesday, then today is LaborDay–.
- The inverse statement is –If today is not Labor Day, then tomorrow isnotTuesday‖.
- The contra-positive statement is –If tomorrow is not Tuesday, then today is not Labor Day

Laws oflogic:

pUp =p (idempotent law)

pUq =q Up (Commutative law)

(pUq)Ur =p U(q Ur) (Associative law)

pUF =F (identity law)

p*U*T* =*p(universal law)

Thus, (a set of propositions, U) is a discrete structure

With respect to combination of conjunction operator i.e. logical AND operator disjunction operator i.e. logical OR (U) , we can generate distributive laws. Thus, isanother discrete structure where . *P* denotes the setof propositions.

Ø(***p*** U***q***)=Ø***p***∩Ø***q***(De - Morgan law)

Ø(***p*** ∩***q***) =Ø***p***UØ***q***(De - Morgan law)

.

Example: Show that $(p \cap q) \rightarrow q$ is a tautology

1. $Ø(p \cap q)$ Uq	Implication Law on 0
2.($Øp$U$Øq$) Uq	De Morgan's Law (1st) on 1
3.$Øp$ U ($Øq$Uq)	Associative Law on 2
4.$Øp$ U 1	Negation Law on 3
5.1	Domination Law on 4

Example : Show that $\neg(p \leftrightarrow q) \equiv (p \leftrightarrow \neg q)$

0. $(p \leftrightarrow \neg q)$
1. $\equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$
3. $\equiv \neg(\neg((\neg p \vee \neg q) \wedge (q \vee p)))$
4. $\equiv \neg(\neg(\neg p \vee \neg q) \vee \neg(q \vee p))$
5. $\equiv \neg((p \wedge q) \vee (\neg q \wedge \neg p))$

6. $\equiv \neg((p \vee \neg q) \wedge (p \vee \neg p) \wedge (q \vee \neg q) \wedge (q \vee \neg p))$ Distribution Law
7. $\equiv \neg((p \vee \neg q) \wedge (q \vee \neg p))$ Identity Law
8. $\equiv \neg((q \rightarrow p) \wedge (p \rightarrow q))$ Implication Law
9. $\equiv \neg(p \leftrightarrow q)$ Equivalence Law

Show that $\neg(q \rightarrow p) \vee (p \wedge q) \equiv q$

- 0. $\neg(q \rightarrow p) \vee (p \wedge q)$

1. $\equiv \neg(\neg q \vee p) \vee (p \wedge q)$ Implication Law
2. $\equiv (q \wedge \neg p) \vee (p \wedge q)$ De Morgan's & Double negation
3. $\equiv (q \wedge \neg p) \vee (q \wedge p)$ Commutative Law
4. $\equiv q \wedge (\neg p \vee p)$ Distributive Law
5. $\equiv q \wedge 1$

$\equiv q$ Identity Law Identity Law

Logical Implication-Rules ofInference

Rule of inference	Tautology	Name
$p \to q$ p $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollen
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetic al syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctiv e syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplifica tion
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunctio n
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (p \vee r)$	Resolution

1. It is not sunny this afternoon and it is colder thanyesterday.
2. If we go swimming it issunny.
3. If we do not go swimming then we will take a canoetrip.
4. If we take a canoe trip then we will be home bysunset.
5. We will be home bysunset

p It is sunny this afternoon *q*

It is colder thanyesterday *r*

Wegoswimming

s. We will take acanoetrip

t. We willbe home by sunset (theconclusion)

propositions

1. $\neg p \wedge q$
2. $r \rightarrow p$
3. $\neg r \rightarrow s$
4. $s \rightarrow t$
5. t

hypotheses

	Step	Reason
1.	$\neg p \wedge q$	Hypothesis
2.	$\neg p$	Simplification using (1)
3.	$r \to p$	Hypothesis
4.	$\neg r$	Modus tollens using (2) and (3)
5.	$\neg r \to s$	Hypothesis
6.	s	Modus ponens using (4) and (5)
7.	$s \to t$	Hypothesis
8.	t	Modus ponens using (6) and (7)

Discussion of Direct Proof and IndirectProof

Let n be apositiveinteger. Prove that n^2 is odd if and only if n is odd.

Proof: It is known that an odd integer may be written as $n = 2k + 1$, where $k \in Z$

squaring both sides,

$n^2=(2k+1)^2=4k^2+4k+1=2(2k^2+2k)+1=2m+1$ where $m=(2k^2+2k)$, an integer

Thus, n^2 is odd . To prove the converse, we shall give an indirect proof. To begin with let n^2 be an odd integer. To prove that n is odd. Suppose on the contrary that let n be an even integer. Then we can write as $n=2k$ so that $n^2=(2k)^2=4k^2=2(2k^2)$ an even integer. But this is against our hypothesis that n^2 is odd. Hence, we conclude that n is an odd integer.

Prove that if $3m+2$ is an odd integer, then m is an odd integer.

Proof: Observe that $3m + 2 = 2k + 1$, therefore $3m = 2k - 1$, an odd integer . Now, we have $3m$ an odd integer. From this, we must conclude that m is odd, otherwise $3m+2=3(2k)+2=6k+2=2(3k+1)$, an even integer. this is impossible as it is given that odd integer. $3m+2$ is an odd integer. Hence our assumption that m is even is wrong

Let n be an integer. Prove that n is even if and only if $31n + 12$ is even.

Proof: Suppose that n is an even integer, Thus, **$n = 2k$, where k is an integer.** Therefore,

$31n + 12 = 31 \times 2k = 62k = 2 \times (31 \times k)$ is even. On the other hand suppose that $31n + 12$ is even.

To prove that n isan even integer. We shall prove this by giving an indirectproof.Suppose on the contrary, let n be odd. Then we can write **$n = 2k + 1$, where k is an integer.** Thus,

$31n + 12 = 31 \times (2k + 1) + 12 = 62k + 43 = 62k + 42 + 1 = 2(31k + 21) + 1$

$= 2m + 1$ **where** $m = 31k + 21$ is an odd integer which is impossibleas we ourselveshad assumed that $n$is even. Hence, n must be an eveninteger.

A note onQuantifiers

In a practical situation, we come across a number of quantified statements like

- Live lectures through EDUSAT Programme is open for all engineering college students in Karnataka
- A vehicle having Karnataka state permit is permitted to move on any road in the state.
- There are some bad boys in a class room.
- SomestudentsneverfollowDiscreteMathematicscoursewhoeverteachesthiscourse.
- There is at least one student in some engineering watching this programme coming live from VTU studio, Bangalore.
- Pythagoras Theorem holds well for all right angled triangles and the list go on.

Statements shown in previous slides can be expressed symbolically using quantifiers.

There are two types of quantifiers:

1. **Universal quantifier:** A statement which is universally valid may be explained using an universal quantifier. This is similar to the situation ,namely, for all, for every and foranyetc. Symbolically, it is denoted as $\forall$.

Consider an open statement $p(x)$ which istruefor substitutions from a universe ofdiscourse, then the same may be written in symbolic form as $\forall x\, p(x)$.

2. **Existential Quantifier:** There .are statements which are true only under circumstances. These statements may be symbolically expressed in terms of existential quantifier, denoted by$\exists$. Consider an open statement of the form $q(x)$. Suppose that this statement is true only for some values of x . This may be symbolically written as $\exists x\, q(x)$

3. Note: An open statement of the form $q(x)$ gets A meaning only when x is replaced by a proper value from the universe or universe of discourse, denoted by U .

Examples: Examples:Consideruniverseofdiscourseasthesetofalldaysofaweek.Then$U$= {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}. Consider the statement $p(x)$: x**is a holiday** . It is known that there is a day in a week which is declared as a general holiday (i.e. Sunday), thus, this situation may be written as $\exists x\, p(x)$

Another example; Take the universe as the set of all flowers and consider the statement, –***Flowers arebeautiful***‖. To write this statement in symbolic form, we shall set up $f(x)$: x **is a flower** , then above may be written symbolically as $\forall x\, f(x)$ . .

<u>Tutorial on Quantifiers:</u>

For the universe of integers, let $p(x)$, $q(x)$, $r(x)$, $s(x)$, $t(x)$ be the following open statements.

$p(x)$: $x > 0$

$q(x)$: x **is even**

$r(x)$: x **is a perfect square**

$s(x)$: x **is exactly divisible by 4** t

(x) : x **is exactly divisible by 5**

Write the following statements in symbolic form:

(i) At least one integer is even.

(ii) There exists a positive integer that iseven.

(iii) If x is even, then x is not divisibly by5.

(iv) No even integer is divisible by5.

(v) There exists an even integer divisible by5.

(vi) If x is even and x is a perfectsquare, then $x$is divisibly by 4.

<u>Solution:</u>i) $xq(x)(ii) $x[p(x)Uq(x)](iii) " x [q (x) →Øt(x)]

(iv) "x[Øq(x)→t(x)] **(v)** $x[q(x)→t(x)](vi)"xë(q(x)Ùr(x))® s (x)û

AssignmentQuestions

1. Prove the validity of the statement using laws of inference $(\emptyset p \cup \emptyset q) \rightarrow r \cap s) \cap (r \rightarrow t) \cap (\emptyset t) \rightarrow p$.
2. For all integers n≥24 then n can be written as sum 0f 5‘s and / or7‘s.
3. What is tautology? Checkwhether$(p \cap q) \rightarrow (\emptyset r)$ is a tautology using truthtables.

Outcome

1. Verify the correctness of an argument using propositional and predicate logic and truth tables.
2. Construct proofs using direct proof, proof by contradiction, and indirectproof.

FurtherReading

1. Ralph P. Grimaldi: Discrete and combonotorial Mathematics, 5thEdition,Pearson Education2004.
2. Basavaraj S Anami and venkanna S Madalli: Discrete Mathematics – A Concept based approach UniversitiesPress,2016.
3. Kenneth H. Rosen: Discrete Mathematics and its Applications, 6th Edition, McGraw Hill 2007.
4. JayantGanguly: A Treatise on Discrete Mathematical Structures, Sanguine-Pearson, 2010.
5. D.S. Malik and M.K. Sen: Discrete Mathematical Structures: Theory and Applications, Thomson,2004.
6. Thomas Koshy: Discrete Mathematics with Applications, Elsevier, 2005, Reprint2008.
7. https://en.wikipedia.org/wiki/Rule_of_product
8. https://en.wikipedia.org/wiki/Hasse_diagram
9. https://math.feld.cvut.cz/habala/teaching/dma-e/book4x.pdf

Module – II

STRUCTURE

Introduction

This module gives the overview of principle of counting which consists of permutation and combination. This is used for learning the arrangements and selection of a given problem. It also consists recursive definitions and mathematical induction.

objectives

- Understand the concept of permutation and combination and its application.
- Solve problems involving recurrence relations.
- Proving using the method of mathematical induction

Mathematical induction

Mathematical induction is a technique for proving results or establishing statementsfor natural numbers. This part illustrates the method through a variety of examples.

Definition

Mathematical Induction is a mathematical technique which is used to prove astatement, a formula or a theorem is true for every natural number.

The technique involves two steps to prove a statement, as stated below:

Step 1(Base step): It proves that a statement is true for the initial value.

Step 2(Inductive step): It proves that if the statement is true for then^{th}iteration (ornumber *n*), then it is also true for $(n+1)^{th}$ iteration (or number *n+1*).

How to Do It

Step 1: Consider an initial value for which the statement is true. It is to be shown thatthe statement is true for *n*=initial value.

Step 2: Assume the statement is true for any value of*n=k*. Then prove the statement istrue for *n=k+1*. We actually break *n=k+1* into two parts, one part is n=k (which is already proved) and try to prove the other part

Problem 1

3^n -1 is a multiple of 2 for n=1, 2, ...

Solution

Step 1: For n=1, 3^1-1 = 3-1 = 2 which is a multiple of 2

Step 2: Let us assume 3^n-1 is true for n=k, Hence, 3^k -1 is true (It is anassumption) We have to prove that 3^{k+1}-1 is also a multiple of 2 $3^{k+1} – 1 = 3 × 3^k – 1 = (2 × 3^k) + (3^k –1)$

The first part (2×3^k) is certain to be a multiple of 2 and the second part ($3^k - 1$) is also

true as our previous assumption.

Hence, $3^{k+1} - 1$ is a multiple of 2.

So, it is proved that $3^n - 1$ is a multiple of 2.

Problem 2

$1 + 3 + 5 + ... + (2n-1) = n^2$

Solution

Step 1: For n=1, $1 = 1^2$, Hence, step 1 is satisfied.

Step 2: Let us assume the statement is true for n=k.

Hence, $1 + 3 + 5 + ... + (2k-1) = k^2$ is true (It is an assumption)

We have to prove that $1 + 3 + 5 + ... + (2(k+1)-1) = (k+1)^2$ also holds $1 + 3 + 5 + ... + (2(k+1) - 1)$

$= 1 + 3 + 5 + ... + (2k+2 - 1)$

$= 1 + 3 + 5 + ... + (2k + 1)$

$= 1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$

$= k^2 + (2k + 1)$

$= (k + 1)^2$

So, $1 + 3 + 5 + ... + (2(k+1) - 1) = (k+1)^2$ hold which satisfies the step 2. Hence, $1 + 3 + 5 + ... + (2n - 1) = n^2$ is proved.

Problem 3

Prove that $(ab)^n = a^n b^n$ is true for every natural number *n*

Solution

Step 1: For n=1, $(ab)^1 = a^1b^1 = ab$, Hence, step 1 is satisfied.

Step 2: Let us assume the statement is true for n=k, Hence, $(ab)^k = a^kb^k$ is true (It is an assumption).

We have to prove that $(ab)^{k+1} = a^{k+1}b^{k+1}$ also hold

Given, $(ab)^k = a^kb^k$

Or, $(ab)^k(ab)=(a^kb^k)(ab)$[Multiplyingbothsideby_ab']

Or, $(ab)^{k+1} = (aa^k)\,(bb^k)$

Or, $(ab)^{k+1} =$ (ak+1bk+1)

Hence, step 2 is proved.

So, $(ab)^n = a^nb^n$ is true for every natural number n.

RECURRENCE RELATION

In this chapter, we will discuss how recursive techniques can derive sequences and be used for solving counting problems. The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**. We study the theory of linear recurrence relations and their solutions. Finally, we introduce generating functions for solving recurrence relation.

Definition

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing Fn as some combination of Fi with i<n).

Example: Fibonacci series: Fn= Fn-1+ Fn-2, Tower of Hanoi: Fn= 2Fn-1+ 1

Linear Recurrence Relations

A linear recurrence equation of degree k is a recurrence equation which is in the format xn= A1 xn-1+ A2 xn-1+ A3 xn-1+... Akxn-k (An is a constant and Ak≠0) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations:

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1=a_2=1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1=1$, $a_2=3$	Lucas number
$F_n = F_{n-2} + F_{n-3}$	$a_1=a_2=a_3=1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1=0$, $a_2=1$	Pell number

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is: Fn = AFn-1 +BFn-2 where A and B are real numbers.

