MODULE : 5

INTEGRAL CALCULUS

1. $I = \int_0^2 \int_1^2 (x^2+y^2)\, dx\, dy$

integrate wrt x

$$= \int_0^2 \left[\frac{x^3}{3} + xy^2 \right]_1^2 dy$$

$$= \int_0^2 \left[\left(\frac{8}{3} + 2y^2\right) - \left(\frac{1}{3} + y^2\right) \right] dy$$

$$= \int_0^2 \left[\frac{7}{3} + y^2 \right] dy$$

Integrate wrt y

$$I = \left[\frac{7}{3} y + \frac{y^3}{3} \right]_0^2$$

$$I = \left[\left(\frac{14}{3} + \frac{8}{3} \right) - 0 \right]$$

$$I = \frac{22}{3}$$

$$\underline{I = 22/3}$$

2. $\int_0^1\int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$ $\qquad \int \frac{1}{\sqrt{1-x^2}} = \sin^{-1}$

Integrate w.r.t x.

$\int_0^1 \left[\sin^{-1}x\right]_0^1 \frac{dy}{\sqrt{1-y^2}}$

$\int_0^1 \left[\sin^{-1}1 - \sin^{-1}0\right] \frac{dy}{\sqrt{1-y^2}}$

$\int_0^1 \frac{\pi}{2} \frac{dy}{\sqrt{1-y^2}}$

$\frac{\pi}{2}\int_0^1 \frac{dy}{\sqrt{1-y^2}}$

$= \frac{\pi}{2}\left[\sin^{-1}y\right]_0^1$

$= \frac{\pi}{2}\left[\frac{\pi}{2} - 0\right]$

$I = \frac{\pi^2}{4}$

3. $\int_0^{\pi/4}\int_0^{\pi/2} \sin(x+y)\, dx\, dy$

integrate w.r.t x. $\qquad \cos(\pi/2+\theta) = -\sin\theta$

$I = \int_0^{\pi/4}\left[-\cos(x+y)\right]_0^{\pi/2} dy.$

$I = \int_0^{\pi/4}\left[-\cos\left[\frac{\pi}{2}+y\right] + \cos\left[0+y\right]\right] dy.$

$$I = -\int_0^{\pi/4} -\sin y - \cos y \, dy$$

$$I = -\left[\cos y - \sin y\right]_0^{\pi/4}$$

$$I = -\left[(\cos \pi/4 - \sin \pi/4) - (\cos 0 - \sin 0)\right]$$

$$I = -\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - 1 + 0\right]$$

$$\underline{\underline{I = 1}}$$

4. $$\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$$

~~diff~~ Integrate w.r.t x.

$$= \int_0^1 \left[\frac{x^4}{4} y\right]_0^{\sqrt{1-y^2}} dy$$

$$\left(\left(\sqrt{1-y^2}\right)^2\right)^2$$

$$(1-y^2)^2.$$

$$1 + y^4 - 2y^2$$

$$= \int_0^1 \left[\frac{(\sqrt{1-y^2})^4 y}{4} - 0\right] dy$$

$$= \int_0^1 \frac{(1 + y^4 - 2y^2) y}{4}$$

$$= \frac{1}{4}\left[y + \frac{y^6}{6} - \frac{2y^4}{4}\right]_0^1$$

$$= \frac{1}{4}\left[\left(1 + \frac{1}{6} - \frac{2}{4}\right) - 0\right].$$

$$\underline{I = \frac{1}{24}}$$

always ↑ always y ... Integrate ...

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3 $\int_1^2\int_3^4 (xy+e^y)\,dy\,dx$

Integrate wrt x

$I = \int_1^2 \left[\frac{x^2}{2}y + xe^y\right]_3^4 dy$

$I = \int_1^2 \left[\left(\frac{16}{2}y + 4e^y\right) - \left(\frac{9}{2}y + 3e^y\right)\right] dy$

$I = \int_1^2 \left[\frac{7}{2}y + 1e^y\right] dy$

$I = \left[\frac{7}{2}\cdot\frac{y^2}{2} + 1e^y\right]_1^2$

$I = \left[\left(\frac{7}{2}\cdot\frac{\cancel{4}^2}{\cancel{2}} + 1e^2\right) - \left(\frac{7}{2}\cdot\frac{1}{2} + 1e^1\right)\right]$

$I = \left(7 + e^2 - \frac{7}{4} - e^1\right)$

$I = \frac{21}{4} + e^4 - e^3$

(4.) $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

Integrate wrt y

$\sqrt{x}\cdot\sqrt{x}\cdot\sqrt{x}$

$x\sqrt{x}$

$I = \int_0^1 \left[\frac{y^3}{3} + x^2 y\right]_x^{\sqrt{x}}$

$I = \int_0^1 \left(\frac{(\sqrt{x})^3}{3} - \frac{x^3}{3}\right) + \left((\sqrt{x})^2 - x^3\right)$

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$$I = \int_0^1 \frac{x\sqrt{x}}{3} - \frac{x^3}{3} \Rightarrow \int_0^1 \frac{(\sqrt{x})^{3/2}}{3} - \frac{x^3}{3}$$

$x^{1+1/2} = x^{3/2}$

$\frac{3}{2}+1 = \frac{3+2}{2}$

$$I = \left[\frac{1}{3}\frac{x^{5/2}}{5/2} - \frac{x^4}{12}\right]_0^1$$

$$I = \left[\left(\frac{2}{3}\frac{1}{5} - \frac{1}{12}\right) - 0\right)\right]$$

$$I = \left[\frac{2}{15} - \frac{1}{12}\right]$$

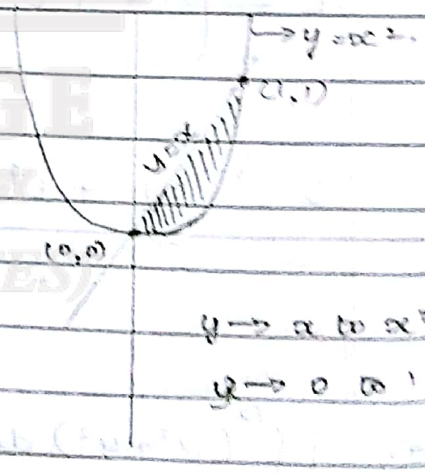
$$I = \frac{3}{35}$$

5. $\iint_R xy(x+y)\,dx\,dy$ over the region R bounded b/w the parabola $y=x^2$ and line $y=x$.

The point of intersection of these two curves are (0, 0) and (1, 1).

$\therefore$ x varies from 0 to 1

$\therefore$ y varies from x^2 to x.



$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x^2y + xy^2)\,dx\,dy.$$

$y \to x$ to x^2

$y \to 0$ to 1

$$= \lim \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\,dy\,dx.$$

$y=x$; $y=x^2$

$y=0$

$x=0$.

Integrate w r t y.

$$\int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

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$$I = \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3} \right) - \left(\frac{x^6}{2} + \frac{x^7}{3} \right) dx \right.$$

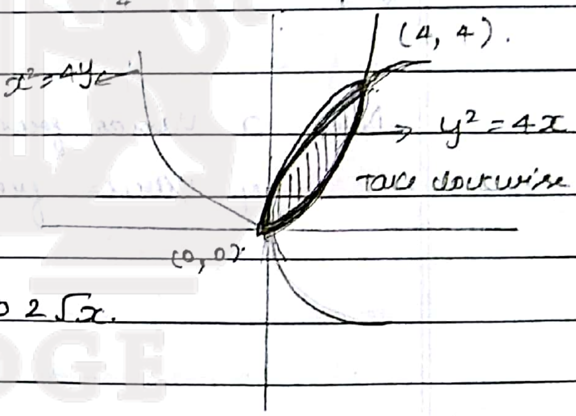
$$I = \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1$$

$$I = \left[\frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24} \right] - 0$$

$$\underline{I = \frac{3}{56}}$$

6. Evaluate $\iint_R y\, dx\, dy$ where R is a region bounded by the parabola $y^2 = 4x$ and $x^2 = 4y$

$y = 2\sqrt{x}$ $\quad \frac{x^2}{4} = y$



when $x = 4 \Rightarrow y = 4$.

The point of intersection of the curves are $(0,0)$ $(4,4)$.

