Graph Coloring

Alessandro Hill

rev. 1.0(AH) – 2024

The Graph Coloring Problem

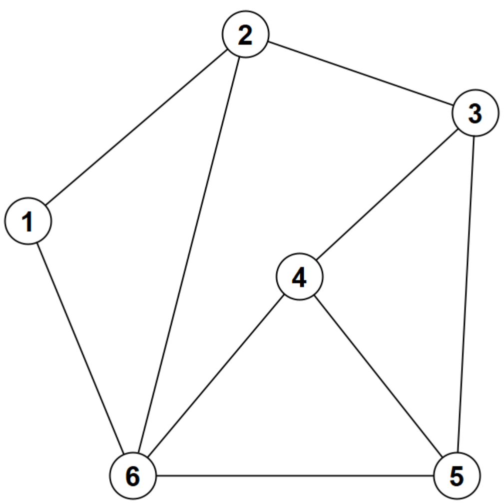
Colorazione dei grafi

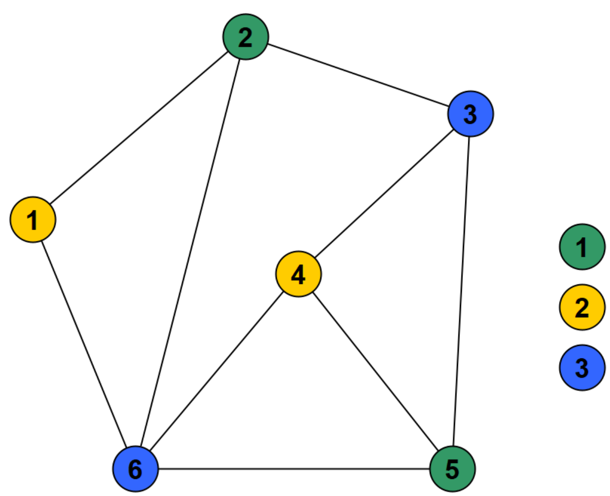
We are given an undirected graph **G** with node set **V** and edge set **E**. Edges in **E** are also called conflicts. The set **C** contains numbers that correspond to colors.

A feasible coloring of **G** is an assignment of its nodes to colors from **C** such that two adjacent nodes have different colors.

The Graph Coloring Problem asks for a feasible coloring of **G** using the minimum number of colors from **C**.*

*NP-hard (extremely difficult to optimize)





Applications:

- Frequency assignment
- Seating assignment
- Scheduling
- Experiment design
- Computer science, etc.

The Graph Coloring Problem

Definition: The chromatic number $\chi(G)$ of graph **G** is the minimum number of colors needed for a feasible coloring of **G**.

Definition: The clique number $\omega(G)$ of graph **G** is the size of a maximum clique.

