Understanding the family of Fourier transform in depth

– From Fourier series to FFT

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1. Fourier series

A **periodic** function $f(t)$ with a period T can be expended to a Fourier series of sines and cosines:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi kt}{T}\right)$$

When $k = 1$, the fundamental frequency of the cosine and sine terms have a period of $2\pi/(2\pi/T) = T$. The formulas for computing the coefficients are given in following:

$$a_0 = \frac{\int_{-T/2}^{T/2} 2f(t)dt}{\int_{-T/2}^{T/2} dt} = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_k = \frac{\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi kt}{T}\right)dt}{\int_{-T/2}^{T/2} \cos^2\left(\frac{2\pi kt}{T}\right)dt} = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi kt}{T}\right)dt$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi kt}{T}\right)dt$$

2. Fourier series (complex form)

To deduct the complex form Fourier series, one needs to borrow Euler's formula:

$$e^{ix} = \cos x + i\sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Hence, the Fourier series shown above can be rewritten as:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \frac{e^{i\frac{2\pi kt}{T}} + e^{-i\frac{2\pi kt}{T}}}{2} + \sum_{k=1}^{\infty} b_k \frac{e^{i\frac{2\pi kt}{T}} - e^{-i\frac{2\pi kt}{T}}}{2i}$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(\frac{a_k}{2} + \frac{b_k}{2i}\right)e^{i\frac{2\pi kt}{T}} + \sum_{k=1}^{\infty}\left(\frac{a_k}{2} - \frac{b_k}{2i}\right)e^{-i\frac{2\pi kt}{T}}$$

Note substituting k by $-k$ in the last term yields:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(\frac{a_k}{2} + \frac{b_k}{2i}\right)e^{i\frac{2\pi kt}{T}} + \sum_{k=-1}^{-\infty}\left(\frac{a_{-k}}{2} - \frac{b_{-k}}{2i}\right)e^{i\frac{2\pi kt}{T}}$$

Define new complex coefficients c_k as:

$$c_k = \begin{cases} a_0/2\,, & k = 0 \\ (a_k - ib_k)/2\,, & k = 1,2,3\ldots \\ (a_{-k} + ib_{-k})/2\,, & k = -1,-2,-3\ldots \end{cases}$$

It's not hard to realize $c_k = c_{-k}^*$. To sum up, the complex form Fourier series expansion is defined as:

$$\boxed{\begin{array}{ll} f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{i\frac{2\pi k t}{T}} & t \in (-T/2\,, T/2) \\ c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i\frac{2\pi k t}{T}} dt & k \in Z \end{array}}$$

3. Fourier transform

To derive the Fourier transform, one needs to assume the period $T \to \infty$. Let $\omega = \frac{2\pi k}{T}$, $\Delta\omega = \frac{2\pi}{T} \to 0$. The complex form Fourier series becomes:

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} c_k\, e^{i\frac{2\pi k t}{T}} \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i\frac{2\pi k t}{T}} dt\, e^{i\frac{2\pi k t}{T}} \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Delta\omega \int_{-T/2}^{T/2} f(t) e^{-i\omega t} dt\, e^{i\omega t} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt\, e^{i\omega t} d\omega \end{aligned}$$

Hence:

$$\boxed{\begin{array}{ll} F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt & \widehat{\omega} \in (-\infty, \infty) \\ f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega & t \in (-\infty, \infty) \end{array}}$$

Unlike the Fourier series expansion, the Fourier transform does not require $f(t)$ to be periodic! One might also notice the frequencies become continuous as the gap of frequencies approaches 0. However, the Fourier transform does require $f(t)$ to be integrable in $(-\infty, \infty)$. One exception is when the given function $f(t)$ is periodic. In such cases, the Fourier transform is still valid after introducing Dirac delta function.

