

# Introduction

**Dan Calderone**

[djcal@uw.edu](mailto:djcal@uw.edu)

[danjcalderone.github.io](https://danjcalderone.github.io)

Post-doctoral scholar (Prof. Ratliff's group)  
University of Washington

PhD: Berkeley, (under Shankar Sastry, 2017)

PostDoc: in AA & EE at UW (Ratliff, Ackimese, 2018-2019)

Lecturer: AA & EE at UW (2019-2022)



## Research Interests:

Game theory & optimization  
applied to transportation networks

## Personal Interests: Math visualization

[danjcalderone.github.io/teaching](https://danjcalderone.github.io/teaching)

[danjcalderone.github.io/dcmath](https://danjcalderone.github.io/dcmath)

# Introduction

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Game theory & optimization  
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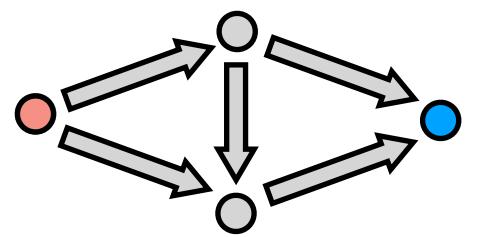
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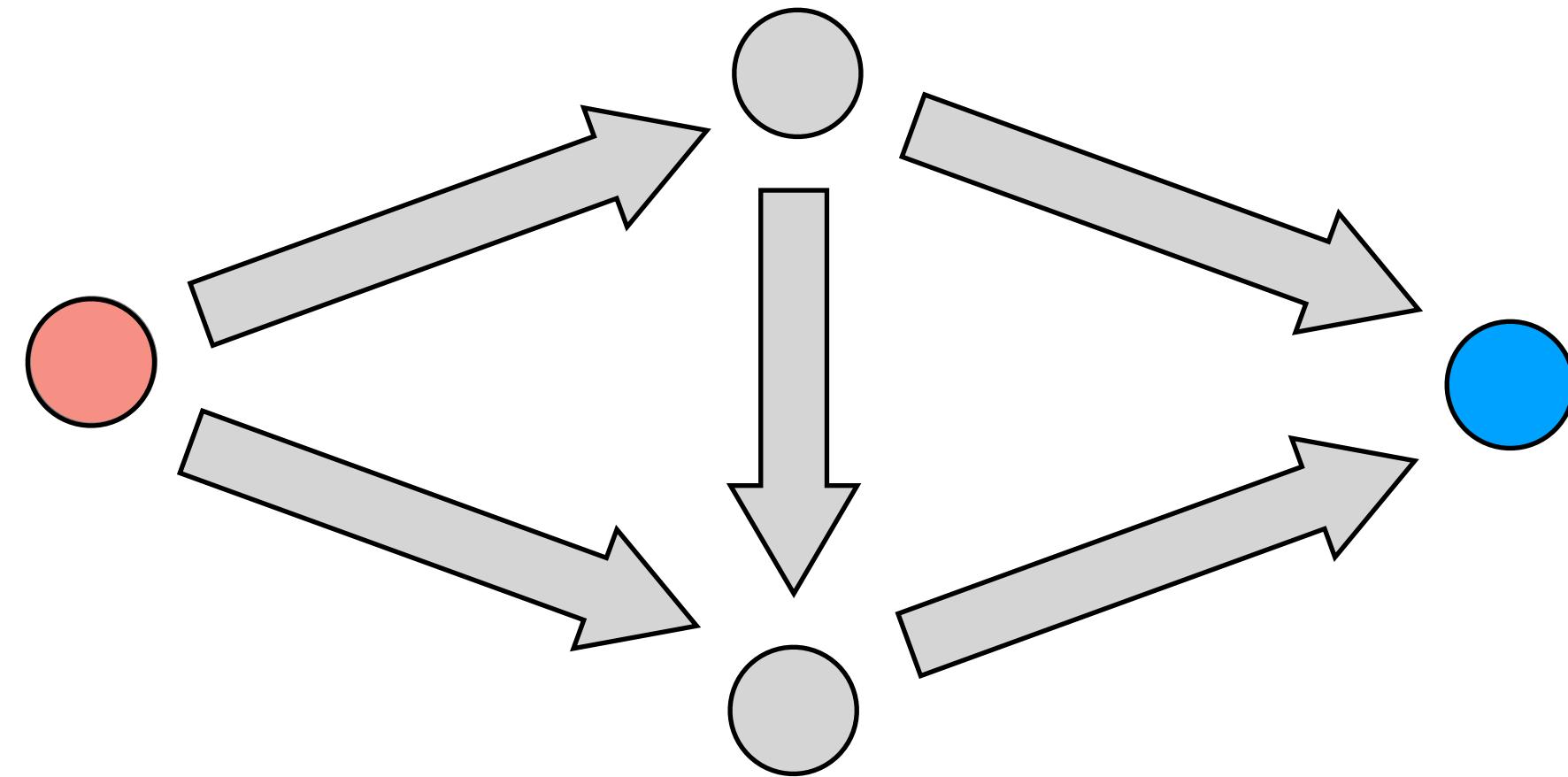
[danjcalderone.github.io/dcmath](https://danjcalderone.github.io/dcmath)

## Potential Games

Routing  
Games



## Routing Games

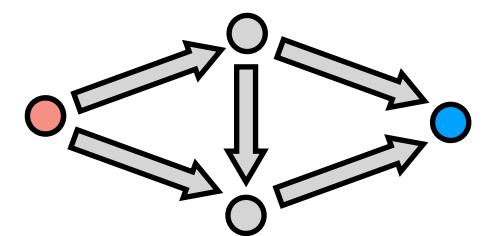


$x$  : edge traffic

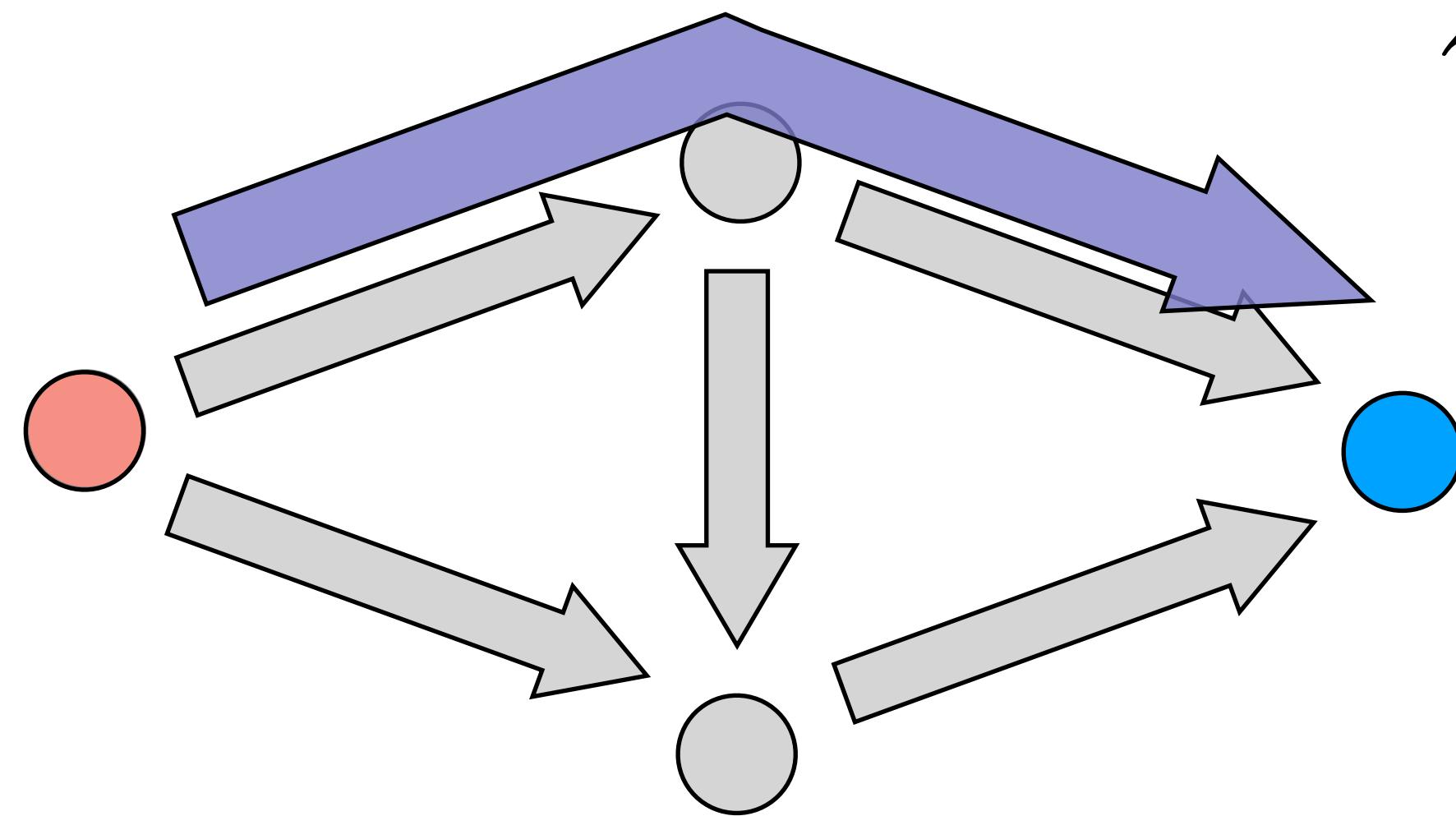
$z$  : route traffic

## Potential Games

Routing  
Games



## Routing Games

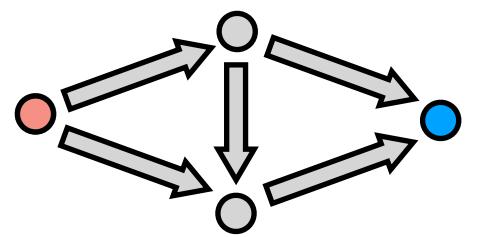


$x$  : edge traffic

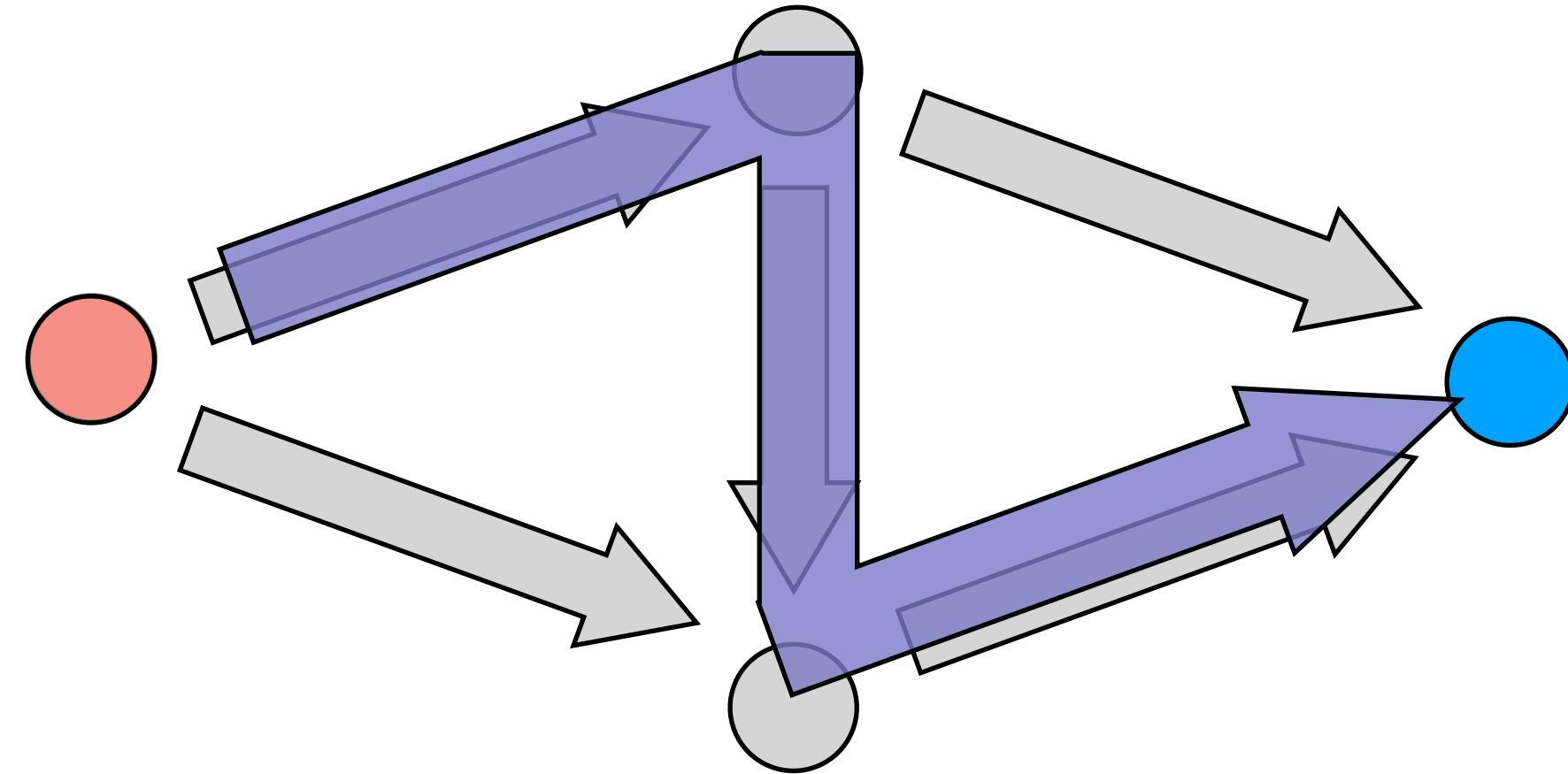
$z$  : route traffic

## Potential Games

Routing  
Games



## Routing Games

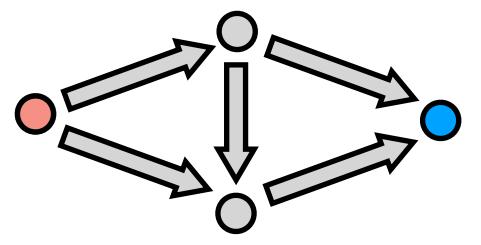


$x$  : edge traffic

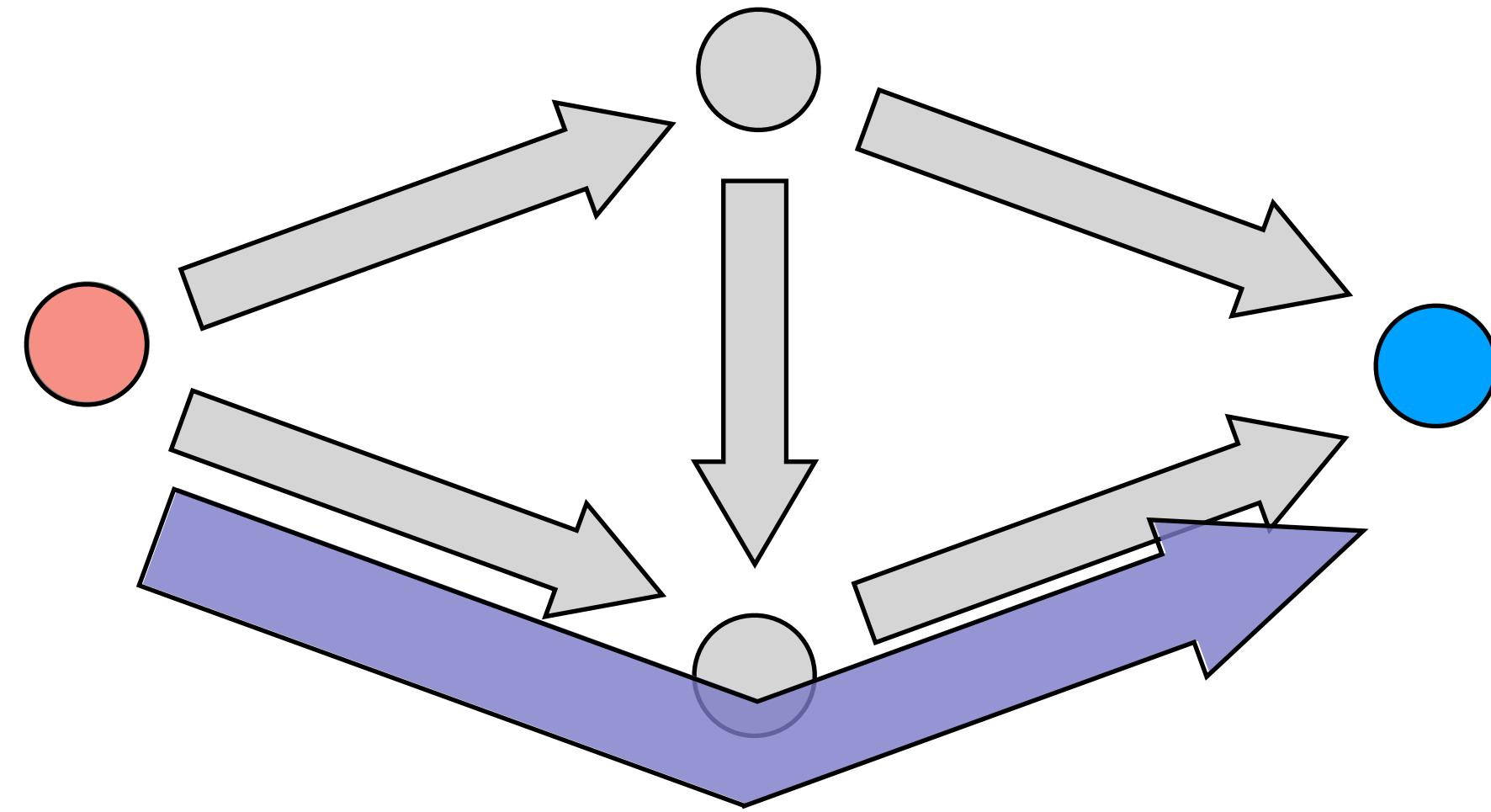
$z$  : route traffic

## Potential Games

Routing  
Games



## Routing Games

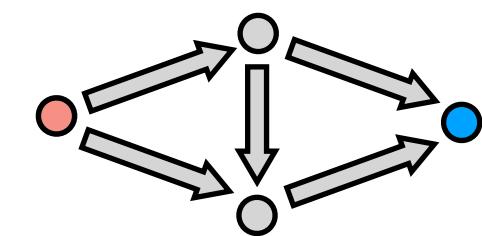


$x$  : edge traffic

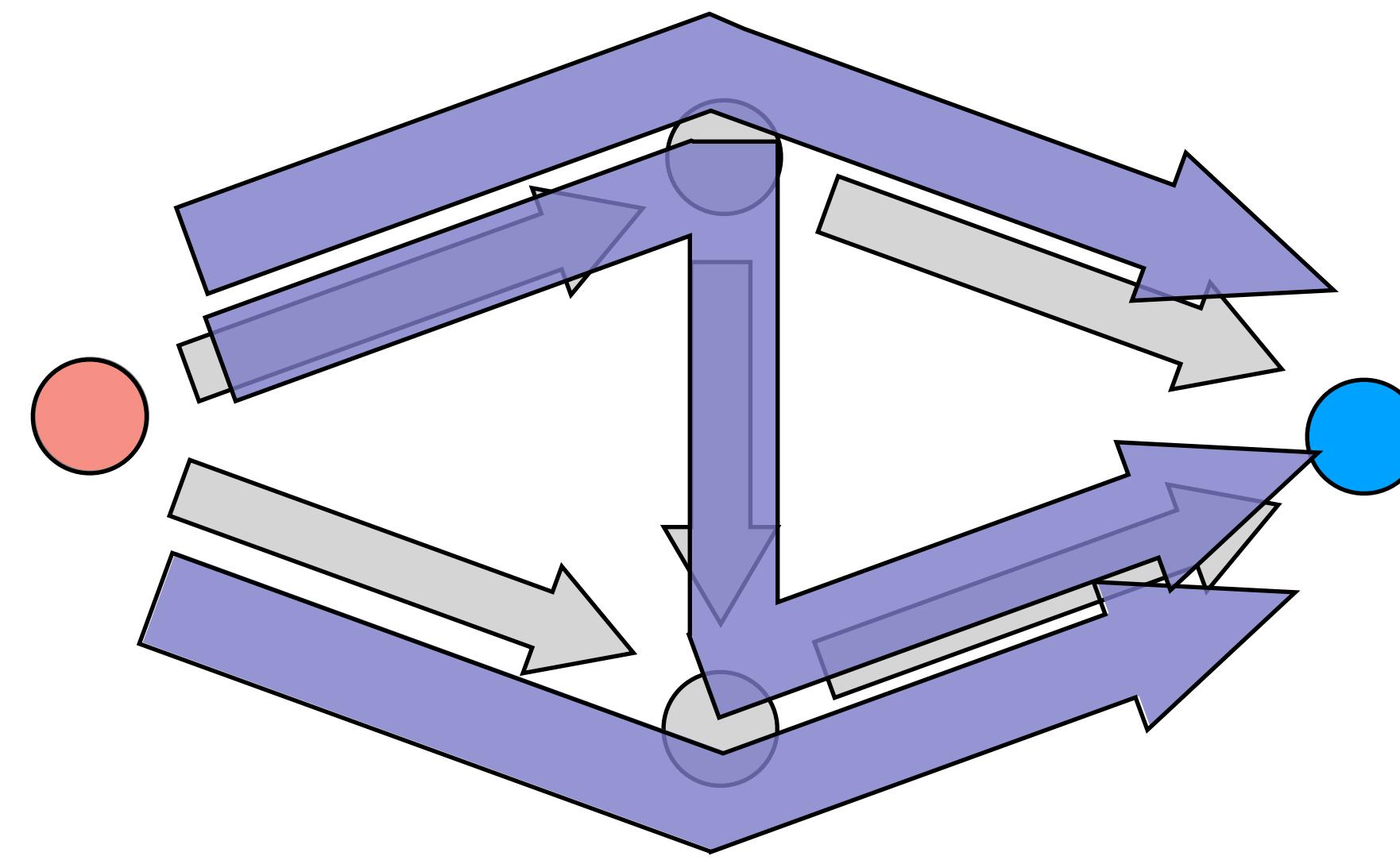
$z$  : route traffic

## Potential Games

Routing  
Games



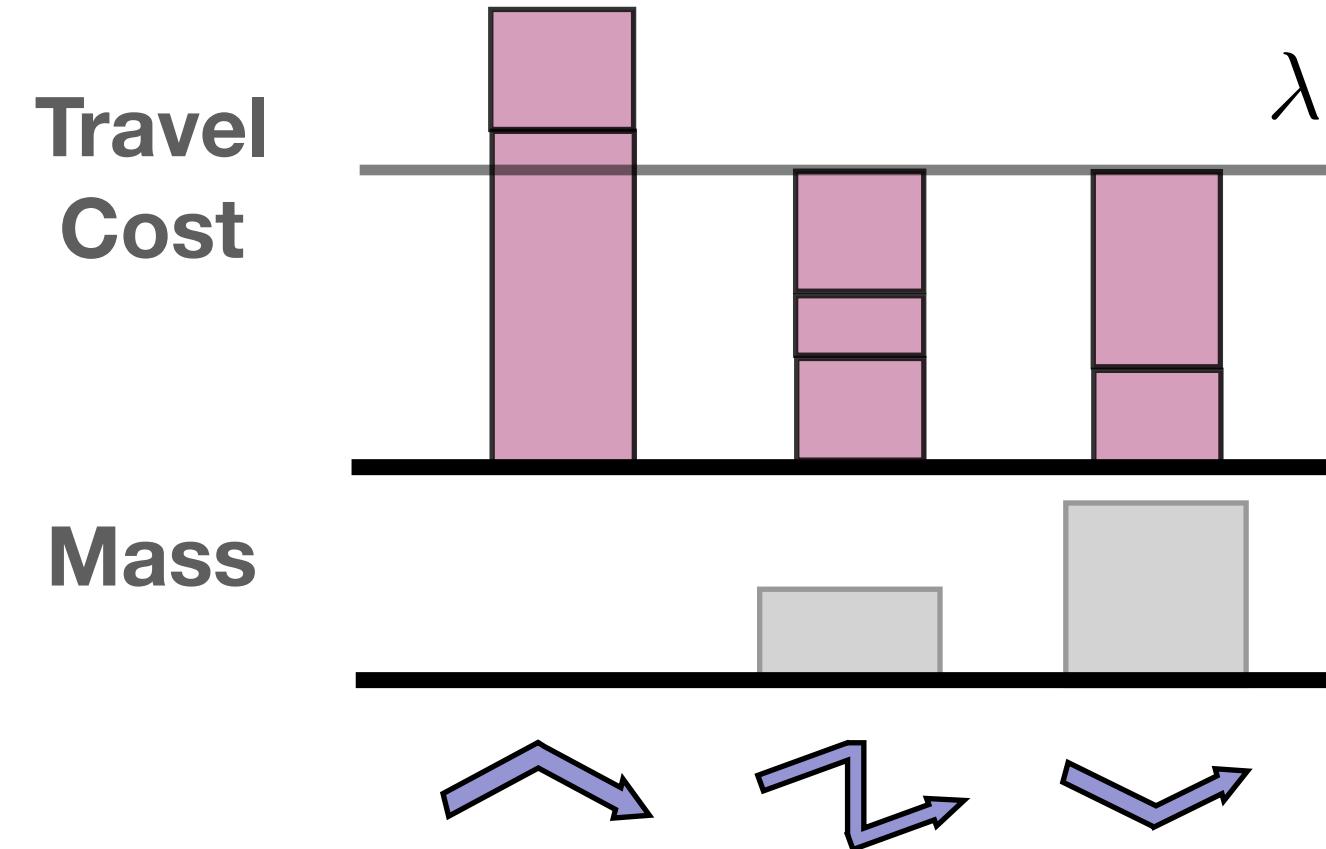
## Routing Games



$x$  : edge traffic

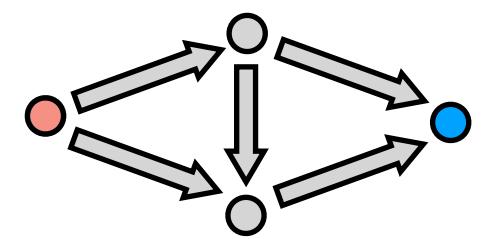
$z$  : route traffic

## Wardrop Equilibrium



## Potential Games

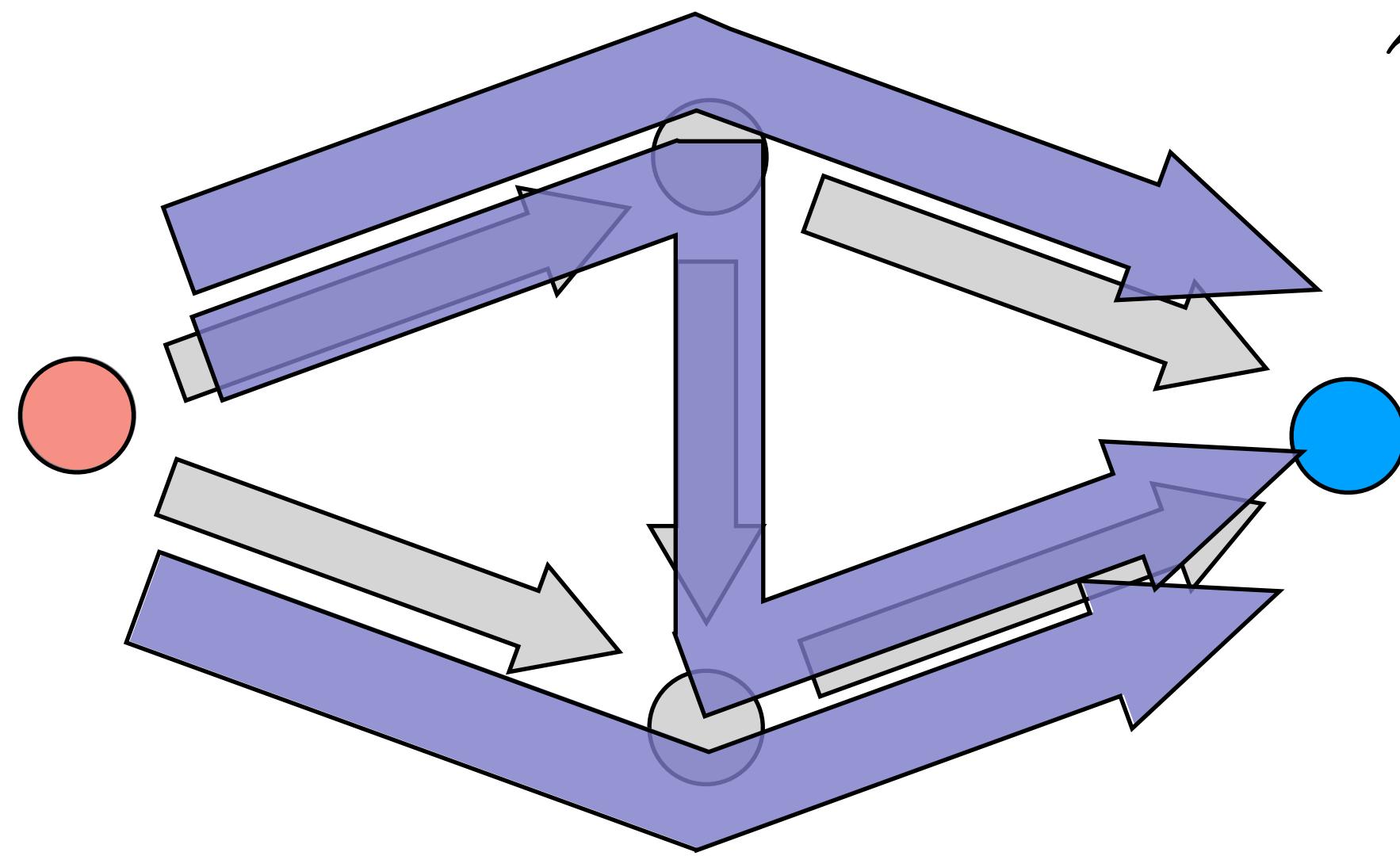
Routing  
Games



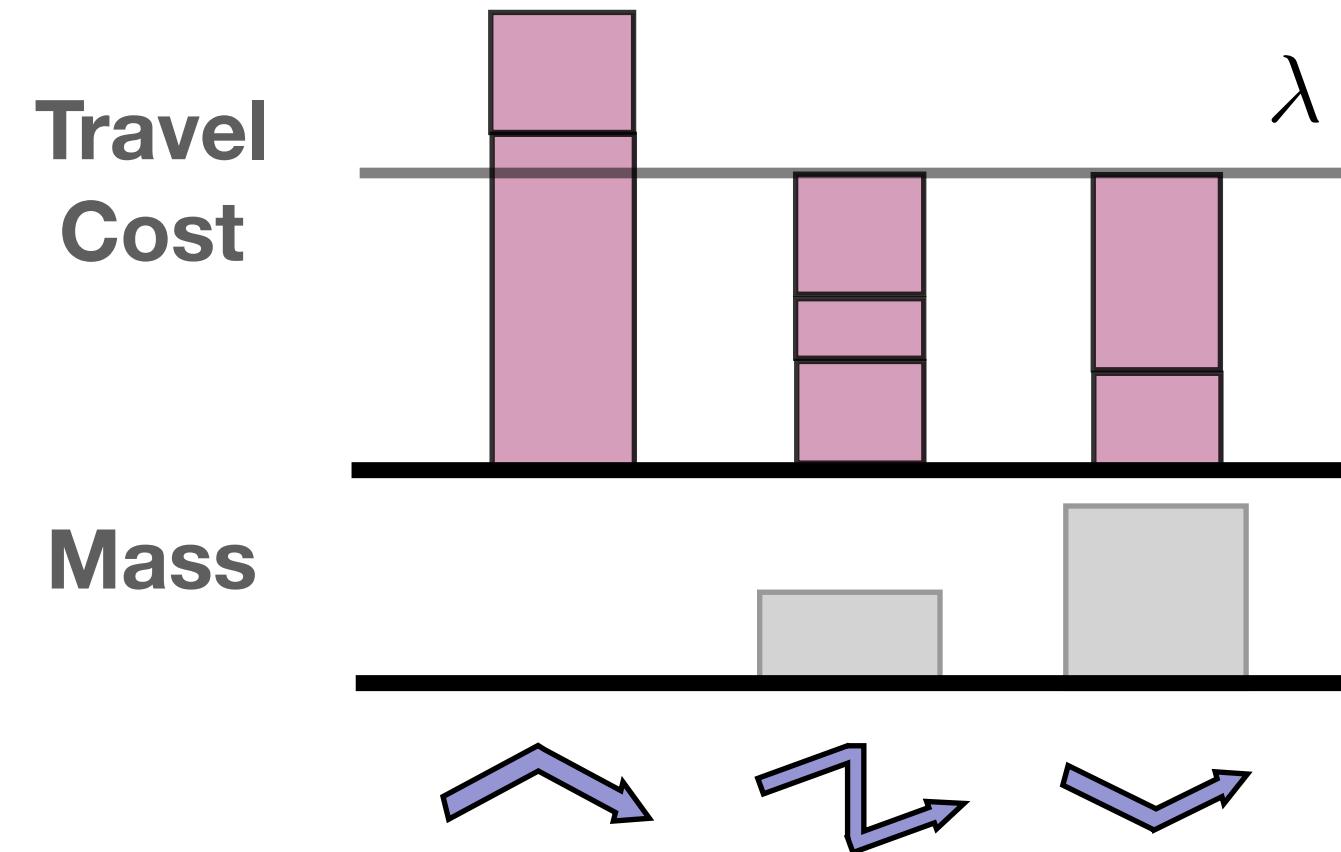
Potential  
Function

$$F(x)$$

## Routing Games



## Wardrop Equilibrium

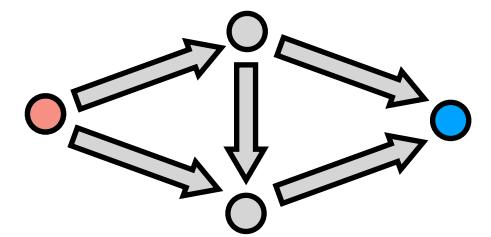


$x$  : edge traffic

$z$  : route traffic

## Potential Games

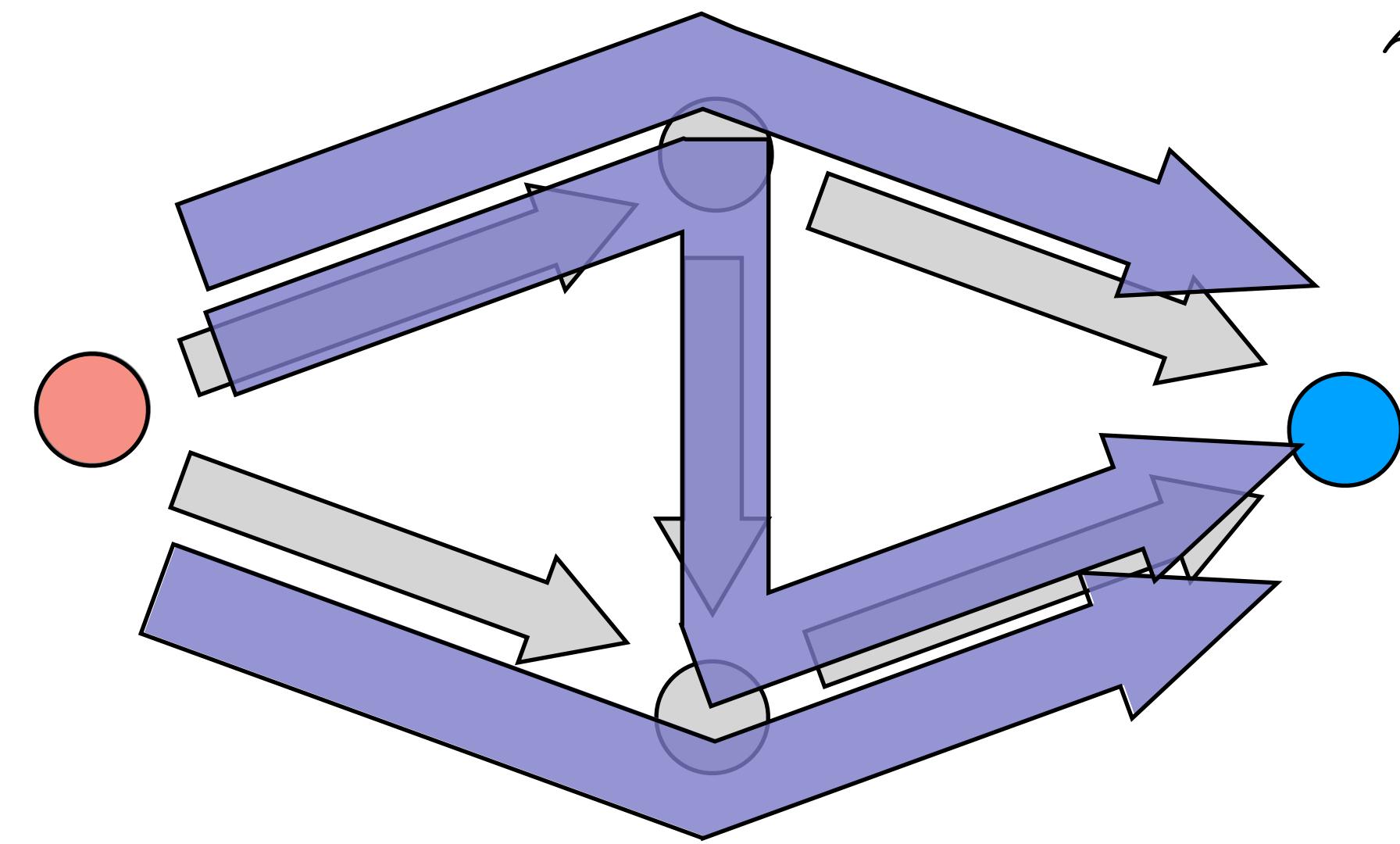
Routing  
Games



Potential  
Function

$$F(x)$$

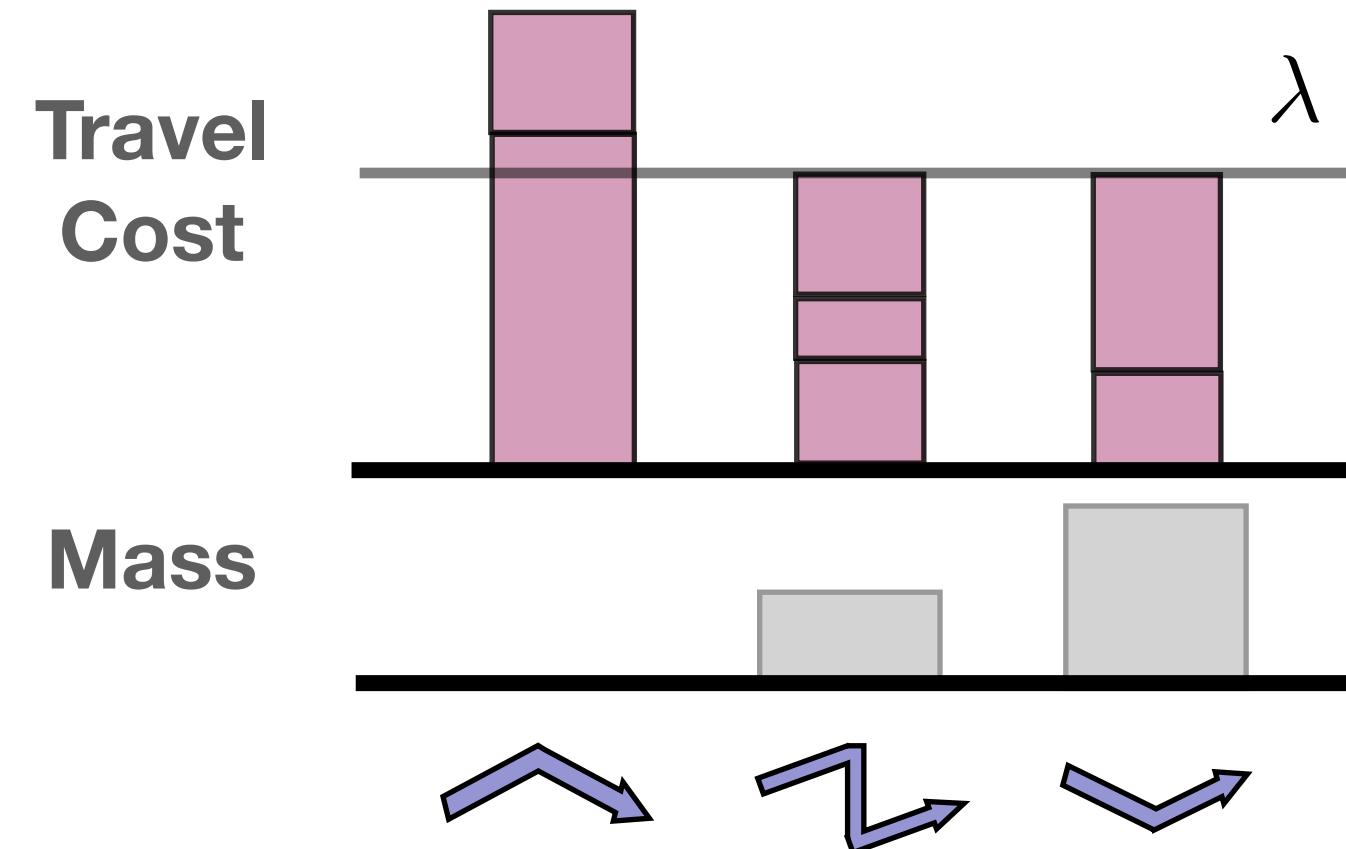
## Routing Games



$x$  : edge traffic

$z$  : route traffic

## Wardrop Equilibrium



$$\min_{x,z} \quad F(x)$$

$$\text{s.t.} \quad 1^T z = m \quad z \geq 0$$

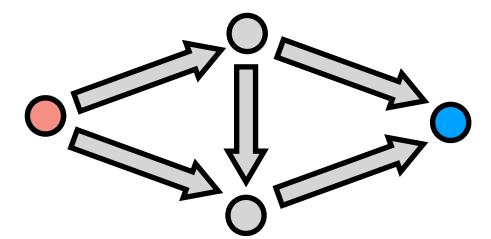
$$x = Rz$$

Mass conservation

Edges in routes

## Potential Games

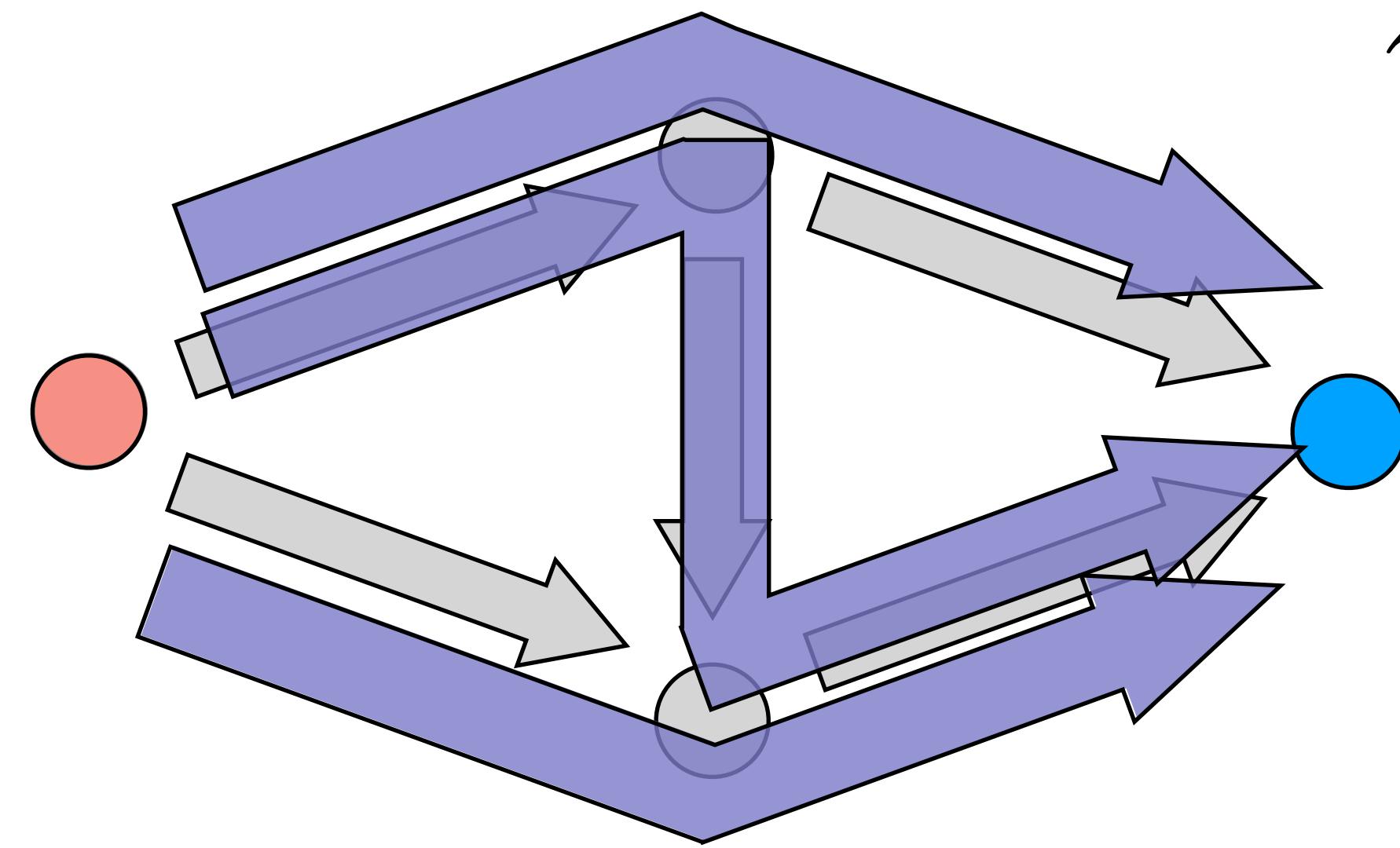
Routing  
Games



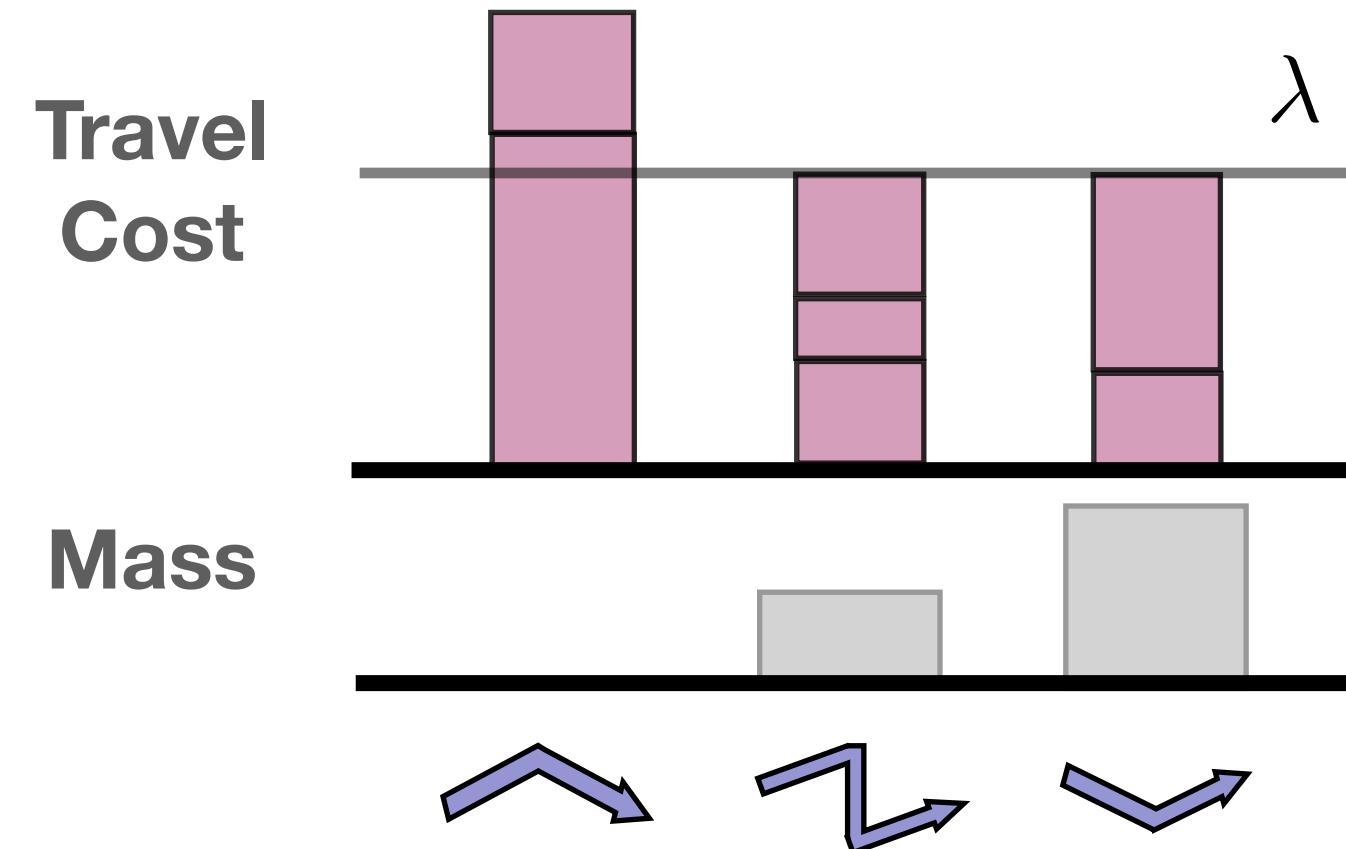
Potential  
Function

$$F(x)$$

## Routing Games



## Wardrop Equilibrium



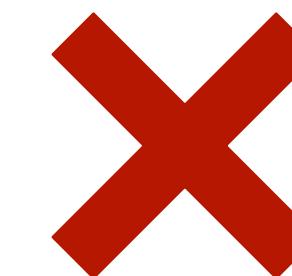
$$\min_{x,z}$$

$$F(x)$$

s.t.

