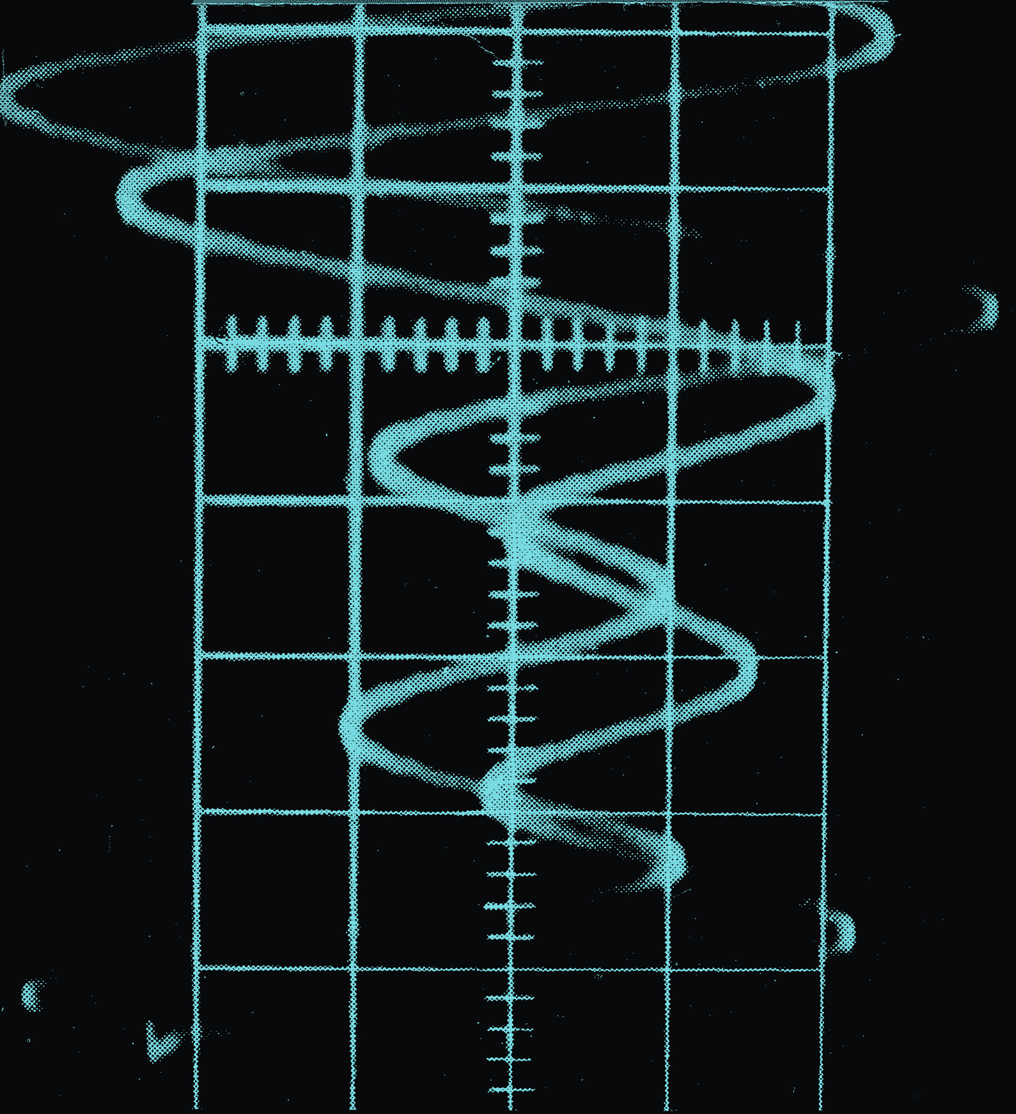

high-speed Analog Computers

Rajko Tomovic and Walter J. Karplus

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HIGH-SPEED ANALOG COMPUTERS

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DOVER PUBLICATIONS, INC., NEW YORK

PREFACE

Immediately following World War II when the newly developed technique of electronic analog computation was first introduced to the public, two types of analog devices appeared to offer approximately equal promise. These were the repetitive or high-speed computers and the one-shot or slow-speed computers. In the following decade, however, the one-shot computer received such overwhelming acceptance and preference among engineers that the term "analog computer" became virtually synonymous with one-shot analog computers. By the late 1950's this trend began to reverse. Possibly as a result of competition by the digital computer and possibly because of the development of new electronic techniques, a resurgence of repetitive computation began.

The primary purpose of this book is to introduce the reader to the electronic devices and circuits which combine to constitute a repetitive computer facility, as well as to survey applications of repetitive analog computers to engineering problems. In view of the predominant place of one-shot analog computers, this book must be regarded as an advanced textbook treating one facet of a broad field. Accordingly it is assumed that the reader already has a certain amount of experience and perspective regarding analog computation. The material generally covered in a one-semester college course in analog computation would be one way of acquiring this background. Many other readers will have gained equivalent experience by contacts with industrial computer facilities.

Because of the relatively advanced nature of the text, it appeared appropriate to include also a considerable amount of material that is pertinent to the entire analog computing field but which is generally omitted from introductory texts. Such topics as computer programming, error analysis, and scaling are treated in terms of their relation to fundamental mathematical disciplines. This presentation is intended to supplement rather than to replace the more "practical" treatments of these subjects in introductory texts. To appreciate fully these topics, the student should have some familiarity with the techniques of advanced calculus and matrix methods. Particular emphasis is placed throughout on contributions of investigators whose work is normally not available to all readers. For example, the work of Bihovski and Tihanof in error analysis is considered in some detail.

The book includes an introduction and three main parts devoted respectively to theory, equipment, and applications. In the introductory chapter the general approach to the subject of analog computation is outlined, and the notation and symbolism used throughout the text are defined.

Part I treats the theoretical fundamentals relative to analog computation. These apply equally to repetitive and one-shot computers. Chapter 2 considers the basic principles underlying programming of differential equations, with special emphasis on the relation between closed-loop methods and Picard's method of successive approximations.

Chapter 3 surveys various methods of error analysis. The concept of sensitivity functions due to Miller and Murray, as well as to Bihovski, is considered in some detail, and two methods for employing analog computers for the experimental determination of these coefficients are presented. The approaches of Macnee, Raymond, and Marsocci, and Miura for the error analysis of linear simulations are compared and summarized.

Chapter 4 is devoted to the selection of suitable scale factors and a consideration of the effect of drift and zero offset in operational amplifiers used in repetitive computers.

In Part II the major components of repetitive differential analyzers are surveyed. Chapter 5 is devoted to linear elements including a comparison of the three major approaches to operational amplifier design, the adaptation of potentiometers to high frequency operation, and the design of adders and integrators. The newly introduced technique of dynamic storage (memory) is described, and a number of complete electronic computer circuits are presented.

Chapter 6 treats nonlinear computer elements, particularly multipliers and function generators. A method of "universal function generation" using voltage comparators, which has been successfully used in Europe, is described in detail. The use of the Hall effect for multiplication is also briefly considered.

Chapter 7 describes output equipment for the display of repetitive solutions by oscilloscopes, two coordinate plotters, and for the measurement of specific points (instantaneous values) of the solution.

Chapter 8 considers auxiliary equipment including generators for the control of the repetition rate, equipment for the imposition of initial conditions, and power supplies.

Part III is devoted to a survey of applications of repetitive differential analyzers. Chapter 9 briefly considers the solution of ordinary differential equations with special emphasis on a comparison of the results obtainable with repetitive and one-shot computers.

Chapter 10 considers the treatment of partial differential equations with particular emphasis on those techniques for which repetitive operation is particularly suitable. These include the DYSTAC technique in which time is discretized while the space derivatives are maintained in continuous form, the DSDT technique involving the replacement of all derivatives by finite difference approximations, and the Monte-Carlo method using random excitations.

Chapter 11 treats in a similar fashion the solution of integral equations. Chapter 12 includes a discussion of the use of statistical techniques in analog computation, conformal transformations, and the application of ultra-rapid differential analyzers.

A portion of the material presented in this textbook appeared originally in the book *Calculateurs Analogiques Répétitifs*, by R. Tomovic, which was published in French by Masson et Cie. in Paris in 1958. Although the general chapter structure of that monograph has been retained, the entire text has been completely revised and rewritten. A considerable amount of new and original material has been added as well.

We take pleasure in acknowledging with thanks the assistance and help of Dr. F. H. Raymond, H. Meissinger, G. Korn, and G. R. Kaelin. A portion of the secretarial work involved in preparing the manuscript was supported by a grant of the Ford Foundation to the University of California at Los Angeles for educational development.

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April 1962

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chapter 1
INTRODUCTION

1.1 General Remarks

The present advanced state of the analog computation art can be attributed directly to the accelerated development of electronic techniques and systems during World War II. Although it is possible to trace the use of special-purpose analog devices back to antiquity, and although such forerunners of modern computers as the Busch mechanical differential analyzer and the power network analyzer became important engineering tools in the late 1920's and during the 1930's, analog computing techniques became generally accepted only when it became possible to perform the calculations electronically.

Specifically it was the perfection of d-c operational amplifiers of sufficient precision and stability that permitted the fashioning of electronic summers, integrators, inverters, and other computing units. Since these electronic devices were considerably more compact and convenient to use than previously existing mechanical differential analyzer components, they were soon universally adopted for general-purpose computations. Almost from the start it became apparent that the wide variety of available electronic components made possible two radically different approaches to analog computations.

One method termed variously "long-time," "slow," or "one-shot" computation makes use of integrators having relatively long time constants, so that typical problem solutions may take several seconds to one or more minutes. Strip-chart recorders or servo-driven two-coordinate plotters can then be employed to record the solution. In the second method known as "short-time," "high-speed," or "repetitive" computation, on the other hand, the integrator time constants are such that complete solutions are obtained in several milliseconds. The entire computation is then repeated ten or more times per second and the result displayed by an oscilloscope. In recent years the sharp distinction between the two methods has to some extent been broken down. Thus several manufacturers of slow computers have equipped their installations with control circuits which make possible repetitive operation. At the same time high-speed computers have found application in areas in which successive computer runs are not identical, so that they cannot be considered as being truly repetitive. For the purpose of this discussion, however, these latter refinements will be considered as exceptions to the over-all generalization, and the terms "high-speed" and "repetitive" will be considered as being equivalent.

Repetitive computers are inherently more complex devices since they require circuitry to discharge all computing capacitors at the end of each solution run and to apply automatically the appropriate initial conditions before each run. Furthermore, in order to obtain reasonably high accuracies with oscilloscopic displays, it is necessary to provide additional electronic circuitry to counteract errors due to the inherent nonlinearity of the cathode-ray deflection system. Repetitive computers possess, however, one important advantage. By means of potentiometers the various parameters of the problem may be varied, and the effect of these variations upon the problem solutions becomes visible at once on the oscilloscope. For long-time computers a separate run lasting a fraction of a minute must be made after each parameter variation. Repetitive computers are therefore particularly suited for problems involving parameter variations and optimization as in engineering design studies. Because of the rapid solution times involved, the bandwidth of the components comprising a repetitive system must be considerably greater than that in a one-shot analog computer. Stray capacitance must therefore be kept to a minimum. This in general makes it difficult to use a patch-bay design in a repetitive computer. Most frequently such computers are arranged in a modular fashion.

1.2 Historical Development of Repetitive Analog Computers

Both one-shot and repetitive computers became available as commercial products soon after World War II. In the United States George A. Philbrick Researches Inc. marketed a relatively inexpensive repetitive differential analyzer which featured the utmost in simplicity of operation but had relatively low accuracy. In the one-shot analog computer field, Reeves Instruments Inc. introduced their REAC, to be followed a short time later by Electronic Associates' PACE, Berkeley's EASE, Goodyear's GEDA, and a host of lesser known models. Because of the advantages attending centralized patch-bay design and because of their greater accuracy, one-shot analog computers seized a great part of the analog computer market and retained this advantage over the following decade.

Spurred by competition between computer manufacturers, engineers strove to produce one-shot computers of ever-increasing accuracy and complexity. In time the term "analog computer" became virtually synonymous with "one-shot electronic differential analyzer."

In the meantime the repetitive analog computer field developed much more slowly. In 1950 Macnee described a relatively high-speed repetitive computer which he had constructed at Massachusetts Institute of Technology. Until the late 1950's this remained the only significant innovation in the repetitive computer art in the United States. In Europe, on the other hand, several large repetitive facilities were constructed. Foremost among these were the analyzers at the Boris Kidrich Institute at Belgrade, the Technical Faculty of Darmstadt, Germany, the University of Bologna, The Institution of Automation and Telemechanics at Moscow, and the National Laboratory of Physics at London. None of these computers, however, became commercial products. Rather they became part of computational service facilities of their respective organizations.

In the late 1950's two new types of repetitive differential analyzers became commercially available in the United States. These were manufactured respectively by G.P.S. Instrument Inc. and Computer Systems Inc. These computers featured the latest advances in electronic design and precision engineering and opened up new avenues of computer application and research. About the same time Telefunken of Germany introduced a repetitive computer of comparable accuracy. As a result of these developments engineers began to take a second look at analog computation and the trend towards one-shot computers was gradually halted if not reversed. Several of the larger one-shot computer manufacturers began to convert their equipment to permit optional repetitive operation and more and more emphasis came to be placed on this type of computer usage.

1.3 The Analog Computer Approach

It is superfluous at this juncture to point out the importance of differential equations in dealing with scientific and engineering problems. The value and efficacy of the mathematical approach to the solution of physical phenomena and technical problems are well established. In most cases, however, practical difficulties exist in arriving at suitable solutions by mathematical means. It is necessary to construct a mathematical model of the phenomena under consideration. Even when the appropriate equations have been formulated new problems appear—how to extract the necessary information without unreasonable expenditure of effort and time.

These practical considerations in the application of mathematics often weigh against the utilization of analytical techniques. It would be easy to enumerate many instances where mathematical approach was abandoned in the course of a scientific research or technical development because it was impossible within the given time to obtain the hoped for results. Frequently it becomes necessary to make simplifying assumptions that tend to limit the value of the solution. It is for these reasons that the development of differential analyzers was undertaken.

Kelvin was among the first to introduce mechanical computational techniques, but his work did not have a direct influence upon subsequent developments. He demonstrated that by using mechanical integrators it is possible to solve differential equations without employing successive approximations. This led to the concept which forms the basis for modern differential analyzers—that by interconnecting computer elements it is possible to generate rapidly solutions of a large class of differential equations.

The basic concepts involved may be summarized as follows:

1. A physical variable is selected to represent within a computer the magnitudes of the variables of the original problem. In differential analyzers it is necessary that the computer variables correspond to the relationship

$$y = f(x) \tag{1.1}$$

where x, the independent variable, is bounded according to

$$a \leq x \leq b$$

whereas $f(x)$, the dependent variable, is a real function bounded by

$$c \leq f(x) \leq d$$

2. Computer "building blocks" are constructed in such a manner that the relationship between the computer variable at the output terminals to that at the input terminals corresponds to a mathematical relationship such as addition, integration, etc.

3. These building blocks are interconnected such that the resulting computer system is governed by the algebraic or differential equations to be solved.

4. Suitable initial conditions and driving functions are applied to the system and the problem solution, in the form of the computer variable existing at some specific point within the system, is displayed or recorded.

The fundamental advantage of this method lies in the fact that the same basic building blocks suffice for a very wide variety of algebraic and differential equations. It is therefore unnecessary to modify the building blocks; one need only modify the manner in which they are interconnected.

Thus the classic problem of solving differential equations regardless of their order, linearity, or their type is reduced to establishing certain simple rules linking the differential equations and the manner of interconnection of the computer elements.

1.4 Selection of the Computer Variable

It is the selection of the computer variable which determines to a large extent the basic characteristics of the differential analyzer. For example, the choice of mechanical rotation leading to the mechanical differential analyzer results in a radically different computer from that which results from a choice of electric voltage as in electronic analog computers. This difference has profound effects upon the accuracy, the speed of solution, convenience of operation, ruggedness, etc.

In principle it is possible in specifying an electronic differential analyzer to select either voltage or current as the computer variable. On the other hand, with regard to the independent variable, no real choice exists and one is limited to time. Theoretically there is no reason to reject either voltage or current as the dependent variable. Up to now, however, practical reasons have weighted in favor of voltage. Voltage sources and voltage measuring instruments can be applied to a circuit without breaking any connections, making them more convenient to use than ammeters and current sources which must be connected in series with circuit elements. Furthermore, vacuum tubes, which until recently were the only devices available for amplification, operate more naturally and satisfactorily as voltage amplifiers. That economical and convenient analog computers have been realized in this manner does not eliminate the possibility of employing current.

The advent of transistors, which approximate true current amplifiers, and that of special magnetic materials has led to a reexamination of this basic question. Fortunately, the transition from voltage to current as the computer variable is conceptually a simple one. By invoking the principle of duality the topological transformation is readily made (series circuits become parallel circuits, loops become nodes, etc.). In place of amplifiers having a high input impedance and a low output impedance, it then becomes necessary to design amplifiers with small input impedances and large output impedances. The general method of computer operation remains virtually unchanged.

1.5 Elements of a Differential Analyzer

As has been pointed out, it is necessary in developing electronic analog computers to construct computing elements such that the relationship between the transient voltage at the output terminals to that at the input terminals corresponds to some specific mathematical operation. In analog computers the elements comprising the computer circuit contain active as well as passive components. For this reason it has become the general practice to refer rather to ensembles of such elements in accordance with the function that they perform in the computer. These assemblages of elements are termed "adders," "integrators," "multipliers," etc. Although it is possible to describe the circuit in different terms, particularly in the synthesis of linear circuits, and although this terminology is by no means indispensable, it has been retained in this book. The major elements constituting electronic analog computers will now be briefly categorized and identified. The reader is referred to introductory texts[1,2,3] for more detailed descriptions of these units.

The first group comprises those elements which are employed to effect linear computing operations. The simplest of these is the potentiometer employed to multiply by a constant less than unity. One terminal of the unit is generally connected to ground. In order to multiply by constants greater than unity and in order to add two or more voltages an active circuit is required. The heart of such a unit is the d-c operational amplifier with appropriate feedback and input resistors. Adders as they are realized in practice always produce a change in polarity between input and output. They therefore automatically introduce a multiplication by -1, making it unnecessary to develop a separate circuit for subtraction. By employing a capacitor instead of a resistor as the feedback impedance in the active circuit, it is possible to combine the operations of summation and integration. Here too, the polarity at the output becomes the reverse of that at the input. Differentiation can be accomplished by employing a capacitor as the input impedance and a resistor as the feedback element. Since the operation of differentiation accentuates noise transients, this operation is avoided wherever possible in analog computation. By suitable programming and rearrangement of equations to be solved it is nearly always possible to obtain satisfactory solutions of linear problems with constant coefficients using only potentiometers, adders, and integrators. By employing more complex input and feedback circuits, that is, series and parallel combinations of resistors and capacitors, a wide variety of linear input-output relations can be realized. In general if the input circuit has a voltage-current transfer function $Z_i(s)$ and the feedback circuit a transfer function $Z_f(s)$, the computer circuit will have a transfer function $-Z_f(s)/Z_i(s)$, where s is the Laplace operator.

The second group of elements comprising analog computers includes those units which are required for the solution of nonlinear algebraic and differential equations. A multiplier accepts two transient voltages as the input and provides an output proportional to the product of these two voltages. Since implicit function generating techniques can be employed to accomplish the operation of division using a multiplier together with an operational amplifier, dividers are usually not provided as separate units in a repetitive analog computer installation. There must be available, however, a number of high gain amplifiers without permanently connected feedback elements. Additional nonlinear equipment is provided for the generation of arbitrary or analytic functions. Prior to the problem solution, the desired function is "memorized," so that the specified input-output relationship is realized when a problem variable voltage is applied to the unit.

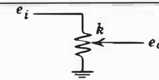
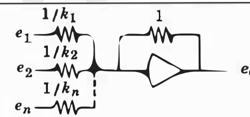
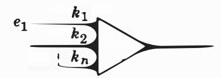
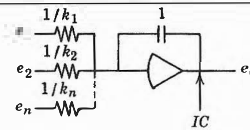
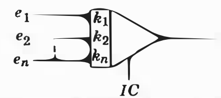
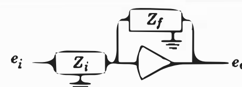
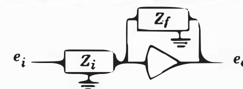
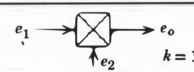
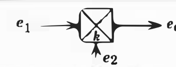
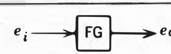
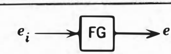
Table 1.1 represents a summary of the basic computer operations. The symbol t is included to emphasize that the independent variable is represented by time. In principle these elements are identical regardless whether they are employed in one-shot or in repetitive differential analyzers. The chief difference lies in the fact that repetitive analog computer components must have a much wider bandwidth extending well beyond 10 kc, and in that the operational amplifier is usually of a different design. The high computing speeds required make it impossible to use servo-driven multipliers and function generators, and rule out many other techniques of multiplication and function generation. In order to complete solutions in the short repetitive cycle, the feedback capacitors and integrators are usually of the order 0.01 μf, whereas input resistors are usually of the order of 100 kilohms. To minimize stray capacitance, potentiometers are often of the single turn rather than the ten-turn variety.

Two sets of notations are commonly used in schematic diagrams of analog computer circuits. One shows the input and feedback elements in detail and is useful when considering the operation of specific computer units; the other is more compact and is generally used in schematics of circuits for the analog solution of entire problems. Both these notations are presented in Table 1.1, and are used when appropriate in this book.

In the third group of computer elements fall those devices that do not take a direct part in the problem solution but which are required auxiliary equipment to permit repetitive operation and suitable display. The display of the computer solutions is generally effected by means of a cathode-ray oscilloscope. This is usually accompanied by a special calibrating system to assist in the accurate identification of coordinates on the tube screen. Alternative output systems described in Chapter 7 employ voltage comparators in conjunction with d-c voltmeters.

In order to establish the basic repetition cycle, which most often varies from 10 to 50 cycles/sec, a voltage generator supplying a rectangular wave is required. This control voltage divides the computing cycle into an

Table 1.1

Computer Operation	Input e_i	Output e_o	Detailed Schematic	Compact Schematic
Constant multiplication	$f(t)$	$kf(t),\ k \leq 1$		
Addition	$f_i(t)$	$-\sum_{i=1}^{n} k_i f_i(t)$		
Integration	$f_i(t)$	$-\int_0^t \left(\sum_{i=1}^{n} k_i f_i(t) \right) dt + IC$		
Complex transfer functions	$f(s)$	$-\frac{Z_f(s)}{Z_i(s)} f(s)$		
Multiplication	$f(t), g(t)$	$kf(t)\, g(t)$		
Function generation	$g(t)$	$f[g(t)]$		
High-gain amplification	$f(t)$	$-gf(t)$		

active period when solutions are obtained, and an inactive period for the resetting of initial conditions. An elaborate electromechanical or electronic switching system accompanies this voltage generator and acts at appropriate times during the repetitive cycle to reset all capacitors to the appropriate initial condition. A separate set of potentiometers is provided to control the initial voltage at the output of each integrator. Finally, power supplies are required to provide the required $B+$ and $B-$ voltages for the operational amplifiers and to furnish d-c voltages to the initial condition potentiometers.

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part I
THEORY

chapter 2
ANALYTICAL FOUNDATIONS

2.1 General Remarks

The programming of mathematical problems on analog computers and the obtaining of solutions are generally accomplished by a number of straightforward and well-defined steps. Little recourse is made to the underlying principles and theories unless an unusually difficult or novel problem is encountered. In order to place the analog method on a firm foundation, however, it is nevertheless desirable to consider in some detail the theoretical aspects of the programming and solving of problems on the computer.

In this chapter the generality of the analog method is first delineated. An effort is made to define the largest ensemble of differential equations which can be solved in principle using the analog method. This problem is not unlike the derivation of existence theorems in analytical mathematics. For theoretical and practical reasons this discussion is divided into two parts: the solution of linear differential equations, and the solutions of nonlinear differential equations. This grouping of equations does not, however, have any direct significance in the application of analog equipment, since nonlinear differential equations do not in general present any additional problems.

The treatments of the linear and nonlinear cases lead directly to the two most commonly used programming methods. These are outlined in some detail. A description follows of two transformation techniques which extend the utility of the analog method to problems that otherwise could not be treated.

FUNDAMENTAL PRINCIPLES

2.2 Linear Differential Equations

Following the development by Raymond,[1] consider the general system of linear differential equations with constant coefficients

$$\begin{vmatrix} b_{11}s, a_{12}, \ldots, a_{1n} \\ a_{21}, b_{22}s, \ldots, a_{2n} \\ - - - - - - - \\ a_{n1}, a_{n2}, \ldots, b_{nn}s \end{vmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{Bmatrix} \tag{2.1}$$

where a_{ij}, b_{ij} are real numbers, $x_1 \ldots x_n$ are solutions, and s is the Laplace operator. In more concise notation, Equation 2.1 becomes

$$|A + Bs|\{x(t)\} = \{f(t)\} \tag{2.2}$$

The basic question is then whether it is possible to realize a computer circuit whose governing equations are exactly identical to those expressed by Equation 2.1.

If in Equation 2.1 the vector $\{x(t)\}$ is chosen arbitrarily, a vector $\{r(t)\}$ can be formed for which

$$\{r(t)\} = |(A + Bs)| \cdot \{x(t)\} - \{f(t)\} \tag{2.3}$$

For every $\{x(t)\}$ there then exists a $\{r(t)\}$ and vice versa. By definition the solution of Equation 2.1 involves making

$$\{r(t)\} = \{r_0(t)\} = 0 \tag{2.4}$$

and finding the corresponding values of

$$\{x(t)\} = \{x_0(t)\} \tag{2.5}$$

where the subscript 0 identifies the vector whose components are solutions of Equation 2.1. The analytic methods for obtaining $\{x_0(t)\}$ are not of present interest. It should be noted, however, that the calculations involved in the transformation

$$\{x(t)\} \rightarrow \{r(t)\} \tag{2.6}$$

is relatively simple and direct, whereas the inverse

$$\{r(t)\} \rightarrow \{x(t)\} \tag{2.7}$$

is much more complex. The introduction of the relationship expressed by Equation 2.3 or the transformation Equation 2.6 is useful because the corresponding circuits as realized on the computer are simple and always realizable, as will now be demonstrated.

One possible circuit corresponding to Equation 2.6 is shown in Figure

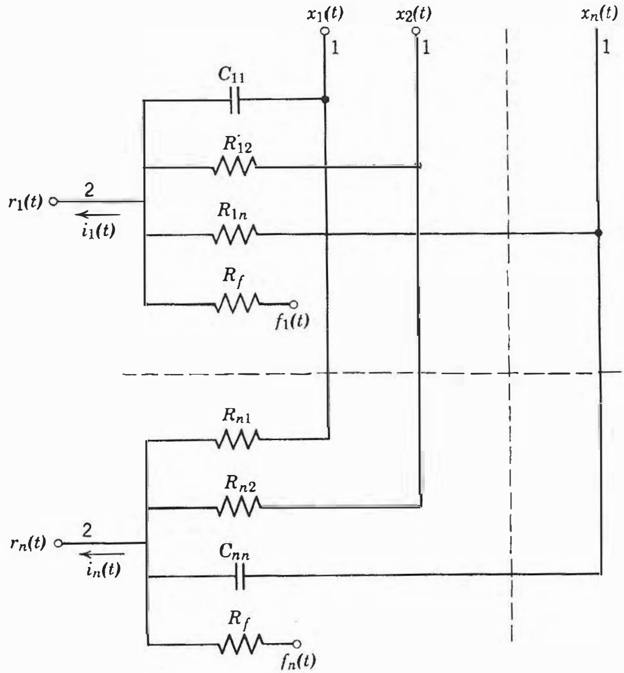

Fig. 2.1 Circuit for the transformation $\{x(t)\} \rightarrow \{r(t)\}$.

2.1. The equations governing the network are obtained by expressing the currents into terminals 2 as a function of the applied voltages as

$$\begin{aligned} i_1(t) &= C_{11}sx_1(t) + \frac{1}{R_{12}}x_2(t) + \cdots + \frac{1}{R_{1n}}x_n(t) - \frac{1}{R_f}f_1(t) \\ &- - - - - - - - - - - - - - - - - - - \\ i_n(t) &= \frac{1}{R_{n1}}x_1(t) + \frac{1}{R_{n2}}x_2(t) + \cdots + C_{nn}sx_n(t) - \frac{1}{R_f}f_n(t) \end{aligned} \tag{2.8}$$

Comparing Equations 2.8 with Equations 2.1 and 2.3, the following correspondences become evident

$$\begin{aligned} i_n(t) &= r_n(t) \\ \frac{1}{R_{ij}} &= a_{ij} \\ \frac{1}{R_f} &= 1 \\ C_{ii} &= b_{ii} \end{aligned} \tag{2.9}$$

The circuit of Figure 2.1 is then analogous to the system of Equation 2.3. This result demonstrates that the synthesis of a computer network for the transformation Equation 2.6 does not present any difficulties. However, this circuit is not satisfying since it would be desirable to find a circuit which automatically realizes the transformation Equation 2.7 subject to Equation 2.4.

In electrical terms, condition 2.4 signifies that both the currents and the voltage at nodes 2 in Figure 2.1 must be equal to zero; for only then is Equation 2.8 similar to the system given by Equation 2.1, except for the trivial case where nodes 2 are grounded. The question now arises if it is possible to make the voltage and current at nodes 2 equal to zero and assure simultaneously a unique relationship between $\{r(t)\}$ and $\{x(t)\}$.

The synthesis of such a circuit presents a more complex problem; one that cannot be solved except by introducing a new type of circuit element into the synthesis procedure. The circuit whose schematic is shown in Figure 2.2 fulfills the required conditions if all elements operate in an ideal fashion. As a first approximation the equation governing this circuit can be expressed as

$$(A + Bs)\cdot\{x(t)\} - \{f(t)\} = \frac{\{x(t)\}}{g} \tag{2.10}$$

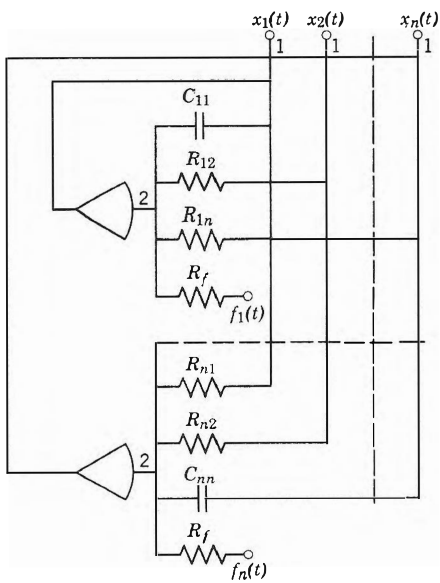

Fig. 2.2 Circuit for the automatic realization of the transformation $\{r(t)\} \to \{x(t)\}$.

where g represents the gains of the amplifiers. Equation 2.1 is approximated more and more closely as g becomes large; but a complete identity is impossible since the gain of the amplifier is necessarily limited. An examination of the circuit of Figure 2.2 shows that the amplifiers introduce feedback into the system. The transformation of the original simple network into a multiloop feedback system obviates the necessity for iteration procedures and leads to an automatic realization of transformation Equation 2.7. Above all, condition 2.4 is fulfilled in a practical manner.

The limitations imposed by the finite amplifier gain ($g \neq \infty$) are not as unfavorable as they may seem at first. It should be remembered that in the great majority of practical problems, the original data upon which the mathematical model is based are lacking in extreme accuracy, and that the final results can never be better than the original data. In other words, in view of the limited accuracy of the results sought in the calculation, it is not necessary to simulate absolutely the system of Equation 2.1.

Until now the discussion has been limited to simple equivalent circuits to demonstrate the principle of the analog method. The over-all generality of this technique is now considered in order to show that a computer circuit can *always* be found corresponding to the category of equations under discussion, subject only to the condition of stability.

Returning to Figure 2.2, consider the ensemble of voltages and currents at nodes 1 and 2. In order to facilitate the derivation, without limiting the generality of the result, let

$$\{f(t)\} = 0 \tag{2.11}$$

In general, using the Laplace transform, the voltage-current relationship of a multipole network having two ensembles of input and output terminals can be expressed by the linear set of equations

$$\begin{aligned} I_1 &= Y_{11}E_1 + Y_{12}E_2 \\ I_2 &= Y_{21}E_1 + Y_{22}E_2 \end{aligned} \tag{2.12}$$

where E and I are functions of s; the Ys are matrices of nth order, the terms of which are rational functions of s. The properties of the matrix

$$Y = \begin{vmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{vmatrix} \tag{2.13}$$

are well known to circuit theorists. The matrices Y_{11} and Y_{22} must be symmetric and Y_{12} is the transpose of Y_{21}. Consider now the behavior of such a four-terminal network if

$$\begin{aligned} I_2 &= 0 \\ E_2 &= 0 \end{aligned} \tag{2.14}$$

From Equation 2.12 it is evident that

$$Y_{21}E_1 = 0 \tag{2.15}$$

It therefore appears that the voltages E_1 cannot be arbitrary but must satisfy a system of differential equations expressed concisely by Equation 2.15. This confirms the above mentioned conclusion that $\{x(t)\}$ and $\{r(t)\}$ as realized on the computer are uniquely related. It is easy to demonstrate that the introduction of nonhomogeneous equations does not change this conclusion.

The second point to be examined is whether the network is physically realizable. The answer to this question is positive in view of the properties of the matrix Equation 2.13; for although Y_{21} and therefore Y_{12} are determined, the choice of Y_{11} is unrestricted. Moreover, since the transform function of the network can be realized by a wide variety of circuits, the result is not unique. It has already been demonstrated how the conditions of Equation 2.15 can be approximated.

It now remains to describe the operation of the circuit of Figure 2.2 taking into account the nonideal properties of physically realizable electronic amplifiers. In this figure the points labeled 1 and 2 correspond to the subscripts in Equation 2.12. In order to relate the characteristics of the amplifiers with the four-terminal network, an nth order matrix H is defined so that the amplifier output voltages and currents are given by

$$E_1' = HE_1 \qquad I_1' = HI_1 \tag{2.16}$$

where E_1' and I_1' refer to the voltages and currents at the outputs of the amplifiers. H then specifies the manner of interconnection of the amplifiers and of the passive elements. The equations of a physically realizable network then take the form

$$(A + Bs)E_1 + \frac{HE_1}{g(s)}\,\phi(s) = 0 \tag{2.17}$$

The factor $\phi(s)$ depends upon the input and output impedances of the amplifiers and upon the choice of the matrix Y_{22}. For $g(s) = \infty$, as has been demonstrated, the preceding equations give an exact solution of the system of differential equations.

This concludes the study of the linear case from the point of view of network theory. The conclusions may be summarized as follows.

It is always possible to carry out the synthesis of a circuit with n degrees of freedom such that the voltages at n terminals satisfy the given system of differential equations. The given mathematical equations and the equations governing the network will be identical if

1. A proportionality exists between the parameters of the network and the coefficients of the given equations.
2. The topology of the circuit is determined by the structure of the given equations.
3. Within the circuit there exist a number of nodes, the voltage of which is maintained at zero, or as small as desired, with the aid of stable feedback loops.

2.3 Nonlinear Differential Equations

The method employed for establishing the relationship between the electrical network and the given system of linear differential equations can be extended to the general case. In this discussion, however, the equivalent circuits will not be derived from matrix considerations, since this method leads to generalizations which are not fully satisfying. Rather, for arbitrary differential equations, equivalent circuits suggested by the form of the equation

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right) \tag{2.18}$$

will be employed. It is interesting to consider briefly in this connection how Kelvin first got the idea of solving differential equations mechanically. He was interested in solving the differential equation

$$\frac{d}{dt}\left[\frac{1}{F(t)}\frac{dy}{dt}\right] + y = 0 \tag{2.19}$$

In the first phase of his work he considered mechanizing the ordinary methods of successive approximations. He therefore connected two mechanical integrators in cascade. At the input of the first he introduced an arbitrary function y_0. The output of the second integrator was therefore

$$y_1 = \int F(t)(C - \int y_0\, dt)\, dt \tag{2.20}$$

If y_1 is recorded and if one introduces this function in place of y_0, the results of a second integration are given by

$$y_2 = \int F(t)(C - \int y_1\, dt)\, dt \tag{2.21}$$

This procedure is repeated until the difference $y_m - y_{m-1}$ becomes negligible so that y_m represents to a certain tolerance the integral of the differential equation.

So far this method is a simple imitation of analytical techniques; but the introduction of feedback loops into such a system completely changes the picture. As demonstrated by Tomovic,[2] this can be best illustrated by making reference to Picard's method of solving differential equations. For the system of differential equations

$$\frac{dy_i}{dt} = f_i(t, y_1, y_2, \ldots, y_n) \qquad i = 1, 2, \ldots, n \tag{2.22}$$

considered in the interval

$$0 \leq t \leq t_0$$

it can be shown that there exists only one system of integrals, $y_i(t)$, which take the values y_i^0 at time $t = 0$. In order to calculate $y_i(t)$, y_i^0 is first approximated.* If y_i^0 is inserted into Equation 2.22, the result is

$$y_i^1(t) = y_i^0 + \int_0^t f_i(\tau, y_1^0, y_2^0, \ldots, y_n^0)\, d\tau \tag{2.23}$$

In a second integration, y_i^1 takes the place of y_i^0, so that

$$y_i^2(t) = y_i^0 + \int_0^t f_i(\tau, y_1^1, y_2^1, \ldots, y_n^1)\, d\tau \tag{2.24}$$

The resulting solution has the form

$$y_i^m(t) = y_i^0 + \int_0^t f_i(\tau, y_1^{m-1}, y_2^{m-1}, \ldots, y_n^{m-1})\, d\tau \tag{2.25}$$

When

$$\lim\, (y_i^m - y_i^{m-1}) = 0 \tag{2.26}$$

the integrals of the system become

$$\lim_{m \to \infty} y_i^m(t) = y_i(t) \tag{2.27}$$

If Equation 2.18 is used, the series of successive solutions takes the form

$$y_m(t) = \int_{(n)} \cdots \int_0^t f(\tau, y_{m-1}, y_{m-1}^{(1)}, \ldots, y_{m-1}^{(n-1)})\, d\tau \tag{2.28}$$

The method can also be applied if Equation 2.18 consists of n first order equation by letting

$$\begin{aligned} \frac{dy}{dt} &= y_1 \\ \frac{dy_1}{dt} &= y_2 \\ &\text{– – – – – –} \\ \frac{dy_{n-2}}{dt} &= y_{n-1} \end{aligned} \tag{2.29}$$

* In this discussion the superscripts 0, 1, 2, . . . , m indicate the number of iterations that have been performed. The superscripts (1), (2), . . ., (n) refer to the number of times the variable has been differentiated.

where

$$\frac{dy_n}{dt} = f(t, y, y_1, \ldots, y_{n-1}) \tag{2.30}$$

The solutions of Equation 2.29 then become

$$\begin{aligned} y^{(1)} &= y_1 \\ y^{(2)} &= y_2 \\ &\text{— — — — — —} \\ y^{(n-1)} &= y_{n-1} \end{aligned} \tag{2.31}$$

and satisfy the specified initial conditions.

It should be noted at this point that the synthesis of the circuit corresponding to Equation 2.18 is just as direct as that in the linear case. The only difference is that now nonlinear operational elements must be included in the synthesis. Thus the equivalent circuit constructed using the differential analyzer corresponds directly to the given mathematical equations. The basic similarities and differences between the method of Picard and the analog method are now considered.

The distinctive step in the method of successive approximations lies in the comparison of $y_m(t)$ and $y_{m-1}(t)$ after integration. This involves two successive and distinct phases: the calculation of the integral of Equation 2.18, followed by substitution in this same equation; the second step can be considered purely mechanical. The time involved in carrying out this substitution does not enter into the problem. The separation of the work into two successive phases in the method of successive approximation follows from the way by which one obtains integrals in mathematical analysis. The integral

$$F(t) = \int_0^t f(\tau)\, d\tau \tag{2.32}$$

is obtained as a relationship between two ensembles by means of tabulated functions, series expansion, or by some other method. This process is distinguished by the establishment of a correspondence between the two ensembles f and F in their totality followed by the determination of particular values $F(t_k)$.

In the analog method the exact opposite holds true. Evidently, analog elements cannot operate with general expressions. Rather at each instant of time the physical functions have a specific value. Integration by analog methods takes the following steps:

1. Two sets of transient voltages are generated *simultaneously* such that one set represents the functions $f(t)$ and the other the integrals $F(t)$ of

the functions $f(t)$. It is assumed that the time delays within the electric circuits are negligible.

2. The transfer functions of the computer elements are chosen so as to assure that all the values of the output voltages are made to correspond to the required integral automatically; that is, the output voltages are members of the set $F(t)$.

In practice this implies that the entire ensemble of functions to be integrated are treated identically. This manner of integration eliminates one practical limitation of Picard's method, since all successive integrals can be obtained by the same process and with the same effort. But the analog method goes a step further. The solution of the differential equations is obtained here in a continuous manner by a single process.

These considerations point up the necessity of studying the general

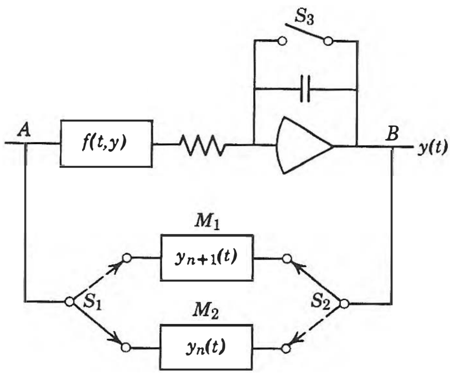

Fig. 2.3 Analog network for iteration process.

relationship between the method of mathematical iteration and the analog computer method. One approach is to regard the iteration process as being modeled on the analog network, which leads to a more rapid conversion than in the classical method of Picard. In this connection the following considerations are of interest. Figure 2.3 shows in block diagram form a circuit for carrying out an iteration process whose speed of conversion is more rapid than Picard's original method but slower than that of the continuous analog solution. The analog memories M_1 and M_2 are alternatively in their "read-in" and "read-out" positions as determined by switches S_1 and S_2. The function of switch S_3 is to reset the initial condition of the integrator after each interval $\Delta\tau$. It is assumed that the switches are properly synchronized and that the switching time is negligible in comparison to the time intervals of interest.

In mathematical terms the process represented in Figure 2.3 involves the solution of the equation

$$\frac{dy}{dt} = f(t, y) \qquad y(0) = y_0 \tag{2.33}$$

The switching frequency of S_1, S_2, and S_3 is such that one iteration step is accomplished in each time interval $\Delta\tau$. The entire iteration process can then be described as

$$y_n(t) = y_0 + \int_0^t f(\tau, y_{n-1})\, d\tau \qquad \tau_{n-1} \leq \tau \leq \tau_n \tag{2.34}$$

and

$$y_{n+1}(t) = y_0 + \int_0^t f(\tau, y_n)\, d\tau \qquad \tau_n \leq \tau \leq \tau_{n+1} \tag{2.35}$$

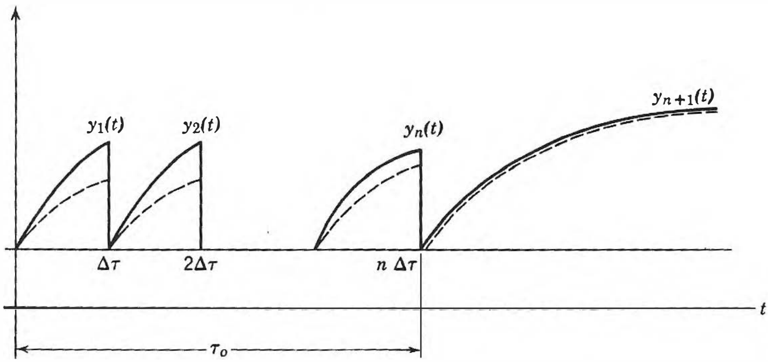

Fig. 2.4 Graphical representation of convergence of solution.

If the necessary conditions of convergence

$$y_{n+1} \to y(t)$$
$$n \to \infty$$

are fulfilled, the analog circuit represents a mechanization of an iteration process in which each step lasts $\Delta\tau$ (instead of $n\,\Delta\tau$ as would result if Picard's original method were mechanized). Moreover, if the switching frequency is increased, the convergence process is correspondingly accelerated, so that if this frequency becomes infinite, points A and B can be short circuited. Thus the transition to the conventional analog feedback network is realized.

Figure 2.4 shows the above described iteration process in graphical form. The time intervals τ_0 is a build-up period after which the true solution and the analog solutions remain in a constant relationship determined by the quality and performance of the computer.

PROGRAMMING TECHNIQUES

2.4 The Computer Program

A major advantage of the analog method lies in the facility and rapidity with which equations can be programmed on the computer. The construction of a logical schematic is simple and direct for all classes of equations that can be solved on the computer. This does not imply that it is unnecessary to precede the actual programming by various simple transformations of the given equations to adapt them to the specific characteristics of the differential analyzer.

The logical approach employed in programming a specific problem on the computer is of greatest importance, for if the manner in which the computer units are to be used is planned in an optimum fashion, the electrical connections and manipulations become very easy. As indicated above, a number of equivalent schemes may exist which can be employed to treat a specific problem. Which of these methods is best depends to a large extent upon the layout and design of the specific differential analyzer to be used. In this section the general features of the two most widely used programming methods are outlined. One of these is based upon the matrix representations of equations such as Equation 2.1, and is particularly useful for linear differential equations with constant coefficients. The second method is based upon formulations of the type of Equations 2.18 and 2.28, and is more applicable to the nonlinear case. The method that is preferable depends upon the design features of the specific machine; thus in general one method cannot be considered as being superior to the other.

For the linear differential analyzer, the matrix method has certain advantages. Since the equations always have the same form, the units comprising the computer network can be permanently interconnected so that only the specified coefficients and initial conditions need be introduced separately. Since the rows, the columns, and the diagonals of the matrices can be adjusted separately, the problem of selecting suitable scale factors becomes relatively easy on such a computer. If the specified equations are not already in the necessary form, their reduction to a system of the type of Equation 2.1 is very straightforward. In general the following steps are followed:

1. With the aid of substitutions of the type of Equations 2.29 and 2.30, the problem is formulated in the form of n first-order equations.
2. The resulting matrix is transformed in such a manner that the coefficients of the derivatives are located only along the principal diagonals.

If these steps are followed, the solution of the problem on the computer is limited to the introduction of the specified coefficients. Condition 2 assures that the network will be stable. Gutenmaher[3] and Honnell[4] have developed practical rules for the adaptation of differential equations with constant coefficients for programming on a differential analyzer organized to facilitate matrix programming. Such an analyzer has fixed internal connections and employs potentiometers for coefficient settings. Parodi[5] has demonstrated that such systems will be stable if the equations are formulated so that all terms on the main diagonal are positive and greater in absolute value than the terms in the corresponding rows and columns. The following is a simple illustration of his method.

Example 1. Given the system of differential equations

$$x^{(2)} + 2x^{(1)} + 3x - y^{(2)} = 0$$

$$x^{(1)} + 4y^{(2)} - 3y + y = 0$$

let

$$x^{(1)} = z$$

$$y^{(2)} = v$$

so that

$$z^{(1)} + 2z + 3x - v^{(1)} = 0$$

$$z + 4v^{(1)} - 3v + y = 0$$

$$x^{(1)} - z = 0$$

$$y^{(1)} - v = 0$$

In matrix form the system is represented as

$$\begin{array}{cccc} z & v & x & y \\ \end{array}$$

$$\begin{bmatrix} s+2 & -s & 3 & 0 \\ 1 & 4s-3 & 0 & 1 \\ -1 & 0 & s & 0 \\ 0 & -1 & 0 & s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiplying the first row by 4 and adding row 2 to it, a matrix suitable for machine programming results in

$$\begin{array}{cccc} z & v & x & y \\ \end{array}$$

$$\left[\begin{array}{cccc} 4s+9 & -3 & 12 & 1 \\ 1 & 4s-3 & 0 & 1 \\ -1 & 0 & s & 0 \\ 0 & -1 & 0 & s \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$\begin{array}{cccc} x^{(1)}(0) & y^{(1)}(0) & x(0) & y(0) \end{array}$$

Evidently it is necessary to select suitable scale factors in advance. The reduction and representation of a system of equations with constant coefficients in the above form, with the initial conditions indicated beneath the corresponding columns, are very convenient. In this manner one can indicate next to each row the factors by which it was multiplied or divided, so that the method of transformation of the system of equations is clearly indicated; and errors made in computer programming and setup are readily localized and identified.

For analyzers that are not limited to linear differential equations, the manipulations of the given equations are based upon the method discussed in Section 2.3. The following approach is then employed:

1. The highest derivative of each equation is expressed explicitly as

$$y^{(n)} = f(t, y^{(1)}, \ldots, y^{(n-1)})\, dt \tag{2.36}$$

and integrated

$$y = \int_{(n)} \cdots \int f(t, y^1, \ldots, y^{(n-1)})\, dt \tag{2.37}$$

This formulation is used in analog computers because the derivatives of lower orders can then be obtained by means of integrations. The operation of integration is preferred to that of differentiation because it leads to computer systems which are more accurate, more stable, and less subject to noise.

2. A network corresponding to the expression $f(t, y^{(1)}, \ldots, y^{(n-1)})$ is realized by employing analog computer elements. In this case the realization follows directly from the structure of the specified equations, regardless of their type or order.

3. A feedback link is added to the circuit to effect the correspondence expressed by Equation 2.37. In this way a unique computer system is realized.

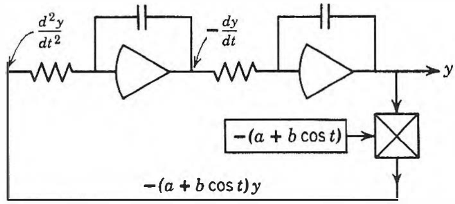

Fig. 2.5 Computer circuit for the equation: $\frac{d^2y}{dt^2}+(a+b\cos t)y = 0.$

4. If the functions $f(t)$ are themselves solutions of known differential equations, they are usually generated by solving simultaneously a complementary set of differential equations rather than by means of special function generators. This approach is generally followed because function generators are usually more expensive and complicated than integrators, and because their accuracy is limited.

Example 2. It is desired to construct the computer circuit for the differential equation

$$\frac{d^2y}{dt^2} + (a + b\cos t)y = 0$$

According to this described method: (1) the equation is rearranged so that

$$\frac{d^2y}{dt^2} = -(a + b\cos t)y$$

(2 and 3) The schematic is now constructed as shown in Figure 2.5. Note that feedback is necessary.

(4) The equation contains the expression $a + b\cos t$. In place of a function generator, the auxiliary equation

$$\frac{d^2y}{dt^2} = -k^2y$$

is employed. Its schematic diagram is shown in Figure 2.6.

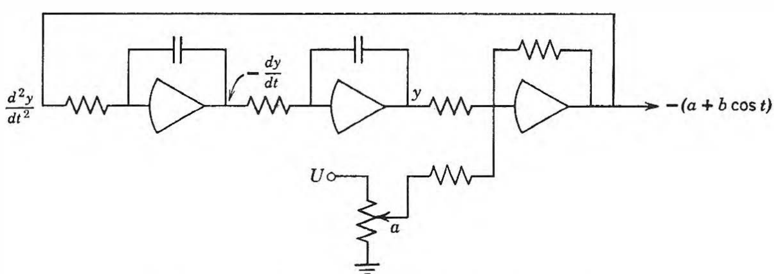

Fig. 2.6 Auxiliary circuit to replace the function generator.

2.5 The Transformation $e^{\lambda t}$

Occasionally it becomes necessary in engineering simulations to instrument equations whose roots are located in the right half of the complex frequency plane. The solutions will then have terms which diverge and grow to infinity as time approaches infinity. Provided solutions are required only over a limited interval of time it may be possible to choose amplitude scale factors in such a manner that the dynamic range of the computing amplifiers is not exceeded during the computer run. Where it is impractical to manipulate the scale factors in this fashion, the transformation $e^{\lambda t}$ may be introduced.

Consider a system of the type of Equation 2.1, with initial conditions

$$\begin{aligned} x_1(0) &= c_1 \\ x_2(0) &= c_2 \\ &-\ -\ -\ -\ - \\ x_n(0) &= c_n \end{aligned} \tag{2.38}$$

The system is transformed in the following manner:

$$\begin{aligned} x_1(t) &= y_1(t)e^{\lambda t} \\ x_2(t) &= y_2(t)e^{\lambda t} \\ &-\ -\ -\ -\ -\ - \\ x_n(t) &= y_n(t)e^{\lambda t} \end{aligned} \tag{2.39}$$

where λ is a real positive number. After substitution in Equation 2.1 the new system becomes

$$\begin{vmatrix} b_{11}s, a_{12}, \ldots, a_{1n} \\ a_{21}, b_{22}s, \ldots, a_{2n} \\ -\ -\ -\ -\ -\ -\ - \\ a_{n1}, a_{n2}, \ldots, b_{nn}s \end{vmatrix} \begin{Bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{Bmatrix} = \begin{Bmatrix} e^{-\lambda t}f_1(t) \\ e^{-\lambda t}f_2(t) \\ \vdots \\ e^{-\lambda t}f_n(t) \end{Bmatrix} \tag{2.40}$$

Divergent solutions are made convergent by choosing a sufficiently large λ. They can then be obtained directly from the differential analyzer regardless of how long the desired solution time in the t domain. The transformation equation 2.39 is then employed to arrive at the desired result. This can be facilitated by the use of special nomograms for converting from $x(t)$ to $y(t)$.

This same method can be employed for simulations in which the solutions of the equations diminish very rapidly, leading to another source of error. In this case λ is given a negative polarity.

2.6 Treatment of Problems in Which $a_n \neq 1$

In Section 2.4 it is shown how the equivalent analog network for differential equations of the type

$$y^{(n)} = f(t, a_0, a_1 y^{(1)}, \ldots, a_{n-1} y^{(n-1)}) \tag{2.41}$$

can be realized. The only assumption regarding the above equation related to the coefficient of highest order derivative; a_n was assumed equal to unity. All other coefficients in the equation were taken fully arbitrarily. They can be linear or nonlinear, constant or time varying. Nor were theoretical restrictions put on the order of the differential equation.

Although it is always possible in principle to reduce any differential equation to the form of Equation 2.41 by dividing by a_n, there are important cases where it is very inconvenient to do so. There are no difficulties involved in setting up analog networks for $a_n \neq 1$ if differentiators are permitted. Since the use of differentiators entails serious noise problems, however, a general procedure is required to synthesize such analog networks using only integrators.

An important class of problems in the area of random processes in automatic control,[6] in optimal filter synthesis, and other fields is reduced to the following linear differential equation with time variable coefficients:

$$\sum_{i=0}^{n} a_i(t) y^{(i)}(t) = \sum_{i=0}^{m} b_i(t) x^{(i)}(t) \qquad m \leq n \tag{2.42}$$

Equation 2.42 describes the response $y(t)$ in the time domain of a linear system with variable coefficients for a specified input $x(t)$. For example, the following equation may arise:

$$5(1 - e^{-t/5}) y^{(2)} + (3 - e^{-t/5}) y^{(1)} + \tfrac{2}{5} y = (3 - e^{-t/5}) x^{(1)} + \tfrac{2}{5} x \tag{2.43}$$

A division of the equation by $5(1 - e^{-t/5})$ would lead to a more complicated computer circuit. Moreover, for $t \to 0$ all terms divided by $5(1 - e^{-t/5})$ would approach infinity. Thus it would be impossible in any case to solve Equation 2.43 in the vicinity of $t = 0$.

A general solution of this problem is proposed by Matiash.[7] To make the notation more compact introduce the operators

$$\begin{aligned} L &= \sum_{i=0}^{n} a_i(t) \frac{d^i}{dt^i} \\ M &= \sum_{i=0}^{m} b_i(t) \frac{d^i}{dt^i} \end{aligned} \tag{2.44}$$

so that Equation 2.42 may be written

$$L[y] = M[x] \tag{2.45}$$

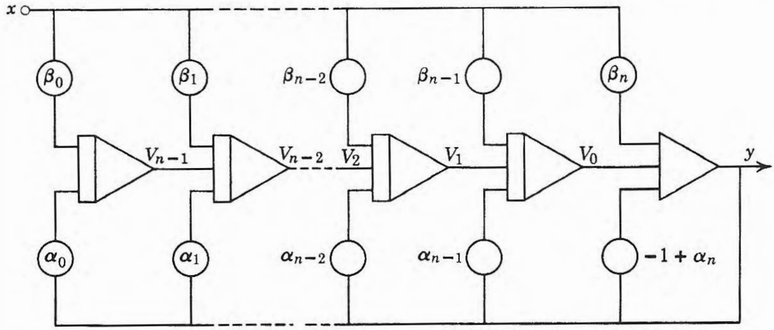

Fig. 2.7 General analog network for Equation 2.42.

Two auxiliary linear operators will also be needed.

$$L_1 = \sum_{i=0}^{n} \alpha_i(t) \frac{d^i}{dt^i}$$
$$M_1 = \sum_{i=0}^{m} \beta_i(t) \frac{d^i}{dt^i} \tag{2.46}$$

where $\alpha_i(t)$ and $\beta_i(t)$ are obtained with the aid of the circuit diagram shown in Figure 2.7. This diagram represents a general analog network capable of solving Equation 2.42 by the use of integrators only and when $\alpha_n(t) = 1$. The relations between $a_i(t)$, $b_i(t)$ and $\alpha_i(t)$, $\beta_i(t)$ must now be established. Referring to Figure 2.7, the integrator outputs are

$$\begin{aligned}
-V_0 &= \alpha_n y + \beta_n x \\
-V_0^{(1)} &= V_1 + \alpha_{n-1} y + \beta_{n-1} x \\
-V_1^{(1)} &= V_2 + \alpha_{n-2} y + \beta_{n-2} x \\
&- - - - - - - - - - - - - - - \\
-V_{k-1}^{(1)} &= V_k + \alpha_{n-k} y + \beta_{n-k} x \qquad k = 1, 2, \ldots, n-1 \\
-V_{n-1}^{(1)} &= \alpha_0 y + \beta_0 x
\end{aligned} \tag{2.47}$$

These equations permit the calculation of V_k. Starting with V_2,

$$V_2 = -V_1^{(1)} - \alpha_{n-2} y - \beta_{n-2} x \tag{2.48}$$

In order to calculate V_2 only in terms of $\alpha_i y$ and $\beta_i x$, the expression $V_0^{(1)}$ is differentiated

$$-V_0^{(2)} = V_1^{(1)} + (\alpha_{n-1} y)^{(1)} + (\beta_{n-1} x)^{(1)} \tag{2.49}$$

Combining

$$V_2 = V_0^{(2)} + (\alpha_{n-1}y)^{(1)} + (\beta_{n-1}x)^{(1)} - \alpha_{n-2}y - \beta_{n-2}x \qquad (2.50)$$

and using the first of Equations 2.47,

$$V_2 = -(\alpha_n y)^{(2)} - (\beta_n y)^{(2)} + (\alpha_{n-1}y)^{(1)} + (\beta_{n-1}x)^{(1)} - \alpha_{n-2}y - \beta_{n-2}x \qquad (2.51)$$

A generalization of this procedure gives

$$V_k = -\sum_{s=0}^{k}(-1)^{k-s}(\alpha_{n-s}y)^{(k-s)} - \sum_{s=0}^{k}(-1)^{k-s}(\beta_{n-s}x)^{(k-s)} \qquad (2.52)$$

By replacing V_k with $k = n - 1$ in Equation 2.47, the differential equation solved by the analog network is expressed as

$$\sum_{i=0}^{n}(-1)^i(\alpha_i y)^i = -\sum_{i=0}^{m}(-1)^i(\beta_i x)^{(i)} \qquad m \leq n \qquad (2.53)$$

Using the operators $L_1{}^*$ and $M_1{}^*$ conjugate to L_1 and M_1

$$L_1{}^*[y] = \sum_{i=0}^{n}(-1)^i(\alpha^i y)^{(i)}$$

$$M_1{}^*[x] = \sum_{i=0}^{n}(-1)^i(\beta_i x)^{(i)} \qquad (2.54)$$

Equation 2.53 can be expressed as

$$L_1{}^*[y] = -M_1{}^*[x] \qquad (2.55)$$

A comparison of Equations 2.53 and 2.42 provides the required relation between the coefficients in operator form

$$L_1{}^*[y] = L[y]$$

$$-M_1{}^*[x] = M[x] \qquad (2.56)$$

or since $L[y]$ and $M[x]$ are given,

$$L_1[y] = L^*[y]$$

$$M_1[x] = -M^*[x] \qquad (2.57)$$

taking into account that for each linear operator k the relation

$$(k^*)^* = k \qquad (2.58)$$

applies. Thus in the circuit diagram of the analog computer solution of Equation 2.42 the coefficients $\alpha_i(t)$, $\beta_i(t)$ should be set corresponding to the conjugate operators of the left- and right-hand sides. The explicit expression for calculating these coefficients is easily obtained by performing the differentiations required by the operators. Thus

$$\beta_{n-k} = \sum_{i=0}^{k} (-1)^{m-i} \frac{(m-i)!}{(m-k)!(k-i)!} b_{m-i}^{(k-i)} \qquad (k = 0, 1, 2, \ldots, m)$$

$$\alpha_{n-k} = \sum_{i=0}^{k} (-1)^{n-i} \frac{(n-i)!}{(n-k)!(k-i)!} a_{n-i}^{(k-i)} \qquad (k = 0, 1, 2, \ldots, n) \tag{2.59}$$

Example 1. Given the differential equation

$$5(1 - e^{-t/5})y^{(2)} + (3 - e^{-t/5})y^{(1)} + \tfrac{2}{5}y = (3 - e^{-t/5})x^{(1)} + \tfrac{2}{5}x$$

In order to set up the equivalent analog network without differentiators, the conjugate expressions of the left- and the right-hand side must be found using Equation 2.54

$$L_1[y] = L^*[y] = 5(1 - e^{-t/5})y^{(2)} - 3(1 - e^{-t/5})y^{(1)} + \tfrac{2}{5}(1 - e^{-t/5})y$$
$$-M_1[x] = M^*[x] = -(3 - e^{-t/5})x^{(1)} + \tfrac{1}{5}(2 - e^{-t/5})x$$

Equation 2.59 now specifies the coefficients to be set on the computer

$$\alpha_0 = \tfrac{2}{5}(1 - e^{-t/5})$$
$$\alpha_1 = -3(1 - e^{-t/5})$$
$$\alpha_2 = 5(1 - e^{-t/5})$$
$$\beta_0 = -\tfrac{1}{5}(2 - e^{-t/5})$$
$$\beta_1 = 3 - e^{-t/5}$$

The computer setup corresponding to Figure 2.7 is shown in Figure 2.8. A possible practical realization of this network to be set on the computer is shown in Figure 2.9.

