Multi-Dimensional Continuous Type Population Potential Games

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***Abstract*— We consider an extension of continuous population potential games where the population mass is represented by a distribution over a multi-dimensional continuous type space. Each dimension of the type space corresponds to a subset of strategies in the potential game. The difference between two elements in each population member's type vector encodes their relative preference for strategies in each subset. We define an extension of a Wardrop equilibrium for this type based model and show how to compute the equilibrium by means of optimizing a specific potential function.**

I. Introduction

Continuous population potential games [1] have been a well studied fixture of the game theory community particularly in the traffic assignment problem [2]. In these games, individual agents are modeled as infinitesimal masses in a continuous population choosing from a finite set of strategies. The Nash-like equilibrium concept for these games is known as a *Wardrop equilibrium* [3] and in many cases can be characterized as the optimizer of a potential function within some constrained set [1], [4].

A restriction of standard population potential games is that they treat each member of the population as being identical and anonymous. Several lines of research have sought to alleviate this restriction. One of simplest and most common models is what is called the *variable demand* version of potential games where the total mass playing the game fluctuates with the equilibrium reward value population members receive [5], [6]. One can conceptualize this as some members of the population opting out of playing if the reward is too low. Another more complex class of models considers users as making decisions based on more than one reward criteria. These *multi-criteria equilibrium models* [7]–[9] often involve users considering the trade-off between travel time and monetary cost (tolls). Some models consider population users to belong to discrete type sets with different time-money trade-off parameters [10]–[12]. Another interesting class of models considers population members as a distribution over a continuous parameter space [13]–[17]. The equilibrium concepts in these continuous type models make use of variational inequality formulations to describe both how the population divides itself among routes and how the type space is partitioned into distinct regions in which population members choose specific strategies.

In this work, we propose a multi-dimensional continuous type model where the population is represented by a distribution supported over a multi-dimensional type space. Each population member has a type-vector that encodes their arbitrary preferences for specific strategy options in the potential game. We define an equilibrium concept that extends the traditional Wardrop equilibrium to this continuous type model and define a potential function and the corresponding optimization problem for computing the equilibrium. Our model is related to the bicriterion continuous type models listed above and reduces to the variable demand model in the one-dimensional case. The analysis in this paper bears some resemblance to the the design of multidimensional screening mechanisms or contracts [18], [19].

The rest of the paper is organized as follows. In Section II, we given an overview of congestion game models and classical Wardrop equilibria with several examples. In Section III, we formulate a multivariate type mass distribution model and expand the notion of a Wardrop equilibria to a *multivariate continuous type Wardrop equilibrium*. In Section IV, we give a potential function that can be used to solve for the equilibrium and detail it's properties. We give specific attention to it's first and second derivatives and also compute it's Legendre transform. In Section V, we detail the potential optimization program and show that it's optimizer encodes the expanded Wardrop equilibrium. In Section VI, we give subgradient algorithms for equilibrium computation. In Section VII, we give numerical example and in Section VIII we discuss future work.

II. Congestion Game Formulation

Continuous population potential games are games where individual agents are modeled as infinitesimal masses in a continuous population. Each population member chooses from a finite number of strategies (ex. a set of routes in a transportation network) and the aggregate choices of individuals results in an overall distribution of the population mass over a state space (city wide traffic congestion.) The payoff each member receives depends on their individual strategy and also the overall aggregate population behavior (ex. travel time on a route depends on the route and the traffic in the city). In a slightly broader class of models, population members must select a general group of strategies to consider along with a specific strategy In transportation networks, for example, these groups might correspond to different transportation modes such as driving, taking public transit, or biking. We present a general form for such congestion games and then specific examples.

1) Population Masses: Let $\mathcal{I}$ refer to set of overall options or modes that population members can choose from and let $m_i \in \mathbb{R}_+$ refer to the amount of population that chooses option $i \in \mathcal{I}$ with $\sum_i m_i \leq M$ where $M \in \mathbb{R}_+$ is the total

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mass. Within each option let Ξ_i represent a set of strategies that each member can choose from. For example $\mathcal{I}$ might be a set of transportation options such as driving, bus, walking, etc. and within each option Ξ_i refers to a set of routes individuals can choose. Let $x^i \in \mathcal{R}_+^q$ refer to a population mass vector resulting from the aggregate choices of the population choosing option i and let $x = (x^1, \ldots, x^{|\mathcal{I}|})$ refer to the collected distribution vectors. Depending on the context the elements of x^i may refer directly to the masses that choose each particular strategy ($q = |\Xi_i|$) or it may also refer to a distribution that results from the strategy choices. In the routing case for example x^i will usually refer to a mass distribution over edges in a transportation network.

2) Rewards and Equilibrium: Each mass vector x^i then gets plugged into a reward (or loss) function $\mathbf{r}^i : x^i \mapsto \mathbb{R}^q$. In general, for modeling congestion effects $\mathbf{r}_k^i(\cdot)$ will be strictly decreasing. For a particular, strategy in $\xi \in \Xi_i$, the reward each player gets, $\mathbf{R}^i(x, m, \xi)$ will depend on both the mass distributions m and x and the actual strategy ξ (specifics below) usually taken as some weighted-sum of the elements in $\mathbf{r}^i(x^i)$ (see specific examples below).

The *Wardrop Equilibrium* is the most commonly adopted equilibrium notion in these continuous population games as it can be thought of as a Nash equilibrium for infinitesimal agents; *no infinitesimal player can improve their reward by switching strategies.*

Definition 1 (Wardrop Equilibrium [3])**.** *A mass distribution (m, x) is a Wardrop equilibrium if*

$$\mathbf{R}^i(x, m, \xi) \geq \mathbf{R}^{i'}(x, m, \xi'), \quad \textit{if } supp(\xi) \subset supp(x^i) \quad (1)$$

for $\xi \in \Xi_i$ and any other $\xi' \in \Xi_{i'}$ where $supp(\cdot)$ denotes the support of a distribution.

3) Potential Functions: If the rewards satisfy a differential symmetry condition, $\frac{\partial \mathbf{r}_k^i}{\partial x_{k'}} = \frac{\partial \mathbf{r}_{k'}^i}{\partial x_k}$, then $\mathbf{r}^i(x^i)$ define a conservative vector field and we can construct a potential function $\mathbf{F}^*(x)$ whose gradient encodes the cost information, ie. $\frac{\partial \mathbf{F}^*}{\partial x^i} = \mathbf{r}^i(x^i)^\top$. When this is the case the Wardrop equilibrium conditions can be framed as the KKT optimality conditions of an optimization problem over the space of possible population distributions (detailed below). A commonly used potential function is

$$\mathbf{F}^*(x) = \sum_i \sum_{k=0}^{q} \int_0^{x_k^i} \mathbf{r}_k^i(u)\ du$$

We use the * notation because we will be more interested in the Legendre transform of $\mathbf{F}^*$ (which we will denote $\mathbf{F}$) which can be shown to be

$$\mathbf{F}(r) = \max_x\ r^\top x - \mathbf{F}^*(x) = \sum_i \sum_{k=0}^{q} \int_{\mathbf{r}_k^{i\ -1}(0)}^{r_k^i} \mathbf{r}_k^{i\ -1}(u)\ du$$

We also want to encode how the masses m_i that participate in the game are determined. There are several options for different contexts. First, the simplest option is to assume that the mass that chooses each subgame m_i is fixed and not variable. Second, we can assume that mass is free to shift from one option to another by simply enforcing an overall mass conservation constraint $\sum_i m_i = M$. Implicitly, this assumes that each population member values each option $i \in \mathcal{I}$ equally. Lastly, within traditional methods we can assume there are separate populations that consider each option and how much of each population decides to play is dependent on a demand-type function $\mathbf{p}_i : m_i \mapsto p_i$ that says how much the payoff p_i of option i must be in order for mass m_i to participate in the game. In general, $\mathbf{p}_i(\cdot)$ will be strictly increasing, ie. for more mass m_i to choose i, the payoff p_i must be higher. For this last option, these demand functions can be encoded in a potential of the form.

$$\mathbf{G}^*(m) = \sum_i \int_0^{m_i} \mathbf{p}_i(u)\ du$$

with Legendre transform

$$\mathbf{G}(p) = \min_m\ p^\top m - \mathbf{G}^*(m) = \sum_i \int_{\mathbf{m}_i(0)}^{p_i} \mathbf{m}_i(u)\ du$$

with $\mathbf{m}_i(\cdot) = \mathbf{p}_i^{-1}(\cdot)$. The function $\mathbf{m}_i(p_i)$ is actually a more typical demand function that gives the amount of mass willing to participate in option i for a payoff of p_i.

The primary goals of this paper is to provide a more sophisticated way to model populations that consider tradeoffs between various options and provide appropriate "demand" functions $\mathbf{G}(p)$ (and $\mathbf{G}^*(m)$).

Note that the Legendre transforms are defined so that $\frac{\partial \mathbf{F}}{\partial r_k^i} = \mathbf{r}_k^{i\ -1}(r)^\top$ and $\frac{\partial \mathbf{G}}{\partial p} = m^\top$ and $\frac{\partial \mathbf{F}^*}{\partial r_i}(\cdot) = \frac{\partial \mathbf{F}}{\partial x_i}^{-1}(\cdot)$, $\frac{\partial \mathbf{G}^*}{\partial p_i}(\cdot) = \frac{\partial \mathbf{G}}{\partial m_i}^{-1}(\cdot)$

4) Optimization Formulations: Given the potential functions, equilibria can be computed as maximizers of the following optimization program written in a general form.

$$\max_{x,m}\quad \mathbf{F}^*(x) - \mathbf{G}^*(m) \quad \text{s.t.} \quad A^i x^i = B^i m_i,\ x^i \geq 0\ \forall i \tag{2}$$

Here as discussed above m_i is the total mass choosing option i and the potential $\mathbf{G}^*(m)$ may be replaced by either fixed masses m_i or an overall mass constraint $\sum_i m_i = M$. x^i is the resulting mass distribution and the constraint encodes some mass conservation condition. The dual program is given by

$$\min_{x,m}\quad \mathbf{G}(p) - \mathbf{F}(r) \quad \text{s.t.} \quad \begin{array}{c} {r^i}^\top - {v^i}^\top A^i + {\mu^i}^\top = 0\ \forall i \\ {\mu^i}^\top \geq 0,\ p_i = {v^i}^\top B^i\ \forall i \end{array} \tag{3}$$

Here $r^i \in \mathbb{R}^q$ and r_k^i is a reward value experienced by individuals in mass x_k^i. $\mu^i \in \mathbb{R}_+^q$ and μ_k^i is any inefficiency experienced by population x_k^i. $v_i^\top$ represents some value function related to the conservation constraints and $p_i \in \mathbb{R}$ is the overall reward by mass m_i.

