Module II : PROPERTIES OF INTEGERS.

Mathematical Induction.

Well ordering Principle: Every nonempty subset of Z^+ contains a smallest element. (We often express this by saying that Z^+ is well ordered).

Finite Induction Principle: (Principle of Mathematical Induction): Let $S(n)$ denote an open mathematical statement that involves one or more occurrences of the variable n, which represents a positive integer.

(a) If $S(1)$ is true; and

b) If whenever $S(k)$ is true (for some particular but arbitrarily chosen $k \in Z^+$), then $S(k+1)$ is true.

then $S(n)$ is true for all $n \in Z^+$.

Proof: Let $S(n)$ be such an open statement satisfying conditions (a) and (b). and let $F = \{t \in Z^+ \mid S(t) \text{ is false}\}$. We wish to prove that $F = \phi$. so to obtain a contradiction we assume that $F \neq \phi$. Then by the Well-Ordering Principle F has a least element s. Since $S(1)$ is true it follows that $s \neq 1$, so $s > 1$, and consequently $s-1 \in Z^+$. With $s-1 \notin F$ we have $S(s-1)$ true. So by condition (b) it follows that $S((s-1)+1) = S(s)$ is true contradicting $s \in F$. This contradiction arose from the assumption that $F \neq \phi$. Consequently $F = \phi$.

(1) Prove by Mathematical induction that, for all positive integers $n \geq 1$.

$$1+2+3+4+\cdots+n = \frac{1}{2}n(n+1).$$

Sol: Here, we have to prove the statement

$$S(n): 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$

for all integers $n \geq 1$.

Basis step: We note that $S(1)$ is the statement

$$1 = \frac{1}{2}\cdot 1\cdot(1+1)$$

Which is clearly true. Thus, the statement $S(n)$ is verified for $n=1$.

Induction step: We assume that the statement $S(n)$ is true for $n=k$ where k is an integer ≥ 1; that is, we assume that the following statement is true:

$$S(k): 1+2+3+\cdots+k = \frac{1}{2}k(k+1).$$

Using this, we find that (by adding $k+1$ to both sides)

$$(1+2+3+\cdots k)+(k+1) = \frac{1}{2}k(k+1)+(k+1)$$
$$= (k+1)\left\{\frac{1}{2}k+1\right\}$$
$$= \frac{1}{2}(k+1)(k+2).$$

This is precisely the statement $S(k+1)$.

Thus on the basis of the assumption that $S(n)$ is true for $n=k\geq 1$, the truthness of $S(n)$ for $n=k+1$ is established

(2) Prove that, for each $n \in Z^+$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

OR.

Prove that, for each $n \in Z^+$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1).$$

Sol: Let $S(n)$ denote the given statement.

Basis step: We note that $S(1)$ is the statement

$1^2 = \frac{1}{6} \times 1 \times 2 \times 3$. which is clearly true.

Induction Step: We assume that $S(n)$ is true for $n=k$ where $k \geq 1$; that is, we assume that the following statement is true.

$$S(k): 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6} k(k+1)(2k+1).$$

Adding $(k+1)^2$ to both sides of this, we obtain

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

This is precisely the statement $S(k+1)$.

Thus, the statement $S(k+1)$ is true whenever the statement $S(k)$ is true for $k \geq 1$.

(3) By mathematical induction prove that $(n!) \geq 2^{n-1}$ for all integers $n \geq 1$.

Sol Basis step: For $n=1$, $S(n)$ reads $1! \geq 2^{1-1}$ which is obviously true. Thus, $S(n)$ is verified for $n=1$.

Induction step: We assume that $S(n)$ is true for $n=k$, where $k \geq 1$; that is, we assume that

$$k! \geq 2^{k-1}, \text{ or } 2^{k-1} \leq k! \text{ is true.}$$

$$2^k = 2 \cdot 2^{k-1} \leq 2 \cdot k!$$
$$\leq (k+1) \cdot k! \quad \text{because } 2 < k+1 \text{ for } k \geq 1.$$
$$= (k+1)!$$
$$(k+1)! \geq 2^k.$$

This is precisely the statement $S(n)$ for $n=k+1$. Thus, on the assumption that $S(n)$ is true for $n=k\geq 1$ we have proved that $S(n)$ is true for $n=k+1$.

(4) Prove that every positive integer $n \geq 24$ can be written as a sum of 5's and/or 7's.

Sol Basis step: We note that

$$24 = (7+7) + (5+5)$$

This shows $S(24)$ is true.