Let's take a look at the inverse transform of a Dirac delta function $F(\omega) = \delta(\omega - \omega_0)$. $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{i\omega t} d\omega = \frac{e^{i\omega_0 t}}{2\pi}$. Hence:

$$\boxed{\frac{e^{i\omega_0 t}}{2\pi} \overset{F}{\leftrightarrow} \delta(\omega - \omega_0)}$$

The "widths" in the temporal and frequency domains satisfy the uncertainty principle, i.e., $\Delta T \Delta\omega =$ const. For periodic functions, $\Delta T \to \infty$ and $\Delta\omega \to 0$.

a. The Fourier transform of several commonly used functions

I. Square function $f(t) = \Pi(t/T_p)$

$$f(t) = \Pi(t/T_p) = \begin{cases} 1, & -T_p/2 \le t < T_p/2 \\ 0, & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \Pi(t/T_p)e^{-i\omega t}dt \\
&= \int_{-T_p/2}^{T_p/2} e^{-i\omega t}dt = \frac{e^{-i\omega T_p/2} - e^{i\omega T_p/2}}{-i\omega} \\
&= \frac{2\sin(\omega T_p/2)}{\omega} = T_p \mathrm{sinc}\left(\frac{\omega T_p}{2\pi}\right)
\end{aligned}
$$

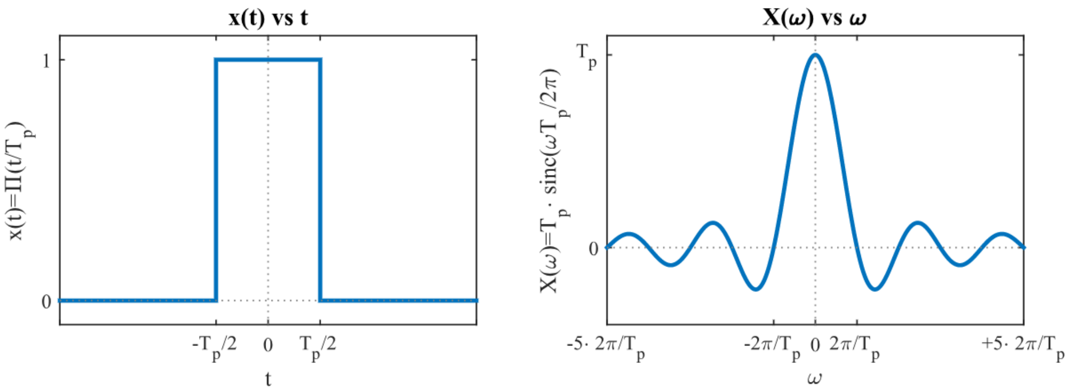

Figure 1. Square function and sinc function are a Fourier pair

II. Unit triangle function $f(t) = \Lambda(t/T_p)$

$$
f(t) = \Lambda(t/T_p) = \begin{cases} 1+t, & -1 \le t < 0 \\ 1-t, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}
$$

$$
F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-1}^{0} (1+t)e^{-i\omega t}dt + \int_{0}^{1} (1-t)e^{-i\omega t}dt = \mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)
$$

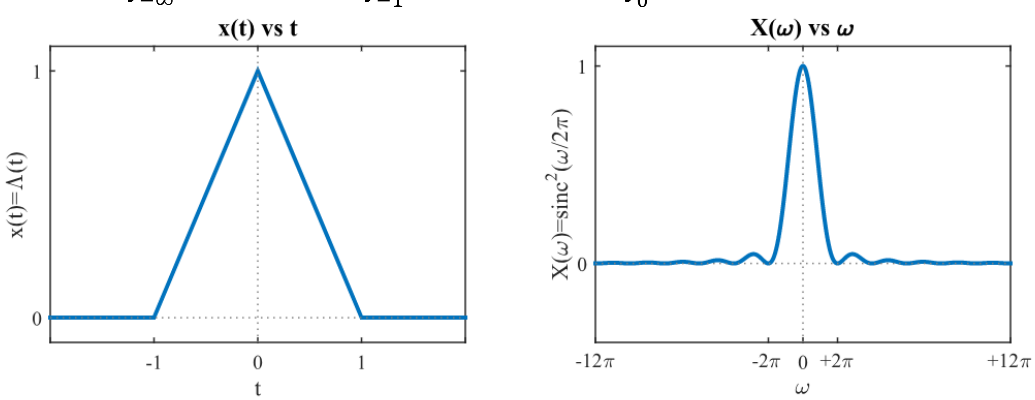

Figure 2. Triangle and sinc squared functions are a Fourier pair

b. Prosperities of the Fourier transform

I. Convolution Theorems

The convolution theorem states that **convolution in time domain corresponds to multiplication in frequency domain, and vice versa**:

$$
\boxed{\begin{array}{ll} \mathcal{F}[x(t) * y(t)] = X(\omega)y(\omega) & (a) \\ \mathcal{F}[x(t)y(t)] = X(\omega) * y(\omega) & (b) \end{array}}
$$

