# Matrices in $\mathbb{R}^{2\times 2}$

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***Abstract*—This tutorial paper gives basic visualizations for real matrices in $\mathbb{R}^{2\times 2}$ wih a primary focus on column geometry. Basic notation and basic column and row geometry are given followed by visualizations of several basic types of matrices. Matrix multiplication is discussed. Transposes and inverses are discussed with a focus on symmetric/skew-symmetric matrices and visualization. Similarity transforms and eigenvalue decompositions are discussed. Explicit algebraic characterization of eigenvalues/eigenvectors along with thorough visualizations. Special attention is then given to complex eigenvalues/eigenvectors including a discussion of rotation matrices. Symmetric and definite matrices are discussed in the context of quadratic forms. Grammians (shape matrices) are discussed; polar decompositions are derived and discussed, and the singular value decomposition is discussed in detail with analogies drawn from complex numbers. The final section of the paper details linear vector fields. The spectral mapping theorem and the matrix exponential are discussed along with stability criteria in continuous and discrete time.**

## I. Introduction

Real matrices in $\mathbb{R}^{2\times 2}$ show up in every corner of modern matheematics. Beyond being useful in their own right, they also provide foundational examples and intuition for studying invertible linear maps (square matrices) in general. In this paper, we give many detailed visualizations of basic structural results about $\mathbb{R}^{2\times 2}$. Of course, it cannot cover all facets of $2\times 2$ matrix structure, but it is meant to be thorough.

The initial section covers basic notation as well as gives basic details of column and row geometry and the image of sets under $2 \times 2$ linear transformations. Our analysis in this paper will focus primarily on column geometry as it is the most natural but row geometry will be discussed as well at several points. Matrix multiplication is discussed briefly as well. We next detail the structure of several basic classes of square $2 \times 2$ matrix structures including diagonal, upper/lower triangular, symmetric/skew-symmetric, rotations/reflections, and nilpotent structures. This section is meant to give a general flavor and build basic spatial intuition.

The next section discusses the geometry of the matrix transpose, ie. the geometry of the rows relative to the columns. While algebraically immediate, this geometry is actually fairly subtle. Particular attention is given to the symmetric and skew-symmetric portions of the matrix. Inverses are then discussed. Whereas transpose are algebraically simple and geometrically complicated, inverses have the opposite flavor (geometrically simple but algebraically complicated).

We then turn our attention to the rich subject of similarity transformations and eigenvalue decompositions. Similarity transforms are visualized with specific attention given to orthonormal similarity transforms (similarity transforms that are also congruent). The characteristic polynomial and it's relation to eigenvalues and left/right eigenvectors is thoroughly visualized and discussed. Formulas for eigenvalues, eigenvectors, and diagonalizations are given. Special attention is given to the complex eigenvalue case and pseudo-diagonal/rotational forms of complex eigenvalue decompositions. We also include a discussion of repeated eigenvalues, Jordan form, and nilpotent matrices. We then present in detail how eigenvalues and both left/right eigenvectors relate to the column geometry of a matrix in the both the real and complex eigenvalues cases. These particular visualizations are detailed and extensive. The complex case is then expanded further to detail it's rotational structure and specific attention is given to true rotation/reflection matrices. We also give specific attention to skew-symmetric matrices as real matrices with purely imaginary eigenvalues. Finally, we conclude the initial eigen-decomposition discussion with a brief discussion of the spectral mapping theorem. We next turn our attention to symmetric matrices in the context of quadratic forms. Quadratic form surfaces and their relationships to symmetric matrix eigenstructure is discussed. Positive-definite, negative-definite, and indefinite matrices are discussed.

The next section of the paper contains a detailed discussion of matrix shapes including the polar decomposition and singular value decomposition. The two Gramian matrices and, more importantly, their square roots, are discussed as the primary two definitions of matrix's positive definite "shape". From there we derive and visualize the polar decomposition in both contexts. Finally, we use the eigen-structure of the Gramian matrices to give the singular value decomposition (the classical construction) and give visualizations. Detailed connections between each of these decompositions as well as the sym/skew-sym decomposition are given as a thorough discussion of analogies with complex numbers in their Cartesian and polar form.

