

# Farka's Lemma

## Convex Optimization

Dan Calderone

**Farka's Lemma** Let  $H \in \mathbb{R}^{m \times n}$  and  $r \in \mathbb{R}^n$ .

Then exactly one of the following two assertions is true.

1. There exists  $v \in \mathbb{R}^m$  such that  $v^\top H = r^\top$  and  $v \geq 0$
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## Core Intuition: Square Case

*“each row of  $H$  is orthogonal to the (other) columns of  $H$  inverse”*

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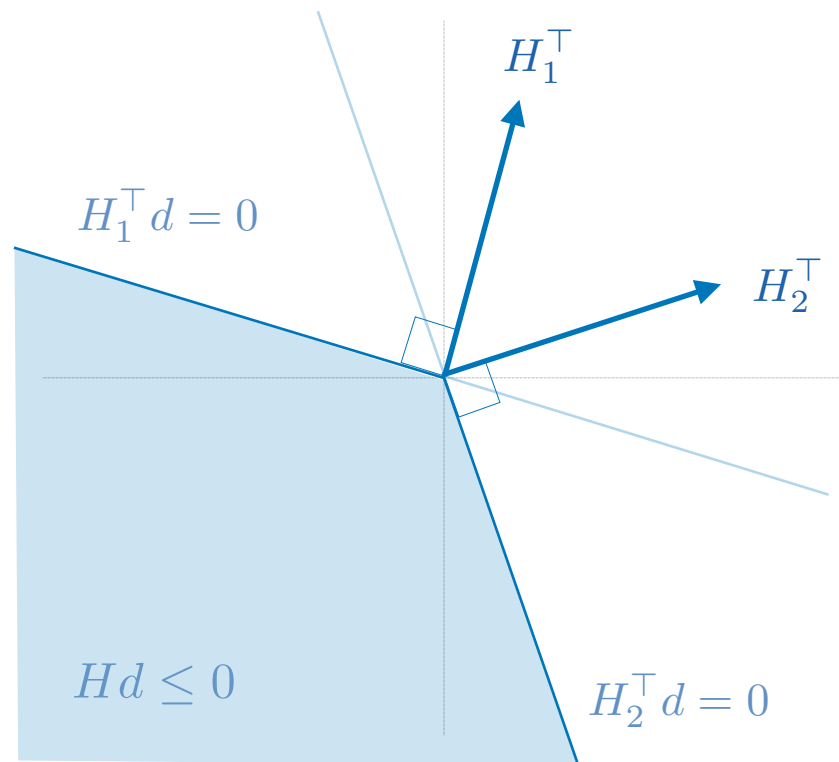
