La-Place's transform - II

Periodic function:

A function $f(t)$ is said to be periodic function of period 'T' if $f(t+nT)=f(t)$ where $n=1,2,3\ldots$

For example, $\sin t$ & $\cos t$ are periodic function of period 2π

Let $f(t)=\cos t$, then $f(t+2\pi)=\cos(t+2\pi)$
$(T=2\pi,\ n=1)$ $\qquad = \cos t$

→ If $f(t)$ is a periodic function of period T, then prove that $L\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$

* Given that $f(t)$ is a periodic function of period 'T'.

$\therefore f(t+nT)=f(t)$

WKT $L\{f(t)\}=\int_0^\infty e^{-st}f(t)\,dt$

$$=\int_{u=0}^{\infty} e^{-su}f(u)\,du$$

$$=\int_{u=0}^{T} e^{-su}f(u)\,du+\int_T^{2T} e^{-su}f(u)\,du+\cdots+\int_{nT}^{(n+1)T} e^{-su}f(u)\,du+\cdots$$

$$=\sum^{\infty}\left[\int_{nT}^{(n+1)T} e^{-su}f(u)\,du\right]$$

Q.

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→

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put $u = t + nT$

$du = dt$

if $u = nT$, then $t = 0$

if $u = (n+1)T$, then $t = T$

$$= \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-s(t+nT)} f(t+nT)\,dt$$

$$= \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-st} e^{-snT} f(t)\,dt \quad [\because \text{from } (1)]$$

$$= \left(\sum_{n=0}^{\infty} e^{-snT}\right)\left[\int_{t=0}^{T} e^{-st} f(t)\,dt\right] \quad \left[G.P \; S_\infty = \frac{a}{1-r}\right]$$

$$= \left[1 + e^{-sT} + (e^{-sT})^2 + \dots\right] \int_{t=0}^{T} e^{-st} f(t)\,dt$$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_{t=0}^{T} e^{-st} f(t)\,dt$$

Q. A periodic function of period $\frac{2\pi}{\omega}$ is defined by $f(t) = \begin{cases} E \sin \omega t, & \text{for } 0 < t < \pi/\omega \\ 0, & \text{for } \pi/\omega < t < 2\pi/\omega \end{cases}$

where E & ω are +ve constants then show that $L\{f(t)\} = \frac{E\omega}{(s^2+\omega^2)} \frac{1}{1-e^{-\pi s/\omega}}$

→ Given that $f(t)$ is a P.F of period $\frac{2\pi}{\omega}$ i.e $T = \frac{2\pi}{\omega}$

WKT $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \int_0^{\pi/\omega} e^{-st} E \sin\omega t \, dt \right\}$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \left\{ \frac{e^{-st}}{(-s)^2+\omega^2} \left(-s \sin\omega t - \omega\cos\omega t\right) \right\}_0^{\pi/\omega}$$

$$= \frac{E}{1^2 - \left(e^{-\pi s/\omega}\right)^2} \left\{ \frac{e^{-s\pi/\omega}}{s^2+\omega^2} \left(0 - \omega(-1)\right) - \frac{1}{s^2+\omega^2}(0-\omega) \right\}$$

$$= \frac{E\omega}{\left(1-e^{-\pi s/\omega}\right)\cancel{\left(1+e^{-\pi s/\omega}\right)}} \frac{\cancel{\left(e^{-s\pi/\omega}+1\right)}}{(s^2+\omega^2)}$$

$$L\{f(t)\} = \frac{E\omega}{\left(1-e^{-s\pi/\omega}\right)\left(s^2+\omega^2\right)} \quad //$$

Q. Obtain the La Place's transform of the rectangular wave $f(t)$ given by the following figure

$$\rightarrow \quad f(t) = \begin{cases} A & 0 < t < T/2 \\ -A & T/2 < t \le T \end{cases}$$

Period $T = T$

W.K.T

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) \, dt$$

$$= \frac{1}{1-e^{-sT}} \left[\int_0^{T/2} e^{-st}(A)\,dt + \int_{T/2}^{T} e^{-st}(-A)\,dt \right]$$

$$= \frac{A}{1-e^{-sT}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{T/2} - \left(\frac{e^{-st}}{-s}\right)_{T/2}^{T}\right]$$

$$L\{f(t)\} = \frac{A}{1-e^{-sT}}\left[\frac{-e^{-sT/2}}{s} + \frac{1}{s} + \frac{e^{-sT}}{s} - \frac{e^{-sT/2}}{s}\right]$$

$$= \frac{A}{s(1+e^{-sT/2})(1-e^{-sT/2})}\left(1-2e^{-sT/2}+e^{-sT}\right)$$

$$= \frac{A\left(1-e^{-sT/2}\right)^2}{s\left(1+e^{-sT/2}\right)\cancel{\left(1-e^{-sT/2}\right)}}$$

$$= \frac{A}{s}\left(\frac{e^{sT/4}-e^{-sT/4}}{e^{sT/4}+e^{-sT/4}}\right)$$

$$= \frac{A}{s}\tanh\left(\frac{sT}{4}\right)$$

Inverse La-Place's transform:

If $L\{f(t)\} = F(s)$ then, the inverse La-place's transform of $F(s)$ is defined as $L^{-1}\{F(s)\} = f(t)$

→ Standard formulae

* $L\{a\} = \frac{a}{s} \Rightarrow L^{-1}\{a/s\} = a$
* $L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
* $L\{e^{-at}\} = \frac{1}{s+a} \Rightarrow L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$
* $L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$
* $L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

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* $L\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

* $L\{\cosh at\} = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

* $L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$ or $\frac{t^n}{\Gamma(n+1)}$

* $L\{H(t-a)\} = \frac{e^{-as}}{s} \Rightarrow L^{-1}\left\{\frac{e^{-as}}{s}\right\} = H(t-a)$

  $L\{H(t)\} = \frac{1}{s} \Rightarrow L^{-1}\left\{\frac{1}{s}\right\} = H(t)$

** $L\{H(t-a)f(t-a)\} = e^{-as}F(s) \Rightarrow L^{-1}\{e^{-as}F(s)\}$

$= H(t-a)f(t-a)$

* $L\{\delta(t-a)\} = e^{-as} \Rightarrow L^{-1}\{e^{-as}\} = \delta(t-a)$

  $L\{\delta(t)\} = 1 \Rightarrow L^{-1}\{1\} = \delta(t)$

* $L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \Rightarrow L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

* $L\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}} \Rightarrow L^{-1}\left\{\frac{1}{(s-a)^{n+1}}\right\} = \frac{t^n e^{at}}{n!}$

Obtain the inverse La-Place's transform of the following:

1. $F(s) = \frac{1+e^{-3s}}{s^2} = L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}$

$= \frac{t}{1!} + H(t-3)f(t-3)$

$\left[\begin{array}{l} \text{let } F(s) = \frac{1}{s^2} \\ f(t) = L^{-1}\left\{\frac{1}{s^2}\right\} = t \\ f(t) = t \\ f(t-3) = t-3 \end{array}\right]$

$\rightarrow\ = t + H(t-3)(t-3)$

Find the inverse La-Place's transform of the following

(i) $e^{-s}\left[\frac{s+2}{(s+1)^4}\right]$ (ii) $\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

(iii) $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$ (iv) $\tan^{-1}(2/s)$

→ $L^{-1}\left\{e^{-as}F(s)\right\} = H(t-a)\,f(t-a)$

$L^{-1}\left\{e^{-s}\left(\frac{s+2}{(s+1)^4}\right)\right\} = H(t-1)\,f(t-1) \longrightarrow (1)$

$F(s) = \frac{s+2}{(s+1)^4}$

$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{(s+1)+1}{(s+1)^4}\right\}$

$= e^{-t}L^{-1}\left\{\frac{s+1}{s^4}\right\}$

$= e^{-t}\left[L^{-1}\left\{\frac{1}{s^3}\right\} + L^{-1}\left\{\frac{1}{s^4}\right\}\right]$

$f(t) = e^{-t}\left[\frac{t^2}{2!} + \frac{t^3}{3!}\right]$

$L^{-1}\left\{e^{-s}\left(\frac{s+2}{(s+1)^4}\right)\right\} = H(t-1)\,e^{-(t-1)}\left[\frac{(t-1)^2}{2} + \frac{(t-1)^3}{3!}\right]$

ii) $F(s) = \frac{se^{-s/2}}{s^2+\pi^2} + \frac{\pi e^{-s}}{s^2+\pi^2}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{se^{-s/2}}{s^2+\pi^2}\right\} + L^{-1}\left\{\frac{\pi e^{-s}}{s^2+\pi^2}\right\}$