The KKT optimality conditions (for both the primal and dual problems) are given by

$$\begin{array}{c} \frac{\partial \mathbf{F}}{\partial r^i} = {x^i}^\top,\quad \frac{\partial \mathbf{G}}{\partial p} = m^\top \quad A^i x^i = B^i m_i, \quad x^i \geq 0 \\ {r^i}^\top - {v^i}^\top A^i + {\mu^i}^\top = 0,\ {v^i}^\top B^i = p_i,\ \mu_i \geq 0, \end{array} \tag{4}$$

and ${\mu^i}^\top x^i = 0$. This last complementary slackness constraint is critical. In particular, a strategy $\xi \in \Xi_i$ can be represented by some type of indicator (or probability) vector with the same dimensions as x^i that also satisfies the primal constraints (with m_i replaced by 1), ie. $A^i\xi = B^i$. ${\mu^i}^\top\xi$ can be interpreted as the inefficiency of strategy ξ and if $\operatorname{supp}(\xi) \subseteq \operatorname{supp}(x^i)$ then ${\mu^i}^\top x^i = 0 \implies {\mu^i}^\top\xi = 0$, ie. ξ is an optimally efficient strategy. We give more detailed specific examples below.

A. Specific Examples:

We now give several compact examples of congestion game setups. For readers unfamiliar with these setups, exact definitions of each matrix along with further references are given in the appendix.

1) Basic Congestion Game: For a basic congestion game, we can take the constraints to be

$$\begin{aligned} \text{Primal:} \quad & \mathbf{1}_i^\top x^i = m_i, \quad x^i \geq 0 \\ \text{Dual:} \quad & r_i^\top - v_i\mathbf{1}_i^\top + \mu_i^\top = 0, \quad \mu_i \geq 0 \end{aligned}$$

Here $x^i \in \mathbb{R}_+^{|\Xi_i|}$ is a mass distribution vector over options in set Ξ_i. For a given mixed strategy $\xi \in \Xi_i$ over the simplex of appropriate dimension, ${\mathbf{1}^i}^\top\xi = 1,\ \xi \geq 0$, the reward is given by $\mathbf{R}^i(m,x,\xi) = r_i^\top\xi$. If $\operatorname{supp}(\xi) \subseteq \operatorname{supp}(x^i)$ the KKT conditions give

$$\begin{aligned} & r_i^\top\xi = v_i\mathbf{1}_i^\top\xi - \mu_i^\top\xi = v_i - \mu_i^\top\xi \\ \text{for} \ \ \xi,\xi' \quad \Rightarrow \qquad & c_i^\top\xi = v_i = c_i^\top\xi' - \mu_i^\top\xi' \leq c_i^\top\xi' \end{aligned}$$

where both ξ,ξ' satisfy $\mathbf{1}^\top\xi = 1,\ \xi \geq 0$ and $\operatorname{supp}(\xi) \subseteq \operatorname{supp}(x)$.

2) Routing Game Edge Formulation: For a traditional routing game edge formulation, the constraints are given by

$$\begin{aligned} \text{Primal:} \quad & E^i x^i = S^i m_i,\ x^i \geq 0 \\ \text{Dual:} \quad & {r^i}^\top - {v^i}^\top E^i + {\mu^i}^\top = 0,\ \mu^i \geq 0,\ {v^i}^\top S^i = p_i \end{aligned}$$

Here the $x_i \in \mathbb{R}^{|\mathcal{E}_i|}$ vectors are mass distributions over edges in a set of graphs $\{\mathcal{G}_i = (\mathcal{V}_i,\mathcal{E}_i)\}_i$, $E^i \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$ is a standard node-edge incidence matrix and $S^i \in \mathbb{R}^{|\mathcal{V}|}$ is a source-sink vector. For a simple path-connected graph, the primal constraints ensure that mass m_i is routed from the source to the sink. For a particular graph $\mathcal{G}_i$, Ξ_i is set of routes each represented by a path indicator vector $\xi \in \Xi_i$ with $\xi \in \mathbb{R}^{|\mathcal{E}|}$ that satisfies $E^i\xi = S^i,\ \xi \geq 0$. The cost or reward on a particular route is given by $\mathbf{R}^i(m,x,\xi) = {r^i}^\top\xi$. At Wardrop equilibrium, any route $\xi \in \Xi_i$ taken by the population, ie. $\operatorname{supp}(\xi) \subset \operatorname{supp}(x^i)$, must be optimal. This is encoded in the KKT conditions as follows

$$\begin{aligned} & {r^i}^\top\xi = {v^i}^\top E^i\xi - {\mu^i}^\top\xi = {v^i}^\top S^i - {\mu^i}^\top\xi = p_i - {\mu^i}^\top\xi \\ \text{for} \ \ \xi,\xi' \quad \Rightarrow \qquad & {r^i}^\top\xi = p_i = {r^i}^\top\xi' - {\mu^i}^\top\xi' \leq {r^i}^\top\xi' \end{aligned}$$

when $\operatorname{supp}(\xi) \subseteq \operatorname{supp}(x)$.

3) MDP Congestion Game: We now consider a Markov decision process congestion game. For each option i, define a set of states $\mathcal{S}_i$ and state-action pairs $\mathcal{A}_i$. For each state-action combination $a \in \mathcal{A}_i$, there is a set of transition probabilities from that state to other states encoded in a column stochastic matrix $P^i \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$ (see the appendix for details). The potential game constraints are given by

$$\begin{aligned} \text{Primal:} \quad & \left[P^i - E^i_{\mathcal{A}_i}\right]x^i = 0,\ \mathbf{1}_i^\top x^i = m_i,\ x^i \geq 0 \\ \text{Dual:} \quad & {r^i}^\top - p_i\mathbf{1}^\top - w_i^\top\left[P^i - E^i_{\mathcal{A}_i}\right] + {\mu^i}^\top = 0,\ \mu_i \geq 0, \end{aligned}$$

where $v_i^\top = [w_i^\top \ p_i]$. Here $x_+ \in \mathbb{R}^{|\mathcal{A}|_i}$ is a population distribution over the state-actions; $E^i_{\mathcal{A}_i}$ is an indicator matrix for which actions are available from each state and P^i is the transition kernel. $w^i \in \mathbb{R}^{|\mathcal{S}_i|}$ is a value function on the states; $p_i \in \mathbb{R}$ is the average-time reward for an optimal policy; and $\mu^i \in \mathbb{R}_+^{\mathcal{A}_i}$ gives the inefficiency of each action. A specific joint state-action distribution (generated by a policy and the corresponding steady state distribution) $\xi \in \mathbb{R}^{|\mathcal{A}|}$ satisfies $\left[P^i - E^i_{\mathcal{A}_i}\right]\xi = 0,\ \mathbf{1}^\top\xi = 1,\ \xi \geq 0$. The KKT conditions give

$$\begin{aligned} & {r^i}^\top\xi = p_i\mathbf{1}^\top\xi + v_i^\top\left[P^i_{\mathrm{i}} - E^i_{\mathrm{o}}\right]\xi - \mu_i^\top\xi = p_i - \mu_i^\top\xi \\ \text{for} \ \ \xi,\xi' \quad \Rightarrow \qquad & {r^i}^\top\xi = p_i = {r^i}^\top\xi' - \mu_i^\top\xi' \leq c_i^\top\xi' \end{aligned}$$

where $\operatorname{supp}(\xi) \subseteq \operatorname{supp}(x)$.

III. Multivariate Type-Based Demand

The goal of this paper is to extend the demand function $\mathbf{G}(p)$ presented above to incorporate multi-dimensional preferences among members of the population.

A. Inspiration: Variable Demand from Mass Distribution

The single dimensional demand function $\mathbf{m}(p)$ can be thought of as arising from a population mass distribution over a 1D type space illustrated below. Each bit of mass's location along the θ-axis gives the payoff required for them to opt-in to the game. The total mass that opts-in is thus given by integrating over $\theta \geq p$ as shown in Fig. 1

$$\mathbf{m}_1(p_1) = \int_{\theta_1 \geq p_1} M(\theta_1)\ d\theta_1 = \int_0^\infty M(\theta_1 = p_1 - s)\ ds$$

Along with providing a rational for a particular demand function, this setup also allows us to model non-homogeneous population preferences. The one dimensional type space and region of integration are illustrated in Fig. 1.

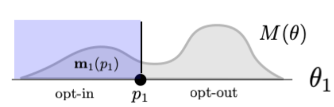

Fig. 1. Integration formulation for 1D demand function. The demand function is given by the CDF up to point p_1

B. Multivariate Type Space & Preference Distribution

We now consider how to extend this demand function to a multi-dimensional parameter space where the paramters represent the population's *preferences* or *type*. The population is now represented by measure $M(\theta)$ over a multi-dimensional continuous type space $\Theta = \mathbb{R}^{|\mathcal{I}|}$. A type vector $\theta \in \mathbb{R}^{|\mathcal{I}|}$ represents a population member's arbitrary preference for actions in each subset. In a routing game, for example, these subsets might refer to different types of transportation options such as biking, riding the bus, or driving.

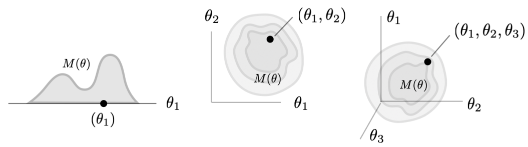

Fig. 2. Multi-dimensional type distributions in 1D,2D, and 3D

Different individuals might have arbitrary preferences for one type of transportation over another. When a population member chooses $\xi \in \Xi_i$, they receive a reward of $\mathbf{R}^i(x, m, \xi) - \theta_i$. In addition, we can model the case where that population member will only choose to play the game if $\mathbf{R}^i(x, m, \xi) - \theta_i \geq 0$, ie. θ_i represents the minimum amount of reward required for the agent to be willing to choose option i.

