Quadratic Forms, Definite Matrices, Congruence Transformations

Linear Algebra:

Winter 2022 - Dan Calderone

Quadratic Form: $f(x) = x^T Q x$ $Q \in \mathbb{R}^{n \times n}$ $Q = Q^T$

Definiteness:	Short	Notation	Definition		Analogy	Eigenvalues
Positive definite:	PD	$Q \succ 0$	$x^T Q x > 0$	$\forall x x \neq 0$	positive orthant	$\lambda_i > 0 \lambda_i \in eig(Q)$
Positive semi-definite	PSD	$Q \succeq 0$	$x^T Q x \ge 0$	$\forall x$	positive orthant w/ boundary	$\lambda_i \ge 0 \lambda_i \in eig(Q)$
Negative-definite	ND	$Q \prec 0$	$x^T Q x < 0$	$\forall x x \neq 0$	negative orthant	$\lambda_i < 0 \lambda_i \in eig(Q)$
Negative semi-definite	NSD	$Q \preceq 0$	$x^T Q x \le 0$	$\forall x$	negative orthant w/ boundary	$\lambda_i \le 0 \lambda_i \in eig(Q)$
Indefinite:			$x^T Q x > 0$	some x	the rest of the space	
			$x^T Q x < 0$	some x		

Note: not a useful definition for general matrices

... condition only says something about the symmetric part of Q

Symmetric/Skew-symmetric Decomposition

$$Q = \frac{1}{2} \underbrace{\left(Q + Q^T\right)}_{\text{symmetric}} + \frac{1}{2} \underbrace{\left(Q - Q^T\right)}_{\text{skew-sym}}$$

$$x^{T}Qx = \frac{1}{2}x^{T}\left(Q + Q^{T}\right)x + \frac{1}{2}x^{T}\left(Q - Q^{T}\right)x$$

$$= \frac{1}{2}x^{T}\left(Q + Q^{T}\right)x + \frac{1}{2}x^{T}Qx - \frac{1}{2}x^{T}Q^{T}x$$
...transpose
$$= \frac{1}{2}x^{T}\left(Q + Q^{T}\right)x + \underbrace{\frac{1}{2}x^{T}Qx - \frac{1}{2}x^{T}Qx}_{=0}$$

$$= \frac{1}{2}x^{T}\left(Q + Q^{T}\right)x$$
...only the symmetric part matters

Eigenvalue condition proof:

...consider eigenvector coordinates

$$x = Vx'$$

since V is invertible...
$$\forall x \iff \forall x'$$

$$x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i}x_{i}'^{2}$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x^{\prime} \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Quadratic Form:

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

Definiteness:

Eigenvalues

 $\lambda_i > 0 \quad \lambda_i \in eig(Q)$

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$

 $\lambda_i < 0 \quad \lambda_i \in eig(Q)$

Eigenvalue condition proof:

Positive definite:

PSD

ND

$$Q \succ 0$$

 $Q \succeq 0$

 $Q \prec 0$

 $x^T Q x \ge 0 \quad \forall \ x$

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

...the rest of the space

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

$$x = Vx'$$

...consider eigenvector coordinates

$$\forall x \iff \forall x'$$

$$x^T Q x = x V D V^T x = x'^T D x' = \sum_{i} \lambda_i x_i'^2$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x^{\prime} \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Negative semi-definite