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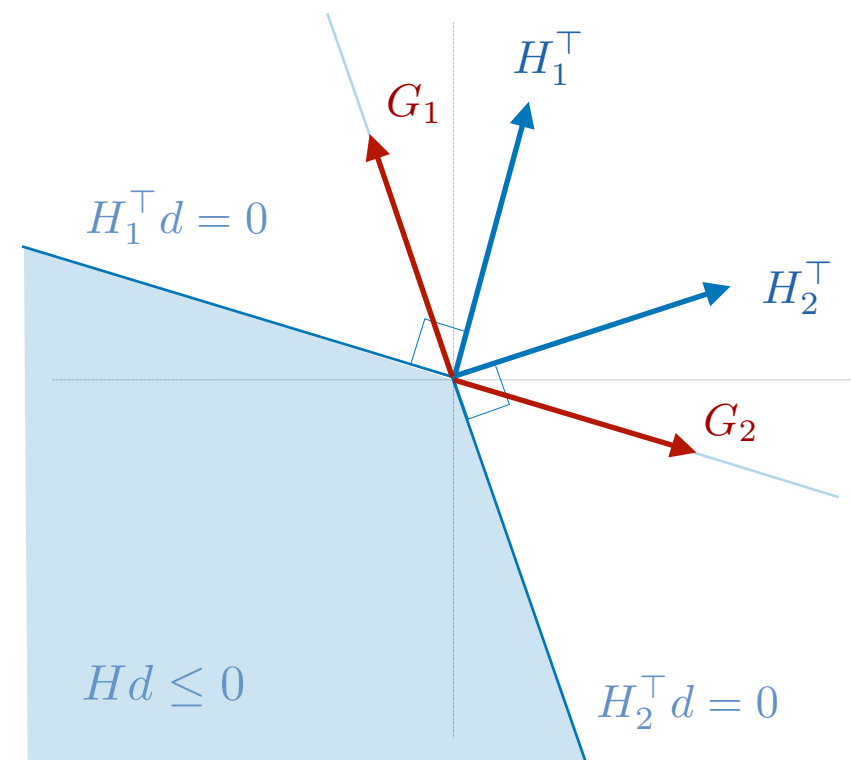
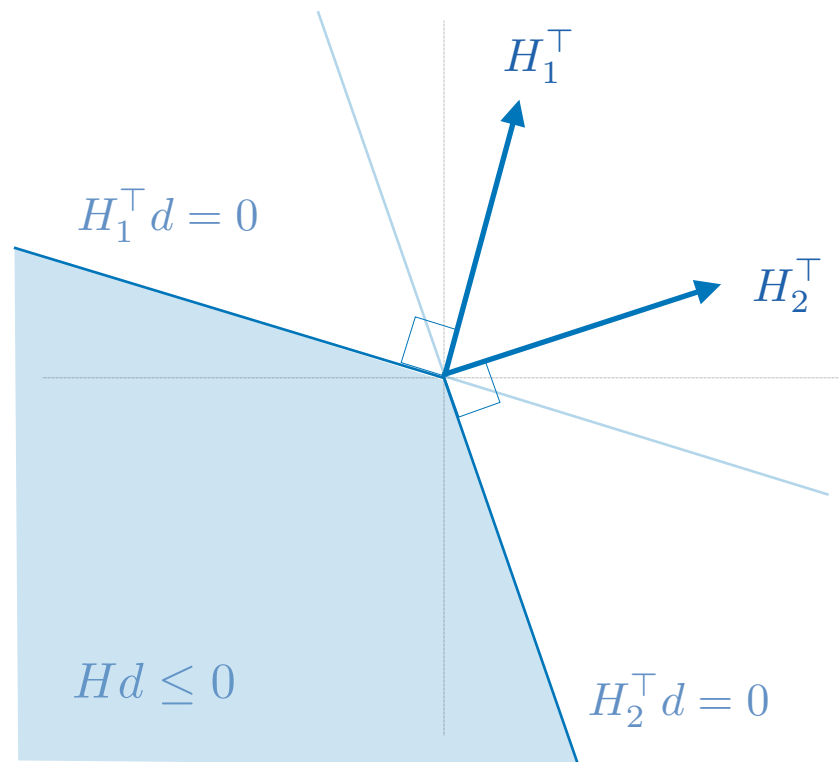
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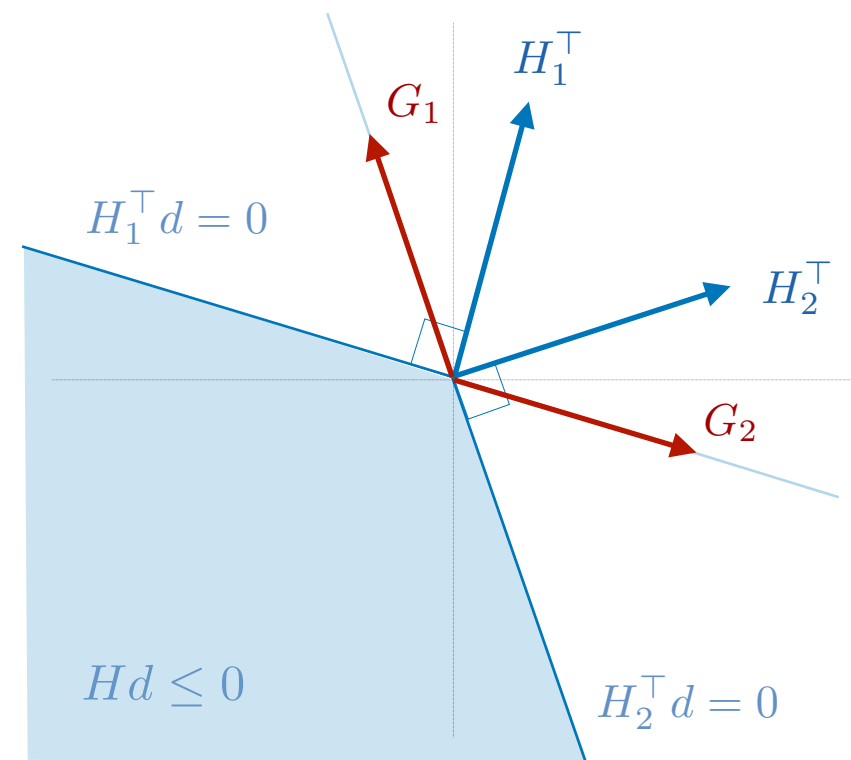