The method described is convenient when the differential equation of the system whose response is sought is given. In certain cases, such as

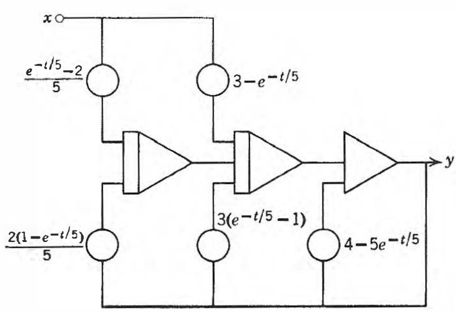

Fig. 2.8 Computer circuit corresponding to Figure 2.7.

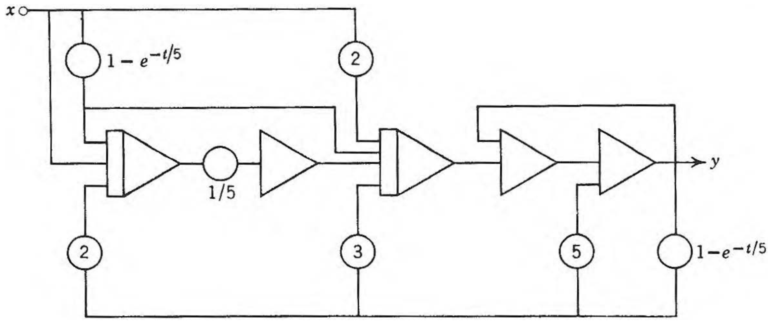

Fig. 2.9 Practical computer realization.

optimum filter design, the weighting function $W(t, \tau)$ may be given instead. The solution of the differential equations may be obtained as

$$y(t) = \int_{-\infty}^{t} W(t, \tau)x(\tau)\, d\tau \tag{2.60}$$

Various approaches have been developed to obtain $a_i(t)$, $b_i(t)$ when $W(t, \tau)$ is given.[6,8,9] The analog network corresponding to the differential equation whose weighting function $W(t, \tau)$ is given is shown in Figure 2.10. It contains two sets of coefficients $a_i(t)$ and $\gamma_i(t)$. The coefficients $a_i(t)$ are calculated from $W(t, \tau)$ by standard methods.[9] The coefficient $b_i(t)$ need not be determined at all in this method; instead the coefficients $\gamma_i(t)$ to be set directly on the analog computer are needed. They are obtained by following a procedure similar to that used above.

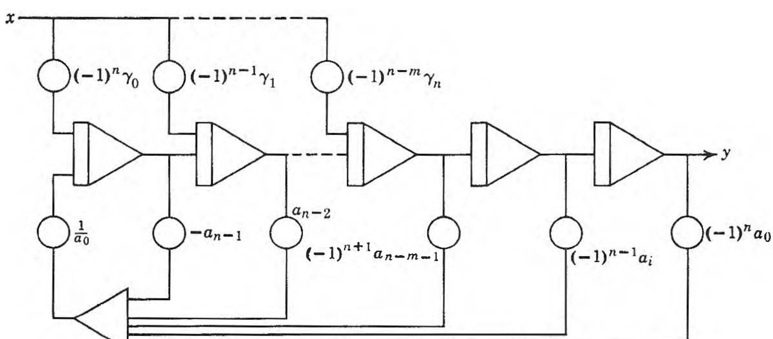

Fig. 2.10 Computer circuit for Equation 2.60.

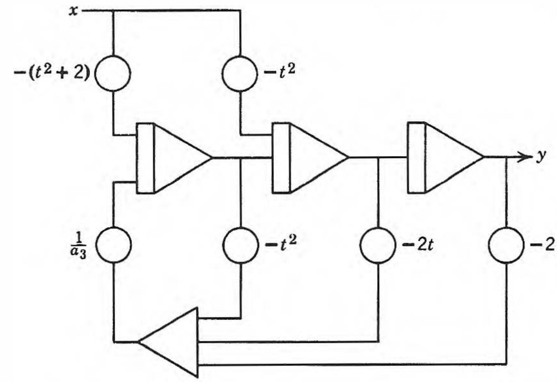

Fig. 2.11 General circuit for weighting function calculation.

It has been proved that

$$\gamma_{n-1-j}(t) = \left[\frac{\partial^j W(t,\tau)}{\partial t^j}\right]_{\tau=t} \tag{2.61}$$

Thus all the information needed to realize the diagram of Figure 2.10 is available.

Example 2. The analog network whose weighting function is

$$W(t,\tau) = t^2 + \tau^2 e^{\tau} e^{-t} - 2\tau t$$

is to be found. Evidently the order of the equation in $y(t)$ is $n = 3$. The basic system of the solution of the homogeneous equation is

$$y_1 = t, \qquad y_2 = t^2, \qquad y_3 = e^{-t}$$

The coefficients $a_i(t)$ are

$$a_0 = 2 \qquad a_1 = 2t \qquad a_2 = t^2 \qquad a_3 = t^2 + 2t + 2$$

The coefficients $\gamma_i(t)$ are calculated by Equation 2.61

$$\gamma_2 = W(t, t) = 0$$

$$\gamma_1 = \left[\frac{W(t,\tau)}{\partial t}\right]_{\tau=t} = -t^2$$

$$\gamma_0 = \left[\frac{\partial^2 W(t,\tau)}{\partial t^2}\right]_{\tau=t} = -t^2 + 2$$

The general diagram for this system is shown in Figure 2.11. A practical diagram is shown in Figure 2.12.

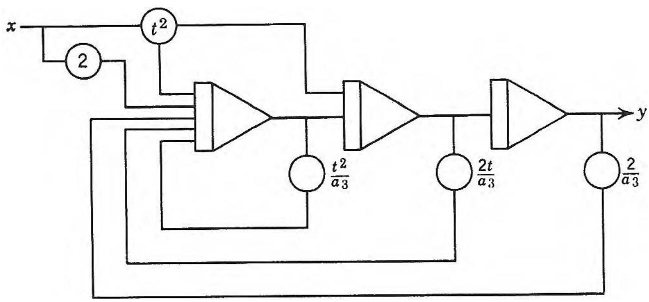

Fig. 2.12 Practical computer diagram.

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chapter 3
ERROR ANALYSIS OF ANALOG COMPUTERS

3.1 General Remarks

The subject of error analysis constitutes a formidable problem in virtually all areas of numerical analysis. Since numerical methods almost always involve approximation, the effect of these inaccuracies in method or equipment upon specific solutions must be determined or predicted. The analog method is not immune from such problems, since deviations from their specified values of the magnitude of passive elements, as well as noninfinite gain, zero-offset, and drift in operational amplifiers, invariably introduce perturbations and errors into the solution. More specifically two more-or-less equivalent questions arise.

1. Given the deviation of the computer elements from their ideal values, what is the resulting dynamic error in the solution?
2. If it is desired to obtain solutions of a specified accuracy, what tolerance or permissible deviation may be assigned to the computer elements?

General answers to these questions are, of course, not obtainable since the effect of error sources in the computer will depend upon the nature of the specific problem being solved. Nevertheless a considerable amount of effort has been expended in developing techniques for carrying out useful error analyses of the analog method. This chapter represents an integrated and unified survey of the more important of these methods.

Since an intimate relationship can be shown to exist between the sensitivity of computer solutions to errors and the stability of solution, Liapunoff's and Poincaré's stability concepts are first presented. Provided the solutions are stable it is theoretically possible, by improving the quality of the analog elements, to attain solutions of any specified accuracy. For unstable systems this is possible only over limited integration times.

The perturbation techniques applied to the solution of problems in mechanics by Poincaré and Liapunoff lead directly to the perturbation method for error analysis of computer solutions.

A general approach to the error analysis is then discussed. This method, which is applicable to nonlinear as well as linear systems, was developed on a theoretical level by Miller and Murray. The main problem in applying this method for the error analysis of computer solutions is to find a quick and practical method of determining the sensitivity coefficients. Two methods using the analog computer itself for the determination of these coefficients have been developed. When an analytic description of the system whose dynamic accuracy is to be determined is available, Meissinger's method of simultaneously solving the differential equations and the equations governing the sensitivity coefficients may be used. If the network to be analyzed for sensitivity coefficients is given in terms of its circuit diagram, Bihovski's method is suitable; for all measurements can then be taken without having to formulate the differential equation of the system.

In essence the sensitivity coefficients determined by these methods are expressions relating the error in the solution to a perturbation in a specific parameter. In this context the following question can be asked: Since the analog computer displays the solution continuously, why not observe the effect of perturbations directly by making small variations in the potentiometer controlling the parameter in question? Evidently this can be done if the perturbation will cause errors in excess of the analog computer accuracy. However, in many instances, very small perturbation effects must be observed so that a direct observation of the error is not possible. With Meissinger's and Bihovski's methods the full dynamic range of the computer is employed to observe the effects of perturbations, even if the perturbations are very small. Also there are a number of instances where sensitivity coefficients are needed for other purposes. For example, they are used in determining the statistical characteristics of the output voltage as a result of a specific tolerance of network components. The optimization of the performance of electrical and electronic networks can likewise be carried out using sensitivity coefficients.

As would be expected, a much more extensive error analysis can be carried out if one limits oneself to linear systems. Under these conditions the effect of such perturbation as the limited frequency response of operational amplifiers can be generalized and examined in detail. Over the years a number of investigators have carried out such studies and arrived at results which were contradictory only because different sets of starting assumptions were employed. The more prominent of these studies are reviewed in the final section of this chapter.

3.2 Relation between Stability and Accuracy

Before discussing the analysis of errors in analog computers it is necessary to review briefly the theory of stability of differential equations. This treatment will be limited to the approaches used by Liapunoff[1] and Poincaré.[2]

Consider a physical system with k degrees of freedom. The dependent variables describing the system behavior are identified as

$$a_1, a_2, \ldots, a_k$$

Assume that these variables are functions of time t, so that they have nonzero derivatives with respect to t, designated by

$$a_1^{(1)}, a_2^{(1)}, \ldots, a_k^{(1)}$$

It is further assumed that $a_1, a_2, \ldots, a_k$ are solutions of k first-order differential equations

$$a_1 = f_1(t), a_2 = f_2(t), \ldots, a_k = f_k(t) \tag{3.1}$$

The behavior of the system described by Equation 3.1 will be termed "nonperturbed."

The perturbations of the system are defined as follows:

$$\begin{aligned} a_1 &= f_1(t) + \Delta q_1, a_2 = f_2(t) + \Delta q_2, \ldots, a_k = f_k(t) + \Delta q_k \\ a_1^{(1)} &= f_1^{(1)}(t) + \Delta q_1^{(1)}, a_2^{(1)} = f_2^{(1)}(t) + \Delta q_2^{(1)}, \ldots, a_k^{(1)} = f_k^{(1)}(t) + \Delta q_k^{(1)} \end{aligned} \tag{3.2}$$

Δq_j, $\Delta q_j^{(1)}$ being real constant. The behavior of the system described by Equation 3.2 will be termed "perturbed."

Let $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$ be real and continuous functions of

$$a_1, a_2, \ldots, a_k \qquad a_1^{(1)}, a_2^{(1)}, \ldots, a_k^{(1)}$$

In the perturbed behavior

$$Q_1 = h(t, \Delta q_1, \Delta q_2, \ldots, \Delta q_n)$$

whereas in the unperturbed case

$$P_1 = h(t)$$

When all Δq_j, $\Delta q_j^{(1)}$ are zero, all

$$Q_1 - P_1, Q_2 - P_2, \ldots, Q_n - P_n$$

are also zero for all values of t. The question now arises as to whether $Q_n - P_n$ will approach an infinitely small value, which will never be exceeded for any $t > t_0$, as $\Delta q_j^{(1)}$ and Δq_j are made to approach zero.

Let $L_1, L_2, \ldots, L_n$ be n given positive numbers. If for any values of L, however small, one can find positive numbers

$$E_1, E_2, \ldots, E_k \qquad E_1^{(1)}, E_2^{(1)}, \ldots, E_k^{(1)}$$

so that inequalities

$$|\Delta q_j| < E_j \qquad |\Delta q_j^{(1)}| < E_j^{(1)} \qquad j = 1, 2, \ldots, k$$

are fulfilled, and, at the same time,

$$|Q_1 - P_1| < L_1, \qquad |Q_2 - P_2| < L_2, \ldots, \qquad |Q_n - P_n| < L_n$$

for all values of $t > t_0$, the unperturbed movement of the system is said to be stable with respect to $Q_1, Q_2, \ldots, Q_n$. In analog computer terms this implies that results of a specified accuracy can be obtained over the entire range of integration.

The connection between stability theory and error analysis is now apparent, since in a stable system the errors can always be kept within a prescribed limit by controlling Δq_j and $\Delta q_j^{(1)}$ (perturbations). The error analyses to follow will therefore be limited to systems which are stable in the above sense. If a system of differential equations

$$\frac{dy_i}{dt} = f_i(t, y_1, y_2, \ldots, y_n) \tag{3.3}$$

is to be set up on the analog computer, the machine equations will have the form

$$\frac{dy_i}{dt} = f_i(t, y_1, y_2, \ldots, y_n, \Delta q_1, \Delta q_2, \ldots, \Delta q_m) \tag{3.4}$$

where $\Delta q_1, \Delta q_2, \ldots, \Delta q_m$ are perturbation terms due to the shortcomings of the computer elements. Preferably

$$|\Delta q_j| < A$$

where A is a small positive number. If the system is stable, the differences between the solutions of Equations 3.3 and 3.4 decrease as A is decreased.

The next problem is to determine whether or not the system is stable. One evident way is to integrate Equations 3.3 and 3.4 and then to analyze the solutions. This is, except in the simplest cases, impossible in analog computation, not only because of calculation difficulties but also because the perturbations introduced by the analog network are not accurately known. In analog computers the initial conditions and the coefficients of the differential equation differ, within specified limits, from those of the unperturbed equations under study; the problem is to determine whether the difference between the solution of the perturbed and unperturbed equations remains within given limits during the entire time of the solution.

A mathematical answer to this problem was given by Liapunoff. Using his "second method" he solved the stability problem without integrating the differential equations of the perturbed system. The result is obtained by studying the character of a special auxiliary function to be established for each case.

Another general mathematical method for stability studies of differential equations was presented by A. N. Tihonow.[3] Consider the system of differential equations

$$\begin{aligned} \frac{dy_i}{dt} &= f_i(t, y_k, z) \qquad i, k = 1, 2, \ldots, n-1 \\ \mu \frac{dz}{dt} &= F(t, y_k, z) \qquad \mu > 0 \end{aligned} \tag{3.5}$$

When $\mu = 0$, the system reduces to

$$\begin{aligned} \frac{dy_i}{dt} &= f_i(t, y_k, z_y) \\ z_y &= \phi(t, y_k) \end{aligned} \tag{3.6}$$

where z_y is one of the roots of the equation

$$F(t, y_k, z) = 0 \tag{3.7}$$

The solutions of the system will coincide when $\lim \mu \to 0$ and when $\mu = 0$, provided $z_y = \phi(t, y_k)$ is an ordinary (in the sense of Poincaré) and stable singularity of Equation 3.7. As demonstrated by Vasilieva[4,5] additional conditions pertain to time derivatives of $y_i(t, \mu)$ and its partial derivative with respect to μ.

Thus stability criteria may serve to predict the accuracy of the computer solutions. That is, if one knows in advance that a system solution is stable (in the sense of Liapunoff) in the presence of perturbations, a satisfactory accuracy is generally obtainable on the analog computer. For as the quality of the analog computing elements is improved, the errors in the solution decrease. The Liapunoff criterion applies to an infinite range of integration (t goes from 0 to ∞), whereas in analog computers t is always limited. Thus, even if a problem is unstable in the sense of Liapunoff it may be solved on an analog computer for a fixed range of independent variables. A suitable transformation to achieve this is described in Section 2.4. On the other hand, although a problem is stable in Liapunoff's sense, it may be so sensitive to perturbations that satisfactory analog solution cannot be obtained. A careful treatment of this problem related to analog computers was presented by Eterman.[6]

3.3 Sensitivity Coefficients and the Sensitivity Equation

Although the preceding criteria are general, they cannot be used in a direct form for the error analysis of analog computers. In analog computers it is desired to obtain an estimate of the error without being forced to go into the analytical formulation of the perturbed equations, at the same time using the analog computer itself as much as possible to obtain the answer. In this manner the explicit mathematical formulation is replaced by measurements on the computer. Such an approach also requires modifications of the standard mathematical approaches to perturbation studies of differential equations. A basic contribution on this subject was presented by Miller and Murray.[7]

Consider first the simple case of the perturbed solution depending upon just one parameter t, and a perturbation term Δq.

$$x = x(t, \Delta q)$$

To express $x(t, \Delta q)$ in terms of the given unperturbed differential equation and Δq, $x(t, \Delta q)$ is expanded in Taylor series.

$$x(t, \Delta q) = x(t, 0) + \left(\frac{dx}{d\,\Delta q}\right)_0 \Delta q + \cdots \tag{3.8}$$

The subscript 0 in Equation 3.8 indicates that the derivative $dx/d\,\Delta q$ is to be taken for $\Delta q = 0$. Evidently, for this expansion to be valid, the solution must analytically depend on Δq. Using Equation 3.8 one can find an approximate value of the perturbed solution by knowing the original solution and the derivative $dx/d\,\Delta q$; this derivative is termed the *sensitivity-coefficient*. This forms the basis for a practical error analysis of analog computers. The main role of sensitivity analysis is then to determine the ways in which partial derivatives, with respect to perturbation parameters, can be found analytically or experimentally. Having calculated these partial derivatives, the errors in the solutions can be determined by using Taylor-series expansions.

Consider a system of linear or nonlinear differential equations of the form

$$F_i(\dot{x}_1, \ldots, \dot{x}_n, x_1, \ldots, x_n, t) = 0 \qquad i = 1, \ldots, n \tag{3.9}$$

The equations governing the perturbed system are

$$G_i(\dot{x}_1, \ldots, \dot{x}_n, x_1, \ldots, x_n, t, \Delta q_1, \ldots, \Delta q_m) = 0 \tag{3.10}$$

The solution of the system 3.10 has the general form

$$x_i = x_i(t, \Delta q_1, \ldots, \Delta q_m) \tag{3.11}$$

By assuming that the solution depends analytically on Δq_m, Equation 3.11 can be expanded in power series

$$x_i = x_i(t, 0, \ldots, 0) + \sum_{k=1}^{m} \left(\frac{\partial x_i}{\partial q_k}\right)_0 \Delta q_k + \frac{1}{2!} \sum_{k=1}^{m} \left(\frac{\partial^2 x_i}{\partial \Delta q_k{}^2}\right)_0 \Delta q_k{}^2$$
$$+ \frac{1}{2!} \sum_{k=1}^{m} \left(\frac{\partial^2 x_i}{\partial \Delta q_{k_1}\, \partial \Delta q_{k_2}}\right)_0 \Delta q_{k_1} \Delta q_{k_2} + \ldots \quad (3.12)$$

The required partial derivatives for first-order approximations are found by differentiating Equation 3.10

$$\sum_{i,j=1}^{n} \frac{\partial G_i}{\partial \dot{x}_j} \frac{\partial \dot{x}_j}{\partial \Delta q_k} + \sum_{i,j=1}^{n} \frac{\partial G_i}{\partial x_j} \frac{\partial x_j}{\partial \Delta q_k} + \frac{\partial G_i}{\partial \Delta q_k} = 0 \quad (3.13)$$

Let

$$u_{j,k} = \frac{\partial x_j}{\partial \Delta q_k} \quad (3.14)$$

and

$$\frac{\partial u_{j,k}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial x_j}{\partial \Delta q_k} = \frac{\partial}{\partial \Delta q_k} \frac{\partial x_j}{\partial t} = \dot{u}_{j,k} \quad (3.15)$$

Equation 3.13 is now written as

$$\sum_{i,j=1}^{n} \left(\frac{\partial F_i}{\partial \dot{x}_j}\right) \dot{u}_{j,k} + \sum_{i,j=1}^{n} \left(\frac{\partial F_i}{\partial x_j}\right) u_{j,k} = -\frac{\partial G_i}{\partial \Delta q_k} \quad (3.16)$$

All derivatives with respect to Δq_k are taken at the point where $\Delta q_k = 0$, so that $\partial F_i/\partial \dot{x}_j$ and $\partial F_i/\partial x_j$ could be substituted for $\partial G_i/\partial \dot{x}_j$ and $\partial G_i/\partial x_j$ in Equation 3.13. Equation 3.16 indicates that the required partial derivatives can be obtained by solving a linear system of differential equations whose homogeneous part is derived from the original equations.

Since for power series development higher order partial derivatives are needed, a generalization of the Equation 3.16 is sought. By the same analytical procedure it can be proved that all higher order partial derivatives take always the same form as the left side of Equation 3.16. The term $\partial G_i/\partial \Delta q_k$ in Equation 3.13 is a function of lower order derivatives, so it can be calculated by recursive procedures. In practical applications, only the first term in the series expansion is taken into consideration, so that higher order partial derivatives are not needed. An explicit expression for the error caused by neglecting the higher (second) order terms in the series expansion is given by Bihovski.[8]

Thus the partial derivatives needed in the power series expansion can be obtained by integrating the system of linear equations regardless whether the original system is linear or nonlinear. This result has been proven apparently independently by Miller and Murray[7] and Bihovski.[9] Miller and Murray also treated the case in which the perturbation term raises the order of the differential equation and demonstrated that this problem can also be reduced to the study of a system of linear equations.

3.4 Error Analysis by Means of the Computer

The basic result of Murray and Miller's work is that the sensitivity coefficients in the case of an analytical dependence of the solution upon a parameter satisfy always a system of linear differential equations. This system is easily obtained by applying Equation 3.16 regardless whether the given equations are linear or nonlinear. Meissinger[10, 11] has shown how the system of the derived linear equations whose solutions are the sensitivity coefficients can be solved by the computer itself so that no analytical tools are needed. This, however, requires that the given and derived differential equations be available in explicit analytical form. There are, however, many cases where this is not the case; rather a given network is to be analyzed. For such situations Bihovski[8] has developed an experimental approach to obtain the sensitivity coefficients by simple measurements without need to express in analytical form the equations governing the system. The theoretical basis of both approaches is again the basic Equation 3.16 which permits the linearization of the problem for small perturbations around a fixed value of the parameter. Bihovski has demonstrated by stressing the physical aspects of this linearization what this actually implies from the point of view of circuit theory. For instance, if nonlinear circuit elements are contained in a given network, the derived network will nevertheless be linear. For transient circuits, this is accomplished by replacing each nonlinear element in the original system by a resistor whose magnitude is varied during the computer run. The resistance-time relationship of this time-varying element is determined by measuring the voltage across and the current through the nonlinear element under normal operating conditions. The corresponding resistor is then varied in a manner that will produce the same voltage-current relationship. Since the superposition theorem is applicable to time-varying but not to nonlinear systems, this transformation permits an experimental determination of sensitivity coefficients by means ordinarily limited to linear systems. Naturally, such an approach will lead to sensitivity coefficients which are applicable only to a specific system excitation.

The method due to Meissinger will be summarized first. By using this method the same analog computer is programmed to solve both the original and the perturbed equations. The method is best explained by application to the error analysis of a simple dynamic system. Consider the linear differential equation

$$\frac{d^2x}{dt^2} + \mu \frac{dx}{dt} + \lambda x = f(t) \qquad x(0) = a \qquad \frac{dx}{dt}(0) = b \tag{3.17}$$

If an error analysis with respect to the parameter λ is desired, partial differentiation yields

$$\frac{\partial^3 x}{\partial \lambda\, \partial t^2} + \mu \frac{\partial^2 x}{\partial \lambda\, \partial t} + \lambda \frac{\partial x}{\partial \lambda} + x = 0 \tag{3.18}$$

Letting $u = \partial x/\partial \lambda$

$$\frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial u}{\partial t} + \lambda u = -x \tag{3.19}$$

Equation 3.17 is identical to Equation 3.19 except for the forcing function. In a similar way, the parameter influence of μ can be determined to be

$$\frac{\partial^2 v}{\partial t^2} + \mu \frac{\partial v}{\partial t} + \lambda v = -\frac{\partial x}{\partial t} \tag{3.20}$$

where

$$v = \frac{\partial x}{\partial \mu}$$

Now, both systems, the perturbed and the original one, are programmed on the computer and the solutions $x(t)$, $u(t)$, or $v(t)$ are observed simul-

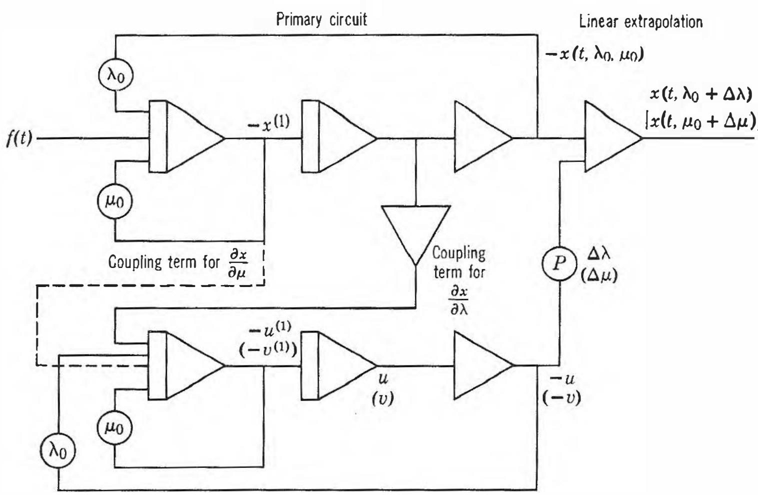

Fig. 3.1 Circuit for obtaining parameter influence by Meissinger's method.

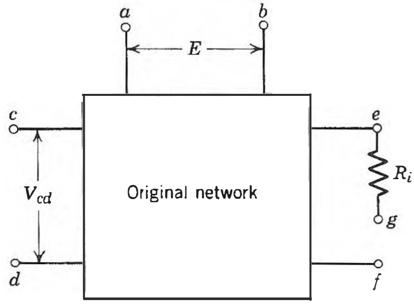

Fig. 3.2 Passive network for Bihovski's method.

taneously. Since the equations for error analysis of μ and λ differ only by the forcing functions, the change from one parameter to the other does not require a basic reprogramming of the computer. In Figure 3.1, the computer setup for this problem is shown. Evidently, the solution of sensitivity equations in this way requires at least double the equipment, or, if the problem is solved sequentially, double the time. There are many instances, however, where such an analysis may be justified.

The second practical approach to error analysis was proposed by Bihovski.[8] This work represents the most complete study of experimental methods for determining the partial derivatives. The theoretical treatment is also based on the sensitivity Equation 3.16. Linear, nonlinear, static, and dynamic systems are analyzed, and experimental measuring methods are proposed to obtain the influence of errors in passive parameters *R*, *L*, *C*, and *M*. A few examples illustrate the general approach.

Consider the passive network shown in Figure 3.2, where E is a d-c voltage source of constant magnitude. The error in voltage at terminals *c–d*, ΔV_{cd}, caused by a resistance variation ΔR_i is to be determined. The sensitivity equation for this case is

$$\Delta V = E\left(\frac{\partial \phi_{ab.cd}}{\partial R_i}\right)_0 \Delta R_i \tag{3.21}$$

It will be shown that the partial derivative $\partial\phi/\partial R_i$ can be determined experimentally in a simple manner. Define the voltage transfer function between terminals *a–b* and *c–d* as

$$\phi_{ab.cd} = \frac{V_{cd}}{V_{ab}}$$

It is easily seen that

$$\phi_{ab.cd} = -\phi_{ab.dc}, \qquad \phi_{ab.cd} = -\phi_{ba.cd}, \qquad \phi_{ab.cd} = \phi_{ba.dc}, \quad \text{etc.} \tag{3.22}$$

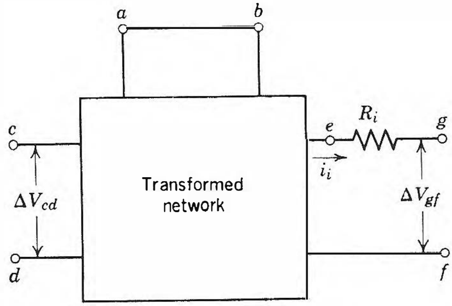

Fig. 3.3 Transformed network corresponding to Figure 3.2.

The transformed network in Figure 3.3 gives the relation between the voltage error ΔV_{cd} due to ΔV_{gf}. The voltage source ΔV_{gf} is an equivalent source resulting from the presence of ΔR_i,

$$\Delta V_{gf} = i_i \Delta R_i \tag{3.23}$$

The insertion of ΔR_i in Figure 3.2 will change the current distribution in the following way:

$$V_{gf} = (i_i + \Delta i_i)\,\Delta R_i \tag{3.24}$$

but the higher order errors will be neglected.

In terms of the defined transfer functions, the error ΔV_{cd} is

$$\Delta V_{cd} = \phi_{gf,cd}\,\Delta V_{gf} \tag{3.25}$$

Inserting Equation 3.23 as well as substituting V_{ef}/R_i for i_i

$$\Delta V_{cd} = \frac{1}{R_i} V_{ef}\phi_{gf,cd}\,\Delta R_i = E\frac{1}{R_i}\frac{V_{ef}}{E}\phi_{gf,cd}\,\Delta R_i \tag{3.26}$$

But

$$\frac{V_{ef}}{E} = \phi_{ab,ef} \tag{3.27}$$

so that

$$\Delta V_{cd} = E\frac{1}{R_i}\phi_{ab,ef}\phi_{gf,cd}\,\Delta R_i \tag{3.28}$$

Comparing this expression with 3.21, the partial derivative is seen to be

$$\left(\frac{\partial\phi_{ab,cd}}{\partial R_i}\right)_0 = \frac{1}{R_i}\phi_{ab,ef}\phi_{gf,cd} \tag{3.29}$$

Thus, instead of employing analytical means to calculate the partial derivatives, two transfer functions have to be experimentally determined and multiplied. The required transfer functions are found by dividing the corresponding voltages in the original and the transformed network (Figures 3.2 and 3.3). The original network with the error source short-circuited gives the effect of voltage E, whereas the transformed network with E and all other voltage sources short-circuited gives the effect of ΔR_i; the results of the two separate measurements are then added. This process makes it possible to measure the effect of ΔV_{gf} using the full voltage range so that errors in measurement are of no great influence.

The same technique can be used for an electrical network in a transient state. Here, two possibilities exist. One is to use the Laplace transform and reduce the problem to that of steady state. However, calculations are then needed to pass from the complex-frequency domain to the time domain. The second way is to determine purely experimentally the partial derivatives, which now are time-dependent variables. By using the approach of original and transformed networks, the Laplace transformed equation of sensitivity coefficients (partial derivatives) can be derived without manipulating the differential equations of the original system. Instead, d-c original and transformed networks are used as before, but all the circuit elements, voltages, and parameters are replaced by corresponding Laplace transformed expressions. The procedure is then reduced to the one used for steady state. Evidently the transfer functions are now complex-frequency variables and cannot be measured, but they can be calculated in the same way as voltage ratios at two specified network terminal pairs.

As an example, consider the effect of additional conductivity G in a network in the transient state. This perturbation could be the result of leakage resistance in a capacitor. The given network is shown in Figure 3.4*a*. Between terminals e, f an additional conductivity ΔG is introduced.

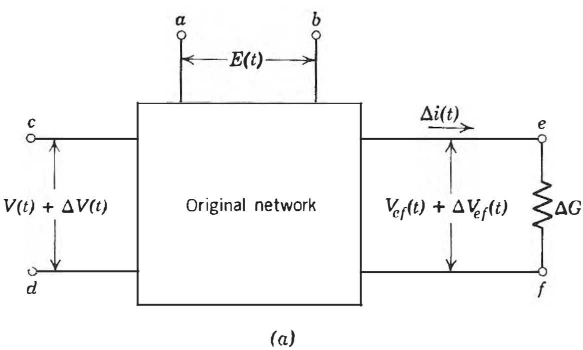

Fig. 3.4 (*a*) Original network for evaluation of effect of leakage conductance.

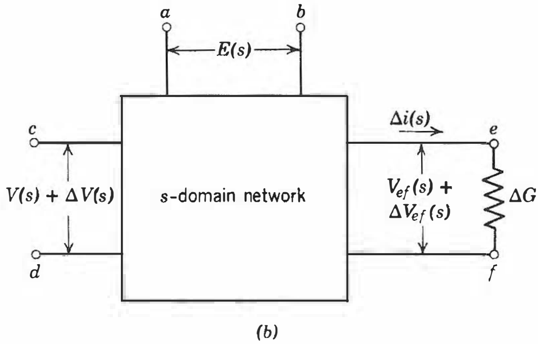

Fig. 3.4 (*b*) *s*-domain behavior.

The Laplace transform behavior of this network is indicated in Figure 3.4*b*. The current change due to ΔG is

$$\Delta i(s) = [V_{ef}(s) + \Delta V_{ef}(s)]\,\Delta G \tag{3.30}$$

or neglecting higher order errors

$$\Delta i(s) = V_{ef}(s)\,\Delta G \tag{3.31}$$

As in steady-state analysis, the effects of voltage $E(s)$ and ΔG will be considered by superposition of two networks, the original and transformed one as shown in Figures 3.5 and 3.6. The effect of ΔG is equivalent to introducing a voltage source

$$\Delta V_i(s) = \frac{\Delta i(s)}{G_i(s)} \tag{3.32}$$

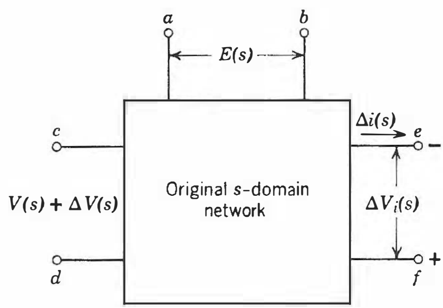

Fig. 3.5 Original network.

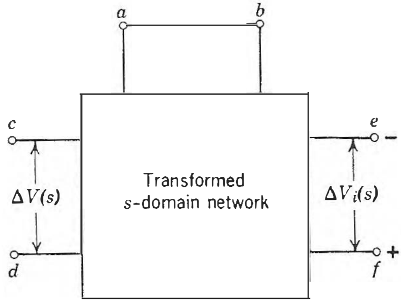

Fig. 3.6 Transformed network.

The desired voltage relation is given as

$$\Delta V(s) = \phi_{ef,cd}(s)\,\Delta V_i(s) \tag{3.33}$$

By replacing $\Delta V_i(s)$ and $\Delta i(s)$ in this expression

$$\Delta V(s) = \frac{1}{G_i(s)} V_{ef}(s)\phi_{ef,cd}(s)\,\Delta G \tag{3.34}$$

Comparing this with the first-order power series expansion as in the steady state, the partial derivative is

$$\left[\frac{\partial V(s)}{\partial \Delta G}\right]_0 = \frac{1}{G_i(s)} E(s)\phi_{ab,ef}(s)\phi_{ef,cd}(s) \tag{3.35}$$

The voltage $V_{ef}(s)$ has been replaced by

$$V_{ef}(s) = E(s)\phi_{ab,ef}(s) \tag{3.36}$$

which applies for zero initial conditions. If this is not so, then

$$\left[\frac{\partial V(s)}{\partial \Delta G}\right]_0 = -\frac{1}{G_i(s)} V_{ef}(s)\phi_{ef,cd}(s) \tag{3.37}$$

where initial condition must be taken into account to obtain $V_{ef}(s)$. Thus, in expressing the partial derivative there is no need to deal with the original and perturbed differential equations. All initial conditions in the given system can be taken as equivalent d-c voltage sources in series with capacitors, resistors, or other elements of the network. The transfer functions of the type $\phi_{ef,cd}(s)$ in the complex-frequency domain are obtained as complex voltage relations at specific network terminals. Naturally, there remains the calculation of the inverse Laplace transform by standard analytical methods.

As a more concrete example, consider a conventional operational amplifier and the analysis by this method of the effect of leakage conductance in parallel with the feedback capacitor. Figure 3.7*a* represents the operational amplifier with ΔG; Figure 3.7*b* represents its Laplace transform network; Figure 3.7*c* is the transformed network for the study of the effect of ΔG. Initial conditions are taken to be zero. By applying Equation 3.37,

$$\left[\frac{\partial e_o(s)}{\partial\,\Delta G}\right]_0 = -\frac{1}{G_i(s)}\,V_{ef}(s)\phi_{ef,cd}(s) \tag{3.38}$$

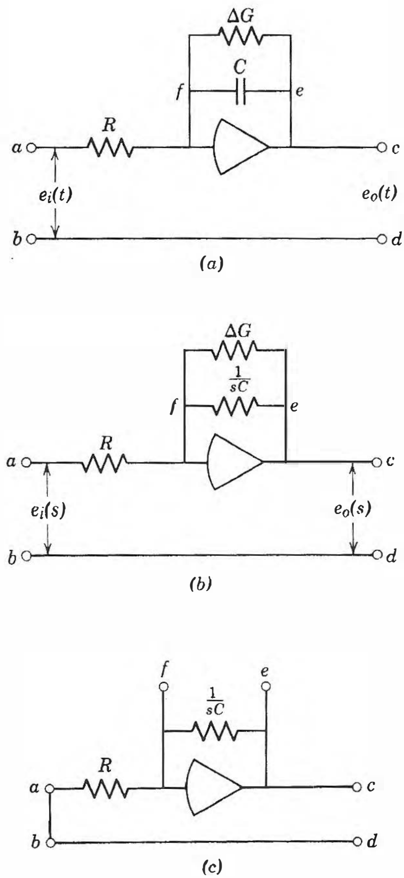

Fig. 3.7 (*a*) Integrator with leakage conductance. (*b*) *s*-domain network. (*c*) Transformed network.

$V_{ef}(s) = e_o(s)$, since the voltage between the input grid and ground is approximately zero. Furthermore

$$G_i = sC \qquad \phi_{ef.cd}(s) = 1 \tag{3.39}$$

Thus

$$\left[\frac{\partial e_o(s)}{\partial \Delta G}\right]_0 = -\frac{1}{sC} e_o(s) \tag{3.40}$$

The inverse transform is

$$\left[\frac{\partial e_o(t)}{\partial \Delta G}\right] = -\frac{1}{C}\int_0^t e_o(t)\, dt \tag{3.41}$$

The effect of ΔG on the output voltage is

$$\Delta e_o(t) = -\frac{\Delta G}{C}\int_0^t e_o(t)\, dt \tag{3.42}$$

or in more conventional terms

$$\Delta e_o(t) = -(\Delta GR)\frac{1}{RC}\int_0^t e_o(t)\, dt \tag{3.43}$$

The sensitivity coefficients can also be determined using purely experimental methods. The general expression for the partial derivatives in the complex-frequency domain has the form

$$\frac{\partial v}{\partial \Delta q}(s) = E(s)\frac{1}{q(s)}\phi_1(s)\phi(s)_2 f(s) \tag{3.44}$$

This expression is obtained by studying the parameter influences of ΔR, ΔL, ΔM, or ΔG. Equation 3.35 is but a special case of 3.44.

For the experimental determination of $\partial v/\partial \Delta q(t)$ it is now necessary to construct three networks: the original network with the transfer function $\phi_1(s)$, the transformed network with transfer function $\phi_2(s)$, and a third network with the transfer characteristic $f(s)$. These networks are to be connected in cascade, and $E(t)$ is to be applied at the appropriate terminals. The partial derivative can then be observed or recorded at corresponding terminals. The networks must be separated by isolating or buffering circuits so that loading effects are eliminated. An example of this experimental method for finding time-varying sensitivity coefficients is shown in Figure 3.8. Figure 3.8*a* shows the original system with an error source ΔL as a result of series inductance in the resistor R_i. The experimental system in Figure 3.8*b* is seen to contain three sections corresponding respectively to the three terms in the Laplace transform expression of the sensitivity coefficient

$$\frac{\partial V(s)}{\partial \Delta L} = E(s)\phi_{ab,\,ef}\phi_{gf,\,cd}\frac{s}{R_i} \tag{3.45}$$

referring to the development for the static system.

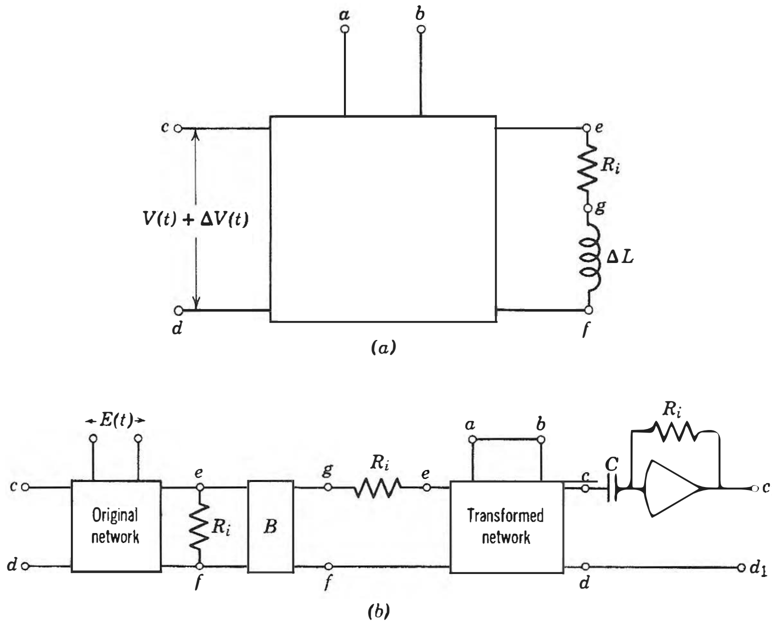

Fig. 3.8 (*a*) Original network for obtaining transient sensitivity coefficients. (*b*) Cascaded networks.

The first section corresponds to the circuits of Figure 3.2, the second to the circuit of Figure 3.3, whereas the third section is a differentiator to introduce the term s/R_i. The capacitor C is selected in accordance with the scale factor desired for the solution. The transient measured at terminals c_1–d_1 then represents the dynamic sensitivity coefficient. Terminals c–d are left open. It appears that Bihovski's method permits the use of the given and the transformed network in all cases in order to obtain sensitivity coefficients. Thus analytical procedures can be consistently avoided and replaced by simple experimental methods.

Bihovski has also developed a combined experimental-analytical method for obtaining the partial derivatives, based on the use of only one network, so that a saving of equipment can be obtained by using additional calculations. The method is based on the observation that Equation 3.44 in the time domain can be expressed in the form of a convolution integral of weighting functions,

$$\frac{\partial V}{\partial\,\Delta q_i}(t) = \mathscr{L}^{-1}[\phi_1(s)\phi_2(s)\ldots\phi_n(s)] = h_1(t)*h_2(t)\ldots*h_n(t) \qquad (3.46)$$

where $h(t)$ is the impulse response of the system and the asterisks designate the operation of convolution. If the network whose error analysis is being studied is available, an impulse function h can be applied at the input and $h_1(t)$ recorded. The same network is then slightly modified to obtain the transformed network with respect to the desired parameter change and $h_2(t)$ is recorded. Now

$$\frac{\partial V}{\partial \Delta q_i}(t) = \int_0^t h_1(t - \tau)h_2(\tau)\, d\tau \tag{3.47}$$

has to be calculated. This can be done either analytically or with the aid of a computer. A repetitive computer can very rapidly and easily generate the convolution integral so that it may be employed for its own error analysis or for the analysis of one-shot analog installations.

Bihovski has also considered the effect of parameter errors in a network from the point of view of probability theory. Since in certain cases it is reasonable to assume that the values of the actual elements in the circuit deviate from the specified values according to a specific probability distribution, it should be possible to specify mathematically the behavior of the over-all circuit and the error distribution within the network as a function of the error distribution of the specific elements. Assuming that the error distribution has the shape of Gaussian error function within the specified limits, a voltage excitation produces

$$\begin{aligned} d_v &= \sum_{i=1}^{n} \left(\frac{\delta\phi}{\delta\, \Delta q_i}\right) d_i \\ D_v^2 &= E^2 \sum_{i=1}^{n} \left(\frac{\delta\phi}{\delta\, \Delta q_i}\right)^2 D_i^2 \end{aligned} \tag{3.48}$$

where d and D are respectively the mathematical probability and the standard deviation squared. This method, too, predicates a knowledge of the partial derivatives.

3.5 Error Analysis of Linear Systems

The general theoretical methods of sensitivity analysis outlined can, of course, be applied to linear problems without difficulty. In the case of linear systems, however, certain approaches that are not applicable to nonlinear systems are possible. In particular, the frequency limitations of the linear computing elements can be considered explicitly. This approach makes use of the transfer functions of the linear computing elements and of closed loop systems in general. In linear systems the application of inputs containing frequency components within certain definite limits will result in solutions containing errors which are functionally related to the known transfer characteristics of the computing elements. Conversely, given the admissible errors in the solution and the transfer function of the computing elements, the range of frequencies which can be handled can be calculated. Thus a class of allowable inputs can be obtained for a specified error in the solution and specified design parameters of the computer units.

The application of error analysis to linear systems is straightforward. Using Laplace transforms, the analysis of the linear system is performed in the complex-frequency domain instead of the time domain, so that the analysis is reduced to the study of algebraic equations rather than differential equations. Since the parameters of the transfer function of the linear computing elements appear as coefficients in the Laplace transform of the equations solved by the machine, their effect on the solution can be studied by sensitivity methods. Consider an mth order linear differential equation

$$\sum_{i=0}^{m} a_i \frac{d^i y}{dt^i} = 0 \tag{3.49}$$

The characteristic equation of Equation 3.49 is expressed as

$$\sum_{i=0}^{m} a_i s^i = 0 \tag{3.50}$$

or

$$f(s) = 0 \tag{3.51}$$

If the coefficients of Equation 3.50 are perturbed by Δa_i, then

$$\sum_{i=0}^{m} (a_i + \Delta a_i) s^i = 0 \tag{3.52}$$

and the corresponding form of Equation 3.51 becomes

$$f(s) + p(s) = 0 \tag{3.53}$$

Designate the roots of Equations 3.51 and 3.53 by β_i and β_i' respectively, so that

$$\beta_i' = \beta_i + \eta_i \qquad i = 1, 2, \ldots, m \tag{3.54}$$

where η_i is a measure of the shift of the roots of Equation 3.51 as a result of the perturbations. If $|\eta_i|$ is small, Equation 3.53 can be approximated by

$$f(\beta_i) + \eta f^{(1)}(\beta_i) + p(\beta_i) + \eta p^{(1)}(\beta_i) = 0 \tag{3.55}$$

Since

$$f(\beta_i) = 0$$

the general expression for shift in the roots in the given and machine equations will be

$$\eta_i = -\frac{p(\beta_i)}{f^{(1)}(\beta_i) + p^{(1)}(\beta_i)} \tag{3.56}$$

Since $f(\beta_i)$ is given, in order to calculate $p(\beta_i)$ the frequency response of the operational amplifiers used in the computer setup must be known. Then the shifts in roots introduced by finite bandwidths of the machine components can be determined by Equation 3.56. This is the basic approach used in error analysis of linear systems. Various authors have obtained different expressions for η_i by starting with different analog representations and approximations of the frequency characteristics of the operational amplifiers.

In the error analysis of linear systems by Macnee,[12] the following assumption regarding the integrator transfer characteristic $g_1(s)$ was made:

$$g_1(s) = -\frac{a_{i-1}}{a_i}\frac{T_0}{1 + sT_0}\frac{1}{1 + sT_1} \tag{3.57}$$

where T_0 is the integrator low-frequency time constant; and T_1 is the integrator high-frequency time constant. The adder transfer characteristic was taken as

$$g_2(s) = \frac{1}{1 + sT_2} \tag{3.58}$$

where T_2 is the adder high-frequency time constant. As to the coefficients, the following restrictions were assumed:

$$\begin{matrix} a_{2i} > 0 \\ a_{2i+1} < 0 \end{matrix} \qquad \frac{a_{i-1}}{a_i} \leq 1 \qquad (RC)_i = 1 \tag{3.59}$$

so that the analog representation of the Equation 3.49 takes the form shown in Figure 3.9.

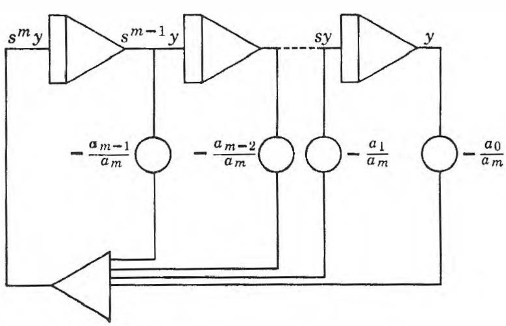

Fig. 3.9 Macnee's[12] network.

The coefficient setting potentiometers were introduced to represent a_{i-1}/a_i. Where these quotients are negative, additional sign changes are, of course, required. It is now possible to find $p(\beta_i)$ in Equation 3.56 by determining the transfer function of the network in Figure 3.9 for the assumed transfer functions of the individual computing elements.

The same basic Equation 3.56 was used by Raymond[13] for the error analysis of Equation 3.49. The starting assumptions in his case were

$$\begin{aligned} a_{2i} &> 0 \\ a_{2i+1} &< 0 \\ g_1(s) &= -\frac{1}{RCs + \dfrac{(1 + RCs)(1 + sT_1)}{g_0}} \end{aligned} \tag{3.60}$$

where g_0 is the gain of the amplifier at zero frequency. Note that $RC \neq 1$. The corresponding circuit is shown in Figure 3.10. The adder transfer function was again taken as $g_2(s)$.

Unlike Macnee and Raymond, Marsocci[14] did not assume any constraints on the coefficients of Equation 3.49. His circuit is shown in Figure 3.11, using

$$\begin{aligned} g_1(s) &= -\frac{1}{s^2T_1 + s\left(1 + \dfrac{T_1}{T_0}\right) + \dfrac{1}{T_0}} \\ g_2(s) &= -\frac{1}{1 + sT_2} \end{aligned} \tag{3.61}$$

Marsocci's expression for η_i gave accurate results when applied to constant coefficient second-order differential equations.

The most general computer setup of Equation 3.49 as well as the most refined transfer functions of the linear computing elements for the calculation of $p(\beta_i)$ were used by Miura and Nayata.[15] This analysis is especially important for repetitive analog computers since it takes into account

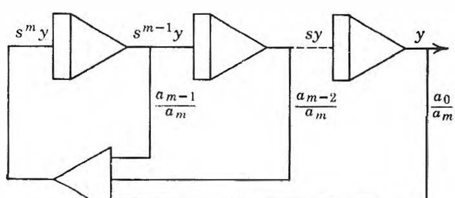

Fig. 3.10 Raymond's network.

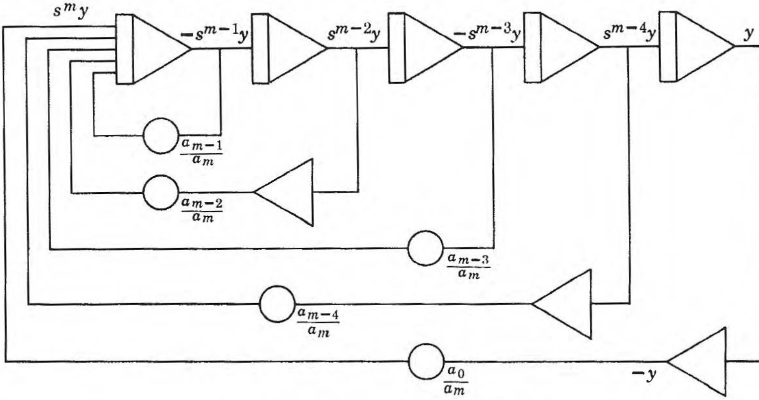

Fig. 3.11 Marsocci's[14] network.

stray capacitances of operational amplifiers as well as the frequency limitations of coefficient-setting elements, which cannot be neglected in high-speed operation. The analog network representation of Equation 3.49 which serves for the calculation of η_i is shown in Figure 3.12. The generalized transfer functions $k_i(s)$ refer to potentiometers and constant coefficient multipliers ($g > 1$) in parallel, including the sign of the integrator.

Another approach was taken by Dow.[16] He established the analytical relation between the transfer functions of the operational amplifiers and equivalent perturbations in the coefficients a_i of Equation 3.49. This opens the possibility of measuring the effects of the equivalent coefficient perturbations on the solution of the given equation. Fuchs[17] developed

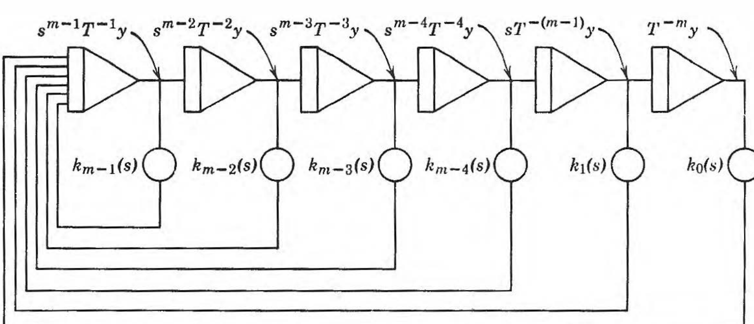

Fig. 3.12 Miura and Nayata's[15] network.

a method for deriving the shift in roots (η_i) given the perturbations Δa_i of the coefficients a_i of Equation 3.49. An attempt was made by Natan[18] to analyze the accuracy of linear analog computers by dividing the inputs into two typical groups: one having wide band power spectra and the other pronounced peaks (typical of sinusoidal waveforms).

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chapter 4
SCALE FACTORS

4.1 General Remarks

In Chapter 2 a number of techniques are described for specifying the manner of interconnecting computing units, as determined by the equation to be solved. Once a satisfactory computer circuit has been synthesized, it becomes necessary to establish a correspondence between voltages at various points in the computer circuit and the dependent variables of the problem, as well as between computer time and the independent variable of the problem. The constants of proportionality which relate the computer variables with the variables of the problem being solved are termed *scale factors*. The selection of optimum scale factors in differential analyzers is important for several reasons.

A judicious choice of amplitude scale factors obviates the possibility of overloading any of the computing elements, a condition which would make the solution completely erroneous. At the same time, the scale factors should be selected so that as much of the permissible voltage range as possible is utilized in the course of a solution. In this way the percentage errors resulting from noise and other random sources of error within the computer are minimized. The voltage ranges most frequently available in differential analyzers are ± 100 volts and ± 50 volts.

In repetitive differential analyzers the time base also has sharply defined upper and lower limits. Too rapid a solution rate introduces errors because of the bandwidth limitations of the computing elements. At the same time the repetition rate fixes a maximum computing interval, generally of the order of several milliseconds, which may be utilized for solution. Additional limitations are introduced by the drift behavior of the operational amplifiers.

In this chapter the establishment of the computer time base—determined by the repetition rate and the available computer elements—is considered. This is followed by the description of two methods of scaling, a direct method and a method involving auxiliary equations. A discussion of the effects of shortcomings of the computer elements, particularly drift in operational amplifiers, on the scaling process concludes the chapter. An attempt is made to treat the problem of scale factoring in a systematic but nonrigorous manner; emphasis is placed upon general principles rather than upon "cook book" procedures.

4.2 Determination of the Time Base

Unlike one-shot computers, repetitive computers almost never operate in real time. Accordingly it is convenient to define at the outset relative units (machine variables) for both the dependent variable (voltage) and the independent variable (time). Most conveniently the base, or significance, of one unit of these variables as measured on suitably calibrated output equipment corresponds to their maximum permissible excursions. With voltage this value is fixed by the characteristic of the operational amplifier and is usually ± 50 volts or ± 100 volts. The base of the independent variable is limited by the repetition rate and can be established in a way that is independent of the repetition rate as follows:

If higher order errors are neglected, the output voltage of an integrator is

$$e = -\frac{1}{RC}\int_0^t f(t)\,dt + k \tag{4.1}$$

For the special case where $f(t)$ has a constant amplitude of one relative unit (equal in magnitude to the dynamic voltage range of the amplifier), and zero initial conditions

$$e = -\frac{1}{RC}\int_0^t dt = -\frac{t}{RC} \tag{4.2}$$

Therefore, e will be equal to one relative unit at time $t = RC$. Letting

$$\frac{t}{RC} = \tau \tag{4.3}$$

where τ is the relative independent variable on the computer, Equation 4.1 becomes

$$e = -\int_0^\tau d\tau = -\tau \tag{4.4}$$

Here the relative coefficient of proportionality of the integrator is regarded as being equal to unity.

With the aid of Equation 4.4 it is easy to determine the base of the independent variable on the computer. In the following discussion it is assumed that a system is available for measuring instantaneous voltages during the repetitive cycle as described in Section 7.3. With the aid of

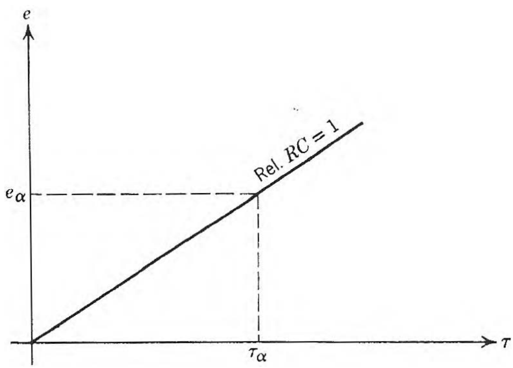


Fig. 4.1 Establishment of relative time base.

such a measuring device the instant of time at which the magnitude of the ramp voltage at the output equals the negative of the applied input voltage is determined. This is indicated in Figure 4.1. The subscript α identifies a specific integrator taken as a reference. Thus if the relative RC is known, the time base is determined. If the time base is known so that the measuring unit has been calibrated, RC is measured. Provided the integrator time constant RC is known accurately, it can be used to determine the relation between the computer time variable τ and real time t.

A second method for establishing the relation between τ and t involves the calibration of the oscilloscope screen by a series of impulses. These impulses must have an exact spacing controlled by a source of constant frequency. Such a calibration system is shown schematically in Figure 4.2.

It is possible to use a differential analyzer without first establishing the exact relation between the real-time variable and the independent variable

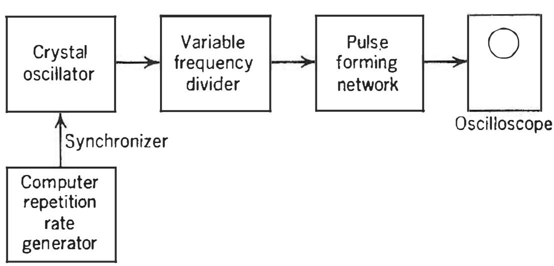


Fig. 4.2 Block diagram of time-calibration pulse generator.

on the computer. The measurement shown in Figure 4.1 yields the quotient

$$\frac{e_\alpha}{\tau_\alpha} = \frac{e_\alpha}{\text{Rel } RC_\alpha \tau_\alpha} = 1 \tag{4.5}$$

The time constant of the integrator used for this measurement is arbitrarily taken as unity. By similar measurements the relative time constants of the other integrators can then be obtained. The use of Equation 4.5 for the determination of the time constant of the integrator usually entails some difficulties. As a result of the presence of zero-offset and of drift, different values for $e_\alpha = +1$ and $e_\alpha = -1$ are obtained, even though their absolute values may be identical. As a rule it is preferable to work with the expression $(e_2 - e_1)/(\tau_2 - \tau_1)$. Good results can then generally be obtained by measuring the intersection of the lines

$$\begin{aligned} e_y &= -\tau + \tfrac{1}{2} \\ e_y &= +\tau - \tfrac{1}{2} \end{aligned} \tag{4.6}$$

This is shown graphically in Figure 4.3, and the corresponding computer schematic is shown in Figure 4.4. The time constant of the integrator can then be expressed as

$$RC = \frac{|y_1|}{\tau_1 - \tau_0} = 1 \tag{4.7}$$

The time constant of the integrator can be controlled in this way with sufficient precision for repetitive differential analyzers. In view of the fact that the system for calibrating time on the oscilloscope can be con-

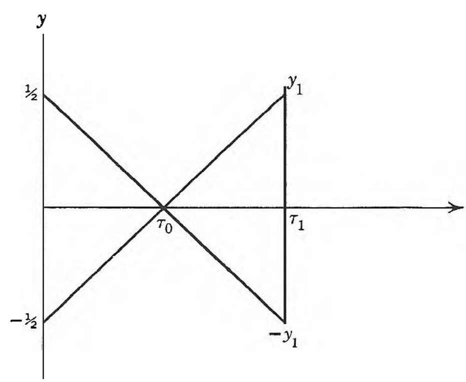


Fig. 4.3 Measurement of *RC* by the intersection of two straight lines.

