# Network QoS
# 371-2-0213

Lecture 11

Gabriel Scalosub

# Outline

# System and Traffic Model

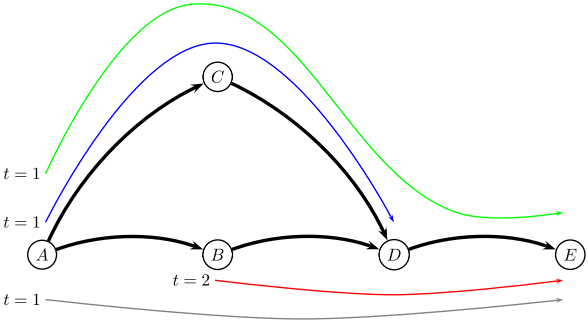
- Slotted time
- Directed graph $G = (V, E)$
  - single rate-1 queue $Q_e$ at link tail
  - unlimited buffer space
  - employs scheduling policy $\mathcal{P}$
- Unit size packets
  - $p = (\pi_p, a_p)$
    - $\pi_p$ simple path in $G$ and $a_p$ arrival time at path source
  - take a single time slot to traverse a link
- $(r, b)$-adversary
  - for any time interval $I$ and edge $e \in E$

$$N(e, I) \leq r\,|I| + b$$

    - $N(e, I)$: # packets injected during interval $I$ with $e \in$ path
- Stability
  - $\sup_t \sum_{e \in E} Q_e(t) < \infty$
    - $Q_e(t)$: queue size at tail of $e$ at time $t$

# Protocols

- Work-conserving (WC)




| | | | |
|---|---|---|---|
| FIFO | Nearest-to-Go (NTG) | Farthest-to-Go (FTG) | Shortest-in-System (SIS) |
| LIFO | Farthest-from-Source (FFS) | Nearest-to-Source (NTS) | Longest-in-System (LIS) |

# Last Week Results

- Stability vs. delay
  - for any $(r < 1, b)$-adversary and any WC protocol, the system is stable iff it has bounded delay
  - "if" direction holds also for $r = 1$
- Farthest-to-go
  - universally stable
- FIFO
  - not universally stable for any rate $r > 0$
- Some protocols are universally stable, and some are not. E.g.,

| Protocol | Universally Stable? |
|---|---|
| FIFO | No |
| LIFO | No |
| Nearest-to-Go (NTG) | No |
| Farthest-from-Source (FFS) | No |
| Farthest-to-Go (FTG) | Yes |
| Nearest-to-Source (NTS) | Yes |
| Shortest-in-System (SIS) | Yes |
| Longest-in-System (LIS) | Yes |

# This Week

- Easy traffic
  - are there rates for which any WC protocol is stable?
    - e.g., sometimes even FIFO might be stable
  - depends on topology?
- Universally stable topologies
  - are there topologies for which any WC protocol is stable?
  - at any rate $\leq 1$?

# Stability Against Low-rate Adversaries

## Theorem

*For any $G = (V, E)$ there exists some $r(G) > 0$ s.t. any WC protocol $\mathcal{P}$ is stable on $G$ against $(r < r(G), b \geq 0)$-adversaries.*

## Proof (for FIFO and $r(G) = 1/d$)

- Assume $r < \frac{1}{d}$
- Prove that every packet has bounded delay $T$ in every link

  $\Rightarrow$ overall bounded delay implies stability
- Given any $\delta \in (0, 1)$ define
  - $w_\delta = \frac{b}{1-r} \cdot \frac{1}{\delta}$
  - $r_\delta = r + \delta(1 - r)$
    - for small enough $\delta$, $r_\delta \in (r, \frac{1}{d})$
- "smooth-out" bursts over intervals $I$ of length $w_\delta$

$$N(e, I) \leq w_\delta r + b = w_\delta r + w_\delta(1 - r)\delta = w_\delta \cdot r_\delta$$

# Stability Against Low-rate Adversaries

## Theorem

*For any $G = (V, E)$ there exists some $r(G) > 0$ s.t. any WC protocol $\mathcal{P}$ is stable on $G$ against $(r < r(G), b \geq 0)$-adversaries.*

## Proof (for FIFO and $r(G) = 1/d$)

- Every packet $p$ arriving at $Q_e$ at $t$, leaves by $T_t = t + w_\delta \cdot r_\delta$
  - By induction on $t$
- Base case: $t \leq d \cdot w_\delta \cdot r_\delta$
  - assume by contradiction $p$ is still in $Q_e$ at $T_t$
  - $Q_e(t) \geq w_\delta \cdot r_\delta + 1$
    - FIFO...
  - $N(e, [0, t]) \leq \lceil d \cdot r_\delta \rceil \cdot w_\delta \cdot r_\delta \leq w_\delta \cdot r_\delta$
    - (number of $w_\delta$ windows $\times$ bound per window) + ($r_\delta < \frac{1}{d}$)
  - contradiction

