Matrix Shape, Polar Decomposition, Singular Value Decomposition

Linear Algebra

Winter 2022 - Dan Calderone

Matrix Shape

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Inner products between

… columns
$$A^T A \in \mathbb{R}^{n \times n} \qquad A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix} = \begin{bmatrix} |A_1||A_1| & \cdots & |A_1||A_n|\cos\theta_{1n} \\ \vdots & & \vdots \\ |A_n||A_1|\cos\theta_{1n} & \cdots & |A_n||A_n| \end{bmatrix}$$

… rows
$$AA^T \in \mathbb{R}^{m \times m}$$
$$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix} = \begin{bmatrix} |a_1||a_1| & \cdots & |a_1||a_n|\cos\theta_{1n} \\ \vdots & & \vdots \\ |a_n||a_1|\cos\theta_{1n} & \cdots & |a_n||a_n| \end{bmatrix}$$

Properties

Positive semi-definite
$$A^T A \succeq 0 \qquad AA^T \succeq 0$$
Proof: $x^T A^T A x = \left\Vert Ax \right\Vert_2^2 \geq 0$

Rank
$$\text{rk}(A) = \text{rk}(A^T) = \text{rk}(A^T A) = \text{rk}(AA^T) = k$$

Proof:
$$Ax = 0 \quad \Rightarrow \quad A^T A x = 0$$
$$A^T A x = 0 \quad \Rightarrow \quad 0 = x A^T A x = \left\Vert Ax \right\Vert_2^2 \quad \Rightarrow \quad Ax = 0$$
$$\mathcal{N}(A) = \mathcal{N}(A^T A) \qquad \textbf{rank-nullity theorem}$$

Matrix Shape

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Inner products between

… columns

$$A^T A \in \mathbb{R}^{n \times n} \qquad A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix} = \begin{bmatrix} |A_1||A_1| & \cdots & |A_1||A_n|\cos\theta_{1n} \\ \vdots & & \vdots \\ |A_n||A_1|\cos\theta_{1n} & \cdots & |A_n||A_n| \end{bmatrix}$$

… rows

$$AA^T \in \mathbb{R}^{m \times m}$$

$$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix} = \begin{bmatrix} |a_1||a_1| & \cdots & |a_1||a_n|\cos\theta_{1n} \\ \vdots & & \vdots \\ |a_n||a_1|\cos\theta_{1n} & \cdots & |a_n||a_n| \end{bmatrix}$$

Diagonalization: $\quad \mathrm{rk}(A) = k$

$$\Sigma \in \mathbb{R}^{k \times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

$$A^T A = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T = \overbrace{\begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix}}^{k \quad n-k} \begin{matrix} k\{ \\ n-k\{ \end{matrix} \overbrace{\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}}^{k \quad n-k} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}$$

Note: need to show same Σ works for $A^T A$ & AA^T … (later)

$$V \in \mathbb{R}^{n \times n}$$

$$V^T V = \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} \begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} = \begin{bmatrix} V'^T V' & V'^T V'' \\ V''^T V' & V''^T V'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$$U \in \mathbb{R}^{m \times m}$$

$$AA^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T = \overbrace{\begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix}}^{k \quad m-k} \begin{matrix} k\{ \\ m-k\{ \end{matrix} \overbrace{\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}}^{k \quad m-k} \begin{bmatrix} - & U'^T & - \\ - & U''^T & - \end{bmatrix} \begin{matrix} \}k \\ \}m-k \end{matrix}$$

$$U^T U = \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} = \begin{bmatrix} U'^T U' & U'^T U'' \\ U''^T U' & U''^T U'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Matrix Shape - positive semi-definite shape

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Matrix shapes   **relative shape of…**   **positive semi-definite**

… columns
$$\boxed{(A^T A)^{\frac{1}{2}}} \in \mathbb{R}^{n \times n} \qquad\qquad (A^T A)^{\frac{1}{2}} \succeq 0$$

… rows
$$\boxed{(A A^T)^{\frac{1}{2}}} \in \mathbb{R}^{m \times m} \qquad\qquad (A A^T)^{\frac{1}{2}} \succeq 0$$

