

Shift, Circulant, & DFT Matrices

Linear Algebra

Winter 2022 - Dan Calderone

Shift Matrix

... for a discrete time periodic signal

$$S \in \mathbb{R}^{T \times T} \quad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

... represents a step forward in time

for $k = 0, 1, \dots, T - 1$

...step by
1 time step

$$x^+ = Sx$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

...step by
t time step

Eigenvalue k

Eigenvector k

(frequency k)

$$e^{i2\pi \frac{k}{T}}$$

$$\begin{bmatrix} e^{i2\pi \frac{0k}{T}} \\ e^{i2\pi \frac{1k}{T}} \\ e^{i2\pi \frac{2k}{T}} \\ \vdots \\ e^{i2\pi \frac{(T-2)k}{T}} \\ e^{i2\pi \frac{(T-1)k}{T}} \end{bmatrix}$$

t-th element
 $\leftarrow e^{i2\pi \frac{tk}{T}}$

Eigenvectors:

$$\begin{bmatrix} e^{i2\pi \frac{0k}{T}} \\ e^{i2\pi \frac{1k}{T}} \\ e^{i2\pi \frac{2k}{T}} \\ \vdots \\ e^{i2\pi \frac{(T-2)k}{T}} \\ e^{i2\pi \frac{(T-1)k}{T}} \end{bmatrix} e^{i2\pi \frac{k}{T}} = \begin{bmatrix} e^{i2\pi \frac{1k}{T}} \\ e^{i2\pi \frac{2k}{T}} \\ e^{i2\pi \frac{3k}{T}} \\ \vdots \\ e^{i2\pi \frac{(T-1)k}{T}} \\ e^{i2\pi \frac{0k}{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\pi \frac{0k}{T}} \\ e^{i2\pi \frac{1k}{T}} \\ e^{i2\pi \frac{2k}{T}} \\ \vdots \\ e^{i2\pi \frac{(T-2)k}{T}} \\ e^{i2\pi \frac{(T-1)k}{T}} \end{bmatrix}$$

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Frequency: $k = 0, 1, \dots, T - 1$

Eigenvalues: $e^{i2\pi \frac{0}{T}}, e^{i2\pi \frac{1}{T}}, e^{i2\pi \frac{2}{T}}, \dots, e^{i2\pi \frac{T-2}{T}}, e^{i2\pi \frac{T-1}{T}},$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi \frac{1(1)}{T}} & e^{i2\pi \frac{1(2)}{T}} & \cdots & e^{i2\pi \frac{1(T-2)}{T}} & e^{i2\pi \frac{1(T-1)}{T}} \\ 1 & e^{i2\pi \frac{2(1)}{T}} & e^{i2\pi \frac{2(2)}{T}} & \cdots & e^{i2\pi \frac{2(T-2)}{T}} & e^{i2\pi \frac{2(T-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi \frac{(T-2)1}{T}} & e^{i2\pi \frac{(T-2)2}{T}} & \cdots & e^{i2\pi \frac{(T-2)(T-2)}{T}} & e^{i2\pi \frac{(T-2)(T-1)}{T}} \\ 1 & e^{i2\pi \frac{(T-1)1}{T}} & e^{i2\pi \frac{(T-1)2}{T}} & \cdots & e^{i2\pi \frac{(T-1)(T-2)}{T}} & e^{i2\pi \frac{(T-1)(T-1)}{T}} \end{bmatrix}$$

Eigenvectors:

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Eigenvectors:

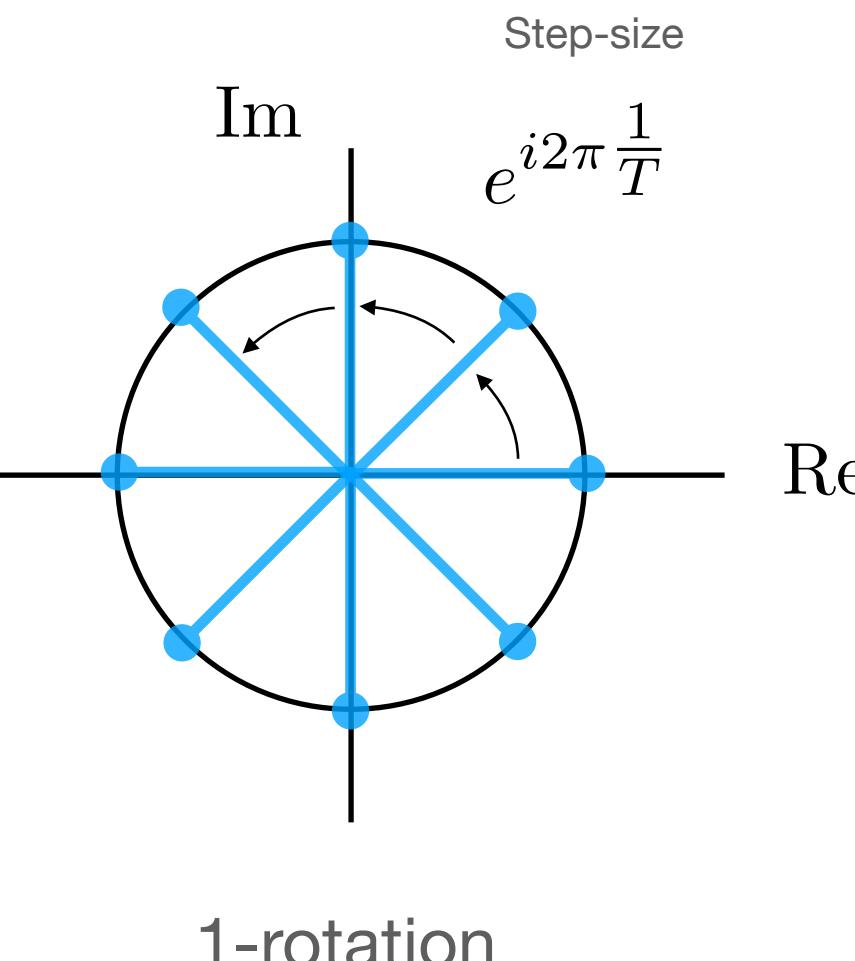
$$\begin{bmatrix} e^{i2\pi \frac{0k}{T}} \\ e^{i2\pi \frac{1k}{T}} \\ e^{i2\pi \frac{2k}{T}} \\ \vdots \\ e^{i2\pi \frac{(T-2)k}{T}} \\ e^{i2\pi \frac{(T-1)k}{T}} \end{bmatrix} e^{i2\pi \frac{k}{T}} = \begin{bmatrix} e^{i2\pi \frac{1k}{T}} \\ e^{i2\pi \frac{2k}{T}} \\ e^{i2\pi \frac{3k}{T}} \\ \vdots \\ e^{i2\pi \frac{(T-1)k}{T}} \\ e^{i2\pi \frac{0k}{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\pi \frac{0k}{T}} \\ e^{i2\pi \frac{1k}{T}} \\ e^{i2\pi \frac{2k}{T}} \\ \vdots \\ e^{i2\pi \frac{(T-2)k}{T}} \\ e^{i2\pi \frac{(T-1)k}{T}} \end{bmatrix}$$

Frequency: $k = 0, 1, \dots, T - 1$

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Eigenvectors:

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...step by
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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

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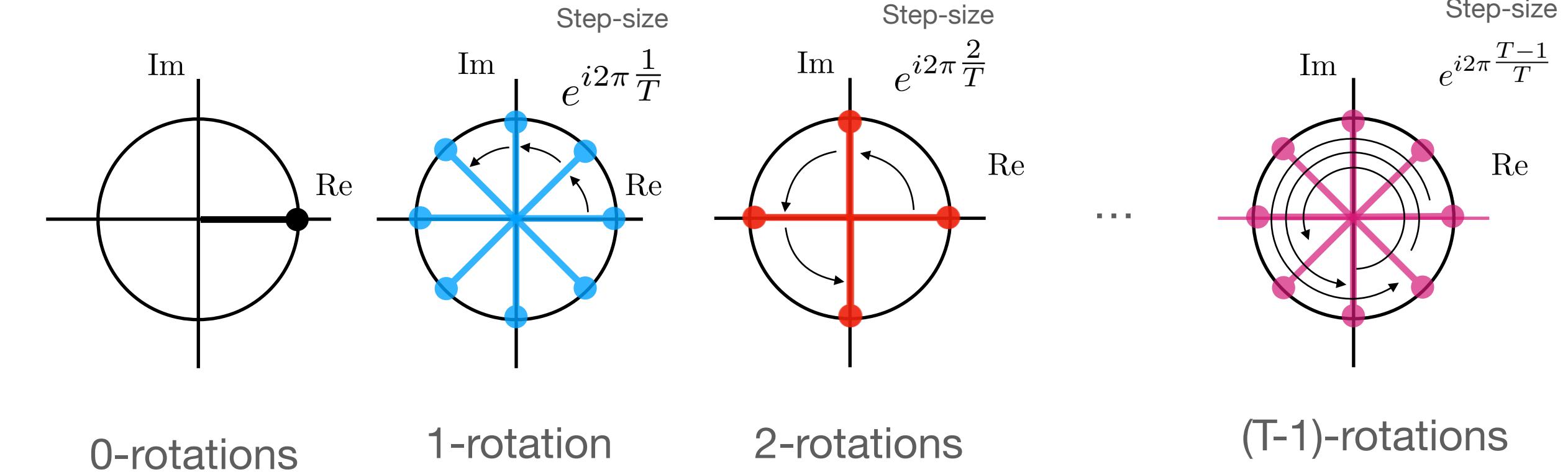
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Ex: for $T = 8$