The final section of this paper details the structure of linear vector fields (ordinary differential equations) in the linear time invariant case. Basic solutions in the form of the matrix exponential in continuous and discrete time are given. The relationship between eigen-structure and trajectories is detailed. Stability criteria in both continous and discrete time are given in terms of eigenvalues as well as various parametric tests for stability. Some of these are classical results while others are somewhat novel.

**Remark 1.** *The primary section missing from this paper is perhaps one focusing on matrix commutators. The authors hope to add this section at some point in the future.*

### A. Prerequisites and Follow-ups

This paper can be read on it's own without much difficulty; however it does assume a familiarity with the notation and vector visualization techniques presented in the following monographs.

- Vector visualizations
- Column geometry

This paper is also meant to be part one of a three part series. The second paper expands many of these results/visualizations to real matrices in $\mathbb{R}^{3\times 3}$; the third paper discusses extensions to general matrices in $\mathbb{R}^{n\times n}$ with visualizations given in $\mathbb{R}^{4\times 4}$. This last paper is (of course) far less thorough since the space $\mathbb{R}^{n\times n}$ is a vast mathematical landscape that has never been fully explored. Any "thorough" discussion would have to include countless specific types of matrices. The visualizations in this last paper are also only meant to be experimental and to give a flavor for how an ambitious student of visualization might seek to extend the the visualization techniques in the first two papers to higher dimensional geometries. As such, they should only be viewed in parallel to the first two monographs. The authors also take no responsibility for any confusion that may result from viewing them. The dissatisfied reader is always heartily encouraged to make improvements or re-fall in love with pure algebraic insight. [1]



## II. Basic Column Geometry

The columns of a matrix $A \in \mathbb{R}^{m\times n}$ are vectors in the co-domain of the linear map

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$

Each individual column $A_j \in \mathbb{R}^m$ tells where the $j$th standard basis vector (in the domain) gets mapped under the transformation. Explicitly $AI_j = A_j$. We can see where a vector $x \in \mathbb{R}^n$ in the domain gets mapped by breaking up $x$ into a linear combination of standard basis vectors (ie. $x = I_1x_1 + \cdots + I_nx_n$), transforming each standard basis vector to the appropriate column, and then recombining. Algebraically, this is given by

$$\begin{aligned} Ax &= A\big(I_1x_1 + \cdots + I_nx_n\big) \\ &= AI_1x_1 + \cdots + AI_n \\ &= A_1x_1 + \cdots + A_nx_n \end{aligned}$$

For a $2\times 2$ matrix, we can expound on the eigen-structure precisely. This is useful for $2\times 2$ matrices specifically but it is also useful for getting intuition for matrices in general.

For the following, we will discuss the $2\times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} - & \bar{A}_1^T & - \\ - & \bar{A}_2^T & - \end{bmatrix}$$

