Travelling Salesman Problems

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rev. 1.0(AH) – 2024

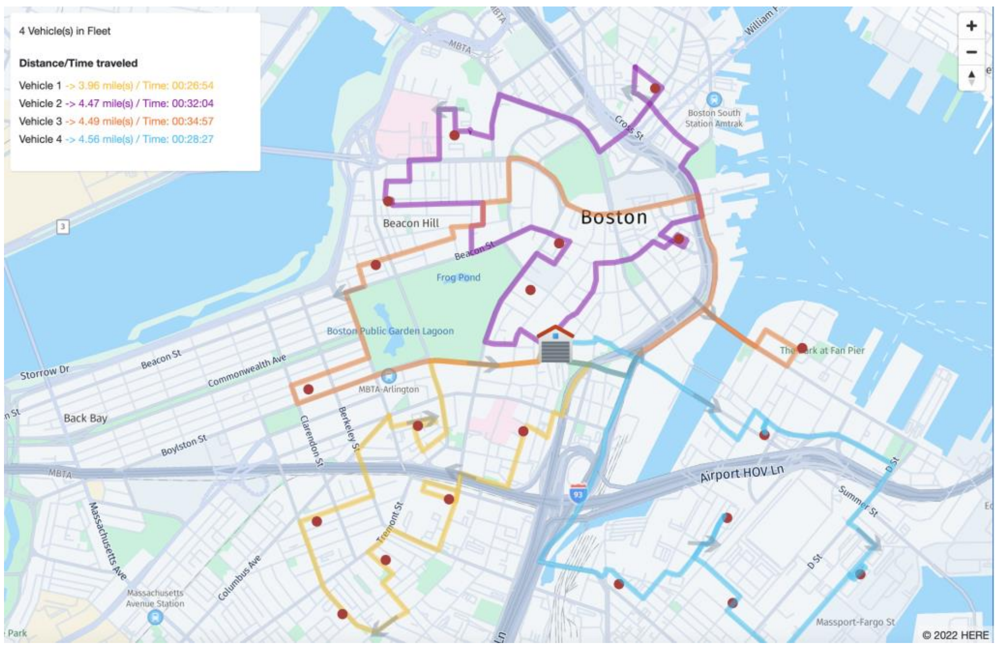
The Traveling Salesman Problem

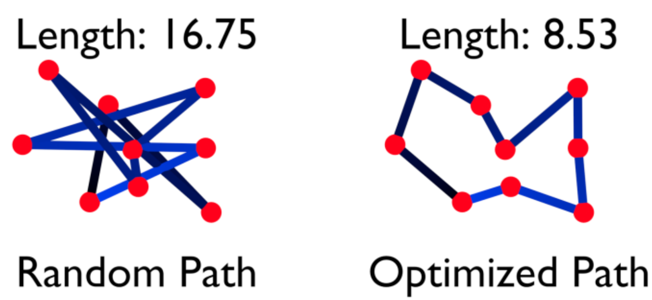
Problema del commesso viaggiatore

We are given a directed base graph with node set N (locations) and arc set A (travel segments). Each arc (i,j) has a non-negative weight $c_{i,j}$ (distance/duration).

> The Travelling Salesman Problem (TSP) asks for a tour (directed cycle, or circuit) of minimum total length that includes every node exactly once.*

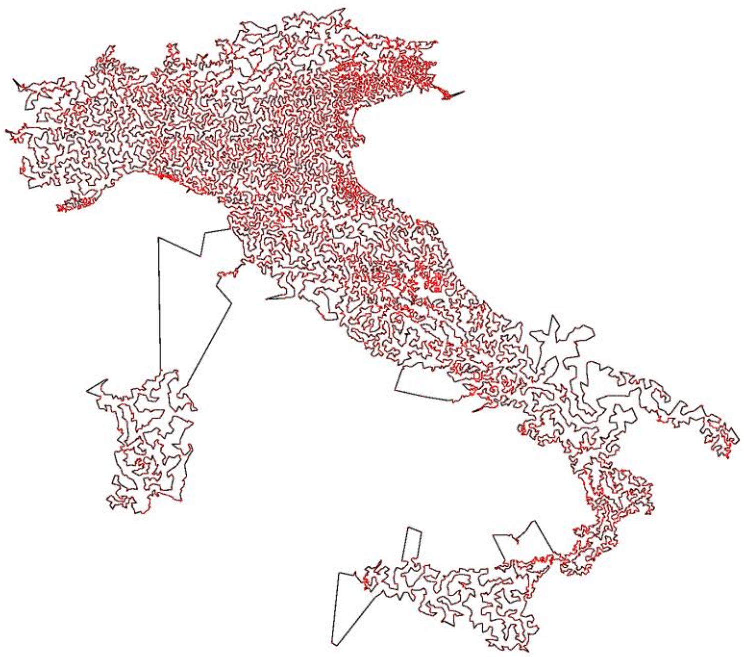
*NP-hard (extremely difficult to optimize)