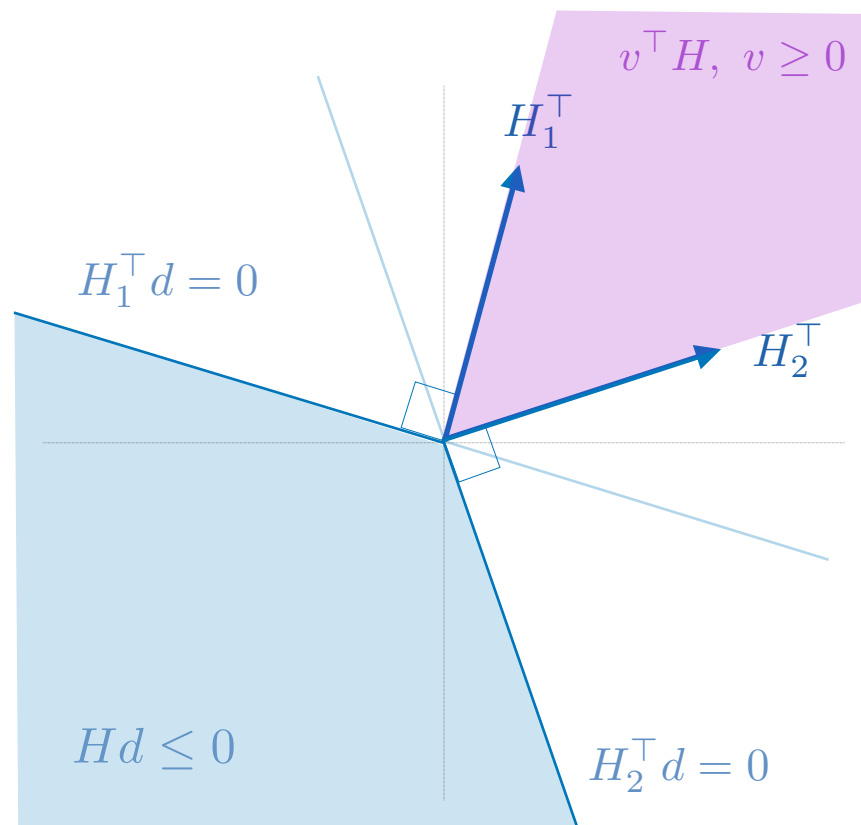
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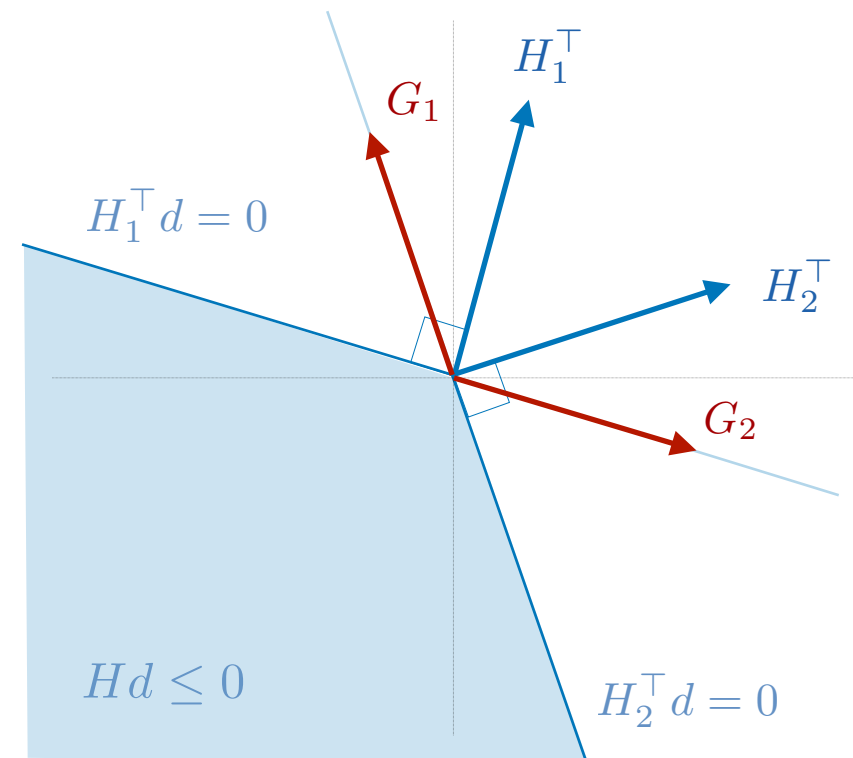
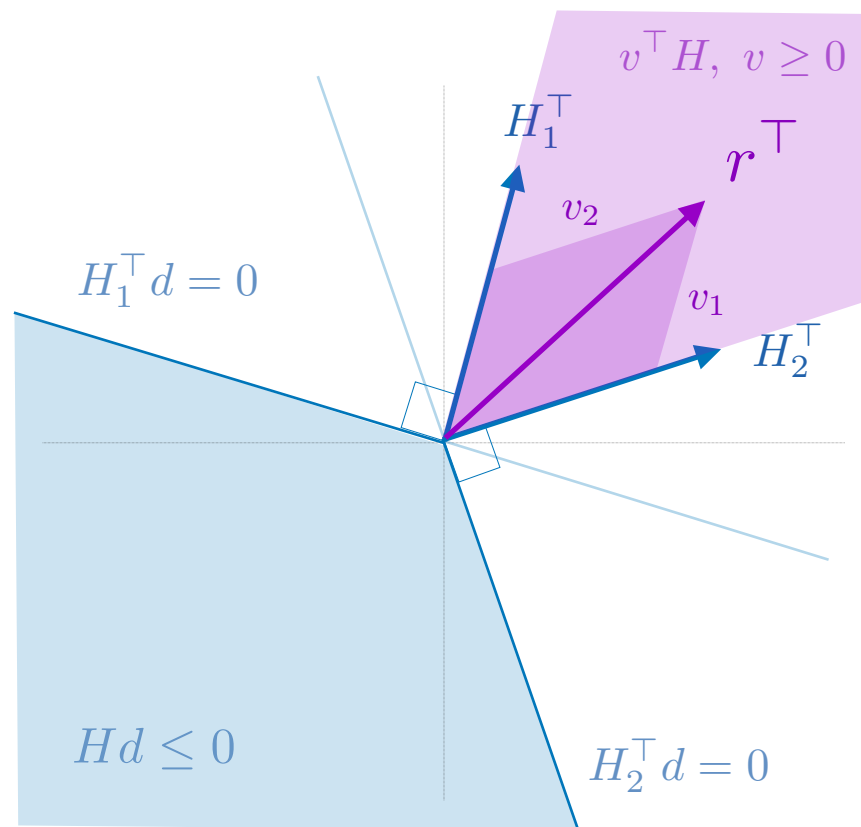
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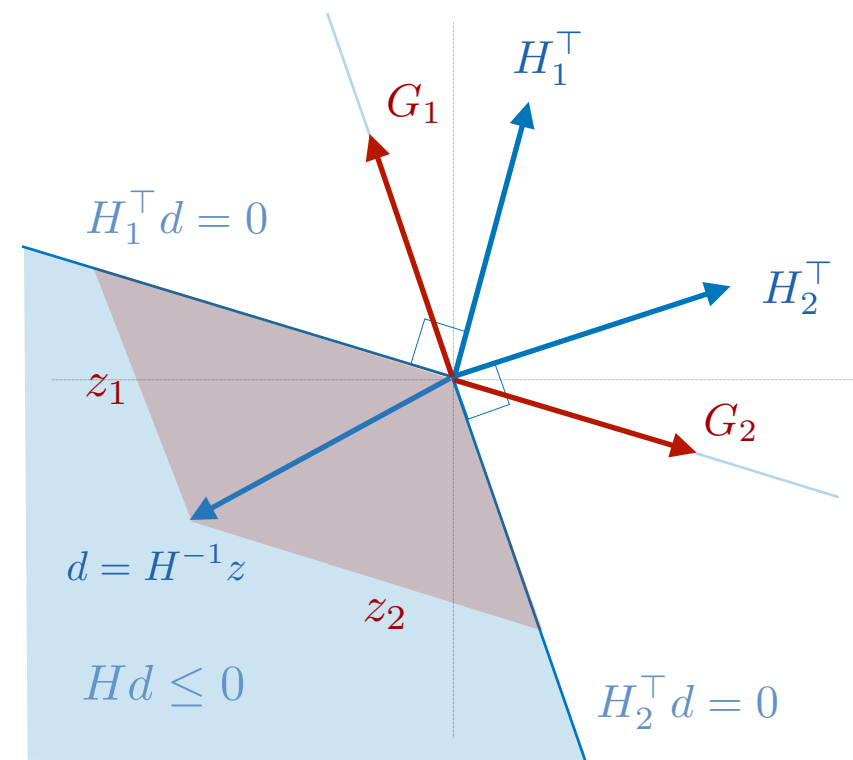
