ESERCIZI

- $$\lim_{n\to+\infty}\left(\frac{n+5}{n+2}\right)^{n} = \lim_{n\to+\infty}\left(1+\frac{3}{n+2}\right)^{n}$$

$$= \lim_{n\to+\infty}\left(1+\frac{3}{n+2}\right)^{\frac{n+2}{3}\cdot\underbrace{\frac{3}{n+2}\cdot n}}$$

$$= \lim_{n\to+\infty}\underbrace{\left[\left(1+\frac{3}{n+2}\right)^{\frac{n+2}{3}}\right]}_{e}^{\frac{3n}{n+2}} = e^{3}$$

- $$\lim_{n \to +\infty} \frac{3n^4 - n! + n^n}{\pi n^{n+1}}$$

$$= \lim_{n \to +\infty} \frac{n^n}{\pi n^{n+1}} = \lim_{n \to +\infty} \frac{1}{\pi n} = 0$$

- $$\lim_{n \to +\infty} \underbrace{(-1)^n}_{\text{limitata}} \cdot \underbrace{\frac{\sqrt{2^n + n^2}}{n!}}_{\to 0} = 0$$

$$\lim_{n \to +\infty} \frac{2n^{n-1} + (n+3)^n}{2n! + 4n^n}$$

$$= \lim_{n \to +\infty} \frac{\cancel{n^n}\left[\overset{\to 0}{\frac{2}{n}} + \left(1 + \frac{3}{n}\right)^n\right]}{\cancel{n^n}\left(\underbrace{\frac{2n!}{n^n}}_{\to 0} + 4\right)}$$

$$= \lim_{n \to +\infty} \frac{\left(1 + \frac{3}{n}\right)^n}{4} = \frac{e^3}{4}$$

- $$\lim_{n\to+\infty} \frac{n^3\left(3^n+2^n\right)}{5^n+n^7}$$

$$= \lim_{n\to+\infty} \frac{n^3\cdot 2^n}{5^n} = \lim_{n\to+\infty} n^3\left(\frac{2}{5}\right)^n$$

$$= \lim_{n\to+\infty} \frac{n^3}{\left(\frac{5}{2}\right)^n} = 0$$

IERI: • $\dfrac{n^k}{a^n} \xrightarrow[n\to+\infty]{} 0$ $\boxed{a>1}$

• $n^k\cdot a^n \longrightarrow 0$ $\quad 0<a<1$

$\Rightarrow \dfrac{n^k}{\left(\frac{1}{a}\right)^n} \longrightarrow 0$

$\hookrightarrow$ poiché $0<a<1$

$\Rightarrow \dfrac{1}{a}>1$

$$\cdot \lim_{n \to +\infty} \overbrace{\frac{\sqrt{n^{\alpha}+n}}{n^2+n}}^{a_n}$$

Se $\underline{\alpha > 1}$: $\quad a_n \sim \dfrac{\sqrt{n^{\alpha}}}{n^2} = \dfrac{n^{\frac{\alpha}{2}}}{n^2}$

- Se $\alpha > 4 \quad a_n \longrightarrow +\infty$
- Se $\alpha = 4 \quad a_n \longrightarrow 1$
- Se $1 < \alpha < 4 \quad a_n \longrightarrow 0$

$$a_n = \frac{\sqrt{n^2+n}}{n^2+n}$$

$\underline{\alpha = 1}$ $$a_n = \frac{\sqrt{2n}}{n^2+n} \xrightarrow[n\to+\infty]{} 0$$

$\underline{\alpha < 1}$ $$a_n = \frac{\sqrt{n}}{n^2+n} \xrightarrow[n\to+\infty]{} 0$$

<u>Conclusione</u>:

$\alpha < 4 \Rightarrow \lim_{n\to+\infty} a_n = 0$, $\alpha = 4 \Rightarrow \lim_{n\to+\infty} a_n = 1$

$\alpha > 4 \Rightarrow \lim_{n\to+\infty} a_n = +\infty$

$$\bullet \lim_{n \to +\infty} \frac{2^n}{n^4+3n^2+1} \cdot \frac{2n^4+3}{(1+|z|)^n}$$

$$\to 2$$

$$= \lim_{n \to +\infty} 2 \underbrace{\left(\frac{2}{1+|z|} \right)^n}$$

Se $\frac{2}{1+|z|} > 1 \iff 2 > 1+|z|$

$\iff |z| < 1$

$\iff -1 < z < 1 \Rightarrow \lim_{n \to +\infty} a_n = +\infty$

- Se $z = \pm 1 \Rightarrow \lim_{n \to +\infty} a_n = 2$

- Se $z < -1 \lor z > 1 \Rightarrow \lim_{n \to +\infty} a_n = \varnothing$

- $$\lim_{n \to +\infty} \frac{2^{2n} (n+2)^{n+1}}{3^n \cdot n^n}$$

$$= \lim_{n \to +\infty} \left(\frac{2^2}{3}\right)^n \frac{\cancel{n}^{n+1} \left(1 + \frac{2}{n}\right)^{n+1}}{\cancel{n^n}}$$

e^2

$\boxed{\left(\frac{2^2}{3}\right)^n}$

Se $2^2 > 3 \iff 2 > \log_2 3 \Rightarrow \lim\limits_{n\to+\infty} a_n = +\infty$