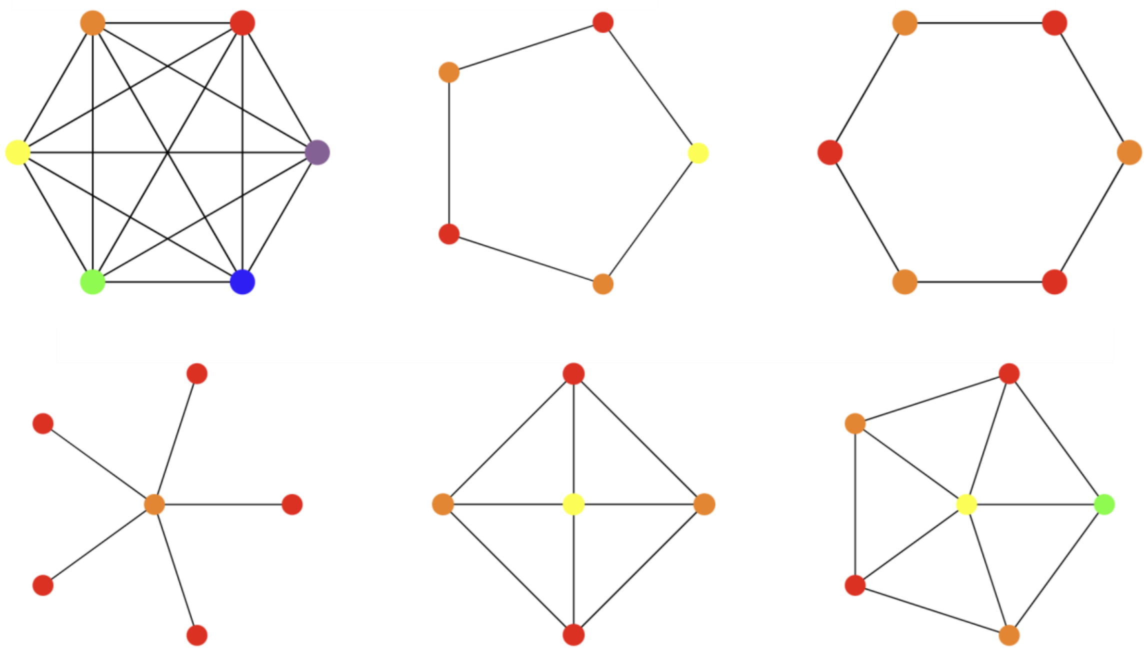
Proposition: $\chi(G) \geq \omega(G)$

Observation: Nodes of same color form an independent set.

In General: Given a clique Q and a feasible coloring with k colors. Then:

$$|Q| \leq \chi(G) \leq k$$

lower bound (LB) upper bound (UB)



The Graph Coloring Problem

IP Model

Binary color assignment variable for each node and color:

$$x_{v,c} = \begin{cases} 1 & \text{if node } v \text{ will be colored with color } c, \\ 0 & \text{otherwise.} \end{cases}$$

Binary color usage variable for each color:

$$y_c = \begin{cases} 1 & \text{if color } c \text{ is used,} \\ 0 & \text{otherwise.} \end{cases}$$

Minimize $\sum_{c \in C} y_c$

Idea: Two connected nodes cannot be colored with the same color.

$$x_{u,c} + x_{v,c} \leq 1 \qquad \forall \{u, v\} \in E,\ c \in C$$

$$\frac{1}{|V|} \sum_{v \in V} x_{v,c} \leq y_c \qquad \forall\, c \in C \qquad \text{(Variable Linking)}$$

$$\sum_{c \in C} x_{v,c} = 1 \qquad \forall\, v \in V \qquad \text{(Color Assignment)}$$

The Graph Coloring Problem

Greedy Heuristic

Input: Undirected graph G=(V,E); Colors C={1,…,|V|}.

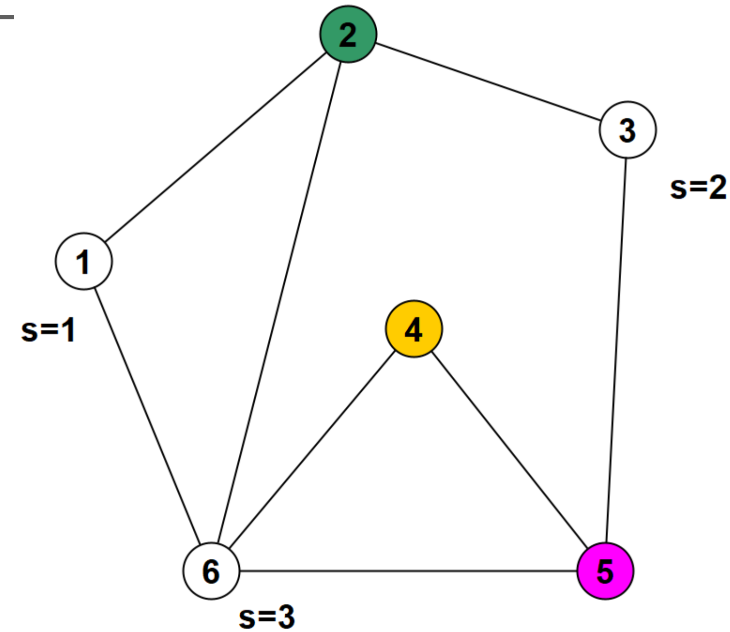
1. Pick an uncolored node v in V.
   → Color v with a “feasible” used color, if possible, else use unused color.