Induction step: We assume that $S(n)$ is true for $n=k$ where $k \geq 24$. Then,

$$k = (7+7+\cdots) + (5+5+\cdots).$$

Suppose this representation of k has r number of 7's and s number of 5's. Since $k \geq 24$, we should have $r \geq 2$ & $s \geq 2$.

Using this representation of k, we find that

$$k+1 = \{\underbrace{7+7+\cdots}_{r}\} + \underbrace{(5+5+\cdots)}_{s}\} + 1.$$

$$= \underbrace{(7+7\cdots)}_{r-2} + (7+7) + \underbrace{(5+5+\cdots)}_{s} + 1.$$

$$= \underbrace{(7+7+\cdots)}_{r-2} + \underbrace{(5+5+\cdots)}_{s+3}.$$

This shows that; $k+1$ is sum of 7's & 5's.

Thus $S(k+1)$ is true.

Recursive Definition:

For describing a sequence two methods are commonly used

(i) Explicit method. (ii) Recursive method.

In explicit method, the general term of the sequence is explicitly indicated.

In first part of recursive method, first few terms of the sequence are sequence must be indicated explicitly. and in the second part the rule which will enable us to obtain new terms of the sequence from the terms already known must be indicated

1. Find an explicit definition of the sequence defined recursively by

$a_1 = 7 \quad a_n = 2a_{n-1}+1$ for $n \geq 2$.

Sol: By repeated use of the given recursive definition, we find that

$$a_n = 2a_{n-1}+1 = 2[2a_{n-2}+1]+1.$$
$$= 2[\{2a_{n-3}+1\}+1]+1.$$
$$= 2^3 a_{n-3} + 2^2 + 2 + 1$$
$$\cdots$$
$$= 2^{n-1} a_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \cdots 2^2 + 2 + 1.$$
$$= 2^{n-1} a_1 + (1 + 2 + 2^2 + \cdots + 2^{n-3} + 2^{n-2}).$$

Using $a_1 = 7$ and the standard result

$$1 + a + a^2 + \cdots + a^{n-1} = \frac{a^n - 1}{a-1} \text{ for } a > 1.$$

this becomes. $a_n = 7 \times 2^{n-1} + (2^{n-1} - 1)$

$$= (8 \times 2^{n-1}) - 1$$

② Obtain the recursive definition for the sequence $\{a_n\}$ in each of the following cases.

(i) $a_n = 5n$ (ii) $a_n = 6^n$. (iii) $a_n = 3n+7$

(iv) $a_n = n(n+2)$ (v) $a_n = n^2$ (vi) $a_n = 2-(-1)^n$.

Sol: i) Here $a_1 = 5$ $a_2 = 10$, $a_3 = 15$ $a_4 = 20$ we can rewrite these as $a_1 = 5$ and $a_n = a_{n-1} + 5$ for $n \geq 2$.

(ii) $a_1 = 6$ $a_2 = 6^2$ $a_3 = 6^3 \ldots$ we can rewrite as

$a_1 = 6$ & $a_{n+1} = 6 \times a_n$ for $n \geq 1$.

(iii). $a_1 = 10 \quad a_2 = 13 \quad a_3 = 16 \quad a_4 = 19 \ldots$

$a_1 = 10 \quad a_n = a_{n-1} + 3.$

(iv). $a_1 = 3 \quad a_2 = 8 \quad a_3 = 15 \quad a_4 = 24$

$a_2 - a_1 = 5 = 2\times 1 + 3 \quad a_3 - a_2 = 7 = 2\times 2 + 3.$

$a_4 - a_3 = 9 = 2\times 3 + 3.$

$a_{n+1} - a_n = 2n + 3.$

$\therefore a_1 = 3 \quad a_{n+1} = a_n + 2n + 3$

(v). $a_1 = 1 \quad a_2 = 4 \quad a_3 = 9 \quad a_4 = 16$

$a_2 - a_1 = 3 = 2\times 1 + 1 \quad a_3 - a_2 = 5 = 2\times 2 + 1.$

$a_{n+1} - a_n = 2\times n + 1$

$\therefore a_1 = 1 \quad a_{n+1} = a_n + 2n + 1$ for $n \geq 1.$

(vi). $a_1 = 3 \quad a_2 = 1 \quad a_3 = 3 \quad a_n = 2 - (-1)^n \quad a_{n+1} = 2 - (-1)^{n+1}.$

$a_{n+1} - a_n = 2(-1)^n.$

$a_1 = 3 \quad a_{n+1} = a_n + 2(-1)^n, \quad n \geq 1.$

3. The fibonacci numbers are defined recursively by $f_0 = 0 \quad f_1 = 1$ & $f_n = f_{n-1} + F_{n-2}$ for $n \geq 2$. Calculate F_2 to F_{10}.

