Delta and Delta-Gamma Hedging

December 3, 2023

Daniyal Shahzad, Jongtaek Lee & Brandon Tam

1 Abstract

This report presents a comprehensive analysis of delta and delta-gamma hedging strategies within the Black-Scholes framework. We look at the foundational theories of delta and gamma hedging and illustrate their roles in offsetting option price changes due to movements in the underlying asset. Our analysis utilizes the Black-Scholes model to construct delta-neutral and gamma-neutral portfolios, aiming to mitigate risks associated with both price changes in the underlying asset and variations in the option's delta. We explore the implications of these hedging strategies through empirical simulations and examine the distribution of profit and loss under different market conditions and assumptions. We study the impacts of varying transaction costs, real-world volatility of the underlying asset, and different values of real-world drift. This report provides an understanding of these hedging techniques and highlights their practical applications in real-world financial scenarios.

2 Introduction

The delta of an option is a measure of how the price of an option changes when the underlying asset price changes by \$1. For example, if an option has a delta of $+0.5$, then the option's price increases by 50 cents for every \$1 increase in the underlying asset price. Mathematically, an option's delta is the partial derivative of the option price with respect to the price of the underlying asset. Delta hedging involves setting up a position in the underlying asset that offsets the delta of an option position. The goal is to make the overall position delta-neutral, meaning that it is not affected by small price movements in the underlying asset.

Delta-gamma hedging takes the hedging strategy a step further by also accounting for the gamma of an option position. Gamma measures the rate of change of delta with respect to the underlying asset price and is an indication of how the delta will change as the price of the underlying asset changes. Mathematically, gamma is the second derivative of the option price with respect to the underlying asset price. While delta hedging focuses on offsetting price changes in the underlying asset, delta-gamma hedging seeks to also manage the risk of changes in the delta itself. This is particularly important for options that are at-the-money, where gamma is highest. Delta-gamma hedging aims to make a portfolio both delta-neutral and gamma-neutral, thereby protecting against price changes in the underlying asset and changes in delta as well.

In this report, we will analyze delta and delta-gamma hedging within the Black Scholes framework. We make the following assumptions:

1. The underlying asset price process is denoted $S = \{S_t\}_{t\geq 0}$ and it follows the Black-Scholes model (see subsection 3.1 for details). Furthermore, the asset's current price is \$10.
2. We have sold 10000 units of an at the money European call option expiring in 0.25 years.
3. There are 63 trading days in a quarter. For simplicity, we divide the quarter year into 63 even periods, ignoring weekends and holidays. Hedging is done once every trading day. Since we are ignoring weekends and holidays, hedging is done every $\frac{0.25}{63}$ years.
4. We can trade in the underlying asset, an at the money European call option on the same asset

with maturity 0.3 years and a risk free bank account with constant force of interest $r = 0.05$.

5. There is a transaction cost of \$0.005 per share for all equity transactions and \$0.005 per option for trades in the hedging option. Moreover, we can only trade integer values of the underlying asset and hedging option.
6. The real world drift, denoted μ, is 0.1. The volatility used for all pricing and hedging, denoted σ, is 0.25. Unless otherwise stated, the pricing volatility is the same as the real world volatility for the underlying asset.

The report is organized as follows: first we present a quick summary of the key theoretical ideas that are required to understand delta and delta-gamma hedging. This includes a brief derivation of the Black Scholes call option pricing formula that we use throughout our simulation and an explanation of the mathematics behind delta and delta-gamma hedging in the Black Scholes framework. Next, we discuss in detail our implementation of delta and delta gamma hedging in Python and present our main results. We start by analyzing the profit and loss distribution under the assumptions presented above. Then, we analyze the hedging positions for different sample paths. Next, we study the impact of the real world drift on the profit and loss distributions. In the final two sections of the analysis, we explore the impact of real world volatility on our earlier results. Finally, we summarize our key findings in the last major section of the report.

3 Theory

3.1 Black Scholes Model

As stated in the introduction, we use the Black Scholes framework in our analysis, which consists of several assumptions. Firstly, the underlying asset $S = \{S_t\}_{t\geq 0}$ satisfies the stochastic differential equation $dS_t = \mu S_t dt + \sigma S_t dW_t$, where μ and σ are constants, $\sigma > 0$ and W_t is a Brownian motion under the real world probability measure $\mathbb{P}$. Secondly, there exists a risk free bank account with continuously compounded interest $r \geq 0$. Thirdly, we assume that the option price process g_t can be written as a function of t and S_t with enough differentiability to apply Ito's lemma. Furthermore, we assume that the market does not admit arbitrage. Finally, the derivation of the Black Scholes pricing formula also assumes no transaction costs and the ability to trade fractional shares. Although we use these assumptions to derive the Black Scholes pricing formula, we do not use these assumptions in our simulation. Throughout the report, we will explore the impact of these violated assumptions on our results.

Using the idea of self financing and the no arbitrage assumption, one can show that the European call option price process $g_t = g(t, S_t)$ with maturity T and strike K satisfies the partial differential equation below:

$$\begin{cases} \partial_t g(t,s) + rs\partial_s g(t,s) + \frac{1}{2}\sigma^2 s^2 \partial_{ss} g(t,s) = rg(t,s) & s > 0, t \in [0,T) \\ g(T,s) = (s-K)_+ \end{cases}$$

By the Feynman-Kac theorem, the partial differential equation above admits the stochastic representation $g(t,s) = e^{-r(T-t)} E^{\mathbb{Q}}[(S_T - K)_+ | S_t = s]$ where $\{S_t\}_{t\geq 0}$ satisfies the stochastic differential equation $dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$ and $\mathbb{Q}$ is the risk neutral measure with the bank account as the numeraire. Solving the stochastic differential equation yields the result that the log returns $\ln(\frac{S_T}{S_t})$ follow a normal distribution with mean $(r - \frac{\sigma^2}{2})(T-t)$ and variance $\sigma^2(T-t)$ under the $\mathbb{Q}$ measure. Using this, we can conclude that the Black Scholes European call option formula is given by:

$$g(t, S_t) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

where $d_\pm = \frac{\ln(\frac{S_t}{K}) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and Φ is the cumulative distribution function of the standard normal distribution.

We will use this formula to price all of the options in our analysis.

3.2 Delta and Gamma Hedging Within the Black Scholes Framework

Recall from subsection 3.1 that the Black Scholes framework assumes the existence of an underlying asset and a risk free bank account. We use these two components to construct a delta neutral position. Let α_t denote the position in the underlying asset S_t at time t and β_t denote the position in the bank account B_t at time t. Assuming that our portfolio is self financing and we sold one unit of an option with price process g_t, the value of the portfolio is given by $0 = \alpha_t S_t + \beta_t B_t - g_t$. Taking derivatives on both sides with respect to the underlying stock price and solving for α_t yields:

$$\alpha_t = \Delta_t^g \tag{1}$$

where Δ_t^g is the delta of the option, as defined in the introduction. In the Black Scholes framework (using the notation introduced in subsection 3.1), $\Delta_t^g = \Phi(d_+)$. This quantity is always non-negative and in the interval $[0, 1]$. In other words, if you are long (have bought) an option, you would hedge by taking a short position in the underlying asset. Conversely, if you shorted an option, you would take a long position in the underlying asset.

It is also important to note that $\Delta_t^g = \Phi(d_+)$ depends on the underlying asset price and is not constant. As a result, delta hedging is not a set-it-and-forget-it strategy. The hedge must be adjusted periodically as the delta changes. This process is known as re-balancing. If re-balancing is done continuously and there are no transaction costs, then delta hedging will result in a profit of zero. In practice, re-balancing can only be done a finite number of times and transactions have costs, so delta hedging is not perfect. In other words, it is possible to realize a profit or a loss even if the option is delta hedged. In this report, we explore some properties of the profit and loss distribution under various market assumptions.

The construction of a delta-gamma hedge requires the underlying asset, the risk free bank account and a second option with price process h_t written on the same underlying asset. Assuming that our portfolio is self financing and we sold one unit of an option with price process g_t, the value of the portfolio is given by $0 = \alpha_t S_t + \beta_t B_t - g_t + \gamma_t h_t$, where γ_t is the position in the hedging option and the other variables are as defined above. Taking first and second derivatives with respect to the underlying asset price yields the system of equations below:

$$\begin{cases} 0 = \alpha_t - \Delta_t^g + \gamma_t \Delta_t^h \\ 0 = -\Gamma_t^g + \gamma_t \Gamma_t^h \end{cases}$$

Solving the system yields:

$$\gamma_t = \frac{\Gamma_t^g}{\Gamma_t^h} \tag{2}$$

$$\alpha_t = \Delta_t^g - \frac{\Gamma_t^g}{\Gamma_t^h} \Delta_t^h \tag{3}$$

where Δ_t^j is the delta of option j and Γ_t^j is the gamma of option j. In the Black Scholes framework (using the notation introduced in subsection 3.1), $\Gamma_t^g = \frac{1}{S_t \sigma \sqrt{T-t}} \phi(d_+)$.

We will use equations (1)-(3) in our analysis.

4 Description of Implementation and Interpretation

4.1 Profit and Loss

Under the assumptions stated in the introduction, both a delta hedge and a delta gamma hedge were implemented in Python. Figure 1 below shows the distribution of the profit and loss for both hedging strategies. The profit distribution when delta gamma hedging is shown on the left and the profit distribution when delta hedging is shown on the right. From the figure, it is clear that the variance of the profits is higher for delta hedging compared to delta-gamma hedging. Moreover, the plot also shows that the mean profit under delta hedging is higher. Unlike the delta hedge profit distribution, the distribution of profits when delta gamma hedging has almost no positive mass. These observations can be verified by direct computation of the mean and standard deviation of the distributions, as shown in Table 1.