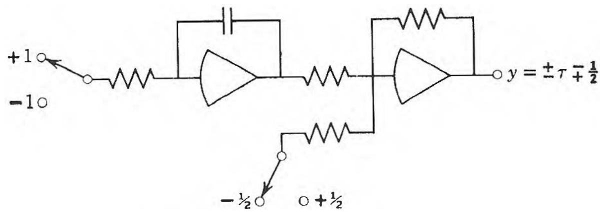


Fig. 4.4 Circuit instrumenting Equation 4.6.

structed with greater precision than the apparatus for measuring instantaneous voltage values, the best solution is to combine the two methods. In this way the more precise system can be used from time to time to calibrate the less precise system.

4.3 Direct Method

Although the excursions of the dependent and independent variables in analytic solutions are generally not limited, computer solutions must remain in a domain bounded by

$$0 \leq \tau \leq \tau_{\max}$$
$$-1 \leq y \leq 1$$

This domain is designated as D_m. The fundamental problem in selecting scale factors is to transpose the domain in which the actual solution is found into D_m. This transposition should be such that a maximum of D_m is utilized. Thus in order to obtain satisfactory solutions it is necessary not only to know the extent of the domain D_m, but it is also necessary to predict the excursions of the problem variables.

The range of the independent variable is generally specified in formulating the problem; the dependent variables and their derivatives are generally not known in advance, however. It is usually impossible, therefore, to specify optimum scale factors at the outset, and a series of iterative adjustments may be necessary. Several cases for which the scale factor problem can be solved directly will first be presented.

The fundamental equations defining the scale factors k_t and k_y are

$$\begin{aligned} t &= k_t\tau \\ y &= k_y Y \end{aligned} \tag{4.8}$$

from which

$$\begin{aligned} \frac{d}{dt} &= \frac{d}{k_t\,d\tau} \\ s &= k_t^{-1}S \end{aligned} \tag{4.9}$$

where τ and the capital letters refer to machine variables. The preceding equations imply that if $k_t > 1$, the time scale is compressed, and then S is divided by the scale factor. When $k_t < 1$, the time scale is spread out and S is multiplied by the scale factor.

Consider now a system of linear differential equations with constant or time-varying coefficients

$$a_n(t)s^n y + a_{n-1}(t)s^{n-1}y + \ldots + a_0 y = 0 \tag{4.10}$$

By substituting in accordance with Equations 4.8 and 4.9,

$$a_n(k_t\tau)k_y k_t^{-n}S^n Y + a_{n-1}(k_t\tau)k_t^{-(n-1)}k_y S^{n-1}Y + \cdots + a_0 k_y Y = 0 \tag{4.11}$$

or dividing by k_y

$$a_n(k_t\tau)k_t^{-n}S^n Y + a_{n-1}(k_t\tau)k_t^{-(n-1)}S^{n-1}Y + \cdots + a_0 Y = 0 \tag{4.12}$$

It appears that the equation remains unchanged when the scale of the dependent variable is changed. The magnitude of the computer dependent variable is determined entirely by the initial conditions that are applied to the system. Thus in this case the maximum available domain D_m can be utilized by adjusting the magnitude of the voltages which energize the potentiometers used to control the initial conditions of the integrators. This makes it possible to optimize the amplitude scale factors by manipulating a single potentiometer. This technique is demonstrated in Figure 4.5. All potentiometers serving to apply initial conditions are energized by the outputs of the two operational amplifiers. A single adjustment of potentiometer P changes simultaneously all the initial conditions, and controls therefore the excursion of the solution. By observing the solution on the oscilloscope it is easy to arrive at an optimum setting of potentiometer P.

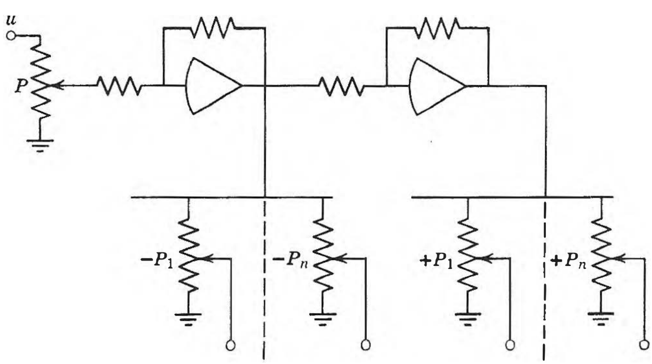


Fig. 4.5 Circuit for scaling initial conditions.

If the equations governing the linear system are presented in matrix form, the transformations involved in scale factoring are readily performed. Consider the system

$$
\begin{array}{c|cccc|l}
 & x_1 & & & x_n \times k_2 & \\
\hline
k_1: & b_{11}s & a_{12} & \cdots & a_{1n} & \\
\hdashline
 & a_{n1} & a_{n2} & \cdots & b_{nn}s & \\
\hline
 & x_1(0) & & & x_n(0) & \searrow k_3
\end{array}
$$

In order to limit all coefficients to

$$-1 \leq a_{nk} \leq 1 \tag{4.13}$$

the rows may be divided by a convenient factor k_1 without affecting the behavior of the system. The multiplication of a column by the factor k_2 changes the scale of those unknowns according to

$$x_n' = k_2 x_n \tag{4.14}$$

and their initial conditions according to

$$x_n'(0) = k_2 x_n(0) \tag{4.15}$$

The multiplication of the main diagonal by the factor k_3 affects all variables as determined by Equation 4.9.

4.4 Method of Auxiliary Equations

Another approach to the transposition from the domain of the mathematical problem to that of the machine solution follows from a consideration of computing element behavior. All computing elements, whether linear or not, have characteristic factors of proportionality. For the adder the proportionality factor is

$$k_a = \frac{R_f}{R} \tag{4.16}$$

where R_f is the feedback resistor and R is the input resistor. For practical reasons k_a is rarely larger than ten for an amplifier without drift stabilization. If there are a number of input resistors

$$k_{an} = \frac{R_f}{R_n} \qquad n = 1, 2, \ldots, p \tag{4.17}$$

For an integrator, the proportionality factor is

$$k_i = \frac{1}{RC} \tag{4.18}$$

For multiple inputs,

$$k_{\text{in}} = \frac{1}{R_n C} \tag{4.19}$$

The permissible ranges of the integrator time constants depend on the noise level within the system and on the drift of the amplifier.

The proportionality factor of nonlinear function generators has the form

$$y = k_f f[k_g g(t)] \tag{4.20}$$

In this case the selection of the proportionality factors depends upon the nature of the specified function. The adjustment and control of the scale factors of function generators are generally simple if a suitable measuring system is available. A ramp voltage $y = \tau$ is applied to the input of the function generator by the integrator whose time constant has been designated as unity. When y reaches unity, the output voltage of the function generator $e_\alpha = f(1)$ is measured. The proportionality factors are then

$$\begin{aligned} k_g &= 1 \\ k_f &= \frac{e_\alpha}{u_\alpha} \end{aligned} \tag{4.21}$$

where u_α is the permissible voltage range. For multipliers

$$y = k_m v(t) g(t) \tag{4.22}$$

In this case the factor k_m is ordinarily taken as

$$k_m = \frac{1}{\max |y|} \tag{4.23}$$

so that

$$k_m \cdot \max |y| = 1 \tag{4.24}$$

in order to avoid overloading any of the computer elements. For example, for a voltage range $u_\alpha = 100$ volts, k_m becomes 0.01.

For a divider

$$y = k_d \frac{y(t)}{g(t)} \tag{4.25}$$

For similar reasons as above, the proportionality factor is generally taken as

$$k_d \cdot \max |y| = k_d \cdot \frac{\max |v(t)|}{\min |g(t)|} = 1 \tag{4.26}$$

If the same voltage range is used and if

$$0 \leq v(t) \leq 100 \text{ volts}$$
$$10 \leq g(t) \leq 100 \text{ volts}$$

k_d is taken as 10.

The method to be presented requires that a proportionality factor be specified for each input of each computer element. The equation governing the computer circuit is then expressed in terms of these factors. Consider as a simple example the differential equation

$$\frac{d^3y}{dt^3} = a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} - a_0 y + a_f f(t) \tag{4.27}$$

The pertinent analog schematic including all factors of proportionality is shown in Figure 4.6. The independent variable is now scaled in accordance with Equation 4.8 so that

$$\frac{d^3Y}{d\tau^3} = k_{41}\, k_{14}\, k_t \frac{d^2Y}{d\tau^2} + k_{13}\, k_{21}\, k_t^2 \frac{dY}{d\tau} - k_{12}\, k_{21}\, k_{31}\, k_t^3\, Y + \frac{k_{11}\, k_{21}\, k_{31}\, k_f\, k_t^3}{k_y} f[k_g(k_t \cdot \tau)] \tag{4.28}$$

A comparison of 4.27 and 4.28 leads to the expression of the coefficients of the given differential equation in terms of the proportionality and scale factors

$$\begin{aligned} a_2 &= k_{41} \cdot k_{14} \cdot k_t \\ a_1 &= k_{13} \cdot k_{21} \cdot k_t^2 \\ a_0 &= k_{12} \cdot k_{21} \cdot k_{31} \cdot k_t^3 \\ a_f &= \frac{k_{11} \cdot k_{21} \cdot k_{31} \cdot k_f \cdot k_t^3}{k_y} \end{aligned} \tag{4.29}$$

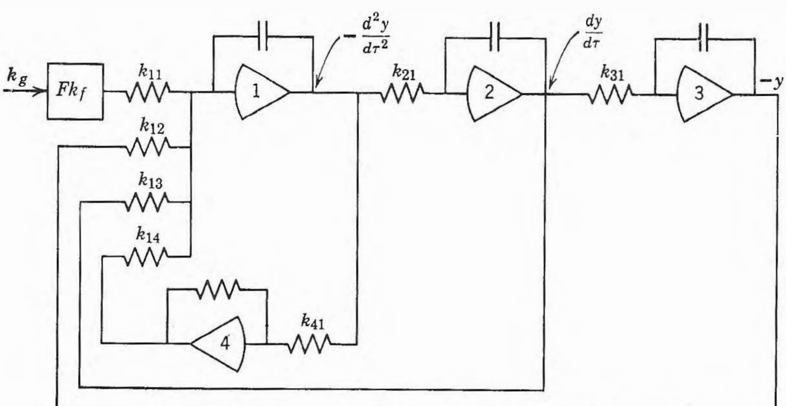


Fig. 4.6 Circuit for Equation 4.27.

There are more unknowns than there are equations. It is therefore necessary to formulate complementary equations to complete the system of Equation 4.29. These equations will depend upon the specific problem and could be, for example,

$$\begin{aligned} k_{11} &= k_{21} = k_{31} = 1 \\ k_g &= 2 \end{aligned} \tag{4.30}$$

4.5 Interrelation of Scale Factors

One limitation of the analog method results directly from imperfections of the computer elements, particularly their limited bandwidth. Another more subtle limitation is due to restrictions which are imposed on the scaling process by certain other shortcomings of these elements. Thus it may be impossible to find sufficiently accurate solutions for certain equations even though the computer schematic appears to be satisfactory.

Consider the differential equation

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = 0 \tag{4.31}$$

Of primary importance in scaling are the relationships

$$d_a = \frac{\max |a_i|}{\min |a_i|} \tag{4.32}$$

Scaling is generally easy if d_a is in the vicinity of unity. When $\min |a_i|$ becomes smaller than the limit of precision of the differential analyzer, say 1%, this expression becomes meaningless because error sources within the system will overshadow the solution. For a 1% accuracy, d_a should be smaller than 100. For large d_a it is often desirable to neglect completely those terms having coefficients smaller than the lower limit.

A second limitation relates to the choice of the independent variable. As stated previously, noise and drift limit the choice of the factor of proportionality of linear computing elements to approximately 10. In other words,

$$d_x = \frac{\max |x|}{\min |\tau|} \leq 10 \tag{4.33}$$

This applies to independently operating elements, that is, to the simulation of differential equations of the first order. In Section 4.6 it is demonstrated in detail that in differential analyzers, the choice of d_x is determined by the structure of the given equations. For repetitive differential analyzers this limitation can be expressed by the empiric relationship

$$\begin{aligned} (d_x)n &\approx \text{const.} \\ d_x &> 1 \end{aligned} \tag{4.34}$$

where n is the order of the differential equation. Equation 4.34 indicates that the choice of d_x is quite restricted.

In practice it is often necessary to effect the transformations d_a and d_x simultaneously. This can lead to difficulties because these transformations are interrelated by the coefficients of the differential equation. Assume, for example, that Equation 4.31 is given in the form

$$1000\frac{d^3y}{dt^3} - 150\frac{d^2y}{dt^2} + 30\frac{dy}{dt} - y = f(t) \tag{4.35}$$

This equation is inconvenient to program because

$$d_a = 1000 \tag{4.36}$$

The transformation

$$t = 10\tau \tag{4.37}$$

leads to

$$\frac{d^3y}{d\tau^3} + 1.5\frac{d^2y}{d\tau^2} + 3\frac{dy}{d\tau} - y = f(10\tau) \tag{4.38}$$

where $d_a < 10$. This demonstrates that it is more convenient to program on an analyzer a system whose coefficients satisfy the conditions

$$a_i \approx k^i$$

$$d_a < 100$$

where k are positive numbers and i are integers. It may be further necessary, after making the above transformation, to compress or spread a solution on the machine. For example, if a scale factor

$$\tau' = 8\tau \tag{4.39}$$

is desired, the Equation 4.38 again becomes inconvenient to program. An analogous situation arises in systems in which the original d_a is of convenient magnitude, say less than 10, and it is desired to make d_x much larger than unity. It then becomes necessary to seek other means of solution, such as modifying the computing time-interval or the obtaining of partial stepwise solutions.

4.6 Effect of Drift upon Scale Factoring

The drift problem in repetitive computers appears in a somewhat disguised manner. However, drift and zero offset of computing elements place a definite limit upon the range of the independent variable in all types of analog computers. These limitations are best expressed in relative units of the integration range. Consider the integrator equation for a step input of magnitude k

$$e_o = \frac{1}{RC} \int_0^t k\, dt \tag{4.40}$$

assuming that no errors are introduced by limited amplifier gain and bandwidth. Drift and noise will always make a given k equal to $k + e_d$, so that the term

$$\frac{1}{RC} \int_0^{t=tm} e_d\, dt = \frac{1}{RC} e_d t_m$$

may become significant for $t_m \gg 1$. A very convenient and general way to express the quality of an integrator in one-shot operation is to fix the upper limit of the product

$$\epsilon \leq \frac{1}{RC} e_d t_m \tag{4.41}$$

in order that the percentage error ϵ stay within given limits. Since e_d is specified as a fixed design characteristic of the operational amplifier, the error is conveniently expressed as

$$\epsilon \leq e_d \gamma \tag{4.42}$$

where

$$\gamma = \frac{1}{RC} t_m$$

Thus, in all analog integrators, the integrator-gain times the upper limit of the integration gives a constant factor γ if ϵ and e_d are fixed. The higher the gain factor $1/RC$, the shorter permissible the integration time, that is, the range of the independent variable. The product γ is a very useful figure of merit of any analog computer. In many problems the range of the independent variable is wide and it is important to run the solution to its final value. Thus, in addition to the percentage error ϵ, the maximum number of independent variable units t_m or the product γ should always be specified. Since γ is constant, it is convenient to set $1/RC = 1$ and to specify the value of t_m. For example, if $\gamma = 2000$, then $t_m = 2000$, which means that 2000 units of the independent variable are the upper range of integration. For

$$R = 1\text{M}, \qquad C = 1\mu f,$$

$$t_m = 2000/\text{sec} \approx \tfrac{1}{2}\,\text{h}$$

which is about the maximum limit of integration in the best machines.

In order to keep the error $\epsilon \leq 0.1\%$, the drift of the operational amplifier, when $t_m = 2000$, must be

$$e_d \leq 50\ \mu\text{v}$$

in accordance with Equation 4.42. This is a stringent condition if one remembers that all causes contributing to zero offset are included in the above figure.

In view of the relatively short computing intervals in repetitive operation, one is at first tempted to jump to the conclusion that drift is not an important factor in repetitive computers. Although it is true that in repetitive computers t_m is much shorter and integrating capacitors are reset in each cycle, the same problems arise in a different form. Returning to Equation 4.40, the product Equation 4.42 must again be constant. But now t_m is fixed by the end of the computing phase of each cycle. A greater γ can be achieved only by using greater gain factors in the linear computing elements. With increasing γ, the drift effect in the solution increases as well. For instance, if the same result $t_m = 2000$, $\epsilon = 0.1\%$ is to be obtained on a compressed scale $t_m' = 10$, then the integrator gain factor $1/RC$ for Equation 4.40 must be 200. It is clear that all drift and noise effects in the output would be increased by the same amount.

As an example, consider the differential equation

$$\frac{d^2y}{dx^2} = -900y \qquad 0 \leq x \leq 30 \qquad \text{or} \qquad x_m \approx 10\pi$$

To permit solution on a repetitive computer, introduce the transformation

$$x = 30x'$$

The equivalent analog network on the repetitive computer must then take the form indicated in Figure 4.7. On a one-shot computer the scale factors could be taken as unity. Then the loop gain would be unity, but t_m would be 30.

In general, the number of cycles of the solution to appear within the computing time of the repetitive machine depends only upon the loop gain

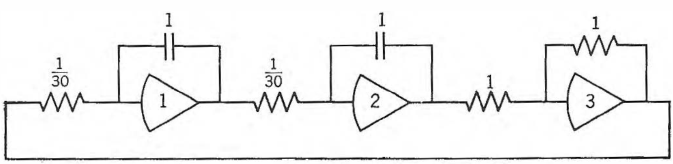


Fig. 4.7 Analog circuit for $\dfrac{d^2y}{dt^2} = +900y.$

of the diagram in Figure 4.5 and not at all on the repetition frequency. Only by increasing the loop gain is the range of the independent variable increased. This is represented in Table 4.1. However, increasing the

Table 4.1

Loop Gain	Solution Range	Solution Range in Periods
100	3π	1.5
1,000	10π	5
10,000	30π	15

loop gain multiplies drift and noise effects by the same factor. Thus, there is no difference in this regard between one-shot and repetitive computers. To obtain the same γ with given ϵ and $\bar{e}_d$, the operational amplifiers must be of the same quality in the one-shot and the repetitive modes of operation. The repetitive machine is not inherently less accurate. The real problem is that it is much more difficult to design computing elements having simultaneously wide bandwidths and low drift and noise levels. However, if γ is reduced, that is to say, if the range of the independent variable is kept within closer limits, the repetitive computer will give the same accuracy for the same problem as the one-shot machine. The use of the feed-forward technique, described in Chapter 5, in the design of operational amplifiers has made it possible to obtain a very wide frequency response with automatic drift correction. Using such equipment, equally accurate results can be obtained in either mode of operation. However, there are as yet no multipliers for high-speed operation with as low drift and noise figures as in one-shot machines. So in nonlinear problems it is likely that for the same range of integration time, less accurate results will be obtained with repetitive computers.

Emms and Brinkman[1] have analyzed the effect of drift upon scaling for several cascaded operational amplifiers. Many computing networks actually consist of loops of cascaded operational amplifiers so that the loop gain must be considered. A representative situation is illustrated in Figure 4.8. The multiple inputs to each amplifier indicate that signals from other points are also fed to the circuit.

As mentioned earlier, the total loop gain in Figure 4.8 is determined by the problem under study so that the proportioning of the total gain to individual amplifiers is to a certain degree in the hands of the operator. In addition to other factors, amplifier drift must be taken into account when dividing the total gain among the individual units. Since these

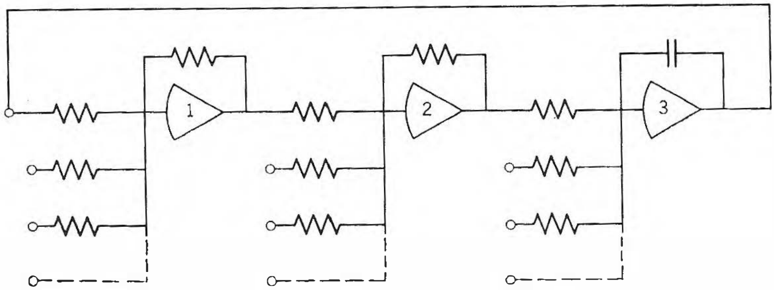


Fig. 4.8 Typical integrator loop.

results are easily derived from the drift relation of a single operational amplifier

$$e_s = -e_d\left(1 + \frac{Z_f}{Z_i}\right) \tag{4.43}$$

they will not be repeated here. For example, in the integrator loop of Figure 4.8 it is wise to concentrate the majority of the loop gains in the amplifier to which the feedback of the integrator is applied (amplifier 1). It can be shown that under otherwise identical conditions, the drift output of the integrator in Figure 4.8 can be reduced by a factor of 1:2 by obeying this rule.

Another situation arises in analog networks when several summing amplifiers are connected in cascade as shown in Figure 4.9. Here again a number of signals are fed to the chain of cascaded amplifiers. The rule to follow now is to apportion the majority of the gains as far forward as possible (first amplifier 1, then 2). The limitation here is evidently set by the range of the linear outputs of the computing elements. Another simple rule which should be observed in regard to drift is to avoid when possible attenuating the signals before summing into amplifiers.

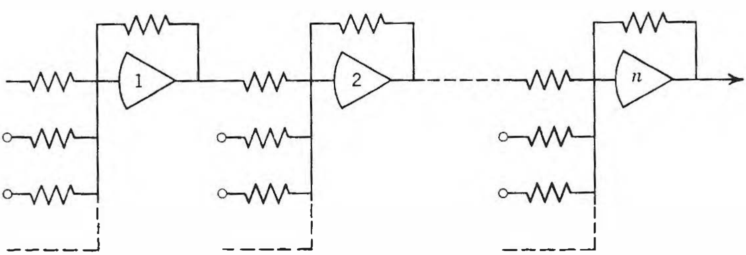


Fig. 4.9 Summing amplifiers in cascade.

Emms and Brinkman also offer another practical suggestion regarding loop balancing. Instead of balancing isolated amplifiers before they are connected into an integrator loop, it is advisable to do so in the loop itself. All inputs should be grounded, but the feedback impedances kept as they are. This means that the integrators should not be zeroed with resistor gains. The balancing procedure should start by observing the output of the amplifier preceding the integrator and zeroing its output by balancing control of the integrator. The meter is then moved forward and the procedure repeated as many times as needed. No switching is then needed at the input grids and small differences in ground potentials in the computer are automatically compensated. However, this does not mean that amplifiers without normal balancing provisions should be built. The standard drift correction of individual amplifiers must be carried out in advance to check their normal performance.

REFERENCE

1. Emms, E. T., and K. H. Brinkman, "The Minimization of Drift in DC Analogue Computers," *Electronic Eng.*, September 1960, Vol. 32, p. 550.

part II
EQUIPMENT

chapter 5
LINEAR ELEMENTS

5.1 General Remarks

Most of the operations necessary for the solution of linear differential equations with constant coefficients are obtained using the circuits shown in general form in Figure 5.1. Depending upon the choice of the impedances Z_i and Z_f, the circuits can be used to perform a wide variety of mathematical operations.

The transfer function of the circuit of Figure 5.1*a* is

$$\frac{e_o}{e_i} = -\frac{Z_f}{Z_i} \tag{5.1}$$

and the circuit of Figure 5.1*b* is governed by

$$e_o = -Z_f\left(\frac{e_1}{Z_1} + \frac{e_2}{Z_2} + \cdots + \frac{e_n}{Z_n}\right) \tag{5.2}$$

provided the gain g of the amplifier is equal to infinity. Thus, the realization of suitable transfer function using Equations 5.1 and 5.2 requires, in addition to the passive circuit elements Z, an electronic amplifier. This amplifier is termed "operational amplifier."

The two most important operational units governed by Equations 5.1

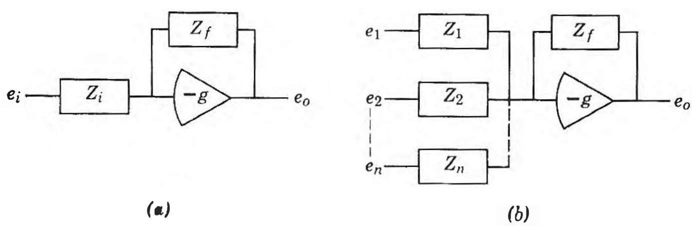


Fig. 5.1 (*a*) Single input operational circuit. (*b*) Multiple input operational circuit.

and 5.2 are the adder and the integrator. Although the basic theory of operation of these units is straightforward and relatively simple, high-accuracy operation demands a detailed consideration of a number of significant error sources. These include particularly errors due to non-infinite gain, limited bandwidth, drift, and input grid current in the operational amplifier. Other errors are due to shortcomings in the passive input and feedback elements. In this chapter the basic principles of operation of adders and integrators are discussed in detail together with a quantitative treatment of principal sources of error. This is followed by a consideration of a relatively novel technique for storing the dependent computer variable from one repetitive cycle to the next. This technique provides dynamic memory capability and greatly increases the utility of the repetitive differential analyzer.

Three basic approaches have been developed for designing operational amplifiers suitable for repetitive operation. These techniques are considered in some detail followed by a brief survey of some practical amplifier realizations. The chapter concludes with a treatment of those imperfections in resistors, capacitors, and potentiometers which are of particular significance in repetitive operation.

5.2 Addition

Basic principles

The operation of the adder can be deduced readily from Equation 5.1. By letting

$$Z_f = R_f$$
$$Z_i = R_i$$

Equation 5.1 becomes

$$e_o = -\frac{R_f}{R_i} e_i \tag{5.3}$$

For several voltage inputs

$$e_o = -R_f \sum_{i=1}^{n} \frac{e_i}{R_i} \qquad i = 1, 2, \ldots, n \tag{5.4}$$

A clear understanding of the operation of the adder can be obtained by referring first to the resistance network of Figure 5.2. According to Kirchhoff's node law

$$\frac{e_1 - e_A}{R_1} + \frac{e_2 - e_A}{R_2} + \cdots + \frac{e_n - e_A}{R_n} = \frac{e_A}{R} \tag{5.5}$$

The principal consideration which limits the applicability of such a network for addition is that proper functioning requires that there exist a

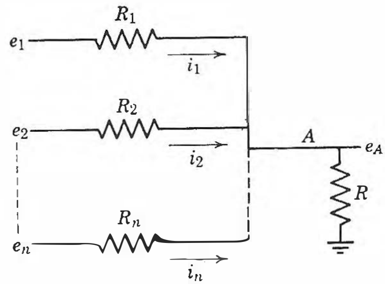


Fig. 5.2 Passive resistance addition network.

linear relationship between e_n and i_n, that is, independent of e_A. Only if e_A is very small does Equation 5.5 become approximately

$$\frac{e_1}{R_1} + \frac{e_2}{R_2} + \cdots + \frac{e_n}{R_n} = 0 \tag{5.6}$$

so that the desired mathematical operation is performed. Although it is not possible to realize a circuit governed by Equation 5.6, it is possible to approximate the relationship to an accuracy which is sufficient for the purposes of analog computation. To this end, the basic network is augmented by an automatic method for maintaining the voltage e_A close to zero. This is shown schematically in Figure 5.3. If for the moment all other sources of error are ignored, the current-law equation at point A can be written as

$$\sum_{i=1}^{n} \frac{e_i}{R_i} - e_A \sum_{i=1}^{n} \frac{1}{R_i} = -\left(\frac{e_o}{R_f} - \frac{e_A}{R_f}\right) \tag{5.7}$$

or letting $e_A = -e_o/g$ and solving for e_o

$$e_o = -\frac{g}{R_f \sum_{i=1}^{n} \frac{1}{R_i} + (g+1)} R_f \sum_{i=1}^{n} \frac{e_i}{R_i} \tag{5.8}$$

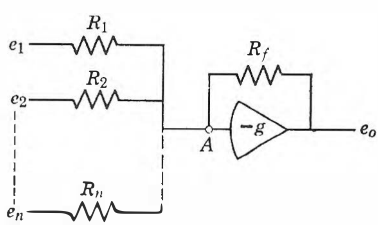


Fig. 5.3 Electronic adder.

Provided the amplifier gain g has a sufficiently large absolute magnitude, $g \gg 1$, Equation 5.8 reduces to

$$e_o = -R_f \sum_{i=1}^{n} \frac{e_i}{R_i} \tag{5.9}$$

as desired.

Errors caused by noninfinite amplifier gain

One error in the operation of addition is a function of the variation of the voltage at point A in Figure 5.3 because $g \neq \infty$. In order to calculate the required gain g, the maximum permissible percentage error for a given number of inputs is specified.

The percentage error is now expressed by the application of Equations 5.8 and 5.4 as

$$\epsilon = 100\left(\frac{e_o - e_o'}{e_o'}\right) = \frac{100}{g}\left(1 + R_f \sum_{i=1}^{n} \frac{1}{R_i}\right) \tag{5.10}$$

where e_0 is the desired output and e_0' the output using imperfect computer elements. So to limit the error to 1%,

$$g \geq 100\left(1 + R_f \sum_{i=1}^{n} \frac{1}{R_i}\right) \tag{5.11}$$

for $R_f = R_i$ and $n = 10$, the gain must be approximately equal to 1000. For $R_f \neq R_i$ and $g = 1000$ it is necessary to satisfy the inequality

$$R_f \sum_{i=1}^{10} \frac{1}{R_i} < 10 \tag{5.12}$$

taking into account the number of terms of the addition and of the relationship R_f/R_i in order to limit the error to 1%.

It is possible to attain the same accuracy in addition without satisfying the inequality of Equation 5.12. Suppose that the gain g is smaller than the required value, so that the error term resulting from the finite gain of the amplifier can not be neglected. It is possible to employ Equation 5.8 to calculate a new value for R_f so as to effect the correction of the systematic error resulting from insufficient gain. This can be expressed as

$$\left(R_f \sum_{i=1}^{n} \frac{1}{R_i}\right) \frac{g}{(1 + g) + R_f \sum_{i=1}^{n} \frac{1}{R_i}} = 1 \tag{5.13}$$

or

$$R_f = -\frac{1 + g}{1 - g} \frac{1}{\sum_{i=1}^{n} \frac{1}{R_i}} \tag{5.14}$$

For example, for $n = 1$, $R_1 = 1M$, $g = 50$, one obtains $R_f = 51/49 = 1.04$. This indicates that R_f must be 4% greater if the error resulting from a gain of only 50 is to be corrected. Although this latter method simplifies the construction of the amplifier and obviates certain instability problems, every change in g and R_i involves a new calculation and a new experimental adjustment of R_f. For this reason in most high quality computers it is preferable to make the gain sufficiently large.

Errors caused by limited bandwidth

Since the arithmetic calculations performed in the differential analyzer involve transient voltages it is insufficient to limit an error analysis to static considerations. It is necessary to consider carefully the operation of the adder as a function of frequency and to determine how its transfer function is changed as the frequency of the input voltage changes. Assume for simplicity that

$$R_f = R_i \qquad \text{and} \qquad g \gg 1 \tag{5.15}$$

Equation 5.8 then becomes

$$e_o = \frac{\sum_{i=1}^{n} e_i}{1 + \dfrac{n}{g(s)}} \tag{5.16}$$

where s is the complex frequency parameter. The frequency response of the amplifier can be expressed with sufficient precision in the form

$$g(s) = \frac{g_0}{1 + sT_\alpha} \tag{5.17}$$

where T_α is a time constant related to the upper cutoff frequency of the amplifier and g_0 is the d-c gain. Inserting this into Equation 5.16 yields

$$e_o = -\frac{\sum_{i=1}^{n} e_i}{1 + \dfrac{n(1 + sT_\alpha)}{g_0}} \tag{5.18}$$

If a new variable T_2 is defined,

$$T_2 = \frac{nT_\alpha}{g_0} \tag{5.19}$$

a simple expression for the transfer function of the adder is obtained as

$$\frac{e_o}{e_i} = -\frac{1}{1 + sT_2} \tag{5.20}$$

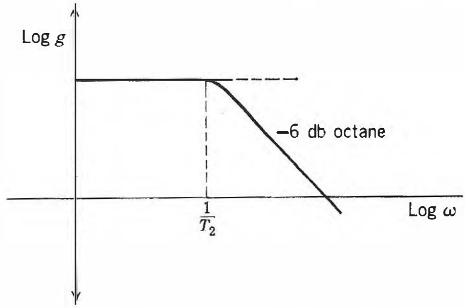


Fig. 5.4 Gain-frequency characteristic of an adder.

If $T_2 = 0$, this reduces to Equation 5.3. A comparison of Equations 5.3 and 5.20 shows clearly the difference between the ideal and the actual response of the adder. Figures 5.4 and 5.5 illustrate respectively the frequency dependence of the gain and of the phase of the unit.

If in 5.8 $R_f = R_i$,

$$e_o = -\frac{g \sum_{i=1}^{n} e_i}{1 + n + g} \tag{5.21}$$

or

$$e_o = -\frac{\sum_{i=1}^{n} e_i \dfrac{g(s)}{1+n}}{1 + \dfrac{g(s)}{1+n}} \tag{5.22}$$

The roots of the equation

$$1 + \frac{g(s)}{1+n} = 0 \tag{5.23}$$

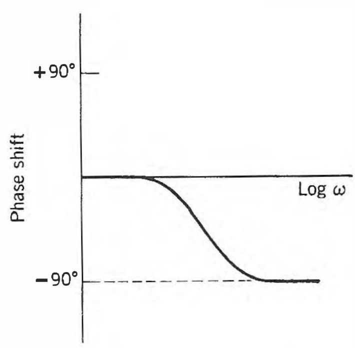


Fig. 5.5 Phase shift-frequency characteristic of an adder.

which lie in the left-hand plane determine the damping of the parasitic signals at the output of the adder when a given voltage is applied at the input. In accordance with Equation 5.20

$$1 + \frac{\dfrac{g_0}{1 + sT_2}}{1 + n} = 0 \qquad (5.24)$$

Solving

$$s = -\omega_2 \left(1 + \frac{g_0}{1 + n}\right) \qquad (5.25)$$

where

$$\frac{1}{T_2} = \omega_2$$

The parasitic signals therefore have the form

$$\exp\left[-\omega_2 t\left(1 + \frac{g_0}{1 + n}\right)\right]$$

This equation indicates that the amplifier will work satisfactorily if: ω_2 the bandwidth is large; g_0 the gain at zero frequency is large; and n the number of terms in the addition is small. The required bandwidth of an adder cannot in general be specified since, as has been shown, the precision of the operation depends on the position of the characteristic roots of the equation of the system to be solved. In repetitive analyzers, amplifiers and adders frequently have an uppercut off frequency of 100 to 600 kc.

Errors caused by drift

Unless special stabilizing circuits are employed, the output voltage of a d-c amplifier whose input is maintained at zero volts will gradually deviate or drift from zero. Since repetitive computers often do not employ stabilization circuits, this source of error must be considered carefully in the design of the differential analyzer. Following a development by Wass[1] the effect of drift upon the precision of an adder will be studied with the aid of the circuit shown in Figure 5.6. The many

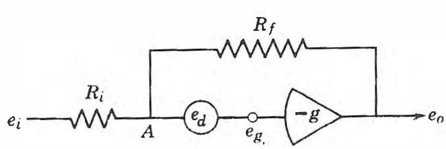


Fig. 5.6 Influence of drift on the operation of an adder.

diverse sources of drift are represented in this figure by a single voltage source e_d at the input terminals of the amplifier. The magnitude of this source is such that it produces at the output of the amplifier the same drift effect as is produced in the actual circuit. Assuming for the moment that the amplifier itself draws no current i_g, the Kirchhoff's law equation for node A is

$$\frac{1}{R_i}(e_i - e_A) = -\frac{1}{R_f}(e_o - e_A) \tag{5.26}$$

where

$$e_A = e_g + e_d \qquad e_g = -\frac{e_o}{g} \tag{5.27}$$

so that

$$e_A = -\left(\frac{e_o}{g} + e_d\right) \tag{5.28}$$

Eliminating e_A in Equation 5.26 and letting $e_i = 0$,

$$\frac{1}{R_i}\left(\frac{e_o}{g} + e_d\right) + \frac{1}{R_f}\left(e_o + \frac{e_o}{g} + e_d\right) = 0 \tag{5.29}$$

If g is large, the term e_o/g is negligible, and the influence of the drift voltage at the output of the amplifier is

$$e_o = -e_d\left(1 + \frac{R_f}{R_i}\right) \tag{5.30}$$

Equation 5.30 demonstrates that the presence of R_f, the feedback resistor, does not in itself reduce or eliminate the drift problem. In commercial amplifiers the drift voltage is of the order of 10 mv/h if no automatic drift correction is provided.

Errors caused by amplifier grid currents

Assume now that the grid of the first tube of the operational amplifier is drawing a grid current i_g. Referring to Figure 5.6 and assuming $e_d = 0$, the node-law equation at node A is under these conditions

$$\frac{1}{R_i}(e_i - e_A) = -\frac{1}{R_f}(e_o - e_A) - i_g \tag{5.31}$$

If

$$e_i = 0 \qquad \text{and} \qquad e_A = -\frac{e_o}{g} \approx 0$$

the output voltage becomes

$$e_o = -i_g R_f \tag{5.32}$$

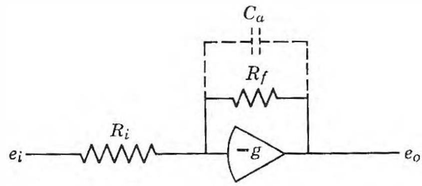


Fig. 5.7 Parasitic capacitance in an adder.

Thus the effect upon the output voltage is the same as if the entire grid current were made to pass through the feedback resistor. For a feedback resistor of 1 megohm and for a desired error of less than 1%, the grid current must be less than 10^{-8} ampere. This is not very difficult to obtain by suitable design of the input stage of the amplifier.

Errors caused by parasitic capacitance

The effect of parasitic capacitances on the operation of an adder can best be described by comparing the transfer function of the adder with and without this capacitance. In Figure 5.7 the parasitic capacitance has been designated by C_a. Taking into account C_a, the transfer function of the adder, which is ideally $(-R_f/R_i)$, becomes

$$\frac{e_o}{e_i}(s) = -\frac{R_f}{R_i(1 + sR_fC_a)} \tag{5.33}$$

Thus the presence of the parasitic capacitance causes certain terms $a_n s^n y$ of the differential equation being solved to become

$$a_n(1 + sR_fC_a)s^n y$$

That is, the order of the differential equation is increased. This may result in divergent solutions of equations which are ordinarily stable. For example, in a second-order harmonic equation, the presence of parasitic capacitance introduces a third root which may have a positive real part. This implies that the product R_fC_a must be made as small as possible.

In case the addition is accomplished within an integrator, the situation is less serious, since the parasitic capacitance then appears in parallel with a feedback capacitor. For this reason the operations of addition and integration are combined wherever possible in differential analyzers.

5.3 Integration

General principles

The equation for the transfer function of an integrator follows readily from the general equation for the operational unit, Equation 5.1. In this

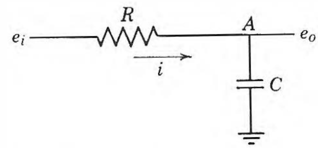


Fig. 5.8 Passive RC integrating network.

instance let

$$Z_f = \frac{1}{Cs}$$

$$Z_i = R_i$$

In the ideal case, therefore,

$$\frac{e_o}{e_i} = -\frac{1}{RCs} \tag{5.34}$$

so that for multiple inputs

$$e_o = -\frac{1}{Cs}\sum_{i=1}^{n}\frac{e_i}{R_i} \tag{5.35}$$

Although this process is in itself quite simple, a more general discussion of integration will be presented. Integration by analog methods requires the use of elements whose input and output relationship is expressed by derivatives. Such an element in its most simple form is the capacitor governed by the equation

$$i = C\frac{de}{dt} \tag{5.36}$$

Figure 5.8 shows a passive circuit that could serve as a first approximation for an integrator. It is evident, however, that since the voltage at point A is not constant, the current which passes through R_i does not depend only upon e_i. Thus the relationship between the voltage and the current through the resistor will cease to be linear as soon as the capacitor acquires a charge. It therefore becomes necessary to introduce an amplifier to maintain the voltage at point A very close to zero. The pertinent circuit is shown in Figure 5.9. Since

$$e_o = -ge_A$$

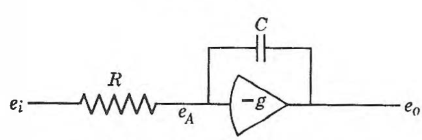


Fig. 5.9 Electronic integrator.

if g is sufficiently large, any variations in e_A will be so small in comparison to the voltage range of e_o that for all practical purposes e_A will be equal to zero. The question now arises as to how large g must be in order that the e_A be "sufficiently" small. In this case it is necessary to analyze the function $\epsilon(t, g)$, that is, to calculate the dynamic error of the integrator as a function of time and gain. Evidently, it is not easy to analyze $\epsilon(t, g)$ for a large number of input functions. It is necessary to limit consideration to certain simple functions for which the difference between a known theoretical solution and the integral furnished by the computer can be determined. A convenient choice for such a test function is the unit step. Once the circuit response to the unit step is known, the convolution integral can be employed to obtain the circuit response to any other analytic function.

Consider the outputs of the circuits of Figures 5.8 and 5.9 when the input is a unit step voltage. The transfer functions will first be expressed without considering the bandwidths of the amplifier and other circuit elements. For the circuit of Figure 5.8

$$\frac{e_o}{e_i} = -\frac{1}{1 + R_iCs} \tag{5.37}$$

For the circuit of Figure 5.9, Equation 5.8 can be applied by letting

$$R_f = \frac{1}{Cs} \qquad n = 1$$

so that

$$\frac{e_o}{e_i} = \frac{g}{1 + (1 + g)R_iC_S} \tag{5.38}$$

Thus the output voltages in Figure 5.8 will have the form

$$e_o = 1 - e^{-t/R_iC} \tag{5.39}$$

Whereas the output in Figure 5.9 will be

$$e_o = 1 - e^{-t/R_i'C'} \tag{5.40}$$

where

$$R_i'C' = (1 + g)R_iC$$

Thus the introduction of the amplifier can be looked upon as increasing the time constant by a factor of $(1 + g)/1$.

To find $\epsilon(t, g)$, expand e_o in series form

$$e_o = \frac{t}{R_iC} - \frac{t^2}{2(R_iC)^2} + \cdots \tag{5.41}$$

By taking only the first two members of the series,

$$e_o = \frac{t}{R_iC} - \frac{t^2}{2(R_iC)^2} \tag{5.42}$$

The percentage error can then be expressed as

$$\epsilon = 100 \frac{\dfrac{t}{R_iC} - \dfrac{t}{R_iC} - \dfrac{1}{2}\left(\dfrac{t}{R_iC}\right)^2}{\dfrac{t}{R_iC}} \tag{5.43}$$

Accordingly, for a specified precision the inequality that must be satisfied is

$$\epsilon < \frac{50t}{(1 + g)R_iC} \tag{5.44}$$

Another function which is convenient to employ in calculating the error is the sine wave

$$y = A \sin \omega t$$

This case has been analyzed in detail by R. J. A. Paul.[2] The output voltage of the electronic integrator for a sine wave input is

$$e_o = -\frac{A\omega g}{1 + \omega^2 K}[Ke^{-t/K} - \sqrt{K^2 + 1/\omega^2} \cos (\omega t + \phi)] \tag{5.45}$$

where

$$K = (1 + g)R_iC$$

$$\phi = \tan^{-1} \frac{1}{\omega K}$$

By comparing Equation 5.45 with the analytic solution

$$e_o = \frac{A}{\omega R_iC}(\cos \omega t - 1) \tag{5.46}$$

it can be seen that since g is finite, the amplitude is reduced by the factor

$$\frac{\omega g K}{(1 + g)\sqrt{1 + \omega^2 K^2}}$$

The exponential coefficient becomes

$$1 - \frac{\omega K}{\sqrt{1 + \omega^2 K^2}} e^{-K/t}$$

and the phase shift becomes

$$\phi = \tan^{-1} \frac{1}{\omega K}$$

Error caused by noninfinite gain

For the integrator the specification of g is determined by the length of time required for solution. To facilitate the analysis of various cases, a relative time scale $T = t/R_iC$ will be used. Assume that for a given analyzer

$$\frac{t_{\max}}{RC} = 2$$

By means of Equation 5.44 the gain necessary for a percentage error $\epsilon = 1\%$ is found to have a minimum value of 100. However, in order to combine the operations of integration and addition in one unit, Equation 5.11 must also be considered. If the gain is taken to be 1000, the amplifier will be able to serve both purposes satisfactorily.

Errors caused by limited bandwidth

In order to evaluate the effect of the frequency characteristics of the operational amplifier upon the operation of an integrator, Equation 5.17 is employed to define the amplifier transfer function. Inserting this expression into Equation 5.8 and letting

$$R_f = \frac{1}{Cs} \qquad \text{and} \qquad n = 1$$

the integrator transfer function becomes

$$\frac{e_o}{e_i} = -\frac{1}{R_iCs + \dfrac{(1 + R_iCs)(1 + sT_1)}{g_0}} \tag{5.47}$$

The cutoff frequency of the integrator is not necessarily the same as that of the amplifier or of the adder. For this reason the symbol T_1 has been introduced for the integrator, T_2 for the adder, and T_α for the amplifier. The transfer function of the ideal integrator is

$$\frac{e_o}{e_i} = -\frac{1}{RCs}$$

as s approaches zero, e_o/e_i approaches infinity. Thus there exists also a lower limit of frequency beyond which the integrator does not operate in an ideal fashion. This limit is determined by the gain of the amplifier at zero frequency

$$g_0 = -\frac{1}{RC\omega_0}$$

or

$$T_0 = \frac{1}{\omega_0} = RCg_0 \tag{5.48}$$

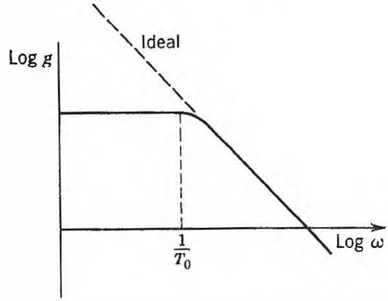


Fig. 5.10 Gain-frequency characteristics of an integrator.

Thus the two constants, T_0 and T_1, both influence the precision of the integrator. The transfer function of integrators can be expressed in terms of the upper and lower cutoff frequencies as

$$\frac{e_o}{e_i} \approx -\frac{g}{(1 + sT_0)(1 + sT_1)} \tag{5.49}$$

Figures 5.10 and 5.11 illustrate typical gain and phase responses of integrators.

Errors caused by amplifier drift and grid current

Just as with the adder, the effect of drift can be calculated by introducing an equivalent drift generator e_d at the input of the amplifier as

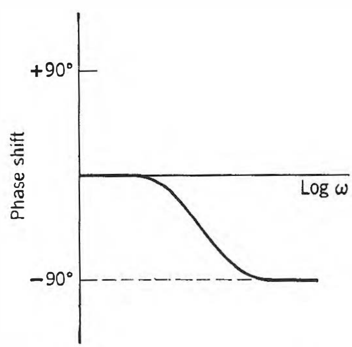


Fig. 5.11 Phase shift-frequency characteristics of an integrator.

shown in Figure 5.6. In this case the feedback element is a capacitor, so that R_f becomes $1/Cs$ from which

$$e_o = -\left(1 + \frac{1}{R_iCs}\right) e_d \tag{5.50}$$

For an integrator, Equation 5.32 becomes

$$e_o = -\frac{i_g t}{C} \tag{5.51}$$

in view of the fact that the capacitors of repetitive computers have magnitudes of the order of 10^{-4} μf and that the integration time is short, the value for i_g which was determined for adders is satisfactory for integrators.

Errors caused by leakage conductance of the capacitor

When feedback capacitor has a leakage conductance or when there is an appreciable amount of leakage resistance due to poor insulation of the amplifier terminals, the operation of the integrator can be represented by the circuit of Figure 5.12. If in Equation 5.8

$$R_f = \frac{R_a}{1 + R_aCs} \qquad \text{and} \qquad n = 1$$

the transfer function of the circuit becomes

$$\frac{e_o}{e_i} = \frac{1}{R_iCs + \dfrac{1}{g} + \dfrac{R_i}{R_a}} \tag{5.52}$$

if the terms $1/R_ag$ and Cs/g are neglected. Equation 5.52 can be expressed as

$$\frac{e_o}{e_i} = -\frac{1}{R_iCs + \dfrac{1}{g'}} \tag{5.53}$$

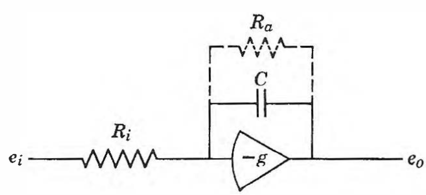


Fig. 5.12 Integrator with leakage conductance.

where

$$\frac{1}{g'} = \frac{1}{g} + \alpha$$

$$\alpha = \frac{R_i}{R_a}$$

Equation 5.53 demonstrates that the effect of the leakage conductance is to reduce the effective gain of the amplifier. This can be seen if the gain of the amplifier with leakage, is expressed as

$$g' = \frac{g}{1 + g\alpha} \tag{5.54}$$

Since α is positive, g' is smaller than g. For example, if

$$R_a = 10^3 \text{ megohms}$$

$$R_i = 1 \text{ megohm}$$

$$g = 1000$$

the effective gain will be $g' = 500$

5.4 Dynamic Memory

In many applications of repetitive analog computers it is useful to be able to register an instantaneous voltage occurring during some part of the repetitive cycle and to reproduce this voltage during the following repetitive cycle. Such an operation is termed dynamic memory and can be effected with the same passive and active components used for other linear operations. Jury[3] describes this approach using equipment with the trade name DYSTAC.

The basic component in a dynamic memory is an ordinary integrator.

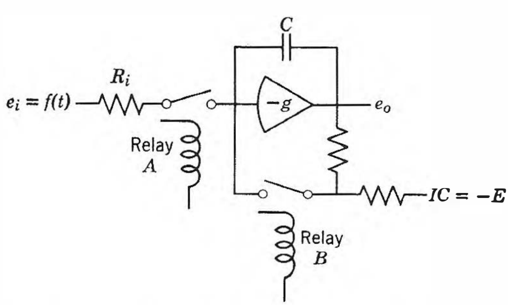


Fig. 5.13 Integrator including initial-condition input and reset-operate relays.

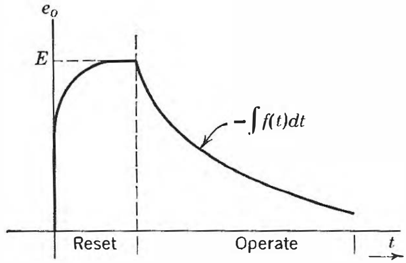


Fig. 5.14 Reset-operate cycle of a typical integrator.

The normal input channels are not used, however. The only input to the memory integrator is through the channel normally used to apply the initial conditions (IC). Figure 5.13 shows a conventional integrator. During the reset mode, relay A is open and relay B is closed. During this period the feedback capacitor is biased to produce an output e_o which is the negative of the voltage imposed at the initial condition input. When the operate cycle is initiated, relay B is opened and relay A is closed so that the voltage input e_i can be integrated. Figure 5.14 shows the transient voltage appearing at the amplifier output during the reset and the operator portions of the repetitive cycle. Since the normal input channels are not used when the unit is employed for dynamic memory, the memory element never integrates but merely tracks the voltage applied at its IC input as long as switch B is closed. When relay B is opened, a discharge path no longer exists for capacitor C, so that the capacitor maintains the voltage it had the instant that switch B was opened. There are four basic ways in which a dynamic memory unit of this type can be used. These are illustrated in Figure 5.15.

In the M mode, the normal computer control-pulse used to control the repetition cycle is used to control relay B. Capacitor C is then charged during the reset cycle and holds this charge during the operate cycle. If the connections to relay B are reversed, it will be closed during the operate portion of the cycle and open during the reset cycle. The Rev-M memory is formed in this way.

Alternatively relay B need not be controlled by the normal computer control-pulse. Rather it can be activated by the output of a voltage comparator, so that it opens or closes at some instant during the operate cycle as determined by the output of the voltage comparator. Thus the instant at which the unit "memorizes" the voltage of its IC input can be made a function of a problem variable. If a ramp voltage is applied to one input of the voltage comparator and d-c voltage of 100 volts to the

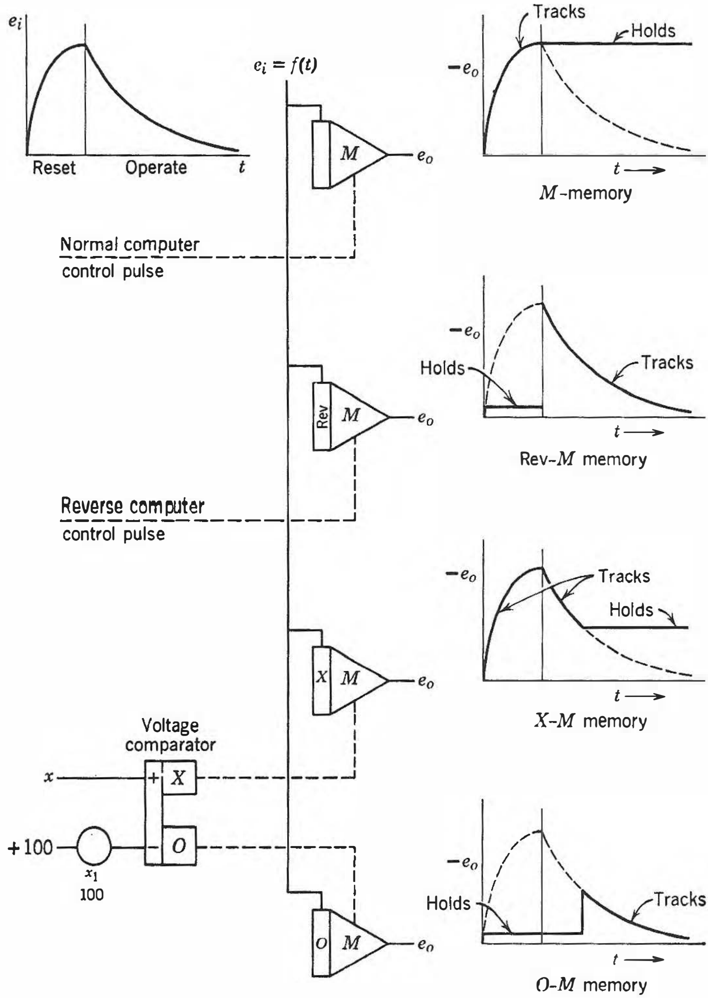


Fig. 5.15 Dynamic memory circuits and output voltage during reset-operate cycle.

other, the instant of time at which memory occurs can be varied throughout the operate cycle by adjusting the slope of the ramp. In the X–M memory, relay B is closed until coincidence is detected in the voltage comparator. Then the relay opens causing the capacitor to hold its charge during the rest of the cycle. In the O–M memory on the other hand, relay B is open until coincidence occurs, at which time it closes and e_o tracks the voltage applied to the IC terminal.

Frequently it is desirable to use two or more of the circuits shown in Figure 5.15 in cascade. For example, by connecting a Rev-M and an M unit in series as shown in Figure 5.16 the so-called ratchet circuit is formed. This circuit makes it possible to memorize a voltage value in one cycle and makes this value available for use in the next cycle. This makes it possible to perform calculations in an iterative fashion as on a digital computer.

The satisfactory operation of memory circuits of the type described depends upon the ability to perform the switching operation of relay B extremely rapidly. Capacitor C must likewise be able to assume a new value in a time which is negligible compared to the repetitive cycle. This is generally not possible with conventional circuits. In Section 8.3 a technique is described for carrying out these operations in a satisfactory manner. Basically this involves the use of solid-state switches and an additional d-c amplifier in the feedback loop.

5.5 Operational Amplifiers

General design techniques

In the preceding discussion the principal factors that must be considered in the selection of the components of the operational computing units have been treated in some detail. The key problem is to choose a suitable operational amplifier. Evidently this choice will be governed to a large extent by the purpose of the specific computer, and therefore no rigid design rules and specifications can be provided. In general, the amplifiers used in the linear operational units of repetitive differential analyzers can be constructed using three basic approaches. In the first, the coupling between successive stages of the amplifier is effected with the aid of a resistance voltage divider so that the d-c as well as transient portions of the input signal are transmitted from one stage to the next. This is by far the most widely used approach and involves the application of conventional d-c amplifier design techniques. The basic elements of such an amplifier using three stages of amplification and a cathode follower are shown in Figure 5.17.

In other amplifiers the coupling between successive stages is effected

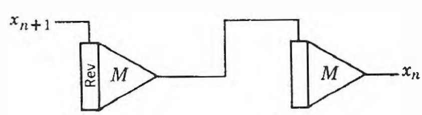


Fig. 5.16 Cascading of dynamic memory circuits.

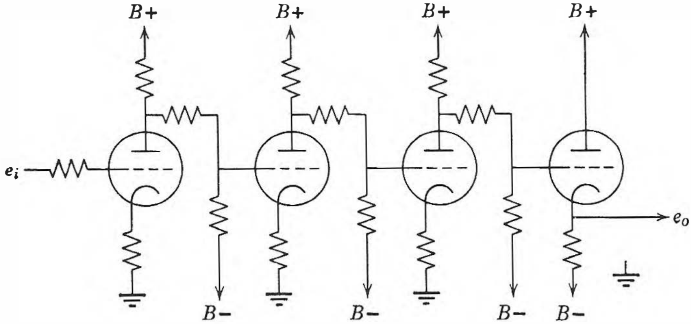


Fig. 5.17 Basic elements of conventional d-c amplifier.

as shown in Figure 5.18. Each coupling capacitor is equipped with a switch which applies the required initial conditions during each "reset" interval. In this manner the operational amplifier retains the properties of an a-c amplifier so that no drift problem arises. Nevertheless, the appropriate d-c level is generated. On the other hand, more components and electronic switches are required, making it more difficult to achieve an adequate bandwidth.

A third way to realize operational amplifiers of sufficient frequency response for repetitive computer operation is to adapt chopper-stabilized operational amplifiers for this purpose. In general, it is too difficult to obtain the desired frequency characteristics by using the standard method of cascading the individual amplifier stages in series as in Figure 5.19.

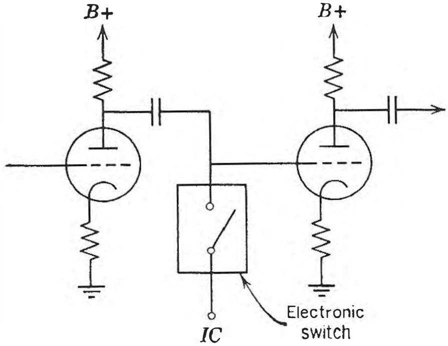


Fig. 5.18 A-c amplifier with auxiliary circuit to permit repetitive operation.

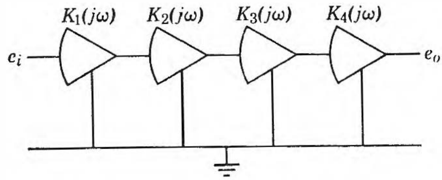


Fig. 5.19 Amplifier stages in cascade.

The transfer function of such an amplifier is the product of the transfer functions of the individual stages.

$$K_t(j\omega) = K_1(j\omega)K_2(j\omega)K_3(j\omega)K_4(j\omega) \tag{5.55}$$

Since accuracy and stability requirements of the operational amplifier set a definite limit on the decrement of $K_t(j\omega)$ with frequency, as well as on the phase margin, it is not easy to keep the time constants of all stages within the required limits.

