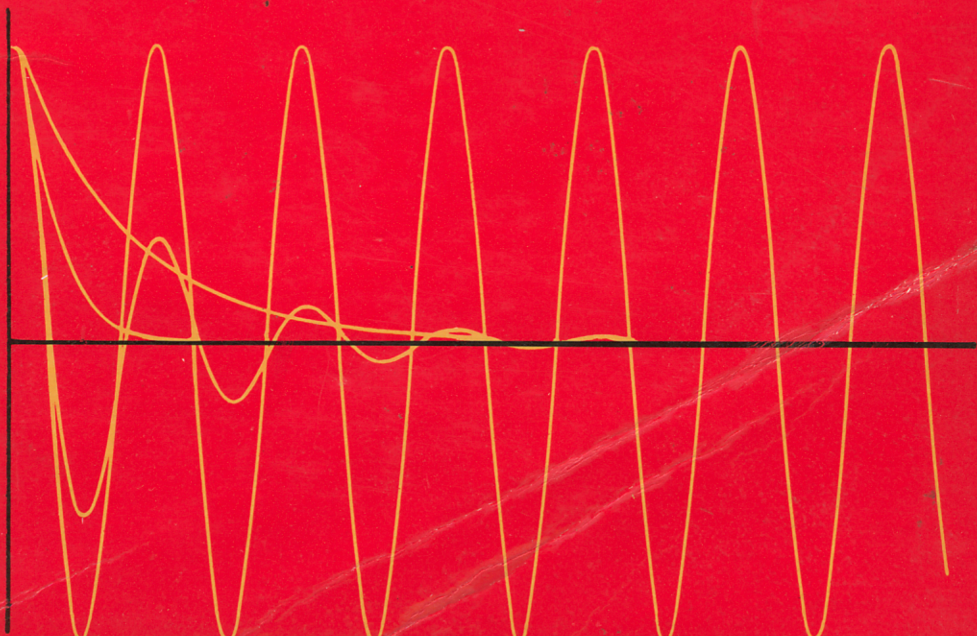
Basic ANALOGUE COMPUTER Techniques

Stewart and Atkinson



Basic Analogue Computer Techniques

by

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PREFACE

The purpose of this book is to provide practical instruction in the basic principles of analogue computing by following a course of selected problems. The choice of such problems is based on experience, gained over several years, in teaching students of engineering and science in technical colleges and grammar school sixth form pupils.

Successful analogue computation requires a good knowledge of the qualitative behaviour of solutions, together with a full appreciation of the capabilities of computing units. The theoretical background of such knowledge is referred to only briefly as it is felt that it is more than adequately treated in the books listed in the bibliography. Instead, emphasis has been placed upon the practical techniques which need to be mastered in order to make full use of the analogue computer. It is hoped such mastery will be attained by solving the problems which are arranged in the form of laboratory experiments. A routine procedure has been developed to solve these problems and, in each experiment, sufficient detail is given to enable students to programme the computer. At the end of each chapter, a supplementary list of problems and answers is given.

As there is a large number of different types of electronic analogue computers in use, no attempt has been made to describe the operation of these machines. It is expected that students will become familiar with their own machine and have access to the appropriate operator's manual. The reference voltages available in computers depend upon whether the machine is a thermionic valve or a transistorized model. Accordingly, a standard method has been adopted of using normalized voltages, i.e., expressing all voltages as a fraction of the reference voltage. This enables the various methods described in the text to be directly applicable to all analogue computers.

The material in the book covers the needs of most first courses in analogue computer programming at college level. Care has been taken to restrict the problems to those involving as few computer units as possible, thus obviating the need for large capacity computers. It is felt that, if the theoretical side is to be backed by practice in the laboratory, then the major part of the course will embody the work on linear operations involved in solving linear differential equations with constant coefficients. Usually, in the time remaining, only a little work on non-linear operations is possible if the student is set the task on the computer himself. The authors, therefore, decided to confine their attention solely to the various uses of the multiplier and have not discussed the use of further non-linear devices.

Since the representation of physical systems by transfer functions plays an important role in industrial computation, an elementary review of the principles involved has been included in chapter 6 with several supporting examples.

Regarding nomenclature, all resistance values, R, used in diagrams are given in megohms, and capacitances, C, in microfarads. Consequently, all products, RC, occurring have the appropriate dimensions, i.e., seconds. The symbols used in the figures are shown on page xi and, although there is still a variety of so-called 'standard symbols', the authors have listed those they have found in practice to be the most commonly used.

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Symbols Representing Basic Computer Units

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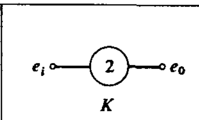
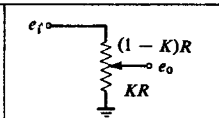
<tr><td>Potentiometer</td><td></td><td></td><td>Pot. No. 2
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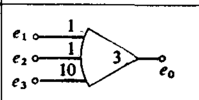
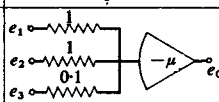
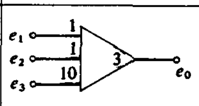
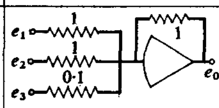
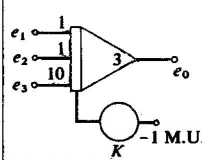
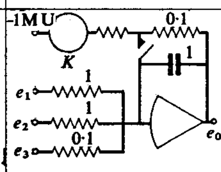
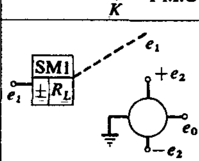
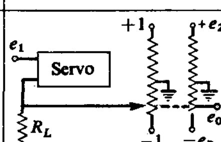
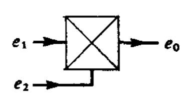
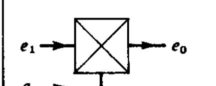
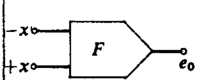
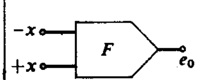
<tr><td>High gain amplifier</td><td></td><td></td><td>Amplifier No. 3
$e_0 = -\mu(e_1 + e_2 + 10e_3)$</td></tr>
<tr><td>Summing amplifier</td><td></td><td></td><td>$e_0 = -(e_1 + e_2 + 10e_3)$</td></tr>
<tr><td>Summing integrator with initial condition</td><td></td><td></td><td>$e_0 = -\int_0^t (e_1 + e_2 + 10e_3)\,dt + K$</td></tr>
<tr><td>Servo-multiplier</td><td></td><td></td><td>$e_0 = +e_1e_2$</td></tr>
<tr><td>Electronic multiplier</td><td></td><td></td><td>$e_0 = -e_1e_2$</td></tr>
<tr><td>Arbitrary function generator</td><td></td><td></td><td>$e_0 = F(x)$</td></tr>
</table>

Note: All voltages are in Machine Units, i.e. normalized. The electronic multipliers and function generators are available in several forms and it is usually sufficient to use the symbols given in the second column.

CHAPTER 1

Basic Linear Units

1.1 The Operational Amplifier

The most important computing unit is the operational amplifier. This consists of a d.c. amplifier together with associated input and feedback impedances (see Figs. 1.1 and 1.2). The d.c. amplifier is a thermionic valve or transistorized unit capable of amplifying voltage signals. Several mathematical operations may be performed using the operational amplifier. The principal ones are (i) summing a number of variable voltages and (ii) producing an integral with respect to time of this sum.

1.2 Use as a Summer

The d.c. amplifier is usually designed to have a negative gain, $-\mu$, where μ is very large, being approximately 10^8 at zero frequency. If ε is the voltage at the amplifier input grid, the output voltage e_0 is

$$e_0 = -\mu\varepsilon. \tag{1.1}$$

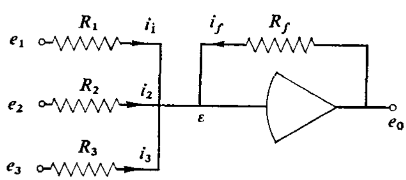


Fig. 1.1

The current flowing into the grid of the first stage of the amplifier shown in Fig. 1.1 is so small that it may be neglected compared with the currents i_1, i_2, i_3 and i_f. Hence Kirchhoff's first law gives

$$i_1 + i_2 + i_3 + i_f = 0 \tag{1.2}$$

and hence by Ohm's law

$$\frac{(e_1 - \varepsilon)}{R_1} + \frac{(e_2 - \varepsilon)}{R_2} + \frac{(e_3 - \varepsilon)}{R_3} + \frac{(e_0 - \varepsilon)}{R_f} = 0. \tag{1.3}$$

Inherently e_1, e_2, e_3, e_0, are voltages of the same order of magnitude. As μ is very large compared with unity it can be seen from Eq. (1.1) that ε is very small compared with e_0, and hence with the other input voltages. Assuming that ε is zero a very good approximation to Eq. (1.3) is

$$\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3} + \frac{e_0}{R_f} = 0 \tag{1.4}$$

giving

$$e_0 = -\left(\frac{R_f}{R_1}e_1 + \frac{R_f}{R_2}e_2 + \frac{R_f}{R_3}e_3\right). \tag{1.5}$$

The quantities R_f/R_1, R_f/R_2, etc., are referred to as the gains associated with the voltages e_1, e_2, etc., respectively. Some computers have a built-in system of resistors providing fixed gains, whilst others are provided with external plug-in resistors from which a wide range of gains may be obtained. The number of inputs available usually varies from about five to ten.

Attention is drawn to the presence of the minus sign which results from the use of an operational amplifier. If the feedback resistor and all input resistors are equal a simple sign reversal of the sum of the input signals is produced.

1.3 Use as an Integrator

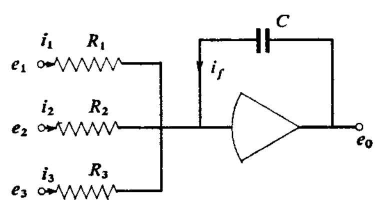


Fig. 1.2

In Fig. 1.2 if Q is the charge on the capacitor C at any instant, then

$$i_f = \frac{dQ}{dt} = C\frac{d}{dt}(e_0 - \varepsilon) \tag{1.6}$$

and

$$i_1 + i_2 + i_3 + i_f = 0, \tag{1.7}$$

thus

$$\frac{e_1 - \varepsilon}{R_1} + \frac{e_2 - \varepsilon}{R_2} + \frac{e_3 - \varepsilon}{R_3} + C\frac{d}{dt}(e_0 - \varepsilon) = 0. \tag{1.8}$$

Again assuming that $\varepsilon = 0$,

$$\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3} + C\frac{de_0}{dt} = 0. \tag{1.9}$$

Solving for de_0/dt and integrating with respect to t gives the output of the integrator at any instant t to be

$$e_0 = -\int_0^t \left(\frac{e_1}{CR_1} + \frac{e_2}{CR_2} + \frac{e_3}{CR_3}\right) dt + e_0(0) \tag{1.10}$$

Note again the sign reversal and the gains associated with each input voltage, i.e. $1/CR_1$, $1/CR_2$ and $1/CR_3$. The term $e_0(0)$ appearing in Eq. (1.10) represents the initial value of e_0. All integrating units have provision for inserting the initial value of the output voltage. This usually involves feeding the reference voltage via a potentiometer, set at the appropriate value, to a specific socket on the integrator. It must be emphasized that the correct initial voltage is obtained by measuring the output of the integrator and adjusting the potentiometer to give the desired voltage.

1.4 The Computing Potentiometer

The computing potentiometer has two uses. First, by applying the computer reference voltage to the potentiometer input, any fraction of this voltage may be obtained, and second, any variable voltage may be multiplied by a positive constant less than or equal to unity.

Care must always be taken when setting potentiometers to correct for loading errors. These become significant when the potentiometer load resistance is not large compared with the resistance of the potentiometer.

Thus to perform the operation $e_1 = 0{\cdot}3e_R$, where e_R is the reference voltage, it is not sufficient to set the potentiometer and obtain $0{\cdot}3e_R$ at its output as shown in Fig. 1.3 (a).

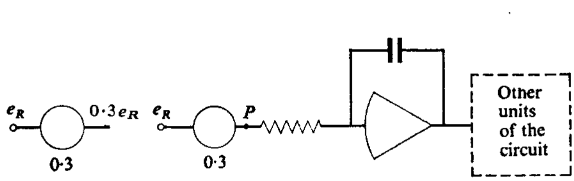


Fig. 1.3

The correct procedure is to set up the complete circuit and adjust the potentiometer so that its output under load conditions reads

$0{\cdot}3e_R$. Figure 1.3 (*b*) shows an example where the potentiometer is followed by an integrator, the point *P* being where voltage measurement is made.

In the more modern types of equipment this potentiometer setting is easily accomplished using a built-in nulling device.

Too much care cannot be given to the question of loading, as incorrect setting of potentiometers is a major source of error in analogue computation.

The reference voltage in computers is usually ± 100 volts or ± 10 volts; the former being found in thermionic machines and the latter in transistorized equipment.

To standardize programming the following convention will be used throughout the rest of this book.

The reference voltage will be referred to as 1 Machine Unit (M.U.) and all other computing voltages will be taken as fractions of this. Thus in a 100-volt machine a computing voltage of 82 volts will be 0·82 M.U. and in a 10-volt machine 7·3 volts will be 0·73 M.U.

1.5 Experiment 1. Use of Summers

1. Set up an operational amplifier to act as a summer. This is usually accomplished by making good a number of connecting links. Details are given in the appropriate computer manual.

2. From the computer reference voltage obtain, with the aid of potentiometers, three voltages 0·05, 0·1 and 0·12 M.U.

3. By suitable choice of input and feedback resistors generate a voltage,

$$E = -(10x + y + z) \tag{1.11}$$

where x, y, z are 0·05, 0·1, 0·12 M.U. respectively. The circuit is as shown in Fig. 1.4. By direct calculation E should be $-0{\cdot}72$.

4. Generate

$$E = x - 10y + z. \tag{1.12}$$

Equation (1.12) may be written $E = -(-x + 10y - z)$ from which it may be seen that $-x$, $+y$, $-z$ are the required input voltages (see Eq. (1.5)). The voltages $-x$ and $-z$ may be obtained by supplying the appropriate potentiometers with -1 M.U.

5. Generate

$$E = -(6x + 1{\cdot}3y + 2{\cdot}4z). \tag{1.13}$$

The circuit is shown in Fig. 1.5.

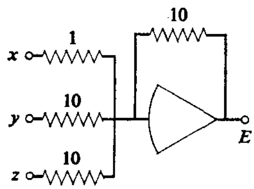


Fig. 1.4

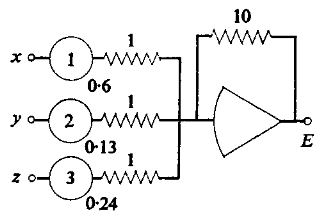


Fig. 1.5

When the gain factor required is non-standard, e.g. 1·3, use the next higher standard value, 10 say, and precede this input by a potentiometer adjusted to offset this increased gain. The setting will be 0·13 in this case. The overall gain is the *product* of the potentiometer 'gain' and the summer gain.

Theoretically, potentiometers 1, 2 and 3 will be set at 0·6, 0·13 and 0·24, but to avoid loading errors it is essential to carry out the test procedure described earlier for computing potentiometers.

6. Generate

$$\text{(i)}\quad E = -(8x + 1{\cdot}7y + 2{\cdot}1z) \tag{1.14}$$

$$\text{(ii)}\quad E = 4{\cdot}3x - 3{\cdot}7y + 5{\cdot}3z \tag{1.15}$$

$$\text{(iii)}\quad E = -7x - 5y + 2{\cdot}8z. \tag{1.16}$$

7. Check the computed values of E by direct calculation.

1.6 Experiment 2. Use of Integrators

1. Set up an operational amplifier to act as an integrator. (Consult the operator's manual for the appropriate connections.)

2. From the computer reference busbar obtain with the aid of potentiometers, the three quantities 0·05, 0·1 and 0·2 M.U. (referred to as x, y and z) respectively.

3. Generate

$$E = -\int_0^t 0{\cdot}1\, dt \text{ M.U.} \tag{1.17}$$

Using the circuit of Fig. 1.6.

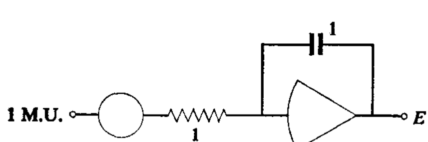


Fig. 1.6

(*Note:* (i) The initial value of E is zero, i.e. the initial condition of the integrator is set to zero. (ii) The input to the integrator, 0·1 M.U., should be checked under loading. (iii) In machines where integrators have fixed gains, an input provided with a gain of unity is selected.)

4. By means of the initial conditions facility already mentioned, generate

$$E = 1{\cdot}0 - \int_0^t 0{\cdot}1\, dt \text{ M.U.} \tag{1.18}$$

5. Generate

(i) $$E = -\int_0^t (x + y + z)\, dt. \tag{1.19}$$

(ii) $$E = \int_0^t (-10x + y + 0{\cdot}1z)\, dt. \tag{1.20}$$

(iii) $$E = -\int_0^t (3x + 0{\cdot}5y + 0{\cdot}08z)\, dt. \tag{1.21}$$

(iv) $$E = -0{\cdot}25 - \int_0^t (3{\cdot}2x - 0{\cdot}4y + 0{\cdot}6z)\, dt. \tag{1.22}$$

6. In each case, apply the signal E to the input of a pen recorder calibrated so that 1 M.U. ≡ full-scale deflection. From the known values of x, y and z check, by calculation, the computed values of E.

In the sections to follow full use will be made in diagrams of the standard computing symbols shown in the frontispiece. This system makes for easier understanding of the more complex computer circuits whilst providing all the necessary information.

1.7 Experiment 3. Simple Applications on Integration

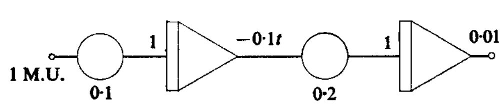


Fig. 1.7

1. Using the circuit shown in Fig. 1.7, generate the functions $y = -0{\cdot}1t$ M.U. and $y = 0{\cdot}01t^2$ M.U. where $0 < t < 10$ s.

Record the two functions by connecting the voltages representing $-0{\cdot}1t$ and $0{\cdot}01t^2$ to the inputs of a pen recorder calibrated so that 1 M.U. gives full-scale deflection.

Some computers have a built-in circuit termed the 'HOLD' circuit which can arrest the computation at any predetermined instant. This provides a means of measuring the output signals without incurring the dynamic errors of a continuous reading voltmeter.

The operator's manual will give full details of the use of the 'HOLD' circuit.

The accuracy of the solution can be checked arithmetically.

2. A practical application of the above is the examination of the motion of a particle travelling with a uniform acceleration. If this acceleration is 0·01 ft/s^2 and the initial velocity is 0·05 ft/s the velocity v ft/s and the distance s ft after any time t seconds are given by

$$v = 0{\cdot}05 + 0{\cdot}01t, \tag{1.23}$$

$$s = 0{\cdot}05t + \tfrac{1}{2}.0{\cdot}01t^2. \tag{1.24}$$

As

$$v = \int_0^t a\,dt + 0{\cdot}05 \tag{1.25}$$

where a is the acceleration and

$$s = \int_0^t v\,dt \tag{1.26}$$

a circuit arranged as shown in Fig. 1.8 will produce the desired solution.

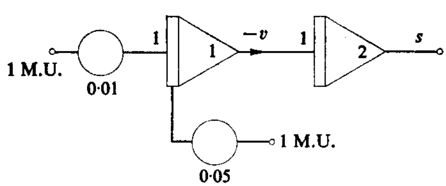


Fig. 1.8

Note that the output of amplifier 1 is $-v$ and the solution should be interpreted accordingly.

Representing a velocity of 1 ft/s by 1 M.U. and a distance of 1 ft by 1 M.U. record the signals $-v$ and s on a pen-recorder calibrated so that 1 M.U. gives full-scale deflection. Once again the HOLD facility may be utilized as an alternative means by which the solution may be measured.

The accuracy of your solutions may be checked by putting values of t in Eqs. (1.23) and (1.24).

CHAPTER 2

Amplitude Scaling

2.1 Scale Factors

The magnitude of each dependent variable of a problem is represented by a voltage at the output of a computing element. This must never exceed 1 M.U. otherwise overloading occurs, warning of which is usually indicated either visually or audibly on the computer.

The constant relating the magnitude of the voltage to one unit of the problem variable is called the amplitude scale factor. For instance in a problem involving a velocity which varies from 0 to a maximum value of 25 ft/s a scale factor K is introduced so that

$$1 \text{ M.U.} = K.25 \text{ ft/s},$$

i.e.

$$K = \tfrac{1}{25} \text{ M.U./ft/s}.$$

The output of the computing element is labelled $v/25$ where v is the velocity. Any voltage at this output, say 0·6 M.U., is translated into the corresponding velocity by multiplying by 25. Hence in the given case the velocity is $0{\cdot}6 \times 25 = 15$ ft/s.

The approximate maximum values of the problem variables are usually obtained from a knowledge of the physical system under examination. The scale factors for each variable are then obtained as the ratio

$$1 \text{ M.U./Estimated Maximum Value.}$$

In practice these estimates are often too high or too low, a fact which is shown by the output voltages from the various computing units. It may thus be necessary to re-scale the problem several times before a satisfactory solution is obtained.

These principles are outlined in the following experiments.

2.2 Experiment 4. A Simple Problem in Dynamics

A particle moving under a uniform acceleration of 2 ft/s^2 starts with a velocity of 10 ft/s. The velocity v and distance s after time t seconds are given by

$$v = 2t + 10 \tag{2.1}$$

$$s = t^2 + 10t \tag{2.2}$$

where v is in ft/s and s in ft.

Obtain graphs of v against t and s against t over a period of 4 s, using the computer and suitable recording apparatus.

1. Before scaling these equations the maximum values of v and s must be estimated. In this simple case they can be calculated exactly as 18 ft/s and 56 ft respectively.

2. The scale factors to produce a maximum of 1 M.U. at the outputs representing v and s are given respectively by

$$1 \text{ M.U.} = K_1 . 18 \quad \text{and} \quad 1 \text{ M.U.} = K_2 . 56,$$

i.e. $$K_1 = \tfrac{1}{18} \text{ M.U./ft/s} \quad \text{and} \quad K_2 = \tfrac{1}{56} \text{ M.U./ft.}$$

These values for K_1 and K_2, while permitting full use of the computer dynamic range, would necessitate awkward arithmetic computation in the interpretation of the computer voltages. This may be prevented by rounding off the scale factors to slightly lower values. Although this does not make full use of the dynamic range the resultant loss in accuracy is usually negligible.

A suitable choice of scale factors is

$$K_1 = 1/20, \quad K_2 = 1/60.$$

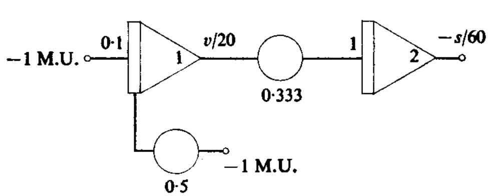


Fig. 2.1

3. The computer circuit for this problem is as shown in Fig. 2.1. Note that the voltages generated on the machine are labelled $v/20$ and $-s/60$. The scaled equations connecting the variables are now

$$v/20 = t/10 + 1/2 \tag{2.3}$$

$$s/60 = t^2/60 + t/6. \tag{2.4}$$

These are the Machine Equations and are obtained by multiplying Eqs. (2.1) and (2.2) by K_1 and K_2 respectively.

4. As in Experiment 3 the simulation of Eq. (2.3) requires the use of an integrator. Its gain G_1 and initial condition setting I_1 may be calculated as follows.

As the output has to be $v/20$ then

$$\frac{v}{20} = -G_1\int_0^t e_1\,dt + I_1 \tag{2.5}$$

but from Eq. (2.3) $$\frac{v}{20} = \frac{1}{10}\int_0^t dt + \frac{1}{2}$$

therefore $$G_1e_1 = -\frac{1}{10}\text{ M.U.}$$

and $$I_1 = \frac{1}{2}\text{ M.U.}$$

The product G_1e_1 may be chosen in many different ways; in Fig. 2.1, e_1 has been taken as -1 M.U. and G_1 as $\frac{1}{10}$.

The distance s is related to the velocity v by the equation

$$s = \int_0^t v\,dt + s_0$$

where s_0 is the initial value of s (zero in this example).

As $v/20$ is available as an input, simulation of Eq. (2.4) requires the use of a further integrator with initial condition zero. Its operation is described by

$$-\frac{s}{60} = -\int_0^t \frac{1}{3}\left(\frac{v}{20}\right)dt. \tag{2.6}$$

The minus signs take account of the inherent sign reversal of the integrator whose gain is seen to be $\frac{1}{3}$.

Assuming only standard gains of 0·1, 1·0, 10·0, etc., to be available this value may be obtained by selecting a gain of 1·0 and preceding integrator 2 by a potentiometer set at 0·333 (remember to allow for loading of this potentiometer).