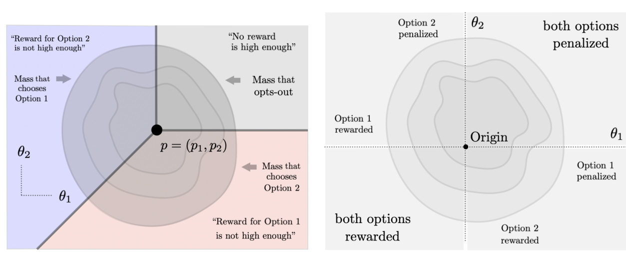

Fig. 3. (Left) Preference regions and division of mass relative to the payoff vector p. (Right) Subjective interpretation of θ relative to the origin: if $\theta_i > 0$, individuals receives a subjective penalty for option i; if $\theta_i < 0$ receive a subjective reward for option i.

Here θ_i represents an extra (subjective) cost an agent gets from choosing option i. Note that there is nothing in this setup that requires θ_i to be positive. Depending on location of mass relative to the origin, members of the population may get a positive or negative subjective reward for choosing specific options.

We will consider two general setups. In the *Class I* case, agents are allowed to opt-out of participating if the reward $\mathbf{R}^i(x, m, \xi)$ for playing $\xi \in \Xi_i$ is not high enough. In the *Class II* case, all agents must participate and only their relative preferences between options matter. This second option is quite useful in many inelastic demand settings. For commuters, for example, choosing not to travel may not be an option but we may still want to model their relative preference for different options. The Class II model will also naturally require us to be less precise in determining the population's preference distribution since only their relative preferences between various options matter.

A specific reward profile $p \in \mathbb{R}^{|\mathcal{I}|}$ (where p_i represents the equilibrium payoff for option i) will divide up the type space up into regions based on preference for the different options. That mass that chooses that option is then given by integrated over those regions. We define these for the Class I and II equilibria in next section.

C. Preference Regions

1) Class I Regions: Given a particular reward vector $p \in \Theta$, we can divide up the type space into different regions $\{\Theta_i(p)\}_{i \in \mathcal{I}}$ where mass in $\Theta_i(p)$ choose option i. Explicitly, we can characterize these regions as follows.

$$\begin{aligned}\Theta_i(p) &= \big\{\theta \in \Theta \;\big|\; p_i - \theta_i \geq p_j - \theta_j,\ p_i - \theta_i \geq 0,\ \forall j \in \mathcal{I} \\ &= \{\theta \in \Theta \mid \theta_i = p_i - s,\ \theta_j \geq p_j - s,\ s \geq 0\ \forall j \in \mathcal{I}\} .. \end{aligned} \tag{5}$$

The second characterization will prove useful in the proofs below. We can also define the portion of the type space that would not choose any action in any subset, ie. $p_i \geq \theta_i$:

$$\Theta_0(p) = \big\{\theta \in \Theta \;\big|\; p_i - \theta_i > 0\ \forall i \in \mathcal{I}\big\}. \tag{6}$$

These type space regions are illustrated in 1D, 2D, and 3D are here.

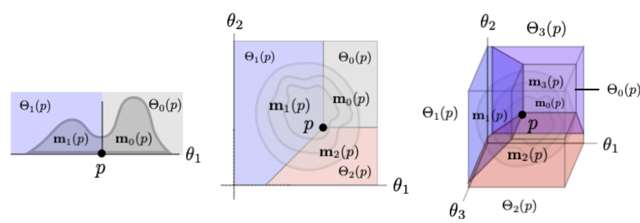

Fig. 4. Class I: $\Theta_i(p)$ regions and resulting $\mathbf{m}_i(p)$ in 1D, 2D, and 3D.

Note that $\Theta = \cup_i \Theta_i \cup \Theta_0$. We now define a map $\mathbf{m} : p \mapsto \mathbb{R}_+^{|\mathcal{I}|}$ that gives the mass in each region: $\mathbf{m}_i(p) = \int_{\theta \in \Theta_i(p)} M(\theta) d\theta$. Given $\Theta_i(p)$, characterized in (5), for some mass distribution $M(\theta)$ with compact support, we can explicitly integrate over these regions

$$\mathbf{m}_i(p) = \int_{\theta \in \Theta_i(p)} M(\theta) d\theta \tag{7a}$$

$$= \int_0^\Omega ds \int_{p_{\mathcal{I}-\{i\}}-s}^\infty d\theta_{\mathcal{I}-\{i\}} M\Big(\theta_i = p_i - s\Big) \tag{7b}$$

$$= \int_0^\Omega ds \int_{p_{-i}-s}^\infty d\theta_{-i} M\Big(\theta_i = p_i - s\Big) \tag{7c}$$

where we have used the short hands

$$M(\theta_i = s') = M(\theta_1, \ldots, \theta_{i-1}, s', \theta_{i+1}, \ldots, \theta_{|\mathcal{I}|})$$
$$\int_{p_{\{ijk\}}-s}^\infty d\theta_{\{ijk\}} = \int_{p_i-s}^\infty \int_{p_j-s}^\infty \int_{p_k-s}^\infty d\theta_i d\theta_j d\theta_k$$

and similarly for subsets of $\mathcal{I}$ other than $\{ijk\}$ such as $\mathcal{I} - \{i\}$. In some cases, we will also use $-ijk$ as a short hand for $\mathcal{I} - \{ijk\}$. Note that the inner integrals in (8) are over θ_j for $j \in \mathcal{I} - \{i\}$ and the outer integral is over s. Ω is just some large constant which is only chosen to be finite so that the potential functions below are not unbounded. We can also give an explicit integral formula for the mass that opts-out, $\mathbf{m}_0(p)$ based on (6)

$$\mathbf{m}_0(p) = \int_{\theta \in \Theta_0(p)} M(\theta)\ d\theta = \int_{p_{\mathcal{I}}}^{\infty} d\theta_{\mathcal{I}} M\big(\theta\big) \tag{8}$$

and note that also $\mathbf{m}_0(p) = M - \sum_i \mathbf{m}_i(p)$

Remark 1. *Since we are working with integrals in many directions, rather than using the typical form* $\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(s_1, s_2) ds_2 ds_1$, *we will mostly use a physics style notation putting the differentials next to the integral symbols, ie.* $\int_{a_1}^{b_1} ds_1 \int_{a_2}^{b_2} ds_2 f(s_1, s_2)$ *so as to be clear which limits of integration apply to which variable. The integral that comes first is the outer integral, etc.*

We can think of $\mathbf{m}(p)$ as a multi-dimensional demand function given the reward vector p. The type space and regions of integration are shown in here in 1D, 2D, and 3D.

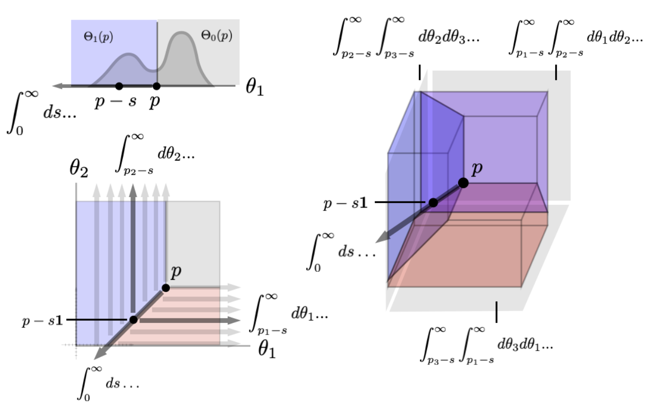

Fig. 5. Class I: Illustration of integrals over $\Theta_i(p)$ in 1D, 2D, and 3D.

2) Class II Regions: We also consider the case where members of the population only care about the differences in reward and none of them opt out. In this case, the mass regions are defined only by the relative difference

$$\begin{aligned}\bar{\Theta}_i(p) &= \big\{\theta \in \Theta \ \big|\ p_i - \theta_i \geq p_j - \theta_j\ \forall j \in \mathcal{I}\big\} &(9)\\ &= \{\theta \in \Theta|\ \theta_i = p_i - s, \theta_j \geq p_j - s\ \forall j \in \mathcal{I}\}\,. &(10)\end{aligned}$$

These regions are illustrated below in 2D and 3D.

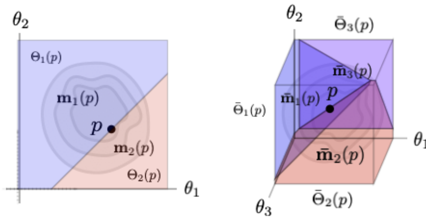

Fig. 6. Class II: $\Theta_i(p)$ regions and resulting $\mathbf{m}_i(p)$ in 2D and 3D.

This setup does not make sense in 1D since all mass will choose the single option. Note there is no region Θ_0 and $\Theta = \cup_i \Theta_i$.

Each mass in this case can be computed using the integral $\bar{\mathbf{m}}_i(p) = \int_{\theta \in \bar{\Theta}_i(p)} M(\theta) d\theta$ which we can explicitly compute as

$$\begin{aligned}\bar{\mathbf{m}}_i(p) &= \int_{\theta \in \bar{\Theta}_i(p)} M(\theta) d\theta &(11a)\\ &= \int_{-\Omega}^{\Omega} ds \int_{p_{-i}-s}^{\infty} d\theta_{-i} M\big(\theta_i = p_i - s\big) &(11b)\end{aligned}$$

Note that a constant shift in all values p_i, ie. $p \to p + \beta\mathbf{1}$ has no impact on the mass values $\bar{\mathbf{m}}_i(p)$. Details of the integration are shown here as well.