$= H(t-1/2)\,f_1(t-1/2) + H(t-1)\,f_2(t-1)$ $\longrightarrow (1)$

$$F_1(s) = \frac{s}{s^2+\pi^2} \qquad F_2(s) = \frac{\pi}{s^2+\pi^2}$$

$$f_1(t) = L^{-1}\{F_1(s)\} = L^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} \qquad f_1(t) = \cos\pi t$$

$$f_2(t) = L^{-1}\{F_2(s)\} = L^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\}$$

$$f_2(t) = \sin\pi t$$

(1) $\Rightarrow$

$$L^{-1}\{F(s)\} = H(t-1)\cos(\pi t - \pi/2) + H(t-1)\sin(\pi t - \pi)$$

$$= H(t-1/2)\sin\pi t - \sin\pi t\, H(t-1)$$

(iv) $\tan^{-1}(2/s)$

$$L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t f(t)$$

$$F(s) = \tan^{-1}(2/s)$$

Diff w.r.t 's' ob.s

$$F'(s) = \frac{1}{1+\frac{4}{s^2}}\left(\frac{-2}{s^2}\right)$$

$$F'(s) = \frac{-2}{s^2+4}$$

$$L^{-1}\{F'(s)\} = -L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\cancel{-}t f(t) = \cancel{-}\sin 2t$$

$$f(t) = \frac{\sin 2t}{t}$$

(iii) $F(s) = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

$= \frac{1}{2}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$

Diff. obs. w.r.t 's'

$F'(s) = \frac{1}{\cancel{2}}\left[\frac{1}{s^2+b^2}\cancel{(2s)} - \frac{1}{s^2+a^2}\cancel{(2s)}\right]$

$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{s}{s^2+b^2}\right\} - L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$

$-t f(t) = \cos bt - \cos at$

$f(t) = \frac{\cos at - \cos bt}{t}$

Q. $L^{-1}\left\{\frac{1}{s}\sin\left(\frac{1}{s}\right)\right\} = \frac{t}{(1!)^2} - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \cdots$

Q. $L^{-1}\left\{\frac{1}{s}\cos\left(\frac{1}{s}\right)\right\} = \frac{1}{1} - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \cdots$

→ WKT

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{3!}\left(\frac{1}{s^3}\right) + \frac{1}{5!}\left(\frac{1}{s^5}\right) - \cdots$

$\frac{1}{s}\sin\left(\frac{1}{s}\right) = \frac{1}{s^2} - \frac{1}{3!}\left(\frac{1}{s^4}\right) + \frac{1}{5!}\left(\frac{1}{s^6}\right) - \cdots$

$L^{-1}\left\{\frac{1}{s}\sin\left(\frac{1}{s}\right)\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3!}L^{-1}\left\{\frac{1}{s^4}\right\} + \frac{1}{5!}L^{-1}\left\{\frac{1}{s^6}\right\} - \cdots$

$= \frac{t}{1!} - \frac{1}{3!}\left(\frac{t^3}{3!}\right) + \frac{1}{5!}\left(\frac{t^5}{5!}\right) - \cdots$

$= \frac{t}{(1!)^2} - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \cdots$

$\rightarrow \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\cos\left(\frac{1}{s}\right) = 1 - \frac{1}{2!}\left(\frac{1}{s^2}\right) + \frac{1}{4!}\left(\frac{1}{s^4}\right) - \cdots$$

$$\frac{1}{s}\cos\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{2!}\left(\frac{1}{s^3}\right) + \frac{1}{4!}\left(\frac{1}{s^5}\right) - \cdots$$

$$L^{-1}\left\{\frac{1}{s}\cos\left(\frac{1}{s}\right)\right\} = L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2!}L^{-1}\left\{\frac{1}{s^3}\right\} + \frac{1}{4!}\left\{\frac{1}{s^5}\right\}$$

$$= 1 - \frac{1}{2!}\frac{t^2}{(2!)} + \frac{1}{4!}\frac{t^4}{(4!)} - \cdots$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \cdots$$

Inverse laplace's transform using partial fractions

Find the inverse laplace's transform of following

(i) $\dfrac{4s+5}{(s+1)^2(s+2)}$

→ $\dfrac{4s+5}{(s+1)^2(s+2)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2} + \dfrac{C}{(s+2)}$ → (1)

$4s+5 = A(s+1)(s+2) + B(s+2) + C(s+1)^2$

put $s=-1$

$1 = B(1) \Rightarrow B = 1$

put $s=-2$

$-3 = C(1) \Rightarrow C = -3$

Comparing obs co-efficient of s^2

$0 = A + C \Rightarrow A = -C$

$A = 3$

(1) ⇒

$L^{-1}\left\{\dfrac{4s+5}{(s+1)^2(s+2)}\right\} = L^{-1}\left\{\dfrac{3}{(s+1)}\right\} + L^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} + L^{-1}\left\{\dfrac{-3}{(s+2)}\right\}$

$= e^{-t} + te^{-t} - 3e^{-2t}$

(ii) $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+2s+5}$ → (1)