5. The desired graphs will be obtained by recording the outputs from integrators 1 and 2 on a chart so calibrated that 1 M.U. gives full-scale deflection. This will be equivalent to 20 ft/s and 60 ft for v and s respectively.

Check these graphs by inserting one or two values for t in Eqs. (2.1) and (2.2).

2.3 Experiment 5. The Discharge of a Capacitor

A capacitor C has initial charge Q_0 and is discharged from time $t = 0$ through a resistance R. Determine the current flowing in the circuit after 6 s in the case where $R = 3M$, $C = 2\mu F$ and $Q_0 = 100$ coulombs.

1. The laws relating the voltage V, current i, and charge Q at time t seconds are

$$i = -\frac{dQ}{dt} = V/R \tag{2.7}$$

and

$$V = Q/C. \tag{2.8}$$

Hence

$$\frac{di}{dt} = -i/RC. \tag{2.9}$$

Initially,

$$i_0 = Q_0/RC. \tag{2.10}$$

2. Since energy is lost the maximum numerical value of the current is

$$Q_0/RC = 100/6 \text{ amp.}$$

The maximum scale factor for i is therefore

$$K = \tfrac{6}{100} \text{ M.U./amp.}$$

To simplify circuit details and to facilitate graph scales take $K = \frac{1}{20}$.

3. The scaled equation corresponding to Eq. (2.9) is thus

$$\frac{d}{dt}\left(\frac{i}{20}\right) = -\frac{1}{6}\left(\frac{i}{20}\right) \tag{2.11}$$

therefore

$$\frac{i}{20} = -\frac{1}{6}\int_0^t \left(\frac{i}{20}\right) dt + \left(\frac{i}{20}\right)_0 \tag{2.12}$$

Solution of Eq. (2.11) on the computer may be obtained using the circuit of Fig. 2.2.

This is obtained by first considering the output of the integrator. If this is taken as $i/20$ then feeding back this voltage and integrating according to Eq. (2.11) the desired result will be achieved. To accomplish this, a gain of 1·0 is used in the integrator in conjunction with a potentiometer set at 0·167. The initial value of i being 100/6, that of $i/20$ is 5/6 and is obtained using a potentiometer set at 0·833.

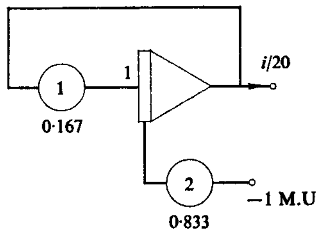


Fig. 2.2

4. Record the solution over a period of about 10 s on a recorder chart calibrated to give a full-scale deflection for 1 M.U. The value of the current flowing after 6 s can then be read from the graph.

5. Obtain the solution to the same problem with C half its original value. (Hint: Choose a new scale factor for i and adjust the potentiometers accordingly.)

2.4 Experiment 6. A Problem in Particle Dynamics with Variable Acceleration

A particle moves in a straight line with an acceleration proportional to time t and directed away from a fixed point O on the line. The acceleration is 1 ft/s^2 when $t = 1$ s and the particle starts from a point P, such that $OP = 25$ ft, with a velocity 18 ft/s towards O. Obtain the graphs of velocity v and distance x against time t. Find v and x when t is 8 s, and the distance covered when the velocity is zero. After how many seconds does the particle reach O?

1. The equations representing the motion are

$$\frac{dv}{dt} = t \tag{2.13}$$

$$v = \int_0^t \frac{dv}{dt}dt - 18 \tag{2.14}$$

$$x = \int_0^t v\,dt + 25 \tag{2.15}$$

where x, v and dv/dt are measured positively in the direction OP.

2. In this problem the greatest numerical values of x, v and dv/dt can be obtained by direct solution of the equations. If t does not exceed 10 s these maximum values are 47 ft, 32 ft/s and 10 ft/s^2 respectively.

3. The maximum scale factors are:

for $\frac{dv}{dt}$, $K_1 = \frac{1}{10}$ M.U./ft/s^2

for v, $K_2 = \frac{1}{32}$ M.U./ft/s

for x, $K_3 = \frac{1}{47}$ M.U./ft.

Suitable working values are:

$$K_1 = \tfrac{1}{10} \text{ M.U./ft/s}^2$$

$$K_2 = \tfrac{1}{50} \text{ M.U./ft/s}$$

$$K_3 = \tfrac{1}{50} \text{ M.U./ft.}$$

4. The machine equations are:

$$\frac{1}{10}\frac{dv}{dt} = \frac{1}{10}t \tag{2.16}$$

$$\frac{1}{50}v = -\frac{1}{5}\int_0^t -\frac{1}{10}\frac{dv}{dt}\,dt - \frac{18}{50} \tag{2.17}$$

$$\frac{1}{50}x = -\int_0^t -\frac{1}{50}v\,dt + \frac{25}{50} \tag{2.18}$$

5. The computer circuit to solve these equations is as shown in Fig. 2.3. Notice the use of a sign reverser to obtain $+x/50$.

6. Record the outputs from integrators 2 and 4 and hence determine the solutions of the problem.

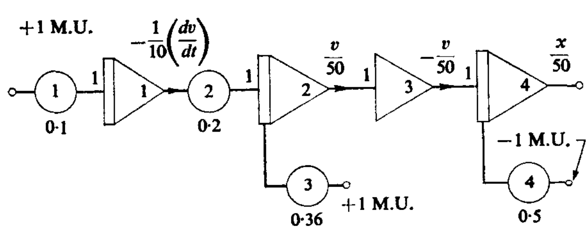


Fig. 2.3

CHAPTER 3

The Solution of Problems Involving Ordinary Differential Equations with Constant Coefficients

3.1 Introduction

Very many physical problems may be represented by mathematical models in the form of differential equations. It must be pointed out however, that in practice, only approximate representation of physical systems by such models is possible. Nevertheless, much important information may be obtained from the use of differential equations and their solution.

In order to facilitate greater understanding of the methods of solution involved in the experiments of this chapter a brief review of the nature of the solutions and method of programming for first and second order differential equations follows.

3.2 First Order Differential Equations

The general first order differential equation with constant coefficients is

$$a\dot{x} + bx = f(t) \tag{3.1}$$

where a and b are constants and $f(t)$ is an arbitrary function.

This equation has the complete solution

$$x = \text{Complementary Function (C.F.)} + \text{Particular Integral (P.I.)}.$$

The particular integral in general takes the form of $f(t)$. The complementary function is the general solution of Eq. (3.1) when $f(t) = 0$, and is

$$x = Ae^{-\frac{b}{a}t}. \tag{3.2}$$

If b/a is positive the magnitude of the C.F. decreases with increasing t, while if b/a is negative it increases.

Equation (3.1) may be re-arranged

$$\dot{x} = -\frac{b}{a}x + \frac{1}{a}f(t) \tag{3.3}$$

Hence

$$x = -\int_0^t \left[\frac{b}{a}x - \frac{1}{a}f(t)\right] dt + x_0 \tag{3.4}$$

where x_0 is the initial value of x.

The computer circuit (Fig. 3.1) used to solve Eq. (3.4) will thus consist of an integrator supplied with the appropriate initial condition and fed with inputs x and $-f(t)$. The latter is obtained from a suitable circuit (not shown) and the former from the output of the integrator itself.

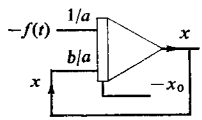


Fig. 3.1

3.3 Second Order Differential Equations

The general linear second order differential equation with constant coefficients is of the form

$$a\ddot{x} + b\dot{x} + cx = F(t). \tag{3.5}$$

When the coefficients a, b, c are all positive this equation represents, ideally, many common physical phenomena, e.g. a spring oscillating in a damping fluid.

Equation (3.5) may be written more conveniently in the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t) \tag{3.6}$$

where ζ and ω_n are constants which play a significant part in the nature of the solution of the equation. Once again the complete solution consists of the sum of a particular integral and a complementary function.

The particular integral, in general, assumes the form of $f(t)$, and the complementary function is the general solution of Eq. (3.6) when $f(t) = 0$ and is

(i) $x = A \sin \omega t$ for $\zeta = 0$.

(ii) $x = e^{-\zeta\omega_n t}[A \cos \omega_n\sqrt{(1 - \zeta^2)}t + B \sin \omega_n\sqrt{(1 - \zeta^2)}t]$ for $\zeta < 1$.

(iii) $x = e^{-\omega_n t}(At + B)$ for $\zeta = 1$.

(iv) $x = e^{-\zeta\omega_n t}[Ae^{\omega_n\sqrt{(\zeta^2 - 1)}t} + Be^{-\omega_n\sqrt{(\zeta^2 - 1)}t}]$ for $\zeta > 1$.

These results are illustrated in Fig. 3.2, the wave forms being termed (i) undamped, (ii) under-damped, (iii) critically-damped and (iv) over-damped.

ζ which clearly controls the rate of decay of stable solutions is called the Damping Ratio. Negative values of ζ lead to exponential terms whose magnitude increase indefinitely with t and hence contribute to unstable solutions.

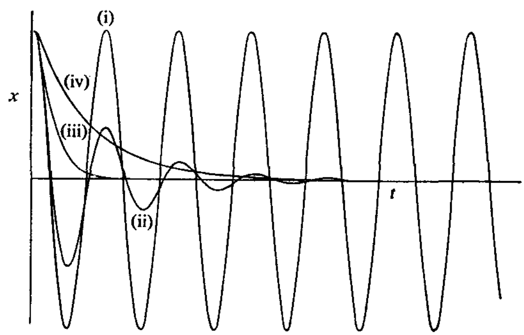


Fig. 3.2

ω_n is the angular frequency in radians per second at which oscillations occur when $\zeta = 0$ and it is defined as the undamped natural frequency. In the case where $0 < \zeta < 1$ the actual frequency of oscillation ω is given by

$$\omega = \omega_n\sqrt{(1 - \zeta^2)}. \tag{3.7}$$

The actual period of oscillation is $2\pi/\omega$ seconds.

Equations such as (3.5) are solved on the computer by successive integration as follows. Solving for the highest derivative present

$$\ddot{x} = -\frac{b}{a}\dot{x} - \frac{c}{a}x + \frac{F(t)}{a}. \tag{3.8}$$

After one integration

$$\dot{x} = -\int_0^t \left(\frac{b}{a}\dot{x} + \frac{c}{a}x - \frac{F(t)}{a}\right) dt + \dot{x}_0 \tag{3.9}$$

where $\dot{x}_0$ is the initial value of $\dot{x}$.

Integrating again

$$x = -\int_0^t - \dot{x}\, dt + x_0. \tag{3.10}$$

The computer circuit to solve Eq. (3.5) is developed by first considering Eq. (3.9). As illustrated in Fig. 3.3 this is accomplished

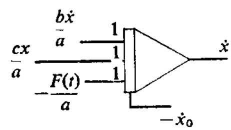


Fig. 3.3

by summing the three quantities appearing on the right-hand side of (3.9) and integrating this sum.

The sign change in passing through the integrator gives $+\dot{x}$ as the output. x is produced from this by use of another integrator.

The output of the first integrator, being $+\dot{x}$, is directly available, after suitable multiplication, for the input signal. The output from the second integrator must be sign inverted before it can be made available for the input $+(c/a)x$ and the other signal $-F(t)/a$ is obtained from within the computer.

Figure 3.4 shows the completed circuit in unscaled form.

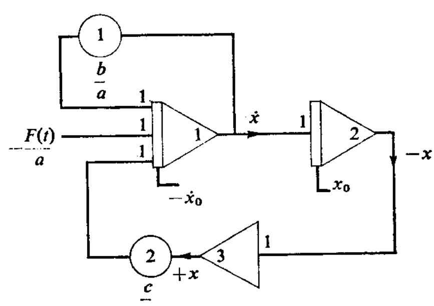


Fig. 3.4

3.4 Estimation of Scale Factors

It must be emphasized that there is no simple method for estimating the maximum values of the variables and their derivatives in differential equations. Consideration must be given both to the initial conditions and the value of the function $f(t)$ occurring on the right-hand side of the equation.

For instance in the first order Eq. (3.1) if $f(t) = 0$

$$\dot{x} = -\frac{b}{a}x \tag{3.11}$$

and the complete solution is

$$x = x_0 e^{-\frac{b}{a}t} \tag{3.12}$$

where x_0 is the initial value of x.

If b/a is positive, the greatest numerical value of x, $|x|_m$, is $|x_0|$ and

$$|\dot{x}|_m = \left|\frac{b}{a}x\right|_m. \tag{3.13}$$

A more detailed application is given in Experiment 7.

In the second order Eq. (3.6), when $f(t) = 0$, $\zeta = 0$ and $x = x_0$, $\dot{x} = 0$ at $t = 0$ then

$$|\dot{x}|_m = \omega_n|x|_m \tag{3.14}$$

$$|\ddot{x}|_m = \omega_n^2|x|_m \tag{3.15}$$

where $|x|_m = |x_0|$.

When $0 < \zeta < 1$ the values of $|\dot{x}|_m$ and $|\ddot{x}|_m$ will be less than those given in Eqs. (3.14) and (3.15), the corresponding relations being

$$|\dot{x}|_m < \omega|x|_m \tag{3.16}$$

$$|\ddot{x}|_m < \omega^2|x|_m \tag{3.17}$$

where $\omega = \omega_n\sqrt{(1 - \zeta^2)}$.

As will be shown in subsequent experiments these results provide a rule of thumb method for estimating the maximum values of the problem variables.

For a differential equation of any order the following empirical rule suggested by A. S. Jackson in his book *Analog Computation* may be used to obtain estimates of the maximum values of x and its derivatives.

3.5 The Equal Coefficient Rule

Consider the nth order differential equation

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1 x^{(1)} + a_0 x = f(t) \tag{3.18}$$

where $x^{(r)} \equiv \dfrac{d^r x}{dt^r}$ for $r = 1, 2, \ldots, n$.

For simplicity take zero initial conditions and

$$f(t) = 0 \quad \text{when } t < 0$$

$$f(t) = A\text{, a constant,} \quad \text{when } t \geqq 0.$$

With these conditions the following results may be obtained from Eq. (3.18)

$$x_m^{(n)} = A/a_n \tag{3.19}$$

and

?

$$x_m = 2A/a_0 \tag{3.20}$$

where x_m and $x_m^{(n)}$ are maxima again referring to absolute magnitude.

Dividing each derivative by its maximum value, Eq. (3.18) becomes on re-writing,

$$A\left\{\frac{x^{(n)}}{A/a_n}\right\} + x_m^{(n-1)}a_{n-1}\left\{\frac{x^{(n-1)}}{x_m^{(n-1)}}\right\} + \dots + a_1 x_m^{(1)}\left\{\frac{x^{(1)}}{x_m^{(1)}}\right\}$$

A/a_{n-1}

$$+ 2A\left\{\frac{x}{2A/a_0}\right\} = f(t). \tag{3.21}$$

This is called the Normalized Equation and the quantities in brackets the Normalized Variables. For instance $x^{(1)}/x_m^{(1)}$ is the normalized first derivative and will not exceed the value 1. The estimates for the maximum values $x_m^{(1)}, x_m^{(2)}, \dots, x_m^{(n-1)}$ are then determined by making the coefficients of the normalized variables equal, with one exception; the coefficient of (x/x_m) will be twice that of the other coefficients.

One restrictive condition applies. The values of $x_m^{(1)}, \dots, x_m^{(n)}$ must form an increasing or a decreasing sequence.

The application of this rule is shown in the following example.

Consider the equation

$$\dddot{x} + 2\ddot{x} + 9\dot{x} + 18x = f(t) \tag{3.22}$$

with $\qquad x = \dot{x} = \ddot{x} = 0 \quad \text{at} \quad t = 0$

and where $f(t) = 0$ for $t < 0, f(t) = 36$ for $t \geqq 0$.

(*a*) From Eqs. (3.8) and (3.9), $\dddot{x}_m = 36$

$$x_m = 72/18.$$

(*b*) The normalized equation thus becomes

$$36\left\{\frac{\dddot{x}}{36/1}\right\} + \ddot{x}_m . 2\left\{\frac{\ddot{x}}{\ddot{x}_m}\right\} + \dot{x}_m . 9\left\{\frac{\dot{x}}{\dot{x}_m}\right\} + 72\left\{\frac{x}{72/18}\right\} = 36. \tag{3.23}$$

(*c*) For equal coefficients of the normalized variables (with the exception of x/x_m)

$$36 = 2\ddot{x}_m = 9\dot{x}_m$$

hence $x_m = 4$, $\dot{x}_m = 4$, $\ddot{x}_m = 18$ and $\dddot{x}_m = 36$.

These agree very well with the actual maximum values which are:

$$x = 3{\cdot}02,\ \dot{x} = 3{\cdot}85,\ \ddot{x} = 10{\cdot}34 \text{ and } \dddot{x} = 36.$$

Once the maximum values of the dependent variable and its derivatives have been found, the corresponding scale factors can be evaluated and the original differential equation written in a scaled form.

To do this, each problem variable is multiplied by its scale factor, and the original differential equation re-written in terms of these new variables. Since the scale factor for any variable is calculated from the relation

$$\text{Scale factor} = 1/\text{Estimated Maximum Value}$$

it will be noted that these new variables are the normalized variables. Hence the scaled equation is equivalent to the normalized equation.

The scaled equation is solved for the highest scaled derivative as in Eq. (3.8), one integration performed, and the L.H.S. rescaled for the lower order derivative. This new equation, suitable for execution on the computer, is the Machine Equation. In the case of the previous example (Eq. (3.22)) the scaled equation is

$$\frac{\dddot{x}}{36} = -\left(\frac{\ddot{x}}{18}\right) - \left(\frac{\dot{x}}{4}\right) - \left(\frac{x}{2}\right) + \frac{f(t)}{36} \tag{3.24}$$

and the machine equation is

$$\frac{\ddot{x}}{18} = -\int_0^t \left[2\left(\frac{\ddot{x}}{18}\right) + 2\left(\frac{\dot{x}}{4}\right) + 2\left(\frac{x}{2}\right) - \frac{f(t)}{18}\right] dt + \left(\frac{\ddot{x}}{18}\right)_0 \tag{3.25}$$

where $(\ddot{x}/18)_0$ is the value of $\ddot{x}/18$ when $t = 0$.

In general since the highest order derivative (in this case $\dddot{x}$) is not present in the machine equation, its scale factor is not needed for the computer circuit. Only occasionally is a graph of this function against t required; in this instance the scale factor is used to enable the appropriate function to be generated.

The following experiments are suggested in order to develop these methods of preparing differential equations for solution on the computer.

3.6 Experiment 7. Solution of a First Order Differential Equation

Given that

$$2\dot{x} + x = 0 \tag{3.26}$$

and $x = 50$ when $t = 0$ obtain a graph of x against t over a period of 10 s.

1. As the coefficient of x is positive the solution of Eq. (3.26) is an exponential decay (compare Eq. (3.2)). The greatest numerical value of x thus occurs when $t = 0$, i.e. $|x|_m = 50$.

Using the relationship given in Eq. (3.13), $|\dot{x}|_m = 25$.

2. The scale factors for x and $\dot{x}$ are respectively

$$K_1 = \tfrac{1}{50} \quad \text{and} \quad K_2 = \tfrac{1}{25}.$$

The scaled equation is

$$2 \times 25\left(\frac{\dot{x}}{25}\right) + 50\left(\frac{x}{50}\right) = 0,$$

i.e.

$$\frac{\dot{x}}{25} + \frac{x}{50} = 0. \tag{3.27}$$

3. Solving for $\dot{x}/25$ and integrating this gives

$$\frac{x}{25} = -\int_0^t \frac{x}{50}\,dt + \left(\frac{x}{25}\right)_0. \tag{3.28}$$

But, since it is desired to generate a signal in the computer equivalent to $x/50$ the machine equation is

$$\frac{x}{50} = -\int_0^t \frac{1}{2}\left(\frac{x}{50}\right)dt + \left(\frac{x}{50}\right)_0. \tag{3.29}$$

4. The simulation of Eq. (3.29) requires one integrator whose output is $x/50$ and input $x/50$. If its gain is G_1 and initial condition I_1, then for this integrator, output

$$\frac{x}{50} = -\int_0^t G_1\left(\frac{x}{50}\right)dt + \left(\frac{x}{50}\right)_0. \tag{3.30}$$

Comparing Eqs. (3.29) and (3.30) leads to the values

$$G_1 = \frac{1}{2} \quad \text{and} \quad I_1 = \left(\frac{x}{50}\right)_0 = 1 \text{ M.U.}$$

G_1 may be obtained by using an integrator gain of unity preceded by a potentiometer set to 0·5.

The output from this integrator may be fed directly into the input to the potentiometer, since it is the required function, $x/50$. Figure 3.5 shows a computer circuit for Eq. (3.29).

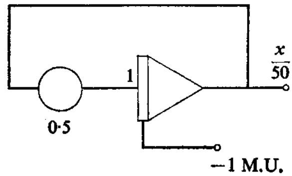


Fig. 3.5

5. Record the output from the integrator and check the accuracy of solution using the values in Table 3.1.

TABLE 3.1

t	0	1	2	10
x	50	30·3	18·4	0·34

6. Using the same initial condition $x_0 = 50$ write out the machine equation for

$$2\dot{x} + x = -20 \tag{3.31}$$

and modify the circuit of Fig. 3.5 to compute the new solution.

The analytical solution in this case is

$$x = 70e^{-t/2} - 20, \tag{3.32}$$

from which the graph of x against t may be checked.

3.7 Experiment 8. Solution of a Second Order Differential Equation

If

$$\ddot{x} + 2\dot{x} + 110x = 0 \tag{3.33}$$

and $x = 25$, $\dot{x} = 0$ when $t = 0$, obtain the graphs of x against t and $\dot{x}$ against t, given that $-25 < x < 25$.

1. The natural undamped frequency of the system represented by Eq. (3.33) is

$$\omega_n = \sqrt{110} = 10{\cdot}49 \text{ rad/s.}$$

The damping ratio is

$$\zeta = \frac{2}{2\sqrt{110}} = 0{\cdot}0954.$$

The actual frequency of oscillation is

$$\omega = \sqrt{110}\sqrt{(1 - 1/110)} = 10{\cdot}44 \text{ rad/s.}$$

2. Using the relations (3.16) and (3.17) which are applicable here,

$$|\dot{x}|_m = 10{\cdot}44 \times 25 = 261$$

and

$$|\ddot{x}|_m = 2725$$

3. The scale factors are:

for x, $K_1 = \frac{1}{25}$
for $\dot{x}$, $K_2 = \frac{1}{261}$
for $\ddot{x}$, $K_3 = \frac{1}{2725}$.

It will be more convenient to take $K_2 = 1/500$ and $K_3 = 1/5000$ as these figures make for easier interpretation of chart recorded values.

4. The scaled equation is

$$5000\left(\frac{\ddot{x}}{5000}\right) + 1000\left(\frac{\dot{x}}{500}\right) + 2750\left(\frac{x}{25}\right) = 0. \tag{3.34}$$

Solving for $\ddot{x}$ and integrating once

$$5000\left(\frac{\dot{x}}{5000}\right) = -\int_0^t \left[1000\left(\frac{\dot{x}}{500}\right) + 2750\left(\frac{x}{25}\right)\right] dt. \tag{3.35}$$

(The initial value of x is zero.)

5. Since it is required to generate a signal equivalent to $\dot{x}/500$ in the computer the machine equation is

$$\frac{\dot{x}}{500} = -\int_0^t \left[2\left(\frac{\dot{x}}{500}\right) + 5{\cdot}5\left(\frac{x}{25}\right)\right] dt. \tag{3.36}$$

From this equation it can be seen that there is no need to scale the $\ddot{x}$ term as no corresponding signal appears in the computer.

6. The computer circuit is constructed as described earlier in the chapter for Eq. (3.5). Any confusion regarding sign inversion is avoided by writing Eq. (3.36) with the negative sign outside the integral. Thus to obtain an output of $\dot{x}/500$ from the first integrator inputs of $\dot{x}/500$ and $x/25$ are necessary. If the respective gain factors are G_1 and G_2 and the initial condition is I_1, then,

$$\frac{\dot{x}}{500} = -\int_0^t \left[G_1\left(\frac{\dot{x}}{500}\right) + G_2\left(\frac{x}{25}\right)\right] dt + I_1. \tag{3.37}$$

Comparing Eqs. (3.36) and (3.37) gives the values

$$G_1 = 2,\ G_2 = 5{\cdot}5, \text{ and } I_1 = 0.$$

The signal $-x/25$ is obtained using a further integrator with gain G_3 and initial condition I_2. Then

$$-\frac{x}{25} = -\int_0^t G_3\left(\frac{\dot{x}}{500}\right) dt + I_2. \tag{3.38}$$

Now since

$$-\frac{x}{25} \equiv -\int_0^t \left(\frac{\dot{x}}{25}\right) dt - 1 \tag{3.39}$$

the values for G_3 and I_2 are found to be

$$G_3 = 20 \text{ and } I_2 = -1.$$

A sign inverter of gain unity provides the necessary signal $+x/25$ for the input to the first integrator. The signal $\dot{x}/500$ is available to be used as the other input. Figure 3.6 shows the resulting circuit when standard gains in association with computing attenuators are used.