The characteristic equation for the above recurrence relation is:

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots:

Case 1: If this equation factors as (x- x1)(x- x1) = 0 and it produces two distinct realroots x1 and x2, then $Fn = ax1^n + bx2^n$ is the solution. [Here, a and b are constants]

Case 2: If this equation factors as $(x- x1)^2 = 0$ and it produces single real root x1, thenFn = a $x1^n$+ bn $x1^n$ is the solution.

Case 3: If the equation produces two distinct real roots x1and x2in polar form x1= r∠θand x2= r∠(-θ), then Fn= r^n(a cos(nθ)+ b sin(nθ)) isthe solution.

Problem 1

Solve the recurrence relation Fn = 5Fn-1 - 6Fn-2 where F0 = 1 and F1 = 4

Solution

The characteristic equation of the recurrence relation is:

$x^2 – 5x + 6=0$,

So, (x-3) (x-2) =0

Hence, the roots are:

x1 = 3 and x2= 2

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is:

$$F_n = ax1^n + bx2^n$$

Here, $F_n = a3^n + b2^n$ (As x1 = 3 and x2=2) Therefore,

$1=F_0 = a3^0 + b2^0 = a+b$

Solving these two equations, we get a = 2 and b = -1

Hence, the final solution is:

$F_n = 2.3^n + (-1) . 2^n = 2.3^n - 2^n$

Problem 2

Solve the recurrence relation $F_n = 10F_{n-1} - 25F_{n-2}$ where $F_0 = 3$ and $F_1 = 17$

Solution

The characteristic equation of the recurrence relation is:

$x^2 - 10x - 25 = 0$,

So, $(x - 5)^2 = 0$

Hence, there is single real root x1 = 5

As there is single real valued root, this is in the form of case

2 Hence, the solution is:

$F_n = a x 1^n + b n x 1^n$

$3 = F_0 = a.5^0 + b.0.5^0 = a$

$17 = F_1 = a.5^1 + b.1.5^1 = 5a+5b$

Solving these two equations, we get a = 3 and b = 2/5

Hence, the final solution is:

$F_n = 3.5^n + (2/5) .n.2^n$

Fundamental Principles of Counting

The Rules of Sum and Product

Our study of discrete and combinatorial mathematics beings with two basic principles of counting: the rules of sum and product. The statements and initial applications of these rules appear quite simple. In analyzing more complicated problems, one is often able to break down such problems into parts that can be solved using these basic Principles. We want to develop the ability to ―decompose‖ such problems and piece together our partial solutions in order to arrive at the final answer. A good way to do this is to analyze and solve many diverse enumeration problems, Taking note of the principles being used.

Our first principle of counting can be stated as follows:

The Rule of Sum:

If a first task can be performed in ***m*** while second task performedIn n ways and the two tasks cannot be performed in continuously, then performing either task can be accomplished in any of***m+n*** ways .

Example 1:

A College library has40textbook on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among 40 + 50 = 90 textbooks in order to learn more about one or the other of these two subjects.

Example 2

The rule can be extended beyond two tasks as long as no pair of tasks can occur simultaneously. For instance, a computer science instructor who has, say, seven different introductory books each on C++, Java and Perl can recommend any one of these 21 books to a student who is interested in learning a first programming language.

Example 3

Suppose a university representative is to be chosen either from 200 teaching or 300 nonteaching employees, and then there are 200 + 300 = 500 possible ways to choose this representative.

The rule of Product:

If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

Example 1:

The drama club of Central University is holding tryouts for a spring play. With six men and eight Women auditioning for the leading male and female roles, by the rule of product the director can Cast his leading couple in 6 X 8 = 48 ways.

Example 2:

A tourist can travel from Hyderabad to Tirupati in four ways (by plane, train, bus or taxi). He can then travel from Tirupati to Tirumala hills in five ways (by RTC bus, taxi, rope way, motorcycle or walk). Then the tourist can travel from Hyderabad to Tirumala hills in 4 X 5 = 20 ways.

Permutations

Continuing to examine applications of rule of product, we turn now to counting linear arrangements of objects. These arrangements are often called permutations when the objects are distinct. We shall Develop some systematic methods for dealing with linear arrangements.

$P(n,r)=\frac{!}{(\ -\ )!}$

Example 1

The number of words of three distinct letters formed from the letters of word ‒JNTU‖ is P (4, 3) = 4!/(4 - 3)!=24. If repetitions are allowed, the number of possible six – letter sequence is 46 =4096

Example 2:

In how many ways can eight men and eight women be seated in a row if (a) any person may sit next to any other (b) men and women must occupy alternate seats (c) generalize this result for n men and n women. Here eight men and eight women are 16 indistinguishable objects.

a) The number of permutations 16 chosen form 16 objects is P (16, 16) = 16! = 20922789890000.

b) Here men and women are distinct (different)

c) Any person may sit: (2n)! Men and women sit alternatively: 2(n!) 2

Example 3

The MASSASAUGA is a brown and white venomous snake indigenous to North America. arranging all of the letters in MASSASAUGA.

We find that there are $\frac{10!}{4!*3!*1!*1!}$=25,200

Possible arrangements are $\frac{7!}{3!}$=840

Among these are In which all four A‘s are together. To get this last result, we considered all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G.

Example 1

A hostess is having a dinner party for some members of her charity committee. Because of the size of her home, she can invite only 11 of the 20 committee members. Order is not important, so she can invite ―the lucky 11‖ in C (20, 11) = 20! / (11! 9!) = 167, 960 ways. However, once the 11 arrive, how she arranges them around her rectangular dining table is an arrangement problem. Unfortunately, no part of theory of combinations and permutations can help our hostess deal with ―the offended nine‖ who were not invited.

Example 2

The number of arrangements of the letters in TALLAHASSEE is

$\frac{11!}{2!*3!*2!*2!}$=831600

How many of these arrangements have no adjacent A‘s? When we disregard the A‘s, there are

$\frac{8!}{2!*2!*2!}$=5040

Ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A‘s. Three of these locations can be selected in c(9,3)= 84 ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are 5040 X 84 = 423,360 arrangements of the letters in TALLAHASSEE with no consecutive A's.

Combinations with Repetition

When repetitions are allowed, we have seen that for n distinct objects an arrangement of size r of these objects can be obtained in n r ways, for an integer r=0. We now turn to the comparable problem for combinations and once again obtain a related problem whose solution follows from our previous enumeration principles.

Example 1:

A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, we can select a dozen donuts in C (20 + 12 - 1, 12) = C(31, 12) = 141,120,525 ways. (Here n = 20, r = 12.)

Example 2:

In how many ways can one distribute 10 (identical) white marbles among six distinct containers? Solving this problem is equivalent to finding the number of nonnegative integer solutions to the equation x1 + x2 + … + x6 = 10. That number is the number of selections of size 10, with repetition, from a collection of size 6. Hence the answer is C(6 + 10 — 1, 10) = 3003.

Assignment Questions

1) $1 + 3 + 5 + ... + (2n-1) = n^2$ solve using mathematical induction

2) The number of words of three distinct letters formed from the letters of word –JNTU‖

3) Solve the recurrence relation Fn = 5Fn-1 - 6Fn-2 where F0 = 1 and F1 = 4

Outcomes

After completing this module one can do

- Problems involving mathematical induction
- Solve recurrence relations
- Understand permutation and combination

Further Reading

1. Ralph P. Grimaldi: Discrete and combonotorial Mathematics, 5th Edition,Pearson Education 2004.

2. Basavaraj S Anami and venkanna S Madalli: Discrete Mathematics – A Concept based approach Universities Press,2016.

3. Kenneth H. Rosen: Discrete Mathematics and its Applications, 6th Edition, McGraw Hill 2007.

4. JayantGanguly: A Treatise on Discrete Mathematical Structures, Sanguine-Pearson, 2010.

5. D.S. Malik and M.K. Sen: Discrete Mathematical Structures: Theory and Applications, Thomson, 2004.

6. Thomas Koshy: Discrete Mathematics with Applications, Elsevier, 2005, Reprint 2008.

7. https://en.wikipedia.org/wiki/Rule_of_product

8. https://en.wikipedia.org/wiki/Hasse_diagram

9. https://math.feld.cvut.cz/habala/teaching/dma-e/book4x.pdf

Module – III

STRUCTURE

Introduction
Objective
Cartesian Product
Functions
Types of functions
 Identity function
 Constant function
 one to one and on to function
Stirling number of the second kind
Pigeon hole Principle
Composition and invertible functions
Hasse Diagram
Assignment Questions
 Outcomes
 Further Reading

Introduction

This module gives the overview of relations and functions, which consists of Cartesian product, types of functions and operations on functions such as composition and invertible functions.

objectives

- Understand the concept of relations and functions.
- Solve problems involving composition of functions.

Relations and Functions

Cartesian product

Let A and B be two sets. Then the set of all ordered pairs (a,b), where a∈A and b∈B, is called the Cartesian Product or Cross Product or Product set of A and B and is denoted

by AXB. Thus AXB = {(a,b)/a∈A and b∈B}

<u>Example:</u>

A = {1,2}, B = {J,M,P,K,V,B}, then

AXB={(1,J),(1,M),(1,P),(1,K),(1,V),(1,B), (2,J),(2,M),(2,P),(2,K),(2,V), (2,B)}

Note:We can have the product of a set A with itself, and this product is

defined as AXA = {(a,b)/a∈A and b∈A}. This product is also denoted by A^2

Problems:

1. A={a,b} B={1,2,3} Find AXB, BXA, AXA, BXB

AXB = {(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)}

BXA = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

AXA = {(a, a), (a, b), (b, a), (b, b)}

BXB = {(,1), (1,2), (1,3), (2,1), (2,2)(2,3), (3,1), (3,2), (3,3)}

2. If A = {1,2,3,4} , B = {2,5}, C = {3,4,7} write down the following

   AXB, BXA, AXC, CXA, BXC, CXB, A∪(BXC), (A∪B)XC, A∩(BXC),

   (A∩B)XC, (AXC)∪(BXC), (AXC)∩(BXC)

   Solution:

   AXB = {(1,2), (1,5), (2,2), (2,5), (3,2), (3,5), (4,2), (4,5)} BxA = {(2,1), (2,2), (2,3), (2,4),(5,1), (5,2),(5,3),(5,4)} BXC = {(2,3), (2,4), (2,7), (5,3), (5,4), (5,7)}

   A∪(BXC) = {1,2,3,4, (2,3), (2,4), (2,7), (5,3), (5,4), (5,7)} A∩B = {2}

   (A∩B)XC = {(2,3), (2,4), (2,7)} (Calculate the other quantities in the same manner)

Note: If A and B are finite sets, then |AXB| = |A||B|, |BXA| = |B||A|, |AXA| = |A||A|

Example:If A has 8 elements and B has 6 elements then |AXB| = 8 x 6 = 48 elements |BXA| = 6 x 8 =48 elements |AXA| = 6 x 6 = 36 elements

Theorem:For any three sets prove the following results

(1) AX(B∪C) = (AXB) ∪ (AXC)

(2) AX(B∩C) = (AXB) ∩ (AXC)

(3) (A∪B)XC = (AXC) ∪ (BXC)

(4) (A∩B)XC = (AXC) ∩ (BXC)

(5) AX(B-C) = (AXB)-(AXC)

(Proof of (1) and (5) are given below others left to the reader to prove)

Proof: (1)

$$(x, y) \in Ax(B \cup C) \Leftrightarrow x \in A, y \in B \cup C \Leftrightarrow x \in A, y \in B \; or \; y \in C$$

$$\Leftrightarrow x \in A, y \in B \; or \; x \in A, y \in C$$

$$\Leftrightarrow (x, y) \in (AxB) \; or \; (x, y) \in (AxC) \Leftrightarrow (x, y) \in (AxB) \cup (AxC)$$

$$Hence \quad Ax(B \cup C) \; \square \; (AxB) \cup (AxC)$$

(5) $$(x, y) \in Ax(B - C) \Leftrightarrow x \in A, y \in B - C \Leftrightarrow x \in A, y \in B, y \notin C$$

$$\Leftrightarrow x \in A, y \in B, x \in A, y \notin C$$

$$\Leftrightarrow (x, y) \in (AxB), (x, y) \notin (AxC)$$

$$\Leftrightarrow (x, y) \in (AxB) - (AxC)$$

$$Hence \; Ax(B-C)=(AxB)-(AxC)$$

Let A and B be two sets. Then a subset of AXB is called a binary relation or a relation from A to B.

Note:

1. If R is relation from A to B, then R is a set of ordered pairs (a,b) where a∈A and b∈B.
2. R is a set of ordered pairs (a,b) where ∈A and b∈B, then R is relation from A to B.
3. If (a,b) R, we mean that –a is related to b by R‖ and is denoted as aRb.
4. R is a relation from A to A, i.e., R is a sub set of AXA, then R is known as binary relation on A
5. If a set A has m elements and B has n elements then the number of relations from A to B are 2^{mn}

Problem:

1. Let A = {1,2,3} and B = {2,4,5} determine the following

|AXB|, Number of relations from A to B, Number of binary relations on A

Solution:

|AxB| =|A||B| = 9

No. of relations from A to B $2^{mn}=2^9=512$

No. of relations from A to A $2^{mm}=2^9=512$

2. Consider the sets A = {0,1,2} and B = {8,9}. Indicate the following sets of ordered pairs are relations from A to B.

(a) R1 = {(0,8),(1,8),(2,9)}

(b) R2 = {(1,8), (1,9), (2,2)}

(c) R3 = {(1,9),(0,8),(8,0)}

Solution: (a) R1 is relation

(b)R2 is not a relation, Because the ordered pair (2,2) does not belong to R2.

(c)R3 is not a relation, Because the ordered pair (8,0) does not belong to R3.

3. Let A = {1,2,3,4} and R be the re ation on A defined by (a,b)∈R if and only if a≤b. Write down R as a set of ordered pairs.

Solution:

First let us write the cross product

AxA = {(1,1), (1,2), (1,3), (1,4), (2,1),

(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(
4,4)} Now the relation R is

R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}

4. Let A = {1,2,3,4,6} and R be the relation on A defined by (a,b)∈R if and only if ais multiple of b. Write down R as a set of ordered pairs.

Solution:

First let us write the cross product

AxA = {(1,1),(1,2),(1,3),(1,4),(1,6),(2,1),(2,2),(2,3),(2,4),(2,6),(3,1),(3,2),(3,3),

(3,4),(3,6), (6,1),(6,2),(6,3),(6,4),(6,6)}

Now the relation R is

={(1,1)(1,2)(2,2)(1,3)(3,3),(1,4),(2,4),(4,4),(1,6),(2,6),(3,6),(6,6)}

Function

For nonempty sets A,B, a function, or mapping, f from A to B, denoted as $f:A\rightarrow B$, is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair.