Negative-definite

Positive semi-definite

$$Q \le 0 \qquad x^T Q x \le 0 \qquad \forall \ x$$

$$\leq 0$$

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$

$$x^T Q x < 0$$
 some

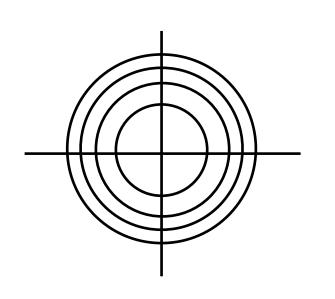
 $x^T Q x > 0$ some x

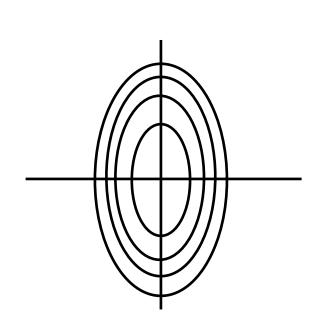
$$x^T Q x < 0$$
 some x

 $Q \succ 0$ Surfaces:

$$Q = I$$

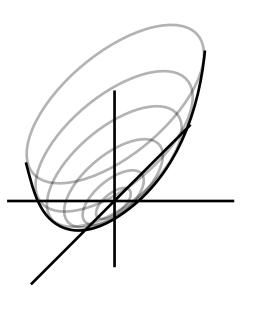
Indefinite:

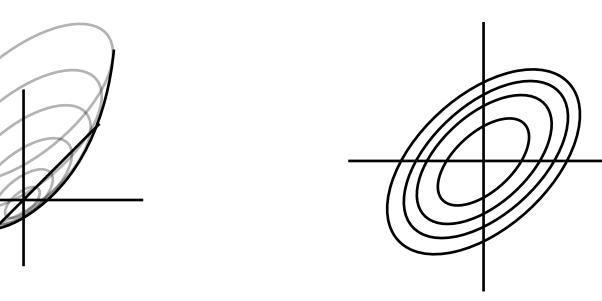


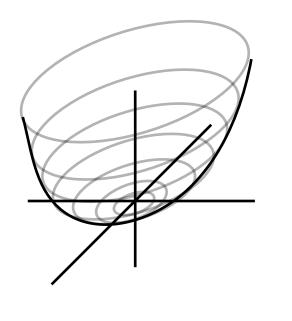


level sets

diagonal







surface

level sets

general

surface

level sets

surface

Quadratic Form:

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

Definiteness:

Analogy

Eigenvalues

Eigenvalue condition proof:

since V is

invertible...

Positive definite:

Negative-definite

$$Q \succ 0$$

 $x^T Q x > 0$

 $x^T Q x < 0$

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

$$\lambda_i > 0 \quad \lambda_i \in eig(Q)$$

$$x = Vx'$$

...consider eigenvector coordinates

Negative semi-definite

ND

$$Q \succeq 0$$

 $Q \prec 0$

$$x^T Q x \ge 0 \quad \forall \ x$$

$$x^T Q x < 0 \quad \forall \ x \quad x \neq 0$$

...negative orthant

...positive orthant

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$

$$\lambda_i < 0 \quad \lambda_i \in eig(\zeta_i)$$

$$Q \leq 0$$
 x^T

$$x^T Q x \le 0 \qquad \forall \ x$$

some x

some x

...the rest of the space

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

$x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i} x_{i}'^{2}$

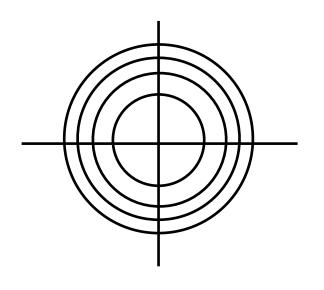
$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

