Perturbation Analysis: KKT Conditions

Original
Stationarity
Condition

$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x, \theta\right) = 0$$

$$g\left(x, \theta\right) = 0$$

$\downarrow$ $\downarrow$

variables parameters

(defines shape
of constraints)

Perturbation Analysis: KKT Conditions

Feasibility Condition
(original constraint)

Original **Stationarity Condition**

$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^{\top}\frac{\partial g}{\partial x}\left(x, \theta\right) = 0$$

$$g\left(x, \theta\right) = 0$$

variables   parameters

(defines shape of constraints)

Perturb Shape of Constraints

$$\theta \rightarrow \theta + \Delta\theta$$

Question:
How does this affect other problem variables at optimum?

Perturbation Analysis: KKT Conditions

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$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^{\top}\frac{\partial g}{\partial x}\left(x, \theta\right) = 0$$

Feasibility Condition
(original constraint)

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$$\downarrow \quad \downarrow$$

variables   parameters

Perturb
Shape of
Constraints

$$\theta \ \to \theta + \Delta\theta$$

Question:
How does this affect
other problem variables
at optimum?

(defines shape
of constraints)

Perturbation Analysis

Perturbed
Feasibility
Conditions

$$g\left(x + \Delta x, \theta + \Delta\theta\right) = 0$$

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$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x, \theta\right) = 0$$

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Question:
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other problem variables
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Perturbation Analysis

Perturbed
Feasibility
Conditions

$$g\left(x + \Delta x, \theta + \Delta\theta\right) = 0 \qquad \approx \qquad g\left(x, \theta\right) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial\theta}\Delta\theta = 0$$

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(original constraint)

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Perturbation Analysis

Perturbed
Feasibility
Conditions

$$g\big(x + \Delta x, \theta + \Delta\theta\big) = 0 \qquad \approx \qquad g\big(x, \theta\big) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

Perturbed
Stationarity
Conditions

$$\frac{\partial f}{\partial x}\Big(x + \Delta x\Big) + \big(\lambda + \Delta\lambda\big)^\top \frac{\partial g}{\partial x}\Big(x + \Delta x, \theta + \Delta\theta\Big) = 0$$

Perturbation Analysis: KKT Conditions

Feasibility Condition
(original constraint)

Original **Stationarity Condition**

$$\frac{\partial f}{\partial x}(x) + \lambda^\top \frac{\partial g}{\partial x}(x, \theta) = 0$$

$$g(x, \theta) = 0$$

↓ ↓

variables parameters

Perturb Shape of Constraints

$$\theta \rightarrow \theta + \Delta\theta$$

Question: How does this affect other problem variables at optimum?

(defines shape of constraints)

Perturbation Analysis

Perturbed **Feasibility Conditions**

$$g(x + \Delta x, \theta + \Delta\theta) = 0 \qquad \approx \qquad g(x, \theta) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

Perturbed **Stationarity Conditions**

$$\frac{\partial f}{\partial x}(x + \Delta x) + (\lambda + \Delta\lambda)^\top \frac{\partial g}{\partial x}(x + \Delta x, \theta + \Delta\theta) = 0$$

$$\frac{\partial f}{\partial x}(x) + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + (\lambda + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i (\lambda_i + \Delta\lambda_i)\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \Delta\lambda_i)\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

Perturbation Analysis: KKT Conditions

Original **Stationarity Condition**

$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^{\top}\frac{\partial g}{\partial x}\left(x,\theta\right) = 0$$

Feasibility Condition
(original constraint)

$$g\left(x,\theta\right) = 0$$

↓   ↓

variables   parameters

Perturb Shape of Constraints

$$\theta \;\rightarrow\; \theta + \Delta\theta$$

Question:
How does this affect other problem variables at optimum?

(defines shape of constraints)

Perturbation Analysis

Perturbed **Feasibility Conditions**

$$g\left(x + \Delta x, \theta + \Delta\theta\right) = 0 \qquad \approx \qquad g\left(x,\theta\right) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

Perturbed **Stationarity Conditions**

$$\frac{\partial f}{\partial x}\left(x + \Delta x\right) + \left(\lambda + \Delta\lambda\right)^{\top}\frac{\partial g}{\partial x}\left(x + \Delta x, \theta + \Delta\theta\right) = 0$$

$$\frac{\partial f}{\partial x}(x) \;+\; \Delta x^{\top}\left[\frac{\partial^2 f}{\partial x^2}\right] \;+\; \left(\lambda + \Delta\lambda\right)^{\top}\left[\frac{\partial g}{\partial x}\right] \;+\; \sum_i \left(\lambda_i + \Delta\lambda_i\right)\Delta x^{\top}\left[\frac{\partial^2 g_i}{\partial x^2}\right] \;+\; \sum_i \left(\lambda_i + \Delta\lambda_i\right)\Delta\theta^{\top}\left[\frac{\partial^2 g_i}{\partial \theta \partial x}\right] \;=\; 0$$

Procedure: 1. Cancel terms…   2. Solve for $\Delta x, \Delta\lambda$ as functions of $\Delta\theta$.