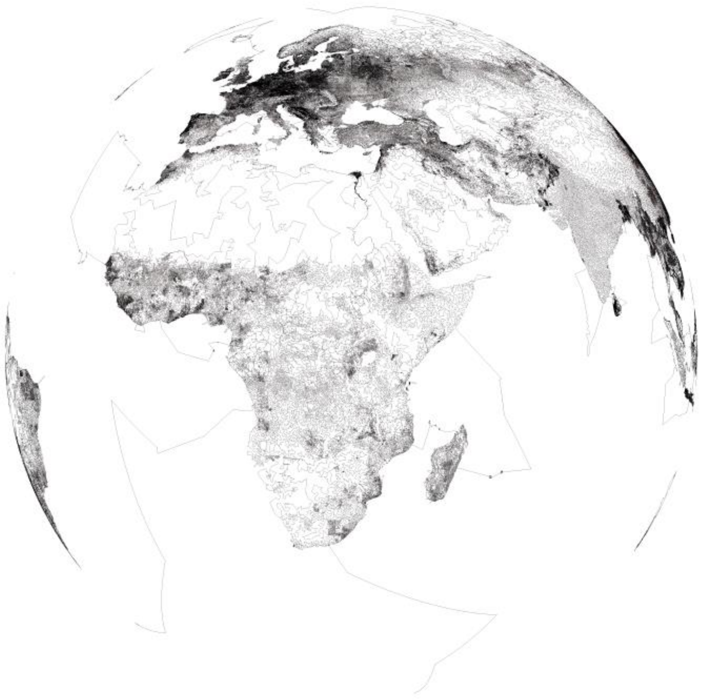
The Traveling Salesman Problem

16,862 Cities in Italy



557,315 km

1,904,711 Cities Worldwide



7,516,353,779 km

Also see http://www.math.uwaterloo.ca/tsp/world/index.html