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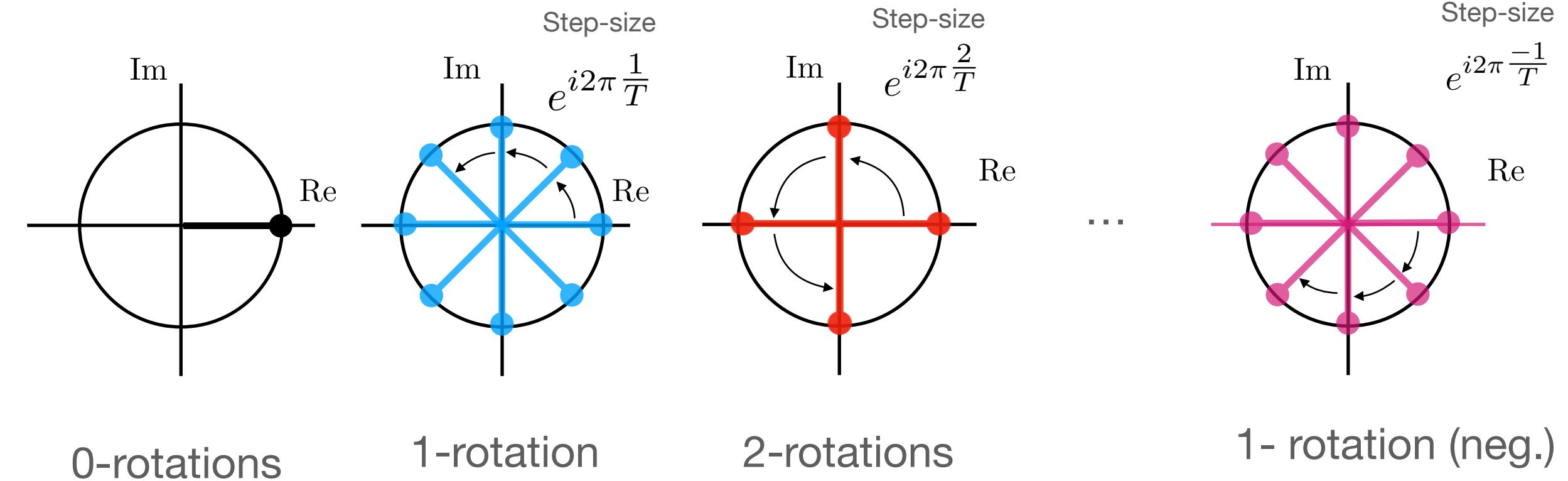
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...step by
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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