$\therefore$ x varies from 0 to 4

y varies from ~~$2\sqrt{x}$ to~~ $x^2/4$ to $2\sqrt{x}$.

$$= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\, dy\, dx$$

$x^2 = 4y$

$y = x^2/4$

$y = \frac{16}{4}$

$\boxed{y = 4}$

$y^2 = 4x$

$\frac{x^4}{16} = 4x$

$\frac{x^3}{64} = 1$

$x^3 = 64$

$\boxed{x = 4}$

Integrate w.r.t y

$$= \int_0^4 \left[\frac{y^2}{2} \right]_{x^2/4}^{2\sqrt{x}}$$

$y^2 = 4x$ $\quad \boxed{y = 2\sqrt{x}}$

$x^2 = 4y$ $\quad \boxed{y = x^2/4}$

$$= \int_0^4 \left[\frac{(2\sqrt{x})^2}{2} - \frac{(x^2/4)^2}{2} \right]$$

$$= \int_0^4 \left[\frac{4x}{2} - \frac{x^4}{32} \right]$$

$$I = \left[\frac{4x^2}{4} - \frac{x^5}{5 \times 32} \right]_0^4$$

$$I = \left[16 - \frac{1024}{160} \right]$$

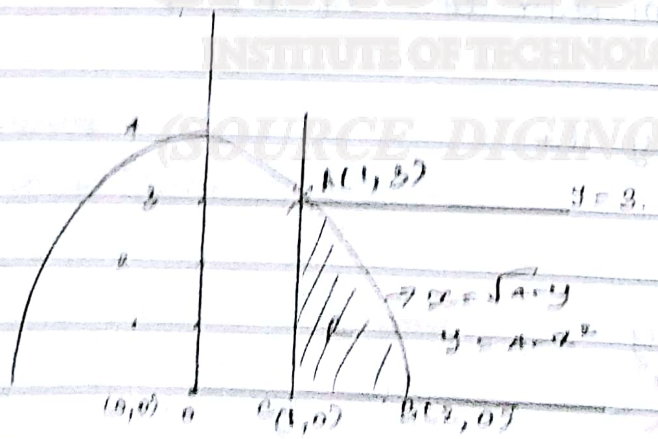
$$I = \frac{48}{5}$$

* Change of order of integration.

1. $\displaystyle\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$.

Now x varies from 1 to $\sqrt{4-y}$

y varies from 0 to 3.



$x = 1$ & $x = \sqrt{4-y}$

$x^2 = 4 - y$

$x = 2$.

∴ x varies from 1 to 2

y varies from 0 to $4 - x^2$

$$I = \int_{x=1}^{2}\int_{0}^{4-x^2} (x+y)\, dy\, dx + \int_0^1\int_0^3 (x+y)\,dy\,dx \;\text{(w.r.t } y)$$

$$I = \int_{x=1}^{2} \left[xy + \frac{y^2}{2}\right]_0^{4-x^2}$$

$2.4.x^2$

$$I = \int_{x=1}^{2} x(4-x^2) + \frac{(4-x^2)^2}{2}\, dx.$$

$$I = \int_{x=1}^{2} 4x - x^3 + \frac{(16+x^4-8x^2)}{2}.$$

$$I = \int_1^2 4x - x^3 + \frac{1}{2}\left[16 + x^4 - 8x^2\right].$$

$$I = \left[\frac{4x^2}{2} - \frac{x^4}{4} + \frac{1}{2}\left(16x + \frac{x^5}{5} - \frac{8x^3}{3}\right)\right]_1^2$$

$$I = \left[2x^2 - \frac{x^2}{4} + 8x + \frac{x^5}{10} - \frac{4x^3}{3}\right]_1^2.$$

$$I = \left[\left(8 - \frac{4}{4} + 16 + \frac{32}{10} - \frac{32}{3}\right) - \left(2 - \frac{1}{4} + 8 + \frac{1}{10} - \frac{4}{3}\right)\right]$$

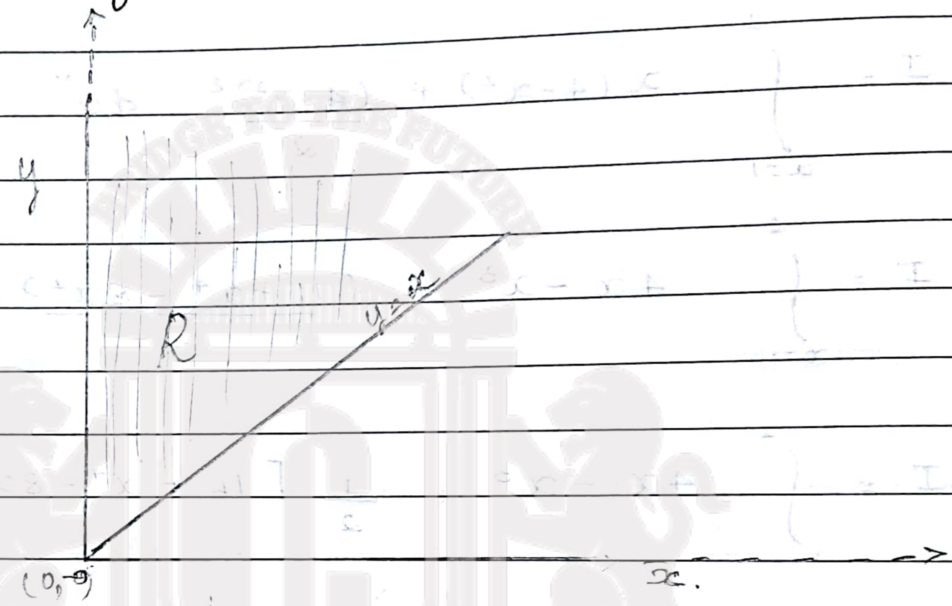
$$I = \left(\frac{233}{15} - \frac{511}{60}\right)$$

$$I = \frac{241}{60}.$$

2. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx.$

Now x varies from 0 to ∞ (const)

y varies from x to ∞ (Variable)



$\therefore$ y varies from 0 to ∞.

x varies from 0 to y.

$$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy \quad \text{(w.r.t } x\text{)}.$$

$$I = \int_0^\infty \left[\frac{e^{-y}}{y} x \, dy \right]_0^y$$

$$I = \int_0^\infty \frac{e^{-y}}{y} \cdot y \, dy.$$

$$I = \int_0^\infty e^{-y} \, dy.$$

(w.r.t y)

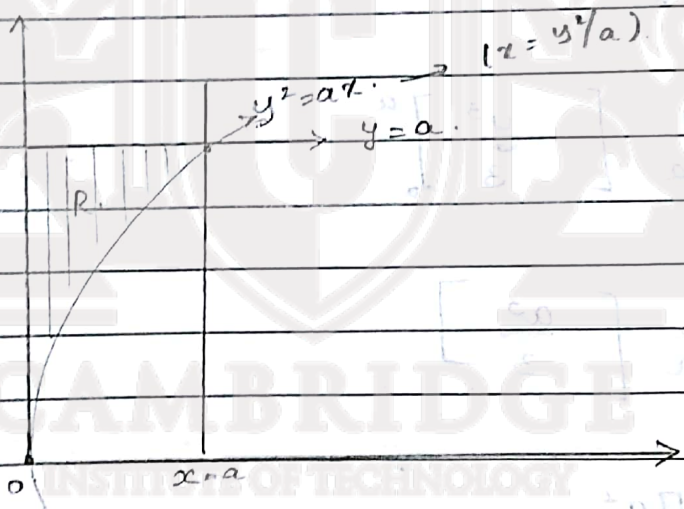
$$I = \left[\frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1} \right]$$

$$\underline{\underline{I = 1}}$$

3 $$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 \, dy \, dx}{\sqrt{y^4 - a^2x^2}}$$

(cons) Now x varies from 0 to a

(var) y varies from $\sqrt{ax}$ to a. or. $y \rightarrow y^2 = ax$ to a.



Y varies from 0 to a

x varies from 0 to y^2/a.

$$I = \int_0^a \int_0^{y^2/a} \frac{y^2}{\sqrt{y^4 - a^2x^2}} \, dx \, dy.$$ (w.r.t x).

Taking y^4 out

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{\sin^{-1}x}{1}.$$

$$I = \int_0^a \frac{y^2}{y^2} \int_0^{y^2/a} \frac{dx}{\sqrt{1 - (ax/y^2)^2}}$$

$$I = \int_0^a \left[\frac{\sin^{-1}(ax/y^2)}{a/y^2}\right]_0^{y^2/a} dy$$

$$I = \int_0^a \left[\frac{\sin^{-1}\left(\frac{a}{y^2}\cdot\frac{y^2}{a}\right)\cdot y^2}{a} - \frac{\sin^{-1}\left(\frac{a}{y^2}\cdot 0\right)\cdot y^2}{a}\right]$$

$$I = \frac{1}{a}\int_0^a y^2\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy$$

$$I = \frac{1}{a}\int_0^a y^2 \; \pi/2 \; dy$$

$$\int_0^\infty \int_0^x x e^{-x^2/y} \, dy \, dx$$

$$I = \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a$$

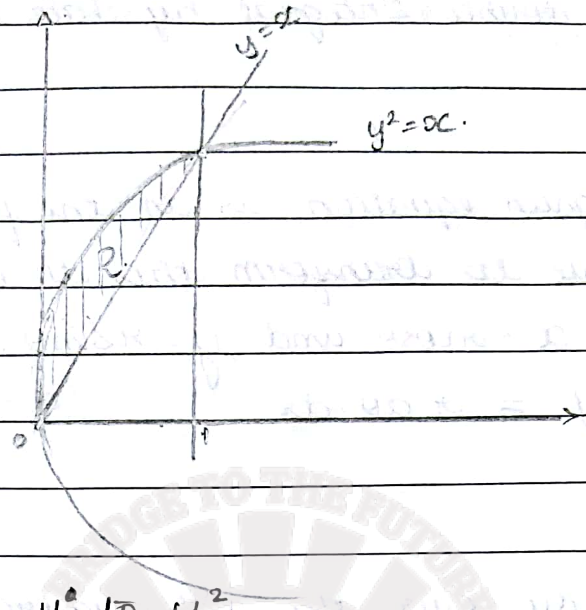
$$I = \frac{\pi}{2a}\left[\frac{a^3}{3}\right]$$

$$I = \frac{\pi a^2}{6}$$

4. $$\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx.$$

Now; x varies from 0 to 1

y varies from $y = x$ to $y = \sqrt{x}$; $y^2 = x$.