Proof of (a):

$$
\begin{aligned}
\mathcal{F}[x(t) * y(t)] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau\right] e^{-i\omega t}dt \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-i\omega t}dt\right] d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-i\omega\tau} \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-i\omega(t-\tau)}d(t-\tau)\right] d\tau \\
&= X(\omega)y(\omega)
\end{aligned}
$$

Proof of (b):

$$
\begin{aligned}
\mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-i\omega t}dt \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega')e^{i\omega' t}d\omega'\right] y(t)e^{-i\omega t}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega') \int_{-\infty}^{\infty} y(t)e^{-i(\omega-\omega')t}dt\, d\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega')Y(\omega-\omega')\, d\omega' \\
&= X(\omega) * y(\omega)
\end{aligned}
$$

For more information, check this link about the properties of Fourier transform, and this link about the Fourier transform of several commonly used functions.

4. Discrete-time Fourier transform (DTFT)

Before discussing the discrete Fourier transform (DFT), one important concept is the so-called discrete-time Fourier transform, which serves as an important bridge between the continuous Fourier transform and the widely used DFT in signal processing. In reality, the measured signal can never be continuous, and is always sampled at a certain interval. Let's assume the sampling interval is T_s. The acquired data points can be expressed as:

$$
f_{\mathrm{m}}(t) = f(t)\sum_{n=-\infty}^{\infty} \delta(t-nT_s)
$$

Here $f(t)$ is the actual continuous signal, n is the indices of acquired points, and the suffix m indicates the parameter decorated corresponds the real measurement. The forward and inverse Fourier transforms then become:

$$
F_{\mathrm{m}}(\omega) = \int_{-\infty}^{\infty} f_{\mathrm{m}}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\, e^{-i\omega t}dt = \sum_{n=-\infty}^{\infty} f(nT_s)e^{-i\omega nT_s}
$$

$$
f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_{\mathrm{m}}(\omega)e^{i\omega t}d\omega
$$

The inverse transform shown above is not the final form. To see this, one needs to examine the expression of the originally acquired signal $f_{\mathrm{m}}(t)$ a bit closer. Let's define:

$$
g(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)
$$

It's not hard to realize $g(t)$ is a periodic function with a period T_s, i.e., $g(t) = g(t-T_s)$. Hence, $g(t)$ can be expended using the Fourier series:

$$g(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{i\frac{2\pi kt}{T_s}}$$

With coefficients:

$$c_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} g(t) e^{-i\frac{2\pi kt}{T}} dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\left[\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right] e^{-i\frac{2\pi kt}{T_s}} dt = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi knT_s}{T_s}} = \frac{1}{T_s}$$

Put c_k back to the Fourier series of $g(t)$, yielding:

$$g(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{i\frac{2\pi kt}{T_s}}$$

Hence, $f_{\mathrm{m}}(t)$ can be expanded as a Fourier series:

$$f_{\mathrm{m}}(t) = f(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \frac{f(t)}{T_s}\sum_{k=-\infty}^{\infty} e^{i\frac{2\pi kt}{T_s}}$$

Performing Fourier transform on this series leads to the following:

$$\begin{aligned} F_{\mathrm{m}}(\omega) &= \int_{-\infty}^{\infty} f_{\mathrm{m}}(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty}\left[\frac{f(t)}{T_s}\sum_{k=-\infty}^{\infty} e^{i\frac{2\pi kt}{T_s}}\right] e^{-i\omega t} dt \\ &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) e^{-i\left(\omega-\frac{2\pi k}{T_s}\right)t} dt \\ &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F\left(\omega-\frac{2\pi k}{T_s}\right) \end{aligned}$$

Note $F\left(\omega-\frac{2\pi k}{T_s}\right)$ is the Fourier transform of the original continuous signal $f(t)$ shifted by phase $\frac{2\pi k}{T_s}$. The summation of $F\left(\omega-\frac{2\pi k}{T_s}\right)$ means the Fourier transform of the measured time-sampled signal is the sum of the Fourier transforms of the original continuous signal shifted by various phases. In other words, sampling in time corresponds to replication in the frequency domain.

