**DESIGN OF DIGITAL FILTERS USING PASCAL’S TRIANGLE**

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**Abstract:** This paper considers the involvement of Pascal’s triangle in the bilinear z-transformation method for converting the transfer function H(s) in the s-domain to the transfer function H(z) in the z-domain, as well as the inverse from H(z) to H(s). Matrix equations are derived by Nguyen, namely the Pascal matrix equation and inverse Pascal matrix equation. These equations are used to find the relationship between the coefficients of the transfer functions in the s-domain and z-domain. Additionally, this paper will mathematically view inverse matrix in the inverse Pascal matrix equation. The inverse matrix is difficult to determine due to the larger size of the matrix. To overcome this problem, some specific matrices are introduced, which will make computing and hand-calculation easier. Both matrix equations are powerful tools for transforming an analogue low-pass filter to a digital filter, and *vice versa*, as well as for transforming a digital filter to another digital filter.

**Keywords:** bilinear z-transform, Pascal’s triangle, frequency transformation, pre-warping frequency, low pass to low pass, low pass to high pass, low pass to band pass, low pass to band stop, low pass to Notch filter, digital to digital filter.

**1.**  **Introduction:**

Nowadays, it is known there are procedures of designing a digital filter (low pass, high pass, band pass, band stop and narrow-band filter), which is starting from a given analog low pass filter, transforming it to an analog filter with the same class of the desired digital filter by using frequency transformations and then applying the bilinear z-transform with pre-warping frequency. On the other hand, Pascal’s triangle, since established, has proved to be a very useful application in mathematics and in other fields. One of the most useful applications of the Pascal’s triangle is to find coefficients and expand the binomial expression. Hence, this paper will consider these wonderful applications of the Pascal’s triangle that are used for the bilinear z- transformation to convert from the transfer function H(s) in s-domain to the transfer function H(z) in z-domain and inverse from H(z) to H(s). From there, an new design algorithm for a digital filter from a given analog low pass filter or a given digital filter by using Pascal matrix equation and inverse Pascal matrix equation is also presented.

**2. Bilinear Z-transformation with pre-warping frequency:**

The effective and popular method that is currently used to convert a transfer function H(s) in s-domain into a transfer function H(z) in z-domain is the bilinear z-transformation. This technique is one to one mapping the poles and zeroes on the left half stable region in s-plane into inside unit circle in z-plane. The main advantage of this method is the transformation to a stable designed analog filter to a stable digital filter which the frequency response has the same characteristics as frequency response of the analog filter. However, this method will give a non-linear relationship between analog frequency ωA and digital frequency ωD and leads to warping of digital frequency response. The bilinear z-transform form the s-domain to z-domain is defined by

Where T is sampling period, substitute **** and **** into equation (1), it will give the relationship between ωA and ωD as shown in equation (2) and this relationship is a non-linear [2] that causing by the ‘tan’ function and the sampling period T and this effects is called warping frequency.

When converting an analog filter to a digital filter using bilinear z-transform method, it will give both filters have the same behavior, but the behavior is not matched at all the frequency in s-domain and digital domain as causing by the warping frequency. One way to overcome the warping frequency is called pre-warping frequency and it is expressed as in equation (3).

The transfer function H(z) of the digital filers (low pass, high pass, band pass, band stop and narrow band) can be obtained from the transfer function H(s) of a designed analog low pass filter. To design a digital filter, first transform a designed analog low pass filter to an analog filter which the same class of the digital filter using the frequency transformations [2],[4] and then apply the equation (3) bilinear z-transform with pre-warping. A block diagram in the Fig. 1 below illustrates converting an analog low pass filter to a low pass, high pass, band pass, band stop and narrow band digital filter using bilinear z-transform with pre-warping.

A designed analog low pass filter H(s)

Frequency transformation

Bilinear z-transformation with pre-wrapping frequency

Digital filter H(z)

Fig. 1 Design a digital filter from an analog filter

The Table I illustrates more details for a block diagram in theFig.1 to transform a designed low pass filter to a desired digital filter using bilinear z-transformation with per-warping frequency.