Perturbation Analysis: KKT Conditions

Original
Conditions
$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x, \theta\right) = 0 \qquad g\left(x, \theta\right) = 0$$

Perturbed Conditions

$$\frac{\partial f}{\partial x}(x) \;+\; \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] \;+\; (\lambda + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right] \qquad\qquad g(x, \theta) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

$$+ \sum_i (\lambda_i + \Delta\lambda_i)\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] \;+\; \sum_i (\lambda_i + \Delta\lambda_i)\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] \;=\; 0$$

Perturbation Analysis: KKT Conditions

Original Conditions
$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x, \theta\right) = 0 \qquad g\left(x, \theta\right) = 0$$

Perturbed Conditions

$$\frac{\partial f}{\partial x}(x) \;+\; \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] \;+\; (\lambda + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right] \qquad\qquad g(x, \theta) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial \theta}\Delta\theta = 0$$

$$+ \sum_i (\lambda_i + \Delta\lambda_i)\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] \;+\; \sum_i (\lambda_i + \Delta\lambda_i)\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial \theta \partial x}\right] \;=\; 0$$

1. Cancel zero-order terms…

Perturbation Analysis: KKT Conditions

Original Conditions

$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^{\top}\frac{\partial g}{\partial x}\left(x,\theta\right) = 0 \qquad g\left(x,\theta\right) = 0$$

Perturbed Conditions

$$\frac{\partial f}{\partial x}(x) \;+\; \Delta x^{\top}\left[\frac{\partial^2 f}{\partial x^2}\right] \;+\; (\lambda + \Delta\lambda)^{\top}\left[\frac{\partial g}{\partial x}\right]$$

$$+ \sum_i \left(\lambda_i + \Delta\lambda_i\right)\Delta x^{\top}\left[\frac{\partial^2 g_i}{\partial x^2}\right] \;+\; \sum_i \left(\lambda_i + \Delta\lambda_i\right)\Delta\theta^{\top}\left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] \;= 0$$

$$g(x,\theta) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial\theta}\Delta\theta = 0$$

1. Cancel zero-order terms…   **2. Cancel second-order terms…**

Perturbation Analysis: KKT Conditions

Original
Conditions
$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x, \theta\right) = 0 \qquad g\left(x, \theta\right) = 0$$

Perturbed Conditions

$$\cancel{\frac{\partial f}{\partial x}(x)} + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + (\cancel{\lambda} + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x}\right]$$

$$\qquad + \sum_i (\lambda_i + \cancel{\Delta\lambda_i})\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \cancel{\Delta\lambda_i})\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\cancel{g(x, \theta)} + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial\theta}\Delta\theta = 0$$

1. Cancel zero-order terms…   **2. Cancel second-order terms…**

$$\Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + \Delta\lambda^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i \lambda_i \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i \lambda_i \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial\theta}\Delta\theta = 0$$

Perturbation Analysis: KKT Conditions

Original Conditions

$$\frac{\partial f}{\partial x}(x) + \lambda^\top \frac{\partial g}{\partial x}(x, \theta) = 0 \qquad g(x, \theta) = 0$$

Perturbed Conditions

$$\cancel{\frac{\partial f}{\partial x}(x)} + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + \cancel{(\lambda + \Delta\lambda)^\top} \left[\frac{\partial g}{\partial x}\right]$$

$$+ \sum_i (\lambda_i + \cancel{\Delta\lambda_i}) \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i (\lambda_i + \cancel{\Delta\lambda_i}) \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\cancel{g(x, \theta)} + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial\theta}\Delta\theta = 0$$