Another way to represent $A$ that will prove useful is

$$A = \begin{bmatrix} m+h & p-k \\ p+k & m-h \end{bmatrix}$$

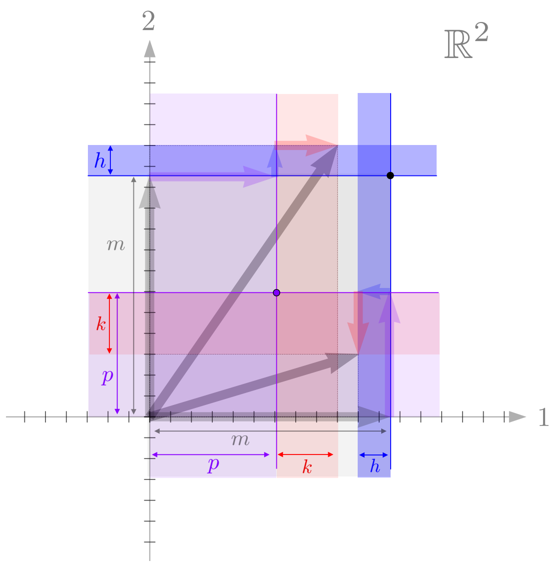
where

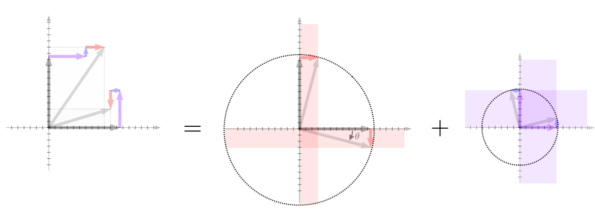
$$\begin{aligned} m &= \tfrac{1}{2}\big(a+d\big), \quad h = \tfrac{1}{2}\big(a-d\big), \\ p &= \tfrac{1}{2}\big(b+c\big), \quad k = h = \tfrac{1}{2}\big(c-b\big) \end{aligned}$$

Note that here $m$ and $p$ are the averages of the diagonal and off-diagonal elements (respectively) and $h$ and $k$ are the differences. This $mhpk$-parametrization is particularly useful for considering limiting cases or special types of matrices. We detail some of these in the image below. Only $m \neq 0$ is a scaled identity matrix; $m, h, p \neq 0$ is symmetric. only $k \neq 0$ is skew symmetric; only $m, k \neq 0$ is a scaled rotation; $h, p \neq 0$ is symmetric with zero trace. (This parametrization also suggests other interesting categories of $2\times 2$ matrices such as $m, p \neq 0$ (symmetric matrices with constant diagonal) or $h, k \neq 0$ (a zero-trace matrix) that has some rotation like properties.) We comment also that this parametrization suggests several matrix decompositions. Specifically, we will consider

$$A = \begin{bmatrix} m & -k \\ k & m \end{bmatrix} + \begin{bmatrix} h & p \\ p & -h \end{bmatrix}$$

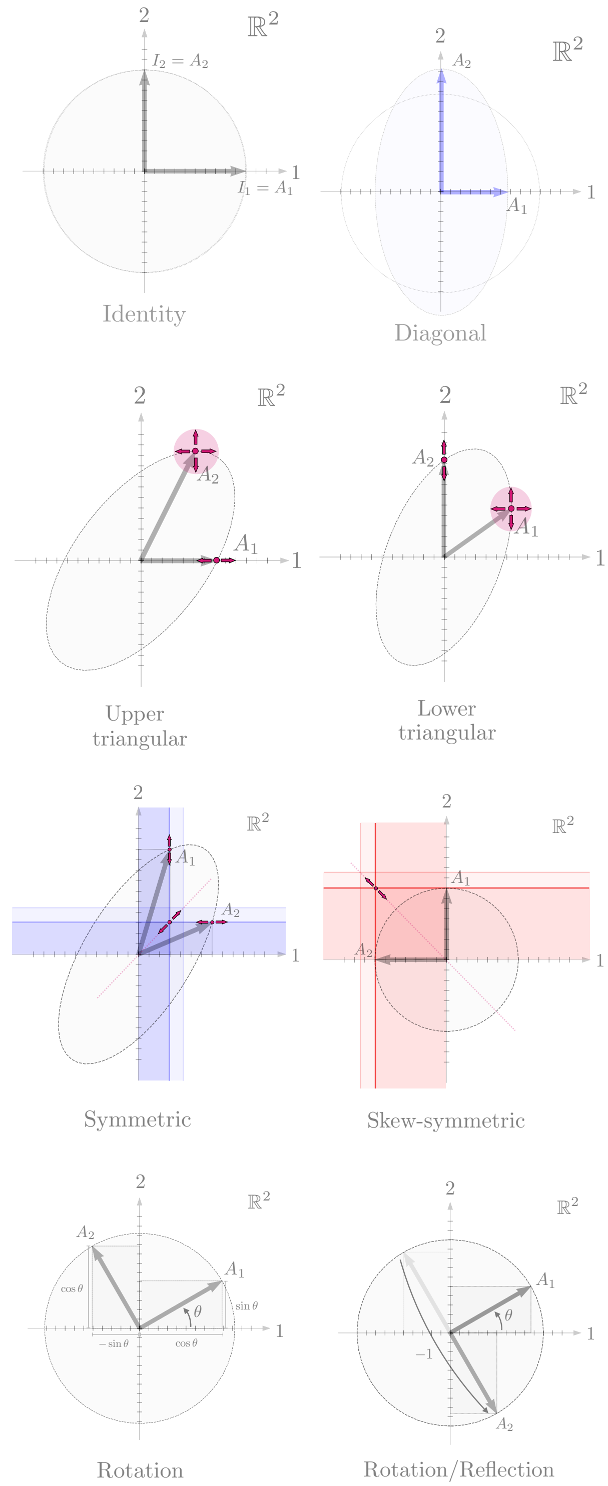
Here we've decomposed $A$ into two orthogonal matrices. the first is a scaled rotation and the second is a symmetric zero-trace matrix that turns out to be a scaled reflection.



