KKT Matrix

Convex Optimization

Major sources:
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Skye Mceowen

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Quadratic Optimization & KKT System

$$\min_x \quad \frac{1}{2}x^\top Q x + c^\top x$$
$$\text{s.t.} \quad Ax = b$$

Primal Variables: $x \in \mathbb{R}^n$

Dual Variables: $v \in \mathbb{R}^m$

$Q \in \mathbb{R}^{n \times n}$ $\quad Q = Q^\top \succ 0$ $\quad c \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$ $\quad b \in \mathbb{R}^m$

fat $(m < n)$,
full row rank (rank $= m$)

Lagrangian: $$\mathcal{L}(x, v) = \frac{1}{2}x^\top Q x + c^\top x + v^\top \big(Ax - b\big)$$

KKT Conditions:

1. Stationarity: $x^\top Q + c^\top + v^\top A = 0$
2. Feasibility: $Ax - b = 0$

KKT System:

$$\begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

KKT Matrix

Solution:

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} -c \\ b \end{bmatrix}$$

KKT Solutions

$$\min_x \quad \frac{1}{2}x^\top Qx + c^\top x$$
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KKT Matrix

Solution:

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q^{-1} - Q^{-1}A^\top\left(AQ^{-1}A^\top\right)^{-1}AQ^{-1} & Q^{-1}A^\top\left(AQ^{-1}A^\top\right)^{-1} \\ \left(AQ^{-1}A^\top\right)^{-1}AQ^{-1} & -\left(AQ^{-1}A^\top\right)^{-1} \end{bmatrix}\begin{bmatrix} -c \\ b \end{bmatrix}$$

*…using **block matrix inversion** (or just directly verify).*

KKT Solutions

$$\min_x \quad \tfrac{1}{2}x^\top Q x + c^\top x$$
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KKT Matrix

Solution:

…projection matrix

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q^{-1}\boxed{\left[I - A^\top\left(AQ^{-1}A^\top\right)^{-1}AQ^{-1}\right]} & Q^{-1}A^\top\left(AQ^{-1}A^\top\right)^{-1} \\ \left(AQ^{-1}A^\top\right)^{-1}AQ^{-1} & -\left(AQ^{-1}A^\top\right)^{-1} \end{bmatrix} \begin{bmatrix} -c \\ b \end{bmatrix}$$

*…using **block matrix inversion** (or just directly verify).*

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KKT Matrix

Solution:

$$x = Q^{-1}A^\top \left(AQ^{-1}A^\top\right)^{-1}\left(AQ^{-1}c + b\right) - Q^{-1}c$$
$$v = -\left(AQ^{-1}A^\top\right)^{-1}\left(AQ^{-1}c + b\right)$$

KKT Matrix Properties

$$\min_x \quad \frac{1}{2}x^\top Qx + c^\top x$$
$$\text{s.t.} \quad Ax = b$$

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KKT System:

$$\underbrace{\begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}}_{\text{KKT Matrix}} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

Matrix Properties:

$$M = \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}$$

M indefinite

$$\operatorname{spec}(M) = \rho_+ \sqcup \rho_-$$

Positive: $\rho_+ = \{\lambda_1, \dots, \lambda_n\} > 0$

Negative: $\rho_- = \{\lambda_{n+1}, \dots, \lambda_{n+m}\} < 0$

Cardinality

$|\rho_+| = n$

$|\rho_-| = m$

Conditioned KKT Matrix

Coordinate Transforms

$$\min_{x} \quad \tfrac{1}{2}x'^\top P^\top Q P x' + c^\top P x'$$
$$\text{s.t.} \quad W^\top A P x' = W^\top b$$

Primal: $x = Px' \in \mathbb{R}^n$

Dual: $v = Wv' \in \mathbb{R}^m$

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Lagrangian: $\mathcal{L}(x', v') = \tfrac{1}{2}x'^\top P^\top Q P x' + c^\top P x' + v'^\top W^\top (APx' - b)$

KKT Conditions:

1. Stationarity: $x'^\top P^\top Q P + c^\top P + v'^\top W^\top A P = 0$
2. Feasibility: $W^\top A P x' - W^\top b = 0$

KKT System:

$$\begin{bmatrix} P^\top Q P & P^\top A^\top W \\ W^\top A P & 0 \end{bmatrix} \begin{bmatrix} x' \\ v' \end{bmatrix} = \begin{bmatrix} -P^\top c \\ W^\top b \end{bmatrix}$$

Conditioning:

$$M \Rightarrow \begin{bmatrix} P^\top Q P & P^\top A^\top W \\ W^\top A P & 0 \end{bmatrix} = \begin{bmatrix} P^\top & 0 \\ 0 & W^\top \end{bmatrix} \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & W \end{bmatrix}$$

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Conditioning: $AQ^{-\frac{1}{2}} = U \left[\Sigma \;\; 0\right] V^\top$ **(SVD)** $\Rightarrow$ $\boxed{P = Q^{-\frac{1}{2}} V \qquad W = U}$

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$$M \Rightarrow \begin{bmatrix} P^\top Q P & P^\top A^\top W \\ W^\top A P & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & \Sigma \\ 0 & I & 0 \\ \Sigma & 0 & 0 \end{bmatrix} \overset{\textbf{Permute...}}{\Longrightarrow} \begin{bmatrix} I & \Sigma & 0 \\ \Sigma & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

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Subblocks:

$$\begin{bmatrix} 1 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} \qquad \begin{array}{c} s(s-1) - \sigma_j^2 = 0 \\ \lambda_{j,j+1} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \sigma_j^2} \end{array}$$

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$$\text{spec} = \underbrace{\Big\{\overset{\times(n-m)}{1}, \tfrac{1}{2} + \overset{\times m}{\sqrt{\tfrac{1}{4} + \sigma_j^2}}}_{+}, \underbrace{\tfrac{1}{2} - \overset{\times m}{\sqrt{\tfrac{1}{4} + \sigma_j^2}}\Big\}}_{-}$$

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