UNIT - IV

Beta and Gamma functions, Laplace Transforms

1. Beta and Gamma functions

Improper Integrals:

An improper integral is a definite integral that has either or both ~~the~~ limits are infinite (or) an integral that approaches to infinity

I Kind $\int_0^\infty e^{-x}\, dx$

$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

II Kind $\int_0^1 \frac{1}{\sqrt{x}}\, dx$

Beta function

The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ $(m, n > 0)$ is called Beta function. It is denoted by $\beta(m,n)$

Alternate form of Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

Limits

If $x=0$, $\sin^2\theta = 0 \Rightarrow \theta = 0$

$x=1$, $\sin^2\theta = 1 \Rightarrow \theta = \pi/2$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} \cdot (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\,(\cos^2\theta)^{n-1}\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\,\sin\theta\,\cos^{2n-2}\theta\,\cos\theta\, d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\, d\theta}$$

This is called trigonometric form of Beta function.

$\beta(m,n) = \beta(n,m)$

Proof :- We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = 1-y$

(or) $y = 1-x$

$x = 1-y$

$dx = -dy$

If $x=0$, $1-y=0 \Rightarrow y=1$

$x=1$ $1-y=1 \Rightarrow y=0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}\, y^{n-1}(-dy)$$

$$\beta(m,n) = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m)$$

$\therefore$ $\beta(m,n) = \beta(n,m)$

Gamma function :-

Def :- The integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ is called the Gamma function. It is denoted by $\Gamma(n)$.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

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Alternate form

$$\Gamma(n) = \int_0^\infty e^{-x} . x^{n-1}\,dx$$

put $x = t^2$

$dx = 2t\,dt$

If $x = 0,\quad t = 0$

$x = \infty,\quad t = \infty$

$$\Gamma(n) = \int_0^\infty e^{-t^2} (t^2)^{n-1}\, 2t\,dt$$

$$= 2\int_0^\infty e^{-t^2}\, t^{2n-2} , t\,dt$$

$$\boxed{\Gamma(n) = 2\int_0^\infty e^{-t^2}\, t^{2n-1}\,dt}$$

Properties of Gamma function

Prove (i) $\Gamma(n+1) = n\,\Gamma(n)$

(ii) $\Gamma(n+1) = n!$, for positive integer n.

proof :-

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} \cdot x^{(n+1)-1}\, dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n}\, dx$$

$$\Gamma(n+1) = \int_0^\infty x^{n} e^{-x}\, dx$$

Integrating

$$= \left[x^n \cdot \int e^{-x} dx - \int \left(n \cdot x^{n-1} \cdot \int e^{-x} dx \right) dx \right]_0^\infty$$

$$= \left(x^n \frac{e^{-x}}{-1} \right)_0^\infty - \int_0^\infty n \cdot x^{n-1} \cdot \frac{e^{-x}}{-1} \, dx$$

$$= \left(-x^n e^{-x} \right)_0^\infty + n \cdot \int_0^\infty e^{-x} x^{n-1} dx$$

$$= (0-0) + n \int_0^\infty \underbrace{e^{-x} \cdot x^{n-1} dx}_{\Gamma(n)}$$

$$\boxed{\Gamma_{n+1} = n \cdot \Gamma_n}$$

(ii) $\Gamma_{n+1} = n \, \underline{\Gamma_n}$

Continuing

$$\Gamma_n = (n-1) \, \Gamma_{n-1}$$

$$\Gamma_{n-1} = (n-2) \cdot \Gamma_{n-2}$$

$$\Gamma_{n-2} = (n-3) \, \Gamma_{n-3}$$

$$\vdots$$

$$\Gamma_3 = 2 \, \Gamma_2$$

$$\Gamma_2 = 1 \, \Gamma_1$$

$$\Gamma_{n+1} = n \cdot (n-1) (n-2) \, \text{-----} \, 3 \cdot 2 \cdot 1 \cdot \Gamma_1$$

$$\Gamma_{n+1} = n! \, \Gamma_1$$

But $\Gamma_1 = \int_0^\infty e^{-x} x^{1-1} dx$ $\quad \because \underline{\Gamma_n} = \int_0^\infty e^{-x} x^{n-1} dx$

$$\Gamma_1 = \int_0^\infty e^{-x} dx$$

$$\Gamma_1 = \left(\frac{e^{-x}}{-1} \right)_0^\infty = -\left[e^{-\infty} - e^{0} \right]$$

$$= -[0-1]$$

$$\boxed{\Gamma_1 = 1}$$

$$\boxed{\Gamma_{n+1} = n!}$$

Relation between Betā and Gamma functions

The relation between Betā and Gamma functions is $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Proof:- We know that