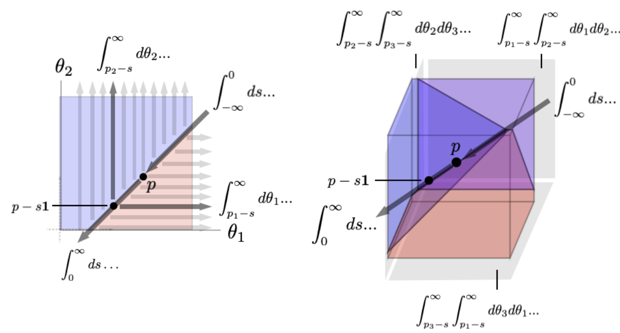

Fig. 7. Class II: Illustration of integrals over $\Theta_i(p)$ in 2D and 3D.

D. Multivariate Type Wardrop Equilibrium

We can now extend the Wardrop equilibrium concept to this continuous type setting.

Definition 2 (Multivariate Type Wardrop Equilibrium)**.** *A set of mass vectors* (x, m) *and reward vector* p *is a* multivariate, continuous type Wardrop equilibrium *if they satisfy the KKT feasibility conditions* (4)*,*

$$\textit{Class I:}\quad m_i = \mathbf{m}_i(p), \quad \textit{Class II:}\quad m_i = \bar{\mathbf{m}}_i(p) \tag{12}$$

where $\mathbf{m}_i(\cdot)$ *and* $\bar{\mathbf{m}}_i(\cdot)$ *are defined by integration over the mass regions defined in* (7) *and* (11) *and*

$$\textit{Class I:}\ R(x, m, \xi) - \theta_i \geq R(x, m, \xi') - \theta_{i'}\ \forall \theta_i \in \Theta_i(p) \tag{13a}$$

$$R(x, m, \xi) - \theta_i \geq 0\ \forall \theta_i \in \Theta_i(p) \tag{13b}$$

$$\textit{Class II:}\ R(x, m, \xi) - \theta_i \geq R(x, m, \xi') - \theta_{i'}\ \forall \theta_i \in \bar{\Theta}_i(p) \tag{13c}$$

and for each $\xi \in \Xi_i$ such that $supp(\xi) \subseteq supp(x^i)$ and any other $\xi' \in \Xi_i$.

IV. Potential Function

We now give the potential function $G(p)$ and $\bar{G}(p)$ and analyze their properties.

A. Potential Definitions

Definition 3 (Multivariate Demand Potential Functions)**.** *The potential functions for the Class I and II cases are given by*

$$\textit{Class I:}\quad \mathbf{G}(p) = \int_0^{\Omega} ds \int_{p_{\mathcal{I}}-s}^{\infty} d\theta_{\mathcal{I}}\ M(\theta) \tag{14}$$

$$\textit{Class II:}\quad \bar{\mathbf{G}}(p) = \int_{-\Omega}^{\Omega} ds \int_{p_{\mathcal{I}}-s}^{\infty} d\theta_{\mathcal{I}}\ M(\theta) \tag{15}$$

We note also an alternative version of each potential that will be useful later on. The Class I potential can be written as.

$$\mathbf{G}(p) = \int_0^{\Omega} ds \int_{\Theta_0(p-s\mathbf{1})} M(\theta)\ d\theta = \int_0^{\Omega} ds\ \mathbf{m}_0(p-s\mathbf{1}) \tag{16}$$

In this way $G(p)$ can be thought of as the integral of a cumulative distribution of the mass in $\Theta_0(p-s\mathbf{1})$ as s increases from 0. This potential function can be illustrated as follows.

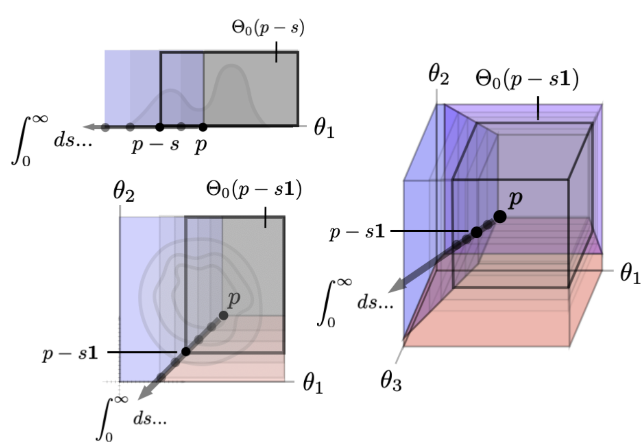

Fig. 8. Interpretation of the Class I potential as given by Eq. (16)

For the Class II potential, a similar construction can be done with region $\bar{\Theta}_1(p+t\mathbf{I}_1)$ takes the place of Θ_0 as t increases from 0. To see this, we can write

$$\begin{aligned}
\bar{\mathbf{G}}(p) &= \int_{-\Omega}^{\Omega} ds \int_{p_1-s}^{\infty} d\theta_1 \int_{p_{-1}-s}^{\infty} d\theta_{-1}\ M(\theta) \\
&= \int_{-\Omega}^{\Omega} ds \int_0^{\infty} dt \int_{p_{-1}-s}^{\infty} d\theta_{-1}\ M(\theta_1 = p_1+t-s) \\
&= \int_0^{\Omega} dt \int_{-\Omega}^{\Omega} ds \int_{p_{-1}-s}^{\infty} d\theta_{-1}\ M(\theta_1 = p_1+t-s) \\
&= \int_0^{\Omega} dt \int_{\bar{\Theta}_1\left(p+t\mathbf{I}_1\right)} d\theta M(\theta) \\
&= \int_0^{\Omega} dt\ \bar{\mathbf{m}}_1(p+t\mathbf{1})
\end{aligned} \tag{17}$$

where $\mathbf{I}_1$ is the first standard basis vector and in the first step we used the coordinate transform $\theta_1 = t + p_1 - s$. This is illustrated as follows.

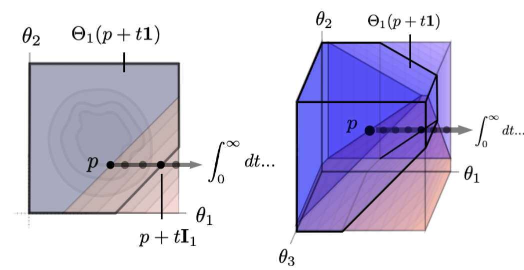

Fig. 9. Illustration of the Class II potential as described in Eq. eqrefeq:Gm1. (Similar forms are possible substituting any p_i for p_1.)

The Class I potential function is for the case where players can opt out of joining the game; Class II potential functions are for the case where all players are required to play. The definition of $\mathbf{G}(p)$ is fairly clean and straightforward. There are many definitions of $\bar{\mathbf{G}}(p)$ that would work, however. In particular, the choice of p_1 as special is arbitrary; any p_i would work just as well. We will develop the theory for Class I potentials first as it is more straightforward. The Class II versions of all the theorems are included in the Appendix.

B. Potential Derivatives (Class I)

We now expound the properties of the first and second derivatives of $G(p)$.

1) 1st Order (Class I): First, we show that the derivative the potential gives the mass in each region of integration. Indeed, the potential function was chosen so that this would be the case.

Proposition 1. *Class I:* $\frac{\partial \mathbf{G}}{\partial p} = \mathbf{m}(p)^\top$ *Class II:* $\frac{\partial \bar{G}}{\partial p} = \bar{\mathbf{m}}(p)^\top$

For Class I, a straightforward calculation gives

$$\frac{\partial \mathbf{G}}{\partial p_i} = \int_0^{\Omega} ds \int_{p_{-i}-s}^{\infty} d\theta_{-i}\ M(\theta_i = p_i - s) = \mathbf{m}_i(p) \tag{18}$$

The proof for the Class II potential is similar. The derivatives of the Class I potential are illustrated in Figure 10 and the Class II potential derivatives are illustrated in Figure 11.

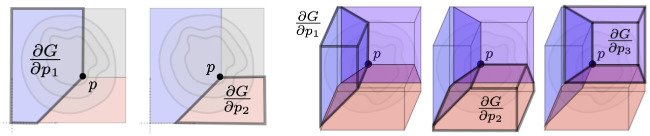

Fig. 10. Class I: Relationship of 1st derivatives to type regions in 2D and 3D.

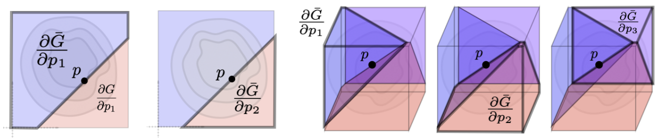

Fig. 11. Class II: Relationship of 1st derivatives to type regions in 2D and 3D.

2) 2nd Order (Class I): In this section, we describe properties of the second derivative for the Class I potential $G(p)$. The parallel theorems (which are slightly more complicated) are given in the Appendix. Much of the structure of potential function (including the inspiration for guessing it) comes from analyzing the second derivative. For a given p, the second derivative values are defined by the boundaries of the regions of integration. Each of the boundary areas is illustrated for $p \in \mathbb{R}^2$ and $p \in \mathbb{R}^3$ in Figures **??** and **??**. Again these pictures are the source of most of the intuition for these proofs.

