

# **Eigenvectors & Eigenvalues:**

## **Linear Algebra**

**Major sources:**

**Winter 2022 - Dan Calderone**

# Eigenvectors & Eigenvalues

Square matrix:  $A \in \mathbb{R}^{n \times n}$

## Eigenvalue/Eigenvector Problem

$A$  transforms  $\mathbb{R}^n$     *...which directions stay unchanged?*     $\rightarrow$  **Eigenvectors**  
                                  ...within those directions...  
                                  *...how much do vectors get stretched*     $\rightarrow$  **Eigenvalues**

## Eigenvector Equation

$$Ax = x\lambda \quad \text{Eigenvector } x \in \mathbb{C}^n \quad \text{Eigenvalue } \lambda \in \mathbb{C}$$

Spans of eigenvectors (& generalized eigenvectors) are called **A-invariant subspaces**

## Eigenvalues:

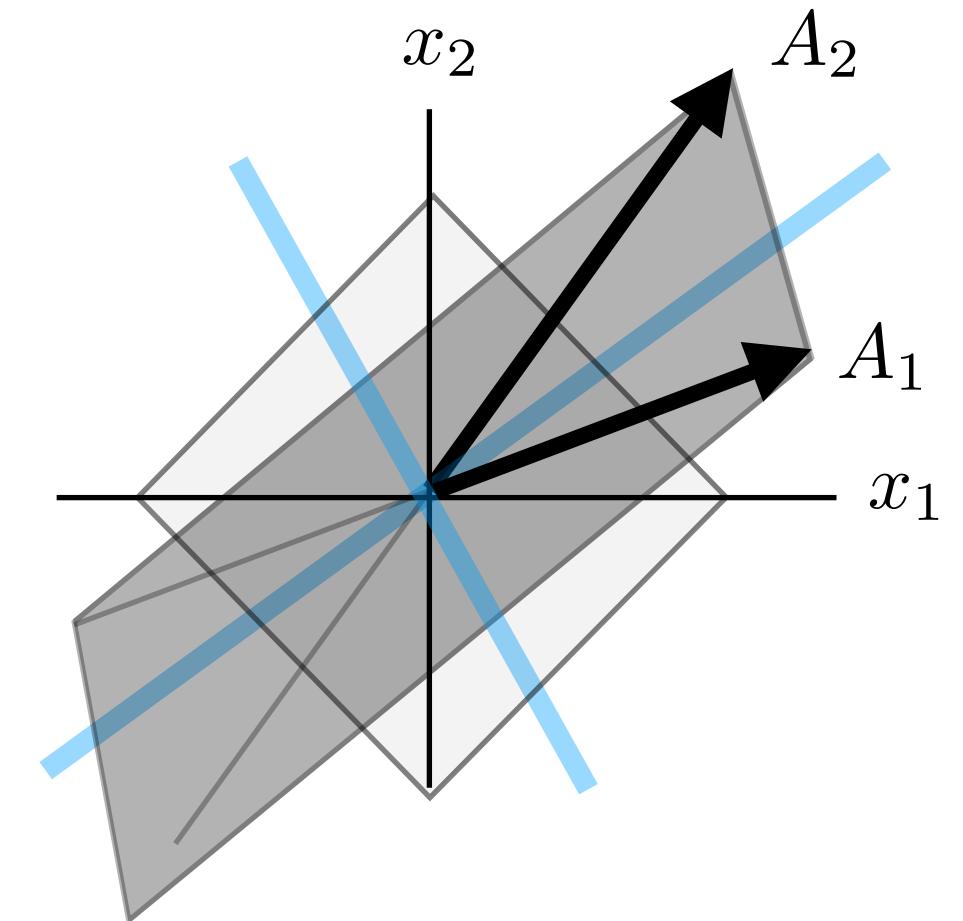
Fundamental property of matrices  
Do **not** change with coordinate/similarity transformations

## Eigenvectors:

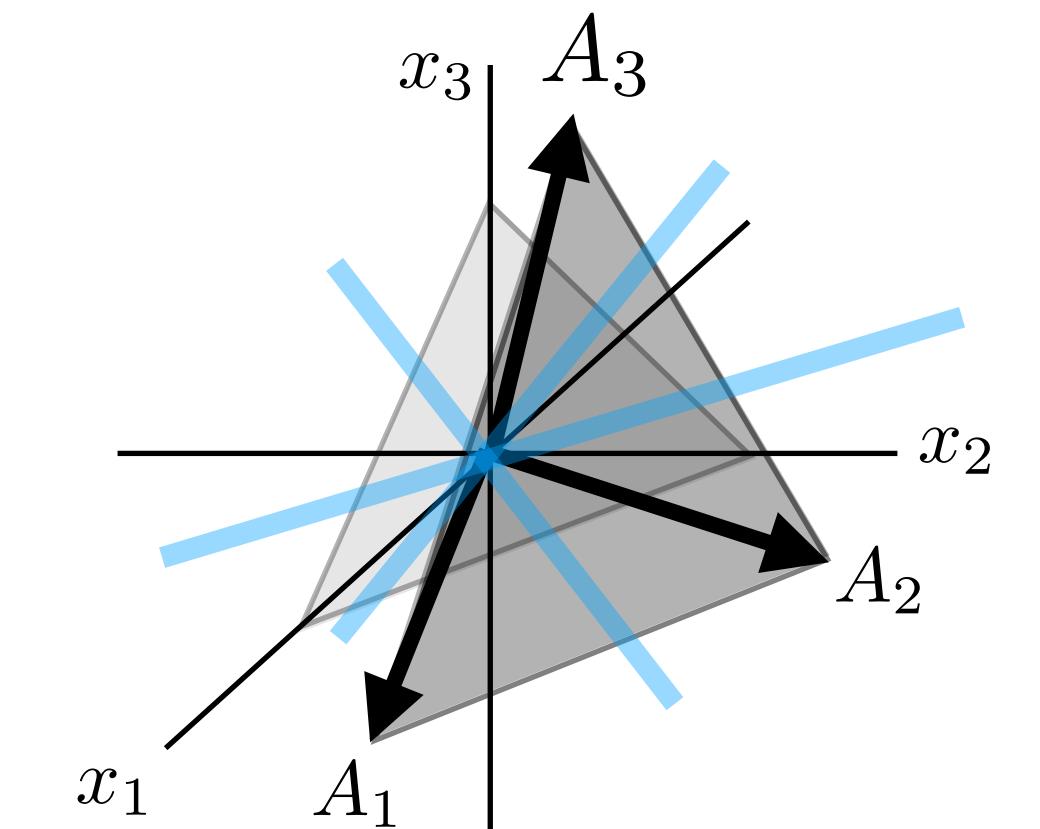
...coordinate dependent (do change with coordinate/similarity transformations)

## Picture Examples:

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$



$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



# Eigenvector/Eigenvalue equation

Square matrix:  $A \in \mathbb{R}^{n \times n}$

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda$$

$$(A - \lambda I)v = 0$$

$$v \in \mathcal{N}(A - \lambda I)$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda$$

$$w^*(A - \lambda I) = 0$$

$$w^* \in \mathcal{N}^L(A - \lambda I) = 0$$

For any eigenvalue, right and left eigenvectors come in pairs since  $A - \lambda I$  drops row and column rank at the same time

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

...how to characterize....  $\rightarrow$

$sI - A$  drops rank only when  $\det(sI - A) = 0$

**Characteristic Polynomial**

$$\text{char}_A(s) = \det(sI - A)$$

**n-th order polynomial**



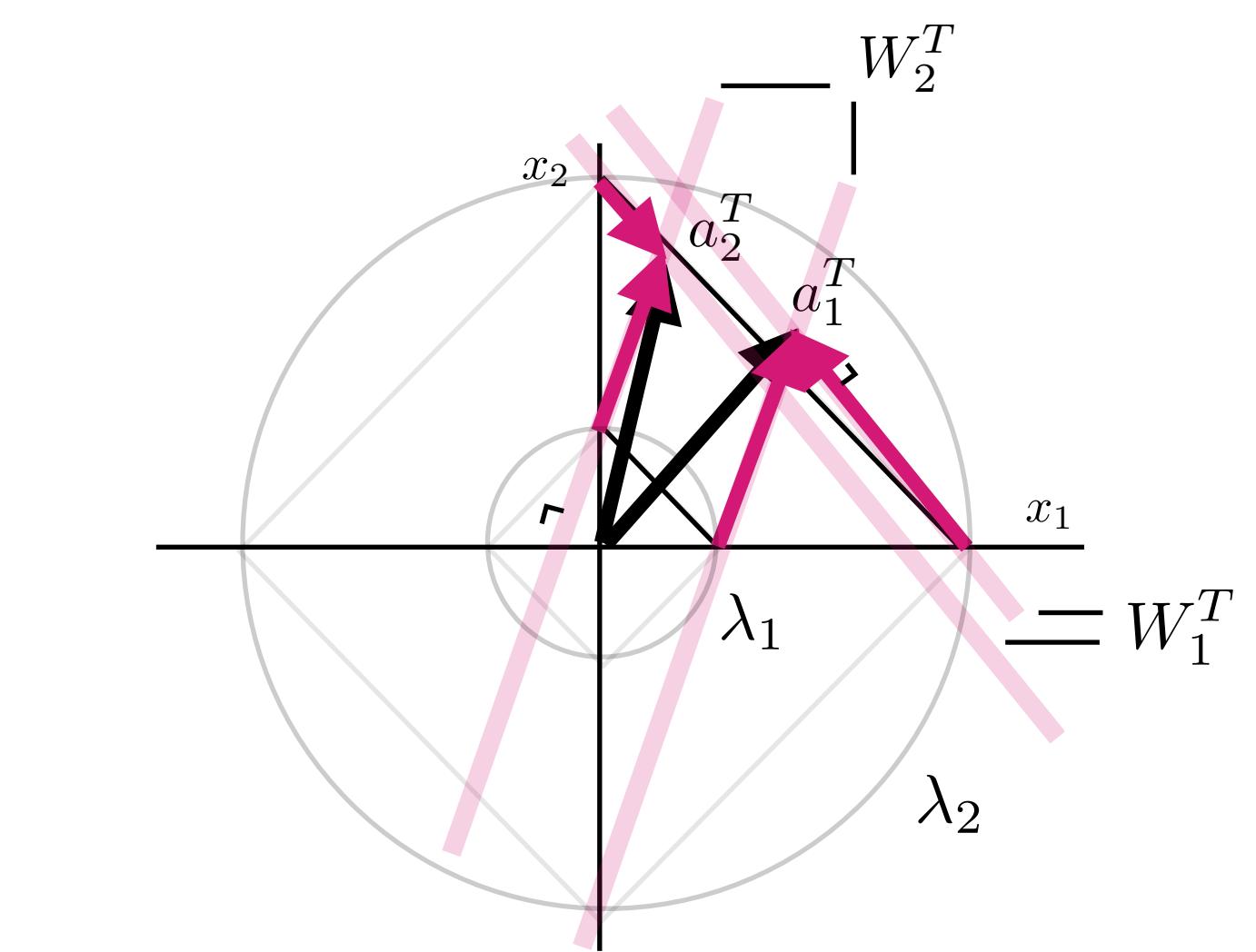
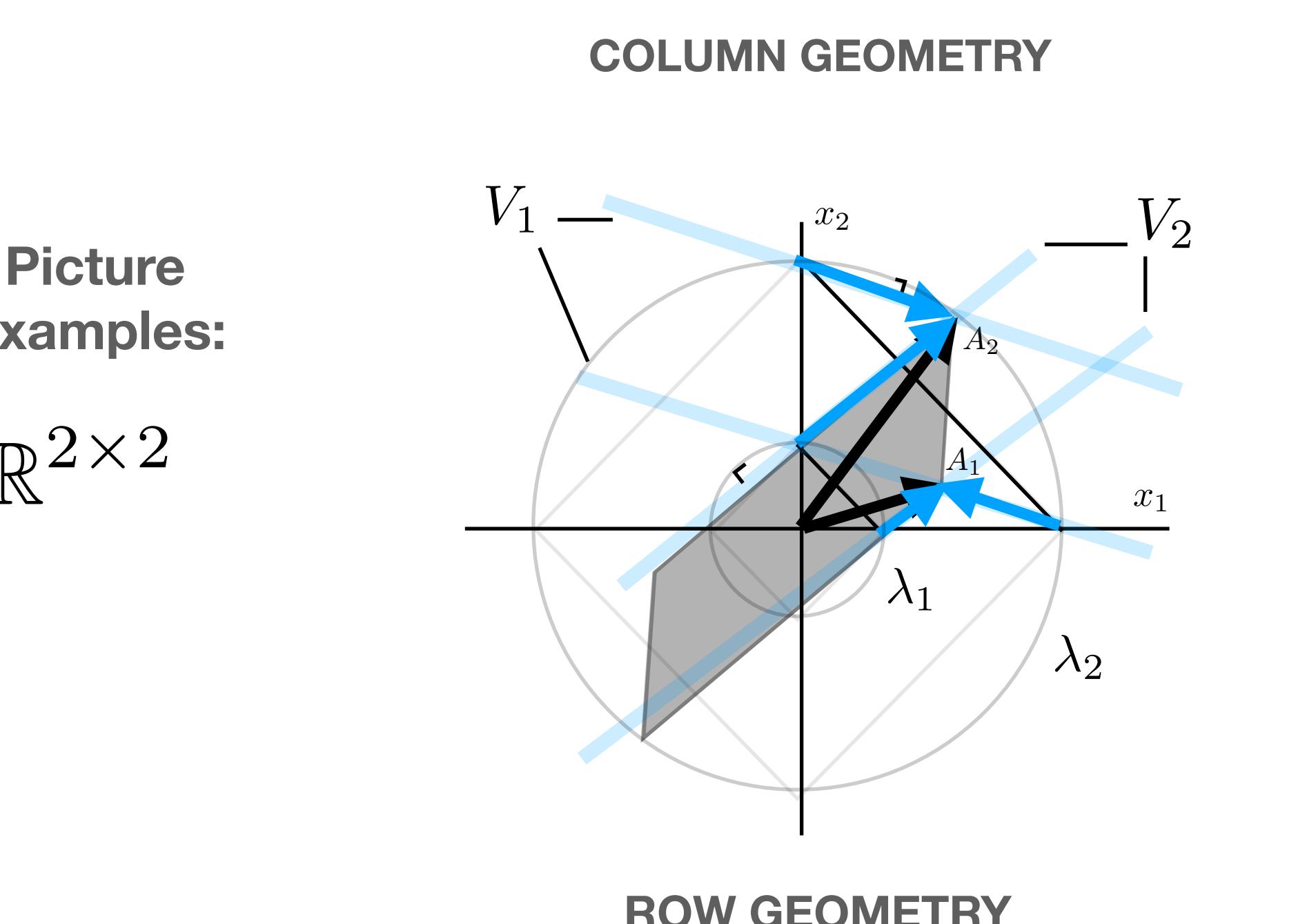
**n roots**

**Roots are eigenvalues:**

$\lambda$  solution to  $\text{char}_A(s) = 0$

Fundamental  
Theorem of Algebra

(see below)



# Eigenvector/Eigenvalue equation

Square matrix:  $A \in \mathbb{R}^{n \times n}$

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda$$

$$(A - \lambda I)v = 0$$

$$v \in \mathcal{N}(A - \lambda I)$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda$$

$$w^*(A - \lambda I) = 0$$

$$w^* \in \mathcal{N}^L(A - \lambda I) = 0$$

For any eigenvalue, right and left eigenvectors come in pairs since  $A - \lambda I$  drops row and column rank at the same time

