ELECTROMAGNETIC THEORY

Electrodynamics is that branch of Physics which deals with the study of the microscopic behavior of charges in motion and their interaction with matters.

- **Scalar and vector fields:**

Every physical quantity can be expressed as a continuous function of the position of a point in a region of space, known as a field. There are two kinds of fields, scalar and vector fields, depending on nature of quantity concerned.

- The Scalar field is represented by continuous scalar function, giving the value of the quantity at each point. Example: Region of electric potential.
- A Vector field is represented by continuous vector function, which is specified by a vector of definite magnitude and direction at each point. Example: Region of Electric field.

- **Gradient of Scalar field:**

Let ϕ(x,y,z) be a Scalar function in a region of space(Scalar field).The Gradient of Scalar function is defined as,

$$\textbf{grad } \phi = \boldsymbol{i}\frac{\partial\phi}{\partial x}+\boldsymbol{j}\frac{\partial\phi}{\partial y}+\boldsymbol{z}\frac{\partial\phi}{\partial z}$$

where i, j, k are the unit vectors along x ,y , z co-ordinate directions.

$$\textbf{grad } \phi = (\boldsymbol{i}\frac{\partial\phi}{\partial x}+\boldsymbol{j}\frac{\partial\phi}{\partial y}+\boldsymbol{z}\frac{\partial\phi}{\partial z})\ \phi \quad = \quad \nabla\phi$$

$$\text{where} \qquad \nabla = (\ \boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{z}\frac{\partial}{\partial z})$$

Note: *The gradient of a scalar function is a vector*, whose magnitude is equal to the rate of change of scalar function ϕ along normal to the level surface.

- For example, the rate of change of electric potential (i.e. vector field) in a region of space (i.e. scalar field) gives Electric field ($\vec{E}$) at that point.

$$\mathbf{E} = \quad -(\ \boldsymbol{i}\frac{\partial V}{\partial x}+\boldsymbol{j}\frac{\partial V}{\partial y}+\boldsymbol{z}\frac{\partial V}{\partial z}) \qquad\qquad \textbf{(or)}$$

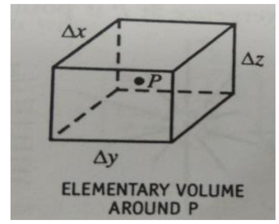
$$\boxed{\vec{\boldsymbol{E}} = -\nabla \mathbf{V}}$$

-ve sign indicates that, the Electric field is directed in the direction of decreasing potential.

- **<u>Divergence of a vector field :</u>**

The divergence of vector field $\vec{A}$ at a point ‘p’ is defined as the outward flux of the vector field per unit volume.

Let ∇v be infinitesimal volume surrounding the point ‘p’, then divergence of vector field $\vec{A}$ at ‘p’ is,



$$\mathbf{div}\,\vec{\mathbf{A}} = \lim_{\Delta v \to 0} \frac{\oiint \vec{\mathbf{A}}.\vec{\mathbf{ds}}}{\Delta \mathbf{v}}$$

The divergence of vector $\vec{A}$ in Cartesian co-ordinates is expressed as,

$$\boxed{\mathbf{div}\,\vec{A} = \nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

Note: <u>*The divergence of a vector field is a scalar quantity.*</u>

- **<u>Physical significance of divergence:</u>**

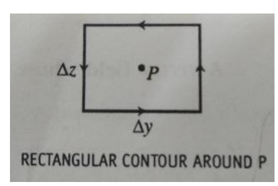
The divergence of fluid velocity $\vec{V}$ at a point $(\nabla.\vec{V})$ represents the quantity of fluid flowing out per second per unit volume at that point.

  - If $(\nabla.\vec{V})$ at a point is –ve, it means that fluid is flowing towards that point. (i.e sink for the fluid).
  - If $(\nabla.\vec{V})$ is +ve , then fluid is flowing outwards. (i.e. source for the fluid at that point)
  - If $(\nabla.\vec{V}) = 0$ at a point, then fluid entering towards the point is equal to the fluid leaving the point. (i.e. no source and sink at that point).

- If $\nabla.\vec{A} = 0$, then $\vec{A}$ is called **<u>solenoidal field</u>**.