The trends observed in Figure 1 are consistent with financial intuition and mathematical theory. Under fully continuous hedging with no transaction costs, the profit and loss should be exactly zero in all simulated sample paths. In other words, all deviations from zero are caused by discretization and transaction costs. Since the delta gamma hedging strategy uses a second order approximation of the option price curve, there is less approximation error compared to the first order approximation used in delta hedging. This explains the smaller variance in the profits.

The large negative means in both hedging strategies can be explained by the transaction costs. Recall that we are assuming a \$0.005 transaction cost per unit for all equity and option transactions. However, the Black-Scholes price does not assume any transaction costs. Moreover, since delta gamma hedging requires trading both the underlying asset and a hedging option, the transaction costs are greater compared to delta hedging, which only requires trading in the underlying asset.

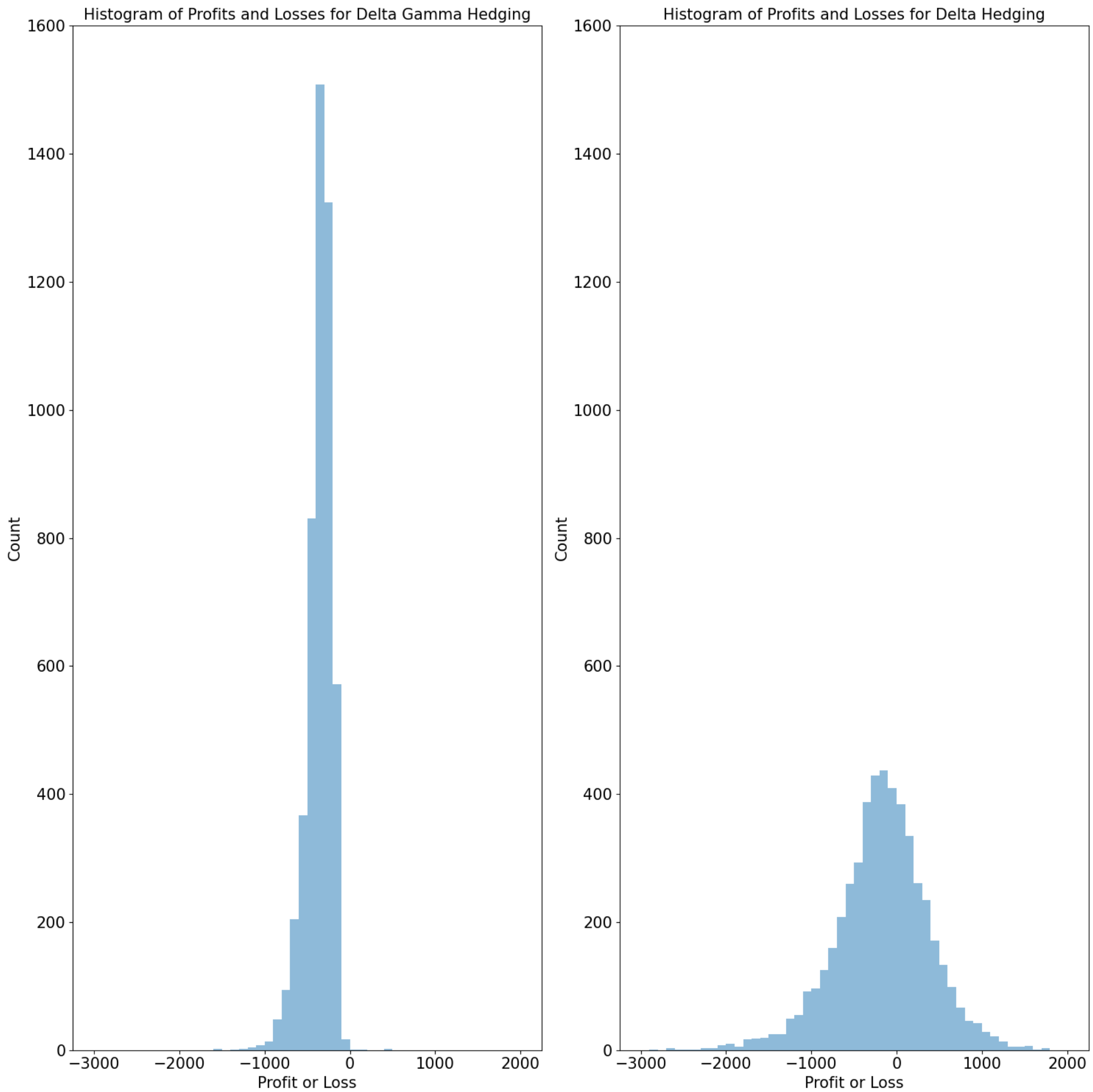

Figure 1: Profit and Loss Under the Two Hedging Strategies

Comparing Figure 2 below with Figure 1 highlights the effects of transaction costs. Figure 2 was generated using the assumptions in the introduction with the transaction cost removed. The left plot corresponds to delta-gamma hedging and the right plot corresponds to delta hedging. From the figure, it is clear that the distributions are centered closer to zero. This observation is verified by direct computation, as shown in Table 1. From Table 1, we can also see that delta gamma hedging has a mean closer to zero, as expected.

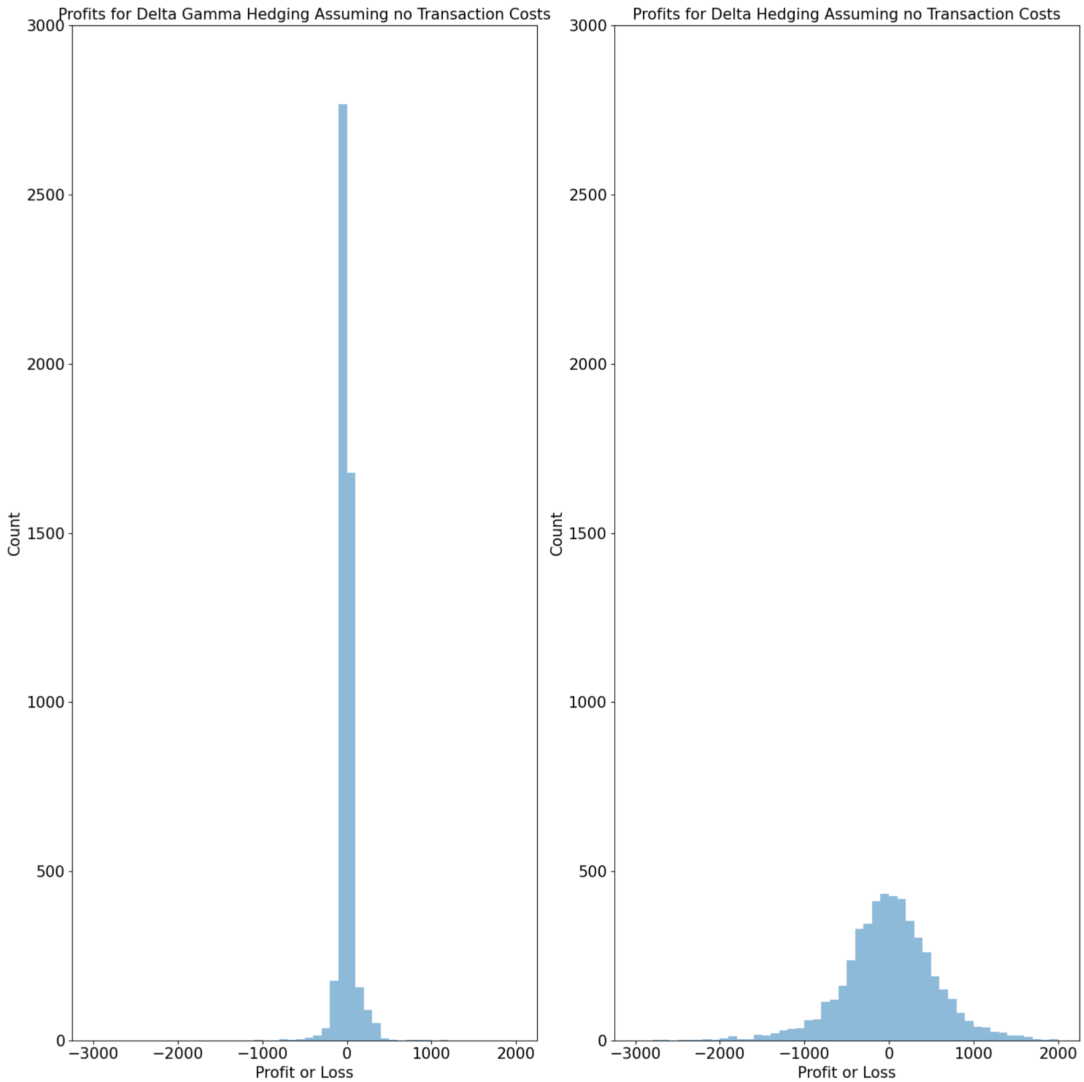

Figure 2: Profit and Loss Under the Two Hedging Strategies Assuming Zero Transaction Costs

	Mean (Fig 1)	Mean (Fig 2)	Standard Deviation (Fig 1)	Standard Deviation (Fig 2)
Delta Hedging	-168.39	5.04	549.58	546.60
Delta-Gamma Hedging	-361.00	-1.45	154.06	86.91

Table 1: Means and Standard Deviations of Profit and Losses for Two Hedging Strategies

4.2 Analyzing Hedging Positions for Different Sample Paths

In this subsection, we analyze the hedging positions for the two sample paths shown in the graph on the left side of Figure 3. The blue line represents a path where the option is ”In the Money” (ITM). Since we have a call option, this means that the underlying asset price is greater than the strike price (which is \$10) at maturity. The orange line represents a path where the option is ”Out of the Money” (OTM). Since we have a call option, this means that the underlying asset price is less than the strike price at maturity.