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

...how to characterize....  $\rightarrow$

$sI - A$  drops rank only when  $\det(sI - A) = 0$

**Characteristic Polynomial**

$$\text{char}_A(s) = \det(sI - A)$$

**n-th order polynomial**



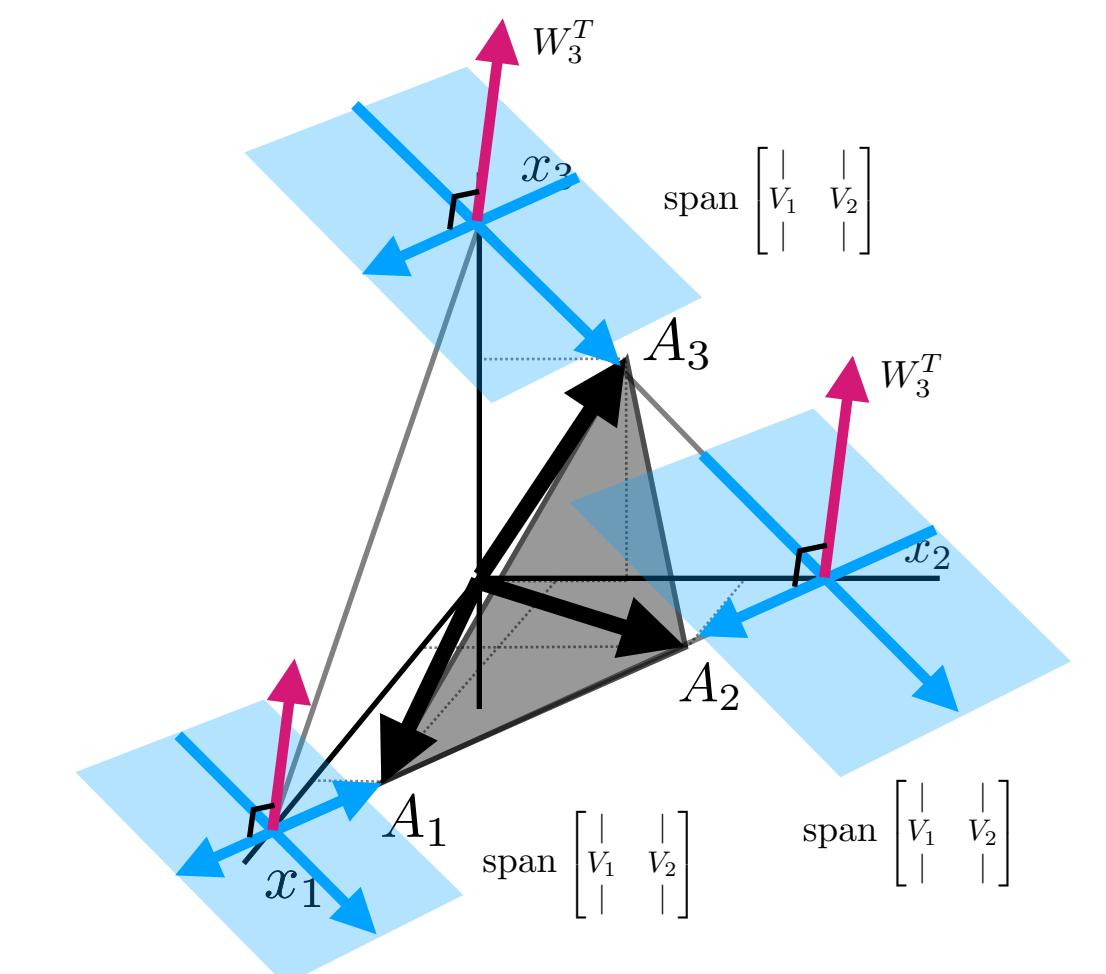
**n roots**

**Roots are eigenvalues:**

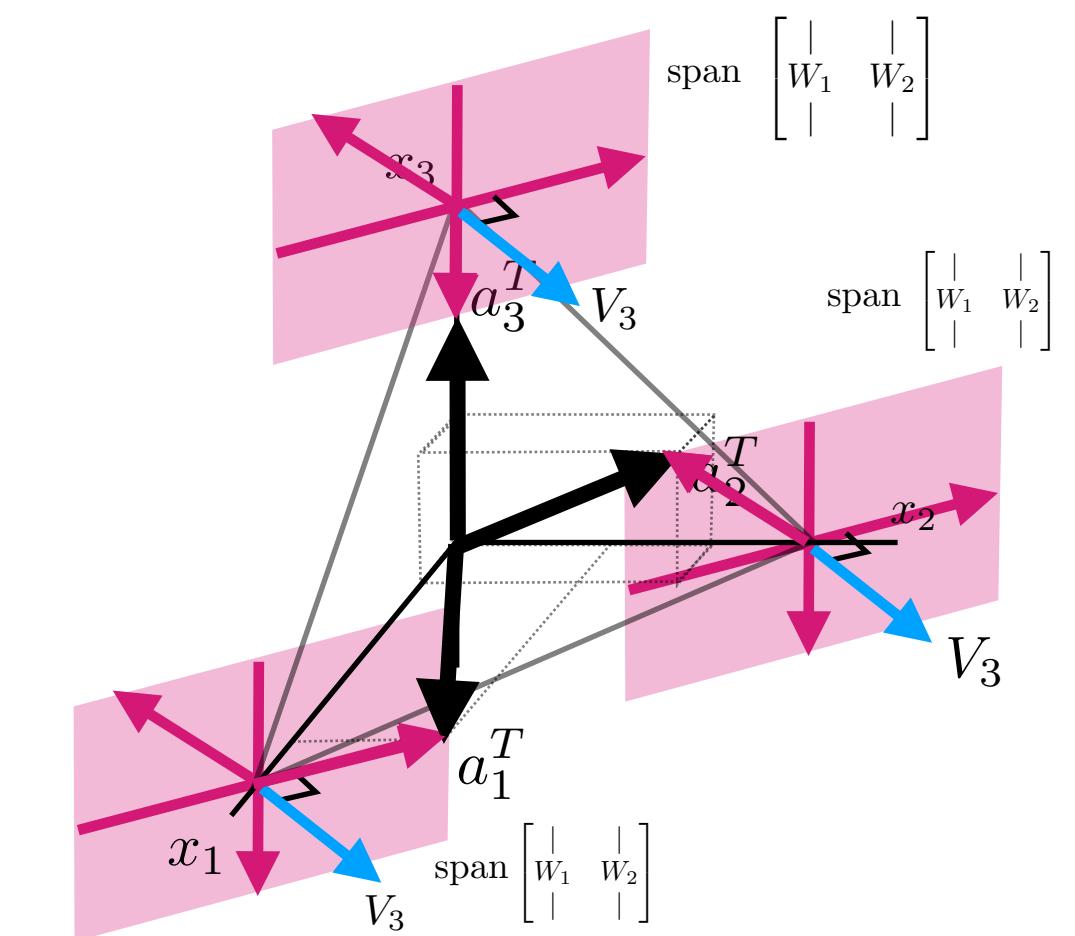
$\lambda$  solution to  $\text{char}_A(s) = 0$

Fundamental  
Theorem of Algebra

COLUMN GEOMETRY



ROW GEOMETRY



(see below)

# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

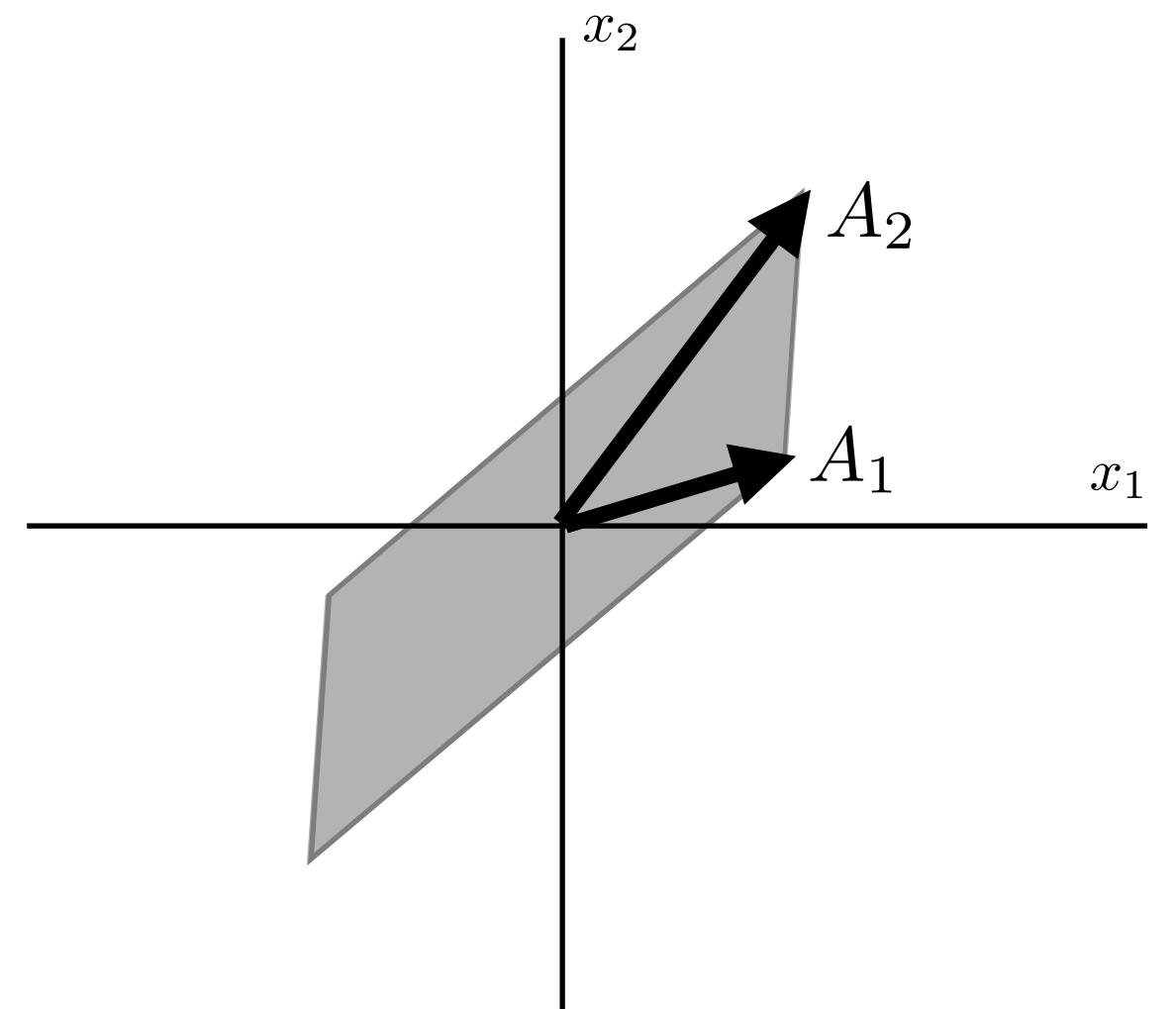
**Characteristic Polynomial**

$$sI - A \text{ drops rank only when } \text{char}_A(s) = \det(sI - A) = 0$$

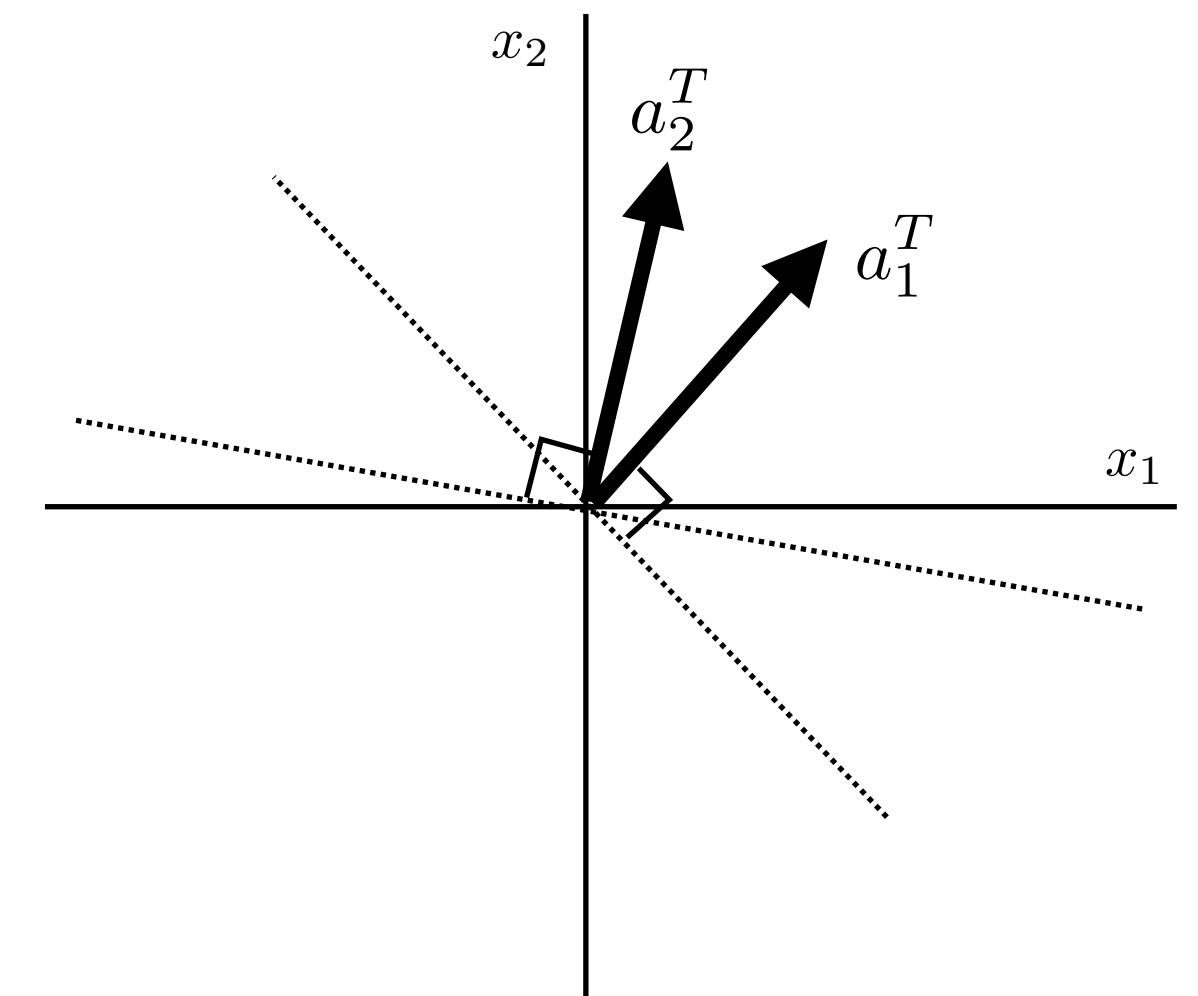
$$A \in \mathbb{R}^{2 \times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \end{bmatrix}$$