Note:

1. A is called domain of f and B is called codomain of f
2. The subset of B consisting of the images of A under f is called range of f and is denoted by f(A)

Problem:

1. Let A = {1,2,3,4}. Determine the following are functions.

$f = \{(2,3),(1,4),(2,1),(3,2),(4,4)\}$

$g = \{(3,1),(4,2),(1,1)\}$

$h = \{(2,1),(3,4),(1,4),(4,4)\}$

Solution:

The element 2 in the domain appeared twice as the first element in the function.

Hence f is not a function.

The element 2 in domain has no image in the co domain. Hence g is not a function.

All the elements in the domain appeared once in the function hence h is a function.

2. If A = {0,±1,±2,±3} and $f:A\rightarrow R$ is defined by $f(x) = x^2-x+1$, $x\in A$, find the range off.

Solution:

Given $f(x) = x^2-x+1$,

Let us calculate, f(0) = 1, f(1) = 1-1+1 = 1, f(-1) = 1+1+1 = 3, f(2) = 4-2+1 = 3, f(-2) = 4+2+1 = 7, f(3) = 9-3+1 = 7, f(-3) = 9+3+1 = 13

Therefore the range of f = f(A) = {1,3,7,13}

3. If $f:R\rightarrow R$ is defined by $f(x) = x^2$, what is the range of f, what is f(Z)? What is f([2,1])?

solution:

Range = $[0,\infty)$

f(Z) = {0,1,4,9,16,…….}

f([-2,1]) = [4,1]

Types of functions

Identity Function

A function f:A→A such that f(a) = a for every a ∈A is called the identity function on A.

In other words: If the image of every element is itself in a function then it is identity function.

Example: f = {(1,1),(2,2),(3,3),(4,4),(5,5)}

Constant Function

A function f:A→B such that f(a) = c for every a ∈A is called the constant function on A

On to Function

A function f:A→B is said to be an onto function if for every element b of B there is an element a of A such that f(a) = b.

In other words: f is an onto function from A to B if every element of B has a preimage in A.

One-to-one Function (or)Injective function

A function f:A→B is said to be one-to-one function if different elements of A have different images in B under f.

In other words: a1,a2∈A with a1=a2 then f(a1)=f(a2)

In other words: $a1,a2 \in A$ with $a1 \neq a2$ then $f(a1) \neq f(a2)$

A function which is both one-to-one and onto is called one-to-one correspondence or a bijective function. In this function every element of A has unique image in B and every element in B has unique preimage in A.

1. Let A = {a1,a2,a3}, B = {b1,b2,b3}, C = {c1,c2} and D = {d1,d2,d3,d4}. Find the nature of the following functions

f1={(a1,b2),(a2,b3),(a3,b}

f2={(a1,d2),(a2,d1),(a3,d}

f3={(b1,c2),(b2,c2),(b3,c}

f4 = {(d1,b1),(d2,b2),(d3,b1),(d4,b2)}

The given function f1 is one – to – one and onto. Hence it has one – to – one correspondence.

The given function f2 is one – to – one not onto.

The given function f3 is not one – to – one but onto.

The given function f4 is not one – to – one and onto.

2. If A = {1,2,3,4}, B = {v,w,x,y,z} f = {(1,v),(2,x),(3,z),(4,y)} and g = {(1,v),(2,v),(3,w),(4,x)} prove that f is one-to-one but not onto and g is neither one-to-one nor onto. Solution:

Every element in A has unique image, hence the function f is one to one. But w in B does not have preimage in A, hence it is not onto.

The elements 1 and 2 in A has same image, hence the function g is not one to one.

Similarly y and z in B does not have preimage in A, hence it is not onto.

3. Let A = {1,2,3,4,5,6} and B = {6,7,8,9,10}. If f : A→B is defined by f = {(1,7), (2,7), (3,8), (4,6), (5,9), (6,9)}. Determine $f^{-1}(6)$ and $f^{-1}(9)$. If B1 = {7,8} and B2 = {8,9,10}, find

$f^{-1}(B1)$ and $f^{-1}(B2)$.

Solution:

$f^{-1}(6) = \{4\}$, $f^{-1}(9) = \{5,6\}$, $f^{-1}(B1) = f^{-1}(7,8) = \{1,2,3\}$ and $f^{-1}(B2) = f^{-1}(8,9,10) = \{3,5,6\}$

4. In eachofthe followingcases, functionf:R→Risgiven. Determinewhetherf is one-to-one oronto. If f is not onto find its range. (i) $f(x)=2x-3$ (ii) $f(x)=x^3$ (ii) $f(x) = x^2$ (iv) $f(x) = x^2+x$ (v) f(x) = sinx

Solution:

(i) $f(x)=2x-3$

Consider

$f(x1)=f(x2)$

$2x1 - 3 = 2x2 - 3$

$2x1 = 2x2$

$x1 = x2$

Hence f(x) is one – to – one

Let f(x) = y

⇒ $2x - 3 = y$

⇒ $X = (y + 3)/2$

$\forall (y+3)/2 \in R$ we get $x \in R$ such that f(x) = y

Hence the given function is onto

(ii) $f(x) = x^3$

Consider

$f(x1)=f(x2)$

$x1^3 = x2^3$

$x1 = x2$

Hence f(x) is one – to – one

Let f(x) = y

$\Rightarrow x^3 = y$

$\Rightarrow X = (y)^{1/3}$

$\Rightarrow (y)^{1/3}$ gives cube roots. We don't consider the complex

numbers. $\forall y \in R$ we get $x \in R$ such that f(x) = y

Hence the given function is onto

Floor and Ceiling Functions

Let x be any real number. Then x is an integer or x lies between two integers. Let x denote the greatest integer that is less than or equal to x, and x denote the least Integer that is greater than or equal to x. Then x is called the floor of x and x is called ceiling of x.

Example:

3.6 + 4.2 = 3+4 =7

3.6+4.2= 7.8 = 7

7.2 - 8.3= 8-9 =1

2.8 x 2.9 = 8.12 =9

Projection

For sets A and B, let $D \subseteq AxB$. Then the function $\pi_A:D \to A$ defined by $\pi_A(a,b) = a$ for all $(a,b) \in D$ is called the projection of D on A, and the function $\pi_B:D \to B$ defined by $\pi_B(a,b) = b$ for all $(a,b) \in D$ is called the projection of D on B.

Example:

Let A = B = R. Determine $\pi_A(D)$ and $\pi_B(D)$ for each of the following

(i) $D = \{(x,y) | x = y^2, 0 \le y \le 2\}$

(ii) $D = \{(x,y) | y = \sin x, 0 \le x \le \pi\}$

(iii) $D = \{(x,y) | x^2 + y^2 = 1\}$

Solution:

Let A=B=R. Determine $\pi_A(D)$ and $\pi_B(D)$ for each of the following sets $D \subseteq AxB$:

1. $D = \{(x,y)|x=y^2, 0 \le y \le 2\}$

$\pi_A(D) = \{x \in R|x=y^2, 0 \le y \le 2\}$ $\pi_B(D) = \{y \in R|y^2=x, 0 \le x \le 4\}$

2. D = {(x,y)|y = sinx, $0 \le x \le \pi$}

π_A(D) = {x∈R| x = $\sin^{-1}$y , $0 \le y$

≤ 1} π_B(D) = {y ∈R| y = sinx, $0 \le$
$x \le \pi$}

3. D = {(x,y)|$x^2 + y^2 = 1$}

π_A(D) = {x∈R| x=$\pm\sqrt{1-y^2}$, $-1 \le y \le$

1} π_B(D) ={y ∈R|y =$\pm\sqrt{1-x^2}$, $-1 \le$
$x \le 1$}

Number of onto functions

Let A and B be finite sets with |A| = m and |B| = n, where m≥n. Then the number of onto functions from A to B is given by the formula Stirling number of the second kind

Note: 1. Given that p(6,4)=1560 and p(7,4)=8400. We can prove that S(6,4) = 65 and

S(7,4) = 350

2. Prove the following: S(5,3) = 25, S(7,2) = 63, S(8,5) = 1050, S(5,4) = 10,
S(8,6) = 266 S(m,1) = 1, S(m,m) = 1 for all m ≥1.

Unary and Binary Operations

Let A be a nonempty set. Then the function $f:A\to A$ is called a unary(or monary) operation on A.

For nonempty sets A and B, function $f:(AxA)\to B$ is called a binary operation on A.

If B=A, then the binary operation is said to be closed

Result

If $|A| = m$ then the number of c osed operations on A are $|A|^{|AxA|}$

Results

If $f:AxA\to B$ is a binary operation, then

1. f is said to be commutative whenever $f(a,b)=f(b,a)$ for all $(a,b)\in AxA$.
2. f is said to be associative whenever $B\subseteq A$ and $f(a,f(b,c))=f(f(a,b),c)$ for all $a,b,c\in A$.
3. An element $x\in A$ is called an identityfor f whenever $f(a,x)=f(x,a)=a$ for all $a\in A$

1. Determine whether the following closed operations f on Z are commutative and /or assosiative

$f(x,y) = x+y-xy$

$f(x,y)=x+y-1$

$f(x,y) = x^y$

(i) $f(x,y) = x+y-xy$

$=y + x – yx$ (since x and y are integers)

$= f(y,x)$

Hence f is commutative.

$f(f(x,y),z) = f(a,z)$ where $a = f(x,y)$

$= a + z - az$

$= f(x,y) + z – z\, f(x,y)$

$= x + y – xy - z \{x + y – xy \}$

$= x + y – xy – zx – zy + xyz \ldots. (1)$

$f(x,f(y,z)) = f(x, b)$ where $b = f(y,z)$

$= x + b – xb$

$= x + f(y,z) – x\, f(y,z)$

$= x + y + z – yz – x\{y + z – yz \}$

$= x + y + z – yz – xy – xz + xyz \ldots.(2)$

Hence (1) = (2)

f is associative

(ii) $f(x,y) = x^y$

$\neq y^x$ since x and y are integers

$\neq f(y,x)$

Hence f is not commutative

- $f(f(x,y),z) = f(a,z)$ where $a = f(x,y)$

$= a^z$

$= f(x,y)^z$

$= x^{yz}$ ———— (1)

$f(x,f(y,z)) = f(x, b)$ where $= f(y,z)$

$= x^b$

$= x^{f(y,z)}$

$= x^{yz}$ ———— (2)

f is associative

Pigeonhole Principle

If m pigeons occupy n pigeonholes, where m>n, then at least one Pigeonhole must contain two or more pigeons in it.

If m pigeons occupy n pigeonholes, then at least one pigeonhole must contain (p+1) or more pigeons, where p =[(m-1)/n].

1. ABC is an equilateral triangle whose sides are of length 1cm each. If we select 5 points inside the triangle, prove that at least two points are such that the distance between them is less than ½ cm.

Solution:

Consider the triangle DEF formed by the midpoints of the sides BC, CA and AB of the given triangle ABC as shown in the diagram. Then the triangle ABC is partitioned into four small equilateral triangles each of which has sides equal to ½ cm. Treating each of these four triangles as pigeonholes and five points chosen inside the triangle as

pigeons, we find by pigeonhole Principle that atleast on triangle must contain two or more points. Hence, the distance between such points is less than ½ cm.

2. Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. When any 3 of these students are chosen to be a debating team from the class, the sum of their shirt numbers is used as the code number of the team. Show that if any 8 of the 20 are selected, then from these 8 we may form two different teams having the same code number.

Solution:

From the 8 of the 20 students selected, the number of a teams of 3 students that can be formed is $^8C_3 = 56$

According to the way in which the code number of a team is determined, we note that the smallest possible code number is 1 + 2 + 3 = 6 and the largest possible code number is 18 + 19 + 20 = 57.

Thus the code numbers vary from 6 to 57 and these are 52 in number.

As such, only 52 code numbers are available consider them as pigeon holes and the 56 teams as pigeons.

By pigeon hole principle at least two different teams will have the same code number.

3. Prove that if 151 integers are selected from the set S = {1,2,3,… ,300}, then the selection must include two integers x,y where x|y or y|x.

Solution:

Let A = {1,3,5,7,………,299}

Then every integer n between 1 and 300 is of the form $n = 2^k a$, where k is an integer ≥ 0

and $a \in A$. Thus every element of S corresponds to some $a \in A$.

The set A has 150 distinct elements and therefore, if 151 elements of S are selected, then at

least two of them say x and y, $x \neq y$ must correspond to the same $a \in A$. Thus, $x = 2^m a$, $y = 2^n a$, for some integers m, $n \geq 0$.

Evidently x divides y if $m \leq n$ and divides x if $n < m$.
This proves the required result.

4. Show that if any seven numbers from 1 to 12 are chosen, then two of them will add to 13. Solution:

Let us consider the sets

(1,12), (2,11), (3,10), (4,9), (5,8), (6,7)

These are the only sets containing two numbers from 1 to 12 whose sum is 13. Since every number from 1 to 12 belongs to one of the above sets, each of the seven numbers chosen must belongs to one of the sets.

Since there are only 6 sets, two of the seven chosen numbers have to belong the same set. Considering the 7 sets as pigeons and 6 sets as pigeonholes, by pigeonhole principle the sum of these two numbers must be equal to 13.

5.Show that if any n+1 numbers from 1 to 2n are chosen, then two of them will have their sum equal to 2n+1.

Solution:

Let us consider the following sets:

(1,2n), (2,2n – 1) , (3, 2n – 2), ……., (n – 1, n + 2), (n, n + 1)

These are the only sets containing two numbers from 1 to 2n whose sum is 2n + 1. Since every number from 1 to 2n belongs to one of the above sets, each of the n + 1 numbers chosen must belong to one of the sets.

Since there are only n sets, two of the n + 1 chosen numbers have to belong to the same set. Considering the 2n + 1 sets as pigeons and 2n sets as pigeonholes, by pigeonhole principle the sum of these two numbers must be equal to 2n + 1.

If 5 colours are used to paint 26 doors, Prove that at least 6 doors will have the same colour.
Solution:

Let us consider 26 doors as pigeons and 5 colours as pigeonholes by generalized pigeonhole principle, at least one of the colours must be assigned 26-1/5+1

Composition of Functions

Consider three non-empty sets A, B, C and the functions $f:A\to B$ and $g:B\to C$. The composition of these two functions is defined as the function $gof:A\to C$ with $(gof)(a) = g[f(a)]$ for all $a\in A$.

1. Let f and g be functions from R to R defined by $f(x) = x^2$ and $g(x)=x+5$. Prove that $gof\neq fog$

Solution:

$(gof)(x) = g[f(x)]$ for all $x\in R$.