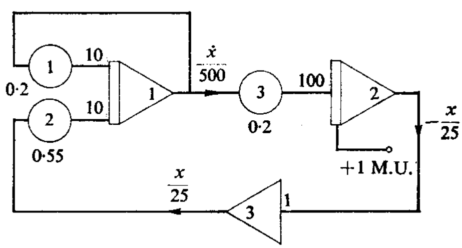


Fig. 3.6

7. Obtain the graphs of x and $\dot{x}$ against t from the signals at the output of amplifiers 1 and 3.

8. Check the accuracy of your solution from the following information:

x:	Frequency	10·44 rad/s
	Initial value	25 units
	Final value	0 units.

TABLE 3.2

Peak	1st	2nd	3rd	4th	5th
x value	+25	−18·5	+13·7	−10·1	+7·5
t value	0	0·3	0·6	0·9	1·2

$\dot{x}$: Frequency 10·44 rad/s
Initial value 0 units/s
Final value 0 units/s.

TABLE 3.3

Peak	1st	2nd	3rd	4th
$\dot{x}$ value	−227	+167	−124	+92
t value	0·15	0·45	0·75	1·05

3.8 Experiment 9. Generation of Sine and Cosine Functions

Generate 5 sin $2t$ and 10 cos $2t$.

1. $x = a \sin \omega t$ is the solution of the differential equation

$$\ddot{x} + \omega^2 x = 0 \tag{3.40}$$

where $x = 0$ and $\dot{x} = a\omega$ at $t = 0$.
When $\omega = 2$, $a = 5$ Eq. (3.40) becomes

$$\ddot{x} + 4x = 0 \tag{3.41}$$

where $x = 0$ and $\dot{x} = 10$ at $t = 0$.

2. From Eq. (3.41) the actual frequency of oscillation, which is equal to the natural undamped frequency, is 2 rad/s.

3. Since $x = 5 \sin 2t$

$$|x|_m = 5, \quad |\dot{x}|_m = 10 \quad \text{and} \quad |\ddot{x}|_m = 20.$$

4. The scale factors are

for x, $K_1 = \frac{1}{5}$
for $\dot{x}$, $K_2 = \frac{1}{10}$
for $\ddot{x}$, $K_3 = \frac{1}{20}$

5. The scaled equation is

$$20\left(\frac{\ddot{x}}{20}\right) + 20\left(\frac{x}{5}\right) = 0. \tag{3.42}$$

6. Solving for $\ddot{x}$ and integrating once

$$20\left(\frac{\dot{x}}{20}\right) = -\int_0^t 20\left(\frac{x}{5}\right) dt + (\dot{x})_0. \tag{3.43}$$

7. As it is required to generate a signal $\dot{x}/10$ in the computer the machine equation is

$$\frac{\dot{x}}{10} = -\int_0^t 2\left(\frac{x}{5}\right) dt + \left(\frac{\dot{x}}{10}\right)_0. \tag{3.44}$$

8. Programming the computer from this equation, integrator 1 has an input of $x/5$ and an output of $\dot{x}/10$. If the gain factor is G_1 and the initial condition I_1, then

$$\frac{\dot{x}}{10} = -\int_0^t G_1\left(\frac{x}{5}\right) dt + I_1. \tag{3.45}$$

Comparing Eqs. (3.44) and (3.45)

$$G_1 = 2, \quad I_1 = \left(\frac{\dot{x}}{10}\right)_0 = 1 \text{ M.U.}$$

The signal $-x/5$ is obtained using a further integrator with gain G_2 and initial condition I_2. Then, since

$$-\frac{x}{5} = -\int_0^t \frac{\dot{x}}{5}\, dt \tag{3.46}$$

and

$$-\frac{x}{5} = -\int_0^t G_2\left(\frac{\dot{x}}{10}\right) dt + I_2 \tag{3.47}$$

it follows that $E_2 = 2, \quad I_2 = 0.$

A sign inverter of gain unity provides a signal $x/5$ for the first integrator and the resulting circuit is as shown in Fig. 3.7.

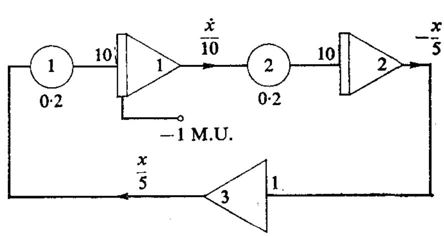


Fig. 3.7

9. The results obtained may be easily checked by measuring the amplitude and frequency of the solution x.

10. From the same circuit derive and record a signal $10 \cos 2t$. The functions $a \sin \omega t$ and $a \cos \omega t$ are frequently used in comput-

ing, particularly as driving functions. It is recommended that students make themselves absolutely familiar with the above procedure.

3.9 Experiment 10. A Second Order Differential Equation with Variable Damping Coefficient

The displacement x of a body acted upon by certain forces satisfies the equation

$$\ddot{x} + r\dot{x} + 9x = 0 \tag{3.48}$$

and $x = 100$ cm, $\dot{x} = 0$ when $t = 0$. r is a variable damping coefficient, which is a measure of the resistive force suffered by the body. Obtain the graphs of x against t for the cases, (i) $r = 1{\cdot}0$, (ii) $r = 6{\cdot}0$, (iii) $r = 10{\cdot}0$.

1. All three cases may be dealt with using a single circuit provided a potentiometer is included in the damping loop and made proportional to r. The amplitude scale factors for the variables can be calculated using the lowest value of r, since this will, in general, lead to the most oscillatory condition and hence the largest amplitudes of x, $\dot{x}$ and $\ddot{x}$.

Since r is positive the system is a decaying one and it will be a good enough approximation to take 100 as the maximum value of x, i.e. $x_m = 100$.

The natural undamped frequency of the equation is $\omega_n = 3$.

The damping ratio $\zeta = r/6 = 1/6$ in the case of least damping. In this case the actual frequency of oscillation is

$$\begin{aligned} \omega &= 3\sqrt{(1 - \tfrac{1}{36})} = 2{\cdot}96 \text{ rad/s.} \\ &= 3 \text{ approximately.} \end{aligned}$$

2. Using the relations (3.16) and (3.17)

$$|\dot{x}|_m = 3 \times 100 = 300 \text{ cm/s}$$

$$|\ddot{x}|_m = 9 \times 100 = 900 \text{ cm/s}^2$$

3. The scale factors are

for x, $K_1 = \frac{1}{100}$
for $\dot{x}$, $K_2 = \frac{1}{300}$
for $\ddot{x}$, $K_3 = \frac{1}{900}$.

4. The scaled equation is

$$900\left(\frac{\ddot{x}}{900}\right) = -300r\left(\frac{\dot{x}}{300}\right) - 900\left(\frac{x}{100}\right). \tag{3.49}$$

5. Integrating this gives

$$\frac{\dot{x}}{300} = -\int_0^t \left[r\left(\frac{\dot{x}}{300}\right) + 3\left(\frac{x}{100}\right)\right] dt + \left(\frac{\dot{x}}{300}\right)_0. \tag{3.50}$$

6. The computer circuit is constructed as described for Experiment 8. Integrator 1 has inputs of $\dot{x}/300$ and $x/100$ and an output $\dot{x}/300$. If the gain factors on the respective inputs are G_1, G_2 and the initial condition is I_1, then

$$\frac{\dot{x}}{300} = -\int_0^t \left[G_1\left(\frac{\dot{x}}{300}\right) + G_2\left(\frac{x}{100}\right)\right] dt + I_1. \tag{3.51}$$

Comparing coefficients in Eqs. (3.50) and (3.51)

$$G_1 = r, \quad G_2 = 3, \quad I_1 = 0.$$

To accommodate the largest value of r, i.e. 10, a potentiometer whose setting is $r/10$ is used in conjunction with an integrator gain of 10 to produce gain G_1. G_2 is similarly obtained by a potentiometer set at 0·3 followed by an integrator gain of 10.

The signal $-x/100$ is obtained by feeding $\dot{x}/300$ through integrator 2. If G_3 and I_3 are the gain and initial condition of this integrator

$$-\frac{x}{100} = -\int_0^t G_3\left(\frac{\dot{x}}{300}\right) dt + I_2 \tag{3.52}$$

but

$$-\frac{x}{100} = -\int_0^t 3\left(\frac{\dot{x}}{300}\right) dt - \left(\frac{x}{100}\right)_0 \tag{3.53}$$

hence $G_3 = 3$, $I_2 = -1$.

A sign inverter provides the signal $+x/100$ for the input to integrator 1. The output of integrator 1 being $\dot{x}/300$ is immediately available as the other input. Figure 3.8 shows the resulting scaled computer circuit.

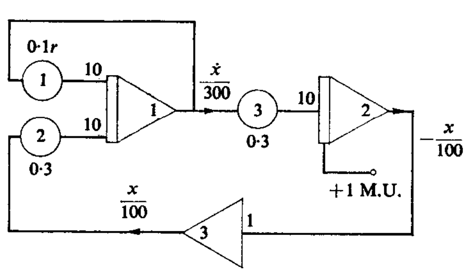


Fig. 3.8

This problem involves the use of three potentiometers and two integrators but in more complex problems it is more usual to employ many more of these units. It is therefore essential that a methodical check procedure be carried out in the computer before attempting to obtain a result. One such procedure will now be outlined and applied to this experiment.

First a static check consisting of two parts is applied. Part one, a pencil and paper check, is made before carrying out any patching. The procedure is as follows:

(*a*) Represent the problem variables and their derivatives by scaled initial condition voltages at the output of each integrator. These voltages are quite arbitrary but are chosen to produce reasonable outputs, they need have no physical significance.

(*b*) On the computer diagram write down these selected scaled voltages next to the outputs of the appropriate integrators.

(*c*) Using these voltages perform the various operations of summing, multiplication by a constant, etc., and hence calculate the voltages at all the input and output terminations of all the computing components. As integrators are not operational in the initial condition position this calculation is not carried beyond the input stage of any integrator.

(*d*) Substitute the assumed initial condition values of the variables into the original (i.e. the unscaled) equation describing the problem and calculate the value of the highest derivative. This value should tally with the voltage level obtained in (*c*) for that point. If the actual voltage levels are too low or zero it is necessary to change some of the assumed values of the variables and repeat the above calculations.

The problem is now patched on the computer and the second part of the static check is carried out. This consists of setting the potentiometer and the initial conditions to the static check values and then measuring the outputs of all the computing components with the switch in the Initial Condition position.

To check that the input resistors to an integrator are of the correct value the following procedure may be followed.

In the given circuit (Fig. 3.9) the gains through the integrator are 1, 10 and 1. Disconnect junction J from the integrator and connect to an amplifier with a feedback resistor of 1. The output of this amplifier should be $1(1) + 10(0{\cdot}02) + (0{\cdot}1) = 1{\cdot}3$.

As this produces overloading the gains need to be reduced by a factor of 10, i.e. a feedback resistor of 0·1.

Having completed the static checks it is often possible to perform various dynamic checks; these however depend upon the information available and it is obviously impossible to generalize.

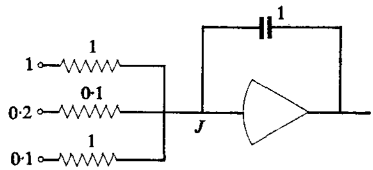


Fig. 3.9

This check procedure will now be applied to Experiment 10. For clarity some form of tabulation is recommended (see Table 3.4).

(1) The potentiometer settings are calculated from the computer diagram Fig. 3.10 and entered on the sheet.

(2) Assume the following values for the variables

$$\dot{x} = 300, \quad x = 100 \quad \text{and} \quad \text{let } r = 1.$$

(3) On a copy of the computer diagram mark in the static check voltages. (These appear in the small rectangles on the diagram.) Notice that integrator 1 is now supplied with an initial condition for the purpose of the check procedure.

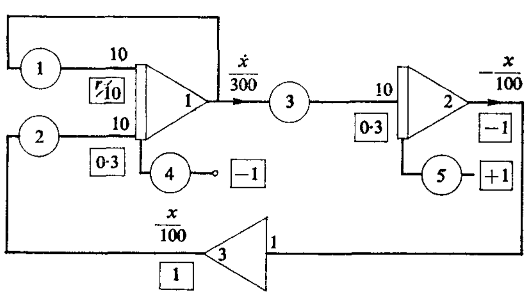


Fig. 3.10

(4) Substitution in the original equation of the problem gives

$$\ddot{x} = -(300r + 900) \text{ ft/s}^2. \tag{3.54}$$

The input to integrator 1 should thus be

$$-\frac{\ddot{x}}{300} = +(r + 3) \text{ M.U.} = +4 \tag{3.55}$$

in the case $r = 1$.

From the circuit above the input at the grid of integrator 1 is $10(0{\cdot}1)(+1) + 10(0{\cdot}3)(+1) = 4$. This checks with the calculated value and indicates that the original equation is correctly modelled by the computer circuit.

(5) As previously stated the computer is now patched and with the above settings the output voltages are checked.

TABLE 3.4

Problem Sheet

Potentiometer	1	2	3	4	5
Setting (r)	0·1	0·3	0·3	1	1

Amplifier	Output	Static check			
		Calculated		Measured	
		Check point (M.U.)	Output (M.U.)	Check point (M.U.)	Output (M.U.)
1	$\frac{\dot{x}}{300}$	$r + 3$	$+1$		
2	$-\frac{x}{100}$	3	-1		
3	$\frac{x}{100}$		$+1$		

3.10 Experiment 11. Solution of Simultaneous Differential Equations when Knowledge of the Problem Variables is Available

Solve using the analogue computer the equations

$$\dot{x} - 2y = 20 \tag{3.56}$$

$$\dot{y} + 2x = 0 \tag{3.57}$$

given that $x = 50$ and $y = 40$ when $t = 0$.

Assume that $-70 < x < 70$ and $-80 < y < 60$, the analytical solution of Eqs. (3.56) and (3.57) being

$$y = 50\sqrt{2}\sin(2t - \pi/4) - 10$$

$$x = -50\sqrt{2}\cos(2t - \pi/4).$$

(1) Using the analytical solution the maximum values of $\dot{x}$ and $\dot{y}$ can be calculated as

$$|\dot{x}|_m = 100\sqrt{2}, \quad |\dot{y}|_m = 100\sqrt{2}.$$

(2) The scale factors are

for x,	$K_1 = 1/70$.	Use 1/100
for $\dot{x}$,	$K_2 = 1/100\sqrt{2}$.	Use 1/150
for y,	$K_3 = 1/80$.	Use 1/100
for $\dot{y}$,	$K_4 = 1/100\sqrt{2}$.	Use 1/150.

(3) The scaled equations are

$$150\left(\frac{\dot{x}}{150}\right) - 200\left(\frac{y}{100}\right) = 20 \tag{3.58}$$

and

$$150\left(\frac{\dot{y}}{150}\right) + 200\left(\frac{x}{100}\right) = 0. \tag{3.59}$$

Thus

$$\frac{\dot{x}}{150} = \frac{4}{3}\left(\frac{y}{100}\right) + \frac{2}{15} \tag{3.60}$$

and

$$\frac{\dot{y}}{150} = -\frac{4}{3}\left(\frac{x}{100}\right). \tag{3.61}$$

(4) Integrating each equation once and rearranging gives the machine equations

$$\frac{x}{100} = -\int_0^t \left[-2\left(\frac{y}{100}\right) - \frac{1}{5}\right] dt + \left(\frac{x}{100}\right)_0 \tag{3.62}$$

$$\frac{y}{100} = -\int_0^t 2\left(\frac{x}{100}\right) dt + \left(\frac{y}{100}\right)_0. \tag{3.63}$$

Note again that a minus sign is placed outside each integral. As previously stated this is to be compared with the inherent sign reversal occurring in the electronic integrators. The signs attached to each term of the integrands of Eqs. (3.62) and (3.63) become the signs required for the corresponding input connection of the computer circuit.

(5) Equations (3.62) and (3.63) can be solved using two integrators. The first supplied with inputs $-y/100$ and $-1/5$ M.U. develops an output of $x/100$ while the second, supplied with an input of $x/100$, produces an output of $y/100$.

If the gain factors on the inputs of the first integrator are G_1 and G_2 and its initial condition is I_1, then its output is

$$\frac{x}{100} = -\int_0^t -\left[G_1\left(\frac{y}{100}\right) + G_2\left(\frac{1}{5}\right)\right] dt + I_1. \qquad (3.64)$$

Similarly if G_3 and I_2 are the gain factor on the second integrator and its initial condition respectively then

$$\frac{y}{100} = -\int_0^t G_3\left(\frac{x}{100}\right) dt + I_2. \qquad (3.65)$$

Comparing Eqs. (3.62) and (3.63) with (3.64) and (3.65) gives

$$G_1 = 2, \quad G_2 = 1, \quad G_3 = 2$$

$$I_1 = \left(\frac{x}{100}\right)_0 = 0{\cdot}5 \text{ M.U.}, \quad I_2 = \left(\frac{y}{100}\right)_0 = 0{\cdot}4 \text{ M.U.}$$

A sign inverter of gain unity converts the output of integrator 2 to the appropriate sign for the input of integrator 1. The complete circuit is as shown in Fig. 3.11.

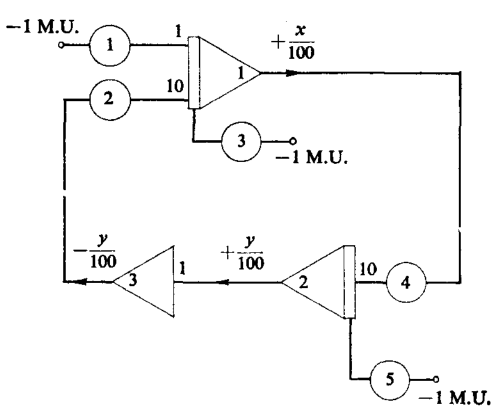


Fig. 3.11

Potentiometer	1	2	3	4	5
Setting	0·2	0·2	0·5	0·2	0·4

(6) Perform the static check procedure described earlier to confirm the voltage levels occurring in the initial set condition.

(7) Record the values of $x/100$ and $y/100$, and test the accuracy of solution using the analytical form given above. The frequency of x and y is 2 rad/s, thus if a number of complete periods are measured

from the graphs of x and y against t the period and hence the angular frequency of oscillation may be checked. Do not simply use large values of t to test the solution, since small percentage errors in the timing can cause large errors in the value of x and y, caused by the resulting phase shift of the curve.

3.11 Experiment 12. Solution of Simultaneous Differential Equations when Prior Knowledge of the Variables is not Available

Obtain the graphical solution of the simultaneous differential equations

$$\frac{dx}{dt} + 2x + y = 0 \tag{3.66}$$

$$\frac{dy}{dt} + x + 2y = 0 \tag{3.67}$$

given that $x = 1, y = 0$ when $t = 0$.

(1) In this problem nothing is known of the maximum values attained by the variables. However estimates may be made in the following way.

Consider the first equation with y regarded as constant at its initial value, i.e. $y = 0$. The equation reduces to

$$\frac{dx}{dt} + 2x = 0 \tag{3.68}$$

and has the solution $x = e^{-2t}$ using the value $x = 1$ at $t = 0$. Consequently the greatest value of x is 1.

If this maximum value of x is used in Eq. (3.66) in place of the variable x, then

$$\frac{dy}{dt} + 2y = -1 \tag{3.69}$$

which has solution $y = \frac{1}{2}(e^{-2t} - 1)$.

Hence $|y|_m = \frac{1}{2}$.

(2) The scale factors are

for x, $K_1 = 1$
for $\dot{x}$, $K_2 = \frac{1}{2}$
for y, $K_3 = 2$
for $\dot{y}$, $K_4 = 1$

using the principle outlined in Experiment 7.

(3) The scaled equations are:

$$2\left(\frac{\dot{x}}{2}\right) + 2(x) + \frac{1}{2}(2y) = 0 \tag{3.70}$$

and

$$\dot{y} + x + 2y = 0. \tag{3.71}$$

(4) The machine equations are

$$x = -\int_0^t \left[2x + \frac{1}{2}(2y)\right] dt + 1 \tag{3.72}$$

and

$$2y = -\int_0^t [2x + 2(2y)]\, dt. \tag{3.73}$$

Referring to previous experiments it can be seen that the gain factors associated with each amplifier input are the actual coefficients of the scaled variables appearing in the machine equation. Thus in Eq. (3.72) the x input to the integrator has an associated gain of 2, this being achieved using a gain of 10 together with a series potentiometer set to 0·2 (potentiometer 1 of Fig. 3.12). In this and future

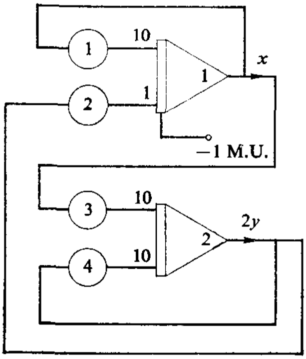


Fig. 3.12

experiments this method will be adopted in programming the computer.

(5) Equations (3.72) and (3.73) are solved in the computer using a circuit containing two integrators as in Fig. 3.12. To obtain the desired gains the potentiometer settings are as follows

Potentiometer	1	2	3	4
Setting	0·2	0·5	0·2	0·2

(6) Perform the static check using some non-zero value of $(2y)_0$.

(7) Record the solutions of x against t and $2y$ against t.

(8) Check the accuracy of solution by direct calculation from the solution

$$x = \tfrac{1}{2}(e^{-t} + e^{-3t})$$
$$y = \tfrac{1}{2}(e^{-3t} - e^{-t}).$$

3.12 Experiment 13. Forced Mechanical Oscillations

A body of mass 8 lb is hung by a spring producing in the spring an extension of 3·2 ft. The upper end of the spring is made to execute a vertical oscillation $z = 12 \sin 4t$, z being measured vertically downwards in feet. The mass is subjected to a frictional resistance whose magnitude in pounds weight is one-half of its velocity in ft/s. Obtain a graph of the displacement x from the equilibrium position. Take g to be 32 ft/s^2.

It is first necessary to obtain the equation describing the physical system.

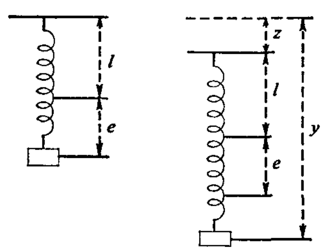


Fig. 3.13

For a body of mass m lb oscillating as stated the forces upon it are

(i) The gravitational force mg.

(ii) The spring force which opposes the displacement and is proportional to the extension of the spring.

(iii) The damping force which opposes the velocity and is proportional to the velocity.

(iv) The excitation z.

Newton's second law of motion states that the rate of change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts. For a constant mass the rate of change of momentum is equal to the product of the mass and acceleration. In the given system the equation of motion is

$$m\ddot{y} = \text{algebraic sum of the forces in the } y \text{ direction,}$$

i.e.

$$m\ddot{y} = -Kg(y - z - 1) - rg\dot{y} + mg. \tag{3.74}$$

In equilibrium (Fig. 3.13(*a*))

$$0 = -Kge + mg$$

i.e.

$$K = \frac{m}{e}.$$

Substituting for K and rearranging, Eq. (3.74) becomes

$$\frac{m}{g}\ddot{y} + \frac{m}{e}(y - 1 - e - z) + r\dot{y} = 0. \tag{3.75}$$

From Fig. 3.13(*b*) it can be seen that the displacement x from the initial equilibrium position is $y - 1 - e$, i.e.

$$y = x + 1 + e.$$

Substituting for y and its derivatives Eq. (3.75) becomes

$$\frac{m}{g}\ddot{x} + r\dot{x} + \frac{m}{e}x = \frac{m}{e}z. \tag{3.76}$$

Substituting the problem values the describing equation is therefore

$$\ddot{x} + 2\dot{x} + 10x = 120 \sin 4t. \tag{3.77}$$

1. The maximum values of the problem variables have now to be found. In practice a working knowledge of the type of problem will in most cases enable a good estimate to be made. For this particular problem a value for the maximum values of x and $\dot{x}$ may be found using the equal coefficient rule and treating the right-hand side of Eq. (3.77) as a constant equal to 120.

This rule gives $|x|_m = \dfrac{2.120}{10} = 24$ (see (3.20))

and

$$|\dot{x}|_m = \frac{120}{1} = 120.$$

The normalized equation is

$$120\left(\frac{\ddot{x}}{120}\right) + 2.\dot{x}_m\left(\frac{\dot{x}}{\dot{x}_m}\right) + 2.120\left(\frac{x}{x_m}\right) = 120 \sin 4t. \tag{3.78}$$

For equal coefficients of the normalized variables

$$2\dot{x}_m = 120, \quad \text{i.e. } \dot{x}_m = 60.$$

2. The computer circuit for solving Eq. (3.78) will consist of the usual circuit for a second order equation together with the circuit for producing the sin $4t$ term.

