Contraction Analysis: A New Perspective in Control System

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Issues to be addressed

- What is contraction Analysis?
- Application areas of Contraction Analysis with Examples and
- Implications of Contraction Analysis

About the Topic

The content of this seminar is prepared based on the teaching of professor Jean-Jacques E. Slotine at MIT whom I admire a lot. Professor Slotine is an author of the famous book "Applied Nonlinear Control". His paper with Winfried Lohmiller entitled "On Contraction Analysis for Nonlinear Systems" introduce the concept of contraction analysis. Some of the examples and their MATLAB simulation used to justify the result of contraction analysis are prepared by my self.

What is Contraction Analysis?

Lyapunov Analysis

- Puts "Virtual classical mechanics" in control
- Use energy like functions

Contraction Analysis

- Puts "Virtual fluid mechanics" in control
- Use differential tools like "differential displacements"
- Global conclusions inferred from local analysis
- Contraction Analysis $\Rightarrow$ Differential Lyapunov Analysis

Consider a dynamic system,

$$\dot{x} = f(x, t)$$

The differential displacement between two trajectories of the system at fixed time is $\delta x = x_2 - x_1$.

The time derivative of the length between the two trajectory is

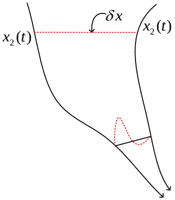

Figure: Arbitrary trajectories

$$\frac{d}{dt}(\delta x^T \delta x) = 2\delta x^T \frac{\partial f}{\partial x}\delta x = 2\delta x^T \frac{\partial f}{\partial x}\delta x$$

If the Jacobian of the system $\frac{\partial f}{\partial x}$ is uniformly negative definite, i.e $\frac{\partial f}{\partial x} < -\alpha I$ for $\alpha > 0$, then $||\delta x|| \longrightarrow 0$ exponentially. So, any two trajectory of the system will tend to each other, i.e the system is contracting.

Necessary and Sufficient Condition for Contraction

Generally for any dynamic system, $\dot{x} = f(x, t)$, and uniformly invertable metric $\theta(x, t)$ (i.e $\theta^T \theta > \alpha I$, for $\alpha > 0$), that maps δx to δz, (i.e $\delta z = \theta \delta x$), any two trajectories of the system tend to each other if the generalized jacobian

$$F = (\theta^{-1})^T (\frac{\partial f}{\partial x}^T \theta^T + \dot{\theta}) + (\dot{\theta} + \frac{\partial f}{\partial x})\theta^{-1}$$

is uniformly negative definite. If $F < -\eta I$ for $\eta > 0$ then $\delta x \longrightarrow 0$ exponentially.
For LTI system θ is constant and $\frac{\partial f}{\partial x} = A$, implying

$$F = P^T A^T + AP < -Q$$

a necessary and sufficient condition for stability of LTI system.

Application of Contraction Analysis

- Nonlinear state estimation/observer design
- Synchronization
- Controller design

Nonlinear state estimation/observer design

Consider the dynamics of Lorenz system,

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= -\beta z + xy\end{aligned}$$

$\alpha > 0, \beta > 0, \rho > 0$, suppose x is measurable, if we just copy the rest two dynamics as observer dynamics and replace with the estimate values

$$\begin{aligned}\dot{\hat{y}} &= \rho x - \hat{y} - x\hat{z}\\ \dot{\hat{z}} &= -\beta\hat{z} + x\hat{y}\end{aligned}$$

The jacobian of the observer is

$$F = \begin{bmatrix} -1 & 0 \\ 0 & -\beta \end{bmatrix}$$

is uniformly negative definite, hence $\hat{y} \to y$ and $\hat{z} \to z$ exponentially.

Consider the dynamics of IPC,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{a}{c} + \frac{\xi}{c}u$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{b}{c} + \frac{\eta}{c}u$$

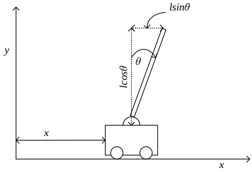

Figure: IPC

where

$$a = (m_p l^2 + I)m_p l x_4^2 \sin x_3 - (m_p l)^2 g \sin x_3 \cos x_3$$
$$b = -(m_p l)^2 (x_4)^2 \sin x_3 \cos x_3 + (m_p + m_c) m_p l g \sin x_3$$
$$c = (m_p + m_c)(m_p l^2 + I) - (m_p l \cos x_3)^2$$
$$\xi = m_p l^2 + I \qquad \eta = -m_p l \cos x_3$$

Nonlinear state estimation/observer design

suppose position of the cart x_1 and angular position of the pendulum x_3 are measurable, an observer for velocity of the cart $x_2 = v_1$ and angular velocity $x_4 = v_2$ can be designed as,

$$\hat{v}_1 = \bar{v}_1 + k_1 x_1 \qquad\qquad \hat{v}_2 = \bar{v}_1 + k_2 x_3$$

$$\dot{\bar{v}}_1 = \frac{\hat{a}}{c} + \frac{\xi}{c}u - k_1 \hat{v}_1 \qquad\qquad \dot{\bar{v}}_1 = \frac{\hat{b}}{c} + \frac{\eta}{c}u - k_2 \hat{v}_2$$

where

$$\dot{\hat{v}}_1 = \frac{\hat{a}}{c} + \frac{\xi}{c}u - k_1(\hat{v}_1 - v_1)$$

$$\dot{\hat{v}}_2 = \frac{\hat{b}}{c} + \frac{\eta}{c}u - k_2(\hat{v}_2 - v_2)$$

Selecting $k_1, k_2 > 0$, ensures that $\hat{v}_1 \to v_1$ and $\hat{v}_2 \to v_2$ exponentially.

