1

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Proof:- $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

RHS

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_0^\infty \int_s^\infty e^{-st} f(t)\,ds\,dt$$

$$= \int_0^\infty \left(\frac{e^{-st}}{-t}\right)_s^\infty f(t)\,dt$$

$$= \int_0^\infty \left(\frac{e^{-\infty} - e^{-st}}{-t}\right) f(t)\,dt$$

$$= \int_0^\infty \left(0 - \frac{e^{-st}}{-t}\right) f(t)\,dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{f(t)}{t}\,dt$$

$$\int_s^\infty F(s)\,ds = L\left\{\frac{f(t)}{t}\right\} \quad \text{LHS}$$

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$

Proof:- $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Let $\int_0^t f(t)\,dt = F(t)$

$F'(t) = f(t)$, $F(0) = 0$

$$L\left\{\int_0^t f(t)\,dt\right\} = L\{F(t)\}$$

$$= \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^\infty F(t)\, e^{-st}\,dt$$

$$= \left[F(t)\left(\frac{e^{-st}}{-s}\right) - \int F'(t) \cdot \frac{e^{-st}}{-s}\,dt \right]_0^\infty$$

$$= (0 - 0) + \frac{1}{s}\int_0^\infty F'(t)\, e^{-st}\,dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt$$

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} \cdot F(s)$$

$\because F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Note:- $L\left\{ \int_0^t \int_0^t f(t)\,dt\,dt \right\} = \frac{F(s)}{s^2}$

Problems

1. Find $L\left\{ t^4 e^{-3t/2} \right\}$

$$f(t) = t^4$$

$$L\{t^4\} = \frac{4!}{s^{4+1}}$$

$$L\{t^4\} = \frac{4!}{s^5} = F(s)$$

$$L\left\{ e^{-3t/2} t^4 \right\} = \frac{4!}{s^5} \qquad s \rightarrow s+3/2$$

$$a = -3/2$$

$\left(\because L\{e^{at} f(t)\} = F(s-a) \right)$
shifting rule

$$L\left\{ e^{-3t/2} t^4 \right\} = \frac{4!}{(s+3/2)^5}$$

(2) Find $L\{e^{2t}\cos 3t\}$

Sol:- $L\{e^{2t}\cos 3t\}$

$$f(t) = \cos 3t$$

$$L\{\cos 3t\} = \frac{s}{s^2+3^2}$$

$$L\{e^{2t}\cos 3t\} = \frac{s}{s^2+3^2}$$

$$s \to s-2$$

$$L\{e^{2t}\cos 3t\} = \frac{s-2}{(s-2)^2+3^2}$$

---

(3) Find $L\{e^{3t}\sin^2 t\}$

Sol:- $L\{e^{3t}\sin^2 t\}$

$$f(t) = \sin^2 t$$

$$f(t) = \frac{1-\cos 2t}{2}$$

$$L\{f(t)\} = \left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$L\{\sin^2 t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$$

$$L\{e^{3t}\sin^2 t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] \quad s \to s-3$$

$$L\{e^{3t}\sin^2 t\} = \frac{1}{2}\left[\frac{1}{S-3} - \frac{S-3}{(S-3)^2+2^2}\right]$$

④ find $L\{e^{-t}\cos^2 3t\}$

Sol :-

$$f(t) = \cos^2 3t$$

$$= \frac{1+\cos 2(3t)}{2}$$

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$f(t) = \frac{1+\cos 6t}{2}$$

$$L\{f(t)\} = \frac{1}{2} L\{1+\cos 6t\}$$

$$= \frac{1}{2}\left[L\{1\} + L\{\cos 6t\}\right]$$

$$L\{\cos^2 3t\} = \frac{1}{2}\left[\frac{1}{S} + \frac{S}{S^2+6^2}\right]$$

$$L\{e^{-t}\cos^2 3t\} = \frac{1}{2}\left[\frac{1}{S} + \frac{S}{S^2+36}\right]$$