2. If all nodes are colored return coloring, else go to 1.

The Graph Coloring Problem

DSatur Heuristic

Definition: For a partial coloring of graph **G**, the saturation of an (uncolored) node is the number of distinct colors of its neighbors.



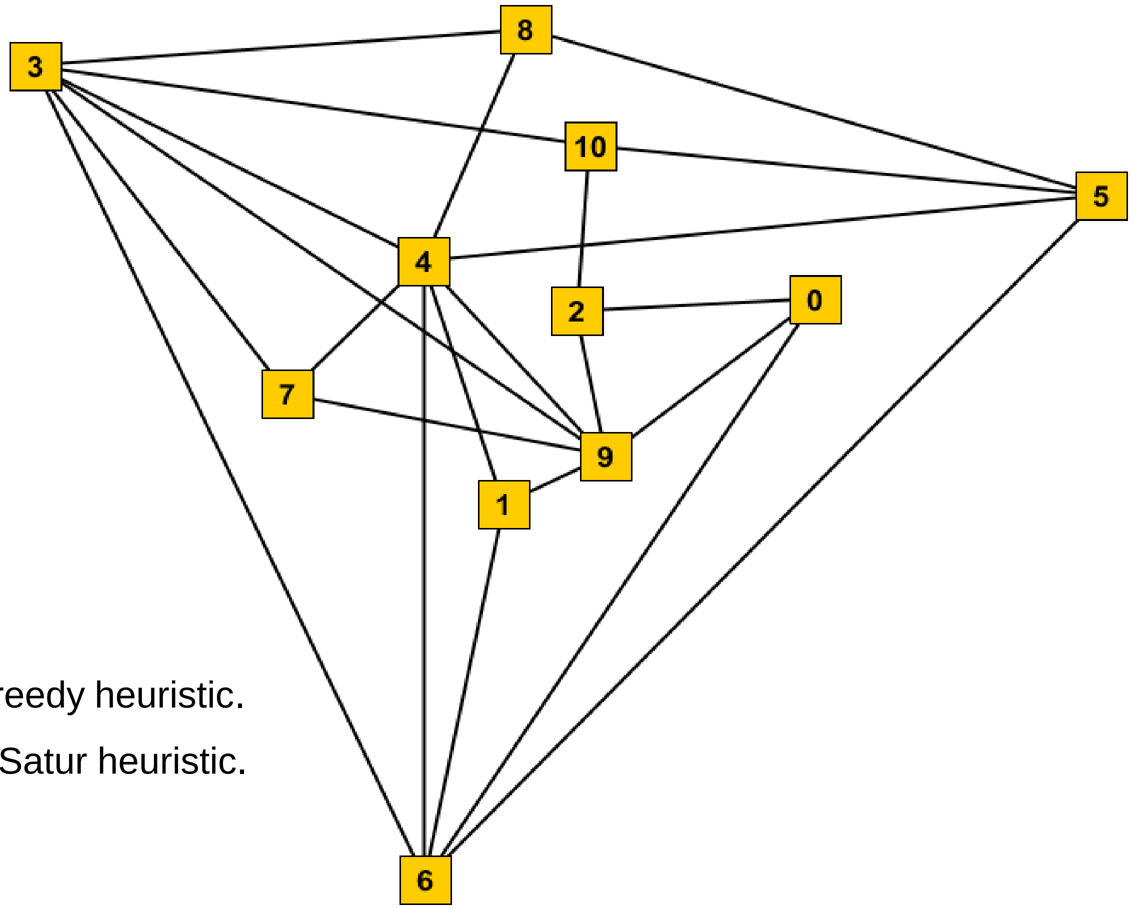
Input: Undirected graph G=(V,E); Colors C={1,…,|V|}.

1. Pick an uncolored node with (1) highest saturation and (2) highest degree.
   → Color node with a “feasible” used color, if possible, else use unused color.