$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$ → (2)

put $s=1$

$8 = A(8) + 0 \Rightarrow A = 1$

Comparing oabs coefficient of s^2

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$0 = A + B \Rightarrow B = -A = -1$

put $s = 0$

$3 = 5(A) + C(-1) \Rightarrow 3 - 5 = -C$

$\Rightarrow C = 2$

$$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{1}{(s-1)}\right\} + L^{-1}\left\{\frac{2-s}{(s^2+2s+5)}\right\}$$

$$= e^t - L^{-1}\left\{\frac{(s+1)-3}{(s+1)^2+4}\right\}$$

$$= e^t - e^{-t} L^{-1}\left\{\frac{s-3}{s^2+4}\right\}$$

$$= e^t - e^{-t}\left[L^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{3}{2} L^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$

$$= e^t - e^{-t}\left(\cos 2t - \frac{3}{2}\sin 2t\right)$$

(iii) $$\frac{s+2}{(s^2+4s+5)^2} = \frac{As+B}{s^2+4s+5} + \frac{Cs+D}{(s^2+4s+5)^2}$$

$$F(s) = \frac{s+2}{(s^2+4s+5)^2} = \frac{s+2}{((s+2)^2+1)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{(s+2)}{((s+2)^2+1)^2}\right\}$$

$$L^{-1}\{F(s)\} = e^{-2t} L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \longrightarrow (1)$$

$$\frac{s}{(s^2+1)^2} = \frac{As+B}{(s+1)} + \frac{Cs+D}{(s+1)^2}$$

$$s = (As+B)(s+1) + Cs + D$$

$$s = As^3 + Bs^2 + As + B + Cs + D$$

comparing on both side co-efficients

$A = 0$, $B = 0$, $A + C = 1$; $D + B = 0$

$C = 1$; $D = 0$

Convolution:

The convolution of two functions $f(t)$ & $g(t)$ is denoted by $f(t) * g(t)$ and is defined as $f(t) * g(t) = \int_{u=0}^{t} f(u) g(t-u)\, du$

Property of convolution

Prove that convolution is commutative.

→ By definition of convolution we have

$$f(t) * g(t) = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

put $t-u = v$

$\Rightarrow$ $u = t-v$

$du = -dv$

if $u = 0$, then $v = t$

$u = t$, then $v = 0$

$$= \int_{v=t}^{0} f(t-v)\, g(v)\, (-dv)$$

$$= \int_{v=0}^{t} f(t-v)\, g(v)\, dv$$

$f(t) * g(t) = g(t) * f(t)$ // Hence proved //

State & prove convolution theorem!

If $L^{-1}\{F(s)\} = f(t)$, $L^{-1}\{G(s)\} = g(t)$ then prove that $L^{-1}\{F(s)\cdot G(s)\} = \int_{u=0}^{t} f(u)\, g(t-u)\, du$

(OR)

$$F(s)\cdot G(s) = L\left\{\int_{u=0}^{t} f(u)\, g(t-u)\, du\right\}$$

(OR)

$$F(s)\cdot G(s) = L\{f(t) * g(t)\}$$

(OR)

$$L^{-1}\{F(s)\cdot G(s)\} = \{f(t) * g(t)\}$$

Proof! Consider RHS.

$$L\left\{\int_{u=0}^{t} f(u)\, g(t-u)\, du\right\} = \int_{t=0}^{\infty} e^{-st}\left[\int_{u=0}^{t} f(u)\, g(t-u)\, du\right] dt$$

$$= \int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st} f(u)\, g(t-u)\, du\, dt \longrightarrow (1)$$

By change of order of integration.

there $u = t$

if $t \to 0$, then $u \to 0$

if $t \to \infty$, then $u \to \infty$

Here $t = u$ to $t \to \infty$

$(1) \Rightarrow$

$$L\left\{\int_{u=0}^{t} f(u)\, g(t-u)\, du\right\} = \int_{u=0}^{\infty}\int_{t=0}^{\infty} e^{-st} f(u)\, g(t-u)\, dt\, du$$

$$= \int_{u=0}^{\infty} f(u)\left[\int_{t=0}^{\infty} e^{-st} g(t-u)\, dt\right] du \longrightarrow (2)$$

put $t-u = v \Rightarrow t = u+v$

$dt = dv.$

$$= \int_{u=0}^{\infty} f(u)\left[\int_{v=0}^{\infty} e^{-s(u+v)} g(v)\,dv\right] du$$

$$= \int_{u=0}^{\infty} f(u)\left(\int_{v=0}^{\infty} e^{-su} e^{-sv} g(v)\,dv\right) du$$

$$= \left(\int_{u=0}^{\infty} e^{-su} f(u)\,du\right) \times \left(\int_{v=0}^{\infty} e^{-sv} g(v)\,dv\right)$$

$$= L\{f(u)\} \times L\{g(v)\}$$

$$= F(s)\cdot G(s)$$

$$= LHS$$

Using convolution theorem, find inverse La-Place's transform of the following

(i) $\dfrac{s^2}{(s^2+a^2)^2}$ (ii) $\dfrac{s}{(s+2)(s^2+9)}$

$$\rightarrow L^{-1}\{F(s)\cdot G(s)\} = \int_{u=0}^{t} f(u)\, g(t-u)\,du \quad \text{—(1)}$$