- **<u>Curl of a vector field:</u>**

If $\vec{A}$ is a vector field at a point ‘p’ and ‘Δs’ ia an infinesinal area around ‘p’ then **curl $\vec{A}$** is defined as



$$\mathbf{curl}\,\overrightarrow{\mathbf{A}} = \lim_{\Delta s \to 0} \frac{\oint [\vec{A}.\vec{dl}]_{max}}{\Delta s}$$

  - In Cartesian co-ordinate system, **curl $\vec{A}$** is expressed as,

$$\mathbf{curl}\,\vec{\mathbf{A}} = \nabla \times \vec{\mathbf{A}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

- <u>*Curl of a vector field is another vector*</u> whose direction is normal to the area around point 'p' when the area is oriented to make the circulation maximum.
- The physical significance of the curl of a vector field is that it provides the maximum value of circulation of the field per unit area, as the area shrinks to zero at the given point.
- For example, when current is passed through a straight conductor, the magnetic field curls around conductor in the same way as **curl** $\vec{H}$.
- If $\nabla \times \vec{A} = 0$, then the vector field has no circulation or turning effect at the point. Such fields (whose $\nabla \times \vec{A} = 0$), are called conservative or irrotational fields.

- **<u>Electric Flux (ϕ) and Flux density (D):</u>**

Electric flux (ϕ) is a scalar field, useful in solving certain electrostatic problems.

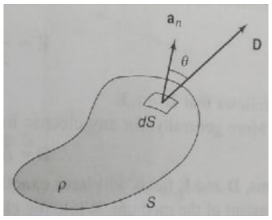
If an amount of Electric flux (dϕ) crosses a small area 'ds', normal to the surface, then the electric flux density (D) at that point is,

$$\vec{D} = \frac{d\phi}{ds}\,\hat{n}$$

where $\hat{n}$ is a unit vector normal to the surface area 'ds'.

- *Electric flux density ($\vec{D}$) is vector field.*

Let a charge of volume charge density (ρ) be enclosed by surface 'S', as shown in the figure.



The electric flux density (D) may vary in magnitude and direction from point to point on the surface 'S'.

At the surface element 'ds', if $\vec{D}$ makes an angle 'θ' with the normal, then electric flux crossing 'ds' is given by,

$$\mathbf{d\phi = D\ ds\cos\theta} = \vec{\boldsymbol{D}}.\overrightarrow{\boldsymbol{ds}}$$

where $\overrightarrow{ds}$ is the vector surface element of magnitude 'ds' and direction along $\hat{n}$.

The total Electric flux through any closed surface 's' is given by,

$$\boldsymbol{\phi} = \int_{s} \vec{\boldsymbol{D}}.\overrightarrow{\boldsymbol{ds}}$$

- **Relation between Electric flux density ($\vec{D}$) and Electric field ($\vec{E}$):**

Electric field ($\vec{E}$) at a point due to a charge configuration is a function of permittivity (ε) of the medium, whereas Electric flux density (D) is independent of the medium.

In general, for an Electric field ($\vec{E}$) in a medium of permittivity ε, the flux density ($\vec{D}$)is,

$$\boxed{\vec{D} = \varepsilon\,\vec{E}}$$

or

$$\boxed{\vec{D} = \varepsilon_0\varepsilon_r\,\vec{E}}$$

Where, $\varepsilon_r \rightarrow$ Di-electric constant (or) relative permittivity of the medium.

➢ **<u>Note:</u>** Unlike $\vec{E}$, $\vec{D}$ is not directly measurable . But, both $\vec{E}$ and $\vec{D}$ vector fields have exactly the same form.

- **<u>Gauss law in Electrostatics :</u>**

Gauss law of electrostatics states that "The total electric flux out of the closed surface is equal to the net charge within the surface".

This can be written in integral form as,

$$\boxed{\oint_{s} \vec{\boldsymbol{D}}.\overrightarrow{\boldsymbol{ds}} = \mathbf{Q}}$$

OR

$$\boxed{\oint_S \vec{E}.\vec{ds} = \frac{Q}{\varepsilon}}$$

('ε' is the permittivity of the medium. $\vec{D} = \varepsilon \vec{E}$)

- **Gauss law in differential form:**

Gauss law in Electrostatics in integral form is given by,

$$\oint_S \vec{D}.\vec{ds} = Q \qquad \rightarrow ①$$

where, Q is the net charge enclosed by the closed surface.