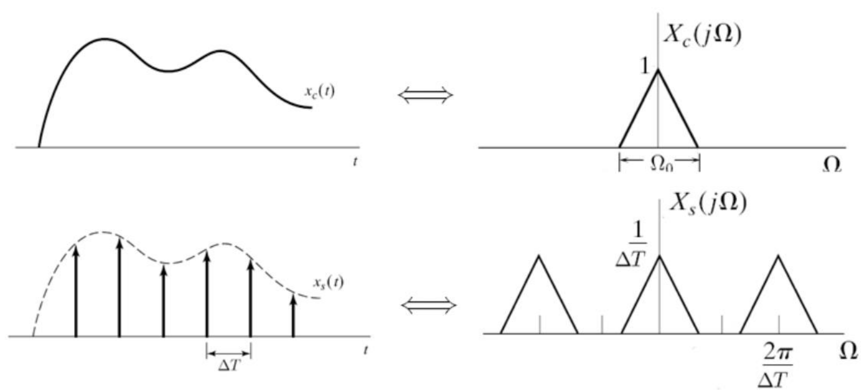

Figure 3. Sampling in time corresponds to replication in the frequency domain

Since in the Fourier transform t and ω are interchangeable, this property can be expended to the following: **sampling in one domain with a sampling interval T_s corresponds to replication in the other domain with period $2\pi/T_s$, and vice versa!**

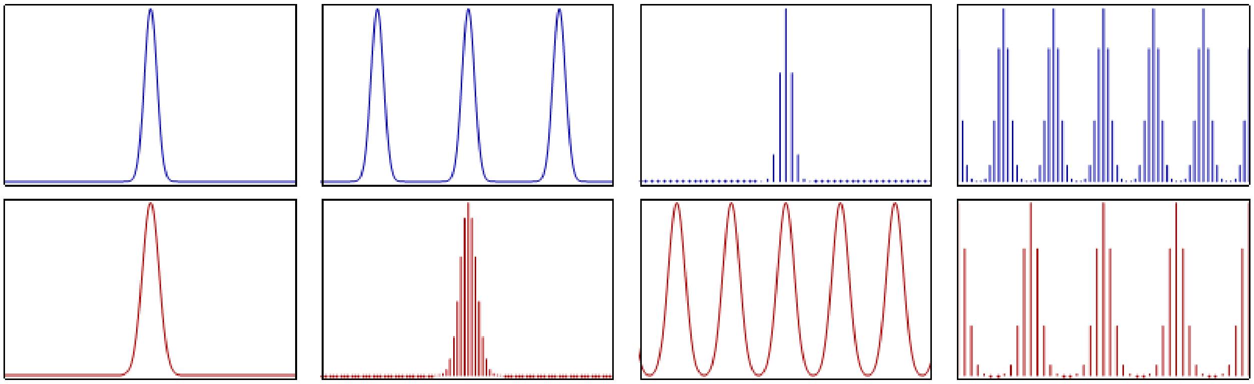

Figure 4. Sampling in one domain corresponds to replication in the other domain, and vice versa. From column 1 to 2: sampling in the red domain corresponds to replication in the blue domain. From column 1 to 3: replication in the red domain corresponds to sampling in the blue domain. From 1 to 4: sampling and replication in the red domain corresponds to replication and sampling in the blue domain

Since $F_{\mathrm{m}}(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F\left(\omega - \frac{2\pi k}{T_s}\right)$, $F_{\mathrm{m}}(\omega)$ is a periodic function with a period of $2\pi/T_s$. This periodicity also explains the frequency aliasing issue known in FFT. Since only information within one period is useful, one can rewrite DTFT forward and inverse transforms in the following form:

$$F_{\mathrm{m}}(\omega) = \sum_{n=-\infty}^{\infty} f(nT_s)e^{-i\omega n T_s}; \quad f(t) = \frac{1}{2\pi}\int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} F_{\mathrm{m}}(\omega)e^{i\omega t}d\omega$$

It's more convenient to deal with the discrete measurement results using the indices of the acquired points n instead of the real time t. Let's define $\widehat{\omega} = \omega T_s$. The following summarizes the final version of forward and inverse DTFT:

$$\boxed{\begin{array}{ll} F_{\mathrm{m}}(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} f(nT_s)e^{-i\widehat{\omega} n} = \sum_{n=-\infty}^{\infty} f_n e^{-i\widehat{\omega} n} & \widehat{\omega} \in (-\pi, \pi) \\ f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_{\mathrm{m}}(\widehat{\omega})e^{i\widehat{\omega} n}d\widehat{\omega} & n = 0,1,2 \ldots N-1 \end{array}}$$

The notation of f_n denotes discrete points.