$$1^T z = m \quad z \geq 0$$

$$x = Rz$$



Mass conservation

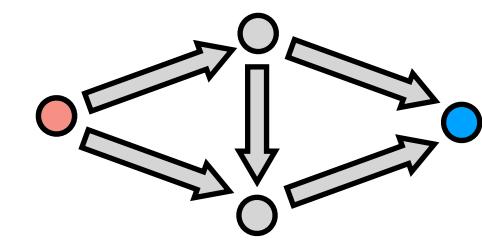
Edges in routes

$x$  : edge traffic

$z$  : route traffic

## Potential Games

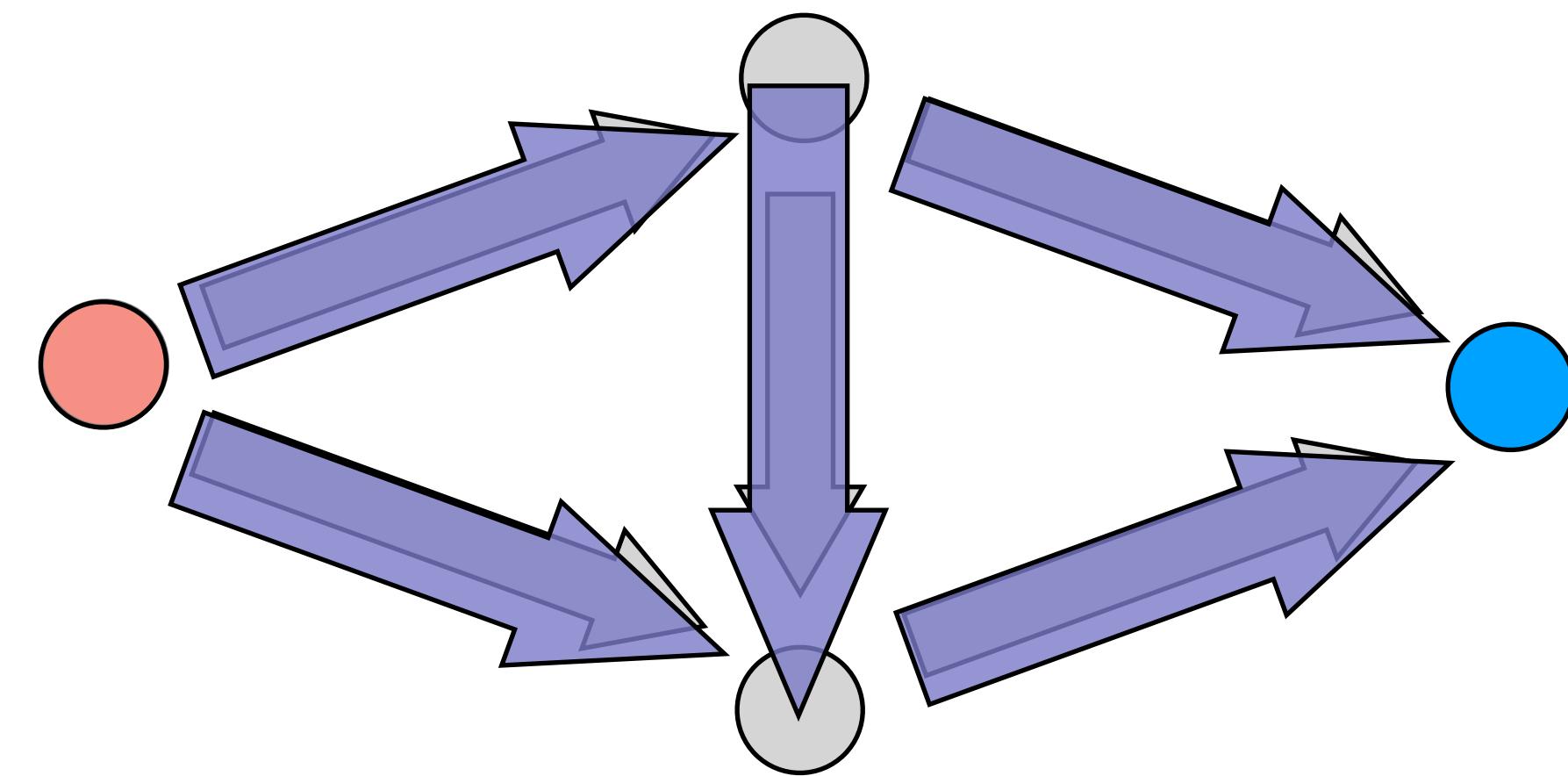
Routing  
Games



Potential  
Function

$$F(x)$$

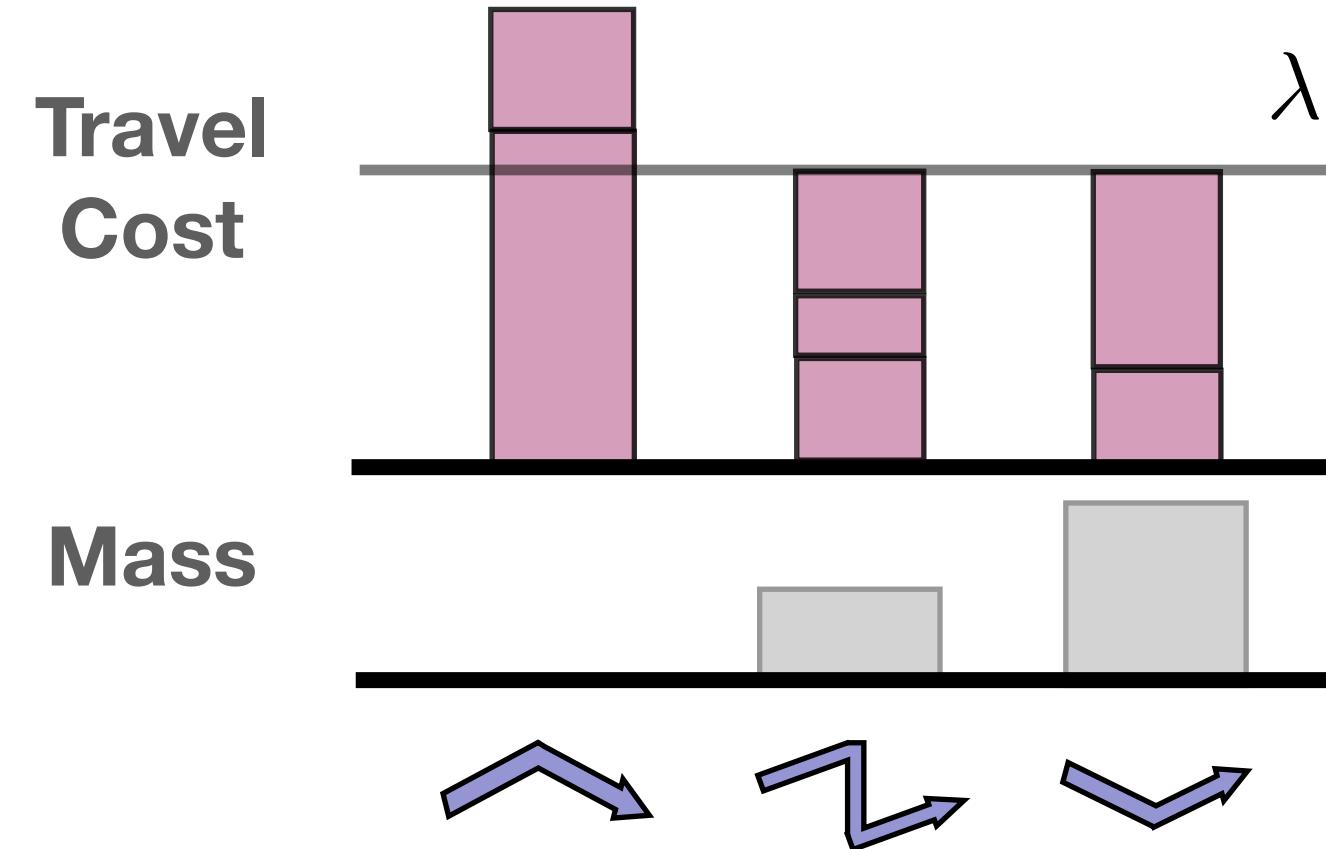
## Routing Games



$x$  : edge traffic

$z$  : route traffic

## Wardrop Equilibrium



$$\min_x F(x)$$

$$\text{s.t. } Ex = Sm, \quad x \geq 0$$

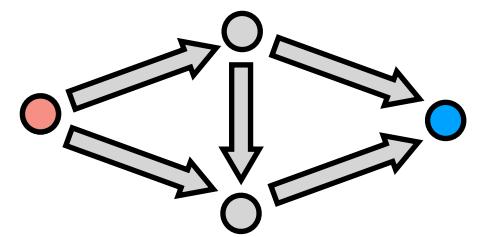
Mass conservation

Graph  
structure

Origin-  
destination

## Potential Games

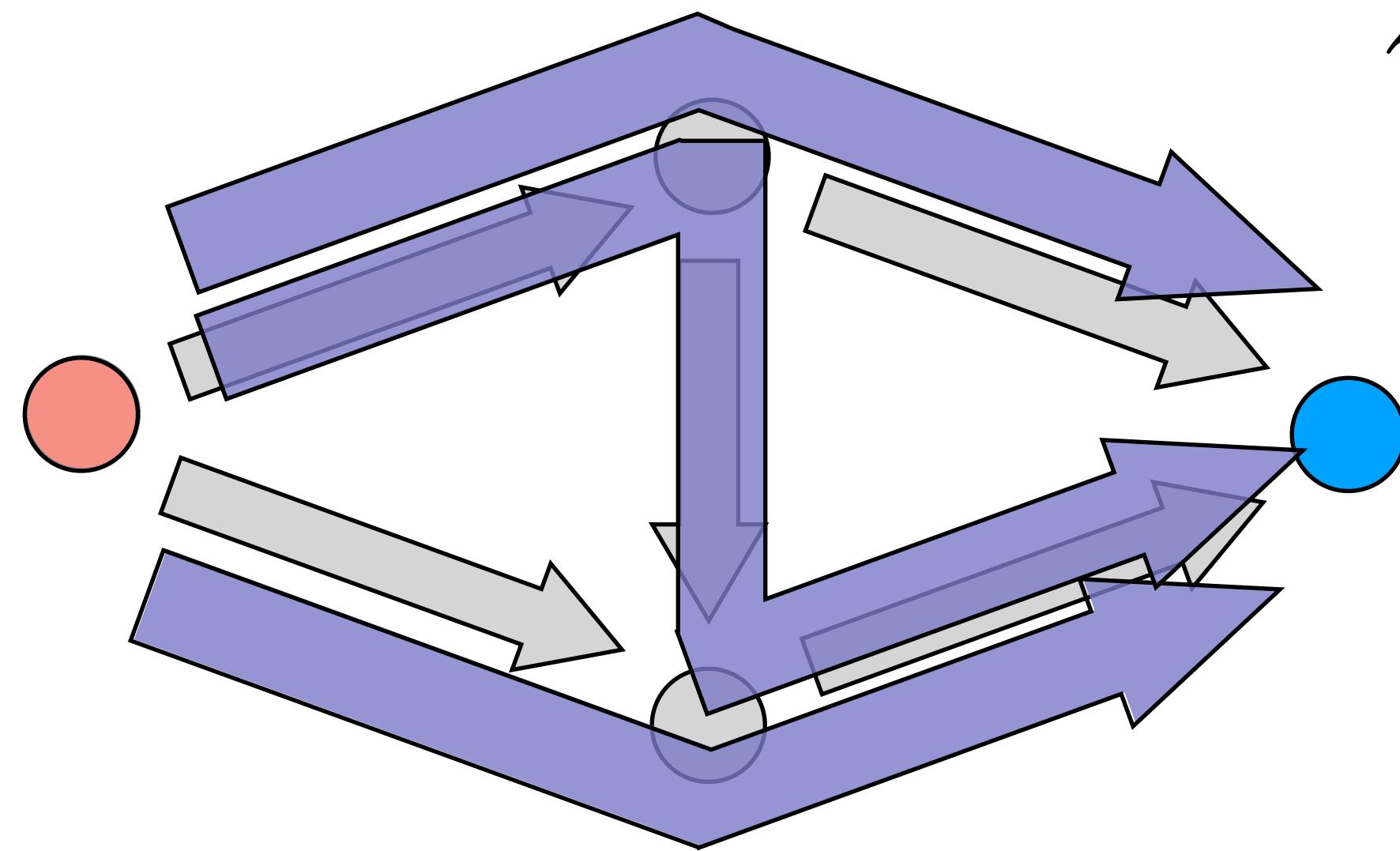
Routing  
Games



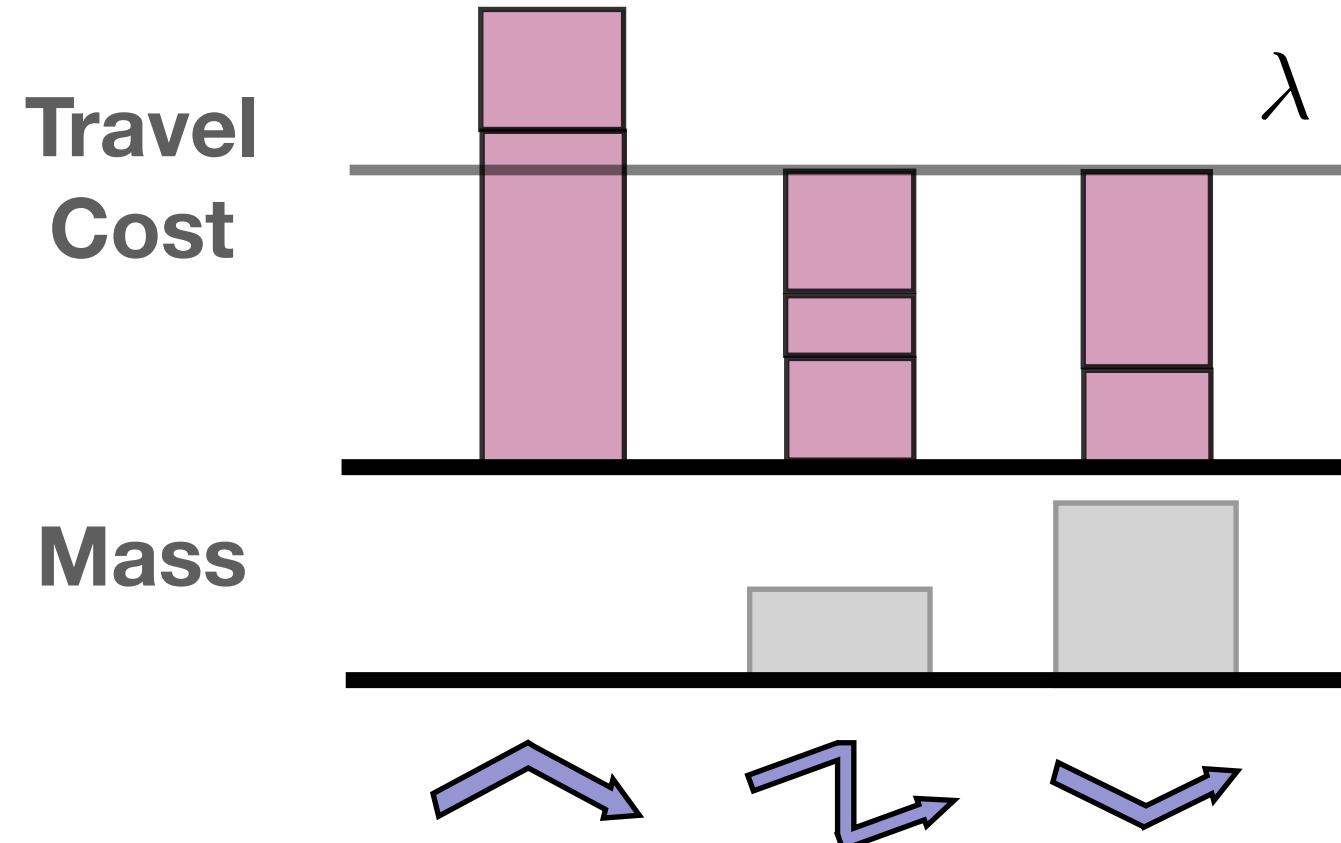
Potential  
Function

$$F(x)$$

## Routing Games



## Wardrop Equilibrium



$$\min_{x,z} F(x)$$

s.t.

$$1^T z = m \quad (\lambda)$$

$$z \geq 0 \quad (\nu)$$

$$x = \mathbf{R}z \quad (w)$$

$x$  : edge traffic

$z$  : route traffic

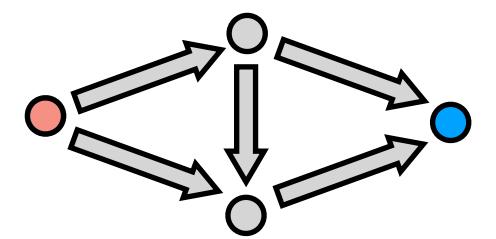
$\lambda$  : travel cost

$\nu$  : route inefficiency

$w$  : edge costs

## Potential Games

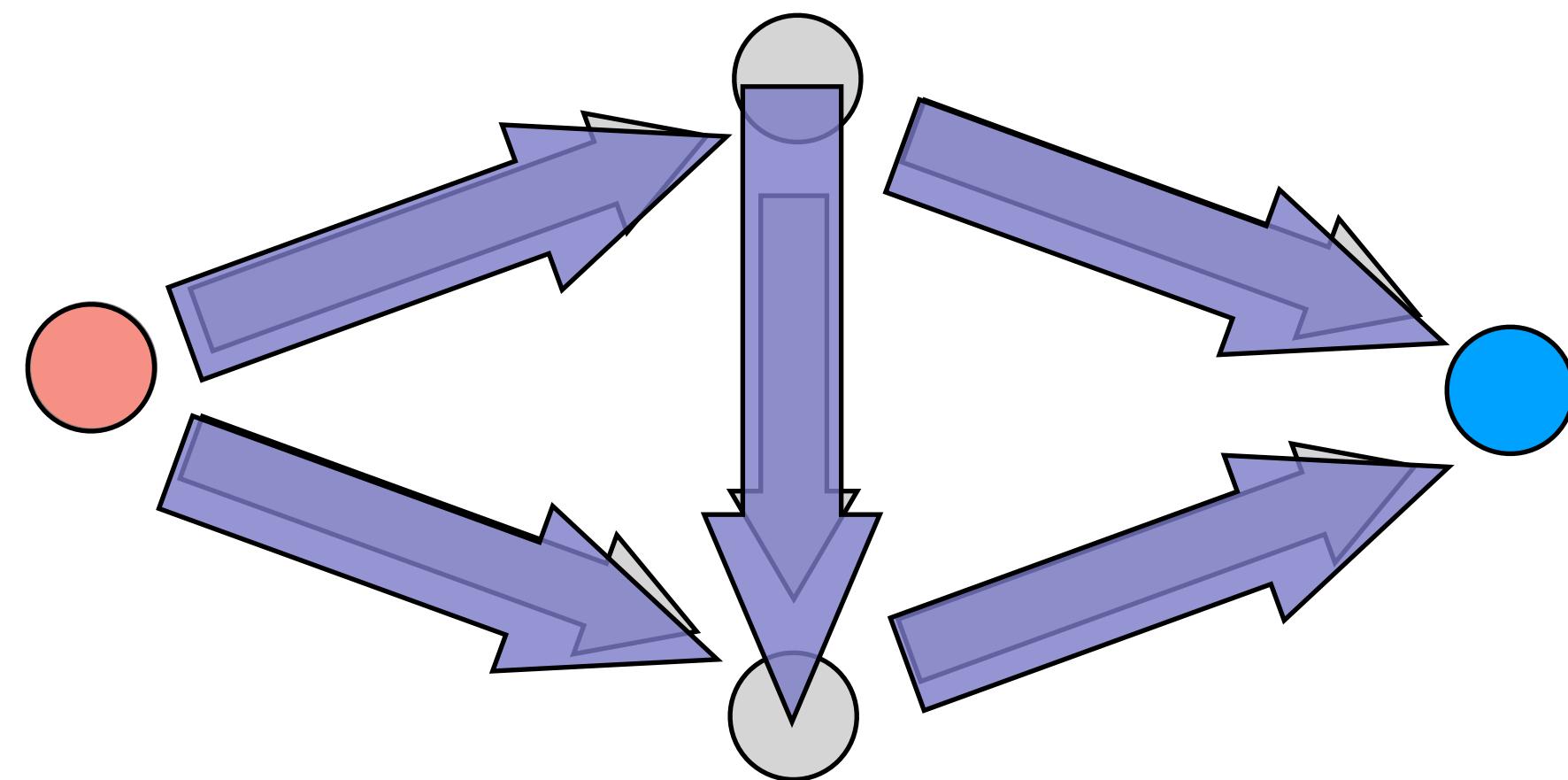
Routing  
Games



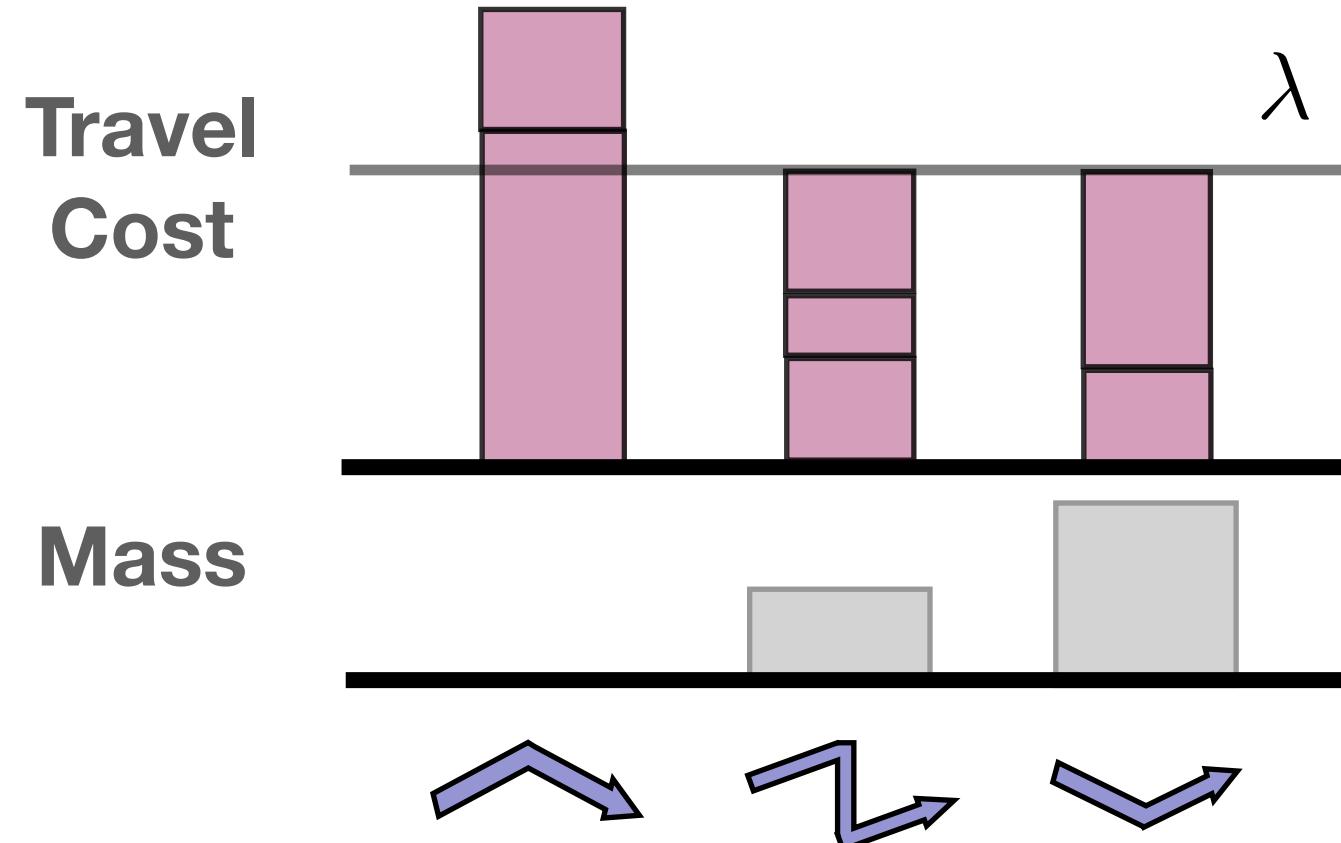
Potential  
Function

$$F(x)$$

## Routing Games



## Wardrop Equilibrium



$$\min_x F(x)$$

s.t.

$$Ex = Sm, \quad v$$

$$x \geq 0 \quad \mu$$

$x$  : edge traffic

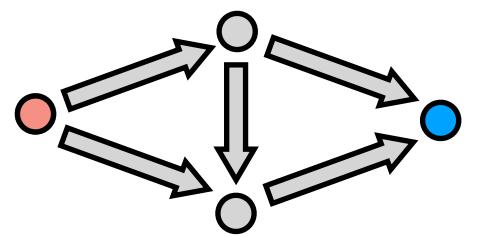
$z$  : route traffic

$\mu$  : edge inefficiency

$v$  : value function

## Potential Games

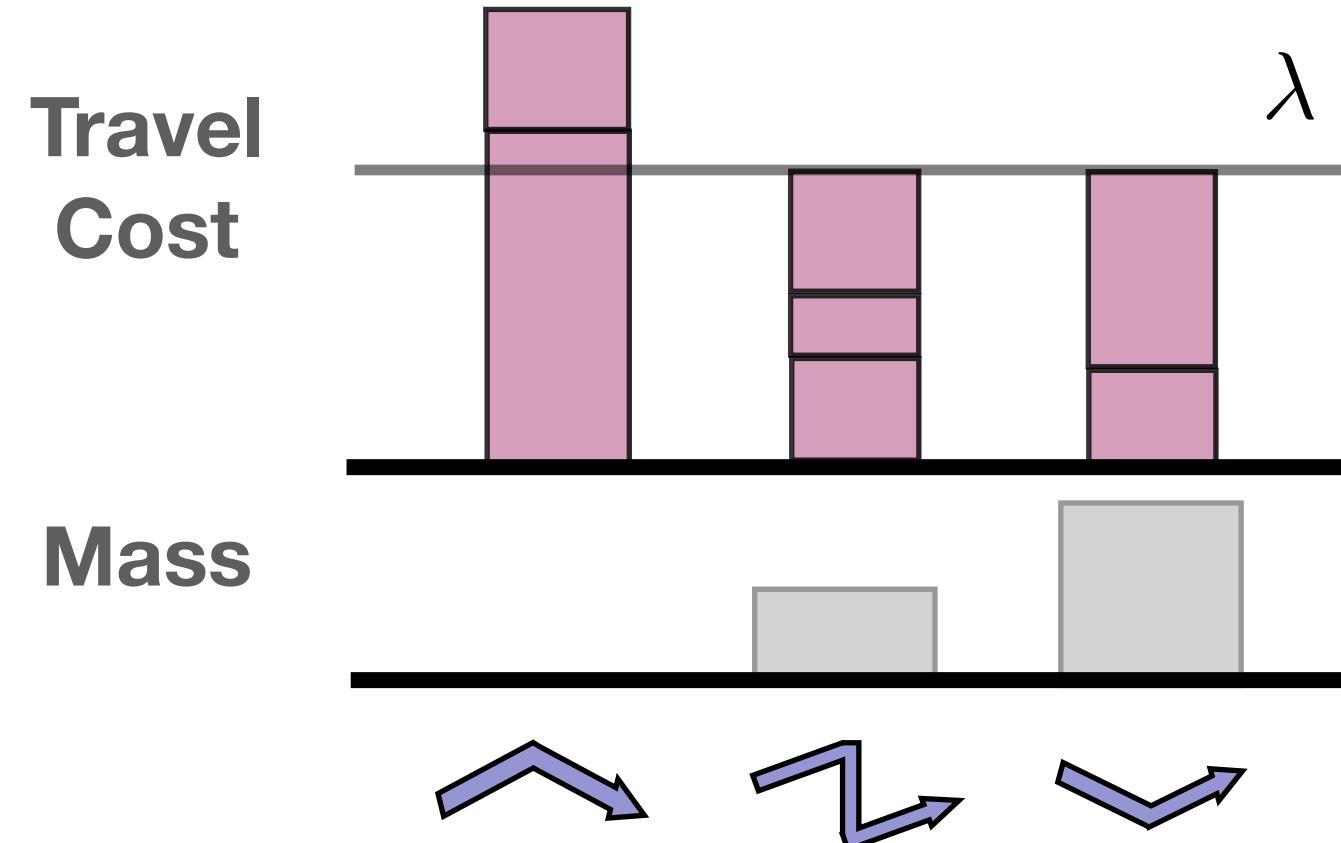
Routing  
Games



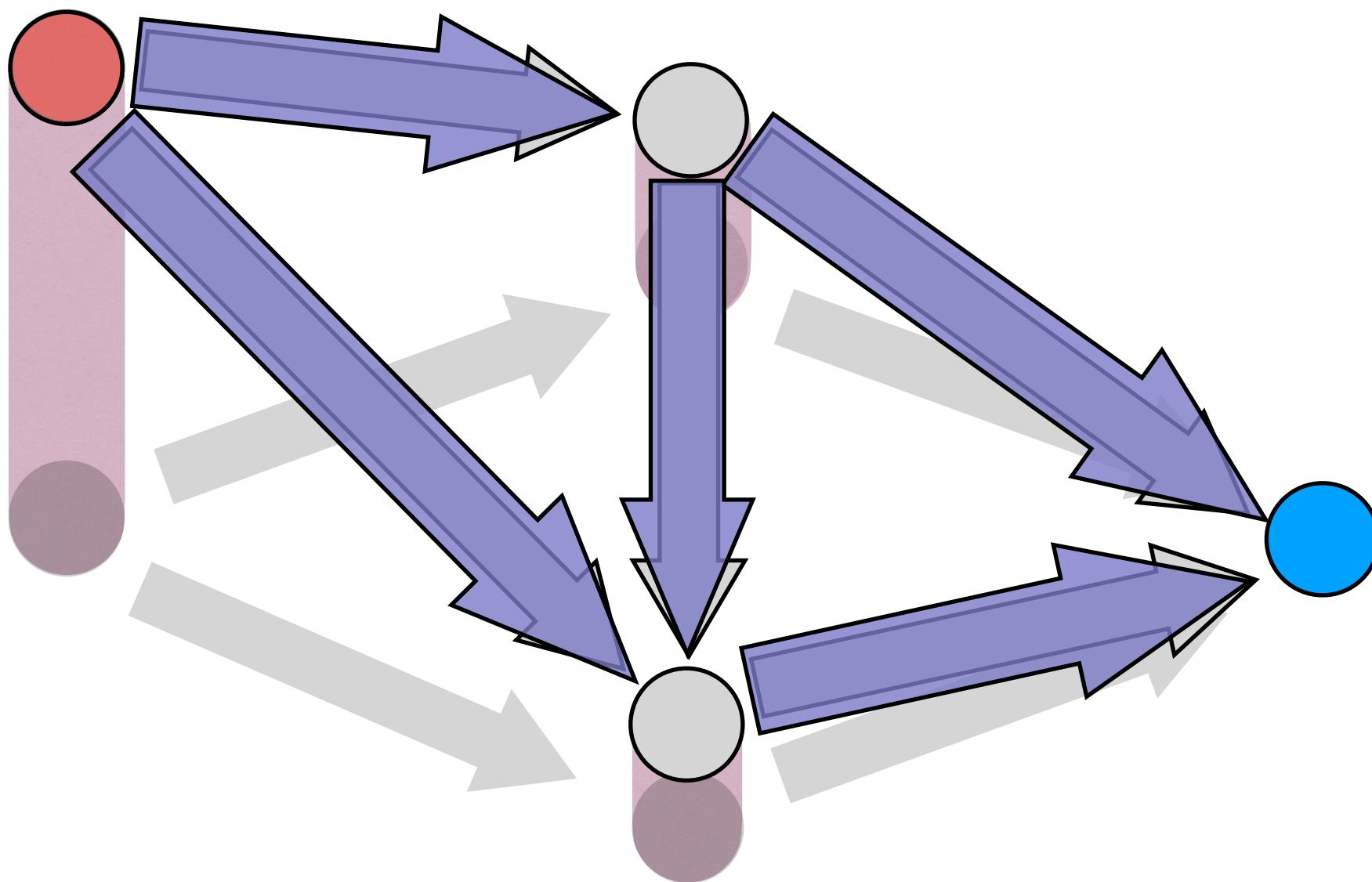
Potential  
Function

$$F(x)$$

## Wardrop Equilibrium



## Routing Games



$$\min_x F(x)$$

s.t.

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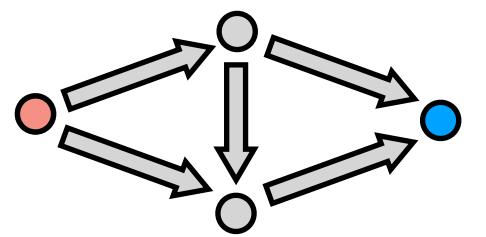
$$x \geq 0 \quad \mu$$

$\mu$  : edge inefficiency

$v$  : value function

## Potential Games

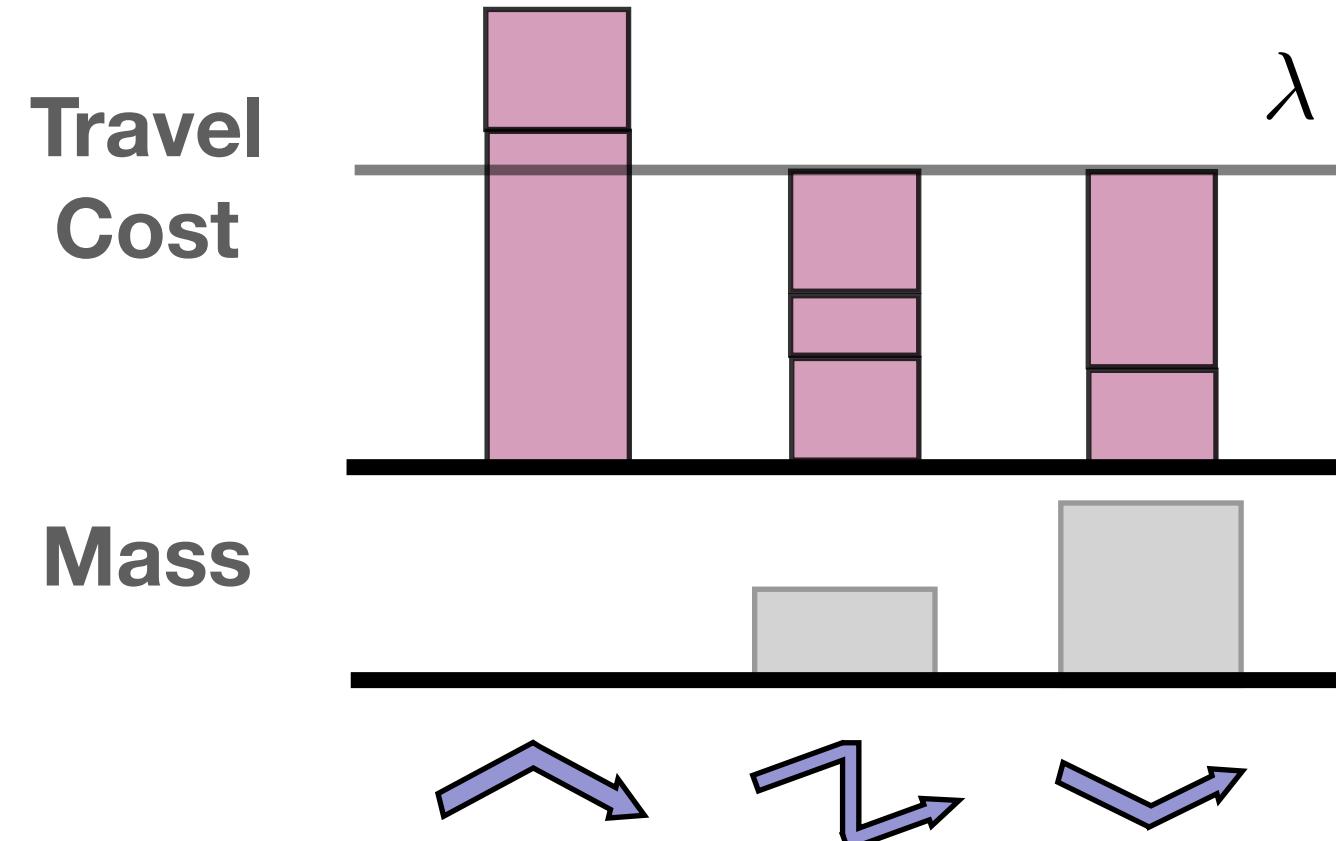
Routing  
Games



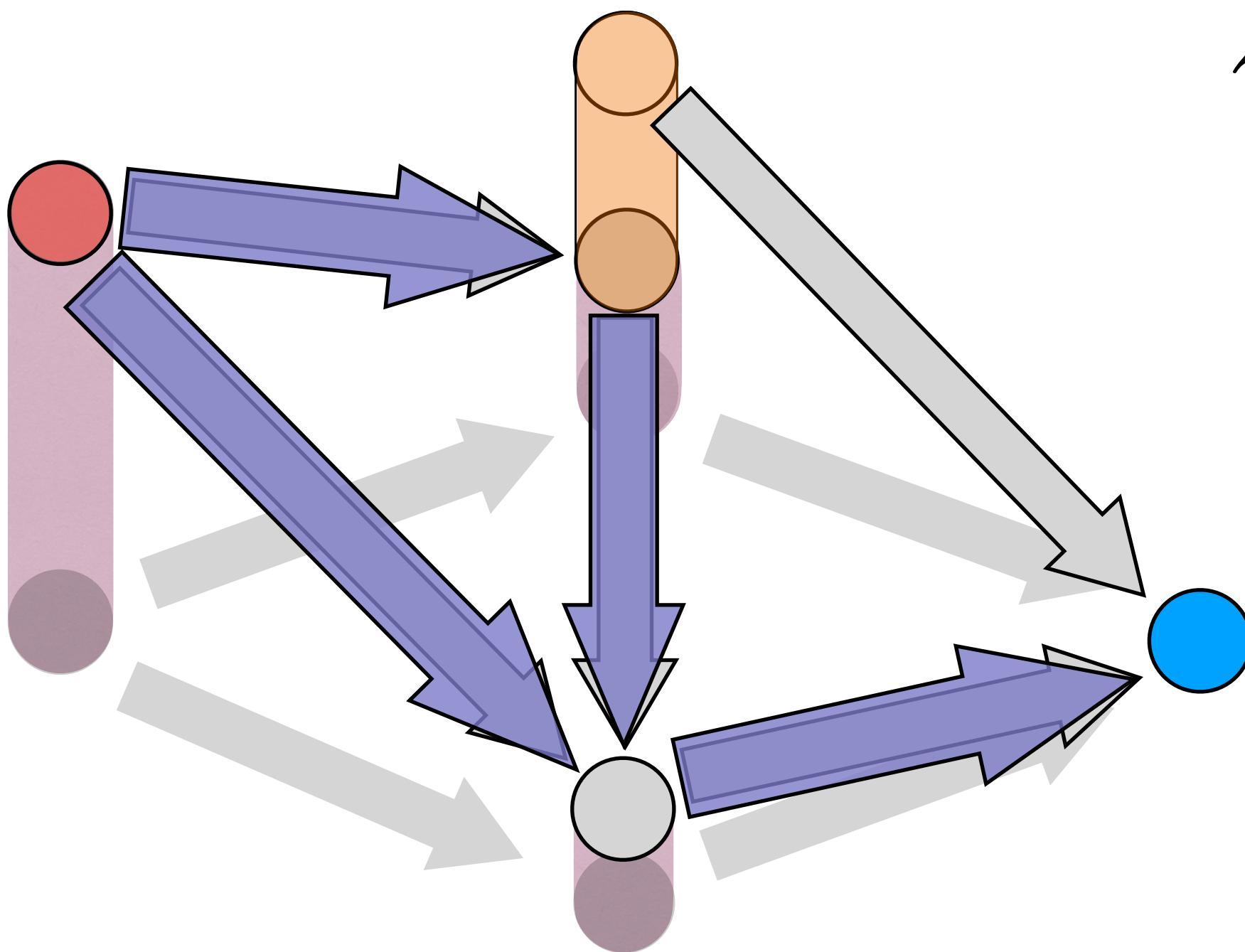
Potential  
Function

$$F(x)$$

Wardrop Equilibrium



## Routing Games



$$\min_x F(x)$$

s.t.

$$Ex = Sm, \quad v$$

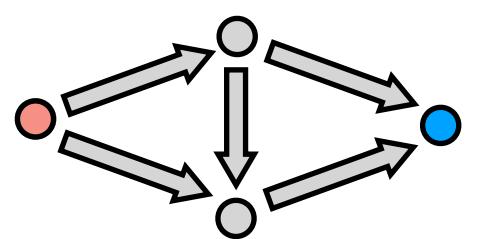
$$x \geq 0 \quad \mu$$

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$v$  : value function

## Potential Games

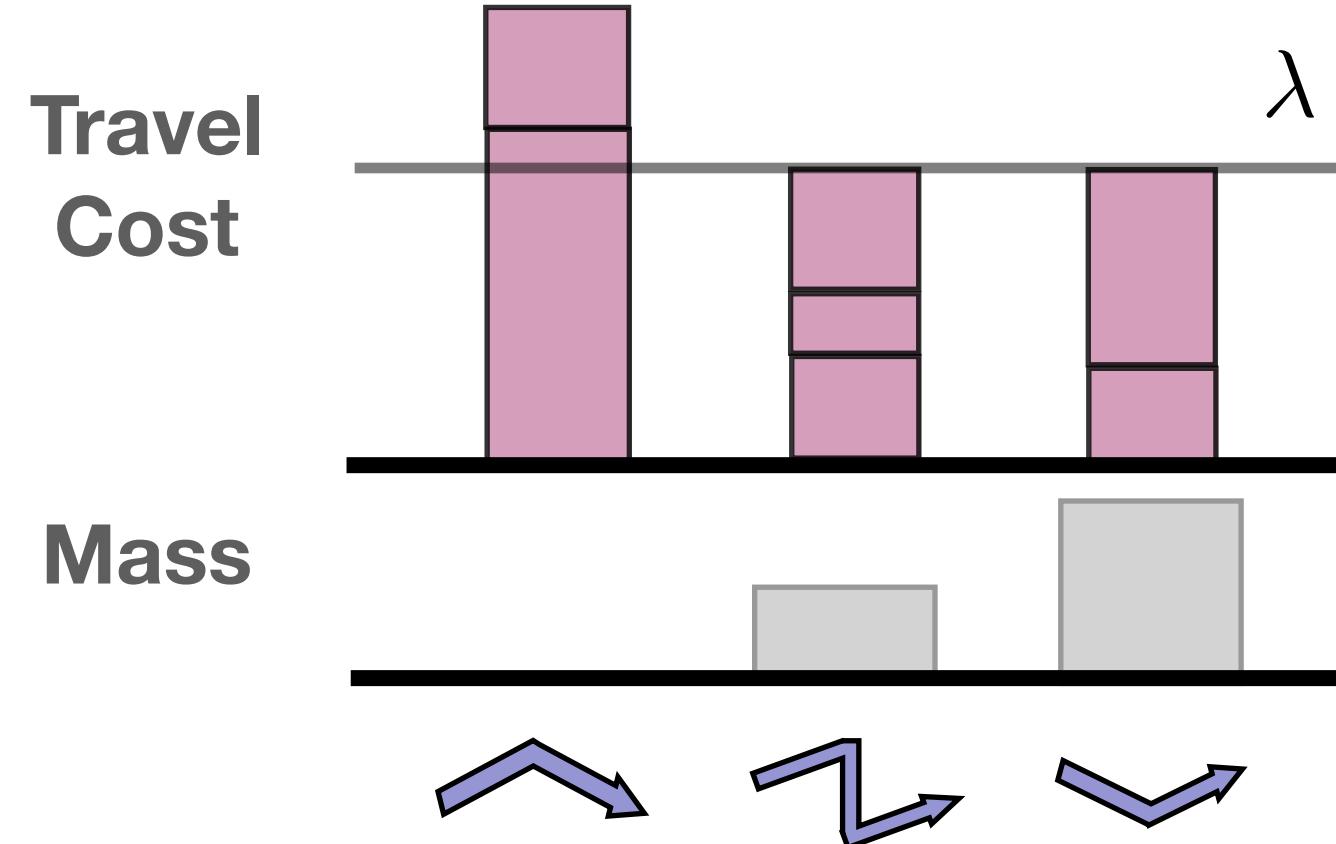
Routing  
Games



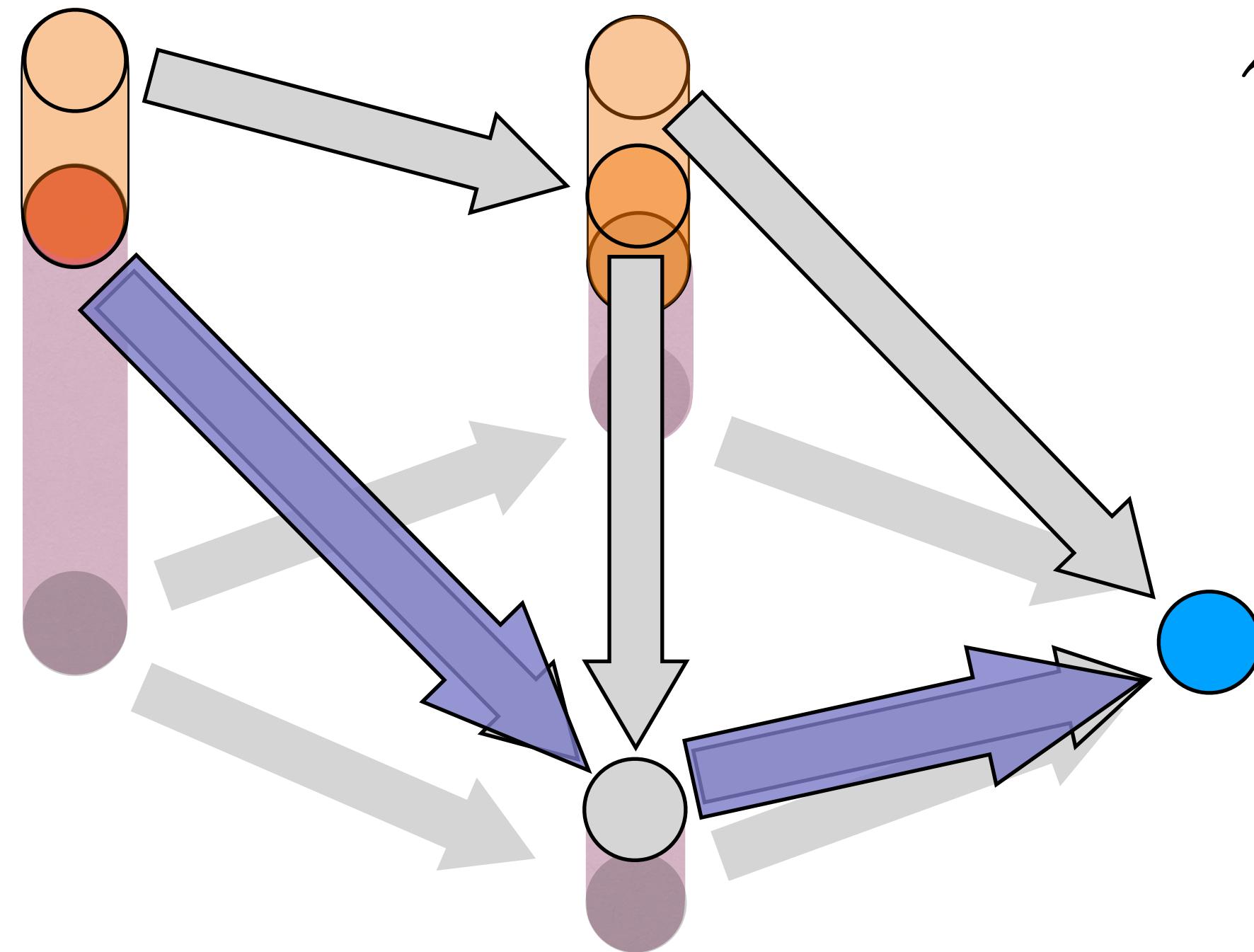
## Potential Function

$$F(x)$$

## Wardrop Equilibrium



## Routing Games



$$\min_x F(x)$$

s.t.

$$Ex = Sm, \quad v$$

$$x \geq 0 \quad \mu$$

$\mu$  : edge inefficiency

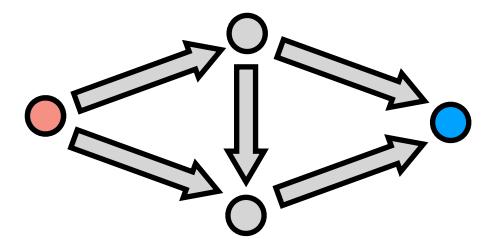
$v$  : value function

$x$  : edge traffic

$z$  : route traffic

## Potential Games

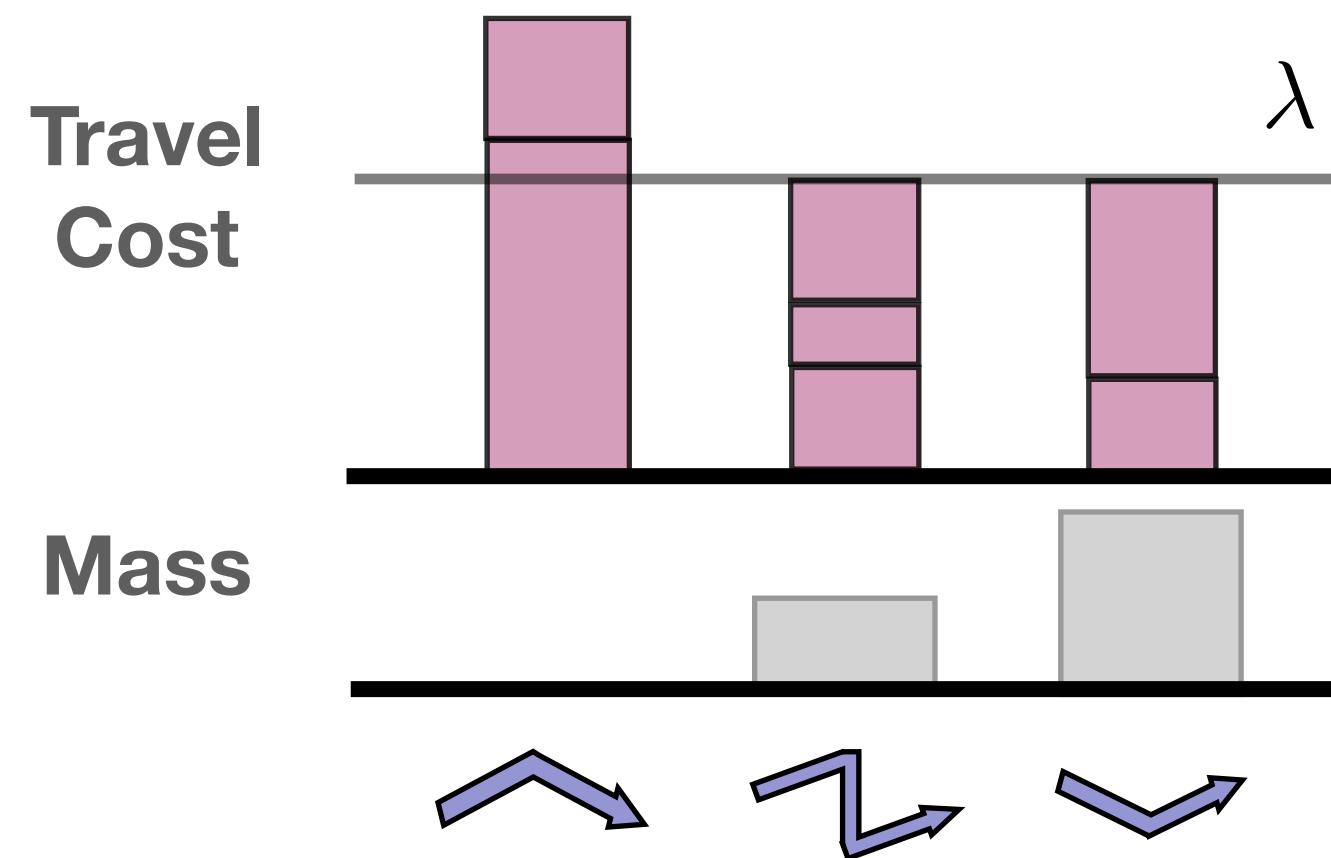
Routing  
Games



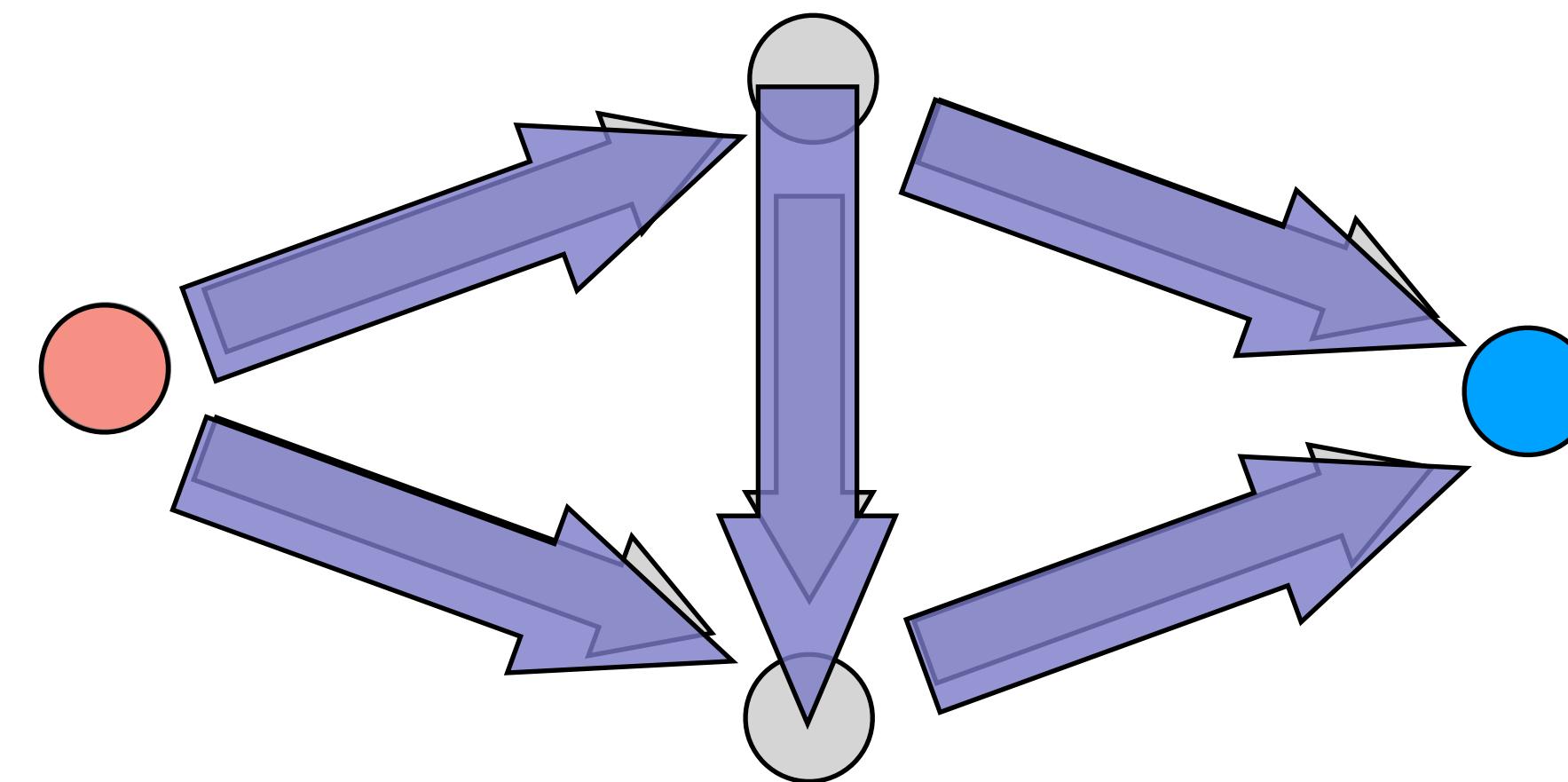
Potential  
Function

$$F(x)$$

## Wardrop Equilibrium



## Routing Games

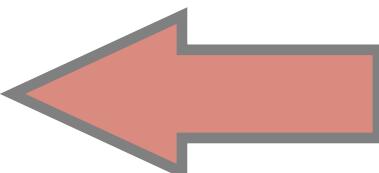


$x$  : edge traffic

$z$  : route traffic

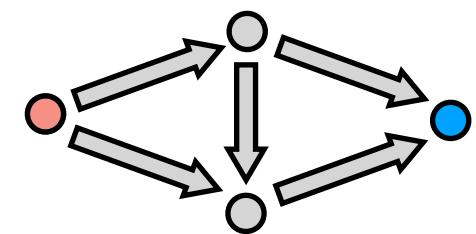
## REFERENCES

- Some theoretical aspects of road traffic research [Wardrop, 1952]
- Studies in the economics of transportation [Beckmann, McGuire, Winsten, 1956]
- The Traffic Assignment Problem: Models and Methods [Patriksson, 2015]

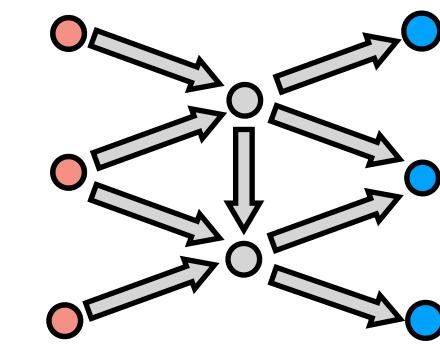


## Potential Games

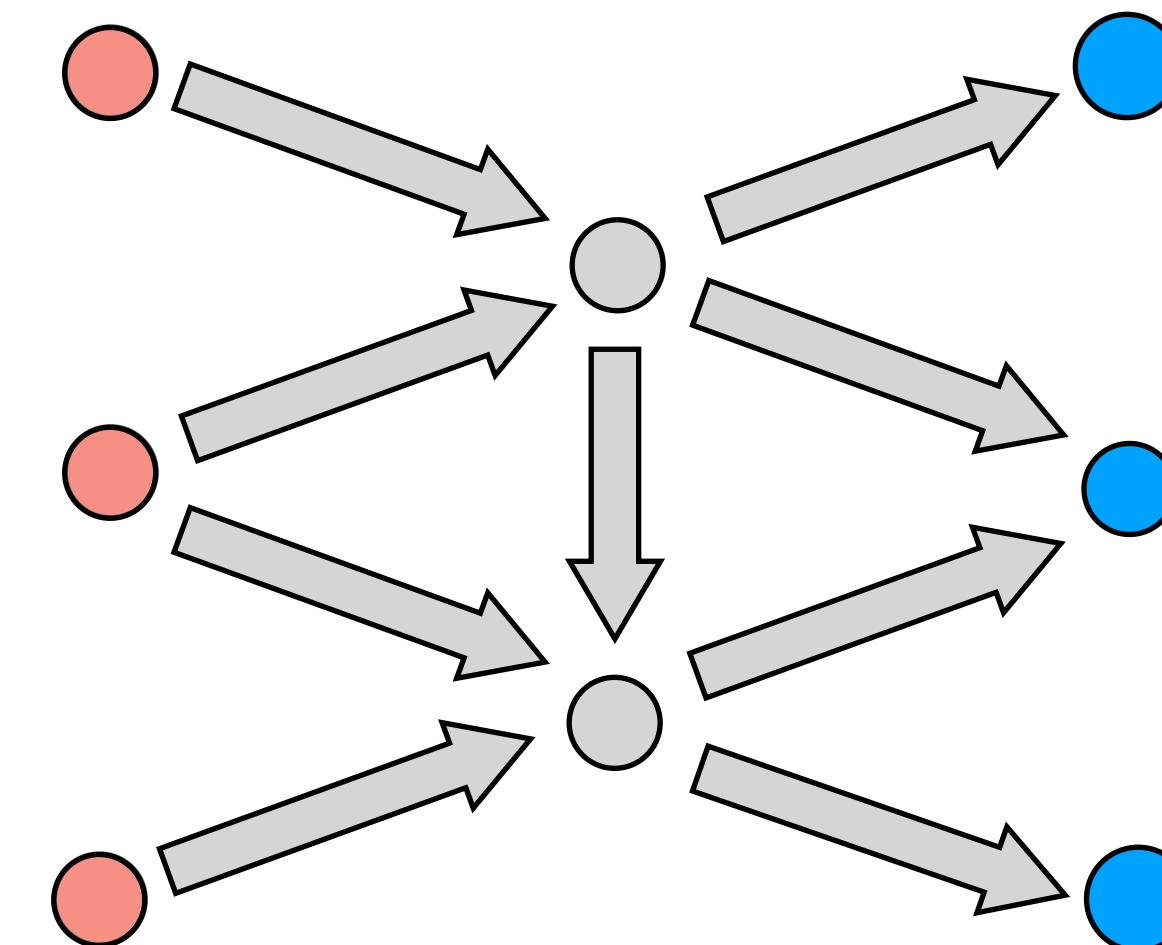
Routing  
Games



Multiple  
sources/  
sinks



## Multiple Source/Sinks



$x$  : edge traffic

$z$  : route traffic

$$\min_x F(x)$$

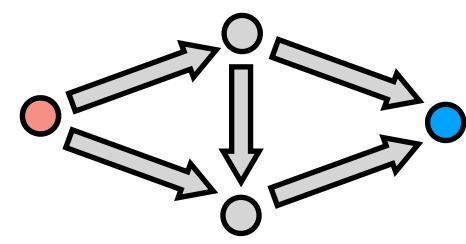
s.t.

$$E_i x_i = S_i m_i, \quad \forall i \quad \boxed{v_i}$$

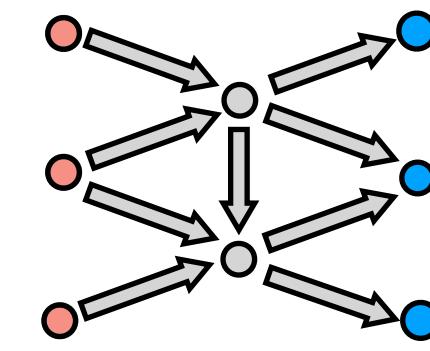
$$x_i \geq 0, \quad \forall i \quad \boxed{\mu_i}$$