If 'ρ' is the volume charge density (i.e charge per unit volume), then the charge enclosed in an elemental volume 'dv' is $\boldsymbol{\rho.\mathrm{dv}}$. Then the total charge inside the closed surface will be,

$$Q = \iiint \rho.\mathrm{dv} \qquad \rightarrow ②$$

From eqn ① *and* ② , the Gauss law is modified as

$$\oint_S \vec{D}.\vec{ds} = \iiint \rho.\mathrm{dv} \qquad \rightarrow ③$$

Now according to divergence theorem,

$$\oint_S \vec{D}.\vec{ds} = \iiint \nabla.\vec{D}\,\mathrm{dv} \qquad \rightarrow ④$$

∴ From eqn. ③ and ④, we get

$$\iiint \nabla.\vec{D}\,\mathrm{dv} = \iiint \rho.\mathrm{dv}$$

This equation holds good for any volume. Hence,

$$\boxed{\nabla.\vec{D} = \rho}$$

This is the statement of Gauss law in differential form.

In terms of $\vec{E}$, Gauss law is

$$\boxed{\nabla.\vec{E} = \frac{\rho}{\varepsilon}}$$

<u>Note:</u> $\nabla.\vec{D} = \rho$ or $\nabla.\vec{E} = \frac{\rho}{\varepsilon}$ This is one of the **Maxwell's equation** for static fields.

- **<u>Current and current density (J):</u>**

Current density is defined as current per unit area of cross section of an imaginary plane held normal to the direction of current.

$$J = \left(\frac{dI}{ds}\right)$$

The total current through the given surface 's' is obtained by,

$$I = \iint \vec{J}.\vec{ds} = \iint \vec{J}\ ds\ Cos\theta$$

<u>Note:</u> Current density is vector quantity.

- **<u>Equation of Continuity :</u>**

The divergence of current density $(\vec{J})$ at that point $(\nabla.\vec{J})$ is equal to the quantity of charge flowing out per second per unit volume, through the small closed surface surrounding that point. By law of Conservation of charges, $(\nabla.\vec{J})$ must be equal to the rate of decrease of charge density (ρ) at that point.

i.e $\nabla.\vec{J} = -\frac{\partial\rho}{\partial t}$

or

$$\boxed{\nabla.\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

This equation is known as Equation of continuity.

- **<u>Magnetic field intensity $(\vec{H})$ and Magnetic flux density$(\vec{B})$</u>**

The source for magnetic field is current elements (I . dl). The magnetic field at a point can be described by two vectors, magnetic field intensity $(\vec{H})$ and magnetic flux density $(\vec{B})$.

- Magnetic field intensity ($\vec{H}$) depends only on the source (i.e current element or moving charges) and it is independent of the medium, whereas magnetic flux density ($\vec{B}$) depends on the medium at that point.
- The two vectors $\vec{B}$ and $\vec{H}$ are related by

$$\boxed{\vec{B} = \mu \vec{H}}$$

OR

$$\boxed{\vec{B} = \mu_0 \mu_r \vec{H}}$$

where , $\mu \rightarrow$ permeability of the medium.

$\mu_o \rightarrow$ permeability of vacuum. $\mu_o = 4\pi \times 10^{-7} Hm^{-1}$

$\mu_r \rightarrow$ Relative permeability of the medium

- 'H' is expressed in units of Am^{-1} whereas 'B' is expressed in tesla (T).
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- **Biot-Savart's Law:**

This law gives the magnetic field intensity at a point due to a current element. Let I be the current in a conductor XY. Let 'p' be a point at a distance of 'r' from a differential element of the conductor of length 'dl'.

According to Biot-Savart law, the magnetic field at 'p' due o 'dl' is given by

$$dH = k \frac{I\, dl\, sin\theta}{r^2}$$

where, θ is the angle between the current element and the line joining the point to the element, and k is the proportionality constant, given by $k = \frac{1}{4\pi}$.

$$\therefore \quad dH = \frac{I\, dl\, sin\theta}{4\pi r^2}.$$

This field acts perpendicular to the plane containing the element and the point 'p'.

In vector form, Biot-Savart law is written as,

$$\overrightarrow{dH} = \frac{I\,\overrightarrow{dl}\,\times \hat{r}}{4\pi r^2}$$

where $\hat{r}$ is the unit vector directed from the element towards ‘p’.