5. Discrete Fourier transform (DFT)

DTFT is one step closer to reality with the introduction of discrete time. However, it still requires n to run from $-\infty$ to ∞. In reality, we can only measure a finite number of points, say $n = 0,1,2 \ldots N-1$ and **implicitly assume the measured signal is periodic outside of this measured scope**.

$$f_{\mathrm{m}}(t) = f(t)\sum_{n=0}^{N-1}\delta(t - nT_s)$$

The forward DFT is simply the sum of a subset of terms in DTFT:

$$F_{\mathrm{m}}(\widehat{\omega}) = \sum_{n=0}^{N-1} f_n e^{-i\widehat{\omega} n}$$

So far, the discussion still happens in the context of DTFT and $\widehat{\omega}$ is still continuous in the range $(-\pi, \pi)$. Recall we made the assumption that the input signal is periodic with a period same as the length of the measurement window (NT_s as measured in time or just N as measured using the number of points).

Also, recall that the periodicity in the temporal domain (with period N) corresponds to sampling in the frequency domain (with sampling interval $2\pi/N$). Hence, $F_{\mathrm{m}}(\widehat{\omega})$ should by sampled at $\widehat{\omega} = 2\pi k/N$, $k = 0,1,2 \ldots N-1$:

$$F_{\mathrm{m}}(\widehat{\omega}) = \sum_{n=0}^{N-1} f_n e^{-i\widehat{\omega}n}\bigg|_{\widehat{\omega}=2\pi k/N, k=0,1,2\ldots N-1}$$

$$\Rightarrow F_k = \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi kn}{N}}, \quad k = 0,1,2 \ldots N-1$$

The inverse transform becomes:

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[F_{\mathrm{m}}(\widehat{\omega})\sum_{k=0}^{N-1}\delta(\widehat{\omega} - 2\pi k/N)\right]e^{i\widehat{\omega}n}d\widehat{\omega} = \frac{1}{N}\sum_{k=0}^{N-1}F_{\mathrm{m}}(2\pi k/N)e^{i\frac{2\pi kn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1}F_k e^{i\frac{2\pi kn}{N}}$$

In summary, the forward and inverse DFT can be shown as:

$$\boxed{\begin{aligned} F_k &= \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi kn}{N}} \qquad k = 0,1,2 \ldots N-1 \\ f_n &= \frac{1}{N}\sum_{k=0}^{N-1} F_k e^{i\frac{2\pi kn}{N}} \qquad n = 0,1,2 \ldots N-1 \end{aligned}}$$

The previous discussion is just one way to derive DFT. Recall that the input signal is assumed to be periodic with a period NT_s. Hence, an alternative derivation approach is to start with the Fourier series expansion of $f_{\mathrm{m}}(t)$:

$$\begin{aligned} F_k = c_k &= \frac{1}{NT_s}\int_{-NT_s/2}^{NT_s/2} f(t)\sum_{n=0}^{N-1}\delta(t - nT_s)\, e^{-i\frac{2\pi kt}{NT_s}}dt \\ &= \frac{1}{NT_s}\sum_{n=0}^{N-1} f(nT_s)e^{-i\frac{2\pi knT_s}{NT_s}} \\ &= \frac{1}{NT_s}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi kn}{N}} \end{aligned}$$

Division by T_s means that the frequencies are in the unit of $1/T_s$. The inverse transform is simply (note $t = nT_s$):

$$f_n = \sum_{k=-\infty}^{\infty} c_k\, e^{i\frac{2\pi kt}{T}} = \sum_{k=-\infty}^{\infty} F_k\, e^{i\frac{2\pi knT_s}{NT_s}} = \sum_{k=0}^{N-1} F_k\, e^{i\frac{2\pi kn}{N}}$$

The last step takes advantage the fact that F_k is periodic due to the sampling process in the temporal domain. Comparing this to the standard DFT format reveals that the coefficient $1/N$ has moved to the equation of F_k, which is simply another standard of defining Fourier transform.

