MODULE – 3

PARTIAL DIFFERENTIAL EQUATIONS :- [PDE]

We denote the partial derivatives as

$$z = f(x, y)$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} \quad r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}.$$

(i) Formation of PDE by eleminating Arbitrary constants.

1. obtain the PDE by eleminating arbitary const. given $z = (x + a)(y + b) \rightarrow ①$

Diff ① partially w.r.t x.

$$\frac{\partial z}{\partial x} = (y + b)(1) \longrightarrow ②$$

Diff ① partially w.r.t y.

$$\frac{\partial z}{\partial y} = (x + a)(1). \rightarrow ③$$

Subs ② & ③ in ①

$$z = \frac{\partial z}{\partial y} \times \frac{\partial z}{\partial x}$$

$\boxed{z = pq} \rightarrow$ req PDE

2. Find the PDE of the family of all spheres whose centre lie on the plane $z = 0$ and have the const radius r.

1) The eq^n of sphere whose centre lie in plane $z=0$ (xy-plane) [with centre $=(a,b,0)$] radius 'r' units is given by

$$(x-a)^2+(y-b)^2+(z-0)^2=r^2 \longrightarrow \text{eq}^n \text{ of sphere.}$$

$$r^2=(x-a)^2+(y-b)^2+z^2 \rightarrow (1)$$

Diff (1) partially w.r.t x.

$$2(x-a)+0+2z\frac{\partial z}{\partial x}=0$$

$\div 2$.

$$(x-a)=-z\frac{\partial z}{\partial x}.$$

$$\boxed{(x-a)=-zp} \rightarrow (2)$$

Diff (1) partially w.r.t y.

$$0=2(y-b)+2z\frac{\partial z}{\partial y}.$$

$\div 2$

$$(y-b)=-z\frac{\partial z}{\partial y}=-zq \rightarrow (3)$$

Subs (2) & (3) in (1)

$$(-zp)^2+(-zq)^2+z^2=r^2$$

$$\boxed{z^2(p^2+q^2+1)=r^2} \rightarrow \text{PDE}$$

3. obtain the PDE for

$z=a\log(x^2+y^2)+b$. by eliminating the arbitary constants.

Scanned by CamScanner

A:- $z = a\log(x^2+y^2)+b \rightarrow ①$

Diff ① partially w.r.t x.

$$\frac{\partial z}{\partial x} = \frac{a(2x)}{x^2+y^2} \rightarrow ②$$

$$\frac{\partial z}{\partial y} = \frac{a(2y)}{x^2+y^2} \rightarrow ③$$

$$\frac{②}{③} \quad \frac{p}{q} = \frac{x}{y}$$

$$\boxed{py - qx = 0} \rightarrow \text{PDE}$$

4. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \rightarrow ①$

Diff ① partially w.r.t x,

$$\cancel{2}\cdot\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{a^2}$$

$$p = \frac{x}{a^2} \rightarrow ②$$

$$\cancel{2}\cdot\frac{\partial z}{\partial y} = \frac{\cancel{2}y}{b^2}$$

$$q = y/b^2 \rightarrow ③$$

~~② & ③ in ①~~

$$\left.\frac{p}{x} = \frac{1}{a^2}, \quad \frac{q}{y} = \frac{1}{b^2}\right\} \rightarrow ④$$

④ in ①

$$2z = \frac{px^2}{x} + \frac{qy^2}{y} = \boxed{px + qy = 2z}$$

PDE

Scanned by CamScanner

(ii) Formation of PDE by eliminating arbitrary functions :-

1. Obtain the PDE by eliminating the arbitrary function given $z = f(x^2-y^2) \rightarrow (1)$

Diff (1) w.r.t x & y

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)\,2x \rightarrow (2)$$

$$\left\{ f' = \frac{\partial f}{(x^2-y^2)} \quad \frac{\partial(x^2-y^2)}{\partial x}\right.$$

$$\frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y) \rightarrow (3)$$

$$\frac{(2)}{(3)} = \frac{p}{q} = \frac{x}{-y}$$

$$qx - py = 0$$

$$\boxed{py + qx = 0}$$

2. $z = y^2 + 2f(1/x + \log y) \rightarrow (1)$

Diff (1) w.r.t x & y.

$$\frac{\partial z}{\partial x} = 2f'(1/x + \log y)\left(\frac{-1}{x^2}\right)$$

$$p = \frac{-2}{x^2} f'(1/x + \log y) \rightarrow (2)$$

$$\frac{\partial z}{\partial y} = 2y + 2f'(1/x + \log y)\frac{1}{y}$$

$$q = 2y + \frac{2}{y} f'(1/x + \log y) \rightarrow (3)$$

$$\frac{(2)}{(3)} = \frac{p}{q} = \frac{-2/x^2\left(f'[1/x + \cancel{\log y}]\right)}{2y + 2/y\, f'(1/x + \cancel{\log y})}$$

$$\frac{p}{q} = \frac{-2/x^2}{2y + 2/y}$$

$$\frac{p}{q} = \frac{-1/x^2}{\frac{y^2+1}{y}}$$

$$p\left(\frac{y^2+1}{y}\right) = -q/x^2 \quad x^2 p y^2 + p x^2$$

$\cancel{2}\ 1/x^2$

$\cancel{2}\left(y + \frac{1}{y}\right)$

$\frac{y^2+1}{y}$

$\frac{2}{x^2} \frac{y}{2}$

(or)

$$\frac{p}{q-2y} = \frac{-\cancel{2}/x^2 f'(1/x + \log y)}{\cancel{2y} + 2/y\, f'(1/x + \log y)}$$

$$\frac{p}{q-2y} = \frac{-y}{x^2}$$

$$px^2 = -qy + 2y^2$$

$$\boxed{px^2 + y(q-2y) = 0} \quad \text{PDE.}$$

3. $z = f(x+ay) + g(x-ay) \longrightarrow (1)$

Diff (1) w.r.t x & y

$t = a^2 r$

$a^2 r - t = 0$

$$\frac{\partial z}{\partial x} = f'(x+ay) + g'(x-ay).$$

$$\frac{\partial z}{\partial y} = f'(x+ay)a + g'(x-ay)(-a)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay)a^2 + g''(x-ay)a^2$$

$$= a^2\left[f''(x+ay) + g''(x-ay)\right]$$

$t = a^2 r$

$\boxed{r a^2 - t = 0}$

4. Obtain the PDE for

$$z = y f(x) + x\phi(y) \rightarrow (1)$$

A:- $p = \frac{\partial z}{\partial x} = y f'(x) + \phi(y) \rightarrow (2)$

$r = \frac{\partial^2 z}{\partial x^2} = y f''(x)$

Diff partially w.r.t y.