- $\overrightarrow{dH}$ is perpendicular to $\overrightarrow{dl}$ and $\hat{r}$.
- $\overrightarrow{dB} = \left(\frac{\mu}{4\pi}\right)\frac{I\,\overrightarrow{dl}\times\hat{r}}{r^2}$

- **<u>Ampere’s law (Integral form):</u>**

This law states that, “the line integral of the tangential component of the magnetic field around a closed path is equal to the current enclosed by the loop”.

i.e $$\oint \vec{H}\cdot\overrightarrow{dl} = I$$

or $$\oint \vec{B}\cdot\overrightarrow{dl} = \mu_0 I$$

Note: The closed loop to which Ampere’s law is applied is called Ampere loop.

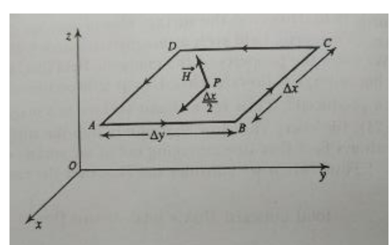
- **Ampere’s law (Differential form):**

Ampere’s law in integral form is given by

$$\oint \vec{H}\cdot\overrightarrow{dl} = I$$

Let us now consider a rectangular path ABCD of infinitesimal size of area (Δs_1) in a plane parallel to XY-plane. Area of the loop = Δs_1= (Δx)(Δy). If J_z is the current density along Z-direction, then the current through the loop is $I_z = J_z(\Delta s_1) = J_z\,(\Delta x)(\Delta y)$.

Therefore from Ampere’s law,



RECTANGULAR CLOSED PATH IN A FIELD

$$\oint \vec{H}\;\cdot\overrightarrow{(dl)}\;= J_z(\Delta s_1)$$

Or $$\frac{\oint \vec{H}\cdot\overrightarrow{dl}}{\Delta s_1} = J_z$$

The assumption that the magnetic field does not vary along the line segment is more valid if we consider the limits Δx→ 0 and Δy→ 0.

$$\therefore \quad \lim_{(\Delta x)(\Delta y)\to 0} \oint \frac{\vec{H}\cdot\vec{dl}}{\Delta s_1} = J_z \qquad \text{----------}\rightarrow (1)$$

As per vector calculus, LHS term of (1) represents curl of H, which is vector field acting normal to Δs_1 (i.e. along Z-direction).

$$\therefore \left[\nabla \times \vec{H}\right]_1 = J_z \hat{k}$$

$\hat{k}$ is unit vector along Z-direction.

Here, $[\nabla \times H]_1$ represents that the curl is in a plane parallel to xy-plane.

Similarly, it can be extended to the planes parallel o yz and zx planes as,

$$\left[\nabla \times \vec{H}\right]_2 = J_x \hat{\imath}$$

$$\left[\nabla \times \vec{H}\right]_3 = J_y \hat{\jmath}$$

Here, $[\nabla \times H]_1$, $[\nabla \times H]_2$ and $[\nabla \times H]_3$ are the components of $\left[\nabla \times \vec{H}\right]$ around the point 'p'.

$$\therefore \nabla \times \vec{H} = J_x \hat{\imath} + J_y \hat{\jmath} + J_z \hat{k}$$

Or $$\boldsymbol{\nabla} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} \qquad \text{------------}\rightarrow (2)$$

where $\vec{J}$ is the current density vector having components $J_x,\ J_y\ and\ J_z$ along the 3 co-ordinates. Equation (2) is represents Ampere's law in differential form.

- Note: $\boldsymbol{\nabla} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}}$ is one of the Maxwell's equations for static fields.

- **<u>Gauss law in magnetism:</u>**

  The magnetic field pattern in a region can be visualized in terms of magnetic flux lines.

  Magnetic flux through a surface is defined as

$$\varphi = \int_s \vec{\boldsymbol{B}}.\vec{\boldsymbol{ds}} \qquad \text{where, B} \rightarrow \text{magnetic flux density.}$$

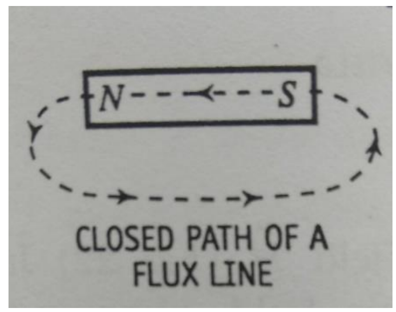
It should be noted that, magnetic flux lines are closed curves, with no starting point or termination point (This is in contrast with lines of electric flux, which originate on +ve charge and terminate on –ve charge).

