Type V : PI of the form $\frac{x^n \cdot V}{f(D)}$

where V is the function of x

Let the given PI of the form

$$\frac{x^n \cdot V}{f(D)} \text{ (or) } \frac{x \cdot V}{f(D)}$$

then $P.I = \frac{1}{f(D)} x^n \cdot V$

$$= \left(x - \frac{f'(D)}{f(D)}\right) \frac{V}{f(D)}$$

Problems

① Solve $y'' - 2y' + y = x\cos x$

Sol:- $(D^2 - 2D + 1)y = x\cos x$

A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1, 1$

$C.F = (c_1 + c_2 x)e^x$

$P.I = \frac{1}{D^2 - 2D + 1} x\cos x$

$$\left(x - \frac{f'(D)}{f(D)}\right)\frac{V}{f(D)}$$

$$P.I = \left(x - \frac{2D-2}{D^2-2D+1}\right)\frac{\cos x}{D^2-2D+1}$$

$$= \left(x - \frac{2\cancel{(D-1)}}{(D-1)^{\cancel{2}}}\right)\frac{\cos x}{D^2-2D+1}$$

$$= \left(x - \frac{2}{D-1}\right)\frac{\cos x}{\underbrace{D^2-2D+1}}$$

put $D^2=-1^2$

$$= \left(x - \frac{2}{D-1}\right)\frac{\cos x}{\cancel{-1}-2D\cancel{+1}}$$

$$= \left(x - \frac{2}{D-1}\right)\frac{\cos x}{(-2)D}$$

$$= -\frac{1}{2}\left(x - \frac{2}{D-1}\right)\frac{1}{D}\cos x$$

$$= -\tfrac{1}{2}\left(x - \frac{2}{D-1}\right)\int \cos x\,dx$$

$$= -\tfrac{1}{2}\left(x - \frac{2}{D-1}\right)\sin x$$

$$= -\tfrac{1}{2}\left(x\sin x - \frac{2\sin x}{\underbrace{D-1}}\right)$$

$$= -\tfrac{1}{2}\left(x\sin x - \frac{2\,(D+1)\sin x}{(D-1)(D+1)}\right)$$

$$P.I = -\frac{1}{2}\left(x\sin x - \frac{2(D+1)\sin x}{D^2-1}\right)$$

Put $D^2 = -1^2$

$$= -\frac{1}{2}\left(x\sin x - \frac{2(D(\sin x) + \sin x)}{-1-1}\right)$$

$$= -\frac{1}{2}\left(x\sin x - \frac{2(\cos x + \sin x)}{-2}\right)$$

$$P.I = -\frac{x\sin x}{2} - \frac{(\cos x + \sin x)}{2}$$

General solution is

$$y = CF + PI$$

$$y = (C_1 + C_2 x)e^{x} - \frac{x\sin x}{2} - \frac{(\cos x + \sin x)}{2}$$

(Or)

II Method

$$P.I = \frac{1}{D^2-2D+1}\, x\cos x$$

$$\boxed{e^{ix} = \cos x + i\sin x}$$

$\cos x =$ Real part of e^{ix}

$= R.P$ of e^{ix}

(4)

$$P.I = \text{Real part of } \frac{1}{D^2-2D+1} x e^{ix}$$

$$= R.P \text{ of } \frac{1}{D^2-2D+1} x e^{ix}$$

$$= R.P \text{ of } e^{ix} \frac{1}{D^2-2D+1} x$$

$a = i$

put $D = D + i$

$$= R.P \text{ of } e^{ix} \frac{1}{(D+i)^2 - 2(D+i) + 1} x$$

$$= R.P \text{ of } e^{ix} \frac{1}{D^2 + i^2 + 2iD - 2D - 2i + 1} x$$

$$= R.P \text{ of } e^{ix} \frac{1}{D^2 \cancel{-1} + 2iD - 2D - 2i \cancel{+1}} x$$

$$= R.P \text{ of } e^{ix} \frac{1}{D^2 + 2iD - 2D - 2i} x$$

$$= R.P \text{ of } e^{ix} \cdot \frac{1}{(-2i)\left[1 - \frac{D^2+2iD-2D}{2i}\right]} x$$

$$= R.P \text{ of } \frac{e^{ix}}{(-2i)} \left[1 - \frac{D^2+2iD-2D}{2i}\right]^{-1} x$$