**Table I**

**Transform a designed low pass filter to a desired digital filter using bilinear z- transformation with per-wrapping frequency**

|  |  |  |  |
| --- | --- | --- | --- |
| Type of transformation | Analog low pass | Frequency transformation | z-domain |
| *Lp to Lp* | ***H(s)*** | ***H(*** | ***H(c* (4)** |
| *Lp to Hp* | ***H(s)*** | ***H(*** | ***H(t* (5)** |
| *Lp to Bp* | ***H(s)*** | ***H(*** | ***H(U* (6)** |
| *Lp to Bs* | ***H(s)*** | ***H(*** | ***H(* (7)** |



In the case of the narrow band filter, if f0 is the center frequency and Q is the quality factor, the coefficients U and L replace by UQ and LQ and the lower frequency fL, the upper frequency fU can be found as below.



**3. The matrix equation:**

This section introduces the bilinear z- transform with pre-warping to convert an analog transfer function H(s) to a digital transfer function H(z) involved the Pascal’s triangle. With the Pascal’s triangle, the matrix equation of the relationship between the digital coefficients and the analog coefficients can be found easy to compute and hand-calculation.

**3.1 The Pascal’s triangle:**

One of the most useful applications of the Pascal’s triangle is to find coefficients and expand the binomial expression (U±L)n and it can be shown in Fig. 2 below.

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | | | | | | | | | | | | |  |
| **0** |  |  |  |  |  | **1** |  |  |  |  |  |  |  |
| **1** |  |  |  |  | ***U*** |  | **± *L*** |  |  |  |  |  |  |
| **2** |  |  |  |  |  | **±*2UL*** |  |  |  |  |  |  |  |
| **3** |  |  |  |  | **±*3*** |  | ***3U*** |  | **±** |  |  |  |  |
| **⁞** |  | **⋰** |  | **...** |  | **...** |  | **...** |  | **⋱** |  |  |  |
| **n** |  |  | **±** |  |  | **...** |  |  | **∓** |  | **±** |  |  |

Fig. 2 The Pascal’s Triangle

**3.2 The transfer function H(s) and H(z):**

The transfer function H(s) of an nth-ordered analog low pass filter in s-domain and the transfer function H(z) of a digital filter (low pass, high pass, band pass, band stop and narrow band filter) in z-domain can be written as equation (12) and (13) below and where ai , bi and Ai, Bi are all real numbers, n is highest order number in the analog low pass filter and N= n for digital low pass and high pass filter and N=2n for digital band pass, band stop filter and narrow band.

**3.3 The matrix equation:**

From Table I and the equations (8) to (13), the relationship between the coefficients Ai and Bi of the designed analog low pass filter and the coefficients ai , bi of the desired digital filter can be found as a matrix equation below



The matrices [a] and [b] in equation (2) have a size of (N+1, 1) and they contain the digital coefficients in numerator and denominator of the transfer function H(z).

The matrix [P] contains the positive and negative binomial coefficients of the Pascal’s triangle in the first, last row and the first, last column corresponding to the edge size and the nth row of the Pascal’s triangle and another element in the matrix [P] can be calculated from its left, diagonal and above element. It has a size of (N+1, N+1). There are two different matrices respectively for low pass filter PLP and PHBS for PHP (high pass), PBP (band pass), PBS (band stop), PNB (narrow band) and they can be found as:

***Lp* → *Lp* :**

***Lp* *Hp , Bp , Bs , Ns : , , ,***

In this paper, the main key is impressed the involving of the Pascal’s triangle of the expansion (U+L)n to find the matrix [Δ]. The matrix [Δ] contains the coefficients Ai, Bi of the analog low pass filter and U, L as shown in the Table II below. In the Table II, it can see that the Pascal’s triangle involved into the matrix [Δ] and for the each element Δi in the matrix [Δ] is the sum of all the elements in the column of Pascal’s triangle multiply with the analog coefficient at the same row, and it can be expressed as the equation (15)