**COLUMN  
GEOMETRY**



**ROW  
GEOMETRY**



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

## Characteristic Polynomial

$$sI - A \text{ drops rank only when } \text{char}_A(s) = \det(sI - A) = 0$$

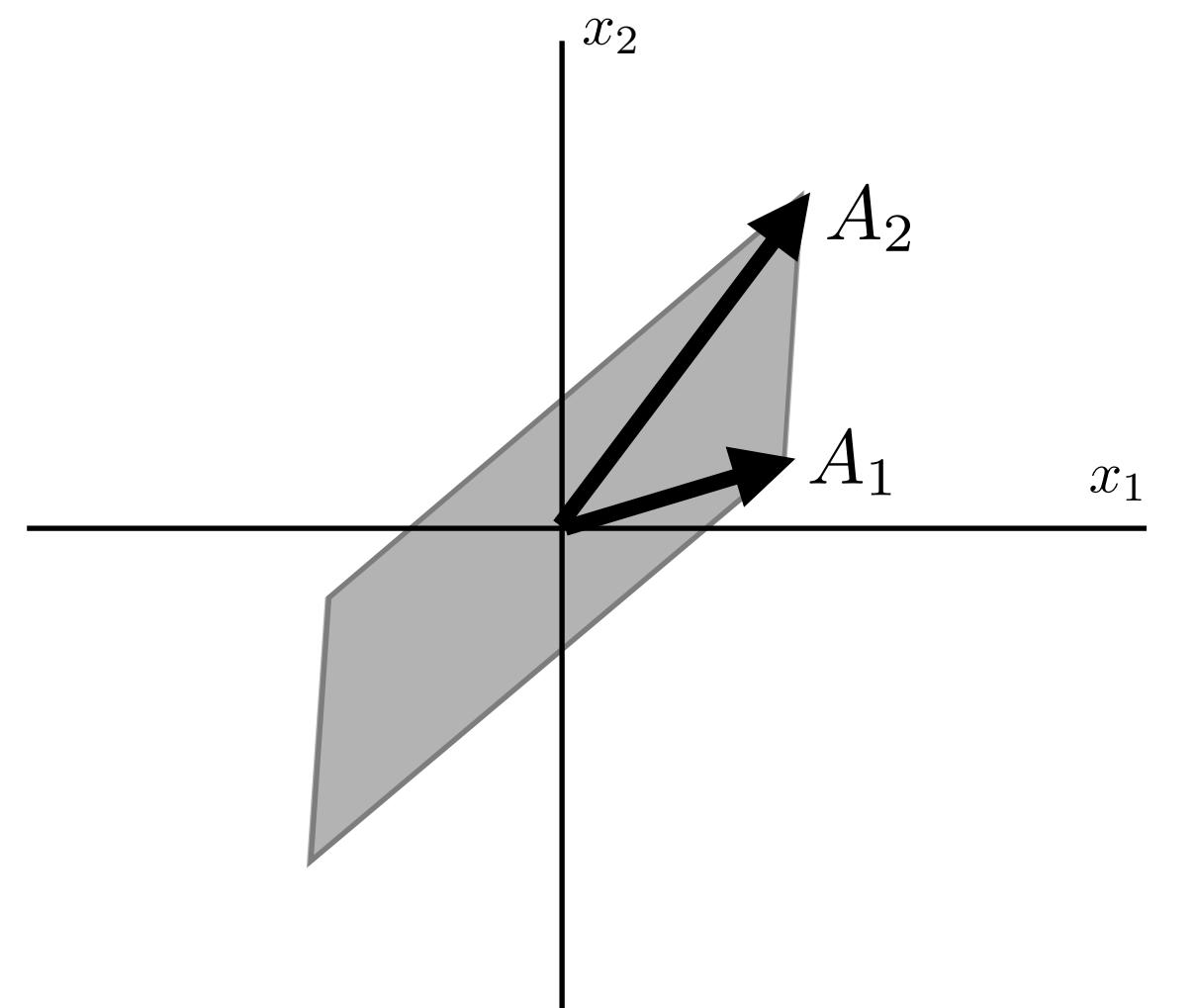
$$A \in \mathbb{R}^{2 \times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \end{bmatrix}$$

## COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$

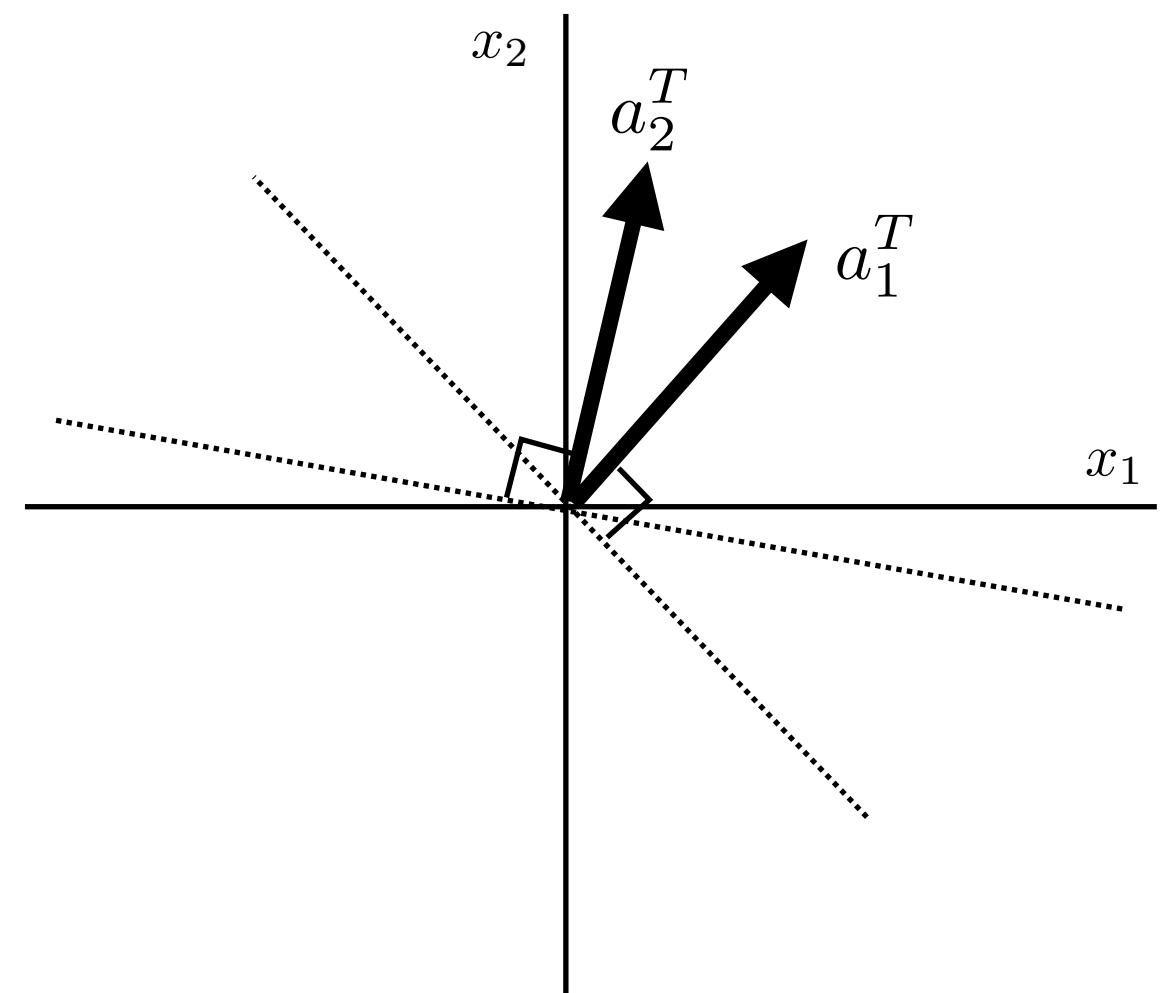
drops rank  
for eigenvalue



## ROW GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$

drops rank  
for eigenvalue



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

## Characteristic Polynomial

$sI - A$  drops rank only when  
 $\text{char}_A(s) = \det(sI - A) = 0$

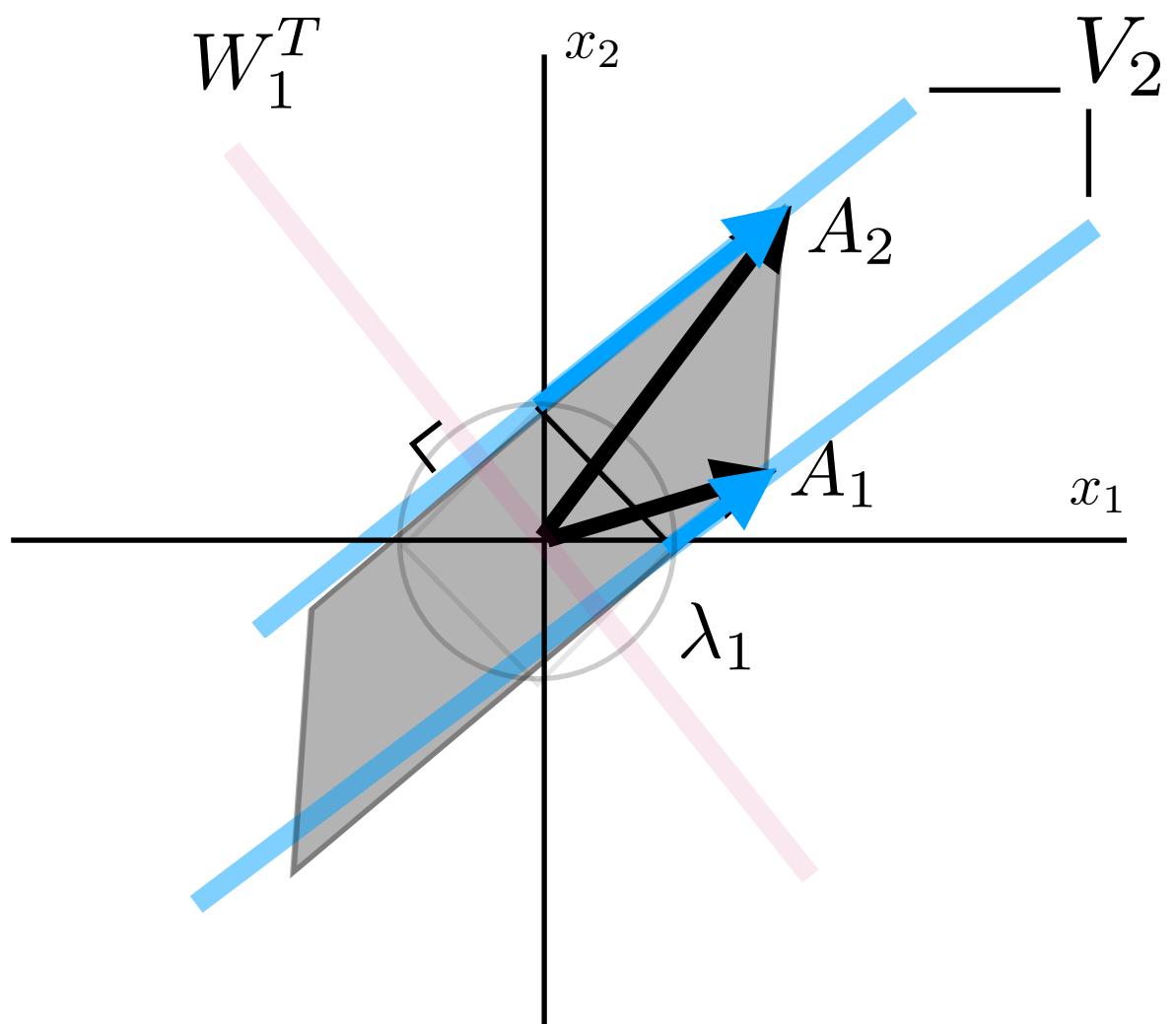
$$A \in \mathbb{R}^{2 \times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \end{bmatrix}$$

### COLUMN GEOMETRY

$$\begin{bmatrix} | & | \\ A - \lambda_1 I & \end{bmatrix}$$

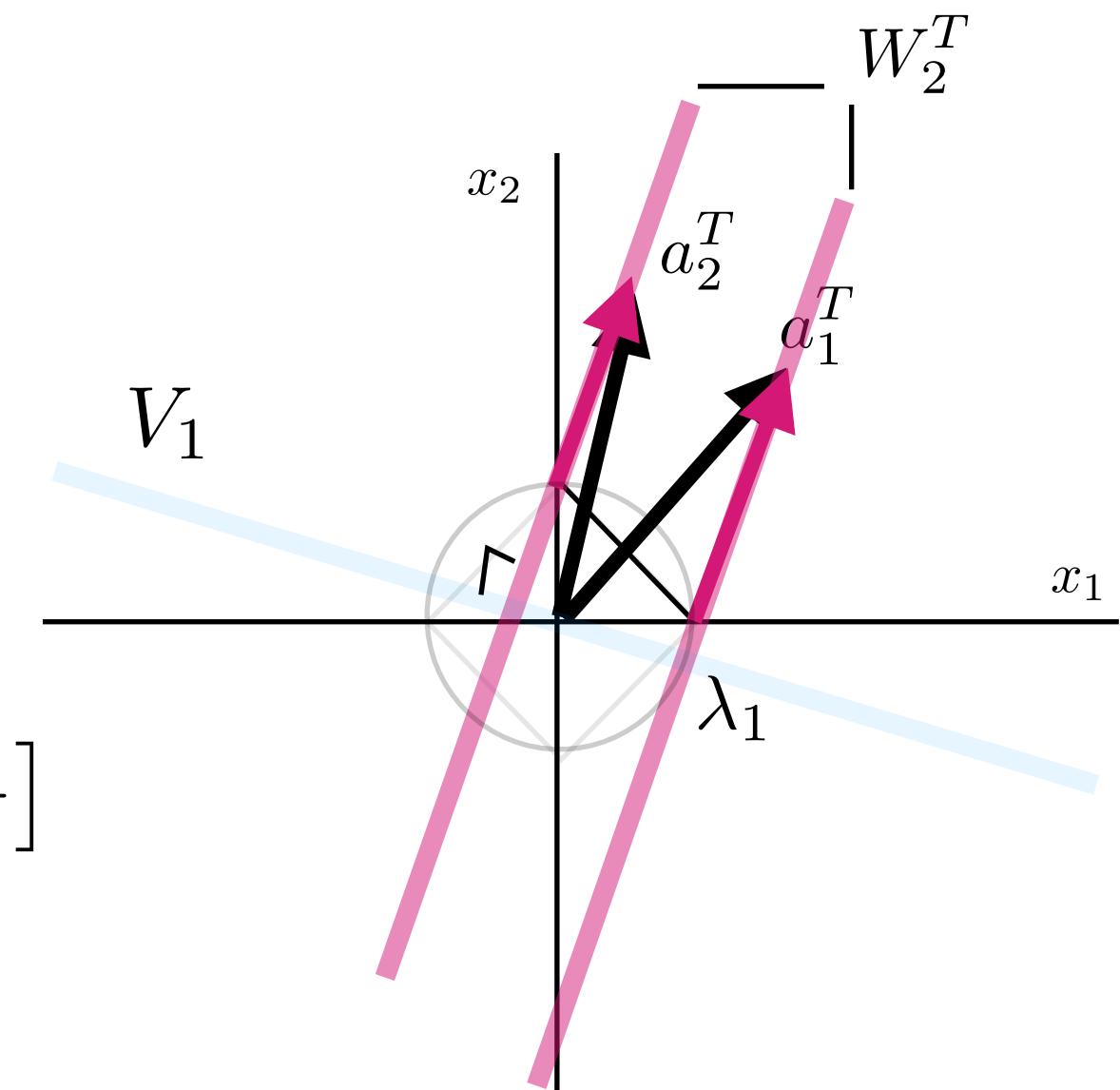
$$\begin{bmatrix} | \\ V_2 \\ | \end{bmatrix} \begin{bmatrix} \lambda_2 - \lambda_1 \\ -W_2^T \end{bmatrix} \begin{bmatrix} - & - \end{bmatrix}$$