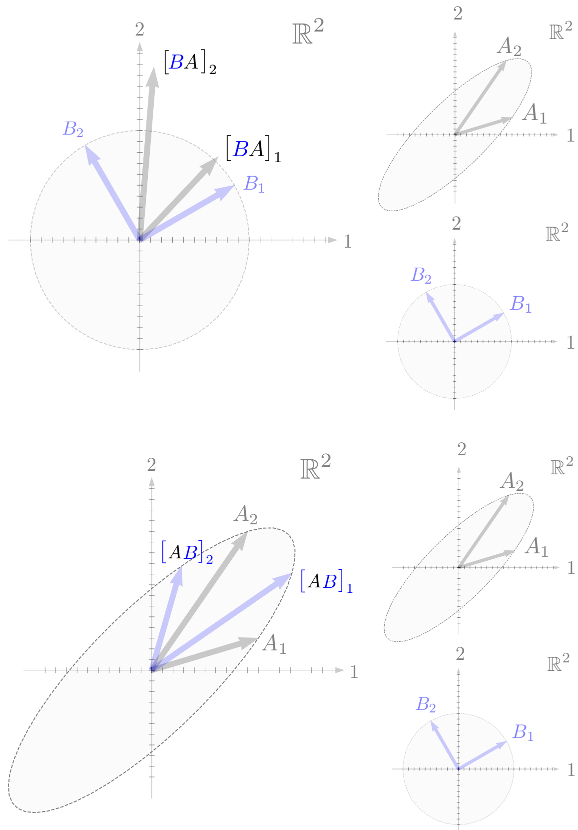

## III. Matrix Types

Identity, diagonal
Upper/lower triangular
Symmetric, Skew-symmetric
Rotations/Reflections



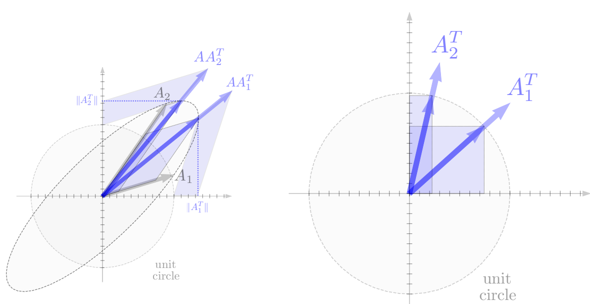

## IV. Matrix Multiplication

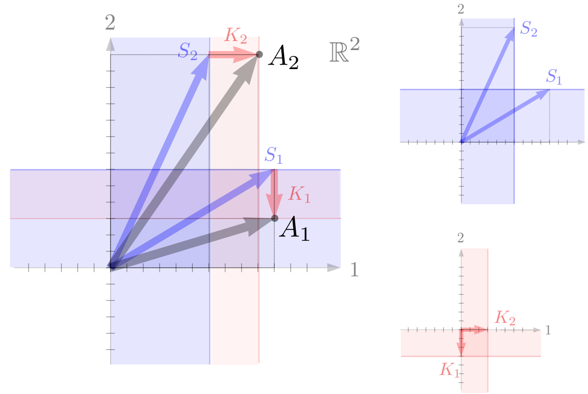
Left multiplication
Right multiplication