If we consider a closed surface, then, all of the magnetic flux entering the closed surface is equal to the flux leaving the closed surface.

Thus, "the total magnetic flux over any closed surface is equal to zero".

i.e. $\varphi = \int_{s} \vec{\boldsymbol{B}}.\overrightarrow{\boldsymbol{ds}} = 0$

This is known as Gauss law in magnetostatics.



Now, as per vector calculus,

$$\nabla \cdot \vec{B} = \lim_{(\Delta v)\to 0} \frac{\int \vec{\boldsymbol{B}}.\overrightarrow{\boldsymbol{ds}}}{\Delta v}$$

$$\therefore\ \boldsymbol{\nabla}.\vec{\boldsymbol{B}} = \boldsymbol{0}$$

- Note: This equation, $\nabla.\vec{B} = 0$ is one of the Maxwell's equations.

- **<u>Faraday's law of electromagnetic induction:</u>**

It states that "the magnitude of the induced emf in a circuit is equal to the rate of change of magnetic flux through it".

$$e = -\frac{d\varphi}{dt}$$

-ve sign indicates that the induced emf opposes the change in flux.

{ If $\vec{E}$ is the electric field, then $(\vec{E} \cdot \overrightarrow{dl})$ gives the work done to move unit +ve charge (+1C) through dl. The total work done to move +1C from a to b is equal to pd or emf between a and b. $\therefore emf = \int_a^b \vec{E} \cdot \overrightarrow{dl}$. In a closed loop, $emf = \oint \vec{E} \cdot \overrightarrow{dl}$ }

The induced emf in a closed loop is given by

$$e = \oint \vec{E} \cdot \overrightarrow{dl}$$

$$\therefore \oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\varphi}{dt} \quad \text{-----------} \rightarrow (1)$$

The magnetic flux φ through an area 'S' enclosed by the loop is,

$$\varphi = \int_{s} \vec{\boldsymbol{B}}.\overrightarrow{\boldsymbol{ds}}$$

Where B is magnetic flux density

$$\therefore \frac{d\varphi}{dt} = \frac{d}{dt}\int_{s} \vec{\boldsymbol{B}}.\overrightarrow{\boldsymbol{ds}}$$

$$\frac{d\varphi}{dt} = \int_{s} \frac{\partial \vec{B}}{\partial t}.\overrightarrow{\boldsymbol{ds}} \quad \text{-----------} \rightarrow (2)$$

From (1) and (2) we get,

$$\oint \vec{E} \cdot \overrightarrow{dl} = -\int_{s} \frac{\partial \vec{B}}{\partial t}.\overrightarrow{\boldsymbol{ds}} \quad \text{----------} \rightarrow (3)$$

Equation (3) is the Faraday's law in integral form.

Now, by Stokes theorem,

$$\oint \vec{E} \cdot \overrightarrow{dl} = \oint_{s} (\nabla \times \vec{E}) \cdot \overrightarrow{ds}$$

Therefore equation (3) can be written as

$$\oint_{s} (\nabla \times \vec{E}) \cdot \overrightarrow{ds} = -\int_{s} \frac{\partial \vec{B}}{\partial t}.\overrightarrow{\boldsymbol{ds}}$$

or $$\oint \left[(\nabla \times \vec{E}) + \frac{\partial \vec{B}}{\partial t}\right] \cdot \overrightarrow{ds} = 0$$

Since, 'ds' $\neq 0$, $$\left[(\nabla \times \vec{E}) + \frac{\partial \vec{B}}{\partial t}\right] = 0$$

Or $$(\boldsymbol{\nabla} \times \vec{\boldsymbol{E}}) = -\frac{\partial \vec{\boldsymbol{B}}}{\partial \boldsymbol{t}} \quad \text{-------------} \rightarrow (4)$$

Equation (4) represents the Faraday's law in differential form.

➢ Note: This equation, $(\nabla \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t}$ is one of the Maxwell's equations.

