Taylor's Series of functions of two variables ①

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$

$$+ \cdots$$

Maclaurin's series of functions of two Variables

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right] +$$

$$\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$

$$\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \cdots$$

① Obtain Taylor's Series of $f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$

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$a = 1. \quad b = -2$

$f(x,y) = x^2y + 3y - 2$ ✓

$f_x = y(2x) + 3(0) - 0$

$= 2xy$ ✓

$f_y = x^2 \cdot 1 + 3(1) - 0$

$= \underline{x^2 + 3}$

$f_{xx} = 2y(1)$

$= 2y$ ✓

$f_{xy} = 2x(1)$

$= 2x$ ✓

$f_{yy} = \underline{0}$

$f_{xxx} = 0$

$f_{xxy} = 2(1)$

$f_{xyy} = 0$

$f_{yyy} = 0$

$f(1,-2) = 1^2(-2) + 3(-2) - 2$

$= -10$

$f_x(1,-2) = 2(1)(-2)$

$= -4$

$f_y(1,-2) = 1^2 + 3 = 4$

$f_{xx}(1,-2) = 2(-2) = -4$

$f_{xy}(1,-2) = 2(1) = 2$

$f_{yy}(1,-2) = 0$

$f_{xxx}(1,-2) = 0$

$f_{xxy}(1,-2) = 2$

$f_{xyy}(1,-2) = 0$

$f_{yyy}(1,-2) = 0$

Taylor's Series is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] +$$

$$\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$

$$+ \cdots$$

$$x^2y + 3y - 2 = -10 + \frac{1}{1!}\left[(x-1)(-4) + (y+2)\cdot 4\right] +$$

$$\frac{1}{2!}\left[(x-1)^2\cdot(-4) + 2(x-1)(y+2)\cdot 2 + (y+2)^2\cdot 0\right] +$$

$$\frac{1}{3!}\left[(x-1)^3\cdot 0 + 3(x-1)^2(y+2)\cdot 2 + 3(x-1)(y+2)^2\cdot 0 + (y+2)^3\cdot 0\right] +$$

$$\cdots$$

$$= -10 + (-4x + 4 + 4y + 8) +$$

$$\frac{1}{2}\left[(x-1)^2(-4) + 4(x-1)(y+2) + 0\right]$$

$$+ \frac{1}{6}\left[6(x-1)^2(y+2)\right] + \cdots$$

② Expand $f(x,y) = e^{x^2+y^2}$ about the point $(1,1)$ upto second degree terms. ④

$(a,b) = (1,1)$

Sol:- $f(x,y) = e^{x^2+y^2}$ $\qquad f(1,1) = e^{1^2+1^2} = e^2$

$f_x = e^{x^2+y^2} \cdot 2x$ ✓ $\qquad f_x(1,1) = e^{1^2+1^2} \cdot 2(1) = 2e^2$

$f_y = e^{x^2+y^2} \cdot 2y$ ✓ $\qquad f_y(1,1) = e^{1^2+1^2} \cdot 2(1) = 2e^2$

$f_{xx} = 2\left[e^{x^2+y^2} \cdot 1 + x \cdot e^{x^2+y^2} \cdot 2x\right]$

$= 2\left[e^{x^2+y^2} + 2x^2 e^{x^2+y^2}\right]$

$f_{xx}(1,1) = 2\left[e^{1^2+1^2} + 2(1)^2 e^{1^2+1^2}\right]$

$= 2\left[e^2 + 2e^2\right]$

$= 6e^2$

$f_{xy} = 2x \cdot e^{x^2+y^2} \cdot 2y$

$= 4xy\, e^{x^2+y^2}$

$f_{xy}(1,1) = 4(1)(1)e^{1^2+1^2}$

$= 4e^2$

$f_{yy} = 2\left[y \cdot e^{x^2+y^2} \cdot 2y + e^{x^2+y^2} \cdot 1\right]$

$= 2\left[2y^2 e^{x^2+y^2} + e^{x^2+y^2}\right]$

$f_{yy}(1,1) = 2\left[2(1)^2 e^{1^2+1^2} + e^{1^2+1^2}\right]$

$= 2\left[2e^2 + e^2\right]$

$= 6e^2$

Taylor's series is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

$$e^{x^2+y^2} = e^2 + \frac{1}{1!}\left[(x-1)\, 2e^2 + (y-1)\, 2e^2\right] + \frac{1}{2!}\left[(x-1)^2 \cdot 6e^2 + 2(x-1)(y-1) \cdot 4e^2 + (y-1)^2 \cdot 6e^2\right] + \cdots$$

(3) Expand $f(x,y) = \sin x \cos y$ in powers of x and y upto third degree.