3. The scale factors are

for x, $K_1 = \frac{1}{24}$. Use $\frac{1}{25}$.
for $\dot{x}$, $K_2 = \frac{1}{60}$. Use $\frac{1}{60}$.
for $\ddot{x}$, $K_3 = \frac{1}{120}$. Use $\frac{1}{120}$.

4. The scaled equation is

$$\frac{\ddot{x}}{120} = -\frac{\dot{x}}{60} - \frac{25}{12}\left(\frac{x}{25}\right) + \sin 4t. \tag{3.79}$$

5. Integrating this gives

$$\frac{\dot{x}}{120} = -\int_0^t \left[\frac{\dot{x}}{60} + \frac{25}{12}\left(\frac{x}{25}\right) - \sin 4t\right] dt + \left(\frac{\dot{x}}{120}\right)_0. \tag{3.80}$$

The machine equations are

$$\frac{\dot{x}}{60} = -\int_0^t \left[2\left(\frac{\dot{x}}{60}\right) + \frac{25}{6}\left(\frac{x}{25}\right) - 2\sin 4t\right] dt + \left(\frac{\dot{x}}{60}\right)_0. \tag{3.81}$$

and also

$$\frac{x}{25} = -\frac{60}{25}\int_0^t -\frac{\dot{x}}{60}\, dt + \left(\frac{x}{25}\right)_0. \tag{3.82}$$

The body starts from rest in the equilibrium position so that

$$\left(\frac{\dot{x}}{60}\right)_0 = \left(\frac{x}{25}\right)_0 = 0.$$

6. The computer circuit is

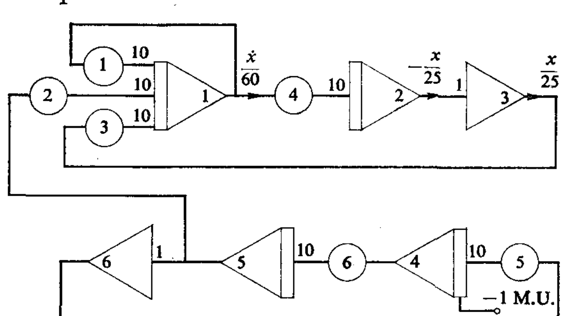


Fig. 3.14

To obtain the desired gains the potentiometer settings are as follows:

Potentiometer	1	2	3	4	5	6
Setting	0·2	0·2	0·42	0·24	0·4	0·4

7. Perform the static check; confirm that the sine wave has the correct period.

8. Record the values of $x/25$. The analytical solution is

$$x = 16e^{-t}\sin(3t + \alpha) - 12\cos(4t - \alpha)$$

where $\tan\alpha = 3/4$.

3.13 Experiment 14. Coupled Circuits

A steady e.m.f. of 12 volts is applied at time $t = 0$ to the circuit shown. The currents in the two branches after t seconds are denoted by x and y and R is 2 ohms. Obtain the graphs of x against t, y against t for the ratio of R/L equal to 5, 1, 0·5.

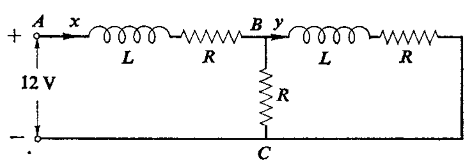


Fig. 3.15

1. Applying Kirchhoff's Laws to this circuit the current flowing in the arm BC is $x - y$ and the following equations are obtained

$$\left.\begin{aligned} L\dot{x} + 2x + 2(x - y) &= 12 \\ L\dot{y} + 2y - 2(x - y) &= 0. \end{aligned}\right\} \qquad (3.83)$$

2. The maximum values of the currents x and y may be estimated from a consideration of the physical principles involved in this circuit. The currents reach a maximum value, i.e. $\dot{x} = 0 = \dot{y}$, and hence

$$\left.\begin{aligned} 4x - 2y &= 12 \\ 4y - 2x &= 0 \end{aligned}\right\} \qquad (3.84)$$

giving $x = 4$, $y = 2$.

3. The scale factors are

for x, 1/4

for y, 1/2

for $\dot{x}$, $\dfrac{L}{16}$ (see (3.13))

for $\dot{y}$, $\dfrac{L}{8}$.

4. The scaled equations are

$$L\frac{\dot{x}}{16} = -\left[\frac{x}{4} - \frac{2}{8}\left(\frac{y}{2}\right) - \frac{12}{16}\right] \tag{3.85}$$

$$L\frac{\dot{y}}{8} = -\left[\frac{y}{2} - \frac{x}{4}\right]. \tag{3.86}$$

5. Integrating gives

$$L\left(\frac{x}{16}\right) = -\int_0^t \left[\frac{x}{4} - \frac{1}{4}\left(\frac{y}{2}\right) - \frac{3}{4}\right] dt + \left(L\frac{x}{16}\right)_0 \tag{3.87}$$

$$L\left(\frac{y}{8}\right) = -\int_0^t \left[\frac{y}{2} - \frac{x}{4}\right] dt + \left(L\frac{y}{8}\right)_0 \tag{3.88}$$

or

$$\frac{x}{4} = -\int_0^t \left[\frac{4}{L}\left(\frac{x}{4}\right) - \frac{1}{L}\left(\frac{y}{2}\right) - \frac{3}{L}\right] dt + \left(\frac{x}{4}\right)_0 \tag{3.89}$$

$$\frac{y}{2} = -\int_0^t \left[\frac{4}{L}\left(\frac{y}{2}\right) - \frac{4}{L}\left(\frac{x}{4}\right)\right] dt + \left(\frac{y}{2}\right)_0. \tag{3.90}$$

Initially the currents x and y are zero. Thus

$$\left(\frac{x}{4}\right)_0 = \left(\frac{y}{2}\right)_0 = 0.$$

6. The computer circuit is

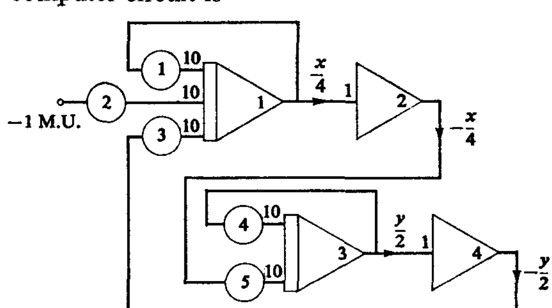


Fig. 3.16

7. To obtain desired gains the potentiometer settings are as follows:

Potentiometer	1	2	3	4	5
Setting	$4/10L$	$3/10L$	$1/10L$	$4/10L$	$4/10L$

8. Perform the static check and then record the values of $x/4$ and $y/2$.

9. The analytical solution is

$$x = 4 - 3e^{-2t/L} - e^{-6t/L}$$
$$y = 2 - 3e^{-2t/L} + e^{-6t/L}$$

3.14 Experiment 15. The Force Developed by a Hawser

A hawser is wrapped around a capstan and a constant force of 200 lb wt. is exerted at its free end. The force developed at the other end depends upon the amount of contact it makes with the capstan and upon the coefficient of friction. Examine the relationship between this force for various angles of contact up to 4π radians (i.e. two full turns) and for coefficients of friction 0·25, 0·2, 0·1, 0·02 and 0·01.

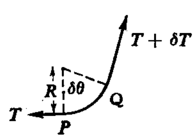


Fig. 3.17

The defining equation of this type of system is found as follows. Consider the forces acting in a small element PQ of length δs. The tangential forces due to the tension in the rope are T and $T + \delta T$ and a frictional force $F\,\delta s$. The normal force acting on PQ is $R\,\delta s$. In the position of limiting equilibrium resolving in the direction of the tangent and normal at P respectively the following equations are obtained

$$\left.\begin{aligned}(T + \delta T)\cos\delta\theta - T - F\,\delta s &= 0\\ (T + \delta T)\sin\delta\theta - R\,\delta s &= 0\end{aligned}\right\} \qquad (3.91)$$

which, as $\delta\theta$ and δT are small, reduce to

$$\left.\begin{aligned}\delta T - F\,\delta s &= 0\\ T\,\delta\theta - R\,\delta s &= 0\end{aligned}\right\} \qquad (3.92)$$

giving $F = dT/ds$, $R = T\,d\theta/ds$ in the limit as $\delta\theta \to 0$.

But
$$F = \mu R$$
$$\therefore \quad \frac{dT}{ds} = \mu T \frac{d\theta}{ds}$$
or
$$\frac{dT}{d\theta} = \mu T. \tag{3.93}$$

1. This equation has a solution of the form

$$T = T_0 e^{\mu\theta} \quad \text{where } T_0 = 200 \text{ (see Experiment 7)}$$

and from this solution the maximum values of T, T_m, may be found. When $\theta = 4\pi$ these are approximately

(*a*) when $\mu = 0{\cdot}25$, $T_m = 46 \times 10^2$
(*b*) when $\mu = 0{\cdot}20$, $T_m = 25 \times 10^2$
(*c*) when $\mu = 0{\cdot}10$, $T_m = 7 \times 10^2$
(*d*) when $\mu = 0{\cdot}02$, $T_m = 2{\cdot}6 \times 10^2$
(*e*) when $\mu = 0{\cdot}01$, $T_m = 2{\cdot}3 \times 10^2$.

2. Examination of these values shows that whilst all solutions may be recorded using the same scale factor greater accuracy would be obtained using a different scale factor in cases (*a*) and (*b*) from that in (*c*), (*d*) and (*e*).

In the former two cases a suitable scale factor would be $10^{-2}/50$ and in the other three $10^{-2}/10$.

3. As $T = \int_0^\theta \mu T \, d\theta$ an analogous equation may be set up on the computer by allowing one second of time in the computer solution to correspond to 1 radian of angle in the problem. The equation is thus

$$T = \int_0^t \mu T \, dt. \tag{3.94}$$

For (*a*) and (*b*) the scaled equation is

$$\frac{10^{-2}T}{50} = -\int_0^t -\mu\left(\frac{10^{-2}T}{50}\right) dt + \left(\frac{10^{-2}T}{50}\right)_0. \tag{3.95}$$

For (*c*), (*d*) and (*e*) the scaled equation is

$$10^{-3}T = -\int_0^t -\mu(10^{-3}T)\, dt + (10^{-3}T)_0. \tag{3.96}$$

4. The computer circuit for both scalings is

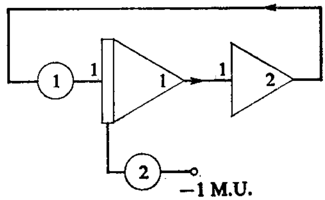


Fig. 3.18

For (*a*) and (*b*) the output of integrator 1 is $10^{-2}T/50$, for (*c*), (*d*) and (*e*) the output is $10^{-3}T$.

5. To obtain the desired gains the potentiometer settings are as follows.

Case	Pot. 1	Pot. 2
(*a*)	0·25	0·04
(*b*)	0·2	0·04
(*c*)	0·1	0·2
(*d*)	0·02	0·2
(*e*)	0·01	0·2

6. Perform the static check and then record the values of T against θ.

7. Check the solutions using $T = 200e^{\mu\theta}$ where μ assumes each of the values given in the problem.

3.15 Further Exercises

1. Set up a circuit to solve the equation

$$\frac{dx}{dt} + \alpha x = 0,$$

with initial condition $x = 1$ at $t = 0$, and where α is in the range $0 \leqq \alpha \leqq 10$.

[Solution: $x = e^{-\alpha t}$.]

2. Set up a circuit to solve the equation

$$\frac{d^2y}{dt^2} + 9y = 10e^{-\alpha t}$$

with initial conditions $y = 0$, $dy/dt = 50$ when $t = 0$. Examine the solution for values of α equal to 0, 1 and 10.

$$\left[\text{Solution: } y = \left\{\frac{50}{3} + \frac{10\alpha}{3(\alpha^2+9)}\right\}\sin 3t - \frac{10}{(\alpha^2+9)}\cos 3t + \frac{10}{(\alpha^2+9)}e^{-\alpha t}\right]$$

3. Given the simultaneous differential equations

$$\begin{aligned} 3\dot{x} + 2x + \dot{y} &= 1 \\ \dot{x} + 4\dot{y} + 3y &= 0, \end{aligned}$$

with the initial conditions $x = 0 = y$ at $t = 0$, use the methods of simplification of Experiment 12 to show that the greatest expected numerical values of x and y are $\frac{1}{3}$ and $\frac{1}{27}$.

Hence program the equations for solution on the computer, set up the corresponding circuit and use the static check procedure before obtaining the graphs of x and y against t.

$$\left[\text{Solution: } \begin{aligned} x &= \tfrac{1}{2} - \tfrac{1}{5}e^{-t} - \tfrac{3}{10}e^{-6t/11} \\ y &= \tfrac{1}{5}e^{-t} - \tfrac{1}{5}e^{-6t/11} \end{aligned}\right]$$

4. Obtain the graphical solution of the equation

$$\ddot{x} + 25x = \cos 5t$$

with initial conditions $x = 0{\cdot}1$, $\dot{x} = 0$ when $t = 0$, over an interval $0 \leqq t \leqq 10$ s. This solution illustrates the case of resonance, the amplitude of x increasing with t. The methods of estimating the problem variables described previously are not applicable; indeed no simple method exists for this type of problem. Assume $|x|_m = 1$ and $|\dot{x}|_m = 5$.

[Solution: $x = 0{\cdot}1(t \sin 5t + \cos 5t)$.]

5. The phenomenon of Beating occurs when two sinusoidal signals of similar frequency are added together, resulting in a signal of approximately the same frequency but modulated at half the difference frequency. Demonstrate this by generating

$$x = 0{\cdot}3 \cos 10t \quad \text{and} \quad y = 0{\cdot}3 \cos 9t$$

and then producing their sum

$$\begin{aligned} z &= 0{\cdot}3(\cos 10t + \cos 9t) \\ &= 0{\cdot}6 \cos 9{\cdot}5t \cos 0{\cdot}5t. \end{aligned}$$

Treat x and y as solutions of differential equations as shown in Experiment 9.

CHAPTER 4

Time Scaling

4.1 Introduction

The solutions to the wide range of problems covered by analogue computers may extend over periods of time ranging from a few microseconds to many hours. In all cases except those involving a solution time of only a few seconds, time scaling of the problem is essential.

The speed of response of the computer and its associated equipment dictates the minimum solution time taken by the machine. The maximum time is essentially determined by convenience in the use of the computer and associated recording equipment.

The need for time scaling is indicated in the scaled equations which are obtained as explained in the previous chapter. From these equations it can be seen that the coefficients of the scaled variables are the net gain factors of the inputs to the appropriate integrators. In those computers fitted with fixed input resistors and feedback capacitors there is an obvious maximum gain (usually 10) which cannot be exceeded, but even where a wide choice of resistors and capacitors is available it is acceptable computer practice to work with gains in the range 1–50. Hence, if in the scaled equations gains outside the acceptable range are indicated time scaling must be applied.

Consider the following example

$$\ddot{x} = 50{,}000 - 50\dot{x} - 40{,}000x \tag{4.1}$$

where $x = 0 = \dot{x}$ initially.

It can be shown that $0 < x < 10$ and $-500 < \dot{x} < 500$.

The scale factors are (i) $\frac{1}{500}$ for $\dot{x}$

(ii) $\frac{1}{10}$ for x.

The scaled equations are

$$\frac{\dot{x}}{500} = -\int_0^t \left[50\left(\frac{\dot{x}}{500}\right) + 800\left(\frac{x}{10}\right) - 100 \right] dt \tag{4.2}$$

$$\frac{x}{10} = \int_0^t 50\left(\frac{\dot{x}}{500}\right) dt. \tag{4.3}$$

Gains of 50, 800, 100 and 50 are required and thus time scaling is necessary.

It can also be seen from this problem that

$$\omega = 200\sqrt{\left(\frac{63}{64}\right)} \quad \text{(see Eq. (3.7)).}$$

Such a high frequency is likely to produce serious errors in recording equipment and also in other computing components.

4.2 Methods of Time Scaling

Time scaling may be performed in two ways. The first of these involves rewriting the problem equation using the following procedure. If T = computer time and t = problem time, let

$$T = \alpha t. \tag{4.4}$$

[*Note:* If $\alpha > 1$ the solution is slowed down by a factor α.
If $\alpha < 1$ the solution is speeded up by a factor α.]

The derivatives become

$$\frac{d}{dt} = \frac{d}{d(T/\alpha)} = \alpha\frac{d}{dT} \tag{4.5}$$

$$\frac{d^2}{dt^2} = \frac{d}{d(T/\alpha)}\left(\alpha\frac{d}{dT}\right) = \alpha^2\frac{d^2}{dT^2} \tag{4.6}$$

and, in general,

$$\frac{d^n}{dt^n} = \alpha^n\frac{d^n}{dT^n}. \tag{4.7}$$

On slowing down the problem stated in (4.1) by a factor of 100 the equation now becomes

$$10^4\frac{d^2x}{dT^2} = 5\times 10^4 - 5\times 10^3\frac{dx}{dT} - 4\times 10^4 x, \tag{4.8}$$

i.e.

$$\ddot{x} = 5 - 0{\cdot}5\dot{x} - 4x, \tag{4.9}$$

the dots now denoting differentiation with respect to T. The ranges of the problem variables are now

$$0 < x < 10, \quad -5 < \dot{x} < 5$$

and the new scale factors are

(i) $\frac{1}{5}$ for $\dot{x}$
(ii) $\frac{1}{10}$ for x.

The scaled equations are

$$\frac{\dot{x}}{5} = -\int_0^T \left[0{\cdot}5\left(\frac{\dot{x}}{5}\right) + 8\left(\frac{x}{10}\right) - 1\right] dT \tag{4.10}$$

$$\frac{x}{10} = \int_0^T \frac{1}{2}\left(\frac{\dot{x}}{5}\right) dT. \tag{4.11}$$

The gains now required are thus 0·5, 8, 1 and 0·5. From Eq. (4.9), $\omega = 2\sqrt{(63/64)}$ which checks that the problem has been slowed by the desired amount.

The second method of time scaling is carried out by changing the rate of integration. This method is preferable to the former as the two scaling operations are kept separate and it also reduces time scaling to a simple mechanical operation.

Consider the integration

$$\int_0^{10} e_i\, dt = e_i\Big[t\Big]_0^{10} = 10e_i, \tag{4.12}$$

where e_i is a constant.

If the independent variable t is changed to T by the relation $T = 5t$, then on substitution in Eq. (4.12)

$$\frac{1}{5}\int_0^{50} e_i\, dT = \frac{e_i}{5}\Big[T\Big]_0^{50} = 10e_i. \tag{4.13}$$

These integrations may be represented graphically (Fig. 4.1).

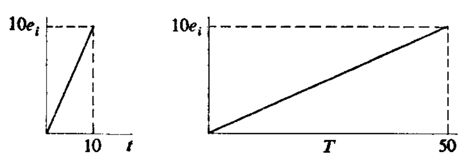


Fig. 4.1

Comparing Eqs. (4.12) and (4.13) the integrals are seen to be of identical form except that the latter is multiplied by a constant of $\frac{1}{5}$.

In a computer an integrator performs the operation

$$e_0 = -\frac{1}{RC}\int_0^{t_1} e_i\, dt + E_0. \tag{4.14}$$

Substituting the new independent variable T this gives

$$e_0 = -\frac{1}{RC}\int_0^{\alpha t_1} e_i \frac{1}{\alpha}\, dT + E_0 \qquad (4.15)$$

$$= -\frac{1}{\alpha RC}\int_0^{\alpha t_1} e_i\, dT + E_0. \qquad (4.16)$$

Thus all that is necessary to change the time scale by a factor of α is to change the integrator time constant from RC to αRC.

As has been shown, solving a differential equation on an analogue computer involves successive integrations. The time scaling in any such equation may thus be changed by altering the integrator time constants as stated. Note also that terms in dx/dt and d^2x/dt^2 must be replaced by $\alpha\, dx/dT$ and $\alpha^2\, d^2x/dT^2$.

In interpreting the results the new time scale factor should of course be borne in mind. The most convenient way of doing this is to relabel the time axis in terms of T/α.

4.3 Summary of Method of Solution of a Differential Equation

At this stage it is convenient to summarize the various steps involved in solving a differential equation.

(1) Estimate the maximum magnitudes of the problem variables and their derivatives.

(2) Determine the appropriate scale factors and write down the scaled equations.

(3) Draw the computer circuit.

(4) If time scaling is found to be necessary change the time constant (RC) of all integrators to the desired value αRC.

(5) Apply the check procedures.

(6) Record the solution and relabel the time axis according to the $t = T/\alpha$ relationship.

4.4 Experiment 16. Slowing Down a Second Order Differential Equation

A vibratory system is described by the equation

$$\ddot{x} + 2000\,\dot{x} + 10^7 x = 0 \qquad (4.17)$$

with initial conditions $x = 2$ and $\dot{x} = 0$. Slow down by a factor of 10^3 and solve.

1. The system is damped oscillatory and thus the maximum value of x may be taken to be approximately 2. Using relations (3.16 and

3.17) the maximum values of $|\dot{x}|$ and $|\ddot{x}|$ are about 6×10^3 and 18×10^6. Hence suitable scale factors are

(*a*) $1/2$ for x
(*b*) $1/10^4$ for $\dot{x}$
(*c*) $1/2 \times 10^7$ for $\ddot{x}$.

2. As

$$\dot{x} = -\int_0^t [2000\dot{x} + 10^7 x]\, dt \tag{4.18}$$

the scaled equations are

$$\frac{\dot{x}}{10^4} = -\int_0^t \left[2000\left(\frac{\dot{x}}{10^4}\right) + 2 \times 10^3\left(\frac{x}{2}\right)\right] dt + \left(\frac{\dot{x}}{10^4}\right)_0 \tag{4.19}$$

and

$$\frac{x}{2} = -\int_0^t -\frac{10^4}{2}\left(\frac{\dot{x}}{10^4}\right) dt + \left(\frac{x}{2}\right)_0. \tag{4.20}$$

3. It can be seen from these equations that gains of 2000 and 5000 would be required. As has been stated, this indicates the need for time scaling. Slowing down by a factor of 10^3 the modified equations will now be

$$\ddot{x} + 2\dot{x} + 10x = 0 \tag{4.21}$$

$$\frac{\dot{x}}{10} = -\int_0^T \left[2\left(\frac{\dot{x}}{10}\right) + 2\left(\frac{x}{2}\right)\right] dT + \left(\frac{\dot{x}}{10}\right)_0 \tag{4.22}$$

and

$$\frac{x}{2} = -\int_0^T -5\left(\frac{\dot{x}}{10}\right) dT + \left(\frac{x}{2}\right)_0. \tag{4.23}$$

4. The computer diagram is

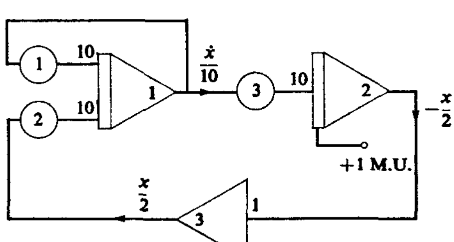


Fig. 4.2

The potentiometer settings are

Potentiometer	1	2	3
Setting	0·2	0·2	0·5

5. Perform the static check. From Eq. (4.21) verify that the frequency is 10^{-3} times the frequency in Eq. (4.17). Record the solution.

4.5 Experiment 17. Speeding up a First Order Differential Equation Given in Problem Form

Radium decays at a rate proportional to the quantity of radium present. In 25 years 0·022 g from a sample of 2 g is found to have decomposed. Produce a curve showing the rate of decay to the point when one-quarter of the original quantity is left.

1. If n = number of grammes of radium at any time t years, then

$$\frac{dn}{dt} = -Kn. \tag{4.24}$$

This gives a solution of the form $n = n_0 e^{-Kt}$ (see Experiment 7). It is given that $n_0 = 2$ and when $t = 25$, $n = 1{\cdot}978$. Hence $1{\cdot}978 = 2e^{-K.25}$, i.e. $\log_e 0{\cdot}989 = -25K$ giving $K = 0{\cdot}00044$.

2. The maximum value of n is 2 and an exactly similar computer equation is obtained if 1 s of time in the computer is made to correspond to 1 year in the problem. The describing equation is thus

$$\frac{dn}{dt} = -0{\cdot}00044n. \tag{4.25}$$

The scale factor for n is $\frac{1}{2}$.

