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Lemma. *Let F_n be the n'th Fibbonacci number. Proof that for all odd n the following equality holds.*

$$F_n = \sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}+k}{2k}$$

Proof. To prove this equality we are going to use double counting tecnique and we are going to count the number of compositions of number n in an odd number of parts.

We first divide the set of partitions in 2, the ones that have 1 as last element and the ones that do not. Those sets are disjoint and each composition is in one of those 2 sets. Let c_n be the number of compositions of number n in an odd number of parts. Then we have that the first set is equal to c_{n-1} since we can add a 1 the each composition of $n-1$ at the end. We also have that the second set is equal to c_{n-2} since we can add 2 to the last number in each composition of $n-2$.

We know that the number of partitions of 1 and 2 in odd parts is both 1, therefore we have that

$$c_n = c_{n-1} + c_{n-2} \qquad c_1 = c_2 = 1$$

And we know that that recurrence is the same as the Fibbonacci recurrence. Therefore $c_n = F_n$.

We will now count that a different way. We will first find the number of ways to compose a number n in k odd parts and then we will sum that for all k. The number of ways to compose n in k odd parts is the same as the ways to compose $n-k$ into even parts which is the same as the number of ways to compose $\frac{n-k}{2}$ (if n is odd and k even we know that the sum of k odd numbers is an even number so the number of compositions for k odd will be 0).

We also know that the number of compositions of a number into k parts is $\binom{n+k-1}{k-1}$ Therefore we find that the number of partitions

$$\sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n-2k+1}{2} + 2k + 1 - 1}{2k + 1 - 1} = \sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n+1}{2} + k}{2k}$$

From where we can conclude that

$$F_n = \sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2} + k}{2k}$$

$\square$