## Potential Games

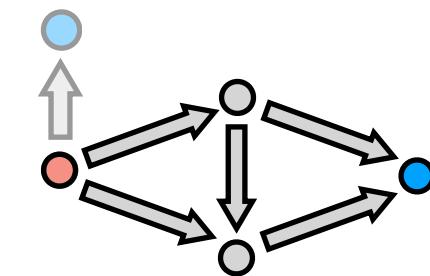
Routing Games



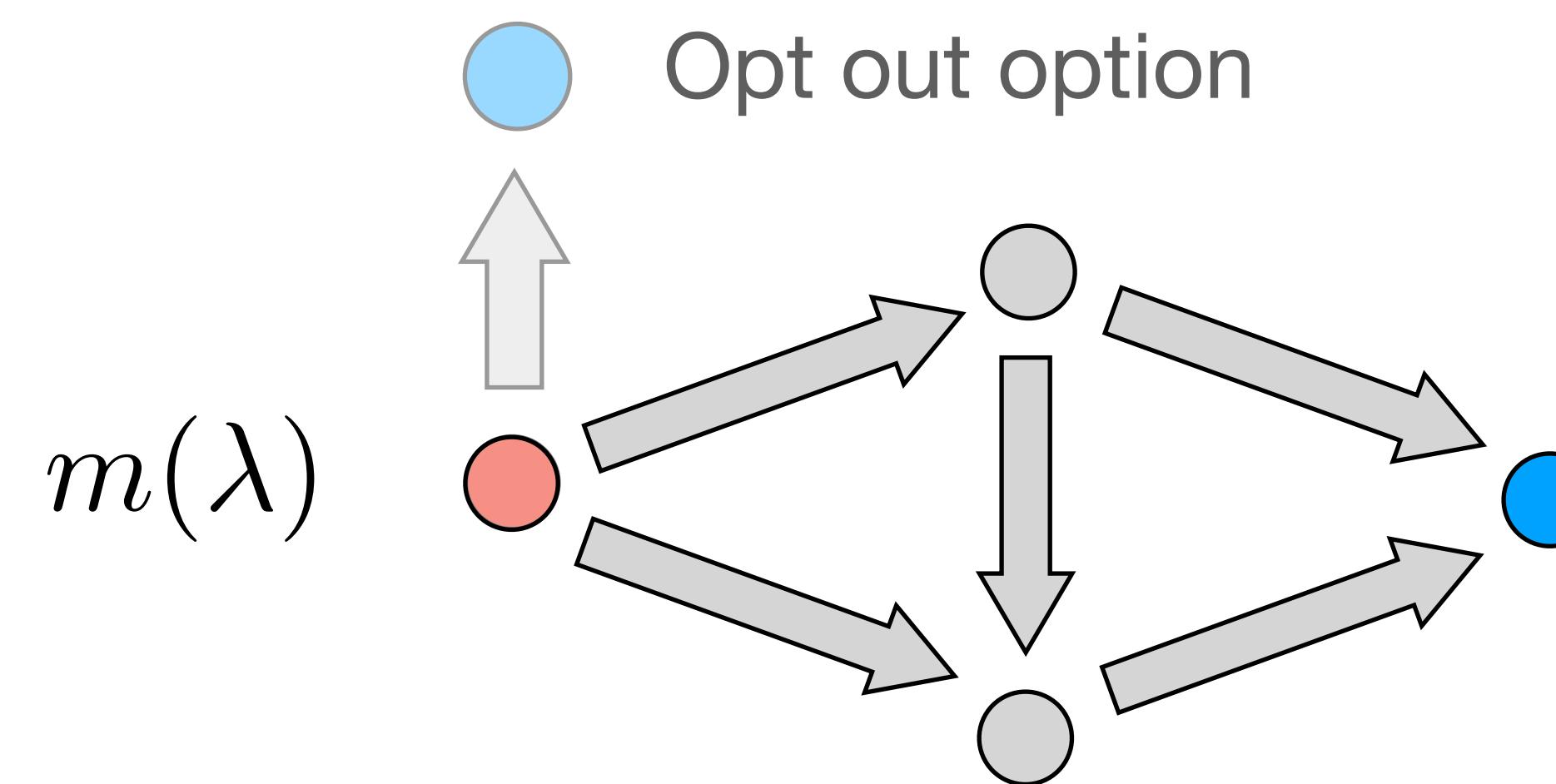
Multiple sources/  
sinks



Variable Demand



## Variable Demand



$$\min_x \quad F(x) + \int_0^m m^{-1}(u) du$$

s.t.

$$Ex = Sm, \quad \boxed{v}$$

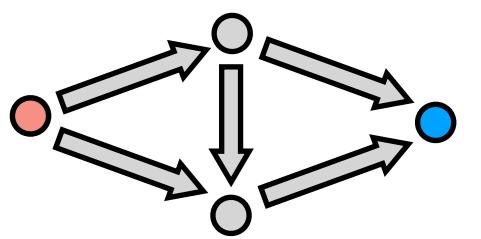
$$x \geq 0 \quad \boxed{\mu}$$

$x$  : edge traffic

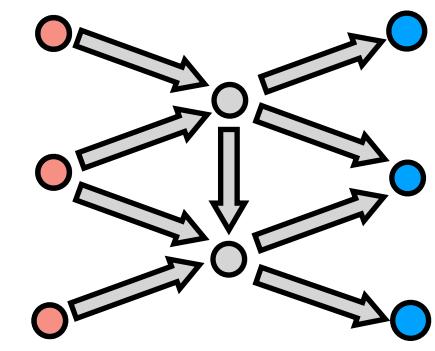
$z$  : route traffic

## Potential Games

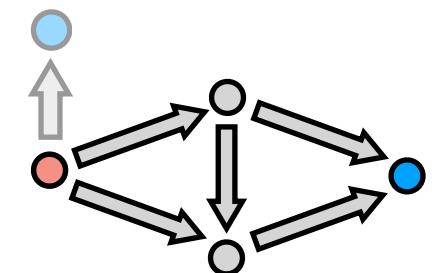
Routing Games



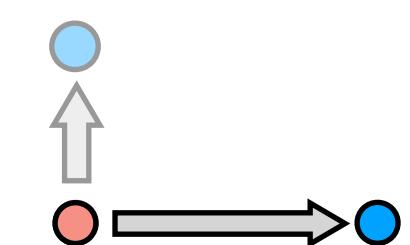
Multiple sources/  
sinks



Variable Demand

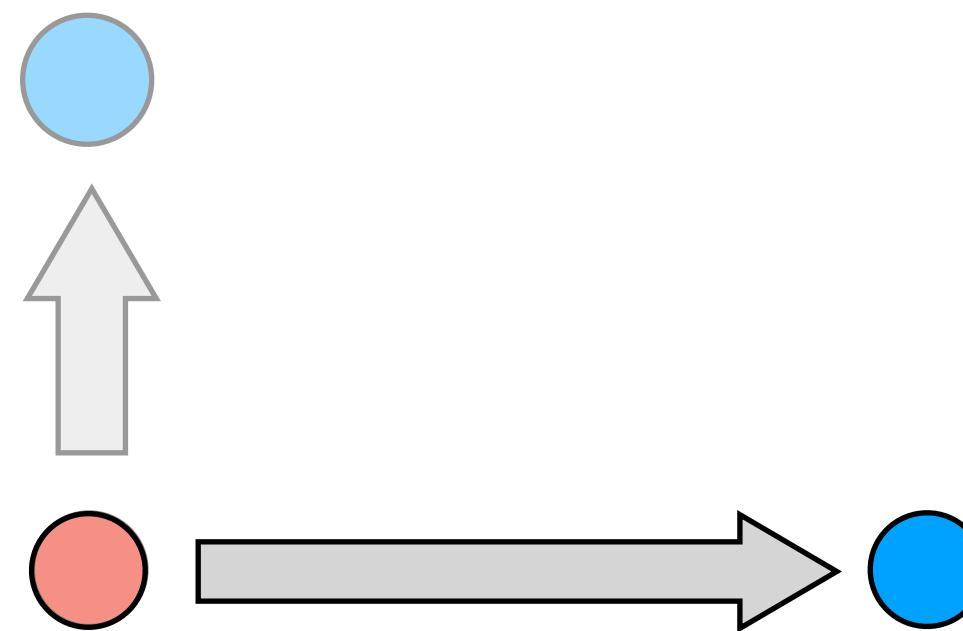


Supply &  
Demand



## Supply & Demand

$$m(\lambda)$$



$$\min_x \quad F(x) + \int_0^m m^{-1}(u) du$$

s.t.

$$Ex = Sm, \quad \boxed{v}$$

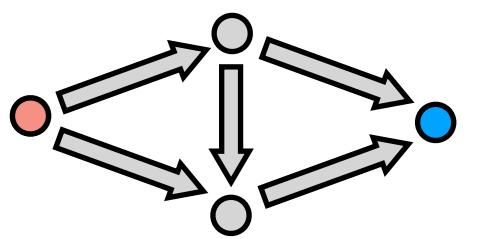
$$x \geq 0 \quad \boxed{\mu}$$

$x$  : edge traffic

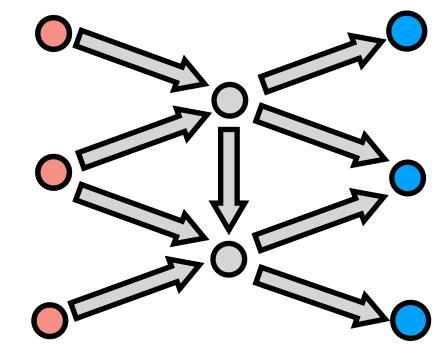
$z$  : route traffic

## Potential Games

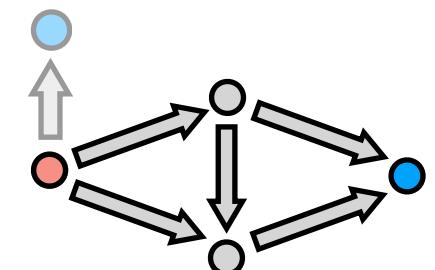
Routing Games



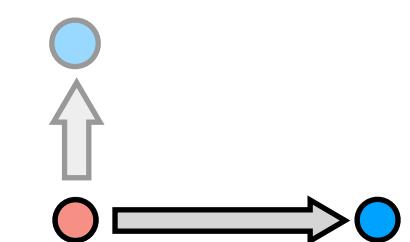
Multiple sources/  
sinks



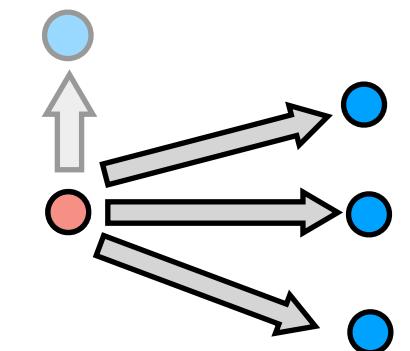
Variable Demand



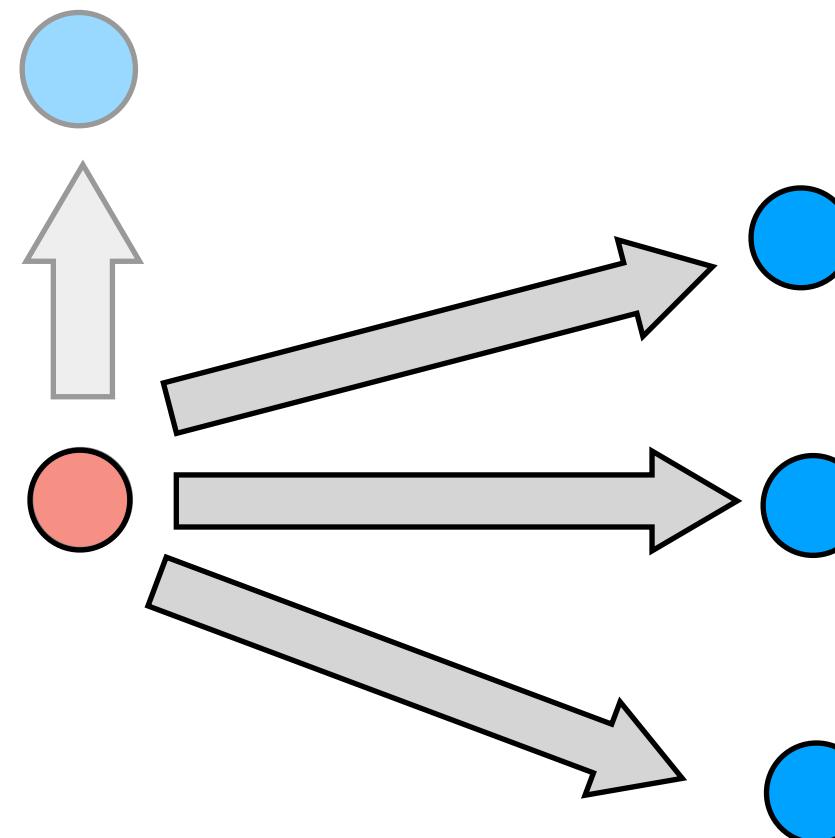
Supply &  
Demand



Cournot Market



## Cournot Market



$$\min_x$$

$$F(x) + \int_0^m m^{-1}(u) du$$

s.t.

$$Ex = Sm, \quad v$$

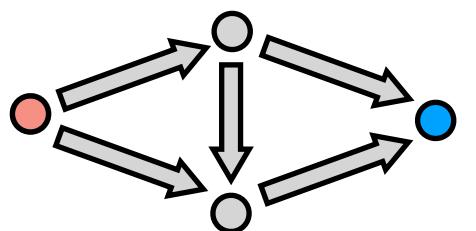
$$x \geq 0, \quad \mu$$

$x$  : edge traffic

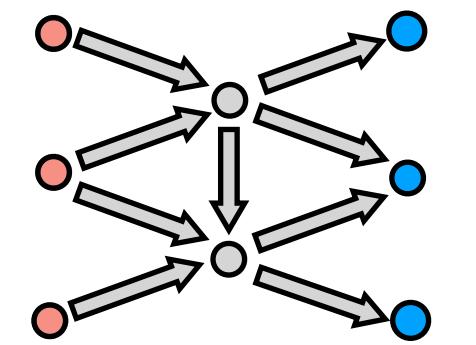
$z$  : route traffic

# Potential Games

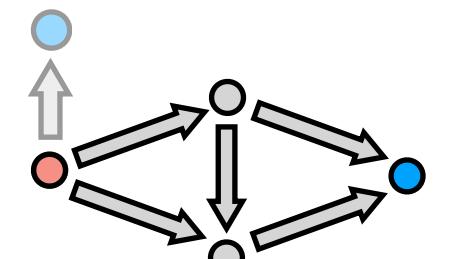
Routing Games



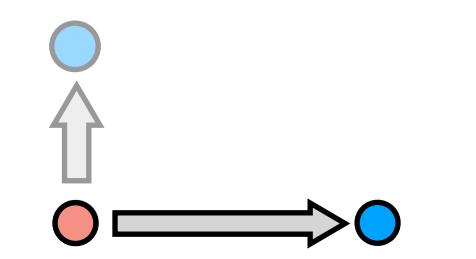
Multiple sources/  
sinks



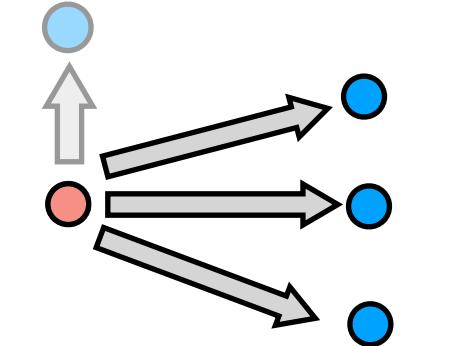
Variable Demand



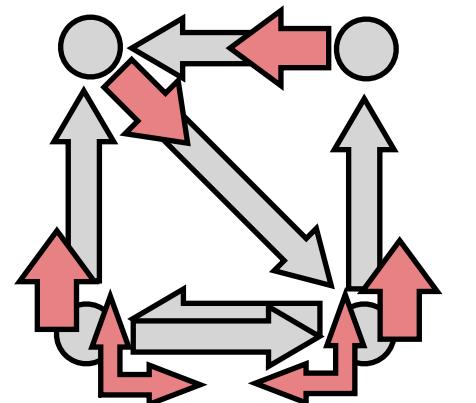
Supply &  
Demand



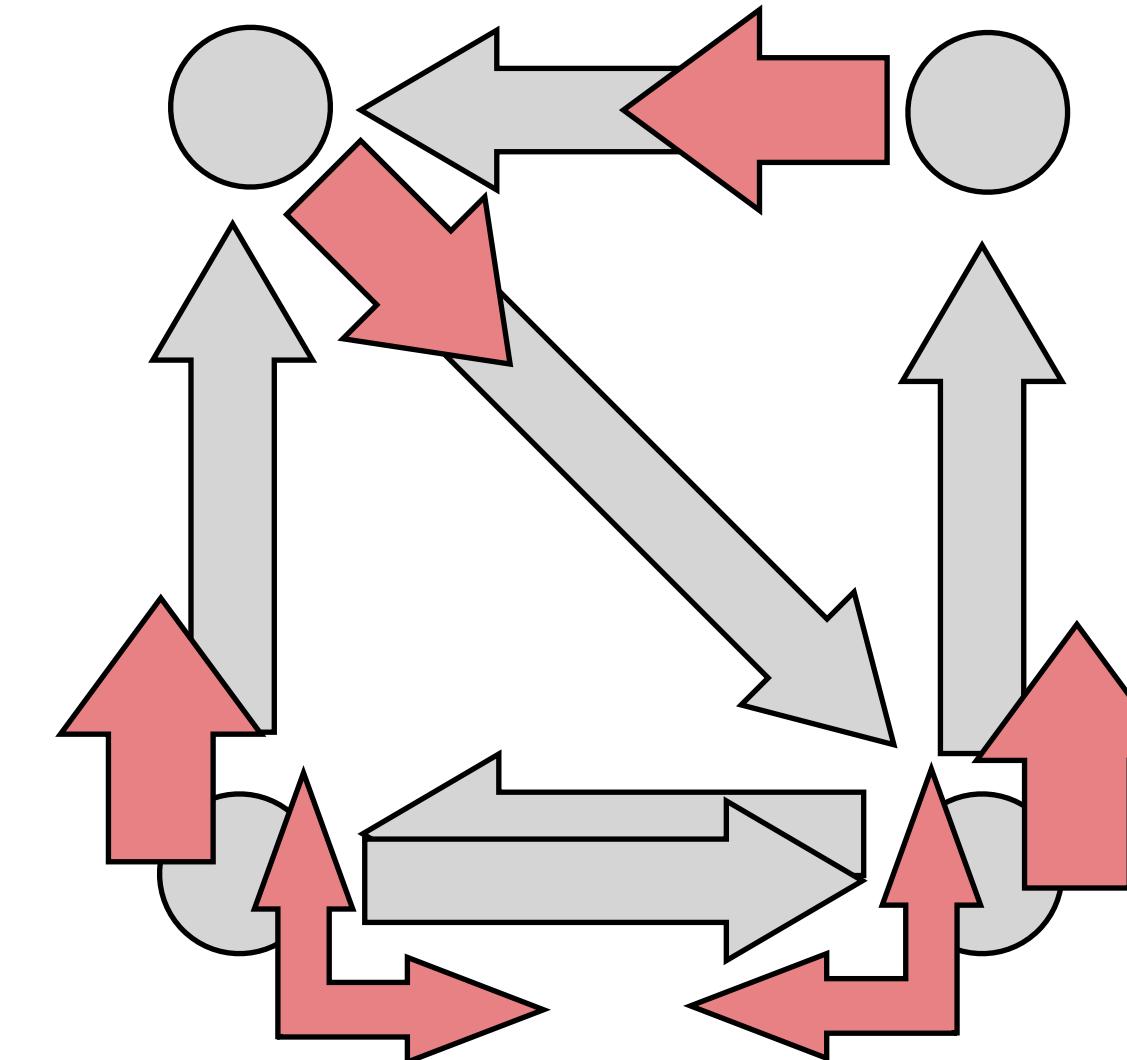
Cournot Market



MDP  
Congestion Game



# Markov Decision Process Congestion Game



$$\min_x \quad F(x)$$

s.t.

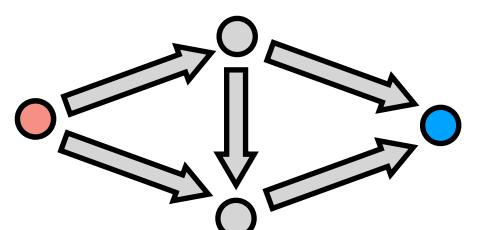
$$1^T x = m \quad (\lambda)$$

$$x \geq 0 \quad (\mu)$$

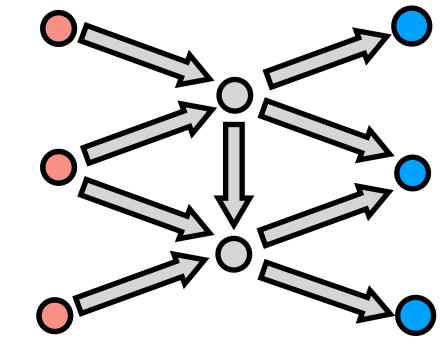
$$EWx = 0 \quad (v)$$

## Potential Games

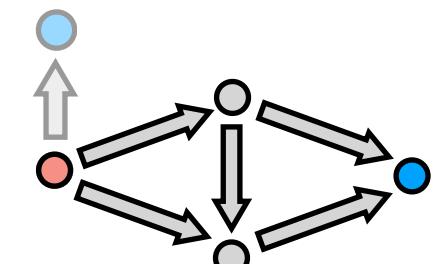
Routing Games



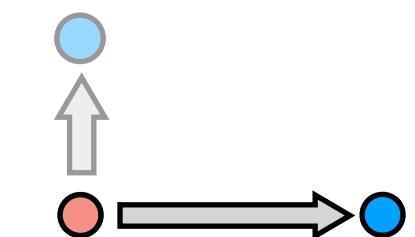
Multiple sources/sinks



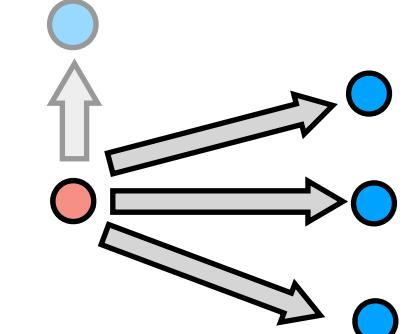
Variable Demand



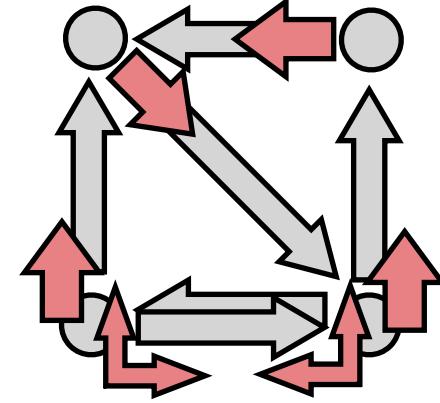
Supply & Demand



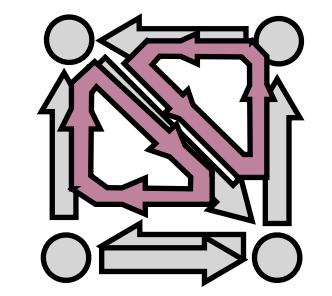
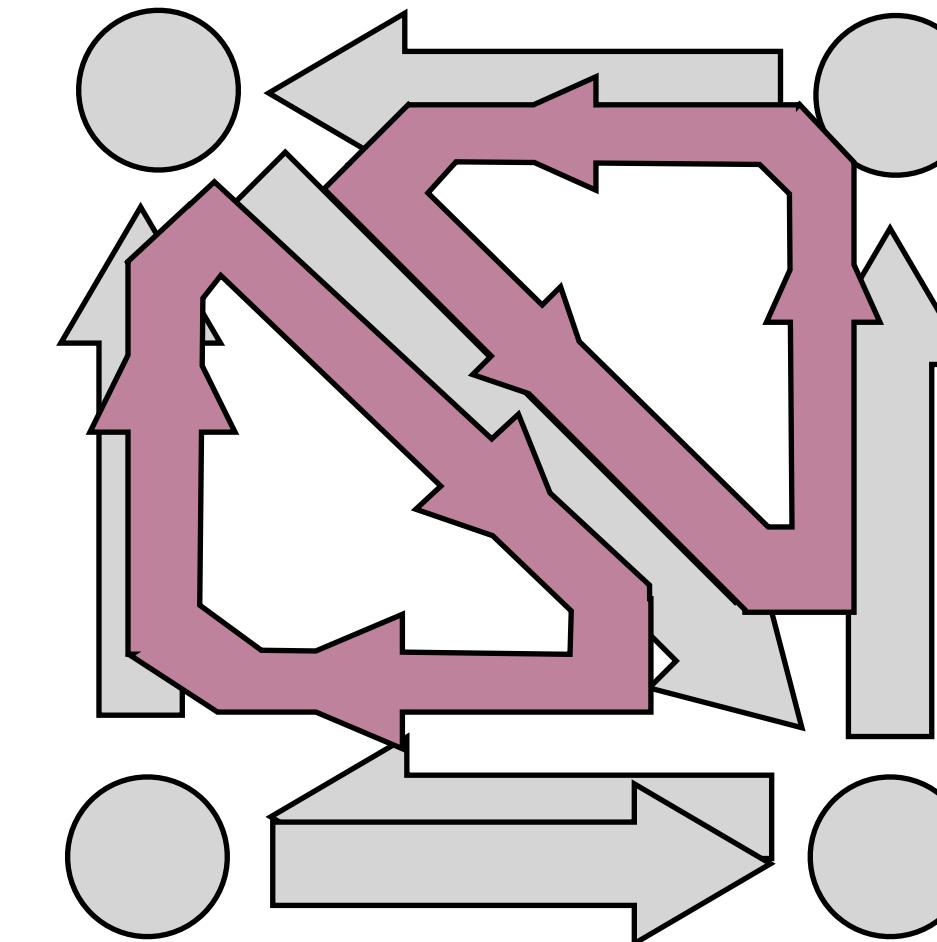
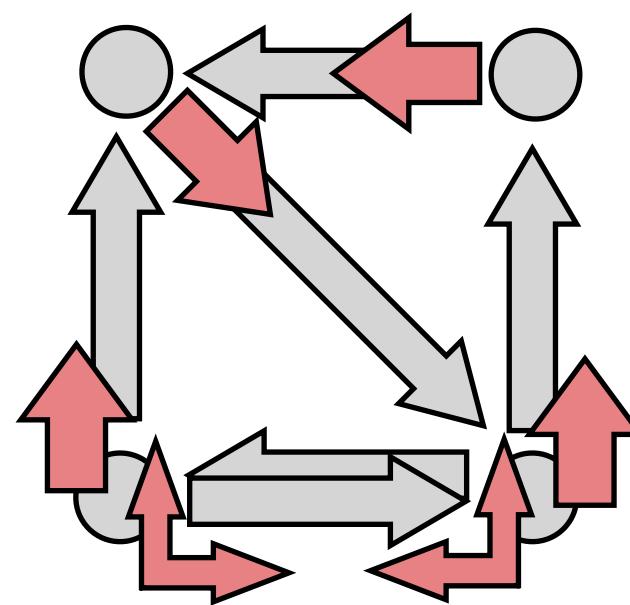
Cournot Market



MDP Congestion Game



## Markov Decision Process Congestion Game



$$\min_x \quad F(x)$$

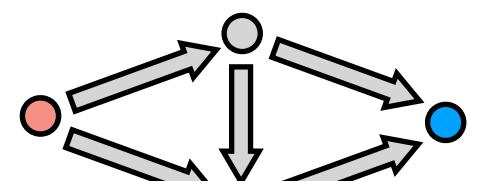
$$\text{s.t.} \quad 1^T x = m \quad (\lambda)$$

$$EWx = 0 \quad (\nu)$$

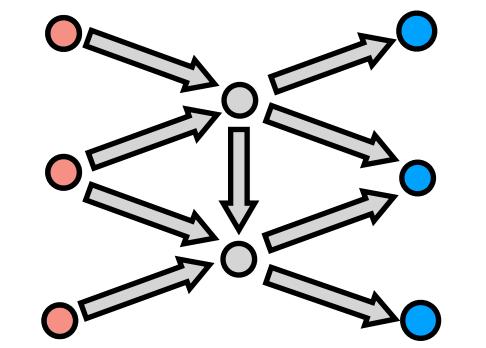
$$x \geq 0 \quad (\mu)$$

## Potential Games

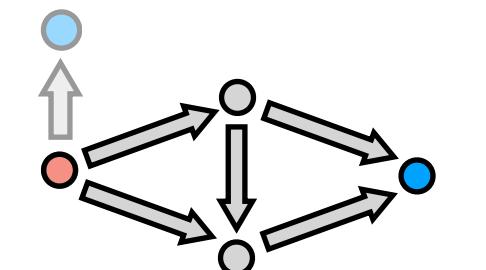
Routing Games



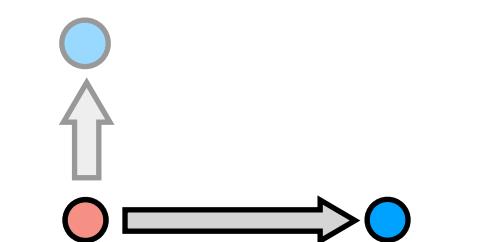
Multiple sources/sinks



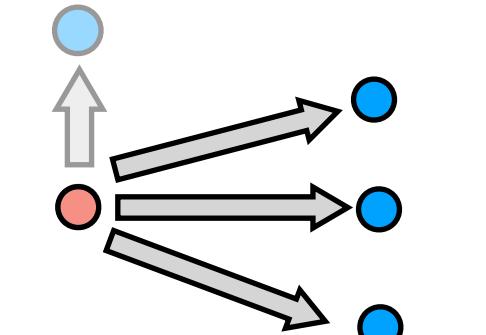
Variable Demand



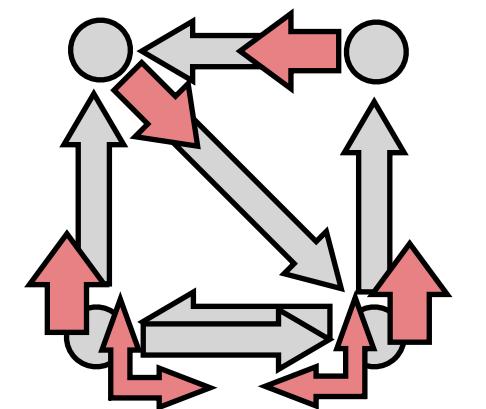
Supply & Demand



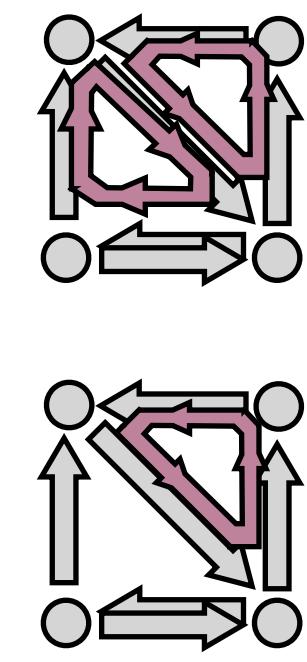
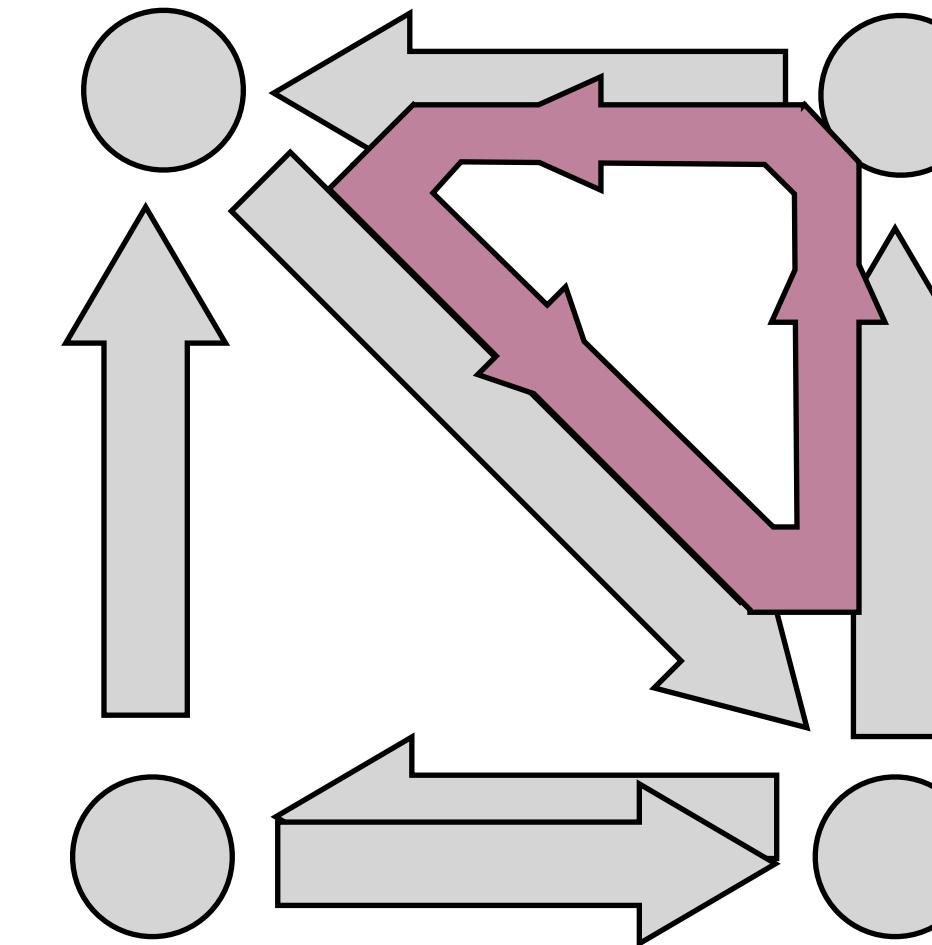
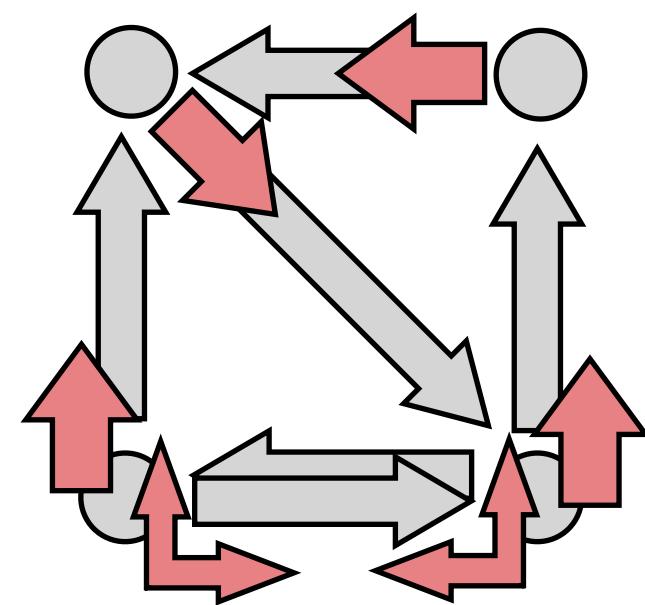
Cournot Market



MDP Congestion Game



## Markov Decision Process Congestion Game



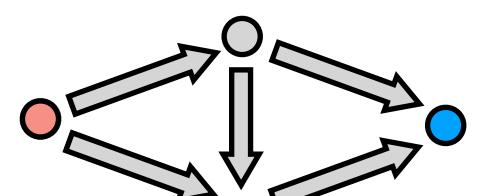
$$\min_x \quad F(x)$$

$$\text{s.t.} \quad \begin{array}{l} 1^T x = m \quad (\lambda) \\ x \geq 0 \quad (\mu) \end{array}$$

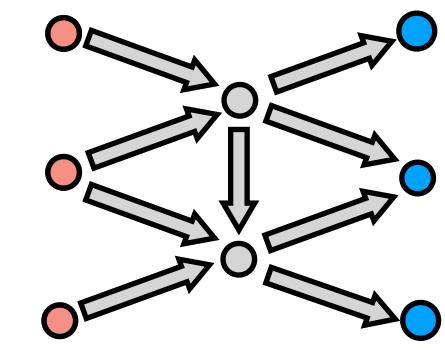
$$EWx = 0 \quad (v)$$

## Potential Games

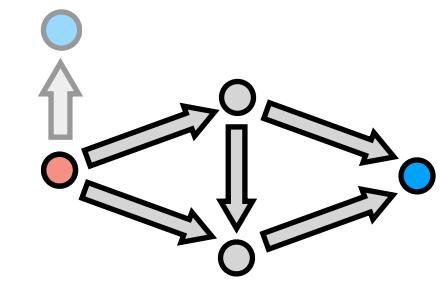
Routing Games



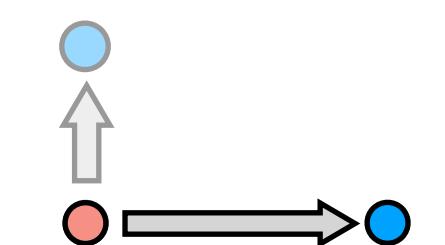
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sinks



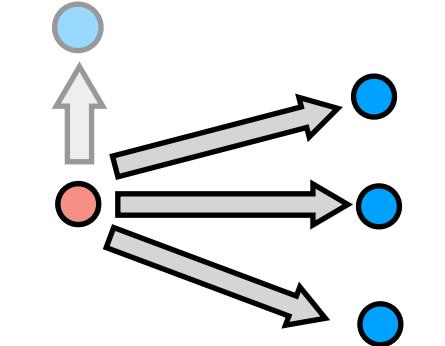
Variable Demand



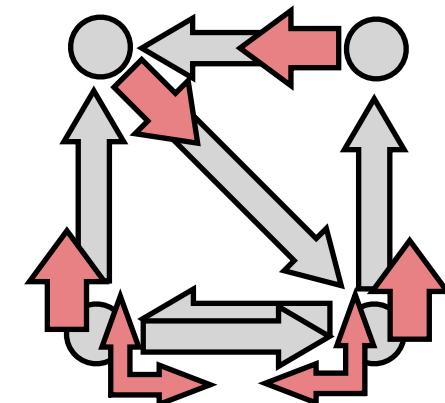
Supply &  
Demand



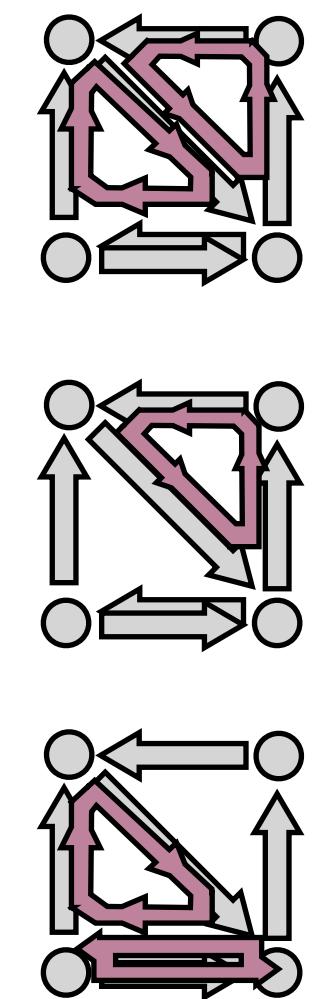
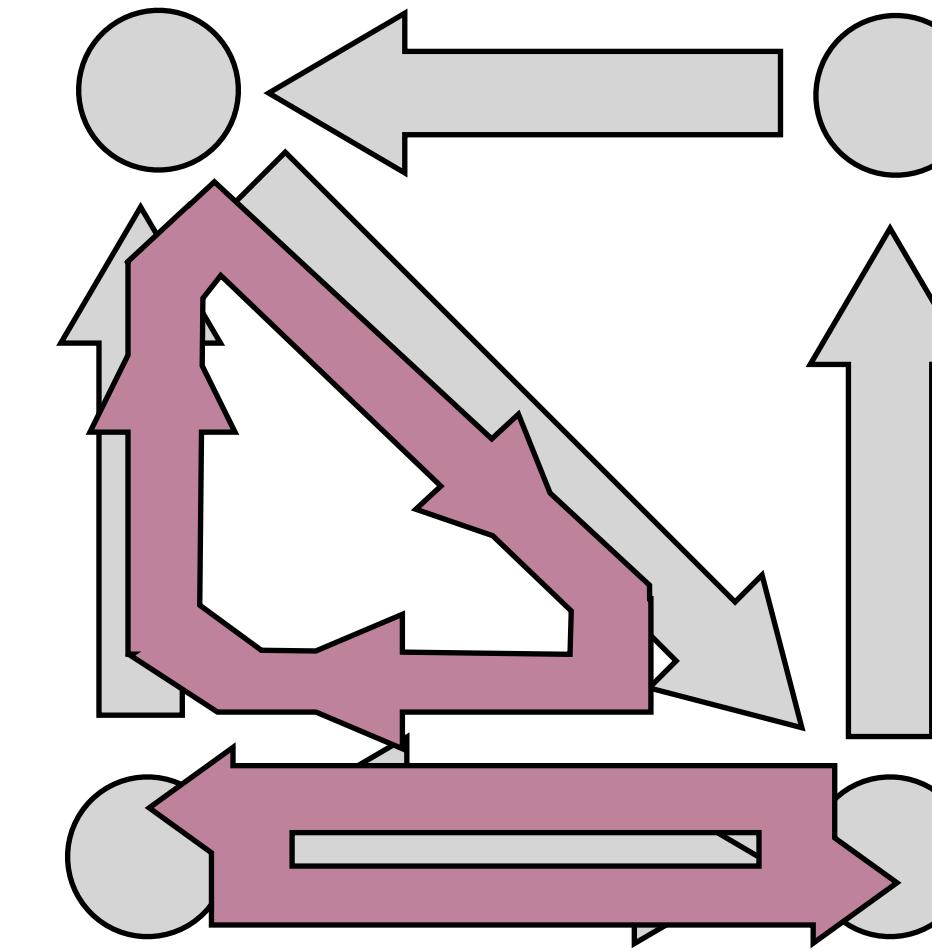
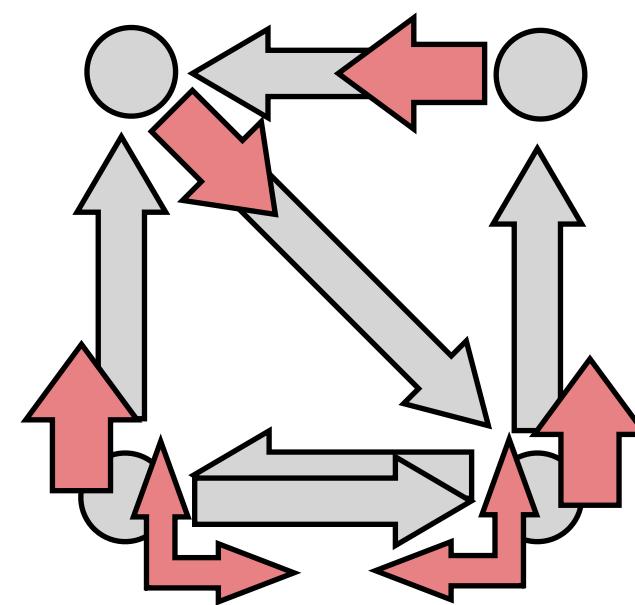
Cournot  
Market



MDP  
Congestion  
Game



## Markov Decision Process Congestion Game



$$\min_x \quad F(x)$$

s.t.

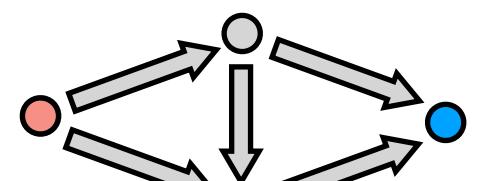
$$1^T x = m \quad (\lambda)$$

$$x \geq 0 \quad (\mu)$$

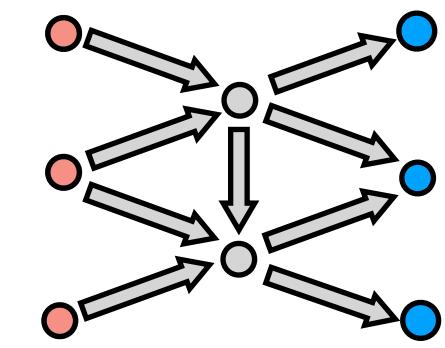
$$EWx = 0 \quad (v)$$

# Potential Games

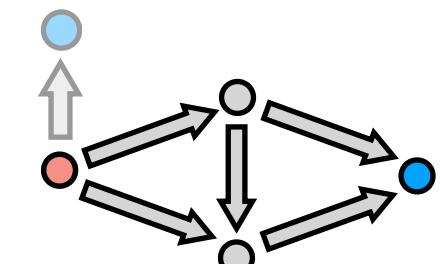
Routing Games



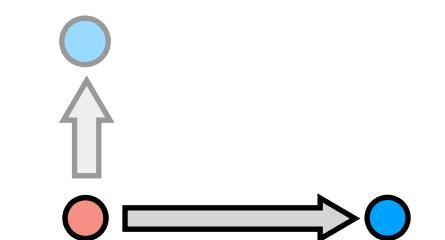
Multiple sources/sinks



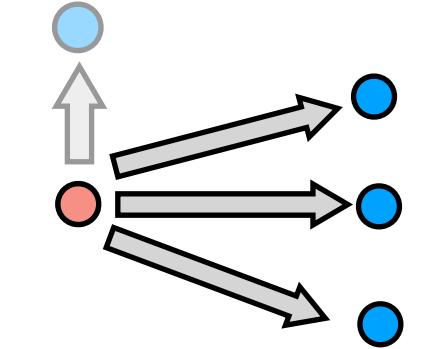
Variable Demand



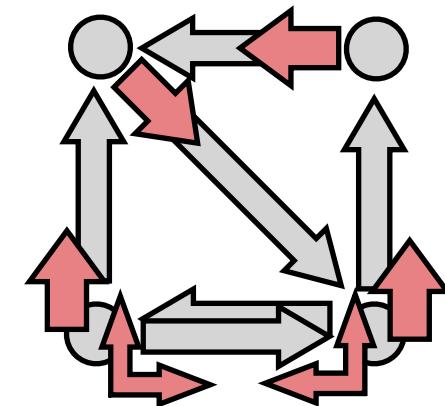
Supply & Demand



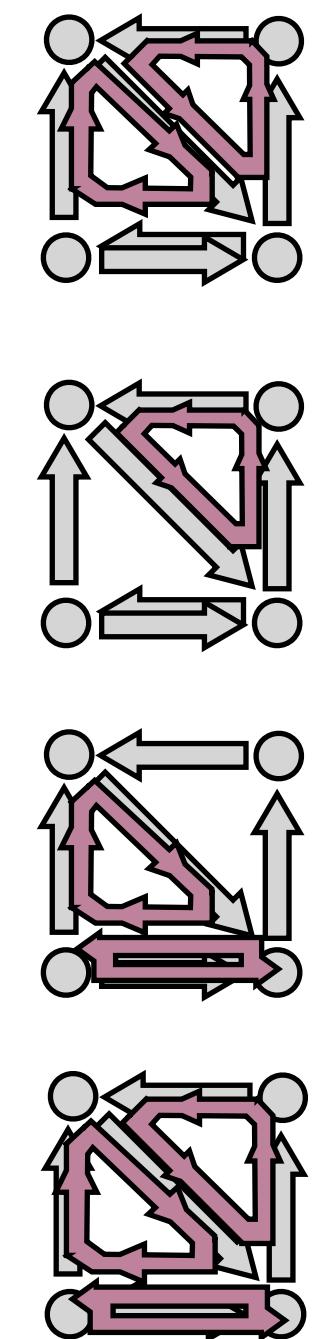
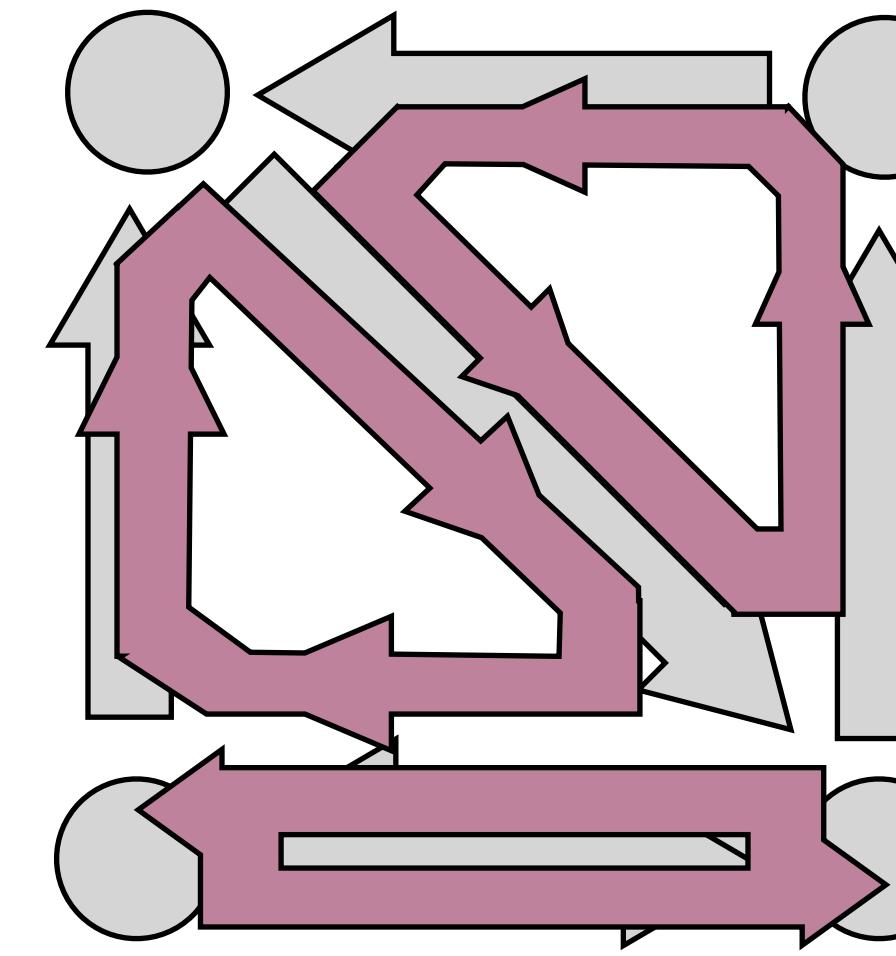
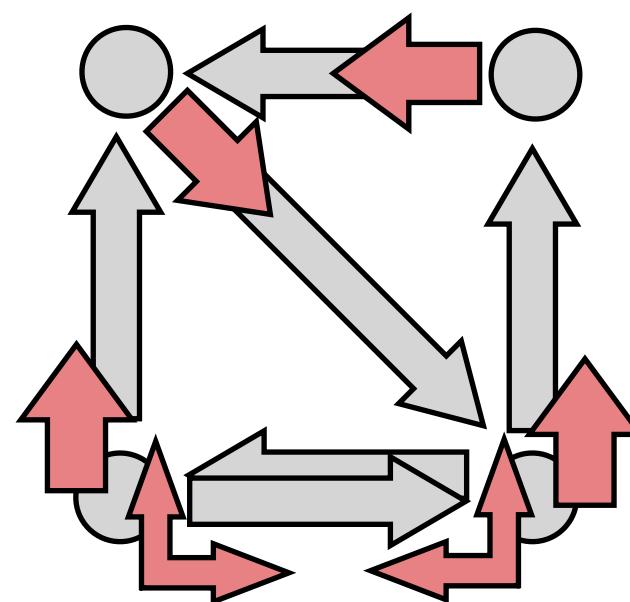
Cournot Market



MDP Congestion Game



# Markov Decision Process Congestion Game



$$\min_x F(x)$$

s.t.

$$1^T x = m \quad \lambda$$

$$EWx = 0 \quad v$$

$$x \geq 0 \quad \mu$$

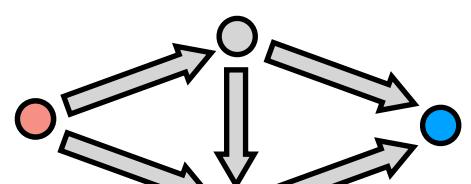
$\lambda$  : average reward

$\mu$  : action inefficiency

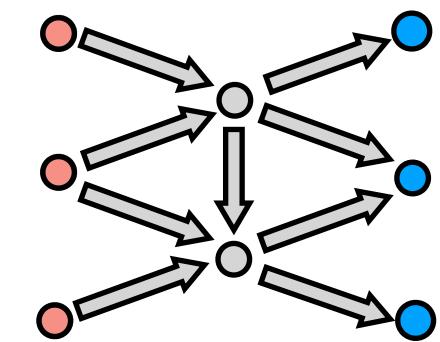
$v$  : value function

# Potential Games

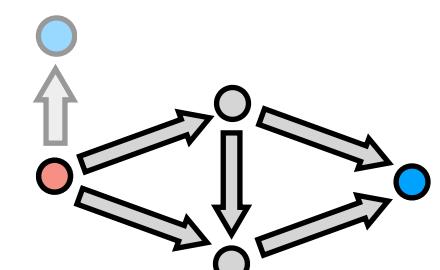
Routing Games



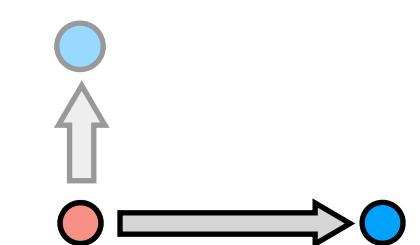
Multiple sources/  
sinks



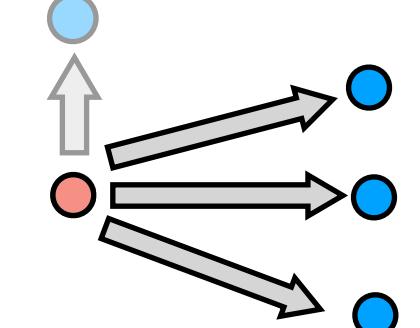
Variable Demand



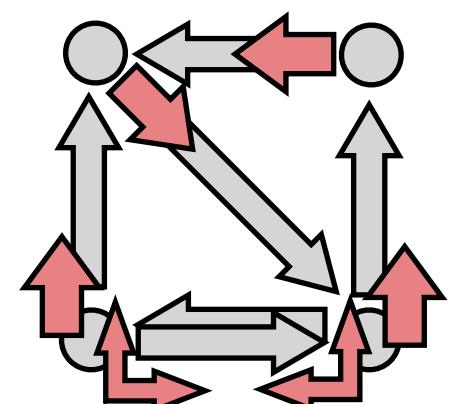
Supply &  
Demand



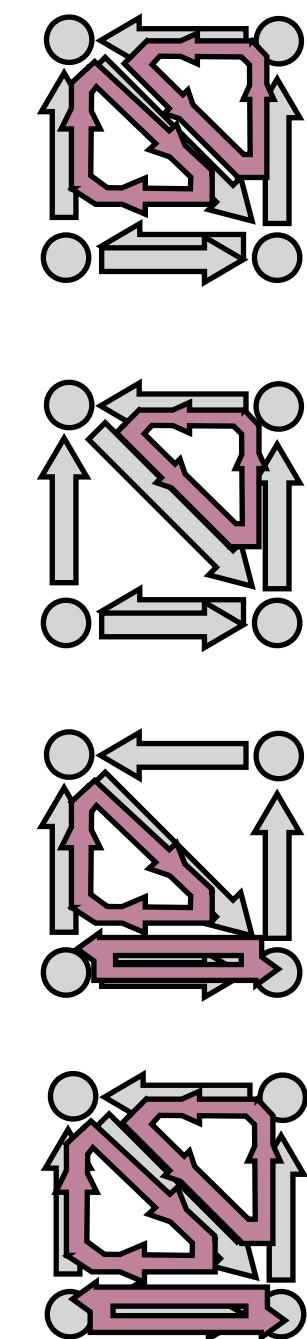
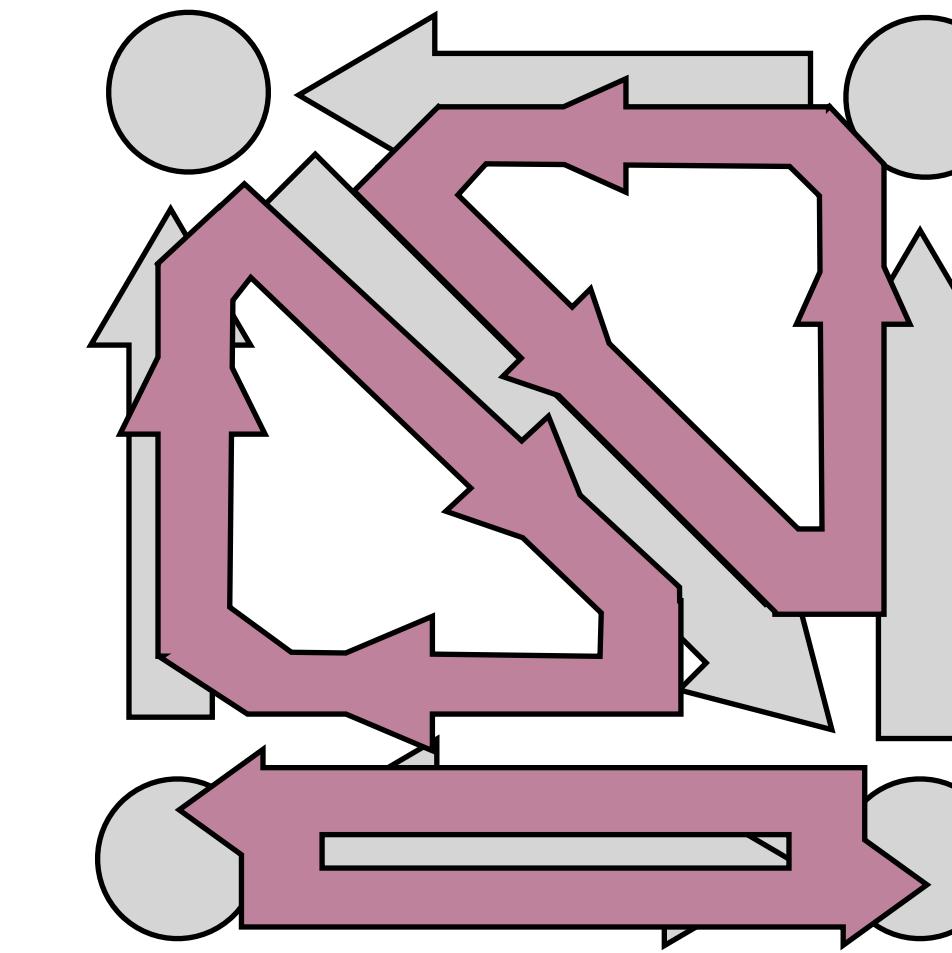
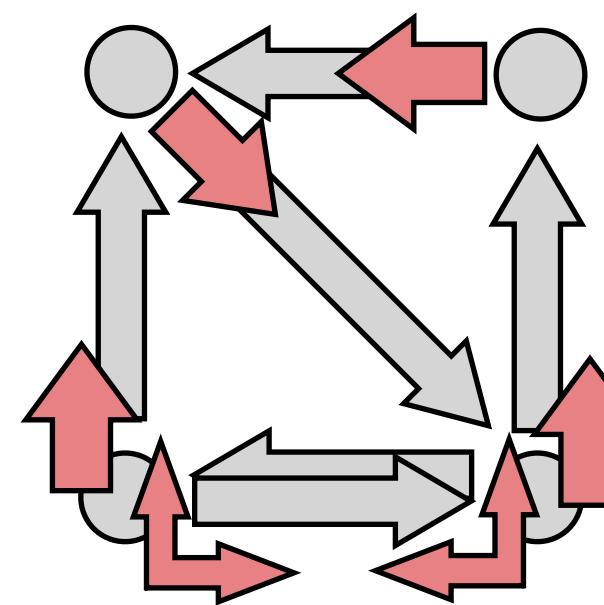
Cournot  
Market



