Laplace Transforms

If $f(t)$ is a real valued function defined on all $t \geq 0$ then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ is defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

provided integral exists.

$$L\{f(t)\} = F(s)$$

Linearity Property

For any two functions $f(t)$ and $g(t)$ and α, β are constants then

$$L\{\alpha \cdot f(t) \pm \beta \cdot g(t)\} = \alpha\, L\{f(t)\} \pm \beta \cdot L\{g(t)\}$$

Laplace transforms of some standard functions

① $L\{a\}$, where a is a constant

We know $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$f(t) = a$

$$= \int_0^\infty e^{-st} a\, dt$$

$$= a \int_0^\infty e^{-st} dt \Rightarrow a\left(\frac{e^{-st}}{-s}\right)_0^\infty$$

$$L\{a\} = -\frac{a}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{a}{s}\left[0-1\right]$$

$$= \frac{a}{s}$$

$$\boxed{L\{a\} = \frac{a}{s}}$$

ly $L\{1\} = \frac{1}{s}$

2 $L\{e^{at}\}$

$f(t) = e^{at}$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{s-a}\left[0-1\right]$$

$$= \frac{1}{s-a}$$

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

ly

$L\{e^{-at}\} = \frac{1}{s+a}$

(3) $L\{\cosh at\}$

$$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2} L\{e^{at}+e^{-at}\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{(s-a)(s+a)}\right] \qquad \left\{\begin{array}{l} L\{e^{at}\} = \frac{1}{s-a} \\ L\{e^{-at}\} = \frac{1}{s+a} \end{array}\right.$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2-a^2}\right]$$

$$\boxed{L\{\cosh at\} = \frac{s}{s^2-a^2}}$$

(4) $L\{\sinh at\}$

$$L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a}{s^2-a^2}\right]$$

$$\boxed{L\{\sinh at\} = \frac{a}{s^2-a^2}}$$

(5) $L\{\cos at\}$

$$L\{\cos at\} = \int_0^\infty e^{-st} \cdot \cos at \, dt$$

$$\int e^{at}\cos bt \, dt = \frac{e^{at}}{a^2+b^2}\left(a\cos bt + b\sin bt\right)$$

$$L\{\cos at\} = \left[\frac{e^{-st}}{(-s)^2+a^2}\left(-s\cos at + a\sin at\right)\right]_0^\infty$$

$$= \left[\frac{\cancel{e^{-\infty}}}{s^2+a^2}\cancel{\left(-s\cos at + a\sin at\right)}^{0} - \frac{e^0}{s^2+a^2}\left(-s\cos 0 + \cancel{a\sin 0}^{0}\right)\right]$$

$$= 0 - \frac{1}{s^2+a^2}(-s) = \frac{s}{s^2+a^2}$$

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

⑥ $L\{\sin at\}$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$\int_0^\infty e^{at} \sin bt \, dt = \frac{e^{at}}{a^2+b^2}(a \sin bt - b \cos bt)$$

$$L\{\sin at\} = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= \frac{e^{-\infty}}{s^2+a^2}(-s \cdot \sin at - a \cos at) - \frac{e^0}{s^2+a^2}(-s \cdot \sin 0 - a \cos 0)$$

$$= 0 - \frac{1}{s^2+a^2}(0 - a)$$

$$= \frac{a}{s^2+a^2}$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

7. $L\{t^n\}$

$$L\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

put $st = x \Rightarrow t = \frac{x}{s}$

$s\,dt = dx$

$dt = \frac{dx}{s}$

If $t = 0, \; x = 0$

$t = \infty, \; x = \infty$

$$L\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \cdot \frac{x^n}{s^n} \cdot \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^n \, dx$$

$$\left[\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx \right.$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^{(n+1)-1} \, dx$$

$$= \frac{1}{s^{n+1}} \cdot \Gamma(n+1)$$

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}.$$

where n is a constant

(8) $L\{t^n\}$, where n is a positive integer

$$L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

$$= \int_0^\infty t^n e^{-st}\, dt$$

$$= \left(t^n \cdot \frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, n \cdot t^{n-1}\, dt$$