$=g(x^2)$

$=x^2 +5$ -----------(1)

$(fog)(x) = f[g(x)]$ for all $x\in R$

$=f(x + 5)$

$=(x + 5)^2$

$= x^2 + 10x+ 25$ --------(2)

$\therefore$ (1) $\neq$ (2)

2. Let f,g,h be functions from R to R defined by $f(x)=x+2$, $g(x)=x-2$, $h(x)=3x$ for all $x \in R$. Find gof, fog, fof, gog, foh, hog, hof, fohog

Solution:

$(gof)(x) = g[f(x)]$ for all $x \in R$

$=g(x + 2)$

$=x + 2 – 2=x$

$(fog)(x) = f[g(x)]$ for all $x \in R$

$=f(x – 2)$

$=x – 2 + 2$

$=x$

$(fof)(x) = f[f(x)]$ for all $x \in R$

$= f(x + 2)$

$= x + 2 + 2$

$= x + 4$

$(gog)(x) = g[g(x)]$ for all $x \in R$

$= g(x – 2)$

$= x – 2 – 2$

$= x – 4$

$(foh)(x) = f[h(x)]$ for all $x \in R$

$= f(3x)$

$= 3x + 2$

$(hog)(x) = h[g(x)]$ for all $x \in R$

$= h(x - 2)$

$= 3(x - 2)$

$= 3x - 6$

$(hof)(x) = h[f(x)]$ for all $x \in R$

$= h(x + 2)$

$= 3(x + 2)$

$= 3x + 6$

$fo(hog)(x) = f\{h[g(x)]\}$ for all $x \in R$

$= f\{h(x - 2)\}$

$= f\{3(x - 2)\}$

$= f\{3x - 6\}$

$= 3x - 6 + 2$

$= 3x - 4$

.

.

3.Let $A = \{1,2,3\}$ and f, g, h, p be functions on A defined as follows

$f=\{(1,2),(2,3),(3,1)\}$, $g=\{(1,2),(2,1),(3,3)\}$, $h=\{(1,1),(2,2),(3,1)\}$, $p=\{(1,1),(2,2),(3,3)\}$.

Find gof, fog, gop, pog, fop, hog, hof, fohog

Solution:

$gof\,(x) = g[f(x)] = \{(1,1), (2,3), (3,2)\}$

fog (x) = f[g(x)] = {(1,3), (2,2), (3,1)} (Find the other quantities) Invertible Functions

A functions f:A→B is said to be invertible if there exists a function g:B→A such that gof = I_A and fog = I_B, where I_A is the identity function on A and I_B is the identity function on B.

Problems

1. Let A={1,2,3,4} and B={a,b,c,d}. Determine whether the following functions from A to B are invertible or not.

   f={(1,a),(2,a),(3,c),(4,d)}
   g={(1,a),(2,c),(3,d),(4,d)}

Solution:

(gof)(1) = g[f(1)] = g(a) = does not exist

(gof)(2) = g[f(2)] = g(a) = does not exist

(gof)(3) = g[f(3)] = g(c) = does not exist

(gof)(4) = g[f(4)] = g(d) = does not exist

(fog)(1) = f[g(1)] = f(a) = does not exist

(fog)(2) = f[g(2)] = f(c) = does not exist

(fog)(3) = f[g(3)] = f(d) = does not exist

$(fog)(4) = f[g(4)] = f(d) =$ does not exist

Hence f and g are not invertible.

2. Let $A = B = R$, Show that $f : A \rightarrow B$ defined by $f(a) = a+1$ for $a \in A$ is invertible. Solution:

By known theorem if f is one – to – one and onto then f is invertible. Let us consider $f(x1) = f(x2)$

$x1 + 1 = x2 + 1$

$x1 = x2$

Hence f is one – to – one Let $f(x) = y$

$x + 1 = y$

$x = y - 1 \in R$

$\forall\, y - 1 \in R = B$ there exists $x \in R = A$.

Hence f is onto

Thus f is invertible.

3. Let A = B = C = R, and f:A→B and g:B→C defined by f(a)= 2a+1, g(b)=b/3 for all a∈A, b∈B. Compute gof and show that gof is invertib e. What is $(gof)^{-1}$

$(gof)(x) = g[f(x)] = g(2x+1) = (2x+1)/3$

It is evident that gof is one – to – one and onto. Hence gof is invertible.

$(gof)(x) = c$

$(2x+1)/3 = c$

$\Rightarrow x = (3c – 1)/2$

$\Rightarrow (gof)^{-1}(x) = (3c – 1)/2$

4. For the functions f and g from R to R given below, verify that $(gof)^{-1} = f^{-1}og^{-1}$ $f(x) = 2x, g(x) = 3x – 2$

$f(x) = 0.5(x+1), \quad g(x) = 0.5(x-1)$

Solution:

$(gof)(x) = g[f(x)] = g(2x) = 6x – 2$

$(gof)^{-1} = (y + 2)/6$-------- (1) $f^{-1} = y/2$ and $g^{-1} = (y+2)/3$

$f^{-1}og^{-1}=f^{-1}[g^{-1}(x)]=f^{-1}[(y+2)/3]=(y+2)/6$ ---- (2)

By (1) and (2)

we get $(gof)^{-1} = f^{-1}og^{-1}$

Theorem 1.

Let f:A→B and g:B→C be any two functions. Then the following are true:

If f and g are one-to-one, so is gof

If gof is one-to-one, then f is one-to-one If f and g are onto, so is gof

If gof is onto, then g is onto

Theorem 2.

Let f : A→B, g : B→C and h: C→D be three functions. Then (hog)of = ho(gof)

Theorem 3

A function f:A→B is invertible if and only if it is one-to-one and onto.

Theorem 4

If a function f:A→B is invertible then it has a unique inverse. Further if f(a)=b, then $f^{-1}(b)=a$ Theorem 5

Let A and B be finite sets with |A| = |B| and f be a function A to B. Then the following statements are equivalent

(i) f is one – to – one (ii) f is onto (iii) f is invertible.

Hasse Diagram

A Hasse diagram is a type of mathematical diagram used to represent a finite partially ordered set, in the form of a drawing of its transitive reduction.

1. Draw the hasse diagram representing the positive divisors of 36?

 The set of all positive divisors of 36 is

D_{36} = { 1,2,3,4,6,9,12,18,36}

The relation R of divisibility (that is aRb if and only if a divides b) is a partial order on this.

We note that ,under R,

1 is related to all elements of D36

2 is related to 2,4,6,12,18,36;

3 is related to 3,6,9,12,18,36;

4 is related to 4,12,36;

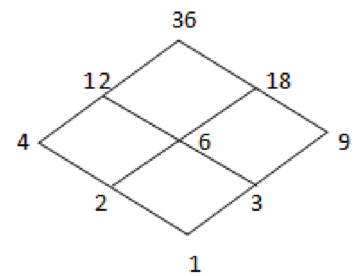
6 is related to 6,12,1,36;

9 is related to 9,18,36;

12 is related to 12 and 36;

18 is related to 18 and 36;

36 is related to 36.



The hasse diagram for R must exhibit all of the above facts. The diagram is as shown above.

2. Let R be a realtion on the set A = {1,2,3,4} defined by xRy if and only if x divides y. Prove that (A,R) is a poset. Draw its hasse diagram?

 From the definition of R, We have

R ={ (x,y) | x,y ∈ A and x divides y}

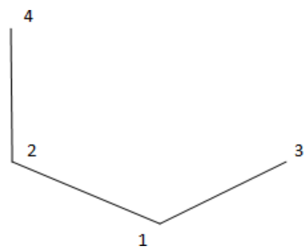
{ (1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}

We observe that (a,a) ∈ R for all a ∈ A. Hence R is reflexive on A.

We verify that the elements of R are such that if (a,b) ∈ R and a ≠ b, then (b,a) ∉ R, Therefore, R is antisymmetric on A.

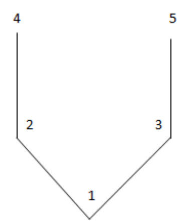
Further , We check that the elements of R are such that if (a,b) ∈ R and (b,c) ∈ R then (a,c) ∈ R. Therefore , A is transitive on A.

Thus, R is reflexive, antisymmetric and transitive. Hence R is a partial order on A; that is (A,R) is a poset.



The Hasse diagram for R is as shown above.

3. Determine the matrix of the partial order whose hasse diagram is given below?



- By examining the given hasse diagram, we find that the corresponding partial order R is defined on the set A = {1,2,3,4,5} and is given by

R = {(1,1),(1,2),(1,4),(1,3),(1,5),(2,2),(2,4),(3,3),(3,5),(4,4),(5,5)}

Consequently, the matrix of R is given below:

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(This matrix can be written down directly by examining the given Hasse diagram. We have written down R explicitly to make things clearer).

Assignment Questions

1) If A = {1,2,3,4} , B = {2,5}, C = {3,4,7} write down the following

AXB, BXA, AXC, CXA, BXC, CXB, A∪(BXC), (A∪B)XC, A∩(BXC), (A∩B)XC, (AXC)∪(BXC), (AXC)∩(BXC)

2) Explain the types of functions.

Outcomes

After completing this module one can

- Understand the relations and functions
- Solve problems involving composition and invertible function.

Further Reading

1. Ralph P. Grimaldi: Discrete and combonotorial Mathematics, 5th Edition,Pearson Education 2004.

2. Basavaraj S Anami and venkanna S Madalli: Discrete Mathematics – A Concept based approach Universities Press,2016.

3. Kenneth H. Rosen: Discrete Mathematics and its Applications, 6th Edition, McGraw Hill 2007.

4. Jayant Ganguly: A Treatise on Discrete Mathematical Structures, Sanguine-Pearson, 2010.

5. D.S. Malik and M.K. Sen: Discrete Mathematical Structures: Theory and Applications, Thomson, 2004.

6. Thomas Koshy: Discrete Mathematics with Applications, Elsevier, 2005, Reprint 2008.

7. https://en.wikipedia.org/wiki/Rule_of_product

8. https://en.wikipedia.org/wiki/Hasse_diagram

9. https://math.feld.cvut.cz/habala/teaching/dma-e/book4x.pdf

Module – IV

STRUCTURE

Introduction

This module gives the overview of principle of inclusion and exclusion where we find the inclusive and exclusive elements. Dearrangement nothing in the write place and rook polynomials. It also consist of recurrence relations with homogeneous and non-homogeneous

objectives

- Understand the concept of inclusion and exclusion.
- Solve problems involving recurrence relations.
- Solving rook polynomial problems.

Principle of Inclusion and Exclusion

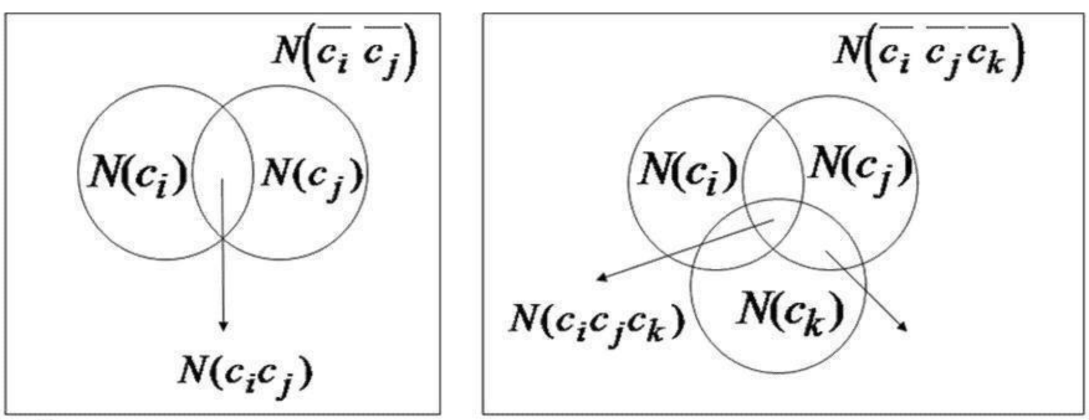
The Principle of Inclusion and Exclusion

In this section we develop some notation for stating this new counting principle. Then we establish the principle by a combinatorial argument. Following this, a wide range of examples demonstrate how this principle may be applied.

The Principle of Inclusion and Exclusion

Let S be a set with $|S| = N$, and let $c_1, c_2, \ldots, c_t$ be a collection of conditions or properties satisfied by some, or all, of the elements of S. Some elements of S may satisfy more than one of the conditions, whereas others may not satisfy any of them.

$N(c_i)$: the number of elements in S that satisfy condition c_i
$N(c_i c_j)$: the number of elements in S that satisfy both of the conditions c_i, c_j, and perhaps some others
$N(\overline{c_i}) = N - N(c_i)$
$N(\overline{c_i}\,\overline{c_j})$: the number of elements in S that do not satisfy either of the conditions c_i or c_j $(\neq N(\overline{c_i c_j}))$



$N(\overline{c_i}\,\overline{c_j}) = N - [N(c_i) + N(c_j)]$

Example 1

Determine the number of positive integers n where 1 n 100 and N is not divisible by 2, 3, or 5.

Here S = {1, 2, 3, . . . , 100} and N = 100. For n S, n satisfies

a) condition c1 if n is divisible by 2.

b) condition c2 if n is divisible by 3, and

c) condition c3 if n is divisible by 5.

Then the answer to this problem is $N = (\bar{c}1\ \bar{c}2\ \bar{c}3)$.
we use the notation *r* to denote the greatest integer less than or equal to r, for any real number r. this function proves to be helpful in this problem as we find that

$N(c_1)$ =100 / 2 = 50 [since the 50 (= 100 / 2)] positive integers 2, 4, 6, 8, . . . , 96, 98 (= 2 . 49), 100 (= 2 . 50) are divisible by 2]; $N(c_2)$ = 100 / 3=331/ 3= 33 positive integers 3, 6, 9, 12, . . . , 96 (=3 . 32), 99(= 3 . 33) are divisible by 3]; $N(c_3)$ = 100 / 5 = 20; N(c1c2) = 100 / 6 = 16 [since here are 16 (= 100 / 6) elements in S that are divisible by both 2 and 3 – hence divisible by lcm (2, 3) = 2 . 3 = 6];

Applying the Principle of Inclusion and Exclusion, we find that

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = S_0 - S_1 + S_2 - S_3 = N - [N(c_1) + N(c_2) + N(c_3)]$$

$$+ \quad [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

= 100 -[50 + 33 + 20] + [16 +10 + 6] - 3 = 26.

(These 26 numbers are 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89, 91, and 97.)

Example 2

Find the number of integers between 1 and 10,000 inclusive, which are divisible by none of 5, 6 or 8.

Let P_1 be the property that an integer is divisible by 5, P_2 the property that an integer is divisible by 6, P_3 the property that an integer is divisible by 8. Let A be the set consisting of the first 10,000 integers. Let A_i be the set consisting of those integers in A with property P_i, for i = 1, 2, 3. The problem is to find the number of integers in $A1\ A\ 2\ A\ 3$. Now $|A_1|$ = 10,000/5 = 2000, $|A_2|$ = 10,000/6 = 1666, $|A_3|$ = 10,000/8 = 1250. Integers in the set $A_1\ A_2$ are divisible by both 5 and 6. Note that an integer is divisible by both 5 and 6 if it is divisible by their lcm {5, 6} = 30.

Also lcm {5, 8} = 40, lcm {6, 8} = 24. then |A1 A2| = 10,000/30 = 333.|A1 A3| = 10,000/40 =250, |A2 A3| = 10,000/24 =416.