$q = \frac{\partial z}{\partial y} = f(x) + x\phi'(y) \rightarrow (3)$

$t = \frac{\partial^2 z}{\partial y^2} = x\phi''(y)$

$s = \frac{\partial^2 z}{\partial x \partial y}$

$\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$ (or) $\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]$

$f'(x) + \phi'(y) \rightarrow (4)$

(2) $\Rightarrow$ $\frac{p - \phi(y)}{y} = f'(x) \rightarrow (5)$

(3) $\Rightarrow$ $\frac{q - f(x)}{x} = \phi'(y) \rightarrow (6)$

(5) & (6) in (4)

$$s = \frac{p - \phi(y)}{y} + \frac{q - f(x)}{x}$$

$$s = \frac{x[p - \phi(y)] + y[q - f(x)]}{xy}$$

$$sxy = px + qy - (x\phi(y) + yf(x))$$

$$\boxed{sxy = px + qy - z}$$

5. obtain the PDE by elimination the arbitary fun^n.

$$z = e^{ax+by} f(ax-by) \rightarrow (1)$$

diff partially w.r.t x.

$$p = \frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)a + f(ax-by).\, e^{ax+by}.\, a..$$

$$q = \frac{dz}{dy} = e^{ax+by}. f'(ax-by)(-b) + f(ax-by) e^{ax+by}..(b)$$

$$p = ae^{ax+by} f'(ax-by) + az \quad \times b$$

$$q = -be^{ax+by} f'(ax-by) + bz \quad \times a$$

$$bp = abe^{ax+by} f'(ax-by) + abz$$

$$aq = -abe^{ax+by} f'(ax-by) + abz$$

$$bp + aq = 2abz$$

$$\boxed{aq + bp = 2abz} \rightarrow \text{PDE}$$

(iii) Formation of PDE for function of the Form $\phi(u,v) = 0$

where u, v are functions of x, y & z is a fuction of x & y

To find the PDE we use the following relation.

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}$$

1. obtain the PDE for a funtion

$$\phi(x+y+z,\ x^2+y^2-z^2) = 0 \rightarrow (1)$$

$$u = x+y+z$$

$$v = x^2+y^2-z^2$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 1+\frac{\partial z}{\partial x} = 1+p \\ \frac{\partial u}{\partial y} &= 1+\frac{\partial z}{\partial y} = 1+q \end{aligned}\right\} (2)$$

$$\left.\begin{aligned} \frac{\partial v}{\partial x} &= 2x - 2z\frac{\partial z}{\partial x} = 2(x-zp) \\ \frac{\partial v}{\partial y} &= 2y - 2z\frac{\partial z}{\partial y} = 2(y-zq) \end{aligned}\right\} (2)$$

$$\therefore \frac{\frac{\partial u}{\partial x}}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}.$$

$$\frac{1+p}{1+q} = \frac{\cancel{2}(x-zp)}{\cancel{2}(y-zq)}$$

$$(1+p)(y-zq) = (1+q)(x-zp)$$

$$y - zq + py - p\cancel{zq} = x - zp + xq - \cancel{zpq}$$

$$py + zp - zq - xq = (x-y)$$

2. Obtain the PDE for the fun^n $\phi(xy+z^2, x+y+z) = 0$. $\longrightarrow$ (1)

$$\phi(xy+z^2, x+y+z) = 0 \rightarrow (1)$$

$$U = xy + z^2$$

$$V = x + y + z$$

$$\frac{\partial U}{\partial x} = y + 2z\frac{\partial z}{\partial x} = y + 2zp$$

$$\frac{\partial U}{\partial y} = x + 2z\frac{\partial z}{\partial y} = x + 2zq$$

$$\frac{\partial V}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p$$

$$\frac{\partial V}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q.$$

$$\frac{\partial U/\partial x}{\partial U/\partial y} = \frac{\partial V/\partial x}{\partial V/\partial y}$$

$$= \frac{y + 2zp}{x + 2zp} = \frac{1+p}{1+q}.$$

$$(1+q)(y+2zp) = (1+p)(x+2zq)$$

$$y + 2zp + qy + \cancel{2zpq} = x + 2zq + px + \cancel{2pqz}$$

$$\boxed{2pz - 2zq + qy - px = x - y}$$

3. Obtain the PDE $\phi(x^2+2yz, y^2+2zx)=0$

$$u = x^2+2yz$$
$$v = y^2+2zx.$$

$$\frac{\partial u}{\partial x} = 2x + 2y\frac{\partial z}{\partial x} = 2(x+yp).$$

$$\frac{\partial u}{\partial y} = 2\left[\frac{\partial z}{\partial y}\cdot y + z\right] = 2(qy+z).$$

$$\frac{\partial v}{\partial x} = 2\left[\frac{\partial z}{\partial x}\cdot x + z\right] = 2[px+z]$$

$$\frac{\partial v}{\partial y} = 2y + 2x\frac{\partial z}{\partial y} \quad 2[y+xq]$$

$$\therefore \frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}$$

$$\therefore \frac{\cancel{2}(x+yp)}{\cancel{2}(qy+z)} = \frac{\cancel{2}(px+z)}{\cancel{2}(y+xq)}.$$

$$(x+yp)(y+xq) = (px+z)(qy+z)$$
$$xy + x^2q + y^2p + \cancel{xypq} = \cancel{xypq} + pxz + qyz + z^2$$
$$x^2q + y^2p - yzq - pxz = -xy + z^2$$
$$\underline{p(y^2-xz) + q(x^2-yz) = z^2 - xy.}$$

4. Obtain the PDE given $f\left(\frac{xy}{z}, z\right)=0$

$$u = \frac{xy}{z}.$$

$$v = z.$$

$$\frac{\partial u}{\partial x} = y\left[\frac{z - x\,\partial z/\partial x}{z^2}\right] = y\left[\frac{1}{z} - \frac{x}{z^2}p\right]$$

Scanned by CamScanner

$$\frac{\partial u}{\partial y} = x\left[\frac{z - y\,\partial v/\partial y}{z^2}\right] = x\left[\frac{1}{z} - \frac{y}{z^2}q\right]$$

$$\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} = p$$

$$\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} = q.$$

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}$$

$$= \frac{y\left[\frac{1}{z} - \frac{x}{z^2}p\right]}{x\left[\frac{1}{z} - \frac{y}{z^2}q\right]} = \frac{p}{q}.$$

$$\therefore \frac{yz - xyp}{xz - xyp} = \frac{p}{q}.$$

$$yzq - \cancel{xypq} = xzp - \cancel{xypq}.$$

$$yzq - xzp = 0.$$

5. $f\left(\frac{z}{x^3}, \ y/x\right) = 0.$

$u = \frac{z}{x^3}, \quad v = y/x.$

$$\frac{\partial u}{\partial x} = \frac{x^3\,\partial z/\partial x - z\,3x^2}{(x^3)^2} = \frac{x^3 p - z\,3x^2}{x^6}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3}\frac{\partial z}{\partial y} = \frac{1}{x^3}q.$$

$$\frac{\partial v}{\partial x} = y \cdot -1 x^{-2} = \frac{-y}{x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{x}.$$

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y}.$$

$$= \frac{\frac{x^3 p - z\,3x^2}{x^6}}{q/x^3} = \frac{-y/x^2}{1/x}.$$

$$\frac{\frac{\cancel{x^2}[xp - 3z\cancel{x}]}{x^4}}{q/x^3} = \frac{-y/x^4}{1/x}$$

$$\frac{\frac{xp - 3z}{x^3}}{q} = \frac{-y/x}{1}$$

$$x\left[\frac{xp - 3z}{x}\right] = -qy.$$

$$xp - 3z = -qy.$$

$$\boxed{xp + qy = 3z}.$$

Solution of non-homogenous PDE by direct integration.