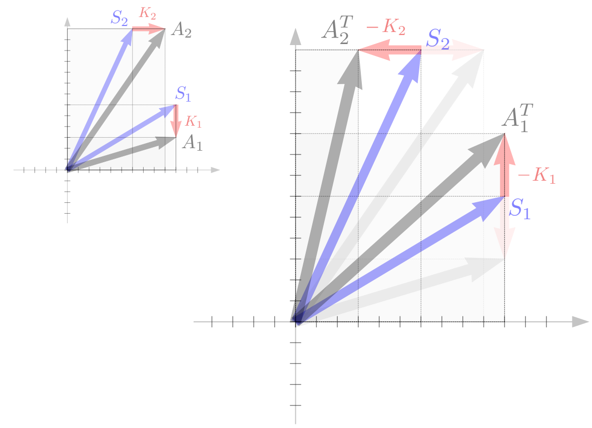



## V. Transposes

Rows in parallel
Range of $A^T$
Sym-skew decomposition




## VI. Determinant & Traces

The determinant of $A$ will also prove important.

$$det(A) = ad - bc = m^2 - h^2 - p^2 + k^2$$

## VII. Inverses

Geometry
Formulas

We also can note the inverse of $A$

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{det(A)} \begin{bmatrix} m-h & -p+k \\ -p-k & m+h \end{bmatrix}$$

## VIII. Similarity Transforms

Visualization
Orthonormal transformations

## IX. Eigenvalue Decomposition

Eigenvalues basics
Characteristic polynomial
Algebraic formulas
Real eigenvalues/eigenvectors

Complex eigenvalues/eigenvectors

The characteristic polynomial of the matrix is given by

$$char_A(s) = s^2-(a+d)s+(ad-bc) = s^2-tr(A)+det(A)$$

Note in the $mhpk$-parametrization this becomes

$$char_A(s) = s^2 - 2ms + m^2 - h^2 - p^2 + k^2$$

The eigenvalues are then given by the roots of the characteristic polynomial which in this case can be computed using the quadratic equation.

$$\lambda_{1,2} = \tfrac{1}{2}tr(A) \pm \sqrt{\left(\tfrac{tr(A)}{2}\right)^2 - det(A)}$$

$$\lambda_{1,2} = \tfrac{a+d}{2} \pm \sqrt{\left(\tfrac{a+d}{2}\right)^2 - ad - bc}$$

$$\lambda_{1,2} = m \pm \sqrt{h^2 + p^2 - k^2}$$

This last formula specifically gives us some direct insight into the structure of the eigenvalues. Below we list these ideas and also whatever generalizations there are to $n \times n$ matrices. First, the two eigenvalues are centered around $m$ which is the trace divided by 2 or the arithmetic mean of the diagonal. This extends to the $n \times n$; the arithmetic mean of the diagonal (ie. $tr(A)/n$) is the centroid of the eigenvalues. Secondly, the geometry of the vectors

$$u_\pm = \begin{bmatrix} m \\ \pm k \end{bmatrix}, \qquad v_\pm = \begin{bmatrix} \pm h \\ p \end{bmatrix},$$

tell us a lot about the matrix and it's eigenvalues. For a symmetric matrix ($k = 0$ ) the eigenvalues are given by

$$\lambda_{1,2} = m \pm \sqrt{h^2 + p^2} = m \pm \left|\left|\begin{bmatrix} h \\ p \end{bmatrix}\right|\right|_2$$

In this case, the definitiness of the matrix is determined by the relative size of $m$ with the norm of $v$. A symmetric $2 \times 2$ matrix is definite if and only if

$$|m| > \left|\left|\begin{bmatrix} h \\ p \end{bmatrix}\right|\right|_2$$

Positive or negative definite depends on the sign of $m$. The discrimant is given by

$$p^2 + h^2 - k^2 = \left|\left|\begin{bmatrix} h \\ p \end{bmatrix}\right|\right|_2^2 - k^2$$