# Stability Against Low-rate Adversaries

## Theorem

*For any $G = (V, E)$ there exists some $r(G) > 0$ s.t. any WC protocol $\mathcal{P}$ is stable on $G$ against $(r < r(G), b \geq 0)$-adversaries.*

## Proof (for FIFO and $r(G) = 1/d$)

- Induction step: consider $t > d \cdot w_\delta \cdot r_\delta$
  - assume by contradiction $p$ is still in $Q_e$ at $T_t$
  - $Q_e(t) \geq w_\delta \cdot r_\delta + 1$
    - FIFO...
  - every packet in $e$'s buffer at $t$ was injected after $t - d \cdot w_\delta \cdot r_\delta$
    - assume the contrary
    - by I.H. it waited in every prior link $\leq w_\delta \cdot r_\delta$ time
    - hence, it was in the system $\leq d \cdot w_\delta \cdot r_\delta$ time
    - it cannot be in the buffer at $t$
  - $N(e, [t - d \cdot w_\delta \cdot r_\delta, t]) \leq w_\delta \cdot r_\delta$
    - same as base case
  - contradiction

$\square$

# Alternative Definition of Adversaries

- Above proof implies an alternative definition of adversaries
- $(w, \varepsilon)$-window adversary
  - for any time interval $I$ of length $w$, and edge $e \in E$

$$N(e, I) \leq \lceil w(1-\varepsilon) \rceil$$

- An adversary is an $(r < 1, b)$-adversary iff it is a $(w, \varepsilon > 0)$-window adversary

$\Leftarrow$ trivial, by choosing $r = (1-\varepsilon)$ and $b = w$

$\Rightarrow$ follows from the proof by choosing a sufficiently small $\delta$:

- $w_\delta = \frac{b}{1-r} \cdot \frac{1}{\delta}$
- $r_\delta = r + \delta(1-r) \qquad (\xrightarrow[\delta \to 0]{} r)$
- set $w = w_\delta$ and $\varepsilon = 1 - r_\delta$

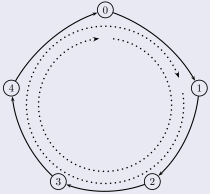
# Unstable Topologies ($r = 1$)

**Theorem (instability of cycles ($r = 1$))**

*LIS is unstable for the n-cycle ($n \geq 3$) against $(1, b)$-adversaries*

**Proof sketch.**

- $i$-loop
  - packet from $i$ to $i$
- $(1, b)$-adversary:
  - phases $P_k$, $k = 1, 2, \ldots$
    - sub-phase $S_{k,1}$: for $nk$ time units inject 1-loops
    - sub-phase $S_{k,2}$: for $nk$ time units: inject 0-loops
- Under LIS, by end of $P_k$
  - 1 packet in node 1
  - $nk - 1$ packets in node 0
- Proof by induction on $k$
- $\lim_{t\to\infty} Q_{(0,1)}(t) = \infty$




□

# Directed Acyclic Graphs (DAGs)

**Theorem**

*A network $G = (V, E)$ is universally stable for $r = 1$ iff $G$ is a DAG*

- Notation
  - for every link $e = (u, v) \in E$, define
    - $d_e$: edge level (max-distance from source node)
    - potential $f(e)$:

$$f(e) = \begin{cases} 1 & u \text{ is a source} \qquad (d_e = 0) \\ 1 + \sum_{u \text{ incoming edges } e'} f(e') & \text{otherwise} \qquad (d_e > 0) \end{cases}$$

  - $A_e(t)$: number of packets in the *network* at $t$ requiring $e$
  - $Q_e(t)$: number of packets in $e$'s queue at $t$
  - assume system is empty at $t = 0$
- Proof idea:
  - show that $A_e(t) \leq b \cdot f(e)$
    - hence, stable ($Q_e(t) \leq A_e(t)$)
  - combine with instability of cycles