$\longrightarrow (1-x)^{-1}$ form

$$= R.P \text{ of } \frac{e^{ix}}{(-2i)}\left[1 + \frac{D^2 + 2Di - 2D}{2i}\right] x$$

$$= R.P \text{ of } \frac{e^{ix}}{(-2i)}\left[x + \frac{1}{2i}D^2(x) + \frac{\cancel{2}Di}{\cancel{2}i}(x) - \frac{\cancel{2}D}{\cancel{2}i}(x)\right]$$

$$= R.P \text{ of } \frac{e^{ix}}{(-2i)}\left[x + 0 + 1(1) - \frac{1}{i}(1)\right]$$

$$= R.P \text{ of } \frac{e^{ix}}{(-2i)}\left[x + 1 - \frac{1}{i}\right]$$

$$\frac{1}{i} = \frac{i}{i^2} = -i$$

$$= R.P \text{ of } \frac{e^{ix}}{-2}\cdot(-i)\left[x + 1 - (-i)\right]$$

$$= R.P \text{ of } \frac{e^{ix}\cdot i}{2}\left(x + 1 + i\right)$$

$$= \frac{i}{2}\cdot R.P \text{ of } (\cos x + i\sin x)\left((x+1) + i\right)$$

$$= \frac{1}{2} R.P \text{ of } \left(i\cos x + i^2 \sin x\right)\left((x+1) + i\right)$$

$$= \frac{1}{2} R.P \text{ of } \left(i\cos x - \sin x\right)\left((x+1) + i\right)$$

$$P.I = -\frac{\cos x}{2} - \frac{(x+1)\sin x}{2}$$

$$P.I = -\frac{\cos x}{2} - \frac{x\sin x}{2} - \frac{\sin x}{2}$$

$$y = CF + PI$$

(2) Solve $y'' + 16y = x\sin 3x$

Sol:- $(D^2+16)y = x\sin 3x$

A.E is $m^2+16=0$

$m^2 = -16$

$m = \pm 4i$

$$CF = c_1\cos 4x + c_2\sin 4x$$

$$P.I = \frac{1}{D^2+16}\, x\sin 3x$$

$$\left(x - \frac{f'(D)}{f(D)}\right)\frac{V}{f(D)}$$

$$= \left(x - \frac{2D}{D^2+16}\right)\frac{\sin 3x}{D^2+16}$$

$a=3$

Put $D^2 = -3^2$

$D^2 = -9$

$$= \left(x - \frac{2D}{D^2+16}\right)\frac{\sin 3x}{-9+16}$$

$$P.I = \left(x - \frac{2D}{D^2+16}\right) \frac{\sin 3x}{7}$$

$$= x\frac{\sin 3x}{7} - \frac{2D(\sin 3x)}{7(D^2+16)}$$

put $D^2 = -3^2$

$$= x\frac{\sin 3x}{7} - \frac{2(\cos 3x)\cdot 3}{7(-9+16)}$$

$$= x\frac{\sin 3x}{7} - \frac{6\cos 3x}{7(7)}$$

$$P.I = \frac{x\sin 3x}{7} - \frac{6\cos 3x}{49}$$

General solution is

$y = CF + P.I$

$$y = (c_1\cos 4x + c_2\sin 4x) + \frac{x\sin 3x}{7} - \frac{6\cos 3x}{49}$$

③ Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Sol: $(D^2 - 2D + 1)y = xe^x \sin x$

A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1, 1$

$CF = (c_1 + c_2 x)e^x$

$$P.I = \frac{1}{D^2-2D+1}\, x e^x \sin x$$

$$= \frac{1}{(D-1)^2}\, \frac{x e^x \sin x}{}$$

$$\left[\frac{e^x (x\sin x)}{\text{Type } \underline{iv}} \right.$$

$$= e^x \cdot \frac{1}{(D-1)^2}\, \underbrace{x \sin x}$$

Put $D = D+1$

$$= e^x \cdot \frac{1}{(D+\cancel{1}-\cancel{1})^2}\, x \sin x$$

$$\left[e^x \cdot \frac{1}{\underbrace{D^2-2D+1}}\, \underline{x \sin x} \right.$$

$$= e^x \cdot \frac{x \sin x}{D^2}$$

$$= e^x \cdot \frac{1}{D}\left(\underbrace{\int x \sin x \, dx} \right)$$

$$= e^x \cdot \frac{1}{D}\left[x(-\cos x) - \int 1\cdot(-\cos x)\,dx \right]$$

$$= e^x \cdot \frac{1}{D}\left[-x\cos x + \sin x \right]$$

$$= e^x \int (-x\cos x + \sin x)\, dx$$

$$= e^x \left(-(-x\sin x - \cos x) - \cos x \right]$$

$$P.I = -e^x \underline{(x \sin x + 2\cos x)}$$

General solution is

$$y = CF + PI$$

$$y = (c_1 + c_2 x)e^{x} - e^{x}(x\sin x + 2\cos x)$$

(4) Solve $y'' - 4y' + 4y = 8x^2 e^{2x}\cos 2x$

Sol:- $(D^2 - 4D + 4)y = 8x^2 e^{2x}\cos 2x$

A.E is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$C.F = (c_1 + c_2 x)e^{2x}$$