Complex Number Analogy:

$$z = \boxed{|z|} e^{i\phi} = a + bi$$

$$|z| = \boxed{(z^* z)^{\frac{1}{2}}} = \sqrt{a^2 + b^2} \; \geq 0$$

complex number "shape", ie. Magnitude

Diagonalization: $\mathrm{rk}(A) = k$

$$(A^T A)^{\frac{1}{2}} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = \overbrace{\begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix}}^{k \quad n-k} {\scriptstyle \begin{matrix} k \\ n-k \end{matrix}} \overbrace{\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}}^{k \quad n-k} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} {\scriptstyle \begin{matrix} k \\ n-k \end{matrix}}$$

$$(A A^T)^{\frac{1}{2}} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T = \overbrace{\begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix}}^{k \quad m-k} {\scriptstyle \begin{matrix} k \\ m-k \end{matrix}} \overbrace{\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}}^{k \quad m-k} \begin{bmatrix} - & U'^T & - \\ - & U''^T & - \end{bmatrix} {\scriptstyle \begin{matrix} k \\ m-k \end{matrix}}$$

$$\Sigma \in \mathbb{R}^{k \times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

Note: need to show same Σ works for $A^T A$ & $A A^T$… (later)

$$V \in \mathbb{R}^{n \times n}$$
$$V^T V = \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} \begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} = \begin{bmatrix} V'^T V' & V'^T V'' \\ V''^T V' & V''^T V'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$$U \in \mathbb{R}^{m \times m}$$
$$U^T U = \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} = \begin{bmatrix} U'^T U' & U'^T U'' \\ U''^T U' & U''^T U'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Matrix Shape - orientation

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Matrix shapes   **orientation of…**   **orthonormal**   **Complex Number Analogy:**

$$z = |z| \boxed{e^{i\phi}} = a + bi$$

… columns $\qquad \boxed{A(A^T A)^{-\frac{1}{2}}} \in \mathbb{R}^{m \times n}$

$$\left((A^T A)^{-\frac{1}{2}} A^T \right) \left(A(A^T A)^{-\frac{1}{2}} \right) = I$$

…if full col rank

$$\boxed{e^{i\phi}}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

… rows $\qquad \boxed{(AA^T)^{-\frac{1}{2}} A} \in \mathbb{R}^{m \times n}$

$$\left((AA^T)^{-\frac{1}{2}} A \right) \left(A^T (AA^T)^{-\frac{1}{2}} \right) = I$$

…if full row rank

complex number orientation, ie. phase

$$\Sigma \in \mathbb{R}^{k \times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

Shape matrix:

… if full column rank

$$(A^T A)^{\frac{1}{2}} = V \Sigma V^T = V' \Sigma V'^T$$

… if full row rank

$$(AA^T)^{\frac{1}{2}} = U \Sigma U^T = U' \Sigma U'^T$$

Orientation matrix:

Note: proof using SVD…

$$A(A^T A)^{-\frac{1}{2}} = U V^T$$

$$(AA^T)^{-\frac{1}{2}} A = U V^T$$

Note: need to show same Σ works for $A^T A$ & AA^T… (later)

$$V \in \mathbb{R}^{n \times n}$$
$$V^T V = \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} \begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} = \begin{bmatrix} V'^T V' & V'^T V'' \\ V''^T V' & V''^T V'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$$U \in \mathbb{R}^{m \times m}$$
$$U^T U = \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} = \begin{bmatrix} U'^T U' & U'^T U'' \\ U''^T U' & U''^T U'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Polar Decomposition

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Polar Decomposition

Complex Number Analogy:

… columns

$$A = \underbrace{A(A^T A)^{-\frac{1}{2}}}_{\text{orthonormal}} \underbrace{(A^T A)^{\frac{1}{2}}}_{\text{PSD}}$$

…if full col rank

$$z = \boxed{|z|}\boxed{e^{i\phi}} = a + bi$$

complex number "shape", ie. Magnitude

$$|z| = \boxed{(z^* z)^{\frac{1}{2}}} = \sqrt{a^2 + b^2}$$

… rows

$$A = \underbrace{(AA^T)^{\frac{1}{2}}}_{\text{PSD}} \underbrace{(AA^T)^{-\frac{1}{2}} A}_{\text{orthonormal}}$$

…if full row rank

complex number orientation, ie. phase

$$\boxed{e^{i\phi}}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\Sigma \in \mathbb{R}^{k \times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

Shape matrix:

Orientation matrix:

… if full column rank

Note: proof using SVD…

Note: need to show same Σ works for $A^T A$ & AA^T… (later)

$$(A^T A)^{\frac{1}{2}} = V\Sigma V^T = V'\Sigma V'^T$$

$$A(A^T A)^{-\frac{1}{2}} = UV^T$$

$$V \in \mathbb{R}^{n \times n}$$

$$V^T V = \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} \begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} = \begin{bmatrix} V'^T V' & V'^T V'' \\ V''^T V' & V''^T V'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

… if full row rank

$$U \in \mathbb{R}^{m \times m}$$

$$(AA^T)^{\frac{1}{2}} = U\Sigma U^T = U'\Sigma U'^T$$

$$(AA^T)^{-\frac{1}{2}} A = UV^T$$

$$U^T U = \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} = \begin{bmatrix} U'^T U' & U'^T U'' \\ U''^T U' & U''^T U'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Singular Value Decomposition (SVD)

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Singular Value Decomposition

… take $\quad U' = AV'\Sigma^{-1} \qquad$ …valid eigenvectors with eigenvalues $\Sigma^2 \qquad AA^T U' = AA^T AV'\Sigma^{-1} = AV'\Sigma^{-1}\Sigma^2$

…orthonormal $\qquad \Sigma^{-1}V'^T A^T AV'\Sigma^{-1} = \Sigma^{-1}V'^T V'\Sigma^2\Sigma^{-1} = I$

$$\underbrace{\begin{bmatrix} U' & U'' \end{bmatrix}}_{U} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U'\Sigma & 0 \end{bmatrix} = \begin{bmatrix} AV' & 0 \end{bmatrix} = \begin{bmatrix} AV' & AV'' \end{bmatrix} = A \underbrace{\begin{bmatrix} V' & V'' \end{bmatrix}}_{V}$$

Diagonalization: $\quad \mathrm{rk}(A) = k$

$$\Sigma \in \mathbb{R}^{k \times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \quad \sigma_i > 0$$

$$A^T A = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T = \overbrace{\begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix}}^{k \quad n-k} \begin{matrix} k \\ n-k \end{matrix} \left\{ \overbrace{\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}}^{k \quad n-k} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} \right\} \begin{matrix} k \\ n-k \end{matrix}$$

…eigenvalues of $\quad (A^T A)^{\frac{1}{2}} \quad (AA^T)^{\frac{1}{2}}$

$$AA^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T = \overbrace{\begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix}}^{k \quad m-k} \begin{matrix} \\ m-k \end{matrix} \left\{ \overbrace{\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}}^{k \quad m-k} \begin{bmatrix} - & U'^T & - \\ - & U''^T & - \end{bmatrix} \right\} \begin{matrix} k \\ m-k \end{matrix}$$

$V \in \mathbb{R}^{n \times n} \quad$ …eigenvectors of $\quad (A^T A)^{\frac{1}{2}}$

$U \in \mathbb{R}^{m \times m} \quad$ …eigenvectors of $\quad (AA^T)^{\frac{1}{2}}$

Singular Value Decomposition (SVD)

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Singular Value Decomposition

… take $\quad U' = AV'\Sigma^{-1}$ …valid eigenvectors with eigenvalues $\Sigma^2 \qquad AA^T U' = AA^T AV'\Sigma^{-1} = AV'\Sigma^{-1}\Sigma^2$

…orthonormal $\qquad \Sigma^{-1}V'^T A^T AV'\Sigma^{-1} = \Sigma^{-1}V'^T V'\Sigma^2\Sigma^{-1} = I$

$$\begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U'\Sigma & 0 \end{bmatrix} = \begin{bmatrix} AV' & 0 \end{bmatrix} = \begin{bmatrix} AV' & AV'' \end{bmatrix} = A \underbrace{\begin{bmatrix} V' & V'' \end{bmatrix}}_{V}$$

$$\underbrace{\begin{bmatrix} U' & U'' \end{bmatrix}}_{U}$$

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = \begin{bmatrix} \overbrace{}^{k} & \overbrace{}^{m-k} \\ | & | \\ U' & U'' \\ | & | \end{bmatrix} \begin{array}{l} k\{ \\ \\ m-k\{ \end{array} \begin{bmatrix} \overbrace{\Sigma}^{k} & \overbrace{0}^{n-k} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} \begin{array}{l} \}k \\ \}n-k \end{array}$$

$$\Sigma \in \mathbb{R}^{k \times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