**Table II**

**The Matrix**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ***A*** |  |  |  | | | | | | | | | | |
| ***0*** |  |  |  |  |  |  |  | ***1*** |  |  |  |  |  |
| ***1*** |  |  |  |  |  |  | ***U*** |  | ***L*** |  |  |  |  |
| ***2*** |  |  |  |  |  |  |  | ***2UL*** |  |  |  |  |  |
| ***3*** |  |  |  |  |  |  | ***3L*** |  | ***3U*** |  |  |  |  |
|  |  |  |  |  |  | **⋯** |  | **⋯** |  | **⋯** |  | **⋱** |  |
| ***n*** |  |  |  |  | **⋯** |  | **⋯** | **⋯** | **⋯** |  | **⋯** |  |  |
| to **,**  to ***N=2n*** | | |  |  |  |  |  |  |  |  |  |  |  |
|  | | |  |  |  |  |  |  | to ***N=n, L=0*** | | | | |
| to ***N=n U=0*** | | | | | | | |  |  |  |  |  |  |

**4. The Pascal matrix equation:**

There is another way [5] to find the matrix [Δ], from Table II and equation (15), it can see clearly that the matrix [Δ] can be calculated by matrix multiplication between the matrix of the coefficients [Ai] and [Bi] of the analog low pass filter and the matrix [T] as shown in the equation (16). Where the matrix [T] is the Pascal’s triangle expansion of (U+L)n with inserting zeroes as show in Table III below.

**(16)**

**Table III**

**The Matrix with the matrix**

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ***n***  ***0*** |  | **The Matrix [T]** | | | | | | | | | | |
|  |
|  | ***0*** | ***0*** | ***0*** | ***0*** | ***0*** | ***1*** | ***0*** | ***0*** | ***0*** | ***0*** | ***0*** |
| ***1*** |  | ***0*** | ***0*** | ***0*** | ***0*** | ***U*** | ***0*** | ***L*** | ***0*** | ***0*** | ***0*** | ***0*** |
| ***2*** |  | ***0*** | ***0*** | ***0*** |  | ***0*** | ***2UL*** | ***0*** |  | ***0*** | ***0*** | ***0*** |
| ***3*** |  | ***0*** | ***0*** |  | ***0*** | ***3*** | ***0*** | ***3U*** | ***0*** |  | ***0*** | ***0*** |
|  |  | ***0*** |  | ***0*** |  | ***0*** |  | ***0*** |  | ***0*** |  | ***0*** |
| ***n*** |  |  | ***0*** |  | ***0*** |  | ***0*** |  | ***0*** |  | ***0*** |  |
|  |  |  |  |  |  |  |  | ***to N=n t=0 U=c*** | | | | |
| ***to N=n c=0 L=t*** | | | | | | |  |  |  |  |  |  |
| to **,  *N=2n*** | |  |  |  |  |  |  |  |  |  |  |  |

Base on equation (14) and equation (16), one new formula can derived and it is called “The Pascal Matrix Equation” as is described in equation (17) and this equation can be used to convert an analog low pass filter to a digital filter:

**(17)**

The procedure of converting an analog low pass filter to a digital filter were studied. The next sections will introduce how to invert a digital filter to an analog low pass filter using the Pascal matrix equation.

**5. The inverse Pascal matrix equation:**

The inverting a digital low pass, high pass, band pass and band stop to an analog filter is a method to transform the coefficients [ai] and [bi] of the transfer function H(z) to the coefficients [Ai] and [Bi] of the transfer function H(s). This method can be done by using the Pascal matrix equation (17), the matrix [Ai] and [Bi] can be found as:  **(18)**

The equation (18) is called “the Inverse Pascal Matrix Equation” and it can be used to find the coefficients of the analog low pass filter from a digital filter.

In this equation, the involving of the inverse matrix [P]-1 and [T]-1 will not make easy computing and hand-calculation for lager matrix size. To overcome this problem, consider some features of the matrix [P] and [T] in the next section.