### ROW GEOMETRY

$$\begin{bmatrix} | \\ A - \lambda_1 I \end{bmatrix}$$

$$\begin{bmatrix} | \\ V_2 \\ | \end{bmatrix} \begin{bmatrix} \lambda_2 - \lambda_1 \\ -W_2^T \end{bmatrix} \begin{bmatrix} - & - \end{bmatrix}$$



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

## Characteristic Polynomial

$sI - A$  drops rank only when  
 $\text{char}_A(s) = \det(sI - A) = 0$

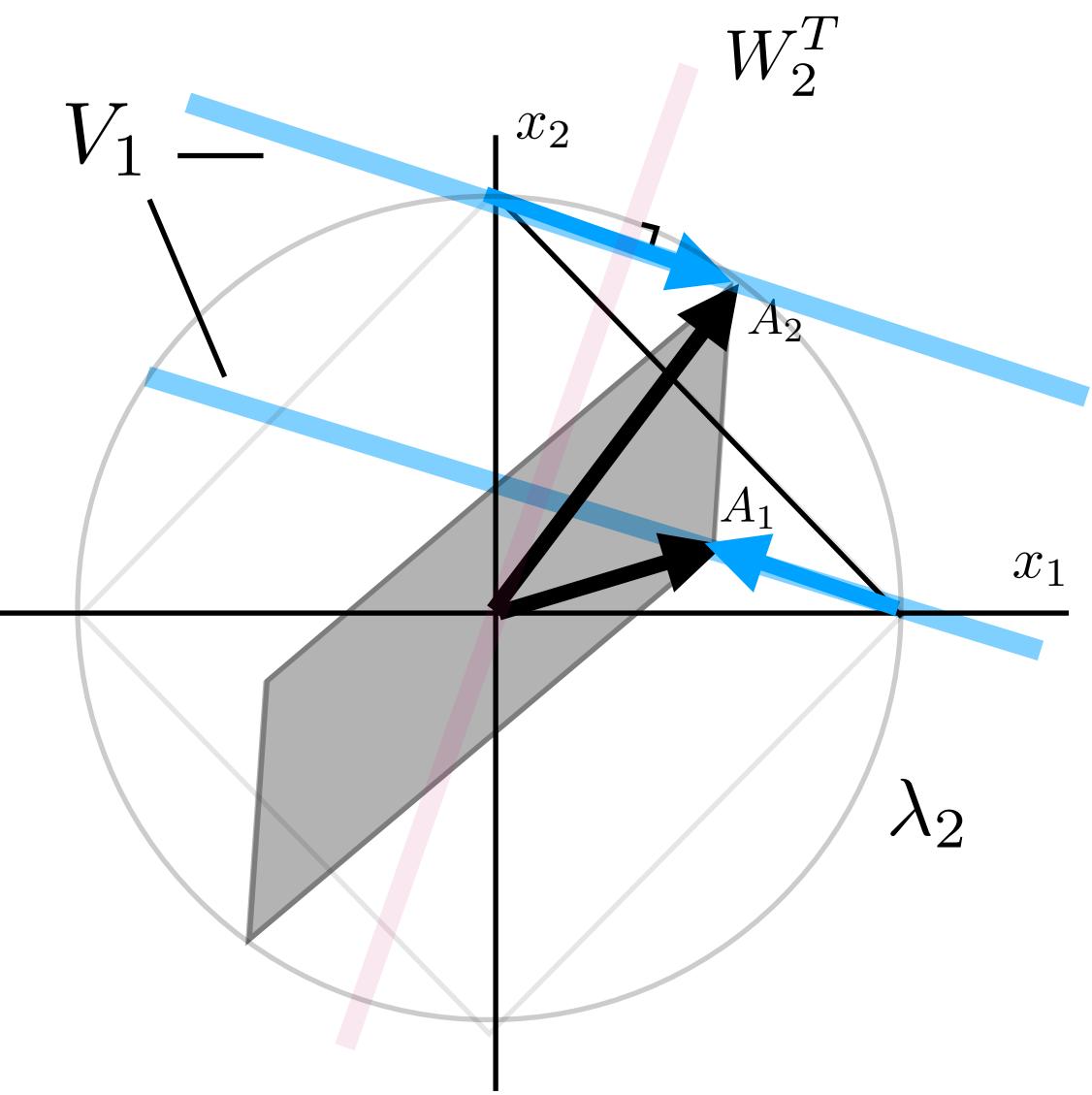
$$A \in \mathbb{R}^{2 \times 2}$$

$$A = \begin{bmatrix} & \\ A_1 & A_2 \\ & \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \end{bmatrix} = \begin{bmatrix} & \\ V_1 & V_2 \\ & \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \end{bmatrix}$$

### COLUMN GEOMETRY

$$\begin{bmatrix} & \\ A - \lambda_2 I & \end{bmatrix}$$

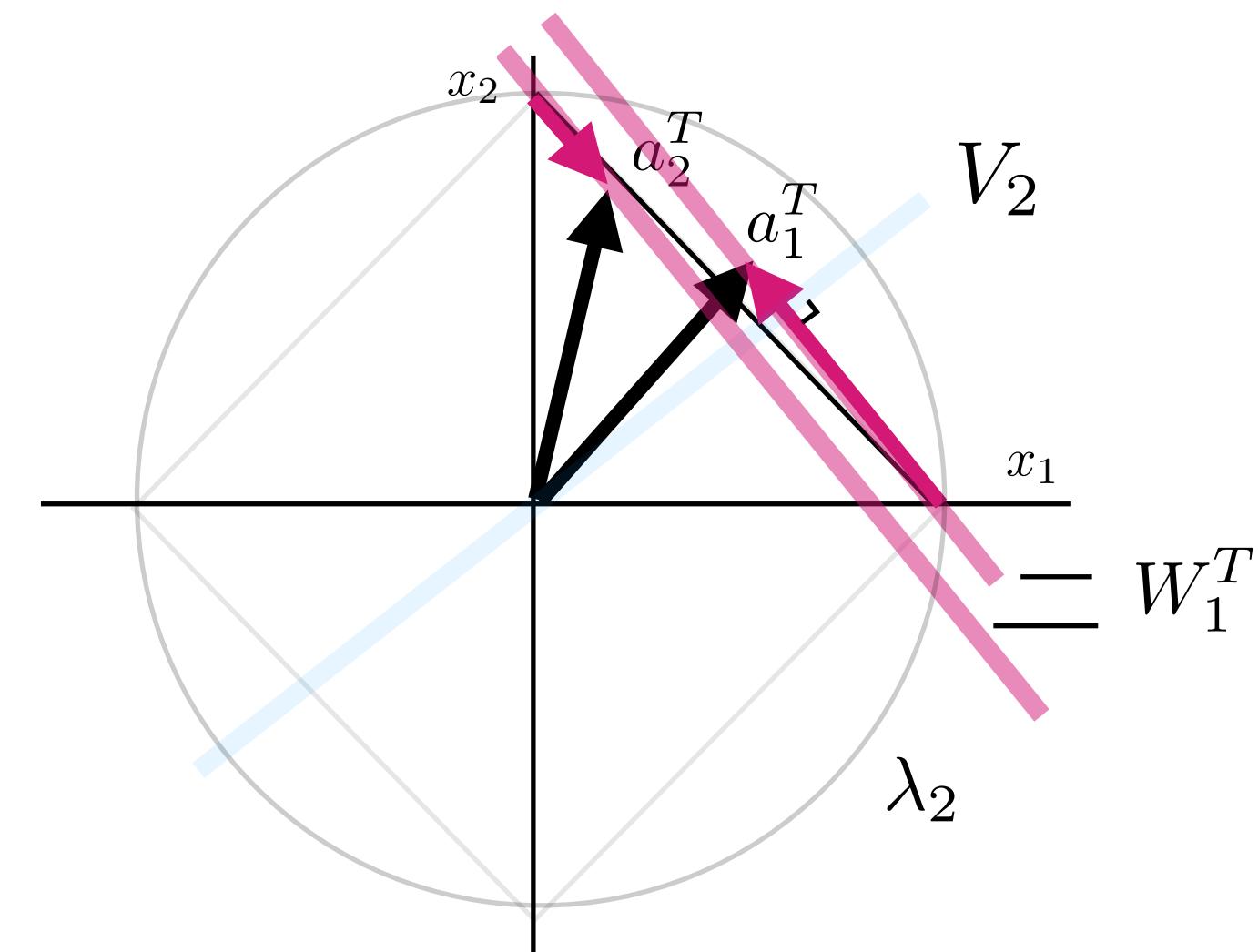
$$\begin{bmatrix} & \\ V_1 & \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & \\ & \end{bmatrix} \begin{bmatrix} -W_1^T & - \end{bmatrix}$$



### ROW GEOMETRY

$$\begin{bmatrix} & \\ A - \lambda_2 I & \end{bmatrix}$$

$$\begin{bmatrix} & \\ V_1 & \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & \\ & \end{bmatrix} \begin{bmatrix} -W_1^T & - \end{bmatrix}$$



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

## Characteristic Polynomial

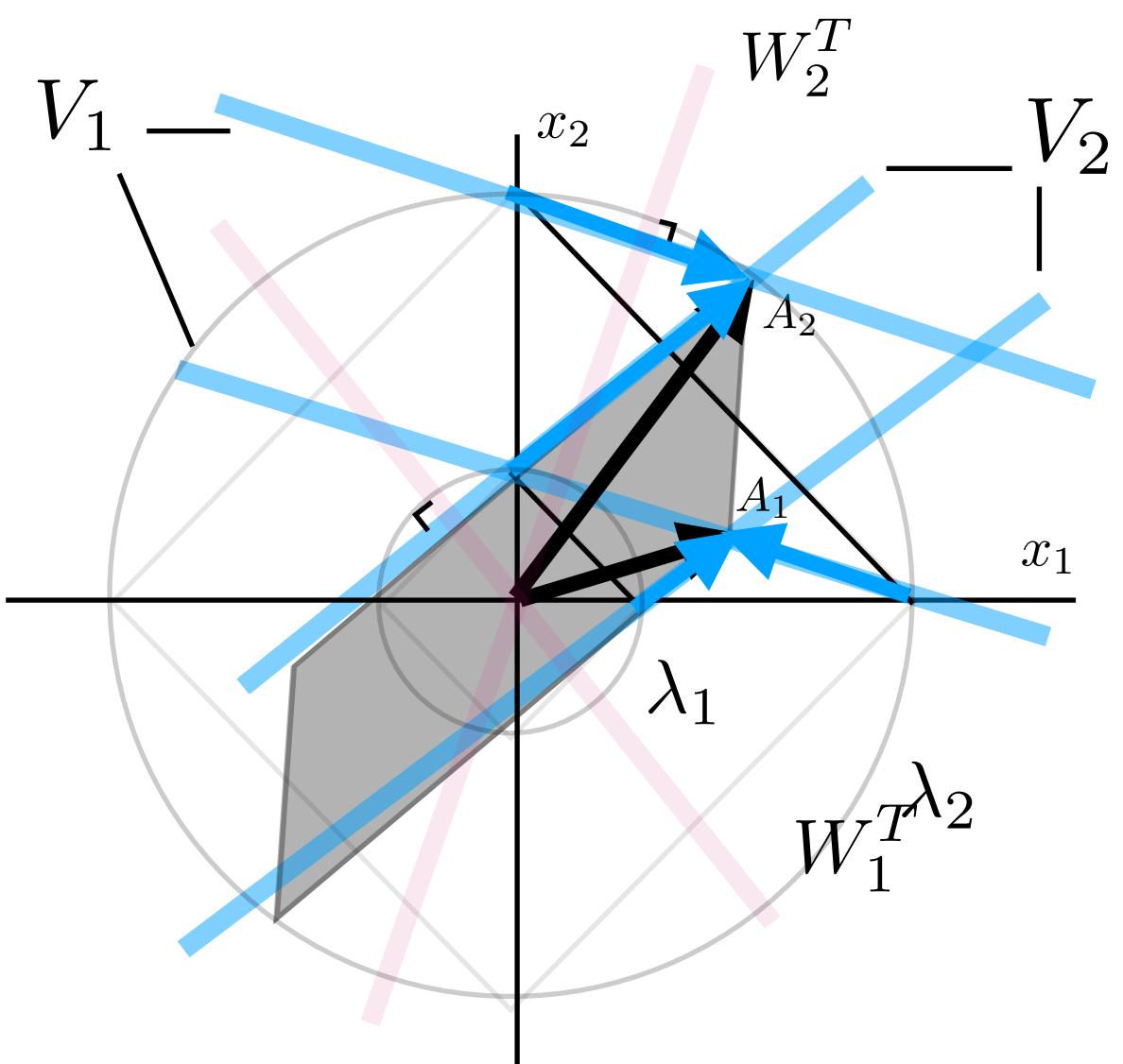
$sI - A$  drops rank only when  
 $\text{char}_A(s) = \det(sI - A) = 0$

$$A \in \mathbb{R}^{2 \times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \end{bmatrix}$$