$x \to y^2$ to y

$y \to 0$ to 1

x varies from y to y^2

y varies from 0 to 1.

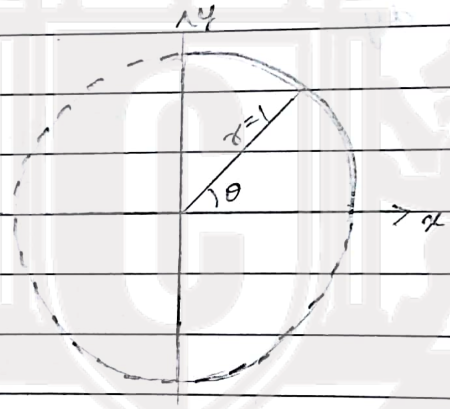
$$I = \int_{y}^{y^2}\int_{0}^{1} xy \, dx \, dy.$$

Evaluation of double Integral by chain of variable into polar coordinates.

Suppose the given equation is in the form (x, y) i.e cartesian form we have to transform this to a polar form by taking substitution $x = r\cos\theta$ and $y = r\sin\theta$. $[\because x^2+y^2=r^2]$

$\therefore \quad dx \cdot dy = r\, d\theta \cdot dr$

1. $\iint xy\, dx \cdot dy$ over the +ve quadrant bounded by the circle $x^2+y^2=1$.



θ varies from 0 to $\pi/2$.
r varies from 0 to 1

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r\cos\theta)(r\sin\theta) \cdot r\, dr\, d\theta$$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^3 \cos\theta\sin\theta\, dr\, d\theta \qquad (w.r.t\ r).$$

$$I = \int_{\theta=0}^{\pi/2} \left(\frac{r^4}{4}\cos\theta\sin\theta\right)_0^1 d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \left[\frac{1}{4}\cos\theta\sin\theta\right].$$

$$I = \frac{1}{4}\int_{\theta=0}^{\pi/2} \frac{2\cos\theta\sin\theta\, d\theta}{2}$$ multiply & ÷ by 2.

$$I = \frac{1}{8}\int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$I = \frac{1}{8}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

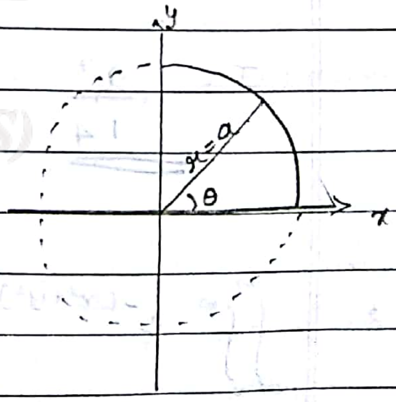
$$I = \frac{1}{8\times 2}\left[-\cos 2(\pi/2) + \cos 2(0)\right]$$

$$I = \frac{1}{16}\left[1 + 1\right] = \frac{2}{16}$$

$$I = \frac{1}{8}$$

2. $\iint xy(x^2+y^2)^{3/2}\, dx\, dy$ over the +ve quadrant of the circle $x^2+y^2=a^2$ by transforming to polar coordinates.

θ varies from 0 to $\pi/2$
r varies from 0 to a.



$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} xy(x^2+y^2)^{3/2}\, dx\, dy$$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\cos\theta \cdot r\sin\theta\,(r^2)^{3/2}\, r\, dr\, d\theta$$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^6 \cos\theta\sin\theta\, dr\, d\theta$$ (w.r.t r)

$$I = \int_{\theta=0}^{\pi/2} \left[\frac{r^7}{7} \cos\theta \sin\theta \right]_0^a d\theta$$

$$\frac{\partial^3 z}{\partial x^2 \partial y} \quad \frac{\partial^2 z}{\partial x \partial y} \quad \frac{\partial z}{\partial y}$$

$$I = \int_{\theta=0}^{\pi/2} \left[\frac{a^7}{7} \cos\theta \sin\theta \right] d\theta$$

$$I = \frac{a^7}{7} \int_{\theta=0}^{\pi/2} \left[\frac{2\cos\theta \sin\theta}{2} \right] d\theta$$

$$I = \frac{a^7}{14} \int_{\theta=0}^{\pi/2} \sin 2\theta \, d\theta$$

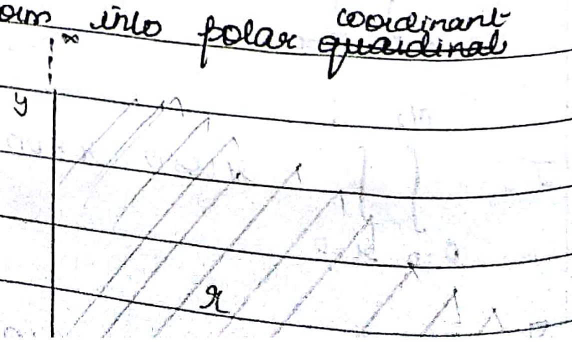
$$I = \frac{a^7}{14} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/2}$$

$$I = \frac{a^7}{14 \times 2} \left[-\cos 2(\pi/2) + \cos 2(0) \right]$$

$$I = \frac{a^7}{28} \left[+1 + 1 \right] = \frac{a^7 \cdot 2}{28}$$

$$\underline{\underline{I = \frac{a^7}{14}}}$$

3. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$. transform into polar ~~quardinal~~ coordinant



Here r varies from 0 to ∞

θ varies from 0 to $\pi/2$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta \quad (\text{w.r.t } r)$$

Substitute $r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

$t = 0 \to \infty$

$$I = \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta$$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} d\theta$$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[-e^{-\infty} + e^{-0}\right] d\theta$$

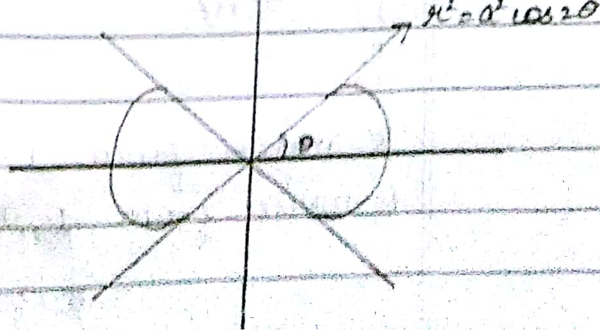
$$I = \frac{1}{2}\int_{\theta=0}^{\pi/2} 1\,d\theta$$

$$I = \frac{1}{2}\left[\theta\right]_0^{\pi/2}$$

$$I = \frac{\pi}{4}$$

4. $\iint_R \frac{dx\,dy}{\sqrt{x^2+y^2+a^2}}$ where R is a region bounded by leminscate $R^2 = a^2\cos 2\theta$

Here, In the 1st quadrant:
r varies from 0 to $a\sqrt{\cos 2\theta}$
θ varies from 0 to $\pi/4$.



$$I = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}.$$

$$I = 4\int_{\theta=0}^{\pi/4} \left[\sqrt{r^2+a^2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$\int \frac{r\,dr}{\sqrt{r^2+a^2}}$ $\quad -\frac{1}{2}+1 \quad \frac{-1+2}{2} = \frac{1}{2}$

let $r^2+a^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

$$I = 4\int_{\theta=0}^{\pi/4} \left[\sqrt{a^2\cos 2\theta + a^2} - a\right] d\theta$$

$\int \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int t^{-1/2} dt$

$\frac{1}{2}\left[\frac{t^{1/2}}{1/2}\right] = t^{1/2}$

$= \sqrt{r^2+a^2}$

$$I = 4a\int_{\theta=0}^{\pi/4} \sqrt{\cos 2\theta + 1} - 1\, d\theta$$

$$I = 4a\int_{\theta=0}^{\pi/4} \sqrt{2\cos^2\theta} - 1\, d\theta$$

$1+\cos 2\theta = 2\cos^2\theta$

$$I = 4a\int_{\theta=0}^{\pi/4} \sqrt{2}\cos\theta - 1\, d\theta$$

$$I = 4\sqrt{2}\,a\left[+\sin\theta - \theta\right]_0^{\pi/4}$$

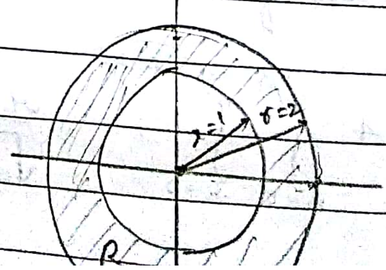
$$I = 4\sqrt{2}\,a\left[\underset{\frac{1}{\sqrt{2}}}{+\sin\frac{\pi}{4}} + \underset{0}{\sin 0}\right] - \left[0 - \frac{\pi}{4}\right]$$

$$\underline{I = 4a\left[1 - \frac{\pi}{4}\right]}$$

5. $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ where R is a all the region angular region bounded by circle $x^2+y^2=1$ and $x^2+y^2=4$.

