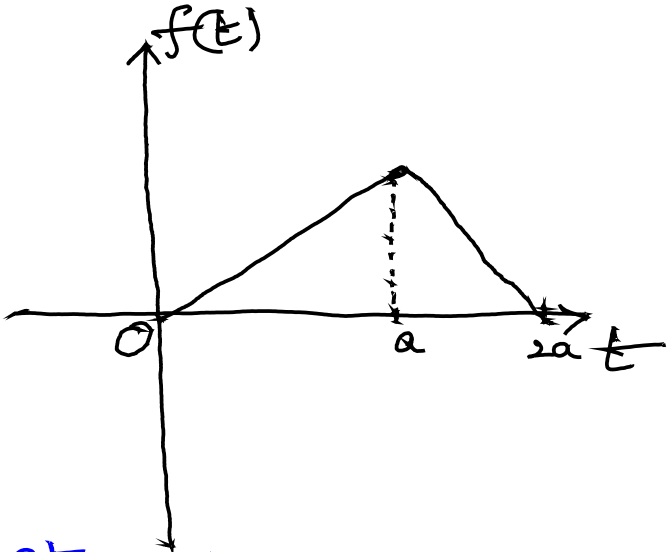
1) Draw the graph of the following function $f(t) = \begin{cases} t, & 0 < t < a \\ 2a-t, & a < t < 2a \end{cases}$ and hence find its Laplace transform.



Sol :-

Period $= T = 2a$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-s(2a)}} \int_0^{2a} e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a t \cdot e^{-st} dt + \int_a^{2a} (2a-t) e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(t \cdot \frac{e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{(-s)^2} \right)_0^a + \left((2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{(-s)^2}\right) \right)_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left(\left(a\cdot\frac{e^{-as}}{-s} - \frac{e^{-as}}{s^2}\right) - \left(0 - \frac{\overset{0}{e}}{s^2}\right)\right)\right] \quad ②$$

$$+ \left(\left(\cancel{(2a-2a)\frac{e^{-2as}}{-s}}^{\,0} + \frac{e^{-2as}}{s^2}\right) - \right.$$

$$\left.\left((2a-a)\left(\frac{e^{-as}}{-s}\right) + \frac{e^{-as}}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\cancel{a\cdot\frac{e^{-as}}{-s}} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \cancel{a\frac{e^{-as}}{s}} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2}\left(-e^{-as} + e^{-2as} - e^{-as} + 1\right)\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[1 - 2e^{-as} + e^{-2as}\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[1^2 - 2\cdot e^{-as}\cdot 1 + (e^{-as})^2\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[(1-e^{-as})^2\right]$$

$$= \frac{1}{s^2\left(1^2-\left(e^{-as}\right)^2\right)} \left(1-e^{-as}\right)^2$$

$$= \frac{1}{s^2\left(\left(1+e^{-as}\right)\cancel{\left(1-e^{-as}\right)}\right)} \left(1-e^{-as}\right)^{\cancel{2}}$$

$$= \frac{1}{s^2}\left(\frac{1-e^{-as}}{1+e^{-as}}\right)$$

$$= \frac{1}{s^2}\,\frac{e^{as/2}}{e^{as/2}}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]$$

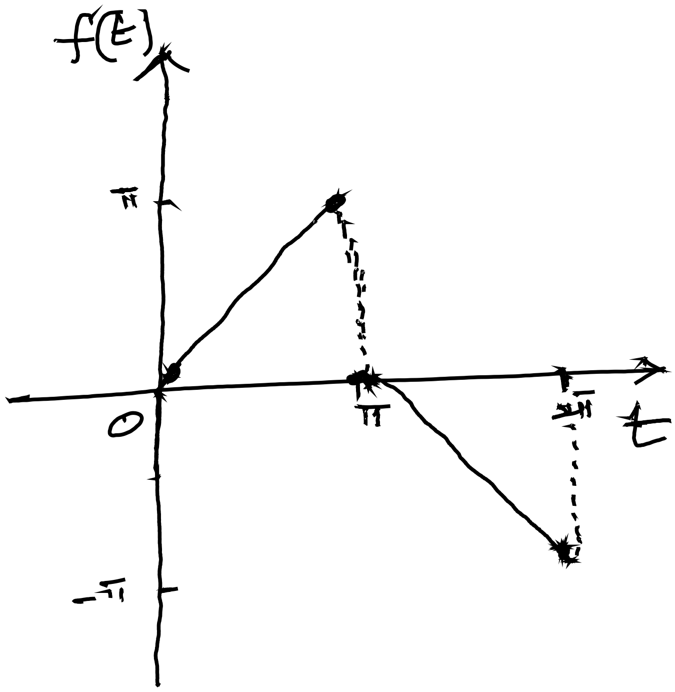
$$= \frac{1}{s^2}\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right]$$

$$= \frac{1}{s^2}\cdot \underline{\tanh\left(\frac{as}{2}\right)}$$

$$\tanh ax = \frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$$

(2) Draw the graph of the periodic function $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$

and hence find its Laplace transform.



