1) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad F(x,y,z) = (x-z, 2x-y+z)$

$M^{E_3}_{E_2}(F) = ?$

$$M^{E_3}_{E_2}(F) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

2) $U \subseteq \mathbb{R}^3$ $\quad U: x+y+z=0$

$F: U \to U \quad F(x,y,z) = (x-y,\ 3y+2z,\ -y-z)$

a) $B = (\underbrace{(1,-1,0)}_{b_1}, \underbrace{(0,1,-1)}_{b_2})$ è base di U

b_1 e b_2 sono lin. ind.

$|B| = 2 = \dim(U)$

$b_1, b_2 \in U$ perché verificano l'eq.

Dobbiamo verificare che

$$F(u) \in U \qquad \forall u \in U$$

$$F(x,y,z) = (x-y,\ 3y+2z,\ -y-z)$$

$$(x-y) + (3y+2z) + (-y-z) = x+y+z = 0 \checkmark$$

b) $M_{B}^{B}(F) = ?$

$$F(b_1) = F(1, -1, 0) = (2, -3, 1) =$$

$$= 2(1, -1, 0) - (0, 1, -1) = (2, -1)_B$$

$$F(b_2) = F(0, 1, -1) = (-1, 1, 0) = (-1, 0)_B$$

$$M_{B}^{B}(F) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

c) $u = (2,3)_B \qquad F(u) \stackrel{?}{=}$

$$M_B^B(F) \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow F(u) = (1,-2)_B = 1(1,-1,0) - 2(0,1,-1)$$

$$= (+1, -3, +2)$$

Esercizio 3

$$f: \mathbb{R}^2 \to \mathbb{R}^2$$

$(1,0) = \alpha(1,1) + \beta(1,-1)$

$$\begin{cases} \alpha + \beta = 1 \\ \alpha - \beta = 0 \end{cases}$$

$f(1,1) = (1,2) \qquad f(1,-1) = (3,4)$

$f(1,0) = ? \qquad f(0,1) = ? \qquad \mathcal{B} = ((1,1),(1,-1))$

$M^{\mathcal{B}}_{\mathcal{E}}(f) = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad (1,0) = \left(\frac{1}{2}, \frac{1}{2}\right)_{\mathcal{B}} \qquad f(1,0) =$

$(0,1) = \left(\frac{1}{2}, -\frac{1}{2}\right)_{\mathcal{B}}$

$f((x,y)_{\mathcal{B}}) = (x+3y, 2x+4y)_{\cancel{\mathcal{E}}}$

$$(1,0) = \left(\frac{1}{2}, \frac{1}{2}\right)_B \qquad (0,1) = \left(\frac{1}{2}, -\frac{1}{2}\right)_B$$

$$f((x,y)_B) = (x+3y, 2x+4y)_E$$

$$f(1,0) = (2,3)_E \qquad f(0,1) = (-1,-1)_E$$

$$M_B^E(f) = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

$$f((x,y)_E) = (2x-y, 3x-y)_E$$

$$M_B^E(f) = \begin{pmatrix} 5/2 & , \\ -1/2 & \end{pmatrix}$$

$$f(1,0) = (2,3) = \alpha(1,1) + \beta(1,-1)$$

$$\begin{cases} \alpha + \beta = 2 \\ \alpha - \beta = 3 \end{cases}$$

$$\alpha = 5/2$$

$$\beta = -\frac{1}{2}$$

Esercizio 4

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$g: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$f(1,2,1) = (2,1,3)$ $\qquad$ $f(1,2,3) = (0,0,0)$

$g(1,2,1) = (2,1,3)$ $\qquad$ $g(1,2,3) = (0,0,0)$

Determiniamo un vettore che completi $(1,2,1)$, $(1,2,3)$ ad una base di $\mathbb{R}^3$

Ad esempio $(1,0,0)$

$f(1,0,0) = (0,0,0)$

$g(1,0,0) = (1,0,0)$

$f(1,0,0) = (0,0,0)$

$f(0,1,0) = \left(\frac{6}{4}, \frac{3}{4}, \frac{9}{4}\right)$

$f(0,0,1) = \left(-1, -\frac{1}{2}, -\frac{3}{2}\right)$

$$(0,1,0) = \alpha(1,2,1) + \beta(1,2,3) + \gamma(1,0,0)$$

$$\begin{cases} \alpha+\beta+\gamma=0 \\ 2\alpha+2\beta=1 \\ \alpha+3\beta=0 \end{cases} \qquad \begin{cases} -2\beta+\gamma=0 \\ -6\beta+2\beta=1 \\ \alpha=-3\beta \end{cases} \qquad \begin{cases} \gamma=-1/2 \\ \beta=-1/4 \\ \alpha=3/4 \end{cases}$$