$$= (0-0) + \frac{n}{s}\underbrace{\int_0^\infty e^{-st}\, t^{n-1}\, dt}$$

$$L\{t^n\} = \frac{n}{s} L\{t^{n-1}\}$$ ✓

$||^{ly}$ $$L\{t^{n-1}\} = \frac{n-1}{s} L\{t^{n-2}\}$$

$$L\{t^{n-2}\} = \frac{n-2}{s} L\{t^{n-3}\}$$

$$\vdots$$

$$L\{t^2\} = \frac{2}{s} L\{t\}$$

$$L\{t\} = \frac{1}{s} \cdot L\{t^0\}$$

$$L\{t^n\} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} \cdot \underbrace{L\{1\}}$$

$$= \frac{n(n-1)(n-2)\cdots 2 \cdot 1}{s^n} \cdot L\{1\}$$

$$L\{t^n\} = \frac{n!}{s^n} \cdot \frac{1}{s}$$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

Formulae

(1) $L\{a\} = \frac{a}{s}$

(2) $L\{e^{at}\} = \frac{1}{s-a}$

(3) $L\{e^{-at}\} = \frac{1}{s+a}$

(4) $L\{\cosh at\} = \frac{s}{s^2-a^2}$

(5) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

(6) $L\{\cos at\} = \frac{s}{s^2+a^2}$

7) $L\{\sin at\} = \frac{a}{s^2+a^2}$

(8) $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, n is a constant

$= \frac{n!}{s^{n+1}}$, where n is a positive integer

Problems

1. Find the Laplace transform of $1+2t^3-4e^{3t}+5e^{-t}$

Sol:- $L\{1+2t^3-4e^{3t}+5e^{-t}\}$

$= L\{1\}+2L\{t^3\}-4L\{e^{3t}\}+5L\{e^{-t}\}$

$= \frac{1}{s}+2\cdot\frac{3!}{s^{3+1}}-4\cdot\frac{1}{s-3}+5\cdot\frac{1}{s+1}$

$= \frac{1}{s}+\frac{12}{s^4}-\frac{4}{s-3}+\frac{5}{s+1}$

2. Find $L\{(t^5+2)^2\}$

Sol:- $L\{(t^5)^2+2^2+2\cdot2\cdot t^5\}$

$= L\{t^{10}+4+4t^5\}$

$= L\{t^{10}\}+L\{4\}+4L\{t^5\}$

$= \frac{10!}{s^{10+1}}+4\cdot\frac{1}{s}+4\cdot\frac{5!}{s^{5+1}}$

$= \frac{10!}{s^{11}}+\frac{4}{s}+\frac{480}{s^6}$ //

③ Find $L\{\sqrt{t}\}$

Sol:- $L\{\sqrt{t}\}$

$= L\{t^{1/2}\}$ $\qquad L\{t^n\} = \frac{\Gamma n+1}{s^{n+1}}$

$= \frac{\Gamma \frac{1}{2}+1}{s^{\frac{1}{2}+1}}$ $\qquad \Gamma n+1 = n\,\Gamma n$

$= \frac{\frac{1}{2}\,\Gamma \frac{1}{2}}{s^{3/2}}$

$\Gamma \frac{1}{2} = \sqrt{\pi}$

$= \frac{1}{2} \cdot \frac{\sqrt{\pi}}{s^{3/2}}$

$L\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2 s^{3/2}}$

④ Find $L\left\{\frac{1}{\sqrt{t}}\right\}$

Sol:- $L\left\{\frac{1}{\sqrt{t}}\right\} = L\left\{\frac{1}{t^{1/2}}\right\} = L\{t^{-1/2}\}$

$= \frac{\Gamma -\frac{1}{2}+1}{s^{-\frac{1}{2}+1}}$

$= \frac{\Gamma \frac{1}{2}}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$

$L\{1/\sqrt{t}\} = \sqrt{\pi/s}$

(5) Find $L\{\cos at \cos bt\}$

Sol:- $f(t) = \cos at \cos bt$

$$\cos at \cos bt = \tfrac{1}{2}[\cos(at+bt) + \cos(at-bt)]$$

$$= \frac{1}{2}[\cos(a+b)t + \cos(a-b)t]$$

$$L\{\cos at \cos bt\} = \tfrac{1}{2}\Big[L\{\cos(a+b)t\} + L\{\cos(a-b)t\}\Big]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(a+b)^2} + \frac{s}{s^2+(a-b)^2}\right]$$