6. Fast Fourier transform (FFT)

First of all, FFT is not yet another form of Fourier transform but the implementing algorithm of DFT. It reduces the time complexity of computing DFT from $\mathrm{O}(n^2)$ to $\mathrm{O}(n\log_2 n)$. The most commonly used FFT algorithm is Cooley–Tukey FFT algorithm, which assumes the number of total data points is a power of 2 and can be recursively broken into two halves. Let the even and odd indices be $2m$ and $2m+1$, $m = 0,1,2 \dots N/2-1$. Hence:

$$
\begin{aligned}
F_k &= \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi kn}{N}} \\
&= \sum_{m=0}^{N/2-1} f_{2m} e^{-i\frac{2\pi k 2m}{N}} + \sum_{m=0}^{N/2-1} f_{2m+1} e^{-i\frac{2\pi k(2m+1)}{N}} \\
&= \sum_{m=0}^{N/2-1} f_{2m} e^{-i\frac{2\pi km}{N/2}} + e^{-i\frac{2\pi k}{N}} \sum_{m=0}^{N/2-1} f_{2m+1} e^{-i\frac{2\pi km}{N/2}}
\end{aligned}
$$

The first term happens to be the DFT of the even-indexed input and the second term is the odd-indexed input multiplied by $e^{-i\frac{2\pi k}{N}}$. In addition, it's not hard to find out:

$$
\begin{aligned}
F_{k+N/2} &= \sum_{m=0}^{N/2-1} f_{2m} e^{-i\frac{2\pi(k+N/2)m}{N/2}} + e^{-i\frac{2\pi(k+N/2)}{N}} \sum_{m=0}^{N/2-1} f_{2m+1} e^{-i\frac{2\pi(k+N/2)m}{N/2}} \\
&= \sum_{m=0}^{N/2-1} f_{2m} e^{-i\frac{2\pi km}{N/2}} - e^{-i\frac{2\pi k}{N}} \sum_{m=0}^{N/2-1} f_{2m+1} e^{-i\frac{2\pi km}{N/2}}
\end{aligned}
$$

Let's define the DFT of the even- and odd-indexed input parts as:

$$
E_k = \sum_{m=0}^{N/2-1} f_{2m} e^{-i\frac{2\pi km}{N/2}}
$$

$$
O_k = \sum_{m=0}^{N/2-1} f_{2m+1} e^{-i\frac{2\pi km}{N/2}}
$$

Hence, we have:

$$
F_k = E_k + e^{-i\frac{2\pi k}{N}} O_k
$$

$$
F_{k+N/2} = E_k - e^{-i\frac{2\pi k}{N}} O_k
$$

Here $k = 0,1,2 \dots N/2-1$, hence, F_k and $F_{k+N/2}$ account for all N terms. This result is the key of the FFT algorithm. The following pseudo-code shows the FFT process:

```
def FFT(x, N, step, shift):
  """
Pseudo code for computing fast Fourier transform in the recursive manner.
Time complexity: O(nlogn); space complexity: O(n)
  :param x: input array with a total number of points equal to the power of 2
  :param N: number of points to compute at the current recursion level
  :param step: step size at the current recursion level
  :param shift: initial shift for locating the odd-indexed part at the current recursion level
  :return FFT results with N data points
  """
  if N == 1:
    return x[shift]
  E = FFT(x, N/2, 2*step, shift)
  O = FFT(x, N/2, 2*step, shift + step)
  for k in range(N/2):
    F[k] = E[k] + exp(-2*i*pi*k/N) * O[k]
    F[k + N/2] = E[k] - exp(-2*i*pi*k/N) * O[k]
  return F
```

7. Spectral leakage and window functions

As discussed, DFS assumes the acquired signal input is periodic with a period equal to the length of the acquisition window. When the window adopted does not exactly match the actual signal periodicity, there are no frequencies in the FFT spectra that correspond exactly to the input signal. Hence, the main FFT frequencies spreads out to various adjacent frequencies, i.e., spectral leakage.

To handle this, one option is to zero pad the signal. Zero-padding increases the visual FFT spectra frequency resolution and helps identifying the spectral peaks. For instance, assume the input signal is a simple cosine function: $f(t) = \cos(5\pi t)$ with a frequency of 2.5Hz. If the defined acquisition window is 1s long, the frequency resolution in the FFT spectra will be 1Hz. Apparently, this resolution is not sufficient to display a 2.5Hz peak. However, if the signal is zero-padded to twice as long, then the FFT spectra frequency resolution increases to 0.5Hz, making a 2.5Hz peak readily revealed.