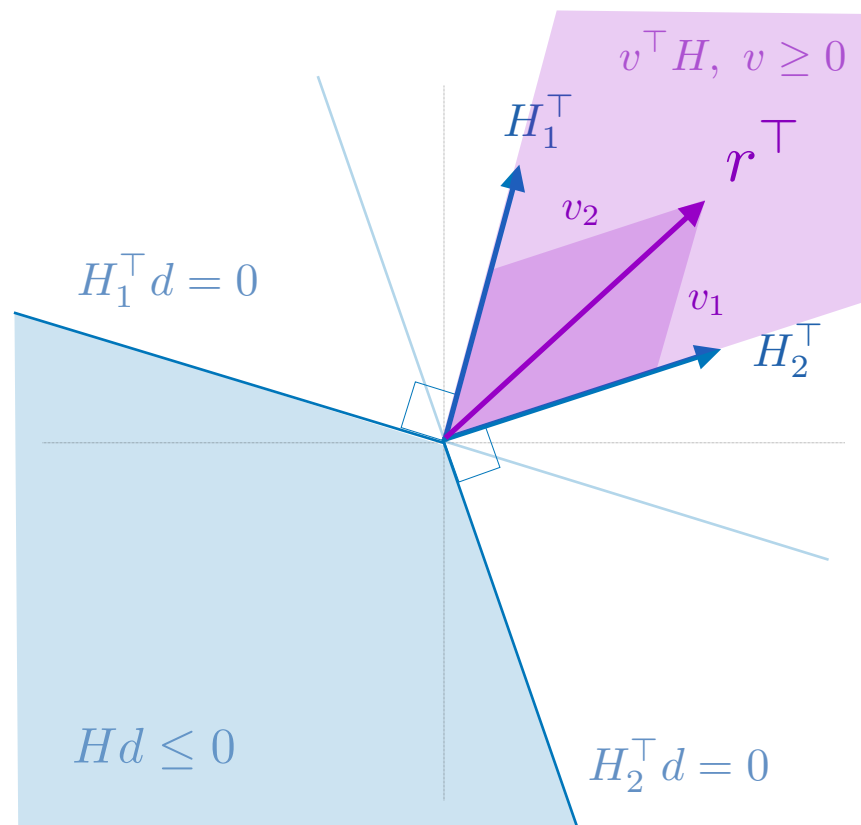
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For square, invertible  $H \quad \exists d \in \mathbb{R}^n$  s.t.  $Hd \leq 0 \quad \Longleftrightarrow \quad d = H^{-1}z$  for  $z \leq 0$