- **<u>Inconsistency or anomaly in Ampere's law:</u>**

Ampere's law in differential form is

$$\nabla \times \vec{H} = \vec{J}$$

Taking divergence on both sides,

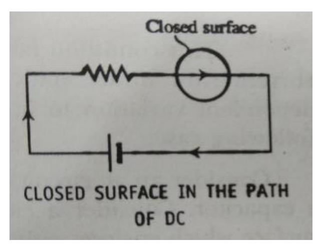
$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J}$$

But, as per the rules of vector analysis, the divergence of curl of a vector field is zero.

$$\therefore \nabla \cdot \vec{J} = 0$$

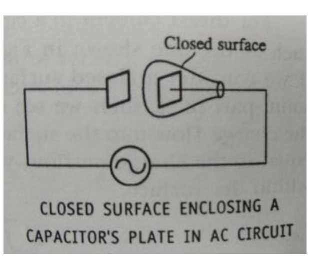
this condition holds good for circuits with current flow under static conditions.

In the circuit shown in the figure, the charge flow into the surface is equal to the charge out flow, i.e. $\nabla \cdot \vec{J} = 0$ which means there is no net charge within the surface.



But this relation $\nabla \cdot \vec{J} = 0$ fails where the fields and currents are varying with time.

Consider an AC circuit with a capacitor. Consider a closed surface which encloses only one plate of the capacitor. Here, current flow into the closed surface is not accompanied by a simultaneous current outflow. (Charges pile up on the capacitor plate). Thus the condition $\nabla \cdot \vec{J} = 0$ fails. In other words, there is an inconsistency in Ampere's law for time varying fields.



- **Displacement current (Maxwell-Ampere's law):**

In order to make Ampere's law work under time varying field conditions, Maxwell suggested the following:

We have from Gauss law in electrostatics,

$$\nabla \cdot \vec{D} = \rho$$

where $\vec{D}$ --> Electric flux density and ρ--> charge density

Let $\vec{D}$ be varying with time.

Differentiating with time we have,

$$\frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}\right) = \frac{\partial \rho}{\partial t}$$

Or

$$\nabla \cdot \frac{\partial \vec{D}}{\partial t} = \frac{\partial \rho}{\partial t} \qquad \text{-------------->} \ (1)$$

But, from the equations of continuity, we have

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \qquad \text{--------------->} \ (2)$$

From (1) and (2) we have,

$$\nabla \cdot \vec{J} = -\nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

or,

$$\nabla \cdot \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] = 0$$

Thus, for time varying field, $\vec{J}$ must be replaced by $\vec{J} + \frac{\partial \vec{D}}{\partial t}$.

The quantity $\frac{\partial}{\partial t}\left(\vec{D}\right)$ has the dimension of $\vec{J}$ and is called as displacement current density.

Therefore, Ampere's law can be written as,

$$\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{D}}}{\partial \boldsymbol{t}}$$

This equation is known as **Maxwell-Ampere's law**.

The right hand side term in the above equation is the total current density consisting of conduction current and displacement current density generated due to time variation of electric flux density $(\vec{\mathrm{D}})$ vector field.

- Note: Displacement current density is $\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$.

$$\text{Hence, displacement current is } I_D = \int_S \vec{J}_D . \overrightarrow{\boldsymbol{ds}}$$

$$\therefore \ I_D = \int_S \frac{\partial \vec{D}}{\partial t} . \overrightarrow{\boldsymbol{ds}}$$

- Displacement current in vacuum or in a dielectric is not a current in the sense of any motion of charges. But it is associated with itself a magnetic field, similar to that of conduction current.

- **<u>Maxwell's equations:</u>**

The differential forms of **Maxwell's equations for <u>time varying fields</u>** are given below:

1) $\nabla \cdot \vec{D} = \rho$ (Gauss law in electrostatics)
2) $(\nabla \times \vec{E}) = -\frac{d\vec{B}}{dt}$ (Faraday's law)
3) $\nabla . \vec{B} = 0$ (Gauss law in magnetism)
4) $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ (Ampere's law)

In case of static fields, the time dependent factors in equations (2) and (3) vanish. The modified **Maxwell's equations for <u>static fields</u>** are given below:

1) $\nabla \cdot \vec{D} = \rho$ (Gauss law in electrostatics)

2) $(\nabla \times \vec{E}) = 0$ (Faraday's law)

3) $\nabla . \vec{B} = 0$ (Gauss law in magnetism)

4) $\nabla \times \vec{H} = \vec{J}$ (Ampere's law)

- Note: A static electric field $\vec{E}$ can exist in the absence of magnetic field $\vec{H}$. For example, a capacitor with static charge produce only $\vec{E}$ but not $\vec{H}$. Likewise, a conductor with constant current 'I' has magnetic field $\vec{H}$ without an electric field $\vec{E}$. However, when fields are time variable, $\vec{H}$ can't exist without $\vec{E}$ and vice versa. i.e. time variable $\vec{E}$ and $\vec{H}$ fields can't exist independently.