…eigenvalues of $\quad (A^T A)^{\frac{1}{2}} \quad (AA^T)^{\frac{1}{2}}$

U' orthonormal basis for $\mathcal{R}(A)$ $\qquad$ V' orthonormal basis for $\mathcal{R}(A^T)$ $\qquad$ $V \in \mathbb{R}^{n \times n}$ …eigenvectors of $\quad (A^T A)^{\frac{1}{2}}$

U'' orthonormal basis for $\mathcal{N}(A^T)$ $\qquad$ V'' orthonormal basis for $\mathcal{N}(A)$ $\qquad$ $U \in \mathbb{R}^{m \times m}$ …eigenvectors of $\quad (AA^T)^{\frac{1}{2}}$

Singular Value Decomposition (SVD)

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Singular Value Decomposition & Polar Decomposition

… full column rank

$$A = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T = \underbrace{A \left(A^T A\right)^{-\frac{1}{2}}}_{\text{orthonormal}} \underbrace{\left(A^T A\right)^{\frac{1}{2}}}_{\text{PD}} = \underbrace{\left(U' V^T\right)}_{\text{orthonormal}} \underbrace{\left(V \Sigma V^T\right)}_{\text{PD}}$$

… full row rank

$$A = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \underbrace{\left(AA^T\right)^{\frac{1}{2}}}_{\text{PD}} \underbrace{\left(AA^T\right)^{-\frac{1}{2}} A}_{\text{orthonormal}} = \underbrace{\left(U \Sigma U^T\right)}_{\text{PD}} \underbrace{\left(U V'^T\right)}_{\text{orthonormal}}$$

…general

$$A = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \underbrace{A \left(A^T A\right)^{\frac{\dagger}{2}}}_{\text{orthonormal}} \underbrace{\left(A^T A\right)^{\frac{1}{2}}}_{\text{PD}} = \underbrace{\left(U' V'^T\right)}_{\text{orthonormal}} \underbrace{\left(V' \Sigma V'^T\right)}_{\text{PD}}$$

$$= \underbrace{\left(AA^T\right)^{\frac{1}{2}}}_{\text{PD}} \underbrace{\left(AA^T\right)^{\frac{\dagger}{2}} A}_{\text{orthonormal}} = \underbrace{\left(U' \Sigma U'^T\right)}_{\text{PD}} \underbrace{\left(U' V'^T\right)}_{\text{orthonormal}}$$

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = \begin{bmatrix} \overbrace{|}^{k} & \overbrace{|}^{m-k} \\ U' & U'' \\ | & | \end{bmatrix} \begin{matrix} k\{ \\ \\ m-k\{ \end{matrix} \begin{bmatrix} \overbrace{\Sigma}^{k} & \overbrace{0}^{n-k} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}$$

$$\Sigma \in \mathbb{R}^{k \times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \quad \sigma_i > 0 \qquad\qquad V \in \mathbb{R}^{n \times n} \quad \text{…eigenvectors of} \quad (A^T A)^{\frac{1}{2}}$$

$$\text{…eigenvalues of} \quad (A^T A)^{\frac{1}{2}} \quad (AA^T)^{\frac{1}{2}} \qquad\qquad U \in \mathbb{R}^{m \times m} \quad \text{…eigenvectors of} \quad (AA^T)^{\frac{1}{2}}$$

SVD & Polar Decomposition

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Related Matrices *…use -1 to represent the Moore-Penrose pseudo inverse*

$$A = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \qquad A\left(A^T A\right)^{-\frac{1}{2}} \left(A^T A\right)^{\frac{1}{2}} = \left(A A^T\right)^{\frac{1}{2}} \left(A A^T\right)^{-\frac{1}{2}} A \qquad = \left(U' V'^T\right)\left(V' \Sigma V'^T\right) = \left(U' \Sigma U'^T\right)\left(U' V'^T\right)$$

