Valuation of American Options - A Discrete Approximation

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Abstract

We study the valuation, exercise boundary, hedging strategies, profit and loss distributions and exercise time distributions of American put options under different risk-free rate and volatility assumptions by approximating real-life continuous-time stock price evolution by discretizing continuous time into small time steps and performing Monte Carlo simulations. We found that increasing the risk-free rate pushes up the exercise boundary and results in a wider distribution of profit and loss, an earlier exercise time and a higher probability of early exercise. Increasing the volatility pushes down the exercise boundary and results in a higher profit and a higher probability of early exercise. If the realized probability is higher than assumed volatility, the profit and loss is higher, the exercise time is higher, and the probability of exercise is higher.

1 Introduction

A put option is a financial derivative that grants the holder the right, but not the obligation, to sell an underlying asset at a specified price called the strike price. Hence, the payoff of an put option is always non-negative. Different types of put options have different restrictions on the time of exercise. For example, an European put option can only be exercised at the time of maturity. In contrast, an American put option can be exercised anytime before the time of maturity. The lack of a fixed exercise time makes analysis complicated. A discrete model for evolution of stock prices and American options is thus a good approach to study American options.

Intuitively, the first thing that one would be interested in such put options would be the determination of their prices. This is not an easy task before the proposal of the Black-Scholes formula by Fischer Black and Myron Scholes in 1973, and the binomial options pricing model by William Sharpe in 1978. The Black-Scholes formula assumes that stock prices follow a geometric Brownian motion under a continuous setting. Although this is realistic, it is often difficult or impossible to perform simulations under such settings. Thus, the binomial options pricing model is often an easier way for analysis of American options.

In the following sections, we employ the binomial options pricing model to study the valuation, early-exercise strategy and profit-and-loss distributions when a trader purchases an American put option. Additionally, we vary stock price evolution parameters to study how he valuation, early-exercise strategy and profit-and-loss distributions respond to changes in assumptions. In Section 2, we provide the theoretical foundations underlying the binomial options pricing model. In Section 3, we employ the binomial options pricing model to numerically value an American option and study the exercise boundary of such an option, that is, the stock price (as a function of time) that will make early exercise optimal. In Section 4, we study the profit and loss by holding such an American put option.

2 Some Theory

Let N be a fixed (large) positive integer, and T be a positive number. Here, N means the number of time steps we take. T refers to the maximum time that we consider. In our context, it would be

the time of maturity of the American option we consider. Thus, we are breaking the continuous time from time 0 up to time T into N time steps.
We let $\Delta t = \frac{T}{N}$, and $t_k = k\Delta t$. In other words, we partition the continuous time from time 0 up to time T into equal small time intervals each of length Δt.
We consider a market characterized by $(\Omega, \mathcal{F}, (\mathcal{F}_{t_k})_{k\in\{0,1,\dots,N\}}, \mathbb{P})$, where Ω and $\mathcal{F}$ are defined in the usual sense. The associated filtration is also made discrete according to our previous discrete partition. $\mathbb{P}$ refers to the true probability measure in the market.
In our market, there is an asset price process $S = (S_{t_k})_{k\in\{0,1,\dots,N}$. The asset price process is given by the stochastic dynamics

$$S_{t_k} = S_{t_{k-1}} e^{r\Delta t + \sigma\sqrt{\Delta t}\epsilon_k},$$

where ϵ_k are independent and identically distributed random variables with $\epsilon_k \in \{+1, -1\}$ and

$$\mathbb{P}(\epsilon_k \pm 1) = \frac{1}{2}\left(1 \pm \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right),$$

with $r \geq 0$ and $\sigma > 0$ being some constants.
There is also a bank account process $B = (B_{t_k})_{k\in\{0,1,\dots,N\}}$, with $B_t = e^{rt}$, i.e. a deterministic process.
Notice that both S and B always take positive values. Thus, both of them can be used as the numéraire in asset pricing.
We first fix N, the number of steps.

2.1 Two-period martingale measure

Consider two-period binomial trees for asset S and bank account B with time step Δt. For simplicity, we only consider two-period binomial trees starting at one of the nodes at time t_{k-1}. But this two-period binomial tree represents every possible two-period binomial tree in the entire binomial tree up to a scaling factor.

Let p_k be the (real) up-probability, which is constant given N and is given by

$$p_k = \mathbb{P}(\epsilon_k = 1) = \frac{1}{2}\left(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right).$$

Note that the (real) down-probability is

$$1 - p_k = 1 - \frac{1}{2}\left(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) = \frac{1}{2}\left(1 - \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) = \mathbb{P}(\epsilon_k = -1).$$

The two-period binomial tree for B is also constructed using the same probabilities as those in the two-period binomial tree of S. Note that B is a deterministic process, so the probabilities assigned are actually arbitrary, and both end nodes take value of $e^{r\Delta t}$ multiplied by the starting value $B_{t_{k-1}}$. Figure 1 shows the two-period binomial tree of S described, and Figure 2 shows the two-period binomial tree for B. Note that B is actually a deterministic process, so "up" or "down" nodes take the same value. The tree of B is constructed this way such that it can be compared with the binomial tree of S.

Note that the bank account B is always positive (given that $B_0 > 0$, so we can use B as numéraire. Let $\mathbb{Q}$ be the martingale measure induced by using the bank account B as a numéraire. Then, we require

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{t_k}}{B_{t_k}} \,\middle|\, \mathcal{F}_{t_{k-1}}\right] = \frac{S_{t_{k-1}}}{B_{t_{k-1}}},$$

or, by letting the up-probability $q_k = \mathbb{Q}(\epsilon_k = 1)$,

$$q_k \cdot \frac{S_{t_{k-1}} e^{r\Delta t + \sigma\sqrt{\Delta t}}}{B_{t_{k-1}} e^{r\Delta t}} + (1 - q_k) \cdot \frac{S_{t_{k-1}} e^{r\Delta t - \sigma\sqrt{\Delta t}}}{B_{t_{k-1}} e^{r\Delta t}} = \frac{S_{t_{k-1}}}{B_{t_{k-1}}}.$$

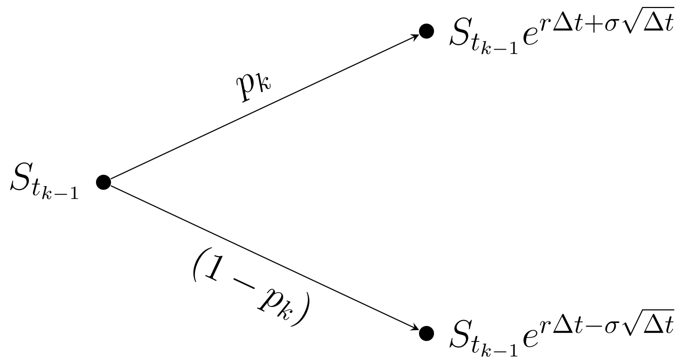

Figure 1: Two-period binomial tree of asset S starting at one of the nodes at time t_{k-1} using real probability

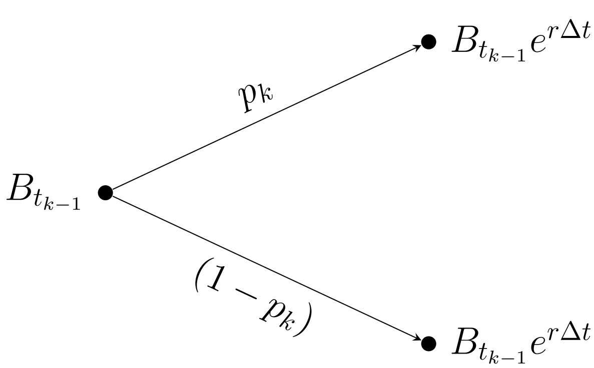

Figure 2: Two-period binomial tree of bank account B starting at one of the nodes at time t_{k-1} using “real probability”

Rearranging terms and elimination yields

$$q_k e^{\sigma\sqrt{\Delta t}} + (1-q_k)e^{-\sigma\sqrt{\Delta t}} = 1,$$

so

$$q_k = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

Moreover, using some algebra we find that

$$1-q_k = \frac{e^{\sigma\sqrt{\Delta t}}-1}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

Now, since $S_{t_k} = S_{t_{k-1}} e^{r\Delta t + \sigma\sqrt{\Delta t}\epsilon_k}$ is always positive due to the exponential term, we can also use asset S as a numéraire. Let $\mathbb{Q}^S$ be the martingale measure induced by using the asset S as a numéraire. Then, we require

$$\mathbb{E}^{\mathbb{Q}^S}\left[\frac{B_{t_k}}{S_{t_k}} \,\middle|\, \mathcal{F}_{t_{k-1}}\right] = \frac{B_{t_{k-1}}}{S_{t_{k-1}}},$$

or, by letting the up-probability $q_k^S = \mathbb{Q}^S(\epsilon_k = 1)$,

$$q_k^S \cdot \frac{B_{t_{k-1}} e^{r\Delta t}}{S_{t_{k-1}} e^{r\Delta t + \sigma\sqrt{\Delta t}}} + (1-q_k^S)\cdot \frac{B_{t_{k-1}} e^{r\Delta t}}{S_{t_{k-1}} e^{r\Delta t - \sigma\sqrt{\Delta t}}} = \frac{B_{t_{k-1}}}{S_{t_{k-1}}}.$$

Rearranging terms and elimination yields

$$q_k^S e^{-\sigma\sqrt{\Delta t}} + (1-q_k^S)e^{\sigma\sqrt{\Delta t}} = 1,$$

so

$$q_k^S = \frac{e^{\sigma\sqrt{\Delta t}}-1}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

Moreover, using some algebra we find that

$$1-q_k^S = \frac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

Hence, we have

$$\mathbb{Q}(\epsilon_k = a) = \begin{cases} \dfrac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} & \text{if } a = 1, \\ \dfrac{e^{\sigma\sqrt{\Delta t}}-1}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} & \text{if } a = -1; \end{cases} \tag{1}$$

and

$$\mathbb{Q}^S(\epsilon_k = a) = \begin{cases} \dfrac{e^{\sigma\sqrt{\Delta t}}-1}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} & \text{if } a = 1, \\ \dfrac{1-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} & \text{if } a = -1. \end{cases} \tag{2}$$