MDP  
Congestion  
Game



# Markov Decision Process Congestion Game



## APPLICATIONS

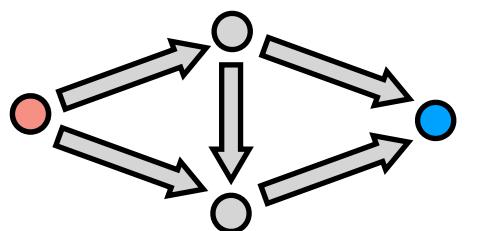
- Ride-sharing drivers planning routes
- Cars circling for street parking
- Air-traffic routing

## PAPERS

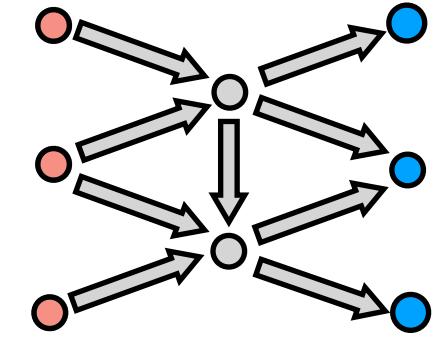
- Markov decision process routing games [Calderone, Sastry, 2017]
- Infinite horizon average cost Markov decision process routing games [Calderone, Sastry, 2017]
- Adaptive constraint satisfaction for Markov decision process congestion games:  
Applications to transportation networks [Li, Calderone, Ratliff, et al. 2021]
- Variable demand and multi-commodity  
flow in Markovian network equilibrium [Yu, Calderone, Ratliff, et al. 2021]

# Potential Games

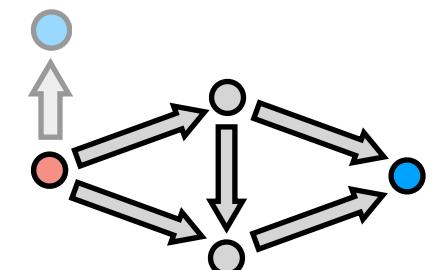
Routing Games



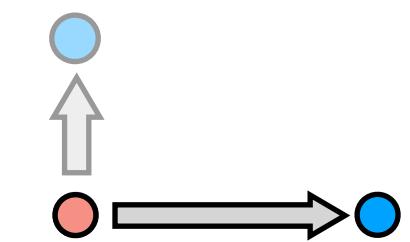
Multiple sources/sinks



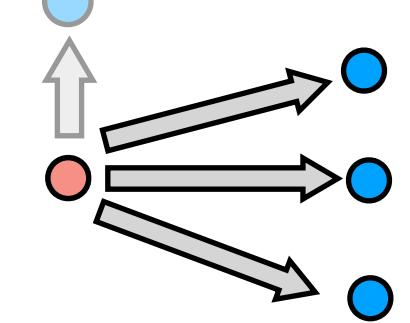
Variable Demand



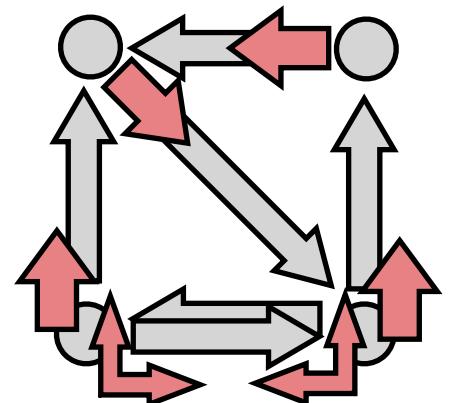
Supply & Demand



Cournot Market

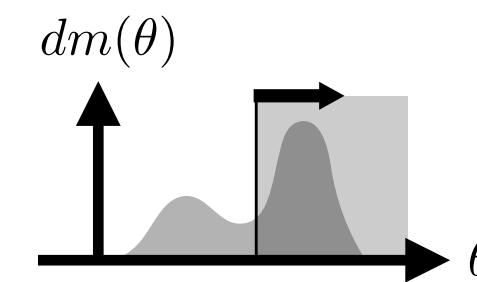


MDP Congestion Game

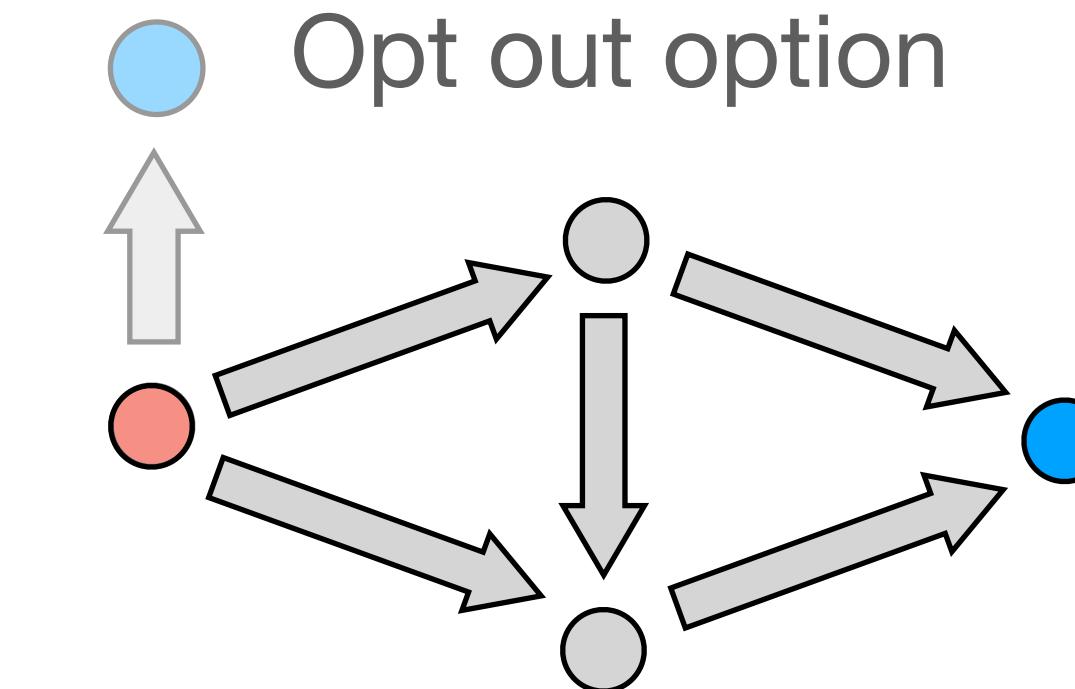


# Variable Demand - Non-Homogeneous Preferences

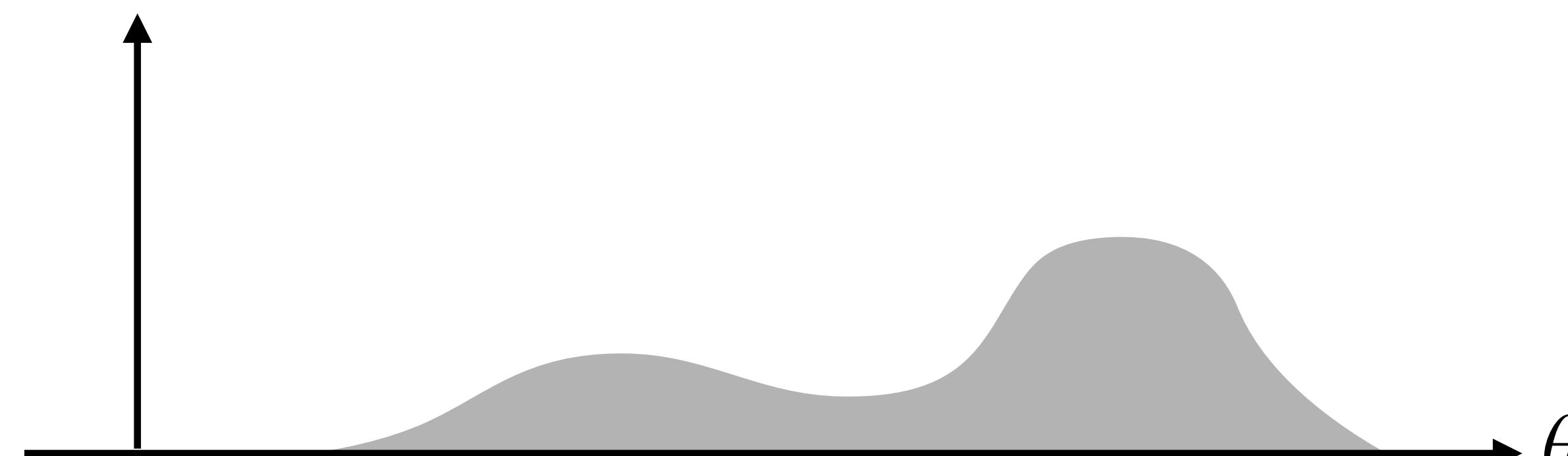
Non-homo-geneous preferences



$$m(\lambda)$$



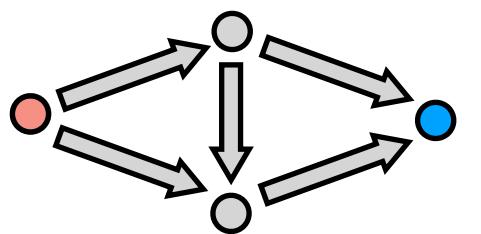
$$dm(\theta)$$



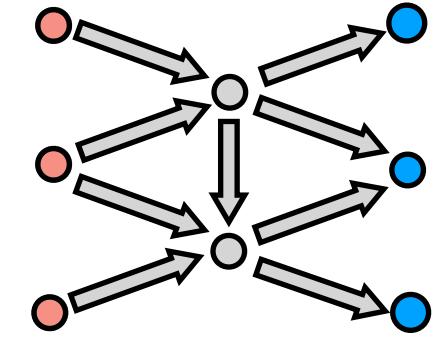
$\theta$  : Max cost  $dm(\theta)$  will pay

# Potential Games

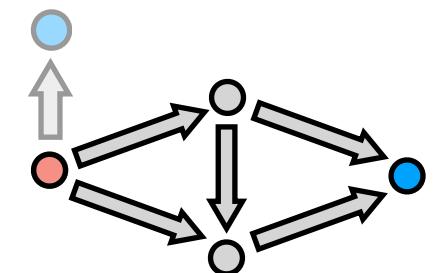
Routing Games



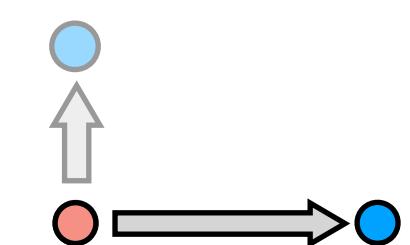
Multiple sources/sinks



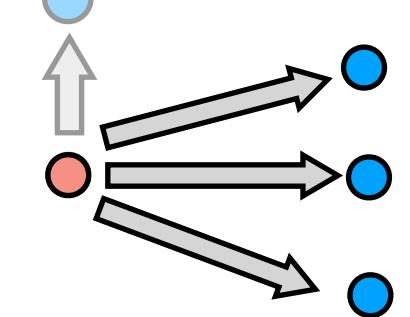
Variable Demand



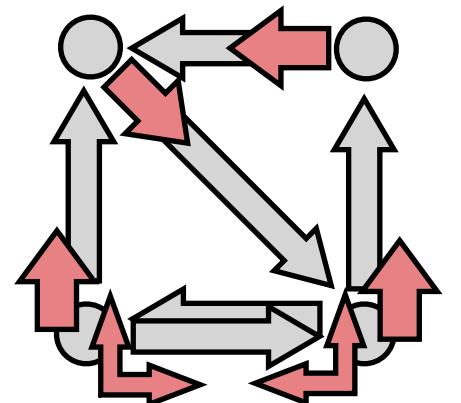
Supply & Demand



Cournot Market

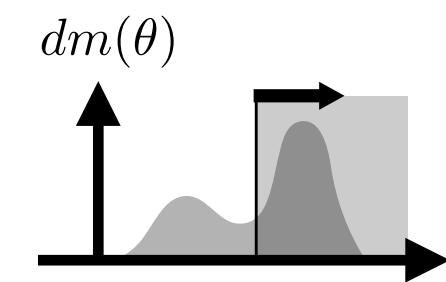


MDP Congestion Game

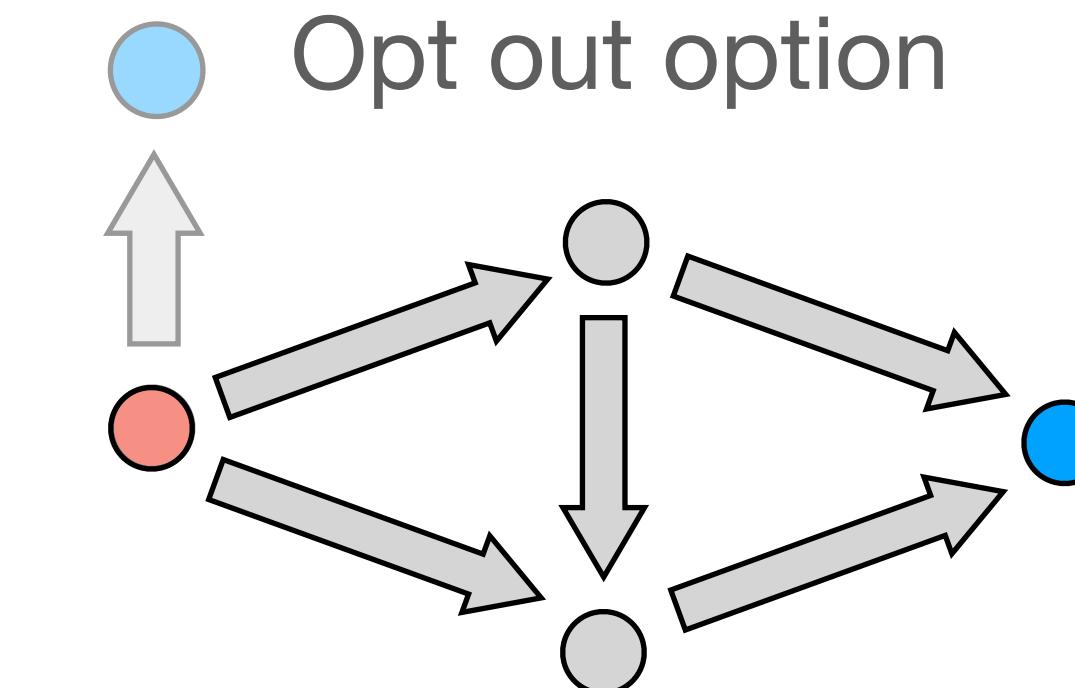


# Variable Demand - Non-Homogeneous Preferences

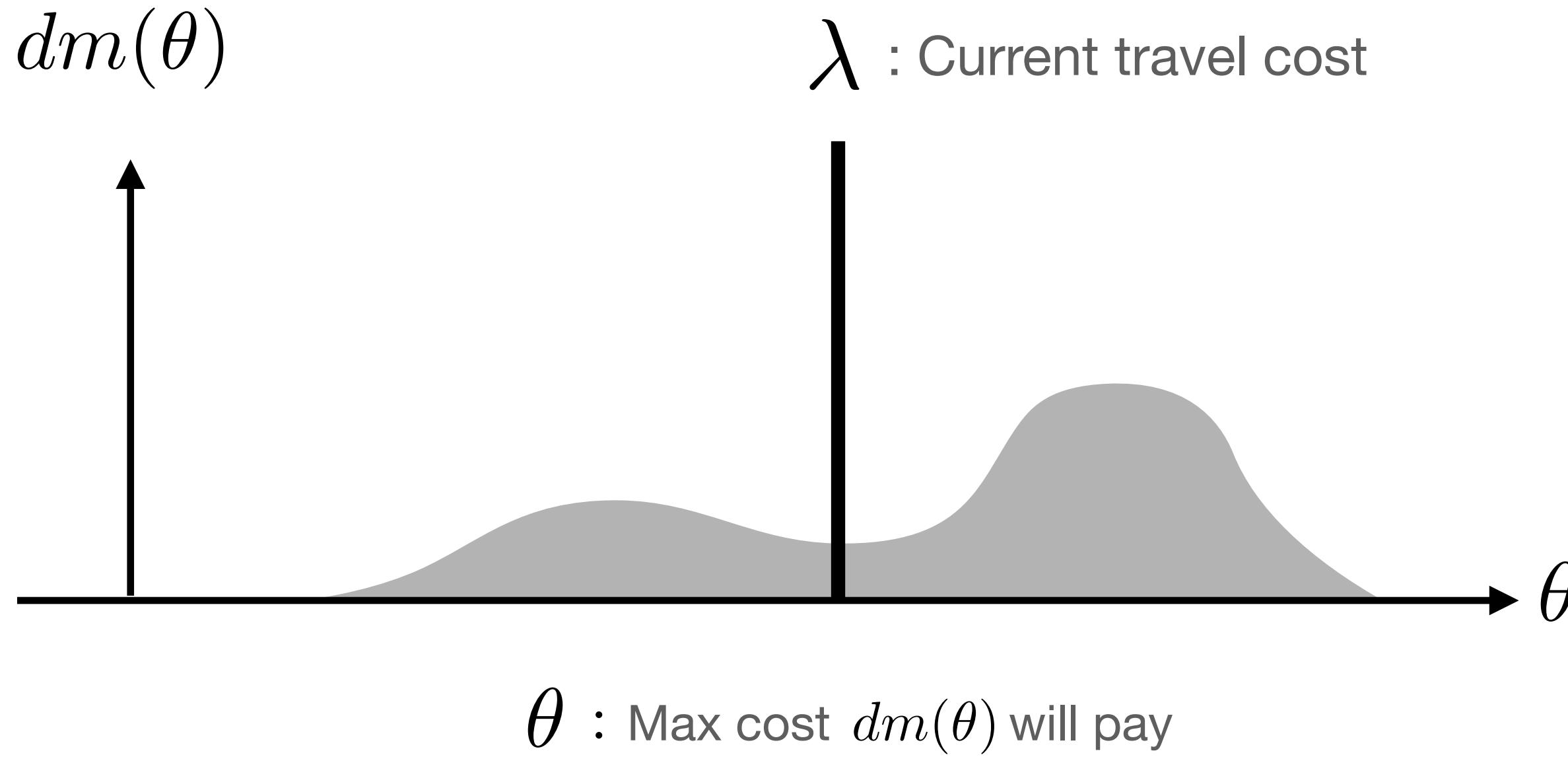
Non-homo-geneous preferences



$m(\lambda)$

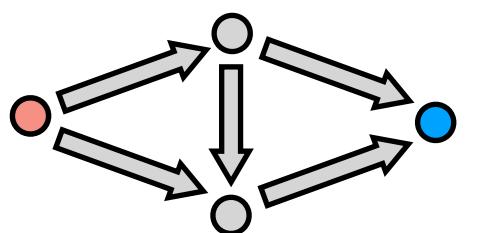


$\lambda$  : Current travel cost

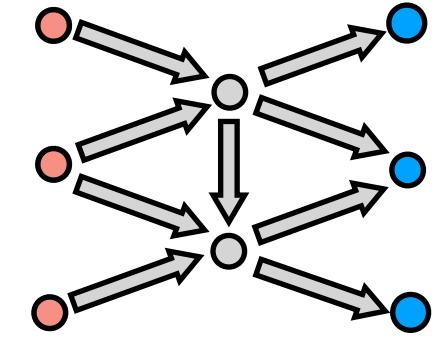


# Potential Games

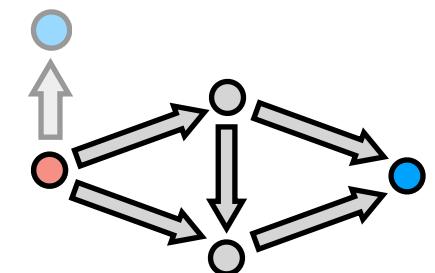
Routing Games



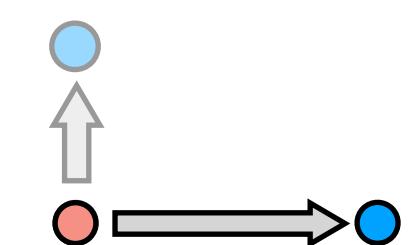
Multiple sources/sinks



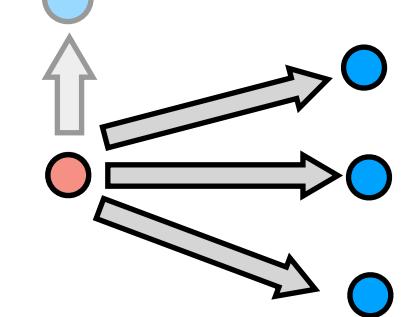
Variable Demand



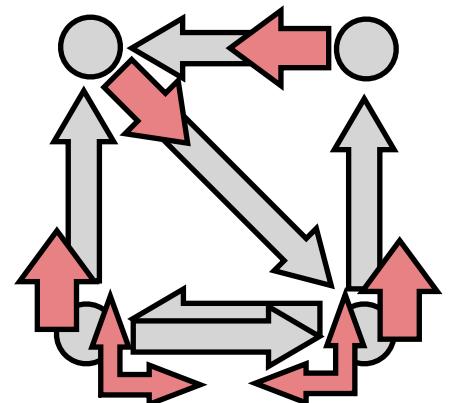
Supply & Demand



Cournot Market

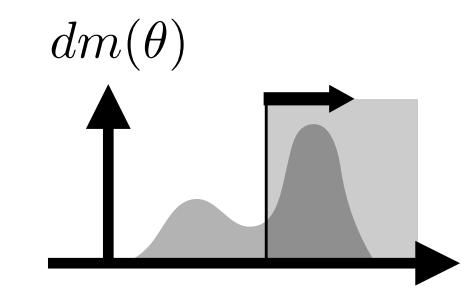


MDP Congestion Game

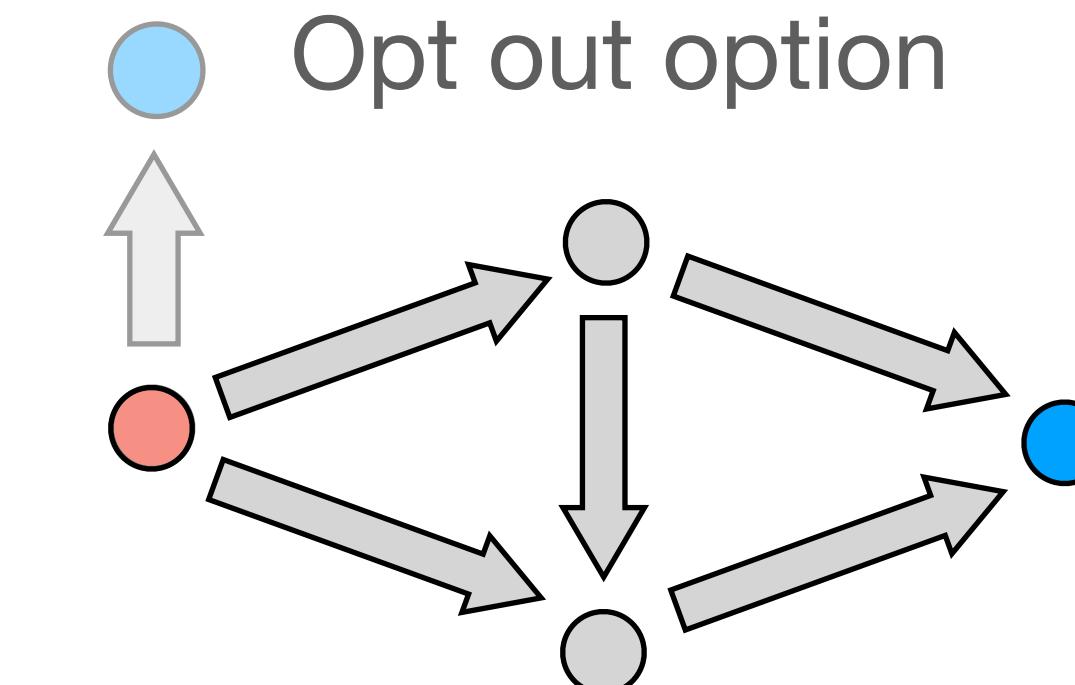


# Variable Demand - Non-Homogeneous Preferences

Non-homo-geneous preferences



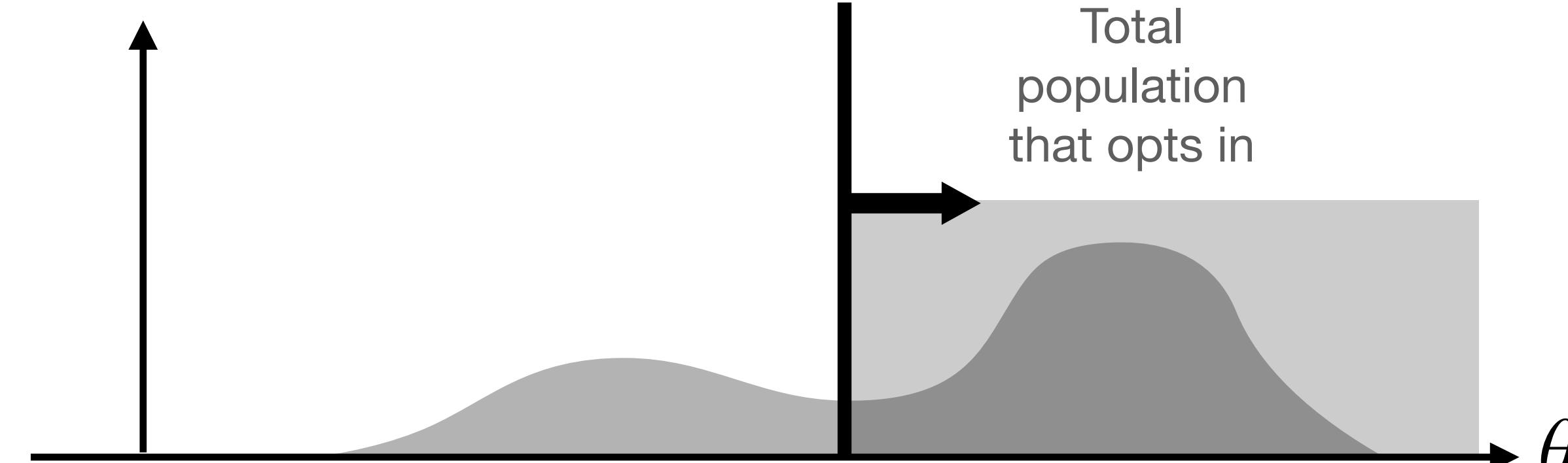
$$m(\lambda)$$



$\lambda$  : Current travel cost

Total population that opts in

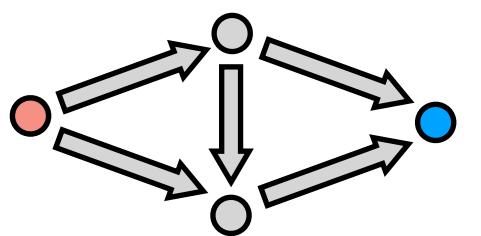
$$dm(\theta)$$



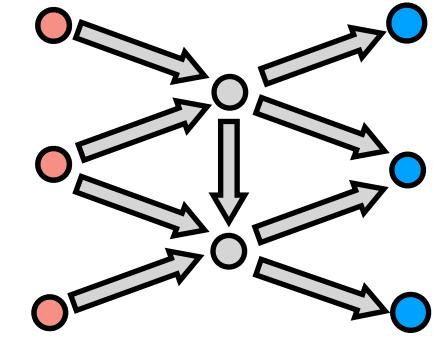
$\theta$  : Max cost  $dm(\theta)$  will pay

# Potential Games

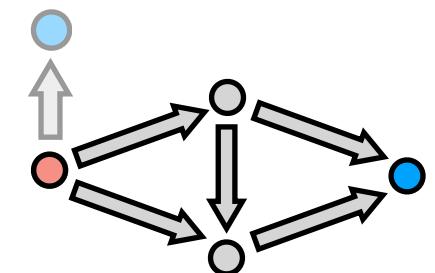
Routing Games



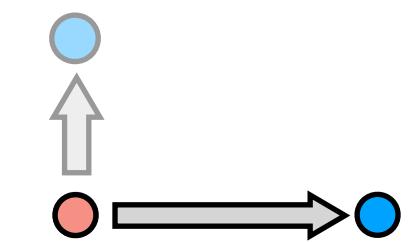
Multiple sources/sinks



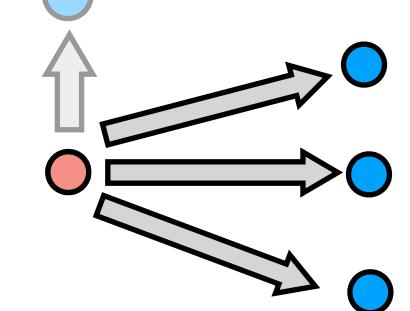
Variable Demand



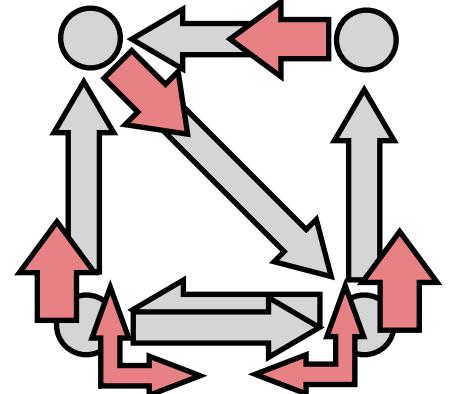
Supply & Demand



Cournot Market

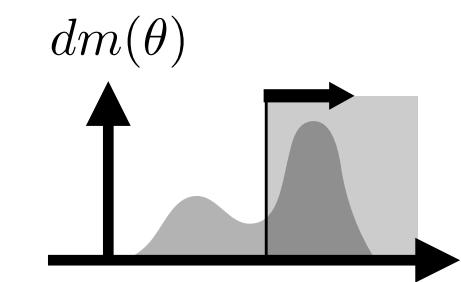


MDP Congestion Game

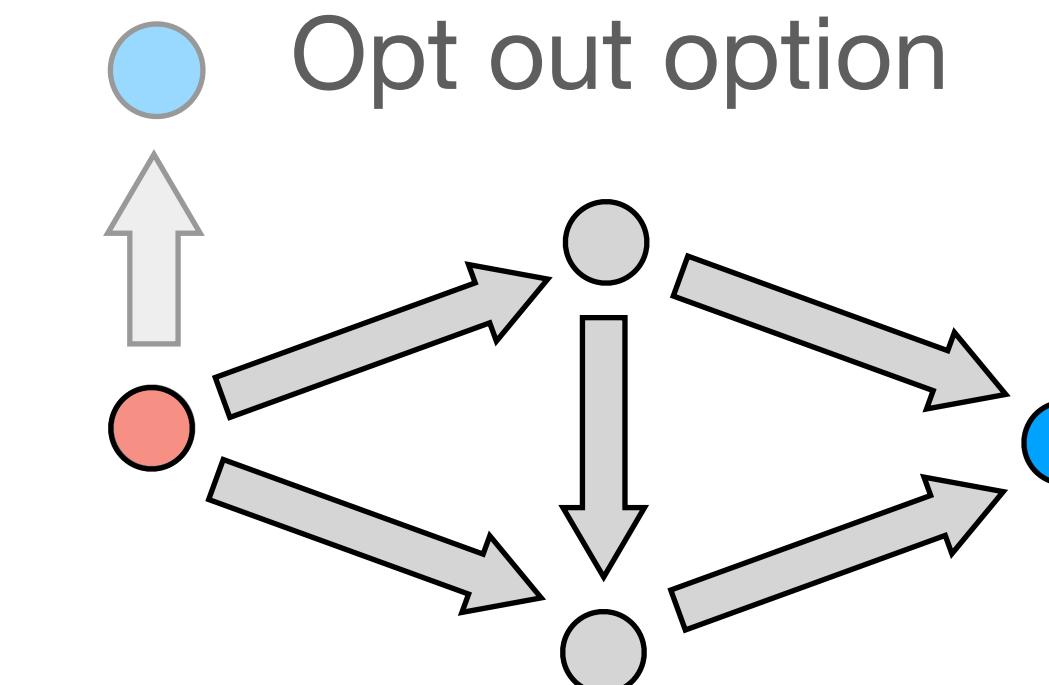


# Variable Demand - Non-Homogeneous Preferences

Non-homo-geneous preferences



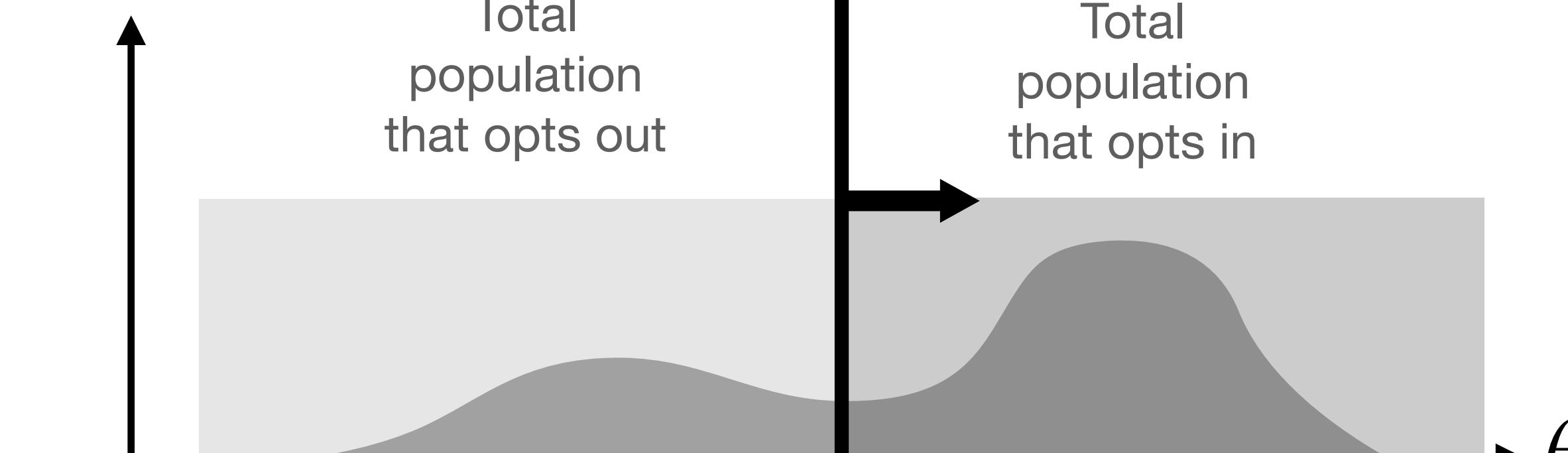
$$m(\lambda)$$



$\lambda$  : Current travel cost

$$dm(\theta)$$

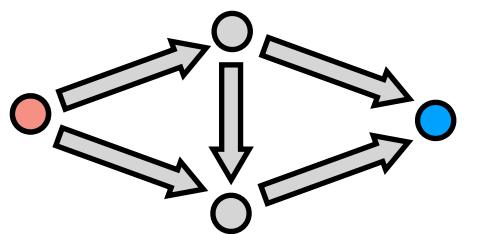
Total population that opts out



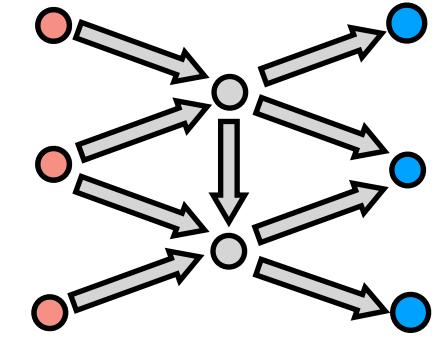
$\theta$  : Max cost  $dm(\theta)$  will pay

# Potential Games

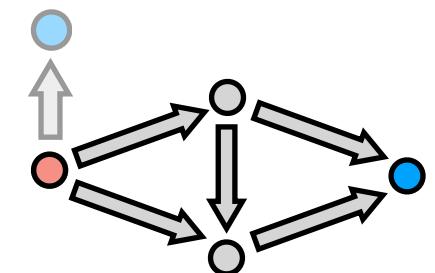
Routing Games



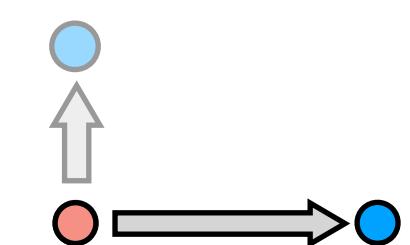
Multiple sources/sinks



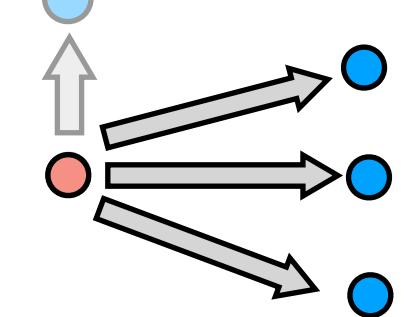
Variable Demand



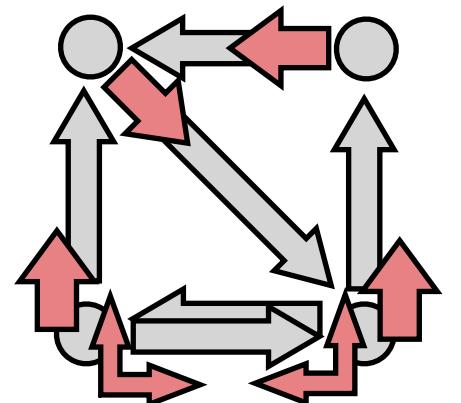
Supply & Demand



Cournot Market

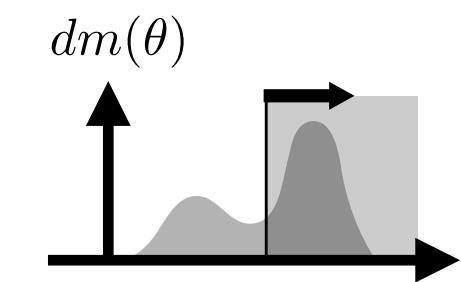


MDP Congestion Game

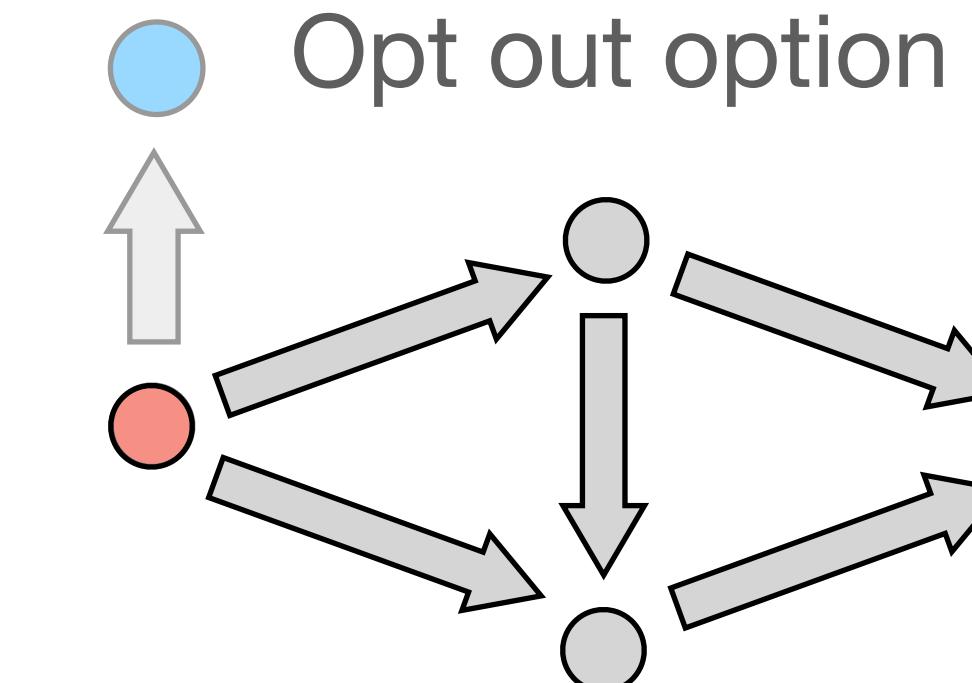


# Variable Demand - Non-Homogeneous Preferences

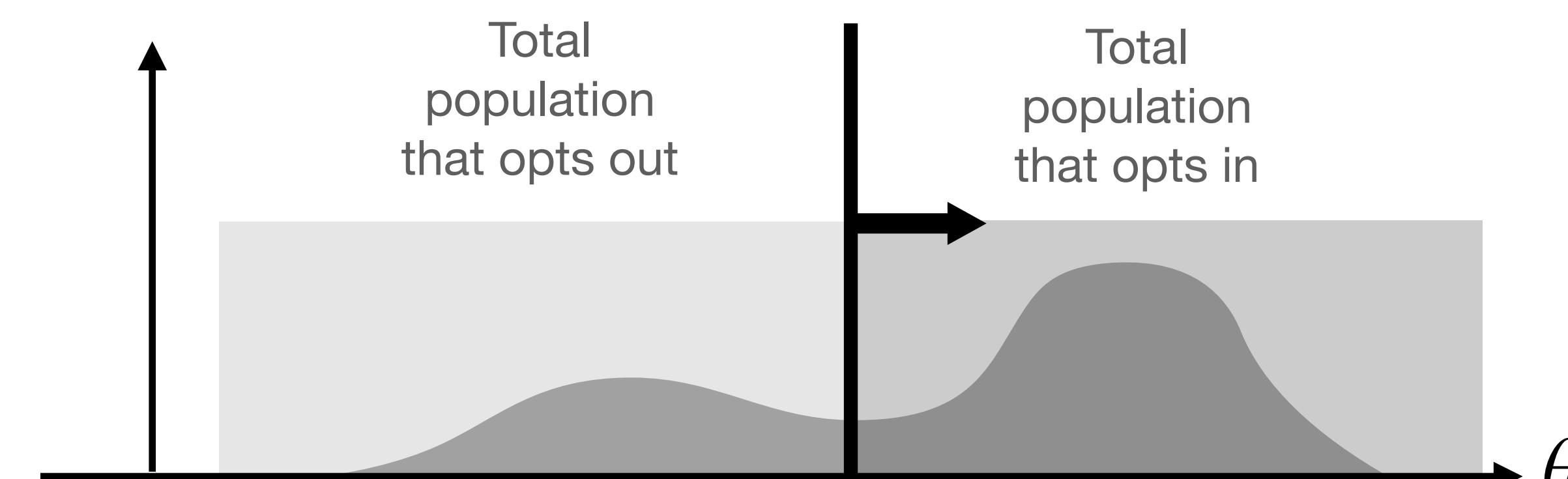
Non-homo-geneous preferences



$$m(\lambda) = \int_{\lambda}^{\infty} dm(\theta)$$



$$dm(\theta)$$



$\lambda$  : Current travel cost

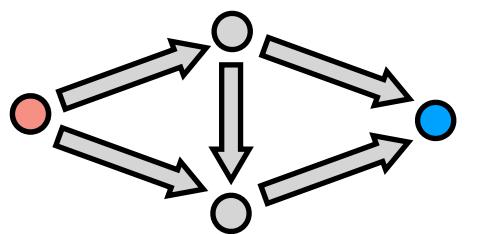
Total population that opts out

Total population that opts in

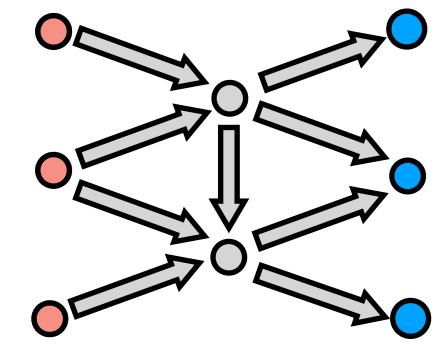
$\theta$  : Max cost  $dm(\theta)$  will pay

# Potential Games

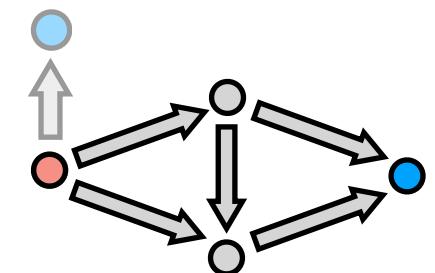
Routing Games



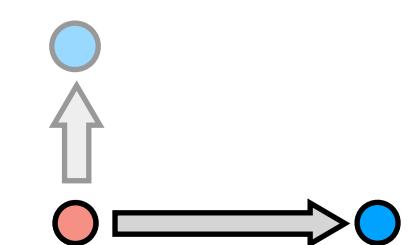
Multiple sources/sinks



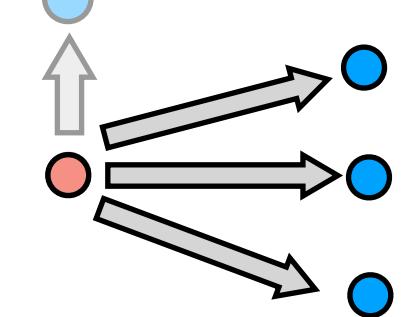
Variable Demand



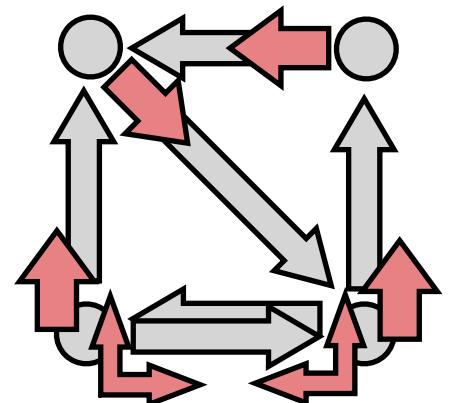
Supply & Demand



Cournot Market

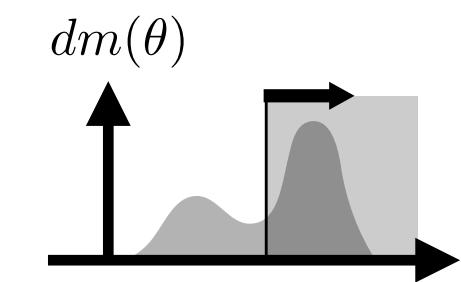


MDP Congestion Game



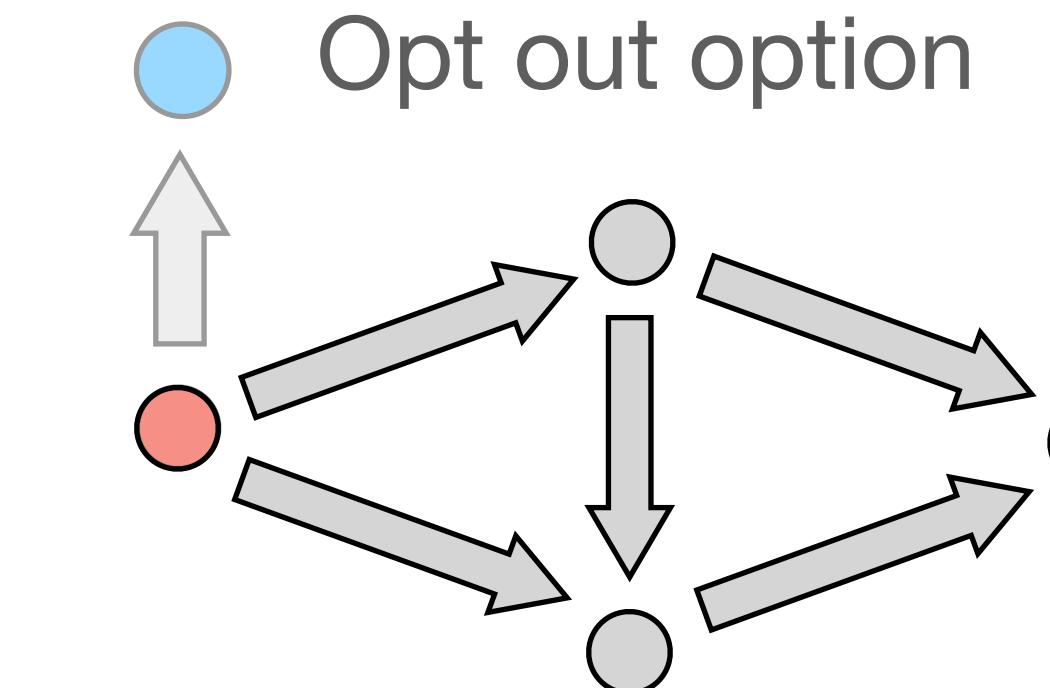
# Variable Demand - Non-Homogeneous Preferences