Proposition 2 (Class I: Second Derivatives)**.**

$$\frac{\partial^2 \mathbf{G}}{\partial p_i^2} = \frac{\partial \mathbf{m}_i}{\partial p_i} = \int_{p_{-i}}^{\infty} d\theta_{-i} M\Big(\theta_i = p_i\Big) - \sum_{j \neq i} \frac{\partial \mathbf{m}_i}{\partial p_j} \tag{19a}$$

$$\frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j} = \frac{\partial \mathbf{m}_i}{\partial p_j} = -\int_0^{\infty} ds \Bigg[\int_{p_{-ij}-s}^{\infty} d\theta_{-ij} \tag{19b}$$

$$M\Big(\theta_i = p_i - s, \theta_j = p_j - s\Big)\Bigg] \quad j \neq i$$

Proposition 3 (Class II: Second Derivatives)**.**

$$\frac{\partial^2 \bar{\mathbf{G}}}{\partial p_i^2} = \frac{\partial \bar{\mathbf{m}}_i}{\partial p_i} = -\sum_{j \neq i} \frac{\partial \bar{\mathbf{m}}_i}{\partial p_j} \tag{20a}$$

$$\frac{\partial^2 \bar{\mathbf{G}}}{\partial p_i \partial p_j} = \frac{\partial \bar{\mathbf{m}}_i}{\partial p_j} = \int_{-\Omega}^{\Omega} ds \Bigg[\int_{p_{-ij}-s}^{\infty} d\theta_{-ij} \tag{20b}$$

$$M\Big(\theta_i = p_i - s, \theta_j = p_j - s\Big)\Bigg], \quad i \neq j$$

Proof. The proofs are shown here only for the Class I potential since they are similar. The $\frac{\partial \mathbf{m}_i}{\partial p_j}$ computation is similar to the computation of $\frac{\partial \mathbf{G}}{\partial p_i}$. To compute, (19a), we first rewrite $\mathbf{m}_i(p)$ using the change of variables $s' = p_i - s$.

$$\mathbf{m}_i(p) = \int_{p_i}^{-\infty} ds' \int_{p_{-i}-p_i+s'}^{\infty} d\theta_{-i} M\Big(\theta_i = s'\Big)$$

Differentiating with respect to p_i gives ∞

$$\begin{aligned}
\frac{\partial \mathbf{m}_i}{\partial p_i} &= \frac{\partial}{\partial p_i} \int_{p_i}^{\Omega} ds' \int_{p_{-i}-p_i+s'}^{\infty} d\theta_{-i} M\big(\theta_i = s'\big) \\
&= -\int_{p_{-i}}^{\infty} d\theta_{-i} M\Big(\theta_i = p_i\Big) + \sum_j \Bigg[\int_{p_i}^{-\infty} ds' \\
&\qquad \int_{p_{-ij}-p_i+s'}^{\infty} d\theta_{-ij} M\Big(\theta_i = s', \theta_j = p_j - p_i + s'\Big)\Bigg] \\
&= -\int_{p_{-i}}^{\infty} d\theta_{-i} M\Big(\theta_i = p_i\Big) + \sum_j \Bigg[\int_0^{\infty} ds \\
&\qquad \int_{p_{-ij}-s}^{\infty} d\theta_{-ij} M\Big(\theta_i = p_i - s, \theta_j = p_j - s\Big)\Bigg] \\
&= -\int_{p_{-i}}^{\infty} d\theta_{-i} M\Big(\theta_i = p_i\Big) + \sum_j \frac{\partial \mathbf{m}_i}{\partial p_j}
\end{aligned}$$

where we have (carefully) applied Leibniz integral rule repeatedly and transformed back into the s-coordinates. □

The above calculations are illustrated visually here in 2D and 3D.

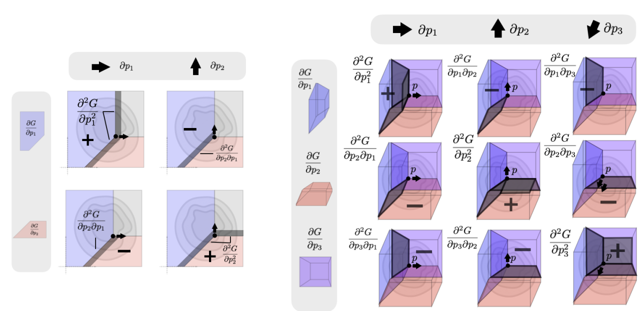

Fig. 12. Class I: Relationship of 2nd derivatives to type regions in 2D and 3D. Perturbing p_i shows the change in each type region (and thus the change in each derivative $\partial \mathbf{G}/\partial p_j$) providing the intuition for Prop. 2 and Props. 4 and 5

Note that the visual intuition for each derivative given by perturbing p in each direction is quite useful in understanding the theorem. For simplicity it can be helpful to consider these formulas in 2-dimensions.

$$\frac{\partial \mathbf{m}_1}{\partial p_2} = \frac{\partial \mathbf{m}_2}{\partial p_1} = \int_0^{\infty} ds \ M\Big(\theta_1 = p_1 - s, \theta_2 = p_2 - s,\Big)$$

The second derivatives of $\bar{\mathbf{G}}$ are illustrated in Fig. IX-B.

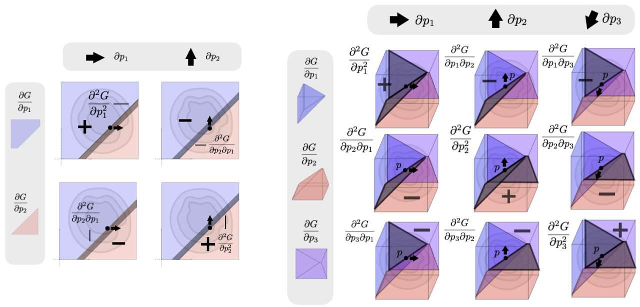

Fig. 13. Class I: Relationship of 2nd derivatives to type regions in 2D and 3D. Perturbing p_i shows the change in each type region (and thus the change in each derivative $\partial \mathbf{G}/\partial p_j$) providing the intuition behind Props. 3, 4 and **??**

Note also that Prop. 2 implies that

$$\sum_j \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j} = \sum_j \frac{\partial \mathbf{m}_i}{\partial p_j} = \int_{p_{-i}}^{\infty} d\theta_{-i} M\Big(\theta_i = p_i\Big) \tag{21a}$$

$$\sum_j \frac{\partial^2 \bar{\mathbf{G}}}{\partial p_i \partial p_j} = \sum_j \frac{\partial \bar{\mathbf{m}}_i}{\partial p_j} = 0 \tag{21b}$$

C. Potential Properties

From the above formulas above, we get several immediate properties of the potential that we list here minimal proofs (since they follow directly from the formulas above.). First, we can check that the vector field, $\mathbf{m}(p)$ is indeed a conservative vector field, the gradient of a potential. Since we've already found the potential, this proposition does not need to be stated. We include it here however since it is the primary source of inspiration for looking for potential in the first place.

In particular these derivatives can be reasoned about graphically as shown in Fig. IV-B.2. This graphical reasoning then gives us the inspiration to look for a potential function. We now state several propositions that follow immediately from the characterization of the second derivative.

Proposition 4. $\frac{\partial \mathbf{m}_i}{\partial p_j} = \frac{\partial \mathbf{m}_j}{\partial p_i}$ *and asdfasdf*

Second, if p lies in the interior of a region of mass, the Hessian is positive definite.

Proposition 5. *The Hessian* $\frac{\partial^2 \mathbf{G}}{\partial p^2}$ *is diagonally strictly dominant with positive diagonal elements and thus is strictly positive definite and* $\mathbf{G}(p)$ *is strictly convex.*

Finally, we use the properties of the second derivative to get further properties of the vector field $\mathbf{m}(p)$. In particular, we use a version of the inverse function theorem for convex functions to show that the $\mathbf{m}^{-1}(\cdot) = \mathbf{p}(\cdot)$ exists on

Proposition 6. *For differentiable* $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$*.defined on a convex domain* $\mathcal{U}$ *is strictly convex, then the gradient* $\frac{\partial \mathbf{F}}{\partial x}$ *has a well defined inverse.*

Proof. https://www.arpm.co/lab/eb-convex-fn-invertible-gradient.html Based on the fact that $\frac{\partial \mathbf{F}}{\partial x}$ is strictly monotone and thus invertible. □

In Section V, we will show that the optimality conditions (4) for Programs (2) and (3) with $G(p)$ defined above correspond to a continuous-type Wardrop equilibrium. First, however, we will derive the Legendre transform of the potential function $\mathbf{G}^*(p)$.

D. Dual Potential Function

The Legendre transform of the objective, ie. the objective of the dual problem is defined by $\mathbf{G}^*(m) = \min_p \quad m^\top p - G(p)$ Solving for the optimizer by differentiating we get $m^\top - \mathbf{m}(p) = 0$ which implies $p = \mathbf{p}(m)$ assuming that $\mathbf{p}(m)$ exists as shown in Prop. 6. Plugging the optimality condition back into the Legendre transform, we get we

Proposition 7. *The Legendre transform of the Class I potential is given by*

$$\begin{aligned} \mathbf{G}^*(m) &= m^\top \mathbf{p}(m) - \int_0^\infty ds \int_{\mathbf{p}(m)_{\mathcal{I}} - s}^{\infty} d\theta_{\mathcal{I}}\ M(\theta) \\ &= m^\top \mathbf{p}(m) - \int_0^\infty \Big[M - \mathbf{1}^\top \mathbf{m}\Big(\mathbf{p}(m) - s\mathbf{1}\Big)\Big]\ ds \end{aligned} \tag{22}$$

It is worth explicitly checking that the derivative of $\mathbf{G}^*(m)$ w.r.t. m is the cost vector $\mathbf{p}(m)^\top$ as desired, ie. $\frac{\partial \mathbf{G}}{\partial m} = \mathbf{p}(m)^\top$.

Proposition 8. $\frac{\partial \mathbf{G}^*}{\partial m} = \mathbf{p}(m)^\top$

Proof.

$$\begin{aligned} \frac{\partial \mathbf{G}^*}{\partial m} &= \frac{\partial}{\partial m}\Big(m^\top \mathbf{p}(m)\Big) - \frac{\partial}{\partial m}\left(\int_0^\Omega M ds\right) \\ &\quad + \frac{\partial}{\partial m}\left(\int_0^\Omega \Big(\mathbf{1}^\top \mathbf{m}\Big(\mathbf{p}(m) - s\mathbf{1}\Big)\Big)\ ds\right) \\ &= \mathbf{p}^\top(m) + m^\top \frac{\partial \mathbf{p}}{\partial m}(m) \\ &\quad + \left(\int_0^\Omega \mathbf{1}^\top \frac{\partial \mathbf{m}}{\partial p}\Big(\mathbf{p}(m) - s\mathbf{1}\Big)\ ds\right)\frac{\partial \mathbf{p}}{\partial m}(m) \\ &= \mathbf{p}^\top(m) + \Big(m^\top - m^\top\Big)\frac{\partial \mathbf{p}}{\partial m}(m) = \mathbf{p}^\top(m) \end{aligned}$$

where in the second to last step we have used the fact that $(*) = \int_0^\Omega \sum_j \frac{\partial \mathbf{m}_i}{\partial p_j}\Big(\mathbf{p}(m) - s\mathbf{1}\Big)\ ds = m_i$ which can be proved by applying Eqn. (21a) to get

$$(*) = \int_0^\Omega \int_{\mathbf{p}_{-i}(m) - s}^{\infty} d\theta_{-i} M\Big(\theta_i = \mathbf{p}_i(m) - s\Big) = m_i$$

□

V. Equilibrium Computation

We now frame the equilibrium computation as an optimization problem. We give both the primal and dual problems but practically the dual problem is much more useful since the dual functions $G(p)$ and $\bar{G}(p)$ are more intuitive and do not require inverting the map $\mathbf{m}(p)$. We can state the following theorem.