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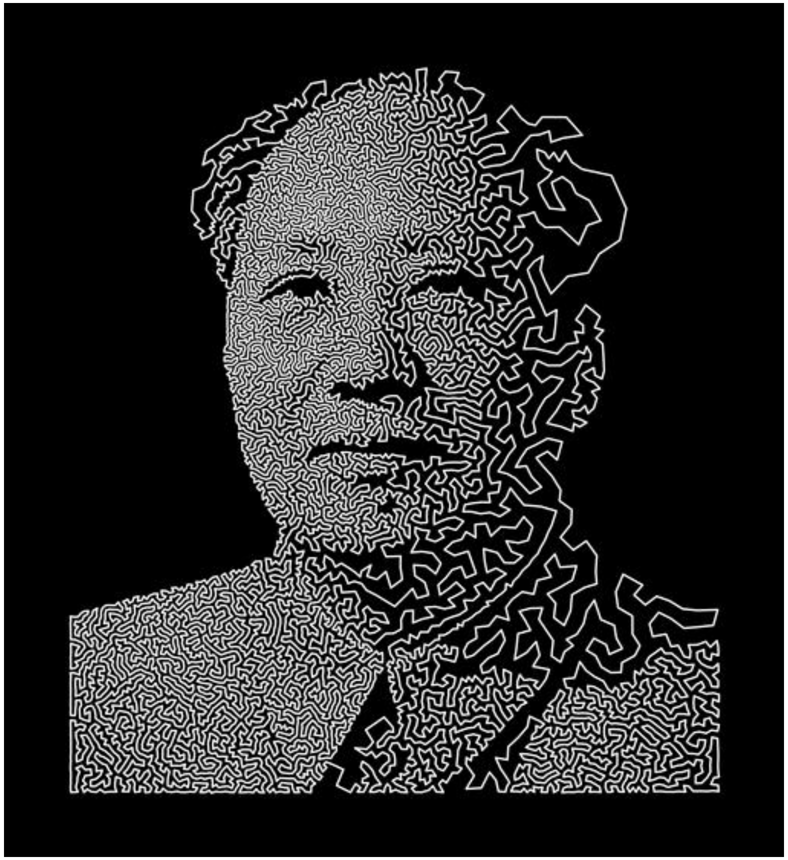
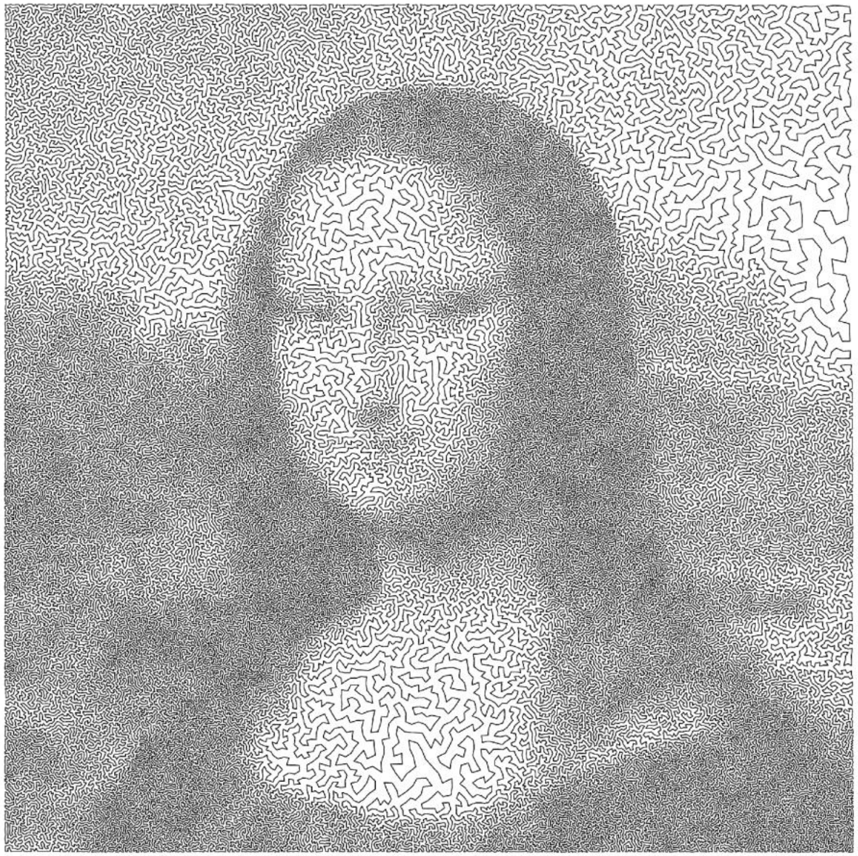
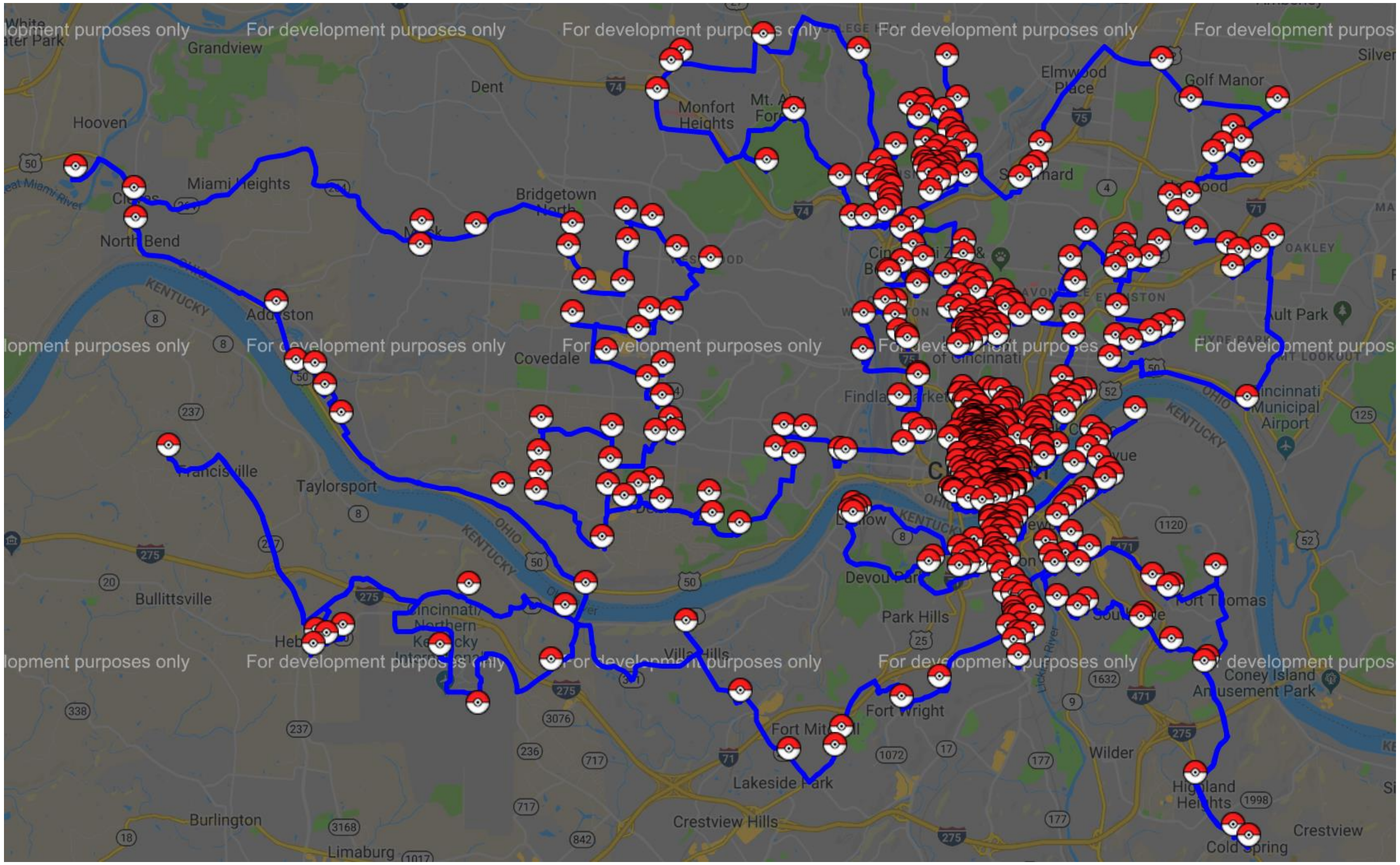