These difficulties can be avoided if the individual amplifier stages are connected in parallel, as shown in Figure 5.20. Here the input signal is not introduced only at the first stage, but also in parallel at the inputs of all other individual amplifying stages. For this reason this method is sometimes termed "feed-forward" amplification. Now the over-all transfer characteristic has the form

$$K_t(j\omega) = \{[(K_1(j\omega) + 1)K_2(j\omega) + 1]K_3(j\omega) + 1\}K_4(j\omega) \tag{5.56}$$

It is much easier to satisfy the frequency requirements for high-speed operation of the operational amplifier in such a design. This approach was applied by Deering,[4] and a detailed analysis of the design of such an amplifier has been presented by Polonnikov.[5] He showed that the decrement $|K(j\omega)| \leq 4$ db/octave and $\phi \leq \pi/2$ now can be maintained

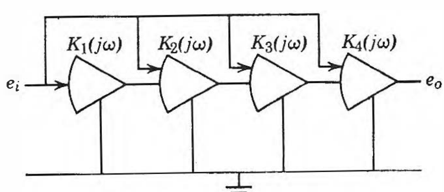


Fig. 5.20 Amplifiers connected to permit "feed-forward" operation.

for all frequencies if the following conditions are fulfilled:

$$T_1 = K_1(0)T_2$$
$$T_2 = K_2(0)T_3 \tag{5.57}$$
$$T_3 = K_3(0)T_{\text{output}}$$

If $T_{\min}$ and $|K(j\omega)|$ are taken equal for all stages, the following relations are obtained:

$$\frac{f_{c4}}{f_{c1}} = \frac{f_{v4}}{f_{v1}} \approx 10K_{\text{output}} \tag{5.58}$$

where f_c is the cutoff frequency and f_v is the break frequency.

Where the gain of the output stage $K_{\text{output}} = 10$ to 50 the bandwidth can be enlarged 100 to 500 times. Under otherwise equal conditions, an accuracy of 0.1% is kept within 20 kc with this design, as opposed to only 100 cycles/sec in the standard design. In this way chopper-stabilized d-c operational amplifiers may be used for repetitive operation with high repetition rates.

In designing the operational amplifier it is necessary to specify an output impedance. This specification must be such that the error resulting from a nonzero output impedance does not exceed the desired value. Suppose that the operational amplifier is loaded by n parallel impedances Z_k, which may include the input impedances of adders, integrators, potentiometers as well as the feedback element of the amplifier itself. The output voltage e_o is then given by

$$e_o = Z_o i + \frac{Z_k}{n} i$$

or

$$\frac{Z_k}{n} i = \frac{e_o(Z_k/n)}{Z_o + (Z_k/n)} \tag{5.59}$$

where Z_o is the output impedance of the operational amplifier, and i is the output current.

The output voltage is thus reduced by the factor

$$\frac{Z_o + (Z_k/n)}{Z_k/n} = 1 + \frac{Z_o n}{Z_k} \tag{5.60}$$

If this difference is to be less than 1%,

$$Z_o < \frac{Z_k}{100n} \tag{5.61}$$

For an adder unit with input impedances Z_i, the effect of the feedback network leads to an output impedance given by

$$Z_o = \frac{Z_o'}{g}\left(Z_f \sum \frac{1}{Z_i} + 1\right) \tag{5.62}$$

where Z_o' is the output impedance of the last stage of the operational amplifier.

Consider, for example, the calculation of the output impedance for an adder with five input resistors for which

$$\begin{aligned} Z_o' &= 4.5 \cdot 10^4 \text{ ohms} \\ R_1 &= 0.1 \text{ megohm} \\ R_2 &= 0.5 \text{ megohm} \\ R_{3,4,5} &= 1 \text{ megohm} \\ R_f &= 1 \text{ megohm} \\ g &= 1500 \end{aligned}$$

Inserting these values in Equation 5.62, $Z_s = 500$. According to Equation 5.61 therefore, such an adder could be loaded by resistances of the order of 50 kilohms without excessive error.

Practical amplifier realizations

A practical example of a wide-band d-c amplifier using automatic drift correction is the amplifier *MV-12 of the Moscow Institute of Automation* as described by Polonnikov[5] and shown in Figure 5.21. There are four parallel signal paths, one of which is chopper-stabilized. The first path includes a standard three-stage a-c amplifier with chopper modulation (M in Figure 5.21) as well as stages V_3, V_4, and V_5; the second path consists of capacitor C_1 and stages V_1, V_2, V_3, V_4, and V_5; the third path includes C_3, V_2, V_3, V_4, and V_5; and the fourth path is comprised of C_3, C_5, V_3, V_4, and V_5. In this design there is no path feeding the input directly to the output stage.

The following are the specifications of the amplifier:

$f_\alpha = 7\text{Mc}$

$g = 1$

$e_i = 10$ volts

Input and output stray capacitance $\approx 50\ \mu\mu$

Input and feedback resistors shunted by $22\ \mu\mu$

Output noise (peak) $= 1$ mv

Drift $= 50\ \mu\text{v}/8$ hr

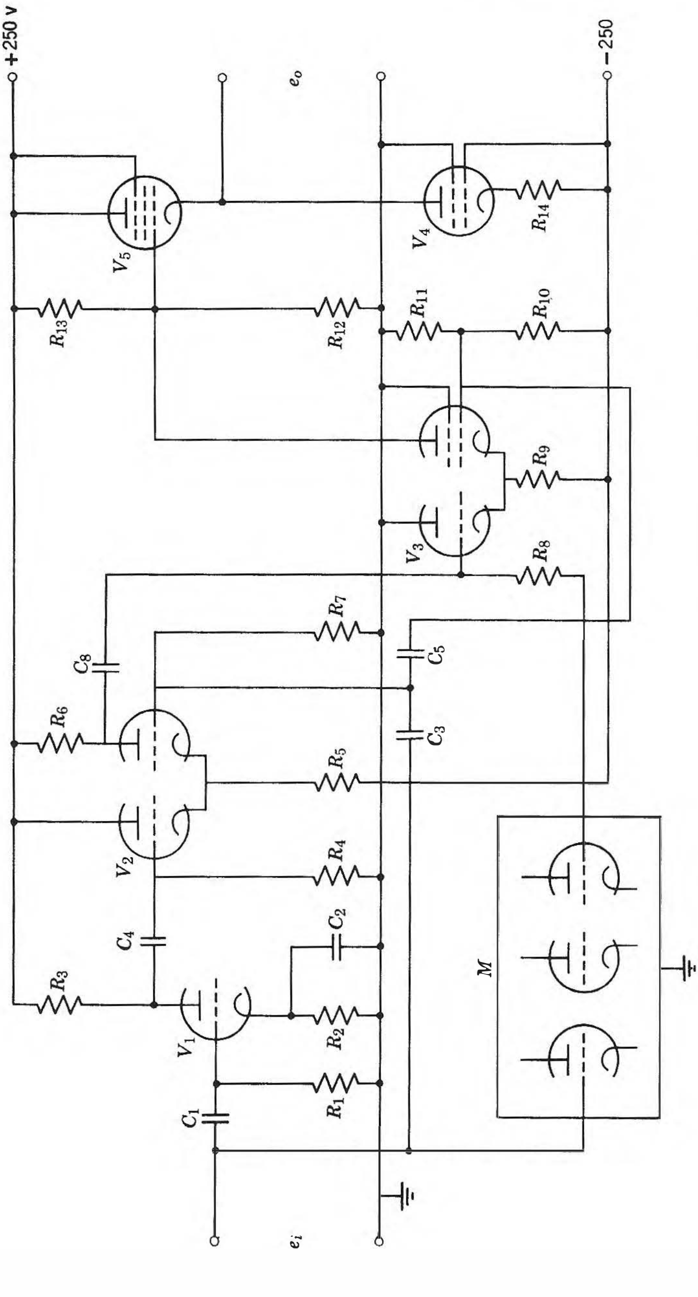


Fig. 5.21 Amplifier of the Institute of Automation and Telemechanics of Moscow.

If the stray capacitance is increased to 500 $\mu\mu$, the bandwidth is reduced to 0.5 Mc. For $e_i = 100$ volts, the bandwidth becomes 1 Mc and 100 kc respectively.

The circuit diagram of the d-c *Amplifier of the Boris Kidric Institute at Belgrade*, designed by D. Mitrovic, is shown in Figure 5.22. The separation of the input and output stages by a cathode follower improves the frequency response and facilitates amplifier adjustments. The characteristics of this amplifier are

$$\begin{aligned} g &= 1500 \\ f_\alpha &= 200 \text{ kc} \\ e_d &= 5 \text{ mv/hr (gain 1)} \\ e_o &= \pm 100 \text{ volts} \end{aligned}$$

For very high performance the amplifier shown in Figure 5.23 is used. By careful choice of components and circuit layout the following performance characteristics are obtained:

$$\begin{aligned} g &= 3000 \\ f_\alpha &= 600 \text{ kc} \\ e_d &= 2 \text{ mv/hr (gain 1)} \\ e_o &= \pm 100 \text{ volts} \\ R_K &= 6 \text{ kilohms} \end{aligned}$$

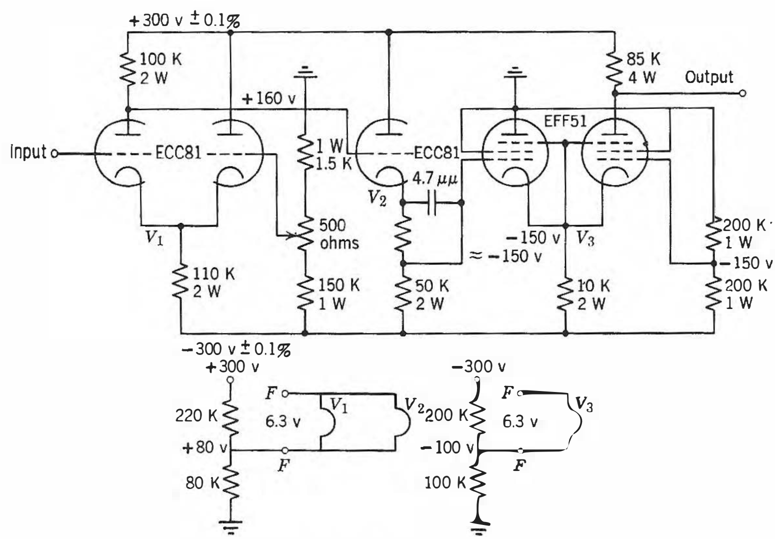


Fig. 5.22 Amplifier of the Boris Kidric Institute, Belgrade.

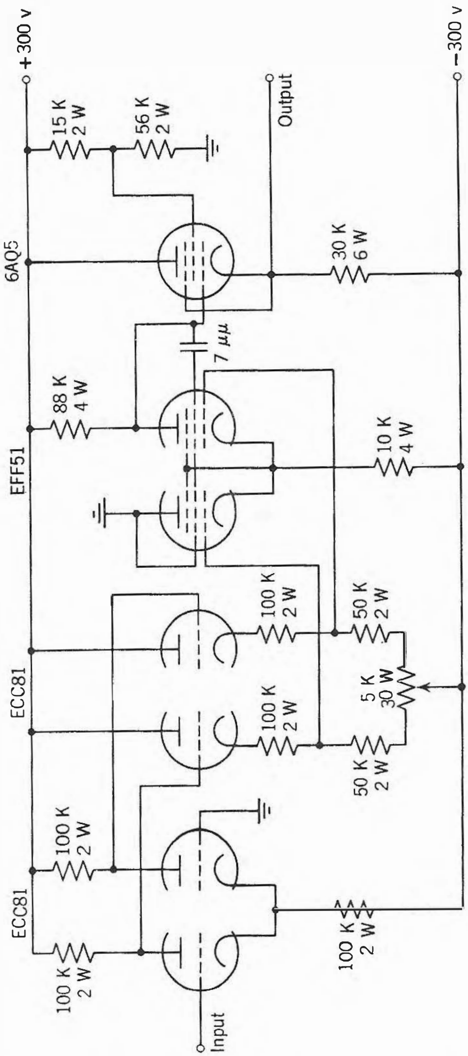
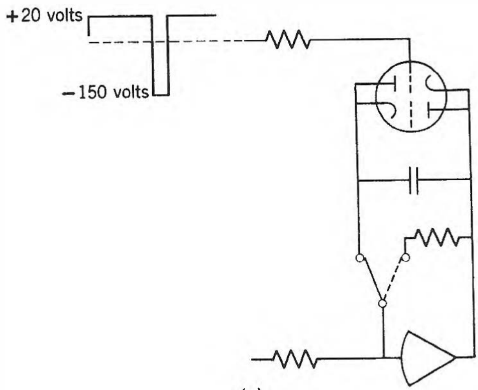


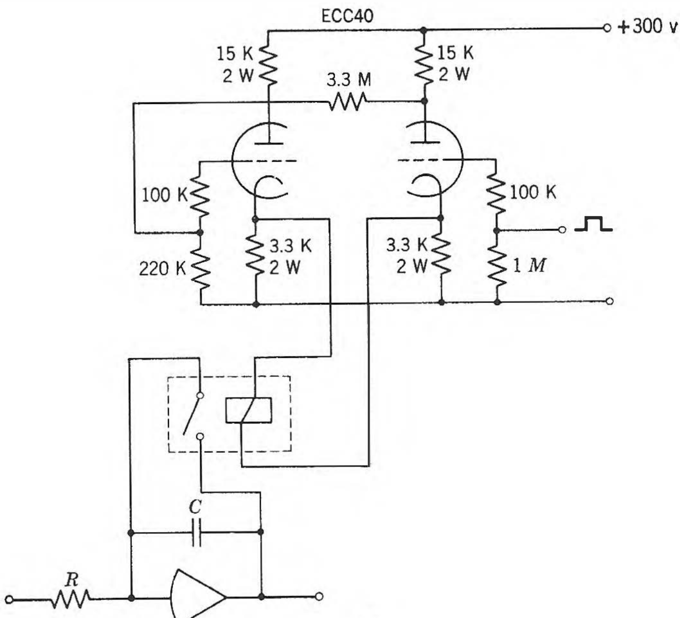
Fig. 5.23 High performance amplifier of the Boris Kidric Institute of Belgrade.

The circuit diagram of an integrator using the amplifier of Figure 5.22 is shown in Figure 5.24*a*. Its characteristics are the following:

$$T_0 = 1.5 \cdot 10^3$$
$$T_1 = 10^{-5}$$
$$Z_o = 75 \text{ ohms (gain 1)}$$
$$e_n = 0.1 \text{ volt}$$



(a)



(b)

Fig. 5.24 (*a*) Integrator using electronic switching. (*b*) Integrator using mechanical switching.

When the input terminal is connected to ground, a noise voltage, e_n, appears at the output of the integrator. This is primarily due to the rectangular-voltage generator which is used to effect the electronic switching. During the time interval during which drift is corrected, the integrator is connected as an inverter, with the aid of a commutator. The electronic switch of Figure 5.24*a* has the advantage of high speed. It can be used for a repetitive work in the range of a few cycles to several kilocycles, but introduces additional noise. For high quality repetitive integration, electromechanical relays are preferred. Figure 5.24*b* shows such an integrator. Each relay has associated with it an electronic circuit so that many integrators can be driven from one source. The maximum practical repetition rates for the mechanical switch are about fifty repetitions per second.

The d-c *Amplifier of the Society of Electronics and Automation, Paris* (as described in Document SEANT 431, September 1955) is shown in Figure 5.25. Its characteristics are

$$\begin{aligned} g &= 30000 \\ f_\alpha &= 100 \text{ kc} \\ i_g &= 2 \cdot 10^{-11} \text{ ampere} \\ e_d &= 5 \text{ mv/hr} \\ e_0 &= \pm 100 \text{ volts} \end{aligned}$$

The *Amplifier for the Repetitive Analyzer of the Technische Hochschule in Darmstadt*, as described by W. Dhen,[6] has several original features which make it very convenient for use in a repetitive analyzer. The complete circuit diagram is shown in Figure 5.26. The circuit uses several a-c amplifier stages. It contains two input terminals that can be

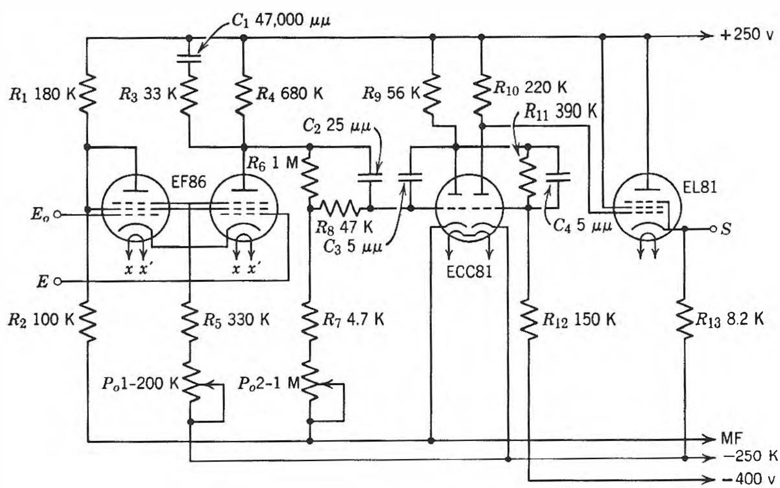


Fig. 5.25 Amplifier of the Society of Electronics and Automation of Paris.

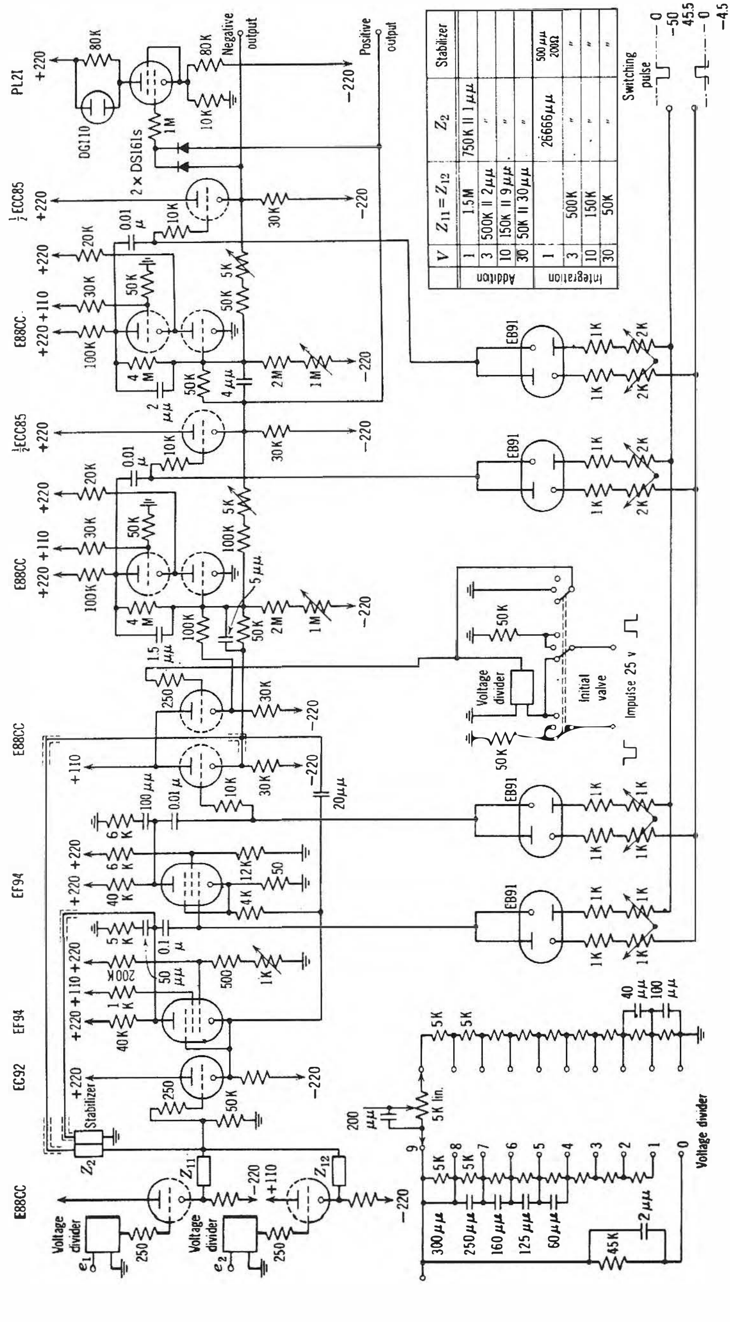


Fig. 5.26 Amplifier of the Technische Hochschule of Darmstadt.

used for addition and integration. The voltage divider at the input consists of a box of decade resistors and a potentiometer. In this way the coefficients can be set with a precision of 0.1%. The output of the voltage divider is applied to a cathode follower. This makes it possible to work at frequencies up to 300 kc. The main amplifier has two stages. The first does not affect the phase. A gain of 20,000 is obtained with the aid of positive feedback that is introduced internally between the cathodes of the pentodes. To each external feedback resistor Z_2 there corresponds a stabilizing circuit which provides a maximum of bandwidth and stability.

The output voltage is not taken directly at one of the terminals of the feedback resistor of the main amplifier; these points are generally not accessible externally. Rather the output voltages are obtained after passing through an inverter (double triode). At this point the initial conditions of the integrator are applied. The output voltage of the first inverter is inverted once more so that positive as well as negative outputs are available. If the amplifier is overloaded, a thyratron fires, and a light is turned on. The output voltage has a range of ± 25 volts. The bandwidth of a typical adder is 80 kc. The amplitude error is less than 1% and the phase shift less than 0.5°. The noise level is always less than 10 mv.

5.6 Errors in Passive Elements

Shortcomings in the three most important passive linear computing elements, which are the fixed resistor, the fixed capacitor, and the resistance potentiometer, can have serious effects on the over-all accuracy of analog computer solutions. Of particular importance in repetitive operation are the shortcomings that affect the dynamic characteristics of the elements. In designing high accuracy analog installations every attempt is of course made to obtain high quality components and to assure their long-term stability by keeping them in a constant-temperature environment. Under many conditions, however, these measures prove inadequate, and a thorough analysis of component errors must be undertaken. Single[7, 8] has developed techniques for the quantitative description of these errors and offers some suggestions for their minimization.

Because of their long-term stability, most computing resistors at present are of the wire-wound type. The dynamic characteristics of wire-wound resistors for frequencies below 100 kc can be approximated by the circuit shown in Figure 5.27. Included in the circuit are the specified d-c resistance R_{dc}, series inductance L, and shunt capacitance C. The impedance as a function of complex frequency can then be expressed as

$$Z = \frac{e_r}{i_r} = \frac{1}{C}\frac{s + R_{dc}/L}{(s^2 + R_{dcs})/(2 + 1)/LC} \tag{5.63}$$

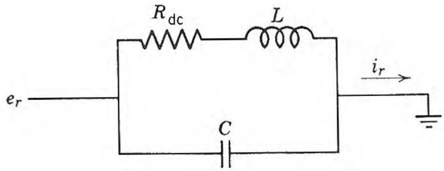


Fig. 5.27 Equivalent circuit of a precision resistor.

For resistors within the conventional range of 100 kilohms to 1 megohm, the time constant L/R_{dc} can be held to less than 10^{-2} sec, whereas the capacitor time constant $R_{dc}C$ will typically be between 2×10^{-6} and 4×10^{-6} sec. This implies that the series inductance can generally be neglected. In adders, the undesired reactances associated with the resistors can be made to cancel if the input and feedback resistors are perfectly matched. For integrators, there is no convenient way of canceling the error caused by shunt capacitance in the resistor.

The precision capacitors used for integration generally deviate from the ideal to a far greater extent than do the precision resistors. Capacitors have a larger temperature coefficient and are generally less stable in value. The electrical characteristics of a typical polystyrene precision capacitor are shown in Figure 5.28. Under certain conditions, a simplified equivalent circuit consisting of the capacitor C in parallel with the leakage resistance R_L can be employed. The leakage resistance must then be assumed to vary inversely with frequency. The capacitance value can be considered constant for many purposes, since this variation with frequency is only about 0.02% per decade. However, this variation must be taken into account in trimming or padding the capacitor to a "precise" specified value.

In selecting potentiometers for use in repetitive analog computer facilities it is necessary to choose between the relatively high-precision but relatively poor frequency response of multiturn potentiometers, and

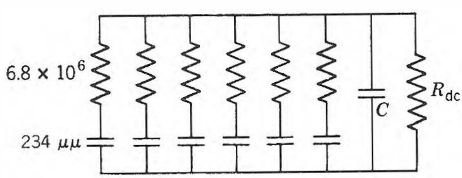


Fig. 5.28 Equivalent circuit of a capacitor.

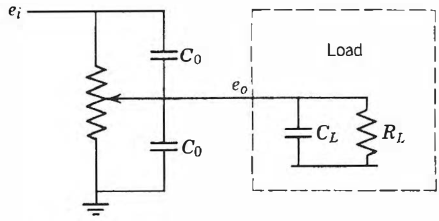


Fig. 5.29 Equivalent circuit of a potentiometer.

relatively low-precision, low-capacitance, single-turn types. In either case, the potentiometers are usually of the copper-mandrel wire-wound construction. In one-shot computers ten-turn potentiometers are used almost exclusively; whereas most, although by no means all, high quality repetitive installations employ single-turn potentiometers. Figure 5.29 shows the equivalent circuit for a typical ten-turn potentiometer. The shunt capacitance C_0 appears to be relatively independent of potentiometer position and of the resistance value of the potentiometer. For a typical ten-turn potentiometer it is approximately 200 $\mu\mu$ in magnitude. For any particular potentiometer displacement it would be possible to add a capacitor from the arm of the potentiometer to the top of the potentiometer to cancel out any phase error resulting from the shunt capacitance. Such compensations are, however, not very practical since the compensating capacitor would then have to vary with both load and displacement. A more complex but more satisfactory method is to employ potentiometers with a number of taps. These permit capacitors to be connected to fixed points along the resistance element as shown in Figure 5.30. Capacitors C_1, C_2, and C_3 are fixed compensating capacitors. Single[8] describes in detail the manner of calculating the value of these compensating capacitors and the improvement that is effected by their use.

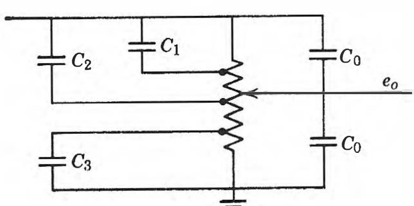


Fig. 5.30 Use of potentiometer taps and auxiliary capacitors to compensate leakage capacitance.

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chapter 6
NONLINEAR OPERATIONS

6.1 General Remarks

Simple combinations of resistors, capacitors, and operational amplifiers suffice to perform virtually all the linear operations required in analog computation. This is not true for nonlinear operations. Here it becomes necessary to introduce additional equipment to effect the nonlinear relationship between the input and output voltage. There are a great number of different methods for realizing such nonlinear functions, and modifications and improvements are constantly being introduced. At the present time, however, it is possible to identify certain dominant trends in nonlinear equipment for analog computers. This is particularly true for repetitive differential analyzers whose large bandwidth requirements eliminate from consideration such commonly used low-speed devices as servomultipliers, servo-driven resolvers, etc. The area of nonlinear operational devices is generally divided into two categories, multipliers and arbitrary function generators.

Well over 50 methods for multiplying one analog voltage by another have been suggested from time to time, and numerous surveys of these schemes have appeared in the literature.[1,2] In this chapter attention is drawn to the devices that have bandwidths sufficiently large to permit their use in repetitive differential analyzers, as well as an accuracy compatible with the rest of the computer system. These operational units fall into three classes.

1. Into the first group fall those methods in which the multiplication is carried out on a continuous ensemble of points within a limited time interval. Clearly the product $e_o(t)$ will also be a continuous function of time; and the precision of the operation will depend exclusively upon the characteristics of the circuit elements. The photoformer and the Hall-effect multiplier are two important members of this class.

2. The second group of multipliers includes those devices in which multiplication is effected on a discontinuous ensemble of points within a limited time interval. In this case the accuracy of the operation depends among other things upon the choice of the ensemble or the sampling rate. The time-division multiplier is an example of such a discontinuous device.

3. In the third class of multipliers no actual multiplication of two voltages is carried out. Rather the product is obtained by the generation of special functions requiring mathematical operations which are more easily realized by analog methods. The quarter-square multiplier is the most widely used device in this class.

The problem of function generation in differential analyzers involves the construction of a device in which a specified relation between input and output voltage can be set manually. Thus the construction of an arbitrary function generator involves the selection of means for introducing the specified functional relationship in such a manner that it is readily translated into an electrical voltage. As with multipliers three methods of arbitrary function generation exist.

1. Continuous function generators.
2. Discontinuous function generators.
3. Function generators employing approximations.

These three classes of devices are reviewed in turn in the next sections. Considerable stress is placed on the technique of "universal function generation" which provides the greatest flexibility using relatively standard components. The chapter concludes with a survey of devices useful for the generation of functions of two dependent variables.

6.2 Continuous-Type Multipliers

The number of electronic multipliers in which two continuous functions are multiplied directly and which have a sufficiently large bandwidth is not great. Only two examples of this type are considered here. One of these is the so-called crossed-fields multiplier described by Macnee.[3] This device is shown in Figure 6.1 and is based on the fact that the force on an electron in a magnetic field is given by

$$F = e(\mathbf{V} \times \mathbf{B}) \tag{6.1}$$

where $\mathbf{V}$ is the electron velocity and $\mathbf{B}$ is the magnetic field intensity; e is the electron charge. Accordingly, an electron beam is made to travel along the axis of a cathode-ray tube; while passing between the first pair of deflection plates, it is given a velocity which is proportional to one of

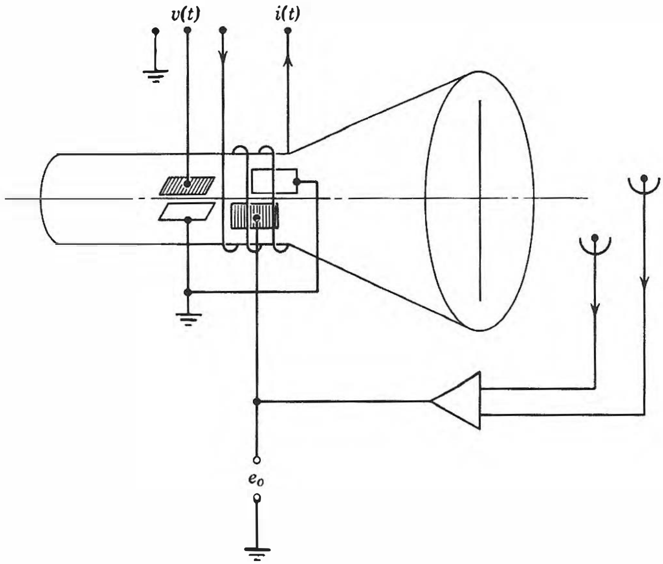


Fig. 6.1 Crossed-fields multiplier.

the variables, represented by the voltage $v(t)$. Around the second pair of deflection plates there is wound a magnetic deflection coil carrying a current $i(t)$ proportional to the other variable $g(t)$ to be multiplied. The beam is therefore deflected by proportional forces according to

$$F = kv(t)i(t) \tag{6.2}$$

which tend to deflect it in a direction normal to the tube axis.

A pair of photocells are employed to detect any horizontal deflection of the beam. The output of these photocells is connected to the input terminals of a differential amplifier whose output is applied to the horizontal deflection plates of the cathode-ray tube. The polarity of this output is such that it acts to counteract the deflecting signal. When such a system is in equilibrium, the output voltage of the amplifier is

$$e_o = kf(t)g(t) \tag{6.3}$$

The original multiplier had an accuracy of the order of 5% and a relatively small bandwidth. Subsequent refinements permitted the attainment of accuracies of 1% with a greatly improved bandwidth. The main difficulty with this design is the magnetic deflection coil, whose appreciable inductance limits the operating frequencies.

A modification of the crossed-field multiplier was proposed by Gundlach.[4] Rather than employing a magnetic deflection coil, this multiplier employs pure electrostatic deflection using a specially constructed deflection-plate structure as shown in Figure 6.2. As in the preceding example, a voltage $v(t)$ is applied to the first pair of deflection plates. The beam has, as it enters the second system of deflection plates, a proportional deflection

$$d_1 = k_1 v(t) \tag{6.4}$$

The second set of deflection plates comprises four hyperbolic plates whose opposite members are electrically connected. A voltage $g(t)$ acts on these plates in such a manner that the electrostatic field is equal to

$$u_2 = k_2 g(t) xy \tag{6.5}$$

where x and y are the coordinates defined by Figure 6.3. When the beam enters the second set of plates, its position is given by the coordinates 0, y, and a deflecting force is applied to it in the horizontal direction which is proportional to

$$d = kv(t)g(t) \tag{6.6}$$

Near the face of the tube there is located a detection system comprising two anodes separated by a slot. The deflection of the beam to one or the other side produces a difference in output current. This difference is amplified by a differential amplifier whose output voltage is applied to a third set of deflection plates in such a manner that the beam is forced back

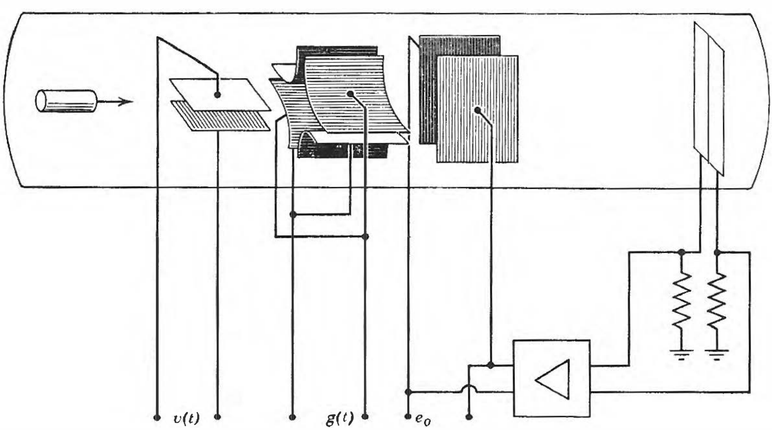


Fig. 6.2 Multiplier using a hyperbolic electrostatic field.

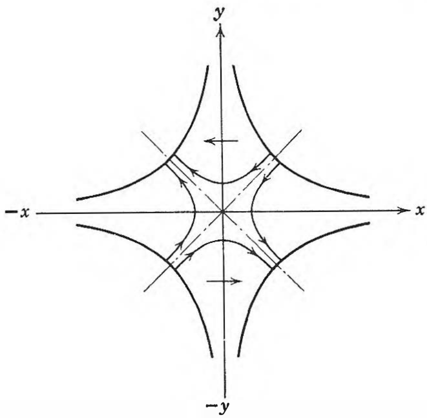


Fig. 6.3 Positions of electrodes in the hyperbolic electrostatic-field multiplier.

to the vertical slot. When the system is in equilibrium, the output voltage of the amplifier is equal to the desired product,

$$e_o = k_m v(t) g(t) \tag{6.7}$$

With such a tube one can attain an accuracy of 0.5% in multiplying constant amplitude sine waves, a bandwidth of 100 to 200 kc, and four-quadrant operation. Although these characteristics make this tube a very attractive repetitive differential analyzer component, the difficulties involved in the manufacture of a special-purpose cathode-ray tube make it a very expensive device.

Another, more recent, approach to the realization of a continuous multiplier led to the development of the so-called Hall-effect multiplier, as described by Lofgren[5] and more recently by Glinski and Landolt.[6] The heart of this multiplier is a semiconductor generating a Hall-effect voltage. This voltage V_y is related to the applied current I_x and the applied magnetic field B_z by

$$V_y = \frac{R_h I_x B_z}{t} \tag{6.8}$$

where R_h is the Hall constant and t is the thickness of the material, as indicated in Figure 6.4.

Among commercial available materials, germanium, silicon, indium arsenide, and indium antimonide have been used as Hall-effect multipliers. Of these, silicon generates the highest Hall voltage at low currents. Glinski, however, cites a number of practical advantages in favor of

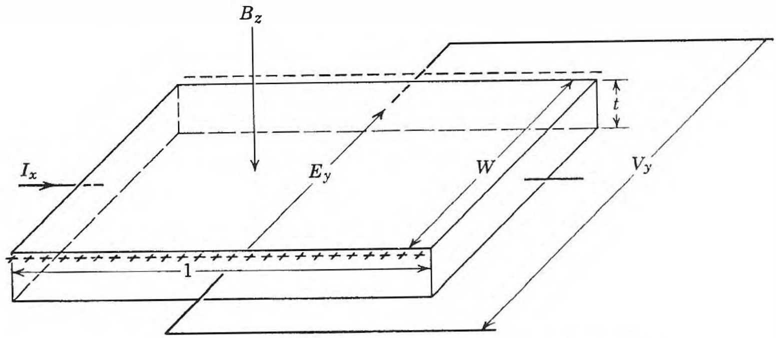


Fig. 6.4 Electric and magnetic fields in a Hall element.

indium arsenide, particularly its low resistivity which makes it convenient to use in conjunction with transistor amplifiers. The basic Hall-effect multiplier circuit is shown in Figure 6.5. Three operational amplifiers are required. In analyzing errors in a specific Hall-effect multiplier for a maximum output voltage of 100 mv, an input current of 250 ma and a magnetic field of 5000 gauss, Glinski reports linearity errors not exceeding 1%, noise less than 5 μv and a frequency response down 3 db at 80 kc. No practical frequency limitation exists in the Hall element itself. The practical limitation in bandwidth is imposed by the frequency response of the coil used to apply the magnetic field.

6.3 Discrete-Type Multipliers

Another approach to analog multiplication is to multiply the two functions at discrete points rather than in a continuous manner. One

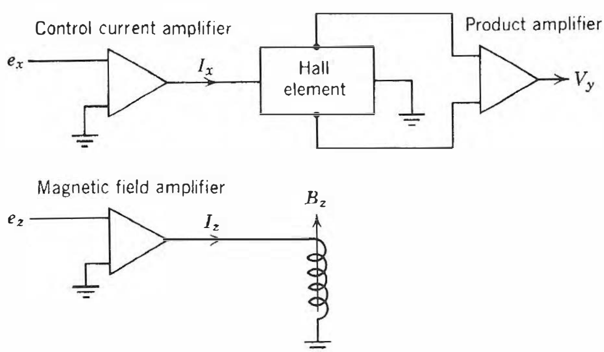


Fig. 6.5 Basic elements of Hall-effect multiplier.

multiplier constructed on this principle involves the generation of rectangular pulses whose height or amplitude is a function of one of the variables, and whose width is a function of the other. These have been described by a number of investigators[6, 7, 8, 9] and are termed "time-division" multipliers. In one-shot differential analyzers, multipliers of this type have attained accuracies of the order of 0.1%. However, since such multipliers require an integrator at their output to translate the area under each impulse into a continuous voltage, they lack the bandwidth required for high-speed operation.

A second type of multiplier of the discrete type is shown in Figure 6.6. Two transient voltages are generated simultaneously such that

$$\begin{aligned} x &= uf(t) \\ z &= yf(t) \end{aligned} \tag{6.9}$$

where x and z are identical linear functions of input voltages u and y. With reference to Figure 6.7, at the instant of time $t = t_1$,

$$f(t_1) = \frac{x}{u} \tag{6.10}$$

and

$$f(t_1) = \frac{z}{y}$$

so that

$$z = \frac{xy}{u} \tag{6.11}$$

if

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt} \ll \frac{df}{dt}$$

That is, if the variables are relatively constant during the sampling interval

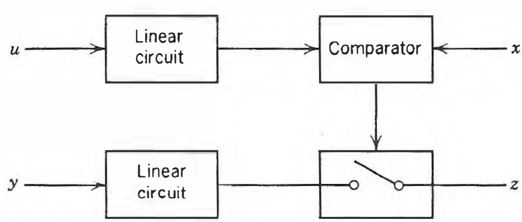


Fig. 6.6 Block diagram of multiplier using two identical linear circuits.

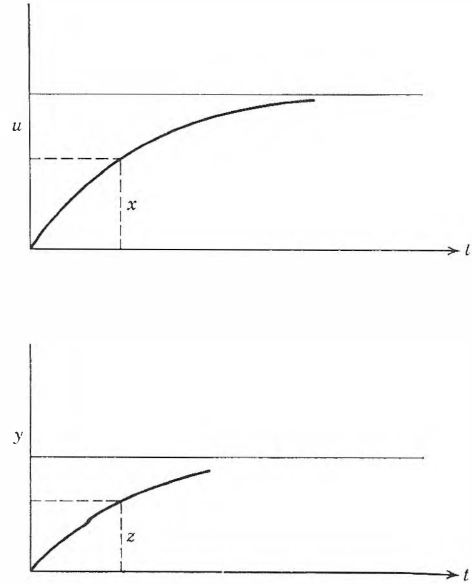


Fig. 6.7 Transient voltages in multiplier of Figure 6.6.

the operations of multiplication and division are effected. A multiplier of this type requires the following elements:

Two identical linear circuits with responses $f(t)$.

A comparator to establish the identity between two voltages.

An electronic switch.

An amplifier to amplify the output signal.

The role of linear circuits is to generate two identical waveforms $f(t)$ which can be linearly modulated by u and y respectively. The waveforms $f(t)$ are generated at high repetition rates with respect to dx/dt and dy/dt. The comparator of Figure 6.6 establishes the equality $x_k = uf(t_k)$. Thus a series of time instants t_k is obtained at which the switch is closed, and corresponding values of other linear circuit voltages $yf(t_k)$ are transmitted to the output. The sampling frequency, that is, the repetition rate of $f(t)$ thus defines in each minor step a product value z_i. Since the outputs of both linear circuits are identical and sampled at the same $t = t_k$, they do not appear in the equation for z. The scale factor is defined by u.

For repetitive computation this method, although simple and reliable, has the disadvantage of low speed. In repetitive computers the slowly varying voltages x, y, or u are themselves repetitive waveforms. Consequently, the sampling frequency, that is, the repetition rate of $f(t)$ must be much higher than the repetition rate. This sets a limit for satisfactory operation of switches, comparator, and other circuit elements involved. Note also that the circuit used to produce $f(t)$ must be brought to the same initial conditions in each sampling step. This requires additional switches working at high switching frequencies.

Different versions of this type of multiplier differ primarily as to the form of the function $f(t)$. Evidently, the linear circuit should be of as simple a form as possible. One frequently used function is

$$f(t) = e^{\alpha t} \tag{6.12}$$

as shown in Figure 6.7. In this case, two identical exponential functions with the same initial conditions are generated during each sampling interval. A more detailed diagram of this multiplier is shown in Figure 6.8. The switches S_1 and S_2 serve to charge the capacitors C to the instantaneous value of the slowly varying input function. A decaying exponential having RC as its time constant is thereby generated during each time interval. A multiplier of this type[10, 11] operates at frequencies up to 400

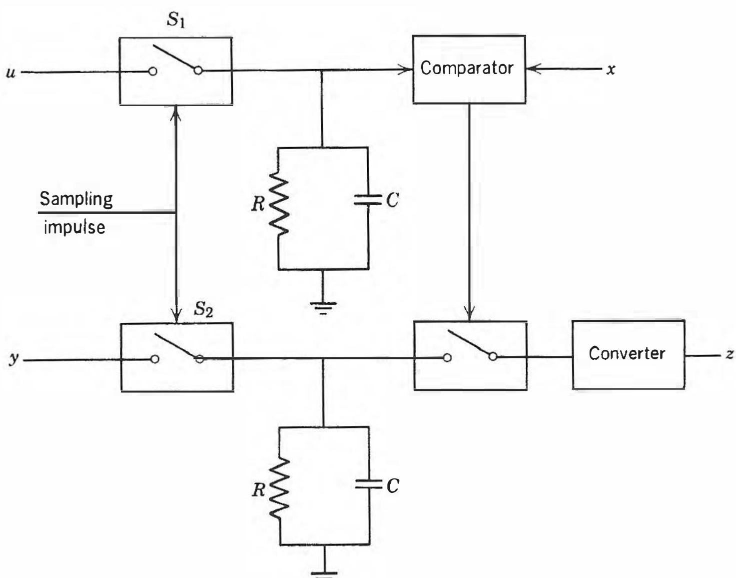


Fig. 6.8 Multiplier using exponential function.

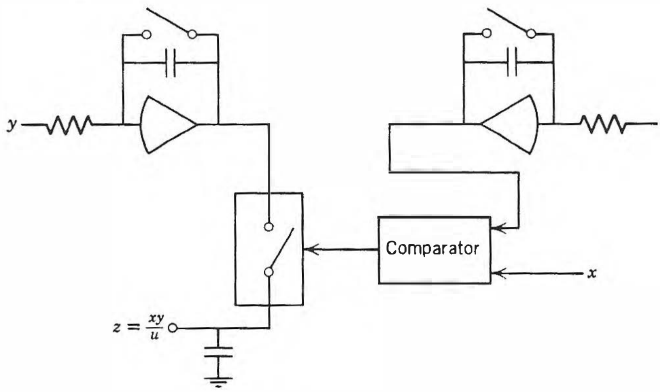


Fig. 6.9 Multiplier using ramp voltage.

cycles/sec with a precision of 0.1%. For a precision of the order of 1% the working frequency can be increased by a large factor.

A similar type of multiplier described by Mitrovic[12] employs

$$f(t) = ut \tag{6.13}$$

The circuit diagram is shown in Figure 6.9. This device is particularly useful for repetitive differential analyzers because it uses only such simple and standardized elements as integrators and amplitude comparators. Since in repetitive computers, the integrators are equipped with switches to reset initial conditions after each computing interval, only one additional switch is necessary.

A third variation of this type of multiplier is due to Isabeau[13] and uses as the function $f(t)$ the solution of the equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = 0 \tag{6.14}$$

The circuit diagram of this multiplier is shown in Figure 6.10. Switches S_1 and S_2 assure the generation of $f(t)$ having always the same initial conditions. This unit can serve simultaneously as a multiplier and as a divider. The results obtained with a prototype model have a precision of the order of 2% and a bandwidth of 1 kc for a sampling frequency of 15 kc.

6.4 Indirect Multipliers

Numerous ways exist which multiplication can be performed with the aid of function generators. Among the more widely used algebraic and

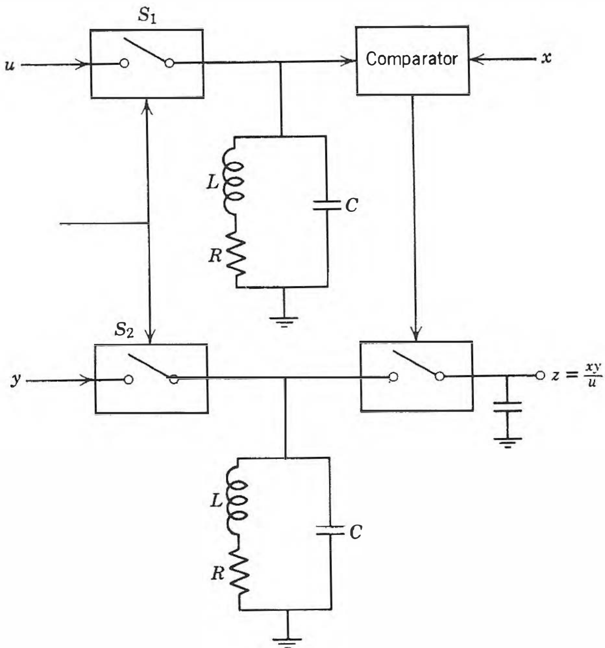


Fig. 6.10 Multiplier using a linear second order differential equation.

transcendental expressions by which the product of the two voltages can be generated are

$$\begin{aligned}
&\text{Logarithmic} &&= \log x + \log y = \log xy \\
&\text{Trigonometric} &&= \cos(x + y) + \cos(x - y) = 2\cos x \cos y \\
&\text{Hyperbolic} &&= \cosh(x + y) - \cosh(x - y) = 2\sinh x \sinh y \\
&\text{Parabolic} &&= (x + y)^2 - (x - y)^2 = 4xy
\end{aligned} \tag{6.15}$$

These methods can be classified by the precision of the multiplication operation and by the number of operational elements that are required. The criterion for comparison of accuracy is the quotient $\Delta\rho/\epsilon$, where ϵ is the maximum value of the absolute error of the function generator, and $\Delta\rho$ is the maximum error of the product. Table 6.1 lists the results of such an accuracy analysis. It is clear the parabolic method is superior from this point of view.

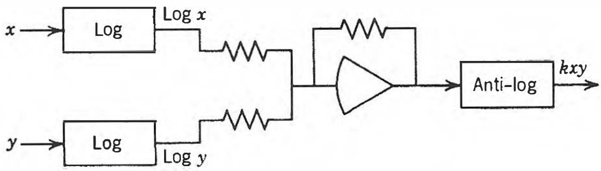


Fig. 6.11 Logarithmic multiplier.

Table 6.1

Type	$\frac{\Delta\rho}{\epsilon}$
Logarithmic	10.5
Trigonometric	3
Hyperbolic	3.7
Parabolic	2

Figure 6.11 shows a multiplier using the logarithmic formula. In Figure 6.12 a multiplier using the parabolic formula is shown. The fundamental problem is the choice of the function generator, since the other elements in the multiplier are the same as those used in linear operations. In practice the diode function generators using segmented approximations to the algebraic or transcendental equations are most often used. This is considered in more detail in Section 6.7.

6.5 Division

Relatively few devices have been developed primarily for analog division. There are a number of reasons for this. In programming a

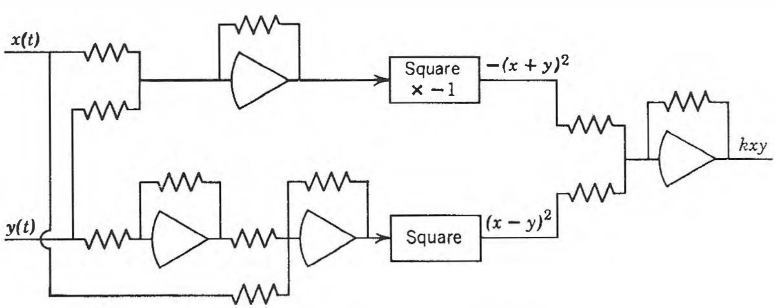


Fig. 6.12 Quarter-square multiplier.

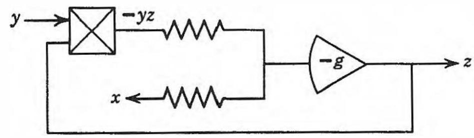


Fig. 6.13 Implicit-function circuit for division.

computer an effort is always made to transform or rearrange equations in such a way that division becomes unnecessary, in order to avoid the difficulties which result when the divisor becomes very small. Moreover, it is always possible with the aid of implicit function generating techniques to employ a multiplier for the purposes of dividing, as shown in Figure 6.13. This circuit solves the implicit equation

$$x - yz = 0 \tag{6.16}$$

so that

$$z = \frac{x}{y}$$

In this case, the high gain amplifier having a gain $-g$ acts effectively as a servo, adjusting the output voltage z so as to make the voltage at its input terminal very nearly equal to zero. In such a feedback system it is always necessary to consider stability conditions. As shown in Figure 6.13 the operation of division is evidently limited for this reason to positive values of y. The polarities of z and x are not restricted.

Other methods of realizing the operation of division involve the use of function generators. These are used to generate the reciprocal of one input function. The output of the function generator is then multiplied by the other variable. As has been shown, several of the commonly used multipliers provide quotients as well as products. For these reasons division is not considered as a fundamental operation in differential analyzers.

6.6 Direct Function Generation

Among the most widely used direct function generators in repetitive analog computers is the so-called photoformer illustrated in Figure 6.14. A portion of the screen of the cathode-ray tube is covered by an opaque mask which has the contour of the desired function. The tube is biased in such a way that the beam falls above the mask in the absence of a correcting signal. However, a photocell, which is sensitive to the light emitted by the beam, generates a voltage which when applied to the vertical deflection plates through an amplifier acts to deflect the beam downward beneath the mask. If the beam is made to sweep in a horizontal direction by a voltage

$$e = g(t) \tag{6.17}$$

the feedback circuit will act to force the beam to follow the upper edge of the mask. Under these conditions the output voltage of the amplifier will be proportional to the specified function

$$y = f[g(t)] \tag{6.18}$$

The mask on the oscilloscope screen then serves to store or memorize the specified functional relationship, while the feedback system and the photoelectric cell act to transform this information into an electrical signal.

The fabrication of an accurate and precise photoformer presents considerable problems. Of particular difficulty are the construction of an accurate mask and the realization of a satisfactory optical system. With very careful construction techniques accuracies of the order of 0.5% with a bandwidth of 100 cycles/sec have been reported by Pederson.[14] Elgeskog[15] has analyzed the errors involved in the construction and application of photoformers in considerable detail. He presents both a static and a dynamic analysis, and he also considers the errors caused by the noise and the optical system. Hancock[16] has demonstrated that it is possible to attain 1% accuracies with a bandwidth of 10 kc and output noise voltages of 10 mv for a dynamic range of ±30 volts.

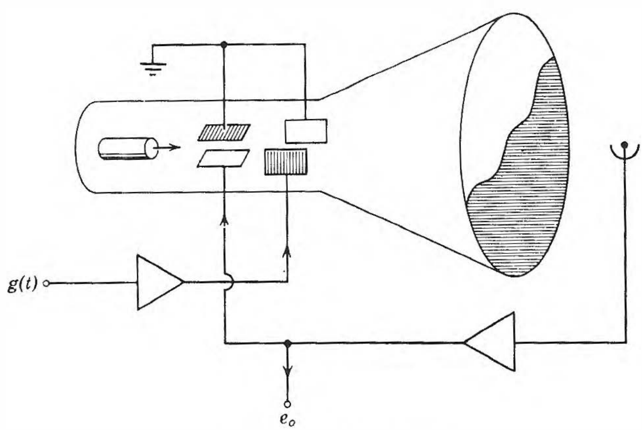


Fig. 6.14 Photoformer.

6.7 Function Generation by Approximations

Within limited intervals, arbitrary functions can be approximated by analytic mathematical expressions which are relatively easy to generate by means of analog computers. The approximation of a given function can be effected in two ways. One method is to employ the equation

$$f(x) = \alpha_1\phi_1(x) + \alpha_2\phi_2(x) + \cdots + \alpha_n\phi_n(x) + R_n(x) \tag{6.19}$$

where $\phi_i(x)$ are usually algebraic or trigonometric polynomials, and $R_n(x)$ is the error resulting from terminating the series after the nth term. The second method is to select a simple function, for example, a polynomial such that it fits the specified function at n points. For example,

$$f(x) = P_n(x) + R_n(x) \tag{6.20}$$

where $P_n(x)$ is an nth order interpolation polynomial, for example, a Lagrange or Newton polynomial.

The simplest method for generating a function $e = f(t)$ at an ensemble of discrete points is to employ potentiometers to "memorize" the values $f(t_i)$. In this method the electrical output is obtained with the aid of a switching system as shown in Figure 6.15. As switches semiconductors,

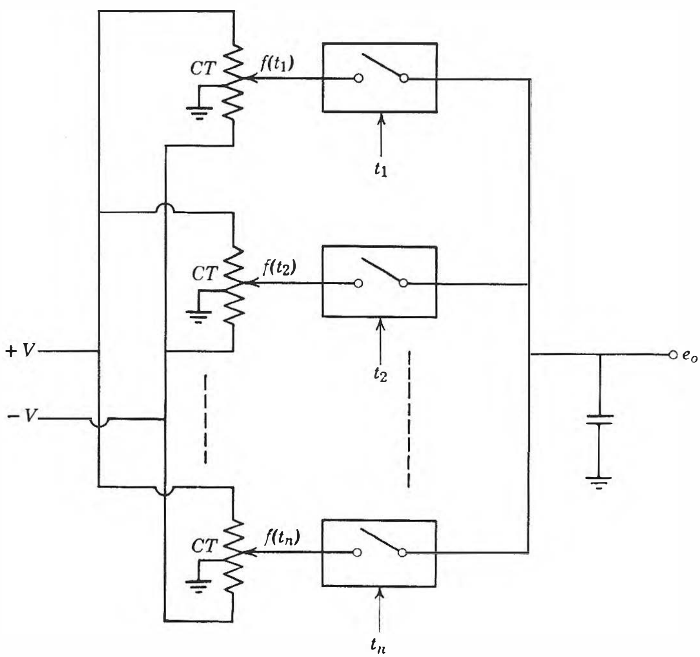


Fig. 6.15 Function generation using an ensemble of discretely spaced points.

vacuum tubes, or gas tubes may be used as determined by the speed of operation of the repetitive differential analyzer. One generator of this kind as described by Wentzel[17] employs dekatron-type tubes and thirty potentiometers. The output of the generator is a series of step functions. If instead of the specified function the potentiometers are adjusted to the time derivatives of the function df/dt a smooth output is obtained upon integration. Such a function generator can attain an accuracy of 0.2% and the entire function can be read out in 75 msec, and can be realized very economically and reliably. Its principal shortcoming is that one cannot generate a function of a function

$$e = f[g(t)] \tag{6.21}$$

Generators employing linear approximations are based upon Equation 6.20. The function which is to be approximated is given for

$$k = 0, 1, 2, \ldots, n$$

points. For interpolation between two successive points one employs

$$f(x) = f(x_{k-1}) + \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} (x - x_{k-1}) \tag{6.22}$$

This equation indicates that as far as the signs of the function and its first differences $\Delta f = f(x_k) - f(x_{k-1})$ are concerned, the following possibilities exist:

I. $f(x_k), f(x_{k-1}) > 0$

$x_k, x_{k-1} > 0$

(*a*) $\Delta f > 0$

(*b*) $\Delta f < 0$

II. $f(x_k), f(x_{k-1}) > 0$

$x_k, x_{k-1} < 0$

(*a*) $\Delta f > 0$

(*b*) $\Delta f < 0$

III. $f(x_k), f(x_{k-1}) < 0$

$x_k, x_{k-1} < 0$

(*a*) $\Delta f > 0$

(*b*) $\Delta f < 0$

IV. $f(x_k), f(x_{k-1}) < 0$

$x_k, x_{k-1} > 0$

(*a*) $\Delta f > 0$

(*b*) $\Delta f < 0$

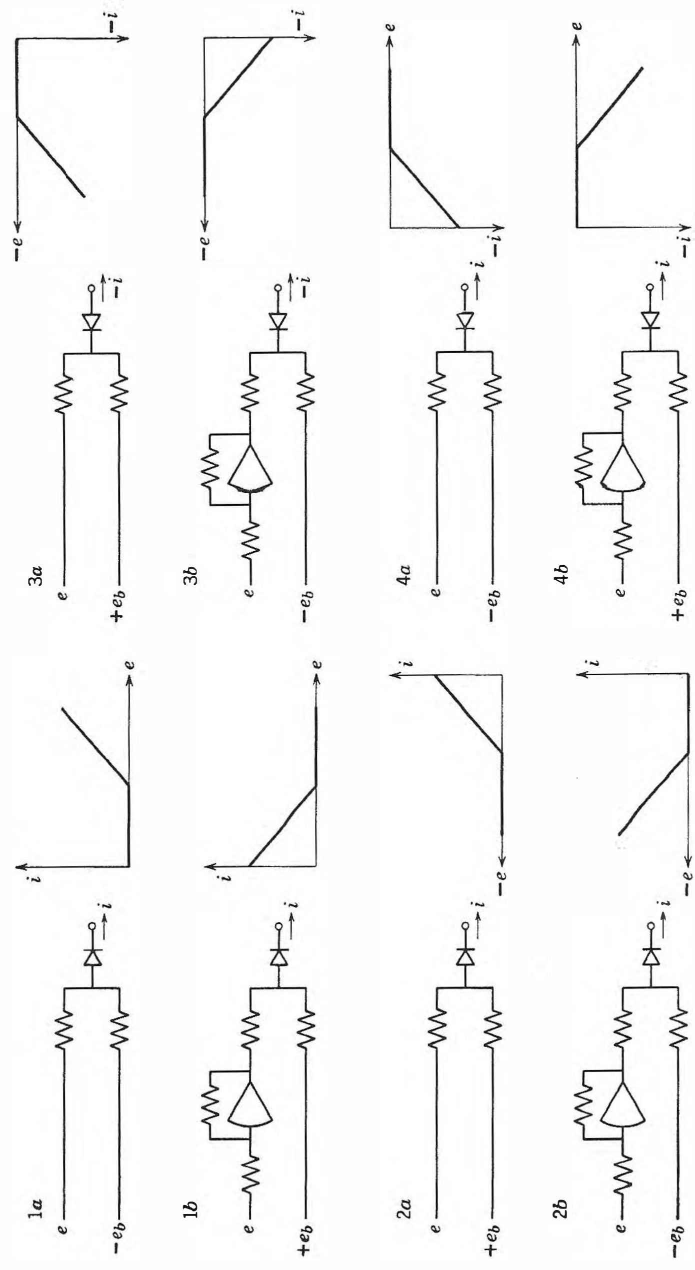


Fig. 6.16 Eight possible diode connections for function generation.

If nonlinear elements are available whose ideal characteristics are

$$R(e) = \begin{vmatrix} \infty & e \leq e_b \\ 0 & e > e_b \end{vmatrix}$$

Equation 6.22 can be realized electrically.

The simple diode approaches these characteristics provided its forward resistance is very low and its reverse resistance very high. Figure 6.16 illustrates a variety of diode circuits which can be employed to realize all of these possibilities. By combining these elements, functions in all four quadrants can be generated. $+e_b$ and $-e_b$ represent positive and negative bias supplies respectively. A description of integration of these circuits in actual function generators follows.