The downside of zero-padding is the presence of sidelobes, which are the results of the multiplication of the original signal with a rectangular window. To address, one can resort to non-rectangular windows, e.g., the Hanning window. Unlike the rectangular window, the Hanning window is more gracefully tapered to zero on both left and right ends, and hence has less sidelobes

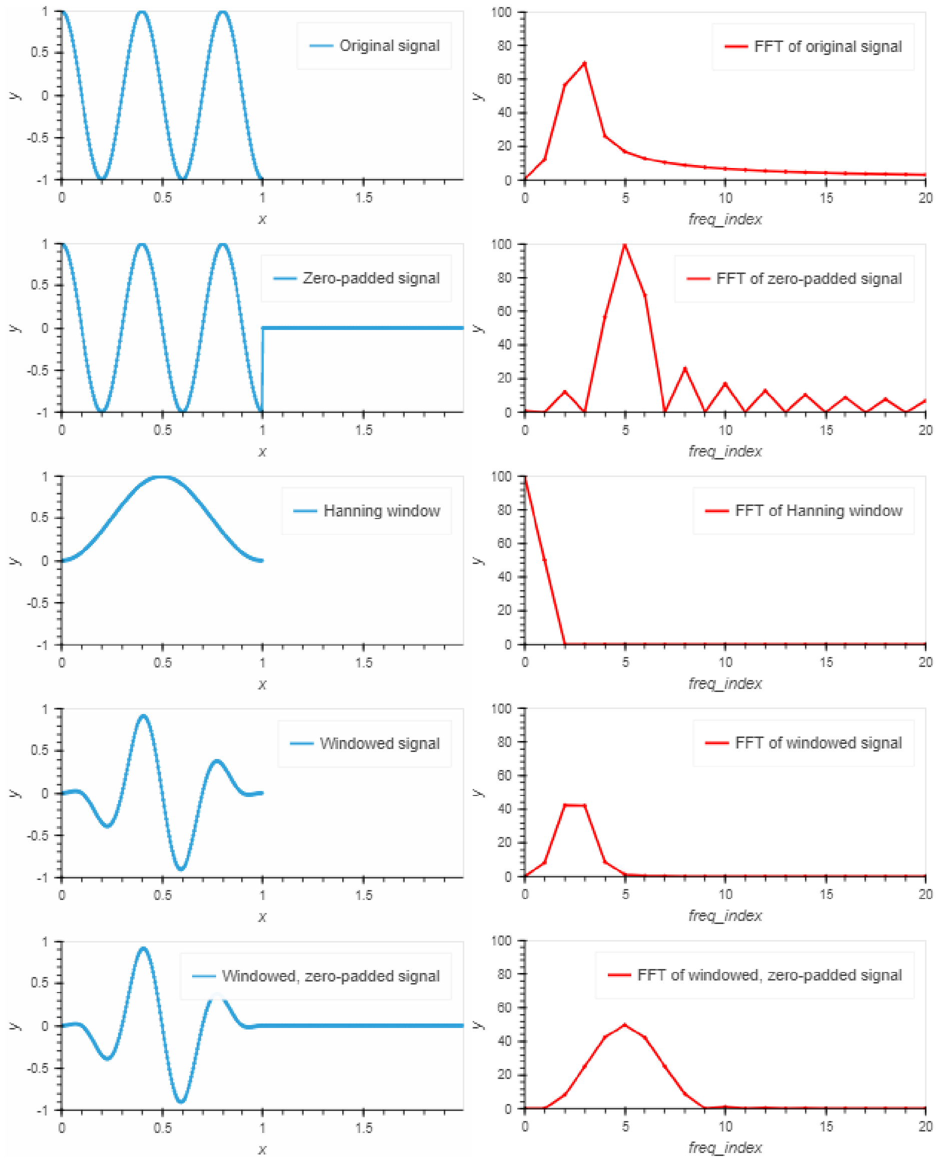

Figure 5. Row 1: FFT of a signal $f(t) = cos(5\pi t)$ whose periodicity does not match that of the acquisition window. The direct FFT result does not show a peak at 2.5 and demonstrates spectra leakage from the main peak to adjacent frequencies. Row 2: zero-padding the original signal increases the FFT spectra frequency resolution and reveals the peak with its real amplitude (now at 5 due to the doubled resolution). However, sidelobes are present. Row 3: FFT of the Hanning window. Row 4: FFT of the original signal modified (multiplied) by the Hanning window. The peak of 2.5 is still not displayed due to the limited frequency resolution. However, spectral leakage is substantially suppressed. Row 5: FFT of the original signal after adding the Hanning window and the zero-padding, the peak (now at 5) is revealed at reduced amplitude and sidelobes are suppressed

In reality, whenever the true frequency falls between the FFT frequency bins, the spectral leakage occurs and the amplitude shown in the FFT spectra is inaccurate. The inaccuracy reaches maximum when the true frequency is in the middle of the bin gap. Zero-padding dramatically resolves this issue.