To solve the given PDE by direct Integration, we add the constant of integration as $f(y)$, when we integrate w.r.t x.

Scanned by CamScanner

We add the constant of integration as $f(x)$ when we integrate w.r.t y.

1. Solve.

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \sin(2x+3y) \rightarrow (1)$$

Integrating (1) w.r.t x treating y as constant.

$$\int \frac{\partial^3 z}{\partial x^2 \partial y} dx = \int \sin(2x+3y)\, dx$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{\cos(2x+3y)}{2} + f(y)$$

again Integrate w.r.t x.

$$\frac{\partial z}{\partial y} = -\frac{\sin(2x+3y)}{4} + f(y)x + g(y)$$

Integrating w.r.t 'y' treating 'x' contant

$$\int \frac{\partial z}{\partial y} \cdot dy = -\frac{1}{4}\int \sin(2x+3y) \cdot dy + x\int f(y) \cdot dy + \int g(y)\, dy.$$

$$\boxed{z = +\frac{1}{4} \frac{\cos(2x+3y)}{3} + x\int f(y) \cdot dy + \int g(y) dy + h(x)}$$

2. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0 \rightarrow (1)$

Integrate (1) w.r.t x treating 'y' as constant.

$$\frac{\partial^2 z}{\partial x \partial y} + 18y^2 \frac{x^2}{2} - \cos\frac{(2x-y)}{2} + f(y) = 0$$

$$\frac{\partial z}{\partial y} + 3y^2 \frac{x^3}{3} - \frac{\sin(2x-y)}{4} + f(y)x + g(y) = 0$$

Integrate w.r.t to y

$$z + \frac{3y^3 x^3}{3} + \frac{\cos(2x-y)}{-4} + x\int f(y)\,dy + \int g(y)\cdot dy + h(x) = 0$$

3. $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$ & $z = 0$ when y is an odd multiple of $\pi/2$.

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y \rightarrow (1)$$

integrate w.r.t x keeping y as const

$$\frac{\partial z}{\partial y} = \sin y(-\cos x) + f(y) \rightarrow (2)$$

integrate w.r.t y keeping x as const

$$z = \cos x \cos y + \int f(y)\cdot dy + g(x) \rightarrow (3)$$

Scanned by CamScanner

given $\frac{\partial z}{\partial y} = -2\sin y$ at $x=0$

(2) $\Rightarrow$

$$-2\sin y = -\sin y[\cos 0°] + f(y)$$

$$= -2\sin y = -\sin y + f(y)$$

$$-2\sin y + \sin y = f(y)$$

$$\boxed{-\sin y = f(y)} \rightarrow (4)$$

(4) in (3)

$$z = \cos x \cos y + \int(-\sin y)dy + g(x)$$

$$z = \cos x \cos y + \cos y + g(x) \longrightarrow (5)$$

given $z=0$, when y is odd multiple of $\pi/2 = (2n+1)\pi/2$

(5) $\Rightarrow$

$$0 = \cos x \cdot \cos(2n+1)\pi/2 + \cos(2n+1)\pi/2 + g(x)$$

$$\boxed{g(x) = 0} \rightarrow (6) \qquad \{\cos(2n+1)\pi/2 = 0$$

(6) in (5)

$\therefore$ Solution is

$$z = \cos x \cos y + \cos y$$

$$\cancel{z = \cos} \quad z = \cos y[\cos x + 1]$$

Scanned by CamScanner

4. $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$ for which $\frac{\partial z}{\partial y} = -2\cos y$ when $x=0$ &

$z=0$ when $y=n\pi$

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y \rightarrow (1)$$

Integrate w.r.t x, y - const

$$\frac{\partial z}{\partial y} = -\cos x \cos y + f(y).$$

w.r.t. y.

$$z = -\cos x \sin y + \int f(y) + g(x). \rightarrow (2)$$

given $\frac{\partial z}{\partial y} = -2\cos y$ when $x=0$.

$$-2\cancel{\cos} y = -\cancel{\cos} y + f(y)$$

$$\boxed{f(y) = \cancel{2}\cos y} \rightarrow (3)$$

(3) in (2).

$$z = -\cos x \sin y + \int \cancel{2}\cos y \, dy + g(x)$$

$$z = -\cos x \sin y + \cancel{2y}^{\sin y} + g(x) \rightarrow (4).$$

given $z=0$, when $y=n\pi$

(4) $\Rightarrow$

$$0 = \cancel{2n\pi}^{0} + g(x).$$

$$\boxed{g(x) = \cancel{-2n\pi}\ 0} \rightarrow (5).$$

(5) in (4)

$$z = -\underline{\cos x \sin y} + \sin y + 0$$

Solution of PDE involving Derivative w.r.t One Independent variable only.

* Since the derivative involved is w.r.t only one variable, we write the given PDE as ODE

$$PDE \rightarrow ODE$$

* Replace C_1 & C_2 of CF by $f(x)$ or $g(y)$
* find $f(x)$ or $g(y)$ by using the given initial conditions.

1. $\dfrac{\partial^2 z}{\partial y^2} = z$ given that when $y=0$, $z=e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$

Since the derivative include one independent variable we convert the given PDE to ODE

$$\frac{d^2z}{dy^2} = z \rightarrow (I)$$

$$\frac{d^2z}{dy^2} - z = 0$$

$$D^2z - z = 0$$

$$(D^2-1)z = 0 \rightarrow (1)$$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$\boxed{z = c_1e^y + c_2e^{-y}} \rightarrow (2)$$

Sol^n of (I) is obtained by

$c_1 \rightarrow f(x)$ $c_2 \rightarrow g(x)$

$$z = f(x)e^y + g(x)e^{-y} \rightarrow (3)$$

Now we find $f(x)$ & $g(x)$ using the given conditions

given $y=0, z=e^{x}$ & $\frac{\partial z}{\partial y} = e^{-x}$.