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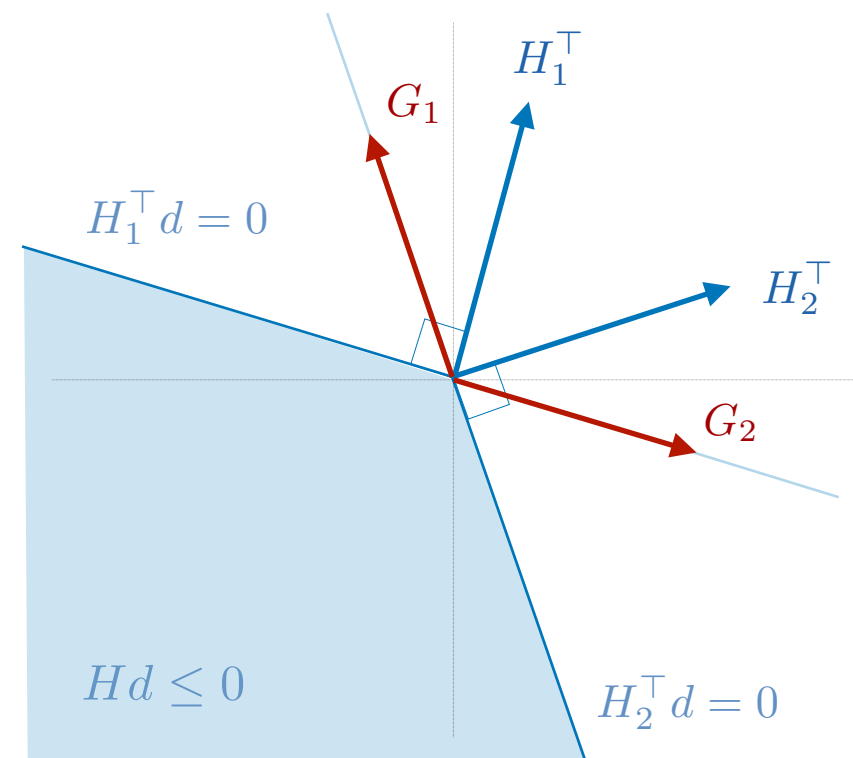
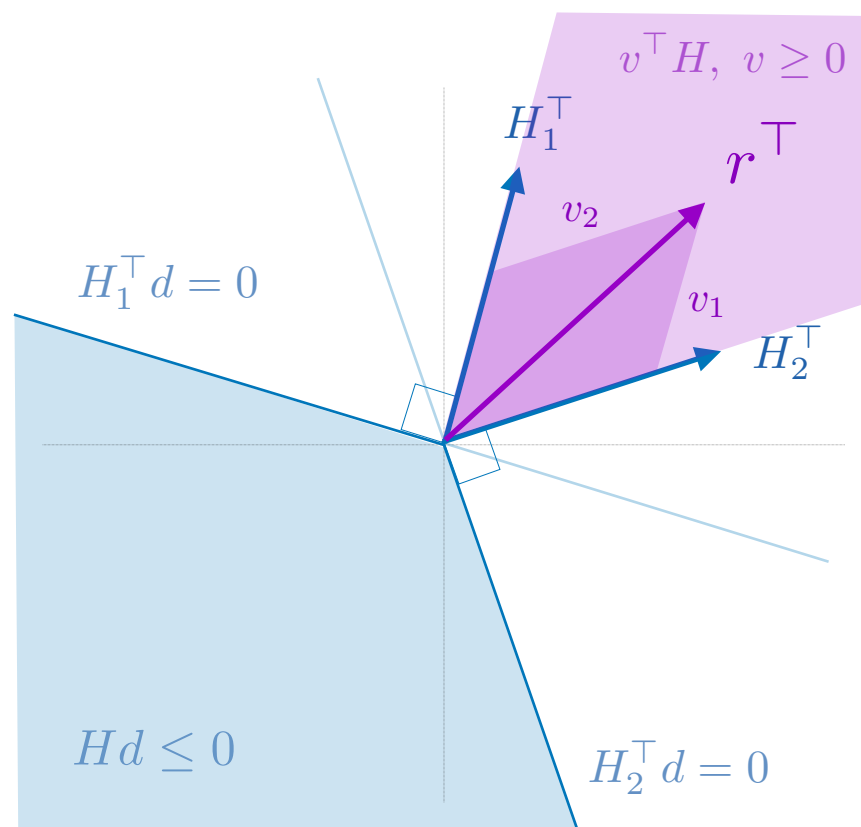
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True