We first consider the delta hedging case. The graph on the right in Figure 3 shows the corresponding asset positions required to maintain a delta hedged portfolio for a sold call option. We denote the units of underlying asset in the hedging portfolio by alpha. From the figure, we can see that alpha for the ITM path (blue line) starts at around 6000 and changes in the same direction as changes in the asset price when we are far from maturity. That is, increases in the underlying asset price correspond to increases in alpha. Furthermore, alpha is quite sensitive to changes in the asset price when the option is far from maturity. In other words, changes in alpha are quite drastic when the asset price changes drastically. As we approach maturity, alpha becomes less volatile. For the ITM path, the alpha value increases and converges to 10000 at maturity. For the OTM path (orange line), alpha starts at around 6000 and changes in the same direction as changes in the asset price throughout the life of the option. As the option approaches maturity and expires worthless, the value of alpha converges to 0.

When delta hedging, the value of alpha is always non-negative. This observation is consistent with the mathematical formulas we derived for delta. When delta hedging, the value of alpha is equal to the delta of the option. The delta of a call option is always positive since call options have a higher payoff when the underlying asset price increases.

Next, we analyze the delta-gamma hedging case. We denote the number of units in the hedging option by gamma. For the ITM path in Figure 4, we see the value of alpha increases as the underlying asset price increases. In other words, a larger position in the underlying asset is required to remain hedged when the asset price increases. Like the delta hedging case, the alpha value increases and converges to 10000 at maturity when the option is in the money. We also observe that the position in the hedging option goes to zero as we approach maturity for the ITM option. This aligns with financial intuition. If the option is in the money close to maturity, that is almost equivalent to owning the underlying asset. Thus, the hedging portfolio should consist of the asset only.

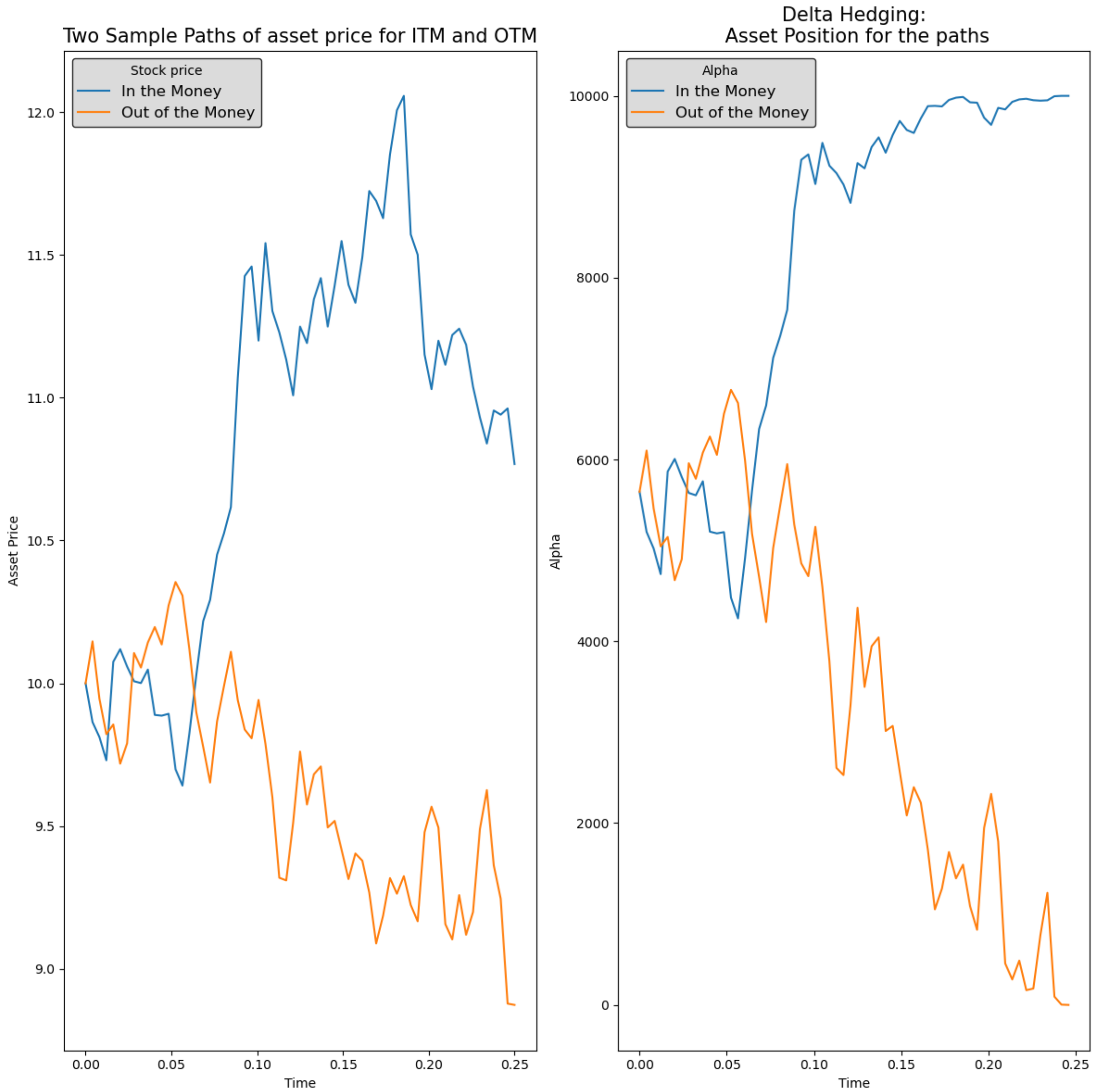

Figure 3: Position in the Underlying Asset for Delta Hedging

Unlike delta hedging, the value of alpha can be negative when delta-gamma hedging. Negative values of alpha are observed both in the ITM and OTM path when the underlying asset price is low. When the asset price is low, the call is unlikely to be exercised so the hedging portfolio consists of negative units of the asset and more units in the hedging option. This relationship between the asset price and the positions in the underlying asset and hedging option are observed until near maturity. At maturity, the value of the out of the money option is zero, so the positions in the underlying asset and the hedging option both go to zero.

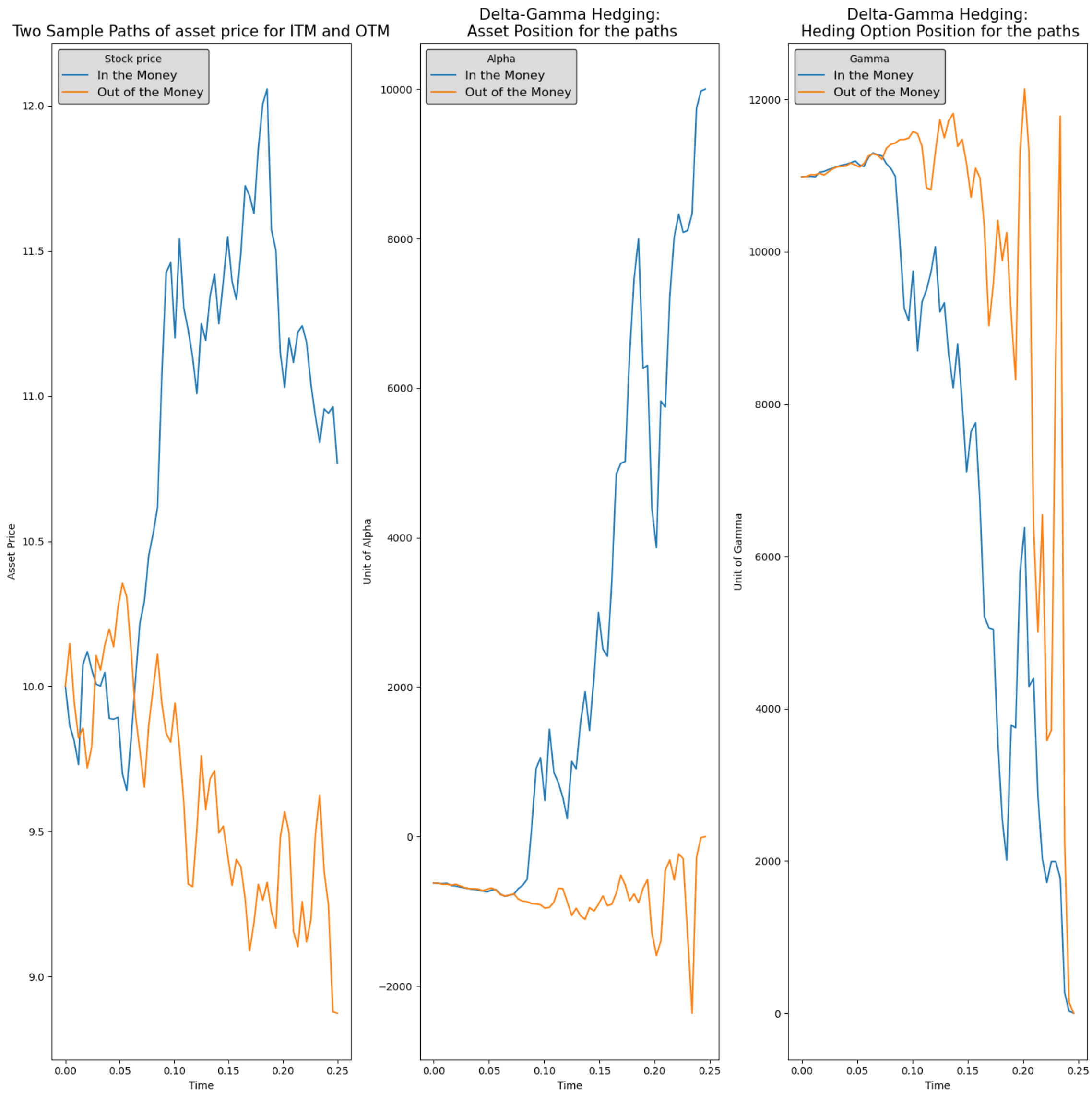

Figure 4: Position in the Underlying Asset for Delta-Gamma Hedging

4.3 The Impact of the Real World Drift

From subsection 3.1, one can see that the real world drift shows up in exactly one place, which is the stochastic differential equation used to model the underlying asset price process. Solving the stochastic differential equation gives the following relationship between distributions of the underlying asset price at time t and $t + dt$ for $dt > 0$:

$$S_{t+dt} = S_t e^{(\mu - \frac{\sigma^2}{2})dt + \sigma(W_{t+dt} - W_t)} \tag{4}$$

where W_t follows a normal distribution with mean 0 and variance dt under the real world probability measure $\mathbb{P}$. Therefore, the log returns $\ln(\frac{S_{t+dt}}{S_t})$ under the real world measure follows a normal distribution with mean $(\mu - \frac{\sigma^2}{2})dt$ and variance $\sigma^2 dt$. In other words, the mean of the log returns (and the underlying asset price) is an increasing function of μ for fixed σ. We can apply this to our profit and loss analysis.