$y=0, z=e^{x}$ in (3)

$$e^{x} = f(x)e^{0} + g(x)e^{0}$$

$$\boxed{e^{x} = f(x) + g(x)} \rightarrow (4)$$

diff (3) partially w.r.t y treating x as constant

$$\frac{\partial z}{\partial y} = f(x)e^{y} - g(x)e^{-y} \rightarrow (5)$$

$$e^{-x} = f(x)e^{0} - g(x)e^{-0}$$

$$\boxed{e^{-x} = f(x) - g(x)} \rightarrow (6)$$

Solving (4) and (6)

$$e^{x} = f(x) + \cancel{g(x)}$$

$$e^{-x} = f(x) - \cancel{g(x)}$$

$$\cancel{2e^{x}} = \cancel{2f(x)}$$

$$e^{x} + e^{-x} = 2f(x)$$

$$\frac{e^{x}+e^{-x}}{2} = f(x)$$

$$f(x) = \cosh x.$$

(4) - (6)

$$2g(x) = e^{x} - e^{-x}$$

$$g(x) = \frac{e^{x}-e^{-x}}{2} = \sinh x$$

$\therefore z = \cosh x\, e^{y} + \sinh x\, e^{-y}$

~~2. Solve $\frac{\partial^2 z}{\partial y^2} = z$ given that~~

2. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$

given that when $x=0$, $z = e^{y}$ & $\frac{\partial z}{\partial x} = 0$

$$\frac{\partial^2 z}{\partial x^2} + z = 0$$

$$\frac{d^2 z}{dx^2} + z = 0 \rightarrow (1)$$

$$D^2 z + z = 0$$

$$(D^2+1)z = 0 \rightarrow (2)$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$z = c_1 \cos x + c_2 \sin x \rightarrow (3)$$

Solution of (1) is obtained by

$c_1 \rightarrow f(y)$ $c_2 \rightarrow g(y)$.

$$\therefore z = f(y)\cos x + g(y)\sin x \rightarrow (4)$$

$\therefore$ case i $x=0$, $z = e^{y}$.

$$e^{y} = f(y) \rightarrow (5)$$

diff (4) partially w.r.t x treating y as constant.

$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x \rightarrow (6)$$

$$\frac{\partial z}{\partial x} = -e^{y}\sin x + g(y)\cos x .$$

case (ii)

at $x=0$, $\frac{\partial z}{\partial x} = 0$.

$$0 = g(y).$$

$$\therefore g(y) = 0 \rightarrow (7)$$

(4) $\Rightarrow$

$$\therefore \boxed{z = e^{y}\cos x}$$

3. Solve $\frac{\partial^3 z}{\partial x^3} + 4\frac{\partial z}{\partial x} = 0$,

given that $z=0$, $\frac{\partial z}{\partial x} = 0$, & $\frac{\partial^2 z}{\partial x^2} = 4$ at $x=0$.

We convert into ODE.

$$\frac{\partial^3 z}{\partial x^3} + 4\frac{\partial z}{\partial x} = 0$$

$$\frac{d^3 z}{dx^3} + 4\frac{dz}{dx} = 0. \rightarrow (1)$$

$$D^3 z + 4Dz = 0$$

$$(D^3 + 4D)z = 0 \rightarrow (2)$$

AE $m^3 + 4m = 0$

$m = \pm 2i, 0$.

$$z = CF = c_1\cos 2x + c_2\sin 2x + c_3 e^{0x}.$$

Solution of (1) is given by.

$c_1 \rightarrow f(y)$, $c_2 \rightarrow g(y)$ $c_3 \rightarrow h(y)$.

$$z = f(y)\cos 2x + g(y)\sin 2x + h(y)e^{0x} \rightarrow (3)$$

Scanned by CamScanner

case (1)

$z = 0$ at $x = 0$

$$\boxed{0 = f(y) + h(y)}$$

∴ case (2) = $\frac{\partial z}{\partial x} = 0$ at $x = 0$.

diff (3) partially w.r.t x

$$\frac{\partial z}{\partial x} = -2f(y)\sin 2x + 2g(y)\cos 2x \rightarrow (4)$$

$0 = 2g(y)$.

$$\boxed{g(y) = 0}$$

diff (4) partially w.r.t x

$$\frac{\partial^2 z}{\partial x^2} = -4f(y)\cos 2x - 4g(y)\sin 2x. \rightarrow (5)$$

$\frac{\partial^2 z}{\partial x^2} = 4$ at $x = 0$.

$4 = -4f(y)$

$$\boxed{f(y) = -1}$$

$0 = f(y) + h(y)$

$h(y) - 1 = 0$

$$\boxed{h(y) = 1}$$

∴ solution is

$$\boxed{z = -\cos 2x + 1e^{0x}}$$

$$\boxed{z = -\cos 2x + 1}$$

$z = 2\sin^2 x$

4. Solve $\frac{\partial^2 u}{\partial x^2} + u = 0$ where u satisfies the condition $u(0,y) = e^{1/2}$

$$\frac{\partial u}{\partial x}(0,y) = 1$$

We convert into ODE.

$$\frac{\partial^2 u}{\partial x^2} + u = 0$$

$$\frac{d^2 u}{dx^2} + u = 0 \rightarrow (1)$$

$$D^2 u + u = 0$$

$$[D^2 + 1]u = 0$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$u = c_1 \cos x + c_2 \sin x.$$

$$c_1 \to f(y) \quad c_2 \to g(y)$$

$$\therefore u = f(y)\cos x + g(y)\sin x. \rightarrow (2).$$

<u>case (i)</u>

at $x = 0$, $u = e^{1/2}$

$$\boxed{e^{1/2} = f(y)}$$

case (ii)

diff (2) partially w.r.t. x.

$$\frac{\partial u}{\partial x} = -f(y)\sin x + g(y)\cos x. \rightarrow (3)$$

$$\frac{\partial u}{\partial x} = 1, \quad \text{at } x = 0.$$

$$1 = -f(y) \times 0 + g(y)\,1$$

$$\boxed{g(y) = 1}$$

$$\therefore \boxed{u = e^{1/2}\cos x + \sin x}$$

5. $\frac{\partial^2 z}{\partial x^2} = a^2 z$.

$x = 0$, $\frac{\partial z}{\partial x} = a\sin y$, $\frac{\partial z}{\partial y} = 0$

$$\frac{d^2 z}{dx^2} = a^2 z.$$

$$\frac{d^2 z}{dx^2} - a^2 z = 0. \rightarrow (1)$$

$$D^2 z - a^2 z = 0$$

$$(D^2 - a^2) z = 0$$

$m^2 = \sqrt{4}$

4

$$D^2 = \pm\sqrt{a^2}$$

$$D = \pm a.$$

$(-2)^2 = 4$

$$z = c_1 e^{ax} + c_2 e^{-ax}.$$

$c_1 \rightarrow f(y)$, $c_2 \rightarrow g(y)$ $(2)^2 = 4$

$$z = f(y)e^{ax} + g(y)e^{-ax} \rightarrow (2)$$

case (i)

$x = 0$ $\frac{\partial z}{\partial x} = a\sin y$.

$$\frac{\partial z}{\partial x} = af(y)e^{ax} + (-a)g(y)e^{-ax}.$$

$$a\sin y = a[f(y) - g(y)]$$

$$f(y) - g(y) = \sin y \rightarrow (3)$$

case (ii)

$$\frac{\partial z}{\partial y} = f'(y)e^{ax} + g'(y)e^{-ax}.$$

$$\boxed{0 = f'(y) + g'(y)} \rightarrow (4)$$

Integrate eq^n (4) w.r.t y.

$$\int f'(y) + \int g'(y) = 0.$$

$$f(y) + g(y) = 0. \rightarrow (5).$$

Solve (3) & (5).

$$f(y) - g(y) = \sin y$$

$$f(y) + g(y) = 0$$

$$2(f(y)) = \sin y.$$

$$f(y) = \frac{\sin y}{2}. \rightarrow (6)$$

(6) in (3) $\Rightarrow$

$$\frac{\sin y}{2} - g(y) = \sin y.$$

$$\frac{\sin y}{2} - \sin y = g(y).$$

$$\frac{\sin y - 2\sin y}{2} = g(y)$$

$$\boxed{g(y) = -\frac{\sin y}{2}}$$

$\therefore$ Solution is:

$$z = \frac{\sin y}{2} e^{ax} + -\frac{\sin y}{2} e^{-ax}.$$

$$z = \frac{1}{2}\left[\sin y\, e^{ax} - \sin y\, e^{-ax}\right]$$

$$z = \frac{\sin y}{2}\left[e^{ax} - e^{-ax}\right] //$$

$\frac{1}{6}$

Scanned by CamScanner

6. $\frac{\partial^2 z}{\partial y^2} = z$.