Let $F(s) = \dfrac{s}{s^2+a^2}$ $G(s) = \dfrac{s}{s^2+a^2}$

$f(t) = L^{-1}\{F(s)\} = \cos at$, $g(t) = L^{-1}\{G(s)\} = \cos at$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \int_{u=0}^{t} \cos au \cdot \cos(at-au)\,du$$

$$= \frac{1}{2}\left[\int_0^t \left(\cos(at) + \cos(2au-at)\right) du\right]$$

$$L^{-1}\left\{\frac{s^2}{s^2+a^2}\right\} = \frac{1}{2}\left[\cos(at)\,(u)_0^t + \left(\frac{\sin(2au-at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{1}{2a}(\sin at - \sin(-at))\right]$$

$$= \frac{t\cos at}{2} + \frac{\sin at}{2a}$$

(a.ii) $F(s) = \frac{s}{s^2+a}$, $G(s) = \frac{1}{s+2}$

$$f(t) = L^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t$$

$$g(t) = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

$$\int_{u=0}^{t} \cos 3u\; e^{-2(t-u)}\,du$$

$$= e^{-2t}\int_{u=0}^{t} e^{2u}\cos 3u\,du$$

$$= e^{-2t}\left[\frac{e^{2u}}{4+9}(2\cos 3u + 3\sin 3u)\right]_0^t$$

$$= e^{-2t}\left[-\frac{2}{13} + \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t)\right]$$

(iii) $\frac{s}{(s^2+4)^2}$ $\quad F(s) = \frac{s}{s^2+4}$ $\quad G(s) = \frac{1}{s^2+4}$

$f(t) = \cos 2t$ $\quad g(t) = \frac{1}{2}\sin 2t$

$$L^{-1}\left\{\frac{s}{s^2+4}\times\frac{1}{s^2+4}\right\} = \int_0^t (\cos 2u)\left(\frac{1}{2}\sin(2t-2u)\,du\right)$$

$$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{2}\int_0^t \sin(2t-2u)\cos 2u\,du$$

$$= \frac{1}{4}\int_0^t \{(\sin 2t) + (\sin(2t-4u))\}\,du$$

$$= \frac{1}{4}\left[\int_0^t \sin 2t\,du + \int_0^t \sin(2t-4u)\,du\right]$$

$$= \frac{1}{4}\left[\sin 2t(u)_0^t - \left(\frac{\cos(2t-4u)}{-4}\right)_0^t\right]$$

$$= \frac{1}{4}\left[t\sin 2t + \frac{1}{4}(\cancel{\cos 2t - \cos 2t})\right]$$

$$= \frac{t}{4}\sin 2t$$

Q.
Verify convolution theorem for the following functions

(a) $f(t) = t, \quad g(t) = \cos t$

(b) $f(t) = \sin t \quad g(t) = e^{-t}$

WKT, By convolution theorem

$$F(s)\cdot G(s) = L\left\{\int_{u=0}^{t} f(u)\,g(t-u)\,du\right\} \rightarrow ①$$

$$F(s) = L\{f(t)\} = L\{t\} = \frac{1}{s^2}$$

$$G(s) = L\{g(t)\} = L\{\cos t\} = \frac{s}{s^2+1}$$

$$LHS = F(s)\cdot G(s) = \frac{\cancel{s}}{s^{\cancel{2}}(s^2+1)} = \frac{1}{s(s^2+1)} \rightarrow ②$$

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$$RHS = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

$$= \int_0^t u \cos(t-u)\, du$$

$$= u\left(\frac{\sin(t-u)}{-1}\right) - 1\left[-\frac{\cos(t-u)}{(-1)^2}\right]_0^t$$

$$= \{(0+1) - (0+\cos t)\} = 1-\cos t$$

$$RHS = L\left\{\int_0^t f(u)\, g(t-u)\, du\right\}$$

$$= L\{1-\cos t\}$$

$$= \frac{1}{s} - \frac{s}{s^2+1} = \frac{s^2+1-s^2}{s(s^2+1)} = \frac{1}{s(s^2+1)} \longrightarrow (3)$$

From (2) & (3) LHS = RHS

Applications of La-Place's transform to solve differential eqn

To solve the DE we use the following formulae.

$$L\{y'(t)\} = sL\{y(t)\} - y(0)$$

$$L\{y''(t)\} = s^2 L\{y(t)\} - sy(0) - y'(0)$$

$$L\{y'''(t)\} = s^3 L\{y(t)\} - s^2 y(0) - sy'(0) - y''(0).$$

Q Solve the following DE's by method of La[illegible] transform

(i) $y''' + 2y'' - y' - 2y = 0$; $y(0) = y'(0) = 0$, $y''(0) = 6$.