Also |A1 A2 A3| = 10,000/120 =83, since lcm {5, 6, 8} = 120.

Now by Principle of Inclusion and Exclusion, the number of integers between 1 and 10,000 that are divisible by none of 5, 6 and 8 equals.

|A1 A2 A3| = |A| – (|A1|+ |A2|+ |A3|) + (|A1 A2|+|A2 A3|+|A3 A1|) – |A1 A2 A3|

10,000 – (2000 + 1666 + 1250) + (333 + 250 + 416) – 83

6000.

Example 3

Determine the number of permutations of the letter J, N, U, I, S, G, R, E, A, T such that none of the words JNU IS and GREAT occur a con e utive letters (that is, permutations such as JNTUISGREAT, ISJNUGREAT, UNJGREATSI etc are not allowed).

Let A be the set of all permutations of the 10 letter given. Let P_1 be the property that a permutation in A contains the word JNU as Conse utive letters, let P_2 be the property that a permutation contains the word IS and let P_3 be property that a permutation contains the GREAT. Let A_i be the set of those perm a ions in A satisfying the property P_i for i = 1, 2,

3. The problem is to find the number of perm tations in Now |A| = 10! = 3,628,800. The set A_1 contains the perm a ions of the 8 symbols JNU, I, S, G, R, E, A, T so $|A_1|$ = 8!. Similarly A_2 contains permu a ions of the 9 symbols J, N, U , IS, G, R, E, A, T so $|A_2|$ = 9! = 362,880. Similarly A_3 contains permutations of the 6 symbols J, N, U , I, S, GREAT so $|A_3|$ = 6! = 720. Also $|A_1\ A_2|$ = 7! = 5040, since $A_1\ A_2$ contains permutations of the 7 symbols JNU, IS, G, R, E, A, T. Also $|A_1\ A_3|$ = 4! = 24, since $A_1\ A_3$ contains permutations of the 4 symbols JNU, I, S, GREAT. Finally $|A_1\ A_2\ A_3|$ = 3! = 6, since $A_1\ A_2\ A_3$ contains the permutations of the three symbols JNU, IS, GREAT. Using the Principle of Inclusion – Exclusion we have

| A1uA2uA3 | = 3,628,800 - (40,320 + 362,880 + 720)

\+ (5040 + 24 +120) - 6

3,230,058

Derangements: Nothing Is in Its Place

We find these ideas helpful in working some of the following Examples.

Example 1

Peggy has seven books to review for the C–H Company, s she hires seven people to review them. She wants two reviews per book, so the first week she gives each person one book to read and then redistributes the books at the start of these end week. In how many ways can she make these two distributions so that he gets two reviews (by different people) of each book?

She can distribute the books in 7! ways the first week. Numbering both the books and the reviews (for the first week) as 1, 2, . . ., 7, for the e ond di tribution she must arrange

These numbers so that none of them is in its nat ral position. This she can do in d7 ways.
By the rule of product, she can make he wo distributions in (7!)d7 = (7!)2(e^{-1}) ways.

The number of derangements of a se wi h n elements is

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right), \textit{ for } n > 1$$

For example

$$D_2 = 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) = 1$$

$$D_3 = 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) = 6\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 2$$

$D_4 = 9$, $D_5 = 44$, $D_6 = 265$, $D_7 = 1854$.

Example 2

A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability in a group of 100 letters (a) no letter is put into the correct envelope (b) exactly 1 letter is put into the correct envelope (c) exactly 98 letters are put into the correct envelope (d) exactly 99 letters are put into the correct envelope (e) all letters are put into the correct envelopes?

The probability of derangements of n objects is D_n/n!. For example

Probability of derangements

n:	2	3	4	5	6	7
Dn/n!:	0.5	0.333	0.375	0 3667	0.36806	0.36786

a) The probability of no letter being p t in the orrect envelope is D_n/100! Because the number of favorable cases (the derangements) is D_n and the total number of favorable cases is n! = 100!.

Example 3

a) List all the derangements of the numbers 1, 2, 3, 4, 5 where the first three numbers are 1, 2, 3 in order.

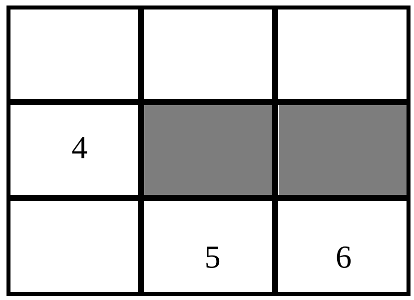
b) List all the derangements of the numbers 1, 2, 3, 4, 5, 6 where the first three numbers are 1, 2, 3 in some order.

c) When 1, 2, 3 are in some order, there are only two derange ents (i) 23,154 and (ii) 31, 254 (other examples include 21,354 and 32,154)

d) There are only for such derangements. For example, ne such set is (i) 231,546 (ii) 312,546 (iii) 231,645 (iv) 312,645 (other example in lude (i) 213,546 (ii) 321, 546 (iii) 213,645 (iv) 321,645)

Rook Polynomials

Consider the six-square –chessboard‖ shown in Figure (Note: The shaded squares are not part of the chessboard.) in chess a piece called a rook or castle is allowed at one turn to be moved horizontally or vertically over as many unoccupied spaces as one wishes. Here a rook in square 3 of the figure could be moved in one turn to squares 1, 2, or 4. A rook at square 5 could be moved to square 6 or square 2 (even though there is no square between squares 5 and 2).



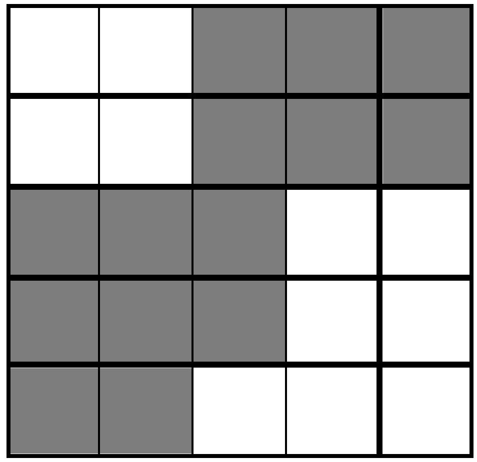
For k Z^+ we want to determine the n mber of way in which k rooks can be placed on the unshaded squares of this chess-board so that no two of them can take each other–that is, no two of them are in the same row or col mn of the chessboard. This number is denoted by r_k or by r_k (C) if we wish to stress ha we are working on particular chessboard C.

For any chessboard, r1 is the number of sq ares on the board. Here $r_1 = 6$. Two nontaking rooks can be placed at the following pairs of positions: {1, 4}, {1, 5}, {2, 4}, {2, 6}, {3, 5}, {3, 6}, {4, 5} and {4, 6}, so $r_2 = 8$ Continuing, we find that $r_3 = 2$, using the Locations {1, 4, 5} and {2, 4, 6}; $r_k = 0$, for k 4

With r0 = 1, the rook polynomial, r (C, x), for the chessboard in Fig 8.6 is defined as r (C, x) = $1 + 6x + 8x^2 + 2x^3$. For each k 0, the coefficient of x^k is the number of ways we can place k nontaking rooks on chessboard C.

What e have done here (using a case-by-case analysis) soon proves tedious. As the size of the board increases, we have to consider cases wherein numbers such as r_4 and r_5 are nonzero. Consequently, we shall now make some observations that will allow us to make use of small boards and somehow break up a large board into smaller subboards.

The chessboard C in Fig 8.7 is made up of 11 unshaded square. We note that C consists of a 2 X 2 subboard C_1 located in the upper left corner and a seven-square subboard C_2 located in the in the lower right corner. These subboards are disjoint because they have no squares in the same row or column of C.



Calculating as we did for our first chessboard, here we find

$r(C_1, x) = 1 + 4x + 2x^2$, $\qquad$ $r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$.

$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 = r(C_1, x) \,.\, r(C_2, x)$.

Hence $r(C, x) = r(C_1, x) \,.\, r(C_2, x)$. B did his occur by luck or is something happening here that we should examine more closely? For example, to obtain r_3 for C, we send to know in how many ways three nontaking rooks can be placed on board C.

These fall into there cases:

a) All three rooks are on subboard C_2 (and none is on C_1): (2)(1) = 2 ways.

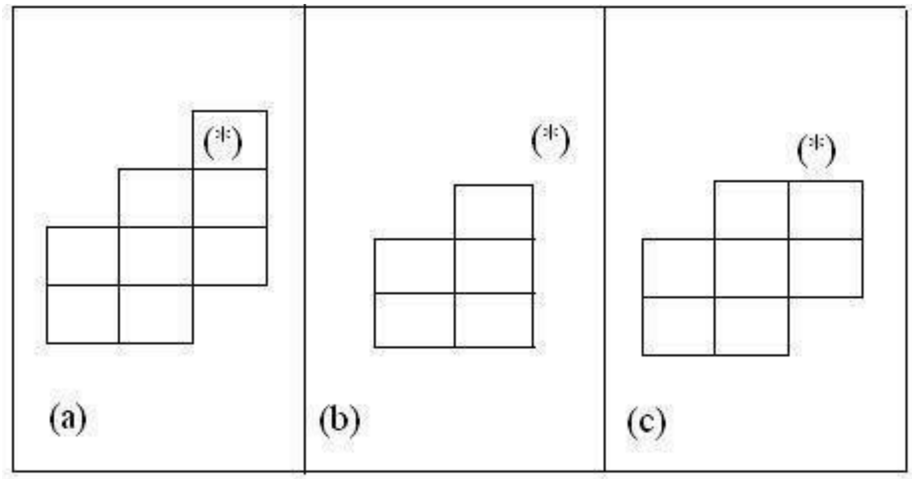
b) Two rooks are on subboard C_2 and one is on C_1: (10)(4) = 40 ways.

c) One rook is on subboard C_2 and two are on C_1: (7)(2) = 14 ways.

Consequently, three nontaking rooks can be placed on board C in (2)(1) + (10)(4) + (7)(2) = 56 ays. Here e see that 56 arises just as the coefficient of x^3 does in the product $r(C_1, x) \,.\, r(C_2, x)$.

In general, if C is a chessboard made up of pairwise disjoint subboards $C_1, C_2, \ldots, C_n$, then $r(C, x)\ r(C_2, x) \ldots r(C_n, x)$.

The last result for this section demonstrates the type of principle we have seen in other results in combinatorial and discrete mathematics: Given a large chessboard, break it into smaller subboards whose rook polynomials can be determined by inspection.



Consider chessboard C in Figure. For k 1, suppose we wish to place k non taking rooks on C. For each square of C, such as the one designated by (*), there are two possibilities to examine.

a) Place one rook on the designated square. Then we remove, as possible locations for the other k – 1 rooks, all other squares of C in the same row or column as the designated square. We use C_s to denote the remaining smaller subboard [seen in Fig.8.8 (b)].

b) We do not use the designated square at all. The k rooks are placed on the sub board Ce [C with the one designated square eliminated — as shown in the Fig. 8.8(c)].

Since these t o cases are all-inclusive and mutually disjoint,

$$r_k(C) = r_k\text{-}1(C_s) + r_k(C_e).$$

From this we see that

$$r_k(C)x^k = r_{k-1}(C_s)x^k + r_k(C_e)x^k. \qquad (1)$$

If n is the number of squares in the chessboard (here n is 8), Then Eq. (1) is valid for all 1 k n, and we write

$$\sum_{k=1}^{n} r_k(C)x^k = \sum_{k=1}^{N} r_{k-1}(C_s)x^k + \sum_{k=1}^{N} r_k(C_e)x^k. \qquad (2)$$

For Eq.(2) we realize that the summations may stop before k = n. We have seen cases, as in Fig. 8.6, where r_n and some prior r_k's are 0. The summations start at k = 1, for otherwise we could find ourselves with the term $r_{-1}(C_s)x^0$ in the first summand on the right-hand side of Eq. (2).

$$\sum_{k=1}^{n} r_k(C)x^k = x\sum_{k=1}^{N} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{n} r_k(C_e)x^k \qquad (3)$$

or

$$1+\sum_{k=1}^{N} r_k(C)x^k = x.r(C_s, x) + \sum_{k=1}^{n} r_k(C_e)x^k + 1,$$

from which it follows that

$$r(C, x) = x.r(C_s, x) + r(C_e, x). \qquad (4)$$

We now use this final equation to determine the rook polynomial for the chessboard shown in part (a) of Figure .Each time the idea in Eq. (4) is used, we mark the special square we are using with (*). Parentheses are placed about each chessboard to denote the rook polynomial of the board.

$$= x^2(1+2x) + 2x(1+4x+2x^2) + x(1+3x+x^2)$$

$$= 3x + 12x^2 + 7x^3 + x(1+2x) + (1+4x+2x^2)$$

$$= 1+8x+16x^2+7x^3.$$

Sequences and Recurrence Relations

EXAMPLE 8.1.2

Consider the following two sequences:

$$S_1 : 3, 5, 7, 9, \ldots$$
$$S_2 : 3, 9, 27, 81, \ldots$$

We can find a formula for the nth term of sequences S_1 and S_2 by observing the pattern of the sequences.

$$S_1 : 2 \cdot 1 + 1, 2 \cdot 2 + 1, 2 \cdot 3 + 1, 2 \cdot 4 + 1, \ldots$$
$$S_2 : 3^1, 3^2, 3^3, 3^4, \ldots$$

For S_1, $a_n = 2n + 1$ for $n \geq 1$, and for S_2, $a_n = 3^n$ for $n \geq 1$. This type of formula is called an **explicit formula** for the sequence, because using this formula we can directly find any term of the sequence without using other terms of the sequence. For example, $a_3 = 2 \cdot 3 + 1 = 7$.

EXAMPLE 8.1.3

Let S denote the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots .$$

For this sequence, the explicit formula is not obvious. If we observe closely, however, we find that the pattern of the sequence is such that any term after the second term is the sum of the preceding two terms. Now

$$\begin{aligned}
\text{3rd term} &= 2 = 1 + 1 = \text{1st term} + \text{2nd term}\\
\text{4th term} &= 3 = 1 + 2 = \text{2nd term} + \text{3rd term}\\
\text{5th term} &= 5 = 2 + 3 = \text{3rd term} + \text{4th term}\\
\text{6th term} &= 8 = 3 + 5 = \text{4th term} + \text{5th term}\\
\text{7th term} &= 13 = 5 + 8 = \text{5th term} + \text{6th term}
\end{aligned}$$

Hence, the sequence S can be defined by the equation

$$f_n = f_{n-1} + f_{n-2} \tag{8.1}$$

for all $n \geq 3$ and

$$\begin{aligned} f_1 &= 1,\\ f_2 &= 1. \end{aligned} \tag{8.2}$$

EXAMPLE 8.1.4

Consider the function $f : \mathbb{N}^0 \to \mathbb{Z}^+$ defined by

$$f(0) = 1,$$
$$f(n) = nf(n-1) \quad \text{for all } n \geq 1.$$

Then

$$f(0) = 1 = 0!,$$
$$f(1) = 1 \cdot f(0) = 1 = 1!,$$
$$f(2) = 2 \cdot f(1) = 2 \cdot 1 = 2 = 2!,$$
$$f(3) = 3 \cdot f(2) = 3 \cdot 2 \cdot 1 = 6 = 3!,$$

and so on. Here $f(n) = nf(n-1)$ for all $n \geq 1$ is the recurrence relation, and $f(0) = 1$ is the initial condition for the function f. Notice that the function f is nothing but the factorial function, i.e., $f(n) = n!$ for all $n \geq 0$.