2.2 Limiting distribution of S_T

Now, we consider the distribution of $\log \frac{S_T}{S_0}$ as $N \to \infty$.
Let

$$X_k := \log \frac{S_{t_k}}{S_{t_{k-1}}} = r\Delta t + \sigma\sqrt{\Delta t}\epsilon_k$$

for $k \in \{1, \ldots, N\}$.
The moment generating function of X is

$$\mathbb{E}[e^{hX}].$$

The Taylor series expansion of e^{hX} is

$$\begin{aligned} e^{hX} &= \sum_{n=0}^{\infty} \frac{(hX)^n}{n!} \\ &= 1 + hX + \frac{1}{2}h^2X^2 + o(h^2), \end{aligned}$$

where

$$\lim_{h \to \infty} \frac{o(h^2)}{h^2} = 0.$$

Thus, the Taylor series expansion of the moment generating function of X is

$$\begin{aligned} \mathbb{E}[e^{hX}] &= \mathbb{E}[1 + hX + \frac{1}{2}h^2X^2 + o(h^2)] \\ &= 1 + h\mathbb{E}[X] + \frac{h^2}{2}\mathbb{E}[X^2] + o(h^2). \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{E}[X] &= (r\Delta t + \sigma\sqrt{\Delta t}) \cdot \mathbb{P}(\epsilon_k = 1) \\ &\quad + (r\Delta t - \sigma\sqrt{\Delta t}) \cdot \mathbb{P}(\epsilon_k = -1) \\ &= \left(r\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}\right) \cdot \frac{1}{2}\left(1 + \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\frac{T}{N}}\right) \\ &\quad + \left(r\frac{T}{N} - \sigma\sqrt{\frac{T}{N}}\right) \cdot \frac{1}{2}\left(1 - \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\frac{T}{N}}\right) \\ &= r\frac{T}{N} + (\mu - r - \frac{1}{2}\sigma^2)\frac{T}{N} \\ &= (\mu - \frac{1}{2}\sigma^2)\frac{T}{N}. \\ \mathbb{E}[X^2] &= (r\Delta t + \sigma\sqrt{\Delta t})^2 \cdot \mathbb{P}(\epsilon_k = 1) \\ &\quad + (r\Delta t - \sigma\sqrt{\Delta t})^2 \cdot \mathbb{P}(\epsilon_k = -1) \\ &= \left(r^2\frac{T^2}{N^2} + 2r\sigma\left(\frac{T}{N}\right)^{1.5} + \sigma^2\frac{T}{N}\right) \cdot \frac{1}{2}\left(1 + \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\frac{T}{N}}\right) \\ &\quad + \left(r^2\frac{T^2}{N^2} - 2r\sigma\left(\frac{T}{N}\right)^{1.5} + \sigma^2\frac{T}{N}\right) \cdot \frac{1}{2}\left(1 - \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\frac{T}{N}}\right) \\ &= r^2\frac{T^2}{N^2} + \sigma^2\frac{T}{N} + 2r(\mu - r - \frac{1}{2}\sigma^2)\frac{T^2}{N^2} \\ &= 2r(\mu - \frac{1}{2}\sigma^2)\frac{T^2}{N^2} - r^2\frac{T^2}{N^2} + \sigma^2\frac{T}{N}. \end{aligned}$$

Notice that

$$\log\frac{S_T}{S_0} = \sum_{k=1}^{N} \log\frac{S_{t_k}}{S_{t_k - 1}} = \sum_{k=1}^{N} X_k.$$

Moreover, $X_1, \dots, X_N$ are independent and identically distributed. Hence, the moment generating function of $\log \frac{S_T}{S_0}$ is

$$
\begin{aligned}
\mathbb{E}[e^{h\sum_{k=1}^{N} X_k}] &= \left(\mathbb{E}[e^{hX_1}]\right)^N \\
&= \left(1 + h\mathbb{E}[X_1] + \frac{1}{2}h^2\mathbb{E}[X_1^2] + o(h^2)\right)^N \\
&= 1 + hN\mathbb{E}[X_1] + \frac{h^2}{2}N\mathbb{E}[X_1^2] + \frac{N(N-1)}{2}h^2\mathbb{E}[X_1]^2 + o(h^2) \\
&= 1 + h(\mu - \frac{1}{2}\sigma^2)T + \frac{h^2}{2}2r(\mu - \frac{1}{2}\sigma^2)\frac{T^2}{N} - \frac{h^2}{2}r^2\frac{T^2}{N} + \frac{h^2}{2}\sigma^2 T \\
&\quad + \frac{h^2}{2}(\mu - \frac{1}{2}\sigma^2)^2T^2 - \frac{h^2}{2}(\mu - \frac{1}{2}\sigma^2)^2\frac{T^2}{N} + o(h^2) \\
&= 1 + h(\mu - \frac{1}{2}\sigma^2)T + \frac{h^2}{2}[(\mu - \frac{1}{2}\sigma^2)^2T^2 + \sigma^2 T] \\
&\quad - \frac{h^2}{2}(\mu - r - \frac{1}{2}\sigma^2)^2\frac{T^2}{N} + o(h^2).
\end{aligned}
$$

As $N \to \infty$, the final term $\frac{h^2}{2}(\mu - r - \frac{1}{2}\sigma^2)^2\frac{T^2}{N} \to 0$. Thus, we have

$$
\lim_{N\to\infty} \mathbb{E}[e^{h\sum_{k=1}^{N} X_k}] = 1 + h(\mu - \frac{1}{2}\sigma^2)T + \frac{h^2}{2}[(\mu - \frac{1}{2}\sigma^2)^2T^2 + \sigma^2 T] + o(h^2).
$$

We claim that it is equal to the Taylor expansion of the moment generating function of some normal random variable with mean a and variance b. The moment generating function of such random variable would be

$$
e^{ah + \frac{b^2}{h^2}}.
$$

The Taylor series expansion at $h = 0$ is

$$
1 + ah + \frac{h^2}{2}a^2 + \frac{h^2}{2}b + o(h^2).
$$

So, claim holds if we set $a = (\mu - \frac{1}{2}\sigma^2)T$ and $b = \sigma^2 T$. Hence, we conclude that as $N \to \infty$, $\sum_{k=1}^{N} X_k = \log \frac{S_T}{S_0}$ follows the $N((\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T)$ distribution. In other words, as $N \to \infty$,

$$
\log \frac{S_T}{S_0} \xrightarrow{d} (\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z,
$$

where $Z \overset{\mathbb{P}}{\sim} N(0, 1)$.

3 Numerical Valuation and Exercise Boundary of American put option

In this section, we evaluate an American put option, assuming that $T = 1, S_0 = 10, \mu = 5\%, \sigma = 20\%, r = 2\%$. We use $N = 5000$ and $K = 10$ (K is the strike price of the American put option). Recall that the payoff of an American put option, exercised at time t ($t \in [0, 1]$) is

$$
(K - S_t)_+ = \max(K - S_t, 0).
$$

Early exercise is possible for American put options.

In Section 2, we argued that both B and S can be used as the numéraire. In this section, we will use both B and S as the numéraire.

3.1 Algorithms for generation of paths and trees

In this section, we briefly introduce our algorithms implemented to generate paths for S, trees for S, B and the tree for valuation of the American put option. We only discuss the algorithms using B as the numéraire here, as using the algorithms using S as the numéraire is basically the same, except that $\mathbb{Q}$ is changed to $\mathbb{Q}^s$.

To generate paths for S using B as the numéraire, we adopt Algorithm 1. To generate a full binomial tree for S using B as the numéraire, we adopt Algorithm 2. To generate a full binomial tree for B, we adopt Algorithm 3.

To find the value of an American option, we start by calculating the payoff of the American option, $(K - S_t)_+$ at each node at time t if it is exercised immediately. The payoff is set to 0 if $K - S_t < 0$ in that node, since in this case the option will never be exercised immediately. Then, starting from the nodes corresponding to the time point one time step before maturity, we find the value of the American option at that node to be

$$\max\left((K - S_t)_+, e^{-r\Delta t}\mathbb{E}(\text{value of option at next time step})\right).$$

Thus, to generate the payoff (value) tree for the American option using B as the numéraire, we adopt Algorithm 4.

Algorithm 1 Path generation for S

```
Set initial value as S_0
for k ∈ {1, 2, ..., 5000} do
    Generate u_k ~ U(0, 1).
    Set ε_k = 1 if u_k < Q(ε_k = 1).
    Set S_k = S_{k-1} e^{rΔt+σ√Δt}.
end for
```

Algorithm 2 Tree generation for S

```
Start with a 5001 × 5001 matrix of 0's, say, S.
Let S[i, k] be the entry of S in the position of row i, column k.
Set initial value as 10.
for k ∈ {1, 2, ..., 5000} do                                   ▷ Column index
    for i ∈ {0, 1, ..., j} do                                  ▷ Row index
        if i=0 then
            S[i, k] = S[i, k − 1] e^{rΔt+σ√Δt}                 ▷ For the "always up" node
        else
            S[i, k] = S[i, k − 1] e^{rΔt−σ√Δt}                 ▷ Recombining tree, always use down is fine
        end if
    end for
end for
```

Algorithm 3 Tree generation for B

Start with a 5001×5001 matrix of 0's, say, M.
Let $M[i,k]$ be the entry of M in the position of row i, column k.
Set initial value as 1.
for $k \in \{1, 2, \dots, 5000\}$ **do** ▷ Column index
  for $i \in \{0, 1, \dots, j\}$ **do** ▷ Row index
    if i=0 **then**
      $M[i,k] = M[i,k-1]e^{r\Delta t}$ ▷ For the "always up" node
    else
      $M[i,k] = M[i,k-1]e^{r\Delta t}$ ▷ For the sake of consistency between trees
    end if
  end for
end for

Algorithm 4 Payoff Tree generation for American option

Start with a 5001×5001 matrix of 0's, say, P.
Let $P[i,k]$ be the entry of P in the position of row i, column k.
Set $P[i, 5001] = (K - S[i, 5001])_+$. ▷ Initial values at time T, or step 5000
Initialize early exercise array, EA, as empty.
for $k \in \{5000, 4999, \dots, 1\}$ **do** ▷ Column index
  for $i \in \{k, k-1, \dots, 1\}$ **do** ▷ Row index
    Compute
$$P_1 = \mathbb{Q}(\epsilon_{k-1} = 1)\frac{P[i-1,k]}{B[i-1,k]} + \mathbb{Q}(\epsilon_{k-1} = -1)\frac{P[i,k]}{B[i,k]}.$$
▷ Expected value of the discounted payoffs
    Find $P_2 = (K - S[i-1, k-1])_+$. ▷ The payoff if option is exercised immediately
    $P[i-1, k-1] = \max(P_1, P_2)$
    if $P_2 > P_1$ **then** ▷ Early exercise is optimal
      Save $S[i-1, k-1]$ in EA. ▷ There might be multiple values per k
    end if
  end for
end for
Take early exercise boundary as largest S in EA at each k.

3.2 Exercise boundary and sample paths

Figure 3 shows the early exercise boundary in green, using S as the numéraire. One sample path where in the option is exercised early at around $t = \frac{1}{2}$ is plotted in blue. Another sample path where in the option is not exercised is plotted in orange.

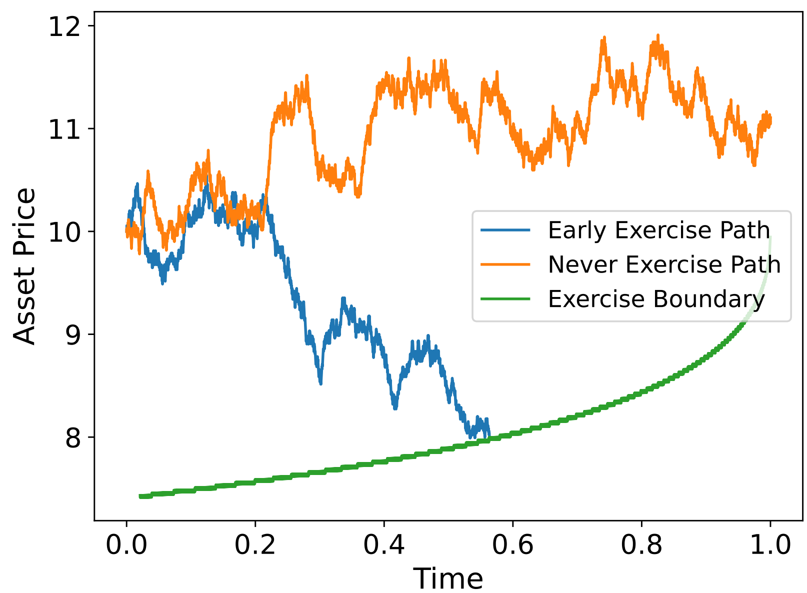

Figure 3: Exercise boundary of the American option and two sample paths of S

Figure 4 shows the early exercise boundary in green, using B as the numéraire. One sample path where in the option is exercised early at around $t = \frac{1}{2}$ is plotted in blue. Another sample path where in the option is not exercised is plotted in orange.