3. As

$$n = -\int_0^t 0{\cdot}00044n\,dt + (n)_0 \tag{4.26}$$

the scaled equation is

$$\frac{n}{2} = -\int_0^t 0{\cdot}00044\left(\frac{n}{2}\right)dt + \left(\frac{n}{2}\right)_0. \tag{4.27}$$

Speeding up by a factor of 10^3 (i.e. 1 s on computer ≡ 10^3 years) a modified equation is obtained:

$$\frac{n}{2} = -\int_0^T 0{\cdot}44\left(\frac{n}{2}\right)dT + \left(\frac{n}{2}\right)_0. \tag{4.28}$$

4. The computer circuit for this is

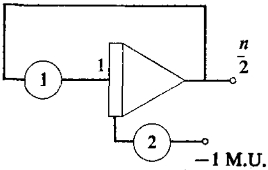


Fig. 4.3

The potentiometer settings are

Potentiometer	1	2
Setting	0·44	1

For a better recording a different time scale factor may be necessary. Repeat the experiment speeding up Eq. (4.25) by factors of 100, 10,000.

4.6 Experiment 18. Xenon Poisoning in a Nuclear Reactor

A reactor which has been operating at a high thermal-neutron flux level is suddenly shut down. Iodine 135 and xenon 135 are then present in amounts $3{\cdot}17 \times 10^{15}$ and $1{\cdot}90 \times 10^{15}$ atoms per unit volume. The half-life periods of the two nuclides are 24,120 and 33,120 hr respectively and it is known that iodine 135 decays to xenon 135 which in turn decays to comparatively stable caesium 135. Obtain graphs showing the concentrations of iodine 135 and xenon 135 as functions of time after shut down.

1. It can be shown that the equation for the concentration of xenon is

$$\frac{dX}{dt} = -\lambda_1 X + \lambda_2 I$$

and that for the concentration of iodine

$$\frac{dI}{dt} = -\lambda_2 I$$

where λ_1 and λ_2 are the decay constants per second for xenon and iodine respectively, and X, I are the concentrations of xenon and iodine respectively in atoms per unit volume. The solution of the first order equation $dy/dt = -Ky$ is $y = y_0 e^{-Kt}$, and applying this to the problem, the describing equations are found to be

$$\frac{dI}{dt} = -\frac{0{\cdot}6931}{24{,}120} I = -2{\cdot}874 \times 10^{-5} I \qquad (4{\cdot}29)$$

and

$$\frac{dX}{dt} = -\frac{0{\cdot}6931}{33{,}120}X + \frac{0{\cdot}6931}{24{,}120}I = -2{\cdot}092 \times 10^{-5}X + 2{\cdot}874 \times 10^{-5}I \tag{4.30}$$

2. The maximum value of I is obviously $3{\cdot}17 \times 10^{15}$. An estimate of the maximum value of X may be obtained in a similar manner to that used in Experiment 12. It is easier however to try say a maximum of 4×10^{15}; if this is too small overloading will occur but it will not be too difficult to re-scale.

3. Using the above values the scale factors are

 (i) for I, $\frac{1}{4} \times 10^{-15}$
 (ii) for X, $\frac{1}{4} \times 10^{-15}$.

4. The machine equations are

$$\frac{10^{-15}}{4}I = -\int_0^t 2{\cdot}874 \times 10^{-5}\left(\frac{10^{-15}}{4}I\right)dt + \left(\frac{10^{-15}}{4}I\right)_0 \tag{4.31}$$

and

$$\frac{10^{-15}}{4}X = -\int_0^t \left[2{\cdot}092 \times 10^{-5}\left(\frac{10^{-15}}{4}X\right) - 2{\cdot}874 \times 10^{-5}\left(\frac{10^{-15}}{4}I\right)\right]dt + \left(\frac{10^{-15}}{4}X\right)_0. \tag{4.32}$$

Speeding up by a factor of 3600, i.e. 1 s of computer time equivalent to 1 hr of the problem time, these equations reduce to

$$\frac{10^{-15}}{4}I = -\int_0^T 0{\cdot}103\left(\frac{10^{-15}}{4}I\right)dT + 0{\cdot}793 \tag{4.33}$$

$$\frac{10^{-15}}{4}X = -\int_0^T \left[0{\cdot}075\left(\frac{10^{-15}}{4}X\right) - 0{\cdot}103\left(\frac{10^{-15}}{4}I\right)\right]dT + 0{\cdot}475 \tag{4.34}$$

5. The computer diagram is

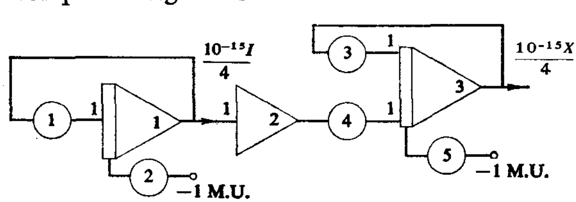


Fig. 4.4

6. The potentiometer settings are

Potentiometer	1	2	3	4	5
Setting	0·103	0·793	0·075	0·103	0·475

7. Carry out the check procedure and obtain the graphs of I against t and X against t.

4.7 Experiment 19. Simultaneous Differential Equations Requiring Time Scaling

Obtain the graphical solutions of the equations

$$\left.\begin{aligned} \ddot{x} + 100\dot{x} + 10^4 x + 10^4 y &= 4 \times 10^4 \\ \ddot{y} + 100\dot{y} + 9 \times 10^4 y + 200\dot{x} - 19 \times 10^4 x &= 22 \times 10^4 \end{aligned}\right\} \tag{4.35}$$

given that $x = y = 0$; $\dot{x} = 200$; $\dot{y} = 400$ at $t = 0$.

1. As in Experiment 12 nothing is known of the maximum values of the problem variables, but estimates may be obtained using a procedure similar to that outlined in that experiment.

Consider the second equation with the x and $\dot{x}$ terms zero. The modified equation is

$$\ddot{y} + 100\dot{y} + 9 \times 10^4 y = 22 \times 10^4. \tag{4.36}$$

Using the equal coefficient rule the following estimates of maximum numerical values are obtained:

$$|\ddot{y}|_m = 22 \times 10^4, \quad |\dot{y}|_m = 22 \times 10^2, \quad |y|_m = \tfrac{44}{9}.$$

(*Note:* These values are calculated on the basis of zero initial conditions and will be only approximate.)

In the problem use $|\dot{y}|_m = 25 \times 10^2$ and $|y|_m = 10$.

Substituting this maximum value of y in the first equation of (4.35) gives

$$\ddot{x} + 100\,\dot{x} + 10^4 x = -6 \times 10^4. \tag{4.37}$$

From this the estimated maximum values are found to be $|\ddot{x}|_m = 6 \times 10^4$, $|\dot{x}|_m = 600$, $|x|_m = 12$ by further use of the equal coefficient rule.

Use in the problem $|\dot{x}|_m = 10^3$ and $|x|_m = 10$.

2. The machine equations may now be written

$$\frac{\dot{x}}{1000} = -\int_0^t \left[100\left(\frac{\dot{x}}{1000}\right) + 100\left(\frac{x}{10}\right) + 100\left(\frac{y}{10}\right) - 40\right] dt + \left(\frac{\dot{x}}{1000}\right)_0 \tag{4.38}$$

$$\frac{x}{10} = -\int_0^t -100\left(\frac{\dot{x}}{1000}\right)dt + \left(\frac{x}{10}\right)_0 \tag{4.39}$$

$$\frac{\dot{y}}{2500} = -\int_0^t \left[100\left(\frac{\dot{y}}{2500}\right) + \frac{9000}{25}\left(\frac{y}{10}\right) + \frac{200}{2{\cdot}5}\left(\frac{\dot{x}}{1000}\right) - \frac{19}{250}\times 10^4\left(\frac{x}{10}\right) - \frac{22\times 10^2}{25}\right]dt + \left(\frac{\dot{y}}{2500}\right)_0 \tag{4.40}$$

$$\frac{y}{10} = -\int_0^t -250\left(\frac{\dot{y}}{2500}\right)dt + \left(\frac{y}{10}\right)_0. \tag{4.41}$$

3. Slowing down by a factor of 100 the modified equations are

$$\frac{\dot{x}}{10} = -\int_0^T \left[\frac{\dot{x}}{10} + \frac{x}{10} + \frac{y}{10} - 0{\cdot}4\right]dT + 0{\cdot}2 \tag{4.42}$$

$$\frac{x}{10} = -\int_0^T -\frac{\dot{x}}{10}\,dT \tag{4.43}$$

$$\frac{\dot{y}}{25} = -\int_0^T \left[\frac{\dot{y}}{25} + 3{\cdot}6\left(\frac{y}{10}\right) + 0{\cdot}8\left(\frac{\dot{x}}{10}\right) - 7{\cdot}6\left(\frac{x}{10}\right) - 0{\cdot}88\right]dT + 0{\cdot}16 \tag{4.44}$$

$$\frac{y}{10} = -\int_0^T -2{\cdot}5\left(\frac{\dot{y}}{25}\right)dT. \tag{4.45}$$

The dots now denote differentiation with respect to T, where $T = 100t$.

4. The computer diagram is

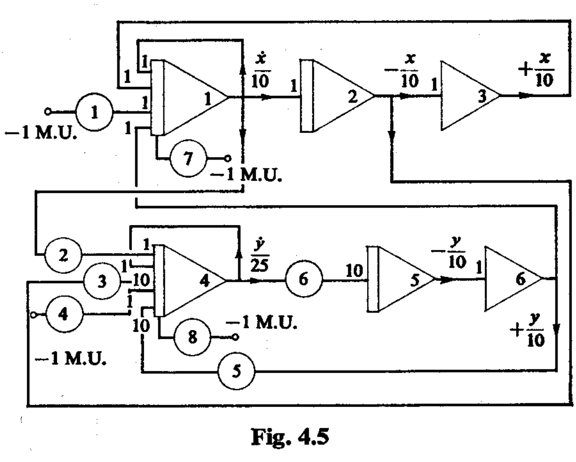


Fig. 4.5

5. The potentiometer settings are

Potentiometer	1	2	3	4
Setting	0·4	0·8	0·76	0·88

Potentiometer	5	6	7	8
Setting	0·36	0·25	0·2	0·16

6. Carry out the check procedure and then obtain the graphs of x against t and y against t.

7. The theoretical solution is

$$x = \sin 200t - \tfrac{1}{2}\cos 200t + \tfrac{1}{2}$$

$$y = 2\sin 200t - 3{\cdot}5\cos 200t + 3{\cdot}5.$$

4.8 Experiment 20. A Suspension Problem

A two-wheeled trailer chassis weighs 960 lbf and its axle 128 lbf. The spring constant between the chassis and axle is 800 lbf/ft and that between the axle and ground (due to the tyres) is 1500 lbf/ft. Examine the effect of various shock-absorber damping coefficients upon the motion of the chassis when the trailer mounts curbs of 2, 4 and 6 in.

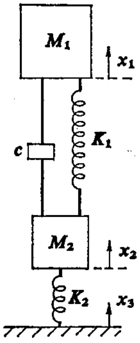


Fig. 4.6

A simplified diagram illustrating the suspension system of one wheel is shown in Fig. 4.6.

$$\begin{aligned} M_1 &= \tfrac{960}{2}\text{ lb} = 480\text{ lbf} \\ M_2 &= \tfrac{128}{2}\text{ lb} = 64\text{ lbf} \\ K_1 &= 800\text{ lbf/ft} \\ K_2 &= 1500\text{ lbf/ft} \\ c &= \text{shock absorber damping coefficient} \\ &\quad\ \text{variable from 10 to 100 lbf s/ft} \\ x_3 &= 0{\cdot}166; 0{\cdot}333; 0{\cdot}500\text{ ft.} \end{aligned}$$

1. Comparing this problem with that in Experiment 13 and using Newton's second law the equations of motion for the two masses are found to be

$$\frac{M_1}{g}\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + K_1(x_1 - x_2) = 0 \tag{4.46}$$

$$\frac{M_2}{g}\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + K_1(x_2 - x_1) + K_2(x_2 - x_3) = 0, \tag{4.47}$$

i.e.

$$15\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + 800(x_1 - x_2) = 0 \qquad (4.48)$$

$$2\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + 800(x_2 - x_1) + 1500(x_2 - x_3) = 0. \qquad (4.49)$$

Initially $x_1 = 0 = x_2$ and $\dot{x}_1 = 0 = \dot{x}_2$.

2. Practical considerations enable a good estimate of the maximum values of x_1 and x_2 to be made in this problem. It will be assumed that the maximum displacements will not exceed twice the value of the input function x_3.

3. Consider Eq. (4.48) without the presence of the terms containing $\dot{x}_2$ and x_2

$$15\ddot{x}_1 + c\dot{x}_1 + 800x_1 = 0. \qquad (4.50)$$

In the case where damping is least $c = 10$, hence

$$15\ddot{x}_1 + 10\dot{x}_1 + 800x_1 = 0 \qquad (4.51)$$

and $\omega_n = \sqrt{\frac{800}{15}} = 10\sqrt{\frac{8}{15}}$.

Then as $|x_1|_m = 1$, $|\dot{x}_1|_m = 10\sqrt{\frac{8}{15}}$. The value $|\dot{x}_1|_m = 5$ ft/s will be used.

Substituting these maximum values in Eq. (4.49)

$$2\ddot{x}_2 + 10\dot{x}_2 - 50 + 800x_2 - 800 + 1500x_2 - 1500x_3 = 0, \qquad (4.52)$$

i.e.

$$2\ddot{x}_2 + 10\dot{x}_2 + 2300x_2 = 850 + 1500x_3 \qquad (4.53)$$

$\omega_n = 10\sqrt{\frac{23}{2}}$ and $|x_2|_m = 1$, $\therefore$ $|\dot{x}_2|_m = 10\sqrt{\frac{23}{2}}$. The value $|\dot{x}_2|_m = 50$ will be used.

4. The scale factors are

for x_1 and x_2, 1
for $\dot{x}_1$, $\frac{1}{5}$
for $\dot{x}_2$, $\frac{1}{50}$.

5. The scaled equations are

$$\frac{\dot{x}_1}{5} = -\int_0^t \left[\frac{c}{15}\left(\frac{\dot{x}_1}{5}\right) - c\frac{2}{3}\left(\frac{\dot{x}_2}{50}\right) + \frac{800}{75}x_1 - \frac{800}{75}x_2\right] dt \qquad (4.54)$$

$$\frac{\dot{x}_2}{50} = -\int_0^t \left[\frac{c}{2}\left(\frac{\dot{x}_2}{50}\right) - \frac{c}{20}\left(\frac{\dot{x}_1}{5}\right) + 23x_2 - 8x_1 - 15x_3\right] dt \qquad (4.55)$$

$$x_1 = -\int_0^t -5\left(\frac{\dot{x}_1}{5}\right) dt \qquad (4.56)$$

$$x_2 = -\int_0^t -50\left(\frac{\dot{x}_2}{50}\right) dt. \qquad (4.57)$$

All initial conditions are zero since the system starts from rest in the equilibrium position.

These equations may be slowed by a factor of 10 to give

$$\frac{\dot{x}_1}{0\cdot5} = -\int_0^T \left[\frac{c}{150}\left(\frac{\dot{x}_1}{0\cdot5}\right) - \frac{c}{15}\left(\frac{\dot{x}_2}{5}\right) + \frac{80}{75}x_1 - \frac{80}{75}x_2\right] dT \quad (4.58)$$

$$\frac{\dot{x}_2}{5} = -\int_0^T \left[\frac{c}{20}\left(\frac{\dot{x}_2}{5}\right) - \frac{c}{200}\left(\frac{\dot{x}_1}{0\cdot5}\right) + 2\cdot3x_2 - 0\cdot8x_1 - 1\cdot5x_3\right] dT \quad (4.59)$$

$$x_1 = -\int_0^T -\frac{1}{2}\left(\frac{\dot{x}_1}{0\cdot5}\right) dT \qquad (4.60)$$

$$x_2 = -\int_0^T -5\left(\frac{\dot{x}_2}{5}\right) dT. \qquad (4.61)$$

6. The computer circuit is

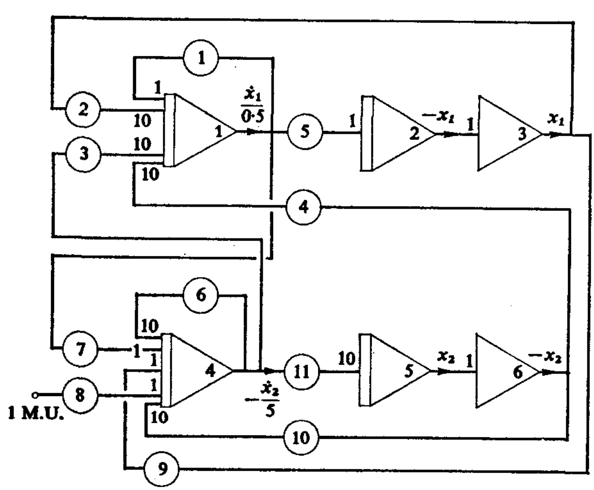


Fig. 4.7

7. The potentiometer settings are

Potentiometer	1	2	3	4	5	6	7	8	9	10	11
Setting	$\frac{c}{150}$	$\frac{8}{75}$	$\frac{c}{150}$	$\frac{8}{75}$	$\frac{1}{2}$	$\frac{c}{200}$	$\frac{c}{200}$	$1{\cdot}5x_3$	$\frac{8}{10}$	0·23	0·5

The initial condition settings are all zero.

8. Carry out the check procedure and for the different values of x_3 examine the effect on the chassis vibration of an increasing damping coefficient c.

4.9 Further Exercises

1. Set up a circuit to solve the equation

$$\ddot{x} + 100\,\dot{x} + 10^6 x = 10^6$$

given that $x = 0 = \dot{x}$ at $t = 0$.

[Approximate solution is $x = 1 - e^{-50t} \sin(10^3 t + 1{\cdot}52)$.]

2. The velocity of a chemical reaction is proportional to the concentration of the reacting substance. Show that if a be the initial concentration of the reagent and x the amount transformed after time t then

$$\frac{dx}{dt} = K(a - x).$$

Set up a suitable circuit to obtain a graph showing the variation of x with t over the first second of the reaction when $K = 5{\cdot}2$ and $a = 90{\cdot}3$.

[Solution: $x = 90{\cdot}3\,(1 - e^{-5\cdot 2}t)$.]

3. Examine the solutions of the equation

$$\ddot{x} + 0{\cdot}06\,\dot{x} + 0{\cdot}011\,x = 0{\cdot}011 \sin \omega t$$

for $\omega =$ (i) 0·08, (ii) 0·09, (iii) 0·1, given that $x = 0 = \dot{x}$ at $t = 0$.
[Solution:

$$x = e^{-0\cdot 03t}[A \cos 0{\cdot}1005t + B \sin 0{\cdot}1005t] + K \sin \omega t + L \cos \omega t$$

where, for

(i) $A = 1{\cdot}195,\quad B = -0{\cdot}5547,\quad K = 1{\cdot}145,\quad L = -1{\cdot}195$
(ii) $A = 1{\cdot}581,\quad B = -0{\cdot}2885,\quad K = 0{\cdot}849,\quad L = -1{\cdot}581$
(iii) $A = 1{\cdot}783,\quad B = +0{\cdot}2364,\quad K = 0{\cdot}297,\quad L = -1{\cdot}783$.]

4. Obtain the graphical solution of the simultaneous equations

$$\ddot{x} + 10^{-2}\dot{x} + 10^{-4}(x - y) = 0{\cdot}5 \times 10^{-4}$$
$$\ddot{y} + 10^{-2}\dot{y} + 10^{-4}(x + y) = 1{\cdot}5 \times 10^{-4}$$

given that $x = -1$, $y = 1{\cdot}5$, $\dot{x} = 10^{-2}$, $\dot{y} = 2 \times 10^{-2}$ at $t = 0$.

$$\left[\begin{array}{l} \text{Solution: } x = \sin 10^{-2}t - 2\cos 10^{-2}t + 1 \\ \qquad\qquad\; y = 2\sin 10^{-2}t + \cos 10^{-2}t + 0{\cdot}5. \end{array}\right]$$

CHAPTER 5

The Use of Non-Linear Units

5.1 Introduction

In the preceding chapters the problems were restricted to those involving linear operations only. Terms such as $y\,dy/dx$ or $\sqrt{y}$ did not occur in the equations to be solved. This limitation may be lifted if certain other non-linear elements are included in the computing equipment. The analogue computer handles non-linear differential equations just as easily as linear ones.

A brief outline of the most important non-linear unit—the multiplier—will be given in this chapter together with some of its major applications. Experiments utilizing these techniques will then be given. For a thorough review of multiplying units the student should consult the computer reference books and the operator's manual for the particular computer at his disposal.

5.2 The Multiplier

A computing potentiometer is the simplest form of multiplier. It will multiply a varying voltage by a constant less than or equal to unity.

To form the product of two varying voltages special units have to be designed. Many forms of these are available but broadly they may be classified as the servo-multiplier type or the electronic multiplier type.

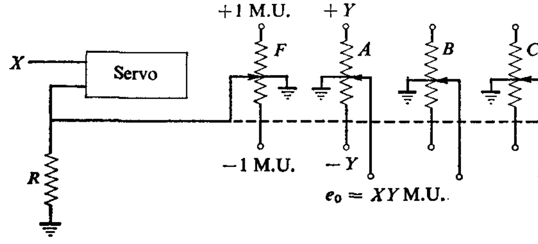


Fig. 5.1

Servo-multipliers are based on the feedback principle (see Fig. 5.1). The multiplier consists of a servo-driven follow-up potentiometer, F, and several multiplier potentiometers (three, A, B, C, have been shown in Fig. 5.1). The wipers of these potentiometers are mechanically ganged together, indicated by the dotted line in the figure.

If an input signal X M.U. causes the feedback potentiometer wiper to rotate θ radians, then

$$\frac{\theta}{\pi} = X. \tag{5.1}$$

Thus if signals $+Y$ and $-Y$ M.U. are applied as shown, the output of the wiper of potentiometer A is

$$e_0 = \frac{\theta Y}{\pi} = XY \text{ M.U.} \tag{5.2}$$

The potentiometers B and C may also be supplied with inputs, say $+U$ and $-U$ and $+V$ and $-V$, to provide further products $+UX$ and $+VX$ respectively. Sign inversions may be obtained by reversing the polarity of the connections of the appropriate multiplying potentiometers.

The static accuracy of the servo-multiplier is largely governed by manufacturing tolerances on the potentiometers, these being significantly small for most applications, but the most severe limitation is the speed of response.

The range of frequencies at which the servo-multiplier will operate accurately is rather low, although if only one of the functions X, Y is of high frequency, this signal may be connected to the function potentiometer A and the slower signal used as input to the servo.

The resistor R is a load compensating resistor, equal in value to the load on the function potentiometer. If R were not present the loading on the potentiometer A caused by the input resistor of the next computing element would cause a small loss in the output voltage from the multiplier. By loading the feedback voltage in an identical way this type of error can be avoided.

Other errors may be due to short-comings in the electrical and mechanical zeroing of the potentiometers F, A, B, C.

Four quadrant multiplication is possible, though if it is not necessary a sign reversing amplifier can be saved by arranging that the servo is driven to the half of the function potentiometer that is supplied with the Y signal.

Electronic multipliers have been designed in many ingenious forms, some of the more familiar types being the Quarter Square, the Time-Division, the Logarithmic, the Photomultiplier, and the Hall Effect. Basically each will give an output $e_0 = -e_1 e_2$ M.U. where e_1 and e_2 are the inputs. Usually the output is developed from a computing amplifier having very low output impedance, while the input impedance varies from one type of multiplier to the next.

The frequency response is much better than in the case of the servo-multiplier and while accuracy was at one time a limiting factor modern electronic techniques are now producing some very accurate multipliers.

5.3 The Dividing Circuit

Division may be performed by inserting a multiplier in the feedback loop of an operational amplifier. Figure 5.2 shows how this is done.

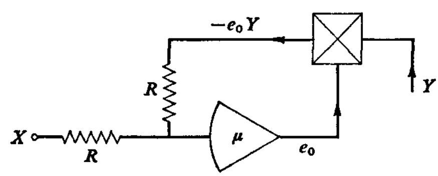


Fig. 5.2

The signal X is fed to an input of a high gain amplifier through a resistor R and another signal Y fed to one input of the multiplier. If the output of the amplifier, e_0, is connected to the second multiplier input the output will be $-e_0 Y$ from Eq. (5.1). This provides the second input to the amplifier through a resistor R.

Assuming the grid of the high gain amplifier to be at earth potential, μ being very large, Kirchhoff's first law gives

$$\frac{X}{R} - \frac{e_0}{R}Y = 0, \tag{5.3}$$

i.e.

$$e_0 = \frac{X}{Y}. \tag{5.4}$$

In this way division of X and Y is achieved. From the last result it is evident that Y should be greater than X, or else voltages greater than one machine unit would result. However, this can be obviated by compensating in the input resistors as in Fig. 5.3.

