Graph Structures & Matrices

Algebraic Graph Theory

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Mathias Colbert Russelson,
Sarah Li
Shahriar Talebi

Spring 2022 - Dan Calderone

Graphs

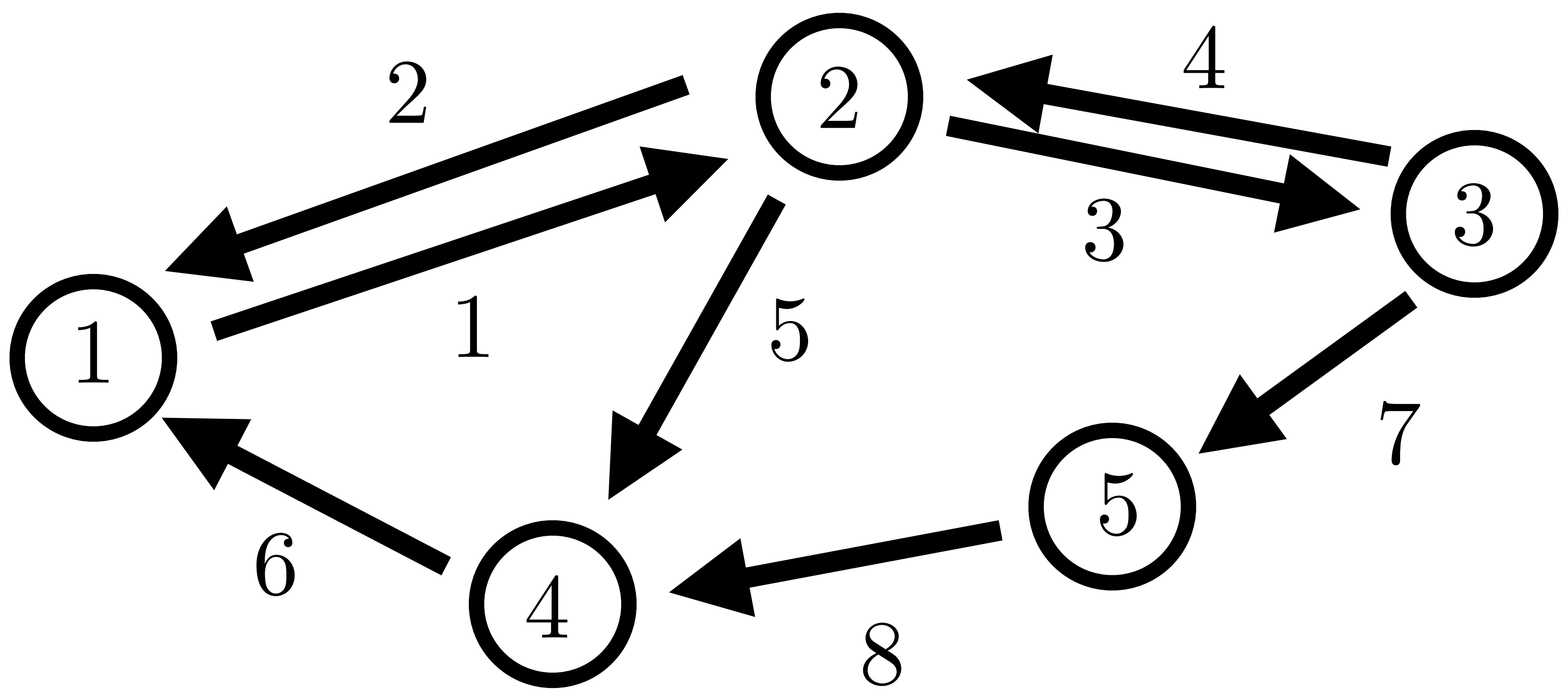
Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