Also see http://www2.oberlin.edu/math/faculty/bosch/tspart-page.html and https://cs.uwaterloo.ca/~csk/other/tsp/

The Traveling Salesman Problem



Also see http://www.math.uwaterloo.ca/tsp/poke/index.html

The Traveling Salesman Problem

IP Model*

Binary arc variables for each arc that could be part of a tour:

$$x_{i,j} = \begin{cases} 1 & \text{if arc } (i,j) \text{ will be used on the tour,} \\ 0 & \text{otherwise.} \end{cases}$$

Minimize $\sum_{(i,j)\in A} c_{i,j}\, x_{i,j}$

Idea: Each node *v* has exactly one incoming arc and one outgoing arc in a tour.

Force node out-arc:

$x_{v,j_1} + \ldots + x_{v,j_k} = 1$ for out-neighbors $j_1, \ldots, j_k$ of v.

Force node in-arc:

$x_{i_1,v} + \ldots + x_{i_k,v} = 1$ for in-neighbors $i_1, \ldots, i_k$ of v.

*May contain sub-tours…

The Traveling Salesman Problem

IP Model**

Subtour Elimination Constraints (SEC):

$\sum_{(i,j)\in A[S]} x_{i,j} \leq |S| - 1$ for each node subset $S \subset V$

Miller-Tucker-Zemlin Constraints (MTZ):

$\mathrm{u_i - u_j + 1 \leq |N| \cdot (1 - x_{i,j})}$ for $\{\mathrm{i,j}\} \subset \mathrm{N}$ (link variables)

$u_1 = 1$

$u_i > 0$ for i in $\mathrm{N} = \{1,2,\ldots\}$ (new node sequence variables)

****No sub-tours…

The Traveling Salesman Problem

Nearest-Neighbor Heuristic (NN) (can fail!)