From another point of view the diode function generator is one form of the general operational computing element described in Section 5.1. Kogan[18] has made an important contribution to the theory of operation of such function generators. His conclusions are summarized briefly below. Consider the operational unit shown in Figure 6.17 having elements such that the current i and i_f are nonlinear functions of the voltages e and e_o

$$\begin{aligned} i &= f_1(e) \\ i_f &= f_2(e_o) \end{aligned} \tag{6.23}$$

No restrictions are placed on the character of the functions $f_1(e)$ and $f_2(e_o)$ except the general hypothesis that only linear (polygonal) approximations will be considered. For a sufficiently large value of amplifier gain g and zero grid current

$$f_2(e_o) = -f_1(e) \tag{6.24}$$

Also in accordance with the stability condition

$$\frac{df_2(e_o)}{de_o} > 0$$

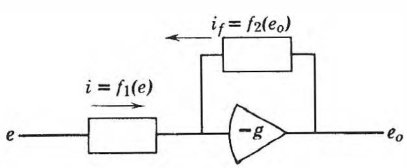


Fig. 6.17 Operational amplifier with nonlinear input and feedback impedances.

The transfer function of the amplifier in differential form can be expressed as

$$\frac{df_2(e_o)}{de_o}\frac{de_o}{de} = -\frac{df_1(e)}{de} \tag{6.25}$$

where

$$\frac{de_o}{de} = \frac{Y_1(e)}{Y_2(e_o)} = S_e \tag{6.26}$$

and

$$\frac{di}{de} = Y(e)$$

When a polynomial approximation is used Equation 6.26 becomes

$$\Delta e_{ok} = -\frac{\Delta Y_{1k}}{\Delta Y_{2k}}\Delta e_k \tag{6.27}$$

where

$$\begin{aligned} \Delta e_k &= e - e_k & \Delta e_{ok} &= e_o - e_{ok} \\ \Delta Y_{1k} &= 0 & e &\leq e_k \\ \Delta Y_{1k} &= \text{const.} & e &> e_k \\ \Delta Y_{2k} &= 0 & e_o &\leq e_{ok} \\ \Delta Y_{2k} &= \text{const.} & e_o &> e_{ok} \end{aligned}$$

The theoretical question now arises where and how to introduce the nonlinear relationships; that is, should the nonlinear functions be the input impedance, the feedback impedance, or should both impedances be nonlinear?

It is known that in function generators the errors become larger as the slope S_e is increased, so that slope must not exceed a certain limit. This restricts the class of functions that may be realized using diode function generators. Let the slope be expressed in the form

$$S_e = -\frac{S_{r1}}{S_{r2}} \tag{6.28}$$

where

$$\begin{aligned} S_{r1} &= \frac{di}{de}\frac{e_{\max}}{i_{\max}} = Y_1\frac{e_{\max}}{i_{\max}} \\ S_{r2} &= \frac{di_f}{de_o}\frac{e_{o\max}}{i_{f\max}} = Y_2\frac{e_{o\max}}{i_{f\max}} \end{aligned} \tag{6.29}$$

Equation 6.29 demonstrates that a specified nonlinear characteristic can be obtained with a smaller variation in slope if the nonlinear elements are placed in both branches.

For

$$\frac{S_{e\max}}{S_{e\min}} \gg 1$$

the given function is divided into two intervals. In the first

$$0 \le S_e \le 1$$

whereas in the second

$$1 \le S_e \le \infty$$

The first part of this nonlinear characteristic is handled by the input impedance of the amplifier, whereas the second part is handled by the feedback impedances.

One problem in using a number of diode function units as a part of a function generator is to prevent interaction between separate units, so that a change in the slope of one unit has a relatively negligible effect upon the operation of other units in the function generator system. One approach to this problem, shown in Figure 6.18, is described in some detail by Dhen.[19] The function element is energized by two regulated voltage supplies e and $-e$ whose output impedances are of the order of 1 ohm. With the aid of potentiometers P_1 and P_2, the break points of the diodes can be placed in any one of the four quadrants. The slope can be made positive or negative, and potentiometer P_3 is employed to adjust the magnitude of the slope. Moreover, since the segments operate in parallel their interaction is very small.

Another approach is shown in Figure 6.19 and discussed in greater detail by Santesmases.[20] Here a cathode follower is employed to provide isolation of the diode segments. In this way slopes up to approximately 80° can be attained.

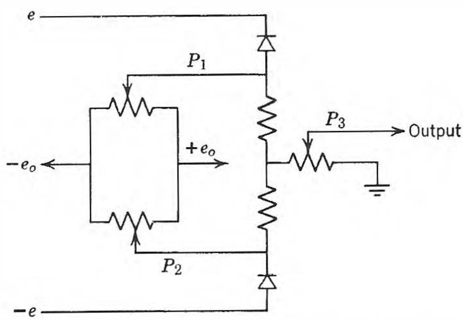


Fig. 6.18 Element of a diode function generator.

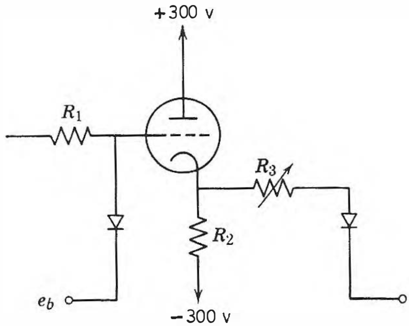


Fig. 6.19 Diode function generator using cathode follower.

More recently a method which provides even greater protection against interaction of individual segments was introduced by Miura.[21] Assume that the function to be generated has the characteristics shown in Figure 6.20*a*. If conventional function generating techniques are to be employed, the individual segments which must be generated are those shown in Figure 6.20*b*. In Miura's method, on the other hand, the segments shown in Figure 6.20*c* are generated. The circuit diagram of this generator is shown in Figure 6.20*d*. Since the slope of each segment is reduced to zero by suitable biasing means as soon as the succeeding segment becomes operative, the slope of each diode section can be adjusted separately without taking into account the slopes of the preceding sections. This greatly facilitates the adjusting procedure. It is claimed that a square wave can be reproduced with such a function generator at 1 kc with a phase shift of no more than 1°.

Other problems in the construction of function generators, in the choice and specification of diodes, in the calculation of resistances and currents, and other details have been treated by Burt[22] and Talancev.[23]

In conclusion, the complete schematic diagrams for the generation of two special functions often encountered in analog computations are presented. Figure 6.21 is the schematic diagram of a squaring circuit described in detail by Gurov.[24] The pertinent equations are the following:

$$e_o = 0.01 e_i^2$$

$$i = -10^{-4} e_i^2 \text{ (ma)}$$

$$i_f = 10^{-2} e_o \text{ (ma)}$$

Two input signals are employed, *e* and $-e$. The resistance calculations are shown in Table 6.2.

Table 6.2

K	e_k, V	i_k, ma	Δe_k	Δi_k	R_k	r_k
0	0	0	10	0.01	1000	—
1	10	0.01	10	0.03	500	1500
2	20	0.04	10	0.05	500	750
3	30	0.09	10	0.07	500	500
4	40	0.16	10	0.09	500	375
5	50	0.25	10	0.11	500	300
6	60	0.36	10	0.13	500	250
7	70	0.49	10	0.15	500	214
8	80	0.64	10	0.17	500	187
9	90	0.81	10	0.19	500	169

As a second example consider the generation of the function

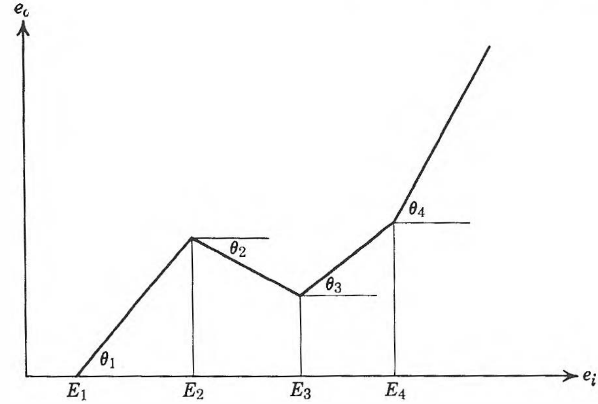
$$y = \sin x \tag{6.30}$$

The schematic of this diode function generator, shown in Figure 6.22, was originally presented by Hartree.[25] The values of the slopes for various values of the input voltage K are given in Table 6.3. With these settings the error never exceeds 10^{-3}.

Table 6.3

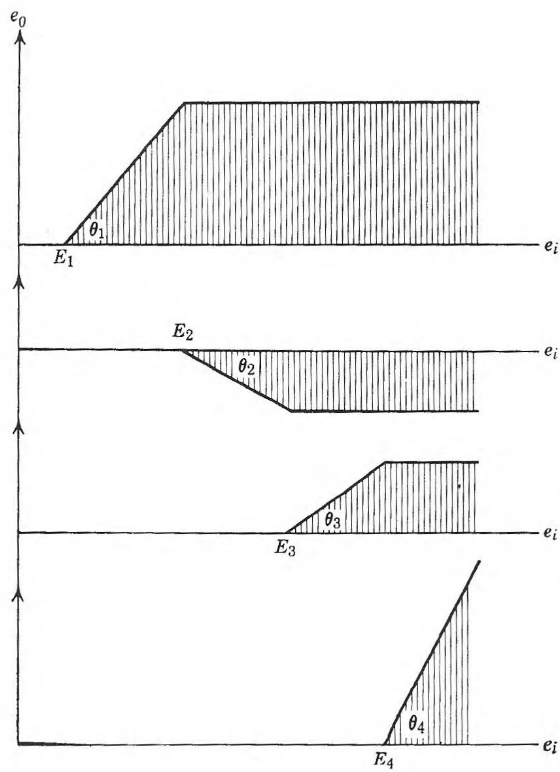
Points	Slope
1/10 radian	
1	0.0245
2	0.0341
3	0.0392
4	0.0435
5	0.0472
6	0.0499
7	0.0528
8	0.0541
9	0.0569
10	0.0575
Final Slope	0.5403

(a)



e_0
θ_1
θ_2
θ_3
θ_4
E_1
E_2
E_3
E_4
e_i

(c)



e_0
θ_1
E_1
e_i
E_2
θ_2
e_i
θ_3
E_3
e_i
θ_4
E_4
e_i

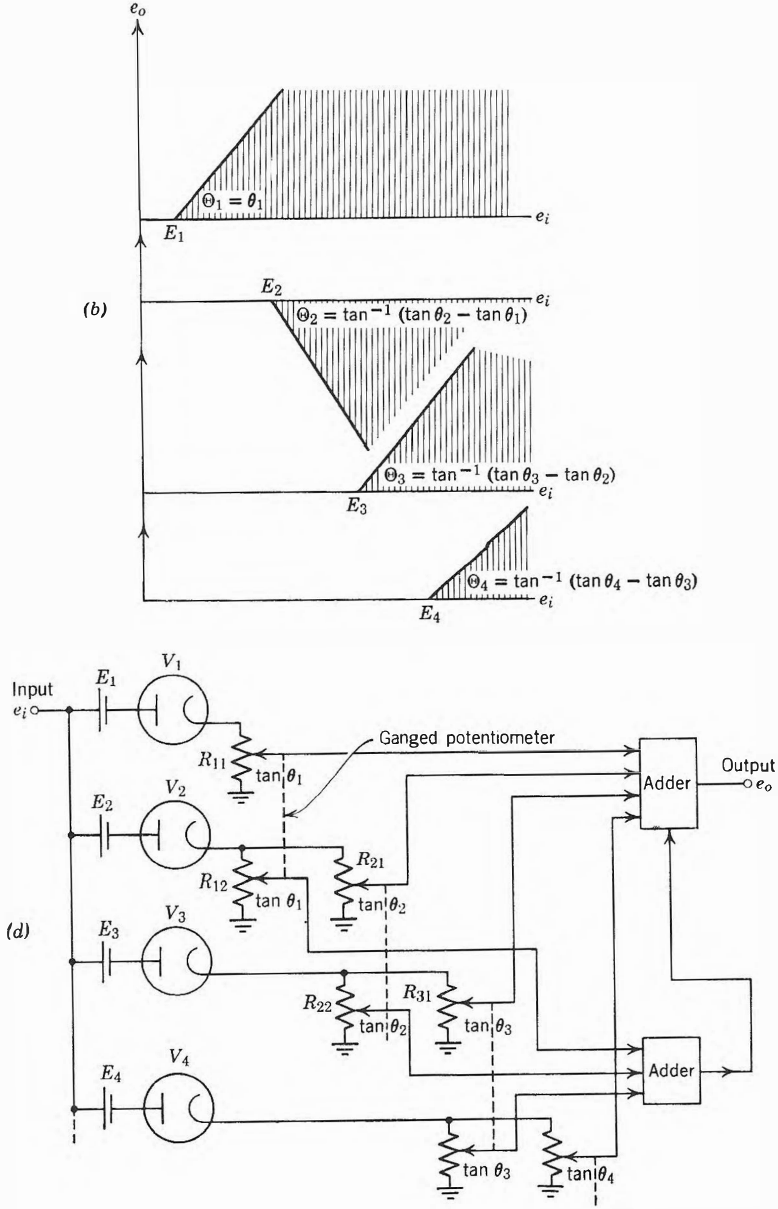


Fig. 6.20 (*a*) Linear approximation of a continuous function. (*b*) Linear approximation by conventional diode function generators. (*c*) Linear approximation by parallel-type diode function generator. (*d*) Circuit diagram of parallel-type diode function generator.

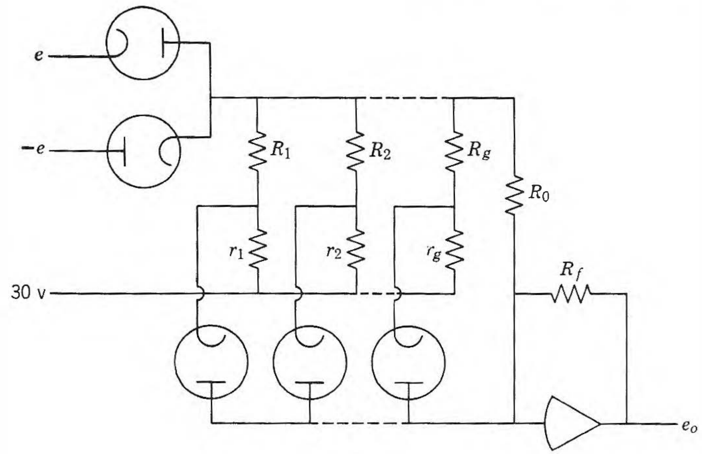


Fig. 6.21 Diode squaring circuit.

Equation 6.19 is also useful for the construction of function generators. It is only necessary to find for $\phi_n(x)$ a form which is easy to realize on the computer. In practice one frequently takes

$$\phi_n(x) = a_n x^n \tag{6.31}$$

Polynomials $a_n x^n$ are obtained by successive integrations of a d-c voltage. In Zanobetti's[26] function generator, polynomials up to the fifth degree are employed. Thus

$$\frac{1}{25}e_o = a_0 + a_1\left(\frac{e}{25}\right) + a_2\left(\frac{e}{25}\right)^2 + a_3\left(\frac{e}{25}\right)^3 + a_4\left(\frac{e}{25}\right)^4 + a_5\left(\frac{e}{25}\right)^5 \tag{6.32}$$
$$-1 \leq a_n \leq +1$$

With such a method it is possible to generate a sinusoid with an error of less than 1%. The chief advantage of this method of function generation is that it does not require the construction of a special function generator for each function. The great shortcoming is that it is limited to simple functions. If one does not know the functional relationship $f[g(t)]$ in advance, the coefficients a_n cannot be calculated.

For approximation by trigonometric polynomials

$$\phi_n(x) = a_n \sin 2n\pi x \tag{6.33}$$

in Equation 6.19. The functions $\sin 2n\pi x$ are easily realized electronically.

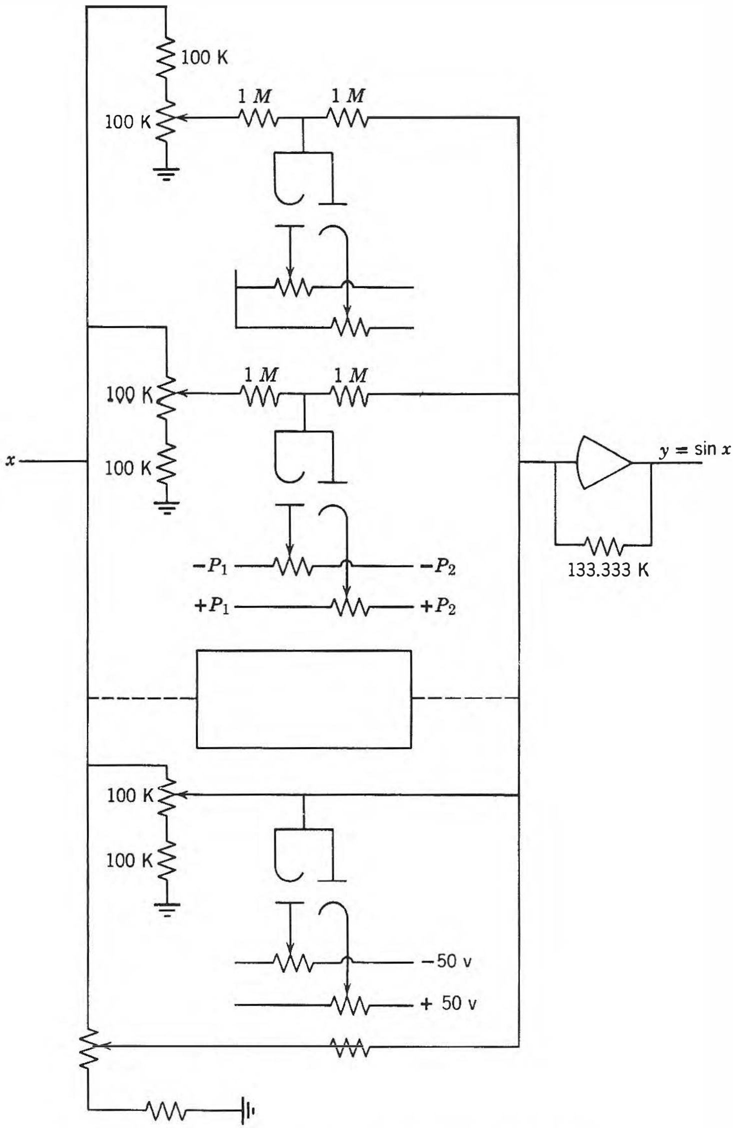


Fig. 6.22 Diode circuit for generation of sin x.

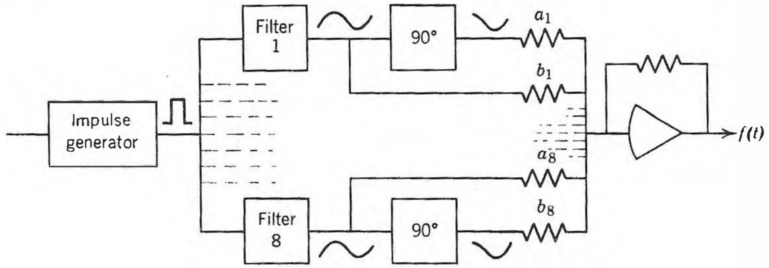


Fig. 6.23 Harmonic generator.

A function generator using the equation

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos 2n\pi x + b_n \sin 2n\pi x) \tag{6.34}$$

was described by Bruk.[27] A maximum of eight harmonics are employed to realize the specified functions. The harmonic components necessary for the application of Equation 6.34 are obtained from a central unit as shown in Figure 6.23. A stable oscillator applies a 10 kc signal to the harmonic generator; these sinusoidal oscillations are then transformed into rectangular impulses, and a series of filters are employed to separate the first eight harmonics; 90° phase shift circuits are then employed to obtain the cosine terms. A simple adding circuit is used to sum the sixteen components.

In order to apply these methods to the generation of functions of the dependent variable $f[g(t)]$ the circuit shown in Figure 6.24 can be employed. The function $f(t)$ is first generated and introduced into a special stage where it is modulated by $g(t)$. The 10 kc sine wave generator controls a sawtooth-voltage generator of the same frequency. The ramp voltages

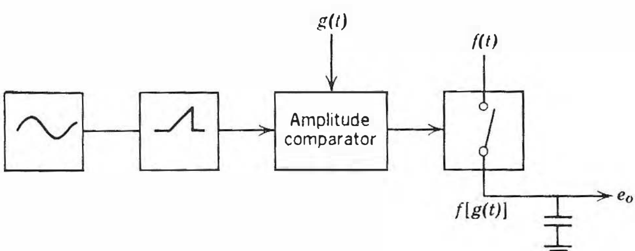


Fig. 6.24 Generator of a function of a function.

obtained in this way are applied to one input of an amplitude comparator. The other input of the comparator is the function $g(t)$. At the instant at which the two inputs are equal in magnitude, the output of the comparator acts to close an electronic switch. The instantaneous value $f[g(t)]$ is then transmitted to the output terminals where it acts to charge a capacitor. This method requires that the sampling rate be large compared to df/dt and dg/dt.

6.8 Universal Nonlinear Operators

A chief advantage of the linear portion of differential analyzers is that all operational circuits are based upon a single relatively simple and economical electronic element—the operational amplifier. This element greatly simplifies the design of the computer and makes it easy to enlarge the capacity of the machine. As has been indicated, however, the performance of nonlinear operations involves special-purpose devices, so that with the possible exception of diode function generators each new nonlinear function poses a new problem. To circumvent this difficulty the so-called universal nonlinear operator was developed.

Specify two arbitrary functions $v(t)$ and $g(t)$, and consider the expression

$$y = v(t)f[g(t)] \tag{6.35}$$

where f is an arbitrary operator. It can readily be shown that from Equation 6.35 all commonly required nonlinear operations can be derived. Only the simplest of these are considered here. These include:

1. f is a linear operator; $v(t)$ and $g(t)$ are arbitrary.

$$y = kv(t)g(t)$$

2. f is a reciprocal function; $v(t)$ and $g(t)$ are arbitrary.

$$y = k\frac{v(t)}{g(t)}$$

3. f is arbitrary; $v(t)$ is constant; $g(t)$ is arbitrary.

$$y = kf[g(t)]$$

4. f is arbitrary; $v(t)$ is constant; $g(t) = at$.

$$y = kf(at)$$

In addition to these, Equation 6.35 can easily be employed to determine more complex transformations, such as, for example,

$$y = kv(t)[g(t)]^{-1/a}$$

or

$$y = kv(t)[g(t)]^{a}$$

Equation 6.35 can be economically realized using a repetitive differential analyzer provided the operator f can be approximated by a discontinuous function. One electronic approach[28] is presented here. Figure 6.25 is a block diagram of such a system.

An ensemble of potentiometers P serves to store in discrete form the relation $f(n)$. The potentiometers are energized by a voltage $v(t)$ which may be constant or transient. The output of each potentiometer is connected to a series electronic switch. The position of this switch is controlled by the output of a corresponding amplitude comparator. The comparison levels of the amplitude comparators are adjusted in a linear fashion. This comparison level is compared to the input function $g(t)$. In this manner the system of amplitude comparators solves the equation

$$g(t) - n\,\Delta u = 0 \tag{6.36}$$

where $n = 1, 2, 3, \ldots, m$, and Δu is the difference between the levels of comparators n and $n - 1$. In this way a series of impulses is obtained which acts to close the electronic switches in such a manner that the

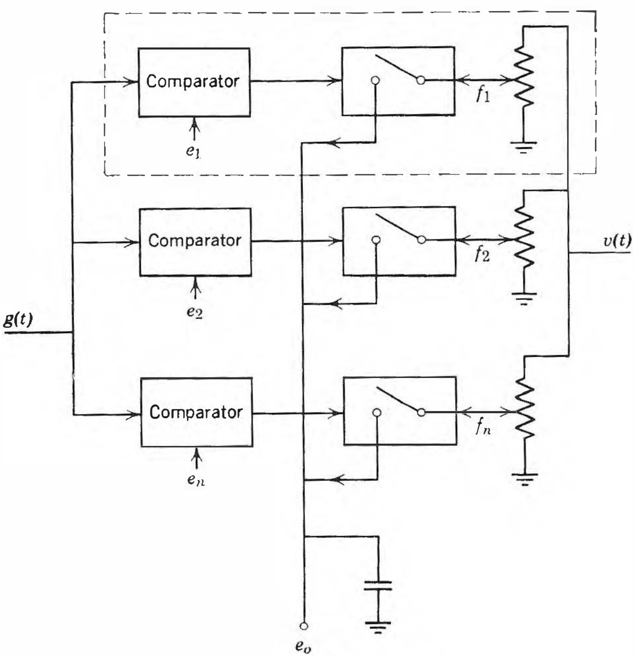


Fig. 6.25 Block diagram of universal nonlinear operator.

output takes the form of a staircase function

$$e_o = \int v(t)f[g(t)] \tag{6.37}$$

An alternative arrangement for universal function generation is shown in Figure 6.26*a*. In this case each potentiometer contributes $\Delta f(n)$ to the output e_o as shown in Figure 6.26*b*. This circuit has the advantage

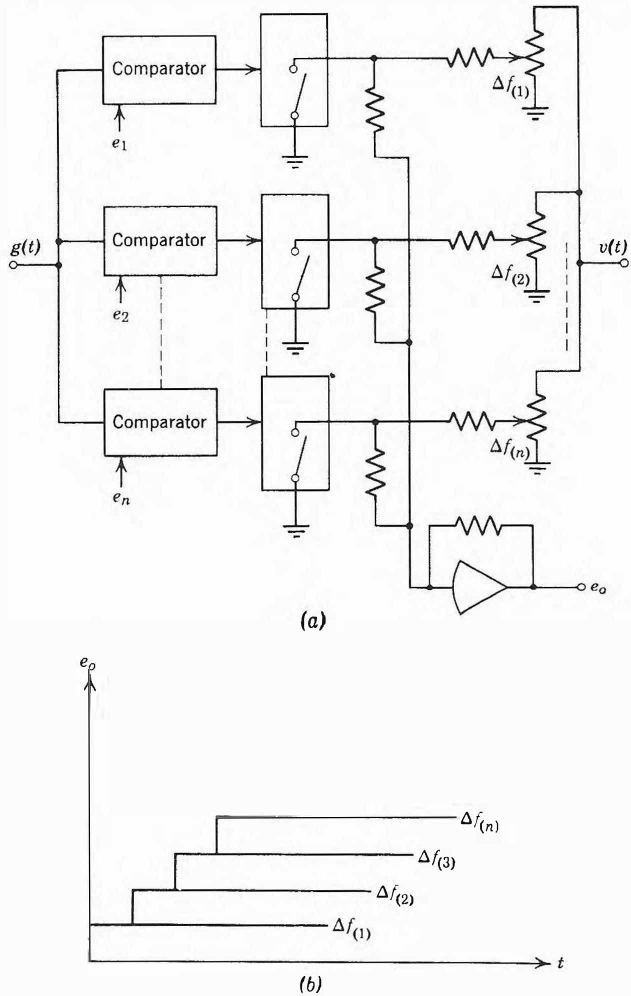


Fig. 6.26 (*a*) Parallel switching arrangement for universal nonlinear operator. (*b*) Output of nonlinear operator with parallel switches.

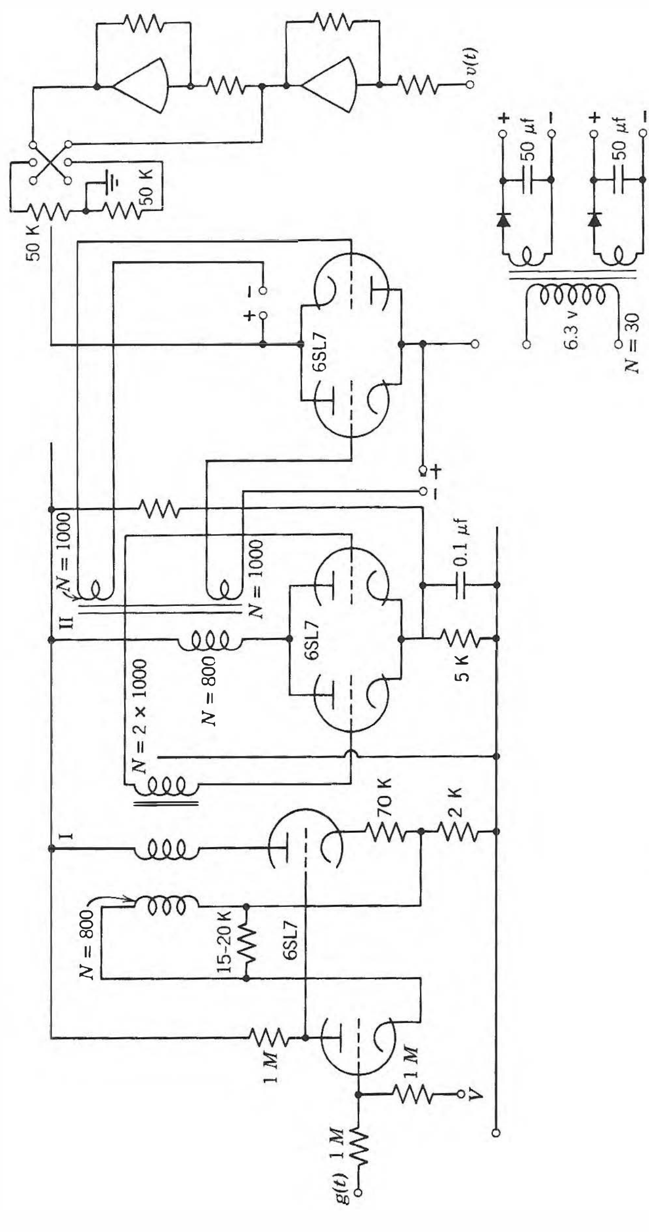


Fig. 6.27 Circuit diagram of a transformation element.

that the output is a steady voltage instead of a series of pulses, which is convenient especially for low operating speeds. Furthermore, the generator can be adjusted and checked in the static state so that function setting is less difficult. On the other hand, as a result of the many adder inputs and parallel electronic switches the signal-to-noise ratio is less favorable than in the series realization. Also in the parallel arrangement the independence of individual ordinate settings is lost, since a change in one stage has an effect on all succeeding settings.

The heart of the universal function generator is the series combination of an electronic switch, an amplitude comparator, and a potentiometer. Such an ensemble of elements is termed a transformation element. Figure 6.27 is a schematic diagram of such a transformation element for use in a differential analyzer with a repetition rate of 50 cycles/sec. In this case all nonlinear operations within the analyzer are effected by identical elements of this type. The basic transformation element for the parallel arrangement is shown in Figure 6.28. Two types of adjustment are provided in the circuit. Potentiometer P_1 serves for hysteresis adjustments and P_2 for the compensation of residual voltages across the electronic switch. Typically, a repetitive differential analyzer might use from 10 to 20 of these transformation elements.

By means of an auxiliary stage it is also possible to obtain a linear

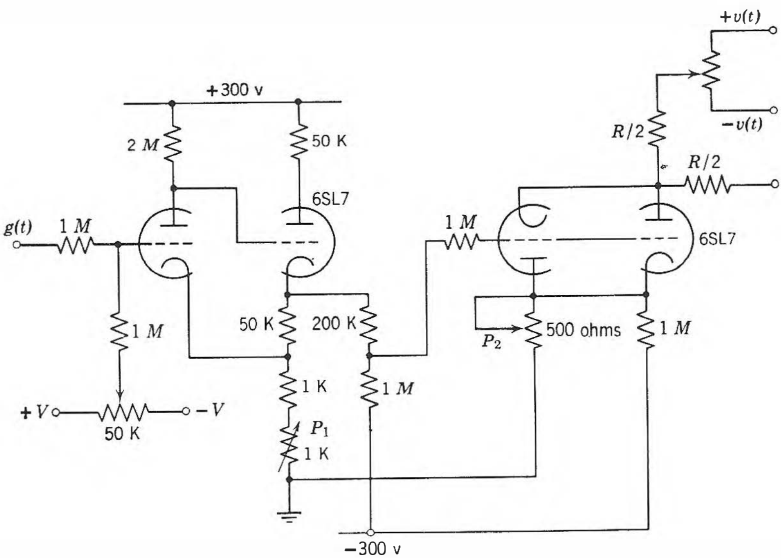


Fig. 6.28 Circuit diagram of universal operator with parallel switches.

approximation to the staircase function. This is accomplished with the aid of an integrator at the output of the function generator. It is desired to obtain at the output of the integrator

$$F(t) = \int_0^t f(y)\,dt \tag{6.38}$$

If the correction term

$$\tfrac{1}{2}\Delta f = \tfrac{1}{2}[f(y_n) - f(y_{n-1})]$$

is added to the staircase function $f(y)$, and the whole expression is integrated:

$$F_1(t) = \int_0^t \Big\lceil\, (f(y) + \tfrac{1}{2}\Delta f)\,dt \tag{6.39}$$

$F_1(t)$ is a linear approximation to $F(t)$. For $y = t$ the exact linear approximation of $F_1(t)$ is obtained, since then the value of the function is known in advance. However, for $y = g(t)$ a higher order error arises, since it is assumed that

$$\Delta f = f(y_{n+1}) - f(y_n) = f(y_n) - f(y_{n-1})$$

This error does not greatly influence the accuracy of the approximation.

A device that automatically provides Δf when the staircase function $f(y)$ is given can be connected to the function generator.[29, 30] A general study of the conditions for linear approximations by the integration of step functions in connection with universal nonlinear operators has been carried out.[31]

The nonlinear relationship Equation 6.37 can also be realized in a manner described by Balchen[32] and shown in Figure 6.29. A mask representing the function to be generated is placed on the screen of the

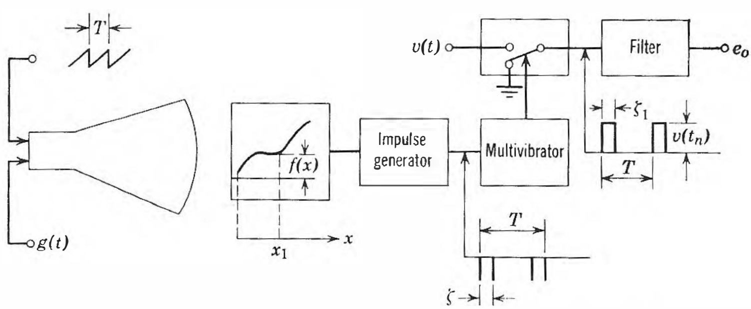


Fig. 6.29 Block diagram of universal operator using cathode-ray tube.

cathode-ray oscilloscope. This mask is completely opaque; light can only pass through two narrow slots, one representing the x-axis and the second the function $f(x)$. The deflection of the beam in the horizontal direction is controlled by the voltage $g(t)$. The vertical deflection plates are energized by a sawtooth voltage having a frequency considerably greater than that of $g(t)$. Accordingly, the beam traverses a large number of times the distance between the x-axis and $f(x)$ in the course of a single horizontal scan. A photoelectric system is placed in front of the oscillograph to produce an electrical impulse each time the beam passes the two slots. The interval between the two impulses in the course of a single scanning period has the form

$$\tau_n = f[g(t_n)]$$

The impulses thus obtained trigger a multivibrator which produces rectangular voltage pulses whose width is proportional to τ_n. The output of the multivibrator controls an electronic switch which in turn acts to interrupt the second variable voltage $v(t)$. The average value of the output current is therefore of the form of e_o in Equation 6.37.

In order to prepare the mask, the function is drawn in indian ink on drafting paper (the line thickness is 0.8 mm), and the drawing is reproduced photographically at a 6:1 scale. The dimensions of the mask on the screen are 82 × 82 mm. The static accuracy of this generator is 0.5%; the bandwidth is 10 cycles/sec; and the dynamic range of the signals at the input and output is ± 100 volts. With certain modifications this generator can be adapted to function generation in all four quadrants. Polimerou[33] has proposed the extension of the photoformer technique to the described method.

Another approach to nonlinear function generation, described by Savant,[34] employs the relationship

$$y = [f(x)]^a[g(y)]^b[h(z)]^c \tag{6.40}$$

Taking logarithms of both sides of the above expression yields

$$\log y = a \log f(x) + b \log g(y) + c \log h(z) \tag{6.41}$$

Since addition and subtraction are readily carried out using standard analog units, the basic problem is the construction of an element that generates the logarithm of a function and the inverse logarithm of a function. Such a transformation element was described by Howard.[35]

6.9 Generators of Functions of Two Variables

Occasionally in analog computations it is desired to generate functions that are themselves functions of two or more dependent variables. Sometimes the pertinent functional relationships are available in analytic form, at other times the functions are arbitrary and presented in graphical form. Examples of the latter condition are the familiar vacuum tube curves. Over the years a wide variety of devices to generate such functions have been suggested. Only a few of these, however, have sufficiently large bandwidths to make them suitable for repetitive operation. The more promising of these methods are described here briefly.

Basically the problem involves the generation of

$$\begin{aligned} z &= f(x, y) \\ a \leq\ &x \leq b \\ c \leq\ &y \leq d \end{aligned} \tag{6.42}$$

The domain of variation of z thus defined will be designated by D.

In an approach introduced by Philbrick,[36] a finite number of values z_k is chosen to be reproduced with the required accuracy. All other points on the z surface are obtained by interpolation. This approach represents a generalization of the familiar diode function generating technique for $y = f(x)$. In this case, however, planar surfaces are used in place of the line segments. This implies that the function generator will consist of a two-dimensional array of logic circuits. The domain D is first divided into rectangular subdomains d, one of which is shown in Figure 6.30. Note that the diagonal separates the subdomain d into two parts. In one subdomain $x < y$, whereas in the other $x > y$. The equation of the diagonal is $y = x$, assuming $\Delta x = \Delta y$. The simple logic circuit of Figure 6.31 is used to produce an output voltage proportional to the value of whichever of the two inputs is smaller than the other. This circuit will thus generate $e_a = \min(x - x_i, y - y_j)$. Having selected one or the other part of the subdomain d, the diode P_3 in Figure 6.31 will start conducting and approximating the corresponding surface curve by a

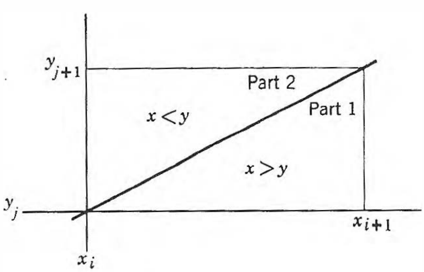


Fig. 6.30 Subdomains d.

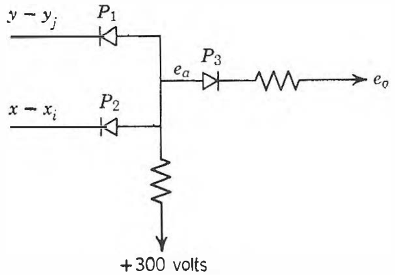


Fig. 6.31 Logic circuit for selecting smaller of two variables.

linear segment. However, this will be true only if min $(x - x_i, y - y_f)$ > 0, because otherwise P_3 is biased in a backward direction. This fact can be conveniently expressed by stating that the output of the circuit in Figure 6.31 is

$$e_o = \max [\min (x - x_i, y - y_j)0] \tag{6.43}$$

where max $[\min (x - x_i, y - y_j)0]$ refers to whichever of the two variables min $(x - x_i, y - y_j)$ or zero is greater. Thus, one first selects the smaller of $x - x_i$ and $y - y_j$, and then starts the interpolation only if the selected variable is greater than zero. The biasing of P_3 is necessary to stop the conduction of P_3 whenever $x - x_i < 0$, or $y - y_j < 0$ so that the different subdomains d can be isolated. In Figure 6.32 the planar surface defined by Equation 6.43 is represented for the subdomain d. The two triangular facets of this planar surface are characteristic of this approximation. The

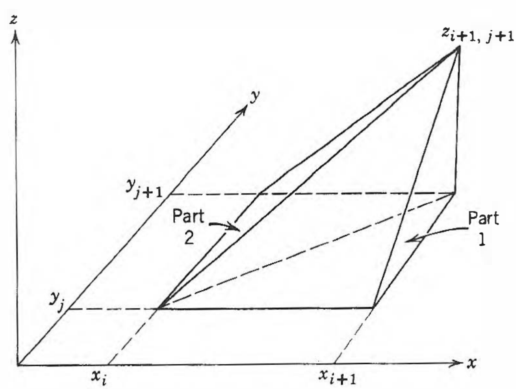


Fig. 6.32 Subdomain d including z coordinate.

PREPARATION OF ASSUMED DATA

		← Z Determined Values/Scaled Equiv's. in volts →					
y_f (+15)	y_5	45 (30 v)	114 (76 v)	144 (96 v)	105 (70 v)	138 (92 v)	111 (74 v)
y_e (+10)	y_4	30 (20 v)	90 (60 v)	123 (82 v)	114 (76 v)	120 (80 v)	99 (66 v)
y_d (+5)	y_3	15 (10 v)	63 (42 v)	96 (64 v)	96 (64 v)	96 (64 v)	81 (54 v)
y_c (0)	y_2	0 (0 v)	39 (26 v)	69 (46 v)	81 (54 v)	75 (50 v)	60 (40 v)
y_b (−5)	y_1	−15 (−10 v)	18 (12 v)	45 (30 v)	54 (36 v)	48 (32 v)	39 (26 v)
y_a (−10)	y_0	−30 (−20 v)	0 (0 v)	24 (16 v)	30 (20 v)	27 (18 v)	21 (14 v)
		x_0	x_1	x_2	x_3	x_4	x_5
		x_a (4)	x_b (8)	x_c (12)	x_d (16)	x_e (20)	x_f (24)

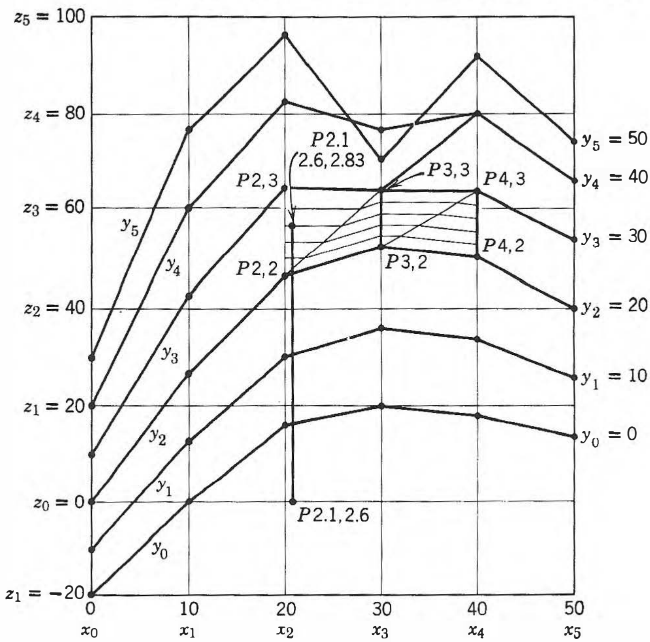


Fig. 6.34 Generation of arbitrary function surface (George A. Philbrick Researches, Inc.).

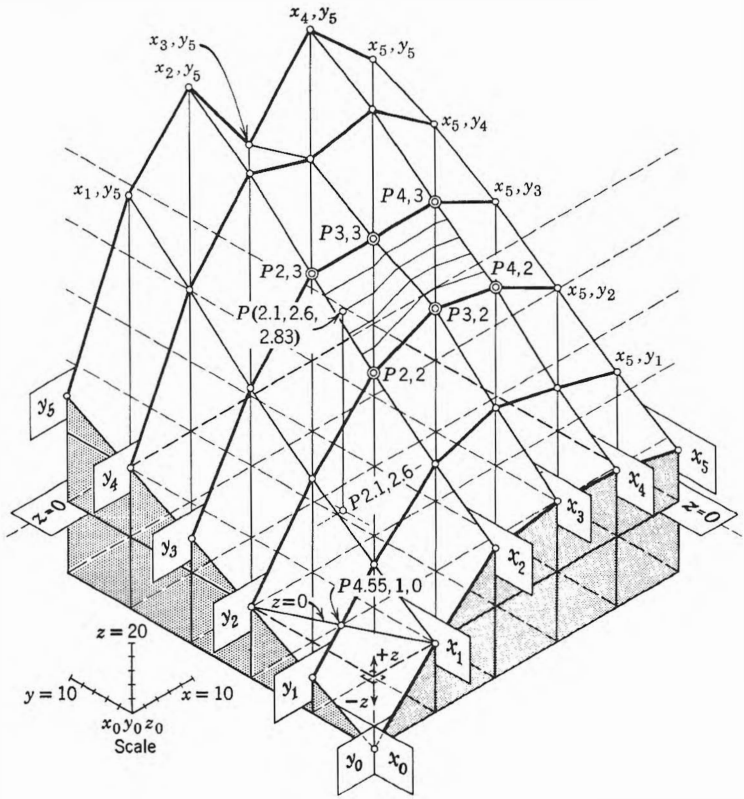


Fig. 6.34 (Continued)

surface is easily constructed by letting $x = 0$, and then $y = 0$, as well as following the selections dictated by Equation 6.43. In part 1 of subdomain d in Figure 6.32, $y = x - x_i$, so that for a specific $x = x_a$ the interpolation is carried out along $z = f(x_a, y)$. The opposite is true in part 2, so that $z = f(x, y_b)$.

Starting from the basic planar surface of Figure 6.32 any curvilinear surface $z = f(x, y)$ can be approximated. The general expression for such an approximation is

$$z(x, y) \approx [z(x, 0) + z(0, y) - z(0, 0)] + \sum_{\substack{i=1 \\ j=1}}^{n,m} z_{ij} \max[\min(x - x_i, y - y_j), 0] \quad (6.44)$$

Starting from given $z(0, 0)$, $z(x, 0)$, $z(0, y)$, the required z_{ij} can be reached by adjusting the slopes of diodes (P_3 in Figure 6.31) at various points. The detailed circuit is shown in Figure 6.33 for $i = 1, \ldots, 6, j = 1, \ldots, 6$. The network has a matrix of 36 elementary diode selection circuits of the type shown in Figure 6.31. The correspondingly biased operational amplifiers serve for the separation of the input voltages x, y into $x_{i+1} - x_i$, $y_{j+1} - y_j$, slices. Very simple operational amplifiers having gains of the order of 100 are adequate for this purpose. A practical example of this approximation is shown in Figure 6.34 together with the set of initial data and the surface obtained. Kehr[37] has effected an important improvement in this technique. The values of $z(x, y)$, instead of being adjusted as voltages and summed as such, are obtained as increments of the slope of z and summed as currents. Thus all settings of z are completely independent of each other.

In the approach just described the formation of the desired surface is obtained by the superposition of the pyramidal elements of Figure 6.32. Petternella and Ruberti[38] have developed a method of generating a series of planar surfaces and using them to approximate $z = f(x, y)$. This represents a direct extension of the one variable linear segment approximation with all its generality and flexibility. The circuit shown in Figure 6.35 is used to generate the planes. The output voltage e_o is

$$e_o = \frac{e_{i1}G_1 + e_{i2}G_2 + EG_E + VG}{G_1 + G_2 + G_0 + G + G_E} \tag{6.45}$$

where G_i are conductances. Equation 6.45 is that of a plane intersecting $e_o = 0$ along the line

$$e_{i1}G_1 + e_{i2}G_2 + EG_E \pm VG = 0 \tag{3.46}$$

and having a slope

$$p = \frac{\sqrt{G_1^2 + G_2^2}}{G_1 + G_2 + G_0 + G + G_E} \tag{6.47}$$

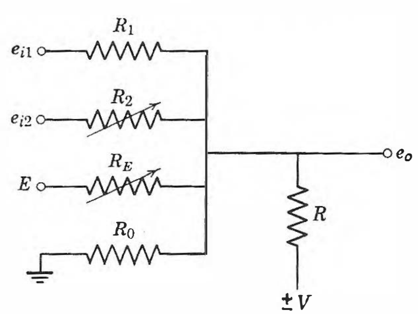
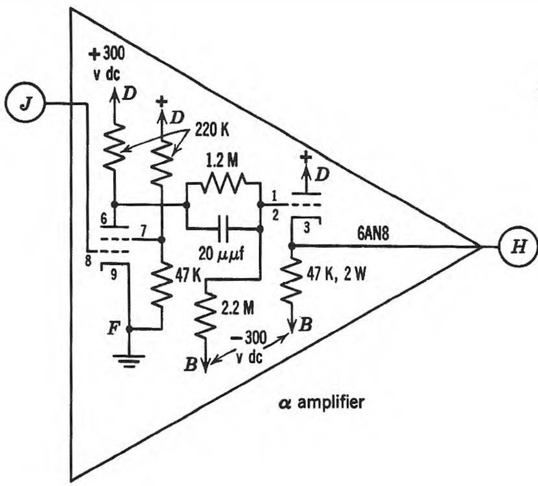


Fig. 6.35 Circuit to generate planes: $z = ax + by + c$.

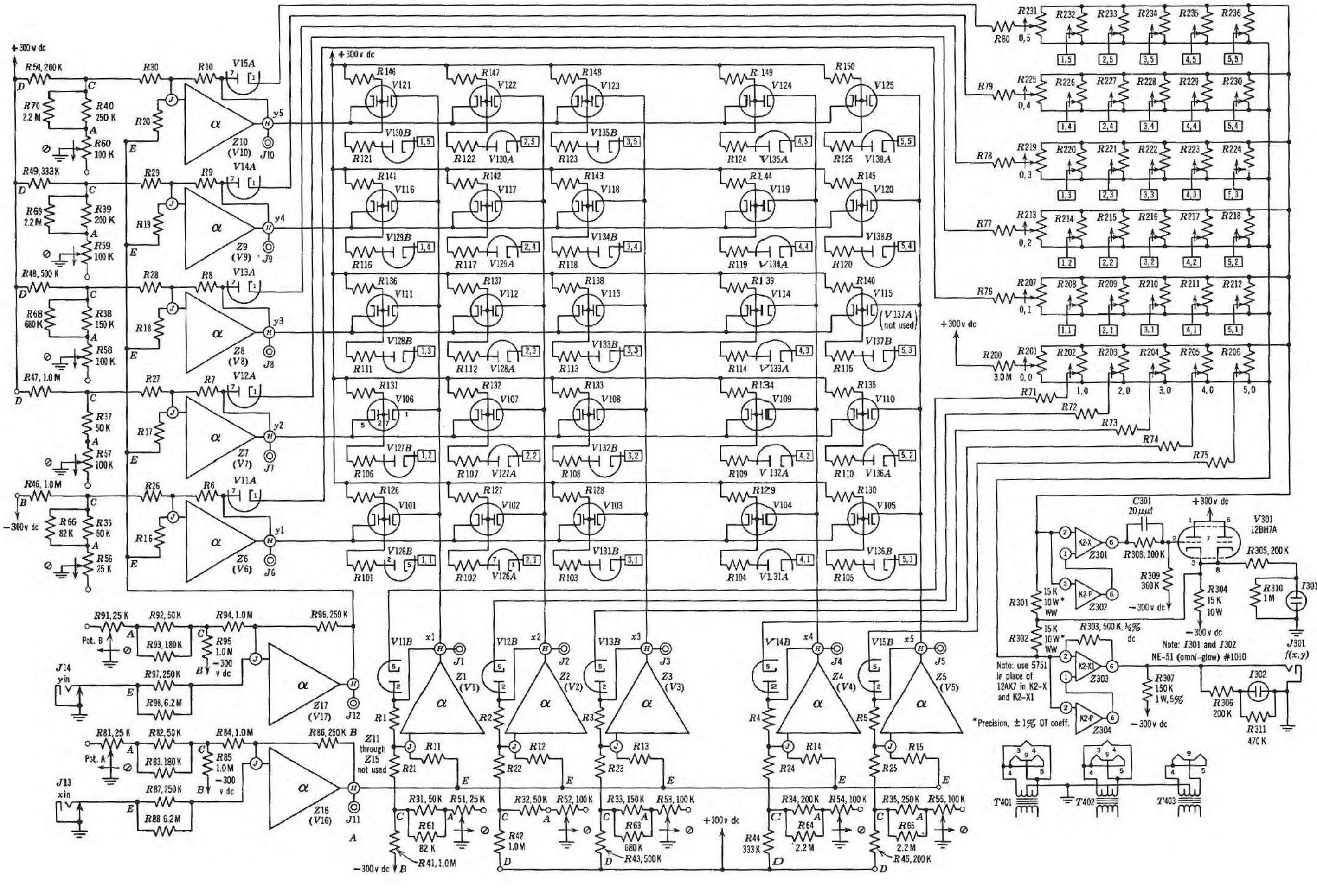


Notes: Letters "A" through "J" at points in the circuitry pertaining to the amplifiers $Z1$ through $Z12$ denote the lugs of the turret assembly on which the circuits are mounted.

$R1$---$R10$, $R11$--$R20$: 250K±0.5% ½W dc
$R21$--$R30$: 1.0M±0.5% ½W dc
$R71$--$R80$, $R101$--$R125$: 100K±0.5% ½W dc
$R126$--$R150$: 500K±0.5% ½W dc
$R201$--$R236$: 1000K pot. WW ±0.5% tot. res., ±1% Lin., ±1°E. rot.

$V1$---$V10$, $V16$, $V17$: 6AN8
$V11$ -- $V15$, $V101$--$V138$: 6AL5
$V301$: 12BH7A

Fig. 6.33 Circuit diagram of generator of functions of two variables (George A. Philbrick Researches, Inc.)



Note: I301 and I302 NE-51 (omni-glow) #1010
Note: use 5751 in place of 12AX7 in K2-X and K2-X1
*Precision, ± 1% OT coeff.
V301 12BH7A
Z11 through Z15 not used

Now by using e_{i1}, e_{i2} either directly or inverted, as well as the positive or the negative reference voltage E, the eight types of approximating planar surfaces can be obtained. These correspond to the eight types of linear segments for curve approximation, as indicated in Figure 6.16. The procedure employed to set up the function generator is similar in principle to that used for one variable. For a given $z = f(x, y)$ the approximating planes are calculated in the form

$$z = ax + by + c \tag{6.48}$$

The voltage $\pm V$ of Figure 6.35 is used for diode biasing to permit the selection of the appropriate plane. The logic circuits used for this purpose are represented in Figures 6.36*a* and *b* and perform the same minimum or maximum operation as before to select the plane that gives a maximum or minimum value of output voltage respectively. Figure 6.36*c* shows a circuit for double selection. Evidently the accuracy of the approximation

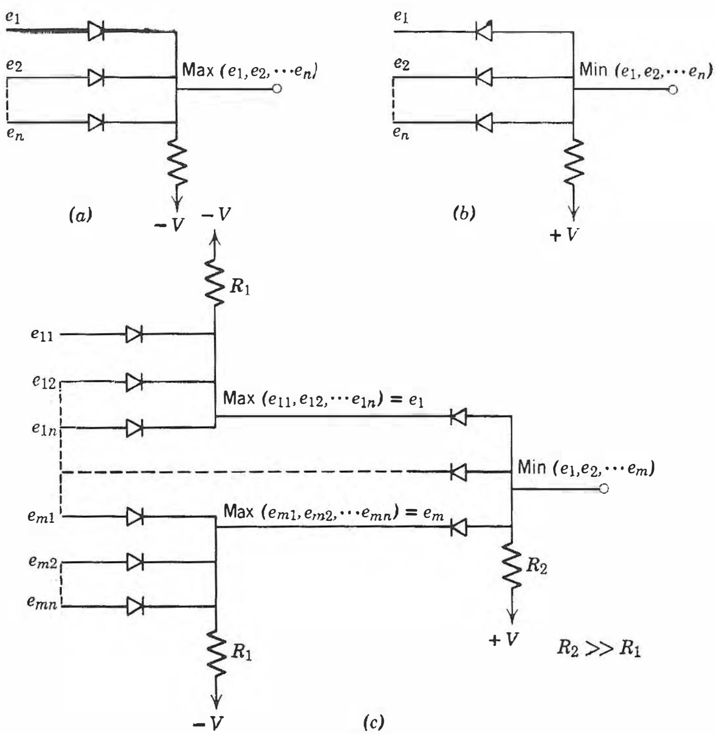


Fig. 6.36 (*a*) Logic circuit for selecting maximum of *n* variables. (*b*) Circuit for selecting minimum of *n* variables. (*c*) Logic circuit for double selection.

depends upon the number of planes used. The device described consisted of two units each having ten planes.

A mathematically different approach has been used by Meissinger.[39] The function $z = f(x, y)$ is represented in parametric form $z = f(x, y_k)$, $k = 1, 2, \ldots, n$, so that the generator has to produce a family of curves with a variable parameter. The parametric method implies two sweeping speeds, one for x and another for y. The advantage of this method is that one-variable function generators can be used. To apply this method, isoclines of the family of curves $z = f(x, y_k)$ must be drawn. A graphical method has been developed to obtain the parametric functional relation to drive the breakpoint reference voltage.

Elliott[40] employs a similar technique for matching families of curves having reasonably regular characteristics. His method involves the use of a diode function generator to match accurately one of the curves of the family. This curve then represents the function of one of the variables. By the addition of suitable voltages to the input and the output of this function generator, the basic curves can be translated in the horizontal or vertical directions. The amount of translation is controlled by the second input variable. Combined horizontal and vertical translations are achieved by adding voltages to the input as well as to the output of the function generator. More sophisticated versions of this technique make it possible to rotate as well as to translate the basic reference curve in order to permit the approximation of a wider variety of families of curves. This makes it unnecessary for the curves to be represented by sets of parallel lines as in Meissinger's method.

The universal function generator described in the preceding section can be adapted to the generation of functions of two variables.[41] This is accomplished by introducing one of the functional relationships to the circuits determining the comparison level of the amplitude comparators. The other functional relationship is introduced as before as settings of potentiometers $P_1 \ldots P_n$.

Several methods have been proposed for the utilization of photographic plates for the storage of the function to be generated. Two such devices have been proposed by Wallman[42] and Wentzel.[43] In both these devices, one of the independent variables must be time. The function $K(x_i, t)$ is stored on a photographic plate. Thirty columns are used for discrete values of x, whereas t is maintained in continuous form. Figure 6.37 shows the method of film preparation. The width of each transparent column represents the function $K(x_i, t)$. The columns are scanned by the light beam from a cathode-ray oscilloscope, and the light pulses are sensed by a photocell. During each minor cycle, the range of the variable t is scanned. After each minor cycle, the process is repeated for the

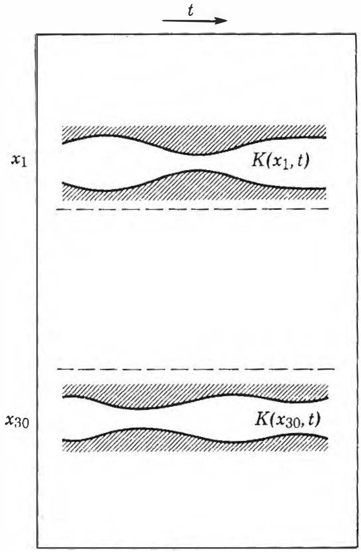


Fig. 6.37 Photographic plate used for function generator.

succeeding value of x_i. The scanning of each column requires 75 msec, so that the entire function is scanned in 3.5 sec. The accuracy of such a device is reported to be of the order of 1%.

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chapter 7
OUTPUT EQUIPMENT

7.1 General Remarks

Since the solution of a problem on a repetitive differential analyzer becomes available immediately and is repeated a number of times per second, the cathode-ray oscilloscope represents the most natural display device. This represents one of the chief advantages of the repetitive computer over one-shot analog computers. Particularly in problems involving the optimization by parameter adjustments, the fact that the effect of a parameter adjustment becomes visible at once greatly simplifies the solution procedure. Photographs of the tube screen provide permanent records of the solution.

The chief difficulty with oscilloscopic displays is that high accuracies are difficult to obtain without the use of auxiliary equipment. Inaccuracies arise from two basic causes:

1. The relation between beam deflection and deflection voltage is generally a nonlinear one. This nonlinearity is particularly pronounced for large beam deflections both in the horizontal and vertical directions. It therefore becomes necessary to apply a correction factor depending upon the coordinates of each point of the solution.
2. The oscilloscope trace generally has an appreciable width, making precise measurements difficult.

Accuracy limitations of oscilloscopic equipment were for many years the limiting factor on the over-all accuracy of repetitive computers and were in some measure responsible for their delayed acceptance. In recent years several manufacturers have developed high precision electronic devices that greatly increase the accuracy of oscilloscopic measurements. A description of one of these is given in some detail in Section 7.2.

For more precise measurements an output system displaying the instantaneous values of the solution voltages at a selected point in the repetitive cycle is very useful. Such a system, which lends itself to digital as well as analog read-out, is described in Section 7.3. The use of a special time-base marker facilitates the accurate calibration of the horizontal (time) axis of oscilloscopic displays. The description of a novel method for employing low-speed output devices for the display of a high-speed solution concludes the chapter.