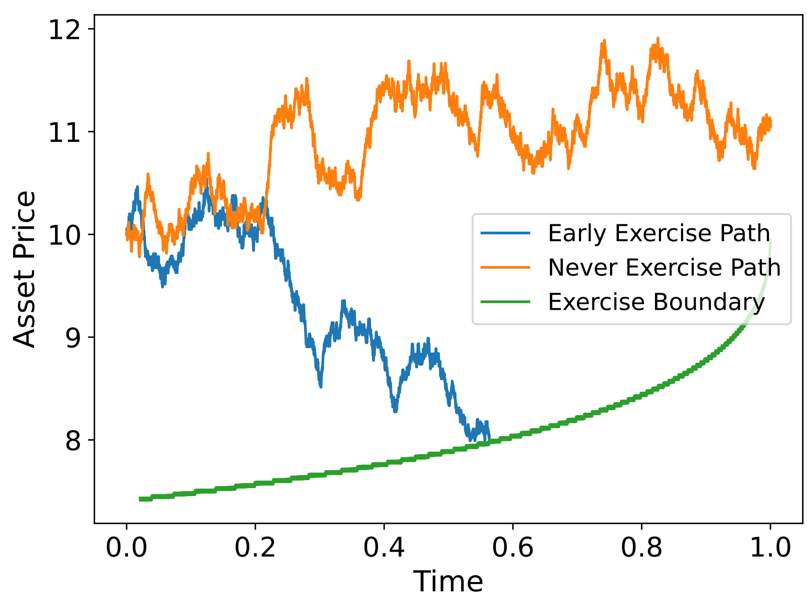

Figure 4: Exercise boundary of the American option and two sample paths of S

Notice that the two exercise boundaries generated using S or B as the numéraire are the same. Indeed, changing the numéraire changes the martingale measure. However, as long as the martingale property holds, the valuation of assets and derivatives is the same. Therefore, using S or B as the numéraire does not matter. From the next section onwards, all results are obtained using S as the numéraire.

3.3 Hedging strategy

We construct a hedging strategy of the American put option, that is, a portfolio that replicates the payoff of the American put option. Such strategy, of course, has to be dynamic. When the hedging strategy at time t_k is implemented, at time t_{k+1}, the payoff of the hedging strategy must replicate the two possible payoffs of the American put option (before any changes to the hedging strategy necessary at time t_{k+1}.

Consider time t_k, when the stock price is S_{t_k}. At this time point, we re-balance our hedging strategy into holding α_{t_k} units of S and holding β_{t_k} units (dollar) in B. Let the value of the American put option that corresponds to the stock price S_{t_k} be P_{t_k}, which can be obtained from the put value tree obtained by Algorithm 4. At time t_{k+1}, S can either go up to $S^u_{t_{k+1}}$ or go down to $S^d_{t_{k+1}}$. Each unit (at time t_k) of B must become $e^{r\delta t}$ at time t_{k+1}. Let $P^u_{t_{k+1}}$ be the value of the American option if the stock price S goes up at time t_{k+1}, and $P^d_{t_{k+1}}$ be the value of the American option if the stock price S goes down at time t_{k+1}. If S goes up, the replicating payoff condition requires

$$\alpha_{t_k} S_{t_k} e^{r\Delta t+\sigma\sqrt{\Delta t}} + \beta_{t_k} e^{r\Delta t} = P^u_{t_{k+1}}; \tag{3}$$

and if S goes down, the replicating payoff condition requires

$$\alpha_{t_k} S_{t_k} e^{r\Delta t-\sigma\sqrt{\Delta t}} + \beta_{t_k} e^{r\Delta t} = P^d_{t_{k+1}}. \tag{4}$$

Notice that $\beta_{t_k} e^{r\Delta t}$ appears in both equations (3) and (4). So, by moving terms around we have

$$\alpha_{t_k} S_{t_k} e^{r\Delta t+\sigma\sqrt{\Delta t}} - P^u_{t_{k+1}} = \alpha_{t_k} S_{t_k} e^{r\Delta t-\sigma\sqrt{\Delta t}} - P^d_{t_{k+1}} = \beta_{t_k} e^{r\Delta t}, \tag{5}$$

which implies

$$\alpha_{t_k} = \frac{P^u_{t_{k+1}} - P^d_{t_{k+1}}}{S_{t_k} e^{r\Delta t}(e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}})} = \frac{P^u_{t_{k+1}} - P^d_{t_{k+1}}}{S^u_{t_{k+1}} - S^d_{t_{k+1}}}. \tag{6}$$

Substituting equation (6) into equation (3) yields

$$\frac{P^u_{t_{k+1}} - P^d_{t_{k+1}}}{S_{t_k} e^{r\Delta t}(e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}})} S_{t_k} e^{r\Delta t+\sigma\sqrt{\Delta t}} + \beta_{t_k} e^{r\Delta t} = P^u_{t_{k+1}},$$

which implies

$$\beta_{t_k} = e^{-r\Delta t}\left[\frac{P^u_{t_{k+1}} - P^d_{t_{k+1}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} e^{\sigma\sqrt{\Delta t}} - P^u_{t_{k+1}}\right]. \tag{7}$$

Since the hedging portfolio in this section considers the replication of the payoff of purchasing an American put option, by holding the hedging portfolio, the holder actually hedges shorting of the American put option. In this case, since the payoff of exercising a purchased American option is $(K-S_t)_+$ if exercised at time t, α (the position in stock) should be negative and β (the position in bank account) should be positive, as all possible payoffs of the American option are $(K-S_t)_+$ for some t, and the payoffs at the up and down nodes are weighted discounted averages of payoffs of such form (or just the payoff if exercised immediately), which has negative coefficient in front of S_t and positive K (corresponding to an investment in bank account in hedging portfolio).

Figure 5 plots α_{t_k} (the positions in asset S) and β_{t_k} (the positions in bank account) as a function of time, under a sample path that results in early exercise of the American put option. It is observed that at the time of exercise, α_{t_k} goes to -1 and β_{t_k} goes to 10. This is because if

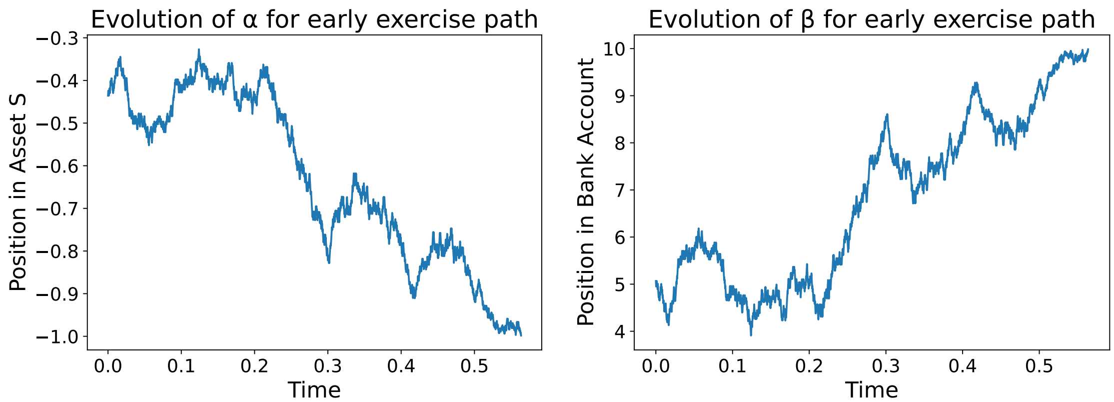

Figure 5: Hedging strategies for an sample path that results in early exercise of the option as a function of t, in units

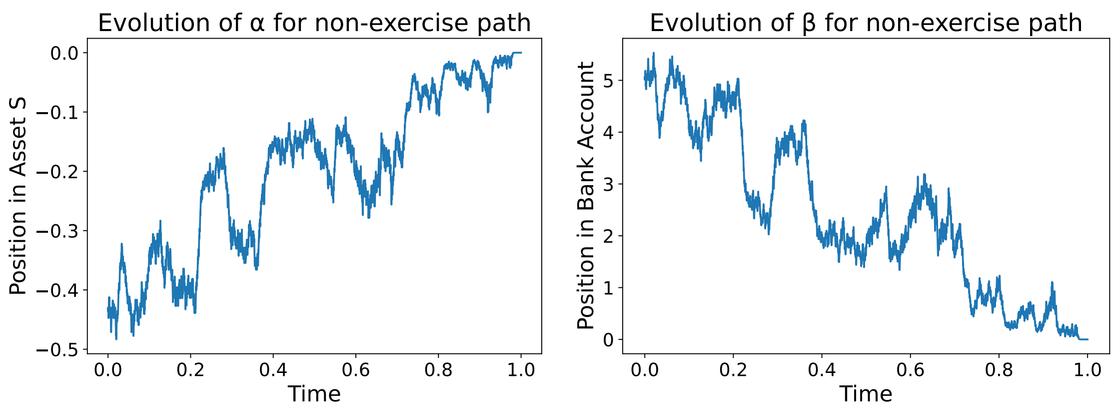

Figure 6: Hedging strategies for an sample path that does not results in exercise of the option as a function of t, in units

early exercise is optimal at time t, the payoff we need to replicate is very close to $K - S_t$ (which is greater than 0), and thus the hedging strategy requires a stock position very close to -1 and a bank position very close to $K = 10$.

Figure 6 plots α_{t_k} (the positions in asset S) and β_{t_k} (the positions in bank account) as a function of time, under a sample path that does not result in exercise of the American put option. It is observed that at $t_k \approx 1$ when the option nearly matures, α_{t_k} and β_{t_k} goes to 0. If the option is never exercised, at t_k very close to 1, the payoff we need to replicate is very close to 0 in both up and down nodes. Hence, $P^u_{t_{k+1}} - P^d_{t_{k+1}} \approx 0$, so $\alpha_{t_k} \approx 0$. Thus, to replicate the 0 payoff, β_{t_k} is also close to 0.

3.4 Change in risk-free rate and volatility

We also investigate the robustness of the exercise boundary with respect to different values of risk-free rate r and volatility σ. We repeat our analysis in previous sections using $r = 0\%, 2\%, 4\%$ and $\sigma = 10\%, 20\%, 30\%$.

If $r = 0\%$, early exercise of the American put option is not optimal. At every time point t, the payoff of exercising the American put option immediately is $K - S_t$. The payoff of holding the American put option until maturity and then exercising it is $K - S_T$.

Now, consider the purchase of an American put option and the investment in a portfolio that

Time	American put	Compare	Portfolio
t	$K - S_t$	$\geq$	$Ke^{-r(T-t)} - S_t$
T	$(K - S_T)_+$	$\geq$	$K - S_T$

Table 1: Comparison of payoffs of purchasing American put option and the portfolio

Time	American put	Compare	Portfolio
t	$K - S_t$	$=$	$K - S_t$
T	$(K - S_T)_+$	$\geq$	$K - S_T$

Table 2: Comparison of payoffs of purchasing American put option and the portfolio when $r = 0$

consists of investing Ke^{-T} in bank account (at time 0) and shorting a unit of stock. Table 1 shows the comparison of payoffs at time t ($t < T$) and time T (maturity of option) of the two strategies.