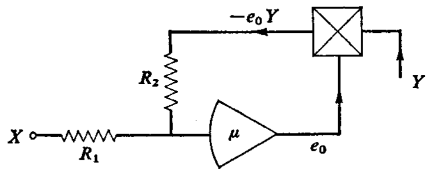


Fig. 5.3

In this case

$$e_0 = \frac{R_2}{R_1}\frac{X}{Y}. \tag{5.5}$$

If X exceeds Y then make R_1 greater than R_2 by the same ratio.

5.4 The Squaring and Square Rooting Circuits

Other applications in which the multiplier may be used are those of squaring and square rooting. For squaring, the multiplier is used with equal inputs X the output being $-X^2$ (see Fig. 5.4).

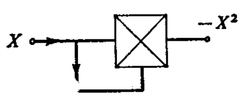


Fig. 5.4

In square rooting the multiplier is used as a squaring device this being placed in the feedback path of an operational amplifier as shown in Fig. 5.5.

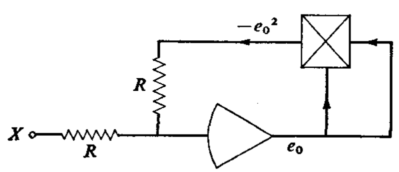


Fig. 5.5

Using a similar argument to that for the division circuit

$$\frac{X}{R} - \frac{e_0^2}{R} = 0, \tag{5.6}$$

i.e.

$$e_0 = \sqrt{X}. \tag{5.7}$$

Squaring devices may be produced other than by using a multiplier. For instance a system of biased diodes may be arranged so as to produce a straight line approximation to a square law. This forms the basis of most squaring units.

The generation of a quotient and of a square root are examples of the implicit function technique. An implicit equation between the input functions and the output function is solved. In the case of division the equation solved is

$$X - e_0 Y = 0 \tag{5.8}$$

and for square-rooting

$$X - e_0^2 = 0. \tag{5.9}$$

More generally the equation

$$F(e_0, Y) + X = 0 \tag{5.10}$$

where F is an arbitrary function covers a wide variety of implicit functions. Figure 5.6 shows the basic circuit for this equation.

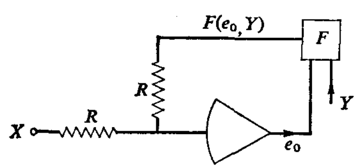


Fig. 5.6

Other forms of non-linear units are available, such as sine and cosine resolvers, which perform these operations on any input voltage; diode limiting units; backlash and hysteresis units; relay comparators; and arbitrary function generators. Although experiments involving the use of these units are beyond the intended scope of this book they nevertheless are extensively employed in many branches of analogue computing.

5.5 Experiment 21. The Use of a Multiplier and Divider

The binomial expansion of $1/(1 + x)$ is $1 - x + x^2 - x^3 + \dots$. Obtain graphs of $1/(1 + x)$ and the first four terms in its expansion, and hence show that errors of more than about 2 per cent of full scale occur when these four terms are used to represent $1/(1 + x)$ when x is greater than about 0·35.

1. The function x can be obtained as a linearly increasing signal by an integrator with a constant input.

To allow a reasonable computing time for recording purposes, suppose that a 10 s run is required. Then x must reach 1 M.U. after 10 s.

Thus let

$$x = \frac{t}{10}$$

where t is problem time.

Two multipliers are used to provide the signals x^2 and x^3. Further sign reversing amplifiers may be needed depending upon the type of multiplier used.

2. The basic computer diagram is shown in Fig. 5.7.

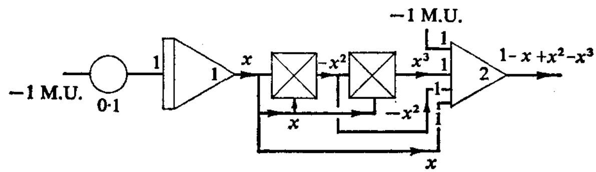


Fig. 5.7

3. Record the output of amplifier 2 for a period of 10 s full-scale calibration. Check the curve for accuracy using suitable values of x in the range $0 \leqq x \leqq 1$.

4. The function $1/(1 + x)$ can be generated using the method of division already discussed, where the divisor is $1 + x$ and the dividend constant equal to unity.

5. The basic computer circuit is shown in Fig. 5.8.

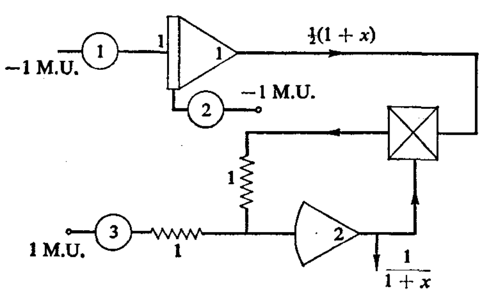


Fig. 5.8

Potentiometer	1	2	3
Setting	0·05	0·5	0·5

In order that the output of amplifier 1 does not exceed 1 M.U. a signal $\frac{1}{2}(1 + x)$ is generated. To compensate for this the numerator is made 0·5 using potentiometer 3.

6. Record the output from amplifier 2 for a period of 10 s on the same scale as the previous graph. Check the curve by calculation using suitable values of x.

7. Compare the two curves and determine the point at which the difference is 2 per cent of full scale.

5.6 Experiment 22. A Further Example on Division

The purpose of this example is to outline a method which may be used when the divisor is zero at time $t = 0$.

Consider the function $y = (\frac{1}{2}\sin t)/t$. Obviously when $t = 0$ overloading might occur in the division circuit. This difficulty is minimized by generating a function of the form Ke^{-at} and adding this to the divisor. When $t = 0$ the divisor is of magnitude K and the exponential function diminishes at a rate depending upon the value of a. It is usual to adjust a so that a voltage is provided until the time divisor is built up. (*Note:* in a time of $4/a$ seconds the exponential output will fall to under 2 per cent of its initial value.) The value of K is made just large enough to prevent overloading.

1. In order to obtain a reasonable length of computing time generate $\frac{1}{20}\sin t$ and $t/10$.

2. The extra term in the divisor (in this case $0{\cdot}01e^{-10t}$) is formed from the circuit of Fig. 5.9.

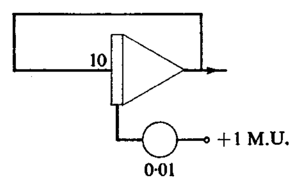


Fig. 5.9

3. The complete circuit is

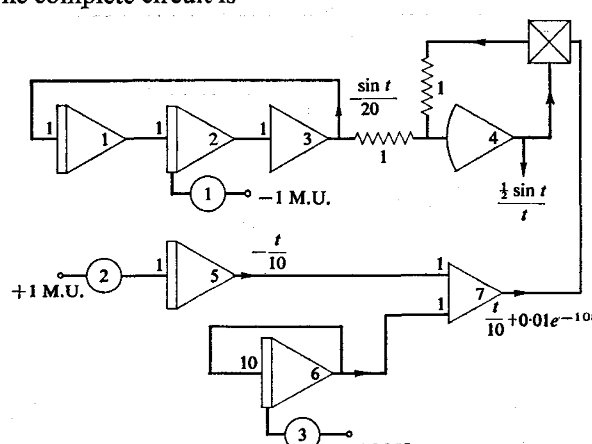


Fig. 5.10

Potentiometer	1	2	3
Setting	0·05	0·1	0·01

5.7 Experiment 23. Use of a Multiplier to Perform the Square-Root Operation

When a heavy cable (W lbf per unit length) is hanging in equilibrium under the action of a horizontal force H lbf as indicated in Fig. 5.11 it can be shown that

$$\frac{d^2y}{dx^2} = \frac{W}{H}\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}. \tag{5.11}$$

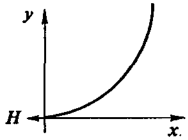


Fig. 5.11

Assuming that the maximum value of y is 20 ft and that of dy/dx is 2 obtain the graphs of y against x for values of

$$\frac{W}{H} = \tfrac{1}{100};\quad \tfrac{1}{200};\quad \tfrac{1}{300}.$$

The values of y and dy/dx when $x = 0$ are both zero.

1. If $x = t$ then the describing equation may be written

$$\ddot{y} = \frac{W}{H}\sqrt{(1 + \dot{y}^2)}. \tag{5.12}$$

2. The scale factors are

for y, $\frac{1}{20}$
for $\dot{y}$, $\frac{1}{2}$.

3. The scaled equations are

$$\frac{\dot{y}}{2} = -\int_0^t -\alpha\sqrt{\left\{\frac{1}{4} + \left(\frac{\dot{y}}{2}\right)^2\right\}}\,dt + \left(\frac{\dot{y}}{2}\right)_0 \tag{5.13}$$

and

$$\frac{y}{20} = -\int_0^t -\frac{1}{10}\left(\frac{\dot{y}}{2}\right)dt + \left(\frac{y}{20}\right)_0 \tag{5.14}$$

where $\alpha = W/H$.

Speeding up by a factor of 10 these now become

$$\frac{\dot{y}}{20} = -\int_0^T -10\alpha\sqrt{\left\{\frac{1}{4} + \left(\frac{\dot{y}}{20}\right)^2\right\}}\,dT + \left(\frac{\dot{y}}{20}\right)_0 \tag{5.15}$$

$$\frac{y}{20} = -\int_0^T -\frac{\dot{y}}{20}\,dT + \left(\frac{y}{20}\right)_0 \tag{5.16}$$

where the dots denote differentiation with respect to T.

4. The computer diagram is

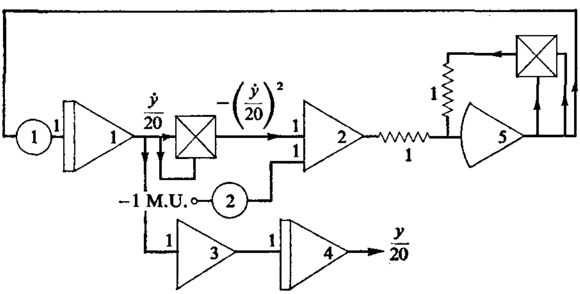


Fig. 5.12

Potentiometer	1	2
Setting	10α	0·25

5. The theoretical solution of Eq. (5.11) is

$$y = \frac{H}{W}\left(\cosh\frac{W}{H}x - 1\right).$$

5.8 Experiment 24. Solution of Mathieus' Equation

This equation describes the behaviour of many physical systems including wave guides, frequency modulation and sinusoidally excited mechanical systems. The equation may be expressed in many different forms one of which is

$$\frac{d^2y}{dt^2} + (a - 2b\cos wt)y = 0. \tag{5.17}$$

The case in which $a = 2b$, $w = 2$ and $0 \leqq a \leqq 5$ provides some interesting studies of stability and instability. In the equation the initial conditions are taken to be $y = 1$ and $\dot{y} = 0$.

1. The equation now becomes

$$\ddot{y} + a(1 - \cos 2t)y = 0 \tag{5.18}$$

from which it can be seen that it is necessary to generate a function $x = 1 - 2\cos 2t$. An accurate method of doing this is by solving a differential equation, thus if

$$x = 1 - \cos 2t \tag{5.19}$$

then

$$\dot{x} = 2\sin 2t \tag{5.20}$$

and

$$\ddot{x} = 4\cos 2t = 4(1 - x). \tag{5.21}$$

As the maximum numerical values of x and $\dot{x}$ are 2 the scale factors are $\frac{1}{2}$ in both cases.

The scaled equations are

$$\frac{\dot{x}}{2} = -\int_0^t \left(4\frac{x}{2} - 2\right) dt + \left(\frac{\dot{x}}{2}\right)_0 \tag{5.22}$$

and

$$\frac{x}{2} = -\int_0^t -\frac{\dot{x}}{2}\, dt + \left(\frac{x}{2}\right)_0. \tag{5.23}$$

2. As the maximum value of x is 2 the maximum values of y and $\dot{y}$ may be estimated from an examination of the equation

$$\ddot{y} + 2ay = 0, \tag{5.24}$$

$y = 1$, $\dot{y} = 0$ at $t = 0$. This has a solution of the form

$$y = \cos \sqrt{2a}t \tag{5.25}$$

whence the maximum value of y is 1 and that of $\dot{y}$ is $\sqrt{10}$. However, as has been stated, some of the solutions of the problem are unstable. A larger maximum value of y is therefore advisable. Let this maximum value be 5. The maximum value of $\dot{y}$ will now be $5\sqrt{10}$. Take this as 20. The scale factors for y and $\dot{y}$ are $\frac{1}{5}$ and $\frac{1}{20}$ respectively.

3. The scaled equations are

$$\frac{\dot{y}}{20} = -\int_0^t \frac{a}{2}\frac{x}{2}\frac{y}{5}\, dt + \left(\frac{\dot{y}}{20}\right)_0 \tag{5.26}$$

$$\frac{y}{5} = -\int_0^t -4\left(\frac{\dot{y}}{20}\right) dt + \left(\frac{y}{5}\right)_0. \tag{5.27}$$

4. The computer diagram is

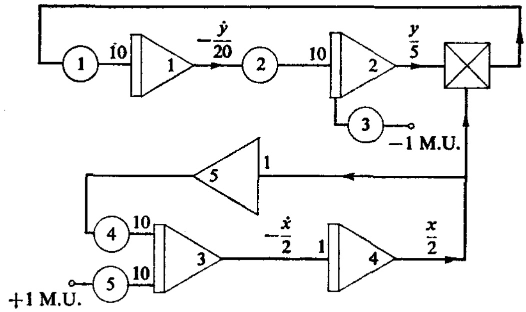


Fig. 5.13

5. The potentiometer settings are

Potentiometer:	1	2	3	4	5
Setting:	$\frac{a}{20}$	0·4	0·2	0·4	0·2

6. Perform the static check and obtain the graphical results which are as shown in Fig. 5.14.

5.9 Further Exercises

1. Water flows through a circular orifice of radius r ft from the side of a cylindrical tank of radius R ft. The velocity of flow is given by $v = \sqrt{(2gh)}$ft/s where h ft is the height of the surface of the water in the tank above the centre of the orifice.

Show that the equation describing the rate at which the surface is falling is

$$\frac{dh}{dt} = -\left(\frac{r}{R}\right)^2 \sqrt{(2gh)}.$$

If $R = 5$ and $h = 100$ initially, show how h varies with t for $r = 0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4$.

$\left[\text{The theoretical solution is } \sqrt{h} = 10 - 4t\left(\frac{r}{R}\right)^2, \text{ taking } g = 32.\right]$

2. Set up a suitable circuit to examine the variation of v with t for the equation $dv/dt + Av^3 - w^2r = 0$ where

(i) $r = \int_0^t v\,dt$,

(ii) $w = 2 \times 10^4$,

(iii) $0 < r < 1$,

(iv) at $t = 0$, $v = 0$ and $r = 0{\cdot}5$,

(v) Max. value of $v = 2 \times 10^4$,

(vi) $0{\cdot}75 \times 10^{-4} < A < 2{\cdot}5 \times 10^{-4}$.

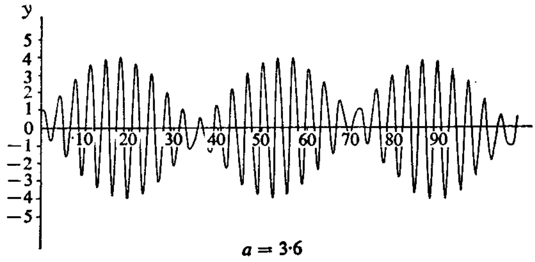


Fig. 5.14 (*a*)

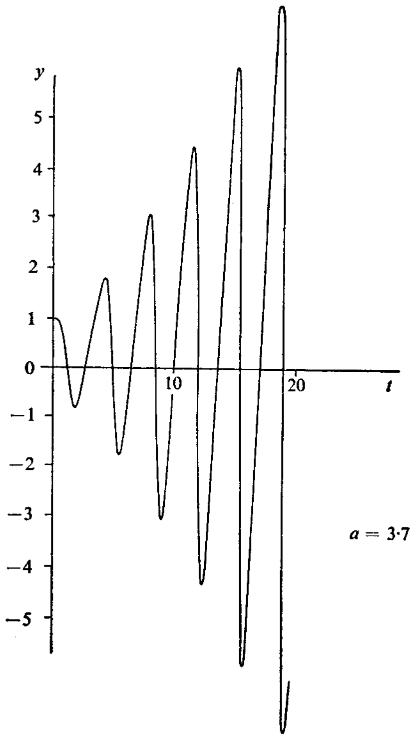


Fig. 5.14 (*b*)

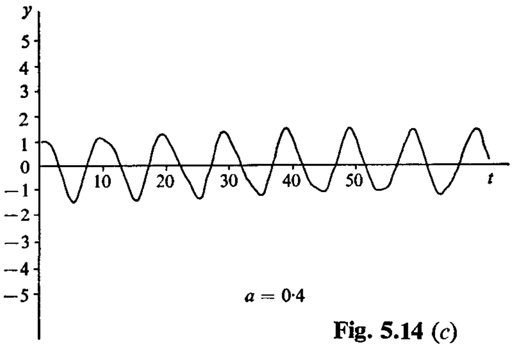


Fig. 5.14 (*c*)

By varying A within the defined limits obtain the type of v–t curves illustrated in Fig. 5.15.

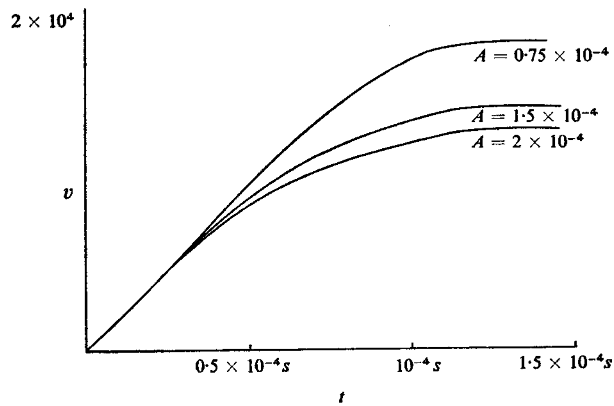


Fig. 5.15

3. An important equation occurring in the theory of feedback oscillators has the form

$$\ddot{y} + \mu(y^2 - 1)\dot{y} + y = 0.$$

This is known as Van der Pol's equation. For small (but arbitrary) initial values of y and $\dot{y}$ the steady state solution is a periodic wave form, varying in character with different values of μ. Assuming the maximum numerical values of y and $\dot{y}$ to be 2 and 4 respectively, show that the machine equations are

$$\frac{\dot{y}}{4} = -\int_0^t \left\{4\mu\left(\frac{\dot{y}}{4}\right)\left[\left(\frac{y}{2}\right)^2 - \frac{1}{4}\right] + \frac{1}{2}\frac{y}{2}\right\} dt + \left(\frac{\dot{y}}{4}\right)_0$$

$$\frac{y}{2} = -\int_0^t -2\frac{\dot{y}}{4}\, dt + \left(\frac{y}{2}\right)_0.$$

Obtain graphs of y against t and $\dot{y}$ against y [an X–Y recorder is essential in this case] for values of μ equal to 0, 0·4, 2. In these experiments use $(\dot{y})_0 = 0$ and $(y)_0 = 0{\cdot}02$. In the case of the phase plane plot (i.e. $\dot{y}$ against y) note that the solution is always limited by a definite bounding curve. This is known as the Limit Cycle and

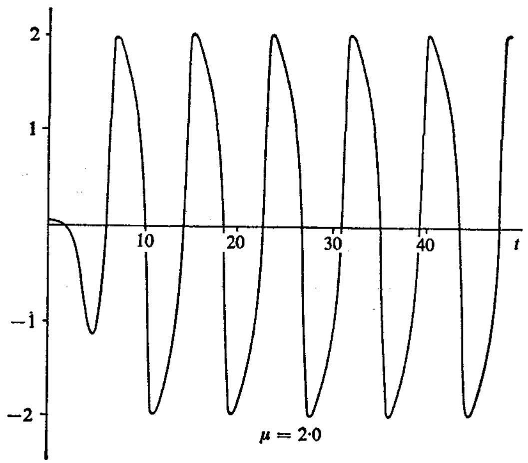


Fig. 5. 16 (*a*)

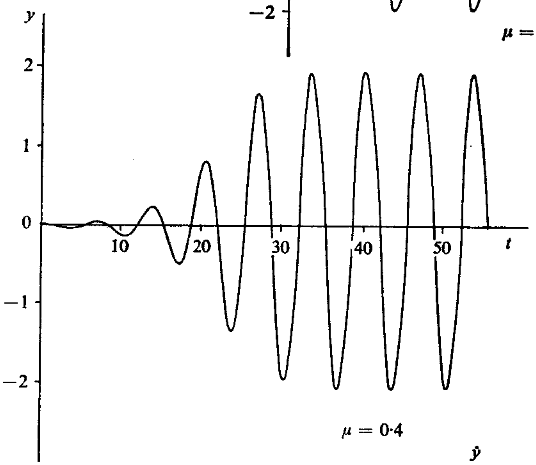


Fig. 5.16 (*b*)

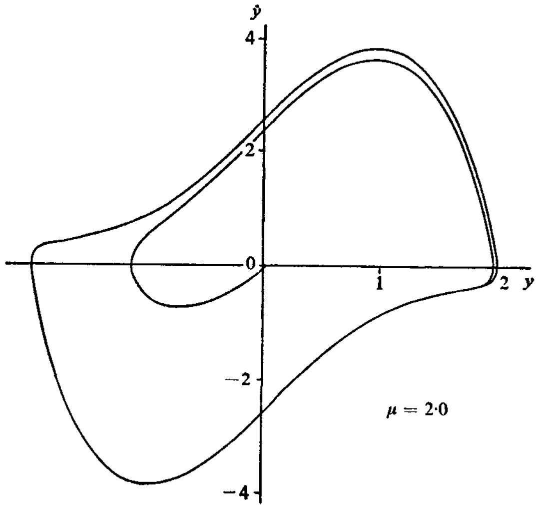


Fig. 5.16 (*c*)

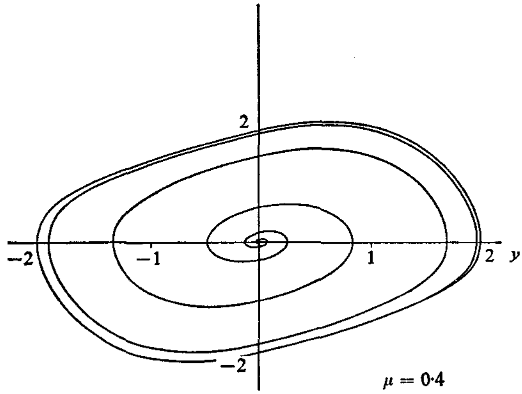


Fig. 5.16 (*d*)

is of great importance in the examination of the stability of oscillatory circuits. For a given value of μ the same limit cycle is obtained irrespective of the initial condition values. Typical results are as shown in Fig. 5.16.

CHAPTER 6

The Use of Transfer Functions

6.1 Transfer Functions

In the study of equations representing physical systems, it is often convenient to consider the relations among the input and output variables of the system. For linear systems these relations are expressed as functions known as Transfer Functions.

Consider, for instance, a stable system governed by the equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t) \tag{6.1}$$

where b/a is positive, and assume all initial conditions to be zero.

Taking Laplace transforms of both sides of Eq. (6.1) produces the algebraic equation

$$(as^2 + bs + c)\bar{x}(s) = \bar{f}(s) \tag{6.2}$$

where s is the Laplace transform variable, in general complex, and $\bar{x}(s)$ and $\bar{f}(s)$ are the Laplace transforms of $x(t)$ and $f(t)$ respectively.

Students unfamiliar with the technique of the Laplace transformation are referred to any standard mathematical text on this subject. An adequate account may be found in Chapter 1 of *Analog Computation* by A. S. Jackson.

The function

$$as^2 + bs + c$$

is called the *characteristic function*, and the equation

$$as^2 + bs + c = 0 \tag{6.3}$$

the characteristic equation (corresponding to the homogeneous case of Eq. (6.1)). The roots of Eq. (6.3), which may be complex, determine the form of the complementary functions in the solution of Eq. (6.1). For example, if the two roots are $s = s_1$ and $s = s_2$ the complementary function is of the form $Ae^{s_1t} + Be^{s_2t}$, A and B being constants.

Since Eq. (6.3) is algebraic in s it yields the ratio

$$\frac{\bar{x}(s)}{\bar{f}(s)} = \frac{1}{as^2 + bs + c}. \tag{6.4}$$

The function on the right-hand side is the transfer function relating $\bar{x}(s)$ and $\bar{f}(s)$ and is the ratio of the Laplace transform of the output of the system divided by the Laplace transform of the input (or driving function). Throughout this chapter, $Y(s)$ will be used to denote a transfer function. In the case of Eq. (6.4),

$$Y(s) = \frac{\bar{x}(s)}{\bar{f}(s)}. \tag{6.5}$$

Thus Eq. (6.1) may be represented as shown in Fig. 6.1 by a block corresponding to the transfer function, with $\bar{f}(s)$ as input and $\bar{x}(s)$ as output.