7.2 Oscilloscopic Displays

George A. Philbrick Researches, Inc. has developed an "electronic graph paper" to provide a convenient and accurate calibration of the entire cathode-ray tube screen. The display system will display simultaneously up to 8 input variables on a rectangular-coordinate grid. All input signals are sampled for display at virtually the same moment, so that their waveforms are seen in their correct time relationship. This sampling occurs at every 62.5 μsec for each of the input signals followed by immediate conversion to corresponding points on the coordinate system. These points are so closely spaced that they appear as a single clear line. A clearly defined coordinate system is displayed full-scale over the entire face of the 17 inch cathode-ray tube at the same time. The horizontal lines indicate voltage, the vertical lines time, both displayed with precision. At a glance, the absolute amplitude and time position of each of the signals can be determined precisely. Because input signal waveforms and the lines of the coordinate system are scanned together and simultaneously by the same vertical sweep, errors stemming from nonlinearity, parallax, and drift are obviated. Any changes affecting signal waveforms will also distort the coordinate lines to exactly the same degree, eliminating the relative error. The heart of the display system is a precision 16 kc crystal oscillator. The frequency of this "clock" is maintained constant to within 0.01%. All the circuits involved in time measurements use triggering pulses derived from this clock. The scanning of the cathode-ray tube is accomplished by a vertical flying-scan system in which the scanning voltage flies from -100 to $+100$ volts at each clock pulse. The high-speed scanning voltage is constantly compared in a coincidence circuit with each of the input signal voltages. Whenever a coincidence occurs a very short brightening pulse is delivered to the cathode-ray tube. Thus a single bright spot occurs on the screen for each signal input every time the high-speed scan passes by it.

To produce the twenty-one horizontal coordinate lines, twenty-one precise reference voltages are introduced to coincidence circuits in exactly the same manner as the eight signal input voltages. Again brightening pulses to the cathode-ray tube produce the spots forming the horizontal lines. The 101 vertical lines are produced by voltage stages applied to the cathode-ray tube grid at precise divisions of the display interval. These voltages stem from the same binary counters that trigger the horizontal sweep voltages. Pulses for the binary counters come directly from the crystal oscillator. To facilitate interpretation every fifth line is intensified. The time required to produce a single display frame by one complete sweep of the face of the cathode-ray tube can be set to 25, 50, 100, 211, 500 msec and 1, 2, 5, 10, 20, and 50 sec. In this way the unit is suitable for slow- as well as high-speed operation. Auxiliary camera units are available to obtain a permanent record of the complete solution.

In Figure 7.1*a* eight solutions are displayed simultaneously. These are the step responses of eight cascaded first-order lags. In Figure 7.1*b* the results of system optimizations in a pneumatic engine control problem is shown. The upper curve biased to +50 volts, indicates flapper valve displacement, the error-sensing device behaving as an undamped spring-mass system. The middle curve indicates pneumatic actuator position. The lower curve biased to −50 volts indicates engine output sensed as pressure. All curves indicate response to a step change in engine pressure.

An alternative method for obtaining high accuracy and precision with oscilloscopic displays is to employ an externally calibrated reference voltage line projected on the screen at the same time as the solution. By varying the reference voltage this line can be moved up and down the oscilloscope screen until it coincides or is tangent to the solution curve in the region of interest. Since this reference line is subject to the same distorting influences as the signal, the magnitude of the reference voltage at the measuring position will be identical to the signal ordinate.

7.3 Measurement of Instantaneous Values

To circumvent the difficulties attending direct measurements on the face of an oscilloscope, separate circuits have been developed to permit the determination of the solution at specific instants of time along the repetitive operating cycle. A schematic diagram of such a system is shown in Figure 7.2. An integrator is employed to produce a sawtooth-voltage wave of the same frequency as the repetition rate of the analyzer. The integrator output is compared with the output voltage of potentiometer P_1 by amplitude comparator A. The output of this comparator will then be a voltage pulse that occurs at a time after the start of the computer cycle, which is determined by the setting of potentiometer P_1. The combination of the integrator, comparator A, and potentiometer P_1 therefore constitutes a linear time-delay circuit. The output of this delay circuit becomes an input to the coincidence detector.

Amplitude comparator B compares the transient voltage $f(t)$ comprising

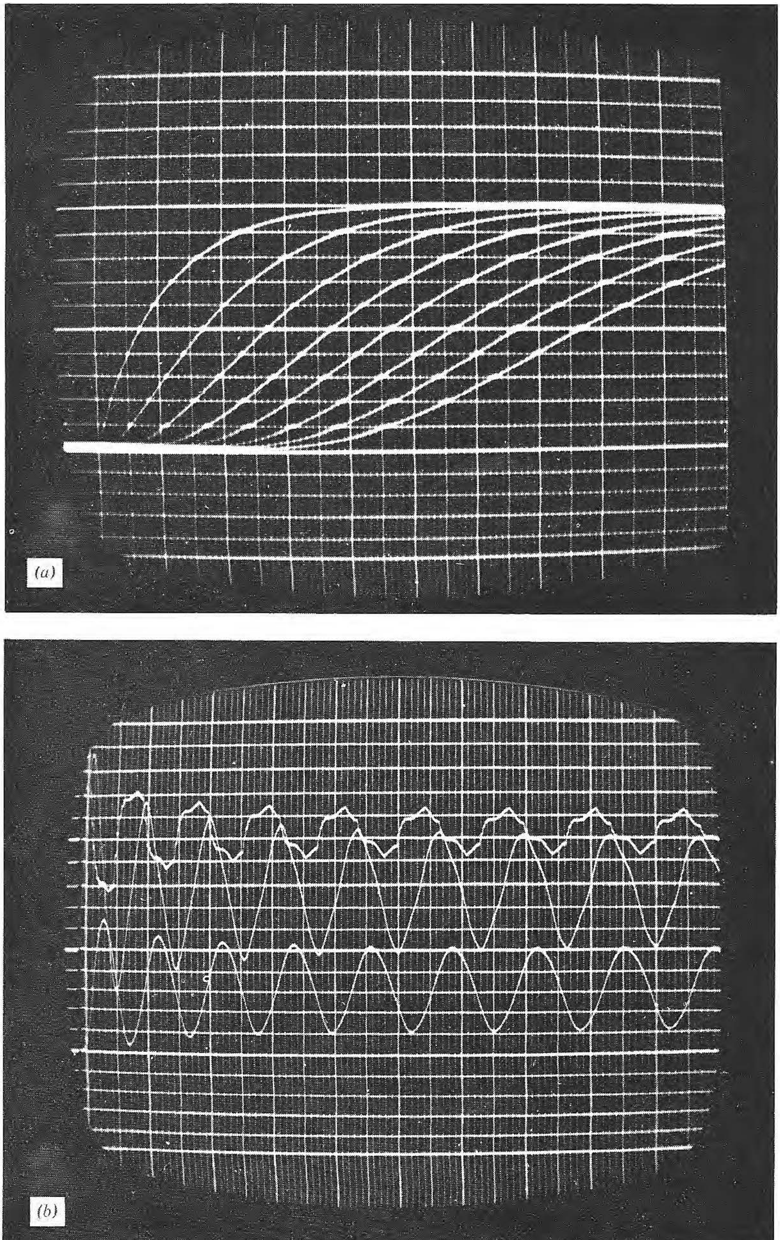

Fig. 7.1 (*a*) Simultaneous display of eight solutions (George A. Philbrick Researches, Inc.). (*b*) Display of system optimization problem (George A. Philbrick Researches, Inc.).

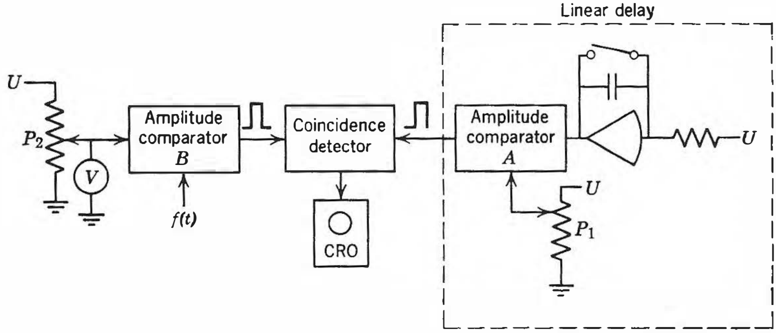

Fig. 7.2 Measurement of instantaneous voltage values using coincidence detector.

the computer solution with the output of potentiometer P_2. When the two inputs are equal, this comparator applies a pulse to the other input of the coincidence circuit. Potentiometer P_2 is adjusted until the desired coincidence is obtained. The voltage output of P_2 as measured by a voltmeter with a high input impedance then constitutes the solution. A cathode-ray oscilloscope can be employed to detect the coincidence signals.

Figure 7.3 illustrates a modification of the measuring system which is particularly suitable for rapid measurements. Rather than seeking a point of coincidence, the value to be measured is read directly on the voltmeter. The output of the amplitude comparator A in Figure 7.2 rather than being applied to a coincidence circuit serves to close an electronic switch. The charge time of capacitor C does not present any problem, since the same solution of the differential equation is repeated a great many times. The forward resistance of the electronic switch is therefore not critical.

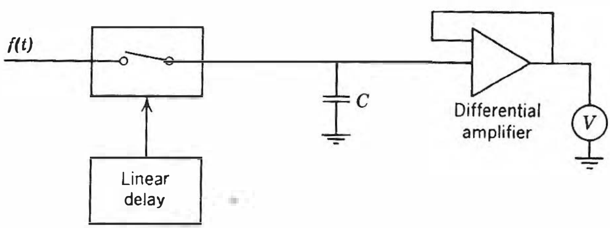

Fig. 7.3 Measurement of instantaneous values of voltage without coincidence circuit.

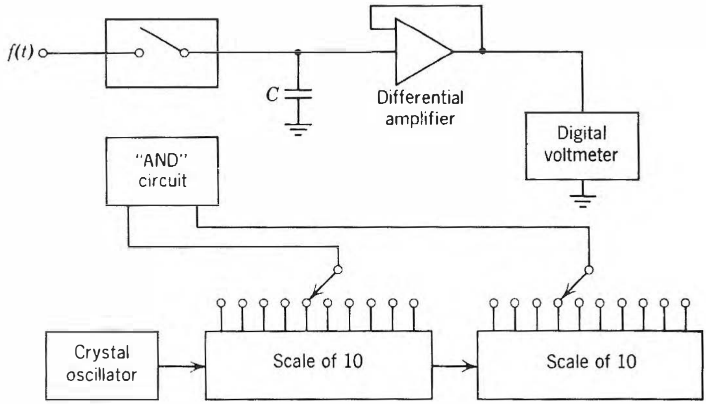

Fig. 7.4 Digital measurement of instantaneous values.

In view of the over-all accuracy of repetitive differential analyzer solutions, measurements at 100, or at the most 1000, equidistant sampling points are generally sufficient. To this end, the circuit in Figure 7.3 can be modified to permit digital measurement of these values. A block diagram of a device for measuring at 100 sampling intervals is shown in Figure 7.4. This permits rapid and precise recording of results. A stable crystal oscillator is included in the instrument to serve as reference for accurate absolute measurements of integrator time constants.

Aleksic[1] has considered in detail the operation and the error analysis of output systems of this type, and Perotto[2] has described a commercial realization. Accuracies of $\pm 0.5\%$ of a maximum ± 100 volt-range are readily obtained. Verification of the accuracy of the amplitude measurements is very simple. It is only necessary to place at the input a constant voltage and to check with a voltmeter if the reading at the input and the

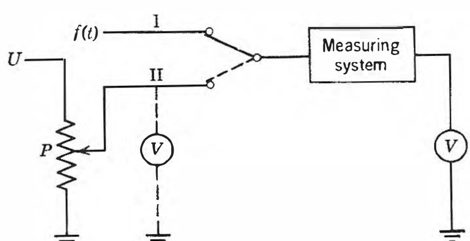

Fig. 7.5 Elimination of the effect of errors in the measuring system.

output of the system is identical. It is interesting to note that one can in this way with little difficulty eliminate completely all errors introduced by this system of measurement. For this purpose the measurement is effected in two steps as shown in Figure 7.5. First, $f(t)$ is measured, and the voltmeter reading is recorded. In the second step a constant voltage is applied to the input. With the aid of potentiometer P the d-c input voltage is adjusted until the voltmeter gives the same reading as in step 1. The voltage indicated by the adjustment dial of potentiometer P is then equal to $f(t)$.

7.4 Time-Base Calibration

In order to attain a maximum of accuracy it is expedient to complement the preceding measuring system with a pulse generator to provide an accurate calibration of the independent variable. Numerous types of timing devices have been developed in the past. One circuit due to Rideout[3] that has been found useful in repetitive differential analyzer applications is shown in Figure 7.6. The impulse frequency is 1250 cycles/sec. The heart of this unit is a stable source of frequency. Since great precisions are not required, a quartz oscillator is not necessary. The oscillator is synchronized with the rectangular voltage wave which controls the operation of the repetitive differential analyzer. The output of the oscillator can be readily applied to the control grid of the cathode-ray oscilloscope which serves to display the solution of the equation.

7.5 Use of Low-Speed Output Devices

Although high-speed display devices such as the cathode-ray oscilloscope are the natural output instruments for repetitive differential analyzers it is possible to obtain satisfactory plots of solutions using devices having much more limited bandwidths, such as servo-driven two-coordinate plotters. Blake[4] has described a method for effecting the necessary frequency transformation. This method does not constitute a change in the scale factor; rather during each repetitive cycle only one solution point is recorded. By making the time interval between the sampling intervals differ from the period T of the solution by a small factor τ the whole solution will be recorded after n repetitive cycles, where $n = T/\tau$. This is illustrated in Figure 7.7.

For example, if

$$T = 0.01 \text{ sec}$$
$$\tau = 10^{-5} \text{ sec}$$
$$n = 1000$$

then after a time $nT = 10$ sec, the entire solution will have been plotted.

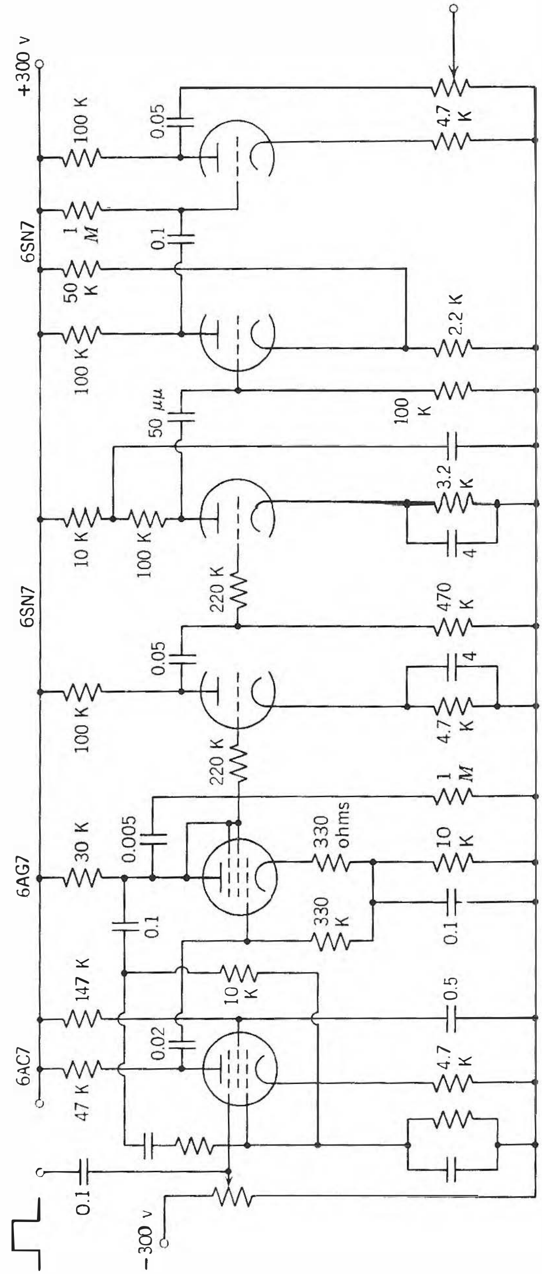

Fig. 7.6 Generator for time pulses (Indian Institute of Science).

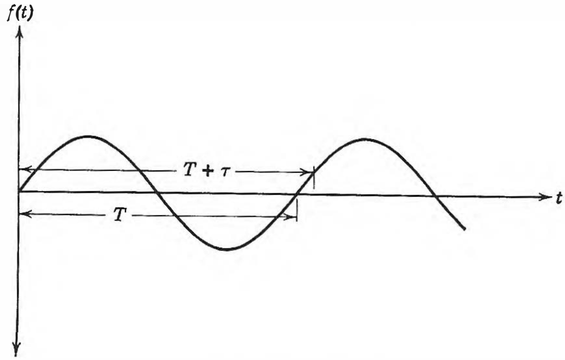

Fig. 7.7 Difference between sine wave period and repetition period for application of low-speed output equipment.

Thus, as far as the recorder is concerned the period of the solution is

$$T_1 = nT = 10 \text{ sec}$$

rather than 0.01 sec. By increasing n, the solution can be "slowed down" even more.

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chapter 8
AUXILIARY DEVICES

8.1 General Remarks

From the point of view of mechanization, repetitive analog computers are more complex than one-shot computers. In addition to the linear and nonlinear computing units and the output or display unit, repetitive computers require a number of units not found in slow computers.

A special rectangular wave generator is required to impose the repetitive cycle. Preferably the frequency of the repetitive cycle should be variable so that an optimum repetition rate can be employed for each specific problem. Additional equipment is required to apply the appropriate initial condition to each integrator at the commencement of each repetitive run. In order that the capacitor attain its specified initial condition by the end of the reset period, its charge-discharge time constant must be short. This is particularly true if it is desired to use the integrator as a dynamic memory as described in Chapter 5.

In this chapter certain theoretical considerations in the choice of the repetition rate are first reviewed. This is followed by a discussion of various methods for imposing initial conditions. The generation of the rectangular wave is next considered in some detail. The chapter concludes with a brief discussion of practical power supply circuits.

8.2 Choice of the Repetition Rate

Practical considerations restrict the repetition rate within rather narrow limits. The repetition frequency must be high enough to produce a sharp and distinct pattern on the display oscilloscope. The upper limit on this frequency is determined by the bandwidth characteristics of the computing elements; for the higher the repetition frequency, the higher the solution frequencies that must be handled with a minimum of error by the computer. Most repetitive differential analyzer facilities employ repetition rates ranging from 10 to 60 cycles/sec.

The choice of the repetition rate determines to a large extent the length of the computing interval during each cycle. The optimum length of this interval is determined in turn by the characteristics of the active elements of the computer. Fuchs[1] discusses the practical consequences of the selection of a specific computing interval and their effect on the accuracy of an over-all solution. One approach to calculating the optimum length of the working interval is to make use of Equation 5.44 repeated here

$$\epsilon < \frac{50t}{(1 + g)R_1C} \tag{8.1}$$

For high gains

$$\frac{t_1}{RC} = \frac{\epsilon g}{50} \tag{8.2}$$

where t_1 is the length of the working interval. For a periodic solution of a differential equation it is generally desirable to be able to view at least five complete solution cycles. Therefore,

$$f_m t_1 = 5 \tag{8.3}$$

where f_m is the frequency of the solution on the machine. f_m is related to the mathematical solution f according to

$$f_m = a_\tau f \tag{8.4}$$

where a_τ is the time scale factor.

In choosing the working interval, the bandwidth of the computing elements must also be considered. Attention is limited to the linear case, that is, to the equation

$$y^2 = -\omega_n{}^2 y \tag{8.5}$$

Using the approach of Sections 3.5 and 5.3, it can be demonstrated that as a result of the bandwidth limitation of the operational amplifier, the solution of Equation 8.5 becomes

$$y = \exp\left[\omega_n{}^2\left(T_1 + \frac{T_2}{2}\right) - \frac{1}{T_0}\right] \cos \omega_n t \tag{8.6}$$

where T_0 and T_1 are the lower and upper cut-off frequencies of the integrator and T_2 is the upper cut-off frequency of the adder. In order that the error resulting from this source be eliminated, it is necessary that

$$\left[(2\pi f_m)^2\left(T_1 + \frac{T_2}{2}\right) - \frac{1}{T_0}\right]t_1 = 0 \tag{8.7}$$

By combining Equations 8.2, 8.3, and 8.7,

$$f_m = \frac{10^{-4}\epsilon}{T_1 + T_2/2} \text{ cycles/sec} \tag{8.8}$$

Provided T_1, T_2, g and ϵ are specified, f_m can be determined. From this the working interval t_1 can be found. Vitenberg[2] has demonstrated in this connection that under certain conditions the output power capabilities of the computing units must be increased as the repetition rate is increased.

Paul[3] has shown that one of the effects of the bandwidth limitations of the operational amplifier is a divergence in the solution of the simple second-order differential equation of harmonic motion. Where the bandwidth of the adder is known, and assuming that the integrators introduce no systematic error, the question arises as to how large the working interval may be without excessive errors caused by the presence of the divergent terms. Assuming that the maximum permissible error due to divergence is 1%, the following approximate relation can be derived:

$$\frac{f_2}{f_m} = 100\pi f_m t \tag{8.9}$$

or

$$f_m = \frac{1}{\sqrt{100\pi}} \frac{\sqrt{f_2}}{\sqrt{t}} \tag{8.10}$$

where f_2 is the bandwidth of the adder. Letting N be the number of cycles of the sinusoid to be observed within the working interval

$$N = f_m t \tag{8.10a}$$

In terms of the bandwidth of the adder and of the integration time

$$N = 0.0564\sqrt{f_2}\sqrt{t} \tag{8.11}$$

For example, if

$$f_2 = 10^4 \text{ cycles/sec} \qquad t_1 = 1 \text{ sec}$$

it is possible to observe $N = 5.64$ periods without excessive error.

8.3 Initial Conditions

In repetitive differential analyzers it is not practical to employ the same method for imposing initial conditions as in nonrepetitive type computers, for this would involve a very complex switching system. Furthermore, it is necessary to adapt the initial conditions circuits to the type of operational amplifier that has been selected. As pointed out in Section 5.5 three different approaches exist to the design of operational amplifiers for repetitive computers.

If an a-c amplifier of the type shown in Figure 5.18 is employed, the initial condition can be imposed by rectangular voltage waves. The

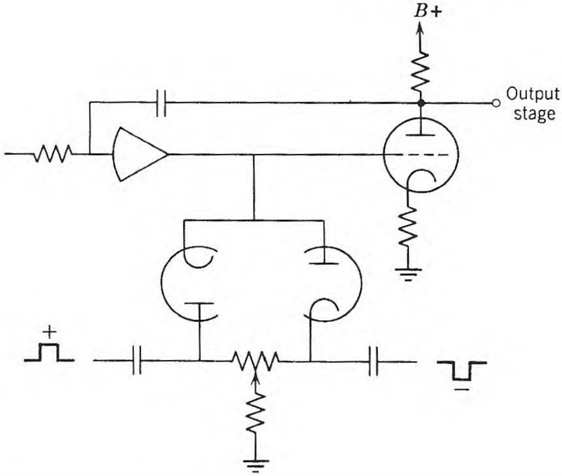

Fig. 8.1 Application of initial conditions with the aid of rectangular pulses.

amplitude of this rectangular voltage pulse is made equal to the initial condition desired of the integrator and is applied to the output stage of the amplifier. A circuit due to Macnee,[4] suitable for this purpose, is shown in Figure 8.1. Another method for applying initial conditions is shown in Figure 8.2. The block indicated by + can be an adder or an integrator. The potentiometer P which is employed to set the initial conditions is fed by a constant d-c voltage, U.

In the latter case it is important not to lose sight of an important practical consideration. Consider, for example, the equation

$$\frac{d^2y}{dt^2} - a_1 \frac{dy}{dt} - a_0 y = 0 \tag{8.12}$$

with the initial condition

$$\frac{dy}{dt}(0) = c_1 \qquad y(0) = 0$$

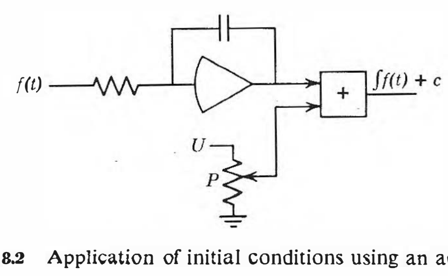

Fig. 8.2 Application of initial conditions using an adder.

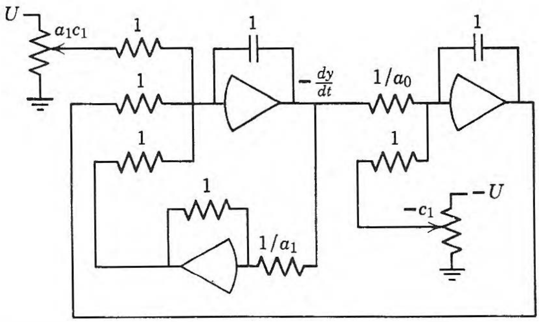

Fig. 8.3 Application of initial conditions in a differential equation using an adder.

The computer diagram is shown in Figure 8.3. Initial conditions must be introduced at two points: at the input of the integrator which generates the function y, and at the integrator generating the first derivative. At the latter point the value a_1c_1 must be applied. This consideration can be readily generalized. If one is treating nth order equation with initial conditions

$$\frac{d^{n-1}y}{dt^{n-1}} = c_{n-1} \tag{8.13}$$

it is necessary to apply the term

$$\frac{d^n y}{dt^n} = \sum_0^{n-1} a_{n-1}c_{n-1} \tag{8.14}$$

The omission of the additional input will, of course, lead to serious errors.

A third way of introducing initial conditions is shown in Figure 8.4.

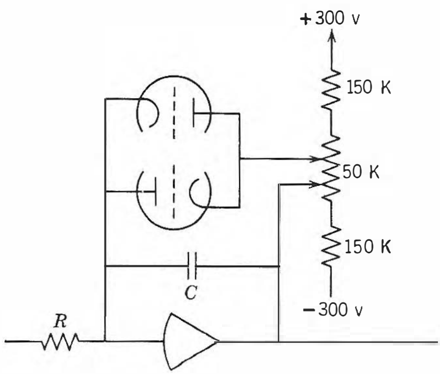

Fig. 8.4 Electronic switch for application of initial conditions.

In this manner nonzero initial conditions can be introduced without making use of succeeding stages. However, the resistance appearing in series with this switch increases the time constant governing the rate of the charging of the capacitor, so that repetition rates are relatively limited. Koerner and Korn[5] recommend that a cathode follower or some other impedance transforming device be used in series with the switch.

In Section 5.4 a technique is described for including dynamic memory in the linear portion of the repetitive differential analyzer. In this technique, the initial condition input terminal of the integrator is used in a novel manner. This application makes it necessary to reset or change the voltage on the capacitor in a time which is negligible compared to the duration of one repetitive cycle. Accordingly, it is necessary to reduce to an absolute minimum the effective impedance in the charge-discharge circuit of the feedback capacitor. This is accomplished as shown in Figure 8.5 by inserting a buffer amplifier in the feedback path. This amplifier acts effectively as an impedance matching device in the initial condition circuit. The impedance of the initial condition circuit is now seen to be the output impedance of the buffer amplifier plus the forward impedance of the solid-state switch *S*. This total impedance is of the order of 50 ohms, so that the *RC* time constant of the resetting operation is 0.5 μsec. This makes it possible to reset from one voltage extreme to the other in 240 μsec. An additional benefit derived from the use of the buffer amplifier is that the relay which is commonly required in series with the input resistor is no longer needed. The smallest input resistor normally used in computer operation is of the order of 100 K. Since the total impedance of the reset circuit is of the order of 50 ohms, when the integrator is in the reset mode, any current flow through the input resistor introduces negligible error. Units of this type have been incorporated in at least one large commercial analog system.[6]

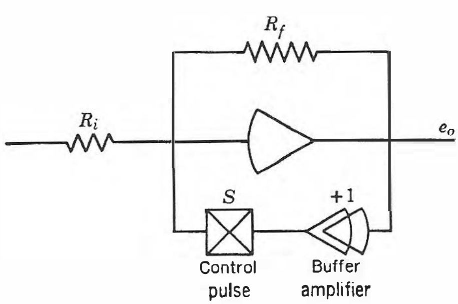

Fig. 8.5 Use of buffer amplifier to shorten reset period.

8.4 Rectangular Voltage Impulse Generator

The design of the rectangular voltage-impulse generators for repetitive differential analyzers is governed by a number of considerations. In certain computers the voltage wave controls only the electronic switches used to discharge the integrator capacitor. In the a-c type of operational amplifier, however, it is necessary to provide electronic switching at each coupling capacitor. As already indicated, the rectangular voltage wave is utilized in this case to apply the initial conditions. In specifying the voltage generator it is therefore necessary to consider whether the wave must have a positive as well as negative polarity and whether its amplitude has to be controlled or not.

The construction of the voltage generator is relatively simple if the d-c type operational amplifier is chosen. Then a parallel electronic switch can be employed to discharge the integrator capacitor. In order to assure proper operation of this switching system it is only necessary that the positive and negative amplitudes of the rectangular voltage wave be larger than the dynamic voltage range of the computer. Thus the amplitude of the rectangular wave does not have any influence on the accuracy of the computations.

Care must be taken that the leading edge of the rectangular wave be sufficiently steep to assure precision in the commencement of the computing interval. Figure 8.6 illustrates a voltage comparator that defines the working interval and the reset interval by the intersection of the functions

$$\begin{aligned} y &= c \\ y &= A \sin \omega t \end{aligned} \tag{8.15}$$

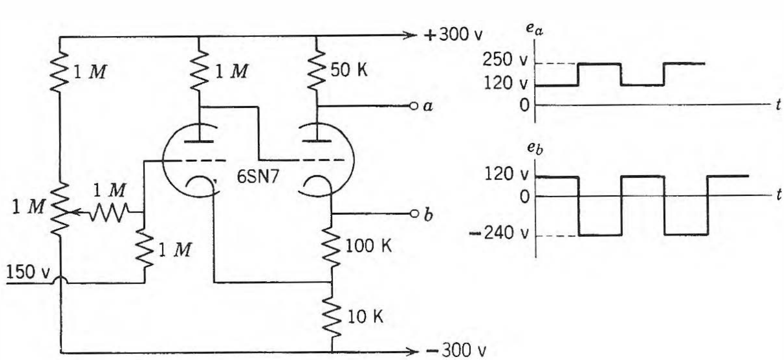

Fig. 8.6 Rectangular impulse generator.

This generator has the advantage that the width of the computing interval can readily be adjusted to the optimum value by changing the reference voltage. Frequently the rectangular impulse generator is accompanied by a frequency divider which permits the pulse frequency to be divided by a factor of two. This facilitates a change in the time base of the computer.

Brubaker and Eckes[7] describe a digital control unit for a repetitive analog computer. This unit combines a 10 kc crystal oscillator with inexpensive preset decimal counters and simple digital circuitry to generate accurate timing pulses which perform the following functions:

1. Reset the repetitive differential analyzer at 100 cycles/sec, 50 cycles/sec, 25 cycles/sec, 10 cycles/sec, or on external triggering.
2. Actuate external equipment after a preset number of 1000 to 10,000 successive runs.
3. Furnish sampling pulses to sampling readout devices at push-button selected sampling times t_1 and t_2 sec after the start of each individual computer run.
4. Furnish variable-brightness oscilloscope timing markers at 1000 cycles/sec and 100 cycles/sec.

8.5 Power Supply

From the point of view of accuracy, the power supplies of repetitive differential analyzers do not present any intrinsic difficulties. The stabilization of d-c voltages with a precision of the order of 0.1% is well within the capabilities of commercial available equipment. It is often desirable to stabilize as well the cathode voltage supply of the amplifier in order to minimize drift.

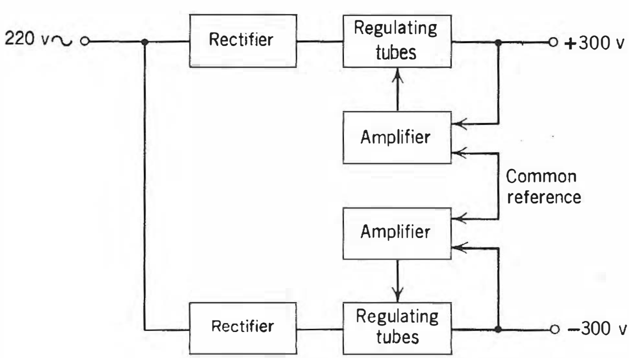

Fig. 8.7 Block diagram of high voltage supply.

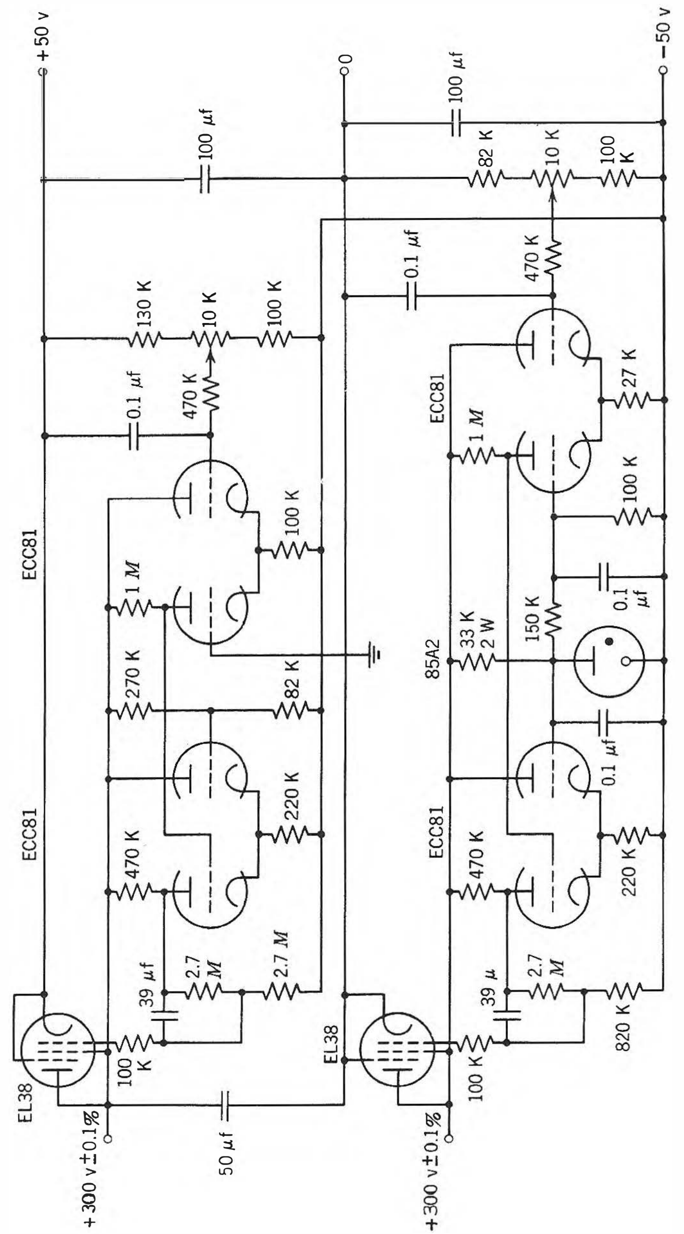

Fig. 8.8 Stabilized reference voltage supply.

In relatively large installations, the requirement for large load currents may present economic problems in the design of regulated voltage supplies. The use of units of rotating machinery is generally inadvisable since their time constants are relatively slow and their regulation therefore inadequate. This is particularly true when one is working with high repetition rates. For this reason it is necessary to employ an electronic voltage source.

Two approaches exist to the design of regulated d-c voltage supplies.

1. One unit which contains all voltage supplies and which has a load capability corresponding to the total consumption can be included in the computer.
2. A number of independent voltage supplies can be provided, one for the linear elements, for the nonlinear elements, etc.

Although one centralized power supply unit is generally the more economic solution, the use of separate supplies makes it easier to utilize and adjust computing elements and to enlarge the installation by adding more equipment.

The stabilized high-voltage supplies in repetitive computers are of conventional design. However, some specific characteristics should be taken into account. Since most d-c voltages needed must be of both polarities, it is convenient to design two identical stabilizers as one unit, so that positive as well as negative outputs can be obtained. The reference voltage should be common for the two stabilized supplies so that drift variations in the output are symmetrical. A block diagram of such a unit with a pair of symmetrical outputs is shown in Figure 8.7. The requirement for reference voltages of the order of ± 100 volts or ± 50 volts presents some difficulties in establishing suitable voltage levels in this type of a unit. To overcome this difficulty and provide better voltage regulation, the circuit shown in Figure 8.8 can be used. The high voltage supply for ± 50 volts is taken from two stabilizers of the type shown in Figure 8.7. These stabilizers employ a common reference tube and have an output ripple at no load of 1 mv peak to peak.

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part III
APPLICATIONS

chapter 9
ORDINARY DIFFERENTIAL EQUATIONS

The purpose of the examples assembled in this chapter is twofold. They are selected to demonstrate the practical limits of the applicability and accuracy of repetitive differential analyzers. Accordingly, whenever possible, solutions obtained by repetitive differential analyzers and non-repetitive analog computers are compared. In addition, an attempt is made to demonstrate the diversity of possible applications of repetitive differential analyzers. Here the objective is not to enter into detailed analyses of specialized domains of application, but rather to provide a brief survey.

9.1 First-Order Equation

The simplest test of a differential analyzer is to employ the integrators to integrate a step voltage

$$e_o = \frac{1}{RC}\int_0^t e(t)\,dt \tag{9.1}$$

When $e(t) = \text{const.} = 1$, as indicated in Chapter 4, the measurement of the output of the integrator can serve to establish the relative time constant of the computer. Since the linearity of the integrator is determined by the gain of the operational amplifier and is thus known in advance, this experiment can also serve to test the measuring instruments or output equipment.

Using the equation

$$e_o = \frac{t}{RC} \tag{9.2}$$

that is,

$$\tau_\alpha = \frac{t_1}{RC} = u_\alpha = 1 \tag{9.3}$$

the following results were obtained with a typical computer:

u_α	100 volts	50 volts	25 volts	10 volts	−10 volts	−25 volts	−50 volts	−100 volts
$\tau = 1$	12.0	11.9	11.9	12	11.8	11.8	11.9	11.8

Since in this specific case

$$RC = 10^{-2} \text{ sec}$$

twelve units in the time scale of the measuring system corresponded to one unit of the independent machine variable, that is, to 10^{-2} sec. The measuring system employed for this purpose was that described in Section 7.3.

9.2 Second-Order Equation

In addition to the general deductions which can be drawn by solving the equation of harmonic motion on a computer, this equation can be used to check the operation of the integrators and the output system. Consider the equation

$$\frac{d^2y}{dx^2} + y = 0 \qquad 0 \leq x \leq 10 \tag{9.4}$$

with the initial conditions

$$y(0) = 1 \qquad \frac{dy}{dx}(0) = 0$$

The range of the independent variable x implies that the solution is to be observed in the interval 0 to 3π. Let the relative time constant of the machine be

$$(RC)' = 1$$

with

$$R = 1 \text{ megohm} \qquad C = 0.01\ \mu\text{f}$$

The integrator time constant in this case is to be

$$(RC)' = 10$$

Accordingly, the integrator elements become

$$R = 0.1 \text{ megohm} \qquad C = 0.01\ \mu\text{f}$$

A second way of interpreting the scale of the independent variable x is to effect the transformation

$$x = 10\tau \tag{9.5}$$

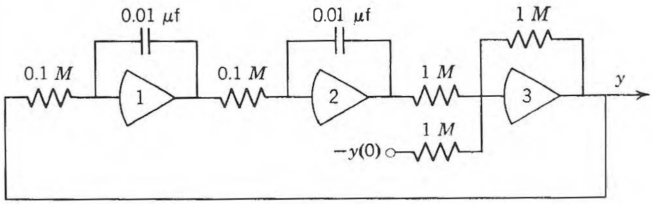

Fig. 9.1 Circuit for second-order differential equations.

If the independent machine variable is taken to have a range of

$$0 \leq \tau \leq 1$$

for

$$(RC)' = 1$$

the oscilloscope display will show the same number of solution cycles.

The solution of Equation 9.4 is

$$y = A \sin (x + \phi) \tag{9.6}$$

from which

$$A = 1$$

$$\phi = 90^\circ$$

The machine diagram for the solution of Equation 9.4 is shown in Figure 9.1. The results obtained on a typical repetitive differential analyzer[1] are

$$(RC)' = \frac{2\pi}{T_m} = \omega_m = 0.994$$

	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$\frac{7\pi}{2}$
y	+61.6 volts	−62.0 volts	+60.6 volts	−60.5 volts

9.3 Fourth-Order Equation

Consider the differential equation

$$\frac{d^4y}{dt^4} + 2.9\frac{d^3y}{dt^3} + 2.7\frac{d^2y}{dt^2} + 0.7\frac{dy}{dt} - 0.1y = 0 \tag{9.7}$$

where

$$y(0) = -10, \quad \frac{dy}{dt}(0) = 0 \quad \frac{d^2y}{dt^2}(0) = -2.1, \quad \frac{d^3y}{dt^3}(0) = 2.99$$

$$0 \leq t \leq 10$$

This equation was solved analytically, using a one-shot analog computer and a repetitive analog computer.[1,2] The analytic solution is

$$y = -10e^{0.1t} + te^{-t} \tag{9.8}$$

The measured results at four different instants of time are tabulated in the following:

	t	1	3	5	10
y volts	Repetitive	−10.65	−13.10	−16.35	−27.00
	One-shot	−10.60	−13.30	−16.50	−26.80
	Analytical	−10.68	−13.35	−16.47	−27.20

The error as a percentage of the maximum voltage is

	t	1	3	5	10
ϵ	Repetitive	−0.1	−0.73	−0.44	−0.74
	One-shot	−0.4	−0.4	−0.8	1.5

Figure 9.2 shows the computer setup for solving Equation 9.7 on a repetitive differential analyzer.

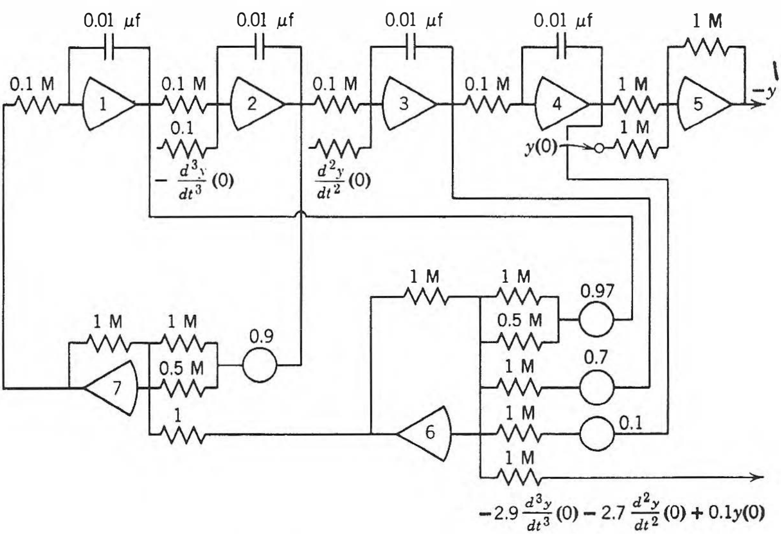

Fig. 9.2 Circuit for fourth-order differential equations.

9.4 Nonlinear Equations

In the study of the dynamic stability of electric power networks there arises a nonlinear differential equation which governs the oscillations of the rotor of a synchronous machine when the system load is varied. In such a motor the equation governing the angle between the vector flux of the rotor and the vector flux of the stator has the form

$$\frac{d^2y}{dt^2} + A \sin y = f(t) \tag{9.9}$$

As an example, this equation was treated using the following parameter values and initial conditions:

$$A = 2 \qquad f(t) = 1$$

$$y(0) = 0 \qquad \frac{dy}{dt}(0) = 0$$

$$0 \leq t \leq 2.5$$

A comparison of the solution of this equation obtained analytically, on a repetitive computer and on a one-shot computer,[3] is given in the accompanying table.

	t	1	1.5	2	2.5
y	Repetitive	25° 13′	45° 50′	60° 44′	65° 53′
	One-shot	24° 48′	44°	59° 18′	65°
	Analytical	24°	44° 30′	60° 30′	66°

The computer circuit is shown in Figure 9.3.

Mathieu's differential equation is often used to demonstrate the application of repetitive differential analyzers, since, as can be seen from the

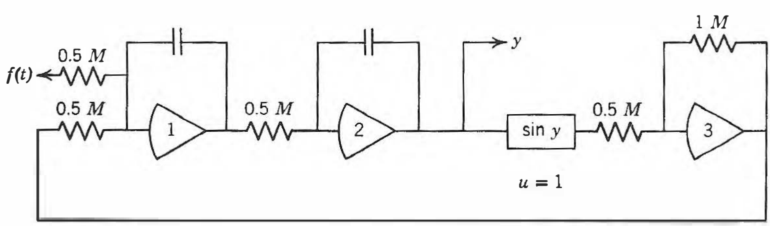

Fig. 9.3 Circuit for Equation 9.9.

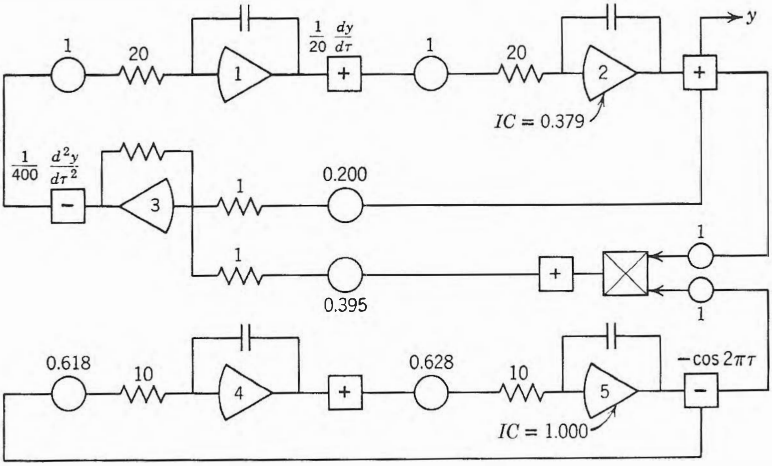

Fig. 9.4 Circuit for Mathieu's equation.

computer circuit Figure 9.4, it constitutes a test of the linear as well as the nonlinear computing elements of the analyzer. Following a method reported by W. B. Dhen using the Darmstadt repetitive differential analyzer, the equation

$$\frac{d^2y}{dx^2} + (a - p\cos 2x)y = 0 \tag{9.10}$$

is solved using two sets of parameter values and initial conditions

1. $p = 16$	2. $p = 16$
$a = 8.115$	$a = 14.182$
$y(0) = 0.395$	$y(0) = 0.892$
$\frac{dy}{dx}(0) = 0$	$\frac{dy}{dx}(0) = 0$

Solutions are obtained for the interval

$$0 \leq x \leq \pi$$

Taking as the range of the machine variables

$$0 \leq \tau \leq 1$$
$$0 \leq Y \leq 1$$

the scale factors are

1. $k_t = \pi$	2. $k_t = \pi$
$k_y = 1.041$	$k_y = 0.963$

If the following transformation is effected,

$$x = \pi\tau$$

Equation 9.10 becomes

$$\begin{aligned} &1.\ \frac{d^2Y}{d\tau^2} + (80.1 - 158\cdot\cos 2\pi\tau)\,Y = 0 \\ &2.\ \frac{d^2Y}{d\tau^2} + (140 - 148\cdot\cos 2\pi\tau)\,Y = 0 \end{aligned} \tag{9.11}$$

Dividing these equations by 400, a form suitable for programming on the analyzer is attained.

$$\begin{aligned} &1.\ \frac{1}{400}\frac{d^2Y}{d\tau^2} + (0.200 - 0.395\cdot\cos 2\pi\tau)\,Y = 0 \\ &\qquad Y(0) = 0.379 \\ &\qquad \frac{dY}{d\tau}(0) = 0 \\ &2.\ \frac{1}{400}\frac{d^2Y}{d\tau^2} + (0.350 - 0.395\cdot\cos 2\pi\tau)\,Y = 0 \\ &\qquad Y(0) = 0.926 \\ &\qquad \frac{dY}{d\tau}(0) = 0 \end{aligned} \tag{9.12}$$

In the computer diagram in Figure 9.4 the + and − signs were introduced because the operational amplifiers which were employed had a positive as well as a negative output (see Figure 5.26). The solution for the parameter values indicated earlier is periodic in form and is illustrated in Figure 9.5. The maximum error at the end of the computing interval was 3%.

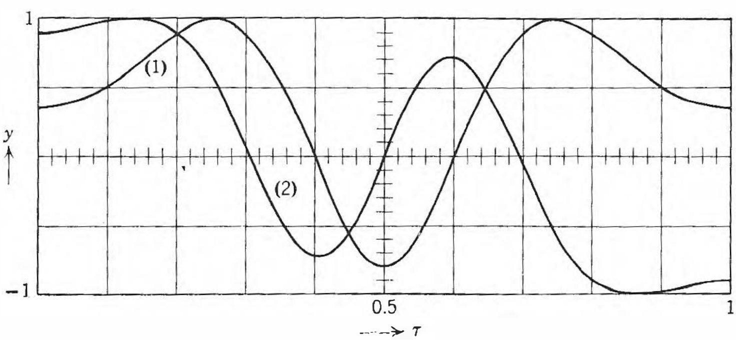

Fig. 9.5 Computer solution of Mathieu's equation.

9.5 Stability Studies

The equations treated in this example arise in the study of the stability of airplanes in flight. The movement of an airplane about its center of gravity is defined by three speeds of rotation: roll, pitch, and yaw. When linearized, the problem becomes a system of linear differential equations of fourth order. As a specific example consider the system

$$
\begin{aligned}
\frac{d\beta}{dt} &= -0.15\beta - r + 0.191\phi \\
\frac{dr}{dt} &= 14\beta - 0.679r - 0.267\frac{d\phi}{dt} \\
\frac{d^2\phi}{dt^2} &= -44.3\beta + 1.2r - 5.25\frac{d\phi}{dt}
\end{aligned}
\tag{9.13}
$$

$$0 \le t \le 2.5$$

Before programming the problem on the analyzer, the transformation

$$\beta_k = 10\beta$$

is made. This results in the system

$$
\begin{aligned}
\frac{d\beta_k}{dt} &= -0.15\beta_k - 10r + 1.91\phi \\
\frac{dr}{dt} &= 1.4\beta_k - 0.679r - 0.267\frac{d\phi}{dt} \\
\frac{d^2\phi}{dt^2} &= -4.43\beta_k + 1.2r - 5.25\frac{d\phi}{dt}
\end{aligned}
\tag{9.14}
$$

$$0 \le t \le 2.5$$

The computer diagram for this system of equations is shown in Figure 9.6. The active time interval on the differential analyzer for a relative integrator time constant

$$(RC)_r = 1 \qquad R = 1 \text{ megohm} \qquad C = 0.01\ \mu\text{f}$$

is

$$0 \le \tau \le 1.2$$

In this case the following values were taken for the integrators:

$$(RC)' = 2 \qquad R = 0.5 \text{ megohm} \qquad C = 0.01\ \mu\text{f}$$

This problem is one example of an application of a repetitive differential analyzer in which simple numerical solutions are not the most important objective. The first question to be answered is: Is the system stable?

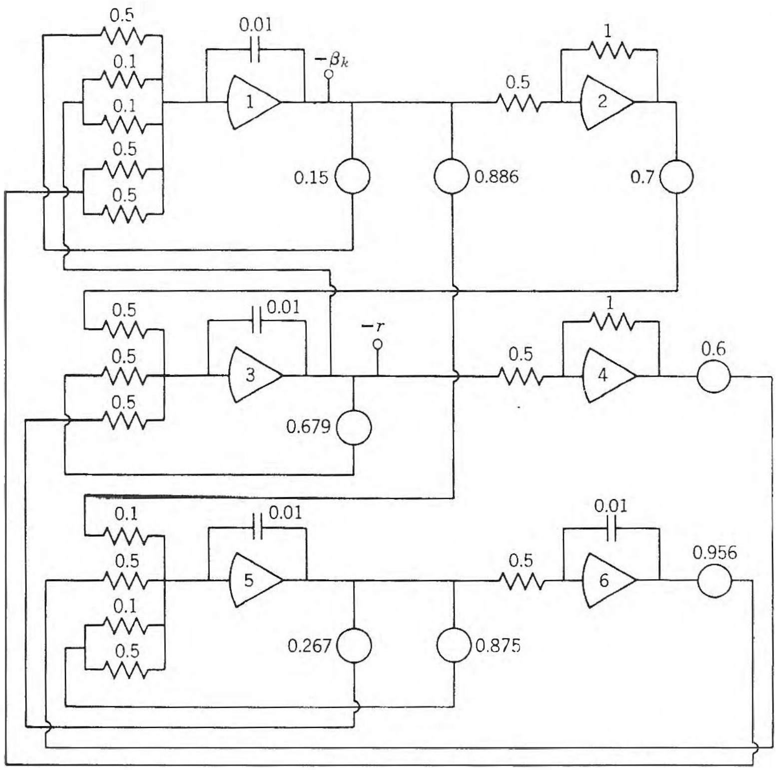

Fig. 9.6 Circuit for aircraft stability problem.

Another very important question of interest to the designer is: Over what range may the coefficients of the system (that is, the design characteristics of the airplane) vary in order that the solution remain in this zone of stability? It is also important to know which factors influence appreciably the transient response of the aircraft when it is exposed to perturbation forces.

For the specific coefficient values selected, the terms

$$1.4\beta_k \qquad \text{and} \qquad 5.25\,\frac{d\phi}{dt}$$

are most influential in regard to the transient response. This can easily be demonstrated with a repetitive differential analyzer.

The control surfaces of the aircraft are never completely rigid; there exists, therefore, an additional degree of freedom. Taking this into consideration, the system of equations becomes one of fifth order.

$$\begin{aligned}
\frac{d\beta_k}{dt} &= -0.15\beta_k - 10r + 1.91\phi \\
\frac{dr}{dt} &= 1.4\beta_k - 0.679r - 0.267\frac{d\phi}{dt} - 1.7\delta - 0.07\frac{d\delta}{dt} \\
\frac{d^2\phi}{dt^2} &= -4.43\beta_k + 1.2r - 5.25\frac{d\phi}{dt} \\
0.2\frac{d\delta}{dt} &= 2.6\beta_k - 60\delta
\end{aligned} \tag{9.15}$$

A comparison of this system with that of the fourth order system demonstrates the effect of the control surfaces on the stability. A differential analyzer study demonstrated that in this instance this effect was not negligible.

Another very useful application of the repetitive technique is in the indirect determination of coefficients. Suppose, for example, that measurements of the transient response of the actual aircraft are available, and that this information is to be utilized in computer studies. On the analyzer the coefficients can be readily varied until the analog solutions correspond to the solutions observed on the aircraft. In this way an approximate measure of the aircraft parameters is obtained.

9.6 Fourier Series

Paul and Thomas[4] have provided another comparison of the application of repetitive and nonrepetitive differential analyzers. One interesting example they furnish is the development of a periodic function $\phi(t)$ in Fourier series

$$\begin{aligned}
a_0 + a_1 \sin \omega t + a_2 \sin 2\omega t + \cdots + a_n \sin n\omega t \\
b_1 \cos \omega t + b_2 \cos 2\omega t + \cdots + b_n \cos n\omega t
\end{aligned} \tag{9.16}$$

The coefficients of the series are determined by the equations

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} \phi(t)\, d\omega t \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} \phi(t) \sin n\omega t\, d\omega t \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} \phi(t) \cos n\omega t\, d\omega t
\end{aligned} \tag{9.17}$$

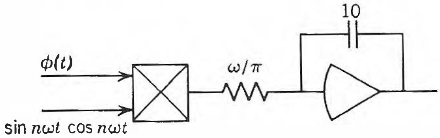

Fig. 9.7 Circuit for generating Fourier Series.

For analog purposes it is convenient to rewrite these equations in the form

$$a_n = \frac{\omega}{\pi}\int_0^T \phi(t)\sin n\omega t\,dt$$
$$b_n = \frac{\omega}{\pi}\int_0^T \phi(t)\cos n\omega t\,dt \tag{9.18}$$

where T is the period of the function $\phi(t)$. When T is equal to the repetition rate of the analyzer solution, the coefficients of the series are the integrals of the products $\phi(t)$ and sin $n\omega t$ or cos $n\omega t$.

The function $\phi(t)$ is obtained electrically with the aid of a function generator. The sinusoidal functions are obtained by resolving a harmonic differential equation as described earlier. The machine diagram for the solution of this problem without the function generator is shown in Figure 9.7. A Fourier analysis of a sawtooth-voltage wave on a repetitive differential analyzer provided the following results:

Harmonic	Analog Solution		Analytical Solution	
n	a_n	b_n	a_n	b_n
1	2.57	<0.001	2.54	0*
2	1.28	<0.001	1.27	0
3	0.875	<0.001	0.846	0
4	0.665	<0.001	0.635	0
5	0.535	<0.001	0.508	0

9.7 Spark Generator Simulation

Another example demonstrating that a numerical solution is not always the objective of a differential analyzer study is the simulation of the spark generator shown in Figure 9.8. The repetitive differential analyzer was employed to determine the appropriate charge time of the capacitor C_1 and to determine if the influence of the generator and the parasitic network parameters could be neglected. If the complete circuit is to be considered, the problem is characterized by a fourth-order differential equation; if the above effects are neglected, a second-order

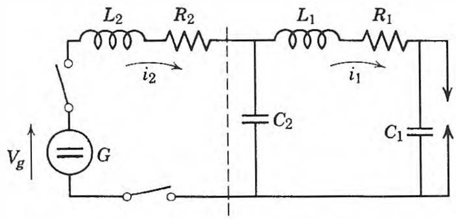

Fig. 9.8 Equivalent circuit of spark gap.

differential equation arises, which is more easily solved by standard techniques. The equations instrumented on the differential analyzer were

$$\frac{d^2i_1}{d\tau^2} = -5.95\frac{di_1}{d\tau} - 6.25i_1 + 0.42i_2$$

$$\frac{d^2i_2}{d\tau^2} = -5.95\frac{di_2}{d\tau} - 0.496i_2 + 0.42i_1 \tag{9.19}$$

Figure 9.9 shows the schematic diagram employed in this solution, and

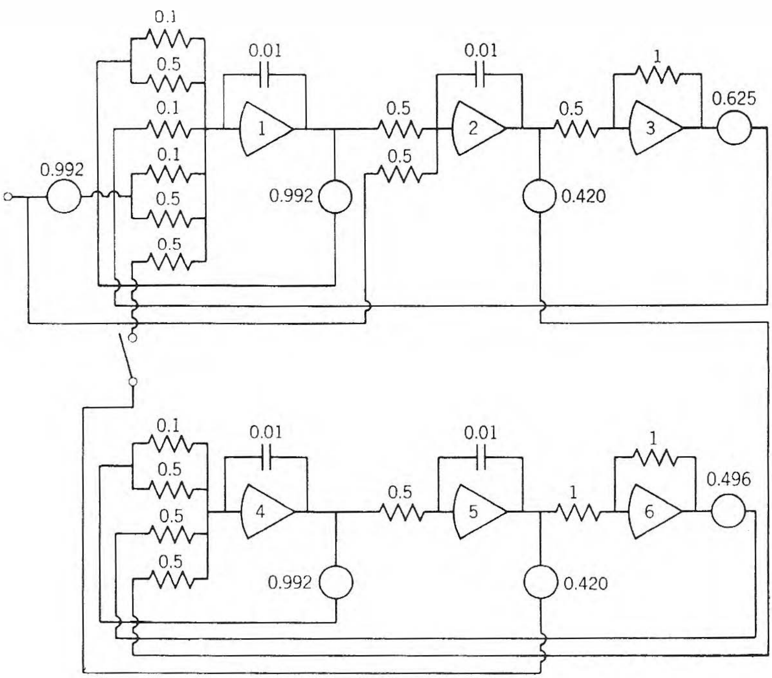

Fig. 9.9 Computer circuit for spark gap problem.

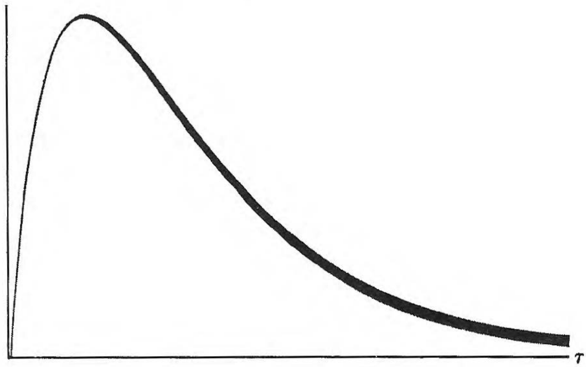

Fig. 9.10 Oscilloscope display of spark gap problem solution.

Figure 9.10 is a copy of the oscilloscopic solution. In the latter figure two solutions are shown simultaneously, one in which the parasitic parameters were neglected, and the other in which these parameters were included. This display demonstrates clearly that in this case the simpler of the two methods of solution suffices.