- Se $2 = \log_2 3 \Rightarrow \lim\limits_{n\to+\infty} a_n = +\infty$

- Se $2 < \log_2 3 \Rightarrow \lim\limits_{n\to+\infty} \underbrace{\left(\frac{2^2}{3}\right)^n}_{} \cdot n e^2 = \emptyset$

$$\bullet \lim_{n \to +\infty} \frac{(n^2+1)^n}{n^{2n}} = \lim_{n \to +\infty} \frac{\cancel{n^{2n}}\left(1+\frac{1}{n^2}\right)^n}{\cancel{n^{2n}}}$$

$$= \lim_{n \to +\infty} \left[\underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{} \right]^{\frac{1}{n}}$$

$$= \lim_{n \to +\infty} e^{\frac{1}{n}} = 1$$

- $$\lim_{n\to+\infty} \frac{n!}{e^n \cdot n^3}$$

Usiamo il criterio del rapporto:

$$\boxed{\frac{a_{n+1}}{a_n}} = \frac{(n+1)!}{e^{n+1}(n+1)^3} \cdot \frac{e^n \cdot n^3}{n!}$$

$$= \frac{\cancel{n!}(n+1)}{e\,\cancel{n^3}\left(1+\frac{1}{n}\right)^3} \cdot \frac{\cancel{n^3}}{\cancel{n!}} = \frac{n+1}{e\left(1+\frac{1}{n}\right)^3} \xrightarrow[n\to+\infty]{} +\infty$$

Quindi: $\lim_{n\to+\infty} a_n = +\infty$

- $\lim_{n\to+\infty} \frac{n^n}{(n+1)^n}\left(\frac{2}{|\alpha|}\right)^n = \lim_{n\to+\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}\left(\frac{2}{|\alpha|}\right)^n$

$$= \lim_{n\to+\infty} \frac{1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to \frac{1}{e}}}\left(\frac{2}{|\alpha|}\right)^n$$

- Se $|\alpha|<2 \iff -2<\alpha<2$: $\lim_{n\to+\infty} a_n = +\infty$

- Se $\alpha = \pm 2 \Rightarrow \lim_{n\to+\infty} a_n = \frac{1}{e}$

- Se $\alpha<-2 \vee \alpha>2 \Rightarrow \lim_{n\to+\infty} a_n = 0$

$$\lim_{n\to+\infty} \frac{n^2+n^\alpha}{\sqrt{n^3+n^{2\alpha}}}$$

Se $\alpha>2$ $\quad a_n \sim \dfrac{n^\alpha}{n^\alpha} \longrightarrow 1$ per $n\to+\infty$

Se $\alpha=2$ $\quad a_n \sim \dfrac{2n^2}{n^2} \longrightarrow 2$ per $n\to+\infty$

Se $\frac{3}{2}<\alpha<2$ $\quad a_n \sim \dfrac{n^2}{n^\alpha} \longrightarrow +\infty$ per $n\to+\infty$

Se $\alpha=\frac{3}{2}$ $\quad a_n \sim \dfrac{n^2}{\sqrt{2}\, n^{3/2}} \longrightarrow +\infty$ per $n\to+\infty$

Se $\alpha < \frac{3}{2}$ $\quad a_n \sim \frac{n^2}{n^{3/2}} \longrightarrow +\infty$

• $$\lim_{n\to+\infty} \frac{\left(\sqrt{n^4+n^3}-\sqrt{n^4-n^3}\right)}{n+n^2} \cdot \frac{\left(\sqrt{n^4+n^3}+\sqrt{n^4-n^3}\right)}{\sqrt{n^4+n^3}+\sqrt{n^4-n^3}}$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^4}+n^3-\cancel{n^4}+n^3}{(n+n^2)\left[\underbrace{\sqrt{n^4+n^3}+\sqrt{n^4-n^3}}_{=2n^2(1+o(1))}\right]}$$

$$= \lim_{n\to+\infty} \boxed{\frac{n}{n+n^2}}$$

$$a_n = \frac{n}{n+n^{\alpha}}$$

Se $\alpha > 1 \Rightarrow a_n \sim \dfrac{n}{n^{\alpha}} \longrightarrow 0$ per $n \to +\infty$