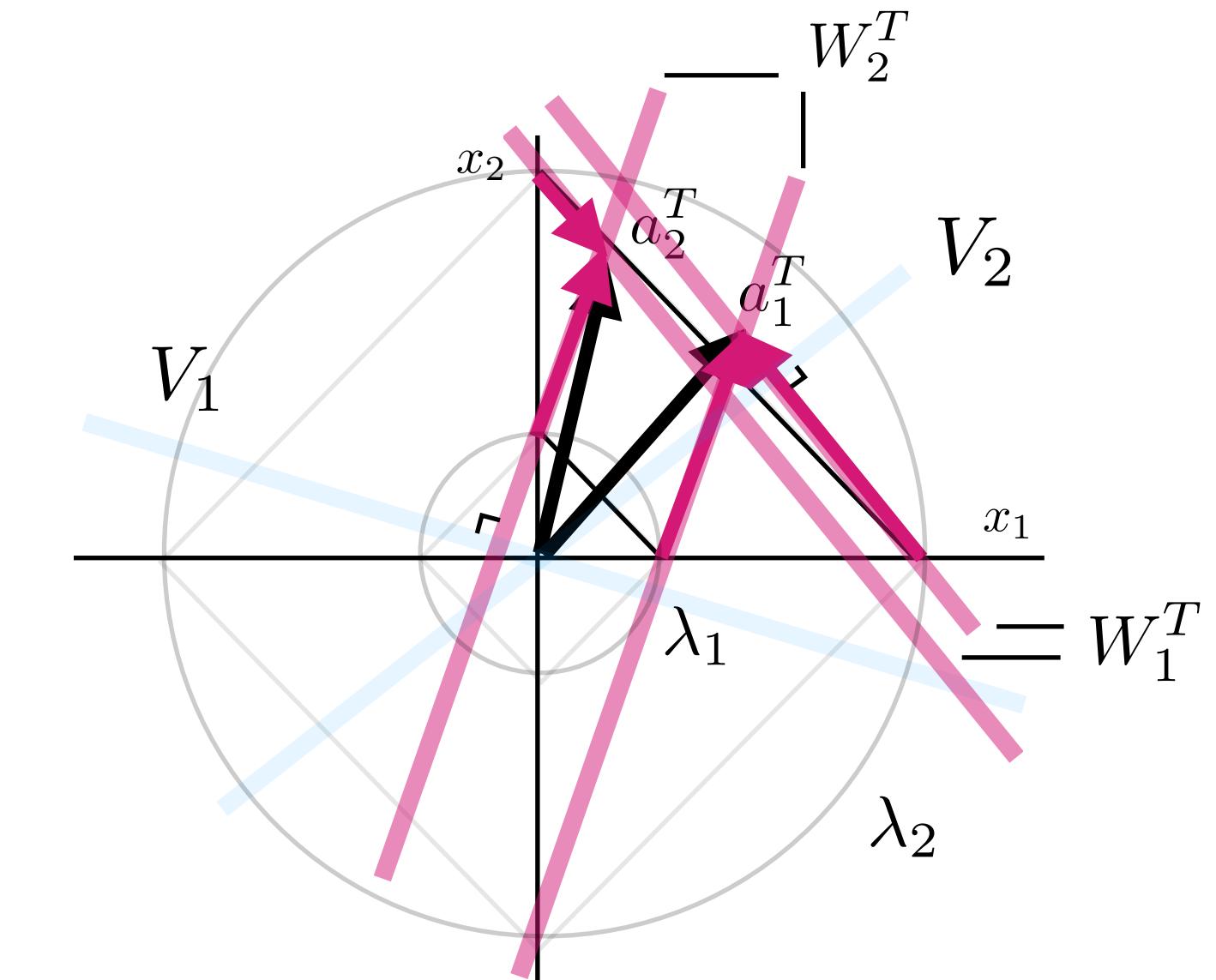
### COLUMN GEOMETRY

$$\begin{bmatrix} A & \lambda I \end{bmatrix}$$



### ROW GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

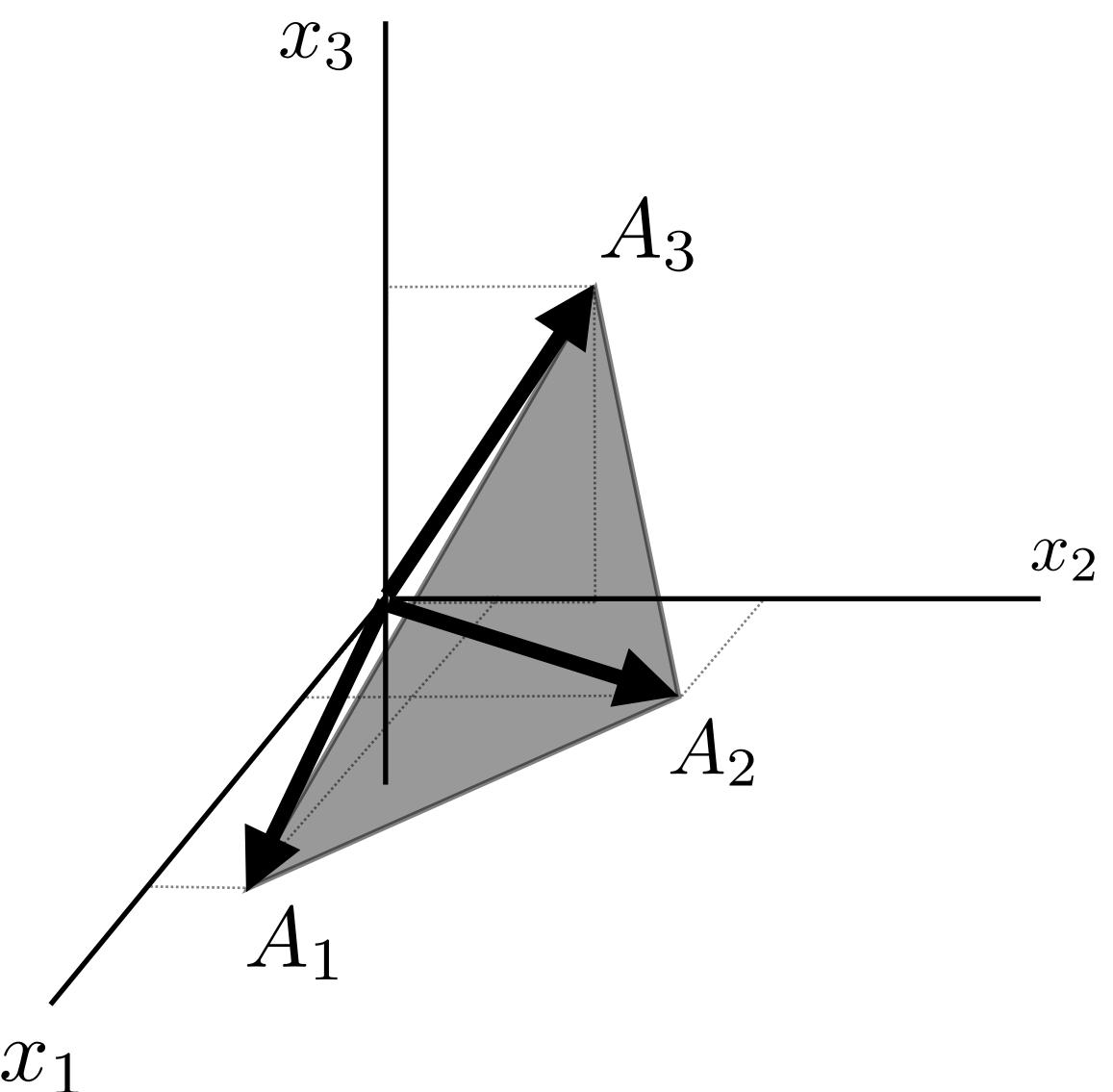
**Characteristic Polynomial**

$$sI - A \quad \text{drops rank only when} \\ \text{char}_A(s) = \det(sI - A) = 0$$

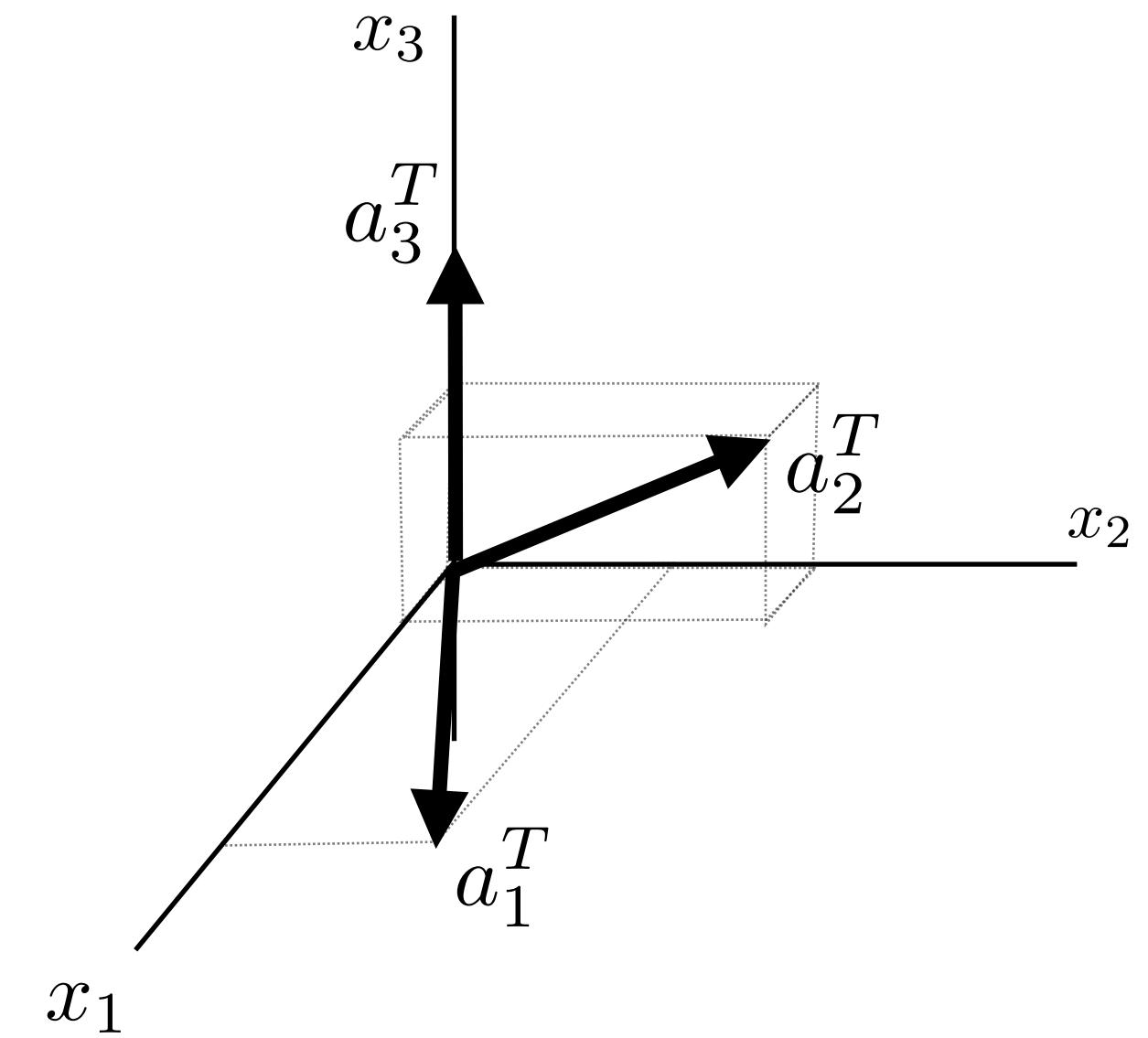
$$A \in \mathbb{R}^{3 \times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \\ -a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \\ -W_3^T & - \end{bmatrix}$$

**COLUMN  
GEOMETRY**



**ROW  
GEOMETRY**



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

## Characteristic Polynomial

$sI - A$  drops rank only when  
 $\text{char}_A(s) = \det(sI - A) = 0$

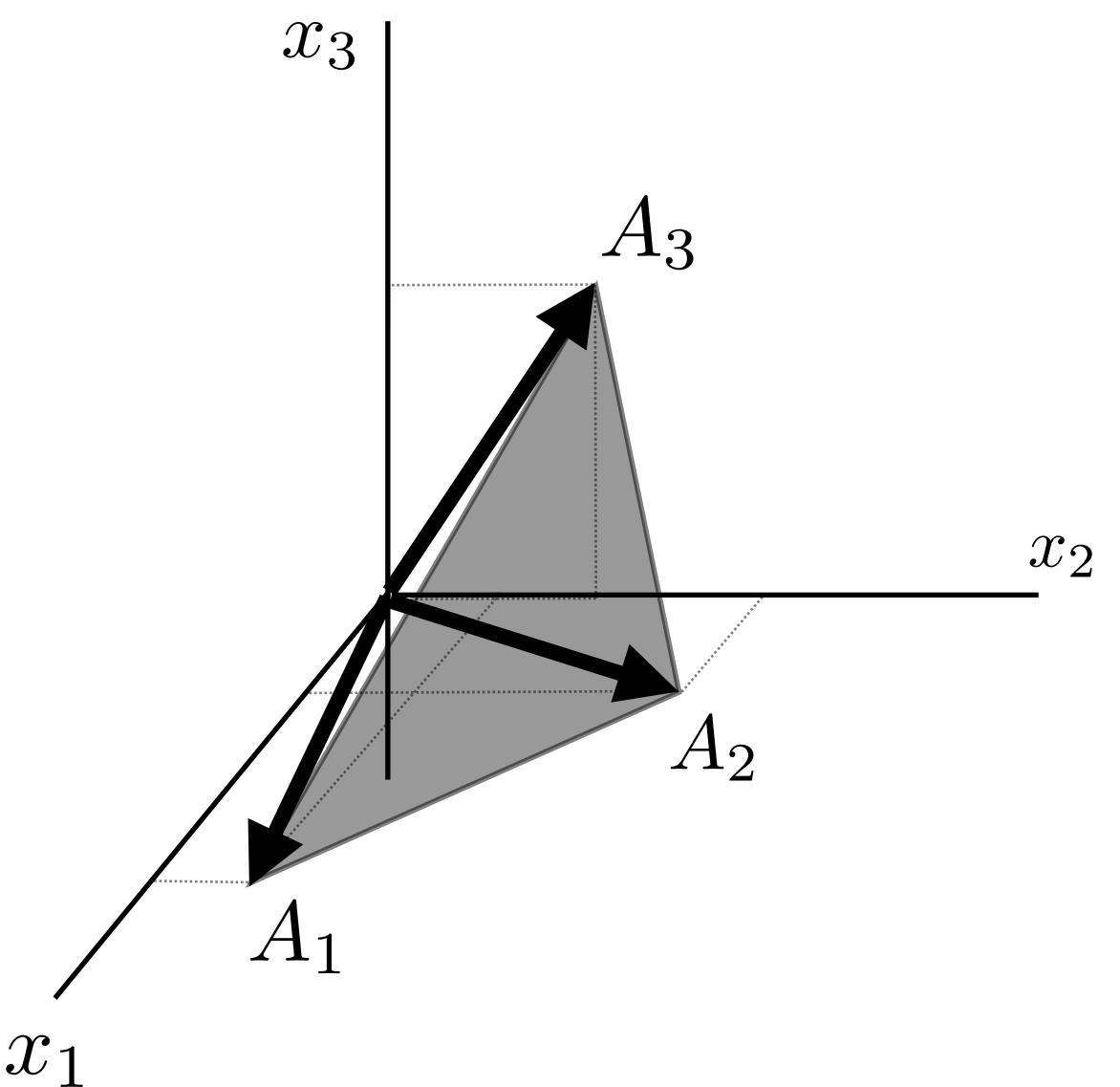
$$A \in \mathbb{R}^{3 \times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \\ -a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \\ -W_3^T & - \end{bmatrix}$$

## COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$

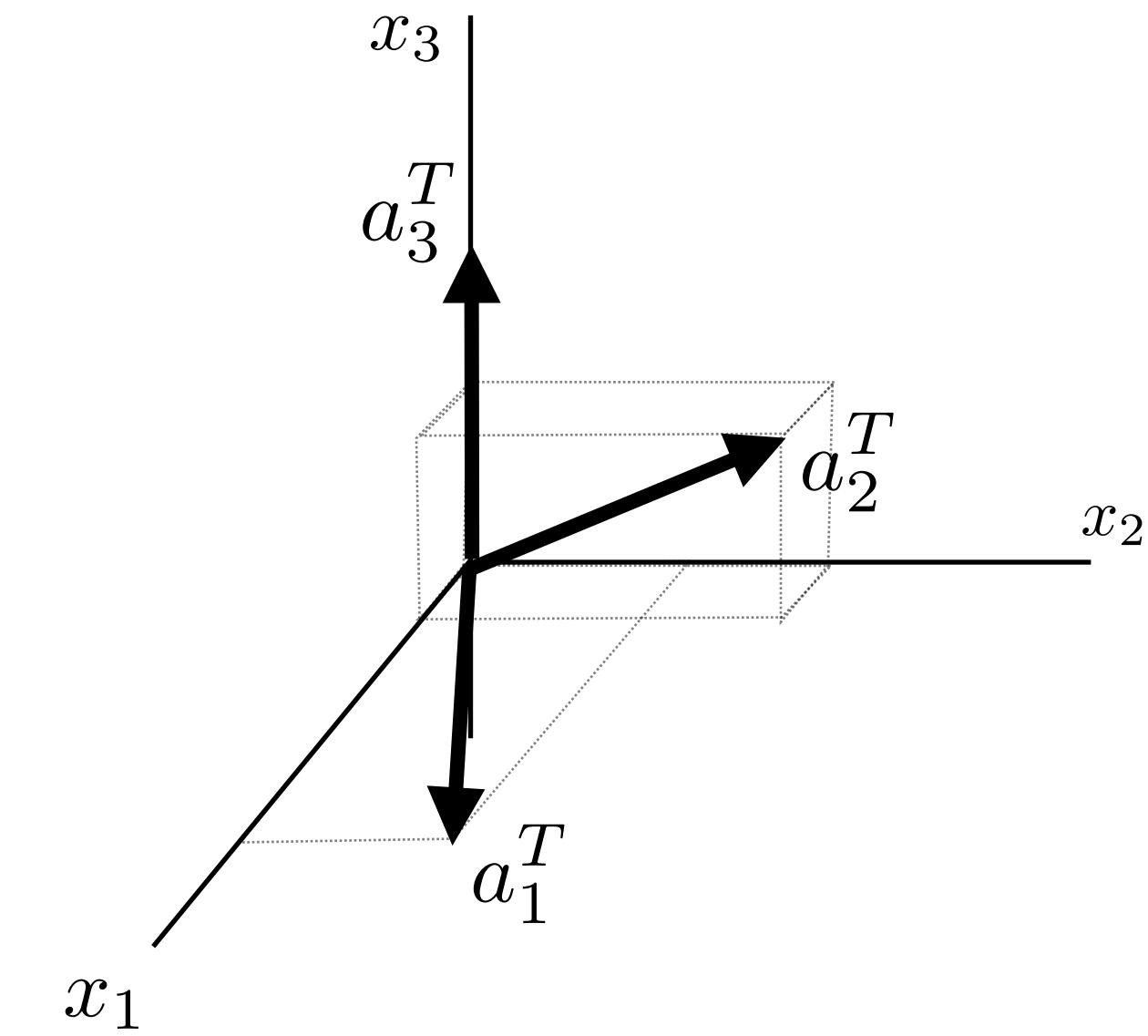
drops rank  
for eigenvalue



## ROW GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$

drops rank  
for eigenvalue



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

# Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

# Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^*A = w^*\lambda \quad \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