9.8 Nonlinear Mechanical System

As an example of a practical study of nonlinear phenomena the simulation of the mechanical system shown in Figure 9.11, as discussed by Rideout,[5] will be described. The connection between the two axes of rotation is nonlinear in nature as shown in Figure 9.12, while the viscous friction has a parabolic character as shown in Figure 9.13. The equations governing this system have the form

$$
\begin{aligned}
q(t) &= J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1\theta_1 + F(\theta_1 - \theta_2) + G\theta_1|\theta_1| \\
0 &= J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2 + F(\theta_1 - \theta_2)
\end{aligned}
\tag{9.20}
$$

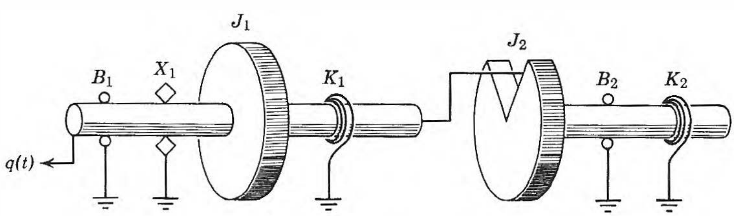

Fig. 9.11 Nonlinear mechanical system.

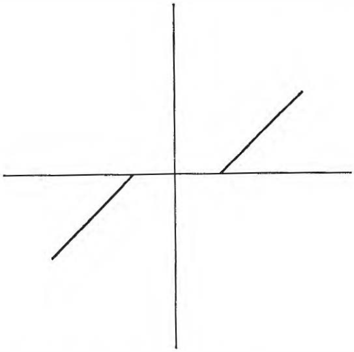

Fig. 9.12 The function $F(\theta_1 - \theta_2)$.

For programming on a differential analyzer, these equations are rewritten as

$$\frac{d^2\theta_1}{dt^2} = \frac{q(t)}{J_1} - \frac{B_1}{J_1}\frac{d\theta_1}{dt} - \frac{K_1}{J_1}\theta_1 - \frac{1}{J_1}F(\theta_1 - \theta_2) - \frac{G}{J_1}\theta_1|\theta_1|$$
$$\frac{J_2}{J_1}\frac{d^2\theta_2}{dt^2} = -\frac{B_2}{J_1}\frac{d\theta}{dt} - \frac{K_2}{J_1}\theta_2 + \frac{1}{J_1}F(\theta_1 - \theta_2) \tag{9.21}$$

The numerical values of the parameters employed are

$$\frac{B_1}{J_1} = 0.5 \qquad \frac{B_2}{J_1} = 0.2 \qquad \frac{1}{J_1} = 1.0$$
$$q(t) = \text{unit step}$$
$$\frac{K_1}{J_1} = 1.5 \qquad \frac{K_2}{J_1} = 0.4 \qquad \frac{G}{J_1} = 1.0$$

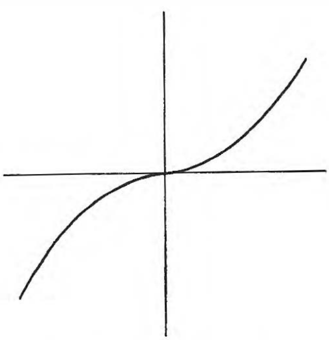

Fig. 9.13 The function $G\theta_1|\theta_1|$.

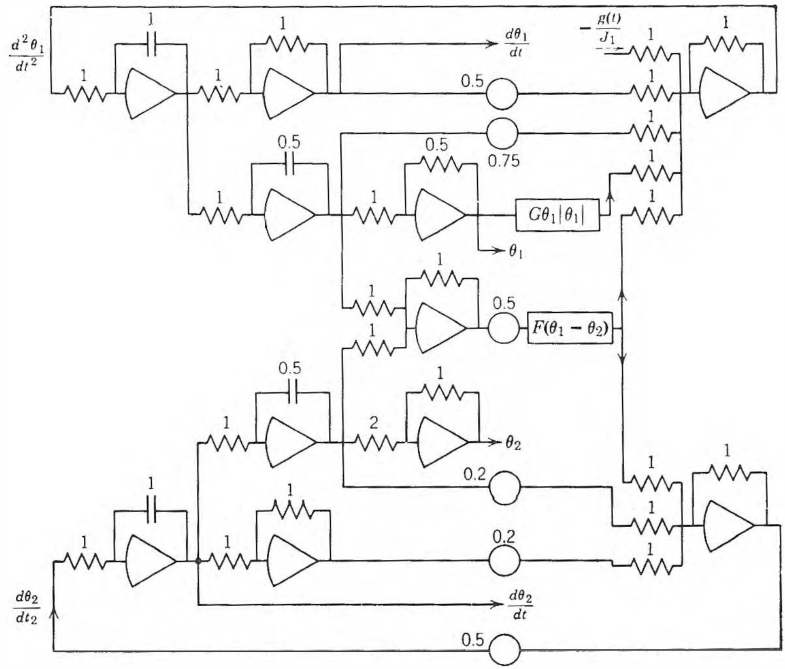

Fig. 9.14 Computer circuit for nonlinear mechanical system.

The machine diagram is shown in Figure 9.14, and Figure 9.15 illustrates the solution as photographed on the oscilloscope.

9.9 Two-Point Boundary Value Problems

An example that clearly illustrates the advantage of the repetitive differential analyzer in permitting the immediate display of the effect of parameter variations in the solution of differential equations was provided by Vitenberg.[6] The problem involved was a boundary-value problem rather than the more commonly treated initial-value problems.

The simplest and most direct method of attacking such a problem on the analog computer is by trial-and-error. Consider, for example, that it is desired to find the solution of a differential equation of second order which passes through the two points A and B. By the method of trial-and-error the initial condition upon the first derivative of y is varied until two solutions are found one of which passes on one side of point B and the other which passes on the other side of point B. By interpolating the exact solution can be rapidly obtained.

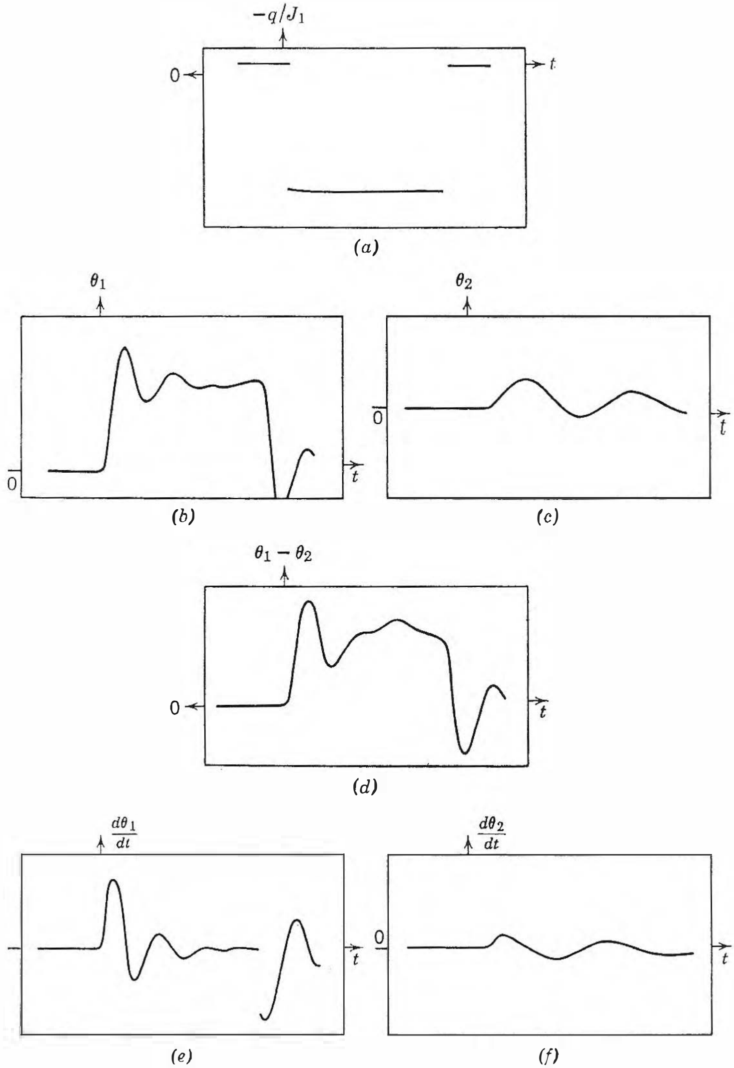

Fig. 9.15 Computer solutions of nonlinear mechanical system.

For higher order equations it is desirable to speed up this trial-and-error method. This can be accomplished by means of special electronic circuitry. Let $y_1(x)$ be the functions which satisfy the system of differential equations

$$\frac{dy_i}{dx} = f_i(y_1, y_2, \ldots, y_n, x) \qquad i = 1, 2, \ldots, n \tag{9.22}$$

and which do not satisfy the boundary conditions

$$\begin{aligned} y_i(x_0) &= a_i \qquad i = 1, 2, \ldots, k \\ y_p(x_1) &= b_p \qquad p = k + 1, k + 2, \ldots, n \end{aligned}$$

The differences between $y_i(x)$ and the boundary conditions are represented by ϵ_i. In the repetitive mode, it is possible to display on the oscilloscope

$$\sum_{i=1}^{n} |\epsilon_i|$$

by the construction of a special electronic minimizer. With the aid of such a device, the problem

$$\begin{aligned} \frac{dz}{dt} &= 0.2 - 4y - 0.01t^2 \\ \frac{dv}{dt} &= 0.1z \\ \frac{du}{dt} &= \frac{0.1y}{2 - 0.01t^2} \\ \frac{dy}{dt} &= 0.1u \end{aligned} \tag{9.23}$$

was solved for the boundary conditions

$$u(0) = z(0) = v(10) = z(10) = 0$$

The solution was obtained in one minute with an accuracy of 5%. This device was also employed to solve the equations

$$\begin{aligned} \frac{d^2x}{dt^2} &= -E\frac{dx}{dt} \qquad & E &= cH_\tau(y)G(v_\tau) \\ \frac{d^2y}{dt^2} &= -E\frac{dy}{dt} - g \qquad & v_\tau &= \sqrt{x^2 + y^2}F(y) \end{aligned} \tag{9.24}$$

with boundary conditions

$$x(0) = y(0) = y(t_1) = 0 \qquad x(t_1) = 3750$$

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chapter 10 PARTIAL DIFFERENTIAL EQUATIONS

10.1 General Remarks

The solution of problems governed by partial differential equations constitutes a special challenge to the analog computer. Such problems generally have two or more independent variables, whereas analog computers are restricted to one independent variable—time. In this chapter various techniques for attacking these problems are reviewed, with special emphasis on applications of repetitive differential analyzers.

Probably the most widely occurring partial differential equation is Laplace's equation

$$\nabla^2 \phi = 0 \tag{10.1}$$

where ∇^2 is the Laplacian operator, which in two Cartesian dimensions takes the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{10.1a}$$

This equation is relatively rarely treated by means of electronic analog computers because such passive analog simulators as the electrolytic tank and resistance networks are particularly effective in treating problems of this type. Equation 10.1 is one of the family of elliptic partial differential equations whose solutions are not time dependent.

The parabolic partial differential equation

$$\nabla(\sigma \, \nabla\phi) = S \frac{\partial \phi}{\partial t} \tag{10.2}$$

occurs in the study of transient heat transfer, the flow of fluids through porous media, and a wide variety of diffusion phenomena. σ and S are field parameters which may be functions of time, space, or the potential function ϕ. For constant parameters and in one-space dimension Equation 10.2 becomes

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \phi}{\partial t} \tag{10.3}$$

Equation 10.2 and modified forms thereof have been widely studied by means of passive resistance-capacitance network analyzers and more recently by means of electronic analog computers.

Other equations that are of frequent interest include the hyperbolic wave equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2} \tag{10.4}$$

occasionally simulated by means of inductance-capacitance network analyzers, and the biharmonic equation

$$\nabla^4 \phi = \frac{\partial^2 \phi}{\partial t^2} \tag{10.5}$$

governing the vibration of beams. The latter equation can be treated by passive networks including resistors, inductors, capacitors, and transformers or by electronic differential analyzers. A detailed discussion of the application of analog simulation techniques to the study of field problems characterized by partial differential equations was presented by Karplus.[1]

In order to treat transient field problems of the type of Equation 10.2 to 10.5 by electronic differential analyzer techniques, it is essential to effect a transformation to reduce the number of independent variables. This is accomplished by means of finite difference techniques in which a continuous variable such as x, y, or t is replaced by an array of discretely spaced points. Solutions are then obtained for these points and interpolation techniques are used to construct continuous equipotential or streamlines. When an independent variable is discretized in this manner, the corresponding partial derivative is replaced by an algebraic expression. In analog computer terms this means that the operation of integration is replaced by additions and subtractions. Taking as a typical example Equation 10.3, three basic possibilities exist in the application of finite difference approximations:

1. The variable x is discretized while the variable t is kept in continuous form. The left side of Equation 10.3 then becomes an algebraic expression, and the right side remains unchanged.

2. The x variable is kept in continuous form while the time variable is discretized. The left side of Equation 10.3 then remains unchanged and the right side becomes an algebraic term.
3. Both the x and the t variables are discretized so that all terms in Equation 10.3 become algebraic expressions.

All three of these techniques have been used with some success in analog simulations. In this chapter these three techniques are briefly summarized in turn. The chapter concludes with a description of the Monte Carlo technique—a method for solving field problems that does not involve discretization.

10.2 Discrete-Space-Continuous-Time

The most widely used approach to the simulation of transient field problems involves the replacement of all derivatives with respect to space variables by means of finite difference expressions. By using Equation 10.3 as an example,

$$\frac{\phi_{x+\Delta x,t} + \phi_{x-\Delta x,t} - 2\phi_{x,t}}{\Delta x^2} = \frac{1}{\alpha}\frac{\partial \phi_{x,t}}{\partial t} \tag{10.6}$$

The second derivative with respect to x has been replaced by the familiar "second central difference." This is obtained by writing

$$\left.\frac{\partial \phi}{\partial x}\right|_{x+\frac{1}{2}\Delta x,t} = \frac{\phi_{x+\Delta x,t} - \phi_{x,t}}{\Delta x} \tag{10.7a}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{x-\frac{1}{2}\Delta x,t} = \frac{\phi_{x,t} - \phi_{x-\Delta x,t}}{\Delta x} \tag{10.7b}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\Delta x}\left(\left.\frac{\partial \phi}{\partial x}\right|_{x+\frac{1}{2}\Delta x,t} - \left.\frac{\partial \phi}{\partial x}\right|_{x-\frac{1}{2}\Delta x,t}\right) \tag{10.7c}$$

where Δx represents the distance between adjacent grid points in the x direction.

The passive analog simulator for Equation 10.3 is constructed by recognizing the formal similarity between Equation 10.6 and the equation obtained by writing the Kirchhoff's current-law equation for an electrical node formed by two resistors and a capacitor to ground. The general analog for Equation 10.2 then takes the form of a rectangular network of resistors in a one-, two-, or three-dimensional array with a capacitor linking each node point to ground.

One application of electronic analog computers to problems of this type involves the simultaneous solution of Equation 10.6 at each node in the space domain. For this purpose an integrator and an adder are

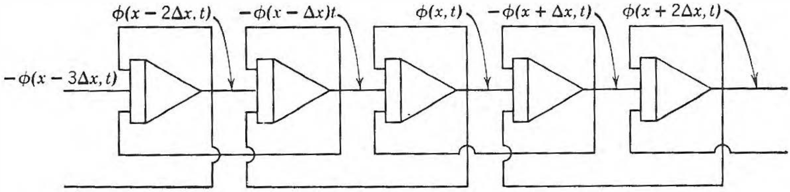


Fig. 10.1 Portion of discrete-space-continuous-time analog of diffusion equation.

required at each grid point. An economy in equipment can be effected if solutions of alternate polarity are acceptable at alternate points in space. The operations of integration and addition can then be combined so that a single integrator performs in effect the computation

$$\phi_{x,t} = -\frac{\alpha}{\Delta x^2}\int_0^t (\phi_{x+\Delta x,t} + \phi_{x-\Delta x,t} - 2\phi_{x,t})\,dt \tag{10.8}$$

The computer system will then take the form shown in Figure 10.1 and require n operational amplifiers for n grid points. The initial condition applied to each integrator will correspond to the specified value of the corresponding point in space at time $t = 0$; and the transient voltages as recorded at the output of each integrator will correspond to the transient field potentials at the corresponding points in the system being simulated. Howe and Haneman[2] have described the application of this technique to a variety of transient field problems.

10.3 Continuous-Space-Discrete-Time

Jury[3] has described an application of the dynamic memory circuits described in Chapter 5 to the solution of Equation 10.3. In this method the space variable x is kept in continuous form while the time variable t is approximated by a series of discrete steps Δt in length. The finite difference approximation of Equation 10.3 then becomes

$$\frac{\partial^2\phi_{x,t}}{\partial x^2} = \frac{\phi_{x,t} - \phi_{x,t-\Delta t}}{\alpha\Delta t} \tag{10.9}$$

The term $\phi_{x,t-\Delta t}$ has been "remembered" from previous calculations, so that $\phi_{x,t}$ is the only unknown. For the next time increment the equation solved in the computer is

$$\frac{\partial^2\phi_{x,t+\Delta t}}{\partial x^2} = \frac{\phi_{x,t+\Delta t} - \phi_{x,t}}{\alpha\Delta t} \tag{10.10}$$

This procedure is repeated for n increments of time that may be of interest. The computer circuitry used to solve Equation 10.9 is therefore

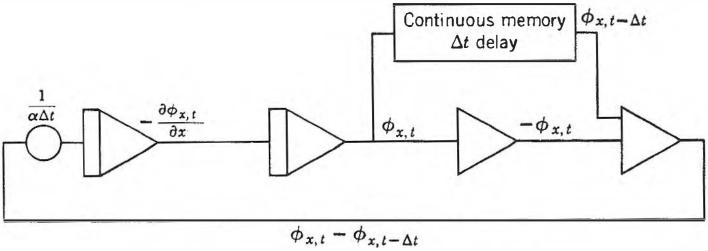


Fig. 10.2 Continuous-space-discrete-time analog of diffusion equation.

used for n computer runs. The larger n the longer the time required for solution. In this respect this approach differs from that presented in Section 10.2 in which a refinement of the finite difference net, by reducing the net interval, involves additional node points and therefore additional computer equipment in direct proportion to the number of nodes. A general circuit illustrating this approach is shown in Figure 10.2. Note that the computer time variable represents the problem space variable x.

The chief problem in the application of this method is the generation of the function $\phi_{x,t-\Delta t}$. Unlike in the discrete-space-discrete-time method described in the following section, a continuous memory is required. That is, the output of the memory circuit must be a function of computer time identical to the function of time at the input of the memory circuit during the preceding repetitive cycle. Jury[3] proposes the use of an interpolation technique using a number of x and 0 memories of the type described in Section 5.4. The value of $\phi_{x,t}$ is memorized at a set of discrete values of x (that is, at a series of computer-time steps). Using Newton's forward interpolation formula

$$f(x) = A + Bx + C\frac{x(x-1)}{2!} + D\frac{x(x-1)(x-2)}{3!} + \cdots \qquad (10.11)$$

these memorized values are then combined to produce the desired continuous function during the succeeding repetitive cycle. Evidently to achieve a reasonably accurate continuous memory a large number of x and 0 memories are required. This counteracts to a large extent the advantage gained by retaining the space variable in continuous form.

10.4 Discrete-Space-Discrete-Time

When transient field problems are treated on a digital computer, all independent variables must be discretized, since a digital machine is capable only of solving algebraic expressions. In approximating a time derivative by a finite difference expression, two possibilities exist termed "forward difference" and "backward difference" approximations. Equations employing forward differences are generally solved explicitly, a process which is relatively simple computationally but which has the inherent possibility of computational instability. If the ratio of the time increment to the space increment is improperly chosen, round-off errors made in the course of the solution will gradually build up until they overshadow the solution, thus making it worthless. In order to obtain satisfactory solutions by this method it is necessary to make the time increment relatively small, that is, to take many time-consuming steps. On the other hand, if backward differences are employed, the computational procedure is an implicit one having no possibility of computational instability but requiring the solution of a large number of simultaneous equations at each time increment—also leading to long computer runs. An analog technique which takes advantage of the ability of electrical networks to solve simultaneous algebraic equations instantaneously was introduced by Liebmann.[4] As in the technique described in Section 10.3, the time derivative of Equation 10.2 is approximated by a backward difference formula. In this case, however, the space derivative is also discretized, so that the approximation of Equation 10.2 becomes

$$\frac{\phi_{x+\Delta x,t} + \phi_{x-\Delta x,t} - 2\phi_{x,t}}{\Delta x^2} = \frac{\phi_{x,t} - \phi_{x,t-\Delta t}}{\alpha \Delta t} \tag{10.12}$$

Equation 10.12 when rewritten as

$$\frac{(\phi_{x+\Delta x,t} - \phi_{x,t})}{\Delta x^2} + \frac{(\phi_{x-\Delta x,t} - \phi_{x,t})}{\Delta x^2} + \frac{(\phi_{x,t-\Delta t} - \phi_{x,t})}{\alpha \Delta t} = 0 \tag{10.13}$$

is seen to correspond term by term to the Kirchhoff's node law equation of an electrical node formed by three resistors Δx^2, Δx^2, and $\alpha\,\Delta t$ in magnitude. A circuit for a one-dimensional field problem governed by Equation 10.2 is shown in Figure 10.3. The voltages $\phi_{x,t}$ are the unknowns, whereas the voltages $\phi_{x,t-\Delta t}$ are the values obtained in the preceding step in the solution.

The memorization of the solutions $\phi_{x,t}$ at each node of the network and the application of these voltages to the lower terminals of the potentiometers in Figure 10.3 are readily accomplished by an electronic computer with repetitive capabilities. This can be done, for example, using the combination of Rev-M and M memories shown in Figure 5.16; a separate memory circuit must be used for each network node. Since only algebraic expressions are being handled, the frequency-response requirement on the analog computer are not critical. It is therefore possible to employ

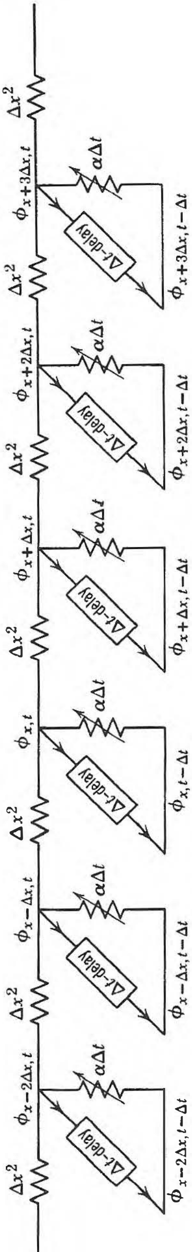


Fig. 10.3 Discrete-space-discrete-time analog of diffusion equation.

this technique using slow electronic differential analyzers equipped with a special control circuit to permit repetitive operation. Such a control system was described by Gilliland.[5]

Karplus[6] has described a method for extending the discrete-space-discrete-time technique to a wide variety of additional field problems including those governed by Equations 10.4 and 10.5 and modified forms thereof. This approach requires active circuitry in the form of at least one differential amplifier at each node, but is similar in principle to that described previously. To simplify the subsequent notation, a finite difference grid in only one-space dimension and in time as shown in Figure 10.4 is considered. The point $\phi_{x,t}$ is designated as ϕ_0 while surrounding points are designated as ϕ_1 to ϕ_{10}.

The term $\partial^2\phi/\partial x^2$ in Equation 10.4 and the term $\partial^4\phi/\partial x^4$ in Equation 10.5 are approximated by averaging the second central difference expressions for time $(t_0 + \Delta t)$ and $(t_0 - \Delta t)$. The second derivatives with respect to time are approximated as differences centered on point 0, and all terms are expressed as differences with respect to ϕ_0. Upon rearrangement Equations 10.3, 10.4, and 10.5 become respectively

$$\frac{\partial^2\phi_0}{\partial x^2} - \frac{1}{\alpha}\frac{\partial\phi_0}{\partial t} = \frac{\phi_1 - \phi_0}{\Delta x^2} - \left(\frac{2}{\Delta x^2} + \frac{1}{\alpha\,\Delta t}\right)(\phi_2 - \phi_0) + \frac{\phi_3 - \phi_0}{\Delta x^2} = 0 \tag{10.14}$$

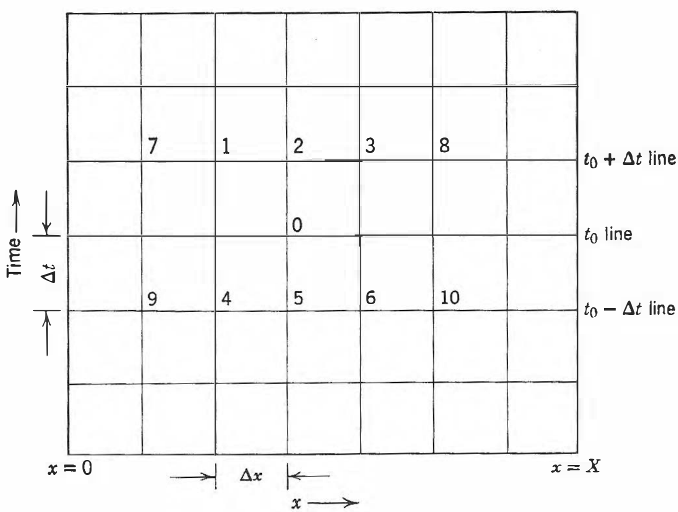


Fig. 10.4 Finite difference grid.

$$\frac{\partial^2\phi_0}{\partial x^2} - k\frac{\partial^2\phi_0}{\partial t^2} = \frac{\phi_1 - \phi_0}{2\Delta x^2} - \left(\frac{1}{\Delta x^2} + \frac{k}{\Delta t^2}\right)(\phi_2 - \phi_0) + \frac{\phi_3 - \phi_0}{2\Delta x^2}$$

$$+ \frac{\phi_4 - \phi_0}{2\Delta x^2} - \left(\frac{1}{\Delta x^2} + \frac{k}{\Delta t^2}\right)(\phi_5 - \phi_0) + \frac{\phi_6 - \phi_0}{2\Delta x^2} = 0 \quad (10.15)$$

$$\frac{\partial^4\phi_0}{\partial x^4} - k\frac{\partial^2\phi_0}{\partial t^2} = -\frac{2}{\Delta x^4}(\phi_1 - \phi_0) + \left(\frac{3}{\Delta x^4} - \frac{k}{\Delta t^2}\right)(\phi_2 - \phi_0) - \frac{2}{\Delta x^4}(\phi_3 - \phi_0)$$

$$- \frac{2}{\Delta x^4}(\phi_4 - \phi_0) + \left(\frac{3}{\Delta x^4} - \frac{k}{\Delta t^2}\right)(\phi_5 - \phi_0)$$

$$- \frac{2}{\Delta x^4}(\phi_6 - \phi_0) + \frac{1}{2\Delta x^4}(\phi_7 - \phi_0) + \frac{1}{2\Delta x^4}(\phi_8 - \phi_0)$$

$$+ \frac{1}{2\Delta x^4}(\phi_9 - \phi_0) + \frac{1}{2\Delta x^4}(\phi_{10} - \phi_0) = 0 \quad (10.16)$$

The analog network is derived from a recognition of the formal similarity of Equations 10.14 to 10.16 and the Kirchhoff's law equations governing a network of electrical resistors. Each term in these equations can be represented by a current flowing into node 0 from adjacent nodes 1 to 10 through resistors R_1 to R_{10}. This is illustrated in Figure 10.5. Some of the resistors in this figure have negative magnitudes. In these networks the potentials in lines t_0 and $(t_0 - \Delta t)$ constitute specified initial conditions, whereas the potentials in line $(t_0 + \Delta t)$ must be determined. An application of certain theorems of numerical analysis indicates that Equations 10.14 to 10.16 will be computationally stable regardless how large Δt is.

To solve a given initial-value problem, typical nodes as shown in Figure 10.5 are interconnected in a one-dimensional, two-dimensional, or three-dimensional array, depending upon the number of space variables involved. Operational amplifiers are employed at each node to realize the negative resistors and to enforce the implicit relationship specified by Equations 10.14 to 10.16. The latter task can be accomplished by using a high-gain amplifier without a feedback resistor to force the nodes in line $(t_0 + \Delta t)$ to assume their correct values virtually instantaneously. Typical active node circuits for Equations 10.14, 10.15, and 10.16 are shown in Figure 10.6, p. 207. The voltages $-V_0$ and V_5 are d-c voltages applied in accordance with the solutions (potential at node 2) at the first and second preceding time increments. These values can be considered to be the initial conditions for each step in the computation. A variety of circuit topologies can be realized by suitable rearrangements of Equations 10.14 to 10.16. For

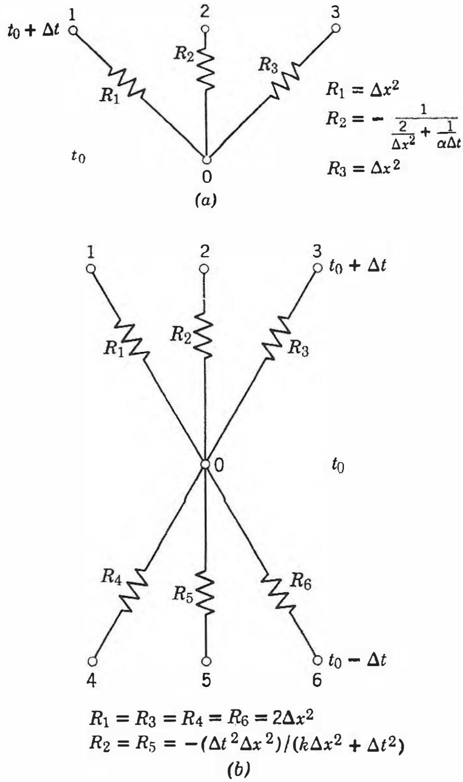
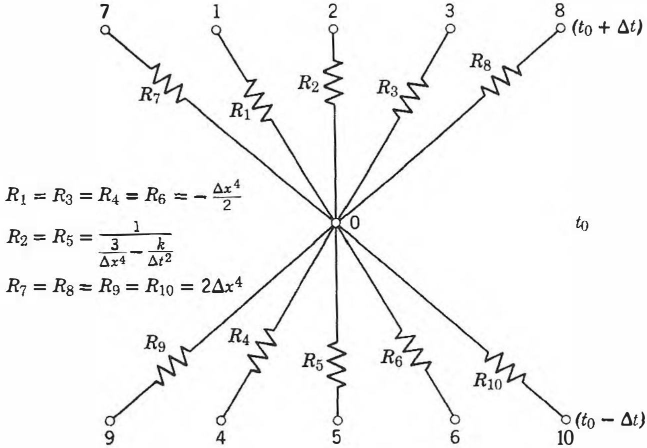


Fig. 10.5 Typical node of resistance network analog. (*a*) Diffusion equation. (*b*) Wave equation.

example, a single differential amplifier at each node can be made to suffice for all node modules.

To introduce the initial values at each step in the computation, dynamic memory units as shown in Figure 5.16 can be employed. Alternatively, a digital computer can be used as the memory. In the latter case corrections for drift and zero offset in the d-c amplifiers can be included in the digital program so that relatively inexpensive analog components can be employed; furthermore, the availability of the digital computer facilitates the solution of field problems in which the field parameters are functions of the field potentials (that is, nonlinear problems). Such a hybrid system is shown in Figure 10.7, p. 208. In this way a relatively modest digital computer in combination with an analog network can handle problems



(*c*)

Fig. 10.5 (*c*) Biharmonic equation.

that would otherwise require very large and elaborate digital facilities for their simulation.

10.5 The Monte-Carlo Method

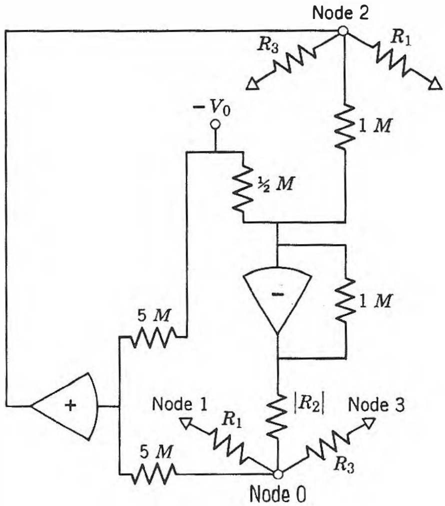
A well-established technique in digital computation involves the use of a sequence of random numbers as computer inputs. By making a sufficiently large number of computer runs, and by taking a weighted average of the results of these runs, convergence to the desired solution can be obtained. This technique can be applied in analog computation by exciting the computer system with a random noise generator—a voltage generator whose output amplitude is governed by a stochastic expression. The repetitive mode of operation is useful in performing such calculations because a large number of separate runs, each with a random noise input, can be completed in a reasonably short time. An application of this technique to the solution of partial differential equations was presented by Chuang, Kazda, and Windeknecht[7] in an excellent paper.

The boundary value problems, for which the Monte-Carlo method is applicable, belongs to a family of generalized Dirichlet problems of the form

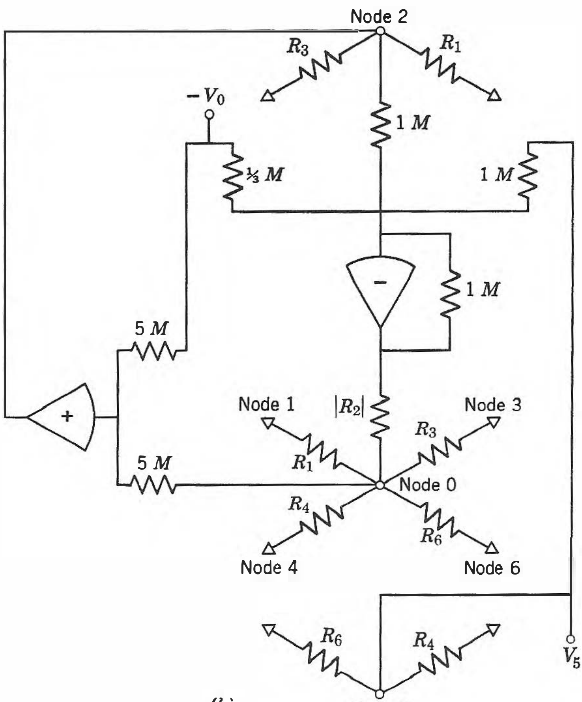
$$D_1 \frac{\partial^2 \phi}{\partial {x_1}^2} + D_2 \frac{\partial^2 \phi}{\partial {x_2}^2} - K_1 \frac{\partial \phi}{\partial x_1} - K_2 \frac{\partial \phi}{\partial x_2} = 0 \qquad (10.17)$$

$$\phi = F(x_1, x_2) \text{ on the boundary } C$$

where K_1 and K_2 are arbitrary functions of the independent variables x_1 and x_2 respectively. The boundary C is an arbitrary finite closed curve,


Node 2
R_3
R_1
$-V_0$
1 M
½ M
1 M
5 M
+
−
Node 1
R_1
$|R_2|$
Node 3
5 M
R_3
Node 0

(a)


Node 2
R_3
R_1
$-V_0$
1 M
⅓ M
1 M
1 M
5 M
+
−
Node 1
$|R_2|$
Node 3
R_3
R_1
5 M
Node 0
R_4
R_6
Node 4
Node 6
R_6
R_4
V_5
Node 5

(b)

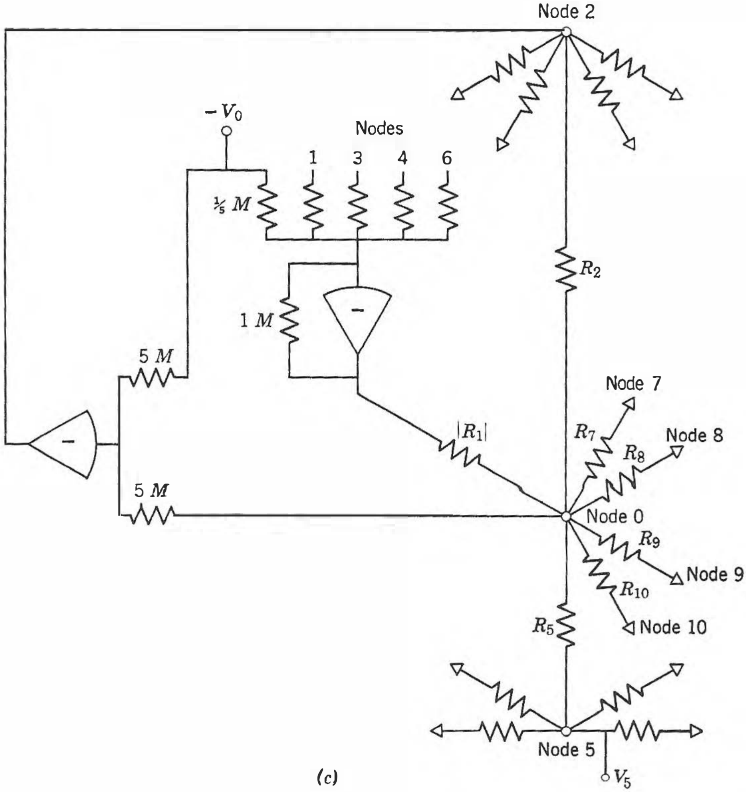


Fig. 10.6 Typical active DSDT node modules. (*a*) Diffusion equation. (*b*) Wave equation. (*c*) Biharmonic equation.

a Jordan curve; D_1 and D_2 are constants. This method provides a solution in the form of the field potential ϕ existing at some point within the field. It does not provide equipotential curves as do the simulation techniques discussed previously.

In principle the Monte-Carlo method involves starting a computer run at the point of interest within the field and commencing a random walk. That is, a sequence of small steps are taken such that each step begins at the end of the preceding step but proceeds in a random direction. Eventually each such random walk will reach some point on the field boundary C. The potential at this boundary point is recorded, and a new random walk is commenced. Provided enough such random walks are taken, and provided each step in the walk is sufficiently small, it can be shown that

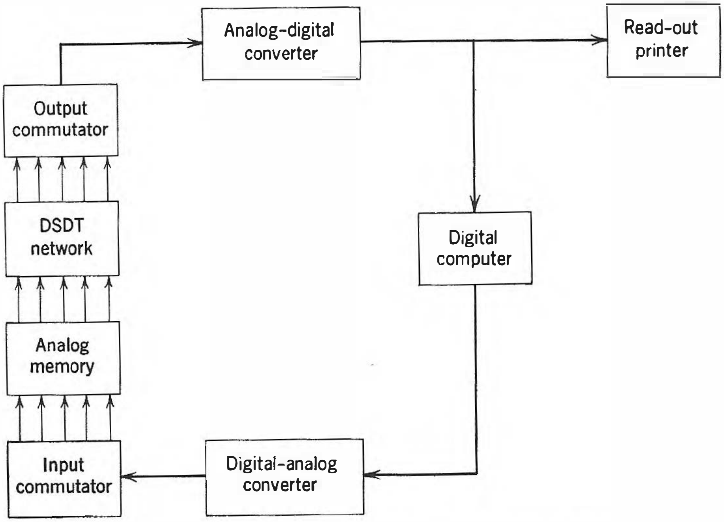


Fig. 10.7 Block diagram of hybrid DSDT system using a small digital computer to memorize initial conditions for each time increment.

the weighted average of the boundary intersections will converge to the potential existing at the point of interest within the field. In the computer method to be described, this random walk is actually performed by the beam of an oscilloscope. When this beam contacts the field boundary as defined by a mask placed over the face of the oscilloscope, the random walk is terminated and a new one commenced.

Some auxiliary circuitry is required to cause the oscilloscope beam to describe the random walk required. It has been shown that when an electric circuit is subjected to an exciting input voltage of Gaussian white noise, the response current of the circuit is a Markoff stochastic process and that consequently the conditional probability density function of the current satisfies the equations governing random walk phenomena. For the solution of two-dimensional generalized Dirichlet problems, two independent resistance-inductance circuits with nonlinear resistance are required. The equations describing the responses of these circuits subjected to two independent Gaussian white noise sources $F_1(t)$ and $F_2(t)$ are

$$\begin{aligned} \frac{dy_1}{dt} + K_1 &= F_1(t) \\ \frac{dy_2}{dt} + K_2 &= F_2(t) \end{aligned} \tag{10.18}$$

where K_1 and K_2 are functions of y_1 and y_2. The voltages generated by solving Equation 10.18 on the computer are employed to drive the oscilloscope beam in the x and y directions. Accordingly, the solution of a Dirichlet problem of the type described by Equation 10.17 takes the following steps:

1. Two resistance-inductance circuits governed by Equations 10.18 are simulated. In general these governing equations will be nonlinear.
2. Two independent voltage sources of white noise, F_1 and F_2, with adjustable spectral density, D_1 and D_2 respectively, are employed to drive the two resistance-inductance circuits.
3. The voltage outputs y_1 and y_2 of the two nonlinear circuits are used to drive the horizontal and vertical inputs respectively of a cathode-ray oscilloscope.
4. The prescribed boundary C is introduced in such a way as to make detectable the impingement of the oscilloscope beam on the boundary.

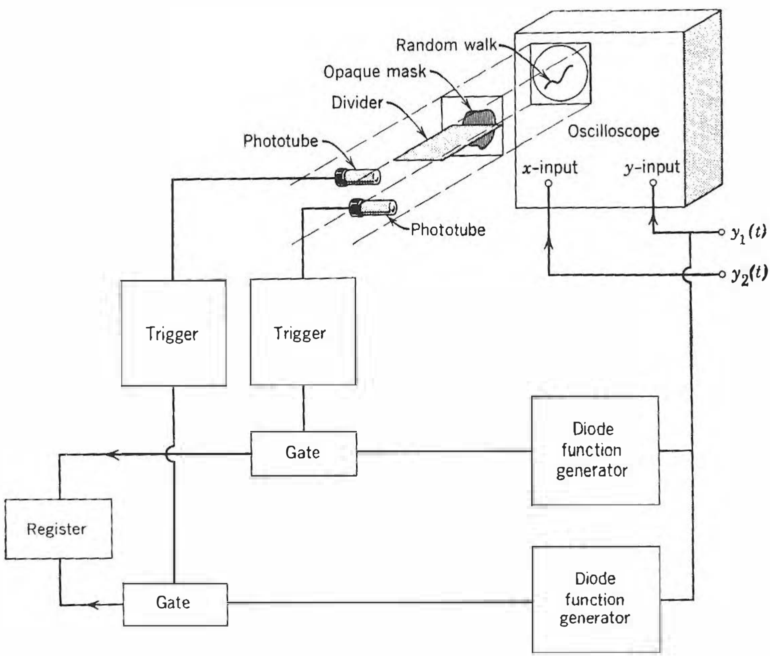


Fig. 10.8 Block diagram of computer system for solving Laplace's equation using Monte-Carlo method (Chuang, Kazda and Windeknecht[7]).

5. A value of $F(Q)$ is generated to correspond to any point Q on C at which the beam makes contact with the boundary.

6. A continuous record of successive generations of $F(Q)$ is kept or, alternatively, successive values of $F(Q)$ are continuously recorded. In either event the average of all these values must be determined.

7. At each occurrence of incidence, the oscilloscope beam is reset to the initial point inside the boundary C, the point at which the solution of Equation 10.17 is desired and at which each random walk of the oscilloscope must commence.

8. This cycle of operations is repeated until such time as the value of the average of the summed boundary value $F(Q)$ ceases to change.

A block diagram and some physical detail of a computer system for the solution of Laplace's equation, Equation 10.1, are shown in Figure 10.8. In this case the potential distribution along the boundary was such that C could be divided in two parts by a straight dividing line, such that the arc length (which is a function of x_1 and x_2) of each part is a single valued function of the length of the dividing line. These values are designated as $\phi[C_1(x_1)]$ and $\phi[C_2(x_2)]$ respectively. Under these conditions two diode function generators suffice to obtain the value $F(Q)$. The incidence of the beam on either of the two sections of the boundary is detected by means of one of two phototubes, which trigger a gate, which in turn permits the output of two diode function generators to apply voltages determined by the boundary point coordinates to be applied to the $F(Q)$ register.

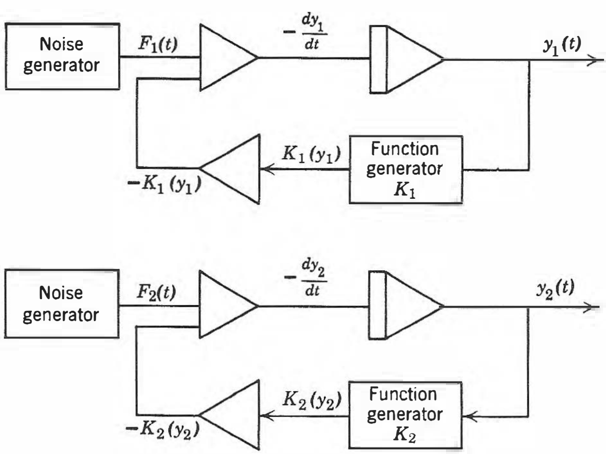


Fig. 10.9 Circuits for generating random walk excitations.

The generation of the variables y_1 and y_2 in Equation 10.18 is effected by a circuit of the type shown in Figure 10.9. This circuit requires two adders, an integrator, and a function generator. Low-frequency noise sources of Gaussian amplitude distributions having the necessary power spectra are commercially available. Such devices generally employ a gas tube subjected to a cross-magnetic field as the source, the power of which is distributed along a band of frequency in the audio range. This power band is then narrowed by a band-pass filter. For the application described, however, an additional requirement must be placed upon the two noise sources. Their power spectra must display an unusual degree of stability over a fairly long period of time. For this reason the outputs of the noise generators are recorded on magnetic tape and played back as required. Solutions of one- and two-dimensional boundary value problems using approximately 300 random walks per experimental point have been published. Good agreement between experimental and analytical solutions is generally obtained.

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chapter 11
INTEGRAL EQUATIONS

11.1 General Remarks

As described in the preceding chapter, a wide variety of methods and techniques for the simulation of fields governed by partial differential equations have been developed. Analog methods of various kinds have been applied successfully to such problems for well over fifty years. This is not true for another important class of problems arising in physics and engineering—those governed by integral equations. Here the application of analog techniques represents a relatively recent development and is still in the formative or experimental stage.

Like partial differential equations, integral equations have more than one independent variable and can be used to characterize the potential distribution in a field. The integral formulation differs from partial differential equation formulation in that the boundary conditions applying to the field are included explicitly in the formulation. It thus represents the entire physical behavior of the problem in a very compact form. In addition there are a number of problems which cannot be represented in terms of differential equations. This is so, for example, if the behavior of the field at some point depends upon the potential at a point some distance away and not just on the potentials at neighboring points.

One of the most widely occurring integral equations, sometimes known as Fredholm's equation of the first kind, has the form

$$F(x) = \int_a^b K(x, y)f(y)\, dy \tag{11.1}$$

The function $K(x, y)$ is known as the kernel and may be a sine or cosine function as in Fourier transforms, e^{-sy} as in Laplace transforms, a Bessel function, a Hankel function, a Legendre function, etc. In applying such an equation to a field problem, the limits a and b generally represent the boundaries of the field along the y coordinate. In Laplace transforms, in which y is time, the limits are zero and infinity respectively.

An often occurring variation of Equation 11.1 is Fredholm's equation of the second kind

$$f(x) = g(x) + \lambda \int_a^b K(x, y) f(y)\, dy \tag{11.2}$$

In this case $g(x)$ is a known function, and a and b are again fixed points at which $f(y)$ satisfies the boundary conditions. In Volterra's equation, the upper limit b is replaced by the dependent variable x.

As in the simulation of fields governed by partial differential equations, the chief problem is to adapt the analog computer which has only one independent variable, time, to problems containing two independent variables. A description of two approaches follow. In one of these, two different computing speeds are employed. That is, the y variable is made to pass through its entire range a great number of times for each time that the x variable goes through its range once. This effectively amounts to a discretization of the x variable and the obtaining of a family of solutions at successive values of x. The second method to be described represents a mechanization of the analytic technique of successive approximations. This method requires an analog memory to store the result of one iteration and use it as the input for the next step. The chapter concludes with several examples of specific simulations.

11.2 Use of Two Computing Speeds

Suppose it is desired to generate a function of two independent variables

$$z = F(x, t) \tag{11.3}$$

If attention is limited to an ensemble of discretely spaced points

$$x_i \qquad i = 1, 2, \ldots, n$$

the desired values of z can be obtained electrically by applying to the analog computer electrical voltages having two computing speeds. One group of these voltages, for example t, are made to vary at a rate such that the condition

$$\frac{\delta F(x, t)}{\delta t} = r(t) \qquad 0 \leq t \leq t_1 \tag{11.4}$$

is satisfied, while at the same time

$$\frac{\delta F(x, t)}{\delta x} = 0 \qquad x = x_i \tag{11.5}$$

In other words, the variable t passes through its entire range of variation for each discrete change x_i. In this way a system is realized electrically which has two independent variables, one of which is in discrete form.

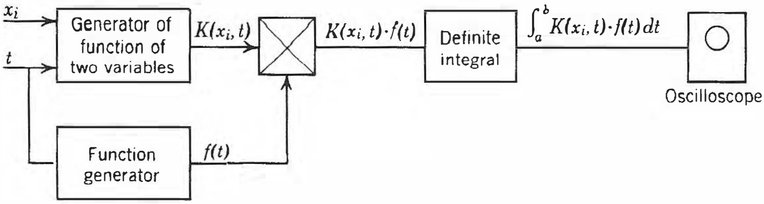

Fig. 11.1 Block diagram for circuit using two computing speeds.

Using this principle Wallman[1] has proposed a method for solving integral equations with the aid of repetitive differential analyzers. Equation 11.1 is expressed as

$$F(x) = \int_a^b K(x_i, t)f(t)\,dt \qquad i = 1, 2, \ldots, n \tag{11.6}$$

From experience it is known that one can take $n \leq 100$ without excessive error. As a first approximation, a function $f_1(t)$, which is known to represent an approximate solution of Equation 11.1, is specified. Inserting $f_1(t)$ in 11.6,

$$F(x) = \int_a^b K(x_i, t)f_1(t)\,dt + R_1(x) \tag{11.7}$$

where $R_1(x)$ is an error term. It is assumed that a repetitive differential analyzer having a very high repetition rate is employed. Consider, for example, a repetition rate such that the integral in Equation 11.7 for a specific value of x_i is calculated in 1 msec. After each integration over the t domain x_i is changed to $x_i + 1$. In this fashion the complete solution (x) is obtained in 0.1 sec if $n = 100$ is taken. This over-all process can be repeated often enough so that the plot of $f(x)$ can also be observed as a stationary display on an oscilloscope.

If the given function $f(x)$ has been placed on the oscilloscope screen, the described method permits the direct observation of the difference $R_1(x)$. In this way a direct indication is obtained as to whether $f_1(t)$ satisfies Equation 11.1 and to what extent it deviates from it. By varying the function $f_1(t)$, as generated by a function generator, the minimum value of $R_i(x)$ can be found. The block diagram of the instrumentation of this method is shown in Figure 11.1. A chief problem in the realization of the system is the construction of a suitable generator of functions of two independent variables. Several possible designs of such devices are described in Section 6.9.

11.3 Method of Successive Approximations

The well-known iterative method of solution of Equation 11.2 is expressed as

$$y_{m+1}(x) = f(x) + \lambda \int_a^b K(x, t) y_m(t)\, dt \tag{11.8}$$

In order to automate this process it is necessary to employ a suitable analog memory to introduce automatically into the machine the values $y_m(t)$ and to memorize $y_{m+1}(t)$. Such a memory was described by Vitenberg[2] and Kozak.[3] The heart of this analog memory is an array of capacitors with electromechanical commutators. This switching system acts to connect the capacitors sequentially to the load in such a manner that successive instantaneous values of the function are stored on the capacitors. In a subsequent cycle the switches act to apply the capacitors to a load in the same sequence, so that the function is "read out" in discrete steps. In Vitenberg's case memories using up to 50 capacitors were employed. Each integration of t required 5 sec, so that the function $y_m(x_i)$ was calculated in 4 to 5 min. The automatic iteration process of solution of the integral equation required approximately 1 hr.

A purely electronic and therefore much more rapid memory for the solution of integral equations was proposed by Fisher.[4] Figure 11.2 is a block diagram of this system. At the beginning of the solution a first approximation $y_1(t)$ has been assumed. This function is fed into the analog memory A. With the aid of a standard analog multiplier, an integrator, and an adder the expression

$$y(x_1) = f(x_1) + \lambda \int_a^t K(x_1, t) y_1(t)\, dt \tag{11.9}$$

is evaluated. At the same time that $y_1(t)$ is read out of memory A, it is

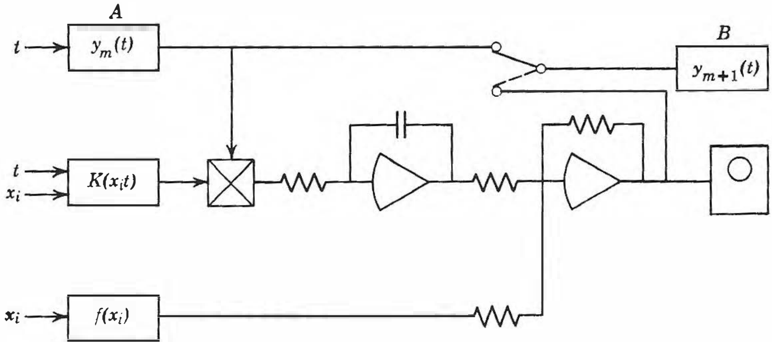

Fig. 11.2 Electronic iterative method.

fed into memory B. At the end of the first minor cycle of integration, memory B contains the value $y(x_1)$ at the point corresponding to x_1. By means of a commutator switch, memories A and B are made to change roles and a new cycle is started. In this new cycle the function $y_2(t)$, which was obtained with the aid of the integration Equation 11.9 during the preceding cycle, is employed. After n minor cycles the complete solution is stored in the memory. The accuracy of the computation is, of course, dependent on the accuracy of the analog memory and on the mathematical approximations introduced.

An analog memory designed for iterative applications with an iteration time of 100 to 400 μsec was described by Bergman.[5] By using such a device, a complete solution of an integral equation can be obtained in 0.1 sec. This speed permits the utilization of the repetitive method to full advantage. From the analog viewpoint the principal advantage of the iterative method over the classical Neumann method is that the successive approximations of Equation 11.9 are made at the same time that the function $y(x)$ is computed. In other words, the integration of an integral equation need not be made in the entire interval

$$a \leq t \leq b$$

before passing on to a new iteration. Rather, an arbitrary subinterval within the domain $b-a$ can be employed. A theoretical analysis of the iteration method demonstrates that that method will always be convergent when the Neumann method is convergent. Fisher has made a detailed error analysis in the application of differential analyzers for the solution of integral equations by the method described.

A chief difficulty associated with Fisher's technique is that it requires complex and special electronic equipment that is generally unavailable in standard installations. In an effort to overcome this difficulty, Tomovic[6] has combined the previously stated theoretical approach with the use of a repetitive differential analyzer with the universal nonlinear computing element described in Chapter 6. This approach is illustrated in Example 3 in Section 11.4. Nelson and Fried[7] have described the use of dynamic memory circuits as given in Section 5.4 to a similar problem.

Another approach to the solution of integral equations involves transforming them into sets of simultaneous algebraic equations. Parezanovic has demonstrated that integral equations can be solved on repetitive differential analyzers in this manner. Example 4 is an illustration of that method.

11.4 Examples of Solutions

Several examples of practical applications of repetitive differential analyzers to the solution of integral equations are now presented. In view of the difficulties in realizing universal generators of functions of two variables, most often special forms of $K(x, t)$ are employed.

Example 1. Macnee[8] describes a method of evaluating the Fourier integral

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t} f(t)\, dt \tag{11.10}$$

when the function $f(t)$ is known. If only the real domain is considered, the function generator must provide

$$\left.\begin{matrix}\sin \omega t \\ \cos \omega t\end{matrix}\right\} \qquad \begin{matrix} 0 \le t \le T \\ \omega_0 \le \omega \le \omega_b \end{matrix}$$

Provided the parameter ω is fixed, these functions can be generated by solving the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{11.11}$$

In this application the parameter ω is changed after each cycle, and the solution of Equation 11.11 is effected sixty times per second. The slow change of ω is obtained by electromechanical means as indicated in Figure 11.3. The moving arms of potentiometers P_1, P_2, and P_3 are displaced at a rate which is 1/100 of the repetition frequency of the differential analyzer. Potentiometer P_2 multiplies approximately

$$\omega_0\left(\frac{\theta_i}{2\pi}\right)y \qquad i = 1, 2, \ldots, 100$$

and P_1 once more by the same factor so that its output is

$$\omega_0{}^2\left(\frac{\theta_i}{2\pi}\right)^2 y$$

The moving arms of P_1, P_2, and P_3 are ganged so that in reality the equation being solved is

$$\frac{d^2y}{dt^2} + \omega_0{}^2\left(\frac{\theta_i}{2\pi}\right)^2 y = 0 \tag{11.12}$$

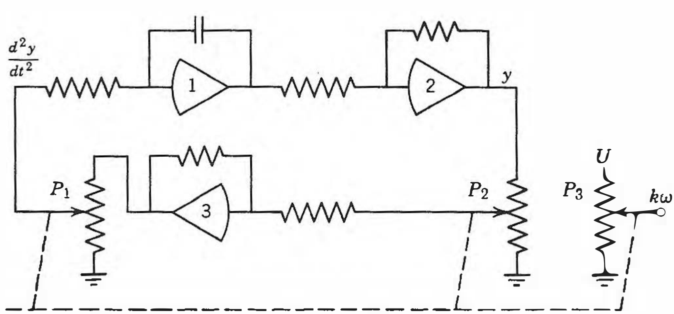

Fig. 11.3 Circuit for evaluating Fourier integral.

The integration for the entire interval θ_i takes $\frac{5}{3}$ sec, and the solution is displayed on a cathode-ray tube having a long persistence screen. The Fourier integral for the functions

$$f(t) = \frac{\pi}{T}\left[1 + \cos\left(\frac{\pi}{T}\right)t\right] \qquad 0 \leq t \leq T \tag{11.13}$$

and

$$f(t) = \frac{T}{6} e^{-6t/T} \qquad 0 \leq t \leq T \tag{11.14}$$

was calculated, and even in the most unfavorable cases errors were less than 10%.

Example 2. Vitenberg[2] describes the application of his electromechanical memory to the solution of integral equations by successive approximations using a slow analog computer. He solved the equation of a homogeneous string acted on by periodic forces. This problem leads to the integral equation

$$y(x) = \int_0^\pi K(x, t)p(t)\, dt + \int_0^\pi K(x, t)y(t)\, dt \tag{11.15}$$

with the kernel

$$K(x, t) = \begin{cases} \dfrac{t(\pi - x)}{\pi} & t \leq x \\[2ex] \dfrac{x(\pi - t)}{\pi} & t \geq x \end{cases} \tag{11.16}$$

The maximum error of the machine solution was 1.5%. The integral equation governing heat conduction,

$$y(x) + \frac{1}{4\pi}\int_0^2 [y(t) - y(x)]\, dt = \begin{cases} \dfrac{\sin x}{2\pi} & 0 \leqslant x \leqslant \pi \\[2ex] 0 & \pi \leqslant x \leqslant 2\pi \end{cases} \tag{11.17}$$

was solved with errors of less than 4%.

Example 3. Consider the integral equation

$$y(x) = \frac{3x}{2} - \frac{7}{6} + \int_0^1 (x - t)y(t)\, dt \tag{11.18}$$

The analog network for the solution of this equation is shown in Figure 11.4. In order to solve the equation on the machine, the variable x was quantized into 16 steps, $x = x_i$, $i = 1, 2, \ldots, 16$. The part of the network indicated by dotted lines generates the expression $(x_i - t)$.

