INTEGRALI di FUNZIONI RAZIONALI

$$\int_a^b \frac{P_m(x)}{Q_n(x)}\,dx$$

dove $P_m(x)$ = polinomio di grado m

$Q_n(x)$ = polinomio di grado n

Consideremo $n > m$

In particolare per noi $n = 2$

Ci sono 3 casi:

1) Q_2 ha 2 radici distinte

$\left(\text{se } Q_2(x) = ax^2 + bx + c \longrightarrow \Delta = b^2 - 4ac > 0 \right)$

__Es__ $$\int_4^6 \frac{x+1}{x^2-4x+3}\,dx$$

- Scompongo

$$x^2-4x+3=(x-3)(x-1)$$

- (scomposizione in fratti semplici).

Cerco $A, B \in \mathbb{R}$ t.c.

$$\frac{x+1}{(x-3)(x-1)}=\frac{A}{x-3}+\frac{B}{x-1}$$

Cerco A e B t.c.

$$\frac{x+1}{(x-3)(x-1)} = \boxed{\frac{A}{x-3} + \frac{B}{x-1}}$$

$$\Rightarrow \frac{Ax - A + Bx - 3B}{(x-3)(x-1)} = \frac{x+1}{(x-3)(x-1)}$$

$$\Leftrightarrow x(A+B) - A - 3B = x+1$$

$$\Rightarrow \begin{cases} A+B=1 \\ -A-3B=1 \end{cases} \quad \begin{cases} A = 1-B \\ -1+B-3B=1 \end{cases} \quad \begin{cases} A=2 \\ B=-1 \end{cases}$$

$$\Rightarrow Q_2(x) = \frac{2}{x-3} - \frac{1}{x-1}$$

$$\Rightarrow \int Q_2(x)\,dx = \int \left(\frac{2}{x-3} - \frac{1}{x-1} \right) dx$$

$$= \Big[2\log(x-3) - \log(x-1) \Big]_4^6$$

2) Q_2 ha una sola radice $(\Delta = 0)$

$\Rightarrow Q_2(x) = (ax+c)^2$

ES $\displaystyle\int_1^2 \frac{x+1}{(x-3)^2}\,dx$

pongo $t = x-3$

$dt = dx$

$x = t+3$

$$\Rightarrow \int_{-2}^{-1} \frac{t+4}{t^2}\,dt$$

$$= \int_{-2}^{-1} \frac{1}{t}\,dt + \int_{-2}^{-1} \frac{4}{t^2}\,dt = \left[\log|t| - \frac{4}{t}\right]_{-2}^{-1}$$

$\sim\sim\sim$

3) Q_2 non ha radici reali $(\Delta < 0)$

$$\int \frac{2x+1}{2+x^2}\,dx = \underbrace{\int \frac{2x}{2+x^2}\,dx}_{\left[\log(2+x^2)\right]} + \underbrace{\int \frac{1}{2+x^2}\,dx}_{*}$$

$$* \int \frac{1}{2+x^2}\,dx = \frac{1}{2}\int \frac{1}{1+\frac{x^2}{2}}\,dx = \underbrace{\frac{1}{2}\int \sqrt{2}\cdot\frac{\frac{1}{\sqrt{2}}}{1+\left(\frac{x}{\sqrt{2}}\right)^2}\,dx}_{\frac{\sqrt{2}}{2}\left[\arctan\left(\frac{x}{\sqrt{2}}\right)\right]}$$

$$\int \frac{1}{x^2+2x+3}\,dx$$

$$x^2+2x+3+1-1 = (x+1)^2+2$$

$$\int \frac{1}{2+(x+1)^2}\,dx = \frac{1}{2}\int \frac{\sqrt{2}\cdot\frac{1}{\sqrt{2}}}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2}\,dx$$

$$= \frac{\sqrt{2}}{2}\left[\arctan\left(\frac{x+1}{\sqrt{2}}\right)\right]$$

ESERCIZI VARI

$$\int_2^8 \frac{dx}{2\sqrt{x}+x\sqrt{x}} = 2\int_2^8 \frac{dx}{2\sqrt{x}\,(2+x)}$$