$y=0, \quad z = e^{x}, \quad \frac{\partial z}{\partial y} = e^{-x}$.

$\frac{d^2 z}{dy^2} - z = 0. \longrightarrow (1)$.

$D^2 z - z = 0$

$(D^2 - 1)z = 0$

$m^2 = 1$

$m = \pm 1$

$\boxed{z = c_1 e^{y} + c_2 e^{-y}} \longrightarrow (2)$

$c_1 \to g(x) \quad c_2 \to h(x)$.

$z = g(x) e^{y} + h(x) e^{-y} \longrightarrow (2A)$.

<u>case (i)</u>

$y = 0, \quad z = e^{x}$

$\boxed{e^{x} = g(x) + h(x)} \longrightarrow (3)$.

case (ii)

$y=0, \quad \frac{\partial z}{\partial y} = e^{-x}$.

diff (2A) partially w.r.t. y.

$\frac{\partial z}{\partial y} = g(x) e^{y} - h(x) e^{-y}$.

$\boxed{e^{-x} = g(x) - h(x)} \longrightarrow (4)$

solve (3) & (4)

$$g(x) + h(x) = e^{x}$$
$$g(x) - h(x) = e^{-x}$$

$$2g(x) = e^{x} + e^{-x}$$
$$g(x) = \frac{e^{x} + e^{-x}}{2}$$
$$\boxed{g(x) = \cosh x} \rightarrow (5)$$

(5) in (3) $\Rightarrow$

$$e^{x} = \cosh x + h(x)$$
$$h(x) = e^{x} - \cosh x$$
$$e^{x} - \left(\frac{e^{x} + e^{-x}}{2}\right) = \frac{2e^{x} - e^{x} - e^{-x}}{2}$$
$$h(x) = \sinh x$$

$$\therefore z = \cosh x \, e^{y} + \sinh x \, e^{-y}$$

$\frac{1}{p} \frac{dy}{d\phi}$

$\frac{dx}{dy}$

Wave eq^n :-

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Heat Equation :-

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution of Wave eq^n

To obtain the solution of wave eq^n

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$ by using

Seperation of Variable method

Let $\boxed{u = XT}$, where $X = X(x)$ $\longrightarrow (2)$ $T = T(t)$

be the solution of (1)

using (2) in (1)

$$\frac{\partial^2 (XT)}{\partial t^2} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$$

$$X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2} \rightarrow \text{PDE} \Rightarrow \text{ODE}$$

$$X \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2} \rightarrow \text{ODE}$$

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2} = K \text{ [constant]}$$

$$\Rightarrow \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = k \quad \& \quad \frac{1}{X} \frac{d^2 X}{dx^2} = k$$

$$\frac{d^2T}{dt^2} - c^2kT = 0 \quad \& \quad \frac{d^2X}{dx^2} - kX = 0$$

$$(D^2 - c^2k)T = 0 \quad , \quad (D^2 - k)X = 0$$

$D = d/dt \rightarrow (3)$ $\qquad D = d/dx \rightarrow (4)$

case (i)

$k = 0$

$(3) \Longrightarrow D^2 T = 0.$

AE $= m^2 = 0$

$m = 0, 0$

$$T = (c_1 + c_2 t)e^{0t}$$

$$\boxed{T = (c_1 + c_2 t)e^{0t}}$$

$(4) \Longrightarrow k = 0.$

$D^2 X = 0.$

$m^2 = 0$

$m = 0, 0$

$$X = (c_3 + c_4 x)e^{0x}$$

$$\boxed{X = c_3 + c_4 x}$$

Solution is given by

$$u = XT$$

$$\boxed{u = (c_3 + c_4 x)(c_1 + c_2 t)}$$

case (ii)

k is positive

let $k = p^2$

$(3) \ \& \ (4) \Longrightarrow$

$$(D^2 - c^2p^2)T = 0 \quad \& \quad (D^2 - p^2)X = 0.$$

$\Rightarrow m^2 - c^2p^2 = 0$ & $m^2 - p^2 = 0$.

$m = \pm cp$ & $m = \pm p$

$T = c_1 e^{cpt} + c_2 e^{-cpT}$ & $X = c_3 e^{px} + c_4 e^{-px}$

Solution $= U = XT$.

$\therefore u = \left[c_3 e^{px} + c_4 e^{-px}\right]\left[c_1 e^{cpt} + c_2 e^{-cpT}\right]$

Case (iii)

k is negative

let $K = -p^2$.

(3) & (4) $\Rightarrow$

$(D^2 + c^2p^2)T = 0$ & $(D^2 + p^2)X = 0$

$AE = m^2 + c^2p^2 = 0$ & $m^2 + p^2 = 0$.

$m = \pm cpi$ $\quad m = \pm pi$

$T = c_1 \cos cpt + c_2 \sin cpt$

$X = c_3 \cos px + c_4 \sin px$.

Solution $= U = XT$

$u = \left[c_3 \cos px + c_4 \sin px\right]\left[c_1 \cos cpt + c_2 \sin cpt\right]$

* **Solution of Heat eq^n.**

Obtain the solution of 1-D Heat eq^n by variable seperable method.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$$

Let sol^n of (1) be.

$$\boxed{u = XT} \rightarrow (2) \qquad X \rightarrow X(x), \quad T \rightarrow T(t)$$

(2) in (1)

$$\frac{\partial (XT)}{\partial t} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$$

$$X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{c^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} = K.$$

$$\frac{1}{c^2 T} \frac{dT}{dt} = K \quad \& \quad \frac{1}{X} \frac{d^2 X}{dx^2} = K$$

$$\frac{dT}{dt} - c^2 K T = 0 \quad \& \quad \frac{d^2 X}{dx^2} - KX = 0.$$

$$[D - c^2 K] T = 0 \rightarrow (3) \qquad [D^2 - K] X = 0 \rightarrow (4)$$

$m = c^2 K$

$D = d/dt \qquad D = d/dx.$

__case i__

$K = 0$	$D^2 X = 0$
$DT = 0$	$m^2 = 0$
$m = 0$	$m = 0, 0.$
$T = c_1 e^{0t}$	$X = (c_2 + c_3 x) e^{0x}.$
$\boxed{T = c_1}$	$X = (c_2 + c_3 x)$