The function generator used is of the universal type and performs the multiplication of two variable voltages at the same time. Moreover, each ordinate of the desired solution $y(t_i)$ can be individually adjusted by means of potentiometers P_i. The output of integrator 3 is

$$e_0 = \int_0^t (x_i - t)y(t)\, dt \tag{11.19}$$

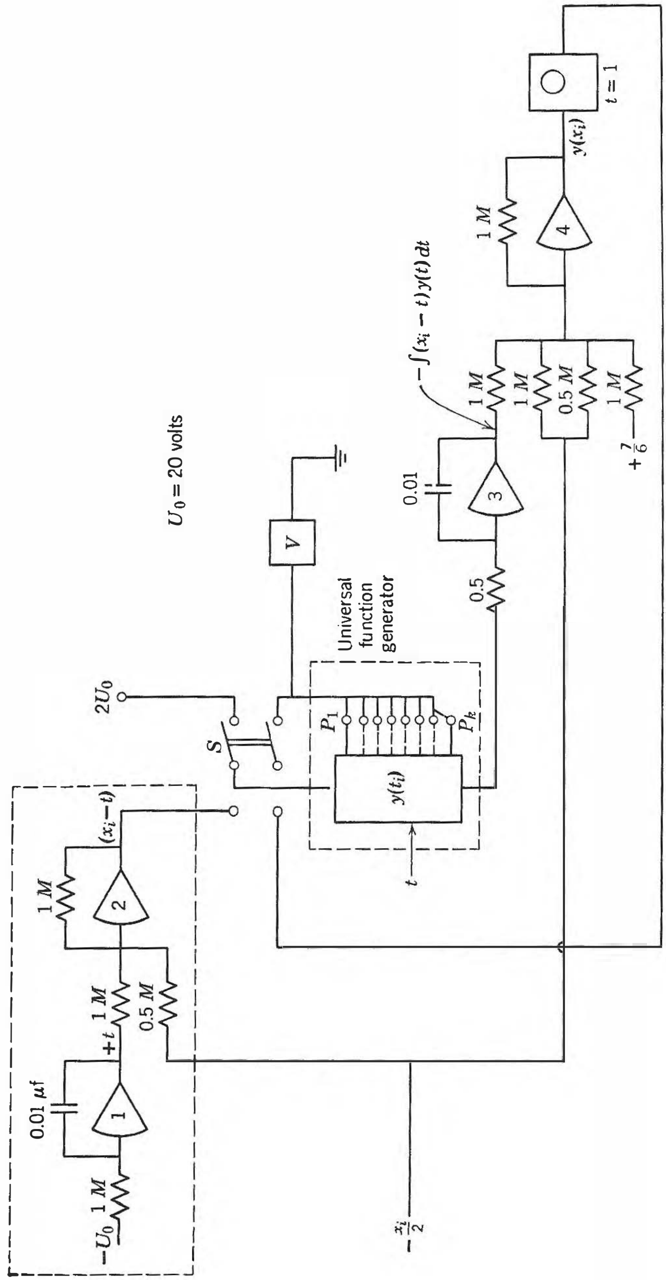

Fig. 11.4 Circuit for solution of Equation 11.18.

The application of Fisher's iteration method is now straightforward. In the first integration $x = x_1$ and $y_1(t) = 0$. The operator then measures at the output of adder 4 the value of the right side of the given integral equations. This value is obtained as a d-c voltage with the aid of a sample-and-hold circuit which is a part of the measuring system. Thus the first ordinate $y_2(x_1)$ is obtained

By using the switch S, a d-c voltage corresponding to $y_2(x_1)$ is set on the first potentiometer P_1 of the function generator. The second minor iteration cycle starts then with

$$y(t) = y_2(t) \qquad t \leq t_1$$

$$y(t) = y_1(t) \qquad t > t_1$$

The next step with $x = x_2$ follows, and the same measurement is repeated at $t = 1$. The second ordinate $y_2(x_2)$ which is a better approximation to $y(t)$ is now available. After n minor iteration cycles $y_2(x_i)$ is available. The major iteration cycle is now repeated, but practice has shown that rarely are more than two major cycles needed.

The solution of Equation 11.18 is

$$y(x) = x - 1 \tag{11.20}$$

The photograph of the solution as obtained on the repetitive analyzer is shown in Figure 11.5.

Example 4. Consider the integral equation

$$y(x) = \sinh x - \int_0^x e^{x-t} y(t)\, dt \tag{11.21}$$

when expressed as a system of simultaneous algebraic equations,

$$i\delta + \sum_{i=1}^{k} i\delta \int_{x_{i-1}}^{x_i} e^{x_i - t}\, dt - \sinh(x_i) = 0 \qquad i = 1, 2, \ldots, 8 \tag{11.22}$$

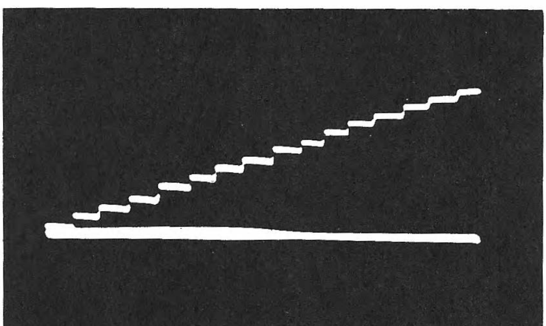

Fig. 11.5 Oscilloscopic display of the output of the circuit of Figure 11.4.

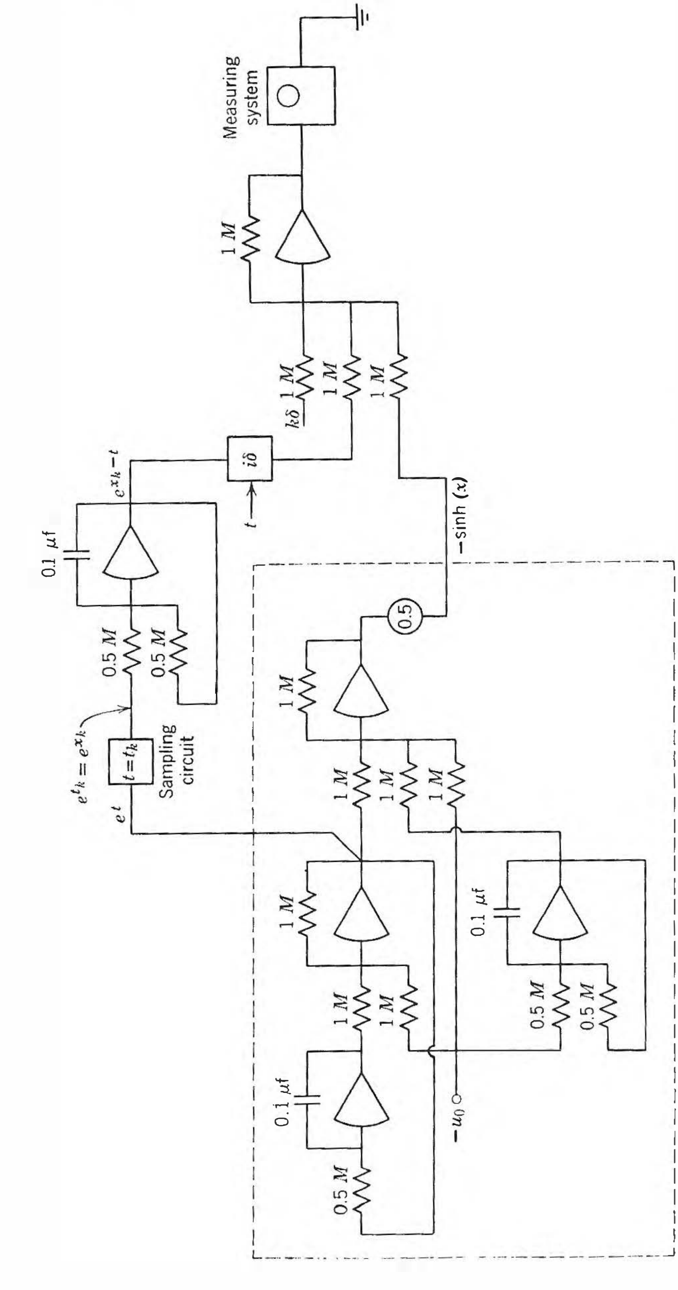

Fig. 11.6 Circuit for the solution of Equation 11.21.

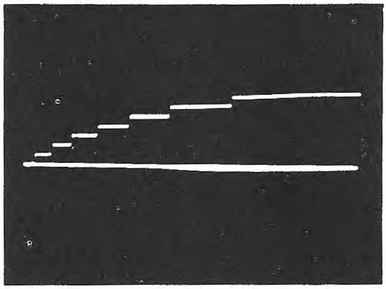

Fig. 11.7 Oscilloscopic display of the output of the circuit of Figure 11.6.

In this expression it is not necessary to take uniform steps Δt of the independent variable t; rather

$$y(x_i) = y(t_i) = y(t_0) + i\delta$$

can be taken. Here δ is a fixed increment in $y(x)$, and corresponding times $t = t_i$ are sought to satisfy Equation 11.21. The analog network to solve Equation 11.21 is shown in Figure 11.6. The portion of the circuit enclosed by dotted lines is employed to generate the kernel e^{x-t} by solving the corresponding differential equation and by varying the initial conditions. The iteration process was performed with the same function generator and measuring system as indicated in Example 3. However, the ordinates of the function generator are preset according to $i\delta$, $i = 1, 2, \ldots, k$, so that the instant $t = t_i$ are adjusted on the set of potentiometers.

First $x = x_0$ is taken, and $y(0)$ is determined. Then $t = t_1$ is sought, such that the output of measuring system indicates zero. The function generator delivers now at $t = t_1$ the ordinate $i\delta$, $i = 1$. Using this ordinate the second time instant $t = t_2$ is determined, and so on. If the system of linear equations thus solved is stable, the iteration process leads to the desired result.

The solution of Equation 11.21 is

$$y = 1 - e^{-x} \tag{11.23}$$

The photograph of the machine solution is shown in Figure 11.7. The ordinates of the solution in Figure 11.7 are obtained with constant increment δ, whereas time instants $t = t_i$ are nonuniformly distributed.

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chapter 12
MISCELLANEOUS APPLICATIONS

12.1 General Remarks

A wide variety of applications of repetitive analog computers exist which do not primarily involve the treatment of differential or integral equations. Some of the more interesting of these are briefly reviewed in this chapter.

The determination of the roots of algebraic equations is a problem that frequently arises in control system analyses and numerous other areas. This problem becomes difficult if some of the roots are complex. In Section 12.2 a method is presented for the determination of such roots in a way that clearly demonstrates the advantages of high-speed repetitive operation.

The next topic treated in this chapter is conformal mapping. Two relatively general methods are described. The first of these is based on trigonometric functions and has the advantage that only linear elements are required. This method, however, requires the plotting of curves in the complex plane and requires equipment for the measurement of instantaneous voltage values during the repetitive cycle. The second method is based on the solution of ordinary differential equations and provides oscilloscopic displays of the mapping directly. Nonlinear equipment in the form of multipliers is required.

Another new area of application of repetitive differential analyzers is in the utilization of statistical techniques in system analysis. In this method random voltages obtained from noise generators are used as computer inputs so that the computer can be employed to observe directly the effect of noise on system behavior. Other applications of statistical techniques involve the determination of statistically specified variations in the system parameters and the generation of complex probability functions using repetitive analog computer equipment.

The chapter closes with a brief description of an ultra-high speed repetitive differential analyzer. In such a machine repetition rates of the order of 1 to 25 kc are employed.

12.2 Roots of Polynomials

There are various analog methods to obtain the roots of polynomials. Many special-purpose machines have been built for this purpose. In another approach standard general-purpose analog machines have been used. An excellent review of references regarding both specialized and general-purpose methods for the solution of polynomials has been presented by Mikailov.[1] Very good insight into the work after 1957 with special attention to high-speed analog computer methods can be found in the work of Petric.[2] In this context, only applications of repetitive computers for the solution of polynomials are considered.

There are several criteria by which an analog method to solve for the roots of polynomials can be judged. A good analog method should be:

1. General, that is, with no restrictions on either roots or coefficients of the polynomial.
2. Noniterative, that is, solutions must be obtained without trial-and-error.
3. Simple, that is, it is desirable that only linear computing elements, operational amplifiers, and linear potentiometers be used.

The work in this direction was started by several authors independently,[3] but a general and satisfactory answer was given by Madic et al.[4] The following examples will serve to illustrate these results.

The general form of a polynomial with real coefficients and complex roots is

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{12.1}$$

where

$$z = x + iy$$

If real and imaginary terms are grouped together, Equation 12.1 can be expressed as

$$f(z) = U(x, y) + iV(x, y) \tag{12.2}$$

The values of x and y that simultaneously satisfy

$$U(x, y) = 0$$
$$V(x, y) = 0$$

are the real and imaginary parts of the roots. Direct representation of the polynomials $U(x, y)$ and $V(x, y)$ on the screen of the cathode-ray tube is not convenient. A much simpler method, fulfilling all the cited criteria, will result if the given polynomial $f(z)$, that is, $U(x, y)$ and $V(x, y)$, is represented in the following form:

$$U(x, y) = f(x) - y^2\left\{\frac{f^{(2)}(x)}{2!} - y^2\left[\frac{f^{(4)}(x)}{4!} - y^2\left(\frac{f^{(6)}(x)}{6!} - \cdots\right)\right]\right\}$$

$$\frac{V(x, y)}{y} = f^{(1)}(x) - y^2\left\{\frac{f^{(3)}(x)}{3!} - y^2\left[\frac{f^{(5)}(x)}{5!} - y^2\left(\frac{f^{(7)}(x)}{7!} - \cdots\right)\right]\right\}$$

As seen, now the variables x and y can be separated and one of them, y, represented in parametric form. In the method of Madic, the variable x is represented by t on the computer, whereas y is the angle of the rotation ω of a set of ganged linear potentiometers. The functions $U(x, y)$ and $[V(x, y)]/y$ are generated by integrators and multiplied by corresponding factors y^2 with the aid of potentiometers. The points $x = t = t_k$, $y^2 = \omega = \omega_k$, at which $U(x, y) = 0$ and $[V(x, y)]/y = 0$ are the roots $s_k = x_k + iy_k$ of $f(z)$. The variable t is obtained by reading the t-coordinate at which the sample-and-hold voltmeter reads zero. The other variable y^2 is read directly on the dial of the ganged linear potentiometers. The computer circuit is shown in Figure 12.1.

Consider the equation

$$z^5 - 9.02z^3 + 21.96z^2 - 21.6199z + 8.8804 = 0 \tag{12.3}$$

The polynomial is determined by

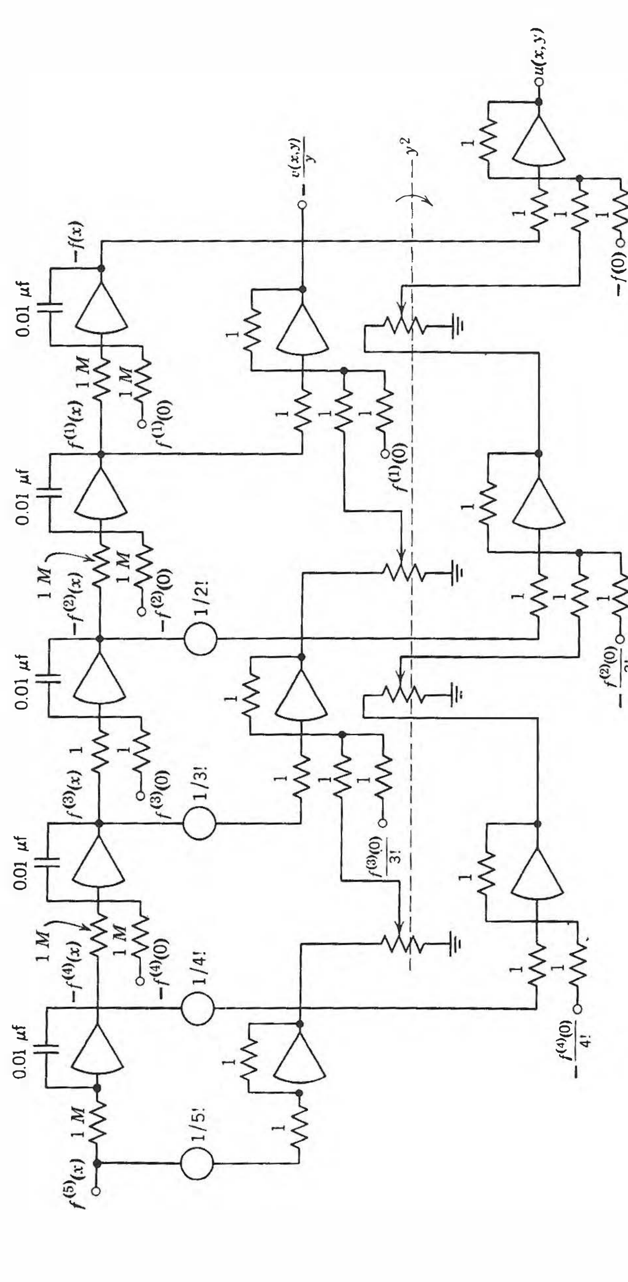
$$f^{(5)}(2) = 120 \tag{12.4}$$

and

$$\begin{aligned} f(0) &= 8.8804 \\ f^{(1)}(0) &= 21.6199 \\ f^{(2)}(0) &= 43.92 \\ f^{(3)}(0) &= 54.12 \\ f^{(4)}(0) &= 0 \end{aligned}$$

The locations of the roots obtained by analog (Figure 12.1.) and analytical method are given in the following table:

	z_1	z_2	z_3	z_4	z_5
Analytical	$1+i\cdot 0.7$	$1-i\cdot 0.7$	$1+i\cdot 0.7$	$1-i\cdot 0.7$	-4
Analog computer	$0.99+i\cdot 0.72$	$0.99-i\cdot 0.72$	$0.99+i\cdot 0.72$	$0.99-i\cdot 0.72$	-4.03



$$u(x,y) = f(x) - y^2\left(\frac{f^{(2)}(x)}{2!} - y^2\,\frac{f^{(4)}(x)}{4!}\right)$$

$$\frac{v(x,y)}{y} = f^{(1)}(x) - y^2\left(\frac{f^{(3)}(x)}{3!} - y^2\,\frac{f^{(5)}(x)}{5!}\right)$$

Fig. 12.1 Circuit for solution for complex roots of fifth-order polynomial.

12.3 Trigonometric Method of Conformal Mapping

This method, as presented by Petric,[5] is based on the fact that differential analyzers can readily solve the second-order harmonic equation

$$\frac{d^2y}{dt^2} + \omega_n{}^2 y = 0 \qquad n = 1, 2, \ldots, k \tag{12.5}$$

The relation defining the mapping is assumed to be in the form of a polynomial

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{12.6}$$

Letting $z = \rho e^{i\theta}$

$$W = \frac{U}{V}(\rho, \theta) + iV(\rho, \theta) \tag{12.7}$$

where

$$U(\rho, \theta) = \sum_{n=0}^{k} a_n \rho^n \cos n\theta$$
$$V(\rho, \theta) = \sum_{n=0}^{k} a_n \rho^n \sin n\theta \tag{12.8}$$

Equations 12.8 can be expressed as

$$U(\rho, \theta) = a_0 + \rho\{a_1 \cos\theta + \rho[a_2 \cos 2\theta + \rho(a_3 \cos 3\theta + \cdots)]\}$$
$$\frac{V(\rho, \theta)}{\rho} = a_1 \sin\theta + \rho\{a_2 \sin 2\theta + \rho[a_3 \sin 3\theta + \cdots]\} \tag{12.9}$$

in which form they are suitable for solution on the differential analyzer.

The variable ρ is represented by the angle of rotation of ganged potentiometers, and θ is represented by the time variable. The analog computer circuit for this conformal mapping is shown in Figure 12.2. It is assumed that all coefficients a_n are positive. Using the measuring system of the repetitive differential analyzer to obtain instantaneous values of wave forms as d-c voltages, a sufficient number of points $U(\rho_n, \theta_n)$ and $V(\rho_n, \theta_n)$ in the W-plane for any given $z = \rho e^{i\theta}$ can be recorded.

As an example, the mapping of the unit semicircle in the left half of a z-plane was performed using the polynomial

$$z^4 + 2z^3 + 1.9z^2 + 1.14z + 0.3869 = 0$$

The result is shown in Figure 12.3 where the dotted line shows the differential analyzer solution.

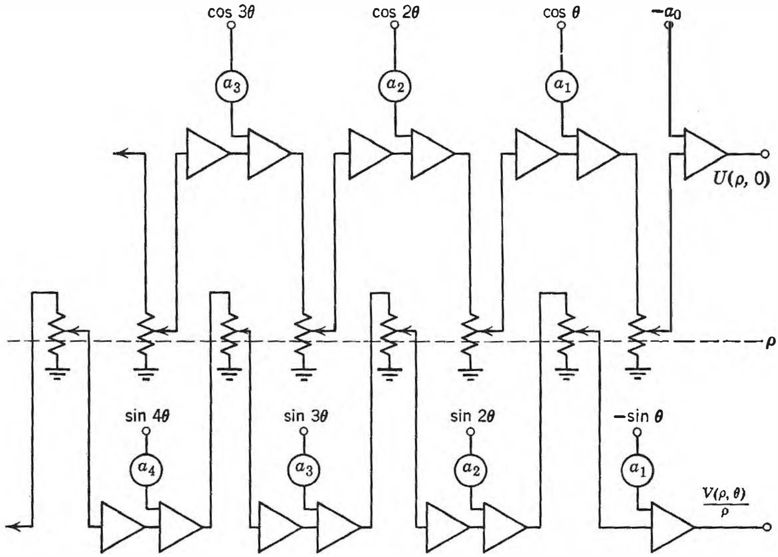


Fig. 12.2 Analog circuit for conformal mapping by trigonometric method.

It is evident that zeros of the polynomial can be found directly by the trigonometric method by looking for those points where

$$U(\rho, \theta) = 0$$
$$V(\rho, \theta) = 0$$

simultaneously. In Figure 12.4*a* and 12.4*b* the real and complex zeros of polynomial

$$z^3 + 0.7z^2 + 0.44z - 0.222 = 0$$

are displayed. The exact values are

$$z_1 = 0.305 \qquad z_{2,3} = 0.49 \pm 0.71i$$

and the machine solutions obtained were

$$z_1 = 0.3 \qquad z_{2,3} = 0.5 \pm 0.7i$$

12.4 Conformal Mapping in the x, y-Plane

A different approach to the analog computer solution of the conformal mapping problem was described by Heinhold.[6] In this method polar coordinates are not explicitly introduced; rather the complex variable is represented in the form $z = x + iy$.

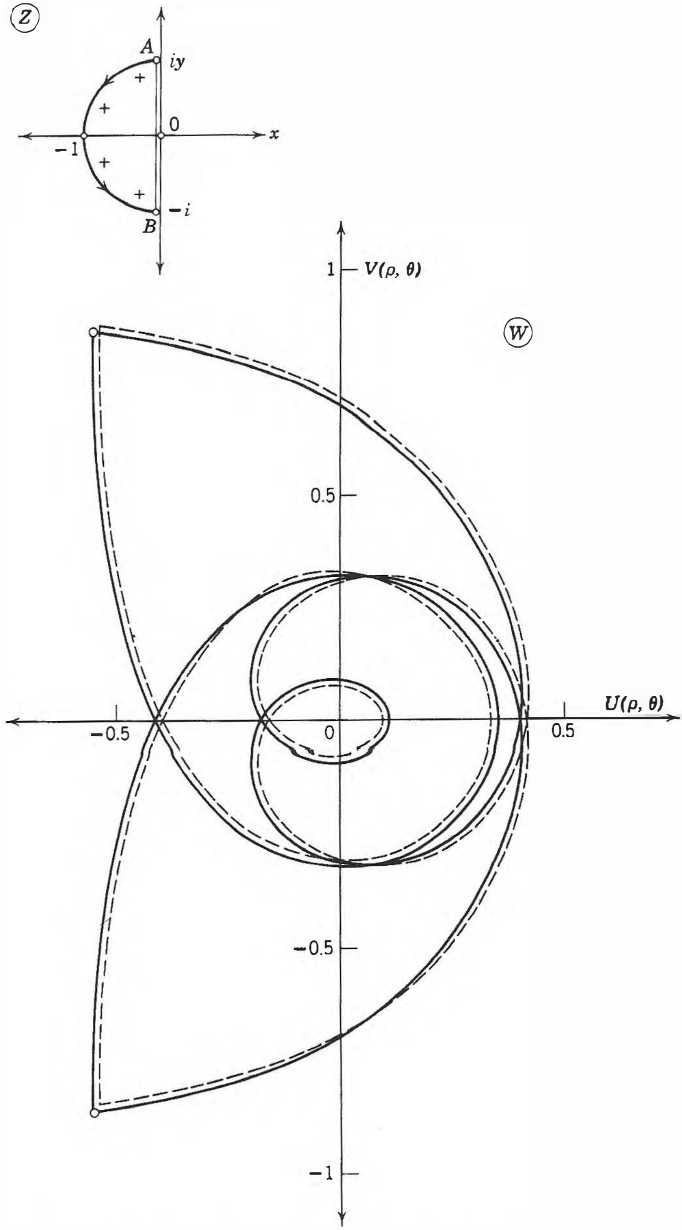


Fig. 12.3 Differential analyzer solution of a conformal mapping.

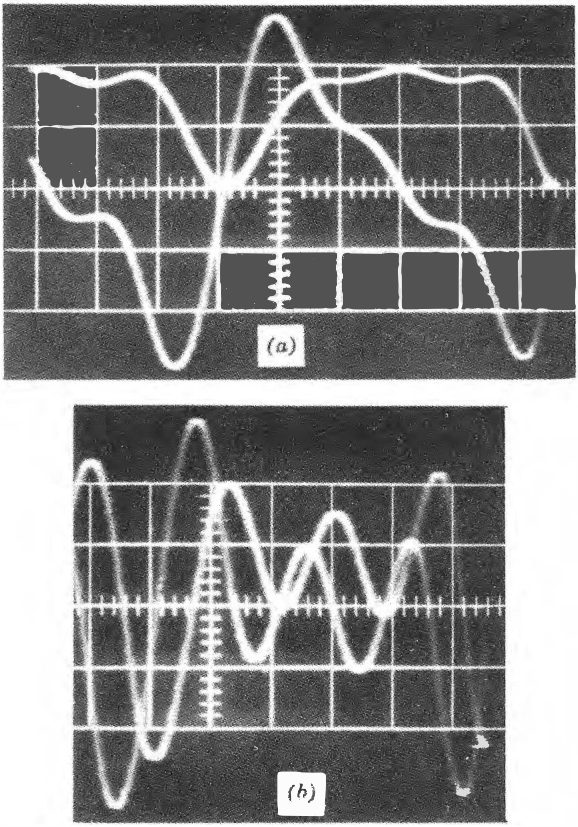


Fig. 12.4 (*a*) Real zero of a third-order polynomial. (*b*) Pair of complex zeros of a third-order polynomial.

Consider first a mapping defined by an explicit relationship. Let

$$w = f(z) = u(x, y) + iv(x, y) \tag{12.10}$$

Since time is necessarily the independent variable in electronic differential analyzers, the problem variables must be represented in the form

$$\begin{aligned} z &= x + iy = x[\phi(t), \psi(t)] + iy[\phi(t), \psi(t)] \\ z &= x(t) + iy(t) \\ u &= u(x, y) = u(t) \\ v &= v(x, y) = v(t) \end{aligned} \tag{12.11}$$

Then

$$w = u(t) + iv(t) \tag{12.12}$$

can be directly represented on the screen of an oscilloscope. The variables $u(t)$ and $v(t)$ are obtained with the aid of the linear and nonlinear computing units of the differential analyzer.

As an example, take

$$w = \tfrac{1}{2}\left(z + \frac{a^2}{z}\right) \qquad a^2 < 1$$

Then

$$u = \frac{x}{2} + \frac{a^2 x}{2(x^2 + y^2)}$$

$$v = \frac{y}{2} - \frac{a^2 y}{x^2 + y^2}$$

The scale factors are derived as follows:

$$0 < t < T$$

$$x^2 + y^2 < 1$$

$$K^2 = \min (x^2 + y^2)$$

$$u^* = k^2 u$$

$$v^* = k^2 v$$

The block diagram of the computer system that performs the preceding transformation is shown in Figure 12.5. The mapping was defined by the set of straight lines

$$x = 0.88;\ 0.82;\ 0.63;\ 0.53;\ 0.41$$

$$y = t$$

The result of the mapping as displayed on the cathode-ray oscilloscope is presented in Figure 12.6.

Alternatively, if the function $w(z)$ satisfies an ordinary differential equation, this fact can be used for further generalization of the mapping problem. The basic relation is

$$\frac{d^n w}{dz^n} = F\left(z, w, \frac{dw}{dz}, \ldots, \frac{d^{n-1} w}{dz^{n-1}}\right) \tag{12.13}$$

with initial values

$$\begin{aligned} w(z_0) &= W_0 = U_0 + iV_0,\ w(z_1) = W_1 = U_1 + iV_1, \ldots \\ w^{n-1}(z_0) &= W_{n-1} = U_{n-1} + iV_{n-1} \end{aligned} \tag{12.14}$$

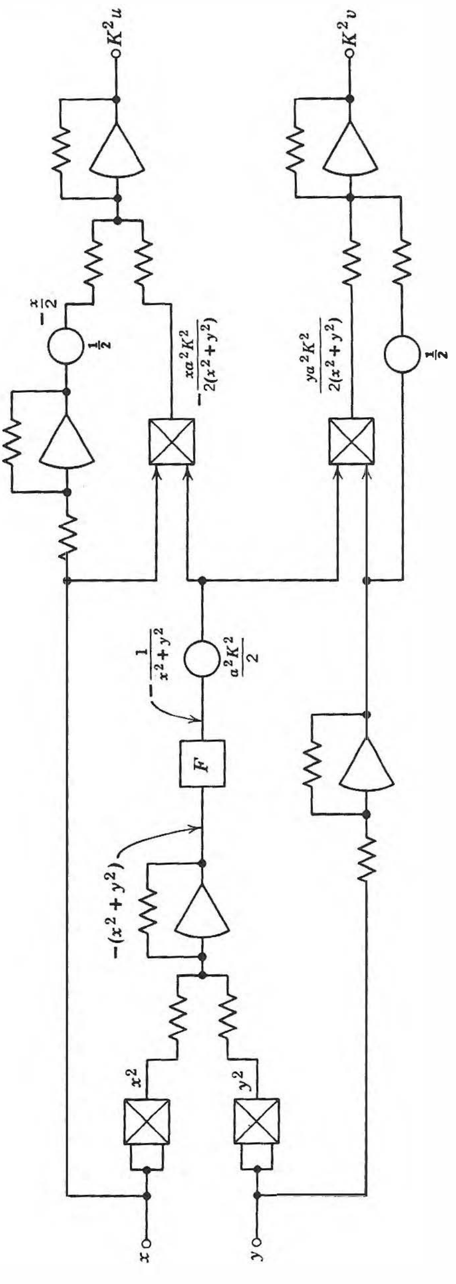


Fig. 12.5 Block diagram of network to perform conformal mapping in the $x - y$ plane.

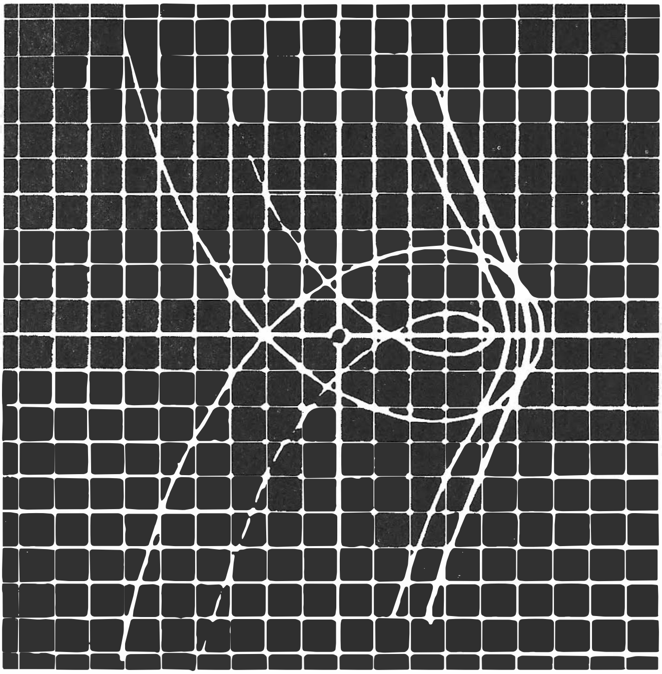


Fig. 12.6 Oscilloscopic display of mapped curves.

From these equations, the general block diagram can be derived. Consider, for example, the mapping defined by Bessel's differential equation

$$w^{(2)} + \tfrac{1}{2}w^{(1)} + \left(1 - \frac{V^2}{z^2}\right)w = 0$$

The curves to be mapped are concentric circles

$$z(t) = Re^{it}$$

The relations between corresponding derivatives are

$$w^{(1)}[z(t)] = \frac{\dot{W}(t)}{\dot{z}(t)}$$

$$w^{(2)}[z(t)] = \frac{\ddot{W}\dot{z} - \dot{W}\ddot{z}}{\dot{z}^3}$$

so that Bessel's equation takes the simple form

$$\ddot{w} - (R^2e^{2it} - V^2)w = 0$$

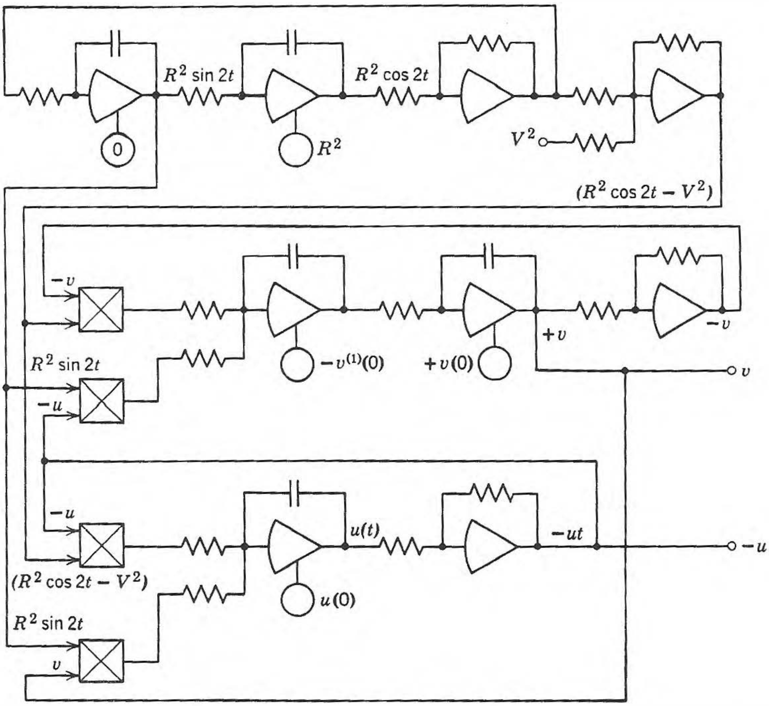


Fig. 12.7 Block diagram of conformal mapping by Bessel's differential equation.

The real and imaginary parts are

$$\ddot{u} - u(R^2 \cos 2t - V^2) + vR^2 \sin 2t = 0$$

$$\ddot{v} - v(R^2 \cos 2t - V^2) - uR^2 \sin 2t = 0$$

The computer circuit is shown in Figure 12.7, and the results are presented in Figure 12.8. The following values were used:

R	R^2	$u(0)$	$v(0)$	$\dot{u}(0)$	$\dot{v}(0)$
0.200	0.040	0.345	0	0	0.169
0.400	0.160	0.479	0	0	0.214
0.785	0.616	0.636	0	0	0.181
1.000	1.000	0.654	0	0	0.092

12.5 Statistical Analysis

The behavior of many engineering and scientific systems can only be characterized with the aid of statistical functions. Using numerical or

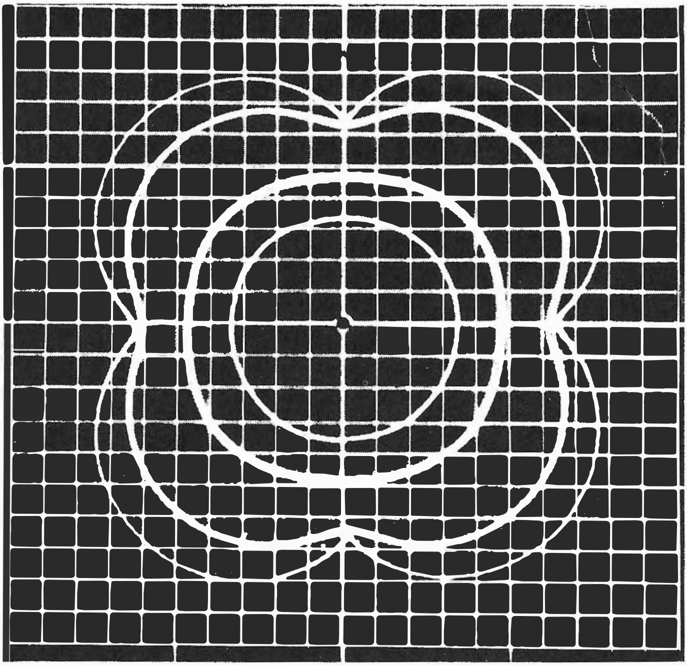


Fig. 12.8 Results of mapping of concentric circles when $w(z)$ is a solution of Bessel's equation.

slow analog computer techniques such problems can present formidable difficulties since it is necessary to repeat the solution many hundreds of times to obtain a satisfactory statistical sample. The repetitive mode of operation is well suited to systems analysis by statistical technique. Using a repetition rate of 50 cycles/sec, random effects on a set of 3000 samples may be analyzed in one minute.

One application of the repetitive computer to statistical analysis is to observe directly the effect of noise on the solution. Vander Velde[7] describes the general approach to problems of this type utilizing equipment newly developed by the G.P.S. Instrument Company. An application of this approach is presented in detail in Section 12.6. In addition to the conventional repetitive analog computer components the following units are required:

1. A voltage having a probability distribution very closely approximating the Gaussian function is generated by means of a noise generator in which the voltage across a gas diode in a magnetic field is amplified

and filtered. A nearly uniform power density spectrum from 20 to 200,000 cycles/sec is obtained in this manner. Low[8] has described the problems involved in the design of such equipment.

2. An rms meter is provided to sample the random voltage variable during the "compute" time of the repetitive cycle and to hold the average value during the reset time of the computer. A calibrated voltmeter indicates the rms value of the random variable.

3. A probability distribution analyzer is provided to determine complete distribution function of the input voltage. For this purpose a voltage comparator and counter are used. If the reference level of the comparator is set to V and the randomly varying voltage $s(t)$ applied to its input at a selected instant $t = t_s$, the comparator emits a pulse whenever $s(t_s) = V$. The counter is preset to a chosen number of runs C_1, after which the computer stops automatically. An estimate of the probability distribution is thus obtained as the ratio of C_2/C_1, where C_2 is the total number of pulses emitted by the comparator. By varying V, a curve of $P(V)$ can be constructed.

Instead of counting the number of pulses the random variable $s(t_s)$ is below the level V, a count proportional to the amount of time $s(t)$ is within the voltage limits $(V_0 + \Delta V/2)$ and $(V_0 - \Delta V/2)$ can be obtained. By dividing this number by the preset number on the counter corresponding to the total observation time t_{max}, an estimate of the probability density $P(V)$ is obtained. The operation of the analyzer is described in Section 12.6.

Another application of the repetitive differential analyzer is in the analysis of sums of distribution functions. In certain experiments, especially of a physio-chemical nature, the result is obtained as a sum of elementary statistical distribution functions (Gaussian, Lorentz, etc.). The problem is to take a number of these elementary distribution functions and to fit the composite experimental curve by adjusting individually their parameters (height, width, position of the peak). Noble et al.[10] have used a repetitive computer to perform these calculations automatically. The synthesis of the experimental distribution from its components is performed by visual adjustments on the cathode-ray tube of the repetitive computer. The distribution-function generator generates ten independent waveforms having the shape of Gaussian or Lorentz distributions. Any selected number up to ten of these components is added and displayed on the cathode-ray tube where a graph of the experimental curve is placed to serve as reference. The operator adjusts the peak, position, height, and width of each component with a set of potentiometers.

The generation of elementary distribution functions is effected as indicated in Figure 12.9. The 60 cycles/sec sinusoidal voltage is amplified sufficiently and clipped to obtain rectangular pulses of the same frequency. A phase shift circuit is used to displace continuously the sinusoidal voltage from 0° to 180°. After the integration of the square waves, isosceles triangles are obtained which serve to produce adjustable distribution curves. Before the triangular waveforms are definitely shaped into symmetrical normal distribution functions, a limiter circuit cuts off the part of the variable voltages below a fixed level. This results in isosceles triangles whose bases are connected by horizontal segments. By varying the amplitude of the square waves at the input of the integrator, the widths of the triangular voltages are changed without influencing their heights and frequency. The peak position of the distribution function is adjusted by means of the phase shift of the sinusoidal input. By increasing the gain of the square wave generator the width of the distribution function is decreased. Finally, the gain adjustment at the output of the shaping network determines the height of the elementary distribution function. The same circuits are used for each of the ten components.

Brubaker and Korn[11] have applied the repetitive differential analyzer to the following problem. By changing the initial conditions of an integrator, whose input is a square wave, in a random manner, a random phase 20 volts peak-to-peak 6 cycles/sec triangle wave is obtained. The probability density estimate for this random process is sought. By definition the desired function is

$$E[(y(t_1)] = \text{Prob}\left[X - \left(\frac{\Delta x}{2}\right) < x(t_1) - X + \left(\frac{\Delta x}{2}\right)\right] \qquad (12.15)$$

where X is a fixed voltage amplitude level. An estimate of E is obtained on the analyzer by sampling $y(t_1)$, at $t = t_1$ in each run, and registering a count if it falls in the voltage band $X - \Delta x/2$ to $X + \Delta x/2$. The number

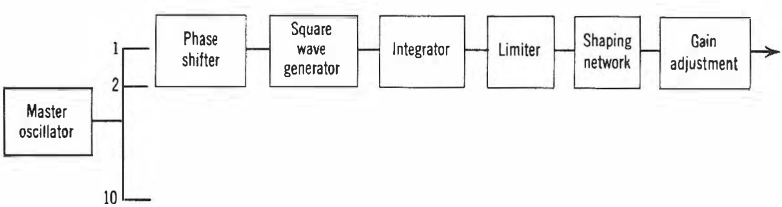


Fig. 12.9 Block diagram of generator of elementary distribution functions (Noble[10]).

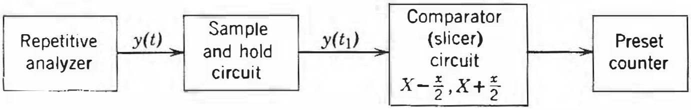


Fig. 12.10 Block diagram of probability function analyzer (Brubaker[11]).

of times $y(t_1)$ is in the voltage band divided by the total number of runs is the desired estimate. The analyzer provides therefore

$$E(y)_{ai} = \frac{1}{n}\sum_{k=1}^{n} K_y(t_1) \qquad k = 1, 2, 3, \ldots, n \tag{12.16}$$

which under certain conditions is an estimate of E. Apart from the repetitive analyzer precision, the estimate of the E will be influenced by the sample size and the statistical dependence of the samples. These conditions must be analyzed separately, in each case taking into account the natural frequencies involved and the bandwidth limitations of the equipment used.

In Figure 12.10 the block diagram of the equipment needed to solve Equation 12.16 is shown. The estimate of the probability density is given in Figure 12.11. The function E, for the triangular wave, would be a constant. The dots around the horizontal line represent the dispersion

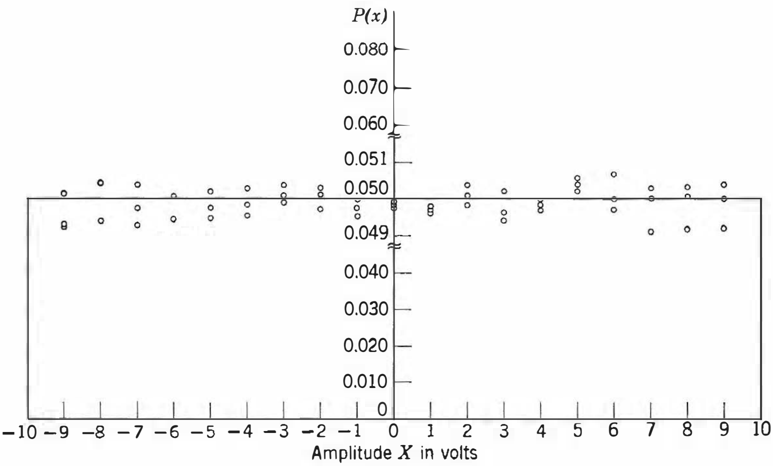


Fig. 12.11 Estimate of probability density (Brubaker[11]).

of points as obtained on the analyzer. The accuracy of the experimental results were 0.1% for $n = 10{,}000$ and $\Delta x = 1$ volt.

12.6 Statistical Evaluation of Guided Missile Performance

As an example of the application of high-speed analog computers in the statistical evaluation of complex systems, the problem of characterizing the "kill" probability of an air-to-air military missile will be described. The authors are indebted to Mr. S. Matsumo and Mr. H. Meissinger of Hughes Aircraft Company for making this material available. The missile system under consideration is shown in block diagram form in Figure 12.12. The tracking system consists of a target-tracking radar mounted on precessible free gyros, a receiver to process angular information, and a gyro torquing system. Nonlinear characteristics related to these components include radome characteristics and the velocity limits of mechanical components. The disturbing input is the target scintillation whose spectal density is a function of missile to target range. The control system includes rate gyros and stabilization networks. Associated with the gyros are drifts which are random in nature; that is, they appear as random initial conditions. The output of the control system is limited because of structural limitations of the missile. The aerodynamics are nonlinear functions of speed and angle of attack. Finally, the target-missile kinematics include random launch errors and random evasive-target maneuvers. In order to evaluate the tactical capability of the missile under the conditions described, analog simulation obviously

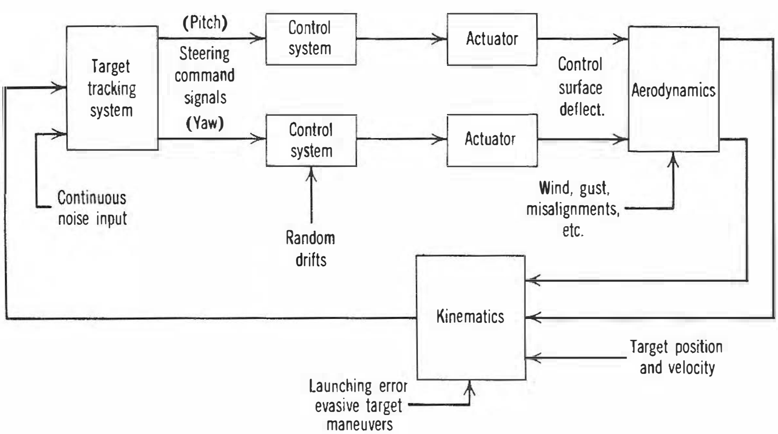


Fig. 12.12 Block diagram of air-to-air missile system.

becomes necessary for analysis. Moreover, because of the statistical confidence required, real-time simulation becomes unreasonably time consuming in obtaining results. Thus, the high-speed repetitive computer with 3000 to 1 time compression, for example, is a necessity.

In a guided missile performance study the ultimate measure of system capability is miss distance, defined as the distance of closest approach to the target. This is often approximated by the distances y and z, as $x \rightarrow 0$; where x, y, z is an orthogonal coordinate system with the target at the origin and the x coordinate directed parallel to the line joining the initial target and missile position. The probability distribution analyzer (PDA) can be used directly as a device for measuring statistical miss distance. A block diagram of this unit is shown in Figure 12.13. Consider a two-dimensional simulation where miss distance is defined as a measure of y as $x \rightarrow 0$. With the y coordinate as input to amplifier 1 and x as range input, $p(x > a)$ may be measured. The zero crossover detector and blocking oscillator generate a pulse when $x = 0$. At such times, comparator z compares the value y with the constant comparator level a. If $y > a$, a pulse appears at the output of blocking oscillator A; if $y < a$ no pulse appears. If this output is counted and the number divided by the number of runs, that is, the number registered by counter A, the probability that $y > a$, that is, $p(y > a)$, is obtained directly. Then as a is varied, a cumulative distribution function can be determined. This function can be plotted on probability graph paper (as shown in Figure 12.14) in which the abscissa is calibrated as a normal distribution function. A straight line would represent a normal cumulative distribution, with a mean equal to the ordinate at abscissa $= 0.5$, and a standard deviation $\sigma =$ ordinate minus the mean at abscissa $= 0.84134$; however, because of system nonlinearities the result is seldom Gaussian as shown. Using another channel in the PDA, $p(a < y < b)$ may be measured by use of an anticoincidence gate. This gate passes a pulse only when a pulse appears at input 1 and no pulse at input 2. Thus if a and b represent the boundaries of a lethal zone $p(a < y < b)$ will be the probability of kill.

In a three-dimensional simulation, miss distribution and cumulative distribution can be determined by measuring the probabilities in bounded or semibounded regions and systematically varying the comparator levels to cover the entire y-z plane. A typical distribution will not necessarily be symmetrical about the y or z axis due to crosscoupling in the pitch and yaw channels. To obtain such two-dimensional density functions a second PDA unit is used for the second (normal) coordinate axis with x input to the zero crossover detector. By utilizing anticoincidence and coincidence gates $p(a > y)$, $p(b > z)$; $p(y > a)$, $p(z > b)$; $p(a < y < b)$, $p(c < y < d)$ may be determined as shown.

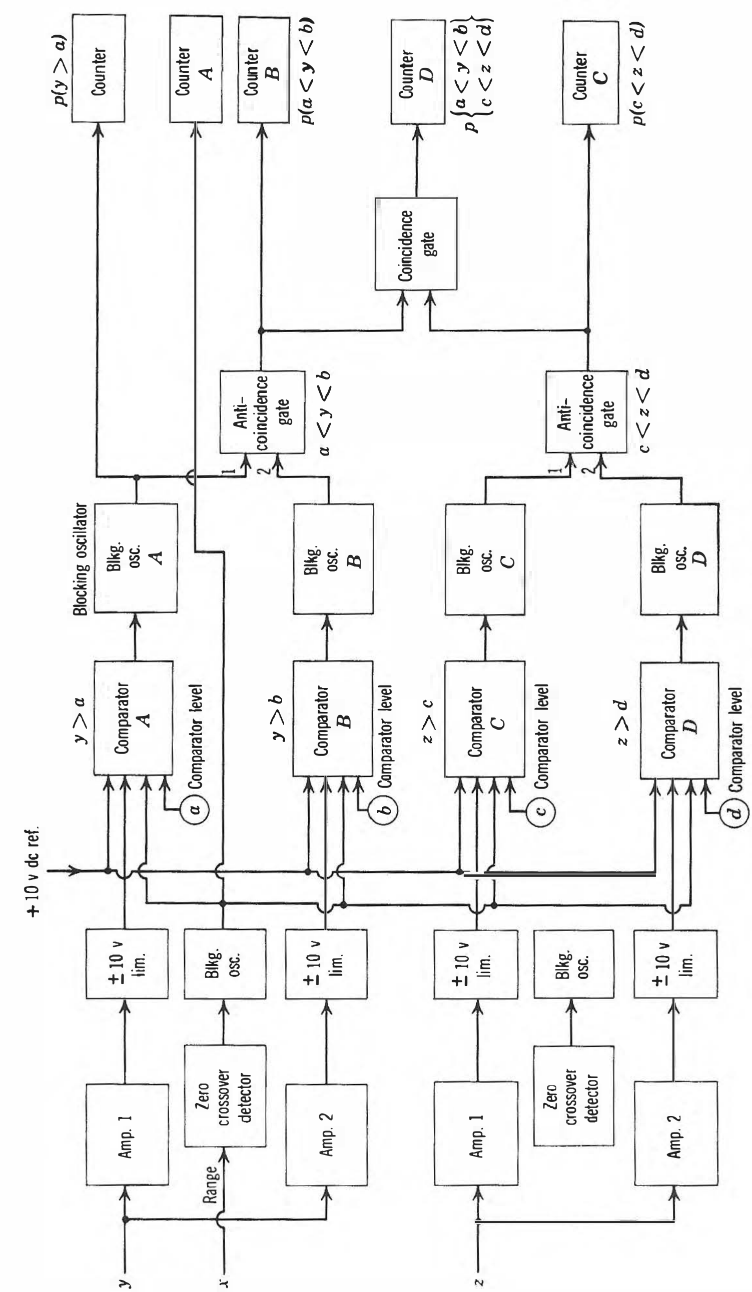


Fig. 12.13 Block diagram of probability distribution analyzer (G.P.S.).

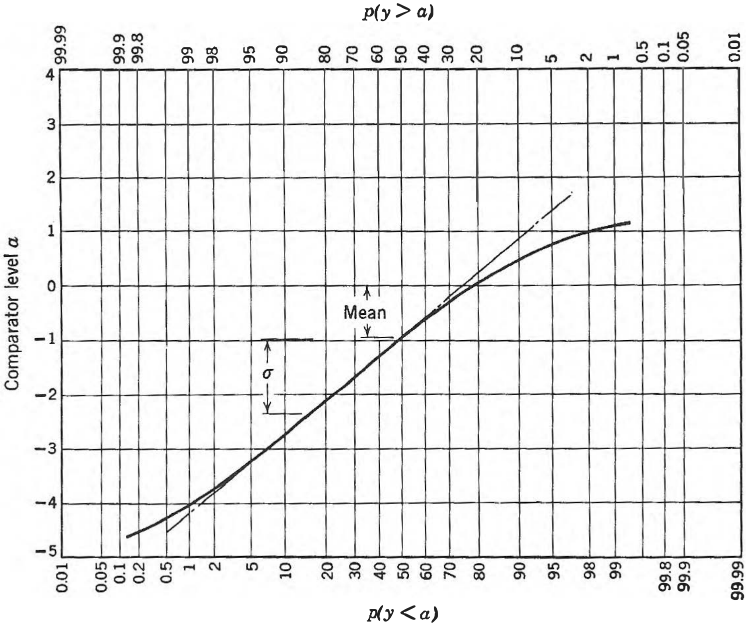


Fig. 12.14 Cumulative distribution function.

One method of determining the probability of target destruction is by obtaining a photographic record of miss dispersion in the y-z plane. This may be accomplished by a time-exposure photograph taken by an oscilloscope camera. Alternatively, a memoscope, which is essentially an oscilloscope with infinite persistence, can be used. The trigger input is taken from the output of the zero crossover detector blocking oscillator. The horizontal and vertical inputs are the y and z distances, respectively. Thus, each miss appears as a point of light on the scope. When a suitable number of samples have been registered, a photograph of the scope face is taken. By using overlap of lethal zones and counting the number of hits or misses, p_k, the probability of kill, is determined. The advantage of obtaining such a photographic record is that different lethal areas representing different target zones or contours can be used with a single record.

A second method of taking p_k data is to use a short-persistence cathode-ray tube in exactly the same manner as indicated; however, a target silhouette or a silhouette of the lethal zone is mounted directly on the

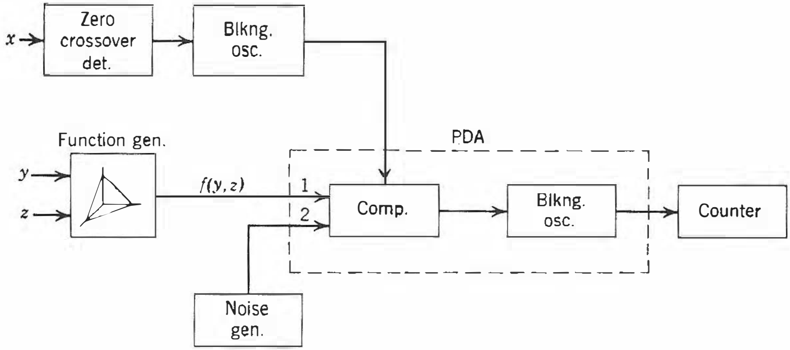


Fig. 12.15 Block diagram for two-dimensional determination of probability.

face of the scope, and a photoelectric tube is used to detect the points of light appearing on the scope within the specified contour. By counting the output of the photoelectric tube, p_k is immediately determined for the target represented on the scope. This method does not take into account statistically weighted areas within the lethal zone, however.

By using function generators, for one-dimensional probability of kill, p_1 in two dimensions can also be determined. In Figure 12.15 $f(y, z)$, the output of the function generator, represents the functional relationship of p_k to miss distance in two dimensions. The theoretical confidence limits of the measured probability are illustrated in Figure 12.16. For

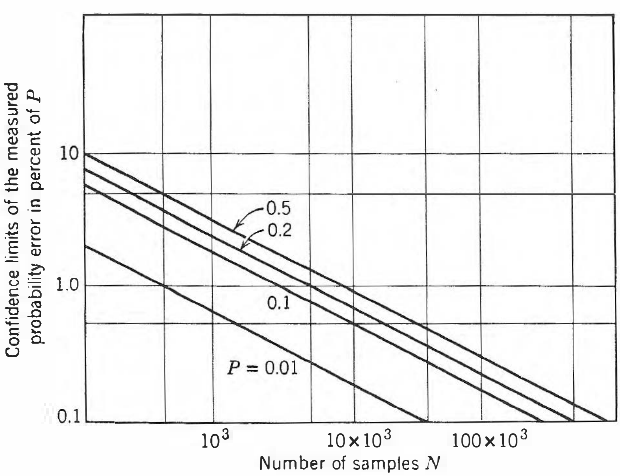


Fig. 12.16 Confidence limits of measured probability.

example, for a probability of 0.5 and with a sample size of 3000 shots the result will be within 2% of 3000 or within 60 counts of the true value. In an actual series of runs, however, the contribution of errors resulting from accuracy limits and faulty computer components becomes very difficult to detect. Various statistical methods are available to measure computer breakdown during successive samples. One crude but simple method would be to take four samples of 500 runs each, instead of taking one sample of 2000 runs, and to monitor the percent error difference between successive cases.

12.7 Ultra-Rapid Repetitive Differential Analyzers

MacKay[12] has described a novel application of the repetitive analog technique. Ordinarily the upper limit of the repetition frequency of commercial repetitive analyzers is of the order of 100 cycles/sec. Recognizing that high-speed operation holds a number of advantages, MacKay constructed an analog computer which operates in an entirely different frequency domain. His objective was to utilize to the fullest the maximum bandwidth of electronic computing elements and to seek the best compromise between operating speed and accuracy. The design of such a machine presents numerous technical problems. Some of these are the same as are present in the design of ordinary repetitive differential analyzers but in more acute form, whereas others are characteristic of the frequency domain involved. In the MacKay machine the repetition frequency is of the order of 1000 to 25,000 cycles/sec.

The linear portion of such an analyzer is more easily designed. It is only necessary to pay greater attention to the attainment of an adequate bandwidth in the computing elements. As indicated in Chapter 3, in the discussion of the accuracy of the integrator, the smaller the integration interval, the smaller may be the gain. Accordingly, for very short working intervals, a single stage of amplification suffices. This greatly facilitates the design of the operational amplifier. In order to realize suitable nonlinear computing elements, function generation and multiplication must be carried out with diode function generators. A quarter-square multiplier using diodes was constructed and observed to have a negligible phase shift up to 50 kc.

The output unit requires major improvements in the commonly used cathode-ray tubes display systems. As a matter of fact, two-dimensional displays are not capable of presenting all the information that becomes available in such an analyzer. This problem is solved by employing "three-dimensional" displays as described by Parker.[13] With the aid of such a display system in conjunction with a repetitive differential analyzer the operator can observe directly on a stationary pattern the solution of equations of the type

$$F(y, y^{(1)}, a, t) = 0 \tag{12.17}$$

in which the parameter a may assume several hundred different values during each cycle. In this case the analyzer must be provided with an electronic switching system that varies each parameter in the equation at a sufficient speed. Actually such an analyzer works with two repetition rates. One parameter of the equation is changed very rapidly during each cycle, thus providing one particular solution; after a certain number of rapid cycles, the whole process is repeated at a lower repetition rate.

The field of application of such a repetitive analyzer has not as yet been fully explored. It is clear, however, that the additional capability provided by the ultra-high repetition rate opens up entirely new avenues of research and simulation. One example provided by Fisher,[14, 15] is cited to demonstrate the utility of the technique in studying differential equations. The equation simulated has the form

$$\frac{dy}{dx} = \lambda \frac{y(x - \mu y + \nu) + \rho}{x(y - \mu^{(1)}x + \nu) + \rho^{(1)}} \tag{12.18}$$

which arises in the area of astro-physics. Examination of the numerator and denominator of the expression shows that there are three types of singularities. These singular solution points generally divide the particular integrals into two or three classes. In this division one class manifests one property, whereas the other has the inverse property. This can be demonstrated conveniently on the differential analyzer.

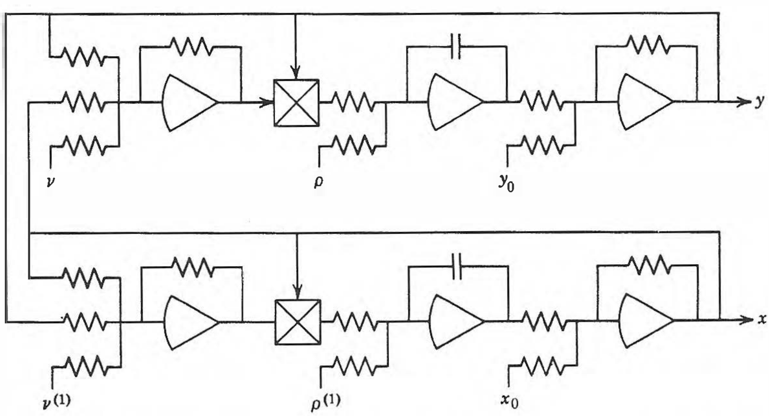


Fig. 12.17 Computer circuit for the solution of differential equations at high speed.

For programming on the analyzer the Equation 12.18 is rewritten as

$$\frac{dy}{dt} = \lambda y(x - \mu y + \nu) + \rho$$
$$\frac{dx}{dt} = x(y - \mu^{(1)}x + \nu^{(1)}) + \rho^{(1)} \tag{12.19}$$

The computer schematic is shown in Figure 12.17.

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