DETERMINANTAL FORMULAS FOR DUAL GROTHENDIECK POLYNOMIALS

1. Problems

In our paper we proved following determinantal identity for dual Grothedieck polynomial $g_\lambda(\mathbf{x})$.

Theorem 1. *The following determinantal identity holds*

$$g_{\lambda/\mu} = \det\left[\varphi^{i-j} h_{\lambda_i - i - \mu_j + j}\right]_{1 \le i,j \le \ell(\lambda)} \tag{1}$$

But this one does not have t-parameters.

Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$. Define $h_n(\mathbf{x} + \mathbf{y}) = h_n(\mathbf{x}, \mathbf{y})$, $h_n(\mathbf{x} - \mathbf{y}) = \sum_{i=0}^{n} h_{n-i}(\mathbf{x}) h_i(-\mathbf{y})$ and $h_n(-\mathbf{y}) = (-1)^n e_n(\mathbf{y})$ (it is just suitable way to write down sums, see ex. below). Note that $h_0 = 1$ and $h_{negative} = 0$. We also write for and integer r: $h_n(\mathbf{x} \pm r) := h_n(\mathbf{x} \pm \mathbf{y}_r)$ where $\mathbf{y}$ vars are specialized(substituted) to 1.

Problem 2 (exercise)**.** Let A, B, C some (may be not finite) alphabets. Prove that

$$\sum_{k \le 0} h_k(A) z^k = \prod_{a \in A} \frac{1}{(1 - za)} \tag{2}$$

$$\sum_{k \le 0} h_k(-A) z^k = \prod_{a \in A} (1 - za) \tag{3}$$

$$\sum_{k \le 0} h_k(A - B) z^k = \frac{\prod_{b \in A} (1 - zb)}{\prod_{a \in A} (1 - za)} \tag{4}$$

$$h_n(A + x - x - B) = h_n(A - B) \tag{5}$$

$$h_n(A + C - (B + C)) = h_n(A - B) \tag{6}$$

Proof. (2) $\sum_{k \le 0} h_k(A) z^k$ is like throwing k balls into $|A|$ holes. Cost of throwing ball is z. Cost of throwing in hole is a. $\frac{1}{(1-za)} = (1 + za + (za)^2 + \dots$

(3) $\sum_{0 \le k} h_k(-A) z^k = \sum_{0 \le k} (-1)^k e_k(A) z^k = \sum_{0 \le k} e_k(-z)^k$. Same as previous, but now only one ball per hole is allowed and cost of throwing is (-z).

(4) $\frac{\prod_{b \in B} (1-zb)}{\prod_{a \in A} (1-za)} = \prod_{b \in B} (1 - zb) \cdot \prod_{a \in A} (1 - za)^{-1} = \sum_{0 \le k} h_k(-B) z^k \cdot \sum_{0 \le k} h_k(A) z^k = \sum_{0 \le k} t^k \sum_{0 \le t \le k} h_{k-t}(A) h_t(-B) = \sum_{0 \le k} h(A - B) z^k$

(5) Let's look at generating function $f(A) = \sum_{0 \le k} h(A) z^k$. $f(A + B) = f(A) * f(B), f(A - B) = f(A)/f(B)(from(4))$. I assume that the order is $h_n(((A + x) - x) - B)$ so $f(((A + x) - x) - B) = ((f(A) \cdot f(x))/f(x))/f(B) = f(A)/f(B) = f(A - B)$

(6) Similarly, $f(A + C - (B + C)) = (f(A) \cdot f(B))/(f(B) \cdot f(C)) = f(A)/f(B)$ $\square$

Verify if the following conjecture holds:

Conjecture 3. Denote $\mathbf{t}_i = \{t_1, \dots, t_i\}$ for $i = 1, \dots, n-1$. For any λ, μ, n such that $\ell(\lambda) \leq n$ and $\ell(\mu) \leq n$

$$g_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \det \left[h_{\lambda_i - i - \mu_j + j}(\mathbf{x}_n + \mathbf{t}_{i-1} - \mathbf{t}_{j-1}) \right]_{1 \leq i,j \leq n} \tag{7}$$

Note, that for

$$\mathbf{t}_r - \mathbf{t}_l = \begin{cases} \mathbf{t}(l,r], & \text{if } r > l \\ 0, & \text{if } r = l \\ -\mathbf{t}(r,l] & \text{otherwise} \end{cases}$$

Proof. Computational experiments shows validity of the formula. □

Verify if the following conjecture holds:

Conjecture 4. Denote $T_i = \{t_1, \dots, t_i\}$ for $i = 1, \dots, n-1$. Verify

$$g_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \det \left[h_{\lambda_i - i - \mu_j + j}(\mathbf{x} + T_{i-1}) \right]_{1 \leq i,j \leq \ell(\lambda)} \tag{8}$$

$$g_{111/1} = x_1^1 x_2^1 + x_1^1 t_2^1 + x_2^1 t_2^1$$

$$det \begin{bmatrix} h_0(x_1, x_2) & h_2(x_1, x_2) & h_3(x_1, x_2) \\ h_{-1}(x_1, x_2, t_1) & h_1(x_1, x_2, t_1) & h_2(x_1, x_2, t_1) \\ h_{-2}(x_1, x_2, t_1, t_2) & h_0(x_1, x_2, t_1, t_2) & h_1(x_1, x_2, t_1, t_2) \end{bmatrix}$$

$$= det \begin{bmatrix} 1 & x_1^1 x_2^1 + x_1^2 + x_2^2 & x_1^1 x_2^2 + x_1^2 x_2^1 + x_1^3 + x_2^3 \\ 0 & x_1^1 + x_2^1 + t_1^1 & x_1^1 x_2^1 + x_1^1 t_1^1 + x_1^2 + x_2^1 t_1^1 + x_2^2 + t_1^2 \\ 0 & 1 & x_1^1 + x_2^1 + t_1^1 + t_2^1 \end{bmatrix}$$

$$= x_1^1 x_2^1 + x_1^1 t_1^1 + x_1^1 t_2^1 + x_2^1 t_1^1 + x_2^1 t_2^1 + t_1^1 t_2^1$$