Solution is given by

$$u = XT.$$

$$\boxed{u = (c_2 + c_3 x)\, c_1}$$

case (ii)

K is positive.

$$k = p^2$$

(3) & (4) $\Rightarrow$

$$(D - p^2c^2)T = 0 \quad \& \quad (D^2 - p^2)X = 0.$$

$$m - p^2c^2 = 0 \qquad m^2 - p^2 = 0$$

$$m = p^2c^2 \qquad m = \pm p.$$

$$T = c_1 e^{p^2c^2t} \qquad X = c_2 e^{px} + c_3 e^{-px}.$$

$$u = XT$$

$$u = \left[c_2 e^{px} + c_3 e^{-px}\right]\left[c_1 e^{p^2c^2t}\right]$$

case (iii)

K is negative

$$k = -p^2.$$

(3) & (4) $\Rightarrow$

$$(D + p^2c^2)T = 0 \quad \& \quad (D^2 + p^2)X = 0$$

$$m + p^2c^2 = 0 \qquad m^2 + p^2 = 0$$

$$m = -p^2c^2 \qquad m = \pm pi$$

$$T = c_1 e^{-p^2c^2t} \qquad X = c_2 \cos px + c_3 \sin px.$$

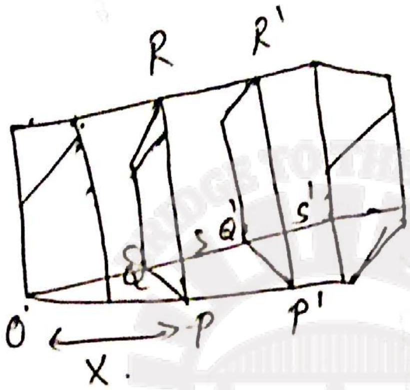
$U = X T.$

$$u = [c_2 \cos px + c_3 \sin px] \cdot [c_1 e^{-p^2 c^2 t}]$$

* Derive 1-D heat eq^n in the form

$$\frac{\partial u}{\partial t} = \frac{c^2 \partial^2 u}{\partial x^2}$$

{Derivative of 1st derivative is second derivative}



A Homogenous slab of crossectional area

Let k – thermal conductivity.

$\rho \rightarrow$ density

$s \rightarrow$ specific heat of material.

$u(x,t) \rightarrow$ Temperature of slab at a distance. x from the origin.

* Consider ~~on~~ the element of the slab b/w the plains PQRS and P'Q'R'S' at x & $x + \delta x$.

{$\frac{m}{v} = \rho$, $m = \rho v$, $v = l \times b \times$ }

* Let δu be the temperature.

* The mass of the element $= \rho A \delta x$.

* The quantity of heat stored in this element $= \rho A \delta x \delta u \cdot s$.

Hence the rate of increase of Heat in the slab is given by

$$R = \rho A \,\delta x \, s \frac{\partial u}{\partial t}$$

if R_1 is the rate of Inflow of heat & R_0 is the Outflow of heat, We've

$$R_1 = -KA\left[\frac{\partial u}{\partial x}\right]_x$$

$$R_0 = -KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x}$$

$$\therefore R = R_1 - R_0$$

$$= -KA\left[\frac{\partial u}{\partial x}\right]_x + KA\left[\frac{\partial u}{\partial x}\right]_{x+\delta x}$$

$$R = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\rho \cancel{A}\, \delta x \, s \frac{\partial u}{\partial t} = K\cancel{A}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\frac{\partial u}{\partial t} = \frac{K}{\rho s} \lim_{\delta x \to 0} \frac{\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore \frac{\partial u}{\partial t} = \frac{K}{\rho s} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\therefore \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

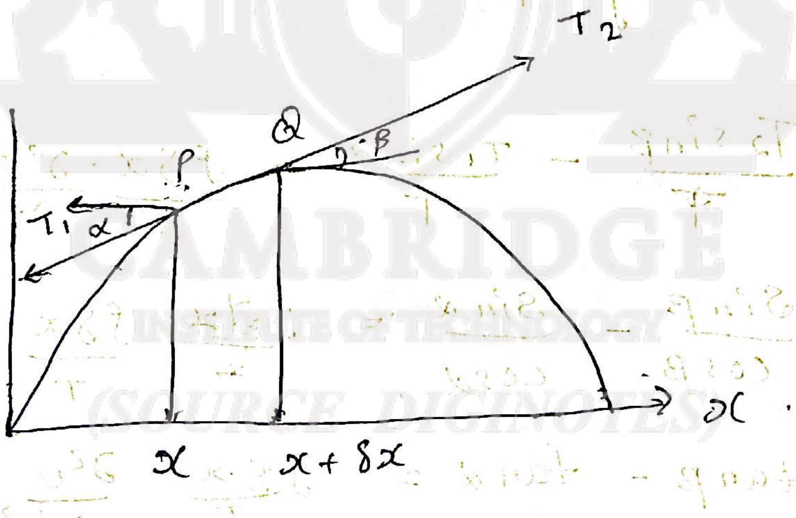
where $c^2 = \frac{K}{\rho s}$.

* Derive 1-D wave eq^n in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

* Consider a string tightly streched b/w two fixed point at a distance 'l' apart.

Let 'ρ' be mass per unit length of the string & 'T' be the tension of string.



* Consider a force acting on small element PQ of length δx.
* Let T_1 & T_2 be the tensions at points P & Q.
* Since there is no motion in horizontal direction we have $T_1 \cos\alpha = T_2 \cos\beta = T \rightarrow (1)$

Let α & β be the angles made by T_1 & T_2 to the Horizontal axis.

The vertical components are given by $-T_1 \sin\alpha$; $T_2 \sin\beta$ [-ve sign becoz T_1 is directed downwards]

Therefore the resultant force acting vertically upwards is

$$T_2 \sin\beta - T_1 \sin\alpha$$

By Newtons second law we've

$$F = m \times a$$

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\,\delta x \times \frac{\partial^2 u}{\partial t^2}$$

$\div$ by T

$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{\rho\,\delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho\,\delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$\frac{dy}{dx} = \tan\theta = \text{slope}$

$$\tan\beta - \tan\alpha = \frac{\rho\,\delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \left[\frac{\partial u}{\partial x}\right]_x = \frac{\rho\,\delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\lim_{\delta x \to 0} \frac{\left[\frac{\partial u}{\partial x}\right]_{x+\delta x} - \left[\frac{\partial u}{\partial x}\right]_x}{\delta x} = \frac{\rho}{L} \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}.$$

$$\frac{T}{\rho} \cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \qquad c = \frac{T}{\rho} //$$

* Solve

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)\,u \rightarrow (1)$$

by the method of seperation of variables.

let sol^{n} of (1) be $u = XY$

$X = X(x), \quad Y = Y(y).$

$$(1) \Rightarrow \frac{\partial (XY)}{\partial x} + \frac{\partial}{\partial y}(XY) = 2\cdot(x+y)\,XY.$$

$$Y \frac{dX}{dx} + X \frac{dY}{dy} = 2(x+y)XY.$$

$\div$ by XY

$$\frac{1}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 2(x+y).$$

$$\frac{1}{X}\frac{dX}{dx} - 2x = -\frac{1}{Y}\frac{dY}{dy} + 2y = K.$$

$$\frac{1}{X}\frac{dX}{dx} - 2x = K \rightarrow (2) \qquad -\frac{1}{Y}\frac{dY}{dy} + 2y = K \rightarrow (3)$$

*.