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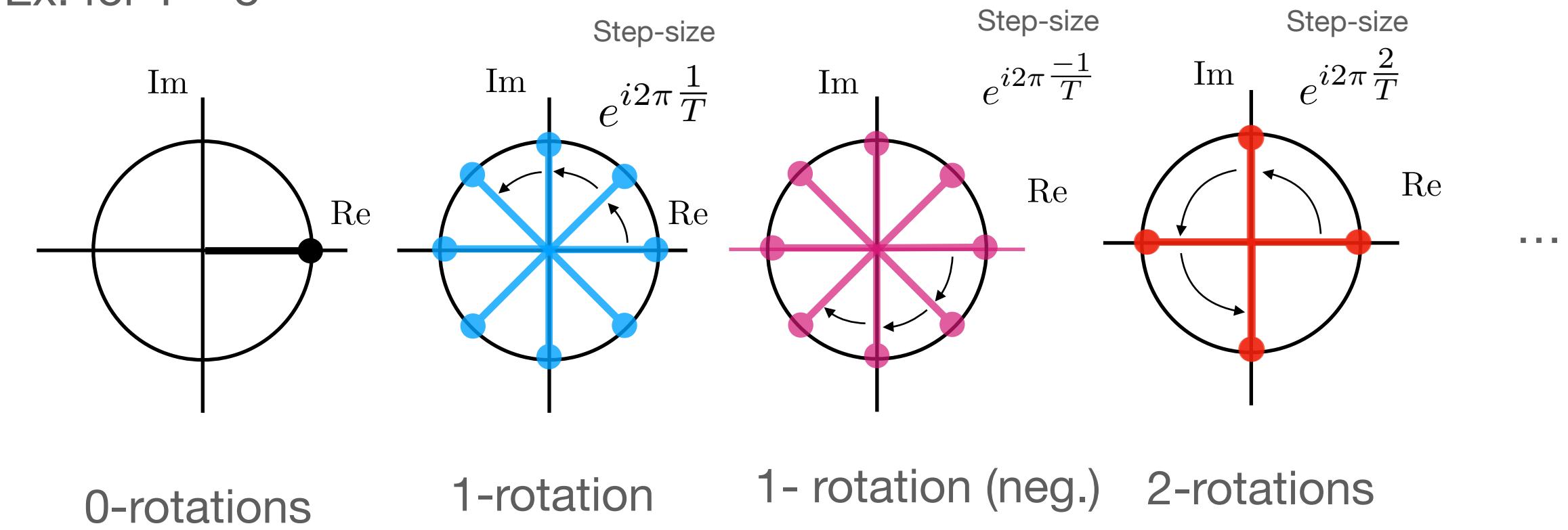
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Diagonalization:

$$S = U \text{dg}(U_1) U^* \quad U^* U = U U^* = I$$

$$S = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{i2\pi\frac{1}{T}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{i2\pi\frac{2}{T}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i2\pi\frac{(T-2)}{T}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{i2\pi\frac{(T-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi\frac{1(1)}{T}} & e^{-i2\pi\frac{1(2)}{T}} & \cdots & e^{-i2\pi\frac{1(-2)}{T}} & e^{-i2\pi\frac{1(-1)}{T}} \\ 1 & e^{-i2\pi\frac{2(1)}{T}} & e^{-i2\pi\frac{2(2)}{T}} & \cdots & e^{-i2\pi\frac{2(-2)}{T}} & e^{-i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{-i2\pi\frac{(T-2)1}{T}} & e^{-i2\pi\frac{(T-2)2}{T}} & \cdots & e^{-i2\pi\frac{(T-2)(-2)}{T}} & e^{-i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi\frac{(T-1)1}{T}} & e^{-i2\pi\frac{(T-1)2}{T}} & \cdots & e^{-i2\pi\frac{(T-1)(-2)}{T}} & e^{-i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Shift Matrix

... for a discrete time periodic signal ... represents a step forward in time

$$S \in \mathbb{R}^{T \times T} \quad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix} \quad \text{...step by } t \text{ time step}$$

...step by
1 time step

$$x^+ = Sx$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Frequency: $k = 0, 1, \dots, T - 1$

Eigenvalues: $e^{i2\pi\frac{0}{T}}, e^{i2\pi\frac{1}{T}}, e^{i2\pi\frac{2}{T}}, \dots, e^{i2\pi\frac{-2}{T}}, e^{i2\pi\frac{-1}{T}}$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Diagonalization:

$$S = U \text{dg}(U_1) U^* \quad U^* U = U U^* = I$$

$$S = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{i2\pi\frac{1}{T}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{i2\pi\frac{2}{T}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i2\pi\frac{(T-2)}{T}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{i2\pi\frac{(T-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi\frac{1(1)}{T}} & e^{-i2\pi\frac{1(2)}{T}} & \cdots & e^{-i2\pi\frac{1(-2)}{T}} & e^{-i2\pi\frac{1(-1)}{T}} \\ 1 & e^{-i2\pi\frac{2(1)}{T}} & e^{-i2\pi\frac{2(2)}{T}} & \cdots & e^{-i2\pi\frac{2(-2)}{T}} & e^{-i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{-i2\pi\frac{(T-2)1}{T}} & e^{-i2\pi\frac{(T-2)2}{T}} & \cdots & e^{-i2\pi\frac{(T-2)(-2)}{T}} & e^{-i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi\frac{(T-1)1}{T}} & e^{-i2\pi\frac{(T-1)2}{T}} & \cdots & e^{-i2\pi\frac{(T-1)(-2)}{T}} & e^{-i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Shift Matrix