1. Initialize empty tour T.

2. Pick arbitrary start node, add to tour T, and use as current node i.

3. Pick the node j that is
   - the closest to the current node i, and
   - not in T.

   Connect i to j, add j to T, and make j the current node.
   (If no node j was found, the heuristic fails.)

4. Go to 3 unless T contains all nodes.

5. Return tour T.

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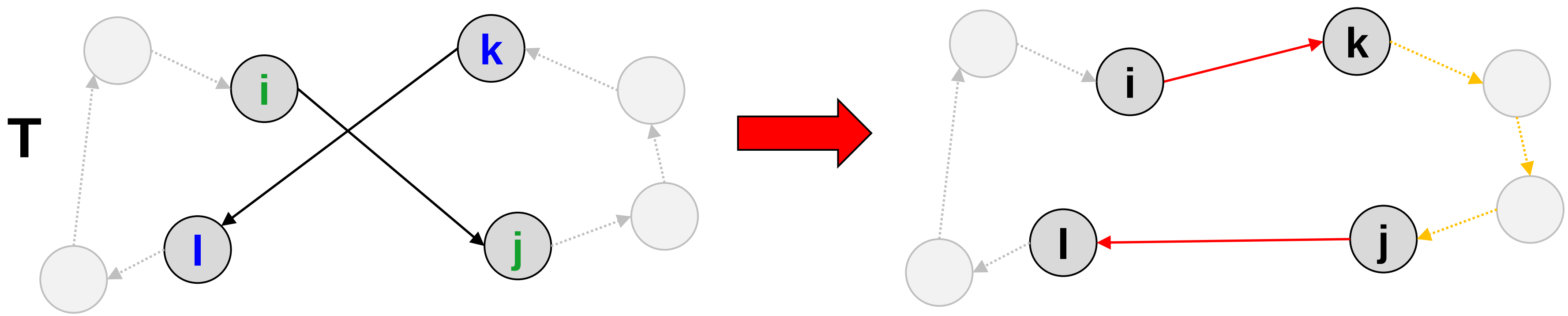
Local Search: "2-Opt"

For a tour T. For two arcs (i,j) and (k,l), let the first 2-opt move be defined as:
- Remove arcs (i,j) and (k,l) from T.
- Insert arcs (i,k) and (j,l) into T.
- Reverse path from j to k in T.

The second 2-opt move inserts (k,i) and (l,j), and reverses path from l to i.)



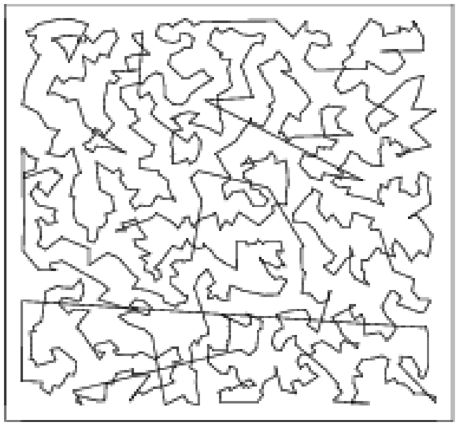
Local Search Heuristic:

INPUT: Feasible tour T.

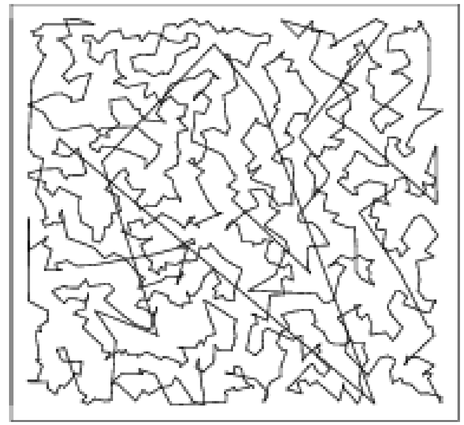
1. Find 2-opt move of highest improvement in T.
2. If improvement > 0 then implement move for T and go to 1.
3. Return tour T.