R varies from 1 to 2

θ varies from 0 to 2π.



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$$I = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} \frac{(r\cos\theta)^2(r\sin\theta)^2}{r^2}\, r.dr\, d\theta$$

$$I = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} \frac{r^5\cos^2\theta\sin^2\theta}{r^2}\, dr\, d\theta$$

$$I = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} r^3\cos^2\theta\sin^2\theta\, dr\, d\theta$$

$$I = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} r^3(\cos\theta\sin\theta)^2\, dr\, d\theta. \quad \text{w.r.t } r.$$

$$I = \int_{\theta=0}^{2\pi} \left[\frac{r^4}{4}\right]_1^2 (\cos\theta\sin\theta)^2\, d\theta.$$

$$I = \int_{\theta=0}^{2\pi} \left[\frac{16}{4} - \frac{1}{4}\right](\cos\theta\sin\theta)^2\, d\theta$$

$$I = \int_{\theta=0}^{2\pi} \frac{15}{4}(\cos\theta\sin\theta)^2\, d\theta.$$

$$I = \frac{15}{4}\int_{\theta=0}^{2\pi} \left(\frac{2\cos\theta\sin\theta}{2}\right)^2 d\theta.$$

$$I = \frac{15}{4.4}\int_{\theta=0}^{2\pi} (\sin 2\theta)^2\, d\theta.$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$$

$$I = \frac{15}{16}\int_{\theta=0}^{2\pi} \sin^2 2\theta\, d\theta.$$

$$I = \frac{15}{16}\int_{\theta=0}^{2\pi} \frac{1-\cos 4\theta}{2}$$

$\sin 4(2\pi)$

$$I = \frac{15}{16\times 2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$I = \frac{15}{32}\left[2\pi - \frac{\sin 4(2\pi)}{4}\right] - \left[0 - 0\right]$$

$$I = \frac{15 \cdot 2\pi}{32}$$

$$I = \frac{15\pi}{16}$$

Evaluation of trip integration.

1. $\int_0^1\int_0^2\int_1^2 xyz^2 \, dx\, dy\, dz$ (wrt x).

$$I = \int_0^1\int_0^2 \left[\frac{x^2}{2} y z^2\right]_1^2 dy\, dz.$$

$\frac{2^2}{1} - \frac{1}{2}$

$\frac{4-1}{2} = \frac{3}{2}$

$$I = \int_0^1\int_0^2 \left[\frac{4}{2} y z^2 - \frac{1}{2} y z^2\right] dy\, dz$$

$$I = \int_0^1\int_0^2 \left[2yz^2 - \tfrac{1}{2} y z^2\right] dy\, dz.$$

$$I = \frac{3}{2}\int_0^1\int_0^2 y z^2 \, dy\, dz$$

$$I = \frac{3}{2}\int_0^1 \left[\frac{y^2}{2} z^2\right]_0^2 dz$$

$$I = \frac{3}{2}\int_0^1 \left[\frac{4}{2} z^2 - 0\right] dz.$$

$$I = \frac{6}{2}\int_0^1 z^2 \, dz$$

$$I = 3\left[\frac{z^3}{3}\right]_0^1$$

$$I = 3\left[\frac{1}{3} - 0\right]$$

$$\underline{\underline{I = 1}}.$$

(w.r.t x).

2. $\int_0^a\int_0^a\int_0^a x^2+y^2+z^2 \; dx\, dy\, dz$

$$I = \int_0^a\int_0^a \left[\frac{x^3}{3} + xy^2 + xz^2\right]_0^a$$

$$I = \int_0^a\int_0^a \left[\frac{a^3}{3} + ay^2 + az^2\right]$$

$$I = \int_0^a \left[y\frac{a^3}{3} + \frac{ay^3}{3} + ayz^2\right]_0^a$$

$$I = \int_0^a \left[\frac{a^4}{3} + \frac{a^4}{3} + a^2z^2\right]$$

$$I = \frac{za^4}{3} + \frac{za^4}{3} + \frac{a^2z^3}{3}\Big|_0^a$$

$$I = \frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3}$$

$$I = \frac{3a^5}{3}$$

$$\underline{\underline{I = a^5}}.$$

3. $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} x+y+z \; dy\, dx\, dz$.

(w.r.t y).

$$I = \int_{-1}^{1}\int_0^{z} \left[xy + \frac{y^2}{2} + yz\right]_{(x-z)}^{(x+z)}$$

$$I = \int_{-1}^{1}\int_{0}^{3} \left(x(x+z) + \frac{(x+z)^2}{2} + (x+z)(z)\right) - \left(x(x-z) + \frac{(x-z)^2}{2} + (x-z)z\right)$$

$$I = \int_{-1}^{1}\int_{0}^{3} \left[\left(x^2 + xz + \frac{x^2+z^2+2xz}{2} + xz + z^2\right) - \left(x^2 - xz + \frac{x^2+z^2-2xz}{2} + xz - z^2\right)\right]$$

$$I = \int_{-1}^{1}\int_{0}^{2} \cancel{2x^2} + \cancel{2xz} + \cancel{x^2} + \cancel{z^2} + 2xz + 2xz + 2z^2 - \cancel{2x^2} + 2xz - \cancel{x^2} + \cancel{z^2} + 2xz$$
$$- \cancel{2xz} + 2z^2$$

$$I = \int_{-1}^{1}\int_{0}^{2} 8xz + 4z^2 \, dx\, dz.$$

$$I = 4\int_{-1}^{1}\int_{0}^{2} 2xz + z^2 \, dx\, dz.$$

$$I = 4\int_{-1}^{1} \left[\cancel{2}\frac{x^2}{\cancel{2}} z + xz^2\right]_0^2 dz$$

$$I = 4\int_{-1}^{1} \left[4z + 2z^2\right] dz$$

$$I = 4\left[\frac{4z^2}{2} + \frac{2z^3}{3}\right]_{-1}^{1}$$

$$I = 4\left[\frac{4}{2} + \frac{2}{3}\right] - \left[\frac{4}{2} - \frac{2}{3}\right]$$

$$I = 4\left[\frac{4}{2} + \frac{2}{3} - \frac{4}{2} + \frac{2}{3}\right]$$

$$I = 4\left(\frac{4}{2} - \frac{2}{3} - \frac{4}{2} + \frac{2}{3}\right)$$

4. $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$ w.r.t y.

$I = \int_0^4\int_0^{2\sqrt{z}} \left[y \right]_0^{\sqrt{4z-x^2}} dx\, dz$

$I = \int_0^4\int_0^{2\sqrt{z}} \sqrt{4z-x^2}\; dx\, dz.$ $a = 2\sqrt{z}$. $(2\sqrt{z})^2 - (x)^2$

$\boxed{\text{FORMULA :-} \int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \sin^{-1}\left(\frac{x}{a}\right)\frac{a^2}{2}.}$

$I = \int_0^4 \left[\frac{x}{2}\sqrt{4z-x^2} + \frac{4z}{2}\sin^{-1}\left(\frac{x}{2\sqrt{z}}\right)\right]_0^{2\sqrt{z}}.$

$I = \int_0^4 \left[\frac{2\sqrt{z}}{2}\sqrt{4z-4z} + \frac{4z}{2}\sin\left(\frac{\cancel{2}\sqrt{z}}{\cancel{2}\sqrt{z}}\right)\right] dz$

$I = \int_0^4 \left(2z\sin^{-1}\left(\frac{\sqrt{z}}{\sqrt{z}}\right) - 0\right) dz.$

$I = \int_0^4 2z\sin^{-1}(1).\, dz$

$I = \int_0^4 \cancel{2}z.\frac{\pi}{\cancel{2}}\; dz$

$I = \pi\left[\frac{z^2}{2}\right]_0^4$

$I = 8\pi$

5 $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \; dz\, dy\, dx$ (w.r.t z)