Non-homo-geneous preferences

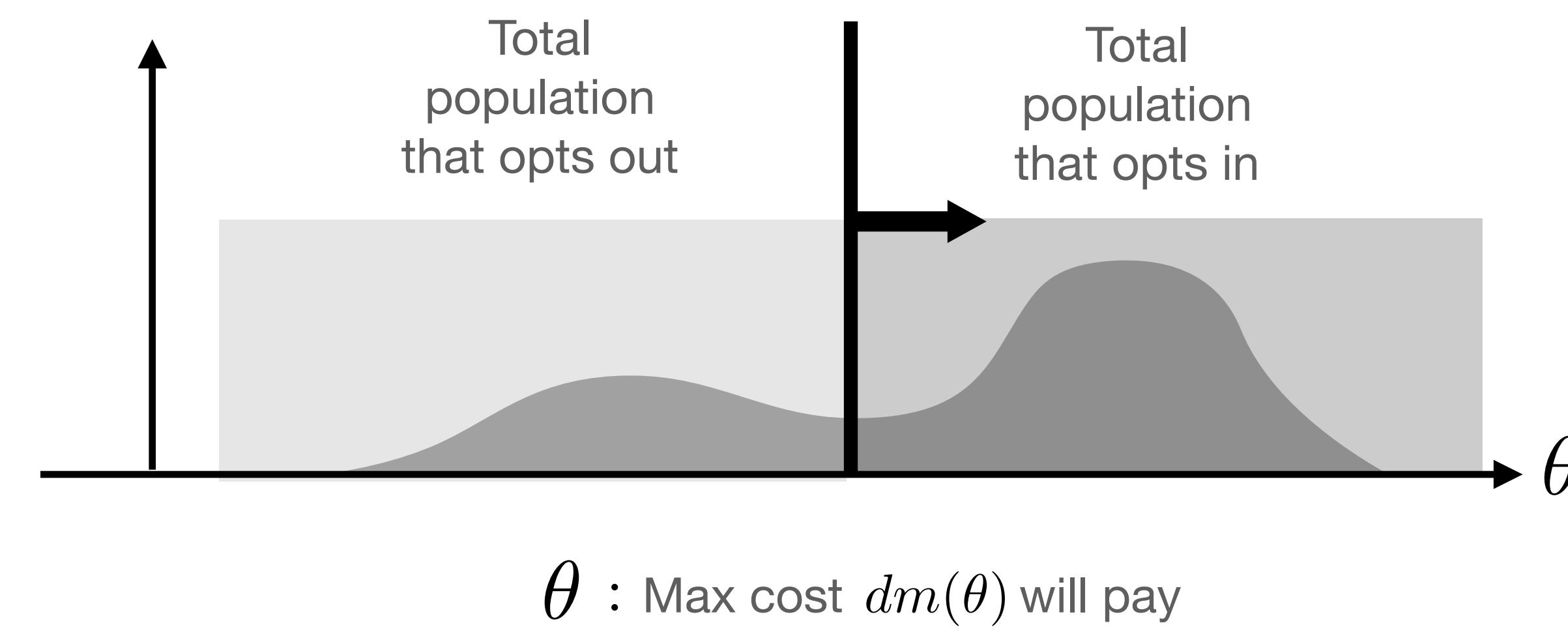


Multi-Variate Preferences

$$m(\lambda) = \int_{\lambda}^{\infty} dm(\theta)$$

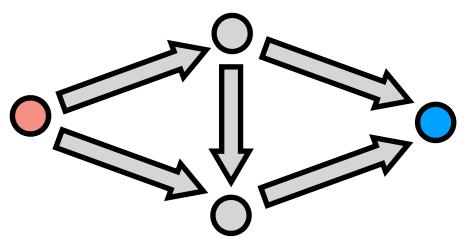


$$dm(\theta)$$

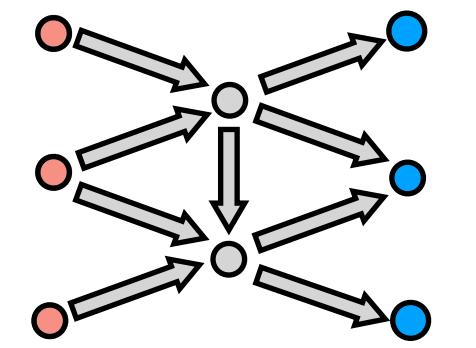


# Potential Games

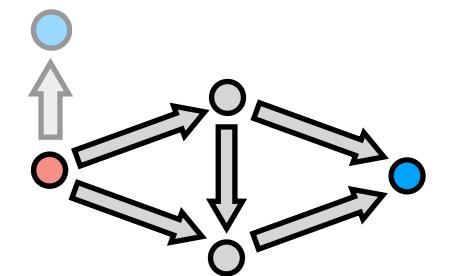
Routing Games



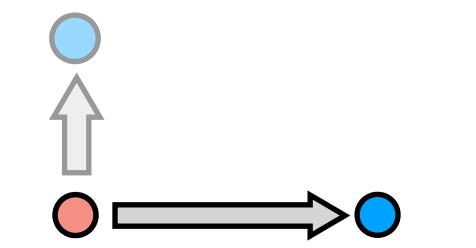
Multiple sources/sinks



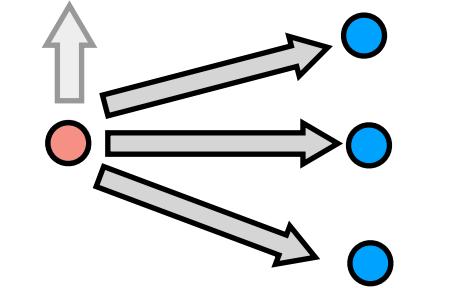
Variable Demand



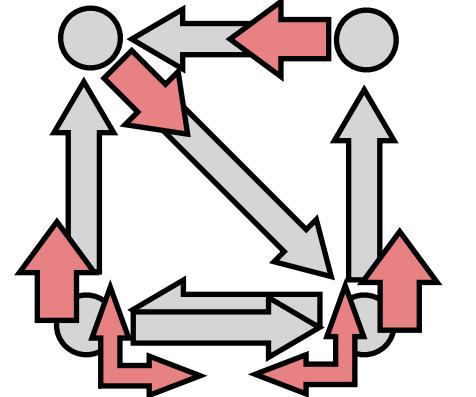
Supply & Demand



Cournot Market

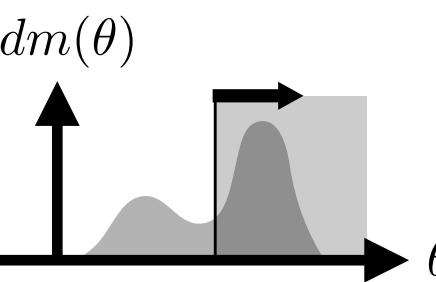


MDP Congestion Game

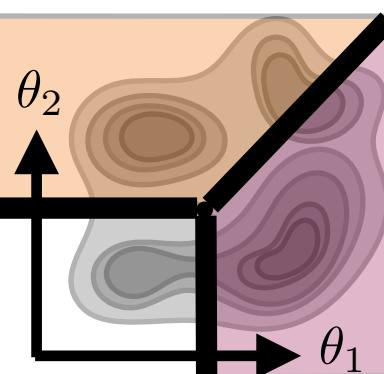


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

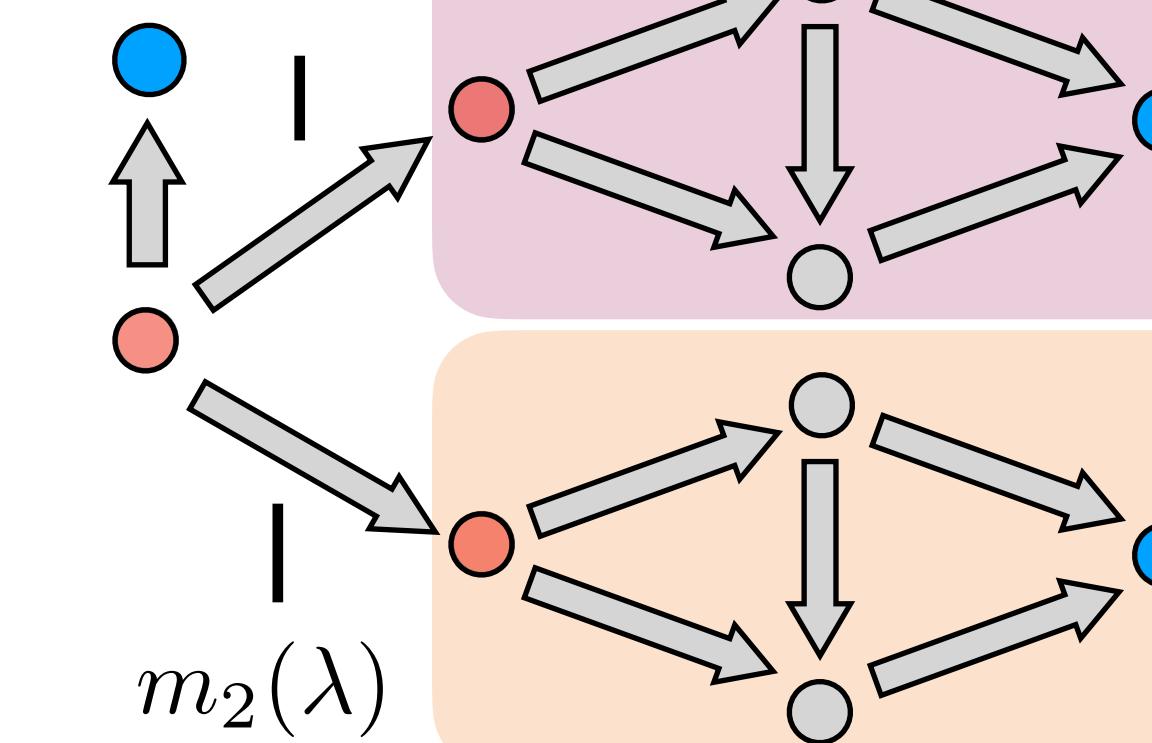
Non-homo-geneous preferences



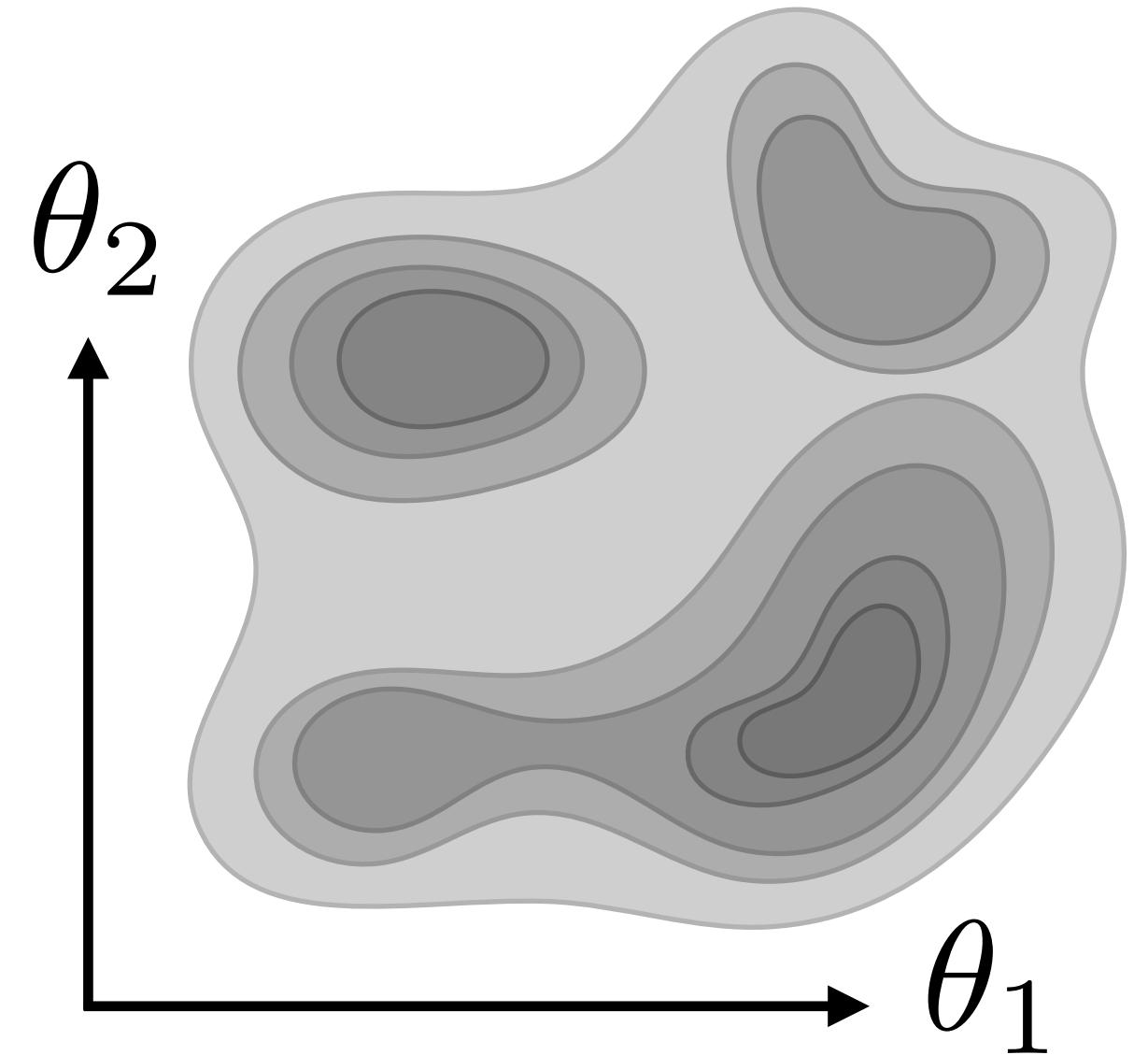
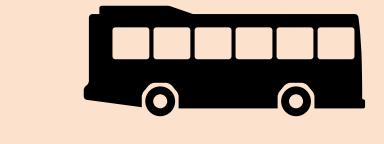
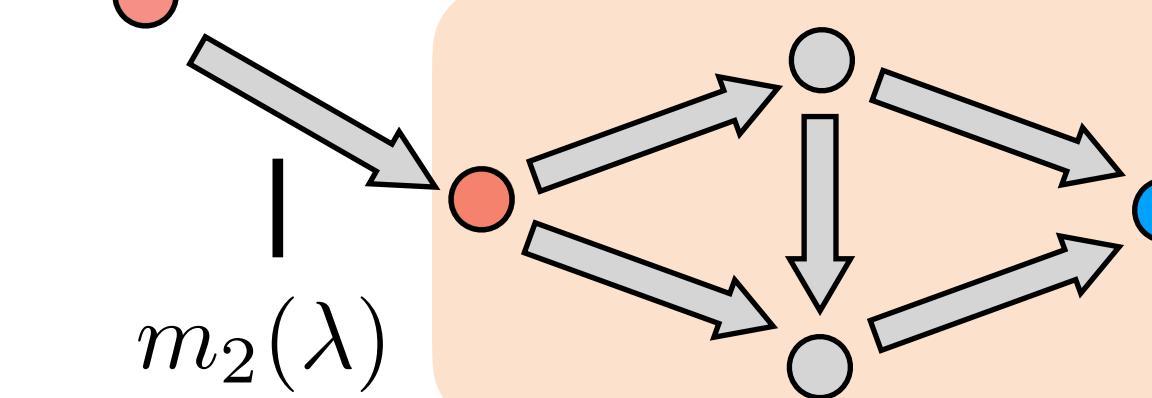
Multi-Variate Preferences



$m_1(\lambda)$

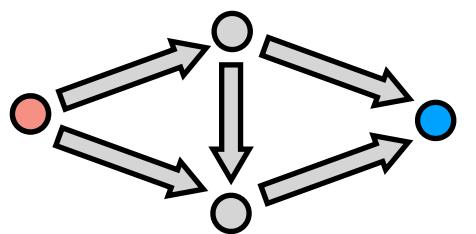


$m_2(\lambda)$

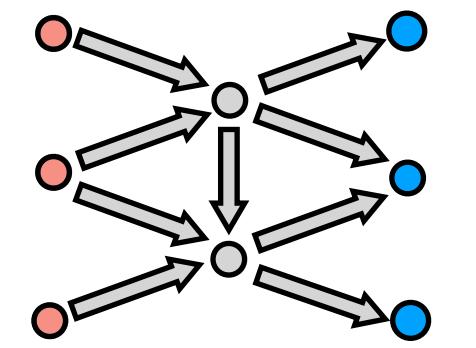


# Potential Games

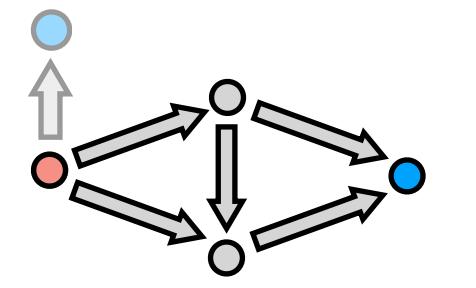
Routing Games



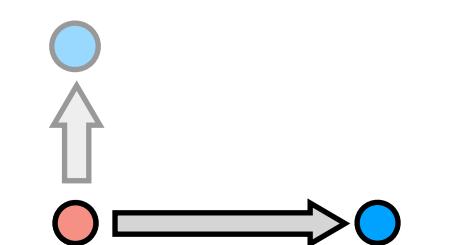
Multiple sources/sinks



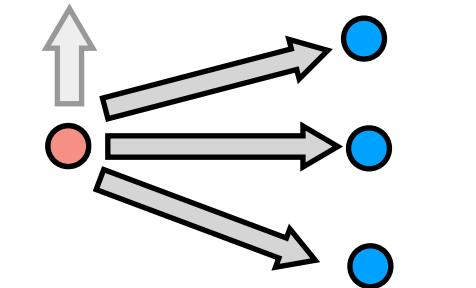
Variable Demand



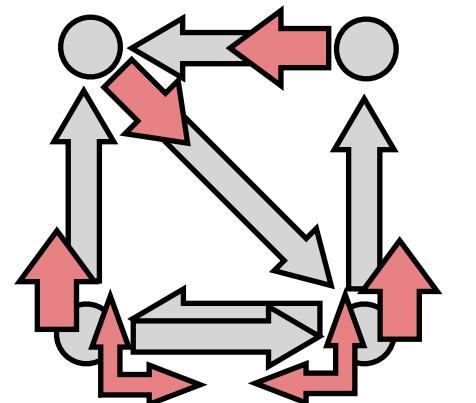
Supply & Demand



Cournot Market

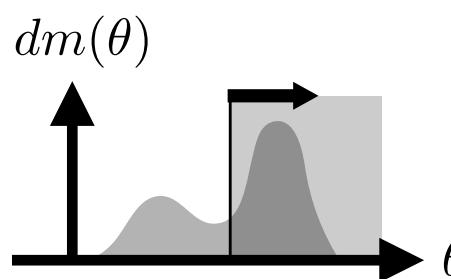


MDP Congestion Game

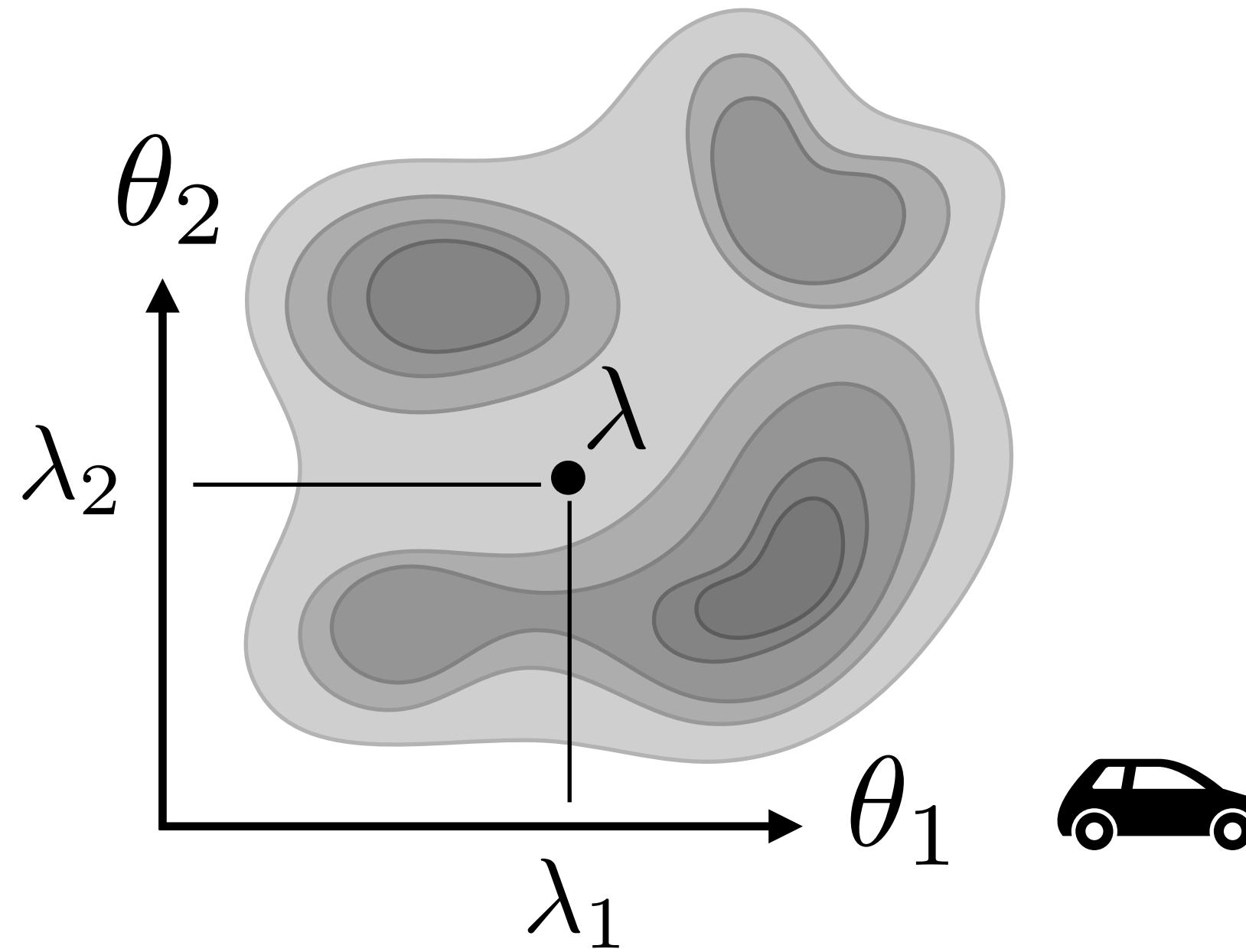
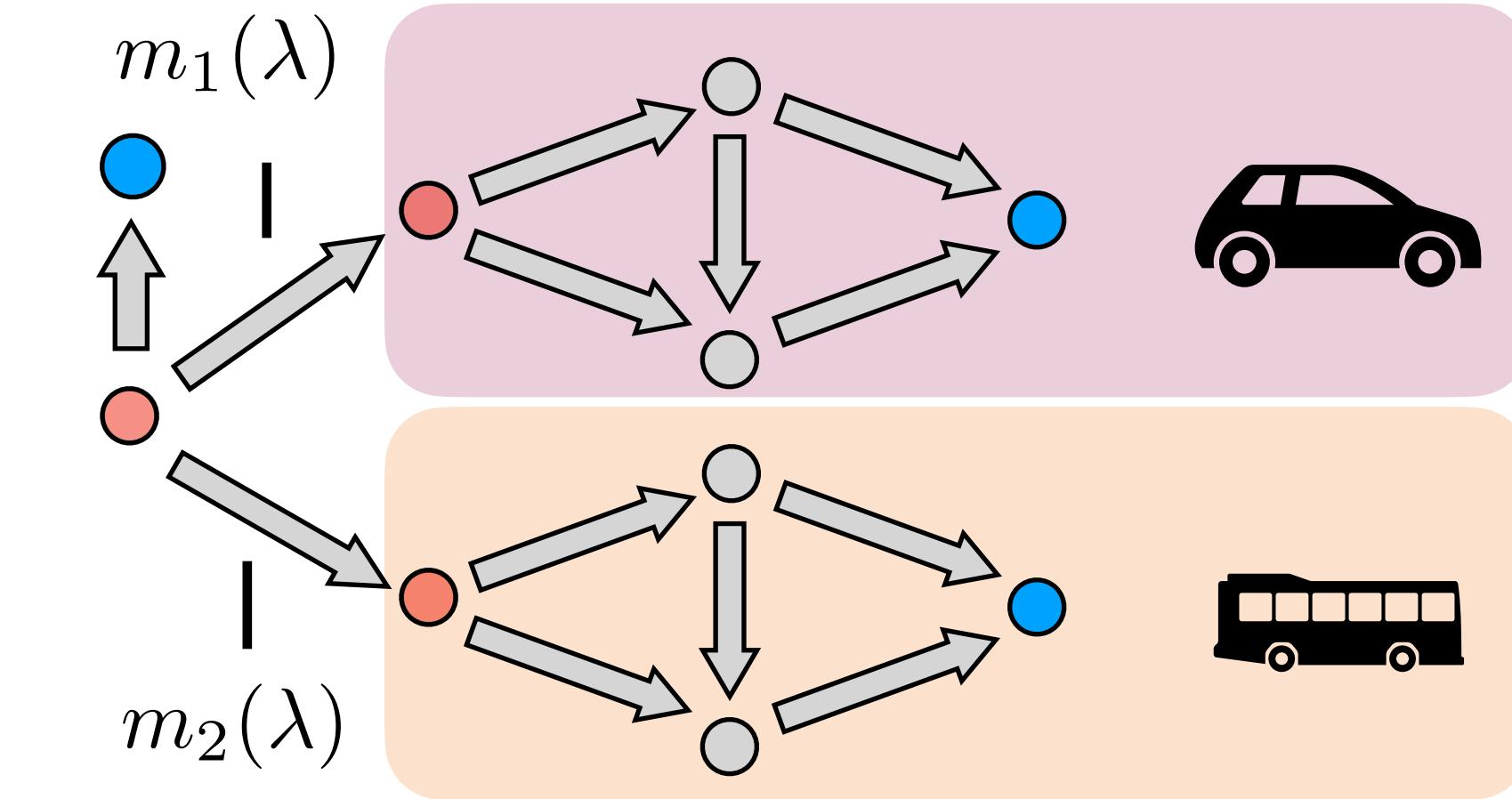
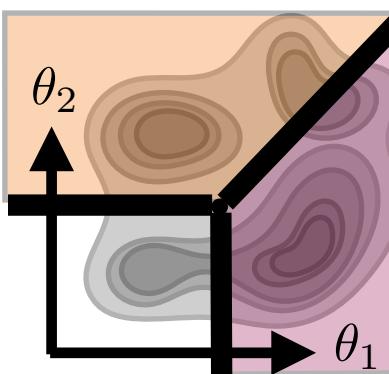


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

Non-homo-geneous preferences

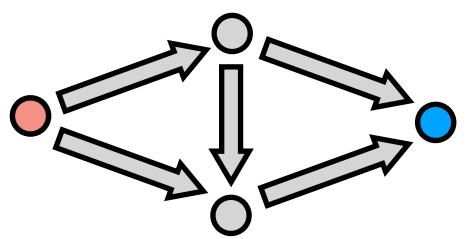


Multi-Variate Preferences

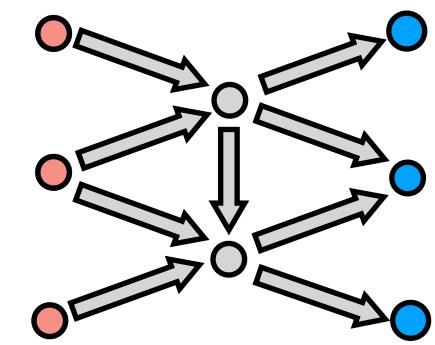


# Potential Games

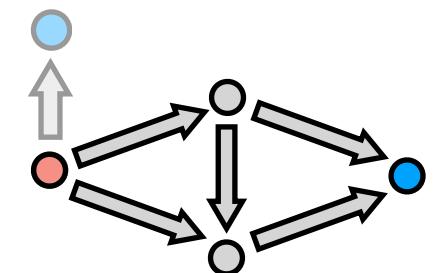
Routing Games



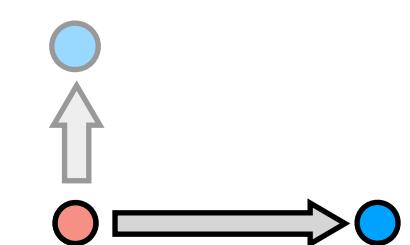
Multiple sources/sinks



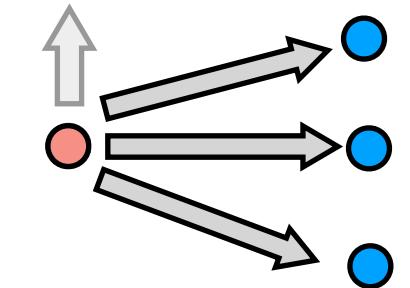
Variable Demand



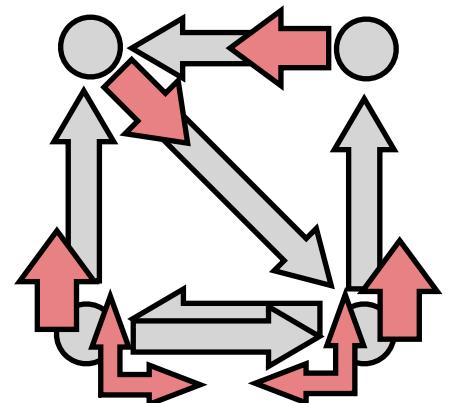
Supply & Demand



Cournot Market

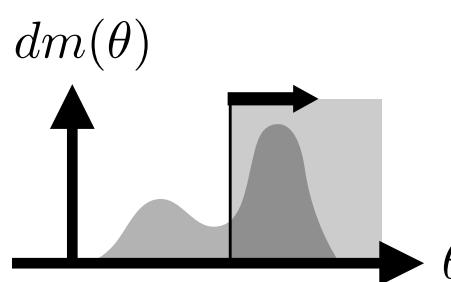


MDP Congestion Game

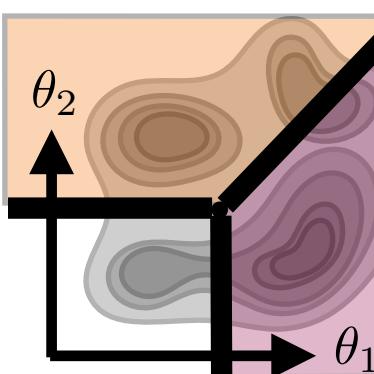


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

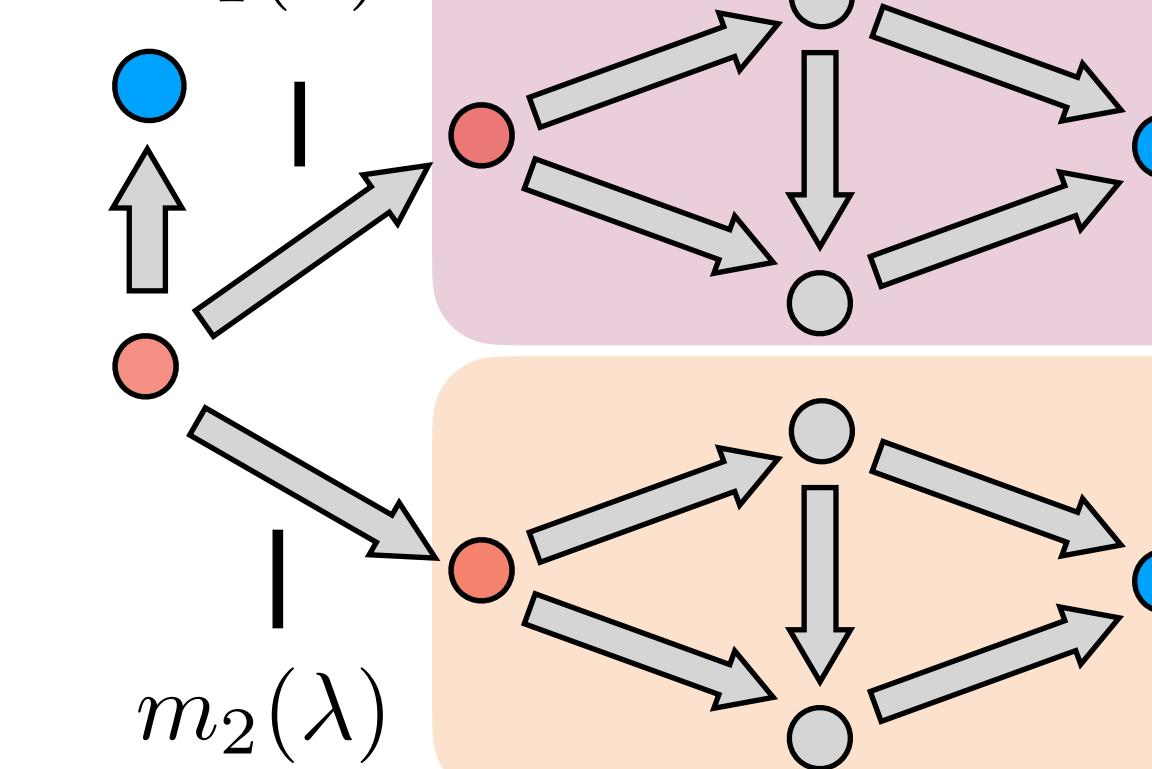
Non-homo-geneous preferences



Multi-Variate Preferences



$$m_1(\lambda)$$

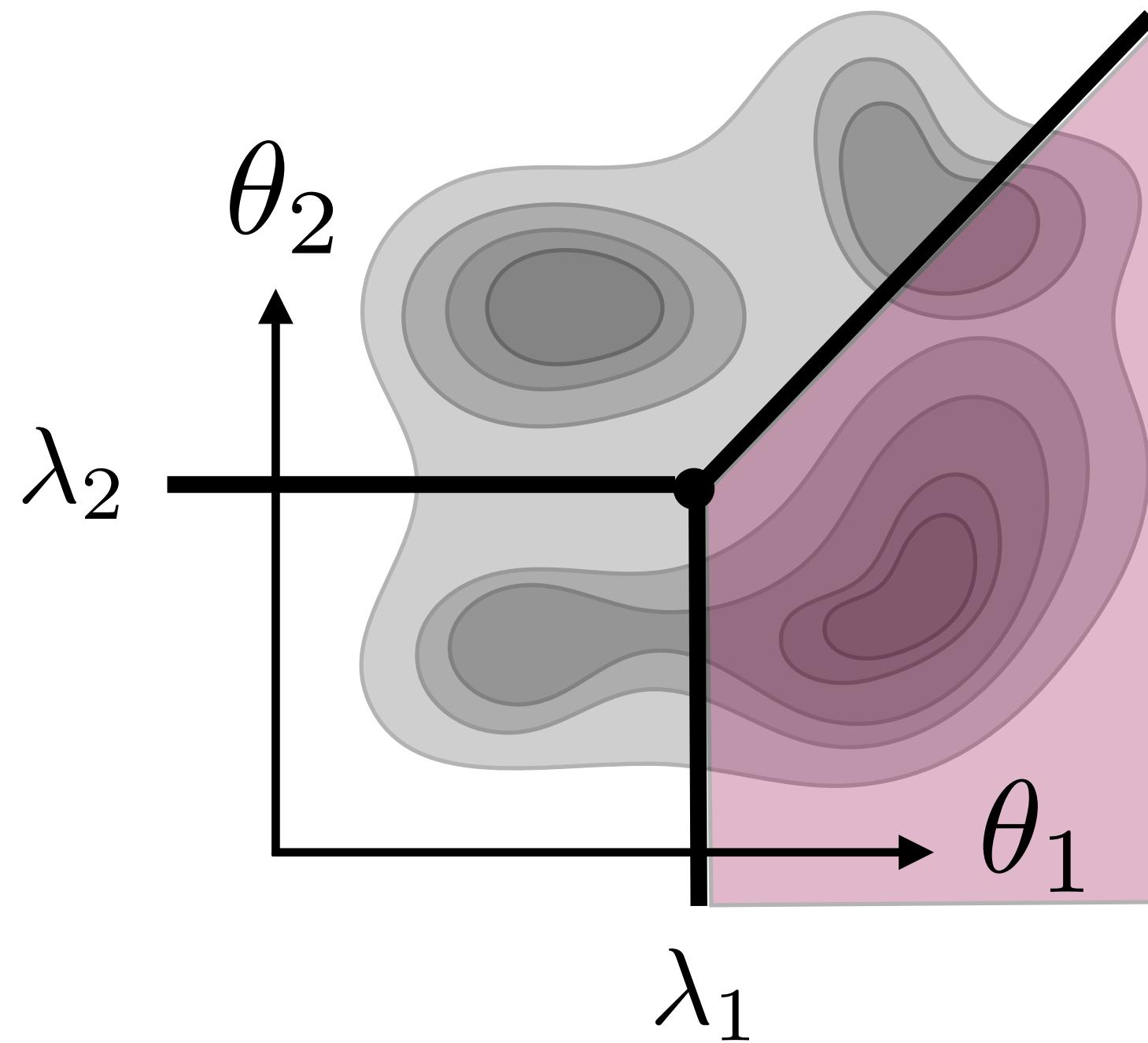


$$m_2(\lambda)$$



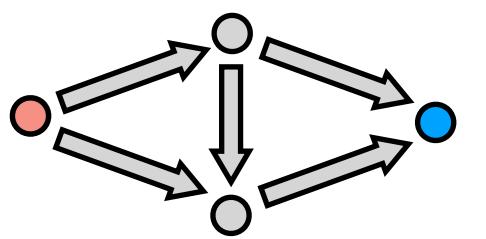
$$m_1(\lambda) = \int_{A_1(\lambda)} dm(\theta)$$

$$A_1(\lambda)$$

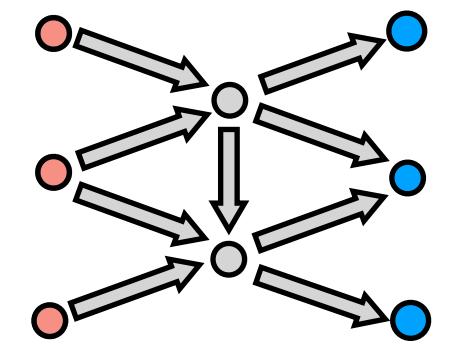


# Potential Games

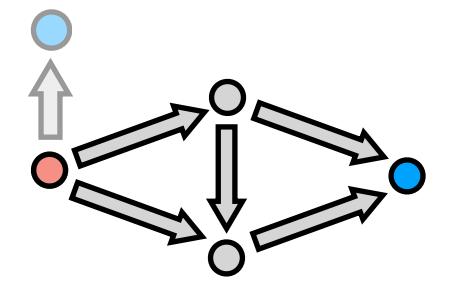
Routing Games



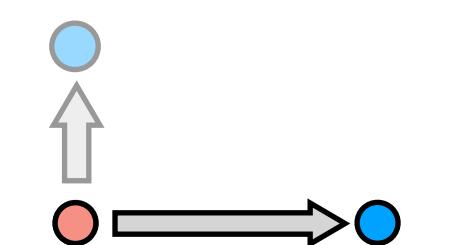
Multiple sources/sinks



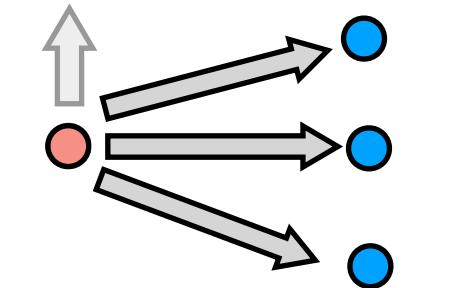
Variable Demand



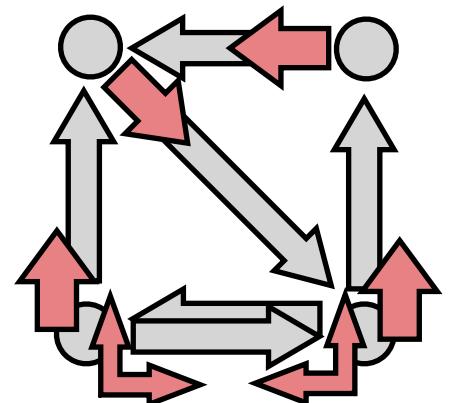
Supply & Demand



Cournot Market

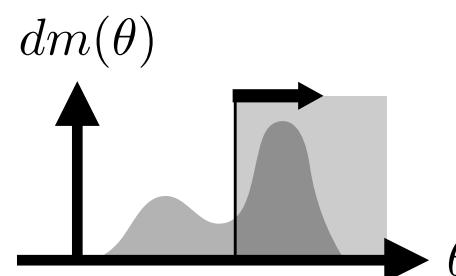


MDP Congestion Game

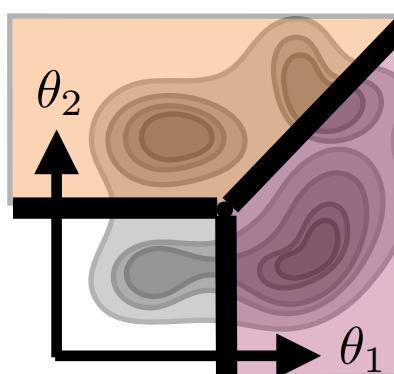


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

Non-homo-geneous preferences



Multi-Variate Preferences



$$A_2(\lambda)$$

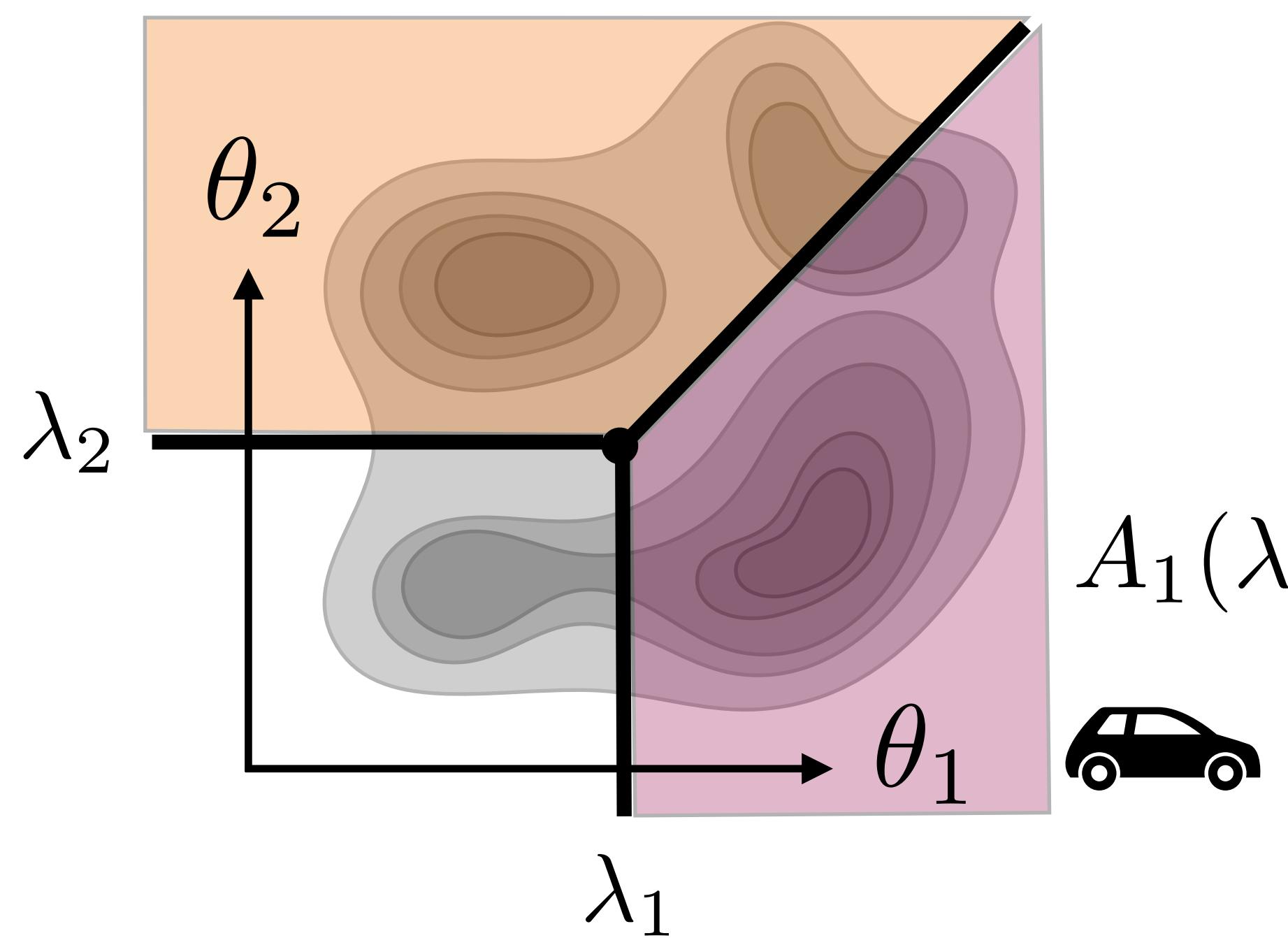
$$m_1(\lambda)$$

$$m_2(\lambda)$$



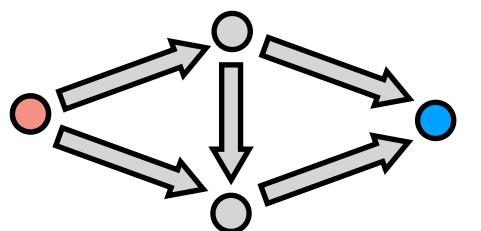
$$m_1(\lambda) = \int_{A_1(\lambda)} dm(\theta)$$

$$m_2(\lambda) = \int_{A_2(\lambda)} dm(\theta)$$

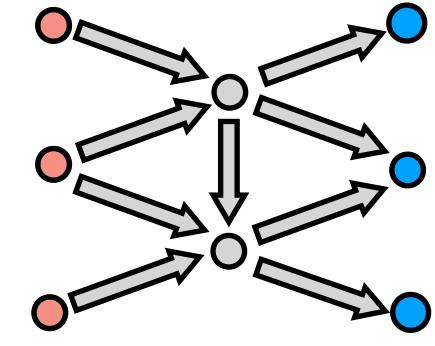


# Potential Games

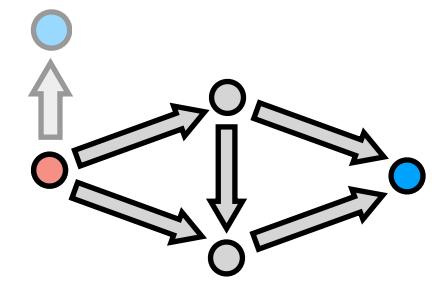
Routing Games



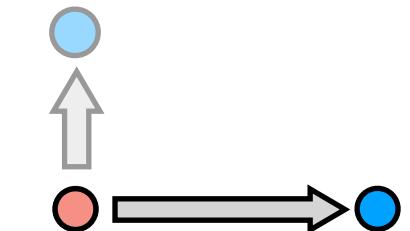
Multiple sources/sinks



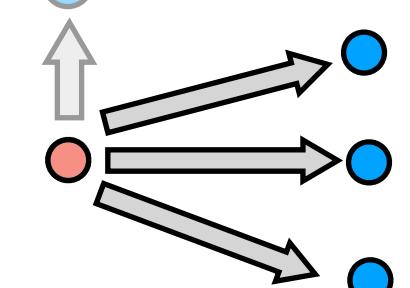
Variable Demand



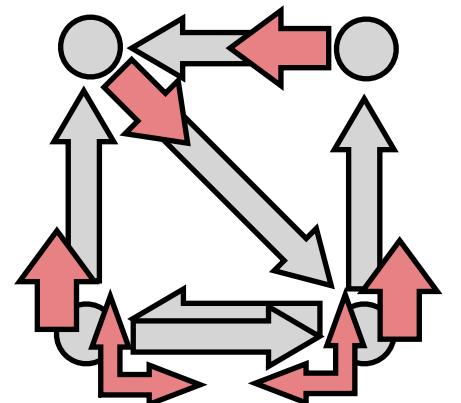
Supply & Demand



Cournot Market

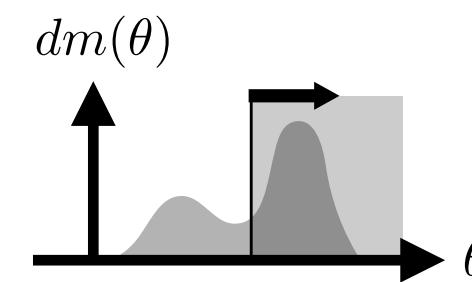


MDP Congestion Game

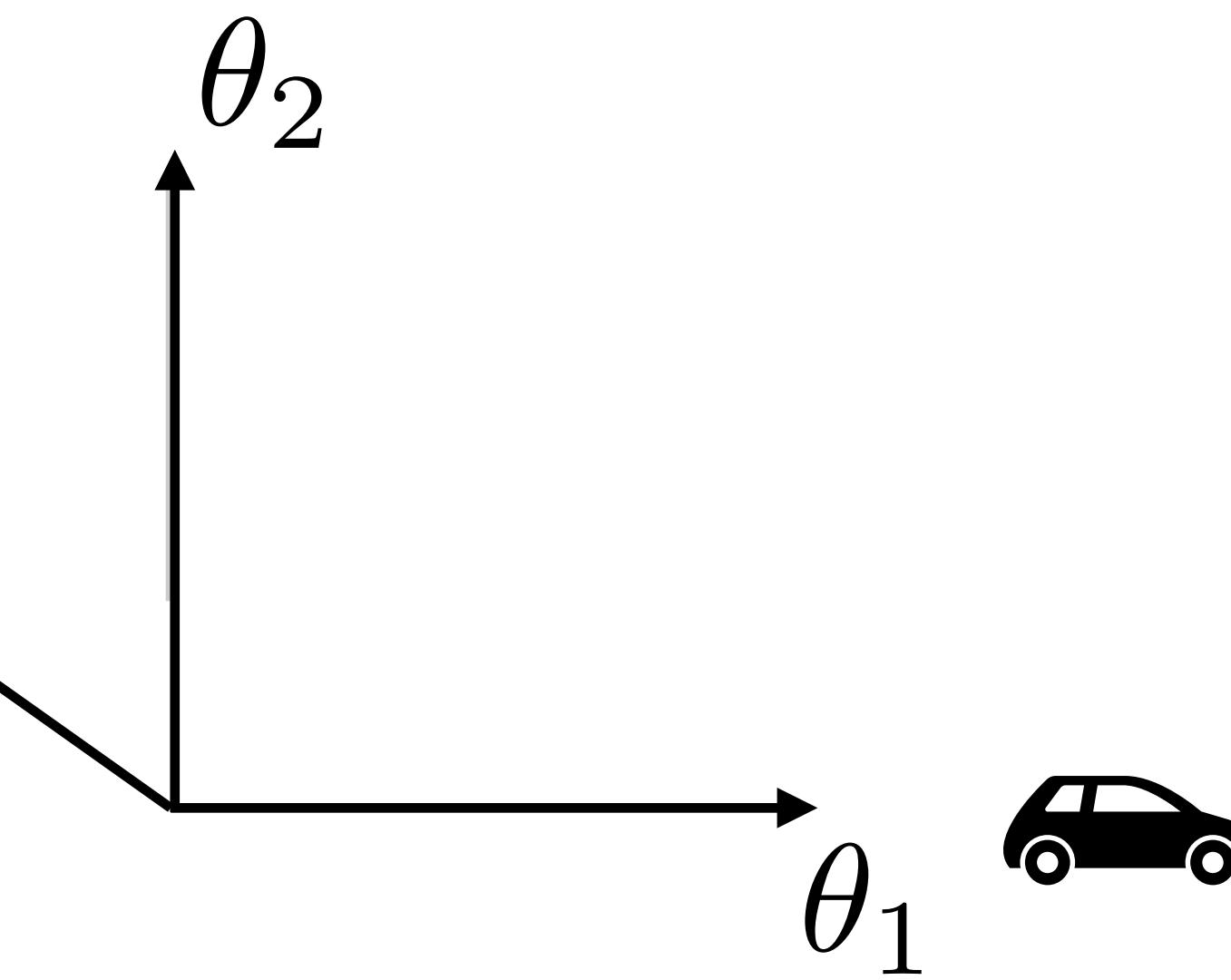
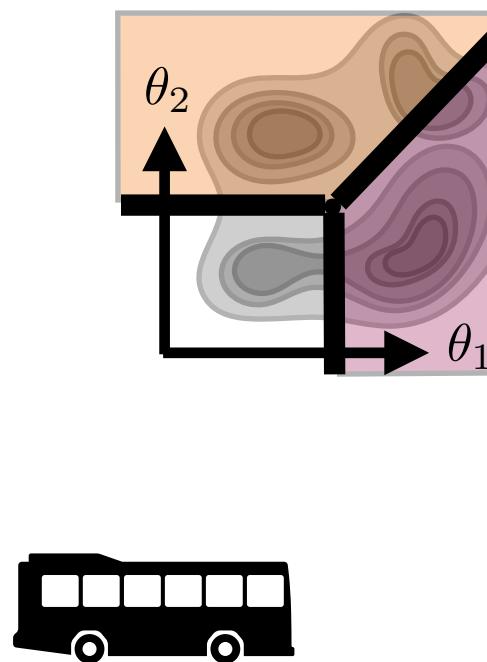