 $\forall x \iff \forall x'$

Surfaces: $Q \prec 0$

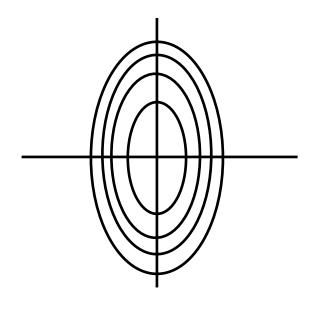
$$Q = I$$



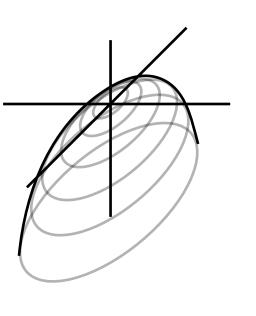
level sets

surface

diagonal

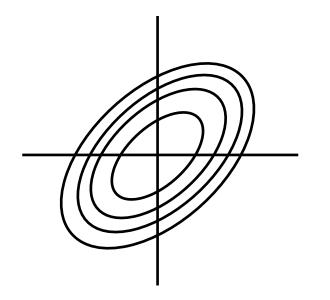




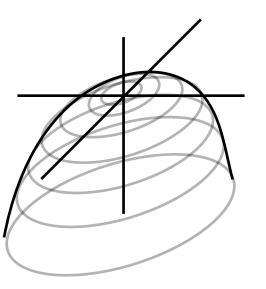


surface

general



level sets



surface

Quadratic Form:

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

 $x^T Q x > 0 \quad \forall \ x \quad x \neq 0$

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$

$$Q = Q^T$$

Definiteness:

Definition Notation

 $x^T Q x \ge 0 \quad \forall \ x$

Positive definite:

Negative-definite

Positive semi-definite

Negative semi-definite

PD

 $Q \succ 0$

PSD

 $Q \succeq 0$

 $Q \prec 0$

ND

NSD

 $Q \leq 0$

 $x^T Q x \le 0 \quad \forall x$

 $x^T Q x > 0$

 $x^T Q x < 0$ some x **Analogy**

...positive orthant

...positive orthant w/ boundary

...negative orthant

...negative orthant w/ boundary

...the rest of the space

Eigenvalues

 $\lambda_i > 0 \quad \lambda_i \in eig(Q)$

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$

 $\lambda_i < 0 \quad \lambda_i \in eig(Q)$

 $\lambda_i \leq 0 \quad \lambda_i \in eig(Q)$

Eigenvalue condition proof:

...consider eigenvector coordinates

$$x = Vx'$$

since V is invertible...

 $\forall x \iff \forall x'$

 $x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i} x_{i}'^{2}$

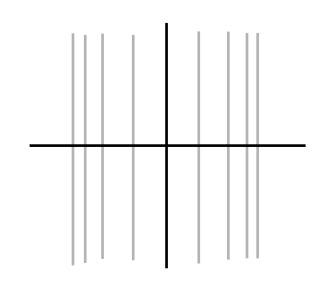
$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

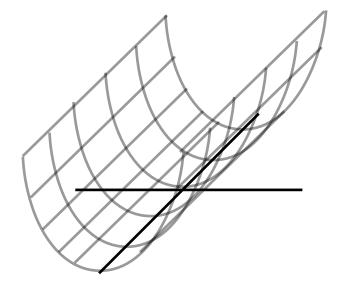
$$x \neq 0$$

Surfaces: $Q \succeq 0$

Indefinite:

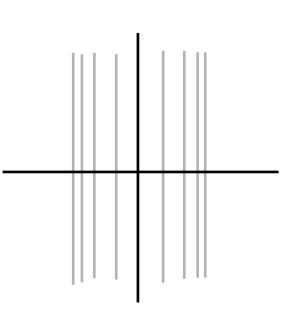
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



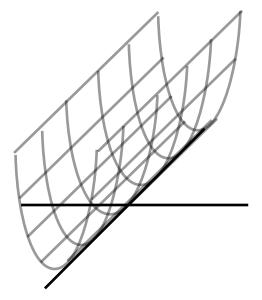


 $Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$

some x

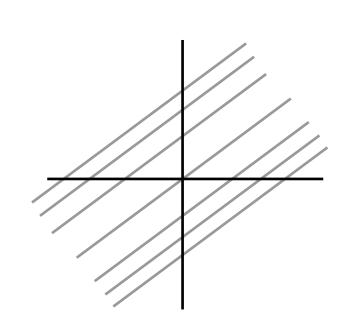


diagonal

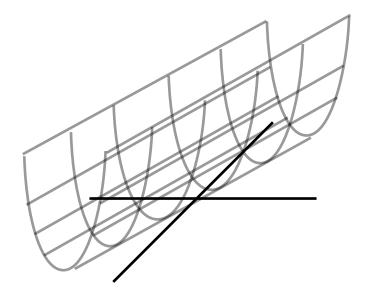


surface

 $Q = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} V^T = \lambda_1 v_1 v_1^T$







surface

level sets

surface

level sets

Quadratic Form:

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

Definiteness:

OI	IUI	

Eigenvalues

Eigenvalue condition proof:

...consider eigenvector coordinates

$$Q \succ 0$$

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

$$x \quad x \neq$$

$$\lambda_i > 0 \quad \lambda_i \in eig(Q)$$

$$x = Vx'$$

Negative semi-definite

$$Q \succeq 0$$

$$x^T Q x \ge 0 \quad \forall \ x$$

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$

invertible...
$$\forall x \iff \forall x'$$

since V is

NSD

$$Q \prec 0$$

 $Q \leq 0$

$$x^T Q x \le 0 \qquad \forall \ x$$

w/ boundary

...the rest of the space

...negative orthant

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

$$x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i} x_{i}'^{2}$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Indefinite:

$$x^T Q x > 0$$

some
$$x$$

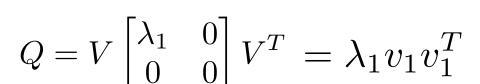
 $Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$

$$Qx < 0$$
 some x

 $x^T Q x < 0$

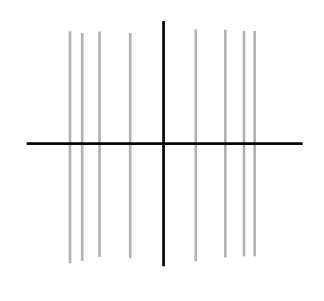
some x

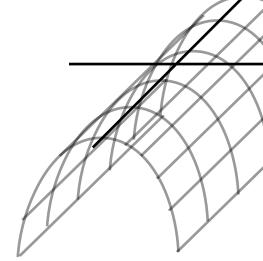
diagonal

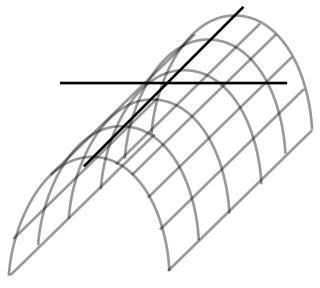


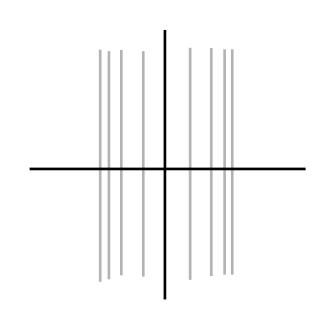
Surfaces:

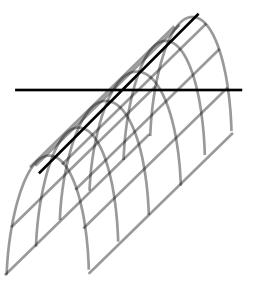
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