Sol

$F_2 = F_1 + F_0 = 1 + 0 = 1$

$F_3 = F_2 + F_1 = 1 + 1 = 2$

$F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = F_4 + F_3 = 3 + 2 = 5$

Similarly

$F_6 = 8, \quad F_7 = 13$

$F_8 = 21 \quad F_9 = 34$

$F_{10} = 55.$

(4) The Lucas numbers are defined by (recursively) $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Evaluate L_2 to L_{10}.

Sol: $L_2 = L_1 + L_0 = 3$. $\quad L_3 = L_2 + L_1 = 4$.

Similarly, $L_4 = 7$, $L_5 = 11$, $L_6 = 18$, $L_7 = 29$, $L_8 = 47$, $L_9 = 76$, $L_{10} = 123$.

(5) For the Fibonacci sequence $F_0, F_1, F_2, \ldots$ Prove that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Sol: For $n=0$ and $n=1$, the required result reads (resp.)

$$F_0 = \frac{1}{\sqrt{5}}(1-1) = 0 \qquad F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1.$$

which are true.

Thus the required result is true for $n=0$ and $n=1$. We assume that the result is true for $n = 0, 1, 2, \ldots, k$ where $k \geq 1$. Then, we find that

$$F_{k+1} = F_k + F_{k-1}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right],$$

using the assumption made

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1+\sqrt{5}}{2}+1\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1-\sqrt{5}}{2}+1\right\}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

this shows that the required result is true for $n=k+1$. Hence by mathematical induction, the result is true for all non-negative integers n.

Rule of Sum. Suppose two tasks T_1 and T_2 are to be performed. If the task T_1 can be performed in m different ways and the task T_2 can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks (T_1 or T_2) can be performed in $m+n$ ways.

ex 1 Suppose T_1 is the task of selecting a prime no <10 and T_2 is the task of selecting an even number <10. Then T_1 can be performed in 4 ways and T_2 can be performed in 4 ways. But, since 2 is both a prime and an even number <10, the task T_1 or T_2 can be performed in $4+4-1=7$ ways.

Rule of Product: Suppose two tasks are to be performed one after the other. If T_1 can be performed in n_1 different ways and for each of these ways T_2 can be performed in n_2 different ways, then both of the tasks can be performed in $n_1 n_2$ different ways.

(1) Suppose a person has 8 shirts and 5 ties. Then he has $8 \times 5 = 40$ different ways of choosing a shirt and a tie.

PROBLEMS

ex (1) Cars of a particular manufacturer come in 4 models, 12 colours, 3 engine sizes, and 2 transmission types. (a) How many distinct cars can be manufactured? (b) Of these how many have the same colour?

Sol (a) By the product rule, it follows that the number of distinct cars that can be manufactured is $4 \times 12 \times 3 \times 2 = 288$.

(b) For any chosen colour, the number of distinct cars that can be manufactured is

$4 \times 3 \times 2 = 24$.

(2) A bit is either 0 or 1. A byte is a sequence of 8 bits. Find (i) the number of bytes. (ii) the number of bytes that begin with 11 and end with 11. (iii) the number of bytes that begin with 11 and do not end with 11. and (iv) the number of bytes that begin with 11 or end with 11.

Sol i) Since each byte contains 8 bits and each bit is a 0 or 1, the number of bytes is $2^8 = 256$

(ii) In a byte beginning and ending with 11, there occur 4 open positions. These can be filled in $2^4 = 16$ ways. Therefore there are 16 bytes which begin and end with 11.

(iii). There occur 6 open positions in a byte beginning with 11. These positions can be filled in $2^6 = 64$ ways. Thus, there are 64 bytes that begin with 11. Since there are 16 bytes that begin and end with 11, the number of bytes that begin with 11 but donot end with 11 is $64 - 16 = 48$.

(iv) As in (iii) the number of bytes that end with 11 is 64. Also the number of bytes that begin and end with 11 is 16. Therefore, the number of bytes that begin or end with 11 is $64 + 64 - 16 = 112$.

(3) Find the number of 3-digit even numbers with no repeated digits.