We first consider the delta hedging case when $\mu < 0$. For negative μ, the mean of the log returns under the real world measure is negative. Since we sold an at the money option, we expect most of our simulated paths to end with the option expiring out of the money. By looking at the orange curve in the right plot of Figure 3 presented in subsection 4.2, one can see that we are constantly selling units of the underlying asset that we purchased earlier (at a loss) throughout the life of the option. This decreases our profit. However, since we sold a call option, our position is more profitable when the option expires out of the money. The effects of our different positions counteract each other. For $\mu > 0$, the option is more likely to expire in the money. From the blue curve in the right plot of Figure 3 presented in subsection 4.2, one can see that we end up with 10000 units of the underlying asset, which we sell at a profit. However, since the option expires in the money, we owe the option holder(s) 10000 units of the stock. Again, our different positions counteract each other. It is not immediately obvious from this simple analysis how our overall profit from the entire position changes with changes in μ. To get a better understanding, we turn to our empirical results. Figure 5 shows the profit and loss distributions for three different values of μ. The left plot corresponds to $\mu = -0.2$, the center plot corresponds to $\mu = 0.2$ and the right plot corresponds to $\mu = 0.6$. The means and 0.25 quantiles are presented in the plot and also summarized in Table 2.

Figure 6 shows the mean and 0.25 quantile for 100 different and equally spaced values of μ between -0.2 and 0.6 (inclusive). The plot shows that the mean profit generally decreases as μ increases, although the correlation does not appear very strong and the relationship is not monotonic. On the other hand, the 0.25 quantile tends to increase as μ increases. This means that the left tails of the losses are generally slightly lighter for higher values of μ. The observations from our empirical results align with our earlier analysis. The effects of the underlying asset position and the short option position counteract each other, so profits do not increase or decrease drastically as μ changes.

Next, we will study delta-gamma hedging. Again, we first consider the case when $\mu < 0$. Since we sold an at the money option, we expect most of our simulated paths to end with the option expiring out of the money. By looking at the orange curves in the center and right plots of Figure 4 presented in subsection 4.2, one can see that we hold a short position in the asset throughout the life of the option. At maturity, we close our position in the asset at a profit. We also hold a long position

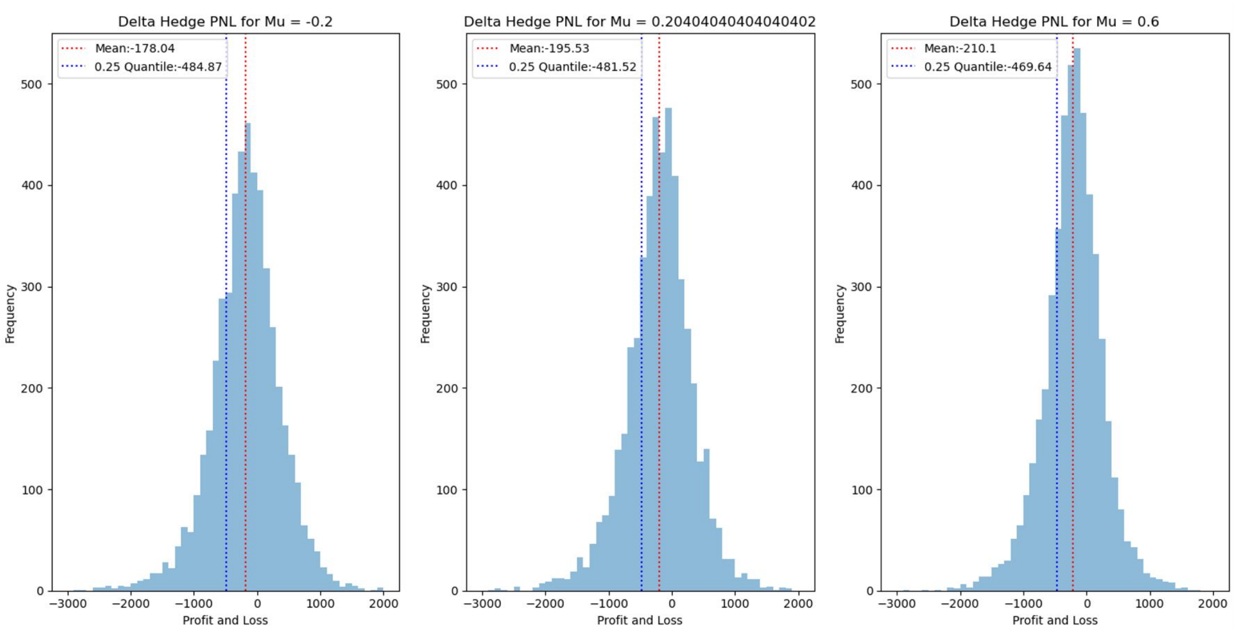

Figure 5: Profit and Loss Under Delta Hedging for Different Values of μ

	Drift = -0.2	Drift = 0.2	Drift = 0.6
Mean	-178.04	-195.53	-210.1
0.25 Quantile	-484.87	-481.52	-469.64

Table 2: Means and 0.25 Quantiles of Profit and Losses for Delta Hedging Under Various Drifts

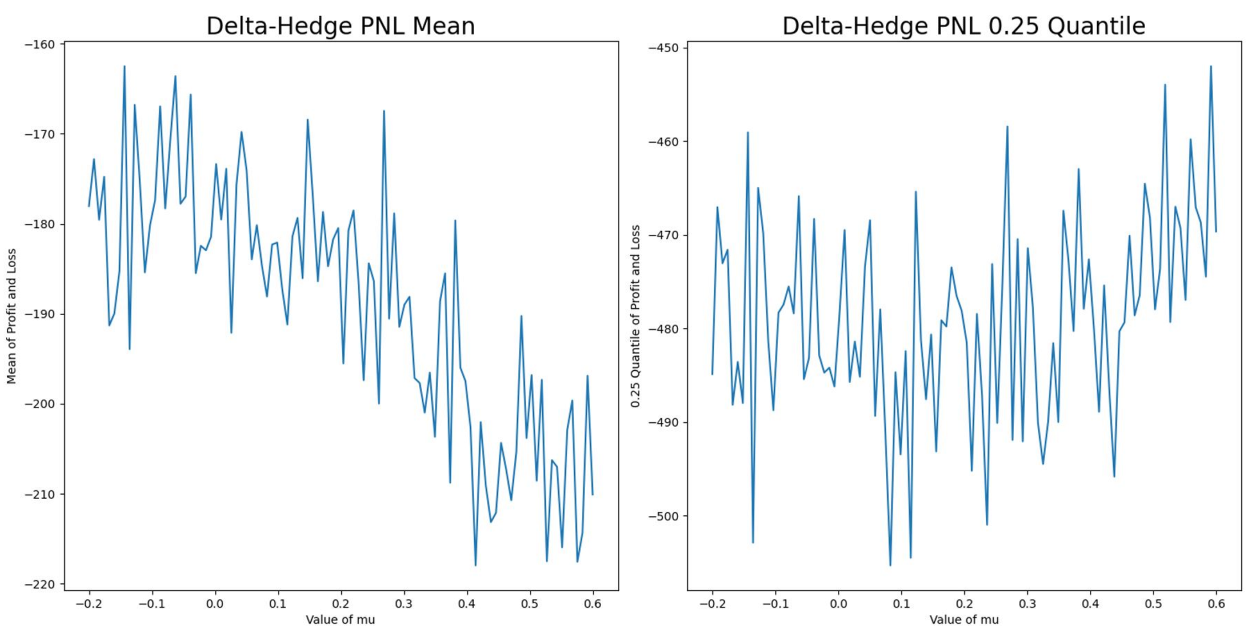

Figure 6: Mean and 0.25 Quantile Under Delta Hedging for Different Values of μ

in the hedging option. Decreasing asset prices have a negative impact on the value of our long call hedging option position. However, since we sold a call option, our position is more profitable when the option expires out of the money. Again, the effects of our different positions counteract each other. For $\mu > 0$, the option is more likely to expire in the money. From the blue curves in the center and right plots of Figure 4 presented in subsection 4.2, one can see that we are constantly purchasing more units of the underlying asset. If the underlying asset price is increasing more on average, we will likely liquidate this position at maturity at a profit. Our position in the hedging option is also more valuable as the underlying asset price increases. However, if the option expires in the money, then we owe 10000 units of the underlying asset to those who purchased the option. Again, the effects of μ on different parts of our position counteract each other, so we analyze the results of our simulation. Figure 7 shows the profit and loss distributions for the same values of μ as Figure 5. The mean and 0.25 quantile is also presented in Table 3.

