T~~EOR~~ (INTEGRAZIONE per PARTI)

Siano $f \in C([a,b], \mathbb{R})$, $g \in C^1([a,b], \mathbb{R})$.

Sia F una primitiva di f.

Allora:

$$\boxed{\int_a^b f(x) g(x)\, dx = \Big[F(x) g(x)\Big]_a^b - \int_a^b F(x) \cdot g'(x)\, dx}$$

DIM

$\forall x \in [a,b]$: $\left(F(x)g(x)\right)' = F'(x)g(x) + F(x)g'(x)$

perché $F' = f$ $= f(x)g(x) + F(x)g'(x)$

Integro entrambi i membri:

$$\int_a^b \left(F(x)g(x)\right)' dx = \int_a^b f(x)g(x)\,dx + \int_a^b F(x)g'(x)\,dx$$

$$\underbrace{\int_a^b \left(F(x)g(x)\right)' dx}_{= \left[F(x)g(x)\right]_a^b}$$

per il Teo fondam

$\times\!\!\times$

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1) $$\int_0^1 \underbrace{x}_{g} \cdot \underbrace{e^x}_{f}\, dx = \left[e^x \cdot x\right]_0^1 - \underbrace{\int_0^1 e^x \cdot 1\, dx}_{\left[e^x\right]_0^1}$$

2) $$\int_{\pi/4}^{\pi/2} \underbrace{x}_{g}\, \underbrace{\cos x}_{f}\, dx = \left[\operatorname{sen} x \cdot x\right]_{\pi/4}^{\pi/2} - \underbrace{\int_{\pi/4}^{\pi/2} \operatorname{sen} x\, dx}_{\left[-\cos x\right]_{\pi/4}^{\pi/2}}$$

3) $\int_2^4 \log x \, dx = [x \cdot \log x]_2^4 - \int_2^4 \cancel{x} \cdot \frac{1}{\cancel{x}} dx$

4) Calcolare una primitiva di $f(x) = \cos^2 x$

$$\int_0^x \cos^2 t \, dt = \underbrace{[\operatorname{sen} t \cos t]_0^x}_{} + \int_0^x \operatorname{sen}^2 t \, dt$$

$$\to = \operatorname{sen} x \cos x + \int_0^x (1 - \cos^2 t) \, dt$$

$$= \operatorname{sen} x \cos x + x - \int_0^x \cos^2 t \, dt$$

$$\frac{\cancel{2}}{\cancel{2}}\int_0^x \cos^2 t \, dt = \frac{\sin x \cos x + x}{2}$$

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \sqrt{1-x^2}\, dx = \left[x\sqrt{1-x^2} \right]_{\frac{1}{3}}^{\frac{1}{2}} - \int_{\frac{1}{3}}^{\frac{1}{2}} x \cdot \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\, dx$$

$$= \dots\dots\dots - \int_{\frac{1}{3}}^{\frac{1}{2}} \left(\underbrace{\frac{1-x^2}{\sqrt{1-x^2}}}_{\sqrt{1-x^2}} - \underbrace{\frac{1}{\sqrt{1-x^2}}} \right) dx$$

$$\frac{\cancel{2}}{\cancel{2}}\int_{\frac{1}{3}}^{\frac{1}{2}} \sqrt{1-x^2}\,dx = \frac{\ldots\ldots + \left[\arcsin x\right]_{\frac{1}{3}}^{\frac{1}{2}}}{2}$$

$$\int_0^1 \arctan x\,dx = \left[x\cdot \arctan x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$- \frac{1}{2}\left[\log(1+x^2)\right]_0^1$$

$$\int_2^4 \log^2 x\,dx = \left[x\cdot \log^2 x\right]_2^4 - \int_2^4 \cancel{x}\cdot \frac{2\log x}{\cancel{x}} \ldots$$

$$\int_0^1 \underbrace{t^2}_{f'} \underbrace{\arctan t}_{g}\, dt = \left[\frac{t^3}{3}\cdot \arctan t\right]_0^1 - \underbrace{\int_0^1 \frac{t^3}{3}\cdot\frac{1}{1+t^2}\,dt}_{\circledast}$$

$$\circledast \quad -\frac{1}{3}\int_0^1 \frac{t^3+t-t}{1+t^2}\,dt$$

$$= -\frac{1}{3}\left[\underbrace{\int_0^1 \frac{t\,\cancel{(1+t^2)}}{\cancel{1+t^2}}\,dt}_{\left[\frac{t^2}{2}\right]_0^1} - \frac{1}{2}\int_0^1 \frac{2t}{1+t^2}\,dt\right.$$

$$\left. -\frac{1}{2}\left[\log(1+t^2)\right]_0^1\right]$$

TEOR (INTEGRAZIONE per SOSTITUZIONE)

Siano I, J int. di $\mathbb{R}$, $f \in C(I, \mathbb{R})$, $\phi \in C^1(J, I)$. Allora:

(1) Se $\alpha, \beta \in J$:

$$\int_\alpha^\beta f(\phi(t))\,\phi'(t)\,dt = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx \qquad (x = \phi(t))$$

(2) Se ϕ è invertibile allora:

$$\int_a^b f(x)\,dx = \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(t))\,\phi'(t)\,dt$$

DIM Dato che f è continua, per il II Teor. fond. ammette una primitiva F.