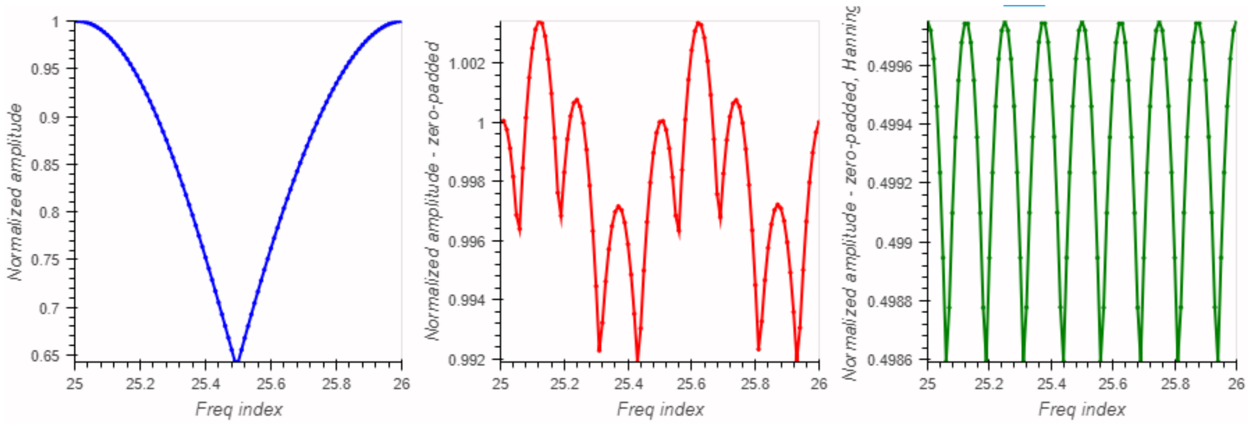

Figure 6. Left: Due to spectral leakage, the FFT amplitude becomes inaccurate whenever the true signal frequency falls between the gap of frequency bins (here 25 and 26). The inaccuracy reaches the maximum when the true signal is in the middle of the gap. Middle: zero-padding (padding length = 7 times original data length) dramatically reduces the amplitude error. Right: same level of amplitude error reduction for the zero-padded and windowed (Hanning) version

A general rule of thumb is the following:

- Use zero-padding to increase the FFT spectra frequency resolution – this will end with more accurate frequencies and amplitudes. The cost is the presence of sidelobes that might contaminate adjacent peaks
- To suppress the sidelobes, one can apply a window function (e.g., the Hanning window). However, in order to determine the original peak amplitude from the windowed FFT results, a scaling factor needs to be applied
- Different window functions have different properties. For instance, the Hanning window has a good balance between the frequency broadening and the amplitude accuracy. However, it might suffer from at most 15% amplitude error. In case accurate amplitudes are desired, the Flattop window should be used (link)

Window Type	Amplitude Correction	Energy Correction
Uniform	1.0	1.0
Hanning	2.0	1.63
Flattop	4.18	2.26
Blackman	2.80	1.97
Hamming	1.85	1.59
Kaiser-Bessel	2.49	1.86

Table 1. The scaling factor of various FFT window functions

8. Summary

a. Fourier family knowledge chain: FS → FT → DTFT → DFT → FFT

b. Fourier family always requires the input to be defined in the range $(-\infty, \infty)$

c. Input and output relationships

   I. FS: continuous, periodic → discrete, aperiodic

   II. FT: continuous, aperiodic → continuous, aperiodic

   III. DTFT: discrete, aperiodic → continuous, periodic (temporal sampling causes frequency replication, which also explains aliasing)

   IV. DFT: discrete, periodic → discrete, periodic

d. Sampling in one domain with sampling interval T_s corresponds to replication in the other domain with a period $2\pi/T_s$, and vice versa

e. Convolution in one domain corresponds to multiplication in the other domain, and vice versa

f. Both zero-padding and use of window functions help with spectra leakage

   I. Zero-padding increases the visual FFT spectra frequency resolution and helps reveal (not create) hidden information – however, sidelobes can appear

   II. Use of special window functions can suppress sidelobes