... for a discrete time periodic signal ... represents a step forward in time

$$S \in \mathbb{R}^{T \times T} \quad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix} \quad \text{...step by } t \text{ time step}$$

...step by
1 time step

$$x^+ = Sx$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Diagonalization:

$$S^t = U \text{dg}(U_1)^t U^* = U \text{dg}(U_t) U^* \quad U^* U = U U^* = I$$

$$S^t = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi \frac{1(1)}{T}} & e^{i2\pi \frac{1(2)}{T}} & \cdots & e^{i2\pi \frac{1(-2)}{T}} & e^{i2\pi \frac{1(-1)}{T}} \\ 1 & e^{i2\pi \frac{2(1)}{T}} & e^{i2\pi \frac{2(2)}{T}} & \cdots & e^{i2\pi \frac{2(-2)}{T}} & e^{i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi \frac{(T-2)1}{T}} & e^{i2\pi \frac{(T-2)2}{T}} & \cdots & e^{i2\pi \frac{(T-2)(-2)}{T}} & e^{i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi \frac{(T-1)1}{T}} & e^{i2\pi \frac{(T-1)2}{T}} & \cdots & e^{i2\pi \frac{(T-1)(-2)}{T}} & e^{i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{i2\pi \frac{1t}{T}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{i2\pi \frac{2t}{T}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i2\pi \frac{(T-2)t}{T}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{i2\pi \frac{(T-1)t}{T}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi \frac{1(1)}{T}} & e^{-i2\pi \frac{1(2)}{T}} & \cdots & e^{-i2\pi \frac{1(-2)}{T}} & e^{-i2\pi \frac{1(-1)}{T}} \\ 1 & e^{-i2\pi \frac{2(1)}{T}} & e^{-i2\pi \frac{2(2)}{T}} & \cdots & e^{-i2\pi \frac{2(-2)}{T}} & e^{-i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{-i2\pi \frac{(T-2)1}{T}} & e^{-i2\pi \frac{(T-2)2}{T}} & \cdots & e^{-i2\pi \frac{(T-2)(-2)}{T}} & e^{-i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi \frac{(T-1)1}{T}} & e^{-i2\pi \frac{(T-1)2}{T}} & \cdots & e^{-i2\pi \frac{(T-1)(-2)}{T}} & e^{-i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Circulant Matrix

... all shifts of a periodic signal...

$$c \in \mathbb{R}^T \quad C \in \mathbb{R}^{T \times T} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix} \quad \text{...step by } t \text{ time step}$$

periodic signal

circulant matrix

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix}$$

$$C = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix}$$

$$\text{Sum of shifts: } C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{T-2} S^{T-2} + c_{T-1} S^{T-1}$$

Frequency: $k = 0, 1, \dots, T - 1$

Eigenvalues: $e^{i2\pi \frac{0}{T}}, e^{i2\pi \frac{1}{T}}, e^{i2\pi \frac{2}{T}}, \dots, e^{i2\pi \frac{-2}{T}}, e^{i2\pi \frac{-1}{T}}$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi \frac{1(1)}{T}} & e^{i2\pi \frac{1(2)}{T}} & \cdots & e^{i2\pi \frac{1(-2)}{T}} & e^{i2\pi \frac{1(-1)}{T}} \\ 1 & e^{i2\pi \frac{2(1)}{T}} & e^{i2\pi \frac{2(2)}{T}} & \cdots & e^{i2\pi \frac{2(-2)}{T}} & e^{i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi \frac{(T-2)1}{T}} & e^{i2\pi \frac{(T-2)2}{T}} & \cdots & e^{i2\pi \frac{(T-2)(-2)}{T}} & e^{i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi \frac{(T-1)1}{T}} & e^{i2\pi \frac{(T-1)2}{T}} & \cdots & e^{i2\pi \frac{(T-1)(-2)}{T}} & e^{i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Discrete Convolution: $y = (c * x) = Cx$

Impulse Response: $c = [c_0 \ c_1 \ \cdots \ c_{T-1}]$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{T-2} \\ y_{T-1} \end{bmatrix} = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

$$y_t = \sum_{\tau} c_{t-\tau} x_{\tau}$$

Note: Impulse response at time t ... selects that column...