$\rightarrow$ $y'''(t) + 2y''(t) - y'(t) - 2y(t) = 0$

Apply L.T obs

$$L\{y'''(t)\} + 2L\{y''(t)\} - L\{y'(t)\} - 2L\{y(t)\} = 0$$

$$s^3 L\{y(t)\} - s^2 \cancel{y(0)} - s\cancel{y'(0)} - y''(0) + 2\left[s^2 L\{y(t)\} - s\cancel{y(0)} - \cancel{y'(0)}\right]$$
$$- \left[s L\{y(t)\} - \cancel{y(0)}\right] - 2L\{y(t)\} = L\{0\}$$

$$L\{y(t)\}\left[s^3 + 2s^2 - s - 2\right] - 6 = 0$$

$$L\{y(t)\} = \frac{6}{(s+2)(s^2-1)} = \frac{6}{(s+2)(s+1)(s-1)}$$

$$y(t) = L^{-1}\left\{\frac{6}{(s+2)(s+1)(s-1)}\right\} \rightarrow ①$$

considering the proper fraction.

$$\frac{6}{(s+2)(s+1)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{s+1} \rightarrow ②$$

$$6 = A(s-1)(s+1) + B(s+2)(s+1) + C(s+2)(s-1)$$

put $s = 1$, $6B = 6$, $\Rightarrow B = 1$.

put $s = -1$, $6 = -2C$ $\Rightarrow C = -3$

put $s = -2$, $6 = 3A$ $\Rightarrow A = 2$

$$② \Rightarrow \frac{6}{(s+2)(s-1)(s+1)} = \frac{2}{(s+2)} + \frac{1}{(s-1)} - \frac{3}{(s+1)}$$

$$① \Rightarrow y(t) = L^{-1}\left\{\frac{2}{s+2} + \frac{1}{s-1} - \frac{3}{s+1}\right\} = 2e^{-2t} + e^{t} - 3e^{-t}$$

The co-ordinates (x,y) of a particle moving along a plaine curve at any pt. t are given by $\frac{dy}{dt} + 2x = \sin 2t$, $\frac{dx}{dt} - 2y = \cos$ where $(t > 0)$, if at $t = 0$, $x = 1$ & $y = 0$ then show that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$ using La Place's transform method.

$\rightarrow$ Given D.E's are

$$\frac{dy}{dt} + 2x = \sin 2t \rightarrow ①$$

$$\frac{dx}{dt} - 2y = \cos 2t \rightarrow ②$$

Also given that

$$x(0) = 1, \quad y(0) = 0$$

Apply L.T for ① & ②

$$L\{y'(t)\} + 2L\{x(t)\} = L\{\sin 2t\}$$

$$sL\{y(t)\} - y(0) + 2L\{x(t)\} = \frac{2}{s^2+4}$$

$$sL\{y(t)\} + 2L\{x(t)\} = \frac{2}{s^2+4} \rightarrow ③$$

$$L\{x'(t)\} - 2L\{y(t)\} = L\{\cos 2t\}$$

$$sL\{x(t)\} - x(0) - 2L\{y(t)\} = \frac{s}{s^2+4}$$

$$sL\{x(t)\} - 2L\{y(t)\} = \frac{s}{s+4} + 1 \rightarrow ④$$

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③ $\times s \Rightarrow$

$$2s\,L\{\cancel{x(t)}\} + s^2 L\{y(t)\} = \frac{2s}{s^2+4}$$

$2t$ ④ $\times 2 \Rightarrow$

$$2s\,L\{\cancel{x(t)}\} - 4L\{y(t)\} = \frac{2s}{s^2+4} + 2$$

$(-) \qquad + \qquad (-) \qquad (-)$

$$L\{y(t)\}\left[s^2+4\right] = -2$$

$$y(t) = L^{-1}\left\{\frac{-2}{s^2+4}\right\}$$

$$y(t) = -\sin 2t$$

$$y'(t) = -2\cos 2t$$

① $\Rightarrow$ $-2\cos 2t + 2x = \sin 2t$

$$2x = \sin 2t + 2\cos 2t$$

$$x = \frac{1}{2}\sin 2t + \cos 2t$$

Now,

$$4x^2 + 4xy + 3y^2 = (2x)^2 + 2(2x)y + 5y^2$$

$= \sin^2 2t + 4\cos^2 2t + 4\sin 2t\cos 2t + 2\left[\sin 2t + 2\cos 2t\right](-\sin 2t) + 5(\sin^2 2t)$

$= 6\sin^2 2t + 4\cos^2 2t + \cancel{4\sin 2t\cos 2t} - 2\sin^2 2t - \cancel{4\sin 2t\cos 2t}$

$= 4(\sin^2 2t + \cos^2 2t)$

$= 4$

A voltage Ee^{-at} is applied at $t=0$, to a ckt of inductance 'L' and resistance 'R' (LR ckt) show that the current at time 't' is $\frac{E}{R-aL}\left(e^{-at}-e^{-Rt/L}\right)$ using LaPlace's transform.