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta \rightarrow (i)$$

$$\Gamma n = 2\int_0^{\infty} e^{-x^2} x^{2n-1} dx$$

$$\Gamma m = 2\int_0^{\infty} e^{-y^2} y^{2m-1} dy$$

$$\Gamma m \cdot \Gamma n = 2\int_0^{\infty} e^{-y^2} y^{2m-1} dy \cdot 2\int_0^{\infty} e^{-x^2} x^{2n-1} dx$$

$$\Gamma m \cdot \Gamma n = 4\int_0^{\infty}\int_0^{\infty} e^{-x^2} \cdot e^{-y^2} \cdot x^{2n-1} \cdot y^{2m-1} dx\,dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dx\,dy$$

Converting into polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

$dx\,dy = r\,dr\,d\theta$

r varies from 0 to ∞

θ varies from 0 to $\pi/2$

$$\Gamma m \cdot \Gamma n = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\,dr\,d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\, \underline{r^{2m-1}} \cdot \cos^{2m-1}\theta\, \underline{r^{2n-1}} \sin^{2n-1}\theta\, \underline{r}\,dr\,d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\, r^{2m-\cancel{1}+2n-\cancel{1}+\cancel{1}} \cdot \cos^{2m-1}\theta \sin^{2n-1}\theta \; dr\,d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\, r^{2m+2n-1} \cos^{2n-1}\theta \cdot \sin^{2m-1}\theta\, dr\,d\theta$$

$$= 2\int_0^{\infty} e^{-r^2} \cdot r^{2(m+n)-1}\,dr \cdot 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

$$\Gamma m \cdot \Gamma n = \Gamma m+n \cdot \beta(m,n)$$

$$\boxed{\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}}$$

1) Show that $\Gamma(1/2) = \sqrt{\pi}$

Sol :- The relation between Beta and Gamma functions is

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Put $m = n = 1/2$

$$\beta(1/2, 1/2) = \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1/2+1/2}$$

$$\beta(1/2, 1/2) = \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1}$$

But $\Gamma 1 = 1$

$$\beta(1/2, 1/2) = \Gamma 1/2 \, \Gamma 1/2 \rightarrow ①$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

$$\beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^{2(1/2)-1}\theta \, \cos^{2(1/2)-1}\theta \, d\theta$$

$$\beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^0\theta \, \cos^0\theta \, d\theta$$

$$= 2\int_0^{\pi/2} d\theta \Rightarrow \beta(1/2, 1/2) = 2(\theta)_0^{\pi/2}$$

$$\beta(1/2, 1/2) = 2\pi/2$$

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \pi \rightarrow (2)$$

From (1) & (2)

$$\Gamma\tfrac{1}{2}\,\Gamma\tfrac{1}{2} = \pi$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \pi$$

$$\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

(2) Show that $\beta(m+1, n) + \beta(m, n+1) = \beta(m, n)$

Sol :- We know that $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

$|||^{ly}$ $\beta(m+1, n) = \dfrac{\Gamma m+1\,\Gamma n}{\Gamma m+1+n}$

$$\beta(m, n+1) = \frac{\Gamma m\,\Gamma n+1}{\Gamma m+n+1}$$

LHS $\beta(m+1, n) + \beta(m, n+1) =$

$$\frac{\Gamma m+1\,\Gamma n}{\Gamma m+n+1} + \frac{\Gamma m\,\Gamma n+1}{\Gamma m+n+1}$$

$\boxed{\Gamma n+1 = n\,\Gamma n}$

$$= \frac{m\,\Gamma m \cdot \Gamma n}{(m+n)\,\Gamma m+n} + \frac{\Gamma m\; n \cdot \Gamma n}{(m+n)\,\Gamma m+n}$$

$$= \frac{\Gamma m \, \Gamma n}{\cancel{(m+n)} \, \Gamma m+n} \cancel{(m+n)}$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$= \beta(m,n)$$