$S \to S+1$

$$= \frac{1}{2}\left[\frac{1}{S+1} + \frac{S+1}{(S+1)^2+36}\right]$$

5 find $L\{e^{-4t}\cos t \sin 2t\}$

Sol :- $f(t) = \cos t \sin 2t$

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

$$\cos t \sin 2t = \frac{1}{2}[\sin(3t) - \sin(-t)]$$

$$f(t) = \frac{1}{2}[\sin 3t + \sin t]$$

$$L\{f(t)\} = \frac{1}{2} L\{\sin 3t + \sin t\}$$

$$= \frac{1}{2}[L\{\sin 3t\} + L\{\sin t\}]$$

$$L\{f(t)\} = \frac{1}{2}\left[\frac{3}{s^2+3^2} + \frac{1}{s^2+1^2}\right]$$

$$L\{e^{-4t}\cos t \sin 2t\} = \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$$

$s \to s+4$

$$= \frac{1}{2}\left[\frac{3}{(s+4)^2+9} + \frac{1}{(s+4)^2+1}\right]$$

⑥ Find $L\{e^{3t}\sin 5t \sin 3t\}$

Sol:- $f(t) = \sin 5t \sin 3t$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin 5t \sin 3t = \frac{1}{2}[\cos(5t-3t) - \cos(5t+3t)]$$

$$= \frac{1}{2}[\cos 2t - \cos 8t]$$

$$L\{\sin 5t \sin 3t\} = \frac{1}{2}[L\{\cos 2t\} - L\{\cos 8t\}]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+2^2} - \frac{s}{s^2+8^2}\right]$$

$$L\{e^{3t}\sin 5t \sin 3t\} = \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+64}\right]$$

$s \to s-3$

$$= \frac{1}{2}\left[\frac{s-3}{(s-3)^2+4} - \frac{s-3}{(s-3)^2+64}\right]$$

① Find $L\{e^{-2t}(2\cos 5t - \sin 5t)\}$

② Find $L\{e^{3t}\sin 6t \cos 2t\}$

1. find $L\{t\cos at\}$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))$$

Sol:- $f(t) = \cos at$

$$L\{f(t)\} = L\{\cos at\}$$

$$= \frac{s}{s^2+a^2} = F(s)$$

$$L\{t \cdot \cos at\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\left[\frac{(s^2+a^2)\cdot 1 - s\cdot(2s)}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$L\{t \cdot \cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

2) find $L\{t \cdot \sinh at\}$

Sol :- let $f(t) = \sinh at$

$$L\{\sinh at\} = \frac{a}{s^2-a^2}$$

$$L\{t \cdot \sinh at\} = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)$$

$$= -\left[\frac{(s^2-a^2)\cdot 0 - a\cdot 2s}{(s^2-a^2)^2}\right]$$

$$= -\left(\frac{-2as}{(s^2-a^2)^2}\right)$$

$$L\{t \cdot \sinh at\} = \frac{2as}{(s^2-a^2)^2}$$

3) find $L\{t^2 \cos at\}$

Sol :- $f(t) = \cos at$

$$L\{f(t)\} = L\{\cos at\}$$

$$= \frac{s}{s^2+a^2} = F(s)$$

$$L\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$L\{t^2\cos at\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \left[\frac{(s^2+a^2)^2\frac{d}{ds}(a^2-s^2)-(a^2-s^2)\frac{d}{ds}(s^2+a^2)^2}{(s^2+a^2)^4}\right]$$

$$= \frac{(s^2+a^2)^2\cdot(-2s) - (a^2-s^2)\cdot 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

$$= \cancel{(s^2+a^2)}\left[\frac{(s^2+a^2)(-2s) - (a^2-s^2)\,4s}{(s^2+a^2)^{\cancel{4}\,3}}\right]$$