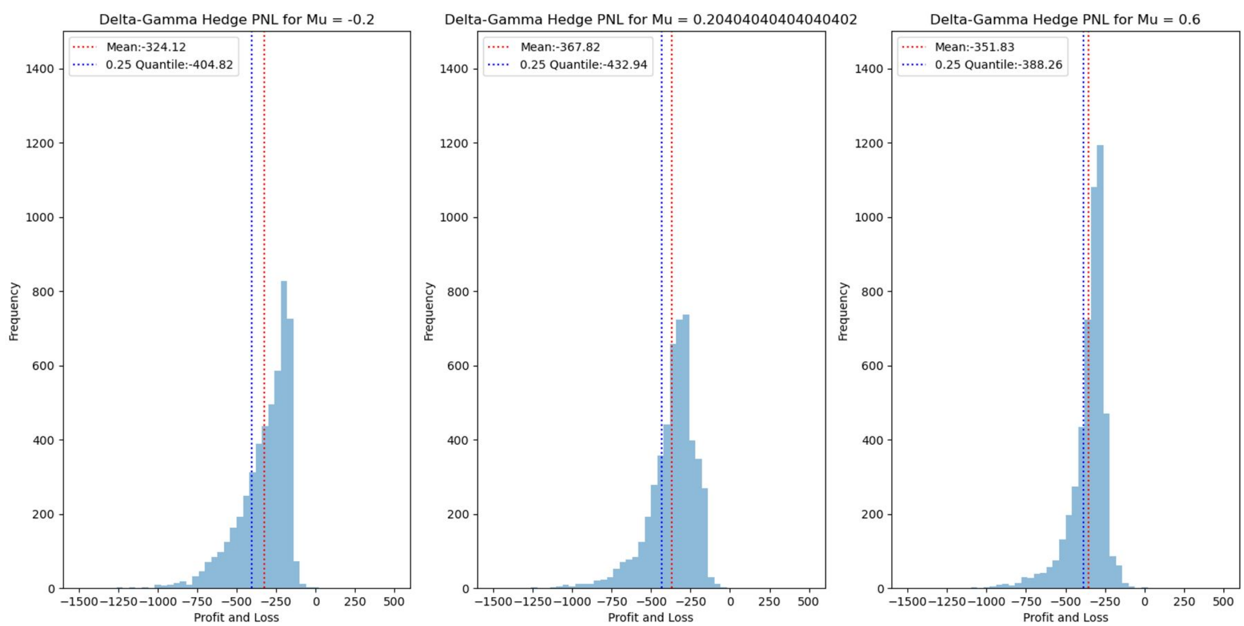

Figure 7: Profit and Loss Under Delta-Gamma Hedging for Different Values of μ

	Drift = -0.2	Drift = 0.2	Drift = 0.6
Mean	-324.12	-367.82	-351.83
0.25 Quantile	-404.82	-432.94	-388.26

Table 3: Means and 0.25 Quantiles of Profit and Losses for Delta-Gamma Hedging Under Various Drifts

Figure 8 shows the mean and 0.25 quantile for 100 different and equally spaced values of μ between -0.2 and 0.6 (inclusive). From the figure, it appears that delta-gamma hedging performs the worst for small positive μ. Also, the mean losses are much larger compared to the delta hedging case due to higher transaction costs.

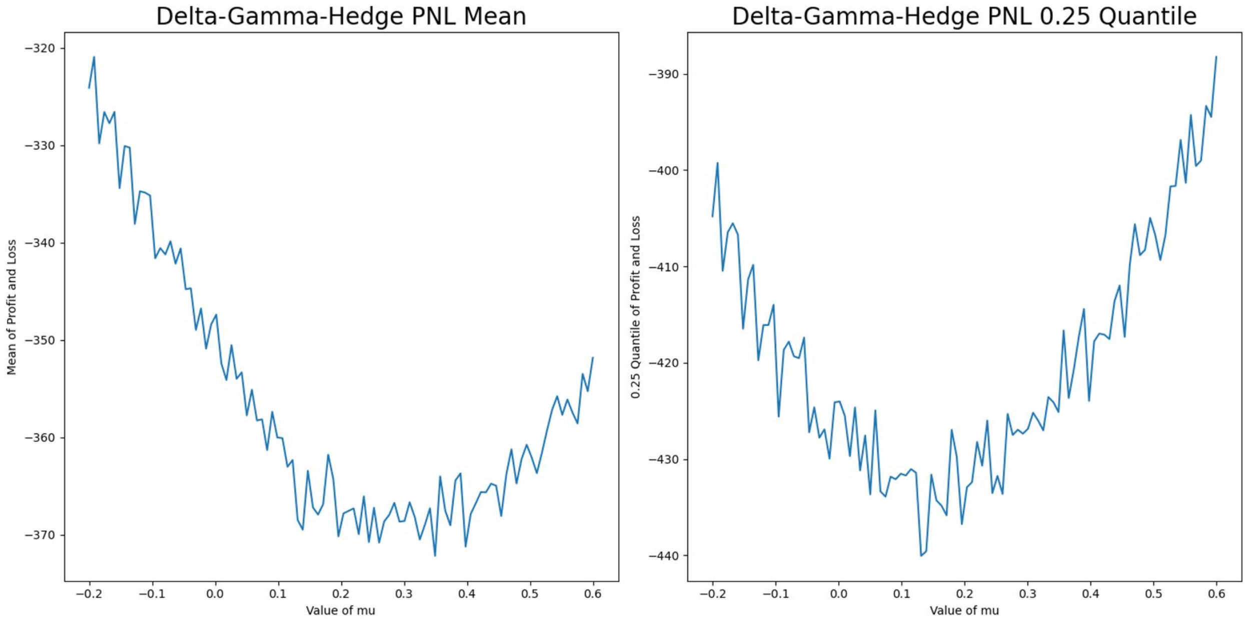

Figure 8: Mean and 0.25 Quantile Under Delta-Gamma Hedging for Different Values of μ

4.4 The Impact of the Real World Volatility on Profit and Loss

In this subsection, we consider the impact of the real world volatility on our profits. When sigma is below the assumed volatility of 0.25, a higher mean value of profit and loss is observed for both delta and delta-gamma strategies. When the real volatility is higher than the assumed volatility, we observe lower mean values for profit and loss in both the strategies. Overall, we can see that the average profit decreases monotonically as real-world volatility increases. This is because option prices increase when volatility of the underlying asset increases (assuming all other factors held constant). When the real-world volatility is higher than the assumed volatility, the option we sold was under-priced, resulting in lower profits. The decreasing average profits can be seen in Figures 9, 10 and 11. Figures 9 and 10 show the distributions of the profit and loss for delta and delta-gamma hedging respectively, with $\sigma_{\text{real}} \in \{0.2, 0.22, 0.24, 0.26, 0.28, 0.3\}$. The mean and interquartile range in Figures 9 and 10 are summarized in Table 4 and 5, respectively. Figure 11 shows the mean value for 28 different and equally spaced values of real-world volatility between 0.18 and 0.32 (inclusive).

In addition to the mean, we also consider the spread of the distributions by looking at the standard deviation and the interquartile range. The interquartile ranges are indicated by blue dashed lines in Figures 10 and 11. The standard deviations for 28 different and equally spaced values of real-world volatility between 0.18 and 0.32 (inclusive) are shown in Figure 12. For delta hedging, the spread of the distribution appears to be quadratic in nature with a minimum when the real volatility is approximately 0.22. The spread of the distribution decreases until 0.22, after which it monotonically increases as the real volatility increases. The hedging strategy appears to perform the best (least variance in profits) when the real-world volatility is close to the pricing volatility.