→ WKT

Diff eqn of LR ckt is

$$L\frac{di}{dt}+Ri=E$$

$$\frac{di}{dt}+\frac{R}{L}i=\frac{Ee^{-at}}{L} \longrightarrow (1)$$

Apply LT obs

$$L\{i'(t)\}+\frac{R}{L}L\{i(t)\}=\frac{E}{L}L\{e^{-at}\}$$

$$s\,L\{i(t)\}-i(0)+\frac{R}{L}L\{i(t)\}=\frac{E}{L}\left(\frac{1}{s+a}\right)$$

since at $t=0$, $i=0$

$$L\{i(t)\}\left[s+\frac{R}{L}\right]=\frac{E}{L}\left(\frac{1}{s+a}\right)$$

$$L\{i(t)\}=\frac{E}{L}\frac{1}{(s+a)}\frac{1}{\left(s+\frac{R}{L}\right)}$$

$$i(t)=L^{-1}\left\{\frac{E}{L}\frac{1}{s+a}\frac{1}{s+\frac{R}{L}}\right\} \longrightarrow (2)$$

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Consider

$$\frac{1}{(s+a)(s+\frac{R}{L})} = \frac{A}{s+a} + \frac{B}{s+\frac{R}{L}}$$

$$1 = A\left(s+\frac{R}{L}\right) + B(s+a)$$

put $s = \frac{-R}{L}$

$$1 = B\left(a-\frac{R}{L}\right)$$

$$\Rightarrow \quad B = \frac{1}{a-\frac{R}{L}}$$

put $s = -a$

$$1 = A\left(\frac{R}{L}-a\right)$$

$$A = \frac{1}{\left(\frac{R}{L}-a\right)}$$

$$\Rightarrow \quad i(t) = \left[\frac{1}{\left(\frac{R}{L}-a\right)} e^{-at} + \frac{1}{\left(a-\frac{R}{L}\right)} e^{-Rt/L}\right] \frac{E}{L}$$

$$i(t) = \frac{E}{R-aL}\left(e^{-at} - e^{-Rt/L}\right)$$

Solve:

$\frac{dy}{dt} + 2y + \int^{t} y\,dt = \sin t$ given that $y(0) = 0$ $y'(0) = 1$, using La-Place's transform method.

$$\rightarrow \quad y'(t) + 2y(t) + \int_0^t y\,dt = \sin t$$

Apply LT obs

$$L\{y'(t)\} + 2L\{y(t)\} + L\left\{\int^{t} y\,dt\right\} = L\{\sin t\}$$

$sL\{y(t)\} + \cancel{y(0)} + 2L\{y(t)\} + \dfrac{L\{y(t)\}}{s} = \dfrac{1}{s^2+1}$

$L\{y(t)\}\left[s+2+\dfrac{1}{s}\right] = \dfrac{1}{s^2+1}$

$L\{y(t)\} = \dfrac{s}{(s^2+1)(s+1)^2}$

$y(t) = L^{-1}\left\{\dfrac{s}{(s^2+1)(s+1)^2}\right\}$

Consider

$\dfrac{s}{(s^2+1)(s+1)^2} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2} + \dfrac{Cs+D}{(s^2+1)}$

$s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 \longrightarrow (2)$

put $s = -1$ $\Rightarrow$ $B = -1/2$

put $s = 0$

$0 = A+B+D \Rightarrow A+D = 1/2$

comparing s^3

$0 = A + C$

comparing s^2

$0 = A + B + D + 2C$

$\Rightarrow C = 0, \quad A = 0, \quad D = 1/2$

(1) $\Rightarrow$

$y(t) = L^{-1}\left\{\dfrac{-1}{2(s+1)^2} + \dfrac{1}{2(s^2+1)}\right\}$

$= -\dfrac{1}{2} e^{-t} L^{-1}\left\{\dfrac{1}{s^2}\right\} + \dfrac{1}{2}\sin t$