Scanned by CamScanner

$$\frac{1}{X}\frac{dX}{dx} - 2x = k.$$

$$\frac{1}{X}\frac{dX}{dx} = k + 2x.$$

$$\int \frac{dX}{X} = \int (k+2x)\, dx.$$

$$\log X + \cancel{\log a} = Kx + x^2 + c_1$$

$$\boxed{\log X = Kx + x^2 + c_1} \Rightarrow e^{(kx + x^2 + c_1)}$$

$\underline{\text{IIIy}}$

$$-\frac{1}{Y}\frac{dY}{dy} + 2y = k$$

$$\int \frac{dY}{Y} = \int (2y - k)\, dy.$$

$$\log Y = y^2 - ky + c_2.$$

$$\boxed{\log Y = y^2 - ky + c_2}$$

$$e^{y^2 - ky + c_2}$$

$$U = XY$$

$$U = e^{(kx + x^2 + c_1)} \times e^{(y^2 - ky + c_2)}$$

$$e^{x^2 + y^2 + k(x-y)}\, e^{(c_1 + c_2)}$$

$$u = C\, e^{x^2 + y^2 + k(x-y)}$$

2. Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \rightarrow (1)$

by method of seperation of variables.

Scanned by CamScanner

$z = XY. \Rightarrow X = X(x)$
$Y = Y(y).$

$$\frac{\partial^2 (XY)}{\partial x^2} - 2\frac{\partial (XY)}{\partial x} + \frac{\partial (XY)}{\partial y} = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} - 2Y\frac{\partial X}{\partial x} + X\frac{\partial Y}{\partial y} = 0.$$

$$Y\frac{\partial^2 X}{\partial x^2} - 2Y\frac{\partial X}{\partial x} = -X\frac{\partial Y}{\partial y}.$$

$\div$ by XY

$$\frac{1}{X}\frac{d^2 X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = -\frac{1}{Y}\frac{dY}{dy} = K$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = K.$$

$$\frac{d^2 X}{dx^2} - 2\frac{dX}{dx} = KX$$

$$\frac{d^2 X}{dx^2} - 2\frac{dX}{dx} - KX = 0.$$

$$(D^2 - 2D - K)X = 0 \longrightarrow (2).$$

$AE = m^2 - 2m - K = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 + 4K}}{2}.$$

$$= \frac{2 \pm 2\sqrt{1 + K}}{2}$$

$$1 \pm \sqrt{1+K}.$$

Sol^n is

$$X = C_1 e^{(1+\sqrt{1+K})x} + C_2 e^{(1-\sqrt{1+K})x}$$

$$-\frac{1}{Y}\frac{dY}{dy} = K$$

$$\int \frac{dY}{Y} = -\int K\,dy$$

$$\log Y = -KY + C_3$$

$$Y = e^{(-KY + C_3)}$$

Solution is $z = XY$

$$Z = XY$$

$$= C_1 e^{(1+\sqrt{1+K})x} + C_2 e^{(1-\sqrt{1+K})x} + e^{(-KY+C_3)}$$

3. Solve

$$\frac{\partial^2 z}{\partial x^2} = xy \quad \text{subject to the condition}$$

$$\log(1+y) = \frac{\partial z}{\partial x} \quad \text{when } x = 1$$

& $z = 0$ when $x = 0$.

Sol:

Note: Since the R.H.S term does not include the Unknown term z, we use the method of direct integration.

$$\frac{\partial^2 z}{\partial x^2} = xy \rightarrow (1)$$

Integrate (1) w.r.t. x treating y as const

$$\int \frac{\partial^2 z}{\partial x^2}\,dx = y\int x\,dx.$$

$$\frac{\partial z}{\partial x} = y\cdot\frac{x^2}{2} + f(y).$$

Scanned by CamScanner

Integrate w.r.t x treating y as const

$$\int \frac{\partial z}{\partial x}\, dx = y/2 \int x^2 dx + f(y) \int dx.$$

$$\therefore z = \frac{yx^3}{6} + f(y)x + g(y) \rightarrow (3).$$

using $\log(1+y) = \frac{\partial z}{\partial x}$ at $x=1$ in (2)

$$\log(1+y) = y/2 + f(y).$$

$$\boxed{f(y) = \log(1+y) - y/2} \rightarrow (4).$$

Subs (4) in (3).

$$z = \frac{yx^3}{6} + [\log(1+y) - y/2]x + g(y) \rightarrow (5)$$

$z=0$ at $x=0$ in (5) $\Rightarrow$

$$0 = 0 + 0 + g(y)$$

$$\therefore g(y) = 0.$$

$$\therefore z = \frac{yx^3}{6} + [\log(1+y) - y/2]\, x \; //$$

4. solve

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} \rightarrow (1)$$

subject to the condition

$\frac{\partial z}{\partial x} = \log x$ when $y=1$

$z = 0$ when $x = 1$

$$\int \frac{\partial^2 z}{\partial x\, \partial y}\, dy = \int \frac{x}{y}\, dy$$

{ Since mixed derivative is present we use direct integrati

We integrate first w.r.t y as we require $\frac{\partial z}{\partial x}$ to use the boundary condition.

$$\frac{\partial z}{\partial x} = x \log y + f(x) \rightarrow (2)$$

again Integrate w.r.t x.

$$z = \log y \frac{x^2}{2} + \int f(x) + g(y) \rightarrow (3)$$

case (i)

$\frac{\partial z}{\partial x} = \log x$ when $y = 1$

$(2) \Rightarrow \log x = x \log(1) + f(x)$

$\therefore \boxed{f(x) = \log x}$

$(3) \Rightarrow z = 0$ at $x = 1$.

$$z = \log y \frac{x^2}{2} + \int f(x) + g(y)$$

$$0 = \log y \frac{x^2}{2} + \int \log x + g(y)$$

$$0 = \log y(1/2) + x\log x - x + g(y) \rightarrow (4)$$

$$\therefore 0 = \frac{\log y}{2} + 0 - 1 + g(y).$$

$$\boxed{g(y) = 1 - \frac{\log y}{2}}$$

$\therefore \quad z = \log y \dfrac{x^2}{2} + \int \log x + 1 - \dfrac{\log y}{2}$ //

5. Solve the eq^n.

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 = \cos(2x-y)$$

by direct integration given that

$z = 0$ when $y = 0$

& $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$.