$A$ has real eigenvalues if and only if

$$|k| \leq \left|\left|\begin{bmatrix} h \\ p \end{bmatrix}\right|\right|_2$$

If $p = h = 0$, then the eigenvalues are given by

$$\lambda_{1,2} = m \pm ki = \sqrt{m^2 + k^2}e^{\pm i\phi}$$

with $\phi = tan^{-1}\left(\frac{k}{m}\right)$ where the second equality gives the polar form. This last characterization shows a close relationship between a matrix of the form

$$A = \begin{bmatrix} m & -k \\ k & m \end{bmatrix} = \sqrt{m^2 + k^2}\begin{bmatrix} \frac{m}{\sqrt{m^2+k^2}} & -\frac{k}{\sqrt{m^2+k^2}} \\ \frac{k}{\sqrt{m^2+k^2}} & \frac{m}{\sqrt{m^2+k^2}} \end{bmatrix}$$
$$= \sqrt{m^2 + k^2}\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

and the complex conjugate pair $m \pm ki$. Indeed much of the intuition behind complex eigenvalues is grounded in understanding matrices of this form. Note that the last two equalites write the matrix as a scaled rotation closely related to the polar form of the eigenvalues.

From the above analysis, it seems profitable to plot $u_\pm$ and $v_\pm$ in a 2D space analogous to the complex plane. For $v_\pm = 0$ this space is precisely a picture of the complex plane and the vectors u$_\pm aretheeigenvaluesofthematrix.When$v$_\pm \neq 0$ we can modify the picture in the following way. Plot the vector $v_\pm$ and the ball it touches. Properties of the eigenvalues of $A$ are then given by what region $u_\pm$ falls in relative to the ball generated by $v_\pm$. These regions are shown in the diagram below

While the above eigenvalue analysis is insightful, we must also consider the eigenvectors in order to get a full picture of the action of a matrix. We will discuss primarily discuss right eigenvectors here, but analogous results apply to left eigenvectors as well.

For an eigenvalue $\lambda$, the right eigenvector is contained in the nullspace of

$$\lambda I - A = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}$$

Row-reducing this matrix gives

$$\underbrace{\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix} \frac{1}{\lambda-a} & 0 \\ 0 & 1 \end{bmatrix}}_{E_1}\begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} = \begin{bmatrix} 1 & -\frac{b}{\lambda-a} \\ 0 & \frac{(\lambda-d)(\lambda-a)-cb}{\lambda-a} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -\frac{b}{\lambda-a} \\ 0 & 0 \end{bmatrix}$$

where the last equation is from $\lambda$ being a root of the characteristic polynomial. We then have the following two characterizations of a right eigenvector for $\lambda$.

$$V_{1,2} \sim \begin{bmatrix} b \\ \lambda_{1,2} - a \end{bmatrix} \sim \begin{bmatrix} \lambda_{1,2} - d \\ c \end{bmatrix}$$

The first characterization here comes from the row reduction done above (where the $1,1$ entry of the matrix is taken as the pivot); the second characterization comes from if the $2,2$ entry of the matrix is taken as the pivot. We note also that each of these vectors is clearly orthogonal to one of the rows of the matrix above Since in a rank-1 $2\times2$ matrix the rows are just scalings of each other, being orthogonal to one row is the same as being orthogonal to other other so we could have have just read off both of these characterizations initially. (Again, note that this rank-1 condition (and thus the above eigenvector characterization) is not true for all $\lambda$ but only when $\lambda$ satisfies the characteristic equation.)

Along with picking vectors orthogonal to both (in this case, either) row, there is another way to read off eigenvectors based on diagonalizing $\lambda I - A$. If we diagonalize $\lambda I - A$ we get

$$\begin{aligned}\lambda I - A &= \begin{bmatrix}\lambda - a & -b \\ -c & \lambda - d\end{bmatrix} \\ &= \begin{bmatrix}| & | \\ V_1 & V_2 \\ | & |\end{bmatrix}\begin{bmatrix}\lambda - \lambda_1 & 0 \\ 0 & \lambda - \lambda_2\end{bmatrix}\underbrace{\begin{bmatrix}- & W_1^T & - \\ - & W_2^T & -\end{bmatrix}}_{V^{-1}}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix}\lambda - a & -b \\ -c & \lambda - d\end{bmatrix} = &\begin{bmatrix}| \\ V_1 \\ |\end{bmatrix}(\lambda - \lambda_1)\begin{bmatrix}- & W_1^T & -\end{bmatrix} \\ &+ \begin{bmatrix}| \\ V_2 \\ |\end{bmatrix}(\lambda - \lambda_2)\begin{bmatrix}- & W_2^T & -\end{bmatrix}\end{aligned}$$