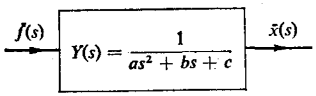


Fig. 6.1

It is to be noted that a transfer function gives no account of the initial conditions of an equation; these were assumed to be zero in the formulation of the transfer function.

When the driving function $f(t)$ is sinusoidal, the particular integral of Eq. (6.1) is also found to be sinusoidal, having the same frequency as $f(t)$ but differing in phase and amplitude from it. If, for instance,

$$f(t) = C \sin \omega t \tag{6.6}$$

then the particular integral takes the form

$$x(t) = D \sin (\omega t + \phi) \tag{6.7}$$

where ϕ is the phase difference between $f(t)$ and $x(t)$ and C and D are constants. Other forms of input may usually be expressed as a Fourier series of sine and cosine terms. Hence it is important to study the behaviour of a system when it is subjected to sinusoidal inputs.

In the solution of an equation representing a stable system the particular integral is essentially the only part left after a large time t has elapsed; for this reason it is termed the *steady state solution.*

One way of measuring the performance of a system is to examine the steady state solution of the describing equation for driving sinusoids covering a whole range of frequencies. In other words the dependence of D and ϕ on ω is observed. This is known as the sinusoidal steady state frequency response, or more usually, the frequency response.

Mathematical convenience is achieved by using a complex driving function of the form

$$f(t) = Ce^{j\omega t} \tag{6.8}$$

$$= C(\cos \omega t + j \sin \omega t), \tag{6.9}$$

where $j = \sqrt{-1}$.
The imaginary part of this function is $C \sin \omega t$ and the resulting complex steady state solution is $He^{j\omega t}$, where H is complex and

$$He^{j\omega t} = |H|\{\cos(\omega t + \arg H) + j \sin(\omega t + \arg H)\}. \tag{6.10}$$

Comparison of the imaginary part of this solution with the right-hand side of Eq. (6.7) gives

$$D = |H| \tag{6.11}$$

and

$$\phi = \arg H. \tag{6.12}$$

Consequently, the variation of D and ϕ with ω can be established from the complex steady state solution.

If the Laplace transform solution of Eq. (6.1) is pursued with $f(t) = Ce^{j\omega t}$, the resulting steady state solution, $x_{ss}(t)$, is found to be

$$x_{ss}(t) = \frac{Ce^{j\omega t}}{a(j\omega)^2 + bj\omega + c}. \tag{6.13}$$

Thus

$$\frac{x_{ss}(t)}{Ce^{j\omega t}} = \frac{1}{a(j\omega)^2 + bj\omega + c}. \tag{6.14}$$

Equation (6.14) determines how the ratio

$$\frac{\text{steady state solution}}{\text{driving function}}$$

varies with frequency. Clearly, this may be obtained by replacing s by $j\omega$ in the transfer function $Y(s)$, giving

$$Y(j\omega) = \frac{1}{a(j\omega)^2 + bj\omega + c}. \tag{6.15}$$

This is called the *frequency-domain transfer function.* This may be written in the exponential form

$$Y(j\omega) = G(\omega)e^{j\phi(\omega)} \tag{6.16}$$

where

$$G(\omega) = |Y(j\omega)|$$
$$= \frac{\text{magnitude of sinusoidal output}}{\text{magnitude of sinusoidal input}} \tag{6.17}$$

and

$$\phi(\omega) = \arg Y(j\omega)$$
$$= (\text{phase angle of output}) - (\text{phase angle of input}). \tag{6.18}$$

$G(\omega)$ and $\phi(\omega)$ are the *gain and phase characteristics* respectively of the transfer function. Consideration of these at various angular frequencies ω will determine the frequency response.

Although the example quoted above deals with a second order differential equation, transfer functions are obtained in the same way for linear differential equations of any order and the definitions of $G(\omega)$ and $\phi(\omega)$ given in Eqs. (6.17) and (6.18) remain valid. In each case the transfer function $Y(s)$ is the ratio $\bar{x}(s)/\bar{f}(s)$.

6.2 Simple Block Diagrams

As mentioned earlier, a system governed by a differential equation may be represented diagrammatically as in Fig. 6.1 by a block, or black box, consisting of the transfer function $Y(s)$ with input $\bar{f}(s)$ and output $\bar{x}(s)$, the transforms of the input (driving) function and the output function (solution) respectively. More complex systems are usually divided into several distinct sections, individually described by a differential equation of reasonably low order, and each represented by a transfer function. The product of the individual transfer functions is the overall transfer function. For instance, if two transfer functions, $Y_1(s)$ and $Y_2(s)$, are linked to their corresponding input and output transforms by the equations

$$\bar{x}_1(s) = Y_1(s).\bar{f}_1(s) \tag{6.19}$$

and

$$\bar{x}_2(s) = Y_2(s).\bar{x}_1(s) \tag{6.20}$$

then the resulting equation relating the input transform $\bar{f}_1(s)$ and the output transform $\bar{x}_2(s)$ is

$$\bar{x}_2(s) = Y_1(s)Y_2(s)\bar{f}_1(s). \tag{6.21}$$

Hence

$$\frac{\bar{x}_2(s)}{\bar{f}_1(s)} = Y(s) = Y_1(s)Y_2(s). \tag{6.22}$$

Any number of transfer functions may be multiplied in this way. Schematically the black boxes representing the various transfer functions are connected in cascade as in Fig. 6.2. For linear systems,

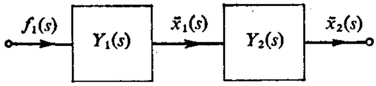


Fig. 6.2

the order in which $Y_1(s)$ and $Y_2(s)$ occur is not important so long as the signal represented by $\bar{x}_1(s)$ is not required. In connecting blocks in this way it is important that one block draws no energy from the preceding one. As far as actual computer circuits are concerned, this implies that electrical loading does not occur. In practice the loading is usually kept down to negligible proportions.

In systems which employ feedback, the output may be passed through a further block to an error sensing device—usually the input grid of an operational amplifier and compared with the input function. Figure 6.3 shows the block diagram for a simple feedback system.

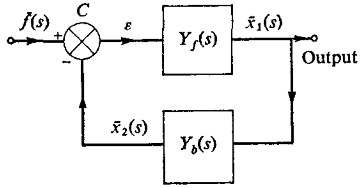


Fig. 6.3

$Y_f(s)$, $Y_b(s)$ are the transfer functions of the forward path and feedback path respectively, and the error device C algebraically sums the inputs to it with the signs shown; that is

$$\varepsilon = \bar{f}(s) - \bar{x}_2(s). \tag{6.23}$$

But

$$\bar{x}_2(s) = Y_b(s)\bar{x}_1(s) \tag{6.24}$$

and

$$\bar{x}_1(s) = Y_f(s)\varepsilon. \tag{6.25}$$

This gives the overall output/input relation

$$\frac{\bar{x}_1(s)}{\bar{f}(s)} = \frac{Y_f(s)}{1 + Y_f(s)Y_b(s)}. \tag{6.26}$$

$Y_f(s)$, $Y_f(s)/[1 + Y_f(s)Y_b(s)]$ are termed the *open loop and closed loop* transfer functions respectively between $\bar{x}_1(s)$ and $\bar{f}(s)$.

6.3 The Use of *RC* Networks to Simulate Transfer Functions

Many methods are available for setting up transfer functions on a computer. One of these is the use of complex *RC* networks as the input and feedback impedances of an operational amplifier.

Let Z_f, Z_i represent the complex impedances in the feedback and input paths of Fig. 6.4. As before, assuming an earth potential

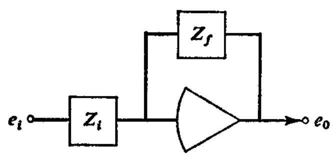


Fig. 6.4

exists at the amplifier grid, it may be shown that

$$\frac{\bar{e}_0(s)}{\bar{e}_i(s)} = -\frac{Z_f(s)}{Z_i(s)} \tag{6.27}$$

where $\bar{e}_0(s)$ and $\bar{e}_i(s)$ are the Laplace transforms of the output and input voltages respectively.

Thus

$$-\frac{Z_f(s)}{Z_i(s)} \equiv Y(s). \tag{6.28}$$

The impedances of a resistance R and capacitance C are R and $1/Cs$ respectively. Hence, for the circuit of Fig. 6.5

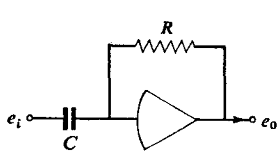


Fig. 6.5

$$Y(s) = \frac{\bar{e}_0(s)}{\bar{e}_i(s)} = -\frac{R}{1/Cs}$$

$$= -RCs \tag{6.29}$$

or

$$\bar{e}_0(s) = -RCs\bar{e}_i(s). \tag{6.30}$$

On taking inverse transforms

$$e_0(t) = -RC\frac{de_i}{dt}. \tag{6.31}$$

The transfer function is $-RCs$ and the operation performed is pure differentiation with gain $-RC$. This result may be arrived at using the elementary method given in Chapter 1 for the integrator.

The differentiating circuit, as it is shown in Fig. 6.5, is not of very great practical use, since, if an unwanted noise signal of high frequency content is present along with the true signal, the output of the differentiator inherits this noise multiplied in magnitude by its angular frequency. The resulting pen recorder trace may exhibit a large amplitude jitter superimposed on the correct solution. In addition, amplifiers connected in this way are liable to run into saturation and instability.

Methods of minimizing these difficulties are based on the filtering off of the high frequency components of the signal and lead to approximate forms of differentiation. Details of this can be found in Experiment 25.

Another commonly occurring circuit is that of Fig. 6.6. This

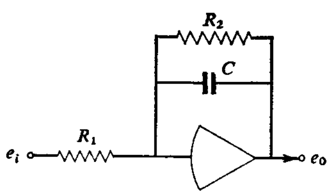


Fig. 6.6

represents the simple lag transfer function

$$Y(s) = -\frac{G}{1 + Ts} \tag{6.32}$$

where $G = R_2/R_1$ and $T = R_2C$.

The result is easily obtained using Eq. (6.27)

$$Z_f = 1\bigg/\left(\frac{1}{R_2} + Cs\right) = \frac{R_2}{1 + R_2Cs}$$

$$Z_i = R_1$$

$$\therefore \quad Y(s) = -\frac{R_2/R_1}{1 + R_2Cs}. \tag{6.33}$$

T is called the lag time constant in seconds, and the transfer function represents the output/input relation between two variables e_0, e_i satisfying the first order differential equation

$$T\frac{de_0}{dt} + e_0 = -Ge_i. \tag{6.34}$$

Other simple transfer functions and their circuits are shown in Fig. 6.7.

6.4 Checking the Accuracy of Transfer Functions

The use of transfer functions in analogue computation involves just as much caıeful checking as do the normal methods for solving differential equations. These checks may be carried out by ascertaining that the gain and phase curves for each transfer function block are correct.

Measurements are taken of $G(\omega)$ and $\phi(\omega)$ for sinusoidal inputs of known character over a wide range of frequency, and the results compared graphically with the calculated values. The two should agree to within the degree of accuracy involved in the computing and measuring equipment.

It is customary to plot G in decibels (i.e. $20 \log_{10} G$) and ϕ in degrees on linear scales against angular frequency ω on a logarithmic scale. The plots are termed Bode Plots.

Another method of presenting the same information is to plot G against ϕ in polar form—called a Nyquist Plot—in which case margins of stability in gain and phase are more easily abstracted.

Other methods of testing transfer functions include the transient response and impulse response techniques.

In the former a step function is applied to the input of the transfer function and the output/time graph recorded and compared against the calculated, while in the latter the input is an impulse function. Essentially these two techniques are used to analyse the system

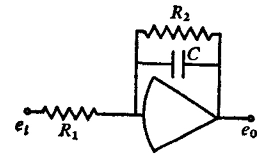
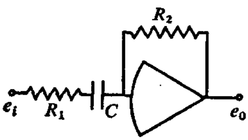
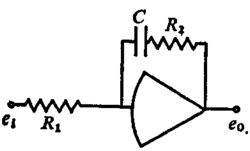
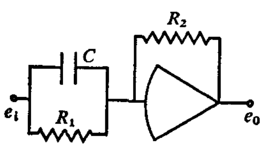
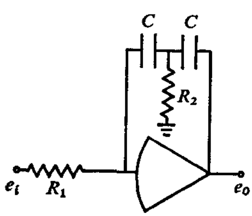
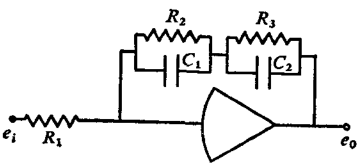
Circuit	$Y(s)$	*Relations*
	$-\dfrac{G}{1+Ts}$	$G=\dfrac{R_2}{R_1}$ $T=R_2C$
	$-\dfrac{Gs}{1+Ts}$	$G=R_2C$ $T=R_1C$
	$-\dfrac{G(1+Ts)}{s}$	$G=\dfrac{1}{R_1C}$ $T=R_2C$
	$-G(1+Ts)$	$G=\dfrac{R_2}{R_1}$ $T=R_1C$
	$-\dfrac{G(1+Ts)}{s^2}$	$G=\dfrac{1}{C^2R_1R_2}$ $T=2R_2C$
	$-\dfrac{G(1+T_1s)}{(1+T_2s)(1+T_3s)}$	$G=\dfrac{(R_2+R_3)}{R_1}$ $T_1=\dfrac{R_2R_3(C_1+C_2)}{R_2+R_3}$ $T_2=R_2C_1$ $T_3=R_3C_2$

Fig. 6.7

under test in the time domain, while the Bode and Nyquist plots are contained in the complex frequency domain. A correspondence may be set up between the frequency response and the impulse response and vice versa so that information on the system in one domain can be transformed into information in the other. Discussion of this is beyond the scope of the present volume.

The ideas put forward so far are now enlarged in the following set of experiments.

6.5 Experiment 25. The Differentiator

Use the three differentiating circuits of Fig. 6.8 to produce the derivative of the function $e^{-t/2}$.

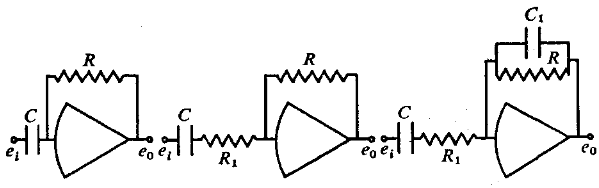


Fig. 6.8

The three types of differentiator shown consist of pure differentiation (*a*), and two approximate circuits using *RC* networks (*b*) and (*c*).

1. The circuit to produce the function $e^{-t/2}$ is obtained as in

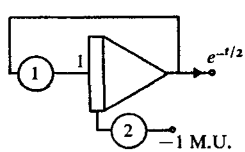


Fig. 6.9

Fig. 6.9. The function required is regarded as the solution of the differential equation

$$\frac{dz}{dt} + \tfrac{1}{2}z = 0, \tag{6.35}$$

with $z = 1$ at $t = 0$.

Potentiometer	1	2
Setting	0·5	1·0

2. In Fig. 6.8(*a*) the operation of the circuit is

$$e_0 = -RC\frac{de_i}{dt} \tag{6.36}$$

and its transfer function,

$$Y(s) = -RCs. \tag{6.37}$$

From Eq. (6.36) it will be seen that on replacing s by $j\omega$,

$$\begin{aligned} G(\omega) &= |Y(j\omega)| \\ &= RC\omega \end{aligned} \tag{6.38}$$

and

$$\begin{aligned} \phi(\omega) &= \arg Y(j\omega) \\ &= -90^\circ. \end{aligned} \tag{6.39}$$

The gain M in decibels (db) is defined as

$$M(\omega) = 20 \log_{10} G(\omega). \tag{6.40}$$

Substituting for G and evaluating M at ω and 2ω gives

$$\begin{aligned} M(2\omega) - M(\omega) &= 20 \log_{10} 2\omega - 20 \log_{10} \omega \\ &= 20 \log_{10} 2 \\ &= 6 \text{ db approximately.} \end{aligned}$$

This is true for all angular frequency ω, and hence the gain-frequency plot for the differentiator increases at 6 db per octave. For example if $RC = 1$ and $\omega = 1$ then $M = 20 \log 1 = 0$ db, and when $\omega = 2$, $M = 6$ db.

The phase characteristics shows a constant phase difference, the output lagging the input by 90 degrees.

Unwanted noise signals are always present in analogue computers and they may well have a high frequency content. Because of the linear dependence of gain on frequency these signals will be amplified by a factor ω when they emerge from a pure differentiator. Thus while their amplitude forms an insignificant part of the true signal on the input side this may not be so at the output. The higher the frequency of the noise signal the larger will be the magnitude of the undesired signal, thereby causing distortion of the true output.

Another difficulty encountered with this ideal circuit is instability caused by using too high a capacitance in the input arm. Especially with computers using plug-in components, the operator should check that the value of C is kept to within the value stated in the manufacturer's rating of the d.c. amplifier.

Choose $RC = 1$ and perform the operation $y_1 = \frac{d}{dt}(e^{-t/2})$ and record y_1.

3. In Fig. 6.8(*b*) the transfer function is

$$Y(s) = -\frac{RCs}{1 + R_1Cs}. \tag{6.41}$$

At low frequencies this approximates to $-RCs$ signifying perfect differentiation (see §6.3 Eqs. (6.30) and (6.31)) while at high frequencies it becomes R/R_1, a constant gain factor.

Choose (i) $RC = 1$, (ii) $R_1C = 0{\cdot}01$.

Basically the time constant R_1C must be chosen to be small compared with the other time constants occurring in the simulation. In this case $R_1C \ll RC$. Usually the time constant is optimized so as to reduce the spurious noise level to a minimum while not interfering significantly with the true differentiation at low frequencies.

A disadvantage of this method is the constant gain at high frequencies.

Perform the operation $y_2 = \frac{-s}{1 + 0{\cdot}01s}(e^{-t/2})$ and record y_2.

4. In Fig. 6.8(*c*) the transfer function is

$$Y(s) = \frac{-RCs}{(1 + R_1Cs)(1 + RC_1s)}.$$

This circuit exhibits the advantages of (*b*) and also a diminishing response at high frequencies when

$$Y(s) = \frac{-1}{R_1C_1s}.$$

The gain–frequency characteristics of the three circuits are shown in Fig. 6.10 for comparison, the values of the constants being taken as

$$RC = 1, \quad R_1C = RC_1 = 0{\cdot}01.$$

The actual values of R, R_1, C and C_1 will depend on whether the computer used is of the fixed component or plug-in type. Care must be taken to ensure that any capacitor placed in series with the input of an amplifier is kept to a minimum, or instability may result.

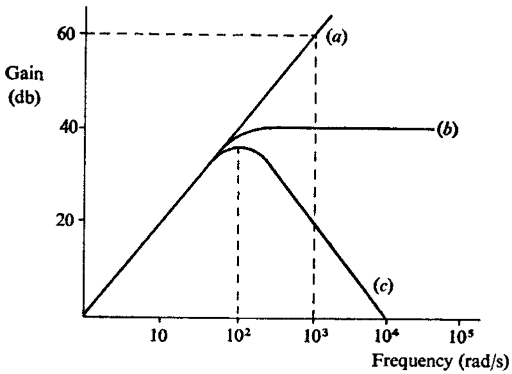


Fig. 6.10

6.6 Experiment 26. The Simple Lag Transfer Function

Examine the frequency response of the transfer function $1/(1 + Ts)$ where $T = 0{\cdot}1$ s.

1. The circuit to produce a transfer function $1/(1 + 0{\cdot}1s)$ is shown in Fig. 6.11. With this arrangement,

$$Y(s) = \frac{R}{R_1} \cdot \frac{1}{(1 + RCs)}. \tag{6.42}$$

Choose R, R_1, C so that $R/R_1 = 1$ and $RC = 0{\cdot}1$.

2. *Calculation of Frequency Response.* The steady state response

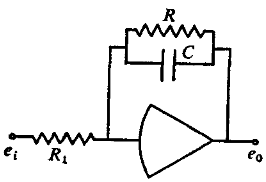


Fig. 6.11

of the transfer function to sinusoidal inputs, that is, the frequency response, consists of the graphs of gain and phase plotted against frequency (the Bode plot).

If $e_i = A \sin \omega t$ then

$$G(\omega) = \left|\frac{\bar{e}_0(j\omega)}{\bar{e}_i(j\omega)}\right|$$

$$= \left| \frac{1}{1 + 0{\cdot}1j\omega} \right| = \frac{1}{\sqrt{(1 + 0{\cdot}01\omega^2)}} \tag{6.43}$$

and the gain in decibels is therefore

$$M = -10 \log_{10} (1 + 0{\cdot}01\omega^2). \tag{6.44}$$

The phase angle

$$\phi(\omega) = \arg\left(\frac{1}{1 + 0{\cdot}1j\omega}\right)$$

$$= -\tan^{-1} 0{\cdot}1\omega. \tag{6.45}$$

For low frequencies $0{\cdot}01\omega^2$ can be neglected compared with unity and then $G = 1$ and $M = 0$ db. For high frequencies $0{\cdot}01\omega^2$ is large compared with unity and then $G = 10/\omega$ and $M = 20 - 20 \log_{10} \omega$ db approximately. The latter gives a straight line graph if ω is plotted on a logarithmic scale passing through the point $M = 20$, $\omega = 1$ and has a slope of about -6 db per octave.

Calculate M and ϕ over the range $\omega = 1$–500 rad/s and plot M against ω on graph paper using a linear scale for M and a logarithmic scale for ω. A useful spacing of points is obtained by evaluating M at $\omega = 1, 3, 7, 10, 30, 70, 100, 300$ and 500 rad/s.

The lines $M = 0$ and $M = 20 - 20 \log_{10} \omega$ are asymptotes to the graph. Their point of intersection is called the *break point* and the frequency at which this occurs, ω_c, the *break frequency*. In general this occurs when $\omega = 1/T$. For the case in question, $\omega_c = 10$ rad/s.

Simple calculation will show that the true curve is 3 db below the break point at the break frequency and that, when $\omega = \frac{1}{2}\omega_c = 5$ rad/s, $M = -1$ db approx., and, when $\omega = 2\omega_c = 20$ rad/s, $M = -7$ db approx. These values together with the two asymptotes allow for quick drawing of the Bode plot for any simple lag.

No such simple process is available for the phase diagram which is plotted linearly in degrees against ω on the same logarithmic scale as in the gain plot. It will however be noticed from Eq. (6.45) that $\phi = 0$ for low frequencies, is $-90°$ for high frequencies and is $-45°$ at the break frequency (see Fig. 6.12).

3. *Measurement of Frequency Response.* Apply sinusoidal inputs to the lag network over the same frequency range as given above and record the output using a transfer function analyser.

This unit provides the sinusoidal signal, called the reference signal,

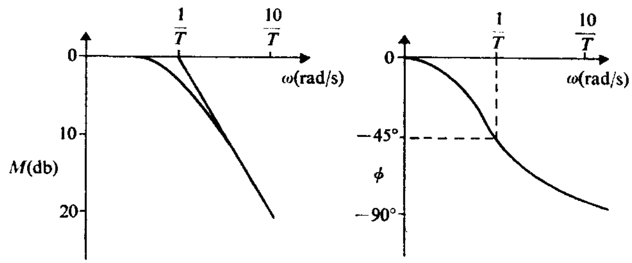


Fig. 6.12

and can measure the 'in phase' and 'quadrature' components p and q respectively, of the output of the network under test. Tabulate p and q against ω.

If a is the amplitude of the reference signal (i.e. $e_i = a \sin \omega t$) then,

$$G = \frac{\sqrt{(p^2 + q^2)}}{a} \tag{6.46}$$

and

$$\phi = \tan^{-1}\left(\frac{q}{p}\right) \tag{6.47}$$

from which the Bode plot may be constructed.

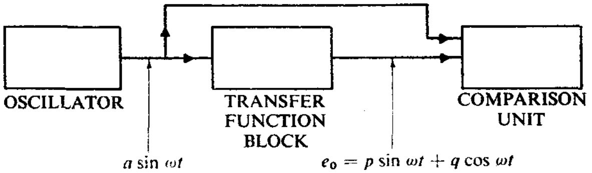


Fig. 6.13

4. Compare the results for the calculated and measured responses. They should agree closely.

5. If a transfer function analyser is not available, students may use a low frequency oscillator to provide the sinusoidal excitation or indeed an oscillatory circuit using computer components (see Experiment 9). The adjacent channels on a pen-recorder provide a means of recording and comparing in amplitude and phase the input and output signals, a separate computer run being required for each value of ω.

6.7 Experiment 27. The Quadratic Transfer Function

Set up a circuit to produce the transfer function $20/(20 + 4{\cdot}2s + s^2)$ and produce calculated and measured Bode plots.