The Traveling Salesman Problem

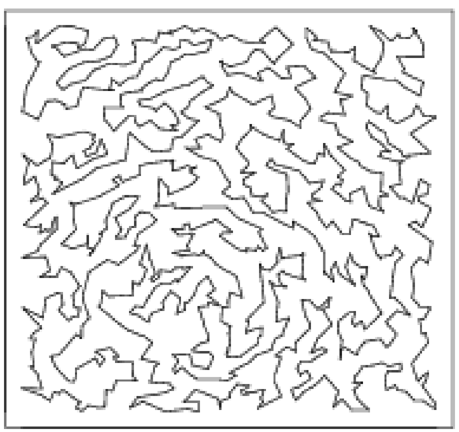
Examples



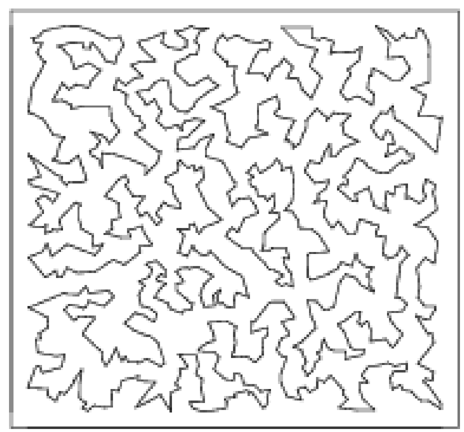
Greedy Tour



Nearest Neighbor Tour



Savings Tour



Optimal Tour

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What is the shortest tour that visits all cities A, …, E?

Travel Distances

Cities	A	B	C	D	E
A	0	3	4	2	7
B	3	0	4	6	3
C	4	4	0	5	8
D	2	6	5	0	6
E	7	3	8	6	0

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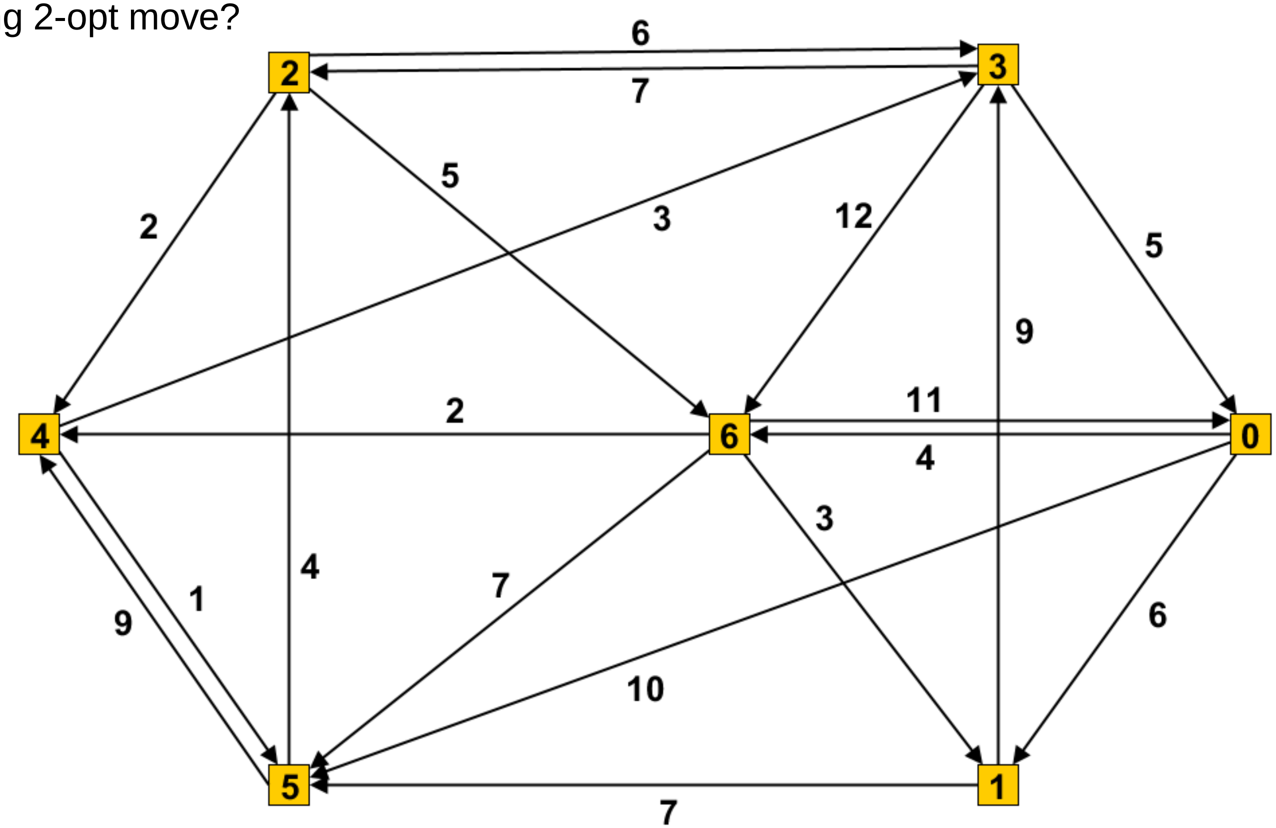
1) Find a TSP tour using the nearest-neighbor heuristic.