Consideriamo $F \circ \phi$:

$$(F\circ\phi)'(t) = F'(\phi(t))\cdot\phi'(t)$$
$$= f(\phi(t))\cdot\phi'(t)$$

Integro tra α e β:

I Teor. fond.

$$\int_\alpha^\beta f(\phi(t))\cdot\phi'(t)\,dt = \int_\alpha^\beta (F\circ\phi)'(t)\,dt = F(\phi(\beta)) - F(\phi(\alpha)) = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx$$

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$$\int_0^1 \underbrace{\sqrt{1-x^2}}\, dx$$

$$= \int_{\boxed{0}}^{\boxed{\frac{\pi}{2}}} \underbrace{\cos t} \cdot \cos t \, dt$$

$x = \operatorname{sen} t$

$(x = \phi(t))$

$\phi'(t) = \cos t$

$$\frac{1}{3}\int_0^1 \frac{3e^{3t}\,dt}{1+e^{3t}} = \frac{1}{3}\left[\log(1+e^{3t})\right]_0^1$$

$x = e^{3t}$

$\phi'(t) = 3e^{3t}$

$$= \frac{1}{3}\int_1^{e^3} \frac{dx}{1+x} = \frac{1}{3}\left[\log(1+x)\right]_1^{e^3}$$

$$\frac{1}{3}\int_0^1 \frac{3e^{3t}\,dt}{1+e^{6t}} = \frac{1}{3}\int_1^{e^3} \frac{dx}{1+x^2} = \frac{1}{3}\left[\arctan x\right]_1^{e^3}$$

cambio

$x = e^{3t}$

$dx = 3e^{3t}\,dt$

$$\int_0^1 \frac{\log(1+\sqrt{x})}{\sqrt{x}+x}\,dx = 2\int_0^1 \frac{\log(1+\sqrt{x})}{2\sqrt{x}\,(1+\sqrt{x})}\,dx$$

Pongo

$t=\sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\,dx$

$$= 2\int_0^1 \frac{\log(1+t)}{1+t}\,dt = 2\cdot\frac{1}{2}\Big[\big(\log(1+t)\big)^2\Big]_0^1$$

2° modo!

$x \quad t=\sqrt{x} \longrightarrow x = t^2 = \varphi(t)$

$dx = 2t\,dt = \varphi'(t)\,dt$

$$\int_0^1 \frac{\log(1+\sqrt{x})}{\sqrt{x}\,(1+\sqrt{x})}\,dx = \int_0^1 \frac{\log(1+t)}{\cancel{t}\,(1+t)}\cdot 2\cancel{t}\,dt$$

$$\int_0^1 \underbrace{2x\sqrt{x+7}}_{f(x)}\,dx$$

pongo

$$t=\sqrt{x+7}$$

$$x=\underbrace{t^2-7}_{\phi(t)}$$

$$dx=\phi'(t)\,dt=$$

$$=2t\,dt$$

$$=\int_{\sqrt{7}}^{\sqrt{8}} 2(t^2-7)\cdot t\cdot \underbrace{2t}_{\phi'(t)}\,dt$$

$\vdots$

finire

$$\int_2^4 \frac{\log^2(2x)+3}{x\log(2x)}\,dx$$

pongo $t=\log(2x)$

$dt = \frac{1}{x}dx$

$$=\int_{\log 4}^{\log 8} \frac{t^2+3}{t}\,dt = \int_{\log 4}^{\log 8}\left(t+\frac{3}{t}\right)dt$$

$$=\left[\frac{t^2}{2}+3\log t\right]_{\log 4}^{\log 8}$$

$$\int_0^1 \frac{e^{3x}}{2+e^x}\,dx$$

pongo $\underbrace{e^x = t}$

$dt = e^x\,dx$

$$= \int_0^1 \frac{e^{2x} \cdot e^x\,dx}{2+e^x}$$

$$= \int_1^e \frac{t^2}{2+t}\,dt = \int_1^e \left(\underbrace{\frac{t^2-4}{t+2}}_{\boxed{t-2}} + \frac{4}{t+2} \right) dt$$

$\rightarrow \left[\frac{t^2}{2} - 2t + 4\log(t+2) \right]_1^e$