2. If all nodes are colored return coloring, else go to 1.

The Graph Coloring Problem

Example:



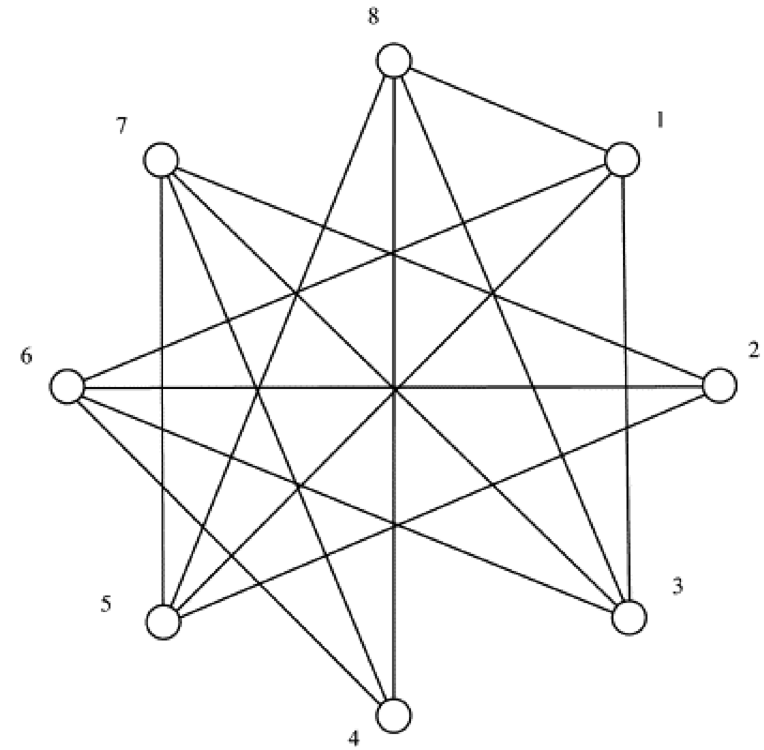
a) Find a feasible coloring using the greedy heuristic.

b) Find a feasible coloring using the DSatur heuristic.

c) Find an optimal coloring using IP.

The Graph Coloring Problem

1) Exercises:



1. Consider the graph with 8 nodes.

2. Find an optimal coloring using IP.
Verify your results visually.

3. Find a coloring using the DSatur heuristic.
Verify your results visually.

4. Add a minimal number of edges to increase the number of needed colors by one.
Verify your results using the IP model.

The Graph Coloring Problem

2) Exercises:

1. Create a random graph with 100 nodes and an arbitrary number of edges in yEd.

Use a convenient layout for your visualization.
Format as necessary.
You can export the edge list using the TGF format.

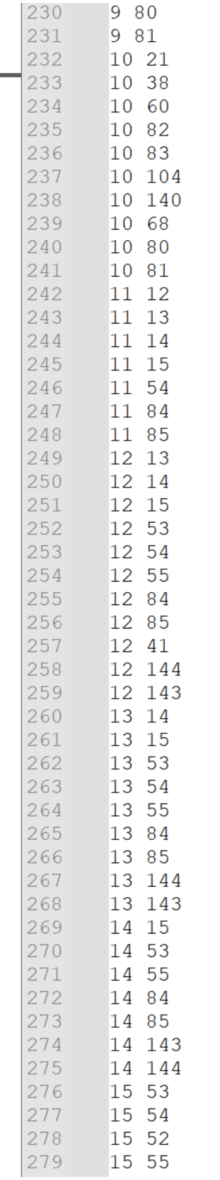
2. Find an optimal coloring using IP.

Verify your results visually.

3. Find an optimal coloring using MiniZinc.

Verify your results visually.

4. For what larger/denser graphs can you still answer the questions above?


```
9 80
9 81
10 21
10 38
10 60
10 82
10 83
10 104
10 140
10 68
10 80
10 81
11 12
11 13
11 14
11 15
11 54
11 84
11 85
12 13
12 14
12 15
12 53
12 54
12 55
12 84
12 85
12 41
12 144
12 143
13 14
13 15
13 53
13 54
13 55
13 84
13 85
13 144
13 143
14 15
14 53
14 55
14 84
14 85
14 143
14 144
15 53
15 54
15 52
15 55
```

The Graph Coloring Problem

3) Exercises:

1. Use the class social network data (Virtuale).

2. Can you find a partition of minimum size such that people in the partition set do not know each other?

Solve the graph coloring problem.

Then nodes of the same color correspond to individuals that do not know each other.

3. Can you visualize your results from 2?