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[xy \frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}}$$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{xy}{2} (a^2-x^2-y^2) \right]$$

$$I = \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xya^2 - x^3y - xy^3 \quad dy\, dz.$$

$$I = \frac{1}{2} \int_0^a \left[\frac{xa^2y^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right]_0^{\sqrt{a^2-x^2}}$$

$$I = \frac{1}{2} \int_0^a \left[\frac{xa^2(a^2-x^2)}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4} \right] dx.$$

$$I = \frac{1}{4} \int_0^a \left[xa^4 - x^3a^2 - x^3a^2 + x^5 - \frac{x(a^4+x^4-2a^2x^2)}{2} \right] dx$$

$$I = \frac{1}{4} \int_0^a \left[xa^4 - 2x^3a^2 + x^5 - x\left(\frac{a^4}{2} + \frac{x^4}{2} - a^2x^2 \right) \right] dx$$

$$I = \frac{1}{4} \int_0^a \left[xa^4 - 2x^3a^2 + x^5 - \frac{xa^4}{2} - \frac{x^5}{2} + a^2x^3 \right] dx$$

$$I = \frac{1}{4} \int_0^a \frac{xa^4}{2} - x^3a^2 + \frac{x^5}{2} \quad dx$$

$$I = \frac{1}{4} \left[\frac{x^2}{2.2} a^4 - \frac{x^4}{4} a^2 + \frac{x^6}{6.2} \right]_0^a$$

$$I = \frac{1}{4}\left[\frac{a^2}{2\cdot 2}\cdot a^4 - \frac{a^4 a^2}{4\cdot} + \frac{a^6}{12}\right]$$

$$\int_{0}^{1}\int_{y^2}^{1}\int_{0}^{1-x} x \, dz \, dx \, dy$$

$$I = \frac{1}{4}\left[\frac{a^6}{4} - \frac{a^6}{4} + \frac{a^6}{12}\right]$$

$$I = \frac{a^6}{48}$$

6. $$\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2+y^2+z^2)\, dx\, dy\, dz.$$ w.r.t x.

$$I = \int_{-c}^{c}\int_{-b}^{b}\left[\frac{x^3}{3} + xy^2 + xz^2\right]_{-a}^{a} dy\, dz$$

$$I = \int_{-c}^{c}\int_{-b}^{b}\left[\frac{a^3}{3} + ay^2 + az^2\right] - \left[\frac{-a^3}{3} - ay^2 - az^2\right] dy\, dz$$

$$I = \int_{-c}^{c}\int_{-b}^{b}\left[\frac{a^3}{3} + ay^2 + az^2 + \frac{a^3}{3} + ay^2 + az^2\right] dy\, dz$$

$$I = \int_{-c}^{c}\int_{-b}^{b}\left[\frac{2a^3}{3} + 2ay^2 + 2az^2\right] dy\, dz.$$

$$I = \int_{-c}^{c}\left[\frac{2a^3}{3}y + \frac{2ay^3}{3} + 2ayz^2\right]_{-b}^{b}.$$

$$I = \int_{-c}^{c}\left[\frac{2a^3b}{3} + \frac{2ab^3}{3} + 2abz^2\right] - \left[-\frac{2a^3b}{3} - \frac{2ab^3}{3} - 2abz^2\right] dz$$

$$I = \int_{-c}^{c}\left[\frac{2a^3b}{3} + \frac{2ab^3}{3} + 2abz^2 + \frac{2a^3b}{3} + \frac{2ab^3}{3} + 2abz^2\right. dz$$

$$I = \left[\frac{2a^3bz}{3} + \frac{2ab^3z}{3} + \frac{2abz^3}{3} + \frac{2a^3bz}{3} + \frac{2ab^3z}{3} + \frac{2abz^3}{3} \right]_{-c}^{c}$$

$$I = \left[\frac{4}{3}a^3bz + \frac{4ab^3z}{3} + \frac{4abz^3}{3} \right]_{-c}^{c}$$

$$I = \left[\left(\frac{4}{3}a^3bc + \frac{4ab^3c}{3} + \frac{4abc^3}{3} \right) - \left(-\frac{4}{3}a^3bc - \frac{4ab^3c}{3} - \frac{4abc^3}{3} \right) \right]$$

$$I = \frac{4}{3}a^3bc + \frac{4ab^3c}{3} + \frac{4abc^3}{3} + \frac{4}{3}a^3bc + \frac{4ab^3c}{3} + \frac{4abc^3}{3}$$

$$I = \frac{8a^3bc}{3} + \frac{8ab^3c}{3} + \frac{8abc^3}{3}$$

$$I = \frac{8}{3}\left[a^3bc + ab^3c + abc^3 \right]$$

7. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy$ (w.r.t z).

$$I = \int_0^1 \int_{y^2}^1 \left[xz \right]_0^{1-x} dx \, dy$$

$$I = \int_0^1 \int_{y^2}^1 x(1-x) \, dx \, dy$$

$$I = \int_0^1 \int_{y^2}^1 x - x^2 \, dx \, dy$$

$$I = \int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{3} \right)_{y^2}^1 dy$$

$$I = \int_0^1 \left[\frac{1}{2} - \frac{1}{3}\right] - \left[\frac{y^4}{2} - \frac{y^6}{3}\right] dy.$$

$$I = \int_0^1 \frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3} \; dy.$$

$$I = \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1$$

$$I = \left[\frac{1}{6} - \frac{1}{10} + \frac{1}{21}\right]$$

$$I = \frac{\cancel{4}}{105} \quad \frac{4}{35}.$$

Gama and Beta function

Gama function is denoted by $\Gamma(n)$ and is defined as –

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \quad \text{or} \quad \Gamma(n) = 2\int_0^\infty e^{-t^2} t^{2n-1} dt$$

Beta function is denoted by $\beta(m,n)$ where $(m,n) > 0$,

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \text{Or}$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

Properties of Γ and β function

1. $\beta(m,n) = \beta(n,m)$

w.k.T

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

let $x = 1-y$

$dx = -dy$

If $x = 0$; $y = 1$

If $x = 1$; $y = 0$

$$\beta(m,n) = \int_1^{\infty} (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \underline{\underline{\beta(n,m)}}.$$

2. $\Gamma(n+1) = n\Gamma(n)$

WKT

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx$$

$$\Gamma(n+1) = \int_0^{\infty} e^{-x} x^{n+1-1}\,dx \longrightarrow \Gamma(n+1) = \int_0^{\infty} e^{-x}_{2} x^{n}_{1}\,dx$$

$$= x^n(-e^{-x})\Big|_0^{\infty} - \int_0^{\infty} -e^{-x}\, n x^{n-1}\,dx.$$

$$= 0 + n\int_0^{\infty} e^{-x} x^{n-1}$$

3. $\Gamma(n+1) = n!$

By property 2 = $\Gamma(n+1) = n\Gamma(n)$ — i

$\Gamma(n) = (n-1)\Gamma(n-1)$

$\Gamma(n-1) = (n-2)\Gamma(n-2)$

$\Gamma(n-2) = (n-3)\Gamma(n-3)$

$\vdots$

$\Gamma(3) = 2\Gamma(2)$

$\Gamma(2) = 1\Gamma(1)$

eq (i) becomes -

$\Gamma(n+1) = n(n-1)(n-2)(n-3) \cdots 2 \cdot 1 \cdot \Gamma(1)$

$\therefore \Gamma(n+1) = n(n-1)(n-2)(n-3) \cdots 2 \cdot 1 = n!$

[By defn: $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\Gamma(1) = \int_0^\infty e^{-x} dx \rightarrow -e^{-x}\Big|_0^\infty = 1$]

Relation between Γ and β functions.

P.T. $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof :-

W.K.T $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx$

$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy.$

$$\therefore \Gamma(m+n) = 2\int_0^\infty e^{-r^2} r^{2m+2n-1}\, dr$$

Now $$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-x^2} e^{-y^2} x^{2m-1} y^{2n-1}\, dx\, dy.$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy.$$

By changing into polar coordinate by taking $x = r\cos\theta$; $y = r\sin\theta$;

$dx\, dy = r\, dr\, d\theta$; $x^2+y^2 = r^2$

$r = 0 \to \infty$ and $\theta = 0 \to \pi/2$.

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1}\, r\, dr\, d\theta.$$

$$= 4\int_0^\infty\int_0^{\pi/2} \left(e^{-r^2} r^{2m-1} r^{2n-1} r\, dr\right)\left(\cos\theta^{2m-1} \sin\theta^{2n-1}\right) d\theta.$$

$$= 2\int_{r=0}^\infty e^{-r^2} r^{2m+2n-1}\, dr \,.\, 2\int_{\theta=0}^{\pi/2} \sin\theta^{2n-1} \cos^{2m-1}$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n).\ \beta(m,n).$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Q. 2. P.T. $\Gamma(1/2) = \sqrt{\pi}$.

WKT by definition of Γ function.

$$\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2m-1}\, dx.$$

eqnt integrals

Taking $m, n = 1/2$.

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}$$

$$\beta(1/2, 1/2) = [\Gamma(1/2)]^2 \qquad \Gamma(1) = 1. \quad \text{—— i}$$

But $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$.

$$\beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta\, d\theta.$$

$$\beta(1/2)(1/2) = 2\int_0^{\pi/2} 1\, d\theta.$$

$$\beta(1/2)(1/2) = 2[\theta]_0^{\pi/2}$$

$$\beta(1/2)(1/2) = 2[\pi/2]$$

$$\beta(1/2)(1/2) = \pi. \quad \text{—— ii}$$

From eq. (i) & ii

eq (i) becomes $[\Gamma(1/2)]^2 = \pi$

$$\underline{\underline{\Gamma(1/2) = \sqrt{\pi}}}$$

W.K.T $\Gamma(n+1) = n\Gamma(n)$.