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

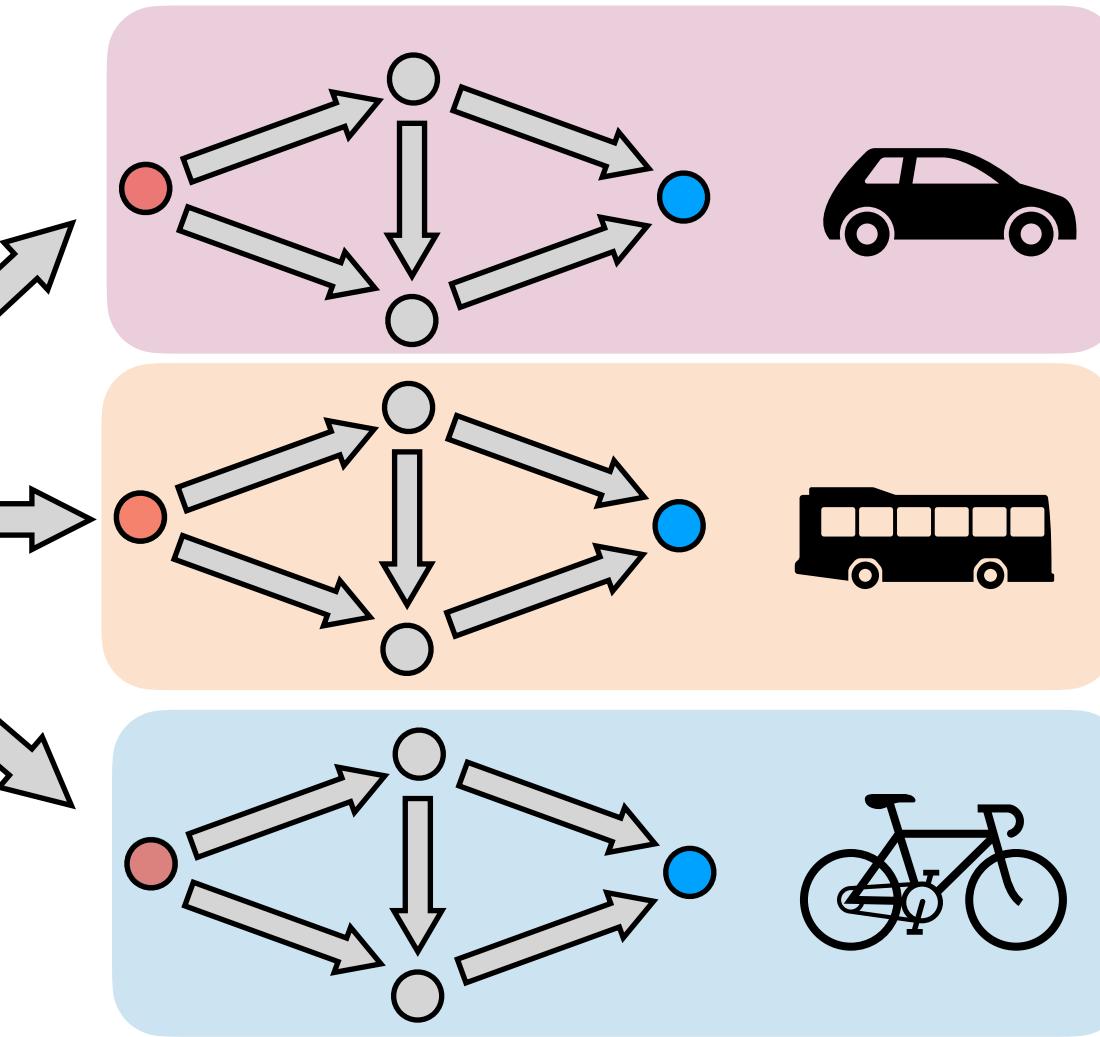
Non-homo-geneous preferences



Multi-Variate Preferences

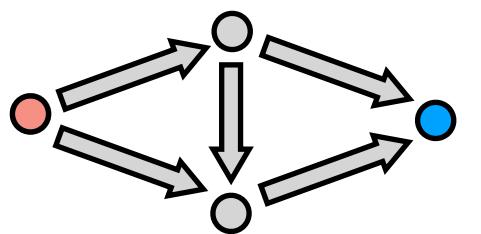


$m_1(\lambda)$   
 $m_2(\lambda)$   
 $m_3(\lambda)$

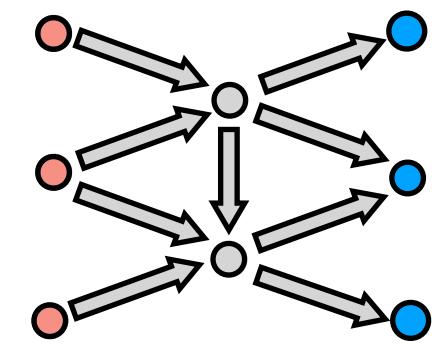


# Potential Games

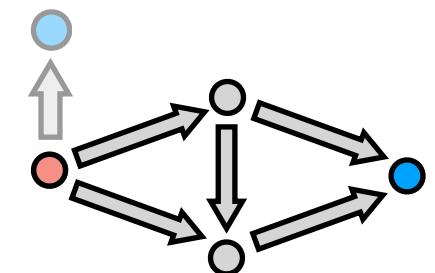
Routing Games



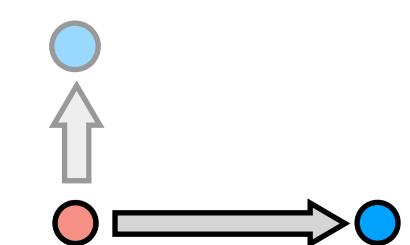
Multiple sources/sinks



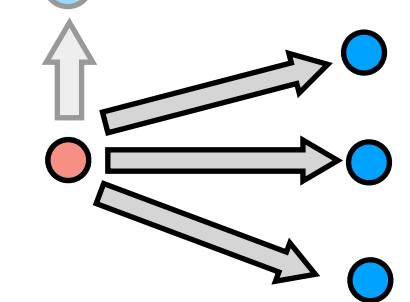
Variable Demand



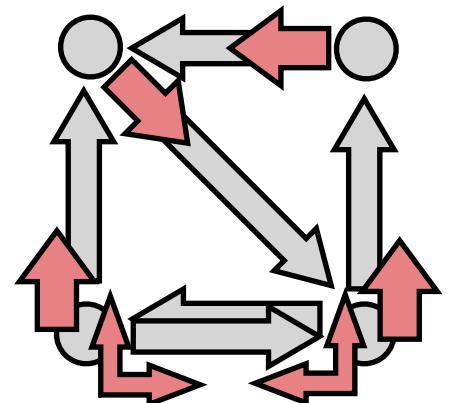
Supply & Demand



Cournot Market

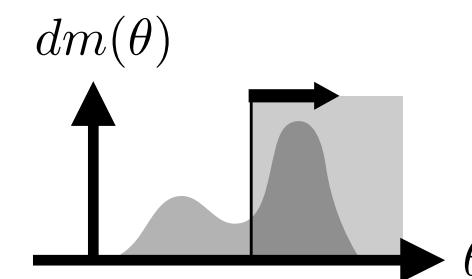


MDP Congestion Game

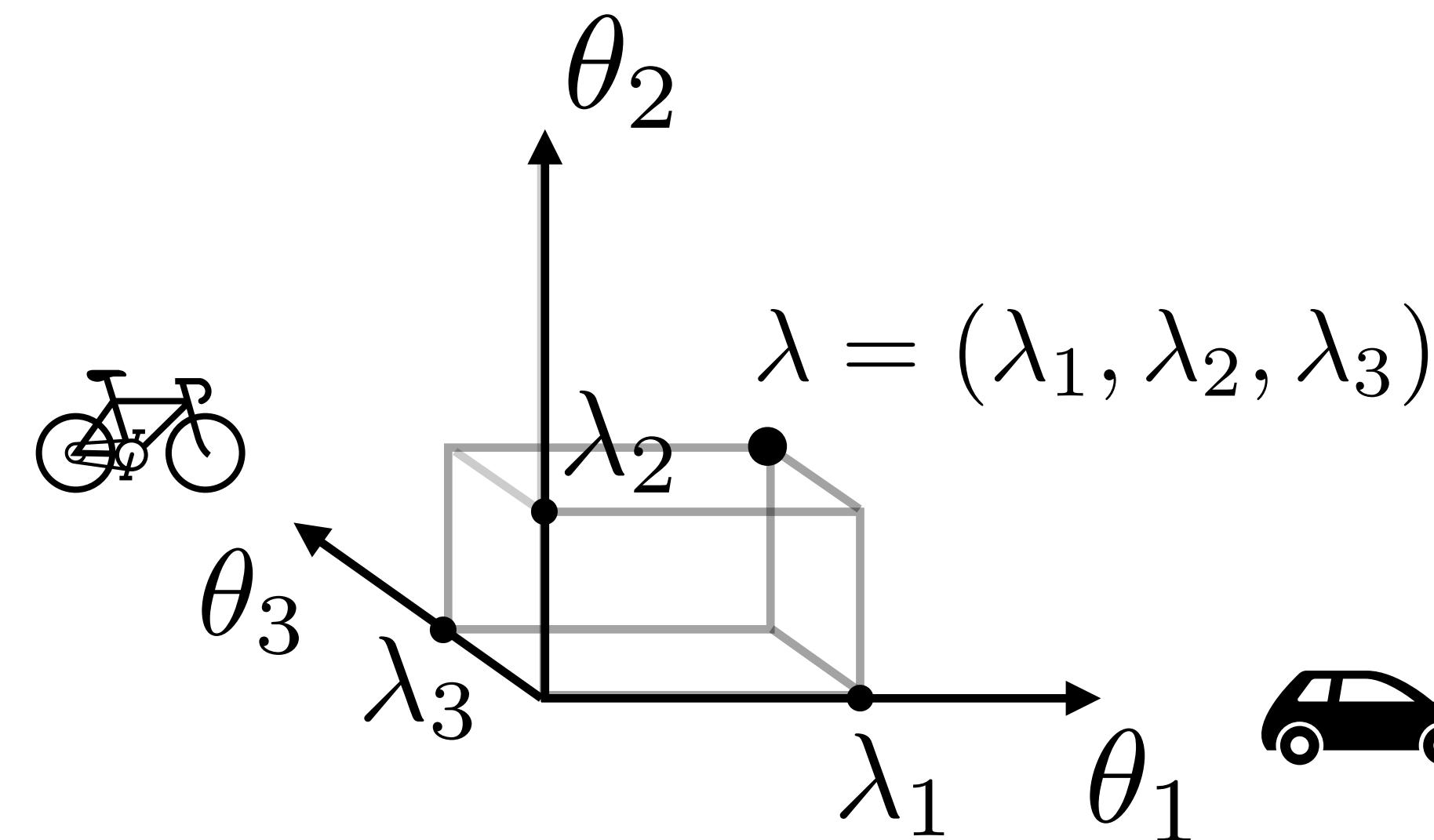
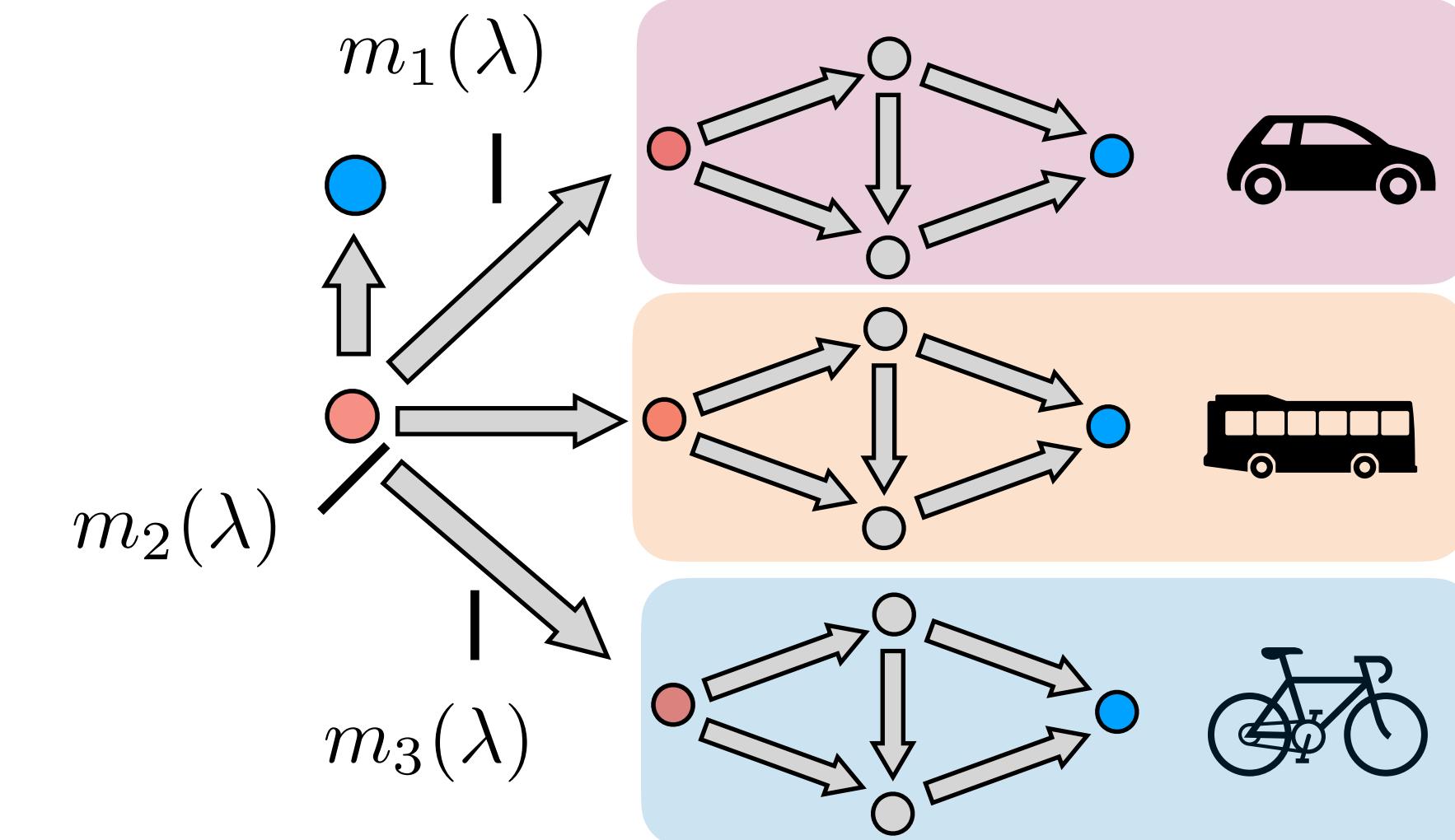
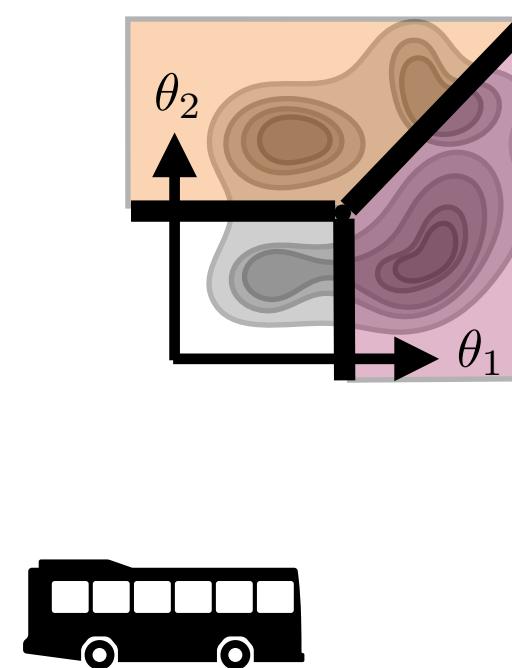


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

Non-homo-geneous preferences

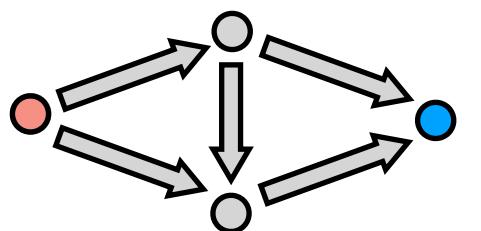


Multi-Variate Preferences

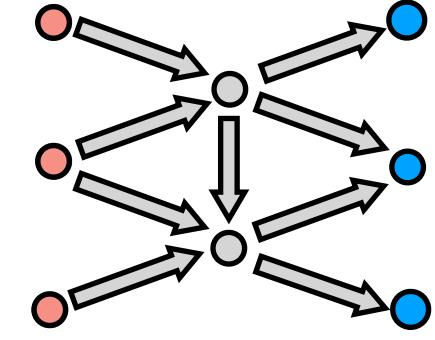


# Potential Games

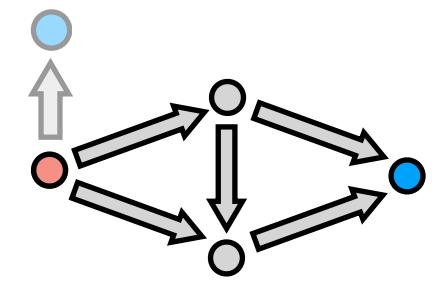
Routing Games



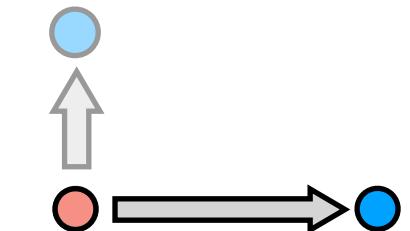
Multiple sources/sinks



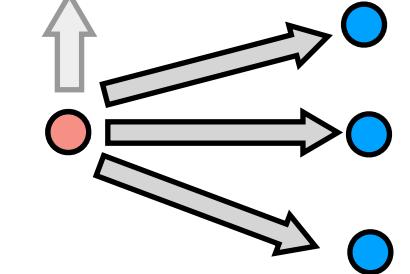
Variable Demand



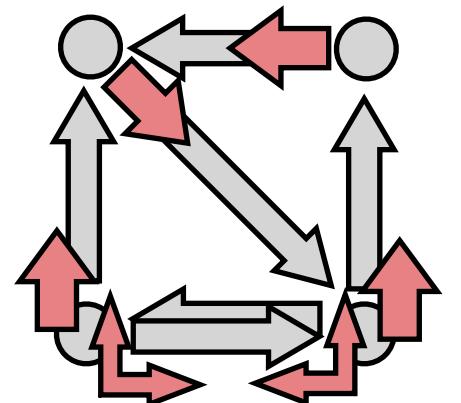
Supply & Demand



Cournot Market

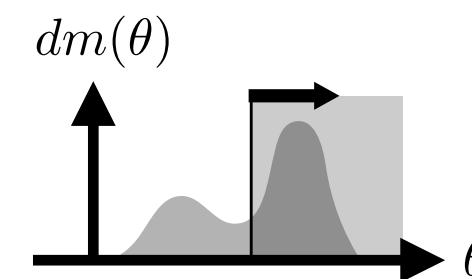


MDP Congestion Game

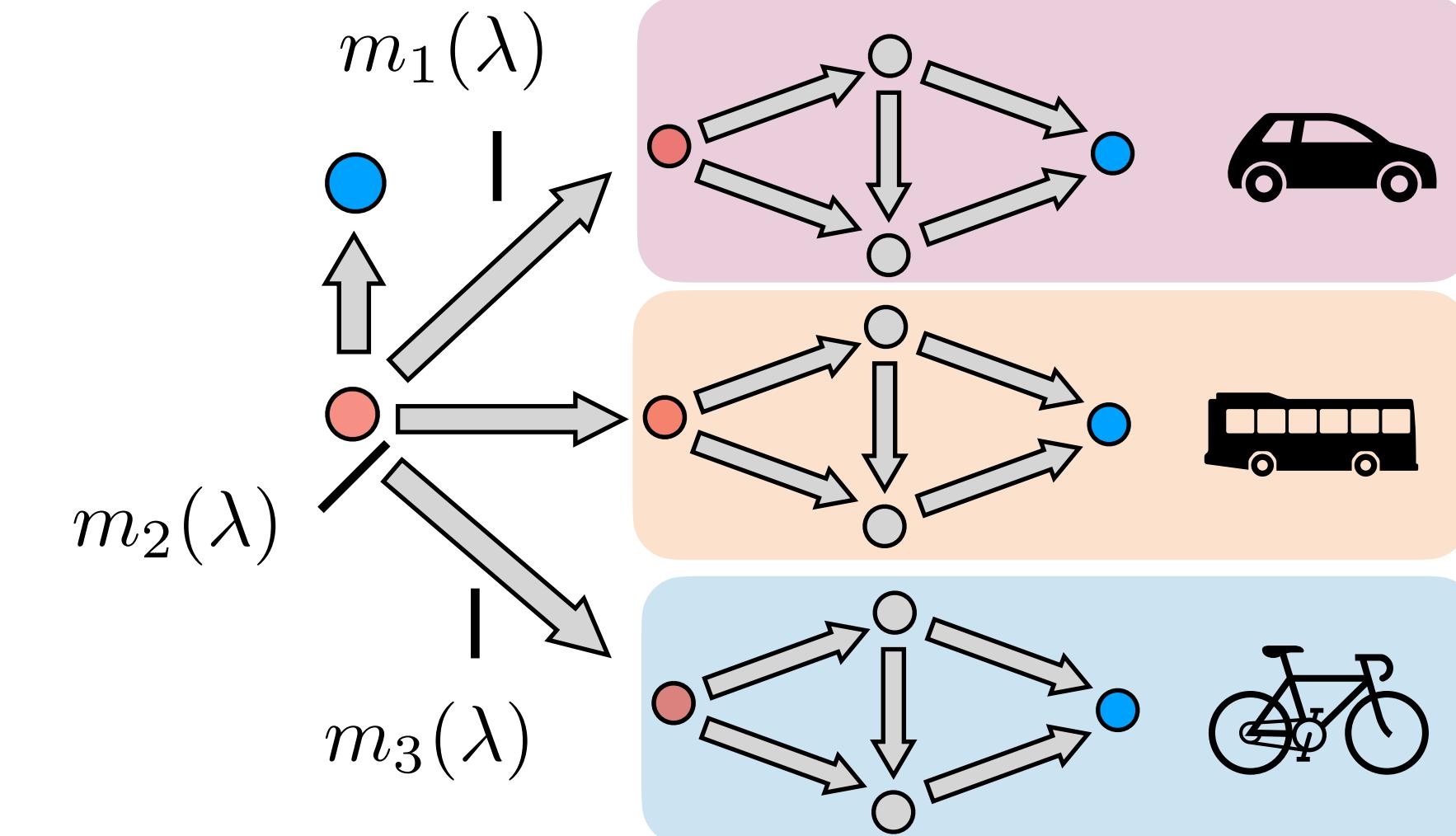
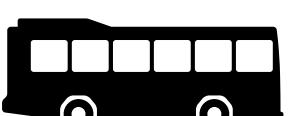
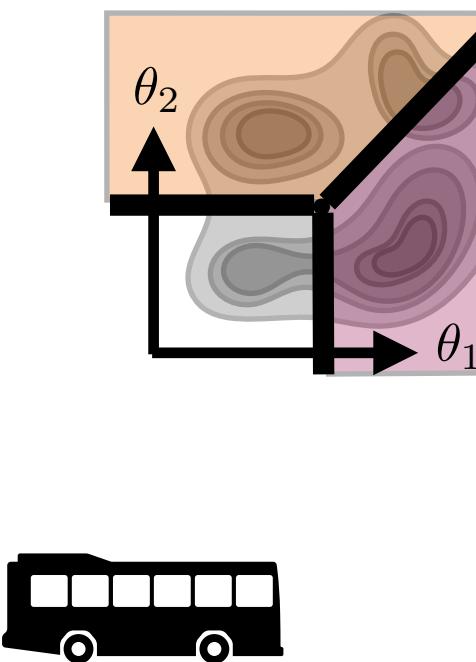


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

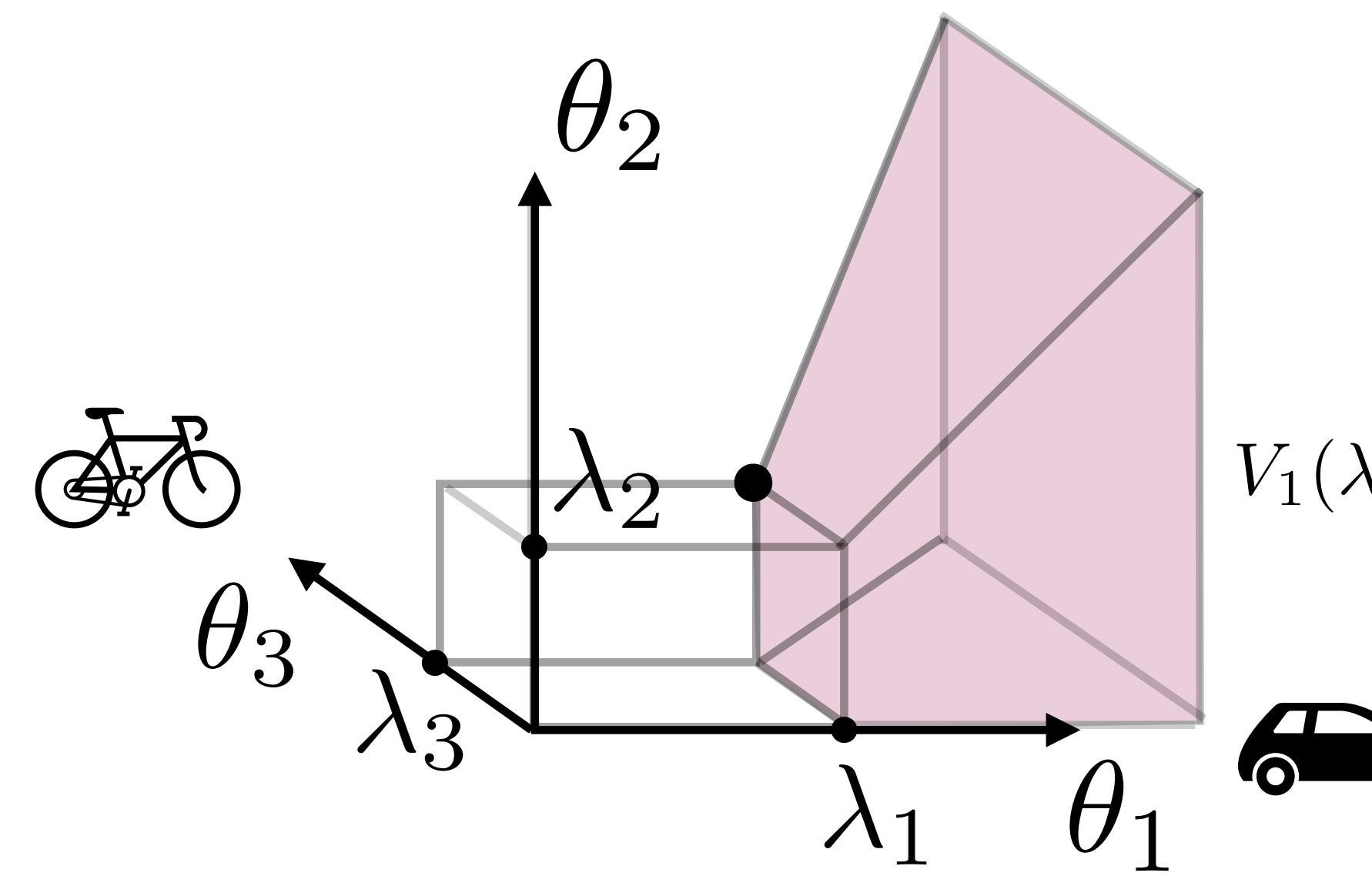
Non-homo-geneous preferences



Multi-Variate Preferences

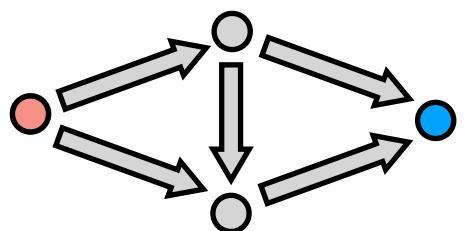


$$m_1(\lambda) = \int_{V_1(\lambda)} dm(\theta)$$

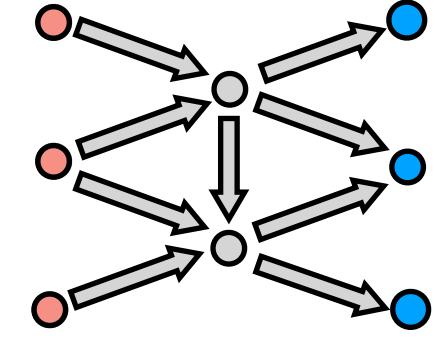


# Potential Games

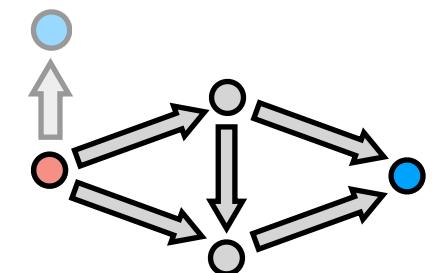
Routing Games



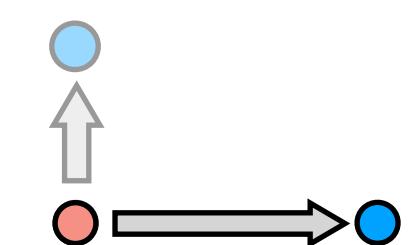
Multiple sources/sinks



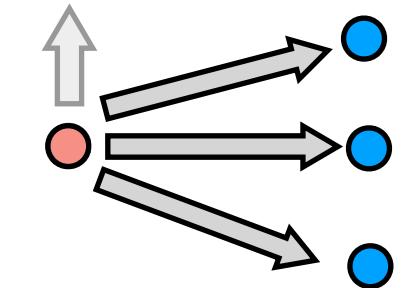
Variable Demand



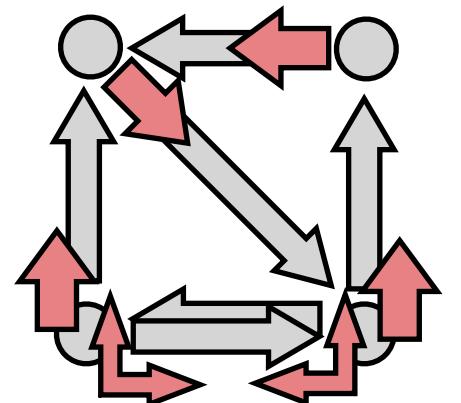
Supply & Demand



Cournot Market

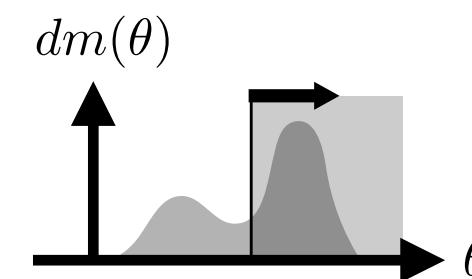


MDP Congestion Game

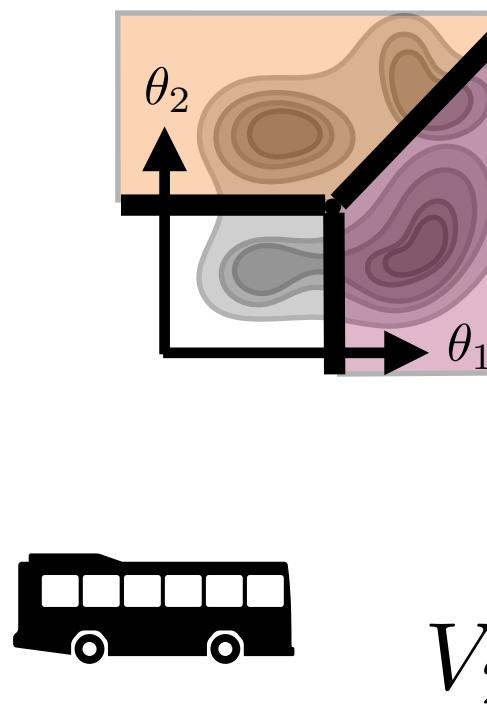


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

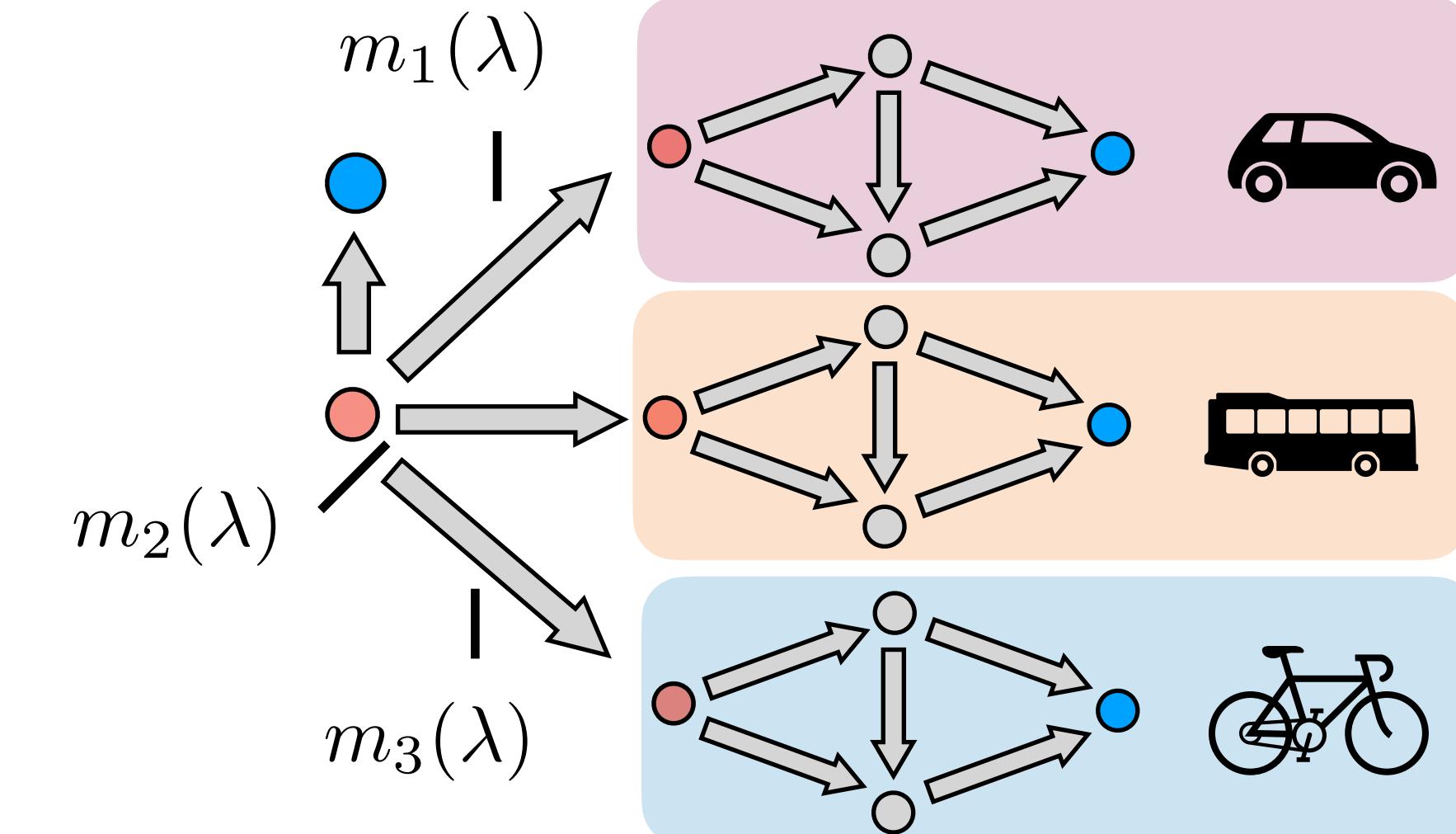
Non-homo-geneous preferences



Multi-Variate Preferences

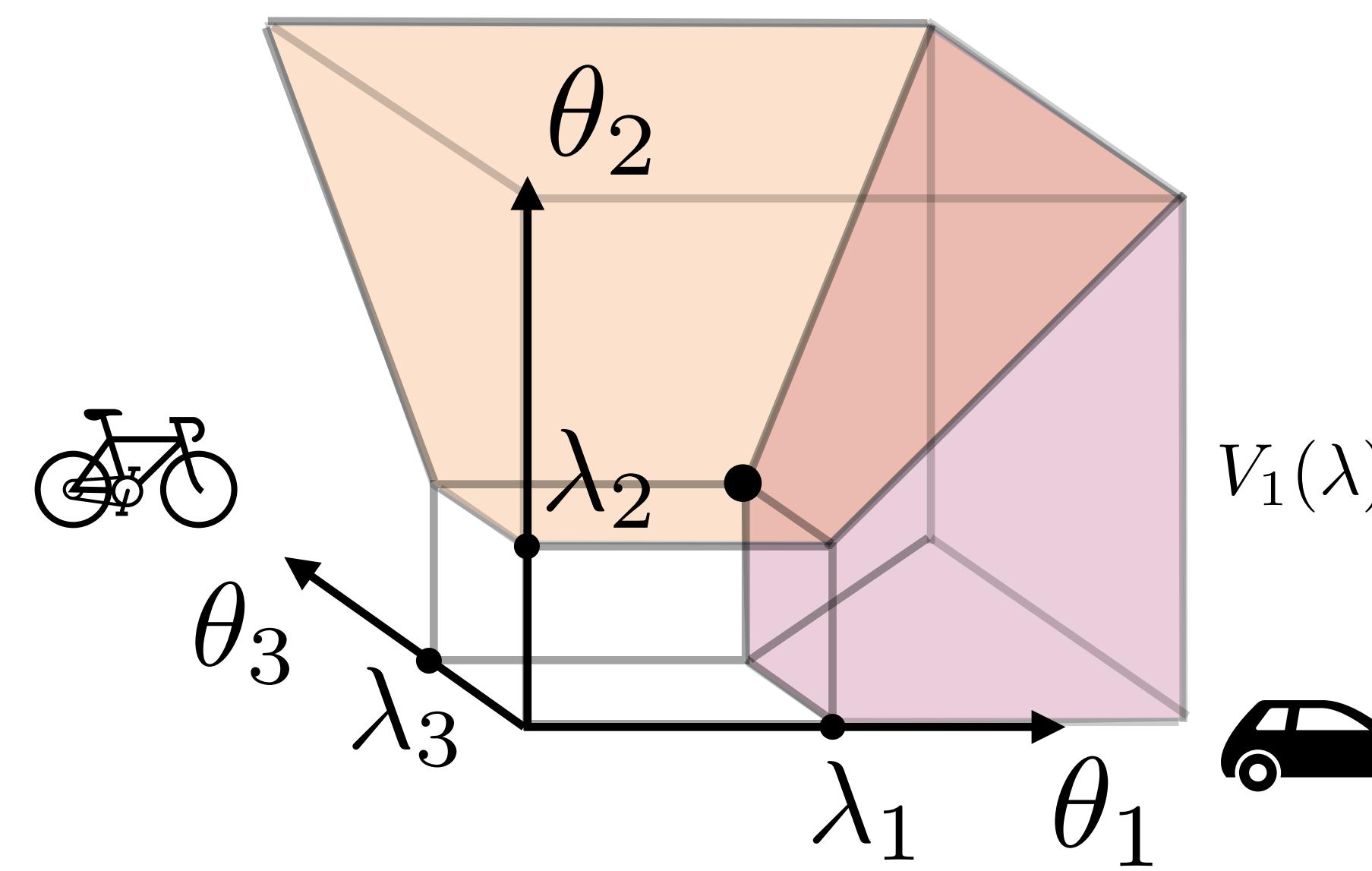


$$V_2(\lambda)$$



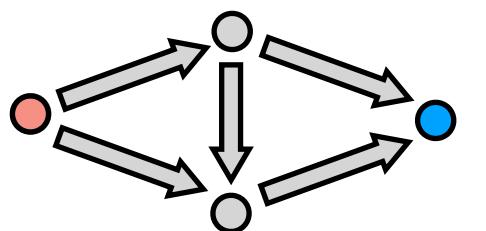
$$m_1(\lambda) = \int_{V_1(\lambda)} dm(\theta)$$

$$m_2(\lambda) = \int_{V_2(\lambda)} dm(\theta)$$

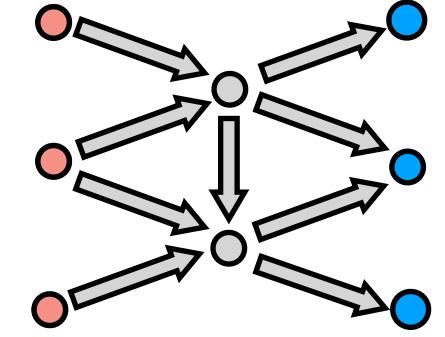


# Potential Games

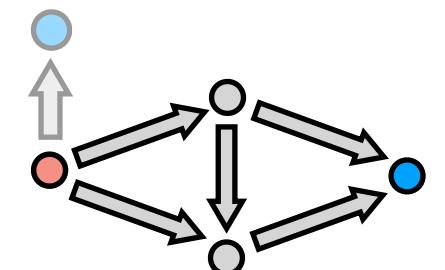
Routing Games



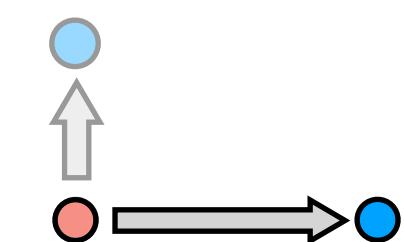
Multiple sources/sinks



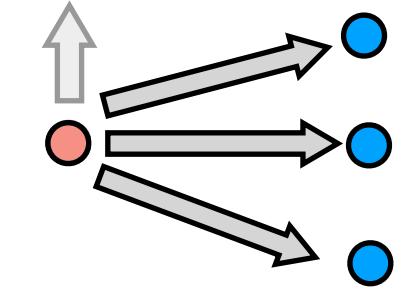
Variable Demand



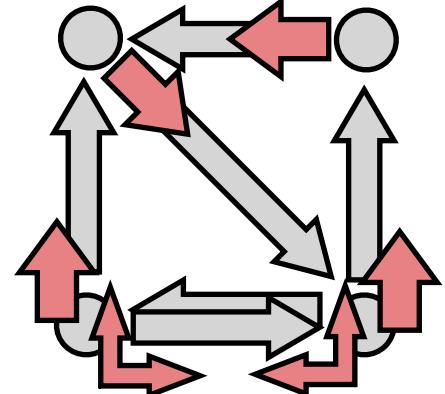
Supply & Demand



Cournot Market

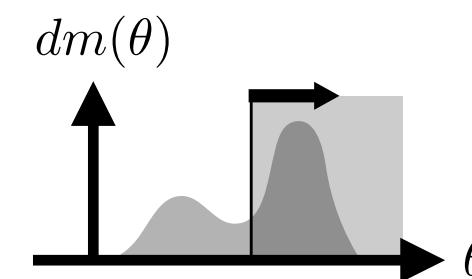


MDP Congestion Game

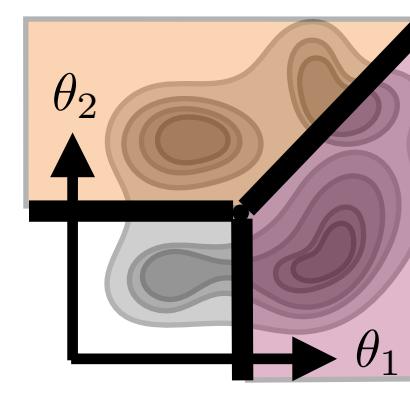


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

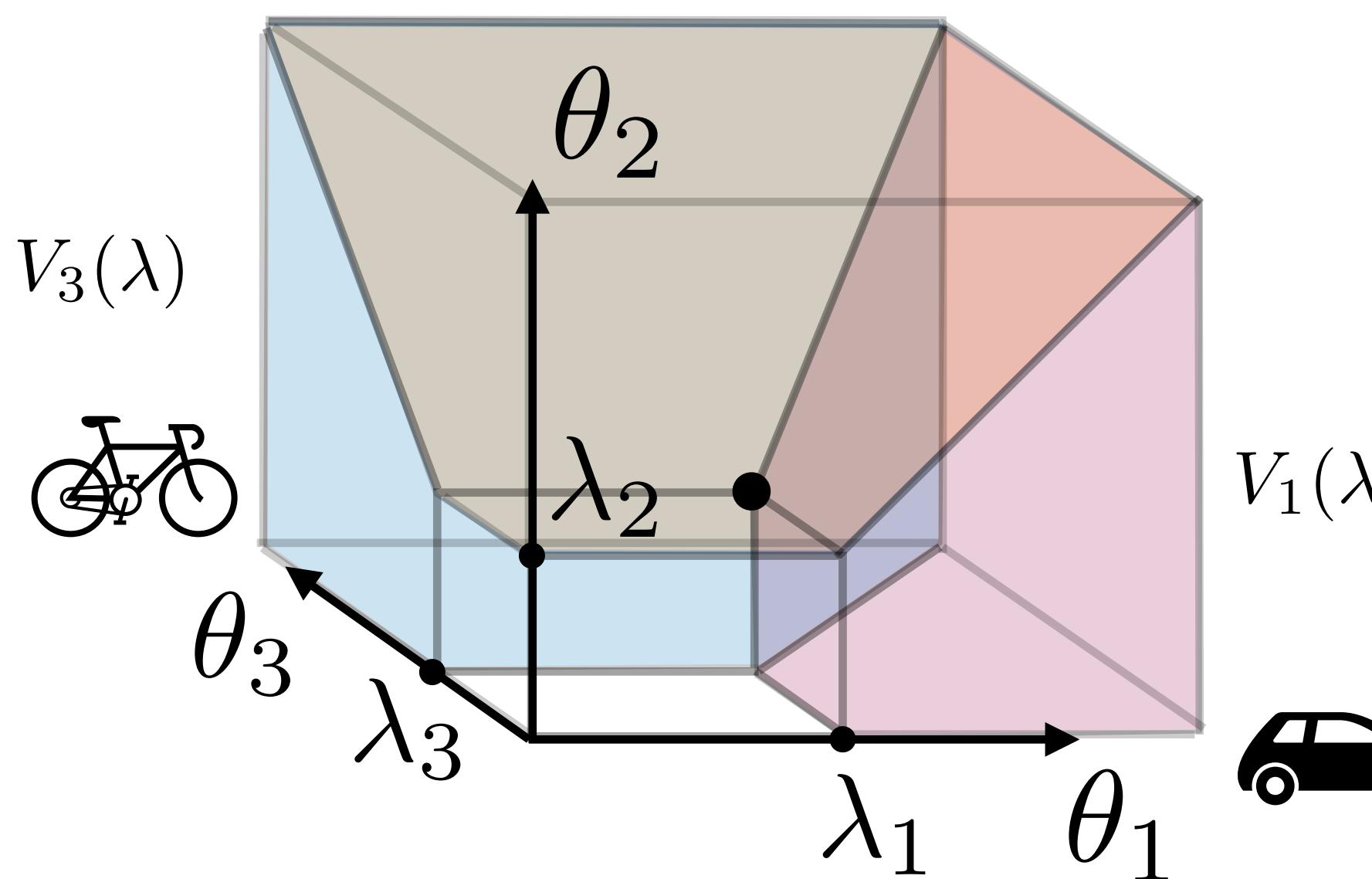
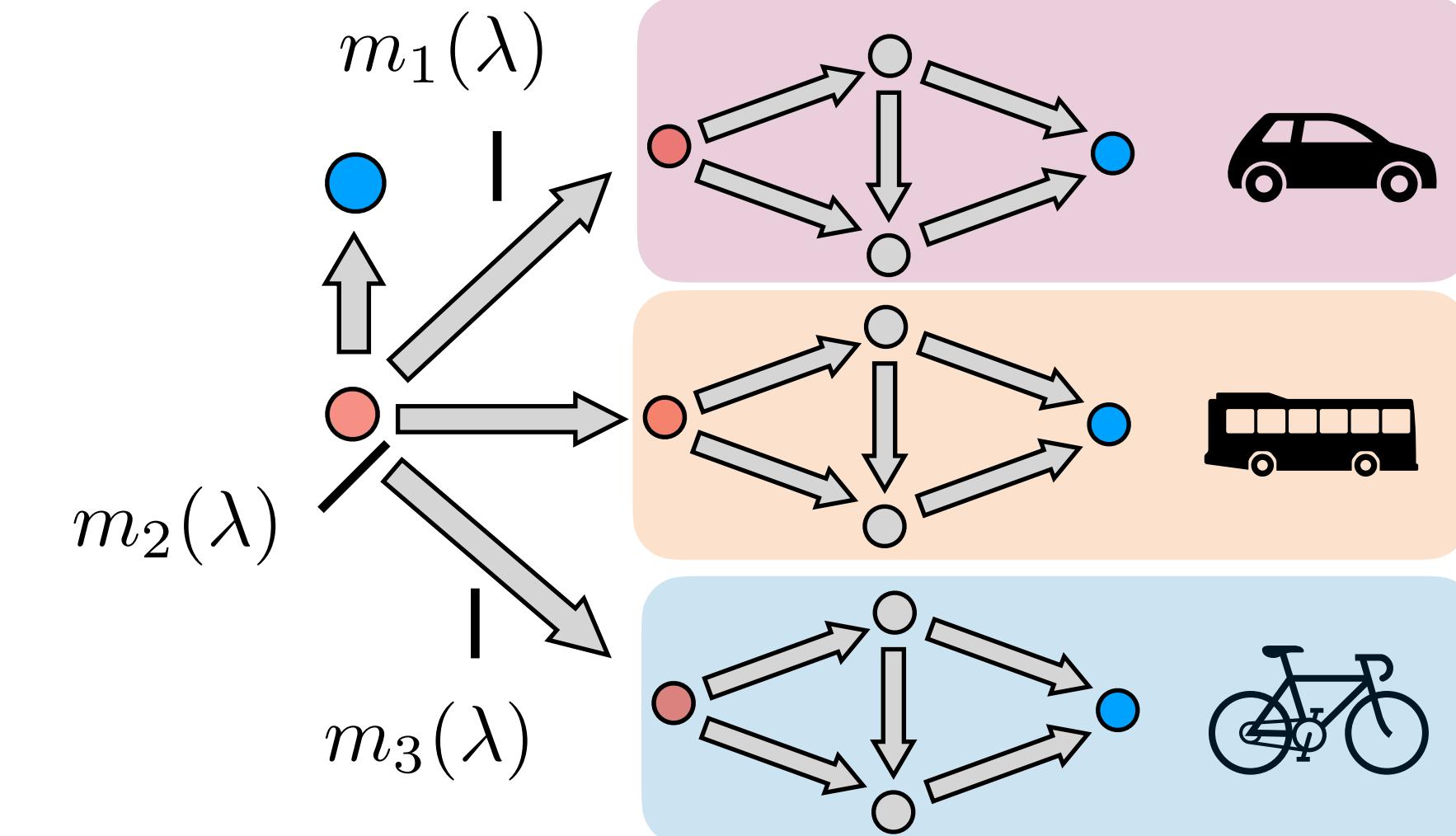
Non-homo-geneous preferences



Multi-Variate Preferences



$$V_2(\lambda)$$

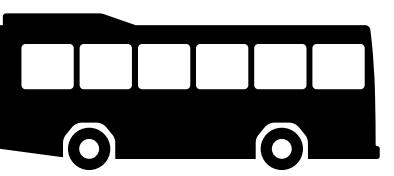


$$m_1(\lambda) = \int_{V_1(\lambda)} dm(\theta)$$

$$m_2(\lambda) = \int_{V_2(\lambda)} dm(\theta)$$

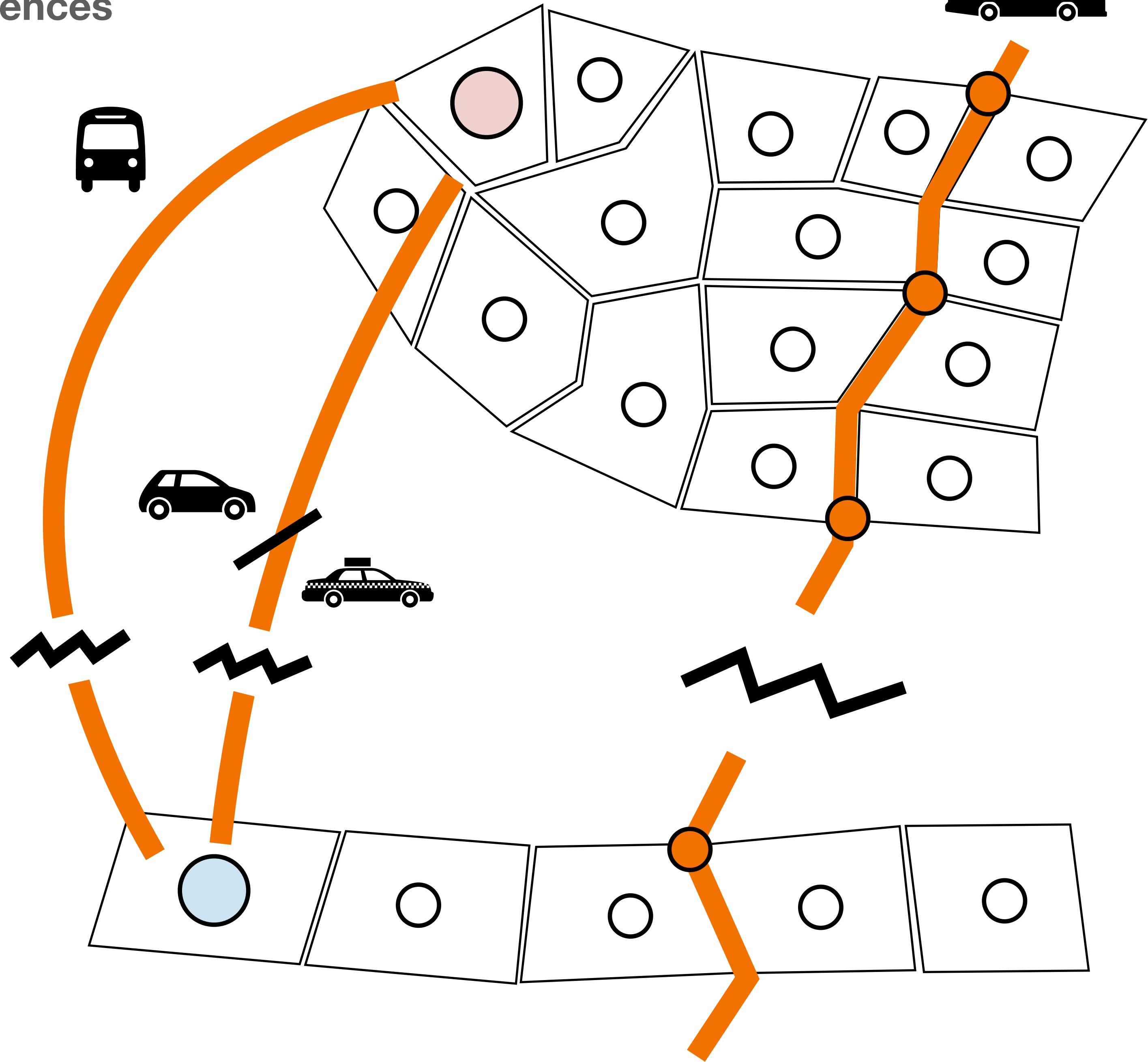
$$m_3(\lambda) = \int_{V_3(\lambda)} dm(\theta)$$

# Multi-Variate Non-Homogeneous Preferences

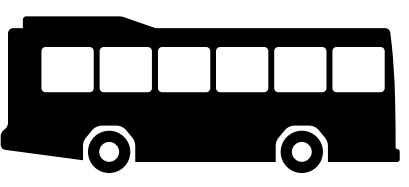


## Primary Modes

- Bus
- Dial-a-ride
- Drive

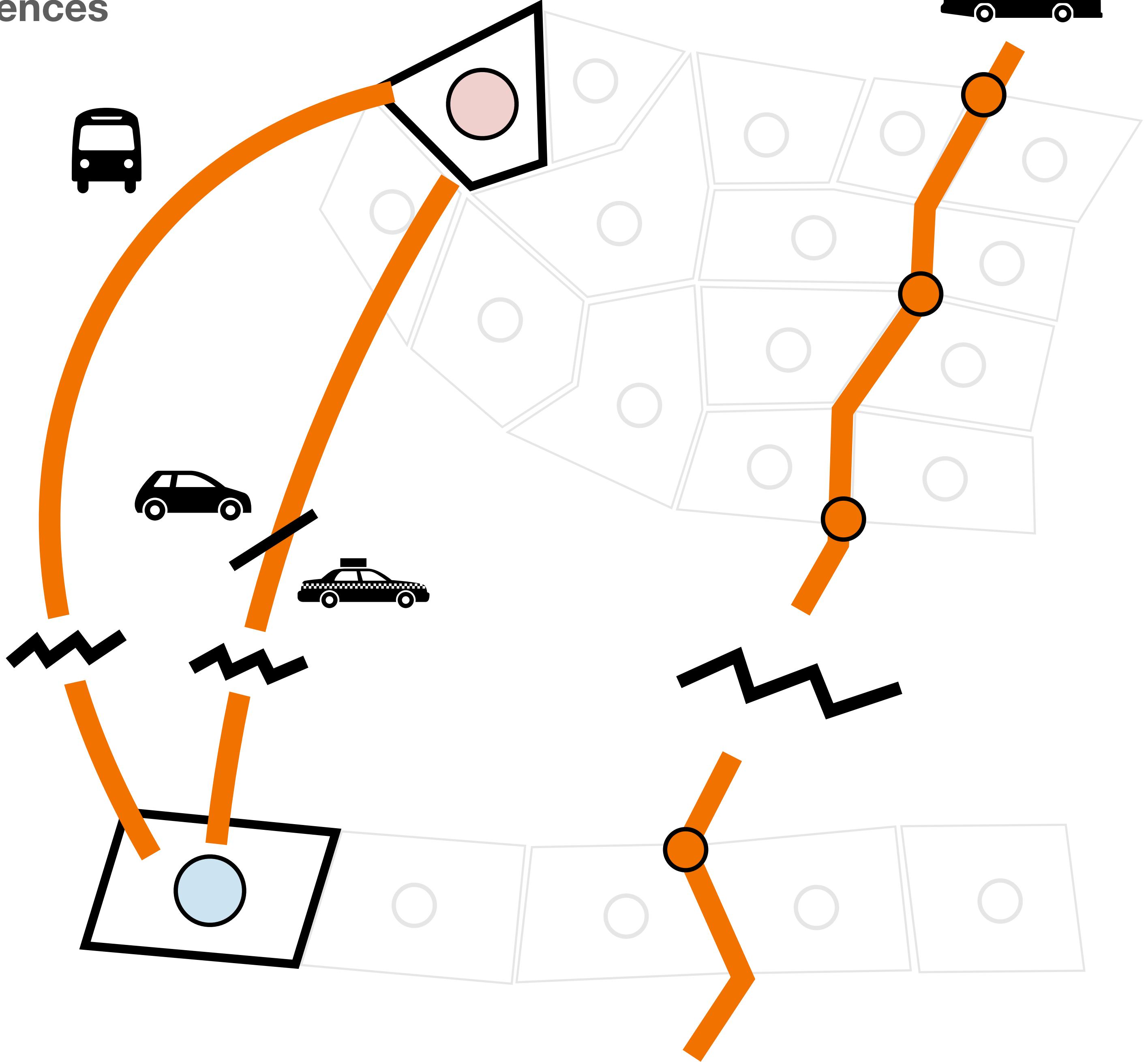
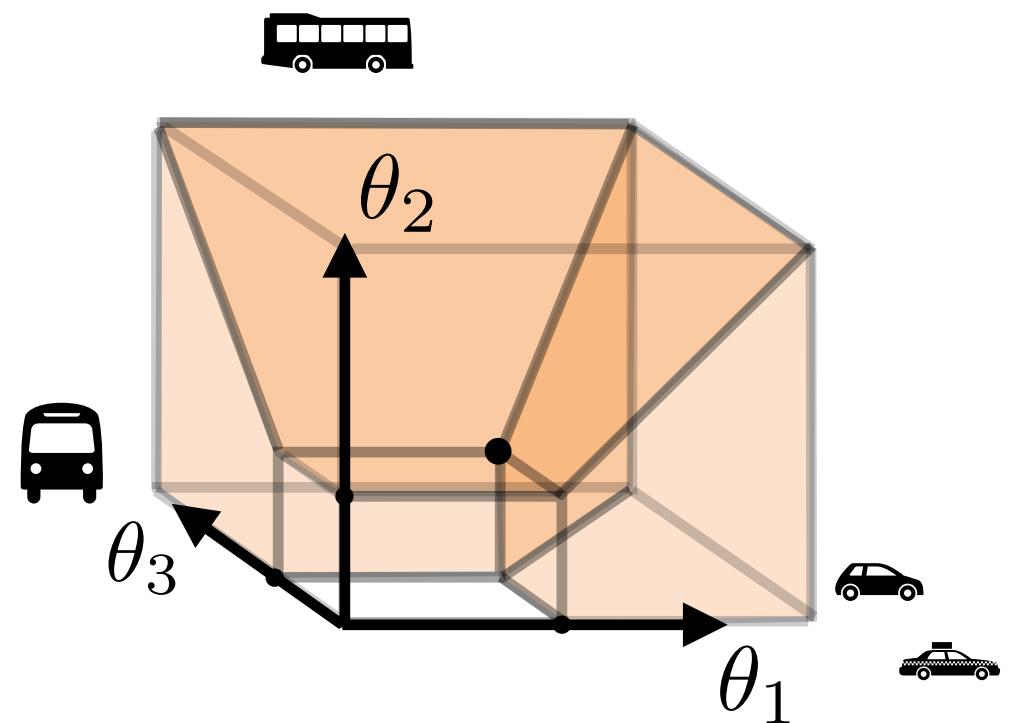


# Multi-Variate Non-Homogeneous Preferences

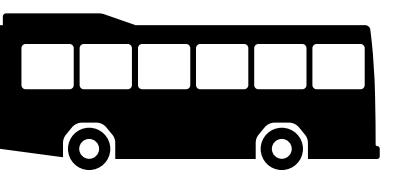


## Primary Modes

- Bus
- Dial-a-ride
- Drive

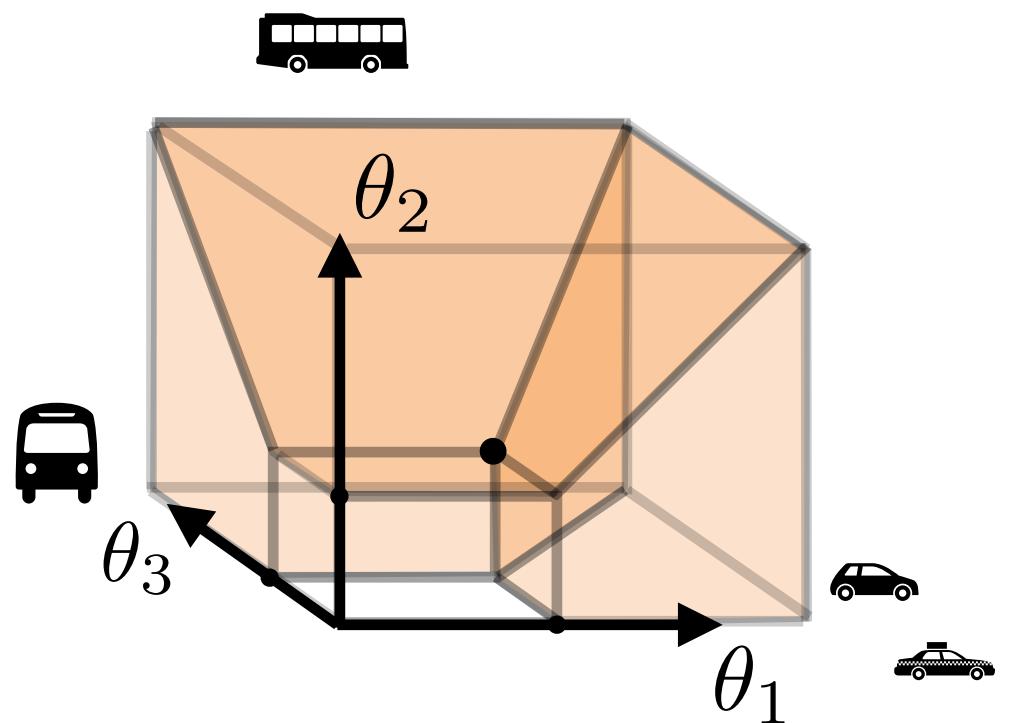


# Multi-Variate Non-Homogeneous Preferences



## Primary Modes

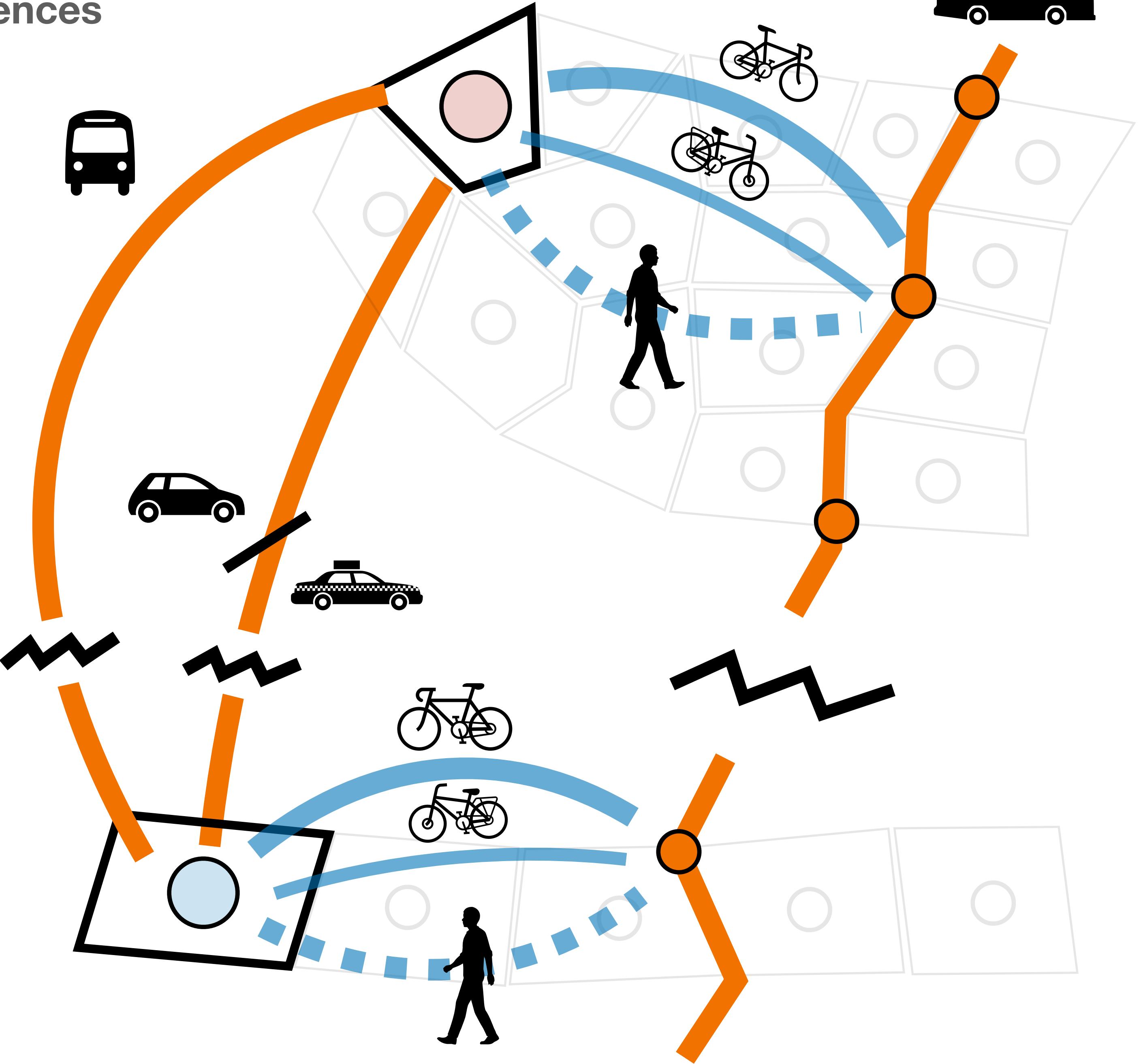
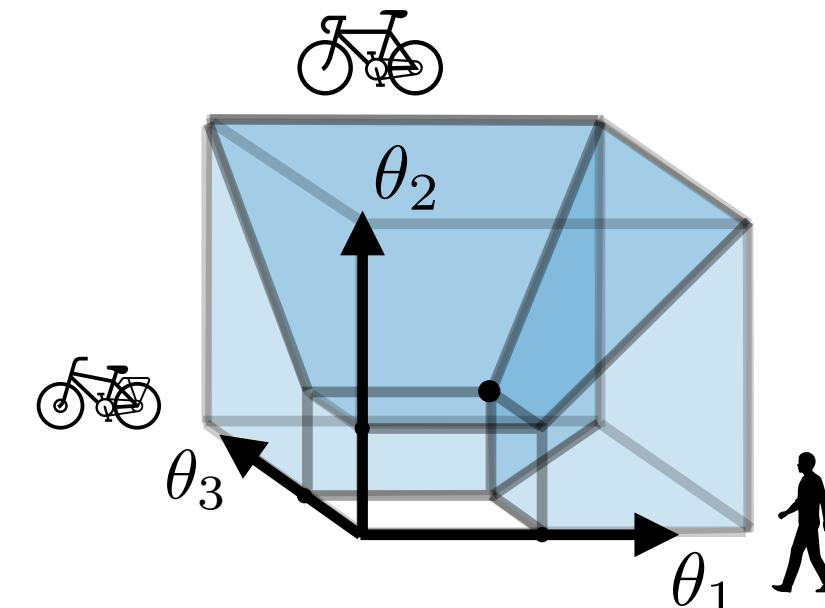
- Bus
- Dial-a-ride
- Drive



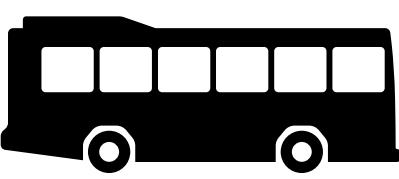
## Secondary Modes

Bus...