$$f(0,1,0) = f\left(\frac{3}{4}(1,2,1) - \frac{1}{4}(1,2,3) - \frac{1}{2}(1,0,0)\right)$$

$$= \frac{3}{4} f(1,2,1) - \cancel{\frac{1}{4} f(1,2,3)} - \cancel{\frac{1}{2} f(1,0,0)} = \left(\frac{6}{4}, \frac{3}{4}, \frac{9}{4}\right)$$

$$g(0,1,0) = \frac{3}{4} g(1,2,1) - \frac{1}{2} g(1,0,0) = \left(1, 3/4, 9/4\right)$$

$$(0,0,1) = \alpha(1,2,1) + \beta(1,2,3) + \gamma(1,0,0)$$

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ 2\alpha + 2\beta = 0 \\ \alpha + 3\beta = 1 \end{cases} \qquad \begin{cases} \gamma = 0 \\ \alpha = -\beta = -1/2 \\ \beta = 1/2 \end{cases}$$

$$f(0,0,1) = -\frac{1}{2} f(1,2,1) + \frac{1}{2} \cancel{f(1,2,3)} =$$

$$= -\frac{1}{2}(2,1,3) = \left(-1, -\frac{1}{2}, -\frac{3}{2}\right)$$

$$g(0,0,1) = -\frac{1}{2} g(1,2,1) + \frac{1}{2} g(1,2,3) = \left(-1, -\frac{1}{2}, -3/2\right)$$

$$f(1,0,0) = (0,0,0)$$

$$f(0,1,0) = \left(6/4,\ 3/4,\ 9/4\right)$$

$$f(0,0,1) = \left(-1,\ -1/2,\ -3/2\right)$$

$$M_E^E(f) = \begin{pmatrix} 0 & 6/4 & -1 \\ 0 & 3/4 & -1/2 \\ 0 & 9/4 & -3/2 \end{pmatrix}$$

$$f(x,y,z) = \left(6/4\, y - z,\ 3/4\, y - \frac{1}{2} z,\ \frac{9}{4} y - 3/2\, z\right)$$

$g(1,0,0)=(\underline{1,0,0})$

$g(0,1,0)=(\underline{1,\tfrac{3}{4},\tfrac{9}{4}})$

$g(0,0,1)=(\underline{-1,-\tfrac{1}{2},-\tfrac{3}{2}})$

$$M_E^E(g)=\begin{pmatrix} 1 & 1 & -1 \\ 0 & \tfrac{3}{4} & -\tfrac{1}{2} \\ 0 & \tfrac{9}{4} & -\tfrac{3}{2} \end{pmatrix}$$

$$g(x,y,z)=\left(x+y-z,\ \tfrac{3}{4}y-\tfrac{1}{2}z,\ \tfrac{9}{4}y-\tfrac{3}{2}z\right)$$

$$M_E^B(f) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$M_E^B(g) = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

Esercizio 5

Stabilire se esiste $f: \mathbb{R}^3 \to \mathbb{R}^2$ t.c.

$$f(1,2,1) = (2,3)$$

$$f(0,1,1) = (1,2)$$

$$f(1,0,-1) = (2,2)$$

$$(1,2,1) - 2(0,1,1) - (1,0,-1) = (0,0,0)$$

$\Rightarrow$ i tre vettori su cui è definita f NON sono indipendenti.

$$(1,2,1)-2(0,1,1)-(1,0,-1)=(0,0,0)$$

$$f(1,2,1)=2f(0,1,1)+f(1,0,-1)$$

$$\begin{array}{c} \| \\ (2,3) \end{array} \overset{?}{=} 2(1,2)+(2,2) \qquad \times$$

$\Rightarrow$ f non esiste !!

Esercizio 6 $\quad \mathbb{R}^3$

$S: x+y=0$

$T=\langle C \rangle \qquad C=((1,1,1),(2,3,1))$

$B=((1,-1,0),(-2,2,1))$

$f: S \to T \qquad f((x,y)_B)=(x+2y,\, y-2x)_C$

$f(3,-3,3)$? $\qquad (3,-3,3)= 9(1,-1,0)+3(-2,2,1)$

$\Vert$

$f((9,3)_B) = (15,-15)_C = 15(1,1,1)-15(2,3,1)$

$= (-15,-30,0)$