$\Gamma(3/2) = \Gamma(1/2+1) = 1/2\,\Gamma(1/2)$

$= 1/2\sqrt{\pi}$

$\Gamma(3/2) \underline{\underline{=}} \sqrt{\pi}/2$.

$\Gamma(5/2) = \Gamma(3/2+1) = 3/2\,\Gamma(3/2)$

$= 3/2 \cdot \sqrt{\pi}/2$

$\Gamma(5/2) \underline{\underline{=}} 3\sqrt{\pi}/4$.

<u>Properties of β, Γ functions</u>

- $\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\beta\,\Gamma(m)\,\Gamma(m+1/2)$.
- $\beta(m, 1/2) = 2^{2m-1}\,\beta(m,n)$
- ✗ $\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}$

Problems

✗ 1. S.T $\underbrace{\int_0^\infty \sqrt{y}\, e^{-y^2} dy}_{I_1} \underbrace{\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy}_{I_2} = \frac{\pi}{2\sqrt{2}}$.

$I_1 = \int_0^\infty e^{-y^2}\sqrt{y}\, dy$.

$I_1 = \int_0^\infty e^{-y^2} y^{1/2}\, dy \longrightarrow i$

Def:- $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$. OR $\Gamma(n) = 2\int_0^\infty e^{-t^2} t^{2n-1} dt$.

By using $\Gamma(n) = 2\int_0^\infty e^{-t^2} t^{2n-1}\, dt$.

or $\frac{\Gamma(n)}{2} =$.

compare (i) and (ii)

$2n-1 = 1/2$

$2n = 3/2$

$n = 3/4$.

$\therefore\ I_1 = \frac{\Gamma(n)}{2}$.

$$\boxed{I_1 = \frac{\Gamma(3/4)}{2}}$$

$I_2 = \int_0^\infty e^{-y^2} y^{-1/2}\, dt$.

By using eq :- $\frac{\Gamma(n)}{2} = \int_0^\infty e^{-t^2} t^{2n-1}\, dt$.

$2n-1 = -1/2$

$2n = -1/2 + 1$

$2n = 1/2$

$n = 1/4$.

$$\therefore\ \boxed{I_2 = \frac{\Gamma(1/4)}{2}.}$$

$I_1 \cdot I_2 = \frac{\Gamma(3/4)}{2} \times \frac{\Gamma(1/4)}{2}$.

$= \frac{\Gamma(3/2)(1/2) \times \Gamma(1/2)\,\Gamma(1/2)}{4}$

$= \frac{\sqrt{\pi}/2 \times \sqrt{\pi} \times \sqrt{\pi} \times \sqrt{\pi}}{4} = \frac{}{8}$.

2. $\int_0^\infty x e^{-x^8} dx \times \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$.

$x = t^{1/8}$

put $x^8 = t$.

$8x^7 dx = dt$.

$x dx = \frac{dt}{8x^6}$

$x dx = \frac{dt}{8(t^{1/8})^6}$

$x dx = \frac{1}{8} \frac{dt}{t^{3/4}}$

$\boxed{x dx = \frac{1}{8} t^{-3/4} dt}$

→ $I_1 = \int_0^\infty x e^{-x^8} dx$.

$I_1 = \frac{1}{8}\int_0^\infty e^{-t} t^{-3/4} dt$

comparing with def(1) of Γ function

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$.

$n-1 = -3/4$

$n = -3/4 + 1$

$n = 1/4$.

$\boxed{I_1 = \frac{1}{8}\Gamma(1/4)}$

→ $I_2 = \int_0^\infty x^2 e^{-x^4} dx$.

put $x^4 = t$

$4x^3 dx = dt$.

$x^2 dx = \frac{1}{4}\frac{dt}{x}$.

$\boxed{x^2 dx = \frac{dt}{4} t^{-1/4} dt}$

$I_2 = \frac{1}{4}\int_0^\infty e^{-t} t^{-1/4} dt$.

By comparing

$n-1 = -1/4$

$n = -1/4 + 1$

$n = 3/4$.

$\therefore \boxed{I_2 = \frac{1}{4}\Gamma(3/4)}$

$I_1 . I_2 = \frac{1}{8}\Gamma(1/4) . \frac{1}{4}\Gamma(3/4)$.

$$= \frac{\pi}{16\sqrt{2}}$$

(*) NOTE :-

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \ d\theta.$$

put $2m-1 = p$ $\quad$ $2n-1 = q$

$2m = p+1$ $\quad$ $2n = q+1$

$m = p+1/2.$ $\quad$ $n = q+1/2.$

$$\therefore \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \ \cos^q\theta \ d\theta.$$

$$\boxed{\int_0^{\pi/2} \sin^p\theta \ \cos^q\theta \ d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right).}$$

(*) If the given problem has

→ $(a+x^n)$ put $x^n = a\tan^2\theta$

→ $(a-x^n)$ then $x^n = a\sin^2\theta.$

Evaluate

3. $\int_0^{\infty} x^{3/2} e^{-x} dx.$

W.K.T $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx.$

$n-1 = 3/2$

$n = 3/2 + 1$

$n = 5/2.$

$= \Gamma(5/2).$

$= 3\sqrt{\pi}/4.$

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4. $\int_0^\infty x^6 e^{-2x} dx$

put $2x = t$

$2 dx = dt$

$dx = \frac{dt}{2}$

$x = 0;\ t = 0$

$x = \infty;\ t = \infty$

$\int_0^\infty \left(\frac{t}{2}\right)^6 e^{-t} \frac{dt}{2}$

$= \frac{1}{2^7} \int_0^\infty t^6 e^{-t} dt.$

w.r.t $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$

$n-1 = 6$

$n = 7.$

$= \frac{1}{2^7} \Gamma(7)$

$\Gamma(n+1) = n!$

$= \frac{1}{2^7} \Gamma(6+1)$

$= \frac{1}{2^7} 6!$

5. $\int_0^1 (\log x)^3 dx.$

put $\log x = -t$; $x = e^{-t}$

$\frac{1}{x} dx = dt.$

$dx = -e^{-t} dt$

if $x = 0$; $\log(0) = \infty - 2t$

$x = 1$; $t = 0.$

$\therefore$ t varies from $-\infty$ to 0

$\int_{-\infty}^0 (-t)^3 \cdot e^{-t} dt.$

$\int_{-\infty}^0 -t^3(-e^{-t}) dt.$

$\int_{-\infty}^0 t^3 e^{-t} dt.$ [Changing the limits]

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$$= -\int_0^\infty t^3 e^{-t}\, dt$$

$n-1 = 3$

$n = 4.$

$= \Gamma(4)$

$= \Gamma(3+1)$

$= \underline{\underline{-3!}}$

6\. S.T $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$.

$$I_1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$$

$$I_1 = \int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta \cdot \cos^0\theta$$ ($\because$ cos is not there so just take as $\cos^0\theta$).

$$\boxed{\text{Formula:- } \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

$p = -1/2 \,;\, q = 0.$

$$I_1 = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$$

$$I_1 = \frac{1}{2}\beta\left(\frac{1/2}{2}, \frac{1}{2}\right)$$

$$\boxed{I_1 = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)}$$

$$I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$$

$$I_2 = \int_0^{\pi/2} \sin^{1/2}\theta \, d\theta.$$

$$I_2 = \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta \, d\theta$$

Using formula $p = 1/2$; $q = 0$

$$I_2 = \frac{1}{2} \beta\left(\frac{3/2}{2} , \frac{1}{2} \right)$$

$$\boxed{I_2 = \frac{1}{2} \beta\left(\frac{3}{4} , \frac{1}{2} \right).}$$

$$I_1 \cdot I_2 = \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{2}\right) \times \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

Using Γ & β relation.

$$I_1 \cdot I_2 = \frac{1}{2} \cdot \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(\frac{1}{4}+\frac{1}{2})} \times \frac{1}{2} \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(3/4 + 1/2)}$$

$$I_1 \cdot I_2 = \frac{1}{4} \left[\frac{\Gamma(1/4)\Gamma(1/2)\cancel{\Gamma(3/4)}\Gamma(1/2)}{\cancel{\Gamma(3/4)}\ \Gamma(5/4)} \right] \qquad \Gamma(1 + 1/4)$$

$$I_1 \cdot I_2 = \frac{1}{4} \left[\frac{\Gamma(1/4)\sqrt{\pi} \cdot \sqrt{\pi}}{\Gamma(5/4)} \right]$$

$$I_1 \cdot I_2 = \frac{\pi}{4} \left[\frac{\Gamma(1/4)}{\Gamma(1/4 + 1)} \right] = \frac{\pi}{4} \left[\frac{\cancel{\Gamma(1/4)}}{\cancel{1/4}\,\cancel{\Gamma(1/4)}} \right] \qquad \Gamma(n+1) = n\Gamma n.$$

$$I_1 \cdot I_2 = \pi$$

7. $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ in term of gamma function.

$$\int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\, d\theta.$$

$$\int_0^{\pi/2} \cos^{1/2}\theta \, \sin^{-1/2}\theta \, d\theta$$ (Formula :- NOTE).