In particular, Table 2 shows the comparison of payoffs at time t ($t < T$) and time T (maturity of option) of the two strategies when $r = 0$. Since the payoffs at time t are the same but the payoff of the American put option at maturity time T is always at least the payoff of the portfolio, it is optimal to not exercise the option at time t but to wait until time T. Intuitively, by exercising early the holder of the option can invest the money into the bank account. But when $r = 0$, the bank account earns nothing, so such early exercise is not worth it. Thus, there is no early exercise boundary and sample paths that result in early exercise for $r = 0\%$.

Figure 7 shows the exercise boundary (if available), a sample path that results in early exercise (if available) and a sample path that does not result in early exercise for different r and σ. Note that a higher r and/or a lower σ will push up the exercise boundary. If r is higher, money in bank account earns more interest, so it is more feasible for early exercise of the option in order to get money instantly for investing in the bank account, even for higher stock prices which means a lower option payoff at time of exercise. If σ is lower, then the value of the option decreases. Exercising the option early to get money to invest in the bank account becomes more feasible, even at higher stock prices (lower option payoff if exercised).

Figure 8 and Figure 9 shows the evolution of α_{t_k} and β_{t_k} for sample paths that result in early exercise of the option. Note that the sample paths only resemble one particular realization of the scenario and may not be representative. We observe that increasing r and decreasing σ results in a "bumpier" evolution of α_{t_k} and β_{t_k} when it is close to time of early exercise.

Figure 10 and Figure 11 shows the evolution of α and β for sample paths that result in early exercise of the option. Note that the sample paths only resemble one particular realization of the scenario and may not be representative. We observe that increasing r and decreasing σ results in a "bumpier" evolution of α_{t_k} and β_{t_k} when it is close to time of maturity.

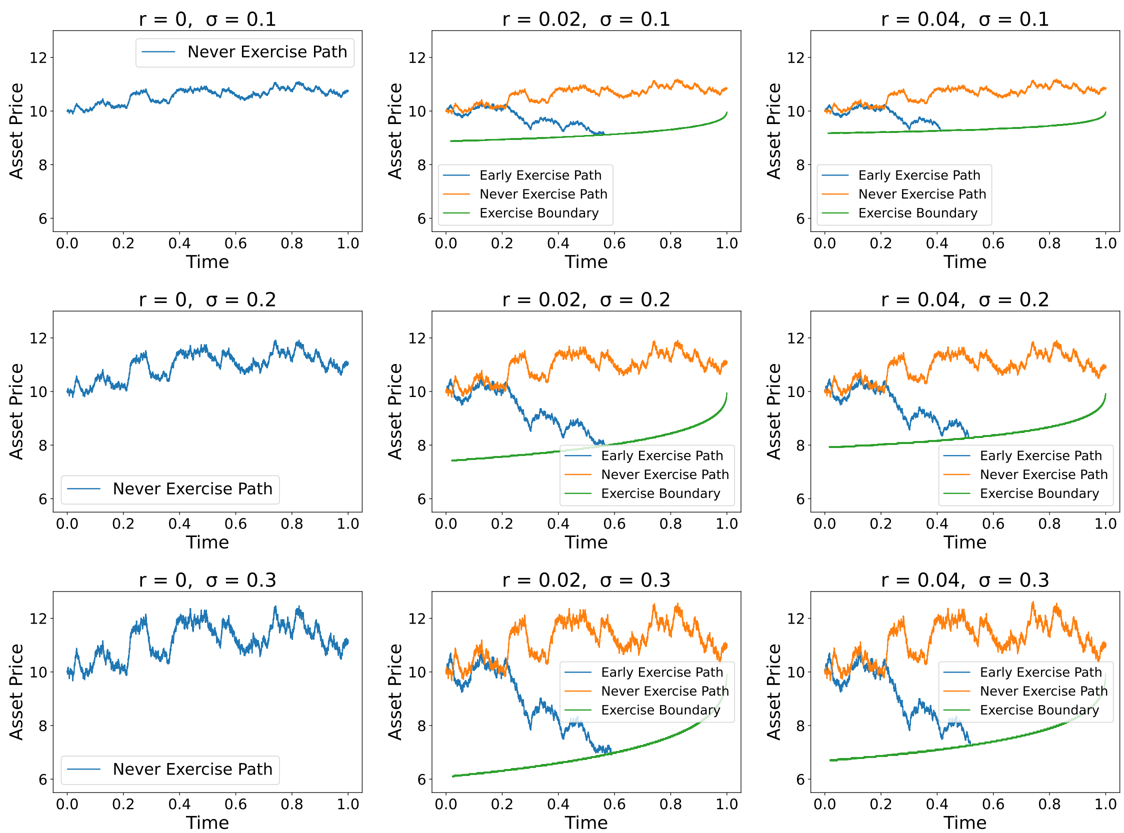

Figure 7: Sample path and exercise boundary (if available) for different r and σ

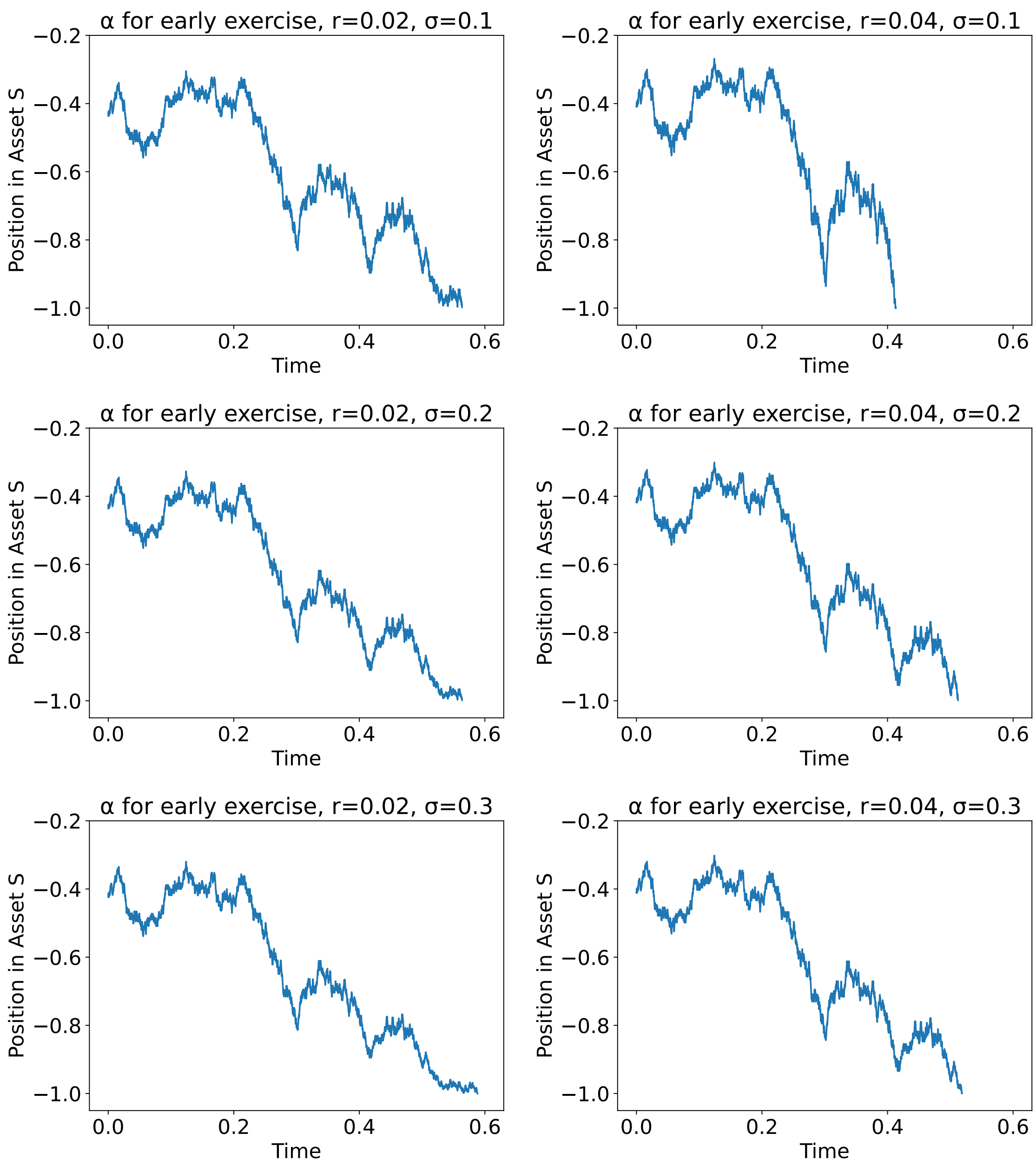

Figure 8: Evolution of α_{t_k} for a sample path that results in early exercise, for different r and σ

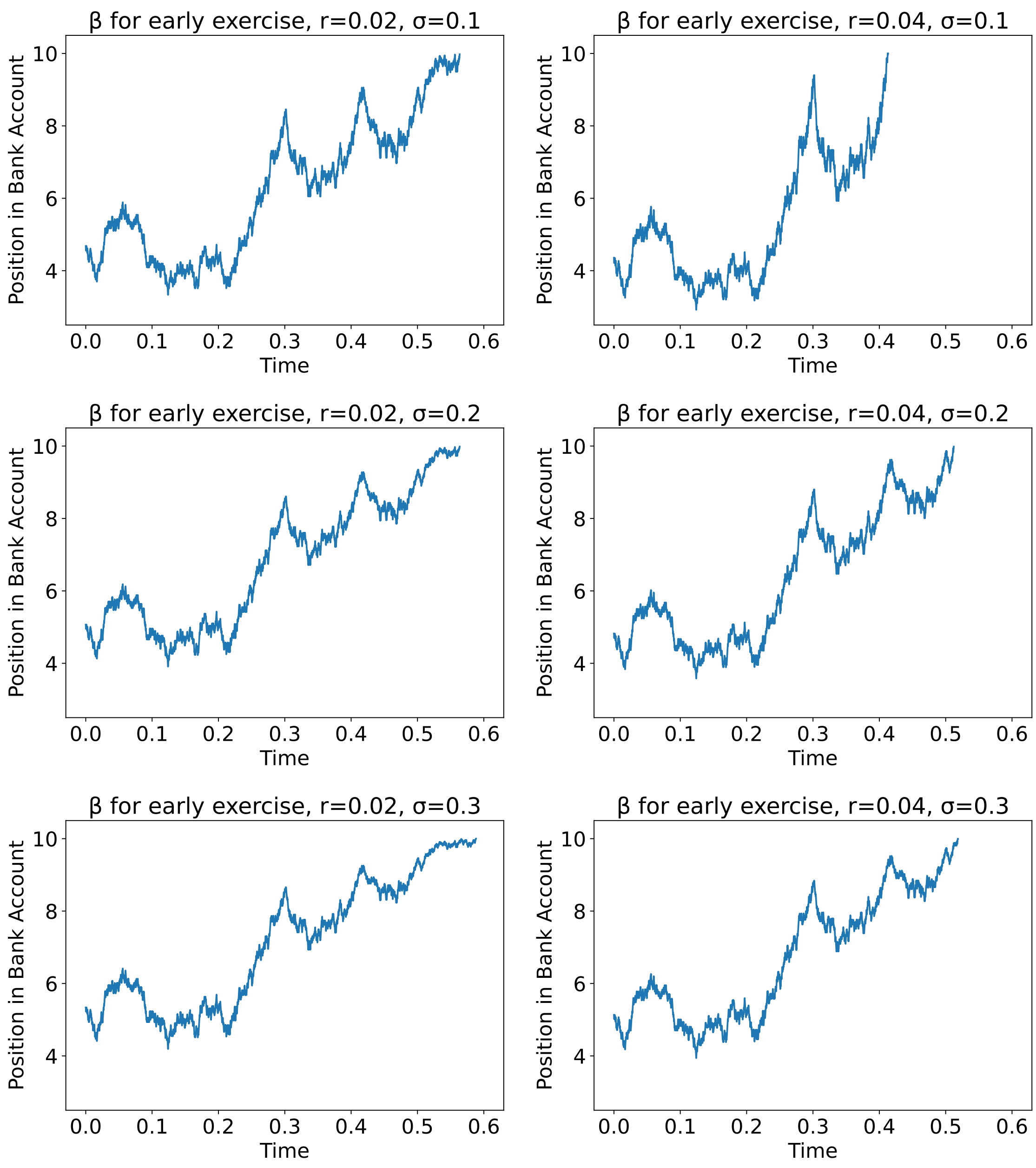

Figure 9: Evolution of β_{t_k} for a sample path that results in early exercise, for different r and σ