Nonlinear state estimation/observer design

Consider the dynamics of beam and ball balance,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{mx_4^2 x_1 - mg\sin x_3}{m\lambda}$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{\tau - 2x_1x_2x_3 - mgx_1\cos x_3}{I + mx_1^2}$$

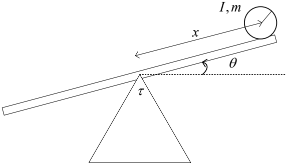

Figure: Ball and Beam System

where

$$\lambda = 1 + \frac{I}{mr^2}$$

suppose position of the ball x_1 and angular position of the beam x_3 are measurable,

Nonlinear state estimation/observer design

A nonlinear observer for linear velocity $x_2 = v_1$ and angular velocity $x_4 = v_2$ can be designed as

$$\begin{aligned}
\hat{v}_1 &= \bar{v}_1 + k_1 x_1 \\
\dot{\bar{v}}_1 &= \frac{m v_2^2 x_1 - mg \sin x_3}{m\lambda} - k_1 \hat{v}_1 \\
\hat{v}_2 &= \bar{v}_1 + k_2 x_3 \\
\dot{\bar{v}}_1 &= \frac{\tau - 2x_1 v_1 x_3 - mgx_1 \cos x_3}{I + m x_1^2} - k_2 \hat{v}_2
\end{aligned}$$

where

$$\begin{aligned}
\dot{\hat{v}}_1 &= \frac{m v_2^2 x_1 - mg \sin x_3}{m\lambda} - k_1(\hat{v}_1 - v_1) \\
\dot{\hat{v}}_2 &= \frac{\tau - 2x_1 v_1 x_3 - mgx_1 \cos x_3}{I + m x_1^2} - k_2(\hat{v}_2 - v_2)
\end{aligned}$$

Selecting $k_1, k_2 > 0$, ensures that $\hat{v}_1 \to v_1$ and $\hat{v}_2 \to v_2$ exponentially.

Synchronization

Consider dynamic systems

$$\dot{x}_1 = f(x_1, t)$$
$$\dot{x}_2 = f(x_2, t)$$

and add a bilateral coupling,

$$\dot{x}_1 = f(x_1, t) + K_1(g_1(x_2) - g_1(x_1))$$
$$\dot{x}_1 = f(x_2, t) + K_2(g_2(x_1) - g_2(x_2))$$

where $g_1(x_1)$ and $g_2(x_2)$ depend only on measured states. The result will be one of the following

- Synchronization or,
- Anti-synchronization or,
- be Leader follower combination (If $K_1 = 0$ or $K_2 = 0$).

Consider two distinct Vander Pol Oscillators

$$\ddot{x} + \alpha_x(x^2 - 1)\dot{x} + \omega_x^2 x = 0$$
$$\ddot{y} + \alpha_y(y^2 - 1)\dot{y} + \omega_y^2 y = 0$$

$\alpha_x, \alpha_y > 0$. Suppose we added a bilateral coupling,

$$\ddot{x} + \alpha_x(x^2 - 1)\dot{x} + \omega_x^2 x = \alpha_x K_1(\dot{y} - \dot{x})$$
$$\ddot{y} + \alpha_y(y^2 - 1)\dot{y} + \omega_y^2 y = \alpha_y K_2(\dot{x} - \dot{y})$$

The generalized Jacobian of the overall system is uniformly negative definite if $K_1 + K_2 > 1$, hence the system is contracting implying that the oscillators will eventually oscillate in synchronous manner.

For the same Van der Pol Oscillators

$$\ddot{x} + \alpha_x(x^2 - 1)\dot{x} + \omega_x^2 x = 0$$
$$\ddot{y} + \alpha_y(y^2 - 1)\dot{y} + \omega_y^2 y = 0$$

$\alpha_x, \alpha_y > 0$. If we change the bilateral coupling to,

$$\ddot{x} + \alpha_x(x^2 - 1)\dot{x} + \omega_x^2 x = -\alpha_x K_1(\dot{y} + \dot{x})$$
$$\ddot{y} + \alpha_y(y^2 - 1)\dot{y} + \omega_y^2 y = -\alpha_y K_2(\dot{x} + \dot{y})$$

In similar way generalized Jacobian of the overall system is uniformly negative definite if $K_1 + K_2 > 1$, hence the system is contracting implying that the oscillators will eventually oscillate in synchronous manner. But x synchronize with $-y$.