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix} = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Circulant Matrix

... all shifts of a periodic signal...

$$c \in \mathbb{R}^T \quad C \in \mathbb{R}^{T \times T} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix} \quad \text{...step by } t \text{ time step}$$

periodic signal

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix}$$

circulant matrix

$$C = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix}$$

Sum of shifts:

$$C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{T-2} S^{T-2} + c_{T-1} S^{T-1}$$

Frequency: $k = 0, 1, \dots, T - 1$

Eigenvalues: $\sqrt{T} U_0^* c \quad \sqrt{T} U_1^* c \quad \sqrt{T} U_2^* c \quad \dots \quad \sqrt{T} U_{T-2}^* c \quad \sqrt{T} U_{T-1}^* c$

Eigenvectors: $U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi \frac{1(1)}{T}} & e^{i2\pi \frac{1(2)}{T}} & \cdots & e^{i2\pi \frac{1(-2)}{T}} & e^{i2\pi \frac{1(-1)}{T}} \\ 1 & e^{i2\pi \frac{2(1)}{T}} & e^{i2\pi \frac{2(2)}{T}} & \cdots & e^{i2\pi \frac{2(-2)}{T}} & e^{i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi \frac{(T-2)1}{T}} & e^{i2\pi \frac{(T-2)2}{T}} & \cdots & e^{i2\pi \frac{(T-2)(-2)}{T}} & e^{i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi \frac{(T-1)1}{T}} & e^{i2\pi \frac{(T-1)2}{T}} & \cdots & e^{i2\pi \frac{(T-1)(-2)}{T}} & e^{i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix}$

Diagonalization:

$$C = \sum_{t=0}^{T-1} c_t S^t = U \left(\sum_{t=0}^{T-1} c_t \text{dg}(U_t) \right) U^* = \sqrt{T} U \text{dg}(U^* c) U^* \quad U^* U = U U^* = I$$

$$C = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi \frac{1(1)}{T}} & e^{i2\pi \frac{1(2)}{T}} & \cdots & e^{i2\pi \frac{1(-2)}{T}} & e^{i2\pi \frac{1(-1)}{T}} \\ 1 & e^{i2\pi \frac{2(1)}{T}} & e^{i2\pi \frac{2(2)}{T}} & \cdots & e^{i2\pi \frac{2(-2)}{T}} & e^{i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi \frac{(T-2)1}{T}} & e^{i2\pi \frac{(T-2)2}{T}} & \cdots & e^{i2\pi \frac{(T-2)(-2)}{T}} & e^{i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi \frac{(T-1)1}{T}} & e^{i2\pi \frac{(T-1)2}{T}} & \cdots & e^{i2\pi \frac{(T-1)(-2)}{T}} & e^{i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix} \sqrt{T} \begin{bmatrix} U_0^* c & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_1^* c & 0 & \cdots & 0 & 0 \\ 0 & 0 & U_2^* c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{T-2}^* c & 0 \\ 0 & 0 & 0 & \cdots & 0 & U_{T-1}^* c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi \frac{1(1)}{T}} & e^{-i2\pi \frac{1(2)}{T}} & \cdots & e^{-i2\pi \frac{1(-2)}{T}} & e^{-i2\pi \frac{1(-1)}{T}} \\ 1 & e^{-i2\pi \frac{2(1)}{T}} & e^{-i2\pi \frac{2(2)}{T}} & \cdots & e^{-i2\pi \frac{2(-2)}{T}} & e^{-i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{-i2\pi \frac{(T-2)1}{T}} & e^{-i2\pi \frac{(T-2)2}{T}} & \cdots & e^{-i2\pi \frac{(T-2)(-2)}{T}} & e^{-i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi \frac{(T-1)1}{T}} & e^{-i2\pi \frac{(T-1)2}{T}} & \cdots & e^{-i2\pi \frac{(T-1)(-2)}{T}} & e^{-i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Circulant Matrix

... all shifts of a periodic signal...

$$c \in \mathbb{R}^T \quad C \in \mathbb{R}^{T \times T} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix} \quad \text{...step by } t \text{ time step}$$

periodic signal

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix}$$

circulant matrix

$$C = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix}$$

Sum of shifts:

$$C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{T-2} S^{T-2} + c_{T-1} S^{T-1}$$

Frequency: $k = 0, 1, \dots, T - 1$

Eigenvalues: $\sqrt{T} U_0^* c \quad \sqrt{T} U_1^* c \quad \sqrt{T} U_2^* c \quad \dots \quad \sqrt{T} U_{T-2}^* c \quad \sqrt{T} U_{T-1}^* c$

