$$\int_0 \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$$

4. Find the fourier transform of $f(x) = x e^{-a|x|}$ where $a>0$

soln :

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\,dx$$

$$= \int_{-\infty}^{0} f(x) e^{iux}\,dx + \int_{0}^{\infty} f(x) e^{iux}\,dx$$

$$= \int_{-\infty}^{0} x e^{ax} e^{iux}\,dx + \int_{0}^{\infty} x e^{-ax} e^{iux}\,dx$$

$$= \int_{-\infty}^{0} x e^{x(a+iu)}\,dx + \int_{0}^{\infty} x e^{-x(a-iu)}\,dx$$

$$F(u) = \left[x \frac{e^{x(a+iu)}}{a+iu} - \frac{e^{x(a+iu)}}{(a+iu)^2} \right]_{-\infty}^{0} +$$

$$\left[x \frac{e^{-x(a-iu)}}{-(a-iu)} - \frac{e^{-x(a-iu)}}{(a-iu)^2} \right]_{0}^{\infty}$$

$$= \left[\frac{-1}{(a+iu)^2} + \frac{1}{(a-iu)^2} \right]$$

$$= \frac{-(a-iu)^2 + (a+iu)^2}{(a^2+u^2)^2} = \frac{-(a^2-u^2-2aiu) + a^2-u^2+2aiu}{(a^2+u^2)^2}$$

$$= \frac{4aiu}{(a^2+u^2)^2}$$

$f(x) = e^{-a^2x^2}$

6. Find fourier sine transform of $e^{-|x|}$.

Soln:

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$= \int_0^\infty e^{-|x|} \sin ux \, dx$$

$$= \int_0^\infty e^{-x} \sin ux \, dx$$ (Since the limit is from 0 to ∞) $\Rightarrow |x| = x$

$$= \frac{e^{-x}}{1+u^2} \left[-\sin ux - u\cos ux \right] \Big|_0^\infty$$

$$= \frac{1}{1+u^2} \{ 0 + u \} = \frac{u}{1+u^2}$$

Find $f(x)$, given that $\int_0^\infty f(x)\cos \alpha x \, dx = \cancel{e} \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

Soln:

$$\int_0^\infty f(x)\cos \alpha x \, dx = F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

Using inverse fourier cosine transform,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\alpha)\cos \alpha x \, d\alpha$$

$$= \frac{2}{\pi} \int_0^1 (1-\alpha)\cos \alpha x \, d\alpha + \int_1^\infty (0)$$

$$= \frac{2}{\pi} \left[(1-\alpha) \frac{\sin \alpha x}{x} - (-1) \frac{-\cos \alpha x}{x^2} \right]_0^1$$

$$= \frac{2}{\pi} \left[-\frac{\cos x}{x^2} + \frac{1}{x^2} \right]$$

$$= \frac{2}{\pi x^2}(1-\cos x) = \underline{\underline{\frac{4\sin^2(x/2)}{\pi x^2}}}$$

* Find Fourier Cosine transform of e^{-ax} & hence deduce that FCT of $x e^{-ax}$. Further evaluate $\int_0^\infty \frac{\cos \lambda x}{x^2+a^2} dx$

$$F_c[f(x)] = \int_0^\infty f(x) \cos ux \, dx$$

$$= \int_0^\infty e^{-ax} \cos ux \, dx$$

$$= \frac{e^{-ax}}{a^2+u^2} \left[-a \cos ux + u \sin ux \right] \Big|_0^\infty$$

$$= \frac{a}{a^2+u^2}$$

$$\int_0^\infty e^{-ax} \cos ux \, dx = \frac{a}{a^2+u^2}$$

diff wrt a on B.S

$$\int_0^\infty -x \, e^{-ax} \cos ux \, dx = \frac{(a^2+u^2)(1) - a(2a)}{(a^2+u^2)^2}$$

$$-\int_0^\infty x e^{-ax} \cos ux \, dx = \frac{a^2+u^2-2a^2}{(a^2+u^2)^2} = \frac{-a^2+u^2}{(a^2+u^2)^2}$$

$$\int_0^\infty x e^{-ax} \cos ux \, dx = \frac{a^2-u^2}{(a^2+u^2)^2}$$

$$F_c[x e^{-ax}] = \frac{a^2-u^2}{(a^2+u^2)^2}$$

Also $F_c[e^{-ax}] = \frac{a}{a^2+u^2}$

By IFT

$$f(x) = \frac{2}{\pi}\int_0^\infty F(u) \cos ux \, du$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+u^2} \cos ux \, du$$

$$\int_0^\infty \frac{\cos ux}{u^2+a^2} dx = \frac{\pi}{2a} e^{-ax}$$

change x to λ & u to x

$$\int_0^\infty \frac{\cos \lambda x}{a^2+x^2} dx = \frac{\pi}{2a} e^{-a\lambda}$$