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 = \cos(2x-y) \longrightarrow (1)$$

Integ ~~diff~~ w.r.t. x. treating y as cons

$$\frac{\partial z}{\partial y} + \cancel{9}^{3} y^2 \cdot \frac{x^3}{\cancel{3}} = \frac{\sin(2x-y)}{2} + f(y)$$

$$z = 3x^2 \frac{y^3}{3} = \frac{1}{2}\left[-\frac{\cos(2x-y)}{2}\right] + \int f(y) + g(x)$$

$$z = x^2y^3 = \frac{-1}{2}\cos(2x-y) + \int f(y)dy + g(x)$$

case i :-

$z = 0 \quad y = 0$.

$$0 = 0 + \frac{1}{2}\cos 2x + g(x).$$

$$\boxed{g(x) = \frac{-1}{2}\cos 2x}$$

Scanned by CamScanner

case ii : $\frac{\partial z}{\partial y} = 0$ $\quad x = 0$

$$0 + 0 = \frac{-\sin y}{2} + f(y)$$

$$f(y) = \frac{\sin y}{2}$$

$$\therefore z = x^2y^3 + \tfrac{1}{2}\cos(2x-y) + \int \frac{\sin y}{2} \cdot dy$$

$$- \tfrac{1}{2}\cos 2x .$$

$$z = x^2y^3 + \tfrac{1}{2}\cos(2x-y) + \tfrac{1}{2}\cancel{\left[\sin y - \int y \cos\right.}$$

$$z = x^2y^3 + \tfrac{1}{2}\cos(2x-y) - \frac{\cos y}{2} - \frac{\cos 2x}{2} .$$

6. Solve $\frac{\partial^2 z}{\partial y^2} = z$.

given that when $y = 0$, $z = e^x$, $\frac{\partial z}{\partial y} = e^{-x}$.

$$\frac{\partial^2 z}{\partial y^2} = z .$$

$$\frac{d^2 z}{dy^2} = z .$$

The derivative includes only one independent variable.

$$D^2 z - z = 0$$

$$(D^2 - 1) z = 0$$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$\cancel{CF = c_1 e^{z} + c_2 e^{-z}}$$

$$\text{PI} = \cancel{\frac{z}{D^2-1}} \quad \cancel{c_1 \to f(y)} \quad \cancel{c_2 \to g(y)}$$

Scanned by CamScanner

$z = c_1 e^{y} + c_2 e^{-y}$

$c_1 \to f(x) \quad c_2 \to g(x)$

$z = f(x) e^{y} + g(x) e^{-y}$.

case (i) $\quad y = 0$,

$z = e^{x}$.

$e^{x} = f(x) + g(x) \longrightarrow (2)$

case (ii).

$; y$

$y = 0$.

$\frac{\partial z}{\partial y} = e^{-x}$

$\frac{\partial z}{\partial y} = c_1 e^{y} - c_2 e^{-y}$

$f(x) e^{y} - g(x) e^{-y}$

$e^{-x} = f(x) - g(x) \longrightarrow (3)$

$\frac{2}{3}$ $(2) + (3)$

$e^{x} + e^{-x} = 2 f(x)$.

$f(x) = \frac{e^{x} + e^{-x}}{2} = \cosh x$.

$g(x) = \frac{e^{x} - e^{-x}}{2} = \sinh x$

$\therefore \quad z = \sinh x \, e^{y} + \cosh x \, e^{-y}$.

7. Solve

$$\frac{\partial^2 z}{\partial x^2} = xy \rightarrow (1)$$

Subject to the condition.

$$\frac{\partial z}{\partial x} = \log(1+y) \quad \text{when } x = \phi.$$

$$\& \; z = 0 \quad \& \quad x = 0.$$

~~ODE~~. Direct Integration.

Integrate (1) w.r.t x treating y as cont

$$\frac{\partial z}{\partial x} = y\frac{x^2}{2} + f(y) \rightarrow (2).$$

Integrate (2) w.r.t x treating y as cont

$$z = \frac{y}{2}\frac{x^3}{3} + \int f(y)x + g(y).$$

Case (i)

$$\frac{\partial z}{\partial x} = \log(1+y) \quad \text{when } x = \phi.$$

$$\boxed{\log(1+y) - \frac{y}{2} = f(y).}$$

Case (ii)

$z = 0$, & $x = 0$.

$$0 = \int \log(1+y) + g(y).$$

$$0 = (1+y)\log(1+y) - (1+y) + g(y)$$

$$(1+y)\left[\log(1+y) - 1\right] + g(y)$$

$$z = \frac{yx^3}{6} + \log$$

8. Form the PDE by eliminating arbitary function from the relation

$$\phi(xy + z^2;\ x + y + z) = 0$$

$$z \Rightarrow z(x, y)$$

$$u = xy + z^2$$

$$v = x + y + z$$

$$\frac{\partial u}{\partial x} = y + 2z\frac{\partial z}{\partial x} \Rightarrow y + 2zp$$

$$\frac{\partial u}{\partial y} = x + 2z\frac{\partial z}{\partial y} \Rightarrow x + 2zq$$

$$\frac{\partial v}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p$$

$$\frac{\partial v}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q$$

$$\frac{\dfrac{\partial u}{\partial x}}{\partial u/\partial y} = \frac{\dfrac{\partial v}{\partial x}}{\partial v/\partial y}$$

$$= \frac{y + 2zp}{x + 2zp} = \frac{1 + p}{1 + q}$$

$$(1+q)(y + 2zp) = (1+q)(x + 2zp)$$

$$y + \cancel{2zp} + qy + \cancel{2zpq} = x + \cancel{2zp} + qx$$
$$\cancel{2zpq}$$

$$x - y = qy - qx.$$

$$\boxed{x - y = q(y - x)}$$

$$\boxed{\frac{x-y}{y-x} = q}$$

9. Find PDE.

$$z = f(y+2x) + g(y-3x) \rightarrow (1)$$

diff partially w.r.t x.

$$p = \frac{\partial z}{\partial x} = f'(y+2x) \times 2 + g'(y-3x) - 3.$$

$$q = \frac{\partial z}{\partial y} = f'(y+2x) + g'(y-3x)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(y+2x)4 + g''(y-3x)\,9.$$

$$t\ \frac{\partial^2 z}{\partial y^2} = f''(y+2x) + g''(y-3x)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[f'(y+2x) + g'(y-3x)\right]$$

$$s = f''(y+2x)2 + g''(y-3x)(-3)$$

$$f''(y+2x)4 + g''(y-3x)9$$
$$+ f''(y+2x)2 + g''(y-3x)(-3)$$

$$p - q \quad f'(y+2x)2 + g'(y-3x)-3$$
$$- f'(y+2x) \mp g'(y-3x)$$

$$p - q = f'(y+2x) - 4g'(y-3x)$$

$$r + s = f''(y+2x)4 + g''(y-3x)9$$
$$+ f''(y+2x)2 + g''(y-3x)(-3)$$
$$6f''(y+2x) + 6g''(y-3x)$$
$$6\left[f''(y+2x) + g''(y-3x)\right]$$

$$r + s = 6T.$$