Sol:- $f(x,y) = \sin x \cos y$ $\quad f(0,0) = \sin 0 \cdot \cos 0 = 0$

$f_x = \cos y \cdot \cos x$ $\quad f_x(0,0) = \cos 0 \cdot \cos 0 = 1$

$f_y = \sin x(-\sin y) = -\sin x \sin y$ $\quad f_y(0,0) = -\sin 0 \cdot \sin 0 = 0$

$f_{xx} = \cos y(-\sin x) = -\sin x \cos y$ $\quad f_{xx}(0,0) = -\sin 0 \cdot \cos 0 = 0$

$f_{xy} = \cos x(-\sin y)$ $\quad f_{xy}(0,0) = -\cos 0 \cdot \sin 0 = 0$

$f_{yy} = -\sin x\,(\cos y)$ $\qquad f_{yy}(0,0) = -\sin 0 \cos 0$
$= 0$

$f_{xxx} = -\cos x \cos y$ $\qquad f_{xxx}(0,0) = -\cos 0 \cdot \cos 0$
$= -1$

$f_{xxy} = -\sin x \cdot (-\sin y)$ $\qquad f_{xxy}(0,0) = \sin 0 \cdot \sin 0$
$= \sin x \sin y$ $\qquad = 0$

$f_{xyy} = -\cos x \cdot \cos y$ $\qquad f_{xyy}(0,0) = -\cos 0 \cdot \cos 0$
$= -1$

$f_{yyy} = -\sin x\,(-\sin y)$ $\qquad f_{yyy}(\ ,0) = \sin 0 \cdot \cancel{\cos}\ \sin 0$
$= 0$

Maclaurin's series is

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x \cdot f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$

$$\sin x \cos y = 0 + \frac{1}{1!}\left[x \cdot 1 + y \cdot 0\right] + \frac{1}{2!}\left[x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0\right] + \frac{1}{3!}\left[x^3 \cdot (-1) + 3x^2 y \cdot 0 + 3xy^2(-1) + y^3(0)\right] + \cdots$$

④ Expand $e^x \log(1+y)$ in powers of x and y upto second degree terms.

Sol: - $f(x,y) = e^x \log(1+y)$

$f(0,0) = e^0 \log(1+0)$
$= 0$

$f_x = \log(1+y) \cdot e^x$

$f_x(0,0) = \log(1+0) \cdot e^0$
$= 0$

$f_y = e^x \cdot \frac{1}{1+y}$
$= e^x (1+y)^{-1}$

$f_y(0,0) = e^0 \cdot \frac{1}{1+0}$
$= 1$

$f_{xx} = \log(1+y) \cdot e^x$

$f_{xx}(0,0) = \log(1+0) \cdot e^0$
$= 0$

$f_{xy} = e^x \cdot \frac{1}{1+y}$

$f_{xy}(0,0) = \frac{e^0}{1+0} = 1$

$f_{yy} = e^x \cdot (-1)(1+y)^{-2}$
$= \frac{-e^x}{(1+y)^2}$

$f_{yy}(0,0) = \frac{-e^0}{(1+0)^2}$
$= \frac{-1}{1}$
$= -1$

Maclaurin's series is

$f(x,y) = f(0,0) + \frac{1}{1!}[x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \cdots$

$$e^x \log(1+y) = 0 + \frac{1}{1!}\left[x \cdot 0 + y \cdot 1\right] + \frac{1}{2!}\left[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)\right] + \cdots$$

$$e^x \log(1+y) = y + \frac{1}{2}\left[2xy - y^2\right] + \cdots$$

⑤ Expand $e^x \sin y$ in powers of x and y upto second degree terms.

Sol:- $f(x,y) = e^x \sin y$

$f_x = e^x \sin y$

$f_y = e^x \cdot \cos y$

$f_{xx} = \sin y \, e^x$

$f_{xy} = e^x \cdot \cos y$

$f_{yy} = e^x(-\sin y)$

$f(0,0) = e^0 \sin 0 = 0$

$f_x(0,0) = e^0 \sin 0 = 0$

$f_y(0,0) = e^0 \cos 0 = 1$

$f_{xx}(0,0) = \sin 0 \cdot e^0 = 0$

$f_{xy}(0,0) = e^0 \cdot \cos 0 = 1$

$f_{yy}(0,0) = -e^0 \sin 0 = 0$

Maclaurin's series is

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x \cdot f_x(0,0) + y \cdot f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$

$$e^{x}\sin y = 0 + \frac{1}{1!}\left[x\cdot 0 + y\cdot 1\right] + \frac{1}{2!}\left[x^{2}\cdot 0 + 2xy\cdot 1 + y^{2}\cdot 0\right] + \cdots$$

$$e^{x}\sin y = y + \frac{1}{2}(2xy + 0) + \cdots$$

$$e^{x}\sin y = y + xy + \cdots$$

Solve

1. Expand $xy^{2} + \cos(xy)$ about $(1, \pi/2)$ upto second degree terms
2. Expand $\sin x \sin y$ in powers of x and y upto third degree terms.
3. Expand $e^{x^{2}-y^{2}}$ in powers of x and y upto second degree terms.

Maxima and Minima of functions of two variables

One variable

$y = f(x)$ be the given function.