**5.1 The features of the Matrix and :**

There are some features of the matrix [P] and they can be used to find the inverse matrix [P]-1. For the matrix low pass [PLP], if multiply the matrix by itself will give a diagonal matrix with all the numbers in the diagonal are equal to 2n[3], and from this the inverse matrix [PLP] can be found as:

**(19)**

The inverse of the matrix [PHBS] can be found similar way and can be written as

**(20)**

From Table III, for low pass filter, let t=0, then L=0, U=c, the matrix [T] becomes a matrix [Tc] has a size of (n+1 ; n+1) and the same for high pass filter, c=0, U=0, L=t, the matrix [T] is the matrix [Tt] as shown:

The matrix [Tc] and [Tt] are the diagonal matrix, so the inverse of them can be obtained by replacements each element in diagonal with its reciprocal as illustrated below

**(21)**

Let a matrix [Th] has a size of (n+1;n+1) and it can be written like a left-half of the Pascal’s triangle with zeroes [T] in Table III as shown below:

The inverting of the matrix [Th] is an upper-left triangular matrix [Th-inv] of the size (n+1; n+1) and it can be found as following steps:

* If the main anti-diagonal is the first, then all the odd anti-diagonal above it are 3rd, 5th, 7th… and corresponding to m=1,2,3,4,…Each element in the mth anti-diagonal can be expressed in the formula as:

**(22)**

* The coefficients K can be calculated as below:
  + All the K in main anti-diagonal are equal to 1.
  + From m= 2, 3, 4, 5… all the K in the first column are replaced with -2, +2,-2, +2… accordingly. And all another K in all anti-diagonals can be found by

**(23)**

* All another elements in the matrix [Th-inv] equal to zero.

The matrix [Th-inv] can be used to inverse from a digital band pass and band stop to an analog low pass filter.

From the equation (18), (19), (20), (21) and the matrix [Th-inv], the inverse Pascal matrix equation for low to low pass, high pass to low pass, band pass to low pass and band stop can be expressed as:

* Inverse a digital low pass filter to an analog low pass filter

**(24)**

* Inverse a digital high pass filter to an analog low pass filter

**(25)**

* Inverse a digital band pass filter to an analog low pass filter

**(26)**

* Inverse a digital band stop filter to an analog low pass filter

**(27)**

The methods covert and invert between the coefficients of an analog low pass filter and a digital filter were studied. The Pascal matrix equation and inverse matrix equation are derived and they are easy to use for hand-calculation and computing. One application of them is transformation a digital to another digital filter.

**5.2 Transform a digital filter to another digital filter:**

This section introduces a new method to design a digital from another filter. Given a digital filter, invert it to an analog low pass filter using inverse Pascal matrix equation and from analog low pass filter converts it to a desired digital filter using Pascal matrix equation. A block diagram in figure.3 illustrates the transformation from a digital to another digital filter.

A given digital filter

Inverse Pascal Matrix equation

Analog Low pass filter

Pascal Matrix equation

A desired digital filter

Fig. 3 Transform a digital filter to another digital filter

Given the coefficients [ag] and [bg] of a digital filter, invert them to the coefficients [Ag] and [Bg] of an analog low pass filter. The equation (18) is rewritten as

**(28)**

Then convert [Ag] and [Bg] to the coefficients [ad] and [bd] of a desired digital filter. The equation (17) is rewritten as

**(29)**

From equation (28) and (29), a new equation, called digital to digital filter Pascal matrix equation, is found, this equation is used for transform a digital to another digital filter.

**(30)**

**6. Design a digital filter using Pascal’s triangle:**

As discussed in the section 4 and 5 in this paper, there are two methods to design a digital filter based on a given analog low pass filter or a given digital filter. A block diagram shown in figure.4 describes a new algorithm to design a digital filter and it can be implemented by a programming (such as MATLAB, C, C++, Assembly program languages) for digital signal processors. Its inherent simplicity could make the algorithm attractive for many applications where it is needed to minimize computational requirements and workloads.