$$A^T = \begin{bmatrix} V' & V'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} = \qquad A^T\left(A A^T\right)^{-\frac{1}{2}} \left(A A^T\right)^{\frac{1}{2}} = \left(A^T A\right)^{\frac{1}{2}} \left(A^T A\right)^{-\frac{1}{2}} A^T \qquad = \left(V' \Sigma V'^T\right)\left(V' U'^T\right) = \left(V' U'^T\right)\left(U' \Sigma U'^T\right)$$
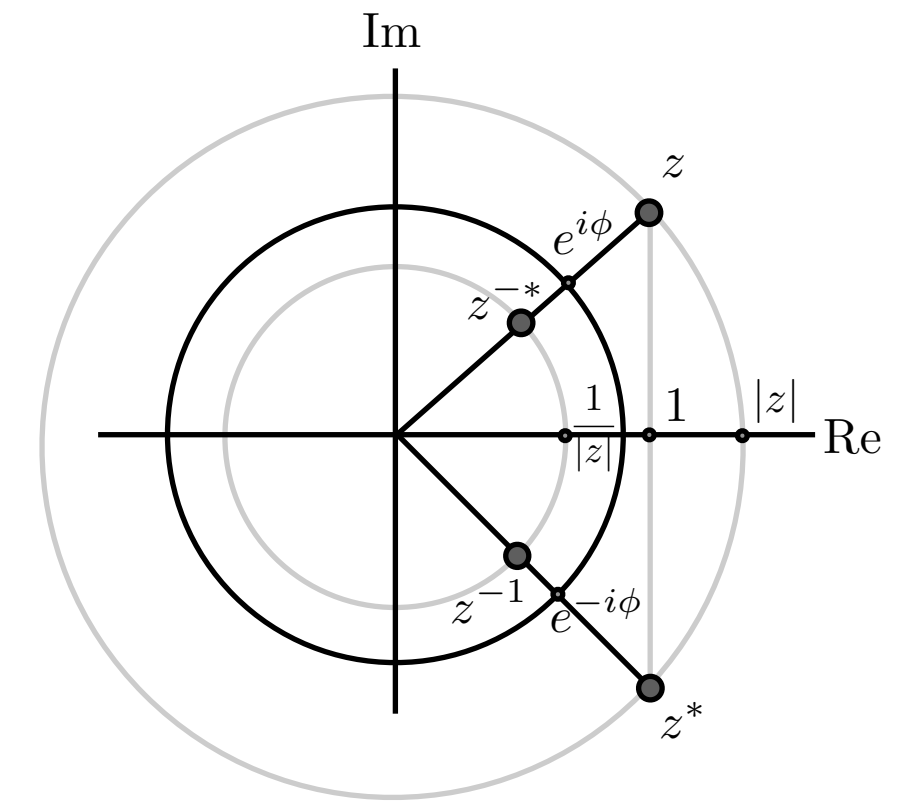
$$A^{-1} = \begin{bmatrix} V' & V'' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} = \qquad A^{-1}\left(A A^T\right)^{\frac{1}{2}} \left(A A^T\right)^{-\frac{1}{2}} = \left(A^T A\right)^{-\frac{1}{2}} \left(A^T A\right)^{\frac{1}{2}} A^{-1} \qquad = \left(V' \Sigma^{-1} V'^T\right)\left(V' U'^T\right) = \left(V' U'^T\right)\left(U' \Sigma^{-1} U'^T\right)$$

$$A^{-T} = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \qquad A^{-T}\left(A^T A\right)^{\frac{1}{2}} \left(A^T A\right)^{-\frac{1}{2}} = \left(A A^T\right)^{-\frac{1}{2}} \left(A A^T\right)^{\frac{1}{2}} A^{-T} \qquad = \left(U' V'^T\right)\left(V' \Sigma^{-1} V'^T\right) = \left(U' \Sigma^{-1} U'^T\right)\left(U' V'^T\right)$$

SVD & Polar Decomposition

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Related Matrices   *…use -1 to represent the Moore-Penrose pseudo inverse*

$$A = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \left(U'V'^T \right)\left(V'\Sigma V'^T \right) = \left(U'\Sigma U'^T \right)\left(U'V'^T \right)$$

$$A^T = \begin{bmatrix} V' & V'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} = \left(V'\Sigma V'^T \right)\left(V'U'^T \right) = \left(V'U'^T \right)\left(U'\Sigma U'^T \right)$$

$$A^{-1} = \begin{bmatrix} V' & V'' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} = \left(V'\Sigma^{-1} V'^T \right)\left(V'U'^T \right) = \left(V'U'^T \right)\left(U'\Sigma^{-1} U'^T \right)$$

$$A^{-T} = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \left(U'V'^T \right)\left(V'\Sigma^{-1} V'^T \right) = \left(U'\Sigma^{-1} U'^T \right)\left(U'V'^T \right)$$