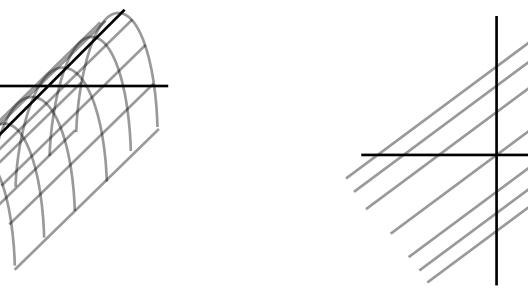


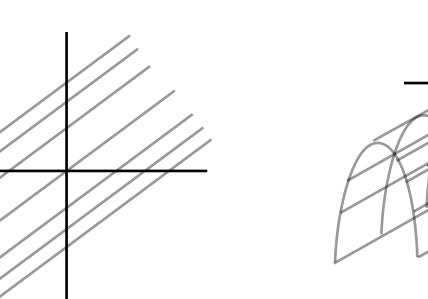


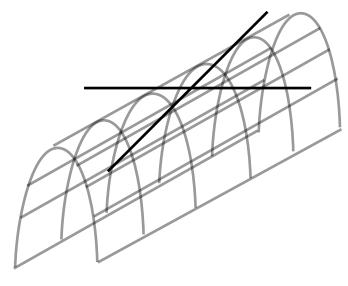












surface level sets

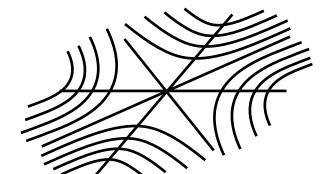
 $Q \leq 0$

level sets

surface

level sets

surface



Quadratic Form:

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

Definiteness:

Short Notation	
----------------	--

Definition

$$Q = Q^T$$

$$Q \succ 0$$

 $Q \succeq 0$

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

$$x \neq$$

Eigenvalues

$$\lambda_i > 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

Positive definite:

Negative-definite

Positive semi-definite

Negative semi-definite

NSD

$$Q \prec 0$$

 $Q \leq 0$

 $x^T Q x \ge 0 \quad \forall \ x$

$$\forall x$$

 $x^T Q x \le 0 \qquad \forall \ x$

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$

 $x^T Q x > 0$ some x $x^T Q x < 0$ some x

Analogy

...positive orthant

...positive orthant w/ boundary

...negative orthant

...negative orthant w/ boundary

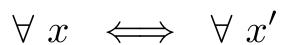
...the rest of the space

Eigenvalue condition proof:

...consider eigenvector coordinates

$$x = Vx'$$

since V is invertible...



$$x^T Q x = x V D V^T x = x'^T D x' = \sum_{i} \lambda_i x_i'^2$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x^{\prime} \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

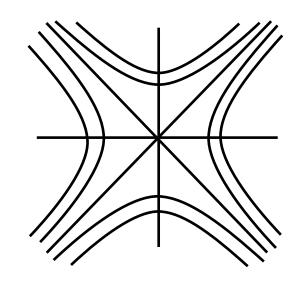
Surfaces:

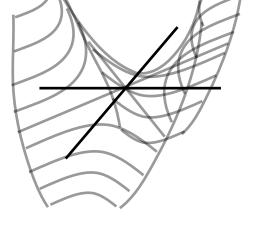
Indefinite:

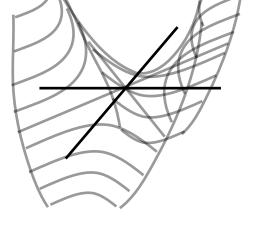


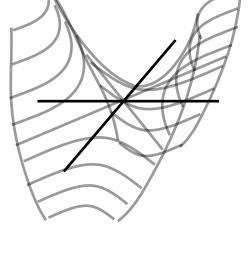
indefinite

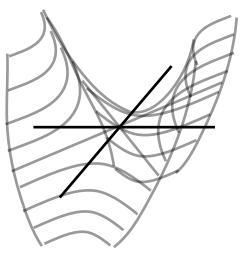
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