Sol :-

period $= T = 2\pi$

$$L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} t \cdot e^{-st}\, dt + \int_{\pi}^{2\pi} (\pi - t)\, e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\left(t\,\frac{e^{-st}}{-s} - 1\cdot\frac{e^{-st}}{(-s)^2}\right)_0^{\pi} + \left((\pi-t)\,\frac{e^{-st}}{-s} - (-1)\,\frac{e^{-st}}{(-s)^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\left(\frac{-\pi e^{-s\pi}}{s} - \frac{e^{-s\pi}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right) + \left((\pi-2\pi)\,\frac{e^{-2\pi s}}{(-s)} + \frac{e^{-2\pi s}}{s^2}\right) - \left(0 + \frac{e^{-s\pi}}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[-\frac{\pi}{s}\,e^{-s\pi} - \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} + \frac{\pi}{s}\,e^{-2\pi s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-s\pi}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left(1 - 2e^{-s\pi} + e^{-2\pi s}\right)\right]$$

3 For the periodic function with period 4, $f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}$

Find its Laplace transform.

Sol :- period $= T = 4$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st} \cdot 3t \, dt + \int_2^4 6 e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[3 \cdot \left(t \frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)^2} \right)_0^2 + 6 \cdot \left(\frac{e^{-st}}{-s} \right)_2^4 \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\left(\frac{3}{s} \cdot \left(-2e^{-2s}\right) - \frac{e^{-2s}}{s^2} \right) - \left(3 \cdot 0 - \frac{1}{s^2} \right) + \left(\frac{-6}{s}\right)\left(e^{-4s} - e^{-2s}\right) \right]$$

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$$= \frac{1}{1-e^{-4s}}\left[\frac{-6}{s}\cancel{e^{-2s}} - \frac{3e^{-2s}}{s^2} + \frac{3}{s^2} - \frac{6}{s}e^{-4s} + \cancel{\frac{6}{s}e^{-2s}}\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{3}{s^2} - \frac{3e^{-2s}}{s^2} - \frac{6}{s}e^{-4s}\right]$$

4 A periodic square wave function $f(t)$ interms of unit step function is written as

$$f(t) = k\left[u_0(t) - 2\cdot u_a(t) + 2\cdot u_{2a}(t) - 2\cdot u_{3a}(t) + \cdots\right]$$

where $u_i(t) = u(t-i)$

then show that $L\{f(t)\} = \frac{k}{s}\tanh\left(\frac{as}{2}\right)$

Sol:- $f(t) = k\{u_0(t) - 2\cdot u_a(t) + 2\cdot u_{2a}(t) \cdots\}$

$$f(t) = k\cdot u_0(t) - 2k\cdot u_a(t) \cdot 2k\cdot u_{2a}(t) \cdots$$

$$L\{f(t)\} = k\cdot L\{u_0(t)\} - 2k\cdot L\{u_a(t)\} + 2k\cdot L\{u_{2a}(t)\} \cdots$$

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$$L\{f(t)\} = \frac{K}{s} \cdot e^{0} - 2K \cdot \frac{e^{-as}}{s} + 2K\frac{e^{-2as}}{s} - 2K\frac{e^{-3as}}{s} + \cdots$$

$u_{?}(t) = u(t-?)$

$u_0(t) = u(t-0) = u(t)$

$u_a(t) = u(t-a)$

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$L\{f(t)\} = \frac{K}{s} - \frac{2K}{s}e^{-as} + \frac{2K}{s}e^{-2as} - \frac{2K}{s}e^{-3as} + \cdots$$

$$= \frac{K}{s}\left[1 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \cdots\right]$$

$$= \frac{K}{s}\left[1 - 2e^{-as}\left(1 - e^{-as} + e^{-2as} - \cdots\right)\right]$$

$$= \frac{K}{s}\left(1 - 2e^{-as}\left(1 - e^{-as} + (e^{-as})^2 - \cdots\right)\right]$$

$1 - x + x^2 + \cdots = (1+x)^{-1}$

$= \frac{1}{1+x}$

$$= \frac{K}{s}\left[1 - 2e^{-as} \cdot \frac{1}{1+e^{-as}}\right]$$

$$= \frac{k}{s}\left[\frac{(1+e^{-as}) - 2e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{k}{s}\left[\frac{1 - e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{k}{s}\,\frac{e^{as/2}}{e^{as/2}}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{k}{s}\,\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}$$

$$L\{f(t)\} = \frac{k}{s}\cdot \tanh\left(as/2\right)$$