Start from a) node 1; b) node 4; c) node 5.

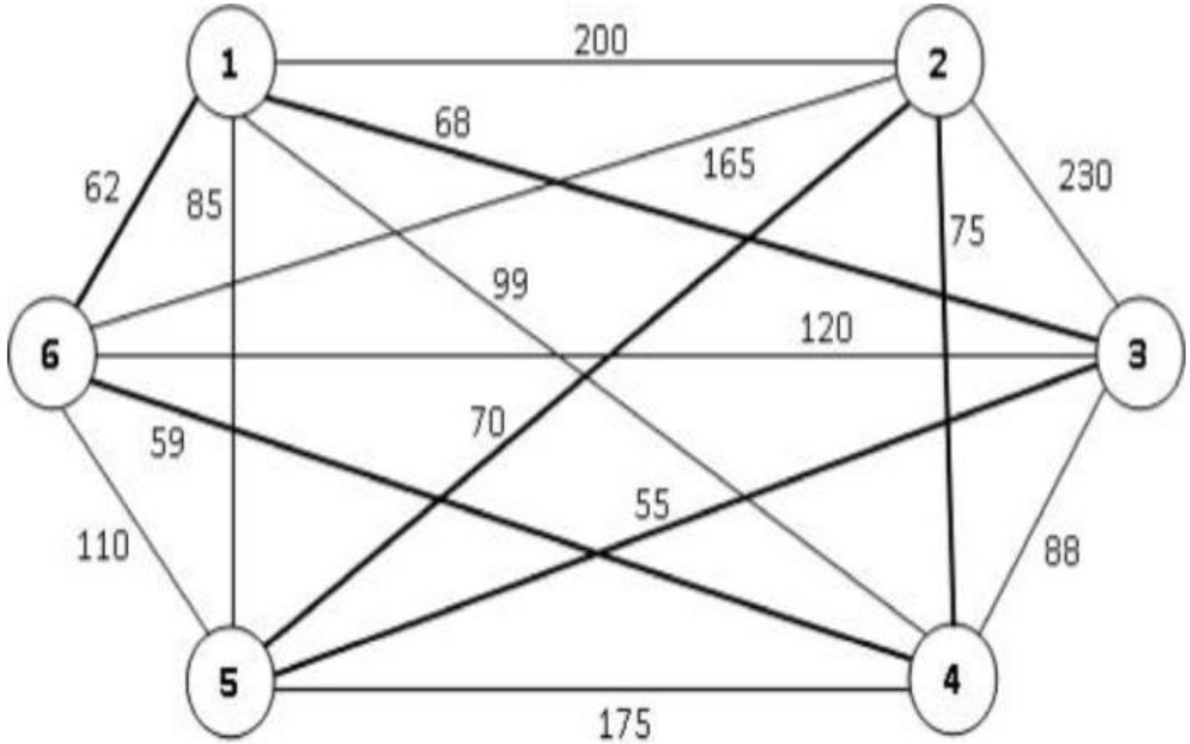
Can we find an improving 2-opt move?



2) Solve the TSP above using IP.