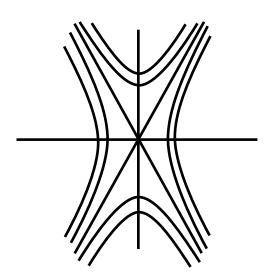




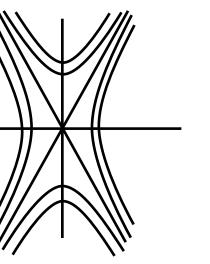


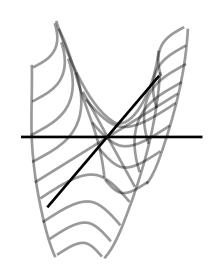




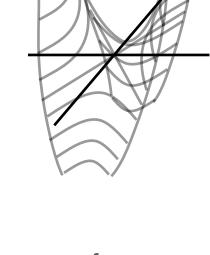


 $Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

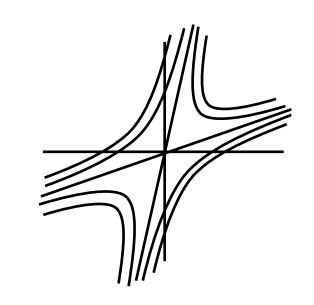




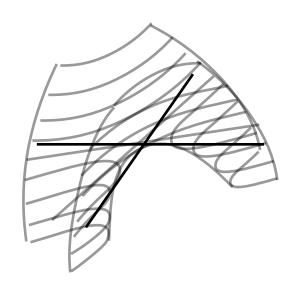
diagonal



$$Q = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^T$$



general



surface level sets

level sets

surface

level sets

surface

Quadratic Form:

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

Definiteness:

Analogy

Eigenvalue condition proof:

Positive definite:

PSD

$$Q \succ 0$$

 $Q \succeq 0$

 $Q \prec 0$

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

 $x^T Q x \ge 0 \quad \forall \ x$

 $x^T Q x > 0$

 $x^T Q x < 0$

$$x \neq 0$$

...positive orthant w/ boundary

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

Eigenvalues

 $\lambda_i > 0 \quad \lambda_i \in eig(Q)$

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$

 $\lambda_i \leq 0 \quad \lambda_i \in eig(Q)$

$$\lambda_i \in \mathrm{elg}(Q)$$

x = Vx'

...consider eigenvector coordinates

$$\forall x \iff \forall x'$$

$$x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i} x_{i}'^{2}$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Negative-definite

Negative semi-definite

Positive semi-definite

NSD

$$Q \leq 0$$
 $x^T Q$

$$x^T Q x \le 0 \qquad \forall \ x$$

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$

some x

some x

...negative orthant w/ boundary

...the rest of the space

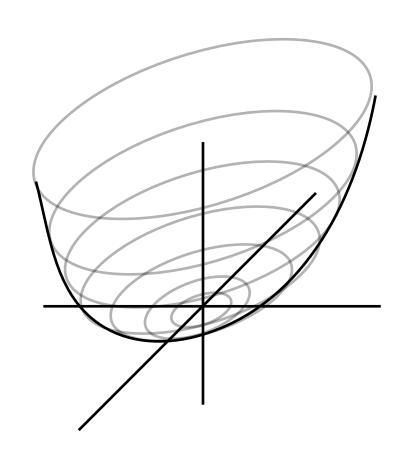
$$T = \begin{bmatrix} | & | \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$Q = VDV^T = \begin{bmatrix} \begin{vmatrix} & & \\ v_1 & v_2 \\ & & \end{vmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$||v_i||_2 =$$

Surfaces:
$$Q \succ 0$$

Indefinite:



 $f(x) = x^T Q x = 1$

$$f\left(\frac{1}{\sqrt{\lambda_{1}}}v_{1}\right) = \frac{1}{\sqrt{\lambda_{1}}}v_{1}^{T}Qv_{1}\frac{1}{\sqrt{\lambda_{1}}}$$

$$= \frac{1}{\sqrt{\lambda_{1}}}\begin{bmatrix} 1 \\ v_{1} \\ 1 \end{bmatrix}^{T}\begin{bmatrix} 1 & 1 \\ v_{1} & v_{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{bmatrix}\begin{bmatrix} - & v_{1}^{T} & - \\ - & v_{2}^{T} & - \end{bmatrix}\begin{bmatrix} 1 \\ v_{1} \\ 1 \end{bmatrix}\frac{1}{\sqrt{\lambda_{1}}}$$

$$= \frac{1}{\sqrt{\lambda_{1}}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{T}\begin{bmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{\sqrt{\lambda_{1}}} = \frac{\lambda_{1}}{(\sqrt{\lambda_{1}})^{2}} = 1$$

surface

level sets

Quadratic Form:

ND

 $f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$

Definiteness:	Short	Notation
Positive definite:	PD	$Q \succ 0$
Positive semi-definite	PSD	$0 \leq 0$

$$\begin{array}{cc} \mathsf{PSD} & Q \succeq 0 \\ \mathsf{ND} & Q \prec 0 \end{array}$$

Negative semi-definite NSD
$$Q \leq 0$$

Indefinite:

Negative-definite

Definition Notation

$$x^{T}Qx > 0 \quad \forall x \quad x \neq 0$$
$$x^{T}Qx \ge 0 \quad \forall x$$

$$x^T Q x < 0 \quad \forall \ x \quad x \neq 0$$

$$x^T Q x \le 0 \qquad \forall \ x$$

$$x^T Qx > 0$$
 some x

$$x^T Q x < 0$$
 some x

Analogy

...positive orthant

...positive orthant w/ boundary

...negative orthant

...negative orthant w/ boundary

...the rest of the space

Eigenvalues

$$\lambda_i > 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

Eigenvalue condition proof:

...consider eigenvector coordinates

$$x = Vx'$$

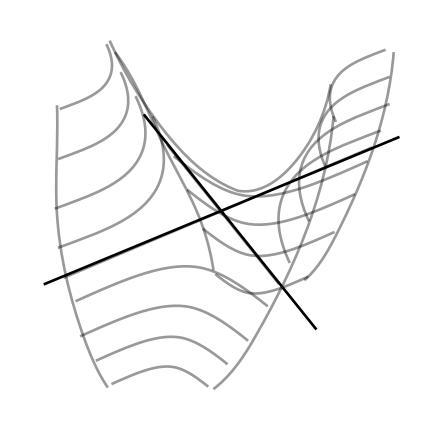
since V is $\forall x \iff \forall x'$ invertible...

$$x^T Q x = x V D V^T x = x'^T D x' = \sum_{i} \lambda_i x_i'^2$$

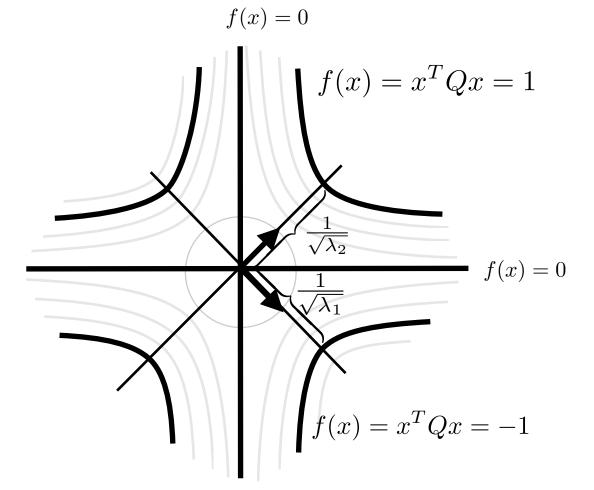
$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Surfaces:
$$Q$$
 indefinite



surface



level sets

$$Q = VDV^{T} = \begin{bmatrix} \begin{vmatrix} & & | \\ v_1 & v_2 \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$f\left(\frac{1}{\sqrt{\lambda_{1}}}v_{1}\right) = \frac{1}{\sqrt{\lambda_{1}}}v_{1}^{T}Qv_{1}\frac{1}{\sqrt{\lambda_{1}}}$$