Se $\alpha = 1 \Rightarrow a_n \sim \dfrac{n}{2n} \longrightarrow \dfrac{1}{2}$ per $n \to +\infty$

Se $\alpha < 1 \Rightarrow a_n \sim \dfrac{n}{n} \longrightarrow 1$ per $n \to +\infty$

• $$\lim_{n\to+\infty} \frac{3^n}{(n+1)^n} \cdot \frac{(n+2)^n}{(|z|+1)^n}$$

$$= \lim_{n\to+\infty} \left(\frac{3}{|z|+1}\right)^n \cdot \underbrace{\left(1+\frac{1}{n+1}\right)^n}_{\to e}$$

• Se $|z|+1<3 \iff |z|<2 \iff -2<z<2 \Rightarrow \lim_{n\to+\infty} a_n = +\infty$

• Se $z=\pm 2 \Rightarrow \lim_{n\to+\infty} a_n = e$

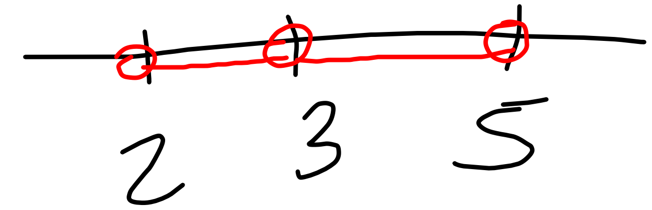
• Se $z<-2 \vee z>2 \Rightarrow \lim_{n\to+\infty} a_n = 0$

LIMITI per FUNZIONI di UNA VARIABILE REALE

Def Sia $J \subseteq \mathbb{R}$. Diciamo che J è un INTERVALLO FORATO se $\exists\, c \in \mathbb{R} \setminus J$ t.c. $J \cup \{c\}$ è un INTERVALLO.

Es. $\mathbb{R}^*$ è un INTERVALLO FORATO poiché $\mathbb{R}^* \cup \{0\} = \mathbb{R}$ è un intervallo

• $\rightarrow (2,3) \cup (3,5)$



è un INT. FORATO

poiché $(2,3) \cup (3,5) \cup \{3\} = (2,5)$ è INTERVALLO

Def (LIMITE) Sia I un intervallo o un intervallo forato di $\mathbb{R}$. Sia $f: I \to \mathbb{R}$, sia $c \in [\inf I, \sup I]$, sia $\ell \in \overline{\mathbb{R}}$. Diciamo che esiste il LIMITE di $f(x)$ per $x \to c$ ed è uguale a ℓ se $\forall\, (a_n)_{n \in \mathbb{N}}$ t.c. $a_n \xrightarrow[n \to +\infty]{} c$, si ha $f(a_n) \xrightarrow[n \to +\infty]{} \ell$.

Def

- Se $l = \pm\infty$, diciamo che f è DIVERGENTE per $x \to c$
- Se $l \in \mathbb{R}$, diciamo che f è CONVERGENTE per $x \to c$.

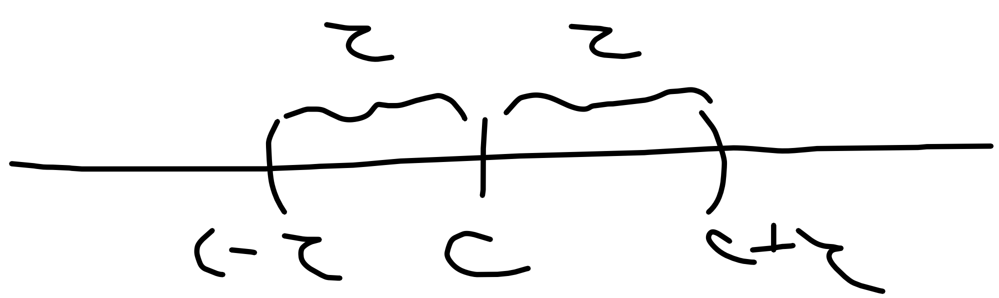
Notazione Scriviamo

$$\lim_{x \to c} f(x) = l \quad \text{o} \quad f(x) \xrightarrow[x \to c]{} l$$

Def (INTORNO)

Sia $c \in \mathbb{R}$, chiamiamo INTORNO di c un intervallo della forma

$$(c-r,\ c+r) \quad \text{per } r>0 \text{ fissato}$$



Dire che x appartiene a un intorno di c è equivalente a dire $\underline{|x-c|<r}$ per qualche $r>0$ fissato.