Basic Concepts of Sets

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One of the most basic and important concepts of mathematics is the set. In order to deal with objects, we need to classify them, and classifications will be according to the sets they belong to.
Definition: A set is an unordered collection of distinct objects. **Notation:** We denote a set by enclosing the objects of the set in braces. the objects themselves (or elements) are set apart one from the other by a comma. We will usually label our sets with capital letters.
Examples:

- A set of countries: $A = \{China, France, Greece, India\}$
- A set of natural numbers: $B = \{2, 6, 3, 104, 19\}$

Sets can include a finite number of objects, but this is not mandatory. Let's introduce the most basic sets of numbers. The set of natural numbers can be defined as set that includes one and rule: If the number n is in the set, then its successor $n+1$ is also in the set. This can be written with mathematical notations: $0 \in A \wedge (n \in A \rightarrow n+1 \in A)$ and the translation is: 0 is the set A and n is in A implies $n+1$ is in A. We denote the set of natural number by $\mathbb{N}$. Another important notation is the colon - which means "such that". Thus the set $\{2n : n \in \mathbb{N}\}$ is the set of all even non-negative integers. Another example can be $\{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$.

Next we would like to construct the set of all integers. In order to do that, we need to introduce some operations on sets:

- Union: $A \cup B = \{x : x \in A \vee x \in B\}$
- Intersection: $A \cap B = \{x : x \in A \wedge x \in B\}$
- Difference: $A \setminus B = \{x \in A : x \notin B\}$

Examples: Let $A = \{1, 2, 3, 4, 5\}, B = \{2, 5, 89\}$

- $A \cup B = \{1, 2, 3, 4, 5, 89\}$
- $A \cap B = \{2, 5\}$
- $A \setminus B = \{1, 3, 4\}$

A set may be empty, in other words, it may include no objects. The notation for the empy set is $\{\ \} = \emptyset$. For example $A \setminus A = \emptyset$.

We are ready to construct now the set of integers:

$$\mathbb{Z} = \mathbb{N} \cup \{-n : n \in \mathbb{N}\}$$

Note that since this course is a pre-academic course, our review is only semi-formal. If we were to do our constructions formally, as in a proper academic course, we would have first chosen a set of axioms in order to create a consistent theory. Without that $-n$ has no meaning, but for example, in group theory, it is defined to be the inverse of n in $\mathbb{Z}$ as a group with the operation $+$.

With the integers we can add and subtract. We can also multiply two integers. But if we would like to compute for example $2 : 3$, the result will not be an integer. Hence we will construct the set of rational numbers:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\} \right\}$$

So the rational numbers are every number that can be expressed as a quotient of two integers. For example $\frac{1}{2}, \frac{6}{17}, \frac{31}{10}$. Indeed every integer is also a rational number, since if $m \in \mathbb{Z}$ then $m = \dfrac{m}{1} \in \mathbb{Q}$.

Definition: Set A is said to be *included* in set B iff every element of A is also an element of B. Formally we write

$$A \subseteq B \Leftrightarrow (x \in A \rightarrow x \in B)$$

Examples:

- $\{3, 5\} \subseteq \{1, 2, 3, 4, 5\}$
- $\{1, 2, 3\} \nsubseteq \{2, 3, 4\}$
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$
- $\{Parallelograms\} \subseteq \{Quadrilaterals\}$

An amazing discovery of the ancients was that there are numbers (or more precisely ratios) that cannot be written as a ratio of two integers. By the famous Pythagorean Theorem, the diagonal of a square with sides of length 1cm should be of length $\sqrt{1^2+1^2} = \sqrt{2}$cm. Indeed $\sqrt{2}$ is not a rational number i.e. it cannot be written as a quotient of two integers.

Proof: For our proof, we shall state a known fact: If $a \in \mathbb{Z}$ is even then a^2 is even and if $b \in \mathbb{Z}$ is odd then b^2 is odd. This can be proved easily and we shall leave that as an exercise to the reader. From that we conclude a is even iff a^2 is even.