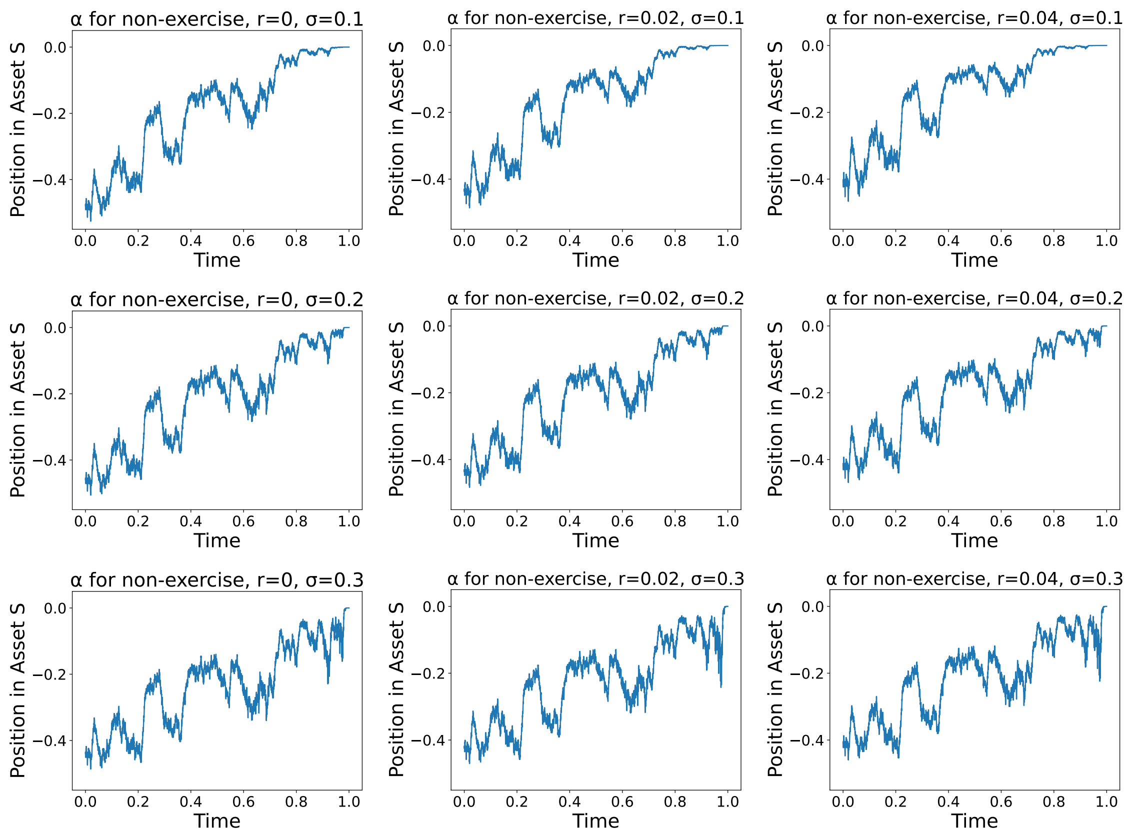

Figure 10: Evolution of α_{t_k} for a sample path that does not result in early exercise, for different r and σ

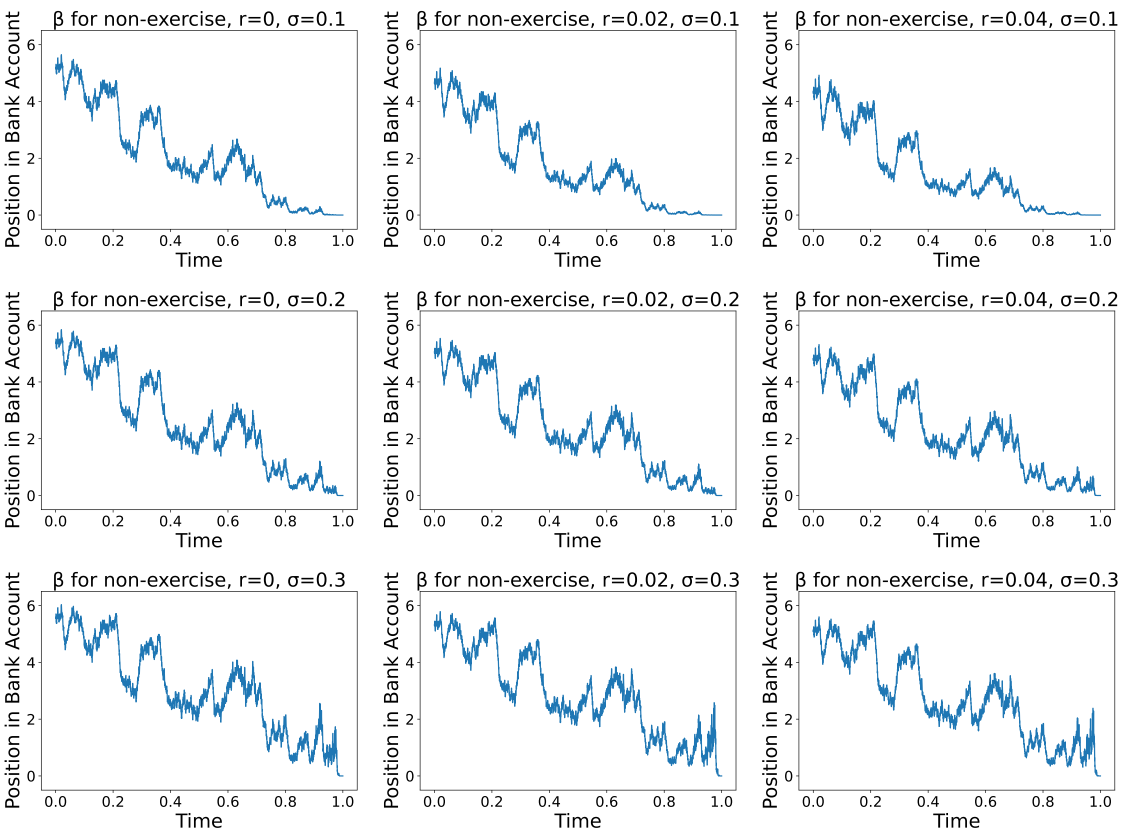

Figure 11: Evolution of β_{t_k} for a sample path that does not result in early exercise, for different r and σ

4 Profit and Loss for a purchased American option

Assume that we purchased the American option using the parameters $T = 1, S_0 = 10, \mu = 5\%, K = 10$. Use $N = 5000$ timesteps as in previous sections.

4.1 Distributions of profit/loss and exercise time

It is very difficult to analytically find the distribution of profit/loss and exercise time for a trader who purchased the option. However, it is possible to approximate it by simulating a large number, say, 10000 times and taking the empirical distribution.
Note that the profit and loss is defined by

$$\text{Present value of payoff of option when exercise (0 if not exercised)} - \text{Initial price of option},$$

and the exercise time is the option exercise time if the option is ever exercised.

We simulate 10000 sample paths using $\sigma = 20\%, r = 2\%$. Then, for each sample path, we check whether the sample path touches the exercise boundary. If yes, the payoff at exercise (first touching time of exercise boundary) is discounted back to time 0. The payoff is obtained from the equation above, and the exercise time is also recorded. Figure 12 shows the histogram and kernel density estimate of the profit and loss distribution if the option is exercise early. Note that the trader sometimes loses money, but most of the time the trader earns some money. It is sensible since we conditioned on the option being exercised. Figure 13 shows the histogram and kernel density estimate of the exercise time of the option if the option is exercised. Note that it is more likely to early exercise towards the maturity, since the value of holding the American option goes down towards maturity due to lower level of fluctuations in stock price before the maturity. Both figures ignore those scenarios when the option is not exercised at all. These scenarios actually happen frequently. 44.9% of the sample paths result in the option being exercised early. This means that the trader will not make any money more than half of the time, and will even lose the initial price of the option.

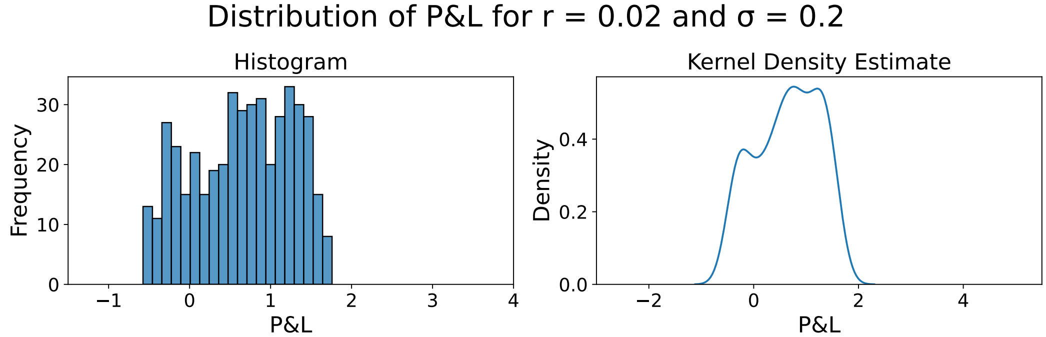

Figure 12: Distribution of the profit and loss for a trader who purchased the option

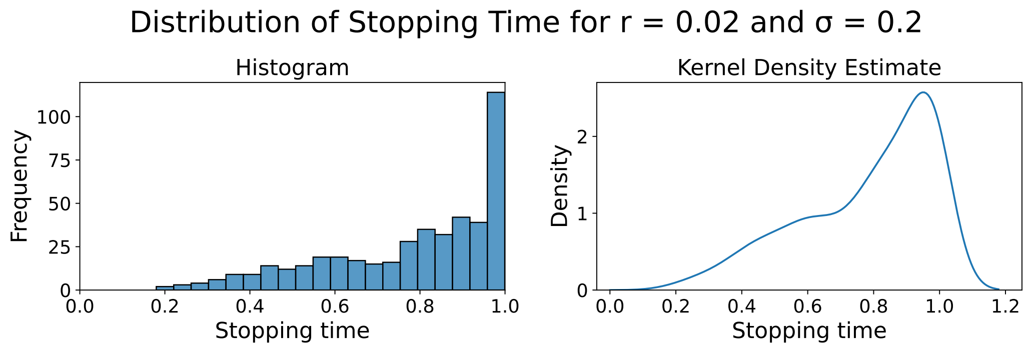

Figure 13: Distribution of the exercise time for a trader who purchased the option

4.2 Robustness of profit/loss and exercise time distributions

To study how the profit and loss and exercise time distribution varies as r and σ varies, we repeat what we performed in Section 4.1 under different risk-free rate of return and volatility using $r = 1\%, 3\%, 5\%$ and $\sigma = 10\%, 30\%, 50\%$. Figures 14, 15 and 16 shows how the profit and loss distribution changes as r and σ vary. Figures 17, 18 and 19 shows how the time of exercise changes as r and σ vary. Table 3 shows how the probability of early exercise varies as r and σ vary.