# Directed Acyclic Graphs (DAGs)

## Lemma

*For all $e \in E$ and $t$, $A_e(t) \leq b \cdot f(e)$*

## Proof (by induction on edge level)

- Basis: $d(e) = 0$, and arbitrary $t$
  - i.e., $e = (u, v)$ leaves a source node $u$
    - $A_e(\cdot) = Q_e(\cdot)$
  - let $t' \leq t$ be maximal such that $Q_e(t') = 0$
    - exists since system is empty at 0
  - Adversary's injections during $(t', t]$
    - $N(e, (t', t]) \leq (t - t') + b$
  - at least $t - t'$ packets traverse $e$ during $(t', t]$
    - by maximality of $t'$ + WC
  - hence, $Q_e(t) \leq b = b \cdot 1 = b \cdot f(e)$
- Induction step: assume true for all $t$ and $e'$ s.t. $d(e') < d(e)$

# Directed Acyclic Graphs (DAGs)

## Lemma

*For all $e \in E$ and $t$, $A_e(t) \leq b \cdot f(e)$*

## Proof (by induction on edge level)

- Choose $t'$ as in the base case
  - hence, $A_e(t) \leq A_e(t') + b$

$$A_e(t') + \overbrace{(t - t') + b}^{r|I|+b} - \overbrace{(t - t')}^{Q_e \neq \emptyset}$$

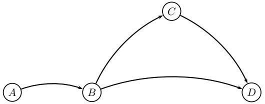
- At $t'$, $Q_e(t') = 0$ (by choice of $t'$)
  - packets contributing to $A_e(t')$ cross some incoming edge of $u$
  - $A_e(t') \leq \sum_{u \text{ incoming edges } e'} A_{e'}(t')$
  - for every incoming edge $e'$ of $u$
    - $d(e') < d(e)$
- Hence,

$$\begin{aligned} A_e(t) &\leq b + A_e(t') \\ &\leq b + \sum_{u \text{ incoming edges } e'} A_{e'}(t') \\ &\leq b + \sum_{u \text{ incoming edges } e'} b \cdot f(e') = b \cdot f(e) \end{aligned}$$

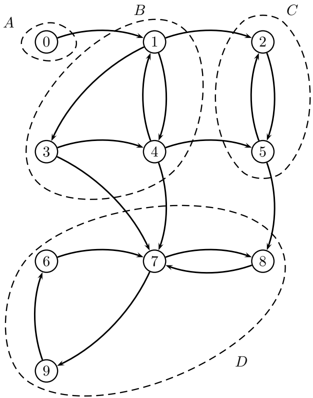
□

# Decomposing Universal Stability

- Goal:
  - find characteristics of networks that guarantee stability ($r < 1$)
  - DAGs for $r = 1$
- Strongly-connected components
- $G$ is a DAG of its strongly-connected components



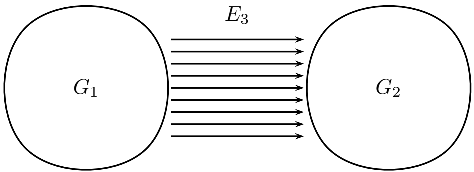

- Stability and connected components
  - if a connected component is not universally stable
    - entire network is not
  - is the converse also true?



# Composing Universal Stability ($r < 1$)

**Theorem**

*If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are universally stable ($V_1 \cap V_2 = \emptyset$), then for any $E_3 \subset V_1 \times V_2$, $G = (V_1 \cup V_2, E_1 \cup E_2 \cup E_3)$ is universally stable.*



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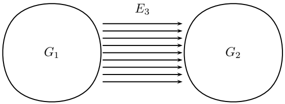
**Corollary**

*A network is universally stable iff every strongly-connected component is universally stable*

- Simple proof by induction on number of connected components
    - "Hard" part is the above theorem...

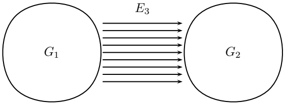
# Composing Universal Stability ($r < 1$)




## Proof of Theorem

- We will use the window-adversary definition
- Assume $G_1$ and $G_2$ are universally stable for $(w, \varepsilon > 0)$-window adversaries
- Goal: show that $G$ is stable for $(w, \varepsilon > 0)$ window adversaries
- Assume some $(w, \varepsilon > 0)$ window adversary
- $G_1$ stable: has some bounded delay $T_1$ (since $\varepsilon > 0$)
  - $\forall t$, any packet injected to $G_1$ at $t$ leaves $G_1$ by $t + T_1$
- Consider any time interval $I$ of length $T_2$
  - how many packets can enter $G_2$ during such an interval?