# Characteristic Polynomial

$sI - A$  drops rank only when  
 $\text{char}_A(s) = \det(sI - A) = 0$

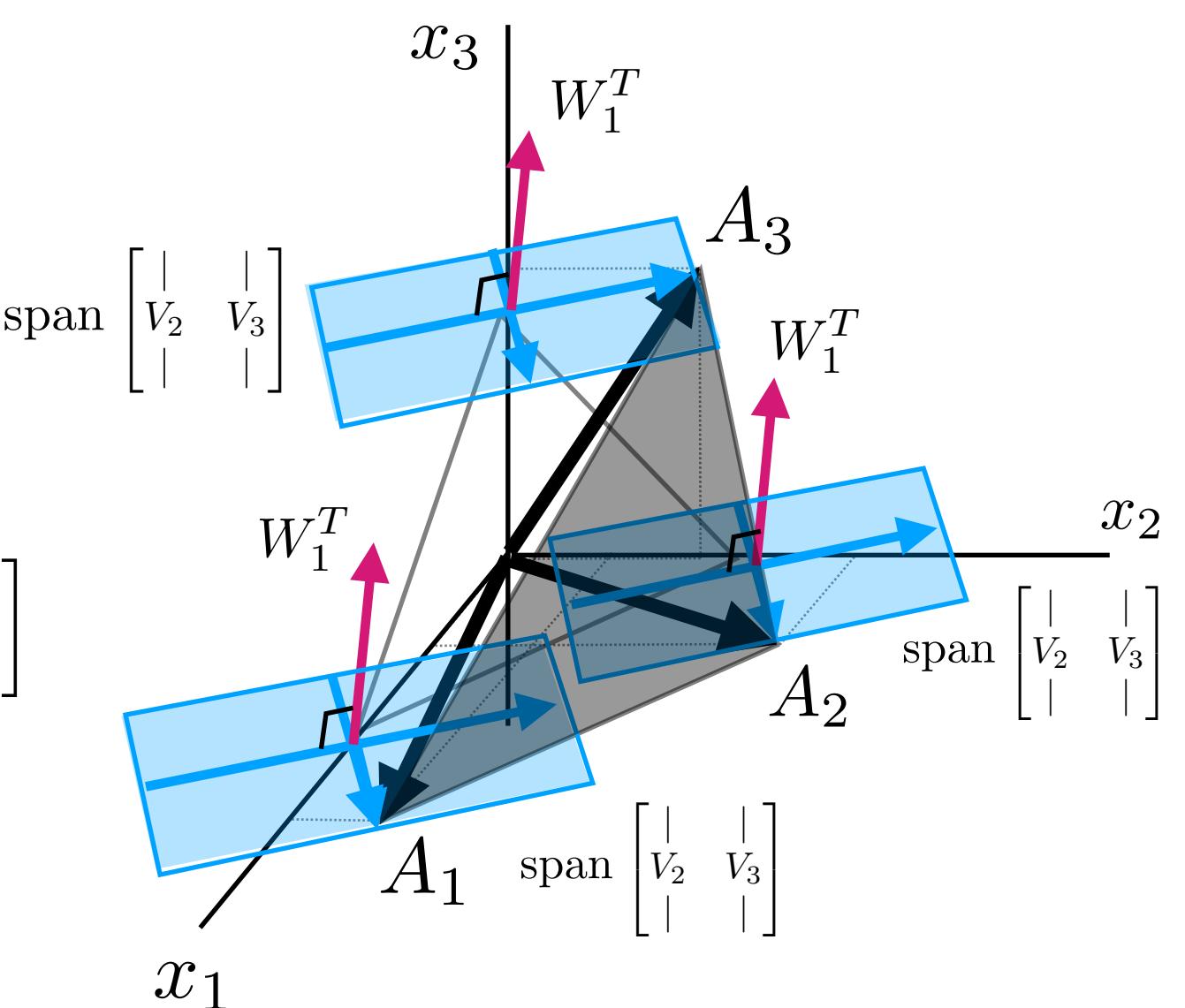
$$A \in \mathbb{R}^{3 \times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \\ -a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \\ -W_3^T & - \end{bmatrix}$$

# COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda_1 I \end{bmatrix}$$

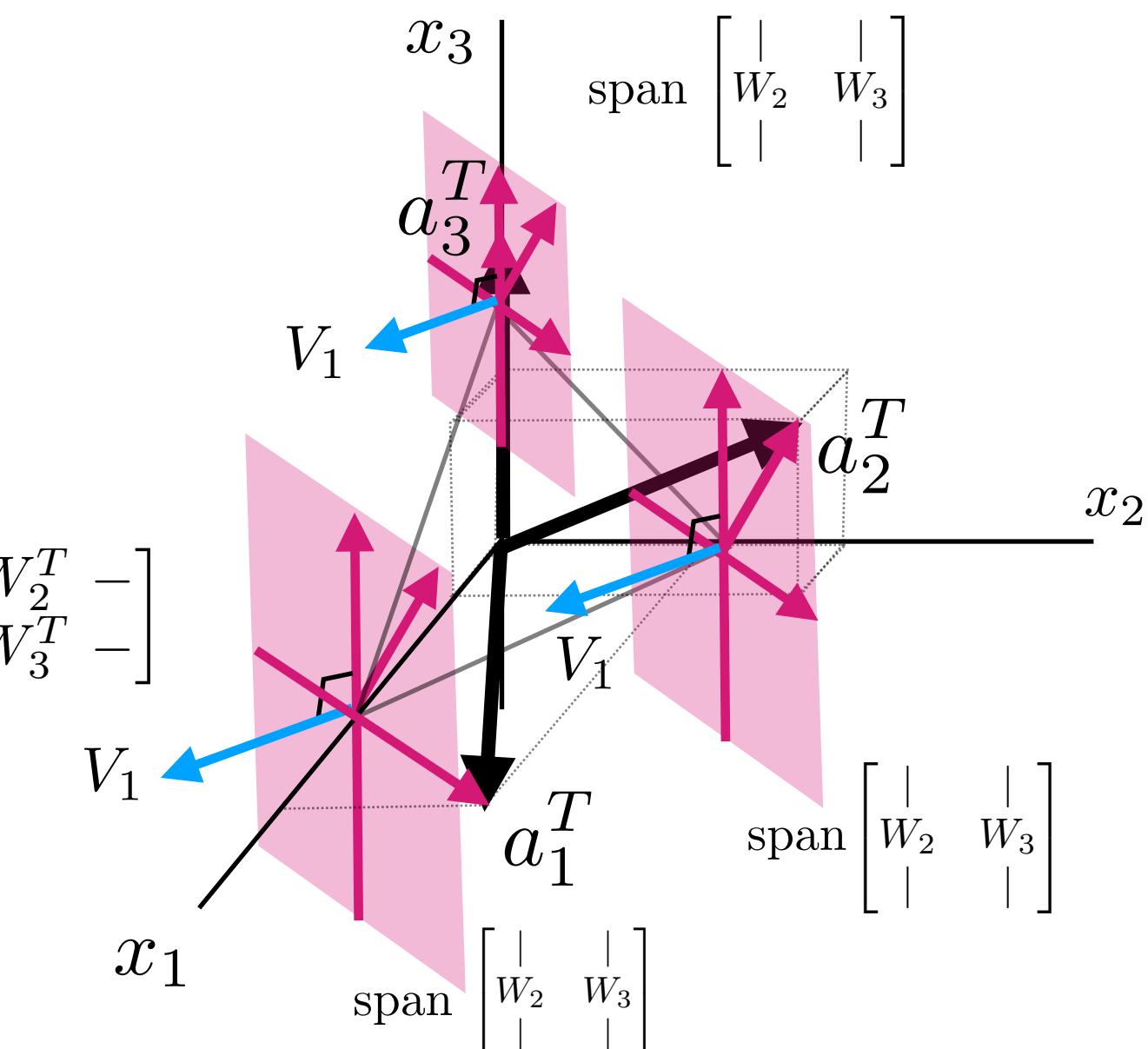
$$\begin{bmatrix} V_2 & V_3 \end{bmatrix} \begin{bmatrix} \lambda_2 - \lambda_1 & 0 \\ 0 & \lambda_3 - \lambda_1 \end{bmatrix} \begin{bmatrix} W_2^T & - \\ W_3^T & - \end{bmatrix}$$



# ROW GEOMETRY

$$\begin{bmatrix} & \\ A - \lambda_1 I & \\ & \end{bmatrix}$$

$$\begin{bmatrix} V_1 & V_2 \\ V_2 & V_3 \end{bmatrix} \begin{bmatrix} \lambda_2 - \lambda_1 & 0 \\ 0 & \lambda_3 - \lambda_1 \end{bmatrix} \begin{bmatrix} -W_2^T \\ -W_3^T \end{bmatrix} =$$



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

## Characteristic Polynomial

$sI - A$  drops rank only when  
 $\text{char}_A(s) = \det(sI - A) = 0$

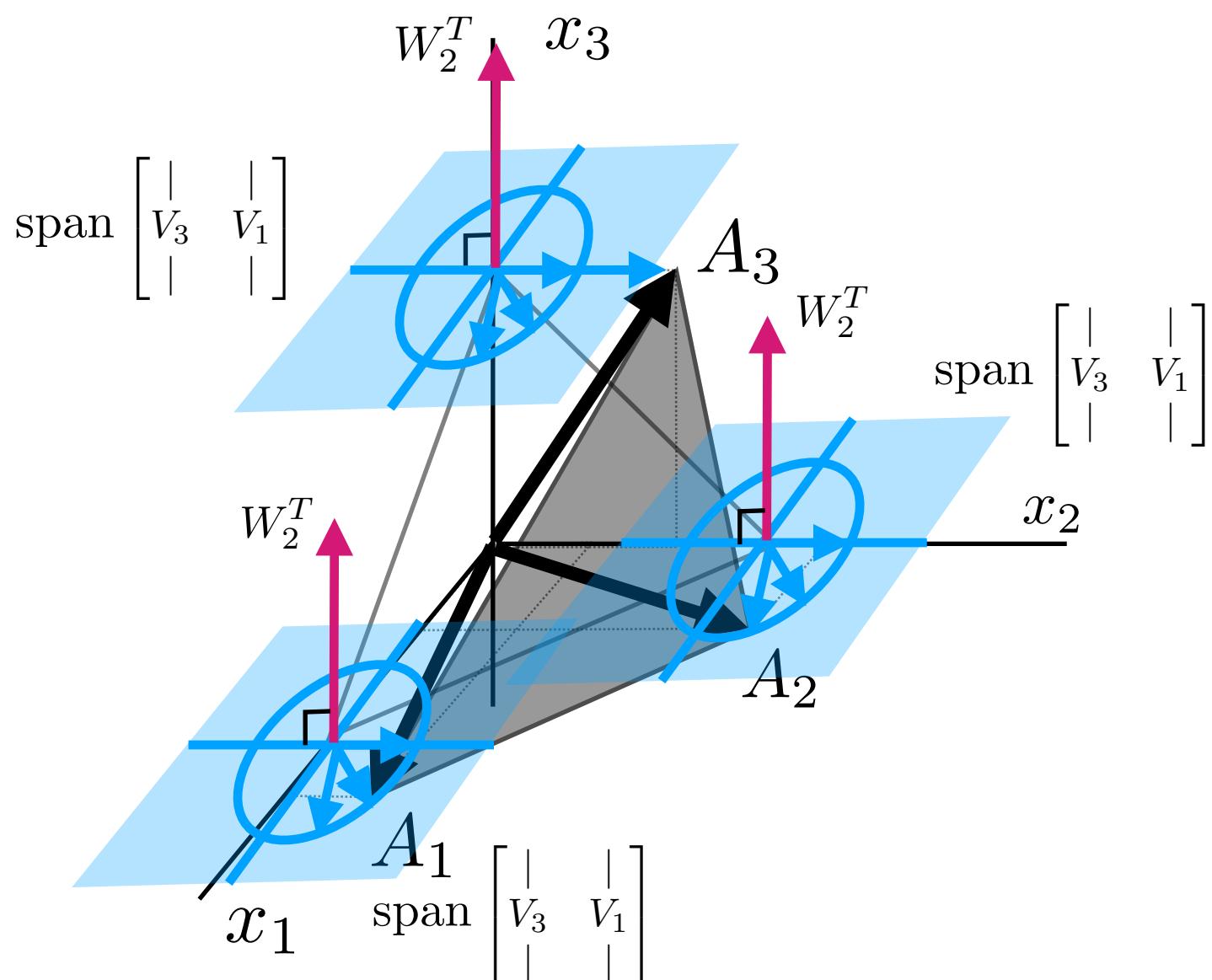
$$A \in \mathbb{R}^{3 \times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \\ -a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \\ -W_3^T & - \end{bmatrix}$$