We note that higher σ results in generally higher profit and higher probability of early exercise. The effect of changing σ on exercise time (conditioning on early exercise) does not appear to be strong. This is because higher σ means larger variations in stock price, and thus it is more likely for the stock price to evolve into a position low enough for early exercise to be feasible, so that exercising the option immediately and investing in the bank account woud be a better choice than holding the option.

Also, higher r results in a wider distribution of profit and loss and an earlier exercise time, as well as a higher probability of early exercise except when $\sigma = 0.5$. This is because a higher return of bank account encourages holders to exercise early to invest money earned into the bank account as long as σ is not too large.

σ \ r	0.01	0.03	0.05
0.1	0.313	0.341	0.417
0.3	0.45	0.454	0.495
0.5	0.543	0.567	0.518

Table 3: Probability of exercising the American option under different r and σ

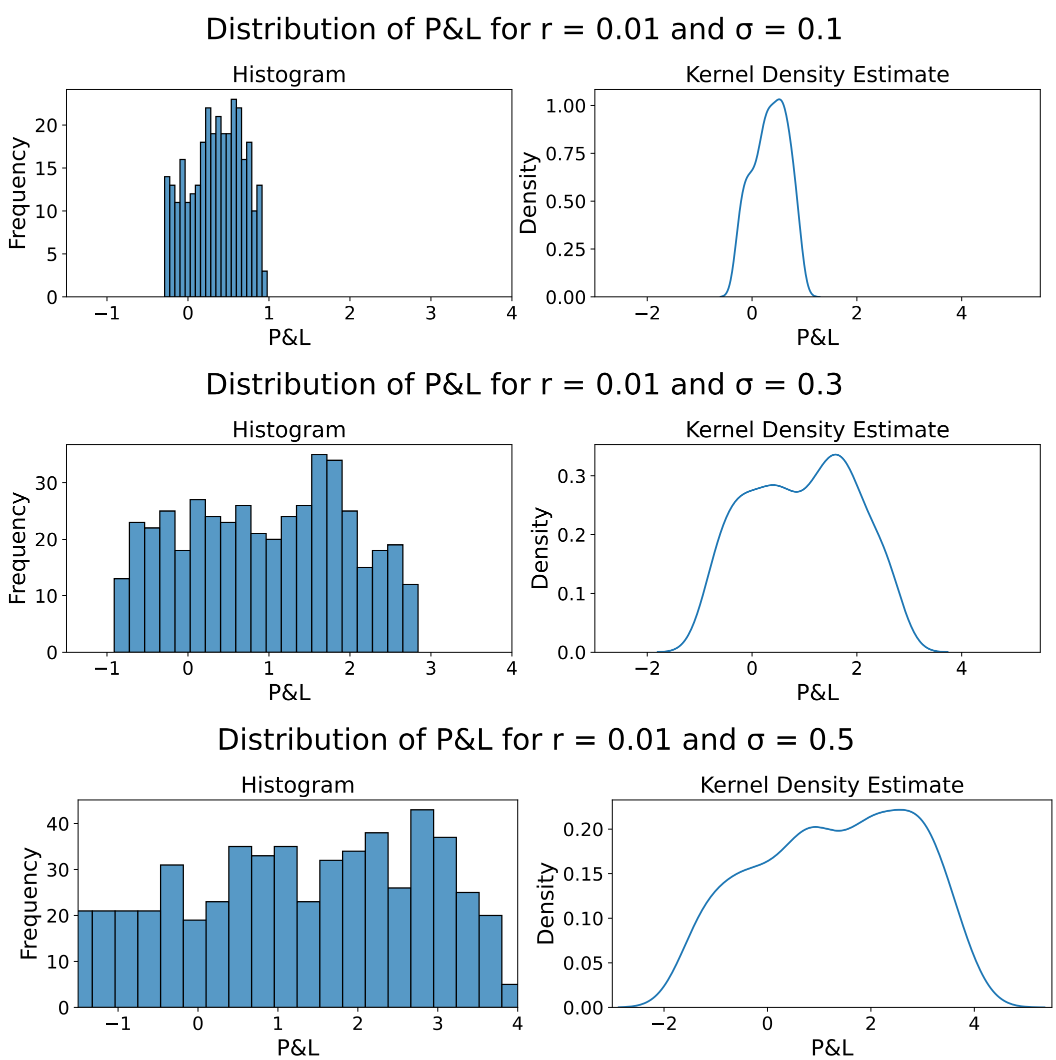

Figure 14: Distribution of the profit and loss for a trader who purchased the option when $r = 0.01$

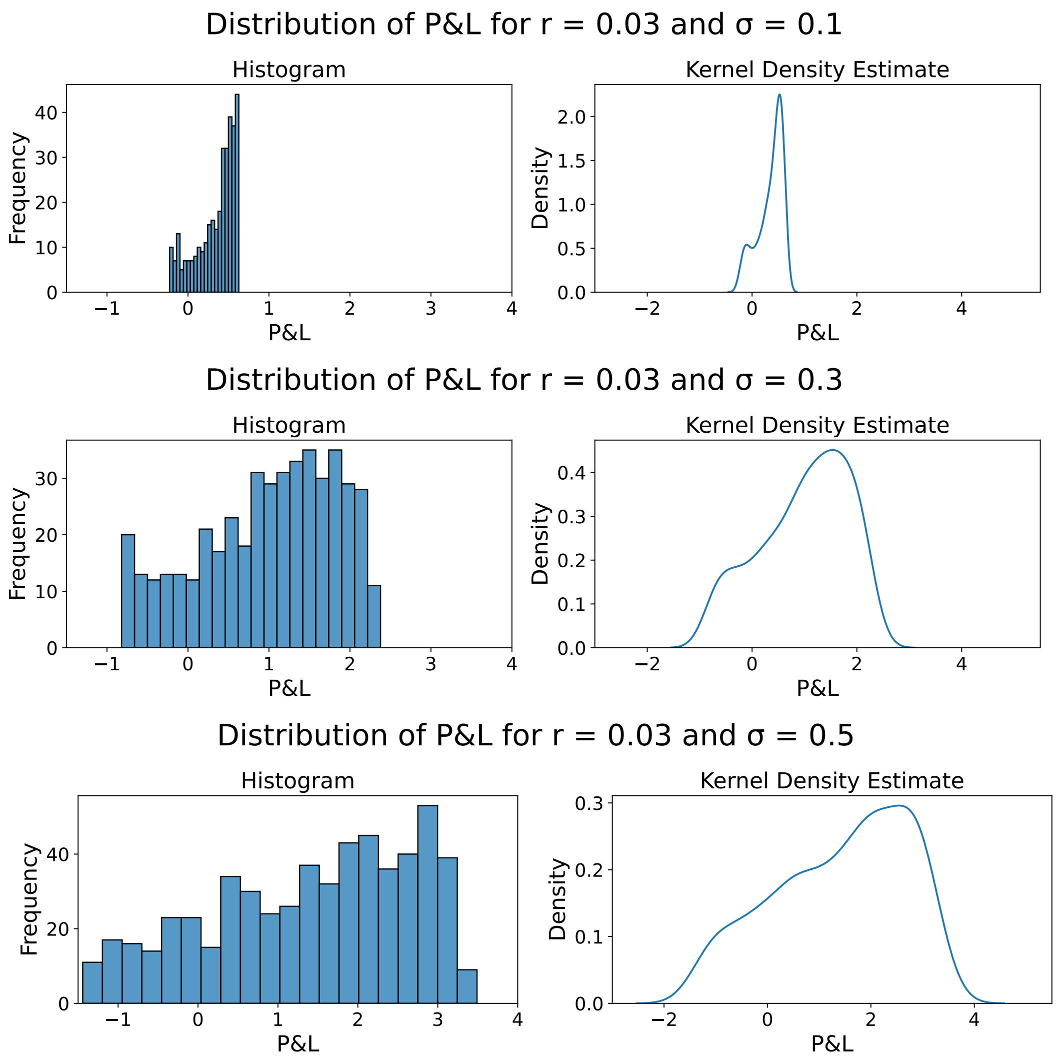

Figure 15: Distribution of the profit and loss for a trader who purchased the option when $r = 0.03$

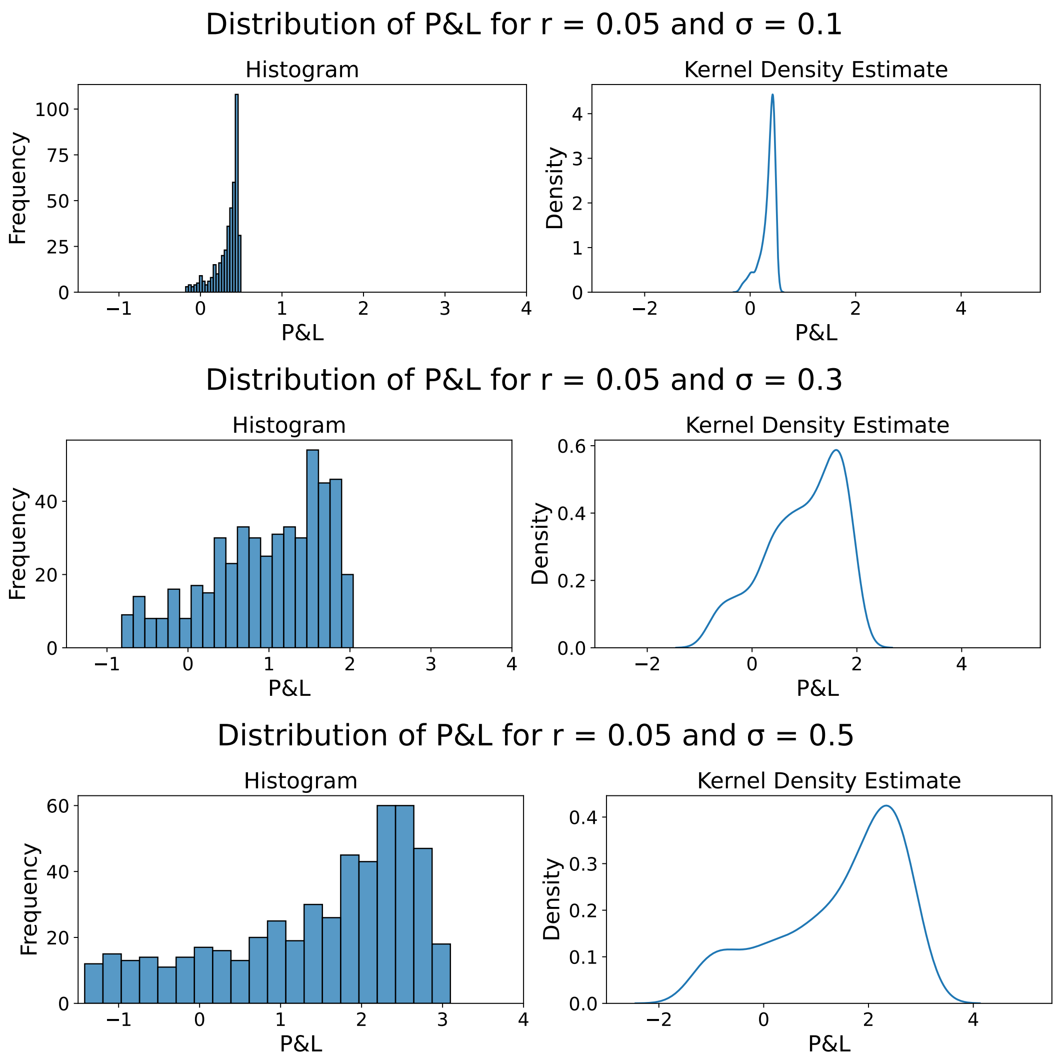

Figure 16: Distribution of the profit and loss for a trader who purchased the option when $r = 0.05$