$$= \frac{1}{\sqrt{\lambda_{1}}}\begin{bmatrix} \begin{vmatrix} 1 \\ v_{1} \end{vmatrix} \end{bmatrix}^{T}\begin{bmatrix} \begin{vmatrix} 1 \\ v_{1} \end{vmatrix} v_{2} \\ \begin{vmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{bmatrix}\begin{bmatrix} - v_{1}^{T} & - \\ - v_{2}^{T} & - \end{bmatrix}\begin{bmatrix} \begin{vmatrix} 1 \\ v_{1} \end{vmatrix} \frac{1}{\sqrt{\lambda_{1}}}$$

$$= \frac{1}{\sqrt{\lambda_{1}}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{T}\begin{bmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{\lambda_{1}}} = \frac{\lambda_{1}}{(\sqrt{\lambda_{1}})^{2}} = 1$$

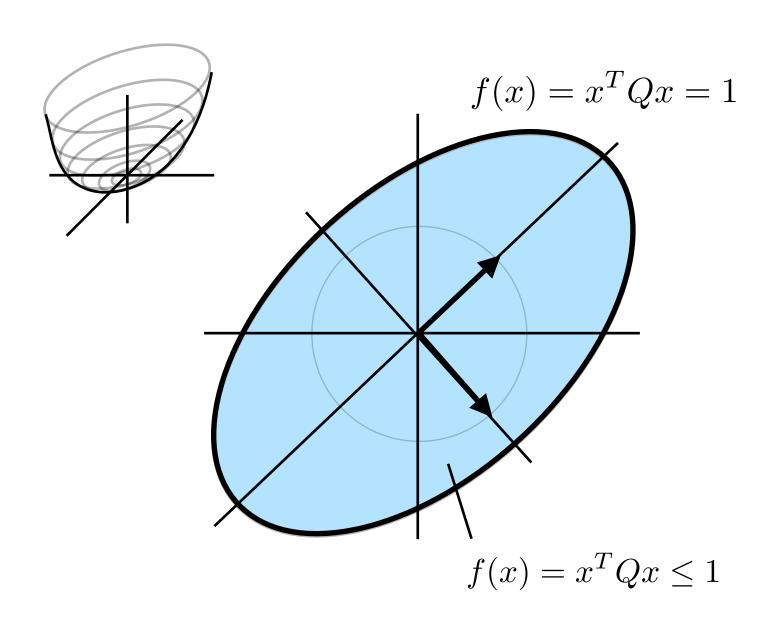
Quadratic Form - Level Sets

Quadratic Form:

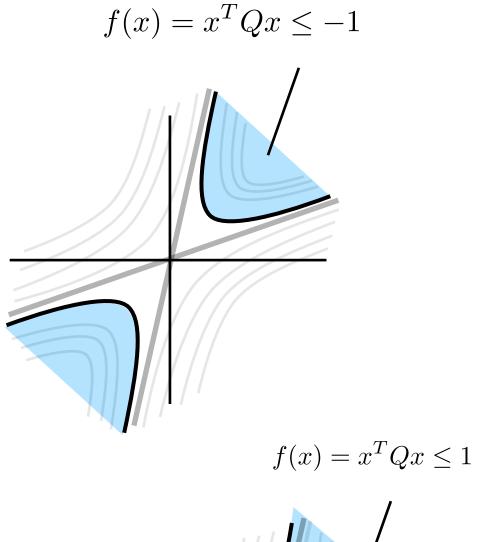
$$f(x) = x^T Q x$$

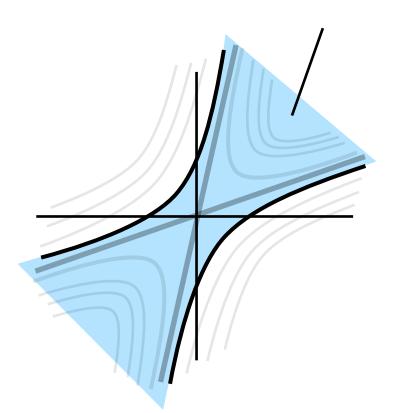
$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

Ellipsoids



Hyperboloids





Cones