If we plug in $\lambda_2$, then the second matrix term in the sum goes to 0 and we get that both columns of the resulting matrix are actually scalings of $V_1$. Similarly if we plug in $\lambda_1$, then the columns become scalings of $V_2$ From this we have that

$$V_1 \sim \begin{bmatrix}\lambda_2 - a \\ -c\end{bmatrix} \sim \begin{bmatrix}-b \\ \lambda_2 - d\end{bmatrix}$$

and that

$$V_2 \sim \begin{bmatrix}\lambda_1 - a \\ -c\end{bmatrix} \sim \begin{bmatrix}-b \\ \lambda_1 - d\end{bmatrix}$$

Note here that the of the eigenvalues/eigenvectors is opposite as opposed to above where it was the same. Note again that the lengths of of the eigenvectors (for each eigenvalue) here are not the same and one would need to work a little harder to show how they differ. Again each of these different subspace characterizations is only the same because $\lambda$ is a root of the characteristic polynomial. Any attempt to show that these vectors have the same span will involve using the fact that $(\lambda - a)(\lambda - d) - cb = 0$. The rank of $\lambda I - A$ dropping and it's relationship to the eigenvectors is given in the following illustration

Plugging in the eigenvalues for $\lambda$ to the above forms gives several specific characterizations.

$$V_{1,2} \sim \begin{bmatrix}b \\ \lambda_{1,2} - a\end{bmatrix} = \begin{bmatrix}p - k \\ -h\end{bmatrix} \pm \begin{bmatrix}0 \\ \sqrt{p^2 + h^2 - k^2}\end{bmatrix}$$

$$V_{1,2} \sim \begin{bmatrix}\lambda_{1,2} - d \\ c\end{bmatrix} = \begin{bmatrix}h \\ p + k\end{bmatrix} \pm \begin{bmatrix}\sqrt{p^2 + h^2 - k^2} \\ 0\end{bmatrix}$$

$$V_1 \sim \begin{bmatrix}\lambda_2 - a \\ -c\end{bmatrix} = \begin{bmatrix}-h \\ -(p + k)\end{bmatrix} - \begin{bmatrix}\sqrt{p^2 + h^2 - k^2} \\ 0\end{bmatrix}$$

$$V_1 \sim \begin{bmatrix}-b \\ \lambda_2 - d\end{bmatrix} = \begin{bmatrix}-(p - k) \\ h\end{bmatrix} - \begin{bmatrix}0 \\ \sqrt{p^2 + h^2 - k^2}\end{bmatrix}$$

$$V_2 \sim \begin{bmatrix}\lambda_1 - a \\ -c\end{bmatrix} = \begin{bmatrix}-h \\ -(p + k)\end{bmatrix} + \begin{bmatrix}\sqrt{h^2 + p^2 - k^2} \\ 0\end{bmatrix}$$

$$V_2 \sim \begin{bmatrix}-b \\ \lambda_1 - d\end{bmatrix} = \begin{bmatrix}-(p - k) \\ h\end{bmatrix} + \begin{bmatrix}0 \\ \sqrt{h^2 + p^2 - k^2}\end{bmatrix}$$

We also

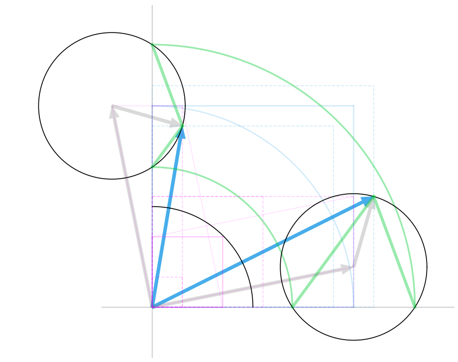
Some of these characterizations are illustrated in the visuals below, but we should also note that $m$ does not appear in any of the formulas. The reason for this is that $m$ gets added to both diagonal elements equally and thus provides the component of $A$ strictly proportional to the identity.