4.5.1 Sequences and Recurrence Relations

Let us consider the function f as given in (8.3). If we write $a_n = f(n)$, then (8.3) translates into the following equation:

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2.$$

That is, a_n is defined in terms of a_{n-1} and a_{n-2}. As remarked previously, such an equation is called a recurrence relation. Moreover, (8.4) translates into $a_0 = 5$ and $a_1 = 7$. These are called the initial conditions for the recurrence relation.

DEFINITION 8.1.5

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ is an equation that relates a_n to some of the terms $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}$ for all integers n with $n \geq k$, where k is a nonnegative integer. The **initial conditions** for the recurrence relation are a set of values that explicitly define some of the members of $a_0, a_1, a_2, \ldots, a_{k-1}$.

The equation

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2,$$

as defined above, relates a_n to a_{n-1} and a_{n-2}. Here $k = 2$. So this is a recurrence relation with initial conditions $a_0 = 5$ and $a_1 = 7$. ∎

EXAMPLE 8.1.9

Number of subsets of a finite set. Let s_n denote the number of subsets of a set A with n elements, $n \geq 0$. In Worked-Out Exercise 9 (Chapter 2, page 144), we proved that

$$s_0 = 1,$$
$$s_n = 2s_{n-1}, \quad \text{if } n > 0$$

Hence, a recurrence relation for the sequence $s_0, s_1, s_2, s_3, s_4, \ldots$ is

$$s_n = 2s_{n-1}, \quad n \geq 1$$

and an initial condition is $s_0 = 1$.

EXAMPLE 8.1.10

Compound Interest. Sam received a yearly bonus and deposited \$10,000 in a local bank yielding 7% interest compounded annually. Sam wants to know the total amount accumulated after n years. Let A_n denote the total amount accumulated after n years. Let us determine a recurrence relation and initial conditions for the sequence $A_0, A_1, A_2, A_3, \ldots$.

The amount accumulated after one year is the initial amount plus the interest on the initial amount. Now A_{n-1} is the amount accumulated after $n - 1$ years. This implies that the amount at the beginning of nth year is A_{n-1}. It follows that the total amount accumulated after n years is the amount at the beginning of the nth year plus the interest on this amount. Because the interest rate is 7%, the interest earned during the nth year is $(0.07)A_{n-1}$. Hence,

$$\begin{aligned} A_n &= A_{n-1} + (0.07)A_{n-1} \\ &= 1.07A_{n-1}, \quad n \geq 1, \\ A_0 &= 10000. \end{aligned}$$

Linear Homogenous Recurrence Relations

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence of numbers. A **linear homogeneous recurrence relation** of order k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \qquad (8.31)$$

where $c_k \neq 0$ and $c_1, c_2, c_3, \ldots,$ and c_k are constants.

Linear Homogenous Recurrence Relations

Consider the following recurrence relations.

(i) $a_n = 3a_{n-1} + a_{n-2}$

(ii) $a_n = 3a_{n-1} + 5$

(iii) $a_n = 3a_{n-1} + a_{n-2} \cdot a_{n-3}$

(iv) $a_n = 3a_{n-1} + a_{n-2} + \sqrt{2}a_{n-3}$

(v) $a_n = 3a_{n-1} + na_{n-2}$

Recurrence relations (i), (ii), (iii), and (iv) are recurrence relations with constant coefficients. Recurrence relation (v), $a_n = 3a_{n-1} + na_{n-2}$, is not a relation with constant coefficients. Notice that (i) is a linear homogeneous recurrence

Linear Homogenous Recurrence Relations

A sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ is said to **satisfy** a linear homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \quad c_k \neq 0 \qquad (8.32)$$

of order k with constant coefficients if $s_n = c_1 s_{n-1} + c_2 s_{n-2} + c_3 s_{n-3} + \cdots + c_k s_{n-k}$.

If a sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ satisfies a linear homogeneous recurrence relation, then the sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ is also called a **solution** of that recurrence relation.

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $c_2 \neq 0$, $n > 1$ be a linear homogeneous recurrence relation with constant coefficients. The equation

$$t^2 - c_1 t - c_2 = 0$$

is called the **characteristic equation** of the recurrence relation.

Theorem 8.2.9: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad n > 1 \tag{8.37}$$

be a linear homogeneous recurrence relation of order 2, where c_1 and c_2 are constants and $c_2 \neq 0$

(i) If the sequences $\{s_n\}$ and $\{p_n\}$ satisfy (8.37), then for any constants b and d, the sequence $\{bs_n + dp_n\}$ satisfies (8.37).

(ii) Let r be a root of the characteristic equation

$$t^2 - c_1 t - c_2 = 0 \tag{8.38}$$

of (8.37). Then the sequence $\{r^n\}$ is a solution of (8.37).

Theorem 8.2.10: Suppose that a sequence $\{d_n\}$ is a solution of the recurrence relation (8.37). If r_1 and r_2 are the distinct roots of the characteristic equation (8.38), then there exist constants b and d, which

Corollary 8.2.11: Suppose that

$$a_0 = d_0, \qquad a_1 = d_1$$

are the initial conditions for the recurrence relation (8.37), where d_0 and d_1 are constants. Further suppose that r_1 and r_2 are the roots of (8.38). If $r_1 \neq r_2$, then there exist constants b and d, which are to be determined by initial conditions, such that the solution of the recurrence relation (8.37) is

$$a_n = br_1^n + dr_2^n, \quad n = 0, 1, \ldots.$$

EXAMPLE 8.2.12

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 7a_{n-1} - 10a_{n-2} \tag{8.41}$$

with initial conditions

$$a_0 = 1$$
$$a_1 = 8.$$

The characteristic equation of the given recurrence relation is:

$$t^2 - 7t + 10 = 0.$$

Next, we find the roots of this equation. Now,

$$t^2 - 7t + 10 = (t - 5)(t - 2)$$

and so

$$(t - 5)(t - 2) = 0.$$

This implies that the roots of the characteristic equation are $t = 5$, and $t = 2$. The roots are distinct. By Theorem 8.2.10, there exist constants c_1 and c_2, which are to be determined from initial conditions, such that

$$a_n = c_1 5^n + c_2 2^n, \quad n \geq 0.$$

We substitute $n = 0$ and $n = 1$, respectively, to obtain

$$a_0 = c_1 + c_2,$$
$$a_1 = 5c_1 + 2c_2.$$

Using the initial conditions, we get

$$c_1 + c_2 = 1,$$
$$5c_1 + 2c_2 = 8.$$

Solving these equations for c_1 and c_2, we get $c_1 = 2$ and $c_2 = -1$. Hence,

$$a_n = 2 \cdot 5^n - 2^n, \quad n \geq 0.$$

Hence, the sequence $\{2 \cdot 5^n - 2^n\}$ is the solution.

Theorem 8.2.13: Suppose that a sequence $\{s_n\}$ is a solution of the recurrence relation (8.37). If r_1 and r_2 are the roots of the characteristic equation (8.38) such that $r_1 = r_2 = r$, then there exist constants b and d, which are to be determined, such that the solution of the recurrence relation (8.37) is

$$s_n = br^n + dnr^n, \quad n = 0, 1, \ldots.$$

To obtain the characteristic equation of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$, $c_k \neq 0$, substitute $a_n = t^n, t \neq 0$, to get

$$t^n = c_1 t^{n-1} + c_2 t^{n-2} + c_3 t^{n-3} + \cdots + c_k t^{n-k}.$$

Thus,

$$t^n = c_1 t^{n-1} + c_2 t^{n-2} + c_3 t^{n-3} + \cdots + c_k t^{n-k}$$
$$\Rightarrow t^n - c_1 t^{n-1} - c_2 t^{n-2} - c_3 t^{n-3} - \cdots - c_k t^{n-k} = 0$$
$$\Rightarrow t^{n-k}(t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k) = 0.$$

Because $t \neq 0$, we have, $t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k = 0$, which is the characteristic equation.

Linear Nonhomogenous Recurrence Relations

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n.$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = 1$.

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n(n^2 + 6n + 5).$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = n^2 + 6n + 5$.

Linear Nonhomogenous Recurrence Relations

Theorem 8.3.5: Let

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n) \qquad (8.62)$$

be a nonhomogeneous recurrence relation, where c_i, $i = 1, 2, \ldots, k$, are constants, $c_k \neq 0$, and $f(n)$ is a nonzero real-valued function. Suppose $\{r_n\}$ is a particular solution of (8.62). Then $\{u_n\}$ is a solution of (8.62) if and only if $u_n = r_n + s_n$, for all n, and $\{s_n\}$ is a solution of the associated homogeneous part, $a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = 0$.

Consider the recurrence relation

$$a_n - 3a_{n-1} = 2^n(4n + 3), \quad n > 1 \qquad (8.94)$$

with initial conditions

$$a_0 = 0,$$
$$a_1 = 14.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n(un + v).$$

Here $d = 3, b = 2, u = 4$, and $v = 3$.

We can solve this recurrence by using the technique of Theorem 8.3.10 and obtaining

$$a_n = c_0 3^n + c_1 2^n + c_2 n 2^n,$$

where c_0, c_1, and c_2 are constants, which are to be determined from the initial conditions.

Assignment questions

1. Define dearrangents. Find the number of deaarangents of 1,2,3,4 using exponential technique.
2. Determine the number of positive integers 1-100 and not divisible by 2 or 3 or 5.

Outcomes

After completing this module one can do

1) Solve rook polynomial problems
2) Understand inclusion and exclusion principle.
3) Solving recurrence relation.

Further Reading

1. Ralph P. Grimaldi: Discrete and combonotorial Mathematics, 5th Edition,Pearson Education 2004.

2. Basavaraj S Anami and venkanna S Madalli: Discrete Mathematics – A Concept based approach Universities Press,2016.

3. Kenneth H. Rosen: Discrete Mathematics and its Applications, 6th Edition, McGraw Hill 2007.

4. Jayant Ganguly: A Treatise on Discrete Mathematical Structures, Sanguine-Pearson, 2010.

5. D.S. Malik and M.K. Sen: Discrete Mathematical Structures: Theory and Applications, Thomson, 2004.

6. Thomas Koshy: Discrete Mathematics with Applications, Elsevier, 2005, Reprint 2008.

7. https://en.wikipedia.org/wiki/Rule_of_product

8. https://en.wikipedia.org/wiki/Hasse_diagram

9. https://math.feld.cvut.cz/habala/teaching/dma-e/book4x.pdf

Module – V

STRUCTURE

Introduction

Objective

Introduction

Types of graphs

Directed graphs

Sub graphs

Isomorphism

The Konigsberg Bridge Problem

Trees

Assignment Questions

Outcomes

Further Reading

Introduction

This module gives the overview of Graphs and Trees, which consists of finding the in degree outdegree, types of graphs and trees.it also includes finding the optimal prefix code for the tree.

objectives

- Understand the concept of graphs and trees.
- Solve problems involving graphs and trees.

INTRODUCTION :

This topic is about a branch of discrete mathematics called graph theory. Discrete mathematics – the study of discrete structure (usually finite collections) and their properties include combinatorics (the study of combination and enumeration of objects) algorithms for computing properties of collections of objects, and graph theory (the study of objects and their relations).

Many problem in discrete mathematics can be stated and solved using graph theory therefore graph theory is considered by many to be one of the most important and vibrant fields within discrete mathematics.

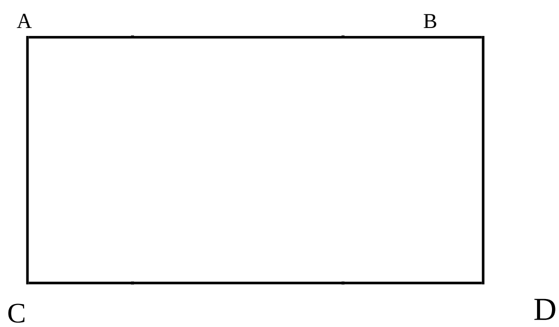


Fig (a) – Graph with five vertices and seven edges

Observe that this definition permits an edge to be associated with a vertex pair (v_i , v_j) such an edge having the same vertex as both its end vertex comes is called a self-loop.

Edge e_1 in fig (a) is a self-loop. Also note that the definition all has more than one edge associated with a given pair of vertices, for example, edges e_4 and e_5 in fig (a), such edges are referred to as _parallel edges‘ graph that has neither self-loops nor **parallel edges** is called a _**simple graph**‘.

“The number of vertices of odd degree in a graph is always even”.

Proof : If we consider the vertices with odd and even degree separately, the quantity in the left side of equation (1.1) can be expressed as the sum of two sum, each taken over vertices of even and odd degree respectively, as follows.

$$\sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k) \qquad (1.2)$$

Since the left hand side in equation (1.2) is even, and the first expression on the right hand side is even (being a sum of even numbers), the second expression must also be even

$$\sum_{odd} d(v_k) = an \; even \; number \qquad (1.3)$$

Because in equation (1.3) each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem.

Types of graphs

A graph in which all vertices are of equal degree is called a **„regular graph‟** (or simply a regular).

DEFINITION:

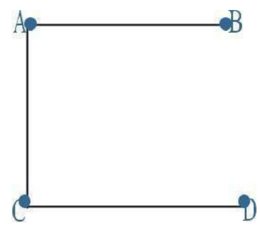
ISOLATED VERTEX, PENDANT VERTEX AND NULL GRAPH

A vertex having no incident edge is called an **‗isolated vertex**‘. In other words, isolated vertices are vertices with zero degree are isolated vertices a vertex of degree one is called a pendent vertex or an end vertex is a pendent vertex. Two adjacent edges are said to be in series if their common vertex is of degree two.

In the definition of a graph G = (V, E), it is possible for the edge set E to be empty. Such a graph without n edge is called a _**null graph**‟. In other words, every vertex in a null graph is an isolated vertex. Although the edge set E may empty the vertex set V must not be empty; otherwise there is no graph. In other words, by definition, a graph must have at least one vertex.