Theorem 1. *Let* (x, m) *and* (r, p, v, μ) *satisfy the KKT conditions* (4) *for optimization programs* (2) *and* (3) *with*

$\mathbf{G}(p)$ or $\bar{\mathbf{G}}(p)$, then (m,x) is a multivariate continuous-type Wardrop equilibrium.

Proof. For any feasible strategy $\xi \in \Xi_i$ such that $A^i\xi = B^i$ we have that $\mathbf{R}^i(x,m,\xi) = p_i - {\mu^i}^\top \xi$ with ${\mu^i}^\top \xi \geq 0$ and if, in addition, $\text{supp}(\xi) \subseteq \text{supp}(x^i)$, then ${\mu^i}^\top \xi = 0$ and $\mathbf{R}^i(x,m,\xi) = p_i$. For any $\theta \in \Theta_i(p)$ or $\theta \in \bar{\Theta}_i(p)$, we have that $p_i - \theta_i \geq p_j - \theta_j$ and thus we have

$$\mathbf{R}_\xi - \theta_i = p_i - \theta_i \geq p_j - \theta_j = \mathbf{R}_\xi + {\mu^i}^\top \xi - \theta_j \geq \mathbf{R}_\xi - \theta_j$$

with $\mathbf{R}_\xi = \mathbf{R}(x,m,\xi)$. For $\theta \in \Theta_i(p)$ we also have $p_i - \theta_i = \mathbf{R}_\xi - \theta_i \geq 0$.

VI. Algorithm - Subgradient Methods

As stated above, the dual problem is much more practical than the primal problem since the primal problem involves inverting the map $\mathbf{m}(p)$ (which in practice could be quite challenging). Many algorithms could work for the dual problem; however, the practitioner would have the most luck with algorithms that do not require specific computation of $\mathbf{G}(p)$ but rather work primarily with the gradient $\partial\mathbf{G}/\partial p$. While it is easier to compute $\mathbf{G}(p)$ then to invert $\mathbf{m}(p)$, computing the derivative $\partial\mathbf{G}/\partial p$ is a simple as integrating the mass in each region $\Theta_i(p)$ or $\bar{\Theta}_i(p)$. Integration of these mass regions can also be done in an approximate manner with decent results as we show in the numerical examples below. In this section, we show a basic implementation of a subgradient algorithm traditionally used in routing games. Many other gradient based algorithms could easily be implemented and further effort could be put into exploiting the structure of the constraints. We give only a simplistic implementation here.

VII. Numerical Example

To illustrate the use of this model, we consider an infinite horizon MDP congestion game modeling ride-sharing in the city of Chattanooga, Tennessee. We divide the city up roughly into neighborhoods according to the map shown. We consider the competition between three different ride-sharing services Uber, Lyft, and a city taxi company. The underlying graph for each MDP is a fully connected graph including self-loops for each node. At each node (neighborhood), drivers have the option of either picking up a rider in that neighborhood s (denoted a_s^{rider}) or transitioning to another neighborhood without a rider (denoted $a_s^{\to s'}$). Each ride-sharing service has some demand profile at each node s associated with action a_s^{rider} that says on average what trips riders want to make. This demand profile is associated with a transition probability $\text{Prob}(s|s')$ which reads as the probability of transitioning to state s from state s' when picking up a rider. These probabilities are shown in the plot below.

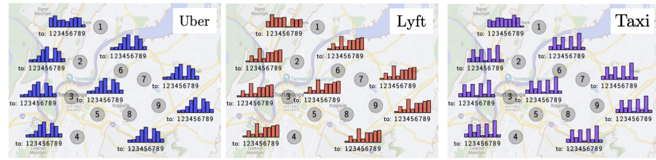

Fig. 14. Transition probabilities $\text{Prob}(s'|s, a_s^{\text{rider}})$ between states when picking up a rider (action a_s^{rider}) for each ride-sharing service (as detailed in Eq. (23)).

The transition kernel is then given by

$$[\,P\,]_{sa} = \begin{cases} \text{Prob}(s|s') & ; \quad a = a_{s'}^{\text{rider}} \\ 1 & ; \quad a = a_{s'}^{\to s} \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{23}$$

Each action is associated with the following reward

$$r_a(x_a) = \begin{cases} \text{FARE} - \text{FUEL} - c_a x_a & ; \quad a = a_{s'}^{\text{rider}} \\ -\text{FUEL} - \epsilon x_a & ; \quad a = a_{s'}^{\to s} \end{cases}$$

where the FARE cost is taken as \$10/mile; the FUEL cost is taken as (\$3 /gal) $\times$ (1 gal/20 mile), and $c_a x_a$ is a congestion cost with congestion factor c_a taken to be (\$1/min)$\times$(2 min/driver). ϵx_a is a small regularization cost when a rider is not picked up (with $\epsilon = 0.01$). For illustration purposes, we use a simple driver preference mass distribution spread over the range $[0,40] \times [0,40] \times [0,40]$ with units of \$ per trip. For approximation purposes, we divide the distribution up into 40 segments per side giving $64,000$ cells each with unit volume. The mass in each cell is sampled uniformly from the interval $[0, 0.01]$. The total mass population of drivers is approximately 320 drivers. We integrate the total mass in each region simply by summing up the mass in each cell where a cell is taken to be in a specific region based on the location of the centroid of the cell relative to the payoff vector p. More complicated mesh and integration schemes could be applied for more accurate results.

Initializing the algorithm at XXX, we show convergence to the type-based Wardrop equilibrium. Figure XXX shows convergence of the reward, value and overall cost variables as well as the corresponding mass distributions at each node with the final distribution being the optimal steady state distributions for each service shown in Figure XXX.

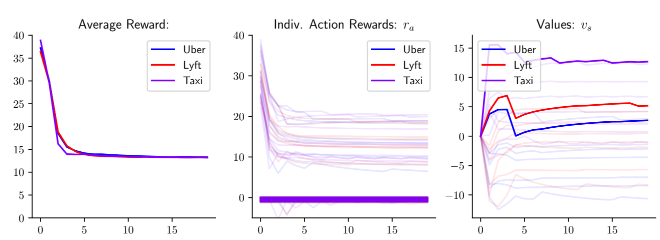

Fig. 15. Convergence of the average reward, the individual action rewards, and the state values. (Note: in this formulation the values encode the difference between the average reward and the immediate reward.)

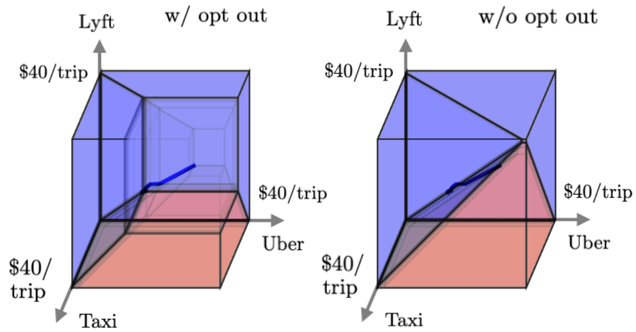

Fig. 18. Illustration of the payoff vector adjusting position in the type space with each iteration of the algorithm for the Class I (Left) and Class II (Right) cases.

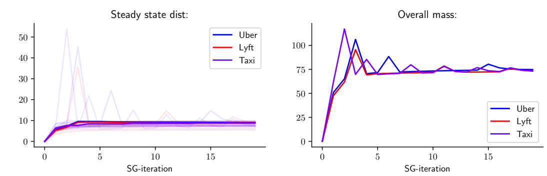

Fig. 16. Convergence of the steady state distribution for each graph and the overall mass for each ride sharing service.

Fig. 17. The equilibrium steady state distributions (Left) and value functions (Right) for the Class I and Class II potentials. Note that the steady state distributions are higher in the Class II case since more mass is required to participate.

The trajectory of the overall reward vector in the type space is also shown. We run these examples for both two and three services and using the opt-out as well as no-opt out models.

VIII. Discussion and Conclusion

In this paper, we develop a potential game model where the population is no longer anonymous but is represented by a distribution over a continuous multi-dimensional type space that encodes a population members arbitrary preference for different subsets of actions. We note that one of the advantages of this model is that it is able to encode a truly wide range of population variations since no ordering on the different options $i \in \mathcal{I}$ is required for all population members. We believe this model can have wide application in population potential games specifically in multi-modal transportation problems. Future work includes analyzing cases where the type parameter does not enter linearly into the total reward as well as potentially an inverse utility learning problem where the mass distribution of the potential game is observed under a variety of conditions and these observations are used to estimate the underlying population type distribution.

IX. Appendix:

A. Congestion Game Notation

1) Routing Game Edge Formulation:: For directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The incidence matrices are given by

$$\left[\ E_{\text{i}}\ \right]_{ve} = \begin{cases} 1 & ; \text{ if edge } e \text{ goes into node } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\left[\ E_{\text{o}}\ \right]_{ve} = \begin{cases} 1 & ; \text{ if edge } e \text{ comes out of node } v \\ 0 & ; \text{ otherwise} \end{cases}$$

and the overall incidence matrix is given by $E = E_{\text{i}} - E_{\text{o}}$.

$$\left[S \right]_v = \begin{cases} 1 & ; \text{ if } v \text{ is the source node} \\ -1 & ; \text{ if } v \text{ is the sink node} \\ 0 & ; \text{ otherwise} \end{cases}$$

A strategy in a routing game can be represented by a route indicator vector $\xi \in \mathbb{R}^{|\mathcal{E}|}$ which says which edges are in that particular route

$$\xi_e = \begin{cases} 1 & ; \text{ if edge } e \text{ is in the route} \\ 0 & ; \text{ otherwise} \end{cases}$$

and routing from the source to the sink is given by the condition $E\xi = S$.