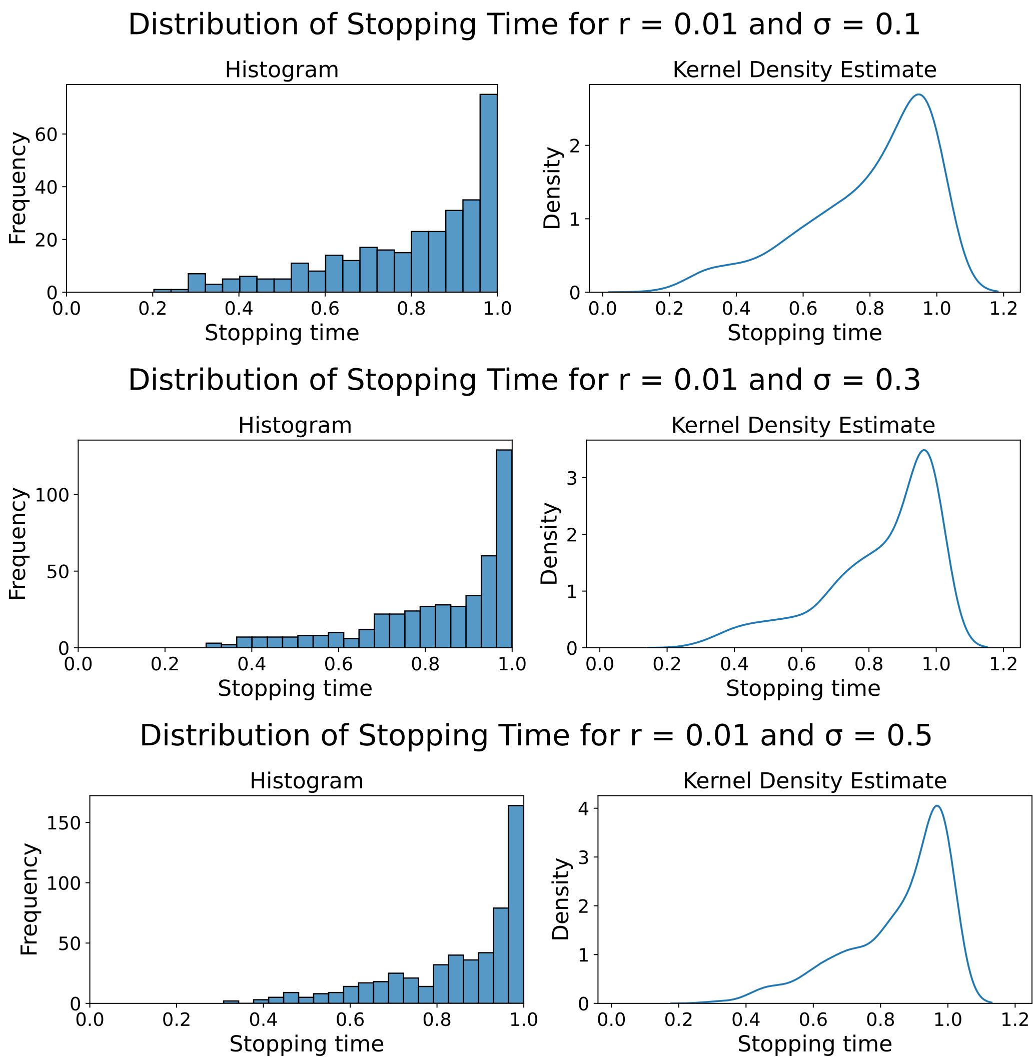

Figure 17: Distribution of the exercise time for a trader who purchased the option when $r = 0.01$

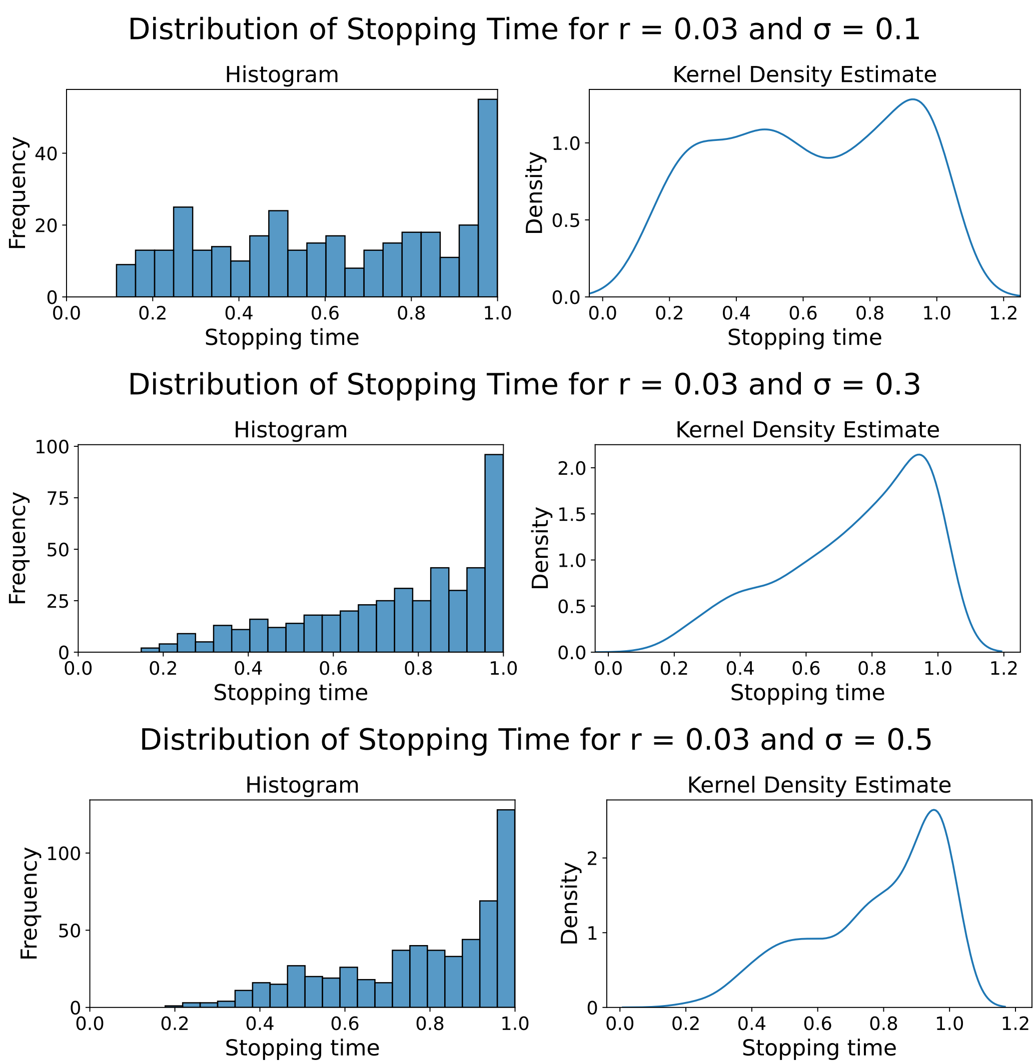

Figure 18: Distribution of the exercise time for a trader who purchased the option when $r = 0.03$

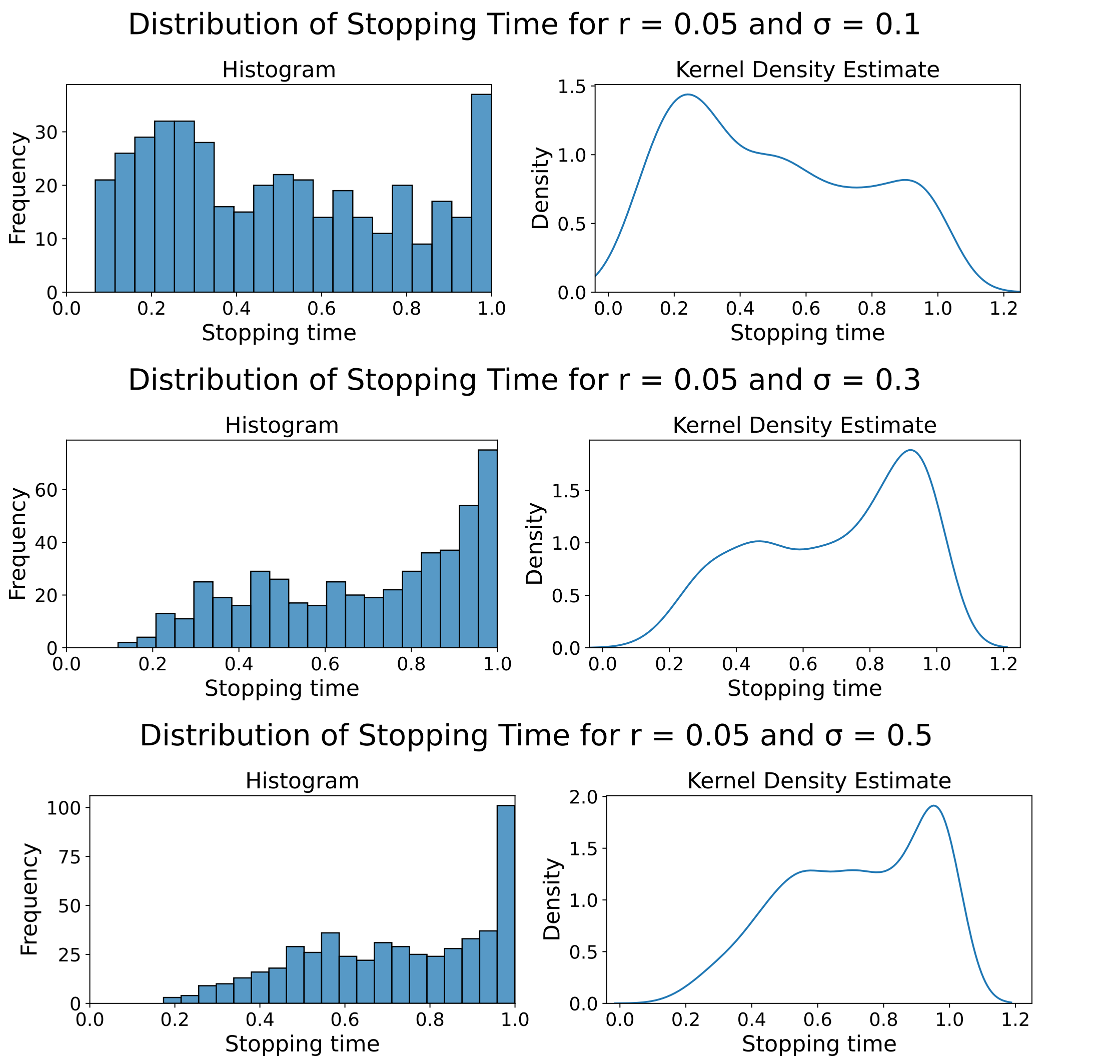

Figure 19: Distribution of the exercise time for a trader who purchased the option when $r = 0.05$

4.3 Mismatch of realized volatility and assumed volatility

In our setting, we purchased with a price assuming that the volatility $\sigma = 20\%$, and the same $\sigma = 20\%$ is used to consider the exercise boundary. However, in real world, the real σ may not be 20%. Intuitively, if the true σ is lower than the assumed σ, then the exercise boundary would be set too low. This means that the profit/loss will be generally lower, while the probability of earlier exercise and exercising the option at all would be lower. Meanwhile, if the true σ is higher than the assumed σ, then the exercise boundary would be set too high. This means the profit/loss will generally be higher, and the probability of earlier exercise and exercising the option at all would be higher.