Sol: Here we consider numbers of the form xyz, where each of x, y, z represents a digit under the given restrictions. Since xyz has to be even z has to be 0, 2, 4, 6 or 8. If z is 0, then x has 9 choices and if z is 2, 4, 6 or 8 (4 choices) then x has 8 choices. (Note that x can not be zero) Therefore, z and x can be chosen in $1 \times 9 + 4 \times 8 = 41$

ways. For each of these ways, y can be chosen in 8 ways. Hence, the desired number is $41 \times 8 = 328$

Permutations: $P(n, r) = \frac{n!}{(n-r)!}$

Generalization. Suppose it is required to find the number of permutations that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of a second type, ... n_k are of k^{th} type, with $n_1 + n_2 + \cdots + n_k = n$. Then, the number of permutations of the n objects is $\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$

PROBLEMS

① Four different mathematics books, five different computer science books and two different control theory books are to be arranged in a shelf. How many different arrangements are possible if (a) the books in each particular subject must all be together? (b) only the mathematics books must be together?

Sol: (a) The mathematics books can be arranged among themselves in 4! ways, the computer science books in 5! ways, the control theory books in 2! ways, and the three groups in 3! ways. Therefore the number of possible arrangements is

$$4! \times 5! \times 2! \times 3! = 34,560.$$

(B) Consider the 4 Mathematics books as one single book. Then we have 8 books which can be arranged in 8! ways. In all of these ways the mathematics books are together. But the mathematics books can be arranged among themselves in 4! ways. Hence, the number of arrangements is

$$8! \times 4! = 967,680.$$

(2) Find the number of permutations of the letters of the word MASSASAUGA. In how many of these, all four 'A's are together? How many of them begin with S?

Sol: The given word has 10 letters of which 4 are A, 3 are S and 1 each are M, U and G. Therefore, the required number of permutations is

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25,200.$$

If in a permutation all A's are to be together, we treat all of A's as one single letter. Then the letters to be permuted read (AAAA), S, S, S, M, U, G (which are 7 in number) and the number of permutations is $\frac{7!}{1!\,3!\,1!\,1!\,1!} = 840.$

For permutations beginning with S, there occur nine open positions to fill, where two are S, four are A, and one each of M, U, G. The number of such permutations is

$$\frac{9!}{2!\,4!\,1!\,1!\,1!} = 7560.$$

③ a) How many arrangements are there for all letters in the word SOCIOLOGICAL?

b) In how many of these arrangements (i) A and G are adjacent? (ii) all the vowels are adjacent?

Sol a) The given word has 12 letters of which 3 are O, 2 each are C, I, L and 1 each are S, A, G. Therefore, the number of arrangements of these letters is

$$\frac{12!}{3!\,2!\,2!\,2!\,1!\,1!\,1!} = 99{,}79{,}200.$$

b) (i) If, in an arrangement, A and G are to be adjacent, we treat A and G together as a single letter, say X, so that we have 3 number of O's, 2 each of C, I, L and one each of S and X, totalling 11 letters. These can be arranged in

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} \text{ ways.}$$

Further the letters A and G can be arranged among themselves in two ways.

Therefore, the total number of arrangements in this case is $\frac{11!}{3!2!2!2!1!} \times 2 = 16,63,200.$

(ii) If, in an arrangement all the vowels are to be adjacent, we treat all the vowels present in the given word (A, O, I) as a single letter, say Y, so that we have 2 each of C and L and 1 each of S, G & Y totalling to 7 letters. These can be arranged in

$$\frac{7!}{2!2!1!1!1!} \text{ ways.}$$

Further since the given word contains 3 O's, two I's and one A, the letters A, O, I (clubbed as Y) can be arranged among themselves in

$$\frac{6!}{3!2!1!} \text{ ways.}$$

Therefore, the total number of arrangements in this case is $\frac{7!}{2!2!1!1!1!} \times \frac{6!}{3!2!1!} = 75,600.$

COMBINATIONS.

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} \text{ for } 0 \le r \le n.$$

PROBLEMS ON COMBINATIONS.

1. A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions

by selecting at least 2 questions from each part.

Sol: The different ways a student can select his 5 questions are.

(i) 3 questions from part A and 2 questions from part B. This can be done in

$C(4,3) \times C(4,2) = 24$ ways.

(ii) 2 questions from part A and 3 questions from part B. This can be done in

$C(4,2) \times C(4,3) = 24$ ways

Therefore, the total number of ways a student can answer 5 questions under given restrictions is $24 + 24 = 48$.

(2) Prove the following identities.

(i) $C(n, r-1) + C(n, r) = C(n+1, r)$.

(ii) $C(m, 2) + C(n, 2) = C(m+n, 2) - mn$.