$$P.I = \frac{1}{D^2 - 4D + 4} 8x^2 e^{2x}\cos 2x$$

$$= 8e^{2x} \frac{1}{D^2 - 4D + 4} x^2\cos 2x$$

Put $D = D+2$

$$= 8e^{2x} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2\cos 2x$$

$$= 8e^{2x} \cdot \frac{1}{D^2 \cancel{+4} \cancel{+4D} \cancel{-4D} \cancel{-8} \cancel{+4}} x^2\cos 2x$$

$$P.I = 8e^{2x} \cdot \frac{1}{D^2}\left(x^2 \cos 2x\right)$$

$$= 8e^{2x} \cdot \frac{1}{D}\left(\frac{1}{D} x^2 \cos 2x\right)$$

$$= 8e^{2x} \cdot \frac{1}{D}\left(\int x^2 \cos 2x \, dx\right)$$

$$= 8e^{2x} \frac{1}{D}\left[\left(x^2\left(\frac{\sin 2x}{2}\right) - \cancel{2}x\left(-\frac{\cos 2x}{\cancel{4}_{2}}\right) + \cancel{2}\left(-\frac{\sin 2x}{\cancel{8}_{4}}\right)\right]$$

$$= 8e^{2x} \frac{1}{D}\left(x^2 \frac{\sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4}\right)$$

$$= 8e^{2x} \int\left(\underbrace{\frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4}}_{\text{Integrate}}\right) dx$$

$$P.I = e^{2x}\left(\cos 2x \,(3 - 2x^2) + 4x \sin 2x\right)$$

General solution is

$$y = CF + PI$$

$$y = (c_1 + c_2 x) e^{2x} + e^{2x}\left(\cos 2x \,(3 - 2x^2) + 4x \sin 2x\right)$$

* Bernoulli's rule for integration

$$\int u\,v\,dx = u \cdot v_1 - u' v_2 + u'' v_3 - u''' v_4 + \text{-----}$$

$u', u'' \ldots$ successive derivatives

$v_1, v_2, \ldots$ are successive integrals

$$\int \underset{u}{x}\,(\underset{v}{\sin x})\,dx$$

$$= x \cdot (-\cos x) - 1 \cdot (-\sin x)$$

$$= (-x\cos x + \sin x)$$

$$\int \underset{u}{x^2}\,\underset{v}{\cos 2x}\,dx =$$

$$\cancel{x^2 \cdot (\sin 2x)}$$

$$= x^2\left(\frac{\sin 2x}{2}\right) - 2x\left(-\frac{\cos 2x}{4}\right) + 2\left(-\frac{\sin 2x}{8}\right)$$

$$= \frac{x^2 \sin 2x}{2} + \frac{2x\cos 2x}{4} - \frac{2\sin 2x}{8}$$

1. Solve $(D^2+2D+1)y = x\cos x$

2. Solve $(D^2+4)y = x\cos 3x$

Initial and Boundary value Problems

Initial value Problem :- (IVP)

Initial value problem in which a solution to a D.E is obtained subject to conditions on the unknown function and its derivatives specified at one value of the independent variable. And such condition is called initial condition

Ex:- $(D^2+2D+4)y = e^{2x}$

$y(0)=1, \quad y'(0)=-1$

$y(1)=1, \quad y'(1)=2$

$y(0)=1$

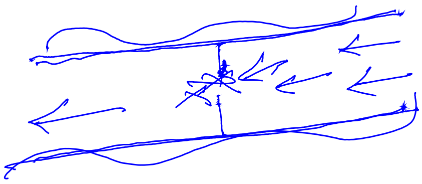
when $x=0$, $y=1$

Boundary Value Problem (BVP)

BVP is one which a solution to a D.E is obtained subject to conditions on the unknown function and its derivatives specified at two (or) more values of the independent variables such condition is called Boundary condition.