- **<u>Electromagnetic waves:</u>**

From Maxwell's equations, it follows that a time varying magnetic field $\vec{H}$ gives rise to an electric field $\vec{E}$ that varies both in space and time. Similarly, a time varying electric field gives rise to magnetic field that varies in space and time. Moreover, from $(\nabla \times \vec{E}) = -\frac{d\vec{B}}{dt}$ and $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, it follows that $\vec{E}$ and $\vec{H}$ must be normal to each other. Thus, no conductor, a loop of a wire or any other medium is necessary. $\vec{E}$ arises directly from time varying $\vec{H}$ and vice versa. These variations propagate as a wave in a direction perpendicular to both $\vec{E}$ and $\vec{H}$ known as electromagnetic waves.

- **<u>Electromagnetic wave equation in differential form:</u>**

From Maxwell's equations, we have,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

and

$$\left(\nabla \times \vec{E}\right) = -\frac{d\vec{B}}{dt}$$

But, we know that D = εE and B = μH.

$$\therefore \nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial}{\partial t}\left(\vec{E}\right) \qquad \text{-----------}\rightarrow (1)$$

$$\left(\nabla \times \vec{E}\right) = -\mu \frac{d\vec{H}}{dt} \qquad \text{-----------}\rightarrow (2)$$

In order to get the wave equation in terms of electric field vector ($\vec{E}$), we must obtain an equation relating the spatial coordinates of $\vec{E}$ to its time coordinate. This is done by eliminating $\vec{H}$ between the two equations (1) and (2). This is done in the following way:

Take curl for both sides of (2).

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) \qquad \text{---------}\rightarrow (3)$$

As per vector analysis,

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

Or, $$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2 \vec{E} \qquad \text{---------}\rightarrow (4)$$

(From Maxwell's equation, $\nabla \cdot \vec{D} = \rho$ (or) $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon}$)

From eq. (3) and (4), we get

$$\nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) \qquad \text{----------}\rightarrow (5)$$

From eq. (1) and (5),

$$\nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$

Or, $$\nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial \vec{J}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Or, $$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2}{\partial t^2}\vec{E} = \mu \frac{\partial \vec{J}}{\partial t} + \nabla\left(\frac{\rho}{\varepsilon}\right) \qquad \text{--------}\rightarrow (6)$$

The LHS of eq. (6) is in the characteristic form of wave equation in terms of $\vec{E}$, for medium with constant μ & ε values (homogeneous and isotropic medium).

If we consider a region, where there are no charges and currents (the sources of $\vec{E}$and $\vec{H}$), then $\rho = 0$ and $\vec{J} = 0$.

Then eq. (6) becomes,

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Or, $$\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

Also, wave propagating along X-direction must be independent of y and z. Then,

$$\boldsymbol{\frac{\partial^2\vec{E}}{\partial x^2} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}} \qquad \text{-----------}\rightarrow (7)$$

Eq. (7) represents a plane EM wave propagating along X-direction.

But the classical wave equation in one dimension in differential form is,

$$\boldsymbol{\frac{\partial^2\vec{y}}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\vec{y}}{\partial t^2}} \qquad \text{--------------}\rightarrow(8)$$

Comparing eq. (7) and (8), we get the velocity of EM waves as

$$\mathbf{v = \frac{1}{\sqrt{\mu\varepsilon}}}$$

In vacuum, $\mu = \mu_0 = 4\pi \times 10^{-7}$ $\mathrm{Hm^{-1}}$ and $\varepsilon = \varepsilon_0 = 8.852 \times 10^{-12} Fm^{-1}$.

$$\therefore \mathbf{v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \times 10^8} \ \mathbf{ms^{-1}}$$

This is same as velocity of light in vacuum.

- **<u>Note:</u>** Equation (7) gives the variation of electric field ($\boldsymbol{\vec{E}}$) in space and time. The associated magnetic field variation is invariably there in the perpendicular plane.