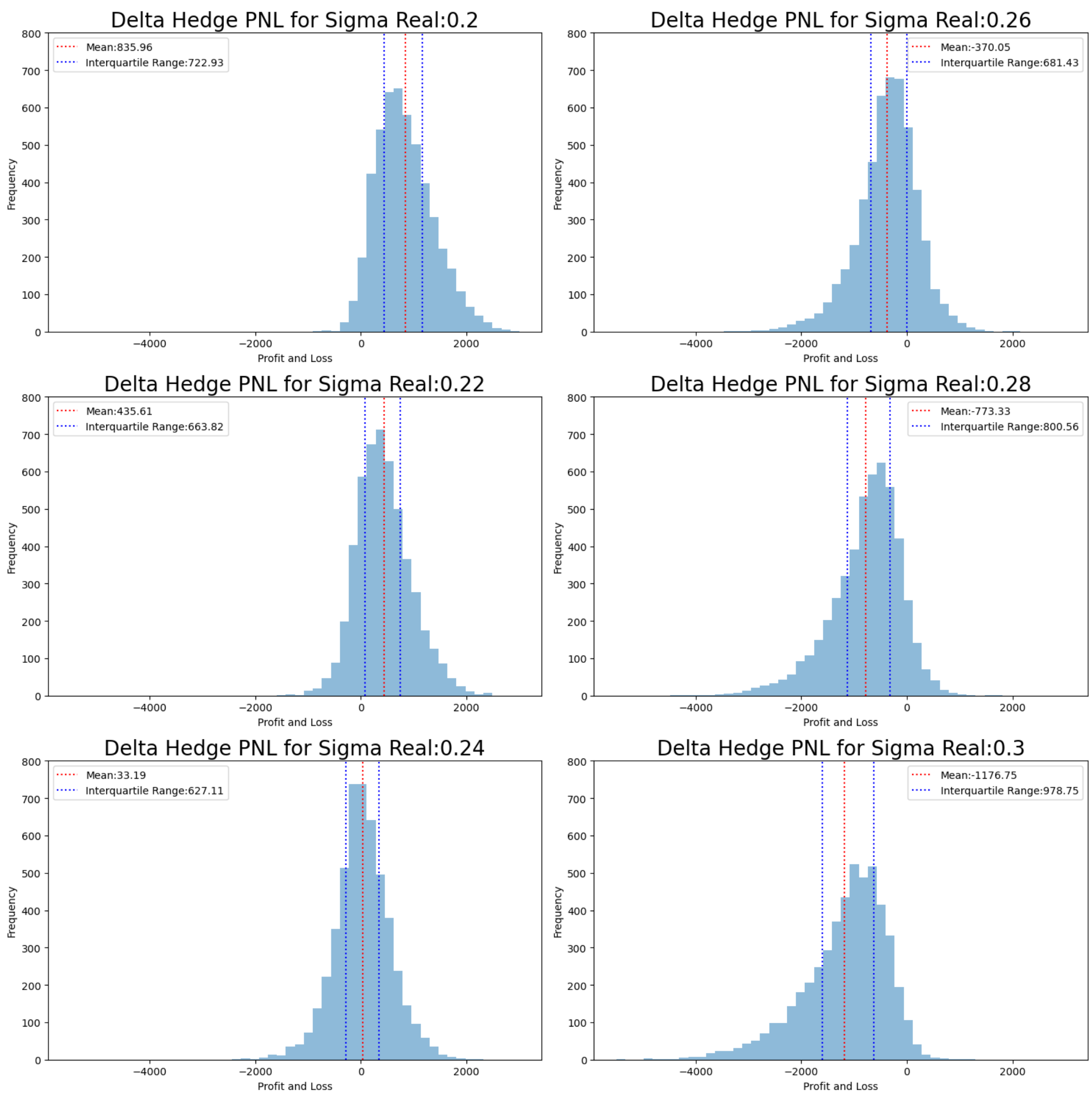

Figure 9: Distribution of Profit and Losses for Delta Hedging Under Different Real World Volatilities

	σ=0.2	σ=0.22	σ=0.24	σ=0.26	σ=0.28	σ=0.3
Mean	835.96	435.61	33.19	-370.05	-773.33	-1176.75
Interquartile Range	722.93	663.82	627.11	681.43	800.56	987.75

Table 4: Means and Interquartile Ranges of Profit and Losses for Delta Hedging Under Various Real World Volatilities

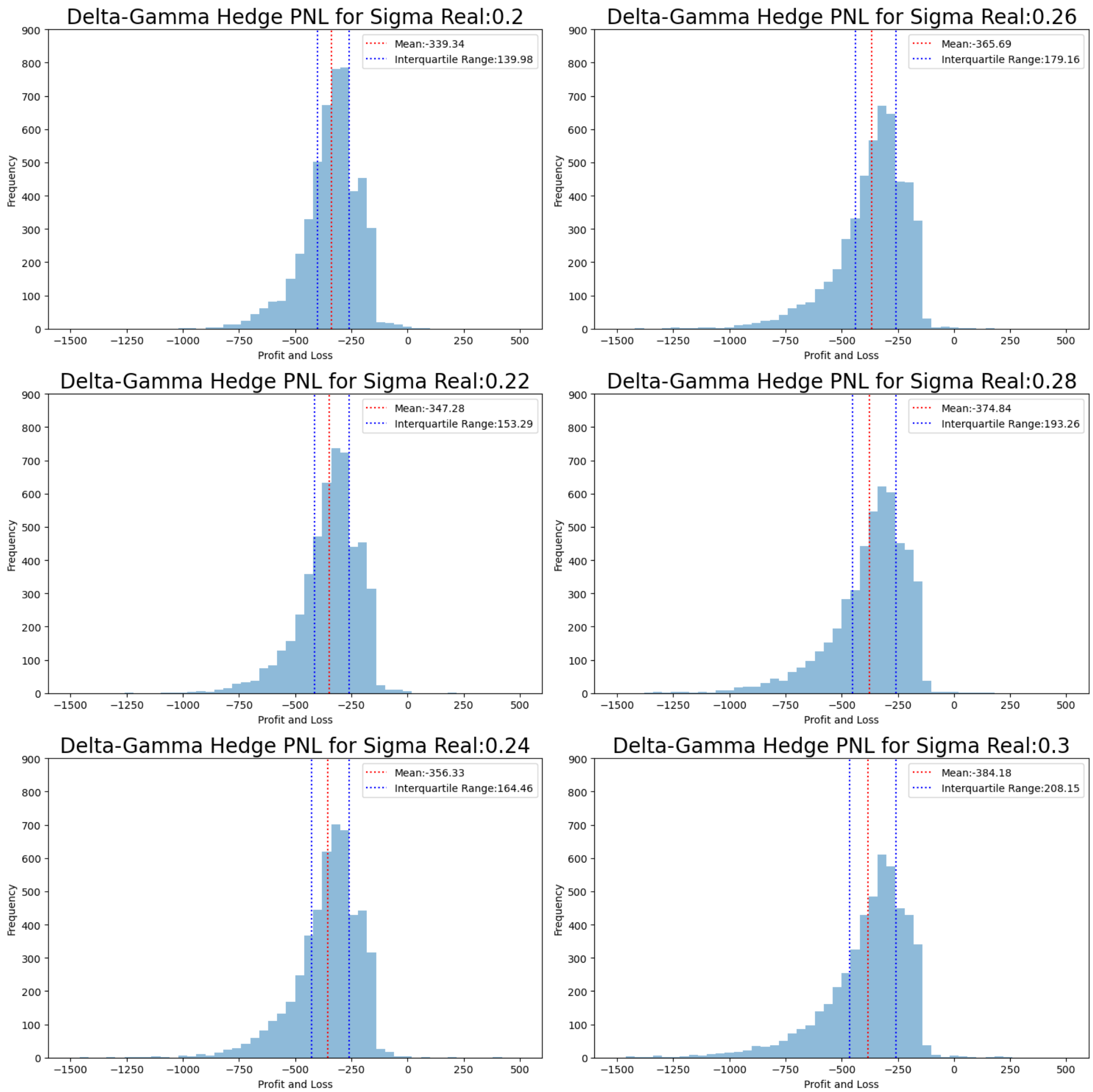

Figure 10: Distribution of Profit and Losses for Delta-Gamma Hedging Under Different Real World Volatilities

	σ=0.2	σ=0.22	σ=0.24	σ=0.26	σ=0.28	σ=0.3
Mean	-339.34	-347.28	-356.33	-365.69	-374.84	-384.18
Interquartile Range	139.98	153.29	164.46	179.16	193.26	208.15

Table 5: Means and Interquartile Ranges of Profit and Losses for Delta-Gamma Hedging Under Various Real World Volatilities

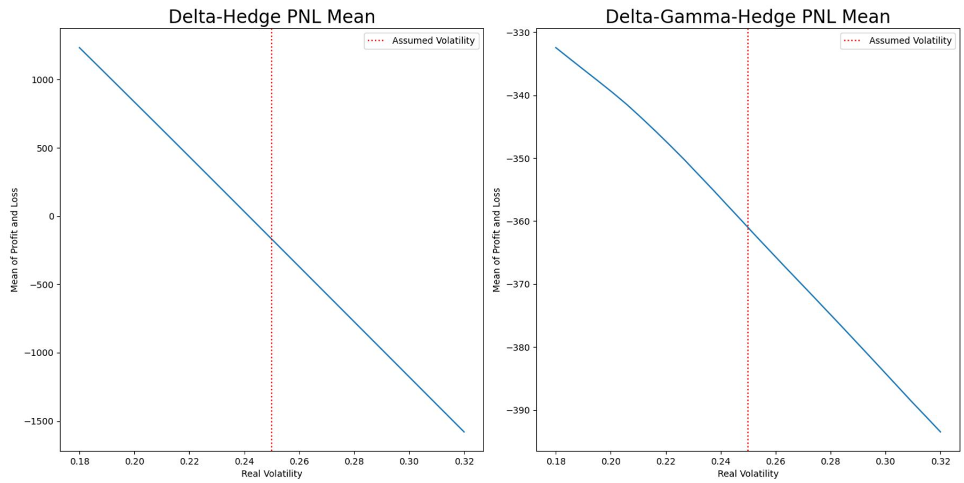

Figure 11: Mean of Profit and Loss Distribution for Different Hedging Strategies

For delta-gamma hedging we can see a monotonic increase in variance of profit and loss as real volatility increases. Unlike the delta hedging case, the variance is not the lowest when the real-world volatility is close to the pricing volatility. This is likely due to the high transaction costs required to maintain a delta-gamma hedged position in a high volatility environment. When comparing the means for the two strategies, it is worth noting that the delta-gamma hedge never has a positive mean profit, even though the option we are selling is overpriced. This is because the hedging option we purchase is also overpriced. Another key observation is that the interquartile ranges for delta-gamma hedging are much smaller for any real world volatility. This makes sense as a delta-gamma hedge uses a higher order approximation of the option price curve, so it is expected that there are fewer large losses or profits.