## COLUMN GEOMETRY

$$\begin{bmatrix} | & | & | \\ A & -\lambda_2 I & | \end{bmatrix}$$

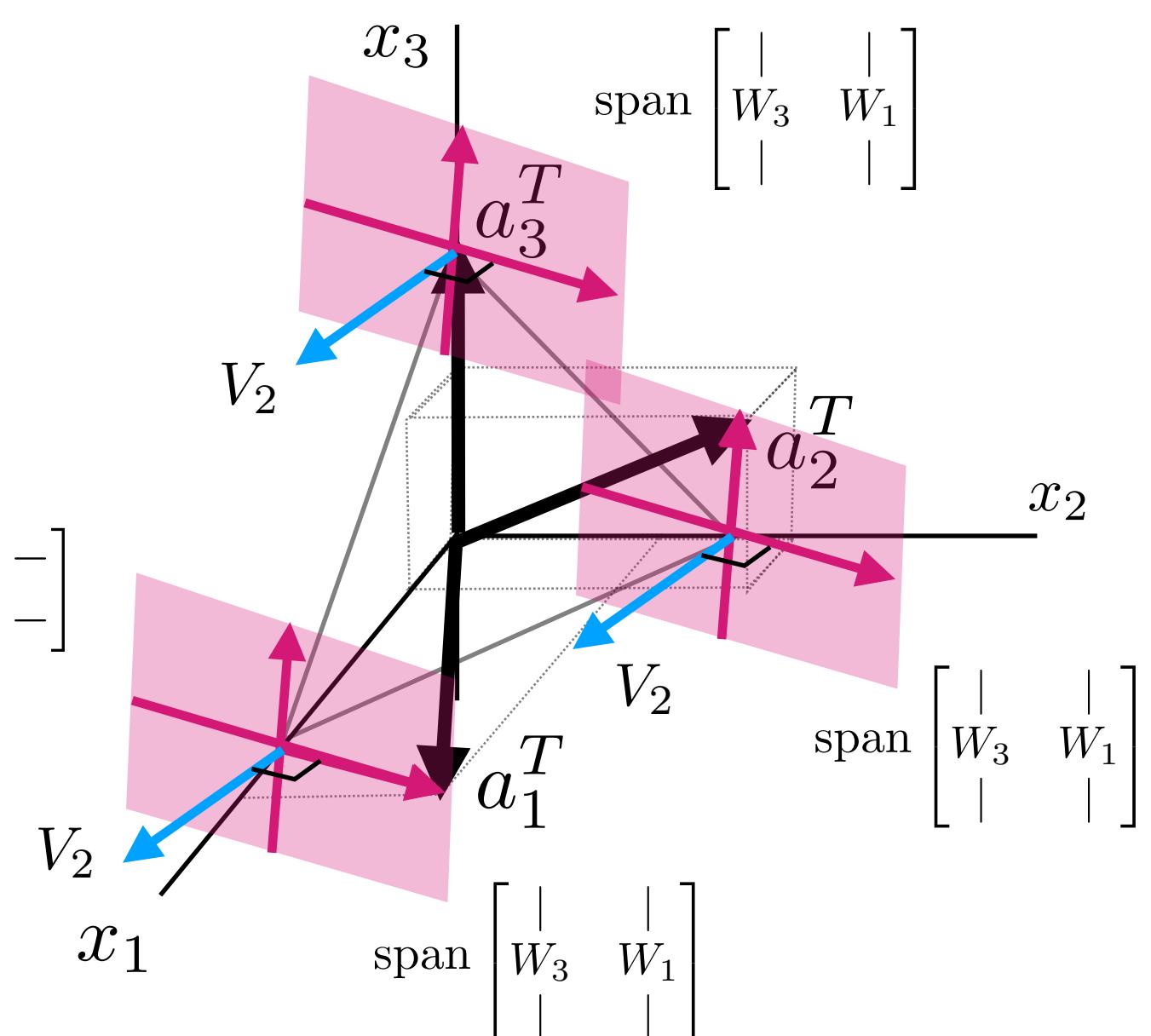
$$\begin{bmatrix} | & | \\ V_1 & V_3 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & \lambda_3 - \lambda_2 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_3^T & - \end{bmatrix}$$



## ROW GEOMETRY

$$\begin{bmatrix} | & | & | \\ A & -\lambda_2 I & | \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & V_3 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & \lambda_3 - \lambda_2 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_3^T & - \end{bmatrix}$$



# Eigenvector/Eigenvalue Picture

For any eigenvalue  $\lambda \in \mathbb{C}$

**Right Eigenvector:**  $v \in \mathbb{C}^n$

$$Av = v\lambda \quad (A - \lambda I)v = 0$$

**Left Eigenvectors:**  $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \quad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of  $s$  where  $A - sI$  drops rank...

## Characteristic Polynomial

$$sI - A \text{ drops rank only when } \text{char}_A(s) = \det(sI - A) = 0$$

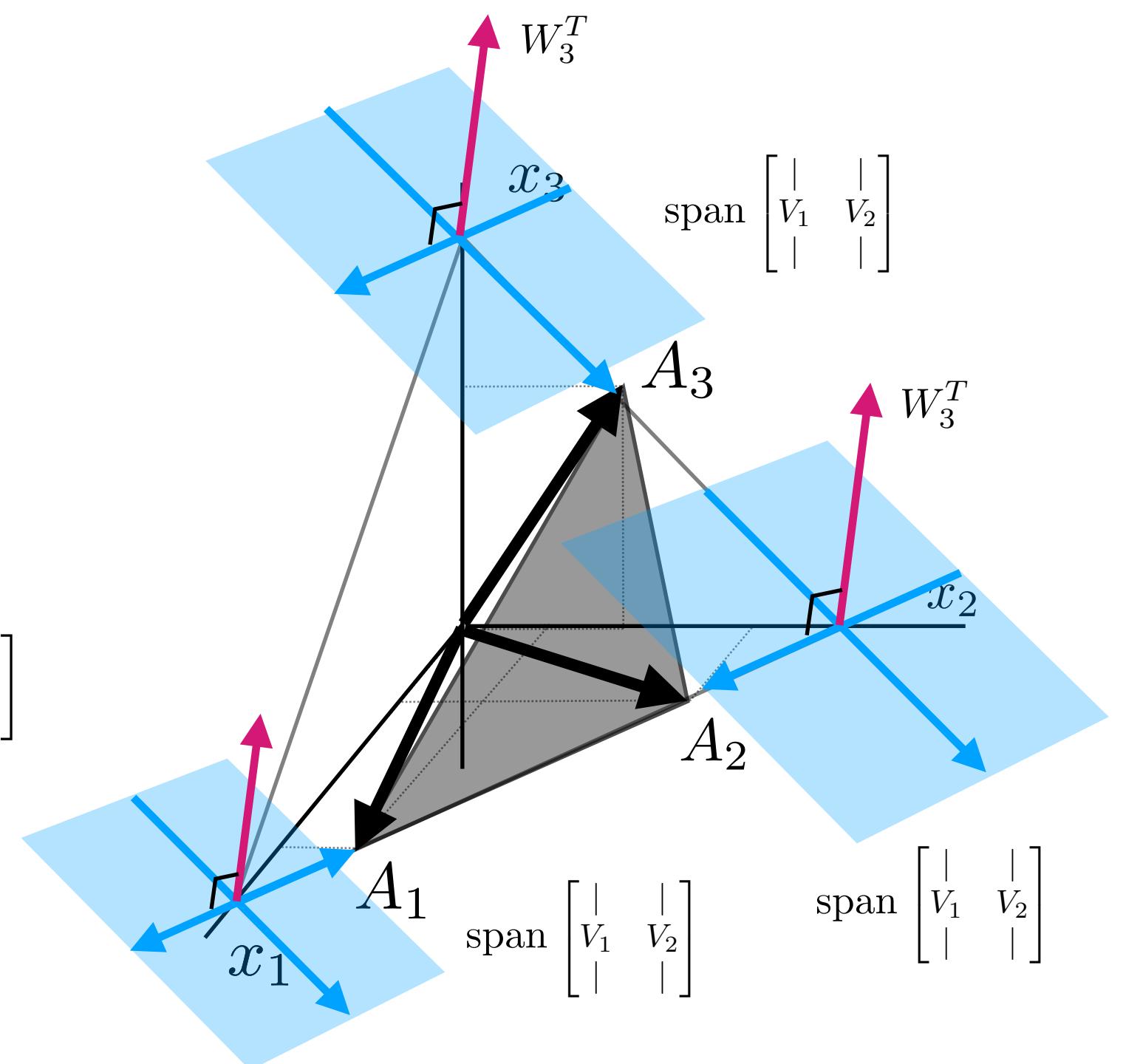
$$A \in \mathbb{R}^{3 \times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \\ -a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \\ -W_3^T & - \end{bmatrix}$$

## COLUMN GEOMETRY

$$\begin{bmatrix} | & | & | \\ A & -\lambda_3 I & | \end{bmatrix}$$

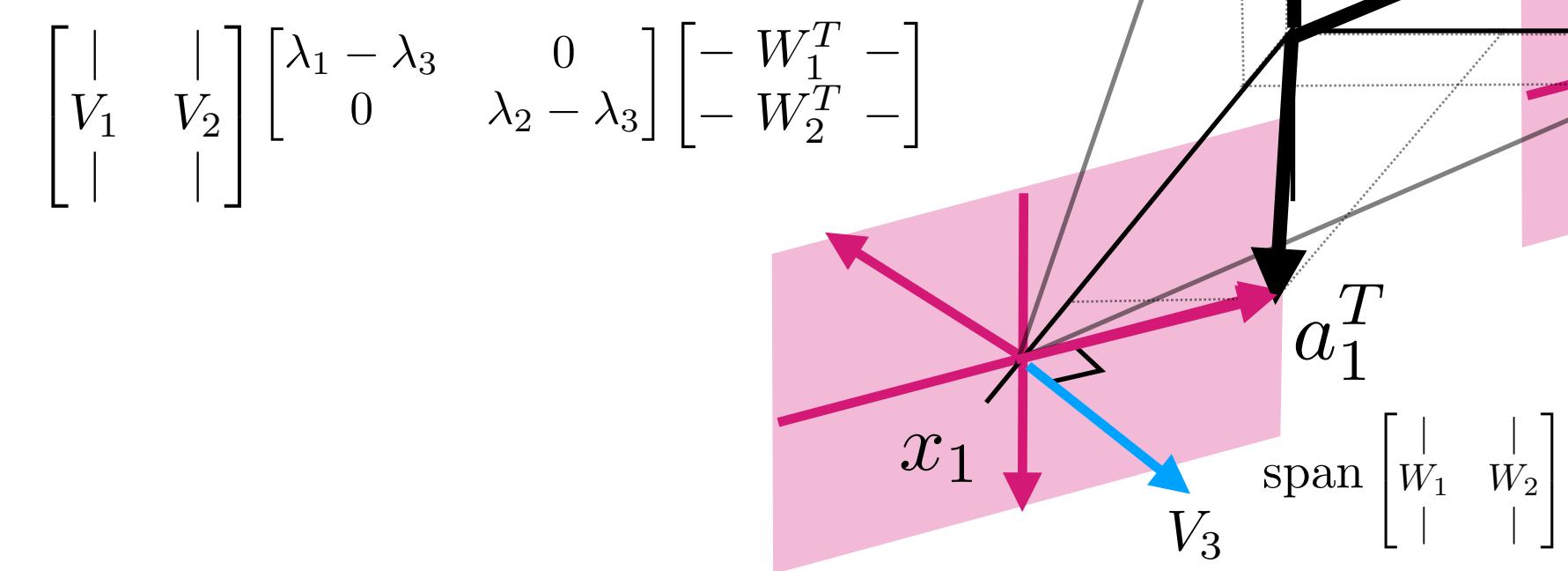
$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_3 & 0 \\ 0 & \lambda_2 - \lambda_3 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \end{bmatrix}$$



## ROW GEOMETRY

$$\begin{bmatrix} | & | & | \\ A & -\lambda_3 I & | \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_3 & 0 \\ 0 & \lambda_2 - \lambda_3 \end{bmatrix} \begin{bmatrix} -W_1^T & - \\ -W_2^T & - \end{bmatrix}$$



$$\text{span} \begin{bmatrix} | & | \\ W_1 & W_2 \\ | & | \end{bmatrix}$$

$$\text{span} \begin{bmatrix} | & | \\ W_1 & W_2 \\ | & | \end{bmatrix}$$

$$\text{span} \begin{bmatrix} | & | \\ W_1 & W_2 \\ | & | \end{bmatrix}$$

# Diagonalization

Square matrix:  $A \in \mathbb{R}^{n \times n}$

Assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Right Eigenvectors:**

$$V = \begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix} \quad AV = \begin{bmatrix} AV_1 \dots AV_n \end{bmatrix} = \begin{bmatrix} V_1 \lambda_1 \dots V_n \lambda_n \end{bmatrix} = \begin{bmatrix} V_1 \dots V_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_D = VD \quad \rightarrow \quad AV = VD$$

**Left Eigenvectors:**

$$W = \begin{bmatrix} -W_1^* - \\ \vdots \\ -W_n^* - \end{bmatrix} \quad WA = \begin{bmatrix} -W_1^* A - \\ \vdots \\ -W_n^* A - \end{bmatrix} = \underbrace{\begin{bmatrix} -\lambda_1 W_1^* - \\ \vdots \\ -\lambda_n W_n^* - \end{bmatrix}}_D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* - \\ \vdots \\ -W_n^* - \end{bmatrix} = DW \quad \rightarrow \quad WA = DW$$

$$W^{-1} = W \quad A = W^{-1}DW$$

Assuming V & W are chosen with compatible orderings and lengths of columns/rows...

$$V^{-1} = W$$

# Diagonalization

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

## Diagonalization

$$A = V D V^{-1}$$

$$\underbrace{\begin{bmatrix} A \\ | \\ V_1 & \cdots & V_n \\ | \end{bmatrix}}_{\text{Right eigen-vectors}} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{D}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

**Right eigen-vectors**      **Eigen-values** (on diagonal)      **Left eigen-vectors**

$$\begin{bmatrix} A \\ | \\ V_i \\ | \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices  
**Dyadic Expansion**

$$\begin{aligned} V^{-1}V &= \begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} W_1^* V_1 & \cdots & W_1^* V_n \\ \vdots & & \vdots \\ W_n^* V_1 & \cdots & W_n^* V_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \end{aligned}$$

...from off diagonal terms  $W_j^* V_i = 0 \quad j \neq i$

$V_i$  orthogonal to all other  $W_j$

$W_i^* V_i = 1$

can be scaled so that  $W_i^* V_i = 1$

...from diagonal terms

$V_i, W_i$

# Diagonalization - Similarity Transform

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$  A is similar to a diagonal matrix

## Diagonalization

$$A = V D V^{-1}$$

$$\left[ \begin{array}{c|c|c|c} A & | & | & | \\ \hline V_1 & \dots & V_n & | \\ | & & | & | \end{array} \right] = \underbrace{\left[ \begin{array}{c|c|c|c} | & & | & | \\ \hline V_1 & \dots & V_n & | \\ | & & | & | \end{array} \right]}_{\text{Right eigen-vectors}} \underbrace{\left[ \begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \underbrace{\left[ \begin{array}{ccc} -W_1^* & - & | \\ | & \vdots & | \\ -W_n^* & - & | \end{array} \right]}_{\text{Left eigen-vectors}}$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

$$\left[ \begin{array}{c|c|c|c} A & | & | & | \\ \hline V_i & & & | \\ | & & & | \end{array} \right] = \sum_i \left[ \begin{array}{c|c|c|c} | & & | & | \\ \hline V_i & & & | \\ | & & & | \end{array} \right] \left[ \begin{array}{c} \lambda_i \\ | \\ | \end{array} \right] \left[ \begin{array}{ccc} -W_i^* & - & | \\ | & \vdots & | \\ -W_i^* & - & | \end{array} \right]$$

Sum of  
rank-1  
matrices  
**Dyadic Expansion**

$$\left[ \begin{array}{c} y'_1 \\ \vdots \\ y'_n \end{array} \right] = \left[ \begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right] \left[ \begin{array}{c} x'_1 \\ \vdots \\ x'_n \end{array} \right] = \left[ \begin{array}{c} \lambda_1 x'_1 \\ \vdots \\ \lambda_n x'_n \end{array} \right]$$

$$x = Vx' \quad y = Vy'$$

$$y = Ax$$

$$Vy' = AVx'$$

$$y' = V^{-1}AVx'$$

$$y' = V^{-1}VDV^{-1}Vx'$$

$$y' = Dx'$$

# Diagonalization - Matrix Multiplication

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

## Diagonalization

Interpretation of  
Matrix Multiplication

$Ax$

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

transforming into eigen-vector coords

**Right  
eigen-  
vectors**

**Eigen-  
values**  
(on diagonal)

**Left  
eigen-  
vectors**

Sum of  
rank-1  
matrices

**Dyadic  
Expansion**

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

# Diagonalization - Matrix Multiplication

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

## Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of  
rank-1  
matrices

**Dyadic Expansion**

Interpretation of  
Matrix Multiplication

$Ax$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$\underbrace{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$  transforming into eigen-vector coords

$\underbrace{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}$  Scaling each coord by eigenvalue

# Diagonalization - Matrix Multiplication

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

## Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of  
rank-1  
matrices  
**Dyadic Expansion**

Interpretation of  
Matrix Multiplication

$Ax$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$\underbrace{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$  transforming into eigen-vector coords

$\underbrace{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}$  Scaling each coord by eigenvalue

$V_1 \lambda_1 W_1^* x + \cdots + V_n \lambda_n W_n^* x$  Transforming back into regular coordinates

# Diagonalization - Matrix Multiplication

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots** If  $x$  is an eigenvector...