We are set now for the proof and we will use the technique of proof by contradiction. Assume that $\sqrt{2}$ is a rational number. That means $\sqrt{2}$ can be written as a ratio of two co-prime integers:

$$\sqrt{2} = \frac{a}{b} \qquad a, b \in \mathbb{Z}$$

We remind you that if a and b are co-prime ($gcd(a,b) = 1$), then $\frac{a}{b}$ is expressed in lowest terms. Square both sides of the equation and multiply by b^2 to get

$$2b^2 = a^2$$

Then a^2 is divisible by 2, so a is even. Hence $a = 2k$, $k \in \mathbb{Z}$. Plugging $a = 2k$ in the last equation we get

$$2b^2 = 4k^2$$

divide by 2 and $b^2 = 2k^2$ implies that b is also even, which contradicts our assumption that a and b are co-prime.

In this present course we will mainly deal with sets of real numbers, which are actually all the number we are familiar with from math class in junior high (not including complex number which you probably met during your high-school math favourite teacher) and in particular intervals of the real axis. Those intervals may or may not include their boundary points. For intervals that include their boundary points we use brackets and call them closed intervals. Intervals that do not include their boundary points are called open intervals and we use parentheses to denote those. **Examples:**

- $[4, 12] = \{x \in \mathbb{R} : 4 \leq x \leq 12\}$
- $(2, 5) = \{x \in \mathbb{R} : 2 < x < 5\}$
- $[-3, 1) = \{x \in \mathbb{R} : -3 \leq x < 1\}$
- $(3, \infty) = \{x \in \mathbb{R} : 3 < x\}$

- $(-\infty, \infty) = \mathbb{R}$

Exercises

1. $\mathbb{R} \setminus \mathbb{R} =$
2. $A \setminus \emptyset =$
3. True or False
   (a) $\mathbb{Z} \in \mathbb{Z}$
   (b) $\emptyset \subseteq \mathbb{Z}$
   (c) $\emptyset \subseteq \emptyset$
   (d) $\emptyset \in \emptyset$
4. Let $A \subseteq B$ and $B \subseteq C$
   (a) Prove $A \subseteq C$
   (b) Prove (a) by contradiction
5. True or False
   (a) $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$
   (b) $\pi \in \mathbb{R} \setminus (\mathbb{N} \cup \mathbb{Z})$
   (c) $\{e, 1, \sqrt{5}\} \in \mathbb{R} \setminus [5, 10]$
   (d) $\{\sqrt{1}, \sqrt{2}, ..., \sqrt{77}\} \subseteq \mathbb{R} \setminus [5, 10]$
6. $[10, 20) \setminus \big((5, 12) \cup [11, 13)\big) =$
7. True or False
   (a) $5 \in \mathbb{N} \cap \mathbb{Z}$
   (b) $2\sqrt{7} \in \mathbb{R} \cap \mathbb{Z}$
   (c) $100 \in \mathbb{Q} \setminus \mathbb{R}$
   (d) $\{7\} \subseteq (\mathbb{Q} \setminus \mathbb{Z}) \cup \mathbb{N}$
8. Prove

(a) $\sqrt{3} \notin \mathbb{Q}$
(b) $0 \neq a \in \mathbb{Q}, b \notin \mathbb{Q} \Rightarrow ab \notin \mathbb{Q}$
(c) $a \in \mathbb{Q}, b \notin \mathbb{Q} \Rightarrow a + b \notin \mathbb{Q}$
(d) $1 + \sqrt{5} \notin \mathbb{Q}$

9. $A = (2, \infty), B = \mathbb{N}, C = [-5, 5], D = [1, 7], E = (-2, 0]$
   (a) $A \cap E =$
   (b) $A \cap B =$
   (c) $C \cup D =$
   (d) $(C \cup D) \cap A =$

Answers

1. $\emptyset$
2. A
3. (a) False
   (b) True
   (c) True
   (d) False
4. Proof
5. (a) True
   (b) True
   (c) False
   (d) False
6. $[13, 20)$
7. (a) True
   (b) False
   (c) False
   (d) True

8. Proof
9. (a) $\emptyset$
   (b) $\mathbb{N} \setminus \{1, 2\}$
   (c) $[-5, 7]$
   (d) $(2, 7]$