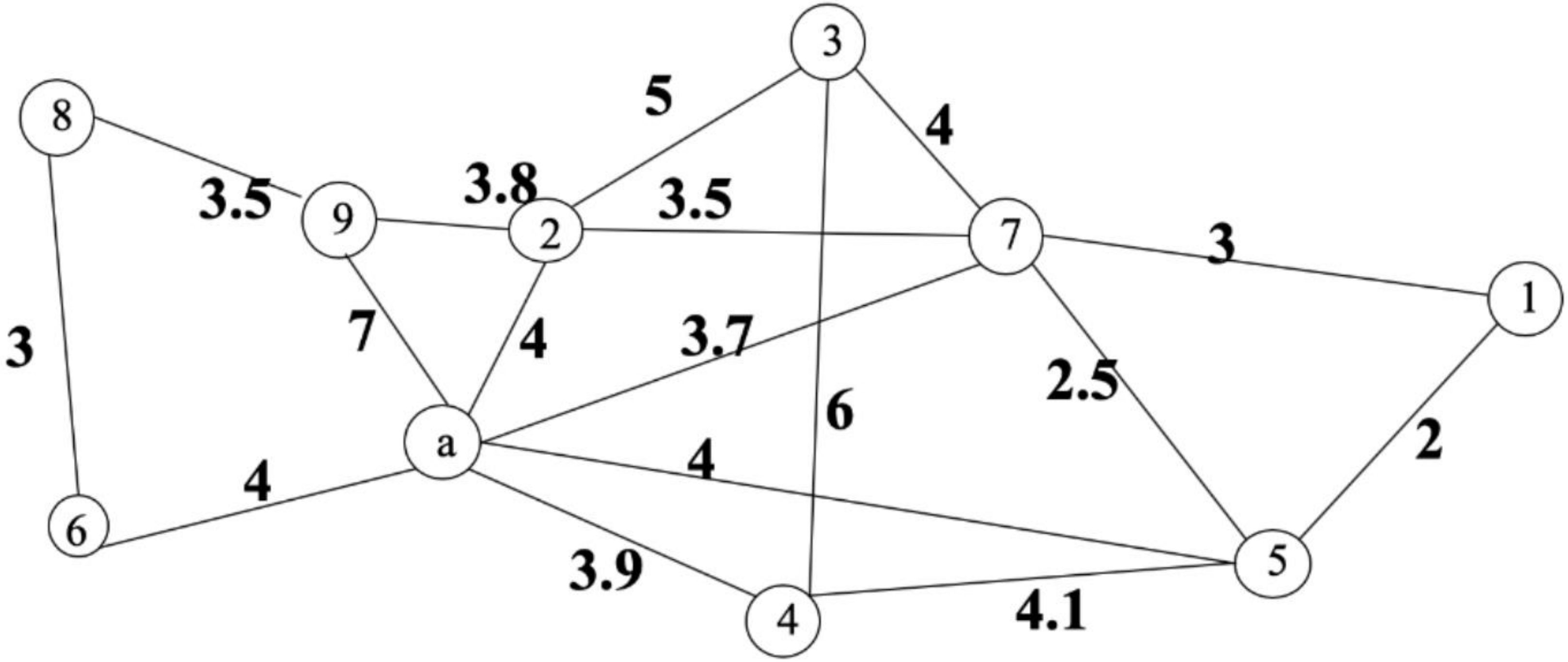
The Traveling Salesman Problem

3) Find a TSP tour using the nearest-neighbor heuristic and compare your results to an IP solution.



The Traveling Salesman Problem

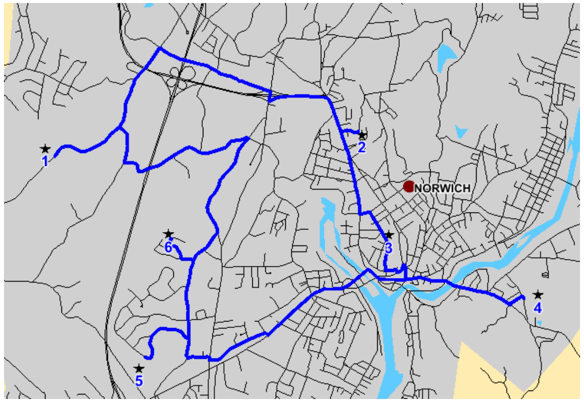
4) Find a TSP tour using the nearest-neighbor heuristic and compare your results to an IP solution.



The Traveling Salesman Problem

5) Group Exercise:

1. Find 5-10 points of interest that we would like to tour in Cesena.
For example, museums, shops, restaurants, bars, cinemas, parks.
They could also all be located within a building.
Other cities are of course also fine.



2. Estimate all the travel distances/durations.
Use an online map system such as Google Maps.
Is there an advanced tool that returns a distance matrix?
You can use walking, cycling, driving to do the tour.
Enter your data in a spreadsheet.

3. Find a short tour using the NN-Heuristic.
Use different starting points if necessary.

4. Build and solve an IP model.
Include either SEC or MTZ inequalities.

5. Visualize your best tour using yEd.
Can you add a map snapshot as background?