Eigenvectors: $U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi \frac{1(1)}{T}} & e^{i2\pi \frac{1(2)}{T}} & \cdots & e^{i2\pi \frac{1(-2)}{T}} & e^{i2\pi \frac{1(-1)}{T}} \\ 1 & e^{i2\pi \frac{2(1)}{T}} & e^{i2\pi \frac{2(2)}{T}} & \cdots & e^{i2\pi \frac{2(-2)}{T}} & e^{i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi \frac{(T-2)1}{T}} & e^{i2\pi \frac{(T-2)2}{T}} & \cdots & e^{i2\pi \frac{(T-2)(-2)}{T}} & e^{i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi \frac{(T-1)1}{T}} & e^{i2\pi \frac{(T-1)2}{T}} & \cdots & e^{i2\pi \frac{(T-1)(-2)}{T}} & e^{i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix}$

Diagonalization:

$$y = (c * x) = Cx = \sqrt{T} U \text{dg}(U^* c) U^* x$$

$$\sqrt{T} U^* y = \text{dg}(\sqrt{T} U^* c) \sqrt{T} U^* x$$

$$C = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi \frac{1(1)}{T}} & e^{i2\pi \frac{1(2)}{T}} & \cdots & e^{i2\pi \frac{1(-2)}{T}} & e^{i2\pi \frac{1(-1)}{T}} \\ 1 & e^{i2\pi \frac{2(1)}{T}} & e^{i2\pi \frac{2(2)}{T}} & \cdots & e^{i2\pi \frac{2(-2)}{T}} & e^{i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi \frac{(T-2)1}{T}} & e^{i2\pi \frac{(T-2)2}{T}} & \cdots & e^{i2\pi \frac{(T-2)(-2)}{T}} & e^{i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi \frac{(T-1)1}{T}} & e^{i2\pi \frac{(T-1)2}{T}} & \cdots & e^{i2\pi \frac{(T-1)(-2)}{T}} & e^{i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix} \sqrt{T} \begin{bmatrix} U_0^* c & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_1^* c & 0 & \cdots & 0 & 0 \\ 0 & 0 & U_2^* c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{T-2}^* c & 0 \\ 0 & 0 & 0 & \cdots & 0 & U_{T-1}^* c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi \frac{1(1)}{T}} & e^{-i2\pi \frac{1(2)}{T}} & \cdots & e^{-i2\pi \frac{1(-2)}{T}} & e^{-i2\pi \frac{1(-1)}{T}} \\ 1 & e^{-i2\pi \frac{2(1)}{T}} & e^{-i2\pi \frac{2(2)}{T}} & \cdots & e^{-i2\pi \frac{2(-2)}{T}} & e^{-i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{-i2\pi \frac{(T-2)1}{T}} & e^{-i2\pi \frac{(T-2)2}{T}} & \cdots & e^{-i2\pi \frac{(T-2)(-2)}{T}} & e^{-i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi \frac{(T-1)1}{T}} & e^{-i2\pi \frac{(T-1)2}{T}} & \cdots & e^{-i2\pi \frac{(T-1)(-2)}{T}} & e^{-i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Circulant Matrix

... all shifts of a periodic signal...

$$c \in \mathbb{R}^T \quad C \in \mathbb{R}^{T \times T} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix} \quad \text{...step by } t \text{ time step}$$

periodic signal

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix}$$

circulant matrix

$$C = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix}$$

Sum of shifts:

$$C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{T-2} S^{T-2} + c_{T-1} S^{T-1}$$

Frequency: $k = 0, 1, \dots, T - 1$

Eigenvalues: $\sqrt{T} U_0^* c \quad \sqrt{T} U_1^* c \quad \sqrt{T} U_2^* c \quad \dots \quad \sqrt{T} U_{T-2}^* c \quad \sqrt{T} U_{T-1}^* c$

Eigenvectors: $U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi \frac{1(1)}{T}} & e^{i2\pi \frac{1(2)}{T}} & \cdots & e^{i2\pi \frac{1(-2)}{T}} & e^{i2\pi \frac{1(-1)}{T}} \\ 1 & e^{i2\pi \frac{2(1)}{T}} & e^{i2\pi \frac{2(2)}{T}} & \cdots & e^{i2\pi \frac{2(-2)}{T}} & e^{i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{i2\pi \frac{(T-2)1}{T}} & e^{i2\pi \frac{(T-2)2}{T}} & \cdots & e^{i2\pi \frac{(T-2)(-2)}{T}} & e^{i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi \frac{(T-1)1}{T}} & e^{i2\pi \frac{(T-1)2}{T}} & \cdots & e^{i2\pi \frac{(T-1)(-2)}{T}} & e^{i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix}$

Diagonalization:

Convolution in time domain...