- Walking
- Biking
- Bike share

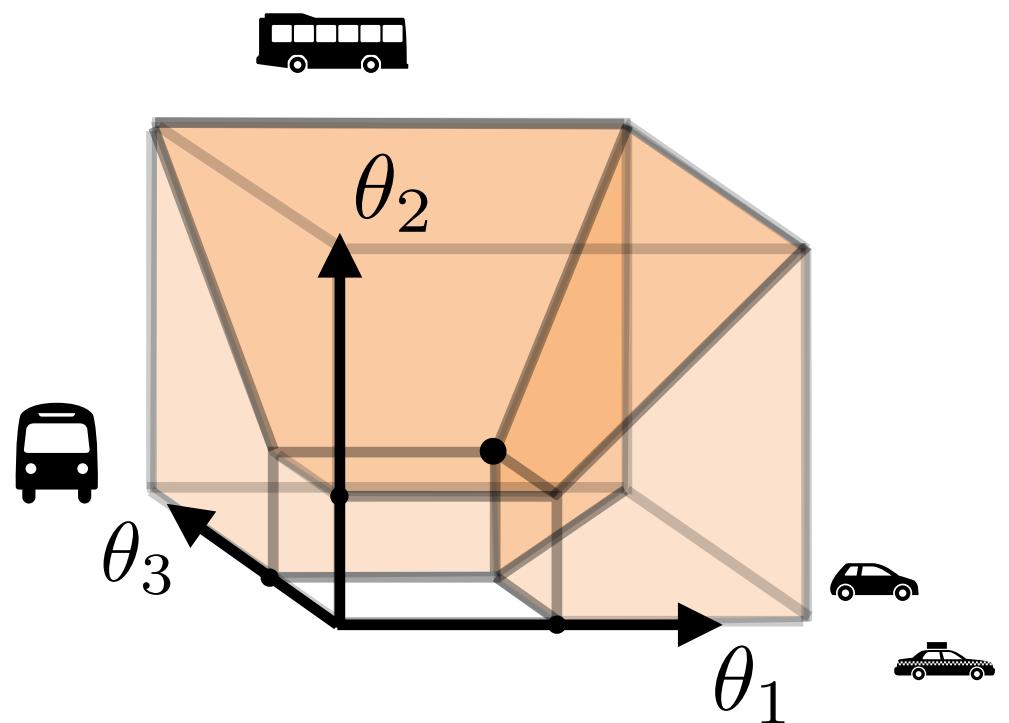


# Multi-Variate Non-Homogeneous Preferences



## Primary Modes

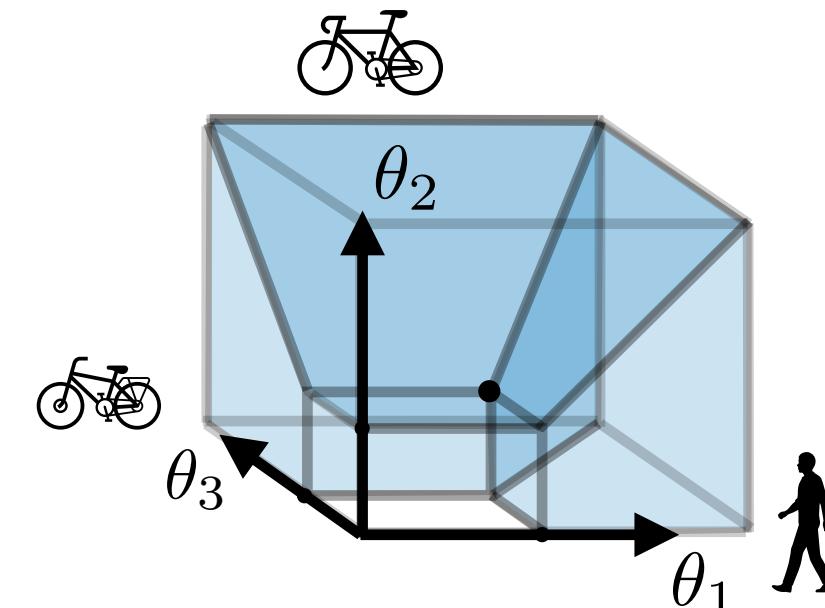
- Bus
- Dial-a-ride
- Drive



## Secondary Modes

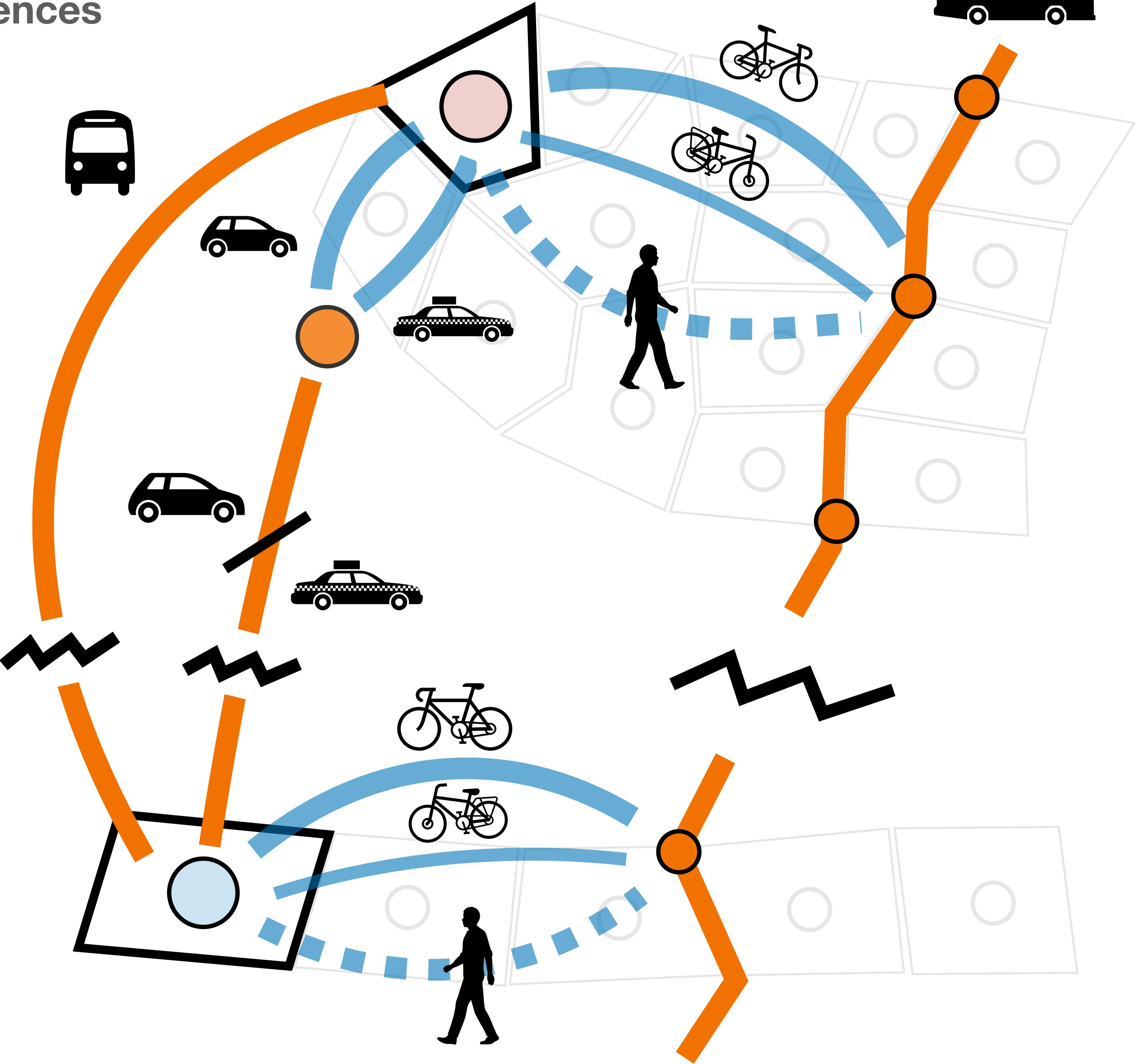
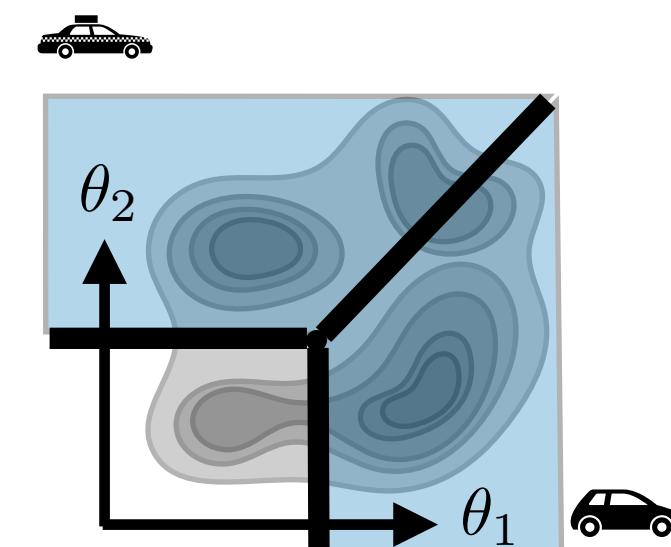
Bus...

- Walking
- Biking
- Bike share



Drive...

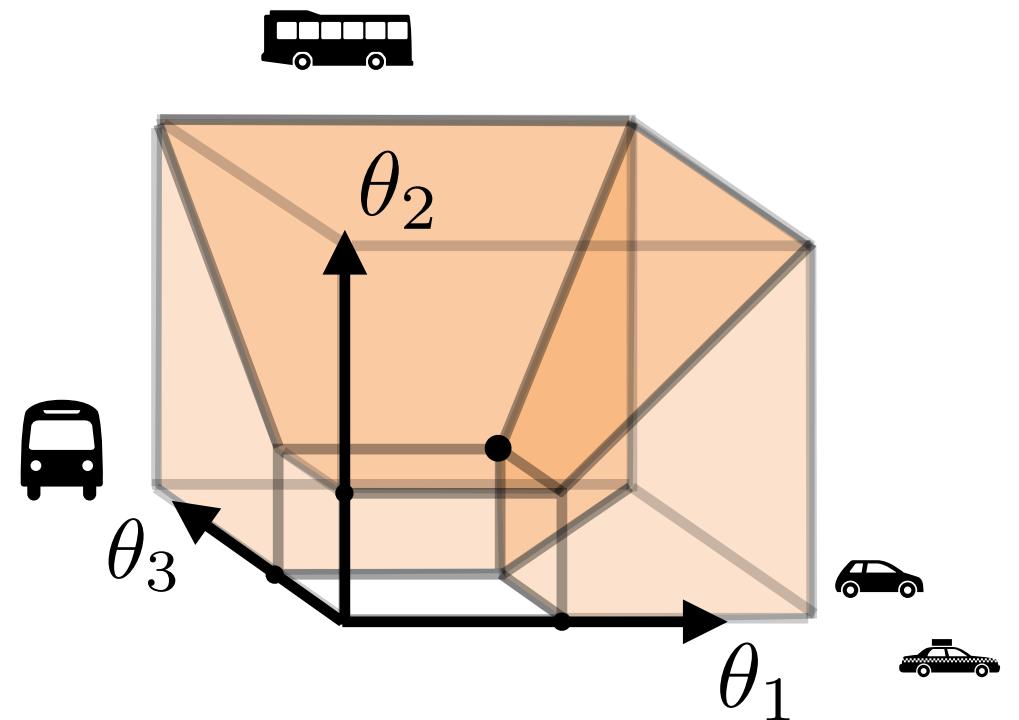
- Personal car
- Ride-share



# Multi-Variate Non-Homogeneous Preferences

## Primary Modes

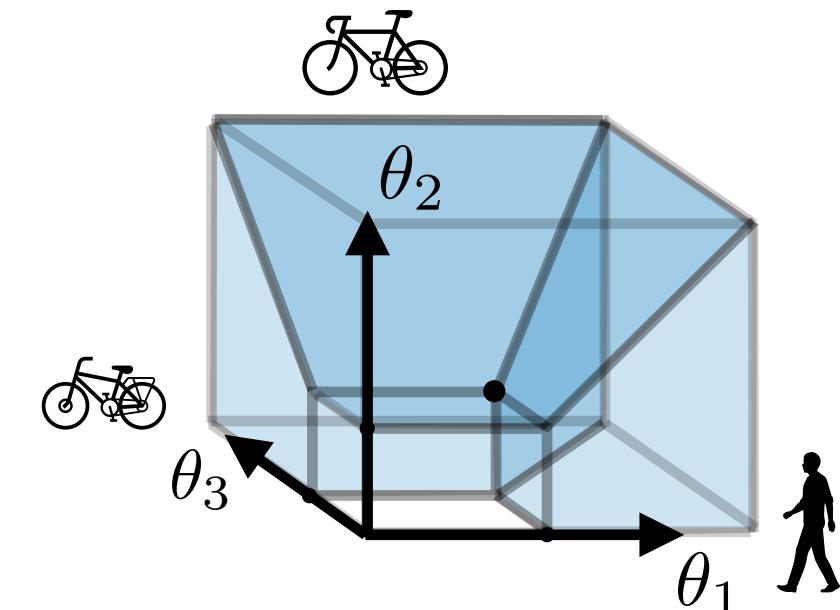
- Bus
- Dial-a-ride
- Drive



## Secondary Modes

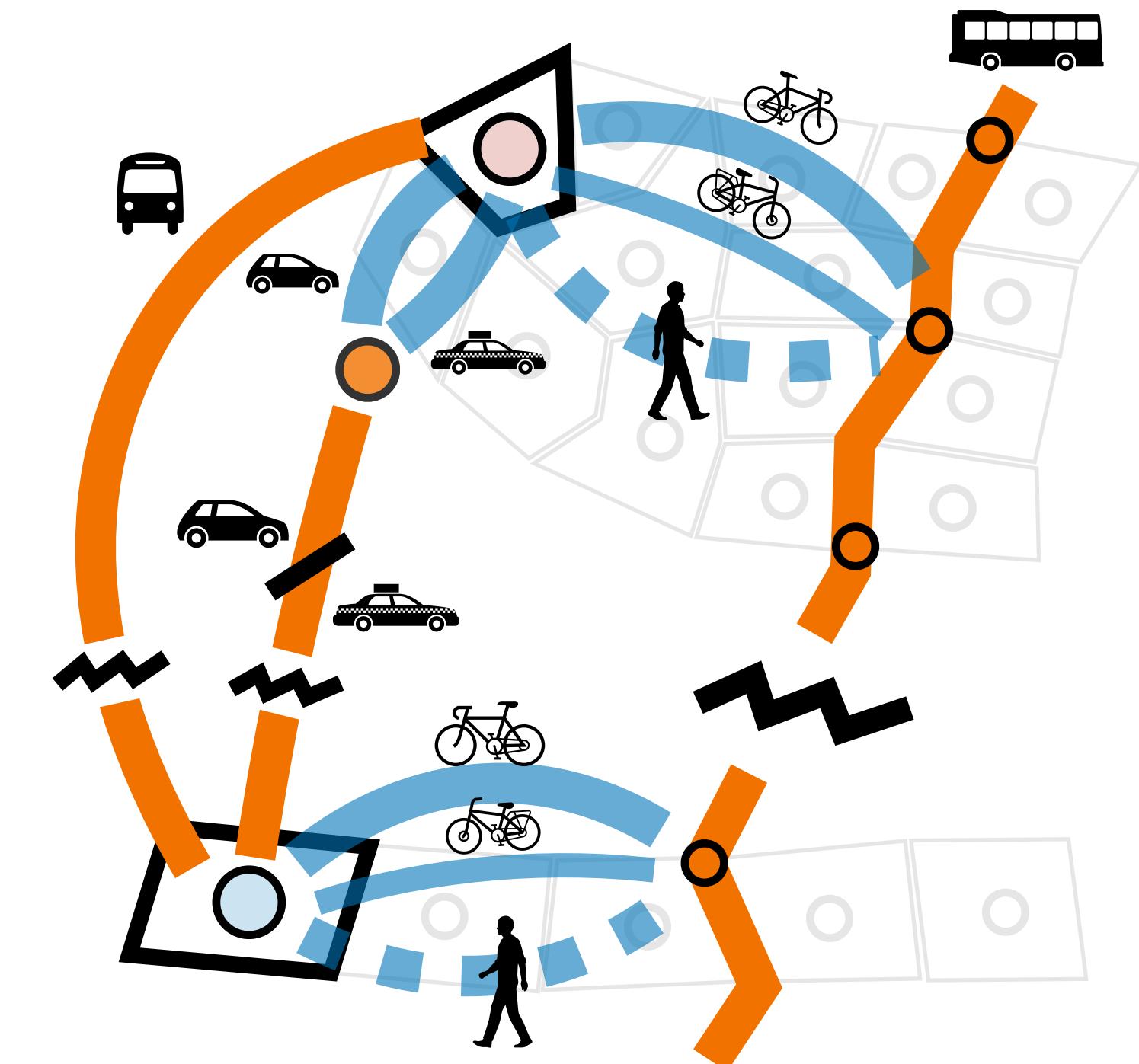
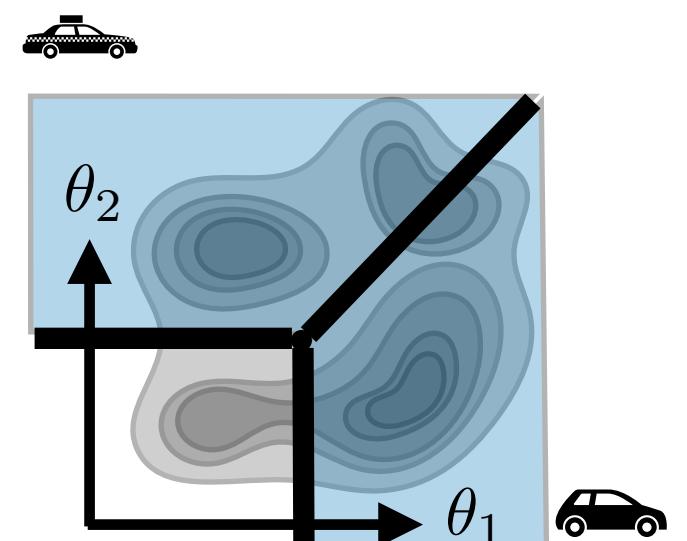
Bus...

- Walking
- Biking
- Bike share



Drive...

- Personal car
- Ride-share



## Individual's Cost

Primary  
Mode  
Preference

Secondary  
Mode  
Preference

+ Travel  
Time

## Equilibrium

Population distribution  
over transport options

*"No one can do any better"*

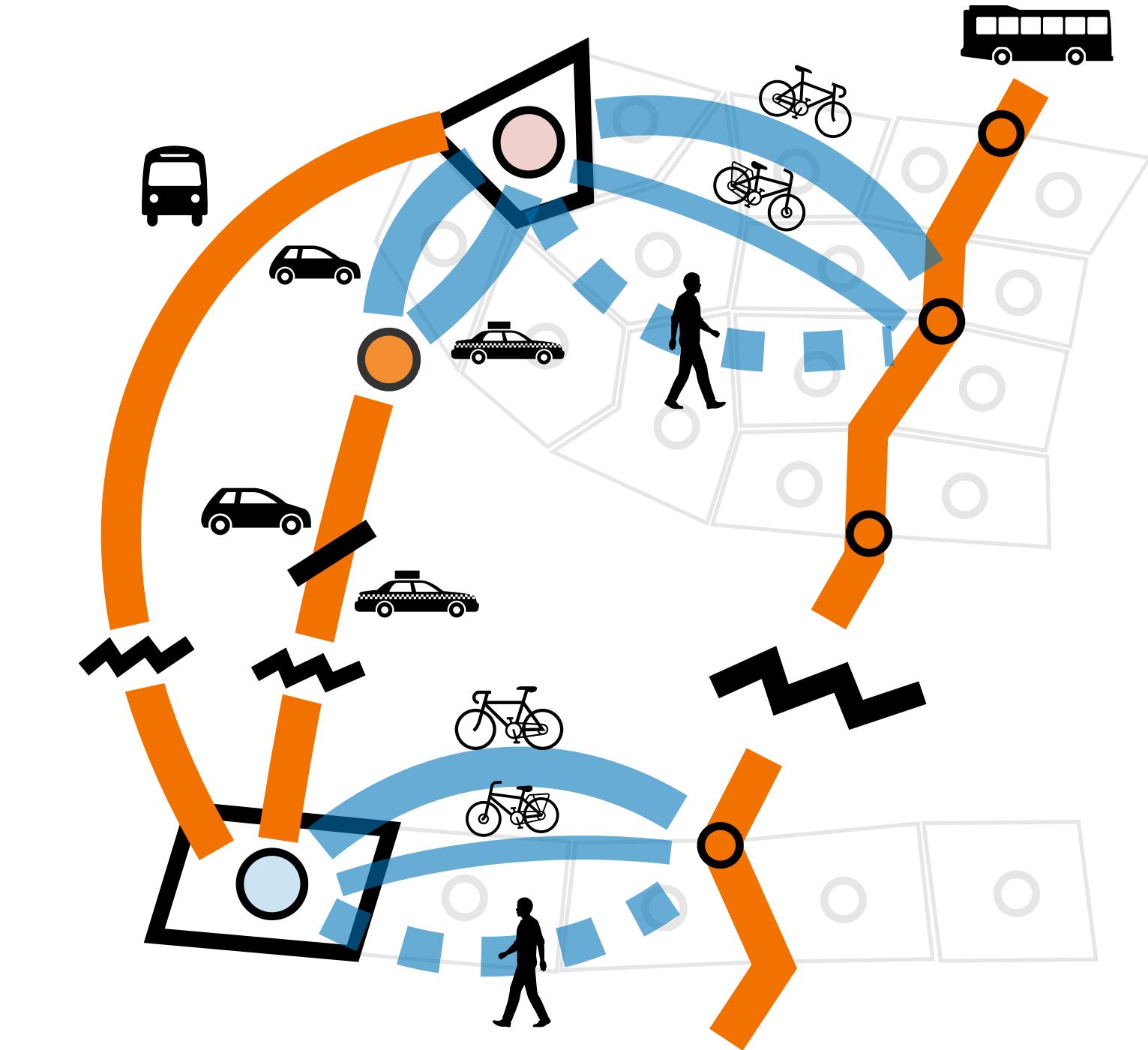
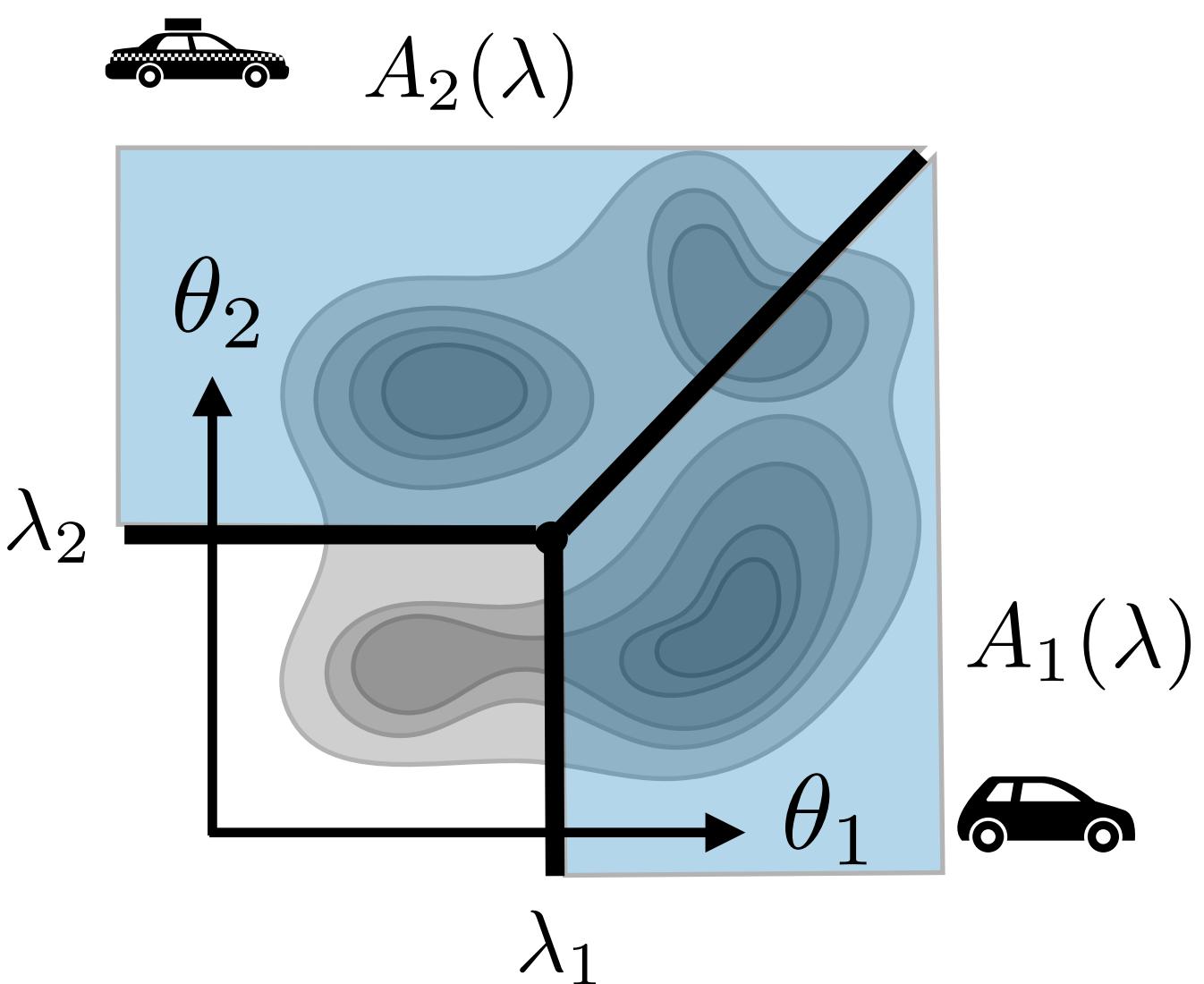
# Multi-Variate Non-Homogeneous Preferences

Computing mass distributions (at each origin point)

Compare options -  
Would need to standardize units  
in survey questions



Survey  
(or guess)



Individual's Cost

Primary Mode Preference

Secondary Mode Preference

Travel Time

Equilibrium

Population distribution over transport options

*"No one can do any better"*

# Multi-Variate Non-Homogeneous Preferences

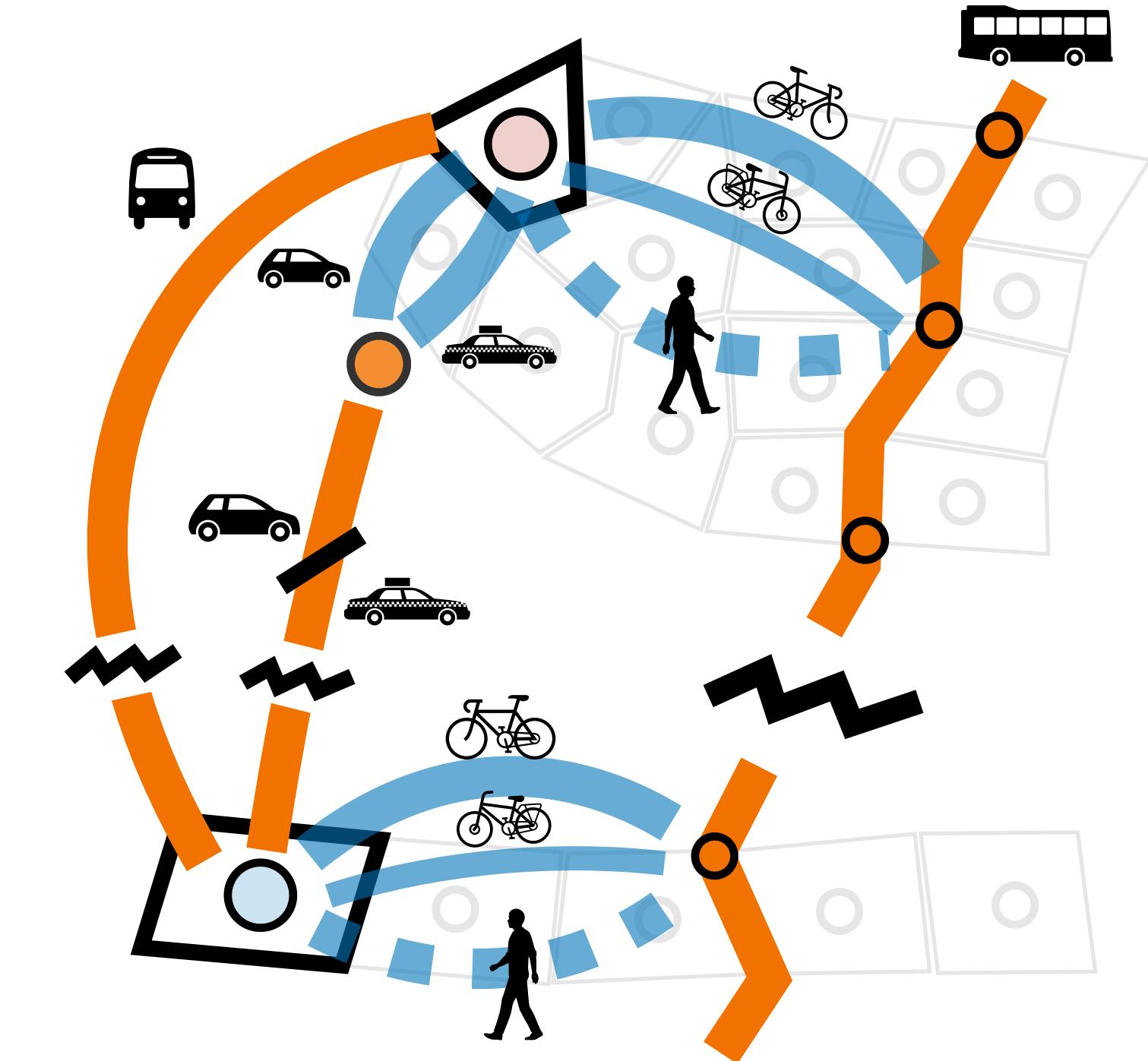
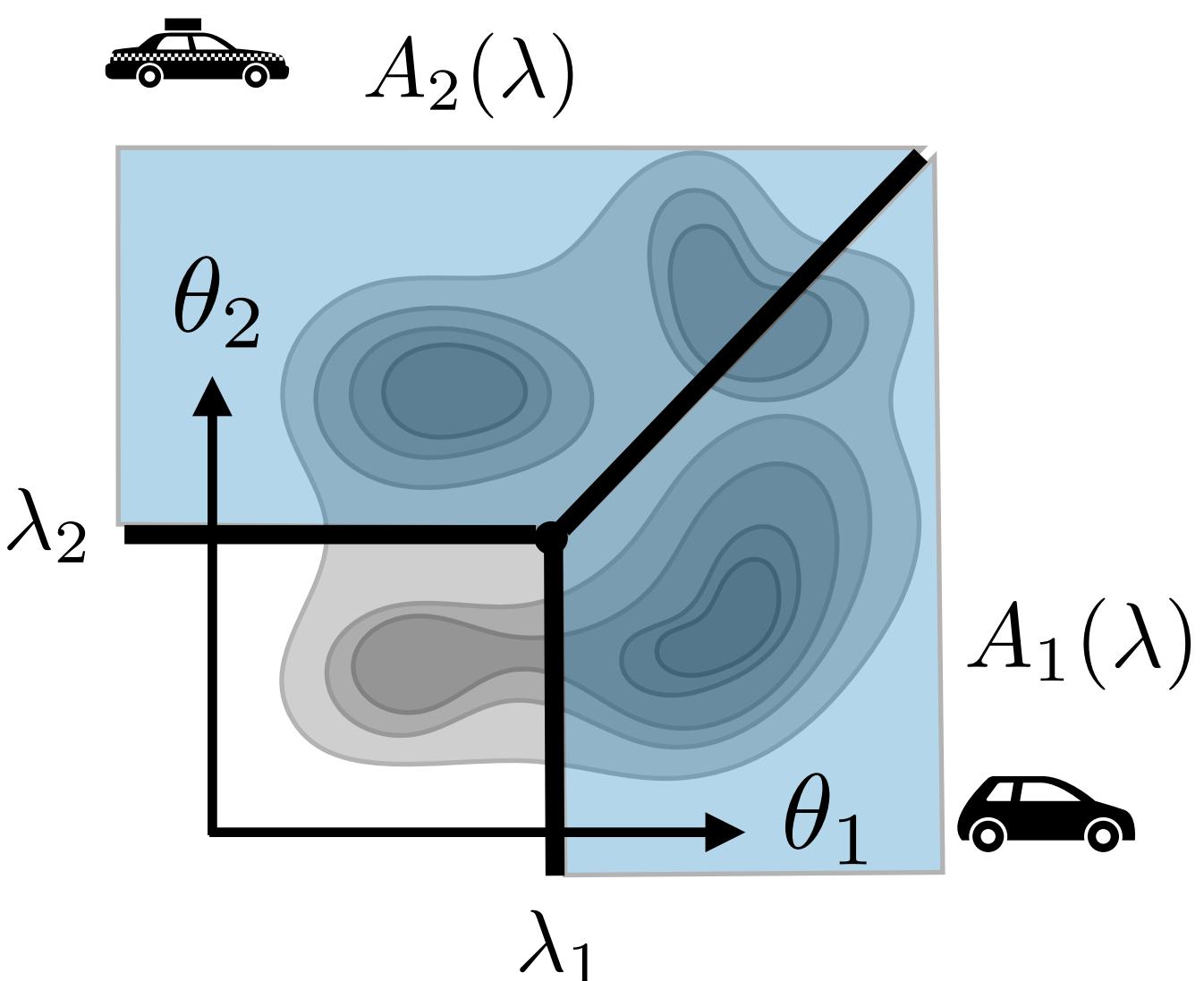
Computing mass distributions (at each origin point)

Compare options -  
Would need to standardize units  
in survey questions

Computation:

Algorithm: **Frank-Wolfe**

*LP for descent direction*



Individual's Cost

Primary Mode Preference

Secondary Mode Preference

+ Travel Time

Equilibrium

Population distribution over transport options

*"No one can do any better"*

# Multi-Variate Non-Homogeneous Preferences

**Computing mass distributions (at each origin point)**

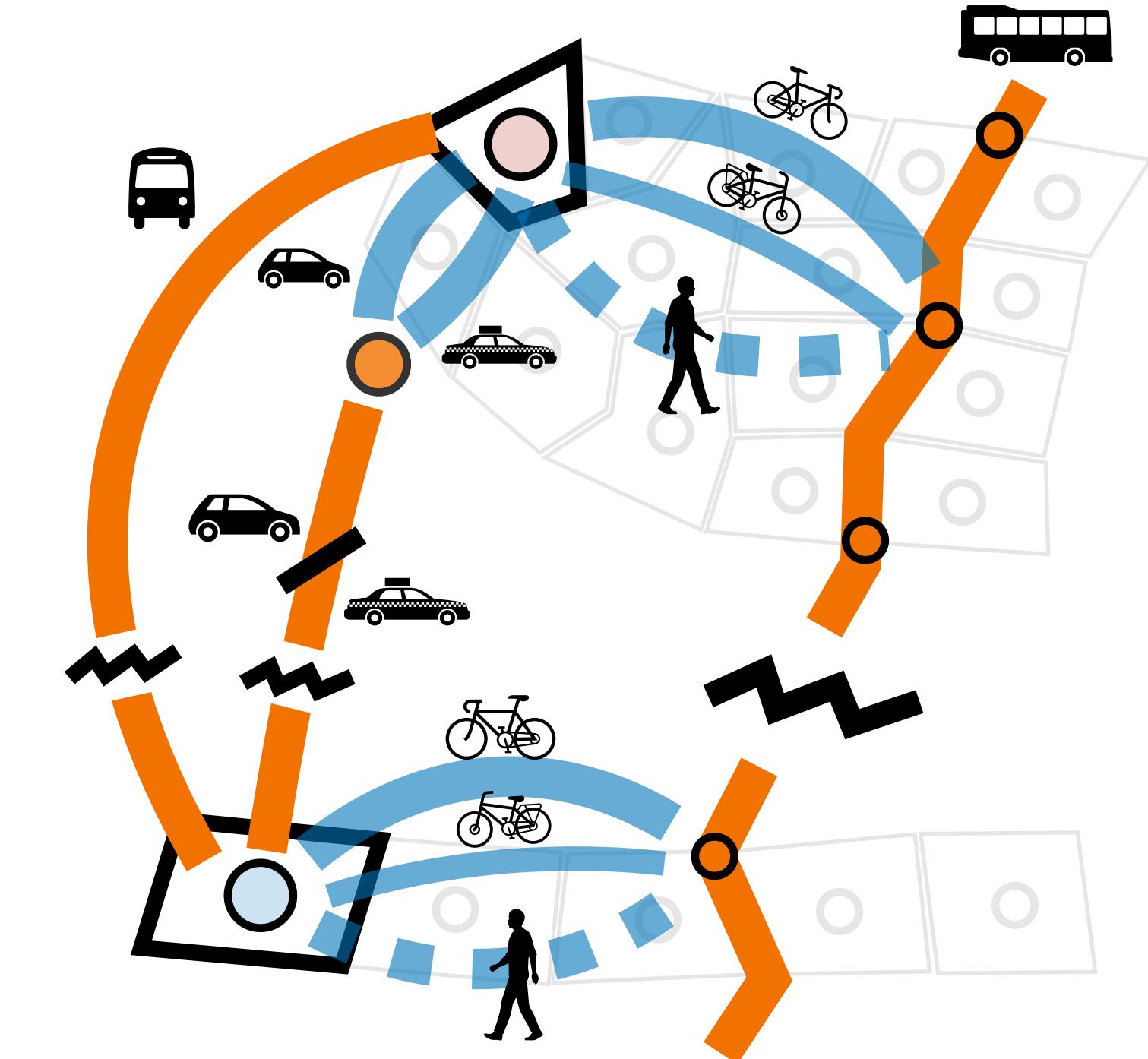
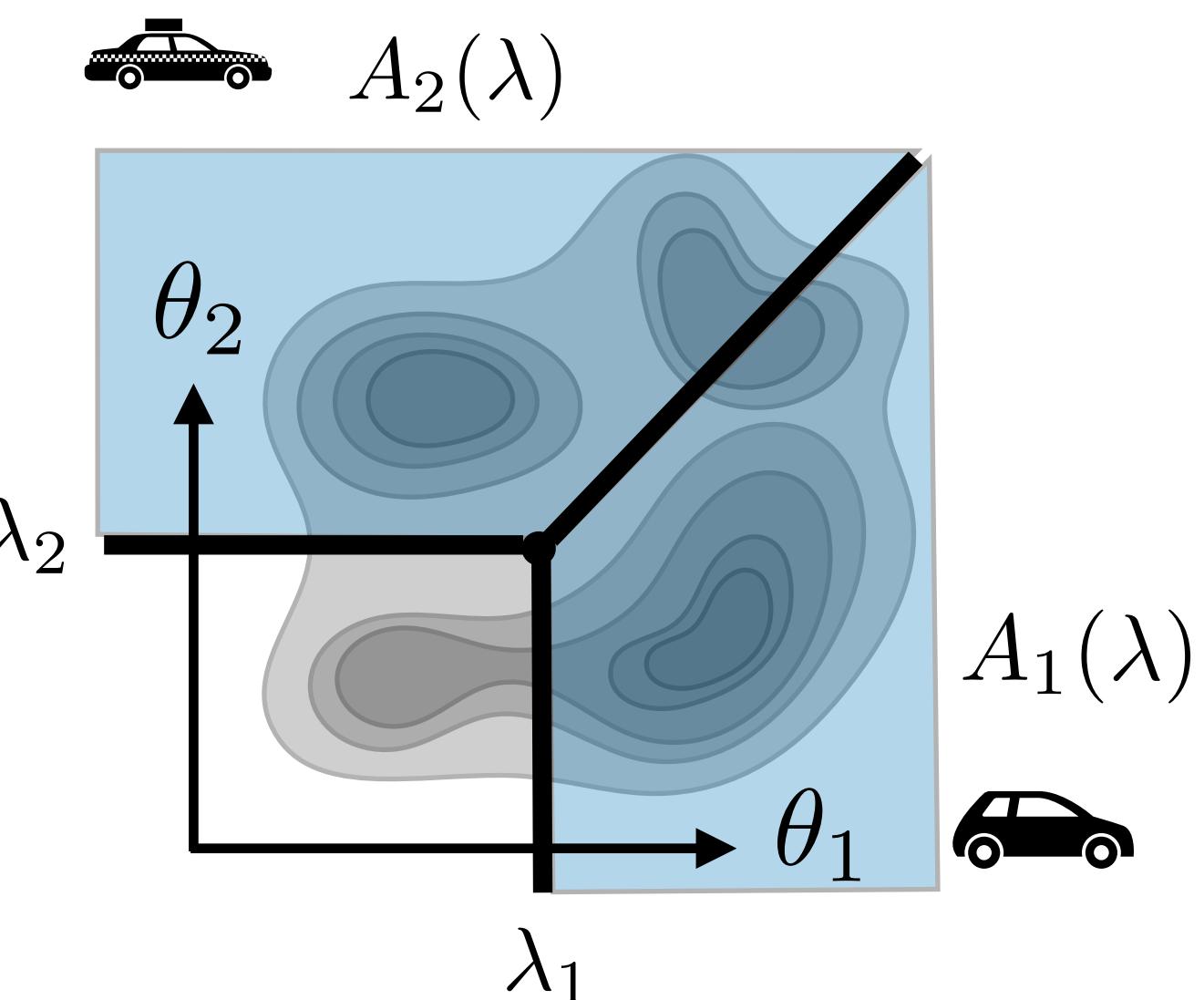
Compare options -  
Would need to standardize units  
in survey questions

**Computation:**

Algorithm: **Frank-Wolfe**

*LP for descent direction*

Size: OD Pairs  $\times$  # Primary Options  $\times$  # Secondary Options



**Individual's Cost**

$$\text{Primary Mode Preference} + \text{Secondary Mode Preference} + \text{Travel Time}$$

**Equilibrium**

Population distribution over transport options

*“No one can do any better”*

# Multi-Variate Non-Homogeneous Preferences

**Computing mass distributions (at each origin point)**

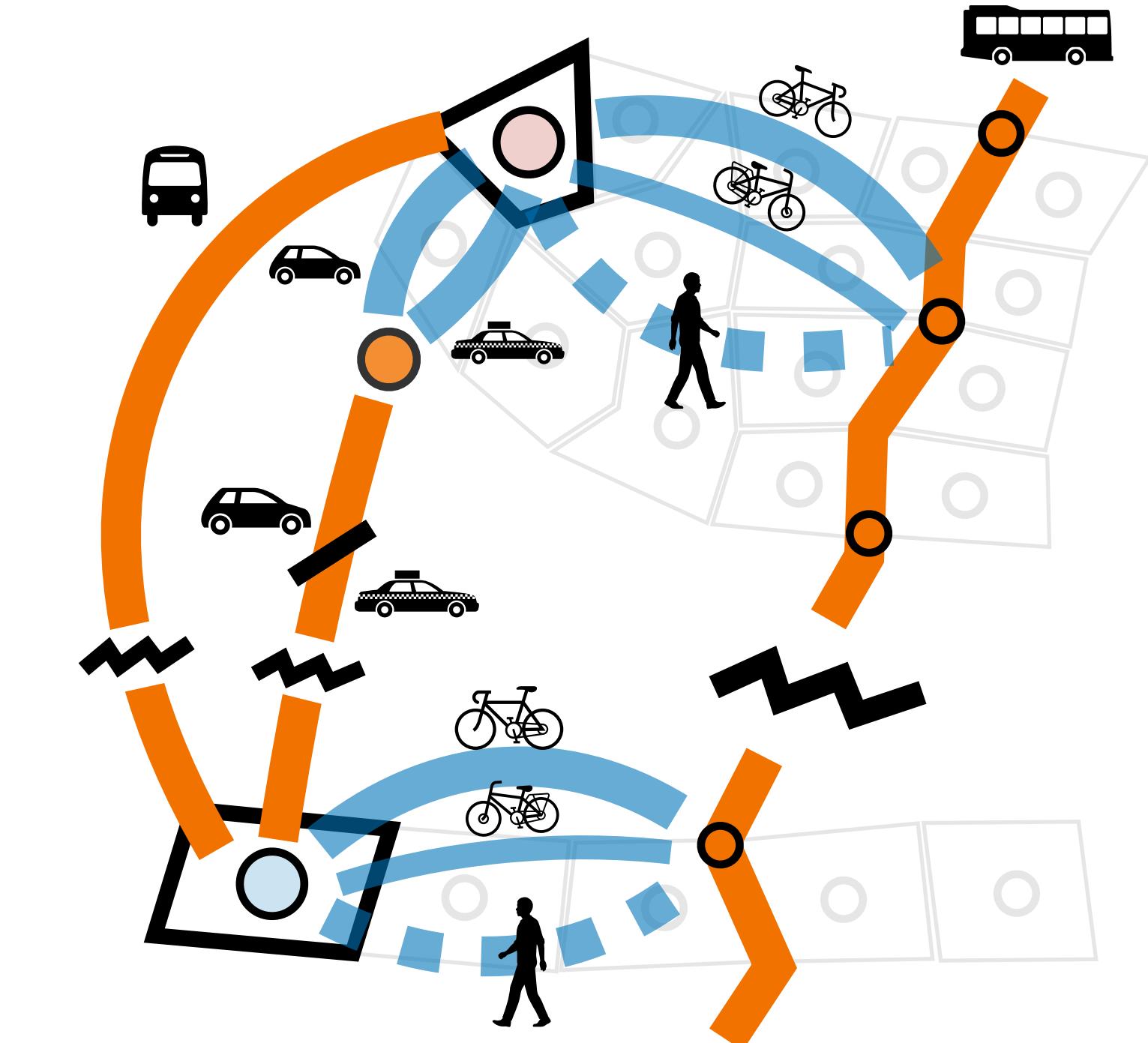
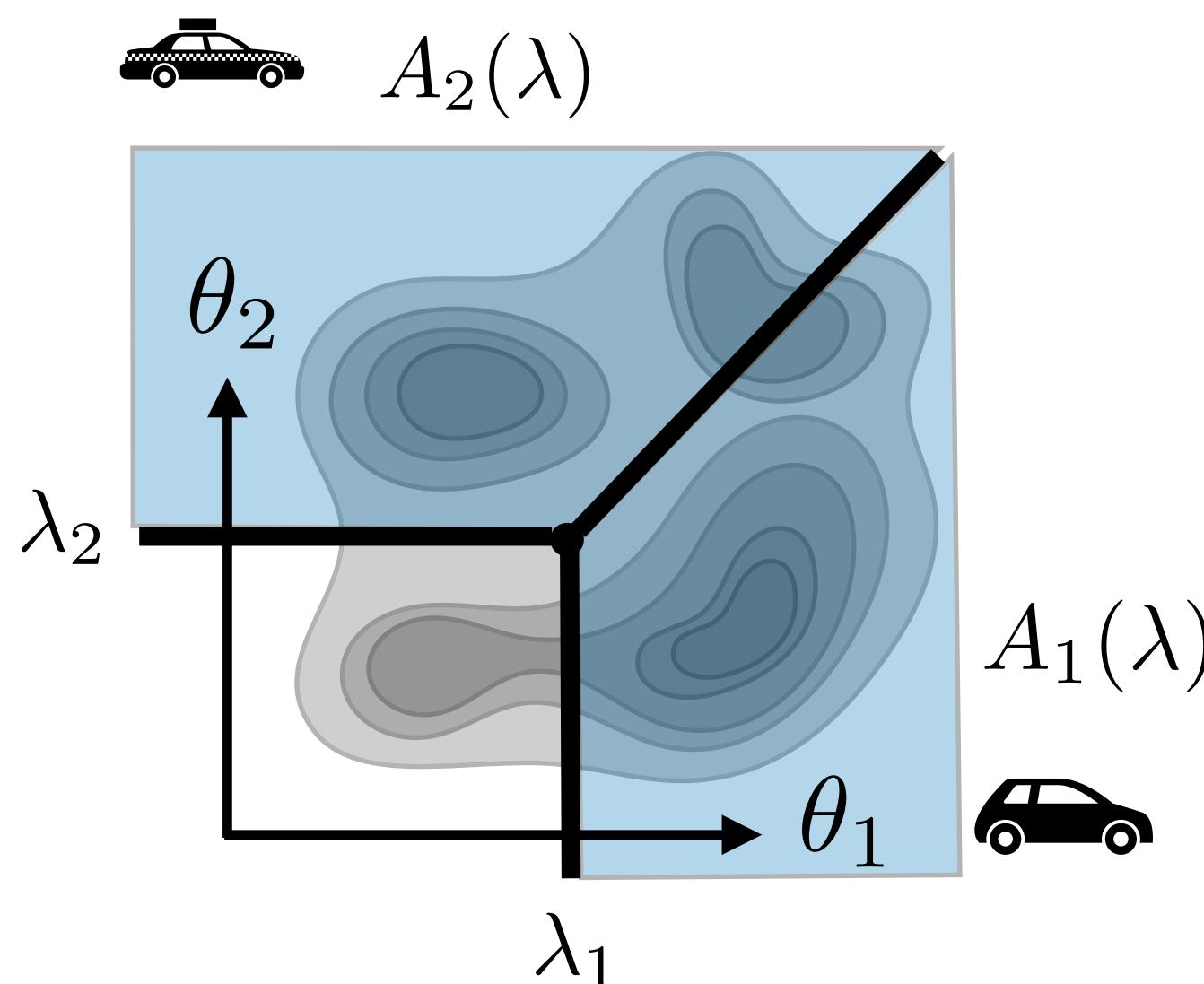
Compare options -  
Would need to standardize units  
in survey questions