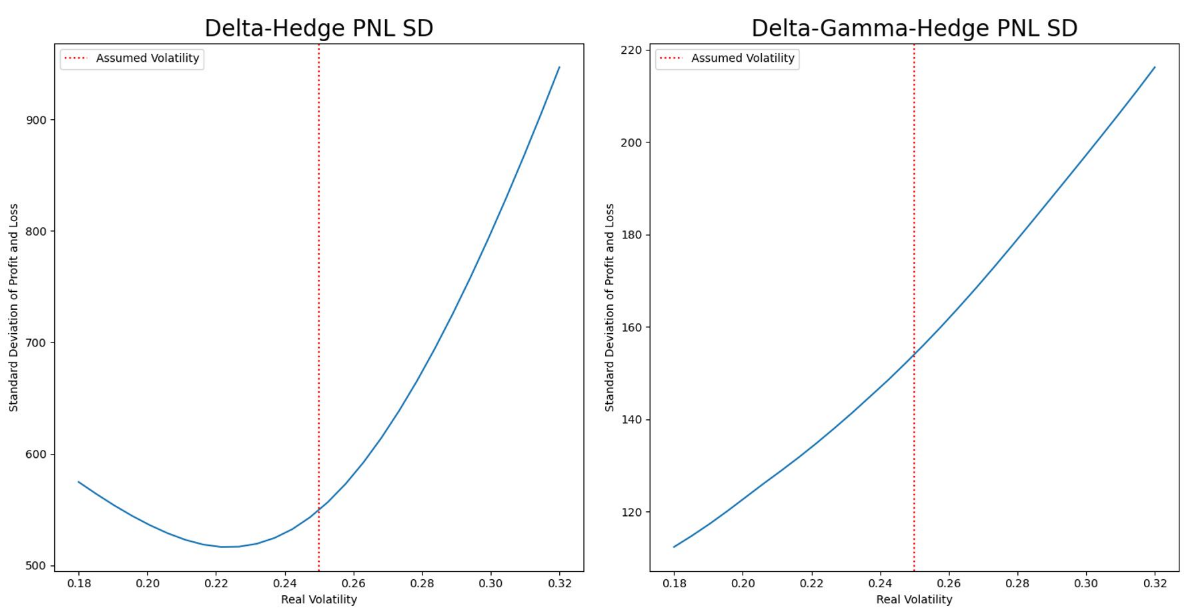

Figure 12: Standard Deviation of Profit and Loss Distribution for Different Hedging Strategies

4.5 The Impact of the Real World Volatility on Hedging Positions

Finally, we consider the impact of real world volatility on our hedging positions. We will analyze the asset paths shown in Figure 13 for the entirety of this subsection. These paths are generated using equation (4) and the assumptions 1-5 in the introduction. We still use a real world drift of 0.1 and a pricing volatility of 0.25, but we vary the real world volatility used to generate the asset prices. Instead of assuming that the real world volatility matches the pricing volatility, we use the real world volatilities $\sigma_{\text{real}} \in \{0.2, 0.22, 0.24, 0.26, 0.28, 0.3\}$. We use the same simulated normal random variables for all the different volatilities, so the paths differ only in the magnitude of the price changes.

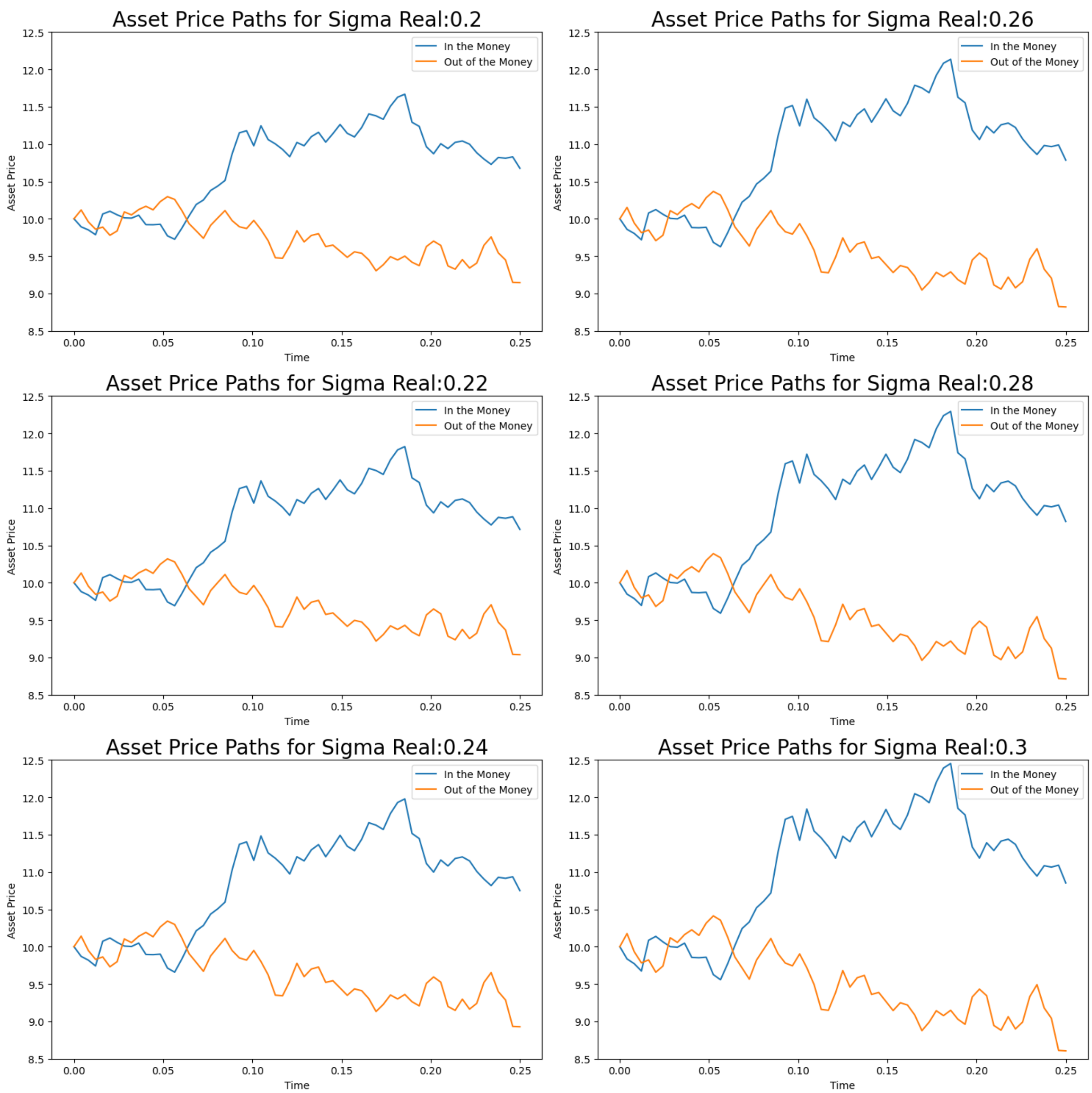

Figure 13: Asset Paths for Different Real World Volatilities

We first compare the positions in the underlying asset for a delta hedge. Positions for select volatilities are shown in Figure 14 below. The left plot corresponds to an in the money path and the right plot corresponds to an out of the money path. Moreover, the blue curve corresponds to a volatility of 0.2, the red curve corresponds to a volatility of 0.24 and the magenta curve corresponds to a volatility of 0.28. When the alpha value is below the starting value (which is just under 6000 for our chosen path), alpha is lower for higher real world volatilities. When the alpha is above the starting value, alpha is higher for higher real world volatilities. This pattern holds regardless of whether or not the option expires in the money.

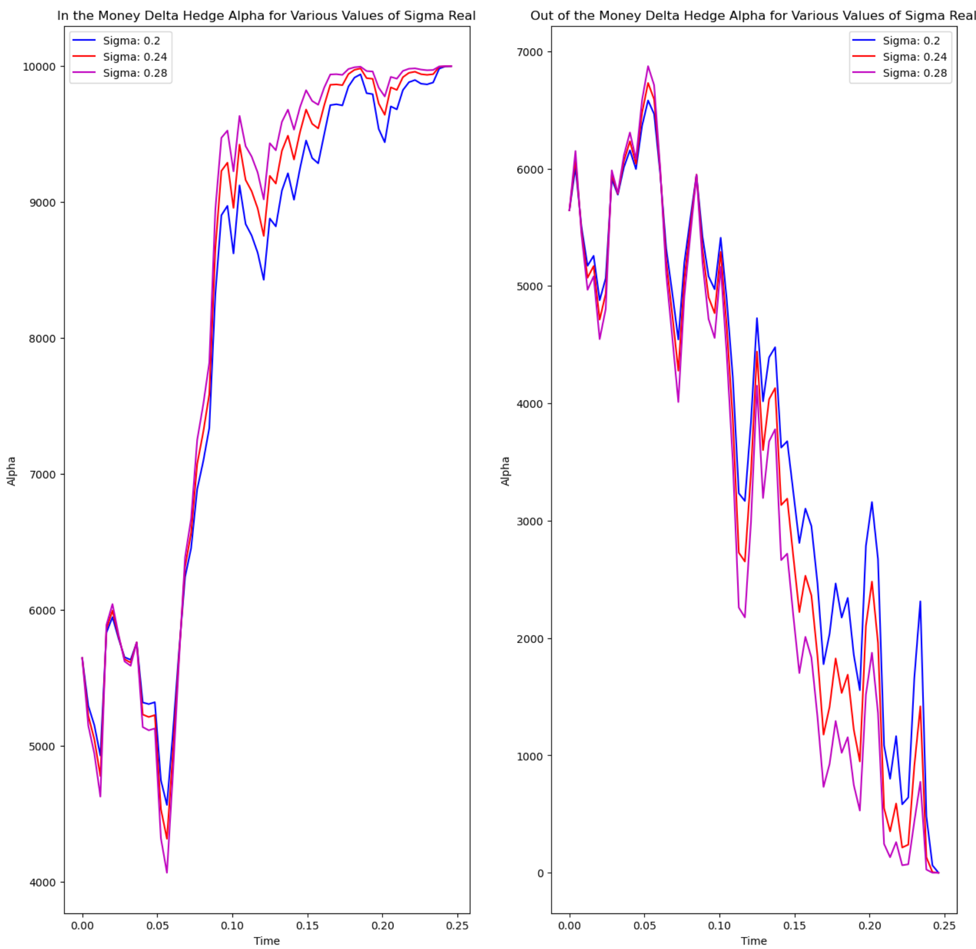

Figure 14: Position in Underlying Asset in a Delta Hedged Portfolio for Different Real World Volatilities

The positions in the underlying asset for a delta-gamma hedge are shown in Figure 15. Again, an in the money path is shown on the left and an out of the money path is shown on the right. From the figure, it is clear that the underlying asset position is higher for higher values of real world volatility. Finally, Figure 16 shows the hedging option positions for a delta-gamma hedge. Both Figure 15 and 16 use the same colouring scheme as Figure 14. From the figure, it is clear that the hedging option position is lower for higher values of real world volatility. Again, this pattern holds for both in the money and out of the money options.