(6) Find $L\{\sin t \sin 2t \sin 3t\}$

Sol:- $f(t) = \sin t \sin 2t \sin 3t$

$$\sin 2t \sin 3t = \tfrac{1}{2}[\cos(2t-3t) - \cos(2t+3t)]$$

$$= \tfrac{1}{2}[\cos t - \cos 5t]$$

$$\left(\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]\right.$$

$$\sin t \underbrace{\sin 2t \sin 3t} = \sin t \cdot \tfrac{1}{2}(\cos t - \cos 5t)$$

$$\sin t \sin 2t \sin 3t = \tfrac{1}{2}[\sin t \cos t - \sin t \cos 5t]$$

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$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin t \sin 2t \sin 3t = \frac{1}{2}\left[\sin 2t - \sin(t-t)\right] + \frac{1}{2}\left(\sin(t+5t) - \sin(t-5t)\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[\sin 2t - 0 - \sin 6t + \sin 4t\right]$$

$$= \frac{1}{4}\left[\sin 2t - \sin 6t + \sin 4t\right]$$

$$L\{\sin t \sin 2t \sin 3t\} = \frac{1}{4}\left[L\{\sin 2t\} - L\{\sin 6t\} + L\{\sin 4t\}\right]$$

$$= \frac{1}{4}\left[\frac{2}{s^2+2^2} - \frac{6}{s^2+6^2} + \frac{4}{s^2+4^2}\right]$$

$$= \frac{1}{4}\left[\frac{2}{s^2+4} - \frac{6}{s^2+36} + \frac{4}{s^2+16}\right]$$

Properties of Laplace Transforms

If $L\{f(t)\} = F(s)$, then $L\{e^{at}f(t)\} = F(s-a)$

(Shifting Property)

Proof:- By definition

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{--- (1)}$$

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-st+at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt \quad \text{--- (2)}$$

From (1) & (2)

$$L\{e^{at}f(t)\} = F(s-a)$$

ly $L\{e^{-at}f(t)\} = F(s+a)$

This is called Shifting Property

Ex :- $L\{e^{2t}\sin 2t\}$

$a = 2$ $\quad L\{\sin 2t\} = \dfrac{2}{s^2+2^2} = F(s)$

$s \to s-2$

$$L\{e^{2t}\sin 2t\} = \frac{2}{(s-2)^2+4}$$

(2) If $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s)) \checkmark$$

Proof :- $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \longrightarrow (1)$$

Differentiating (1) w.r.t 's' on both sides

$$\frac{d}{ds}(F(s)) = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)\, dt$$

(By Leibnitz rule of differentiation under integral sign)

$$\frac{d}{ds}(F(s)) = \int_0^\infty e^{-st}\cdot(-t)\, f(t)\, dt$$

$$\frac{d}{ds}(F(s)) = -\int_0^\infty e^{-st}\, t \cdot f(t)\, dt$$

$$(-1)\frac{d}{ds}(F(s)) = \int_0^\infty e^{-st}\, t \cdot f(t)\, dt$$

$$-\frac{d}{ds}(F(s)) = L\{\underline{t \cdot f(t)}\}$$

This Verifies the result is true for $n=1$

Let us assume that the result is true for $n=k$

$$(-1)^k \frac{d^k}{ds^k}(F(s)) = L\{t^k f(t)\}$$

$$(-1)^k \frac{d^k}{ds^k}(F(s)) = \int_0^\infty e^{-st}\, t^k \cdot f(t)\, dt$$

Again differentiating w.r.t 's' on both sides

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}(F(s)) = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})\, t^k f(t)\, dt$$

$$= \int_0^\infty e^{-st}(-t)\, t^k f(t)\, dt$$

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}(F(s)) = \underline{-} \int_0^\infty e^{-st}\, t^{k+1} f(t)\, dt$$

$$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}} (F(s)) = \int_0^{\infty} e^{-st} t^{k+1} f(t)\, dt$$

$$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}} (F(s)) = L\{ t^{k+1} f(t) \}$$

This is true for $n = k+1$

$\therefore$ By the principle of Mathematical induction the statement is true for all positive integers n.

i.e $$L\{ t^{n} f(t) \} = (-1)^{n} \frac{d^{n}}{ds^{n}} (F(s))$$