$p = -1/2 \quad q = 1/2$.

$$= \frac{1}{2} \beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right).$$ (convert to γ by using relation)

$$\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

$$= \frac{1}{2} \frac{\gamma(1/4)\gamma(3/4)}{\gamma(1/4+3/4)}$$

$$= \frac{1}{2} \pi\sqrt{2}$$

$$= \frac{\pi}{\sqrt{2}}$$

8. $\int_0^{\pi/2} \sin^6\theta \, d\theta.$

$$= \int_0^{\pi/2} \sin^6\theta \, \cos^0\theta \, d\theta$$ Using NOTE formula

$P = 6 \,;\, q = 0$

$= \frac{1}{2} \beta\left(\frac{7}{2}, \frac{1}{2}\right)$.

$\beta(m,n) = \frac{1}{2} \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$. $= \frac{1}{2} \frac{\gamma(7/2)\ \gamma(1/2)}{\gamma(7/2+1/2)}$

$\beta(m,n) = \frac{1}{2} \frac{15\sqrt{\pi}/8 \cdot \sqrt{\pi}}{\gamma(4)}$ $\gamma(3+1) = n!$

$8\times3\times2$

$\beta(m,n) = \frac{1}{2} \cdot \frac{15\pi}{8\times3!} = \frac{15\pi}{16\times3!} = \frac{15\pi}{96}$ $8\times6 = 48\times2$

16×6
96

$\beta(m,n) = \frac{5\pi}{32}$

9. $\int_0^{\pi/2} \sin^4\theta \cos^3\theta \, d\theta$.

$p = 4$; $q = 3$.

$\frac{5}{2} + \frac{4}{2} = \frac{9}{2}$

$= \frac{1}{2} \beta\left(\frac{4+1}{2}, \frac{3+1}{2}\right)$

$\frac{5}{2} + 2$

$= \frac{1}{2} \beta\left(\frac{5}{2}, \frac{4}{2}\right)$

$\frac{5}{2} + \frac{2}{1}$

$\frac{5+4}{2} = \frac{9}{2}$.

$\beta(m,n) = \frac{1}{2} \frac{\gamma(5/2)\gamma(4/2)}{\gamma(9/2)} = \frac{1}{2} \frac{\left[\frac{3\sqrt{\pi}}{4} \times 1\right]}{\gamma(7/2+1)}$.

$\gamma(9/2) \rightarrow$

$7/2 +$

$= \frac{1}{2} \frac{\gamma(5/2)}{7/2 \cdot 5/2 \, \gamma(5/2)}$

$7/2 \, \gamma(7/2)$

$= \frac{2}{35}$

10. $\int_0^1 x^{3/2} (1-x)^{1/2} dx$

By using $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$.

Here $m-1 = 3/2$ $\quad$ $n-1 = 1/2$

$m = 3/2 + 1$ $\quad$ $n = 1/2 + 1$

$m = 5/2$. $\quad$ $n = 3/2$.

$I = \beta(5/2, 3/2)$

$I = \dfrac{\Gamma(5/2)\Gamma(3/2)}{\Gamma(4)} = \dfrac{\frac{3\sqrt{\pi}}{4} \cdot \frac{\sqrt{\pi}}{2}}{3!} = \dfrac{\frac{3\pi}{8}}{6}$

$I = \dfrac{3\pi}{48}$

$I = \dfrac{\pi}{16}$

* 11. $\int_0^2 (4-x^2)^{3/2} dx$.

Here its in the form $(a-x^n)$; so take $x = 2\sin\theta$.

$x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$.

$\frac{2}{2} = \sin\theta$

when $x = 0$; $\theta = 0$

$x = 2$; $\theta = \pi/2$

$I = \int_0^{\pi/2} [4 - (2\sin\theta)^2]^{3/2} \, 2\cos\theta \, d\theta$.

$$I = \int_0^{\pi/2} 4^{3/2} (1 - \sin^2\theta)^{3/2}\, 2\cos\theta\, d\theta$$ $(2^2)^{3/4}$

$$I = \int_0^{\pi/2} 4^{3/2} (\cos^2\theta)^{3/2}\, 2\cos\theta\, d\theta.$$

$$I = 16 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$I = 16 \int_0^{\pi/2} \cos^4\theta\, \sin^0\theta\, d\theta$$

$p = 0;\quad q = 4.$

$$I = \frac{16}{2}\, \beta\left(\frac{1}{2}, \frac{5}{2}\right).$$

$$= 8\, \frac{\gamma(1/2)\,\gamma(5/2)}{\gamma(3).} \Rightarrow 8 \cdot \frac{\sqrt{\pi} \cdot \frac{3\sqrt{\pi}}{4}}{2!}$$

$$= \frac{\cancel{24}^{3}\,\pi}{4 \cdot \cancel{2}}$$

$$= \underline{\underline{3\pi}}$$

$$\int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx$$

$$\int_0^\infty \frac{dx}{1+x^4}$$

Redo I_2 $\int_0^2 x\,(8 - x^3)^{1/3}\, dx.$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x = 0;\ \theta = 0$

$x = 2;\ \theta = \pi/2$

$$I = \int_0^{\pi/2} 2\sin\theta\, (8 - (2\sin\theta)^3)^{1/3}$$

$$I = \int_0^{\pi/2} 2\sin\theta \, (8 - 8\sin^3\theta)^{1/3}.$$

$\cos 2\theta = u$

$\sin 2\theta = 2\sin$

$\cos^2\theta/_2 + \sin^2\theta/_2 - 2\sin\theta/_2 \cos$

$(\cos\theta/_2 - \sin\theta/_2)^2$

$$I = \int_0^{\pi/2} 2\sin\theta \, (2^3)^{1/3} (1 - \sin\theta)$$

$$I = \int_0^{\pi/2} 4\sin\theta \, (1 - \sin\theta).$$

12 $\int_0^2 \cancel{x} (8 - x^3)^{-1/3} \, dx.$

$\frac{2}{3} - 1 = \frac{-1}{3}$

$8 - x^3 \longrightarrow \quad x^3 = 8\sin^2\theta.$

8

$x = 8^{1/3} (\sin^2\theta)^{1/3}.$

$x = (2^3)^{1/3} (\sin^2\theta)^{1/3}.$

$x = 2\sin^{2/3}\theta$

$dx = 2 \cdot \frac{2}{3} \sin^{-1/3}\theta \cos\theta \, d\theta.$

$dx = \frac{4}{3} \sin^{-1/3}\theta \cos\theta \, d\theta.$

if $x = 0 \, ; \, \theta = 0$

$x = 2 \, ; \, \theta = \pi/2.$

$$I = \int_0^{\pi/2} (8 - 8\sin^2\theta)^{-1/3} \frac{4}{3}\sin^{-1/3}\theta \cos\theta \, d\theta$$

$$I = \frac{4}{3} \int_0^{\pi/2} (8)^{-1/3} (1 - \sin^2\theta)^{-1/3} \sin^{-1/3}\theta \cos\theta \, d\theta.$$

$$I = \frac{4}{3} \int_0^{\pi/2} (2^3)^{-1/3} (\cos^2\theta)^{-1/3} \sin^{-1/3}\theta \cos\theta \, d\theta.$$

$$I = \frac{4}{3} \int \frac{1}{2} \cos^{-2/3} \cos\theta \sin^{-1/3}\theta \, d\theta$$

$$I = \frac{4}{6}\int_0^{\pi/2} \cos^{1/3}\theta \ \sin^{-1/2}\theta \ d\theta.$$

$-\frac{2}{3}+1$

$\frac{-2+3}{3}=\frac{1}{3}$

$p = -1/3 \quad ; \quad q = \frac{1}{3}$

$1-\frac{1}{3}=\frac{2}{3}$

$$I = \frac{4}{6}\cdot\frac{1}{2}\ \beta\left(\frac{-1/3+1}{2}, \frac{1/3+1}{2}\right)$$

$\frac{1}{3}+1=\frac{4}{3}$

$$I = \frac{4}{12}\ \beta\left(\frac{2}{6}, \frac{4}{6}\right).$$

$\frac{1}{3}+\frac{2}{3}=\frac{3}{3}$

$$I = \frac{4}{12}\ \frac{\gamma(2/6)\ \gamma(4/6)}{\gamma(1/3+2/3)}.$$

$1+\frac{1}{3}$

$\gamma(1/3)$

$$I = \frac{4}{12}\ \frac{\gamma(1/3)\ \gamma(2/3)}{\gamma(1)}.$$

$$I = \frac{\cancel{4}^{2}}{\cancel{12}\ 3}\cdot \cancel{2}\pi$$

$$I = \frac{2\pi}{3}$$

13 $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$

$\frac{1}{2}-1 = \frac{-1}{2}$

$1+x^4 \longrightarrow \quad x^4 = 1\tan^2\theta$

$x = (\tan^2\theta)^{1/4}.$

$x = \tan^{1/2}\theta.$

$dx = \frac{1}{2}\tan^{-1/2}\sec^2\theta\cdot d\theta.$

if $x=0;\ \theta=0$

$x=\infty;\ \theta=\pi/2$

$$I = \int_0^{\pi/2} \frac{\frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta}{1+\tan^2\theta}$$