$y(t) = \dfrac{1}{2}\left[\sin t - t e^{-t}\right]$

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Solve

Q. $tD^2Y + (1-2t)DY - 2Y = 0 \quad Y(0) = 1,\ Y'(0) = 2$
using La-Place's transform method

Given D.E

$$tY''(t) + Y'(t) - 2tY'(t) - 2Y(t) = 0 \longrightarrow (1)$$

$$L\{tY''(t)\} + L\{Y'(t)\} - 2L\{tY'(t)\} - 2L\{Y(t)\} = 0$$

$$(-1)^1 \frac{d}{ds} L\{Y''(t)\} + sL\{Y(t)\} - Y(0) - 2(-1)\frac{d}{ds} L\{Y'(t)\} - 2L\{Y(t)\} = 0$$

$$\Rightarrow -\frac{d}{ds}\left[s^2 L\{Y(t)\} - sY(0) - Y'(0)\right] + sY(s) - 1 + 2\frac{d}{ds}\left[2L\{Y(t)\} - Y(0)\right] - 2Y(s) = 0$$

$$\Rightarrow -\frac{d}{ds}\left[s^2 Y(s)\right] + 1(1) - 0 + s(Y(s)) - \cancel{1} + 2\frac{d}{ds}\left[sY(s)\right] - 0 - 2Y(s) = 0$$

$$\Rightarrow -s^2\frac{d}{ds}(Y(s)) - 2sY(s) + 8Y(s) + 2s\frac{d}{ds}(Y(s)) + 2Y(s) - \cancel{2Y(s)} = 0$$

$$\frac{d}{ds}(Y(s))\left[-s^2 + 2\cancel{s}\right] = \cancel{s}Y(s)$$

$$\frac{d}{ds}Y(s) = \frac{Y(s)}{-(s-2)}$$

$$\int \frac{Y'(s)}{Y(s)} = \int \frac{-1}{(s-2)}$$

$$\log[Y(s)] = -\log(s-2)$$

$$Y(s) = \frac{1}{s-2}$$

Apply $L^{-1}T$ obs

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s-2}\right\}$$

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$\Rightarrow y(t) = e^{2t}$ //

Q. Determine the response of damped mass-string system under the square wave given by the following equ

$$y'' + 3y' + 2y = u(t-1) - u(t-2), \quad y(0) = 0 = y'(0)$$

Soln:

Applying La-Place's transform.

$$L\{y''(t)\} + 3L\{y'(t)\} + 2L\{y(t)\} = L\{u(t-1)\} - L\{u(t-2)\}$$

$$s^2 L\{y(t)\} - s\,y(0) - y'(0) + 3[sL\{y(t)\} - y(0)] + 2L\{y(t)\} = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$L\{y(t)\}(s^2+3s+2) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$L\{y(t)\} = e^{-s}\left(\frac{1}{s(s+1)(s+2)}\right) - e^{-2s}\left(\frac{1}{s(s+1)(s+2)}\right)$$

$$\{y(t)\} = L^{-1}\left[e^{-s}\left(\frac{1}{s(s+1)(s+2)}\right)\right] - L^{-1}\left[e^{-2s}\left(\frac{1}{s(s+1)(s+2)}\right)\right]$$

$$y(t) = H(t-1)\,f(t-1) - H(t-2)\,f(t-2) \longrightarrow (1)$$

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} \longrightarrow (2)$$

consider

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$\Rightarrow 1 = A(s+1)(s+2) + B(s+2)(s) + C(s)(s+1)$

$\Rightarrow B = -1, \quad C = 1/2, \quad A = 1/2$

Now,

$\Rightarrow L^{-1}\left\{ \frac{1}{2s} + \frac{-1}{s+1} + \frac{1}{2(s+2)} \right\} = f(t)$

$f(t) = \frac{1}{2}(1) - e^{-t} + \frac{1}{2}e^{-2t}$

① $\Rightarrow$

$Y(t) = H(t-1)\left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} \right]$

$- H(t-2)\left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)} \right]$

Q. Solve $y'' + 4y = f(t)$, $y(0) = 0 = y'(0)$

where $f(t) = \begin{cases} 0, & \text{if } t < 3 \\ t, & \text{if } t \geq 3 \end{cases}$

Apply La-Place's obs

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^3 e^{-st}(0)\,dt + \int_3^\infty e^{-st}(t)\,dt$

$= \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\frac{e^{-st}}{(-s)^2} \right]_3^\infty$

$= (0-0) - \left[-\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} \right]$

$L\{f(t)\} = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s} \right) \rightarrow ②$

(1) $\Rightarrow$

$$s^2 L\{y(t)\} - sy(0) - y'(0) + 4L\{y(t)\} = e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$$

$$L\{y(t)\}(s^2+4) = e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$$

$$L\{y(t)\} = \frac{e^{-3s}}{s^2+4}\left(\frac{1}{s^2}+\frac{3}{s}\right)$$

$$y(t) = L^{-1}\left\{\frac{e^{-3s}}{s^2+4}\left(\frac{1}{s^2}+\frac{3}{s}\right)\right\}$$