$$= \frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3}$$

$$L\{t^2 \cos at\} = \frac{2s^3 - 6a^2 s}{(s^2+a^2)^3}$$

$$= \frac{2s(s^2 - 3a^2)}{(s^2+a^2)^3}$$

④ Find $L\{t e^{-2t} \sin 4t\}$

Sol :-

$$f(t) = \sin 4t$$

$$L\{\sin 4t\} = \frac{4}{s^2+4^2}$$

$$L\{t \sin 4t\} = (-1)^1 \frac{d}{ds}\left(\frac{4}{s^2+16}\right)$$

$$= -\left[\frac{(s^2+16)\cdot\frac{d}{ds}(4) - 4\frac{d}{ds}(s^2+16)}{(s^2+16)^2}\right]$$

$$= -\left[\frac{(s^2+16)\cdot 0 - 4\cdot 2s}{(s^2+16)^2}\right]$$

$$= -\left(\frac{-8s}{(s^2+16)^2}\right)$$

$$L\{t \sin 4t\} = \frac{8s}{(s^2+16)^2}$$

$$L\{e^{-2t}\, t \sin 4t\} = \frac{8s}{(s^2+16)^2} \quad s \to s+2$$

$$= \frac{8(s+2)}{[(s+2)^2+16]^2}$$

⑤ Find $L\{t^2 e^{-2t} \sin 3t\}$

Sol:- $f(t) = \sin 3t$

$$L\{\sin 3t\} = \frac{3}{s^2+3^2}$$

$$= \frac{3}{s^2+9}$$

$$L\{t^2 \sin 3t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{3}{s^2+9}\right)\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2+9)\frac{d}{ds}(3) - 3 \cdot \frac{d}{ds}(s^2+9)}{(s^2+9)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{0 - 3 \cdot 2s}{(s^2+9)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$= -6 \frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2 \frac{d}{ds}(s) - s \cdot \frac{d}{ds}(s^2+9)^2}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{(s^2+9)^2 \cdot 1 - s \cdot 4s(s^2+9)}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{(s^2+9)^2 - 4s^2(s^2+9)}{(s^2+9)^4}\right]$$

$$= -6\cancel{(s^2+9)}\left[\frac{s^2+9 - 4s^2}{(s^2+9)^{\cancel{4}\,3}}\right]$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]$$

$$L\left\{e^{-2t} t^2 \sin 3t\right\} = -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]_{s \to s+2}$$

$$= -6\left[\frac{9-3(s+2)^2}{[(s+2)^2+9]^3}\right]$$ //

① Find $L\{t^2 \cos 3t\}$

② Find $L\{t e^{-4t} \sin 3t\}$

③ Find $L\{t \cdot e^{3t} \cos 2t\}$

④ Find $L\{e^{-t} t \cosh t\}$

⑤ Find $L\{t \cdot \sin 3t \cos 2t\}$

① Find $L\left\{\dfrac{e^{-at} - e^{-bt}}{t}\right\}$

Sol:-

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$$

$=$

$$f(t) = e^{-at} - e^{-bt}$$

$$F(s) = L\{f(t)\} = L\{e^{-at} - e^{-bt}\}$$

$$= L\{e^{-at}\} - L\{e^{-bt}\}$$

$$F(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

15

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \Big[\log(s+a) - \log(s+b)\Big]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \underset{s\to\infty}{Lt} \log\left(\frac{s+a}{s+b}\right) - \log\left(\frac{s+a}{s+b}\right)$$

$$= \underset{s\to\infty}{Lt} \log\left(\frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right) - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s+a}{s+b}\right) \qquad \left\{ \begin{array}{l} \text{as } s\to\infty \\ 1/s \to 0 \end{array}\right.$$

$$= \log 1 - \log\left(\frac{s+a}{s+b}\right)$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+a}{s+b}\right)^{-1}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \underline{\underline{\log\left(\frac{s+b}{s+a}\right)}}$$

② find $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

Sol:- $f(t) = \cos at - \cos bt$

$$L\{f(t)\} = F(s) = L\{\cos at\} - L\{\cos bt\}$$

$$F(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$\left\{\begin{array}{l} s \to \infty \\ s^2 \to \infty \\ \frac{1}{s} \to 0 \\ 1/s^2 \to 0 \end{array}\right.$$

$$= \log\left(\frac{s^2+a^2}{s^2+b^2}\right)^{-1/2}$$

$$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

$$= \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$