# Composing Universal Stability ($r < 1$)




## Proof of Theorem

- We would like to find some $0 < \varepsilon' \leq \varepsilon$ satisfying

$$(T_1 + T_2 + w)(1 - \varepsilon) \leq T_2(1 - \varepsilon')$$

  - implies: number of packets entering $G_2$ during interval of length $T_2$ is at most $T_2(1 - \varepsilon')$
  - i.e., traffic entering $G_2$ "generated" by a $(T_2, \varepsilon')$-window adversary
- This is satisfied for $T_2 = \frac{(T_1+w)(1-\varepsilon)}{\varepsilon-\varepsilon'}$
- $G_2$ is stable, hence it has bounded delay
- All traffic entering $G$ has bounded delay $\Rightarrow$ $G$ is stable $\square$

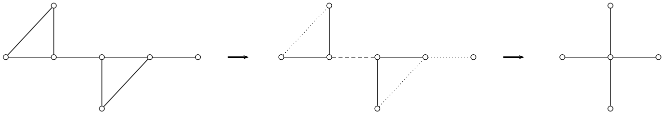
# Universal Stability for $r < 1$

- Are DAGs still the only universally stable topologies?
  - Specifically, are cycles universally stable for $r < 1$?

**Theorem**

*Cycles are universally stable for $r < 1$*

- What are the exact conditions for universal stability for $r < 1$?
- Graph minor of $G$:
  - any graph resulting from contracting/deleting edges



# Universal Stability for $r < 1$

- Are DAGs still the only universally stable topologies?
  - Specifically, are cycles universally stable for $r < 1$?

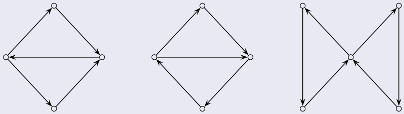
**Theorem**

*Cycles are universally stable for $r < 1$*

- What are the exact conditions for universal stability for $r < 1$?

**Theorem (for networks with no parallel edges)**

*A network is universally stable for $r < 1$ iff it excludes the minors*



- Verifying if $H$ is/isn't a minor of $G$: NP-hard in general
- For fixed minors (as in this case): poly-time solvable

# Open Questions & References

- Are there non trivial (topology,protocol) pairs that are stable?
  - currently only known for universally stable topology/protocol
- Given a protocol P, are there P-stable topologies?
  - topologies stable for protocol P
  - e.g., FIFO-stable topologies?
- Are there protocols/topologies stable only for $r < 1 - \varepsilon$?
  - e.g., only for $r \leq \frac{1}{2}$

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- References

  - Borodin, Kleinberg, Raghavan, Sudan, and Williamson. Adversarial Queueing Theory, Journal of the ACM 48(1):13–38 (2001)
  - Andrews, Awerbuch, Fernández, Leighton, Liu, and Kleinberg. Universal-Stability Results and Performance Bounds for Greedy Contention-Resolution Protocols, Journal of the ACM 48(1):39–69 (2001)
  - Goel. Stability of networks and protocols in the adversarial queueing model for packet routing, Networks 37(4):219–224 (2001)

# An Adversarial Model for Wireless Networks

- Wireless networks
  - dynamic channel conditions
    - dynamic link capacities
    - links disappearing/re-appearing
  - ad-hoc
    - topology constantly changing
    - routing is difficult
- Digraph $G = (V, E)$ with max-degree $\Delta$, slotted time
  - link capacity $c_e(t)$ ($\forall e \in E$ and time $t$):
    - dynamic, changes as a function of $t$
    - controlled by the *adversary*
    - assume: $\forall t$, $c_e(t) \in \{0, 1\}$
    - $E_t = \{e \in E \mid c_e(t) = 1\}$
  - injected packet $p$
    - size $\ell_p$
    - source $s_p$, destination $d_p$
    - path *not specified!!*
    - assume $\ell_p \equiv 1$

# Adversaries and Objective

- Question:
  - how would we define a "reasonable" adversary?
    - no specific path
    - not clear if path ever exists
    - even if it exists now, not clear if it will ever reappear
- Assume the adversary identifies *some* path
  - for any time interval $I$ s.t. $|I| = w$, and $p$ injected during $I$
    - $\Gamma_p$: some specific adversary-path between $s_p$ and $d_p$
  - $N(e, I)$: number of packets $p$ injected during $I$ s.t. $e \in \Gamma_p$
- $(w, \varepsilon)$-window adversary
  - for any time interval $I$ of length $w$

$$N(e, I) \leq (1 - \varepsilon) \sum_{t \in I} c_e(t) \quad (\leq w)$$

**Objective**

Determine *scheduling* and *routing* for each packet

- Concern: (when) can we guarantee stability?