False

**Case 1.**



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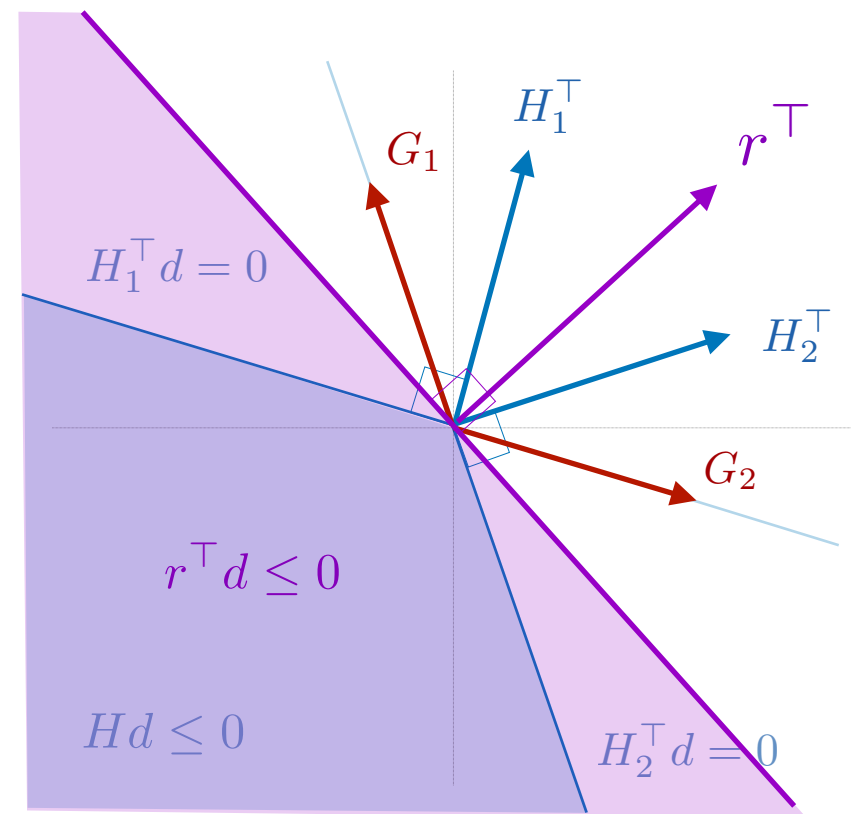
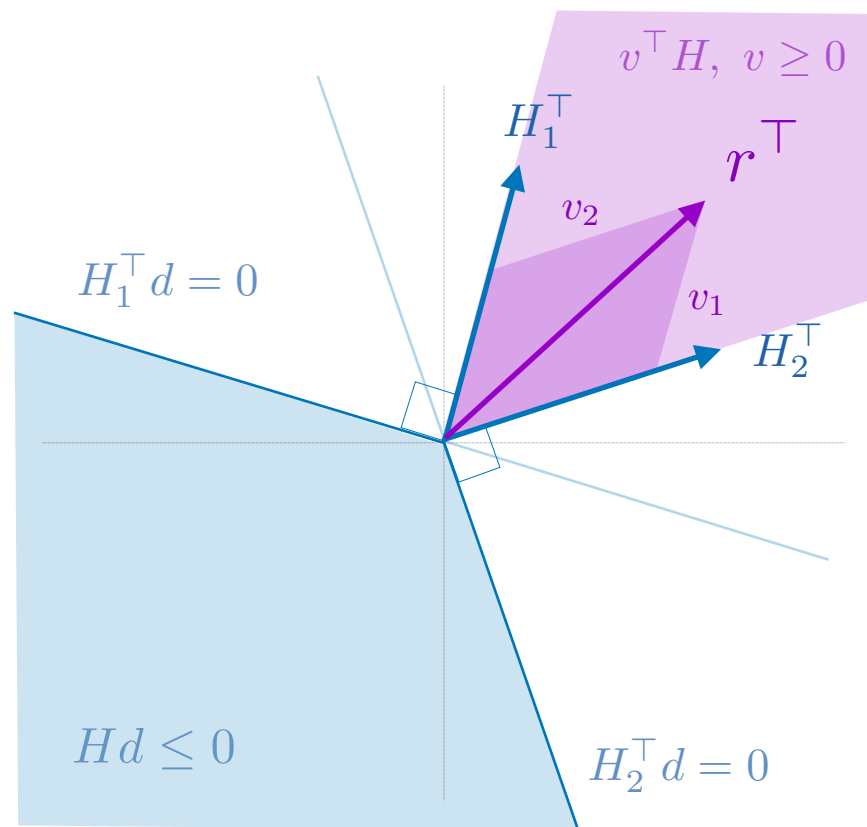
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 $\Rightarrow$ 