What if the systems are different? Can we synchronize Van der Pol Oscillator and a pendulum

$$\begin{aligned} \ddot{x} + \alpha_x(x^2 - 1)\dot{x} + \omega_x^2 x =& 0 \\ \ddot{\theta} + \frac{g}{l}\sin\theta =& 0 \end{aligned}$$

$\alpha_x > 0$. If we use bilateral coupling,

$$\begin{aligned} \ddot{x} + \alpha_x(x^2 - 1)\dot{x} + \omega_x^2 x &= \alpha_x K_1(\dot{\theta} - \dot{x}) \\ \ddot{\theta} + \frac{g}{l}\sin\theta &= K_2(\dot{x} - \dot{\theta}) \end{aligned}$$

In similar way the overall all system is contracting implying that the oscillator and the pendulum will eventually oscillate in synchronous manner (i.e $\theta \Leftrightarrow x$).

Anti-Synchronization of different systems

For the same dynamics if the coupling changes to

$$\ddot{x} + \alpha_x(x^2 - 1)\dot{x} + \omega_x^2 x = -\alpha_x K_1(\dot{\theta} + \dot{x})$$
$$\ddot{\theta} + \frac{g}{l}\sin\theta = -K_2(\dot{x} + \dot{\theta})$$

The oscillator and the pendulum will anti-synchronize (i.e $\theta \Leftrightarrow -x$).

Anti-Synchronization

Consider the same Van der Pol Oscillators

$$\ddot{x} + \alpha_x(x^2 + 2K_1 - 1)\dot{x} + \omega_x^2 x = 0$$
$$\ddot{y} + \alpha_y(y^2 + 2K_1 - 1)\dot{y} + \omega_y^2 y = 0$$

$\alpha_x, \alpha_y > 0, K_1 > 1$. The two oscillators converge to origin exponentially, but If we add bilateral coupling,

$$\ddot{x} + \alpha_x(x^2 + 2K_1 - 1)\dot{x} + \omega_x^2 x = -\alpha_x K_1(\dot{y} + \dot{x})$$
$$\ddot{y} + \alpha_y(y^2 + 2K_1 - 1)\dot{y} + \omega_y^2 y = -\alpha_y K_2(\dot{x} + \dot{y})$$

They will immediately anti-synchronize.

Application in Neuron Synchronization

Consider a second order approximation of a neuron (FitzHugh-Naguma model)

$$\begin{aligned}\dot{\nu} =& c(\nu + \omega - \frac{\nu^3}{3} + I)\\ \dot{\omega} =& \frac{-1}{c}(\nu - a + b\omega)\end{aligned}$$

a, b, c are constants, $c > 0$ and I is external impulse input. The generalized jacobian of the system is clumsy, but if we scale the second equation by c, the generalized jacobian of the system,

$$F = \begin{bmatrix} c(1-\nu^2) & 0 \\ 0 & \frac{-b}{c} \end{bmatrix}$$

is negative definite for large value of $\nu \Rightarrow$ the dynamics is contracting.

Nonlinear Controller Design

Consider the following nonlinear dynamic system,

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -\beta \frac{x_2^3}{3} - \gamma x_2 + u$$

The generalized jacobian of the system

$$F = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -(\beta x_2^2 + \gamma) \end{bmatrix}$$

is uniformly negative definite. Thus the system is contracting. For any input and initial condition every trajectory tends to each other.

Nonlinear Controller Design

Consider the following nonlinear dynamic system,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_2^3 - x_1^2 + u$$

The generalized jacobian of the system is not uniformly negative definite. Transforming the system using constant metric $\theta(x, t)$, the new dynamics will be

$$\dot{x}_1 = -x_2 - x_1^3$$
$$\dot{x}_2 = x_1 - x_2^3 + u$$

with $u = g(x_1, x_2) + r = -2(x_2^2 + x_2^3) + x_2^2 + x_1 - x_1^3 + r^3$, every trajectory tends to each other for any initial condition.

Extending The Concept of Contraction Analysis

Suppose we have designed

- Left Leg movement controller
- Right leg movement Controller
- Right hand movement controller
- left hand movement controller
- Head position controller, etc

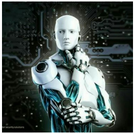

Figure: Humanoid robot

for humanoid robot, and even if their individually performance is great, the overall motion of the robot require the co-ordination.

- Suppose each controllers makes the overall system contracting by design.

$$\dot{x} = f(x,t) + B(x,t)u$$

If each control primitives $u = p_i(x,t)$ make the overall system contracting

Extending The Concept of Contraction Analysis

Contraction analysis can also used in

- Sliding mode controller design $\Leftrightarrow$ no chattering
- Adaptive Controller design

How do to design Contracting system if the real system is not contracting

- Develop one to one mapping transformation matrix $\theta(x, t)$
- Design a virtual contracting system in which the real system is a particular solution.