2) MDP Congestion Game: The stochastic transition kernel of a Markov decision process consists of a set of states and state actions $\mathcal{M} = (\mathcal{S}, \mathcal{A})$. Each action $a \in \mathcal{A}$ is available from only one state $s \in \mathcal{S}$ and is associated with a set of transition probabilities from that state to each of the other states $s' \in \mathcal{S}$. We can use $\mathcal{A}_s$ to represents the actions available from state s and $\mathcal{A} = \sqcup_s \mathcal{A}_s$. The transition probabilities from s to s' when action a is taken are dentoed $\text{Prob}(s'|s,a)$ or just $\text{Prob}(s'|a)$. The states corresponding to each action are encoded in an action indicator matrix $E_{\text{A}} \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{A}|}$

$$\left[\ E_{\text{A}}\ \right]_{sa} = \begin{cases} 1 & ; \text{ if action } a \text{ is available from state } s \\ 0 & ; \text{ otherwise} \end{cases}$$

and the transition probabilities can be encoded in a column stochastic transition kernel matrix $P \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{A}|}$

$$\left[\ P\ \right]_{sa} = \text{Prob}(s|a)$$

A strategy for an MDP can be represented by a stochastic vector $\xi \in \mathbb{R}_+^{|\mathcal{A}|}$ with $\mathbf{1}^\top \xi = 1$, $\xi \geq 0$ that satisfies the stochastic conservation flow constraint $[P - E_{\mathcal{A}}]\xi = 0$. This results in ξ representing a steady-state distribution over the joint state-action space, ie. ξ_a is the steady-state probability of being in state s and taking a. Two more commonly discussed MDP notions, a policy $\pi \in \mathbb{R}^{|\mathcal{A}|}$ and the resulting steady-state distribution over the state space $\rho \in \mathbb{R}^{|\mathcal{S}|}$ can be computed from ξ. Specifically, $\rho_s = \sum_{a\in\mathcal{A}_s} \xi_a$ and $\pi_a = \xi_a/\rho_s$ which can be written compactly in matrix form.

$$\rho = E_{\mathcal{A}}\xi, \quad \mathbf{1} = E_{\mathcal{A}}\pi, \quad \xi = \mathbf{dg}(\rho^\top E_{\mathcal{A}})\pi$$

B. Dual Potential (Class II)

The analysis of the Class II potential is slightly more complicated since we will show that it is not strictly convex. To be more specific, the Hessian is not positive definite everywhere and is not even invertible. One can see this from the following proposition.

Proposition 9. $\frac{\partial^2 \mathbf{G}}{\partial p^2}\mathbf{1} = 0$, *$\left(\frac{\partial^2 \mathbf{G}}{\partial p^2}\right.$ is not full rank.$\left.\right)$*

The fact that $\bar{\mathbf{G}}$ is low rank comes from the fact that $\bar{\mathbf{G}}(p+\alpha\mathbf{1}) = \bar{\mathbf{G}}(p+\alpha'\mathbf{1})$ for any α, α'. This leads us to examine whether or not $\bar{\mathbf{G}}$ is positive definite on the subspace orthogonal to $\mathbf{1}$. To this end, define a coordinate transformation to encode the differences, $u_i = p_i - p_1$ for $i = 2, \cdots, n$ and $u_1 = p_1$ and coordinate transformations $W \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} u_1 \\ u_{-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -\mathbf{1} & I \end{bmatrix}}_{W^{-1}} \begin{bmatrix} p_1 \\ p_{-1} \end{bmatrix}, \qquad \begin{bmatrix} p_1 \\ p_{-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \mathbf{1} & I \end{bmatrix}}_{W} \begin{bmatrix} u_1 \\ u_{-1} \end{bmatrix},$$

Note again that measuring the differences from p_1 is arbitrary; any coordinate of p would work equally well. It is worth also explicitly showing the form of $\bar{\mathbf{G}}$ in the u-coordinates. It is worth writing $\bar{\mathbf{G}}$ explicitly in terms of the u-coordinates.

Proposition 10.

$$\bar{\mathbf{G}}\big|_{Wu} = Mu_1 + \int_{-\Omega}^{\Omega} ds \int_{-s}^{\infty} d\theta_1 \int_{u_{\text{-}1}-s}^{\infty} d\theta_{\text{-}1}\ M(\theta) \tag{24a}$$
$$= Mu_1 + \bar{G}(I_{-1}u_{-1}) \tag{24b}$$

Proof.

$$\begin{aligned}\bar{\mathbf{G}}(p)\big|_{p=Wu} &= \int_{-\Omega}^{\Omega} ds \int_{p_1-s}^{\infty} d\theta_1 \int_{p_{-1}-s}^{\infty} d\theta_{-1}\ M(\theta) \\ &= \int_{-\Omega-p_1}^{\Omega-p_1} ds \int_{-s}^{\infty} d\theta_1 \int_{p_{-1}-p_1-s}^{\infty} d\theta_{-1}\ M(\theta) \\ &= Mu_1 + \int_{-\Omega}^{\Omega} ds \int_{-s}^{\infty} d\theta_1 \int_{u_{-1}-s}^{\infty} d\theta_{-1}\ M(\theta)\end{aligned}$$

where we have first shifted the s variable by p_1 and then we've shifted the limits of integration over s in the last step using the fact that in the above integral $\int_{-\Omega-p_1}^{-\Omega} ds \cdots = Mp_1 = Mu_1$ and $\int_{\Omega-p_1}^{\Omega} ds \cdots = 0$. Note that here we have implicitly used the fact that the total mass is M. □

Note also that in these coordinates, u_1 defines a shift in p by a the vector $\mathbf{1}u_1$. Explicitly, $W = [\mathbf{1}\ I_{-1}]$ (where $I_{-1} \in \mathbb{R}^{n\times(n-1)}$ is the identity with the first column removed) and $p = Wu = \mathbf{1}u_1 + I_{-1}u_{-1}$. In the following proposition we show that u_1 has no impact on $\bar{\mathbf{m}}$.

Proposition 11. $\bar{\mathbf{m}}(p) = \bar{\mathbf{m}}(p + \mathbf{1}\alpha)$

Proof.

$$\bar{\mathbf{m}}_i\big|_p = \int_{-\Omega}^{\Omega} ds \int_{p_{-i}-s}^{\infty} d\theta_{-i} M\big(\theta_i = p_i - s\big) \tag{25a}$$
$$= \int_{-\Omega\text{-}\alpha}^{\Omega\text{-}\alpha} ds \int_{p_{-i}+\alpha\text{-}s}^{\infty} d\theta_{-i} M\big(\theta_i = p_i + \alpha\text{-}s\big) \tag{25b}$$
$$= \int_{-\Omega}^{\Omega} ds \int_{p_{-i}+\alpha-s}^{\infty} d\theta_{-i} M\big(\theta_i = p_i + \alpha - s\big) \tag{25c}$$

□

It follows that $\bar{\mathbf{m}}(Wu) = \bar{\mathbf{m}}(I_{-1}u_{-1})$ (where we here we have simply set $u_1 = 0$). While $\bar{\mathbf{m}}(Wu)$ is not an invertible function of u (since the value of u_1 is undetermined) we can show that $\bar{\mathbf{m}}(I_{-1}u_{-1})$ is an invertible function of u_{-1} and we can use this to compute the Legendre transform.

We have the following immediate characterization of the derivatives.

Proposition 12. $\frac{\partial \bar{\mathbf{G}}}{\partial u} = \frac{\partial \bar{\mathbf{G}}}{\partial p}\frac{\partial p}{\partial u} = \bar{\mathbf{m}}(Wu)^\top W$

Proposition 13. $\frac{\partial \bar{\mathbf{G}}}{\partial u_{-1}} = \frac{\partial \bar{\mathbf{G}}}{\partial p}\frac{\partial p}{\partial u_{-1}} = \bar{\mathbf{m}}(I_{-1}u_{-1})^\top I_{-1}$

Expanding out this gives

$$\begin{aligned}\tfrac{\partial \bar{\mathbf{G}}}{\partial u} = \bar{\mathbf{m}}(Wu)^\top W &= \begin{bmatrix} \sum_i \bar{\mathbf{m}}_i & \bar{\mathbf{m}}_2 & \cdots & \bar{\mathbf{m}}_n \end{bmatrix}\big|_{p=Wu} \\ &= \begin{bmatrix} M & \bar{\mathbf{m}}_2 & \cdots & \bar{\mathbf{m}}_n \end{bmatrix}\big|_{p=Wu}\end{aligned}$$

where in the second step we've plugged in the assumption that $M = \sum_i \bar{\mathbf{m}}_i$. Similarly,

$$\tfrac{\partial \bar{\mathbf{G}}}{\partial u_{-1}} = \bar{\mathbf{m}}(I_{-1}u_{-1})^\top I_{-1} = \begin{bmatrix} \bar{\mathbf{m}}_2 & \cdots & \bar{\mathbf{m}}_n \end{bmatrix}\big|_{p=I_{-1}u_{-1}}$$

The second derivatives are then given by

Proposition 14.

$$\begin{aligned}\tfrac{\partial^2 \bar{\mathbf{G}}}{\partial u^2} &= W^\top \tfrac{\partial^2 \bar{G}}{\partial p^2}\big(Wu\big)W = W^\top \tfrac{\partial \bar{\mathbf{m}}}{\partial p}\big(Wu\big)W \\ \tfrac{\partial^2 \bar{\mathbf{G}}}{\partial u_{-1}^2} &= I_{-1}^\top \tfrac{\partial^2 \bar{G}}{\partial p^2}\big(I_{-1}u_{-1}\big)I_{-1} = I_{-1}^\top \tfrac{\partial \bar{\mathbf{m}}}{\partial p}\big(I_{-1}u\big)I_{-1}\end{aligned}$$