1. The quadratic transfer function occurs frequently in the simulation of linear systems, e.g. the approximate response of a rate gyro. Its general form is

$$Y(s) = \frac{\omega_n^2}{\omega_n^2 + 2\zeta\omega_n s + s^2}. \tag{6.48}$$

This describes how e_0/e_i varies with frequency when e_0 and e_i are defined by the equation

$$\frac{d^2e_0}{dt^2} + 2\zeta\omega_n \frac{de_0}{dt} + \omega_n^2 e_0 = \omega_n^2 e_i \tag{6.49}$$

with $e_0 = 0 = de_0/dt$ at $t = 0$.

2. The gain of the transfer function (6.48) in decibels is

$$\begin{aligned} M(\omega) &= 20 \log_{10} \left|\frac{\bar{e}_0}{\bar{e}_i}\right| \\ &= 20 \log_{10} \left[\frac{\omega_n^2}{\sqrt{[(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2]}}\right] \end{aligned} \tag{6.50}$$

and the phase angle

$$\phi(\omega) = -\tan^{-1} \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}. \tag{6.51}$$

Note that for $\omega \ll \omega_n$ Eqs. (6.50) and (6.51) reduce to

$$M(\omega) = 0 \tag{6.52}$$

and

$$\phi(\omega) = -\tan^{-1}\left(\frac{2\zeta\omega}{\omega_n}\right) = 0. \tag{6.53}$$

When $\omega \gg \omega_n$

$$M(\omega) = 40 \log_{10}\left(\frac{\omega_n}{\omega}\right) \tag{6.54}$$

and

$$\begin{aligned} \phi(\omega) &= -\tan^{-1}\left(\frac{2\zeta\omega_n}{-\omega}\right) \\ &= -180^\circ \text{ approximately.} \end{aligned} \tag{6.55}$$

Relations (6.52) and (6.54) give the low frequency and high frequency asymptotes to the M–ω curve, the latter being a line decreasing at 12 db per octave, while Eqs. (6.53) and (6.55) provide the initial and terminal values of ϕ.

For values of ω near ω_n, the gain curve exhibits a resonance peak when the value of ζ is small. The magnitude of this peak tends to infinity at $\omega = \omega_n$ when ζ tends to zero. In addition, it is seen from Eq. (6.51) that the value of ϕ at $\omega = \omega_n$ is $-90°$ except when $\zeta = 0$.

The foregoing discussion enables some of the characteristics of the Bode plot to be quickly obtained and used at least to check the computed curves.

In the present example $\omega_n^2 = 20$ and $2\zeta\omega_n = 4{\cdot}2$ giving $\omega_n = 4{\cdot}47$ rad/s and $\zeta = 0{\cdot}47$. Put these values into relations (6.50) and (6.51) and calculate M and ϕ against ω over the range $\omega = 1$–500 rad/s. Plot the curves on linear–log graph paper.

3. By means of the method described in the previous experiment determine the Bode plot experimentally. The circuit of Fig. 6.14 may be used for this purpose.

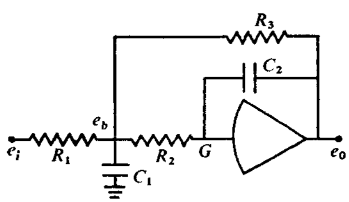


Fig. 6.14

This circuit forms a good example of the use of complex impedances in nodal analysis. Using Kirchhoff's first law and assuming an earth potential at the amplifier grid G,

$$\frac{\bar{e}_i - \bar{e}_b}{R_1} + \frac{\bar{e}_0 - \bar{e}_b}{R_3} - \frac{\bar{e}_b}{R_2} - C_1 s\bar{e}_b = 0 \tag{6.56}$$

and

$$\frac{\bar{e}_b}{R_2} + C_2 s\bar{e}_0 = 0 \tag{6.57}$$

where $\bar{e}_0$, $\bar{e}_i$, $\bar{e}_b$ are the Laplace transforms of the voltages e_0, e_i and e_b. Solving for $\bar{e}_0$ in terms of $\bar{e}_i$ gives

$$\frac{\bar{e}_0(s)}{\bar{e}_i(s)} = -\frac{R_3}{R_1}\,\frac{1}{1 + \left(R_2 + R_3 + \dfrac{R_2R_3}{R_1}\right)C_2 s + C_1C_2R_2R_3 s^2}. \tag{6.58}$$

If for convenience $R_3 = R_1$ then

$$\frac{\bar{e}_0(s)}{\bar{e}_i(s)} = -\frac{1}{1 + (R_1 + 2R_2)C_2 s + C_1 C_2 R_1 R_2 s^2}. \tag{6.59}$$

This is to be compared with $20/(20 + 4{\cdot}2s + s^2)$ giving the relations

$$(R_1 + 2R_2)C_2 = 0{\cdot}21 \tag{6.60}$$

and

$$C_1 C_2 R_1 R_2 = 0{\cdot}05. \tag{6.61}$$

If $C_1 = 0{\cdot}50\ \mu\text{F}$ and $C_2 = 0{\cdot}01\ \mu\text{F}$ then

$$R_1 + 2R_2 = 21 \tag{6.62}$$

and

$$R_1 R_2 = 10. \tag{6.63}$$

Solving for R_1, R_2,

$$R_1 = 1M, \quad R_2 = 10M.$$

The component values in this example have been carefully contrived. In general, although extremely helpful in saving computer equipment and avoiding the necessity for awkward gains, this method has certain disadvantages:

(*a*) The flexibility is decreased as far as system parameter changes are concerned.

(*b*) The complex relations between the component values and parameters often call for computational complexity.

(*c*) Unusual component values frequently result from (*b*). The use of nomographs relating circuit components to parameter values alleviates the second disadvantage.

Table 6.1 gives values of $M(\omega)$ and $\phi(\omega)$ from which the Bode plot for the given transfer function may be obtained.

6.8 Experiment 28. The Synthesis of Transfer Functions

In a certain servo-mechanism the transfer function between the Laplace transforms of the output variable θ_0 and the error signal ε is

$$\frac{\bar{\theta}_0}{\bar{\varepsilon}} = \frac{0{\cdot}5}{s(1 + 0{\cdot}2s)(1 + s)}. \tag{6.64}$$

Set up this transfer function using the *RC* network method and test it for frequency response.

1. It has already been mentioned that linear transfer functions

TABLE 6.1

ω (rad/s)	1	3	7	10	30
$M(\omega)$	0·238	1·553	−6·297	−13·098	−32·958
$\phi(\omega)$	−0·2176	−0·8532	−2·3495	−2·6581	−2·9993
ω (rad/s)	70	100	300	500	
$M(\omega)$	−47·764	−53·969	−73·062	−81·937	
$\phi(\omega)$	−3·0811	−3·0995	−3·1276	−3·1332	

may be multiplied together and manipulated algebraically. Thus relation (6.64) is equivalent to four separate blocks as shown in Fig. 6.15.

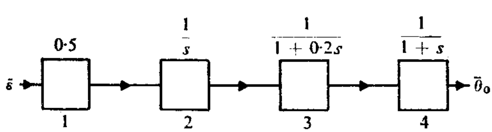


Fig. 6.15

The response to sinusoidal inputs is as usual obtained by replacing s by $j\omega$ in Eq. (6.64) and similarly in each separate block of Fig. 6.15. Table 6.2 shows $G(\omega)$ and $\phi(\omega)$ for each block.

The term 0·5 has gain in decibels $20 \log_{10} 0{\cdot}5 = -6$ and is constant. The phase angle is also constant at zero. The graph corres-

TABLE 6.2

Block	1	2	3	4
$G(\omega)$	0·5	$\frac{1}{\omega}$	$\frac{1}{\sqrt{(1 + 0{\cdot}04\omega^2)}}$	$\frac{1}{\sqrt{(1 + \omega^2)}}$
$\phi(\omega)$	0	−90	$-\tan^{-1} 0{\cdot}2\omega$	$-\tan^{-1} \omega$

ponding to the term $1/s$ has a gain slope of -6 db per octave and cuts the zero db line when $\omega = 1$. The phase angle is constant at $-90°$. The graphs corresponding to both the terms $1/(1 + s)$ and $1/(1 + 0{\cdot}2s)$ have $M = 0$ db as low frequency asymptote and have a high frequency asymptote with slope -6 db per octave. The former has a break point at $\omega = 1$ rad/s and the latter at $\omega = 5$ rad/s. At these frequencies the true response curve lies 3 db below the intersection of the asymptotes.

Figure 6.16 shows the Bode gain plot for each block together with the overall response obtained by adding together the individual M components. The curves are labelled according to the numbering of the blocks in Table 6.2. The overall Bode phase plot may be obtained by adding together ϕ values for each block in a similar way.

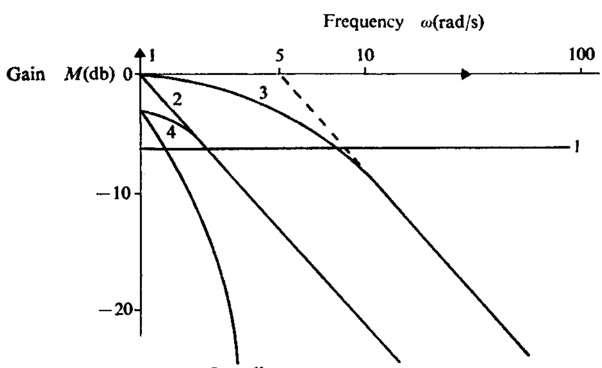


Fig. 6.16

2. To simulate the transfer function the following circuit can be used.

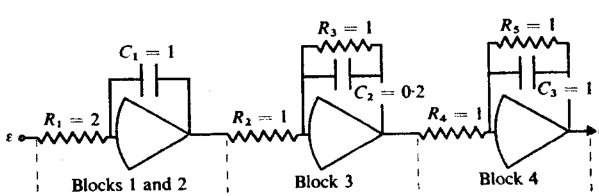


Fig. 6.17

The values of resistance and capacitance are shown in megohms and microfarads respectively. In fixed gain computers it may be necessary to utilize different values of these components, when the following relations must be satisfied.

$$\left.\begin{aligned} R_1C_1 &= 2, \\ R_3/R_2 &= 1 = R_5/R_4, \\ R_3C_2 &= 0{\cdot}2, \\ R_5C_3 &= 1. \end{aligned}\right\} \tag{6.65}$$

For instance in a fixed component transistorized computer the value of the capacitor may be 10 μF.

Then $C_1 = C_2 = C_3 = 10$, and it follows that $R_1 = 0{\cdot}2$, $R_2 = 0{\cdot}02 = R_3$ and $R_4 = 0{\cdot}1 = R_5$.

3. Perform the frequency response test to determine the Bode plot for θ_0/ε and check the graphs by using theoretical values.

6.9 Experiment 29. Simulation of Automatic Control

Set up a circuit to investigate the effects of step changes in the desired output variable of a control system having a process represented by the transfer function

$$\frac{\bar{\theta}_0(s)}{\bar{V}(s)} = Y(s) = \frac{1}{(1 + 0{\cdot}1s)(1 + 0{\cdot}25s)} \tag{6.66}$$

and governed by a three-term controller.

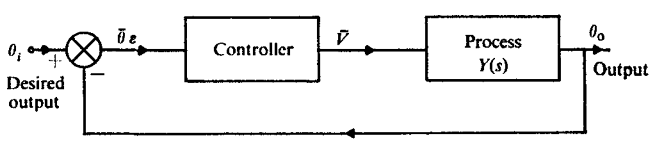


Fig. 6.18

1. The object of an automatic control system is to hold a measured variable such as temperature or pressure at some desired value.

This is achieved by comparing the actual value, θ_0, with the desired value, θ_i, and producing appropriate action, V, from the obtained error signal

$$\theta_\varepsilon = \theta_0 - \theta_i. \tag{6.67}$$

For example, if in a chemical process the desired temperature is 100° C and the measured temperature falls to 95° C, a controller acts in such a way that more heat is provided and the temperature increases. If the measured temperature rises above 100° C the controller reverses its action.

2. A controller can have several physical forms, e.g. pneumatic, hydraulic, electrical, depending in kind on the nature of the controlled variables.

There are three types of control action: (*a*) Proportional, (*b*) Integral, (*c*) Derivative.

In (*a*) the action of the controller V is proportional to the error signal θ_ε, i.e.

$$V = K_P\theta_\varepsilon. \tag{6.68}$$

In (*b*) the rate of action, dV/dt, is proportional to θ_ε, i.e.

$$\frac{dV}{dt} = K_I\theta_\varepsilon \tag{6.69}$$

or

$$V = K_I\int_0^t \theta_\varepsilon\, dt. \tag{6.70}$$

In (*c*) action is proportional to rate of change of error signal, i.e.

$$V = K_D\frac{d\theta_\varepsilon}{dt}. \tag{6.71}$$

Transforming Eqs. (6.68), (6.69), and (6.71) and assuming zero initial conditions, the complete action of the controller in transfer function form is

$$\bar{V} = \left(K_P + \frac{K_I}{s} + K_D s\right)\bar{\theta}_\varepsilon. \tag{6.72}$$

This is shown schematically in Fig. 6.19.

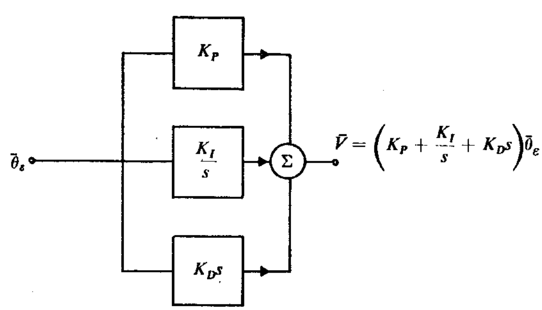


Fig. 6.19

3. Using Eqs. (6.67), (6.68) and (6.72) the following circuit may be obtained to represent the control system.

Amplifier number 1 is used as a differencing element to simulate Eq. (6.67), amplifiers 2, 3 and 4 constitute the controller with an extra gain of 10 provided in the summing amplifier number 5, and finally amplifiers 6 and 7 represent the process dynamics.

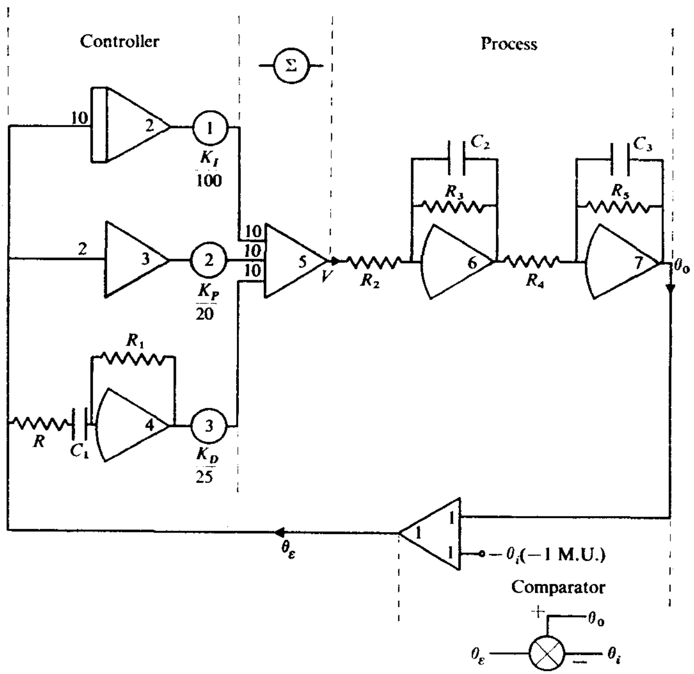


Fig. 6.20

It will be seen that an approximate form of differentiation is used (amplifier 4). The transfer function of this has already been discussed, as have the simple lag circuits of amplifiers 6 and 7.

4. Representative values for the three controller constants are

$$\left.\begin{aligned} 0 \leqq K_P \leqq 20, \\ 0 \leqq K_I \leqq 100, \\ 0 \leqq K_D \leqq 25. \end{aligned}\right\} \tag{6.73}$$

Thus potentiometers 1, 2 and 3 are set to record $K_I/100$, $K_P/20$ and $K_D/25$ respectively. A gain of 10 is incorporated in amplifier 5 leaving amplifiers 2, 3 and 4 with gains of 10, 2 and 2·5 respectively.

In amplifier 4 this amounts to choosing values of R_1 and C_1 such that

$$R_1 C_1 = 2{\cdot}5$$

and ensuring that RC_1 is small in comparison with the lowest time constant, 0·1 s. A value of about 0·001 may be tried.

Further, choose values of R_2, R_3, R_4, R_5 and C_2, C_3 to satisfy the following relations

$$\left.\begin{aligned} \frac{R_3}{R_2} &= 1 = \frac{R_5}{R_4}, \\ R_3C_2 &= 0{\cdot}1, \\ R_5C_3 &= 0{\cdot}25. \end{aligned}\right\} \tag{6.74}$$

5. Initially it is assumed that with $\theta_i = 0$ and $\theta_0 = 0$ the controller is acting to keep the actual output equal to the desired output. Any input signal θ_i will be regarded as representing a *change* in the desired output. Investigate and record the following modes of control using step changes in θ_i to represent sudden changes in desired output.

(*a*) No integral or derivative action (K_I, K_D both zero) using small, medium and large amounts of proportional action (three runs). Notice how as K_P is increased, the steady state error (known as 'offset') is decreased whilst the output tends to become more oscillatory.

(*b*) With each of the above use small and large amounts of integral action (six runs). Notice that the offset is now eliminated at the expense of stability.

(*c*) Use small and large amounts of derivative action with each of the above (eighteen runs). Notice how this improves stability.

6.10 Further Exercises

1. Differentiation can be performed using the implicit function techniques rather than by RC circuits. Simple analysis of Fig. 6.21 will show that

$$\frac{\bar{e}_0(s)}{\bar{e}_i(s)} = \frac{s}{1 + sT}.$$

The advantages are

(*a*) Only standard computing units are used.

(*b*) The appropriate sign of the derivative is generated.

(*c*) The break frequency $1/T$ is easily controllable and prior knowledge of the noise frequencies is not necessary.

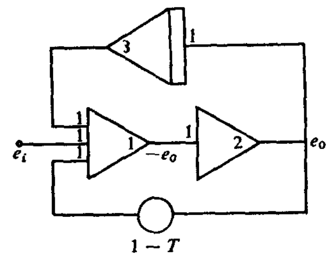


Fig. 6.21

Use this circuit to differentiate $y = \sin 4t$ for various values of T.

2. Set up a circuit to simulate the transfer function

$$Y(s) = \frac{1}{s(1 + 0{\cdot}1s)}.$$

Obtain the Bode diagrams for this circuit experimentally and check them by calculation.

3. If a function of time $t, f_i(t)$, is delayed by T seconds the resulting function $f_0(t)$ is given by

$$f_0(t) = f_i(t - T).$$

It may be shown that the transfer function representing the delay is

$$\frac{\bar{f}_0(s)}{\bar{f}_i(s)} = e^{-sT}$$

where s is the complex operator previously mentioned.

Replacing s by $j\omega$ for sinusoidal inputs the gain

$$G(\omega) = \left|\frac{\bar{f}_0(j\omega)}{\bar{f}_i(j\omega)}\right| = 1 \text{ (constant)}$$

and the phase

$$\phi(\omega) = \arg\frac{\bar{f}_0(j\omega)}{\bar{f}_i(j\omega)}$$

$$= -\omega T$$

which is proportional to frequency. Now

$$e^{-sT} = 1 - sT + \frac{s^2T^2}{2} - \frac{s^3T^3}{6} + \ldots$$

which may be approximated to by the expression

$$-\frac{s-2/T}{s+2/T} = 1 - sT + \frac{s^2T^2}{2} - \frac{s^3T^3}{4} + \ldots.$$

Significant errors arise in the fourth and succeeding terms if sT is not small. The following circuit (Fig. 6.22) has $-(s - 2/T)/(s + 2/T)$ as transfer function. Use it to test the delay of functions

$$(1)\ f_i = \sin 2t, \quad (2)\ f_i = t$$

when $T = 0{\cdot}25$ s.

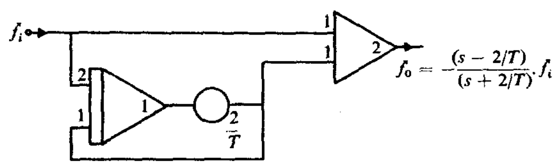


Fig. 6.22

Circuits of more complex form can be produced to provide much better approximations to time delays.

4. In the coplanar study of the homing of a guided missile steering under the proportional navigation system the equations may be

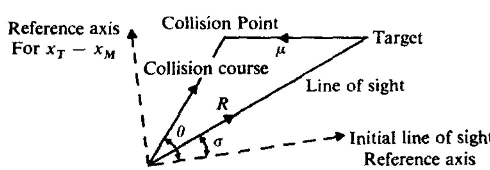


Fig. 6.23

reduced to the following simplified form

$\dot{\theta} = Y(s)\dot{\sigma}$	Missile rate of turn
$\sigma = \dfrac{X_T - X_M}{R}$	Line of sight angle
$R = R_0 - tV\cos\theta_0$	Range (missile to target)
$\ddot{X}_M = V\dot{\theta}\cos\theta_0$	Missile lateral acceleration
$\ddot{X}_T$ as desired	Target lateral acceleration

(θ_0, R_0 are the initial values of θ and R)

$Y(s)$ is a transfer function representing the missile dynamics. It dictates the ability of the missile to perform the programmed steering equation $\dot{\theta} = K\dot{\sigma}$.

In this problem take

$$Y(s) = \frac{K}{1 + Ts}$$

where $0 < K < 10$ and $T = 0{\cdot}5$.

Examine the miss distance (the value of $X_T - X_M$ when $R = 0$) for values of K equal to 0·1, 1, 2 and 10 for a $2g$ target turn, i.e. $\ddot{X}_T = 64$ ft/s^2.

The maximum values of the variables are given in the Table 6.3 and a suggested circuit in Fig. 6.24.

The experiment should be performed for various initial heading errors, i.e. $(\dot{X}_M - \dot{X}_T)_0$.

If a constant signal is applied through an extra input to amplifier 1,

TABLE 6.3

θ	$\dot{\theta}$	σ	R	$\dot{R}$
1 rad	10 rad/s	0·1 rad	20,000 ft	1000 ft/s
X_M	$\dot{X}_M$	X_T	$\dot{X}_T$	$X_T - X_M$
2000 ft	1000 ft/s	2000 ft	1000 ft/s	1000 ft

Potentiometer	1	2	3	4
Setting	$K/10$	$1/10T$	$\dfrac{V\cos\theta_0}{1000}$	$\dfrac{(\dot{X}_M - \dot{X}_T)_0}{1000}$

Potentiometer	5	6	7
Setting	$\dfrac{\ddot{X}_T}{1000}$	$\dfrac{V\cos\theta_0}{20{,}000}$	$\dfrac{R_0}{20{,}000}$

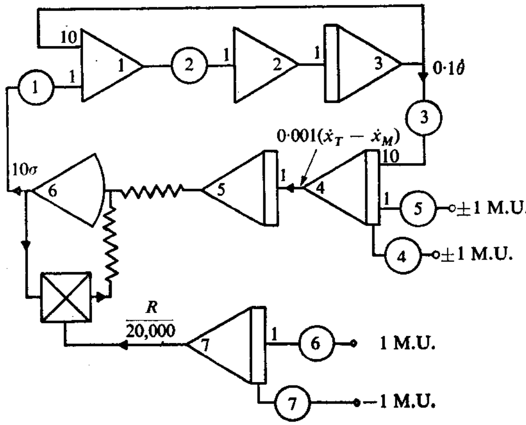


Fig. 6.24

the effect of a bias in the measurement of sight line angle may be investigated.

Amplifiers 1 to 3 represent the transfer function $0{\cdot}1s/(1 + Ts)$ in feedback form. A more realistic representation of the missile dynamics is obtained by the inclusion in the circuit of a term $1/[1 + (2\zeta s/\omega_n) + (s/\omega_n)^2]$ immediately prior to potentiometer 1. The use of *RC* networks to simulate these transfer functions is of course acceptable.

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Basic Analogue Computer Techniques

This is the only practical text to cover a first course in analogue computing. Its purpose is to provide practical instruction in the basic principles and its step by step approach in guiding students through typical problems for computer solution is unique. A set of 29 graded experiments is used to illustrate points of basic theory and practical techniques. The course covers selected problems of the type likely to be encountered by students of engineering, mathematics, and science in universities and technical colleges. To extend the students' experience further, exercises with answers are provided at the end of each chapter. Problems are restricted to those involving as few computer units as possible, thus obviating the need for large capacity computers.

The value of a standardized procedure of scaling is emphasized by application to the very important field of differential equations. The method of programming such equations is introduced, methods of estimating the approximate maximum values of problem variables are outlined, and a routine programming procedure is laid down. A simplified approach, avoiding the use of advanced mathematics, is made to the difficult topic of transfer functions, and important aspects of industrial computation are covered.

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