RHS

(3) Find the value of $\Gamma(-3.5)$

Sol:- $\Gamma(-3.5) = \Gamma -7/2$

We know $\Gamma n+1 = n \Gamma n$

$$\Rightarrow \Gamma n = \frac{\Gamma n+1}{n}$$

$$\Gamma -7/2 = \frac{\Gamma -7/2+1}{-7/2} =$$

$$= \frac{\Gamma -5/2}{-7/2}$$

$$\Gamma -7/2 = -\frac{2}{7} \Gamma -5/2 \rightarrow (1)$$

$$\Gamma -5/2 = \frac{\Gamma -5/2+1}{-5/2} = \frac{\Gamma -3/2}{-5/2}$$

$$\Gamma -5/2 = -\frac{2}{5} \Gamma -3/2 \rightarrow (2)$$

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\tfrac{3}{2}+1\right)}{-\tfrac{3}{2}}$$

$$\Gamma\left(-\tfrac{3}{2}\right) = -\tfrac{2}{3}\,\Gamma\left(-\tfrac{1}{2}\right) \longrightarrow (3)$$

$$\Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(-\tfrac{1}{2}+1\right)}{-\tfrac{1}{2}}$$

$$\Gamma\left(-\tfrac{1}{2}\right) = -2\,\Gamma\left(\tfrac{1}{2}\right) \longrightarrow (4)$$

From (1), (2), (3) & (4) we get

$$\Gamma\left(-\tfrac{7}{2}\right) = \left(-\tfrac{2}{7}\right)\left(-\tfrac{2}{5}\right)\left(-\tfrac{2}{3}\right)(-2)\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{16}{105}\sqrt{\pi} \qquad \text{But } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(-\tfrac{7}{2}\right) = \Gamma(-3.5) = \frac{16\sqrt{\pi}}{105}$$

(4) Evaluate $\int_0^\infty e^{-x^3} \cdot x^3 \, dx$.

Sol:- Put $x^3 = t \longrightarrow x = t^{1/3}$

$$3x^2\,dx = dt$$

$$x^2\,dx = \frac{dt}{3}$$

Limits If $x=0,\ t=0$

$x=\infty,\ t=\infty$

$$= \int_0^\infty e^{-x^3} x \cdot x^2 \, dx$$

$$= \int_0^\infty e^{-t} t^{1/3} \cdot \frac{dt}{3}$$

$$= \frac{1}{3} \int_0^\infty e^{-t} t^{1/3} \, dt$$

$$\boxed{\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} dx}$$

$$= \frac{1}{3} \int_0^\infty e^{-t} t^{\frac{4}{3}-1} dt$$

$$= \frac{1}{3} \Gamma \frac{4}{3}$$

$$\Gamma n+1 = n \Gamma n$$

$$= \frac{1}{3} \cdot \Gamma \frac{1}{3}+1$$

$$= \frac{1}{3} \cdot \frac{1}{3} \Gamma \frac{1}{3}$$

$$= \frac{1}{9} \Gamma \frac{1}{3}$$

⑤ Show that $\int_0^\infty e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a}$

Sol:- $\int_0^\infty e^{-a^2x^2} dx$

Put $a^2x^2 = t \Rightarrow x^2 = \frac{t}{a^2}$

$2a^2x \cdot dx = dt$; $x = \frac{\sqrt{t}}{a}$

$dx = \frac{dt}{2a^2x}$

Limits of $x=0, t=0$

$x=\infty, t=\infty$

$dx = \frac{dt}{2a^2 \cdot \sqrt{t}/a} \Rightarrow dx = \frac{dt}{2a\sqrt{t}}$

$$= \int_0^\infty e^{-t} \frac{dt}{2a\sqrt{t}}$$

$$= \int_0^\infty e^{-t} \frac{dt}{2a} . t^{-1/2}$$

$$= \frac{1}{2a} \int_0^\infty e^{-t} \, t^{-1/2} \, dt$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$= \frac{1}{2a} \int_0^\infty e^{-t} \, t^{1/2 - 1} \, dt$$

$$= \frac{1}{2a} \Gamma \tfrac{1}{2}$$

But $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

$$= \frac{1}{2a} \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2a}$$