$$y = (c * x) = Cx = \sqrt{T} U \text{dg}(U^* c) U^* x$$

Multiplication in frequency domain...

$$\underbrace{\sqrt{T} U^* y}_{\hat{y}} = \underbrace{\text{dg}\left(\underbrace{\sqrt{T} U^* c}_{\hat{c}}\right)}_{\hat{c}} \underbrace{\sqrt{T} U^* x}_{\hat{x}}$$

elementwise...

$$\hat{y}_k = \hat{c}_k \hat{x}_k$$

Discrete Fourier Transform

$$\hat{x} = \sqrt{T} U^* x = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi \frac{1(1)}{T}} & e^{-i2\pi \frac{1(2)}{T}} & \cdots & e^{-i2\pi \frac{1(-2)}{T}} & e^{-i2\pi \frac{1(-1)}{T}} \\ 1 & e^{-i2\pi \frac{2(1)}{T}} & e^{-i2\pi \frac{2(2)}{T}} & \cdots & e^{-i2\pi \frac{2(-2)}{T}} & e^{-i2\pi \frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{-i2\pi \frac{(T-2)1}{T}} & e^{-i2\pi \frac{(T-2)2}{T}} & \cdots & e^{-i2\pi \frac{(T-2)(-2)}{T}} & e^{-i2\pi \frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi \frac{(T-1)1}{T}} & e^{-i2\pi \frac{(T-1)2}{T}} & \cdots & e^{-i2\pi \frac{(T-1)(-2)}{T}} & e^{-i2\pi \frac{(T-1)(-1)}{T}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

$$\hat{x}_k = \sum_{t=0}^{T-1} x_t e^{-i2\pi \frac{tk}{T}}$$

Circulant Matrix - parallels with continuous time

	Discrete Time	Continuous Time
Signals	$c, x, y \in \mathbb{R}^T$	$c(t), x(t), y(t) \quad t \in [0, \Delta t T]$
Eigen functions	$U_k = \begin{bmatrix} e^{i2\pi \frac{0k}{T}} & e^{i2\pi \frac{1k}{T}} & \dots & e^{i2\pi \frac{(T-1)k}{T}} \end{bmatrix}^T \in \mathbb{C}^n$	$e^{i2\pi \omega t}$
Time evolution	$S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$	$\frac{d}{dt} x = \dot{x}$...infinitesimal $\int_0^{\Delta t} e^{\Delta t - \tau}(\cdot) d\tau$...integration over a time step $\int_0^{t\Delta t} e^{t\Delta t - \tau}(\cdot) d\tau$...integration over t time steps
Laplace/ Fourier Transform	$\hat{x} = \sqrt{T} U^* x \quad \hat{x}_k = \sqrt{T} \sum_{t=0}^{T-1} x_t e^{-i2\pi \frac{tk}{T}}$	Laplace $\hat{x}(s) = \mathcal{L}(x(t)) = \int_0^\infty e^{-st} x(t) dt$ Fourier $\hat{x}(\omega) = \mathcal{F}(x(t)) = \int_0^\infty e^{-i2\pi \omega t} x(t) dt$
Inverse Transforms	$x = \frac{1}{\sqrt{T}} U \hat{x} \quad x_t = \frac{1}{\sqrt{T}} \sum_{k=0}^{T-1} \hat{x}_k e^{i2\pi \frac{tk}{T}}$	Inv. Fourier $x(t) = \mathcal{F}^{-1}(\hat{x}(\omega)) = \int_{-\infty}^\infty e^{i2\pi \omega t} \hat{x}(\omega) d\omega$
Convolution	$y = (c * x) = Cx \quad y_t = \sum_{\tau} c_{t-\tau} x_{\tau}$ $y = (c * x) = \sqrt{T} U \text{dg}(U^* c) U^* x$	$y(t) = \int_0^t c(t-\tau) x(\tau) d\tau$ $y(t) = \mathcal{L}^{-1} \left\{ \mathcal{L}\{c\}(s) \cdot \mathcal{L}\{x\}(s) \right\}$ $\hat{y}(s) = \hat{c}(s) \hat{x}(s)$
Plancherel/ Parseval's Theorem	$\langle \hat{y}, \hat{x} \rangle = \hat{y}^* \hat{x} = \frac{\sqrt{T}}{\sqrt{T}} y^* U U^* x = y^* x$	$\langle \hat{y}, \hat{x} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(i\omega)^* \hat{x}(i\omega) d\omega = \int_0^{\infty} y(t)^* x(t) dt = \langle y, x \rangle$