To verify this proposition, we simulate 10000 paths for each $\sigma = 10\%, 15\%, 20\%, 25\%, 30\%$. Figure 20 shows how the profit and loss distribution changes as the real σ varies. Figure 21 shows how the time of exercise distribution changes as the real σ varies. Table 4 shows how the probability of exercise varies as the real σ varies. Indeed, if the real σ is higher, the profit and loss is generally higher, and it is more likely to make a profit. Moreover, the exercise time is generally earlier, and the probability of early exercise is higher. In contrast, if the real σ is lower, the profit and loss is generally lower, and it is more likely to make a loss. Moreover, the exercise time is generally later, and the probability of early exercise is lower. These verify our proposition.

σ	Probability
0.1	0.272
0.15	0.375
0.2	0.424
0.25	0.503
0.3	0.532

Table 4: Probability of exercising the American option under different real σ

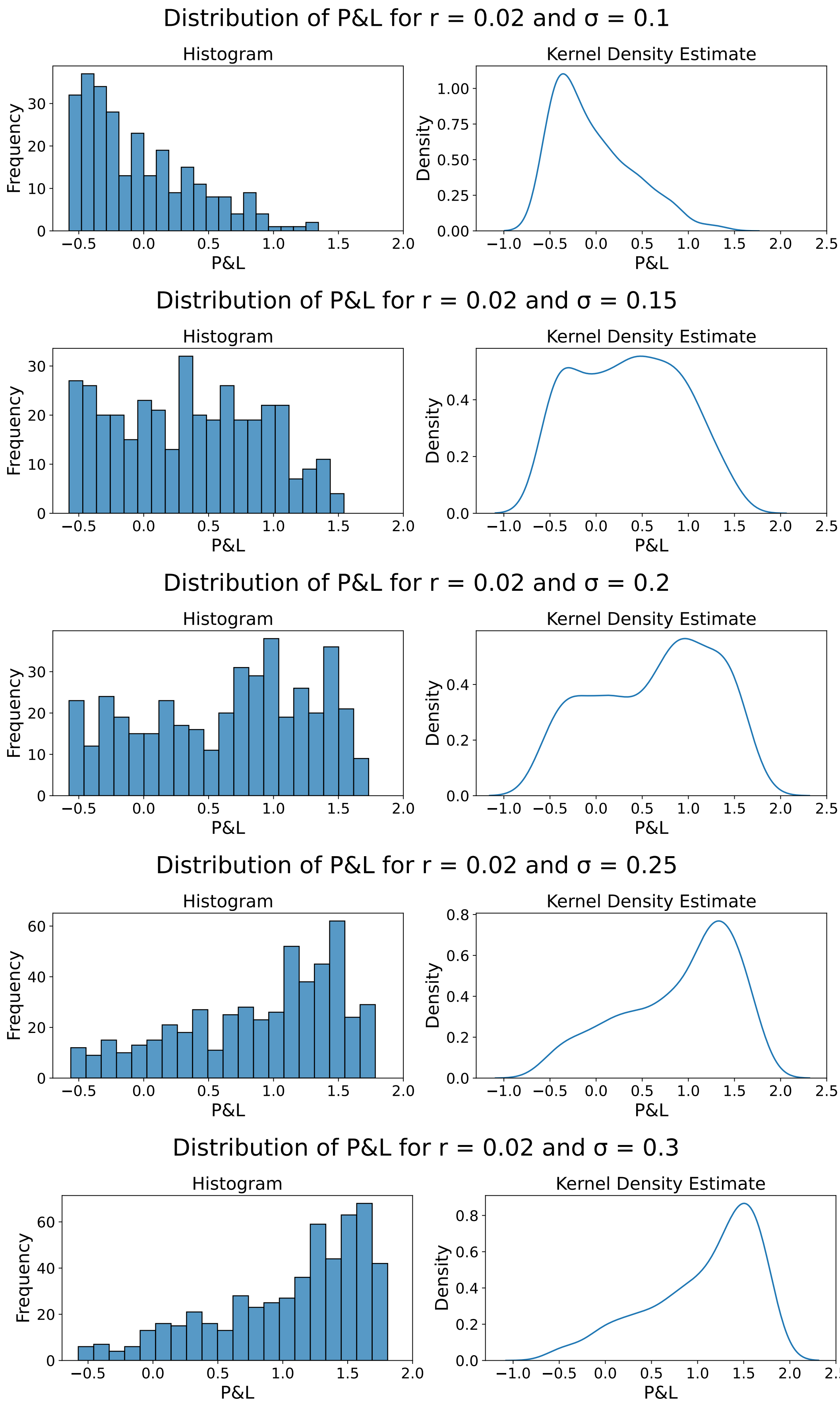

Figure 20: Distribution of the profit and loss under different actual σ

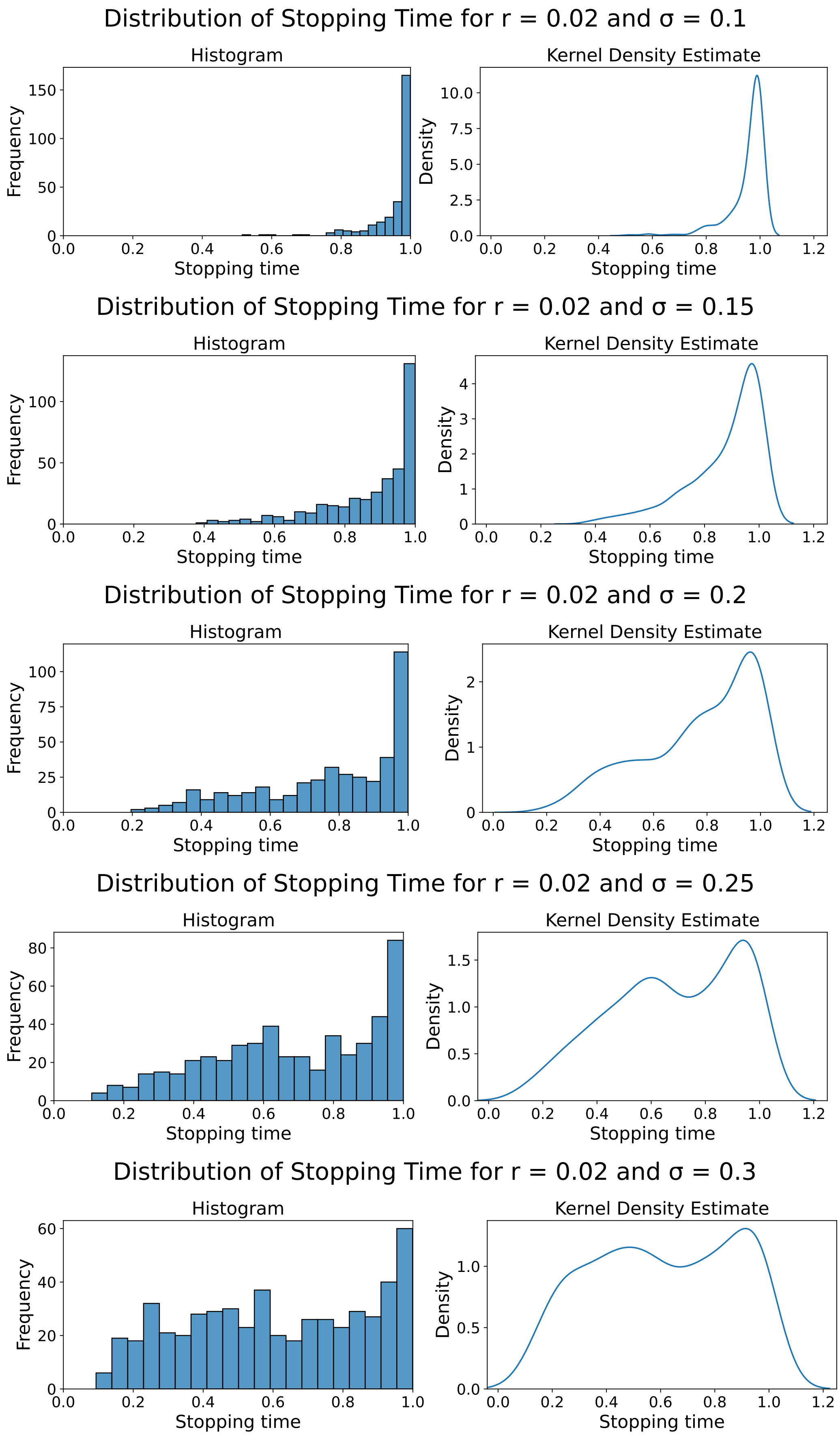

Figure 21: Distribution of the exercise time under different actual σ

5 Conclusion

We derived the martingale measure and the limiting distribution for an asset price process. Using the results, we investigated the valuation of an American put option by considering discrete time steps and simulating possible stock price paths, in order to studying the exercise boundary and hedging strategies. We found that the choice of numéraire does not matter. We also evaluated the impact on exercise boundary and hedging strategies when the volatility and risk-free rate of return changes. We argued that early exercise is not optimal when $r = 0$. Otherwise, increasing risk-free rate of return and decreasing volatility pushes up the exercise boundary. As for the hedging strategies, increasing risk-free rate of return and decreasing volatility results in a "bumpier" evolution of asset and bank positions towards exercise time.

Finally, we investigated the profit and loss as well as the exercise time of a purchased American option. In general, the probability of early exercise is less than half unless the volatility is high. Higher volatility results in generally higher profit and higher probability of early exercise. Higher risk-free rate of return results in a wider distribution of profit and loss and an earlier exercise time, as well as a higher probability of early exercise as long as the volatility is not too high. If the realized volatility is higher than the assumed volatility, the profit and loss is generally higher, the exercise time is generally earlier, and the probability of exercise is generally higher.

6 Attestation

Pak Hop Chan wrote up the whole report here. He wrote up the theory, explained the codes and results, and checked the correctness and the codes. He helped with generation and formatting of some plots.

Daniyal Shahazad did all the initial coding, worked on improving, extending and correcting code and helped with the theoretical foundations.