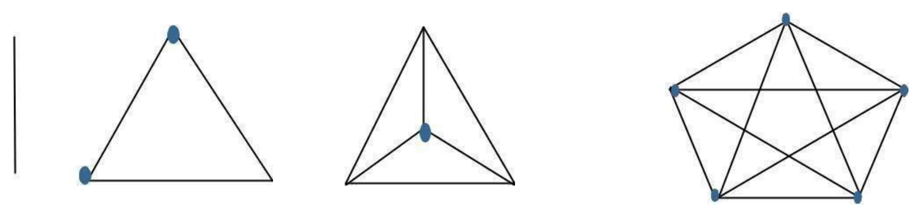
SIMPLE GRAPH :

A graph which does not contain loops and multiple edges is called simple graph.



COMPLETE GRAPH :

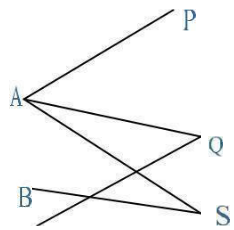
A simple graph of order 2 in which there is an edge between every pair of vertices is called a complete graph (or a full graph). In other words a complete graph is a simple graph in which every pair of distinct vertices are adjacent.



(a) : K_2 (b) : K_3 (c) : K_4 (d) : K_5

BIPARTITE GRAPH

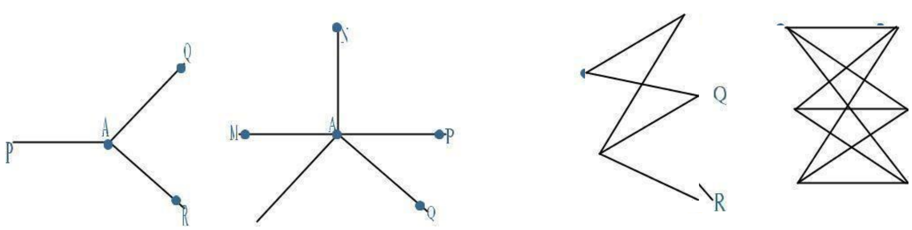
Suppose a simple graph G is such that its vertex set V is the union of two of its mutually disjoint non-empty subsets V_1 and V_2 which are such that each edge in G joins a vertex inV_1 and a vertex inV_2.Then G is called a **bipartite graph**. If E is the edge set of this graph, the graph is denoted by G = (V_1, V_2: E), or G = G(V_1, V_2: E). The sets $V_1$and V_2 are called **bipartites** (or partitions) of the vertex set V.



COMPLETE BIPARTITE GRAPH

A bipartite graph G= {V_1, V_2 ; E} is called a complete bipartite graph, if there is an edge between every vertex in V_1 and every vertex in V_2. The bipartite graph is not a complete bipartite graph. Observe for example that the graph does not contain an edge joining A and S.

A complete bipartite graph G={ V_1, V_2 ; E} in which the bipartites $V_1$and V_2 contain rand s vertices respectively, with r s is denoted by $K_{r,s}$ .In this graph each of r vertices in V= is joined to each of s vertices in V_2 .Thus $K_{r,s}$ has r+ s vertices and rs edges. That is $K_{r,s}$ is of order r+s and size rs. It is therefore a (r+ s,rs) graph.

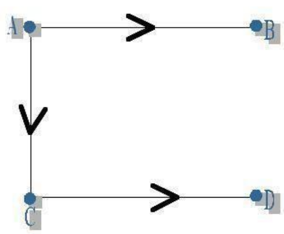


(a) $K_{1,3}$ (b) $K_{1,5}$ (c) $K_{2,3}$ (d) $K_{3,3}$

DIRECTED GRAPHS AND GRAPHS:

DIRECTED GRAPHS :

Look at the diagram shown below. This diagram consists of four vertices A, B, C, D and three edges AB,CD,AC with directions attached to the .The directions being indicated by arrows.



DEFINITION OF A DIRECTED GRAPH :

A directed graph (or digraph) is a pair (V,E), where V is a non empty set and E is a set of ordered pairs of elements taken from the set V.
For a directed graph (V, E), the elements of V are called **Vertices** (points or nodes) and the elements of E are called **"Directed Edges".** The set V is called the **vertex set** and the set E is called the **directed** edge set

The directed graph (V,E) is also denoted

by D=(V,E) or D =D(V,E).

The geometrical figure that depicts a directed graph for which the vertex set is

V={A,B,C,D} and the edge set is

E={AB,CD,CA}={(A,B),(C,D),(C,A)}

IN- DEGREE AND OUT –DEGREE

If V is the vertex of a digraph D, the number of edges for which V is the initial vertex is called the **outgoing degree** or the **out degree** of V and the number of edges for which V is the terminal vertex is called the **incoming degree** or the **in degree** of V. The out degree of V is denoted by d^+ (v) or o d (v) and the in degree of V is denoted by d^- (v) or i d(v).

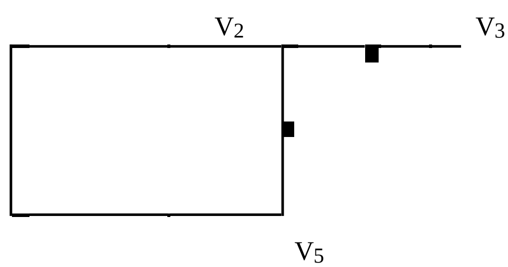
It follows that

. d^+ (v) =0, if V is a sink

d^- (v) =0, if V is a source

. d^+ (v) = d^- (v) = 0, if V

For the digraph shown in figure the out degrees and the in degrees of the vertices are as given below



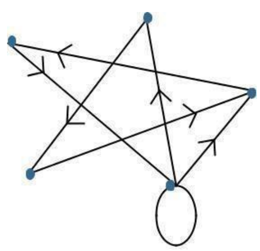
$d^+(v_1) = 2$	$d^-(v_1) = 1$
$d^+(v_2) = 1$	$d^-(v_2) = 3$
$d^+(v_3) = 1$	$d^-(v_3) = 2$
$d^+(v_4) = 0$	$d^-(v_4) = 0$
$d^+(v_5) = 2$	$d^-(v_5) = 1$
$d^+(v_6) = 2$	$d^-(v_6) = 1$

We note that ,in the above digraph, there is a directed loop at the vertex v_3 and this loop contributes a count 1 to each of $d^{+}(v_3)$ and $d^{-}(v_3)$.

We further observe that the above digraph has 6 vertices and 8 edges and the sums of the out-degrees and in-degrees of its vertices are

$$d^{+}(v_j) \,\square\, 8, d^{-}(v_i) \,\square\, 8$$

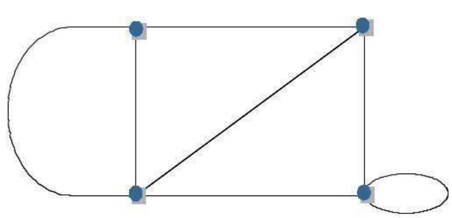
Example 1: Find the in- degrees and the out-degrees of the vertices of the digraph shown in Figure



vertices	V_1	V_2	V_3	V_4	V_5
D^+	1	1	1	3	1
d^-	1	1	1	2	2

Handshaking property :

Let us refer back to degree of the graph shown in figure we have, in this graph,



$d(V_1) = 3$, $d(V_2) = 4$, $d(V_3) = 4$, $d(V_4) = 3$

Also,the graph has 7 edges, we observe that $\deg(V_1) + \deg(V_2) + \deg(V_3) + \deg(V_4) = 14 = 2 \times 7$

Property: The sum of the degrees of all the vertices in a graph is an even number, and this number is equal to twice the number of edges in the graph.

This property is obvious from the fact that while counting the degree of vertices, each edge is counted twice (once at each end).

In an alternative form, this property reads as follows:

For a graph G = (V,E)

deg(v)=2|E|

The aforesaid property is popularly called the ***„handshaking property'***

Because, it essentially states that if several people shake hands, then the total number of hands shaken must be even, because just two hands are involved in each hand shake.

Theorem : In every graph the number of vertices of odd degrees is even

<u>**Proof :**</u> Consider a graph with n vertices. Suppose K of these vertices are of odd degree so that the remaining n-k vertices are of even degree. Denote the vertices with odd

degree by $V_1, V_2, V_3, \ldots\ldots, V_k$ and the vertices with even degree by $v_{k+1}, v_{k+2}, \ldots\ldots, v_n$ then

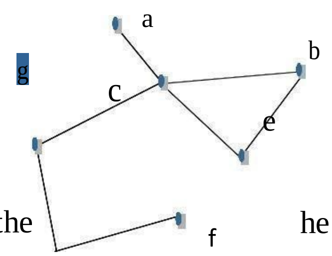
the sum of the degrees of vertices is

$$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) \qquad (1)$$

In view of the hand shaking property, the sum on the left hand side of the above expression is equal to twice the number of edges in the graph. As such, this sum is even. Further, the second sum in the right hand side is the sum of the degrees of the vertices with even degrees. As such this sum is also even. Therefore, the first sum in the right hand side must be even; that is, $deg(V_1) + deg(V_2) + \text{--} + deg\ (V_k) = \text{Even}$—(ii)

But, each of $deg(V_1)$, $deg(V_2)$,…….,$deg\ (V_k)$ is odd. Therefore, the number of terms in the left hand side of (ii) must be even; that is, K is even

Example : For the graph shown in fig 1.15 indicating the degree of each vertex and verify the handshaking property



Solution : By examining the he degrees of its vertices are as given below:

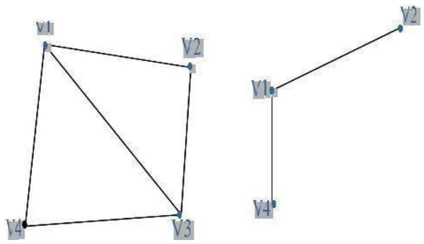
deg (a) = 1, deg (b) = 2, deg (c) = 4, deg (d) = 2, deg(f) = 1, deg (e) = 2, deg (g) = 2
We note that e is an isolated vertex and h is a pendant vertex.
Further, we observe that the sum of the degrees of vertices is equal to 16. Also, the graph has 8 edges. Thus, the sum of the degrees of vertices is equal to twice the number of edges.
This verifies the handshaking property for the given graph.

SUBGRAPHS

Given two graphs G and G_1, we ay that G_1 is a ***sub graph*** of G if the following conditions hold:

(1). All the vertices and all the edge of G_1 are in G.

(2). Each edges of G_1 has the same end vertices in G as in G_1.



Essentially, a sub graph is a graph which is a part of another graph. Any graph isomorphic to a sub graph of a graph G is also referred to as a sub graph of G.

Consider the two graphs G_1 and G shown in figures 1.16(a) and 1.16(b) respectively, we observe that all vertices and all edges of the graph G_1 are in the graphs G and that every edge in G_1 has same end vertices in G as in G_1.Therefore G_1 is a sub graph of G .In the diagram of G ,the part G_1 is shown in thick lines.

The following observation can be made immediately.

i) Every graph is a sub-graph of itself.

ii) Every simple graph of n vertices is a sub graph of the complete graph K_n.

iii) G_1 is a subgraph of a graph G_2 and G_2 is a subgraph of a graph G,then G_1 is a subgraph of a graph G.

iv) A single vertex in a graph G is a subgraph of a graph G.

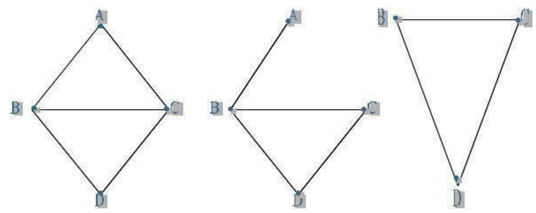
v) A single edge in a graph G together with its end vertices,is a subgraph of G

SPANNING SUBGRAPH :

Given a graph G=(V, E), if there is a subgraph $G_1=(V_1,E_1)$ of G such that $V_1=V$ then

G_1 is called a spanning subgraph of G.

In other words , a subgraph G_1 of a graph G is a spanning subgraph of G whenever the vertex set of G_1 contains all vertices of G. Thus a graph and all its spanning subgraphs have the same vertex set. Obviously every graph is its own spanning subgraph.



For example, for the graph shown in figure1, the graph shown in figure 2 is a spanning subgraph whereas the graph shown in figure 3 is a subgraph but not a spanning subgraph

INDUCED SUBGRAPH

Given a graph G=(V,E), suppose there is a subgraph $G_1=(V_1,E_1)$ of G such that every edge {A,B} of G, where AB V_1 is an edge of G_1 also .then G_1 is called an induced subgraph of G (induced by V_1) and is denoted by < V_1 >.

It follows that a subgraph $G_1=(V_1,E_1)$ of a graph G=(V,E) is not an induced subgraph of G, if for some A,B V_1,there is an edge{A,B} which is in G but not in G_1.

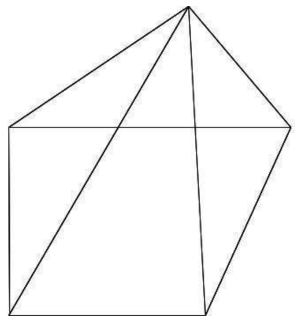
COMPLEMENT OF A SUBGRAPH

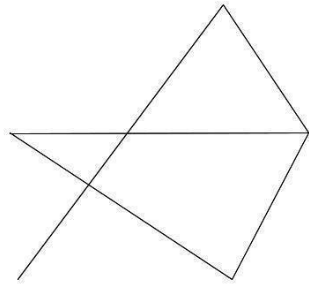
Given a graph G and a subgraph G_1 of G, the subgraph of G obtained by deleting from all the edges that belongs to G_1 is called the complement of G_1 in G; it is denoted by G-G_1 or G_1

In other ords ,if E_1 is the set of all edges of G_1 then the complement of G_1 in G is given by G_1 = G-E_1.We can check that G_1=G-G_1.

For example :

Consider the graph G shown in figure 1.Let G_1 be the subgraph of G shown by thick lines in this figure. The complement of G_1 in G, namely G_1, is as shown in figure 2





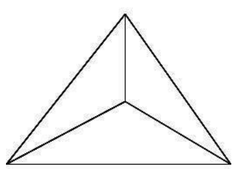
COMPLEMENT OF A SIMPLE GRAPH

Earlier we have noted that every simple sub graph of order n is a subgraph of the complete graph Kn .If G is a simple graph of order n ,then the complement of G in K_n is called the **complement of G**, it is denoted by G. Thus, the complement G of a simple graph G with n vertices is that graph which is obtained by deleti~~ng~~ those edge of Kn which belongs to G. Thus G $=K_n$-G $=K_n$- G.

Evidently K_n, G and G h ve the same vertex set and two vertices are adjacent in G if and only if they are not adjacent in $\overline{G}$. Obviously, G is also a simple graph and the complement of G is G that is G **= G**

In figure the complete graph K_4 is shown. A simple graph G of order 4 is shown in figure The complement G, of G is shown in figure

Observe that G, G & K_4 have the same vertices and that the edges of G are got by deleting those edges from K_4 which belong to G.