For all  $d \in \mathbb{R}^n$  s.t.  $Hd \leq 0$ ,  $r^\top d \leq 0$

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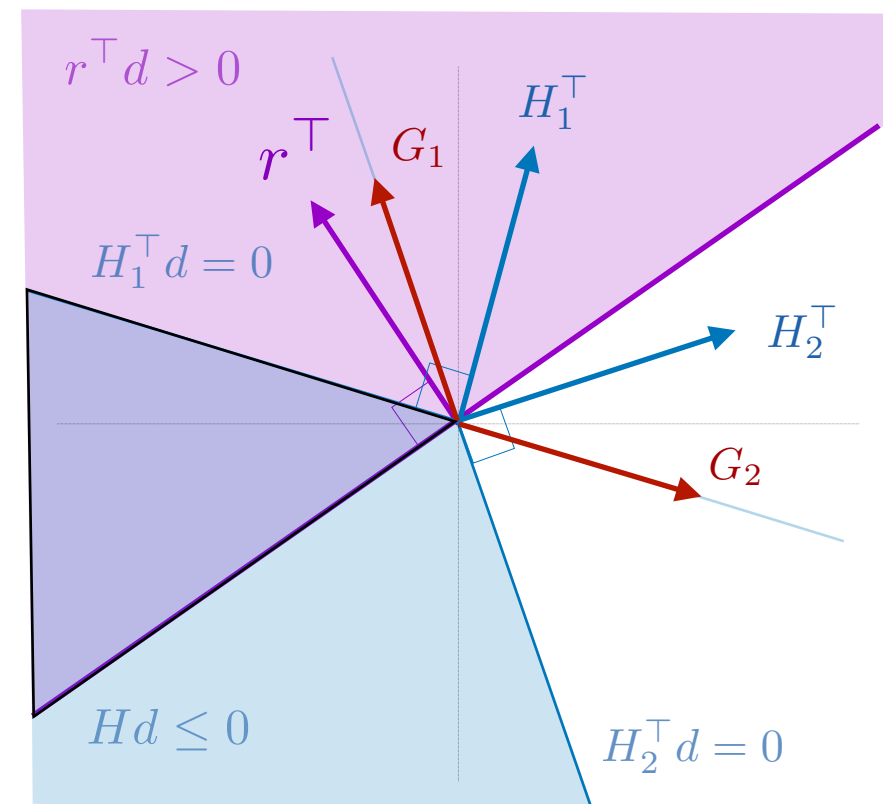
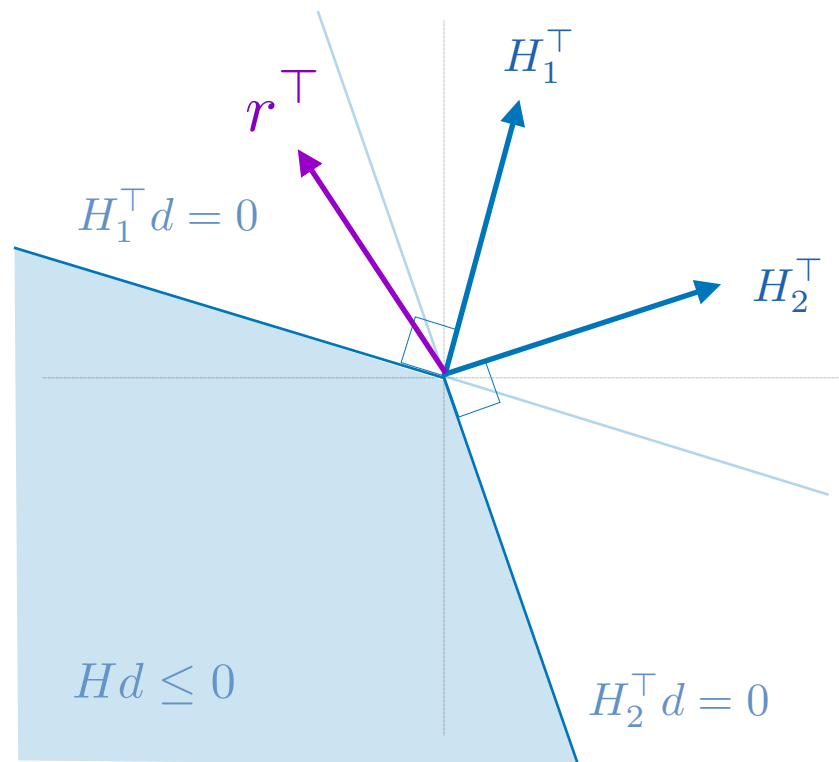
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False

True

**Case 2.**

not

$\Rightarrow$

For all  $v \in \mathbb{R}^m$  s.t.  $v^\top H = r^\top$ ,  $v \not\geq 0$