1. Cancel zero-order terms…   **2. Cancel second-order terms…**

$$\Delta x^\top \left[\frac{\partial^2 f}{\partial x^2}\right] + \Delta\lambda^\top \left[\frac{\partial g}{\partial x}\right] + \sum_i \lambda_i \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2}\right] + \sum_i \lambda_i \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x}\right] = 0$$

$$\frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial\theta}\Delta\theta = 0$$

organize into a system of equations…

$$\begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x\partial\theta} \\ \frac{\partial g}{\partial\theta} \end{bmatrix} \Delta\theta$$

$$Q = \frac{\partial^2 f}{\partial x^2} + \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x^2}$$

$$A = \frac{\partial g}{\partial x}$$

Perturbation Analysis: KKT Conditions

$$\frac{\partial f}{\partial x}(x) + \Delta x^\top \frac{\partial^2 f}{\partial x^2} + \lambda^\top \left[\frac{\partial g}{\partial x}(x,\theta) + \frac{\partial^2 g}{\partial x^2}\Delta x + \frac{\partial^2 g}{\partial x \partial \theta} \right.$$

$$+ \Delta \lambda^\top \left[\frac{\partial g}{\partial x}(x,\theta) + \frac{\partial^2 g}{\partial x^2}\Delta x + \right.$$

Original Conditions

$$\frac{\partial f}{\partial x}\left(x\right) + \lambda^\top \frac{\partial g}{\partial x}\left(x,\theta\right) = 0 \qquad g\left(x,\theta\right) = 0$$

Perturbed Conditions

$$\frac{\partial f}{\partial x}(x) + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2} \right] + (\lambda + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x} \right]$$

$$+ \sum_i (\lambda_i + \Delta\lambda_i)\Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2} \right] + \sum_i (\lambda_i + \Delta\lambda_i)\Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x} \right] = 0$$

$$g(x,\theta) + \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial\theta}\Delta\theta = 0$$

1. Cancel zero-order terms…   **2. Cancel second-order terms…**

$$\Delta x^\top \left[\frac{\partial^2 f}{\partial x^2} \right] + \Delta\lambda^\top \left[\frac{\partial g}{\partial x} \right] + \sum_i \lambda_i \, \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2} \right] + \sum_i \lambda_i \, \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial\theta\partial x} \right] = 0 \qquad \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial\theta}\Delta\theta = 0$$

organize into a system of equations…

$$\begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x \partial \theta} \\ \frac{\partial g}{\partial\theta} \end{bmatrix} \Delta\theta$$

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Perturbation Analysis: KKT Conditions

$$\frac{\partial f}{\partial x}(x) + \Delta x^\top \frac{\partial^2 f}{\partial x^2} + \lambda^\top \left[\frac{\partial g}{\partial x}(x,\theta) + \frac{\partial^2 g}{\partial x^2}\Delta x + \frac{\partial^2 g}{\partial x \partial \theta} \right.$$

Original
Conditions

$$\frac{\partial f}{\partial x}(x) + \lambda^\top \frac{\partial g}{\partial x}(x,\theta) = 0 \qquad g(x,\theta) = 0$$

$$+ \Delta\lambda^\top \left[\frac{\partial g}{\partial x}(x,\theta) + \frac{\partial^2 g}{\partial x^2}\Delta x + \right.$$

Perturbed Conditions

$$\frac{\partial f}{\partial x}(x) + \Delta x^\top \left[\frac{\partial^2 f}{\partial x^2} \right] + (\lambda + \Delta\lambda)^\top \left[\frac{\partial g}{\partial x} \right]$$

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1. Cancel zero-order terms...   **2. Cancel second-order terms...**

$$\Delta x^\top \left[\frac{\partial^2 f}{\partial x^2} \right] + \Delta\lambda^\top \left[\frac{\partial g}{\partial x} \right] + \sum_i \lambda_i \Delta x^\top \left[\frac{\partial^2 g_i}{\partial x^2} \right] + \sum_i \lambda_i \Delta\theta^\top \left[\frac{\partial^2 g_i}{\partial \theta \partial x} \right] = 0$$

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$$\begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} Q & A^\top \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x \partial \theta} \\ \frac{\partial g}{\partial \theta} \end{bmatrix} \Delta\theta$$

$$Q = \frac{\partial^2 f}{\partial x^2} + \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x^2}$$

$$A = \frac{\partial g}{\partial x}$$

$$= \begin{bmatrix} Q^{-1} - Q^{-1}A^\top (AQ^{-1}A^\top)^{-1} AQ^{-1} & Q^{-1}A^\top (AQ^{-1}A^\top)^{-1} \\ (AQ^{-1}A^\top)^{-1} AQ^{-1} & -(AQ^{-1}A^\top)^{-1} \end{bmatrix} \begin{bmatrix} \sum_i \lambda_i \frac{\partial^2 g_i}{\partial x \partial \theta} \\ \frac{\partial g}{\partial \theta} \end{bmatrix} \Delta\theta$$