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{\tan^{-1/2}\theta \cancel{\sec^2\theta} \, d\theta}{\cancel{\sec^2\theta} \, d\theta}$$

$$I = \frac{1}{2}\int_0^{\pi/2} \tan^{-1/2}\theta \, d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta} \, d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$1 + \frac{1}{2} =$

$p = -1/2 \; ; \; q = 1/2$

$$I = \frac{1}{2} \cdot \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$\frac{\pi}{2\sqrt{2}}$

$$I = \frac{1}{4} \frac{\Gamma(1/4)\,\Gamma(3/4)}{\Gamma(1)} = \frac{\pi\sqrt{2}}{4} \quad \frac{1}{2} \; \frac{1}{2} \; \frac{\pi}{2\Gamma}$$

$\sqrt{\pi} \; \sqrt{} \quad \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2} \quad 3/2 \quad 3/2$

$$\boxed{\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}}$$

Application of Integral calculus

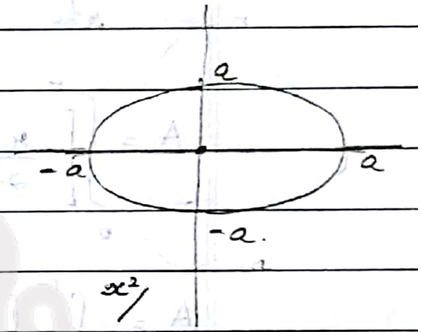
NOTE:- Surface Area $= \iint dx\,dy$

Volume $= \iiint dx\,dy\,dz$

1. Find the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration.

Area $= 4A_1$

$= 4\iint dx\,dy$.



$= 4\int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} dx\,dy$. (w.r.t y).

$x^2/$

$\therefore x \to 0 - a$.

$= 4\int_0^a\int_0^{b/a\sqrt{a^2-x^2}} dy\,dx$

$\frac{y^2}{b} = 1 - \frac{x^2}{a^2}$; $y = b\sqrt{1 - x^2/a^2}$

$y = \frac{b}{a}\sqrt{a^2-x^2}$

$= 4\int_0^a \left[\, y \,\right]_0^{b/a\sqrt{a^2-x^2}} dx$

$= 4\int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2}\right] dx$

($\because$ Formula)

$x/2\sqrt{a^2-x^2} + \sin^{-1}(x/a)\,a^2/2$

$= 4.\left[\frac{b}{a}\left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)\right]_0^a$

$= 4\left[\frac{b}{a}\left(\frac{a}{2}(0). - \frac{a^2}{2}\sin^{-1}(1)\right) - 0\right]$

$= \cancel{4}\frac{b}{a}\left(\frac{a^2}{\cancel{2}} \cdot \frac{\pi}{\cancel{2}}\right)$

$= \frac{b}{a}a^2\pi$

2. Area Enclosed by curve $r = a(1+\cos\theta)$ b/w $\theta = 0$ and $\theta = \pi$.

Here θ varies from $0 \to \pi$.

r varies from $0 \to a(1+\cos\theta)$.

$r = a(1+\cos\theta)$

when $\theta = 0$

$r = a(1+\cos\theta)$

$r = 0 \quad a(1+\cos\theta)$

$$A = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r \, d\theta \cdot dr \quad \text{w.r.t } r$$

$$A = \int_{r=0}^{a(1+\cos\theta)} \int_0^{\pi} r \, dr \, d\theta$$

$$A = \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$A = \int_0^{\pi} \left[\cancel{a^2+a^2\cos^2\theta} \; \frac{a^2(1+\cos\theta)^2}{2} - 0 \right]$$

$(1+\cos\theta) = 2\cos^2 \theta/2$

$$A = \frac{1}{2}\int_0^{\pi} a^2 \, (2^2\cos^4 \theta/2).$$

$$A = \frac{a^2 \cdot 4}{2}\int_0^{\pi} \cos^4 \theta/2 \, d\theta$$

$\frac{\theta}{2} = t$

$d\theta = 2\,dt$

$\theta = 0 \, ; \, t = 0$

$\theta = \pi \, ; \, t = \pi/2$

$$A = \frac{4a^2}{2}\int_{t=0}^{\pi/2} \cos^4 t \; 2 \cdot dt.$$

$$A = \frac{8a^2}{2}\int_0^{\pi/2} \cos^4 t \, dt.$$

$$A = \frac{8a^2}{2}\int_0^{\pi/2} \sin^0 t \cos^4 t \, dt.$$

$p = 0 \, ; \quad q = 4$

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$A = 4a^2 \cdot \frac{1}{2} \beta\left(\frac{1}{2}, \frac{5}{2}\right)$

$A = 2a^2 \frac{\gamma(1/2)\, \gamma(5/2)}{\gamma(3)}$

$\gamma(3) = \gamma(2+1) = 2!$

$A = 2a^2 \frac{\sqrt{\pi} \cdot \frac{3\sqrt{\pi}}{4}}{2} = \frac{\cancel{2}a^2 3\pi}{\cancel{8}_4}$

$A = \frac{3a^2\pi}{4}$.

8. Volume of tetrahedron bounded by the planes $x=0;\ y=0;\ z=0$, ~~Eqn~~ $x/a + y/b + z/c = 1$

Volume $= \iiint dx\, dy\, dz$

$\rightarrow \frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b}$

$z = c\left[1 - \frac{x}{a} - \frac{y}{b}\right]$

$\therefore$ z varies from 0 to $c\left[1 - \frac{x}{a} - \frac{y}{b}\right]$

y varies from 0 to $b\left[1 - \frac{x}{a}\right]$.

$\rightarrow \frac{y}{b} = 1 - \frac{x}{a} - \frac{z}{c}$.

x varies from 0 to a.

$y = b\left[1 - \frac{x}{a} - \frac{z}{c}\right]$

Put $z = 0$

Volume $= \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} \int_{c=0}^{c(1-x/a-y/b)} dx\, dy\, dz$. (w.r.t z)

$y = b\left[1 - \frac{x}{a}\right]$

$\rightarrow$ put $y=0;\ z=0$.

$\frac{x}{a} = 1$

$x = a$.

$V = \int_0^a \int_0^{b(1-x/a)} \left[z \right]_0^{c(1 - \frac{x}{a} - \frac{y}{b})} dy\, dx$

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$$v = \int_0^a \int_0^{b(1-x/a)} c\left[1 - \frac{x}{a} - \frac{y}{b}\right] dx\, dy \quad (w.r.t\ y)$$

$$v = \int_0^a c\left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} dx$$

$$v = c\int_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{x}{a}b\left(1 - \frac{x}{a}\right) - \frac{1}{2b}\left[b\left(1 - \frac{x}{a}\right)\right]^2\right] dx$$

$$\frac{b}{2}\left(1 + \frac{x^2}{a^2} - 2\frac{x}{a}\right)$$

$$\frac{bx^2}{2a^2} - \frac{2bx}{2a}$$

X $$\left(v = c\int_0^a \frac{b - bx}{a}\frac{}{a} - \frac{xb}{a} + \frac{x^2 b}{a^2}\right) X$$

$$v = c\int_0^a b\left(1 - \frac{x}{a}\right)\left[b - \frac{xb}{a} - \frac{1}{2b}\,b\left(1 - \frac{x}{a}\right)\right]$$

$$v = c\int_0^a b\left(1 - \frac{x}{a}\right)\left[1 - \frac{x}{a} - \frac{1}{2b}\left[b - \frac{xb}{a}\right]\right)$$

$$v = c\int_0^a b\left(1 - \frac{x}{a}\right)\left[1 - \frac{x}{a} - \frac{1}{2} + \frac{x}{2a}\right] X$$

$$v = c\int_0^a b - \frac{bx}{a} - \cancel{\frac{xb}{a}} + \frac{x^2 b}{a^2} - \frac{b^2}{2b} - \frac{x^2 b}{2a^2 b} + \cancel{\frac{bx}{2ab}}\ dx$$

$$v = c\int_0^a \frac{b}{2} - \frac{xb}{a} + \frac{1}{2}\frac{x^2 b}{a^2}\cdot dx \quad (w.r.t\ x)$$

$$v = c\left[\frac{b}{2}x - \frac{x^2}{2}\frac{b}{a} + \frac{1}{2}\frac{x^3}{3}\frac{b}{a^2}\right]_0^a$$

$$v = c\left[\frac{ab}{2} - \frac{a^2}{2}\frac{b}{a} + \frac{1}{6}\frac{a^3 b}{a^2}\right]$$

$$V = c\left[\frac{ab}{2} - \frac{ab}{2} + \frac{ab}{6}\right]$$

$$\underline{\underline{V = \frac{abc}{6}}}$$