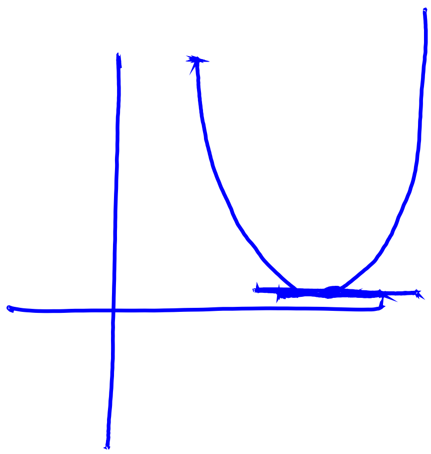
$y' = \frac{dy}{dx}$ $\qquad \frac{dy}{dx} = 0$

Stationary points

$\frac{d^2y}{dx^2}$ (or) $f''(x)$

1) $f''(x) > 0$, at 'a' ~~then~~ the function has minimum value at 'a

2) $f''(x) < 0$, at 'a' then the function has maximum value at 'a'



Definition

A function $f(x,y)$ is said to have a maximum value at a point (a,b) if there exists a neighbourhood $(a+h, b+k)$ where h & k are very small such that

$$f(a,b) > f(a+h, b+k)$$

If $f(a,b) < f(a+h, b+k)$ then $f(x,y)$ is said to have minimum value at (a,b)

Maximum and Minimum values are called extreme values of the function.

Procedure to find Maximum (or) Minimum Value of function of two variables

1. Let $f(x,y)$ be the given function. Find f_x & f_y and equate to zero.
2. We get stationary points (or) critical points
3. Find $r = \frac{\partial^2 f}{\partial x^2}$

   $s = \frac{\partial^2 f}{\partial x \partial y}$

   $t = \frac{\partial^2 f}{\partial y^2}$
4. If $rt - s^2 > 0$, and $r < 0$ at (a,b) then f has maximum value at (a,b)
5. If $rt - s^2 > 0$ and $r > 0$ at (a,b) then f has minimum value at (a,b)
6. If $rt - s^2 < 0$, then f has a saddle point (Maximin (or) Minimax)

7. $rt - s^2 = 0$, then, failure case.
Further investigation is needed to determine the nature of the function

Problems

① Find the maximum and minimum values of the function $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Sol:- $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x = 3x^2 + 3y^2 \cdot 1 - 30x - 0 + 72$

$f_y = 0 + 6xy - 0 - 30y$
$= 6xy - 30y$
$=$

For maxima (or) minima

$f_x = 0$

$3x^2 + 3y^2 - 30x + 72 = 0$

$f_y = 0$

$6xy - 30y = 0$
$6y(x-5) = 0$
$6y = 0 \Rightarrow y = 0$
$x - 5 = 0 \Rightarrow x = 5$

$x = 5,\ y = 0$

Put $x = 5$ in $f_x = 0$

$3(5)^2 + 3y^2 - 30(5) + 72 = 0$
$75 + 3y^2 - 150 + 72 = 0$
$3y^2 - 3 = 0$
$y^2 - 1 = 0 \Rightarrow y^2 = 1$
$\Rightarrow y = \pm 1$

points
~~(5,1) & (5,-1)~~
(5,1) & (5,-1)

Put $y=0$ in $f_x=0$

$3x^2+3(0)^2-30x+72=0$

$3x^2-30x+72=0$

$x^2-10x+24=0$

$x^2-6x-4x+24=0$

$x(x-6)-4(x-6)=0 \implies (x-4)(x-6)=0$

$x=4, 6$

points $(4,0)$ & $(6,0)$

stationary points are $(5,1)$ $(5,-1)$ $(4,0)$ & $(6,0)$

$f_x = 3x^2+3y^2-30x+72$

$f_y = 6xy-30y$

$r = \frac{\partial^2 f}{\partial x^2} =$

$r = 6x+0-30$

$r = 6x-30$

$s = f_{xy} = 0+6y$

$s = 6y$

$t = f_{yy} = 6x-30$

	(4,0)	(6,0)	(5,1)	(5,-1)
$r = 6x - 30$	$-6 < 0$	$6 > 0$	0	0
$s = 6y$	0	0	6	-6
$t = 6x - 30$	$-6 < 0$	$6 > 0$	0	0
$rt - s^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$

at $(4,0)$ r is < 0, $rt - s^2 > 0$

f has maximum at $(4,0)$

Maximum value is

$$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$f(4,0) = 4^3 + 3 \cdot 4 \cdot 0 - 15(4)^2 - 15(0) + 72 \cdot 4$$

$$= 64 + 0 - 240 - 0 + 288$$

$$= 352 - 240$$

$$= 112$$

Max value $= 112$

at $(6,0)$, $r > 0$, $rt - s^2 > 0$

$\therefore$ f has minimum value at $(6,0)$

Min value is

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f(6,0) = 6^3 + 3(6)0^2 - 15(6)^2 - 15(0)^2 + 72(6)$

$= 108$

Min value $= 108$

at $(5,1)$ and $(5,-1)$. $rt - s^2 < 0$,

$\therefore$ It is a saddle point.