Pongo

$t=\sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\,dx$

$$= 2\int_{\sqrt{2}}^{\sqrt{8}} \frac{dt}{2+t^2} = \frac{\cancel{2}}{\cancel{2}}\int_{\sqrt{2}}^{\sqrt{8}} \frac{dt}{1+\left(\frac{t}{\sqrt{2}}\right)^2}$$

$$= \left[\sqrt{2}\ \arctan\left(\frac{t}{\sqrt{2}}\right)\right]_{\sqrt{2}}^{\sqrt{8}}$$

$$\circ\ \frac{1}{2}\int_0^1 \frac{2e^{2x}}{1+e^{2x}+e^{4x}}\,dx$$

Pongo $t = e^{2x}$

$dt = 2e^{2x}dx$

$$=\frac{1}{2}\int_1^{e^2} \frac{dt}{1+t+t^2}$$

Considero $1+\underbrace{t+t^2}=\left(t+\frac{1}{2}\right)^2-\frac{1}{4}+1$

$$=\left(t+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\frac{1}{2}\int \frac{dt}{\frac{3}{4}+\left(t+\frac{1}{2}\right)^2} = \frac{1}{2}\cdot\frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}}{\underbrace{1+\left(\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right)^2}}\,dt$$

$$= \frac{\sqrt{3}}{3}\left[\operatorname{arctan}\left(\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right)\right]$$

$$\int_0^{\frac{\pi}{2}} e^{\operatorname{sen}^2 x}\,\underbrace{\operatorname{sen}^3 x\cos x}_{\frac{1}{2}\operatorname{sen}^2 x\; 2\,\underbrace{\operatorname{sen} x\cos x}}\,dx$$

Pongo $t = \underline{\operatorname{sen}^2 x}$

$dt = 2\operatorname{sen} x\cos x\,dx$

$$= \frac{1}{2}\int_0^1 e^t\cdot t\,dt \underset{\text{per parti}}{=} \frac{1}{2}\left[e^t\cdot t\right]_0^1 - \frac{1}{2}\int_0^1 e^t\,dt = \frac{1}{2}\left[e^t\cdot t - e^t\right]_0^1$$

$$\int_1^e \frac{1+\log x}{x(\log^2 x+3)}\,dx \qquad \text{Pongo } t=\log x$$

$$dt = \frac{1}{x}\,dx$$

$$= \int_0^1 \frac{1+t}{t^2+3}\,dt = \underbrace{\int_0^1 \frac{1}{t^2+3}\,dt}_{\ldots\ \arctan\left(\frac{t}{\sqrt{3}}\right)\Big]_{\ldots}^{-}} + \frac{1}{2}\underbrace{\int_0^1 \frac{2t}{t^2+3}\,dt}_{\left[\frac{1}{2}\log(t^2+3)\right]_{\ldots}^{\ldots}}$$

$$\int_0^1 \frac{\log(t+1)}{(2+t)^2}\,dt \overset{\text{per parti}}{=}$$

$$= \left[-\frac{1}{2+t}\log(t+1)\right]_0^1 + \overbrace{\int_0^1 \frac{1}{2+t}\cdot\frac{1}{t+1}\,dt}^{*}$$

* Scrivo: $\dfrac{1}{(2+t)(t+1)} = \boxed{\dfrac{A}{2+t} + \dfrac{B}{t+1}}$

$\Longleftrightarrow At + A + 2B + Bt = 1 \Longleftrightarrow \begin{cases} A+B=0 \\ A+2B=1 \end{cases} \Rightarrow \begin{cases} A=-B \\ B=1 \end{cases} \begin{cases} A=-1 \\ B=1 \end{cases}$

$t(A+B)+A+2B$

$$\Rightarrow \circledast = \int \left(-\frac{1}{2+t} + \frac{1}{t+1} \right) dt$$

$$= \left[-\log(2+t) + \log(t+1) \right]$$

$$\int 3x \arctan(x+1)\,dx \overset{\text{per parti}}{=}$$

$$= \left[\frac{3}{2}x^2 \arctan(x+1)\right] - \underbrace{\int \frac{3}{2}x^2 \cdot \frac{1}{1+(x+1)^2}}_{(*)}$$

$$(*)\ \frac{3}{2}\int \cancelto{1}{\frac{x^2+2x+2}{x^2+2x+2}} - \underbrace{\frac{2x+2}{x^2+2x+2}}\ \longrightarrow \left[\log(x^2+2x+2)\right]$$