Expanding this out one can check that

$$\frac{\partial^2 \bar{\mathbf{G}}}{\partial u^2} = \begin{bmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & \frac{\partial^2 \bar{\mathbf{G}}}{\partial p_{-1}^2} \end{bmatrix}\Bigg|_{p=Wu} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\partial^2 \bar{\mathbf{G}}}{\partial p_2^2} & \cdots & \frac{\partial^2 \bar{\mathbf{G}}}{\partial p^n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial^2 \bar{\mathbf{G}}}{\partial p_2^2} & \cdots & \frac{\partial^2 \bar{\mathbf{G}}}{\partial p^n} \end{bmatrix}\Bigg|_{p=Wu}$$

and we can see that for $i, j = 2, \ldots, n$

$$\frac{\partial^2 \bar{\mathbf{G}}}{\partial u_{-1}^2} = \begin{bmatrix} \frac{\partial^2 \bar{\mathbf{G}}}{\partial p_{-1}^2} \end{bmatrix}\Big|_{p=I_{-1}u_{-1}} = \begin{bmatrix} \frac{\partial^2 \bar{\mathbf{G}}}{\partial p_2^2} & \cdots & \frac{\partial^2 \bar{\mathbf{G}}}{\partial p^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \bar{\mathbf{G}}}{\partial p_2^2} & \cdots & \frac{\partial^2 \bar{\mathbf{G}}}{\partial p^n} \end{bmatrix}\Bigg|_{p=I_{-1}u_{-1}}$$

Specifically, we have that

$$\frac{\partial^2 \bar{\mathbf{G}}}{\partial u_i^2} = \frac{\partial \bar{\mathbf{m}}_i}{\partial p_1}\bigg|_{I_{-1}u_{-1}} - \sum_{j=2}^{n} \frac{\partial \bar{\mathbf{m}}_i}{\partial p_j}\bigg|_{I_{-1}u_{-1}}$$

and $\frac{\partial^2 \bar{G}}{\partial p_i \partial p_j} = -\frac{\partial \bar{\mathbf{m}}_i}{\partial p_j}\Big|_{p=I_{-1}u_{-1}}$. The structure of the second derivatives is illustrated here.

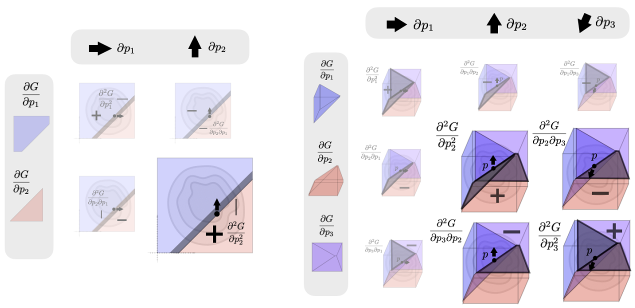

Fig. 19. Class II: Illustration of the second derivative in the u-coordinates. Note that $\frac{\partial^2 \bar{\mathbf{G}}}{\partial u_{-1}^2} = \left[\frac{\partial^2 \bar{\mathbf{G}}}{\partial p_{-1}^2}\right]\Big|_{p=I_{-1}u_{-1}}$ and also that while $\frac{\partial^2 \bar{\mathbf{G}}}{\partial p^2}$ is not positive definite, the sub-block corresponding to the u_{-1} coordinates is.

Here we have isolated the portion of $\bar{G}$ that depends on the differences between elements of p and can show that relative to those elements the Hessian is positive definite (on the interior of the mass region).

Proposition 15. *$\frac{\partial^2 \bar{G}}{\partial u_{-1}^2}$ is diagonally strictly dominant and thus $\bar{G}'$ is strictly convex and from PROP XXX invertible.*

The result of the above analysis is that the map $\mathbf{m}_{-1}(I_{-1}\cdot)$ is an invertible function of u_{-1} where we have implicitly chosen $u_1 = 0$. We will use the notation $\mathbf{u}_{-1} : \bar{m}_{-1} \mapsto u_{-1}$ to refer to the inverse of the map $\mathbf{m}_{-1}(I_{-1}\cdot)$.

The above analysis allows us to compute the Class II Legendre transform in the u coordinates.

$$\bar{\mathbf{G}}^*(\bar{m}) = \min_u \ \bar{m}^\top W u - \bar{G}(Wu) \tag{26}$$

Differentiating to solve gives

$$\begin{aligned}
\bar{m}^\top W &= \bar{\mathbf{m}}(Wu)^\top W \\
\left[\sum_i \bar{m}_i \quad \bar{m}_2 \quad \cdots \quad \bar{m}_n\right] &= \left[\sum_i \bar{\mathbf{m}}_\mathbf{i} \quad \bar{\mathbf{m}}_2 \quad \cdots \quad \bar{\mathbf{m}}_n\right]\Big|_{Wu} \\
\left[\sum_i \bar{m}_i \quad \bar{m}_2 \quad \cdots \quad \bar{m}_n\right] &= \left[M \quad \bar{\mathbf{m}}_2 \quad \cdots \quad \bar{\mathbf{m}}_n\right]\Big|_{Wu}
\end{aligned}$$

The solution is then given by

$$\sum_i \bar{m}_i = M, \quad \bar{m}_{-1} = \bar{\mathbf{m}}_{-1}(Wu) = \bar{\mathbf{m}}_{-1}(I_{-1}u_{-1}) \tag{28}$$

where the second condition can also be written as $u_{-1} = \mathbf{u}_{-1}(\bar{m}_{-1})$. Plugging back into the Legendre transform and using the forms in (24) and , we get

$$\begin{aligned}
\bar{\mathbf{G}}^*(\bar{m}) &= Mu_1 + \bar{m}_{-1}^\top \mathbf{u}_{-1}(\bar{m}_{-1}) - Mu_1 - \bar{\mathbf{G}}\big|_{I_{-1}\mathbf{u}_{-1}(\bar{m}_{-1})} \\
&= \bar{m}_{-1}^\top \mathbf{u}_{-1}(\bar{m}_{-1}) - \bar{\mathbf{G}}\big|_{I_{-1}\mathbf{u}_{-1}(\bar{m}_{-1})} \\
&= \bar{m}_{-1}^\top \mathbf{u}_{-1}(\bar{m}_{-1}) \\
&\quad - \int_{-\Omega}^{\Omega} dt \, \left[M - \mathbf{1}^\top \bar{\mathbf{m}}_{-1}\big(I_{-1}\mathbf{u}_{-1}(\bar{m}_{-1}) + t\mathbf{1}\big)\right]
\end{aligned}$$

Note the similarities in form with (22) where the mass $\bar{m}_1$ has replaced m_0. Since $\bar{\mathbf{G}}^*(\bar{m})$ is only a function of $\bar{m}_{-1}$, we can abuse notation slightly and write $\bar{\mathbf{G}}^*(\bar{m}_{-1})$. We can compute explicitly that the derivative of $\bar{\mathbf{G}}^*(\bar{m}_{-1})$ is the cost vector $\mathbf{u}_{-1}(\bar{m}_{-1})^\top$ as expected.

Proposition 16. $\frac{\partial \bar{\mathbf{G}}^*}{\partial \bar{m}_{-1}} = \mathbf{u}_{-1}(\bar{m}_{-1})^\top$

Proof.

$$\begin{aligned}
\frac{\partial \bar{\mathbf{G}}^*}{\partial \bar{m}_{-1}} &= \frac{\partial}{\partial \bar{m}_{-1}}\left[\bar{m}_{-1}^\top \mathbf{u}_{-1}(\bar{m}_{-1})\right] - \frac{\partial}{\partial \bar{m}_{-1}}\left[\int_{-\Omega}^{\Omega} M ds\right] \\
&\quad + \frac{\partial}{\partial \bar{m}_{-1}}\left[\int_0^\infty \mathbf{1}^\top \bar{\mathbf{m}}_{-1}\Big(I_{-1}\mathbf{u}_{-1}\big|_{\bar{m}_{-1}} + t\mathbf{1}\Big)\, dt\right] \\
&= \mathbf{u}_{-1}^\top\big|_{\bar{m}_{-1}} + \bar{m}_{-1}^\top \frac{\partial \mathbf{u}_{-1}}{\partial \bar{m}_{-1}}\bigg|_{\bar{m}_{-1}} + \left[\int_{-\Omega}^{\Omega} \mathbf{1}^\top \cdot \right. \\
&\quad \left. \frac{\partial \bar{\mathbf{m}}}{\partial p}\Big(I_{-1}\mathbf{u}_{-1}\big|_{\bar{m}_{-1}} + t\mathbf{1}\Big) I_{-1} \, dt\right] \frac{\partial \mathbf{u}_{-1}}{\partial \bar{m}_{-1}}\bigg|_{\bar{m}_{-1}} \\
&= \mathbf{u}_{-1}^\top\big|_{\bar{m}_{-1}} + (\bar{m}_{-1}^\top - \bar{m}_{-1}^\top)\frac{\partial \mathbf{u}_{-1}}{\partial \bar{m}_{-1}}\bigg|_{\bar{m}_{-1}} \\
&= \mathbf{u}_{-1}^\top(\bar{m}_{-1})
\end{aligned}$$

where in the second to last step we have used the fact that for $i \neq 1$ $(*) = \int_{-\Omega}^{\Omega} \sum_{j \neq 1} \frac{\partial \bar{\mathbf{m}}_i}{\partial p_j}\big(I_{-1}\mathbf{u}_{-1}(\bar{m}_{-1}) + t\mathbf{1}\big) I_{-1} \, dt = \bar{m}_i$ which can be proved by applying Eqn. (21b) and (20b) to get

$$\begin{aligned}
(*) &= \int_{-\Omega}^{\Omega} \frac{\partial \bar{\mathbf{m}}_1}{\partial p_j}\Big(I_{-1}\mathbf{u}_{-1}(\bar{m}_{-1}) + t\mathbf{1}\Big) I_{-1} \, dt \\
&= \int_{-\Omega}^{\Omega} dt \int_{-\Omega}^{\Omega} ds \left[\int_{\mathbf{u}_{-1j}+t-s}^{\infty} d\theta_{-1j} \right. \\
&\quad \left. M\Big(\theta_1 = t - s, \theta_j = \mathbf{u}_j(\bar{m}_{-1}) + t - s\Big)\right] = \bar{m}_i
\end{aligned}$$

□

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