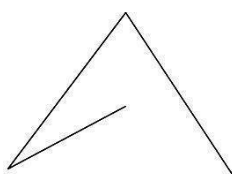


Fig. (b): G Fig. (a): K_4

Fig. (b): G

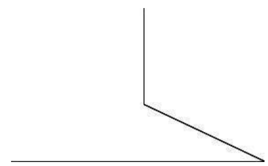


Fig. (c): G

EULER CIRCUITS AND EULER TRAILS.

Consider a connected graph G. If there is a circuit in G that contains all the edges of G. Than that circuit is called an **Euler circuit** (or Eulerian line, or Euler tour) in G. If there is a trail in G that contains all the edges of G, than that trail$_{com}$isalled an **Euler trail**.

Recall that in a trail and a circuit no edge can appear more than once but a vertex also. can appear more than once. This property is carried to Euler trails and Euler Circuits

Since Euler circuits and Euler trails include all edge, then automatically should include all vertices as well. A connected graph that contains an Euler circuit is called a **Semi Euler graph** (or a Semi Eulerian graph).

1. Find all positive integers n (>=2) for which the complete graph K_n contains an euler circuit. For which n does K_n have an euler trail but not an euler circuit?
   - For n = 2 , the graph K_n contains exactly one edge. This edge together with its end vertices constitiutes an euler trail. In this case K_n cannot have an euler ciruit. For n>=3, K_n contains an euler circuit if and only if n-1 (which is the degree of every vertex in K_n) is even; that is if and only if n is odd.

ISOMORPHISM

Consider two graphs G = (V, E) and G‘ = (V‘, E‘) there exists a function

f : V,V‘ such that (i) f is a none to one correspondence and(ii) f r all vertices A, B of G

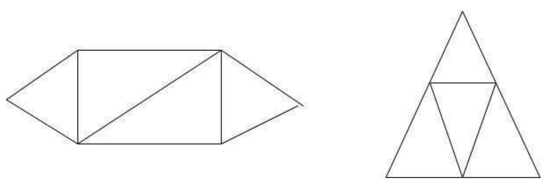
{A, B} is an edge of G if and only if { f(A), f(B)} is an edge of G‘, then f is called as **isomorphism** between G and G‘, G and G‘ are **isomorphic graphs**.

In other words, two graph G and G‘ are said to be isomorphic (to each other) if there is a one to one correspondence between their vertices and between their edges such that the adjacency of vertices is preserved such graphs will have the same structures, differing only in the way their vertice nd edges are labelled or only in the way they are represented geometrically for any purpose, we regard them as essentially the same graphs.

When G and G‘ are isomorphic we write G G‘

Where a vertex A of G corresponds to the vertex A‘ = f(A) of G‘ under a one to one correspondence f : G G‘, we write A A‘ Similarly , we write {A, B} {A‘, B‘} to mean that the edge AB of G and the edge A‘B‘ of G‘ correspond to each other, under f.

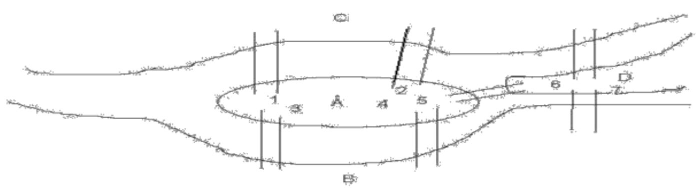
Example 1: Show that the following graphs are not isomorphic.



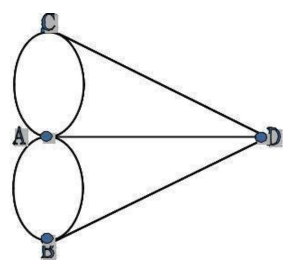
Solution: We note that each of the two graphs has 6 vertices and nine edges. But, the first graph has 2 vertices of degree 4 where as the second graph has 3 vertices of degree 4. Therefore, there cannot be anyone-to-one correspondence come between the vertices and between the edges of the two graphs which preserves the adjacency f vertices. As such, the two graphs are not isomorphic.

THE KONIGSBERG BRIDGE PROBLEM

Euler (1707-- 1782) became the father of graph theory as well as topology when in 1736 he settled a famous unsolved problem of his day called the Konigsberg bridge problem. The city of Konigsberg was located on the Pregel river in Prussia, the city occupied two island plus areas on both banks. These region were linked by seven bridges



A park in Konigsberg 1736



The Graph of the Konigsberg bridge problem

The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point one can easily try to solve this problem empirically but all attempts must be unsuccessful, for the tremendous contribution of Euler in this case was negative.

In proving that the problem is unsolvable, Euler replaced each land area by a point and each bridge by a line joining the corresponding points these by producing a ‗graph‖ this graph is sho n in fig(1.2) where the points are labeled to correspond to the four land areas of fig(1.1) showing that the problem is unsolvable is equivalent to showing that the graph of fig(1.2) cannot be traversed in a certain way.

In proving that the problem is unsolvable, Euler replaced each land area by a point and each bridge by a line joining the corresponding points these by producing a ―graph‖ this graph is shown in fig(1.2) where the points are labeled to correspond to the four land areas of fig(1.1) showing that the problem is unsolvable is equivalent to showing that the graph of fig(1.2) cannot be traversed in a certain way.

Trees

- Connected graph without circuits is called a tree. Graph is called a forest when it does not have circuits. A vertex of degree 1 is called a terminal vertex or a leaf, the other vertices are called internal nodes. Examples: Decision tree, Syntactic derivation tree.
- Any tree with more than one vertex has at least one vertex of degree 1. Any tree with n vertices has n – 1 edges. If a connected graph with n vertices has n – 1 edges, then it is a tree

Rooted Trees

- Rooted tree is a tree in which one vertex is distinguished and called a root. Level of a vertex is the number of edges between the vertex and the root. The height of a rooted tree is the maximum level of any vertex. Children, siblings and parent vertices in a rooted tree. Ancestor, descendant relationship between vertices

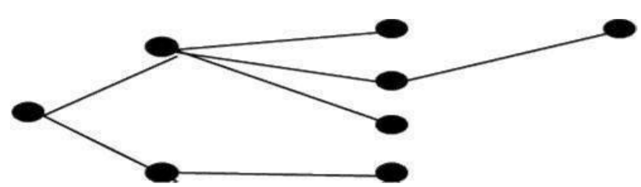
- Binary tree is a rooted tree where each internal vertex has at most two children: left and right. Left and right sub trees.

Spanning Trees

- A sub graph T of a graph G is called a spanning tree when T is a tree and contains all vertices of G. Every connected graph has a spanning tree. Any two spanning trees have the same number of edges. A weighted graph is a graph in which each edge has an associated real number weight. A minimal spanning tree (MST) is a spanning tree with the least total weight of its edges.

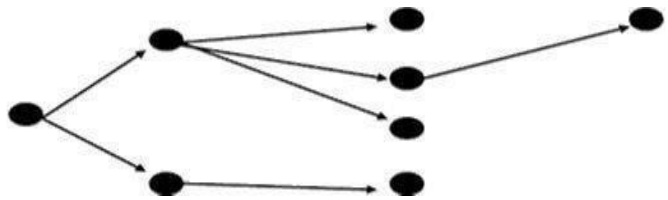
Trees: Definition & Applications

A tree is a connected graph with no cycles. A forest is a graph whose components are trees. An example appears below. Trees come up in many contexts: tournament brackets, family trees, organizational charts, and decision trees, being a few examples.



Directed Trees

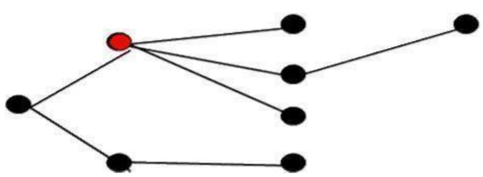
A directed tree is a digraph whose underlying graph is a tree and which has no loops and no pairs of vertices joined in both directions. These last two conditions mean that if we interpret a directed tree as a relation, it is irreflexive and asymmetric. Here is an example.

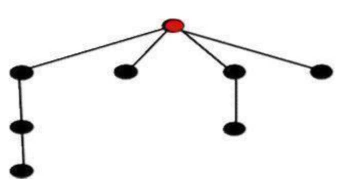


Theorem: A tree T(V,E) with finite vertex set and at least one edge has at least two leaves (a leaf is a vertex with degree one). Proof: Fix a vertex a that is the endpoint of some edge. Move from a to the adjacent vertex along the edge. If that vertex has no adjacent vertices then it has degree one, so stop. If not, move along another edge to another vertex. Continue building a path in this fashion until you reach a vertex with no adjacent vertices besides the one you just came from. This is sure to happen because V is finite and you never use the same vertex twice in the path (since T is a tree). This produces one leaf. Now return to a. If it is a leaf, then you are done. If not, move along a different edge than the one at the first step above. Continue extending the path in that direction until you reach a leaf (which is sure to happen by the argument above).

Rooted Trees

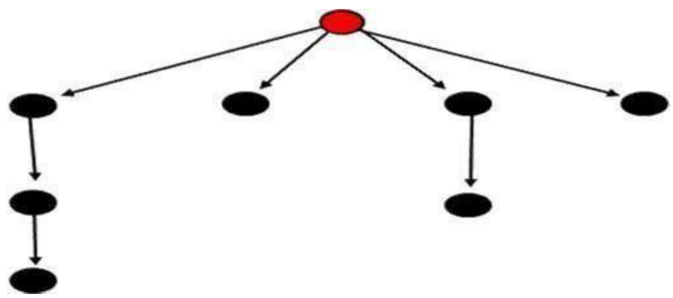
Sometimes it is useful to distinguish one vertex of a tree and call it the root of the tree. For instance we might, for whatever reasons, take the tree above and declare the red vertex to be its root. In that case we often redraw the tree to let it all –hang down‖ from the root (or invert this picture so that it all ―grows up‖ from the root, which suits the metaphor better)





Rooted Directed Trees

It is sometimes useful to turn a rooted tree into a rooted directed tree T′ by directing every edge away from the root.



Rooted trees and their derived rooted directed trees have some useful terminology, much of which is suggested by family trees. The level of a vertex is the length of the path from it to the root. The height of the tree is the length of the longest path from a leaf to the root. If there is a directed edge in T′ from a to b, then a is the parent of b and b is a child of a. If there are directed edges in T′ from a to b and c, then b and c are siblings. If there is a directed path from a to b, then a is an ancestor of b and b is a descendant of a.

Binary & m-ary Trees

We describe a directed tree as binary if no vertex has outdegree over 2. It is more common to call a tree binary if no vertex has degree over 3. (In general a tree is m-ary if no vertex has degree over m+1. Our book calls a directed tree m-ary if no vertex has outdegree over m.) The directed rooted tree above is 4-ary (I think the word is quaternary) since it has a vertex with outdegree 4. In a rooted binary tree (hanging down or growing up) one can describe each child vertex as the left child or right child of its parent.

Trees: Edges in a Tree

Theorem: A tree on n vertices has n–1 edges. Proof: Let T be a tree with n vertices. Make it rooted. Then every edge establishes a parent-child relationship between two vertices. Every child has exactly one parent, and every vertex except the root is a child. Therefore there is exactly one edge for each vertex but one. This means there are n–1 edges.

Theorem: If G(V,E) is a connected graph with n vertices and n–1 edges is a tree.

Proof: Suppose G is as in the statement of the theorem, and suppose G has a cycle. Then we can remove an edge from the cycle without disconnecting G (see the next slide for why). If this makes G a tree, then stop. If not, there is still a cycle, so we can remove another edge without disconnecting G. Continue the process until the remaining graph is a tree. It still has n vertices, so it has n–1 edges by a prior theorem. This is a contradiction since G had n–1 vertices to start with. Therefore G has no cycle and is thus a tree.

(Why can we remove an edge from a cycle without disconnecting the graph? Let a and b be vertices. There is a simple path from a to b. If the path involves no edges in the cycle, then the path from a to be is unchanged. If it involves edges in the cycle, let x and y be the first and last vertices in the cycle that are part of the path from a to b. So there is a path from a to x and a path from y to b. Since x and y are part of a cycle, there are at least simple two paths from x to y. If we remove an edge from the cycle, at least one of the paths still remains. Thus there is still a simple path from a to b.)

Constructing an Optimal Huff man Code

An optimal Huffman code is a Huffman code in which the average length of the symbols is minimum. In general an optimal Huffman code can be made as follows. First we list the frequencies of all the codes and represent the symbols as vertices (which at the end will be leaves of a tree). Then we replace the two smallest frequencies f 1 and f2 with their sum f1 + f2, and join the corresponding two symbols to a common vertex above them by two edges, one labeled 0 and the other one labeled 1. Than common vertex plays the role of a new symbol with a frequency equal to f1 + f2. Then we repeat the same operation with the resulting shorter list of frequencies until the list is reduced to one element and the graph obtained becomes a tree.

1. Consider the prefix code:

a : 111, b : 0,c : 1100, d : 1101, e : 10

Using this code, decode the following sequences:

i 1001111101

ii 10111100110001101

iii 1101111110010

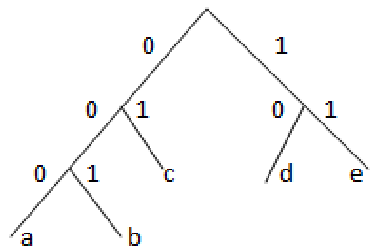
- Keeping the given code in mind, we first split the given sequences into appropriate number of parts. Then the decoded version of the sequence is written down.

  Splitting: 10 0 111 1101, Decode: e b a d

  Splitting: 10 111 10 0 1100 0 1101, Decode: e a e b c b d

  Splitting: 1101 111 1100 10, Decode: d a c e

2. Obtain the prefix code represented by the following labeled complete binary tree.



- The leaves of the given tree are represented by the symbols a,b,c,d,e. These leaves are identified by the sequences as indicated in the following table.

Leaf :	a	b	c	d	e
Sequence:	000	001	01	10	11

This table determines the required prefix code as

P = { 000, 001, 01, 10, 11}

Assignment Questions

1) Explain the types of graphs.
2) What is optimal tree? Explain the Huffman code procedure.

Outcomes

After completing this module one can

- Understand the concept of graphs and trees.
- Solve problem involving graphs and trees.

5.10 Further Reading

1. Ralph P. Grimaldi: Discrete and combonotorial Mathematics, 5th Edition,Pearson Education 2004.

2. Basavaraj S Anami and venkanna S Madalli: Discrete Mathematics – A Concept based approach Universities Press,2016.

3. Kenneth H. Rosen: Discrete Mathematics and its Applications, 6th Edition, McGraw Hill 2007.

4. Jayant Ganguly: A Treatise on Discrete Mathematical Structures, Sanguine-Pearson, 2010.

5. D.S. Malik and M.K. Sen: Discrete Mathematical Structures: Theory and Applications, Thomson, 2004.

6. Thomas Koshy: Discrete Mathematics with Applications, Elsevier, 2005, Reprint 2008.

7. https://en.wikipedia.org/wiki/Rule_of_product

8. https://en.wikipedia.org/wiki/Hasse_diagram

9. https://math.feld.cvut.cz/habala/teaching/dma-e/book4x.pdf