# Designing a Protocol

- AQT lesson: scheduling matters
  - be careful not to cause overload downstream
- How can this be implemented "locally"?
  - necessary: no permanent global view of the graph/routes
  - forward only if doesn't cause "local" overload
- Note: not work conserving!!
  - in particular, no delay guarantees
- A classical result in stochastic network analysis
  - Tassiulas&Ephremides: load-balancing locally is optimal
- Per-destination queueing
  - $\forall v \in V$: maintain a queue $Q_{v,d}$ for each destination $d \in V$
  - $q^t_{v,d}$: size of $Q_{v,d}$ in time $t$

# The MaxWeight Protocol

**Algorithm 1** MaxWeight($\beta$): at any time $t$

accept all arrivals at $t$
**for** every $(v, u) \in E_t$ **do**
  $d^*_{(v,u)} \leftarrow \arg\max_d \left\{ (q^t_{v,d})^\beta - (q^t_{u,d})^\beta \right\}$
  **if** $q^t_{v,d^*} - q^t_{u,d^*} \geq \Delta$ **then**
    forward a $d^*_{(v,u)}$-packet from $v$ to $u$
  **end if**
**end for**
drop all packets that have arrived at their destination

"constant" $\beta \geq 1$

**Theorem (for large enough $\beta$)**

*MaxWeight($\beta$) ensures the system is stable*

# Proof Idea

- Proof method: potential function
  - recall AAP
    - $|O \setminus A| \leq \Phi \leq O(\log n)\,|A|$
  - here: a “classical” drift argument

$$P(t) = \sum_{v,d \in V} (q^t_{v,d})^{\beta+1}$$

- Proof idea:
  - show that $P(t)$ is always bounded:
    - if queues become too large, $P(t)$ decreases (fast)!!
    - a.k.a.: *negative drift*
  - stability: since $\sum_{v,d \in V} q^t_{v,d} \leq P(t)$

# The Key (Technical...) Lemma

## Lemma

*Every packet $p$ injected at $t$ can be associated with packets forwarded along $\Gamma_p$ s.t. the sum of potential changes due to these forwardings until time $(t + w - 1)$ is at least*

$$O(q^{\beta-1}) - \frac{\varepsilon}{1-\varepsilon}(\beta+1)q^{\beta}$$

*where $q$ is height of $p$'s injection queue at $t$.*

## Corollary

*$\exists q^* = q^*(n, w, \varepsilon)$ s.t. if $q \geq q^*$ then the sum of potential change is less than $-\varepsilon q^{\beta}$.*

- I.e., the *decrease* is $\Omega(q^{\beta})$.

# Proof Sketch

- A packet is only forwarded to a smaller queue
  $\Longrightarrow$ potential increase is only due to injections
- By contradiction: adversary makes the system unstable

⊛ $\Longrightarrow$ the potential increases unboundedly
- There's no point to repeat a configuration
  - configuration $S_t$: state of all the queues at $t$
  - for all $t < t'$, $S_t \neq S_{t'}$
    - otherwise, we can just "remove" the period between $t$ and $t'$
- Number of configurations with max-queue size $q$: $q^{(n^2)}$
  $\Longrightarrow$ $t_q = O(q^{(n^2)})$: first time a queue reaches load $q$
    - after $q^{(n^2)}$ steps there's an injection to a queue with load $q$

# Proof Sketch

- A packet is only forwarded to a smaller queue
  $\Longrightarrow$ potential increase is only due to injections
- By contradiction: adversary makes the system unstable

  ⊛ $\Longrightarrow$ the potential increases unboundedly
- Overall potential inc. during $[t_q, t_q + w)$ can be shown to be
  - $O(t_q) = O(q^{(n^2)})$
- By lemma, potential dec. during $[t_q, t_q + w)$
  - $\Omega(q^\beta)$
  - for $q \geq q^*$
- for $\beta > n^2$ and large enough $q$ the difference is $< 0$
  - I.e., a *negative drift*
- Contradiction!

# Aftermath and References

- Aftermath
  - results extend also to arbitrary edge capacities/packet lengths
  - works also for $\beta = 1$
    - a much more complex proof...
  - various related models / results
    - both generalizations and special cases

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- References

  - Andrews, Jung, and Stolyar. Stability of the Max-Weight Routing and Scheduling Protocol in Dynamic Networks and at Critical Loads, STOC 2007, pp. 145–154
  - Lim, Jung, and Andrews. Stability of the Max-Weight Protocol in Adversarial Wireless Networks, Infocom 2012, pp. 1251–1259
  - Tassiulas and Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks, IEEE Transactions on Automatic Control 37(12):1936–1948 (1992)