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

# Diagonalization

# Interpretation of Matrix Multiplication

*AVi*

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} =$$





# Right eigen-vectors

# Eigen-values (on diagonal)

# Left eigen- vectors

$$[A] = \sum_i [V_i] [\lambda_i] [-W_i^* -]$$

# Sum of rank-1 matrices

# Dyadic Expansion

$$[A \mid x] = [V_1 \mid \dots \mid V_n] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ -W_n^* & - \end{bmatrix} [x \mid]$$

Orthogonal to all other left eigenvectors

# Diagonalization - Matrix Multiplication

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots** **If  $x$  is an eigenvector...**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

## Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Sum of  
rank-1  
matrices

**Dyadic Expansion**

Interpretation of  
Matrix Multiplication

$AV_i$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

**Orthogonal to all other left eigenvectors**

**Scaled by specific eigenvalue**

# Diagonalization - Matrix Multiplication

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots** **If  $x$  is an eigenvector...**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

## Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

**Right eigen-vectors**      **Eigen-values (on diagonal)**      **Left eigen-vectors**

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices  
**Dyadic Expansion**

Interpretation of Matrix Multiplication

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

**Orthogonal to all other left eigenvectors**

**Scaled by specific eigenvalue**

**Select out that specific eigenvector**

$\lambda_i V_i$

# Diagonalization (non-unique) case 1: ordering

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Permutation Matrix**  $P \in \mathbb{R}^{n \times n}$

**Diagonalization**

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

$$\left[ \begin{array}{c} A \\ | \\ V_1 \dots V_n \\ | \end{array} \right] = \left[ \begin{array}{ccc} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{array} \right] \underbrace{\left[ \begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \left[ \begin{array}{ccc} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{array} \right] =$$

$$\left[ \begin{array}{ccc} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{array} \right] \underbrace{\left[ \begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right]}_{\text{Shuffling eigenvalues and eigenvectors}} \left[ \begin{array}{ccc} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{array} \right]$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

Sum of  
rank-1  
matrices

**Dyadic Expansion**

$$\left[ \begin{array}{c} A \\ | \\ V_i \\ | \end{array} \right] = \sum_i \left[ \begin{array}{c} | \\ V_i \\ | \end{array} \right] [\lambda_i] \left[ \begin{array}{ccc} - & W_i^* & - \end{array} \right]$$



Order of sum does not matter...

# Diagonalization (non-unique) case 1: ordering

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Permutation Matrix**  $P \in \mathbb{R}^{n \times n}$

**Diagonalization**

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

**Ex.**  $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$   $P^T P = I$

$$\left[ \begin{array}{c} A \\ | \\ V_1 \quad \cdots \quad V_n \\ | \end{array} \right] = \underbrace{\left[ \begin{array}{ccc} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right]}_{\text{Right eigen-vectors}} \underbrace{\left[ \begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \underbrace{\left[ \begin{array}{ccc} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{array} \right]}_{\text{Left eigen-vectors}} = \left[ \begin{array}{ccc} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right] \left[ \begin{array}{c} P \\ | \\ P^T \end{array} \right] \left[ \begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right] \left[ \begin{array}{c} P \\ | \\ P^T \end{array} \right] \left[ \begin{array}{ccc} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{array} \right]$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

Sum of  
rank-1  
matrices

**Dyadic Expansion**

**Shuffling eigenvalues and eigenvectors**

$$\left[ \begin{array}{c} A \\ | \\ V_i \\ | \end{array} \right] = \sum_i \left[ \begin{array}{c} | \\ V_i \\ | \end{array} \right] [\lambda_i] \left[ \begin{array}{ccc} - & W_i^* & - \end{array} \right]$$



Order of sum does not matter...

# Diagonalization (non-unique) case 1: ordering

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Permutation Matrix**  $P \in \mathbb{R}^{n \times n}$

**Diagonalization**

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

Ex.  $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$   $P^T P = I$

$$\left[ \begin{array}{c} A \\ | \\ V_1 & \cdots & V_n \\ | \end{array} \right] = \left[ \begin{array}{c|c|c} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right] \underbrace{\left[ \begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \underbrace{\left[ \begin{array}{ccc} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{array} \right]}_{\text{Left eigen-vectors}} = \left[ \begin{array}{c|c|c} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right] \left[ \begin{array}{c} P \\ | \\ P^T \end{array} \right] \left[ \begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right] \left[ \begin{array}{c} P \\ | \\ P^T \end{array} \right] \left[ \begin{array}{ccc} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{array} \right]$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

**Shuffling eigenvalues and eigenvectors**

$$\left[ \begin{array}{c} A \\ | \\ V_i \\ | \end{array} \right] = \sum_i \left[ \begin{array}{c} | \\ V_i \\ | \end{array} \right] [\lambda_i] \left[ \begin{array}{ccc} -W_i^* & - \end{array} \right]$$

Sum of rank-1 matrices  
**Dyadic Expansion**



Order of sum does not matter...

# Diagonalization (non-unique) case 1: ordering

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Permutation Matrix**  $P \in \mathbb{R}^{n \times n}$

**Diagonalization**

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Ex.  $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$   $P^T P = I$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

Sum of  
rank-1  
matrices

**Dyadic Expansion**

**Shuffling eigenvalues  
and eigenvectors**

$$\begin{bmatrix} A \\ | \\ V_i \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



Order of sum does not matter...

# Diagonalization (non-unique) case 1: ordering

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Permutation Matrix**  $P \in \mathbb{R}^{n \times n}$

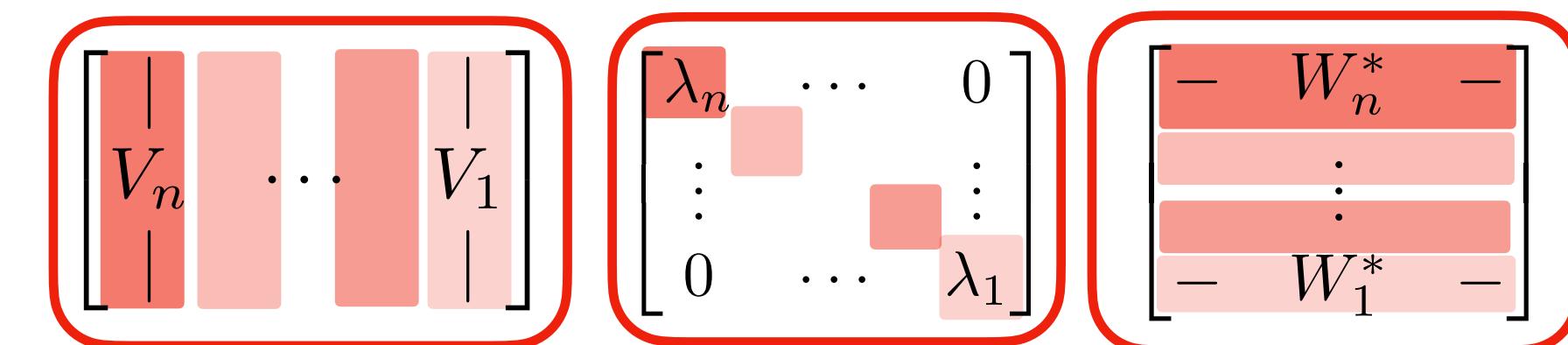
**Diagonalization**

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Ex.  $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$   $P^T P = I$



**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

Sum of  
rank-1  
matrices

**Dyadic Expansion**

**Shuffling eigenvalues  
and eigenvectors**

$$\begin{bmatrix} A \\ | \\ V_i \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



Order of sum does not matter...

# Diagonalization (non-unique) case 2: scaling

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Diagonalization**

diagonal matrices  
commute...

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 \frac{\gamma_1}{\gamma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \frac{\gamma_n}{\gamma_n} \end{bmatrix}}_{\text{Scaling eigenvectors}} \begin{bmatrix} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{bmatrix}$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

**Scaling eigenvectors**

$$\begin{bmatrix} A \\ | \\ V_i \\ | \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of  
rank-1  
matrices  
**Dyadic Expansion**



Order of sum does not matter...

# Diagonalization (non-unique) case 2: scaling

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Diagonalization**

diagonal matrices  
commute...

$$A = V D V^{-1}$$

$$\left[ \begin{array}{c|c} A & \\ \hline V_1 & \cdots & V_n \end{array} \right] = \underbrace{\left[ \begin{array}{c|c} | & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right]}_{\text{Right eigen-vectors}} \underbrace{\left[ \begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \underbrace{\left[ \begin{array}{ccc} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{array} \right]}_{\text{Left eigen-vectors}} =$$

$$\left[ \begin{array}{c|c} | & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right] \left[ \begin{array}{ccc} \gamma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \gamma_n \end{array} \right] \left[ \begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right] \left[ \begin{array}{ccc} \frac{1}{\gamma_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{\gamma_n} \end{array} \right] \left[ \begin{array}{ccc} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{array} \right]$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

**Scaling eigenvectors**

$$\left[ \begin{array}{c|c} A & \\ \hline V_i \end{array} \right] = \sum_i \left[ \begin{array}{c|c} | & | \\ V_i & \\ | & | \end{array} \right] [\lambda_i] \left[ \begin{array}{ccc} - & W_i^* & - \end{array} \right]$$

Sum of  
rank-1  
matrices  
**Dyadic Expansion**



Order of sum does not matter...

# Diagonalization (non-unique) case 2: scaling

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Diagonalization**

diagonal matrices  
commute...

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} | & & | \\ V_1\gamma_1 & \cdots & V_n\gamma_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & \frac{1}{\gamma_1}W_1^* & - \\ | & \vdots & | \\ - & \frac{1}{\gamma_n}W_n^* & - \end{bmatrix}}_{\text{Scaling eigenvectors}}$$

$$V'$$

$$V'^{-1}$$

**Right eigen-vectors**

**Eigen-values**  
(on diagonal)

**Left eigen-vectors**

Sum of  
rank-1  
matrices

**Dyadic Expansion**

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



Order of sum does not matter...

# Spectral Mapping Theorem

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Diagonalization**  $A = VDV^{-1}$   $A^k = VD^kV^{-1}$

**Powers of A** 
$$\begin{aligned} A^k &= VDV^{-1} \times VDV^{-1} \times \dots \times VDV^{-1} \\ &= VDV^{-1}VDV^{-1} \dots VDV^{-1} \\ &= VD^kV^{-1} \end{aligned} \quad = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

## Polynomials of A

polynomial  $\Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \dots + \alpha_1 s + \alpha_0 1$   $\Psi(A) = V\Psi(D)V^{-1}$

plugging in A...

$$\begin{aligned} \Psi(A) &= \alpha_k A^k + \alpha_{k-1} A^{k-1} + \alpha_{k-2} A^{k-2} + \dots + \alpha_1 A + \alpha_0 I \\ &= \alpha_k VD^kV^{-1} + \alpha_{k-1} VD^{k-1}V^{-1} + \alpha_{k-2} VD^{k-2}V^{-1} + \dots + \alpha_1 VDV^{-1} + \alpha_0 VV^{-1} \\ &= V(\alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \dots + \alpha_1 D + \alpha_0 I)V^{-1} \\ &= \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \Psi(\lambda_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Psi(\lambda_n) \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix} \end{aligned}$$

# Spectral Mapping Theorem

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Diagonalization**  $A = VDV^{-1}$   $A^k = VD^kV^{-1}$

**Powers of A** 
$$\begin{aligned} A^k &= VDV^{-1} \times VDV^{-1} \times \dots \times VDV^{-1} \\ &= VDV^{-1}VDV^{-1} \dots VDV^{-1} \\ &= VD^kV^{-1} \end{aligned} \quad = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

## Polynomials of A

polynomial  $\Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \dots + \alpha_1 s + \alpha_0 1$

$$\Psi(A) = V \left( \alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \dots + \alpha_1 D + \alpha_0 I \right) V^{-1}$$

**Spectral Mapping Theorem** for  $f(s)$  analytic

$$\lambda \in \text{eig}(A) \rightarrow f(\lambda) \in \text{eig}(f(A))$$

$A, f(A)$  have the same eigenvectors

$$\begin{aligned} \Psi(A) &= V \Psi(D) V^{-1} \\ &= \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \Psi(\lambda_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Psi(\lambda_n) \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix} \end{aligned}$$

# Spectral Mapping Theorem

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Diagonalization**  $A = VDV^{-1}$

$$A^k = VD^kV^{-1}$$

$$\begin{aligned}\textbf{Powers of A} \quad A^k &= VDV^{-1} \times VDV^{-1} \times \dots \times VDV^{-1} \\ &= VDV^{-1}VDV^{-1} \dots VDV^{-1} \\ &= VD^kV^{-1}\end{aligned}$$

**Polynomials of A**

$$\text{polynomial} \quad \Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \dots + \alpha_1 s + \alpha_0 1$$

$$\Psi(A) = V \left( \alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \dots + \alpha_1 D + \alpha_0 I \right) V^{-1}$$

**Spectral Mapping Theorem** for  $f(s)$  analytic

$$\lambda \in \text{eig}(A) \rightarrow f(\lambda) \in \text{eig}(f(A))$$

$A, f(A)$  have the same eigenvectors

$$= \begin{bmatrix} V \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

**Specific Useful Case: Matrix Exponential**

$$\begin{aligned}e^{At} &= I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k\end{aligned}$$

$$\textbf{Derivative: } \frac{d}{dt} (e^{At}) = Ae^{At}$$

- can see from polynomial definition
- related to definition of e

# Spectral Mapping Theorem

Square matrix:  $A \in \mathbb{R}^{n \times n}$  assume  $\text{char}_A(s) = \det(sI - A)$  has **n distinct roots**

**Eigenvalues:**  $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

**Diagonalization**  $A = VDV^{-1}$

$$A^k = VD^kV^{-1}$$

$$\begin{aligned}\textbf{Powers of A} \quad A^k &= VDV^{-1} \times VDV^{-1} \times \dots \times VDV^{-1} \\ &= VDV^{-1}VDV^{-1} \dots VDV^{-1} \\ &= VD^kV^{-1}\end{aligned}$$

**Polynomials of A**

$$\text{polynomial} \quad \Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \dots + \alpha_1 s + \alpha_0 1$$

$$\Psi(A) = V(\alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \dots + \alpha_1 D + \alpha_0 I)V^{-1}$$

**Spectral Mapping Theorem** for  $f(s)$  analytic

$$\lambda \in \text{eig}(A) \rightarrow f(\lambda) \in \text{eig}(f(A))$$

$A, f(A)$  have the same eigenvectors

$$= \begin{bmatrix} V \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

**Specific Useful Case: Matrix Exponential**

$$\begin{aligned}e^{At} &= I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k\end{aligned}$$

$$\begin{aligned}e^{At} &= Ve^{Dt}V^{-1} \\ &= \begin{bmatrix} V \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}\end{aligned}$$