Ex:— ① $(D^2+2)y = \cos 2x$

$y(0)=1, \quad y(\pi)=-1$



① Solve $y''-6y'+9y=0$, given $y(0)=1$, $y'(0)=2$

(IVP)

Sol:— $(D^2-6D+9)y=0$

A.E is $m^2-6m+9=0$

$m^2-3m-3m+9=0$

$m(m-3)-3(m-3)=0 \Rightarrow m=3,3$

$y=(c_1+c_2x)e^{3x} \rightarrow ①$

Gen Sol $y = (c_1 + c_2 x) e^{3x} \longrightarrow (1)$

14

Given $y(0) = 1$

when $x = 0$
$y = 1$

$1 = (c_1 + c_2(0)) e^0$

$1 = (c_1 + 0) 1 \implies \boxed{c_1 = 1}$

$(1) \implies y = \underbrace{(c_1 + c_2 x)} \cdot e^{3x}$

Diff w r t x

$y' = (c_1 + c_2 x) e^{3x} \cdot 3 + e^{3x}(0 + c_2)$

$y' = 3e^{3x}(c_1 + c_2 x) + c_2 e^{3x}$

Given
$y'(0) = 2$
when $x = 0$
$y' = 2$

$2 = 3e^0(c_1 + c_2(0)) + c_2 e^0$

$2 = 3(1 + 0) + c_2$

$2 = 3 + c_2$

$\implies \boxed{c_2 = -1}$

Sub c_1 & c_2 in (1)

$y = (1 + (-1) \cdot x) e^{3x} \implies y = (1 - x) e^{3x}$

2. Solve the boundary value problem

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 8x^2, \text{ given}$$

$$y(0) = 1 \quad \text{and} \quad y(1) = 1$$

Sol :-

$$(D^2 + 4D + 4)y = 8x^2$$

A.E is

$$m^2 + 4m + 4 = 0$$

$$m^2 + 2m + 2m + \cancel{4} = 0$$

$$m(m+2) + 2(m+2) = 0$$

$$(m+2)(m+2) = 0$$

$$m = -2, -2$$

$$CF = (C_1 + C_2 x)e^{-2x}$$

$$P.I = \frac{1}{D^2 + 4D + 4} 8x^2$$

$$= 8 \cdot \frac{1}{D^2 + 4D + 4} x^2$$

$$= \cancel{8}^{2} \cdot \frac{1}{\cancel{4}\left(1 + \frac{D^2 + 4D}{4}\right)} x^2$$

$$= 2 \cdot \left(1 + \frac{D^2 + 4D}{4}\right)^{-1} x^2$$

$(1+x)^{-1}$ form

$$P.I = 2\left[1 + \frac{D^2+4D}{4}\right]^{-1} x^2$$

$$= 2\left[1 - \left(\frac{D^2+4D}{4}\right) + \left(\frac{D^2+4D}{4}\right)^2\right] x^2$$

$$= 2\left[1 - \frac{D^2}{4} - \frac{\cancel{4}D}{\cancel{4}} + \frac{1}{16}\left(D^4 + 16D^2 + 8D^3\right)\right] x^2$$

$$= 2\left[1 - \frac{D^2}{4} - D + \frac{1}{\cancel{16}}\cancel{16}\, D^2\right] x^2$$

$$= 2\left[1 + D^2 - \frac{D^2}{4} - D\right] x^2$$

$$= 2\left[1 + \frac{3D^2}{4} - D\right] x^2$$

$$= 2\left[x^2 + \frac{3}{4} D^2(x^2) - D(x^2)\right]$$

$$= 2\left[x^2 + \frac{3}{4} \cdot 2 - 2x\right]$$

$$PI = 2\left[x^2 + \frac{6}{4} - 2x\right]$$

Gen sol is $y = CF + PI$

$$y = (c_1 + c_2 x)e^{-2x} + 2\left(x^2 - 2x + 3/2\right) \rightarrow (1)$$

Given $y(0)=1$

when $x=0,\ y=1$

① $\Rightarrow$

$$1=(c_1+c_2\cdot 0)\,e^{-2(0)}+2(0-0+3/2)$$

$$1=c_1+\cancel{2}(3/\cancel{2})$$

$$\Rightarrow \boxed{c_1=-2}$$

Given $y(1)=1$

when $x=1,\ y=1$

① $\Rightarrow$

$$1=(c_1+c_2(1))\,e^{-2(1)}+2\left((1)^2-2(1)+3/2\right)$$

$$1=(c_1+c_2)\,e^{-2}+2(1-2+3/2)$$

$$1=(c_1+c_2)\,e^{-2}+2(-1+3/2)$$

$$1=(c_1+c_2)\,e^{-2}+\cancel{2}(1/\cancel{2})$$

$$(c_1+c_2)\,e^{-2}=1-1$$

$$(c_1+c_2)\,e^{-2}=0 \longleftarrow ②$$

Sub $c_1=-2$ in ② $\Rightarrow$ $(-2+c_2)\,e^{-2}=0$

$$-2+c_2=0$$

$$\boxed{c_2=2}$$

Sub c_1 & c_2 in (1)

$$y = (-2 + 2x)\, e^{-2x} + 2\left(x^2 - 2x + 3/2\right)$$