**Computation:**

Algorithm: **Frank-Wolfe**

*LP for descent direction*

Size: OD Pairs  $\times$  # Primary Options  $\times$  # Secondary Options  $\times$  Primary Mode Sub-problem Size



**Individual's Cost**

$$\text{Primary Mode Preference} + \text{Secondary Mode Preference} + \text{Travel Time}$$

**Equilibrium**

Population distribution over transport options

*"No one can do any better"*

# Multi-Variate Non-Homogeneous Preferences

**Computing mass distributions (at each origin point)**

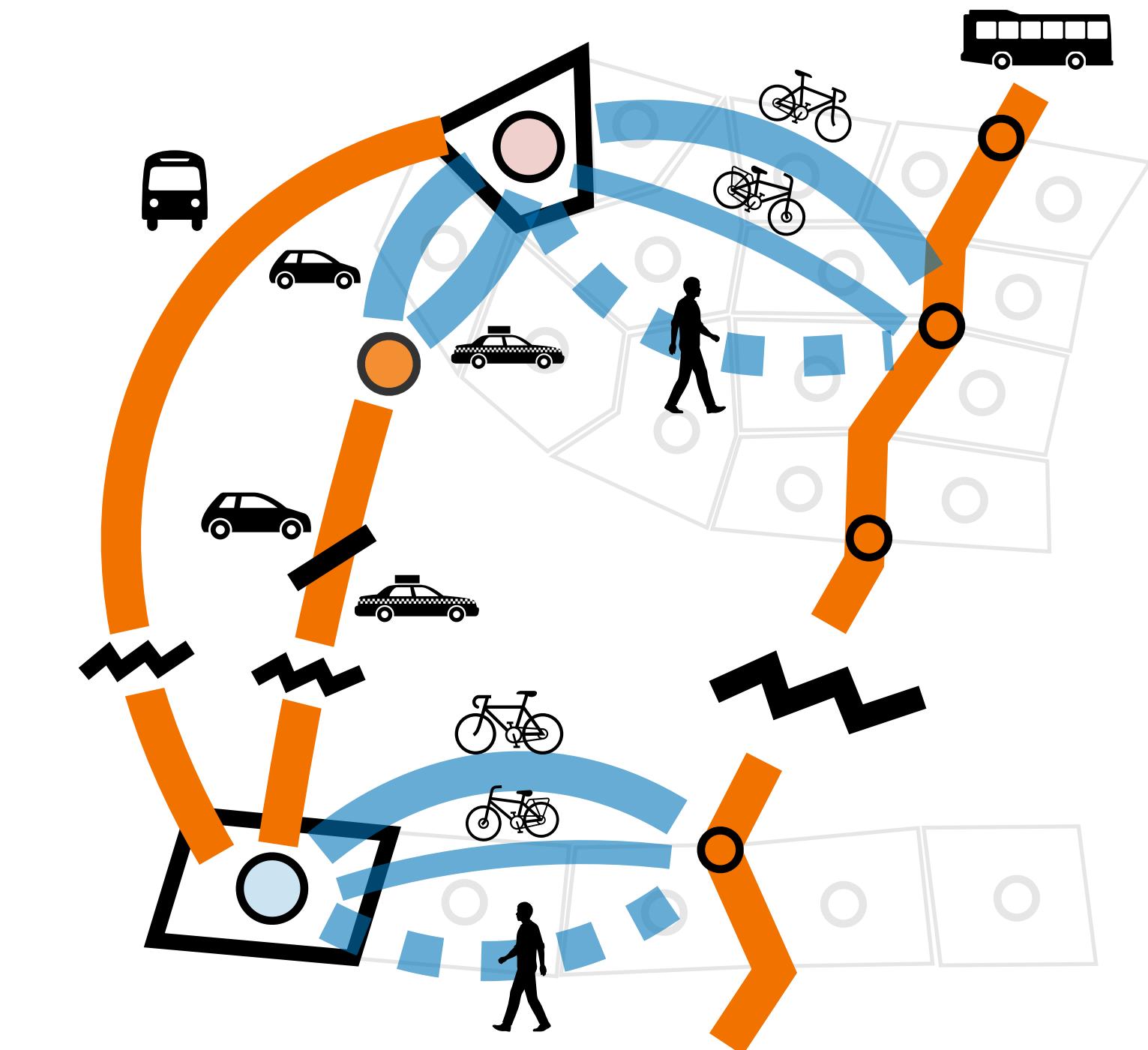
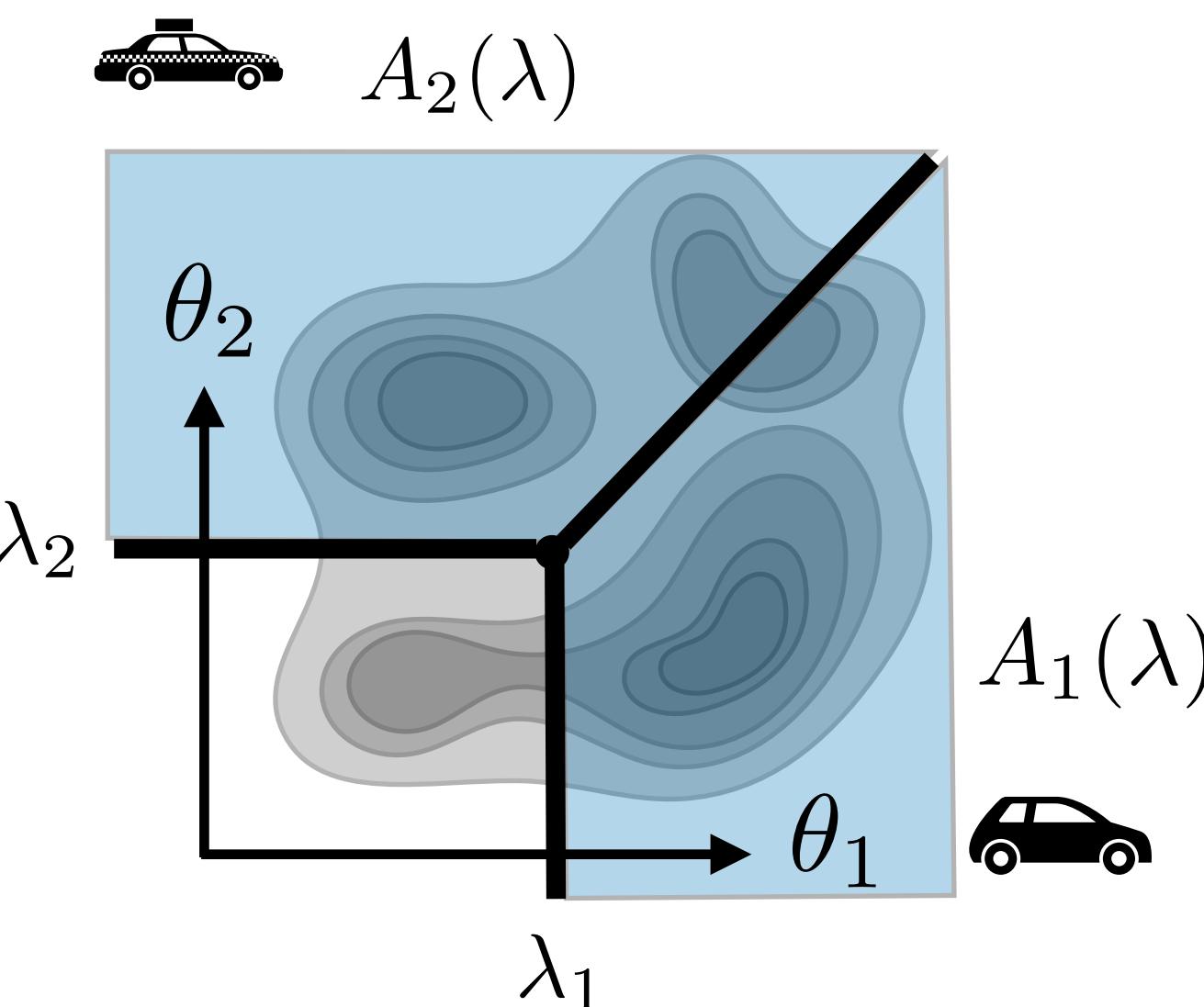
Compare options -  
Would need to standardize units  
in survey questions

**Computation:**

Algorithm: **Frank-Wolfe**

*LP for descent direction*

Size: OD Pairs  $\times$  # Primary Options  $\times$  # Secondary Options



**Individual's Cost**

$$\text{Primary Mode Preference} + \text{Secondary Mode Preference} + \text{Travel Time}$$

**Uses:**

1. **Prediction**
2. **Incentive design** - parking prices, etc
3. **Route design — difficult**  
*(Monte Carlo search, probabilistic planning,...)*

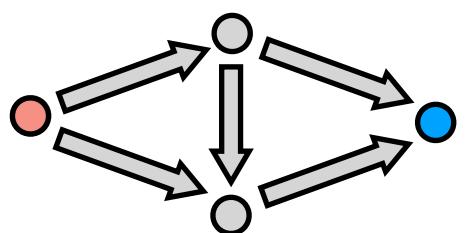
**Equilibrium**

Population distribution over transport options

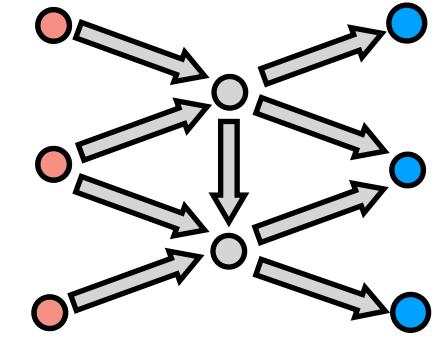
*“No one can do any better”*

# Potential Games

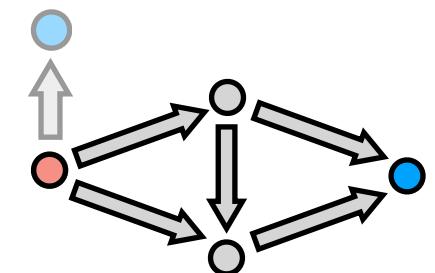
Routing Games



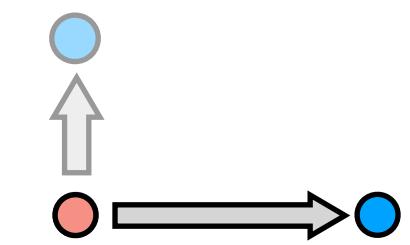
Multiple sources/sinks



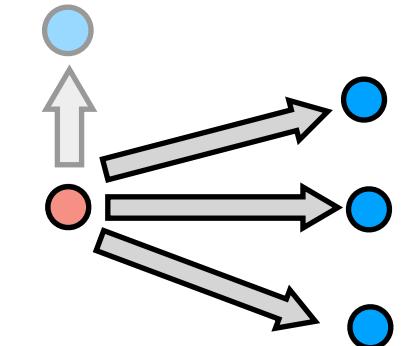
Variable Demand



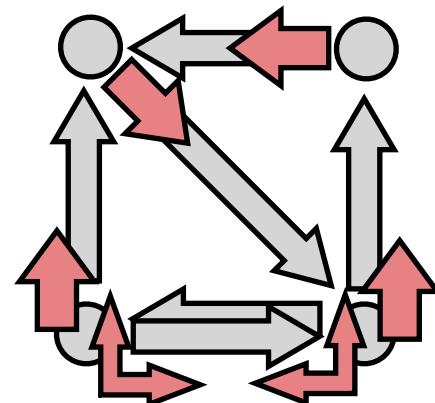
Supply & Demand



Cournot Market

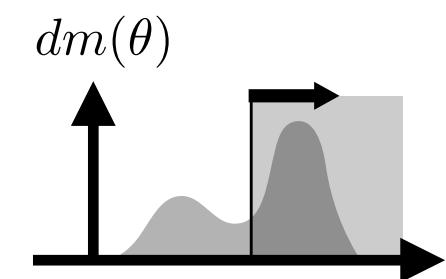


MDP Congestion Game

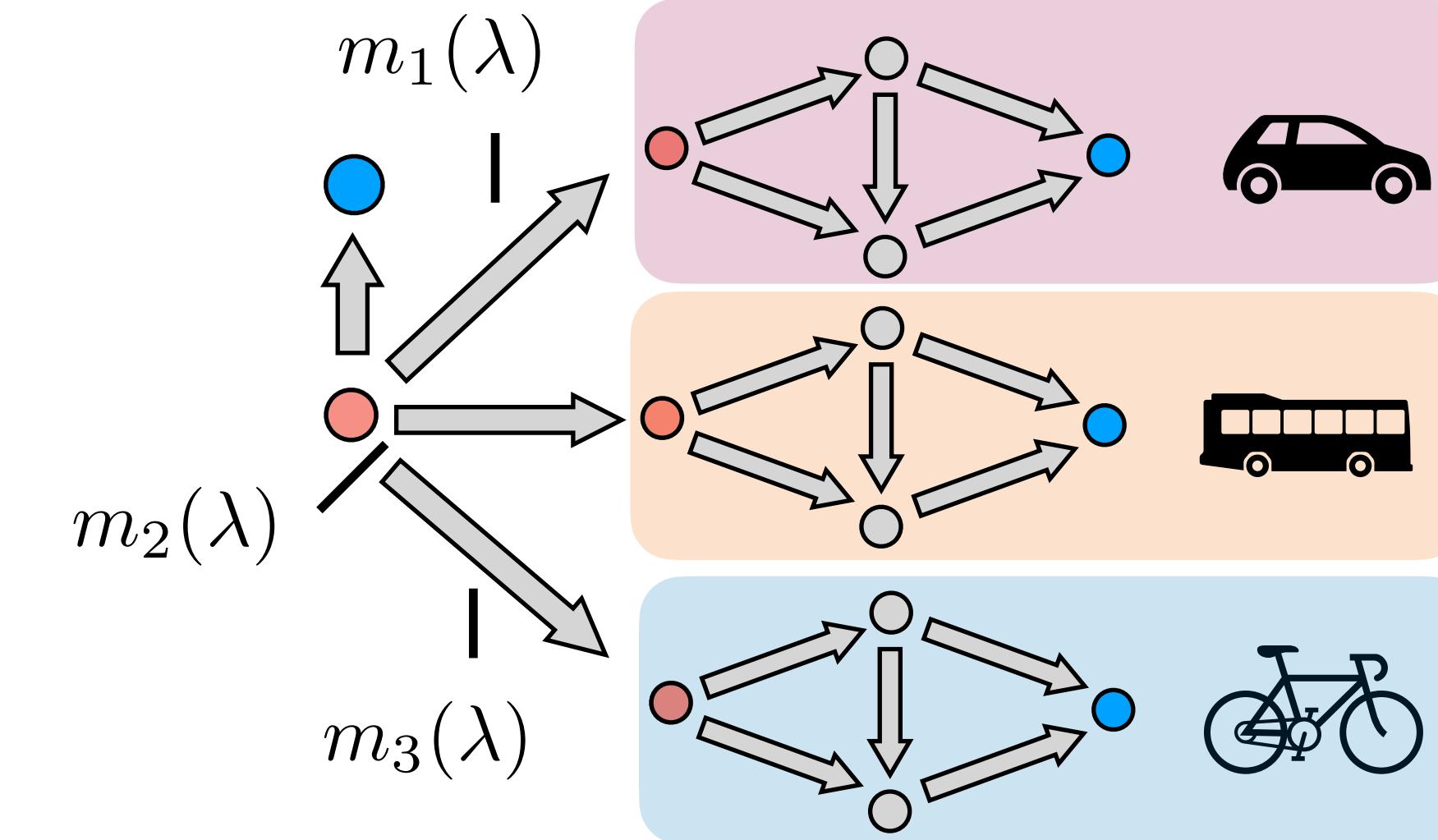
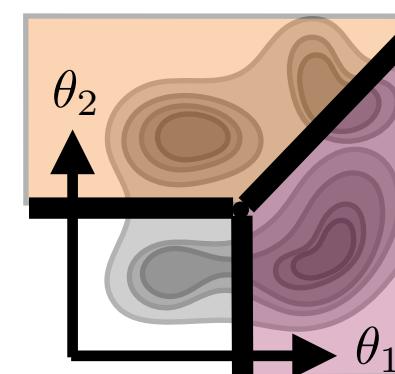


# Variable Demand - Multi-Variate Non-Homogeneous Preferences

Non-homo-geneous preferences



Multi-Variate Preferences



## APPLICATIONS

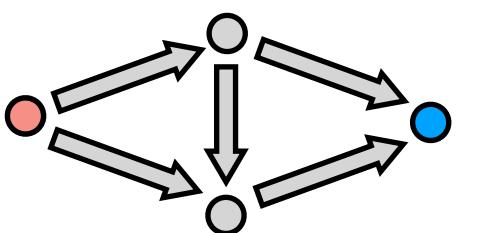
- Multi-modal transportation networks
- Non-homogeneous supply/demand

## PAPERS

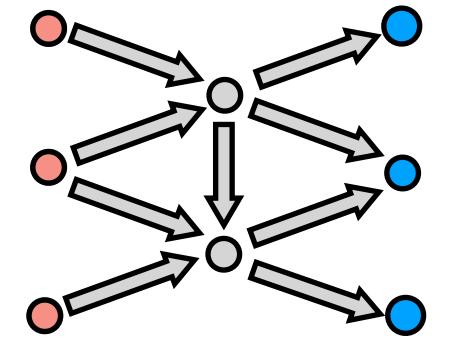
- External-cost continuous-type Wardrop equilibria in routing games  
[Calderone, Dong, Sastry, 2017]
- Multi-dimensional continuous type population potential games  
[Calderone, Ratliff, 2019]

# Potential Games

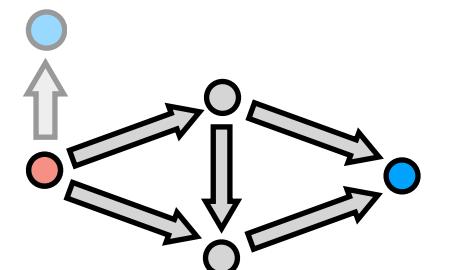
Routing Games



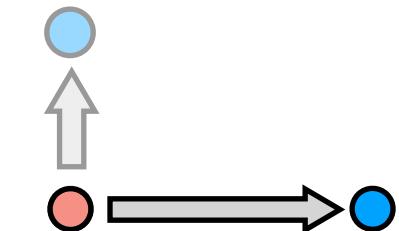
Multiple sources/sinks



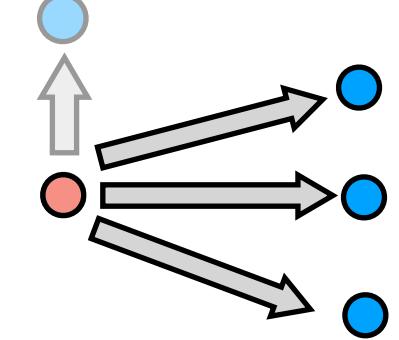
Variable Demand



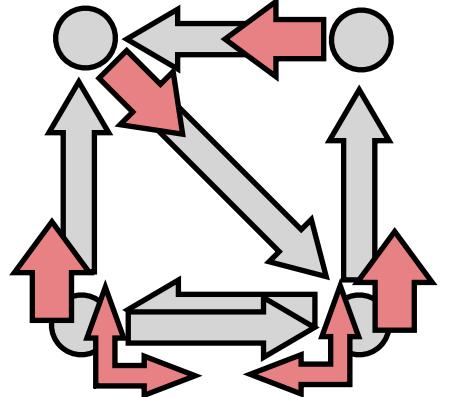
Supply & Demand



Cournot Market

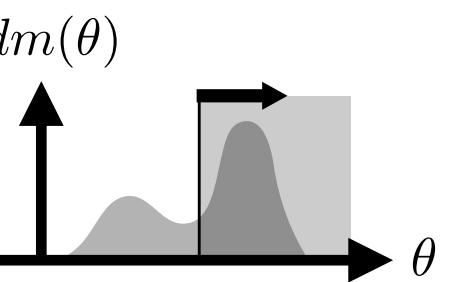


MDP Congestion Game

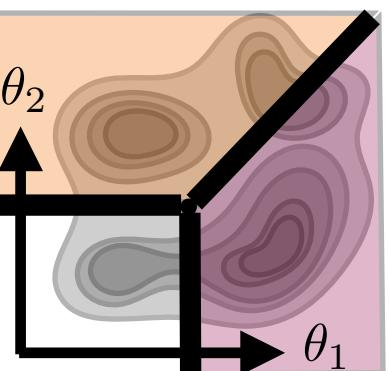


# Braess Paradox

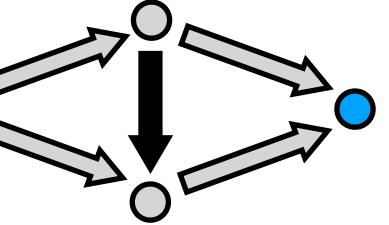
Non-homo-geneous preferences



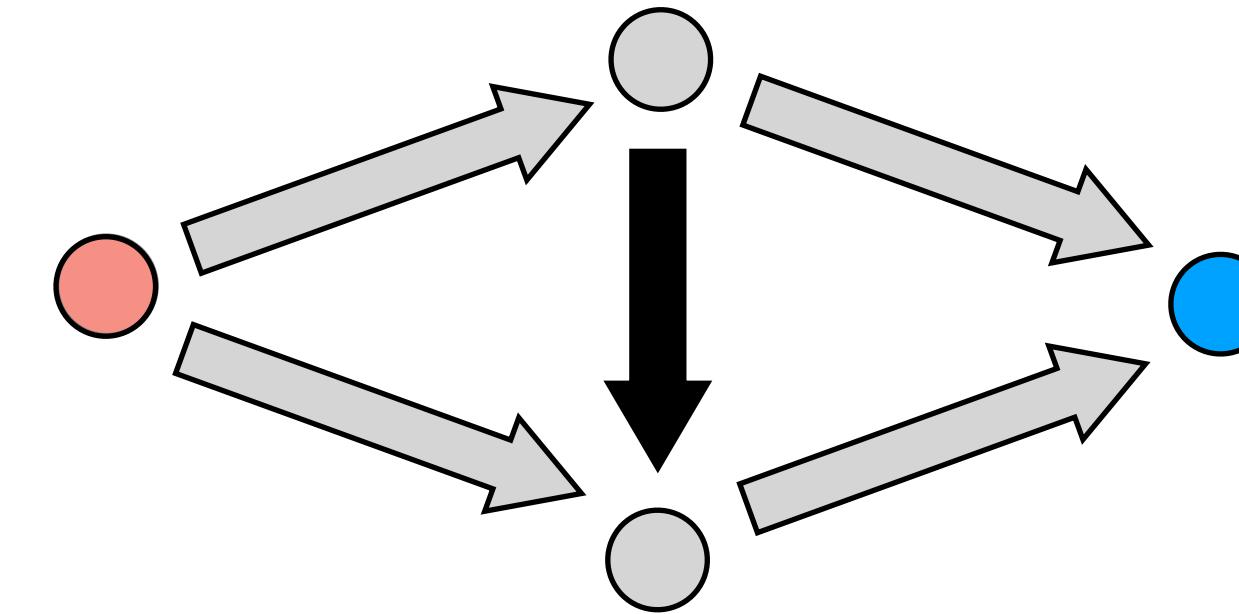
Multi-Variate Preferences



Braess Paradox

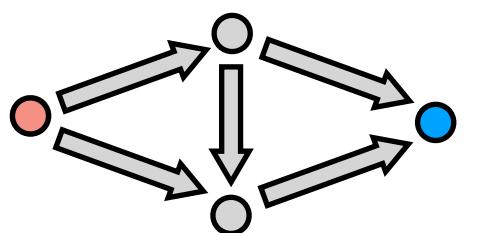


Braess Paradox

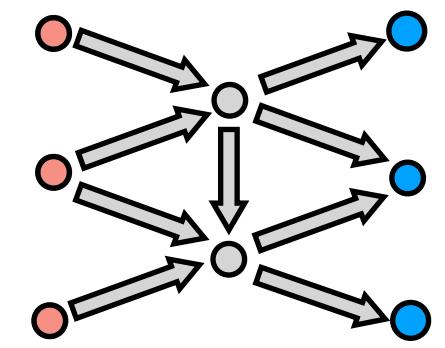


# Potential Games

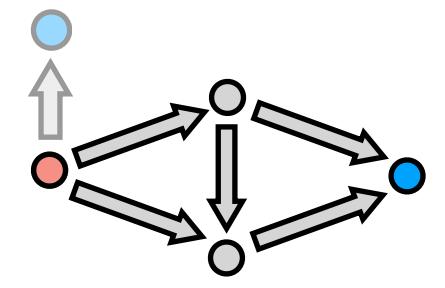
Routing Games



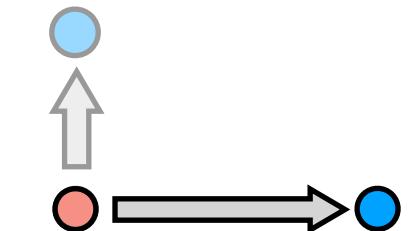
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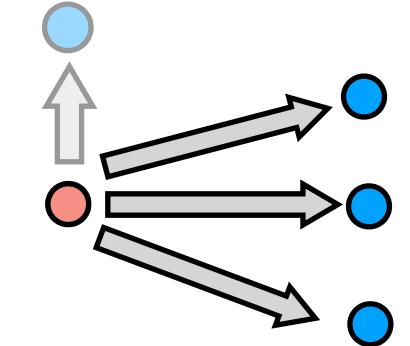
Variable Demand



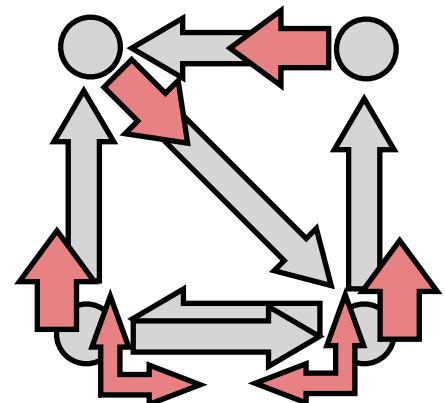
Supply & Demand



Cournot Market

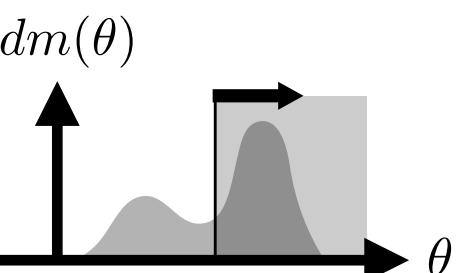


MDP Congestion Game

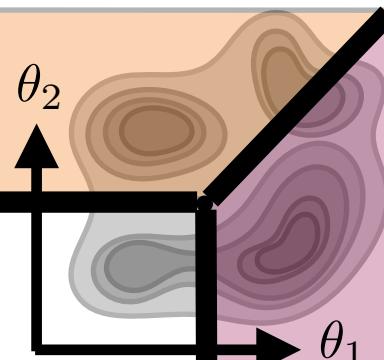


# Braess Paradox

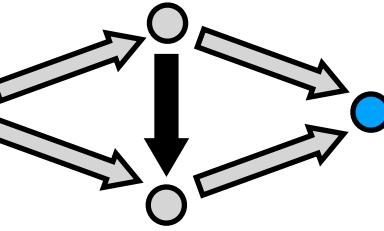
Non-homo-geneous preferences



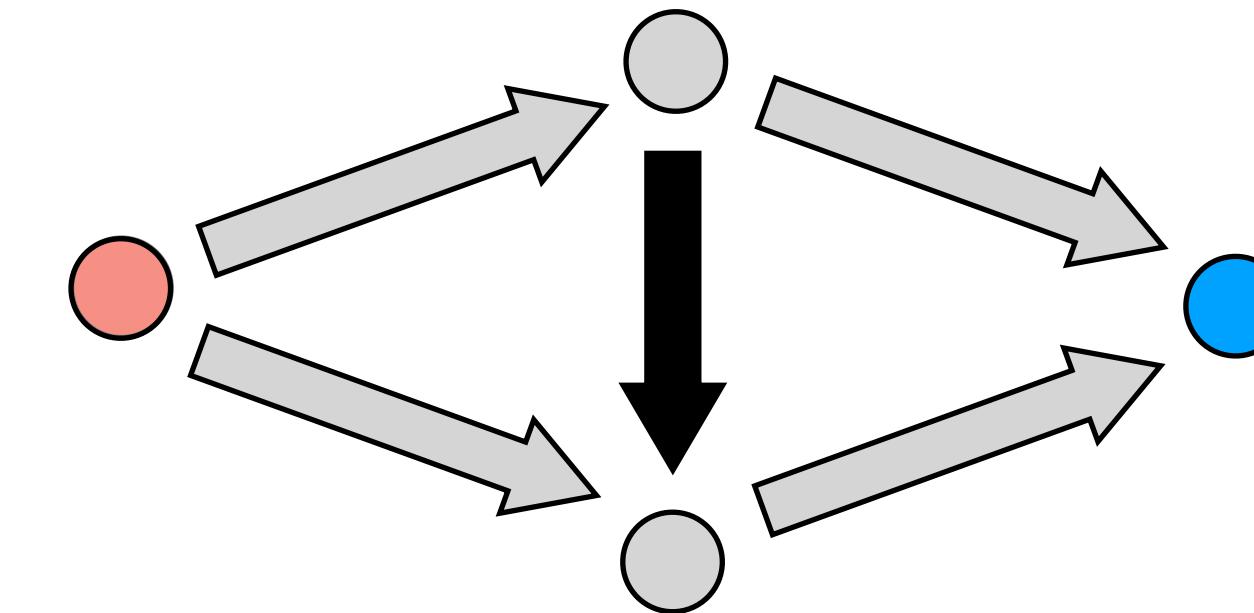
Multi-Variate Preferences



Braess Paradox



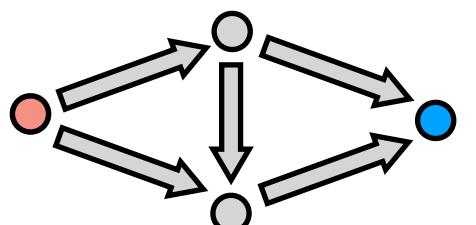
Braess Paradox



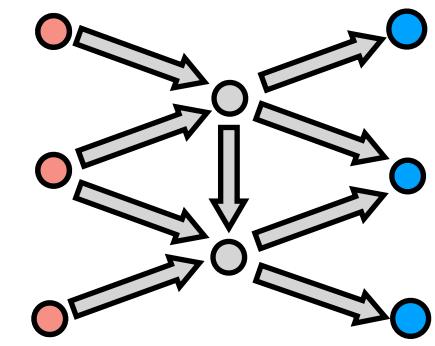
**Adding center road can make traffic worse!**

# Potential Games

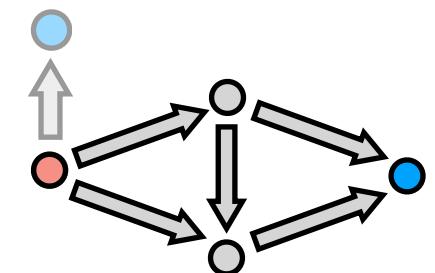
Routing Games



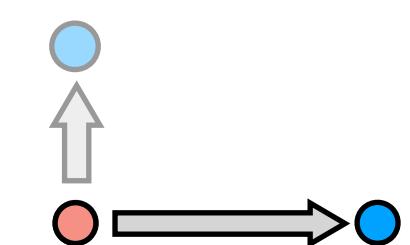
Multiple sources/sinks



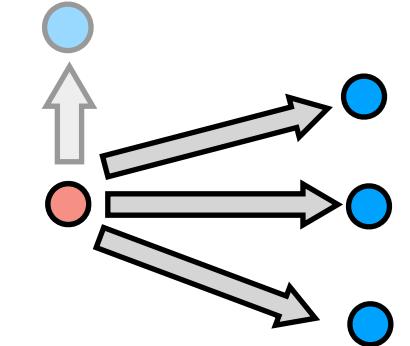
Variable Demand



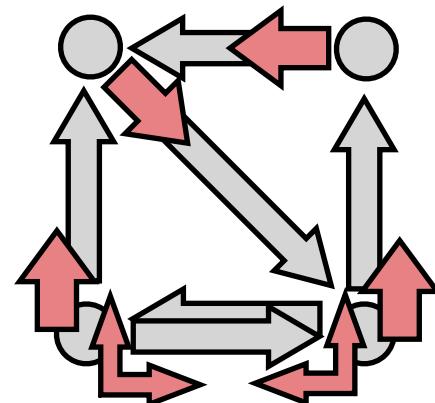
Supply & Demand



Cournot Market

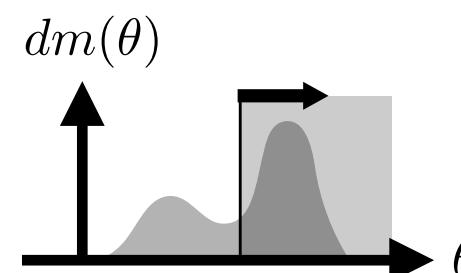


MDP Congestion Game

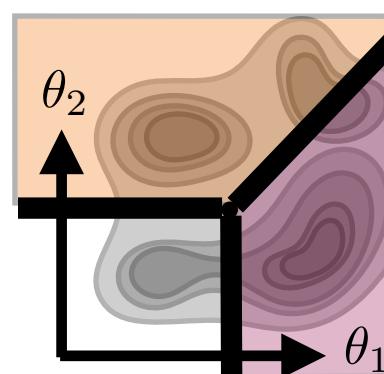


# Braess Paradox

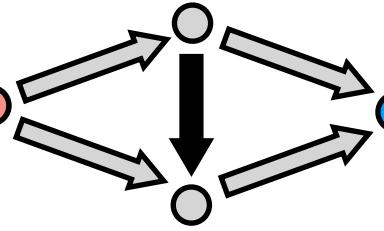
Non-homo-geneous preferences



Multi-Variate Preferences



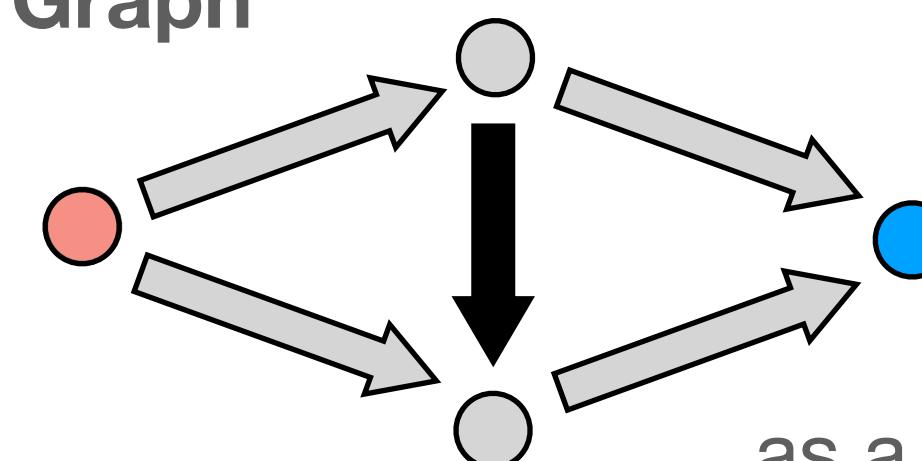
Braess Paradox



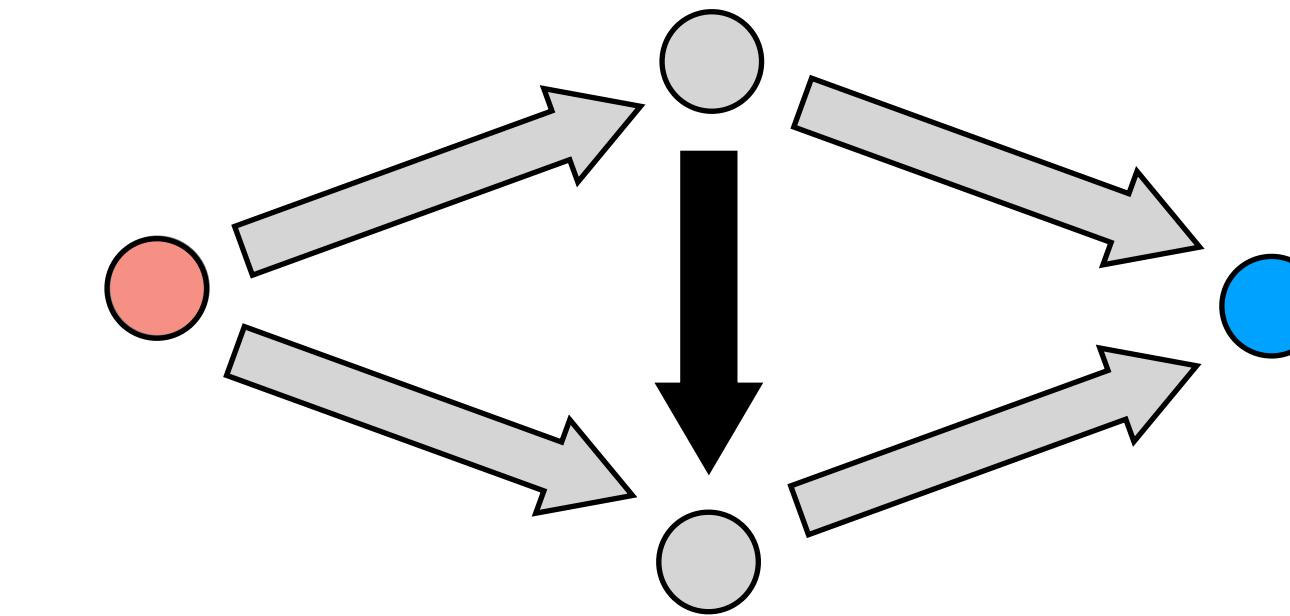
Network Characterization:

Every network has

**Braess Graph**

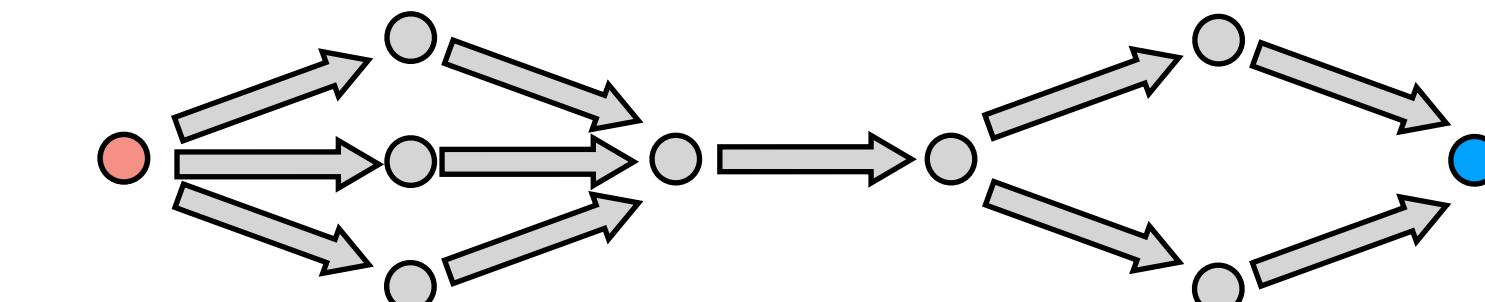


Braess Paradox



OR is

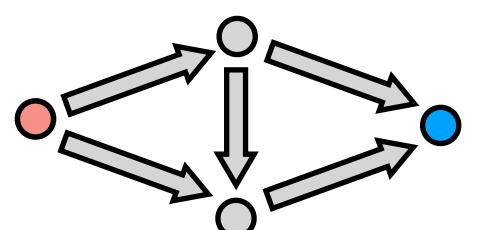
**Series-Parallel Graph**



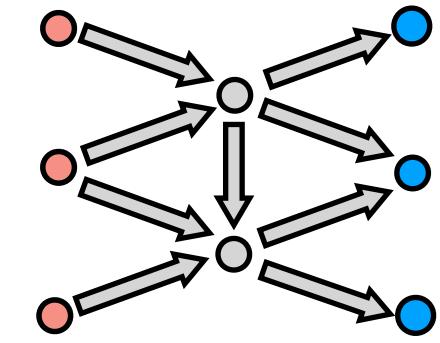
as a subgraph

# Potential Games

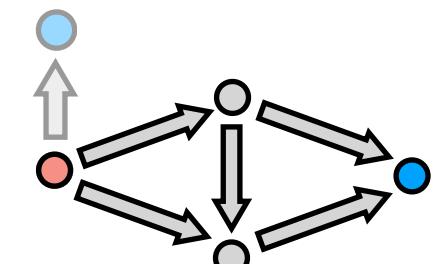
Routing Games



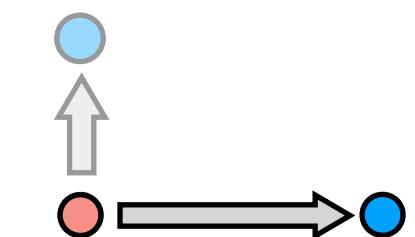
Multiple sources/sinks



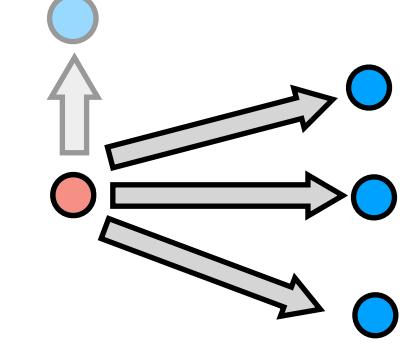
Variable Demand



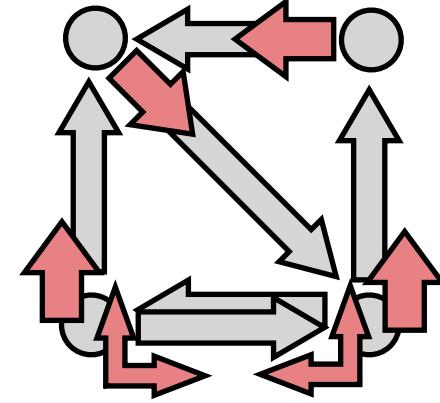
Supply & Demand



Cournot Market

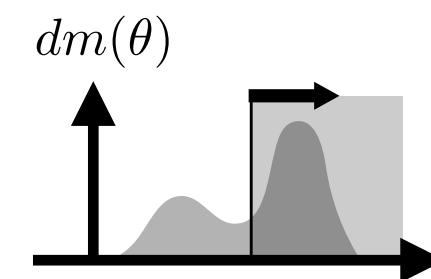


MDP Congestion Game

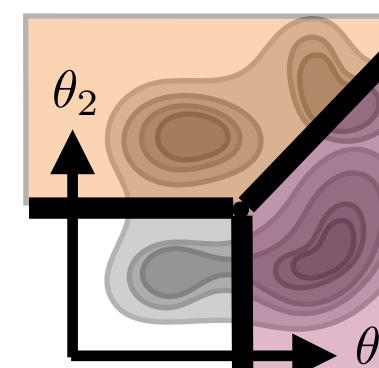


# Braess Paradox

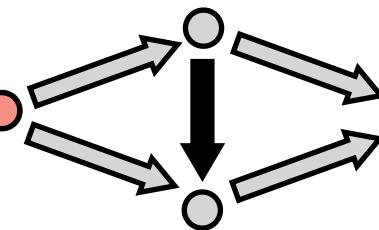
Non-homo-geneous preferences



Multi-Variate Preferences



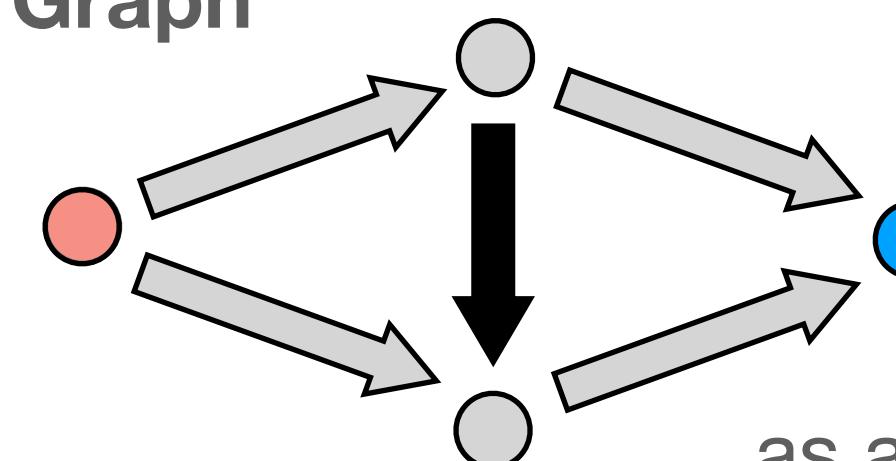
Braess Paradox



**Network Characterization:**

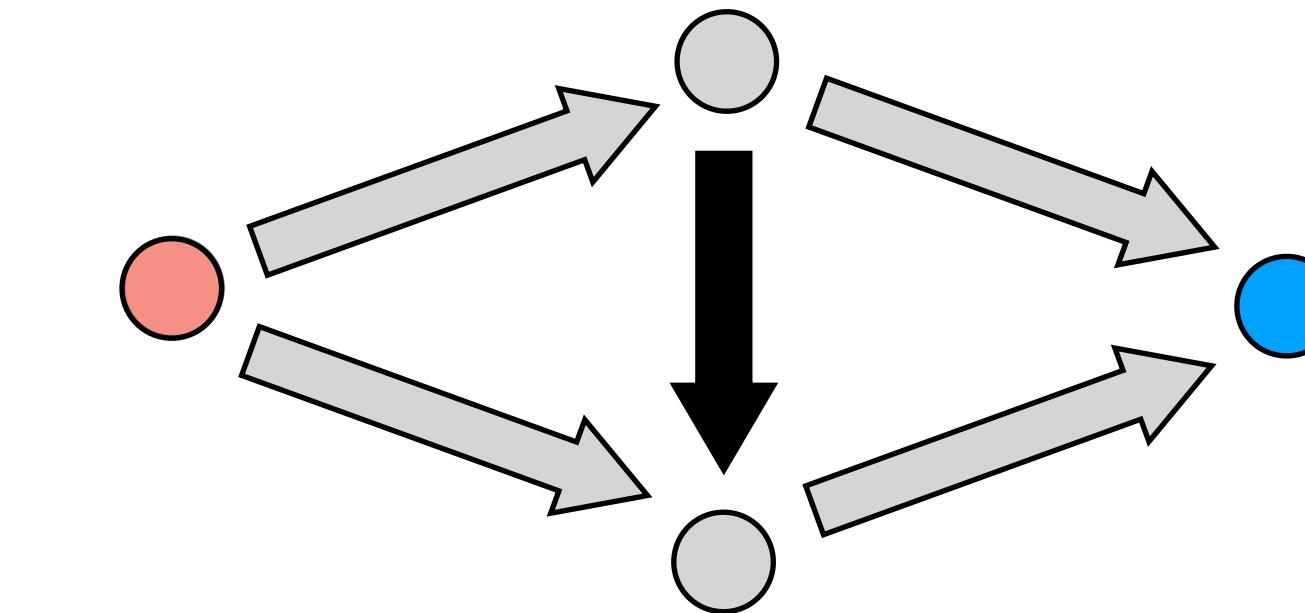
Every network has

**Braess Graph**



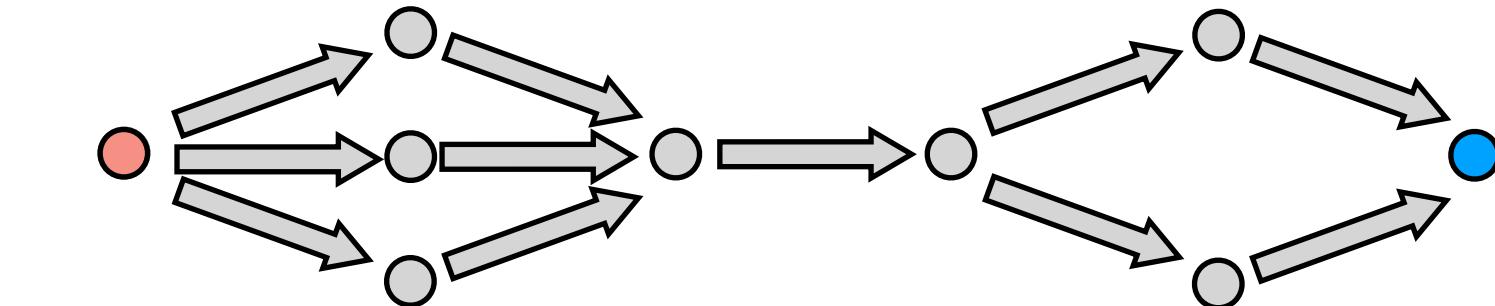
as a subgraph

Braess Paradox



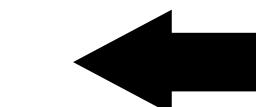
**Adding center road can make traffic worse!**

**Series-Parallel Graph**



OR is

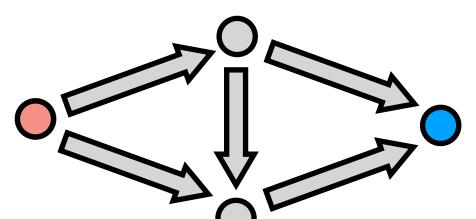
Series - Parallel graphs cannot suffer from Braess paradox



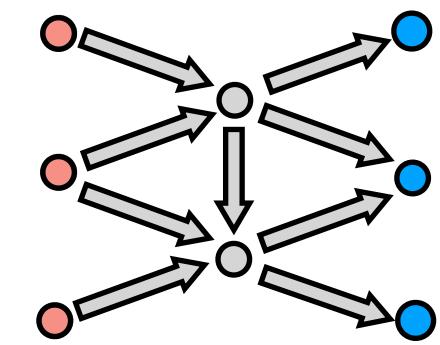
linear-algebraic characterization/proof

# Potential Games

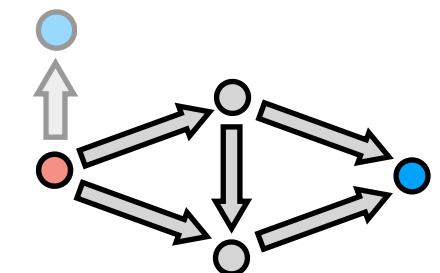
Routing Games



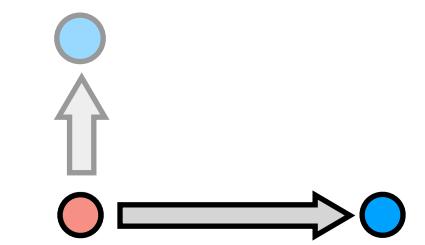
Multiple sources/sinks



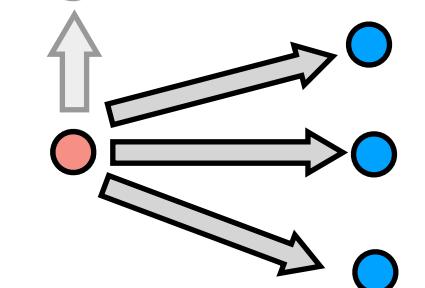
Variable Demand



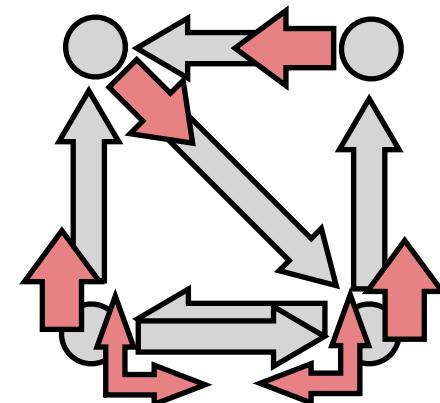
Supply & Demand



Cournot Market

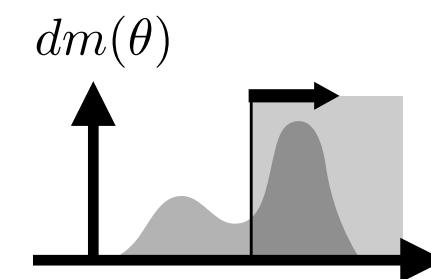


MDP Congestion Game

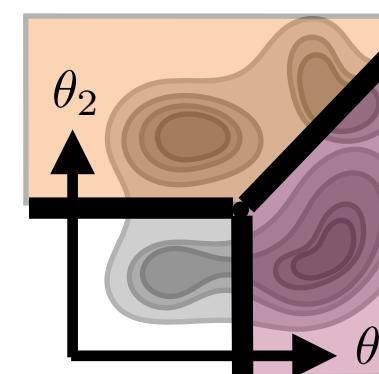


# Braess Paradox

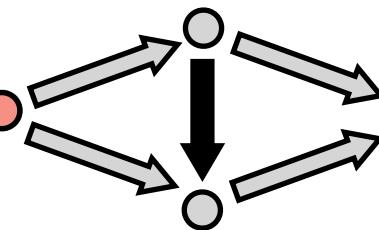
Non-homo-geneous preferences



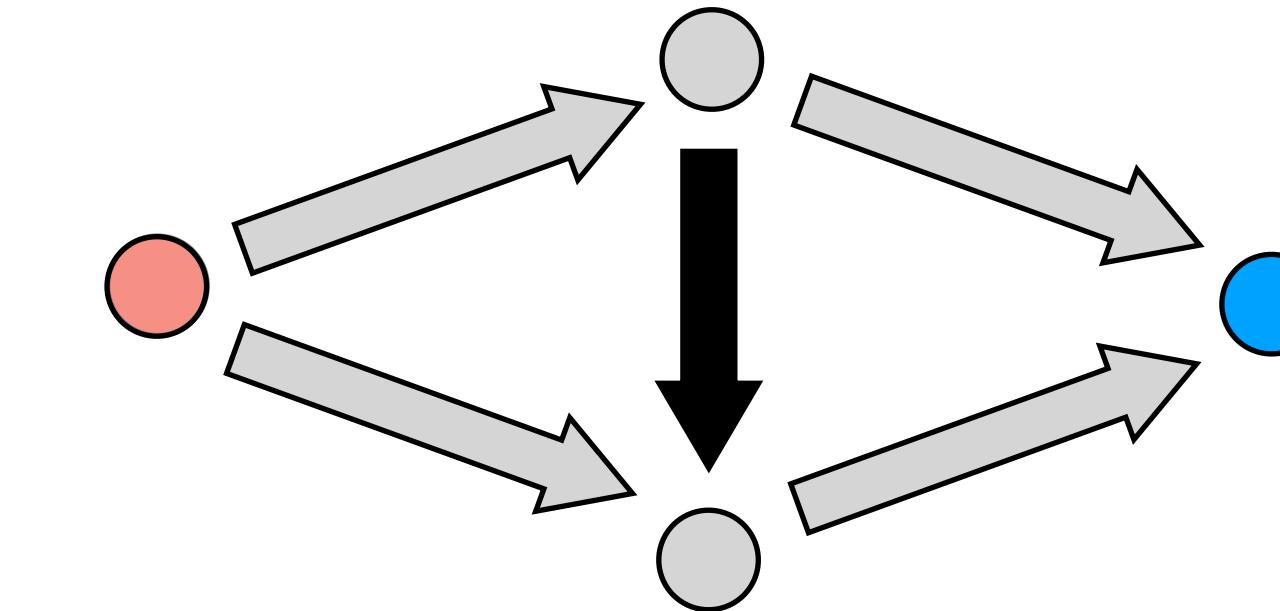
Multi-Variate Preferences



Braess Paradox



Braess Paradox



**Adding center road can make traffic worse!**

## REFERENCES

- Über ein Paradoxon der verkehrsplanung [Braess, 1969]
- Topology of series-parallel networks [Duffin, 1965]
- Network topology and the efficiency of equilibrium [Milchtaich, 2006]

## PAPERS

- Sensitivity analysis for Markov decision process congestion games [Li, Calderone, Ratliff, 2019]
- Algebraic characterization of Braess paradox:  
Network efficiency in series-parallel and Braess networks [Calderone, Ratliff, in prep]