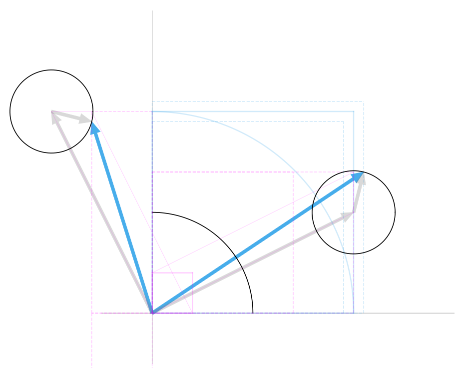
$$A = m\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} + \begin{bmatrix}h & p - k \\ p + k & h\end{bmatrix}$$

Thus for any value of $m$ the matrix has the same eigenvectors. Since adding a scaling of the identity shifts the eigenvalues but does not change the eigenvectors this is to be expected.

We now suggest a way to visualize the eigenstructure related to the column geometry of a $2 \times 2$ matrix. This visualization is quite dense and so we will build it up in stages.

We first look at the case where both eigenvalues are real.





## X. Complex Eigenvalues

Rotation shape
Rotation angle

## XI. Rotation Matrices

## XII. Definite Matrices

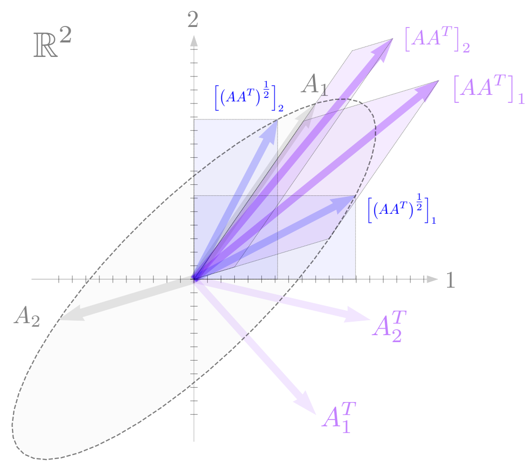
Quadratic forms
Eigenvectors (orthonormal)
Eigenvalues (real)
Positive, Negative, Indefinite

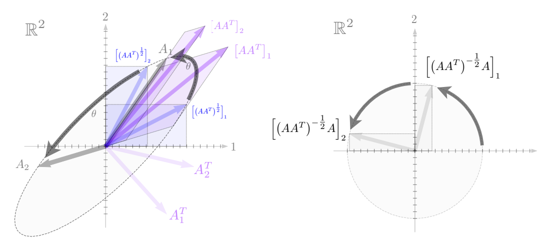
## XIII. Matrix Shapes

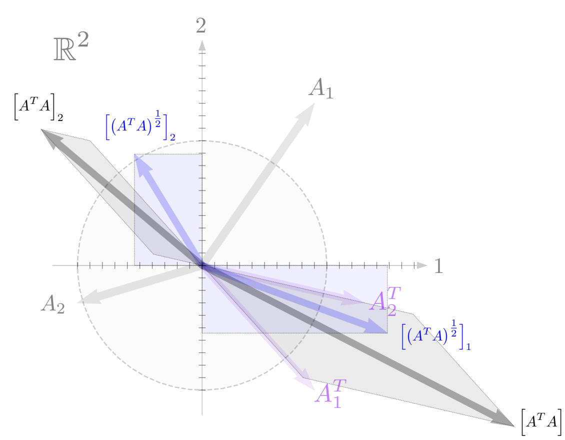
Grammians, rank
Matrix shapes

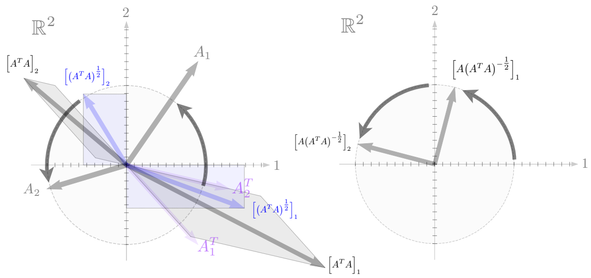
## XIV. Polar Decomposition

Form 1
Form 2
Complex number analogy








# XV. Singular Value Decomposition

## Forms

## Connection to polar decomposition

# XVI. Linear Vector Fields