Proof: (i) $C(n, r-1) + C(n, r) = \dfrac{n!}{(r-1)!\,(n-r+1)!} + \dfrac{n!}{r!\,(n-r)!}$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left\{\frac{1}{n-r+1} + \frac{1}{r}\right\}.$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \cdot \frac{n+1}{r(n-r+1)} = \frac{(n+1)!}{r!\,(n-r+1)!}$$

$$= C(n+1, r)$$

(ii). $C(m,2)+C(n,2) = \dfrac{m!}{(m-2)!\cdot 2} + \dfrac{n!}{(n-2)!\cdot 2}$

$$= \frac{1}{2}\{m(m-1)+n(n-1)\}$$

$$= \frac{1}{2}\{m^2+n^2-m-n\}.$$

$$= \frac{1}{2}(m+n)(m+n-1) - mn.$$

$$= \frac{(m+n)!}{2\,(m+n-2)!} - mn.$$

$$= C(m+n,2) - mn.$$

BINOMIAL AND MULTINOMIAL THEOREMS.

BINOMIAL THEOREM:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

This result is known as the Binomial Theorem for a positive integral index

MULTINOMIAL THEOREM: For positive integers n and k the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_k^{n_k}$ in the expansion of $(x_1+x_2+\dots+x_k)^n$ is $\dfrac{n!}{n_1!\,n_2!\,n_3!\dots n_k!}$

① Find the coefficient of

(i) $x^9 y^3$ in the expansion of $(2x-3y)^{12}$.

(ii) x^{12} in the expansion of $x^3(1-2x)^{10}$.

Sol: We have, by the binomial theorem.

$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} \cdot (2x)^r (-3y)^{12-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} 2^r (-3)^{12-r} x^r y^{12-r}$$

In the expansion, the coefficient of x^9y^3 $(r=9)$

$$\binom{12}{9} 2^9 (-3)^3 = -(2^9 \times 3^3) \times \frac{12!}{9!\,3!}$$

$$= -2^9 \times 3^3 \times \frac{12 \times 11 \times 10}{6}$$

$$= -2^{10} \times 3^3 \times 11 \times 10.$$

(ii) By the Binomial theorem, we have

$$(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} 1^{10-r} (-2x)^r$$

Therefore, $x^3(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (-2)^r x^{r+3}$

The coefficient of x^{12} in this expansion $(r=9)$

$$\binom{10}{9}(-2)^9 = -(10 \times 2^9) = -5120.$$

(c) Determine the coefficient of

(i) xyz^2 in the expansion of $(2x-y-z)^4$.

(ii) $x^{11}y^4$ in the expansion of $(2x^3-3xy^2+z^2)^6$

Sol: By the Multinomial theorem, we note that the general term in the expansion of $(2x-y-z)^4$

is $\binom{4}{n_1,\ n_2,\ n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$.

For $n_1 = 1$, $n_2 = 1$ and $n_3 = 2$, this becomes

$$\binom{4}{1,\ 1,\ 2}(2x)(-y)(-z)^2 = \binom{4}{1,\ 1,\ 2} \times 2 \times (-1) \times (-1)^2 xyz^2$$

This shows that the required coefficient is

$$\binom{4}{1,\ 1,\ 2} \times 2 \times (-1) \times (-1)^2 = \frac{4!}{1!\,1!\,2!} \times (-2)$$
$$= \cancel{-12}\ -24$$

Q(ii) The general term in the expansion of $(2x^3 - 3xy^2 + z^2)^6$ is

$$\binom{6}{n_1,\ n_2,\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}.$$

for $n_3 = 0$ $n_2 = 2$, $n_1 = 3$ this becomes.

$$\binom{6}{3,\ 2,\ 0}(2^3x^9)(3^2x^2y^4) = 72 \times \frac{6!}{3!\,2!\,0!}x^{11}y^4$$

Thus the required coefficient is

$$72 \times \frac{6 \times 5 \times 4}{2} = 4320$$

Combinations with Repetitions.

Suppose we wish to select, with repetition, a combination of r objects from a set of n distinct objects. The number of such selections is given by

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!}$$

(1) In how many ways can we distribute 10 identical marbles among 6 distinct containers?

Sol The selection consists in choosing with repetitions $r = 10$ marbles for, $n = 6$ distinct containers
The required number is

$$C(6+10-1, 10) = C(15, 10) = \underline{3003.}$$

(2) Find the number of nonnegative integer solutions of the inequality $x_1 + x_2 + x_3 + \cdots x_6 < 10$.

Sol We have to find the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + x_3 + \cdots + x_6 = 9 - x_7.$$

Where $9 - x_7 \le 9$ so that x_7 is a non negative integer. Thus, the required number is the number of nonnegative solutions of the equation.

$$x_1 + x_2 + x_3 + \cdots + x_7 = 9.$$

This number is

$$C(7+9-1, 9) = C(15, 9) = 5005.$$