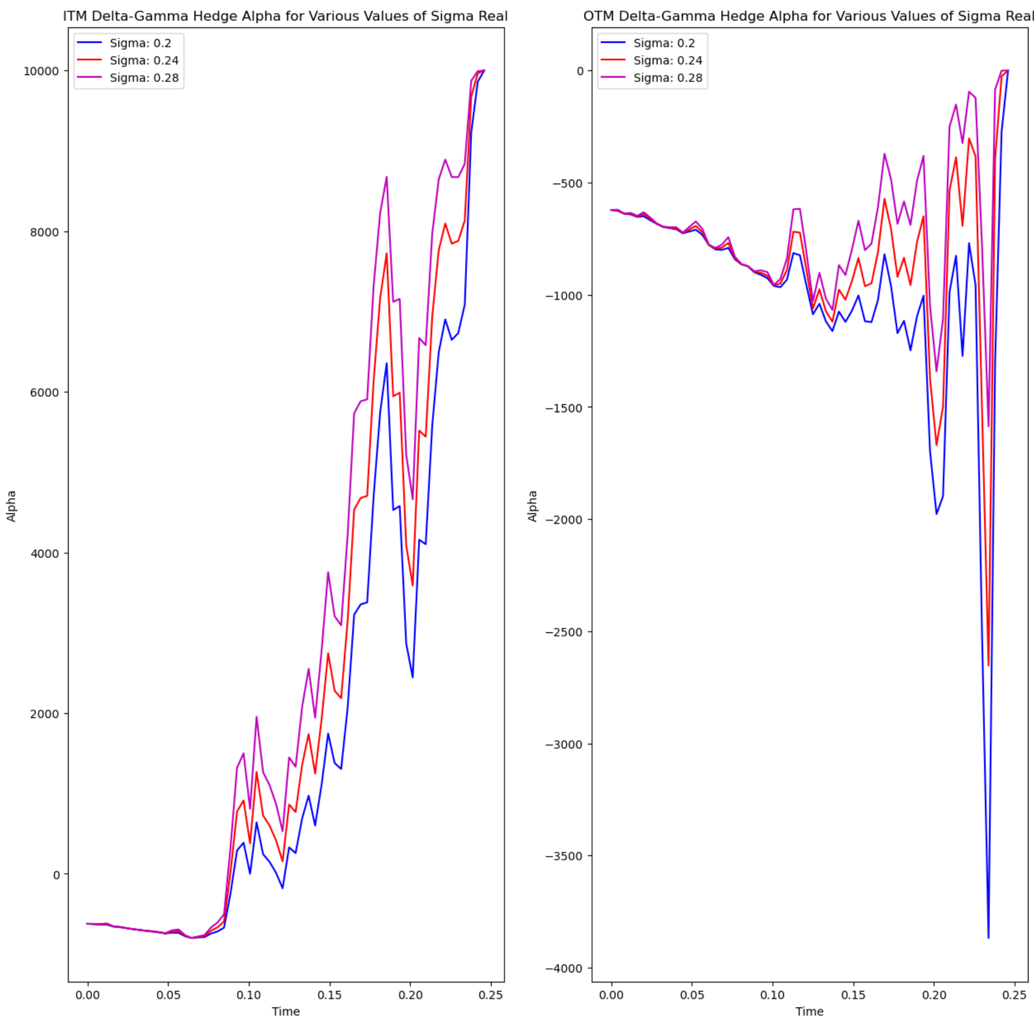

Figure 15: Position in Underlying Asset in a Delta-Gamma Hedged Portfolio for Different Real World Volatilities

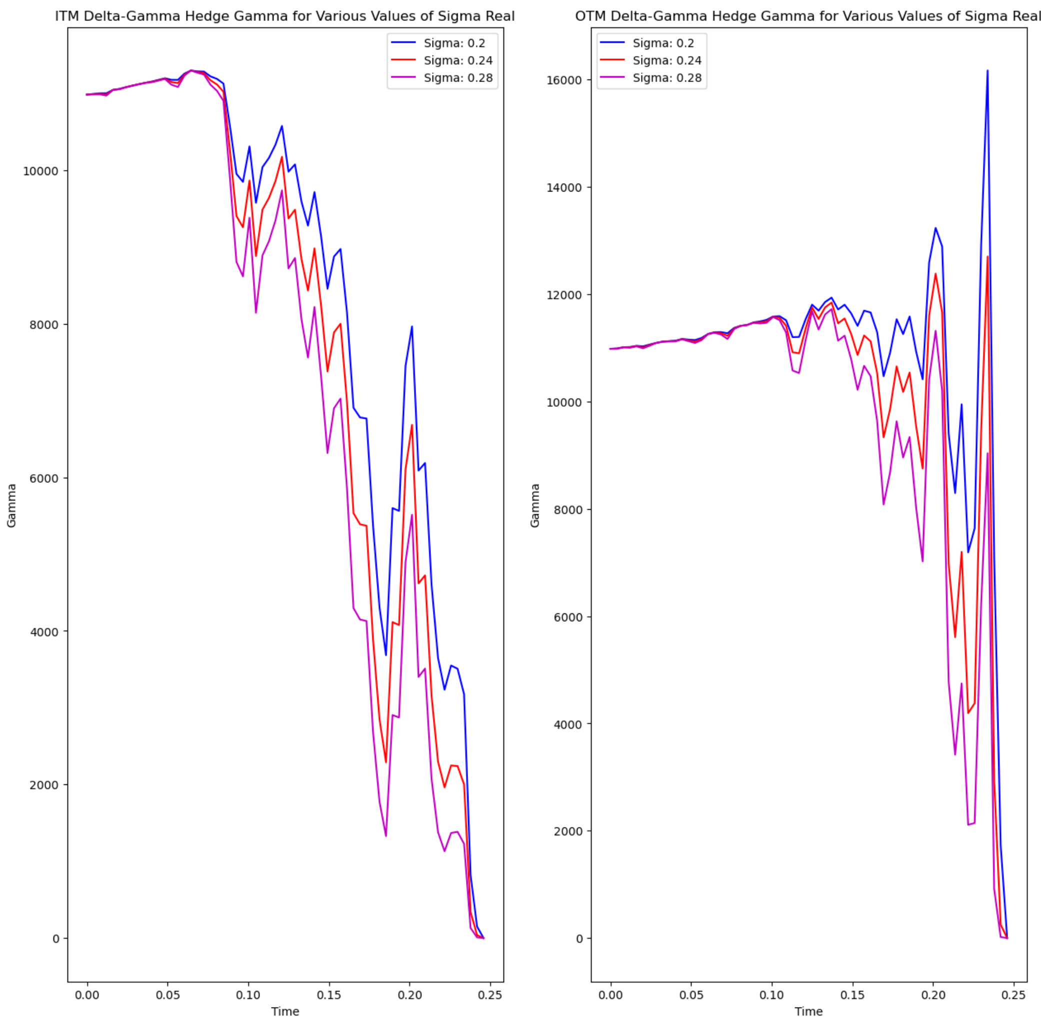

Figure 16: Position in Hedging Option in a Delta-Gamma Hedged Portfolio for Different Real World Volatilities

5 Conclusions

Our analysis of delta and delta-gamma hedging within the Black-Scholes framework highlights the complexities and trade-offs in option pricing and hedging strategies. While both delta and delta-gamma hedging strategies involve frequent re-balancing to manage risk, it is important to note that delta-gamma offers improved precision by addressing both delta and gamma neutrality but introduces heightened complexity and increased transaction expenses.

The influence of the real-world drift on delta and delta-gamma hedging strategies reveals nuanced outcomes. Our findings show a moderate impact of μ on mean profits, with average profits decreasing when μ increases. However, the left tails of the losses are lighter for higher μ values. In delta-gamma hedging, similar dynamics are observed, with less favorable performance for small positive μ and larger mean losses due to higher transaction costs.

Furthermore, our findings underscore the significance of accurately estimating volatility, as differences from assumed values impact profit outcomes. Both delta and delta-gamma hedging strategies exhibit higher mean profits when real volatility is below the assumed level of 0.25 and lower mean values when real volatility exceeds this threshold. Overall, average profits decrease monotonically as a function of real-world volatility in both hedging strategies. Regarding variance, delta-gamma hedging shows lower variance with lower real volatility and higher variance with higher real volatility. Variances for delta hedging follow a more curved shape, with a local minimum when the real volatility is close to the pricing volatility.

In conclusion, the Black-Scholes model provides a solid foundation, but real-world complexities like real world volatilities and varying transaction costs necessitate a nuanced approach to option pricing and hedging. Although the theory suggests that portfolios can be perfectly hedged, this is not the case in practice. Traders must weigh the benefits and drawbacks of delta and delta-gamma hedging, adapting strategies to market conditions and option characteristics.