A given analog low pass filter

A given digital filter

A desired digital filter

Pascal Matrix Equation

Digital to Digital Filter Pascal Matrix Equation

Fig. 4 A new algorithm designing a digital filter

From the figure.4, if given the coefficients [Ai], [Bi] of an nth-ordered analog low pass filter, the equation (17) will be applied to convert them to the coefficients [ai], [bi] of the desired Nth-ordered digital filter. Otherwise, if the coefficients [ag], [bg] of a digital filter are given, then the equation (30) will be applied to get the coefficients of the desired digital filter.

Let consider an example about removing the AC 50 Hz power line effects to the signal of 200 Hz. To solve this problem, two digital filters are needed, one is a low pass filter which can pass the 400 Hz signal and another one is a notch filter which will reject the AC 50 Hz power line. And from the figure. 4, there are two methods to design a digital low pass and a digital notch filter as show below, and for this example, a 3th-ordered Butterworth analog low pass filter with the transfer function H(s) is given.

***Method 1:*** Convert the 3th-ordered Butterworth H(s) to a digital low pass filter has a corner frequency fc=200 Hz and the notch filter has the center frequency f0= 50 Hz with the quality factor Q=50 and the sampling frequency fs =1000 Hz, as shown in fig.4.1.

Pascal Matrix Equation

Digital Low Pass filter

The 3th-ordered Butterworth low pass filter

Pascal Matrix Equation

Digital Notch filter

Fig.4.1 Convert an analog low pass filter to digital filters

* Convert the 3th-ordered Butterworth analog low pass filter to Digital low pass filter, apply the equation (17)











* Convert the 3th-ordered Butterworth analog low pass filter to Digital notch filter, apply the equation (17)



  



***Method 2:*** Convert the 3th-ordered Butterworth to a digital low pass filter has a corner frequency fc=200 Hz and then from the digital low pass filter convert to the notch filter has the center frequency f0= 50 Hz with Q = 50 and the sampling frequency fs =1000 Hz, as shown in fig.4.2.

The 3th-ordered Butterworth low pass filter

Digital Low Pass filter

Digital to Digital Filter Pascal Matrix Equation

Digital Notch filter

Pascal Matrix Equation

Fig.4.2 Convert a digital filter to another digital filter

* Convert the 3th-ordered Butterworth analog low pass filter to Digital low pass filter, apply the equation (17), the coefficients [a] and [b] of the digital low pass filter can be found as below



* Convert Digital low pass filter to Digital notch filter, apply the equation (30)







**7. Conclusion:**

The new method were studied for converse and inverse between an analog low pass filter with the transfer function H(s) and a digital filter (low pass, high pass, band pass, band stop and narrow band) with the transfer function H(z) and from that this method can be applied to transform a given analog low pass or digital filter into another digital filter. The involving of the Pascal’s triangle uses in the Pascal matrix equation and inverse Pascal matrix equation, as presented and demonstrated in the examples, made the work easier for hand-calculation and computing when transforming between s-domain and z-domain. The features of the matrix [P] and [T] are very helpful to find the inverse matrix which is not easy to do with the larger matrix size. The algorithm of this method converse and inverse is so simple due to all operations imply the matrix multiplication and so it is more effective to program and calculation.

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**Biography**

 At first , Phuoc Si Nguyen received a Bachelor's degree in Mathematics and Physics from the University of Can Tho , Vietnam and experienced teaching Mathematics and Physics for highschool students in his hometown . A few years later , he received a bachelor's degree in Electrical and Electronic Engineering from Victoria University of Technology in Melbourne , Australia . Because of the outstanding graduation , he joined the research group in High Efficiency Power Amplifier for Mobile Communication System and he was also a lab assistant and tutor at Department of Communication and Informatics, Victoria University of Technology. His interests are in the field of mathematics and signal processing techniques . Currently, while Maths tutoring for highschool students , he enjoys doing self- study on " How Pascal 's triangle can be applied in design of digital filters" . This idea shows a lot of promise applications which he absorbed in them and hope to share to whom have the same interests.