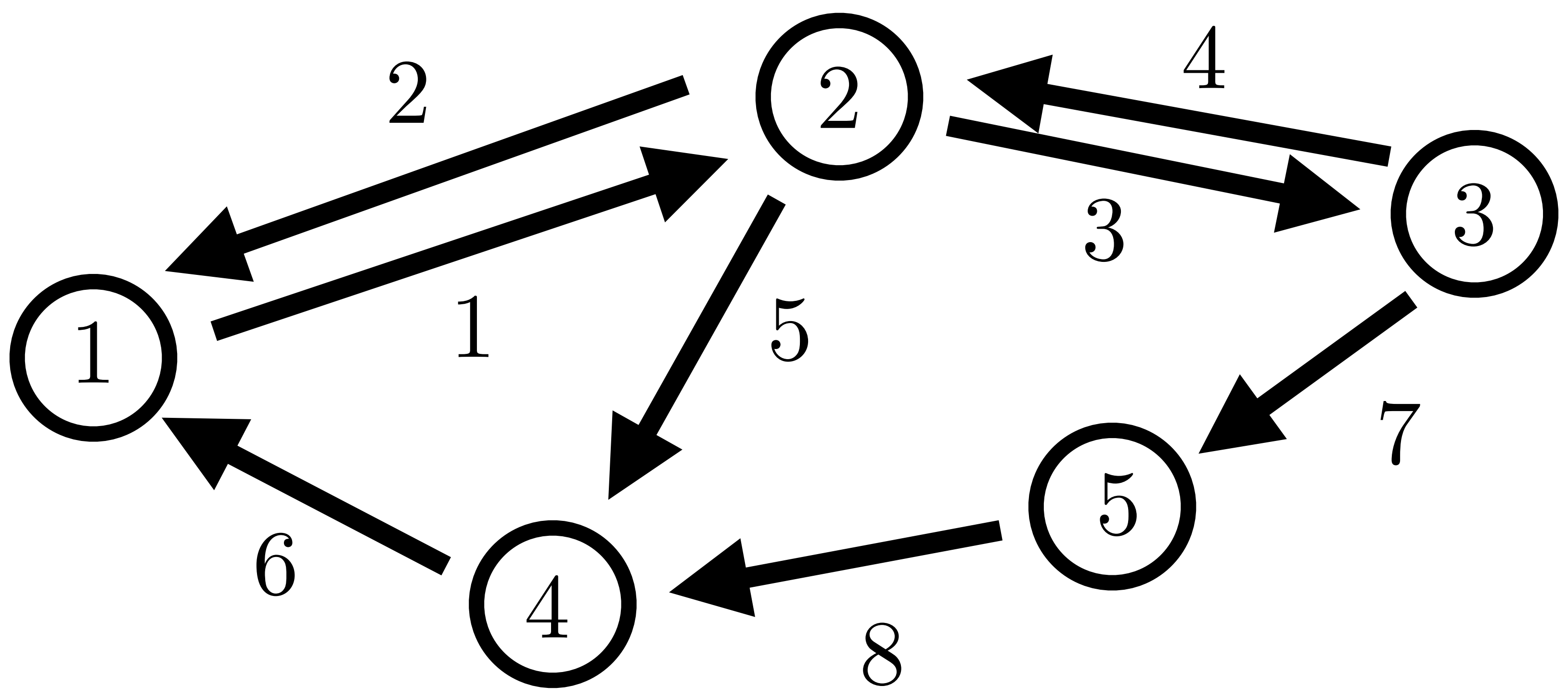
Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix



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$$\Big[\, E \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

edges

$$E = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

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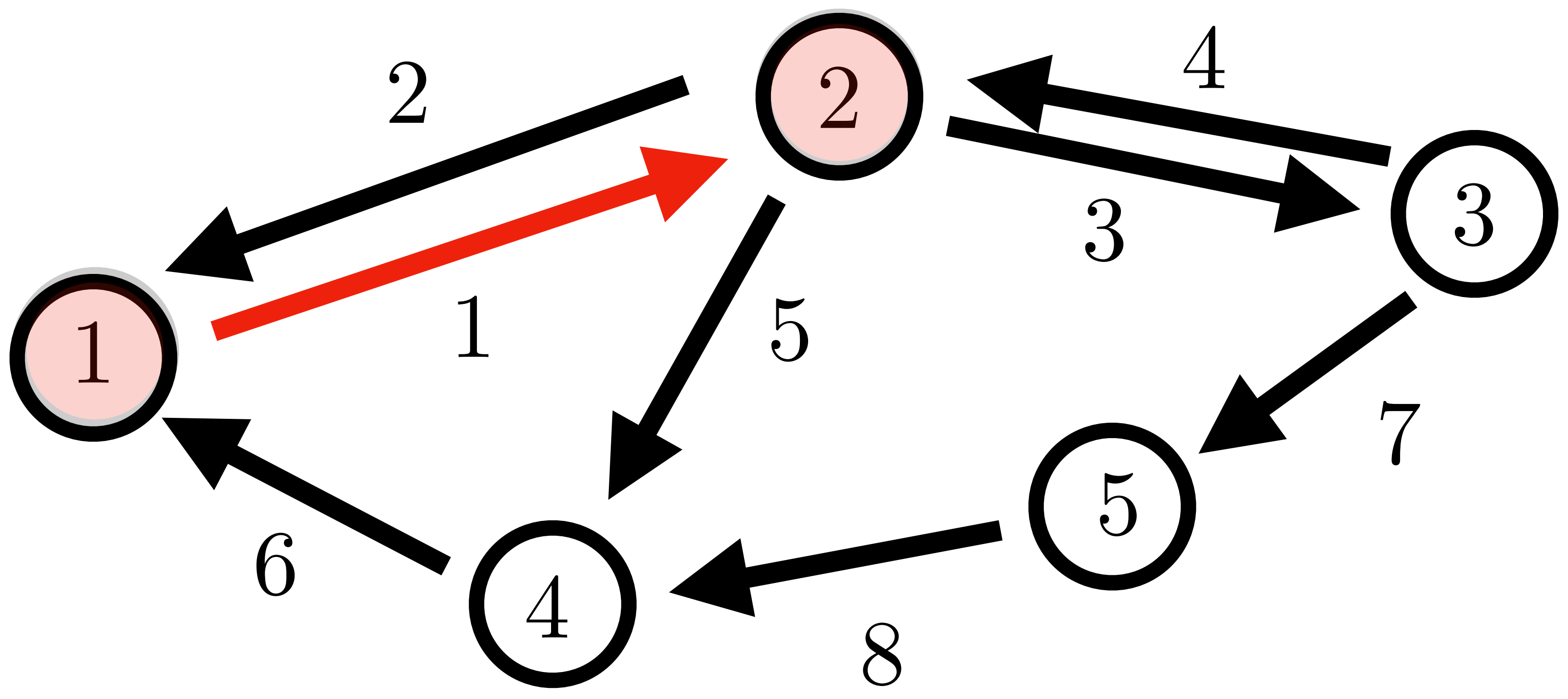

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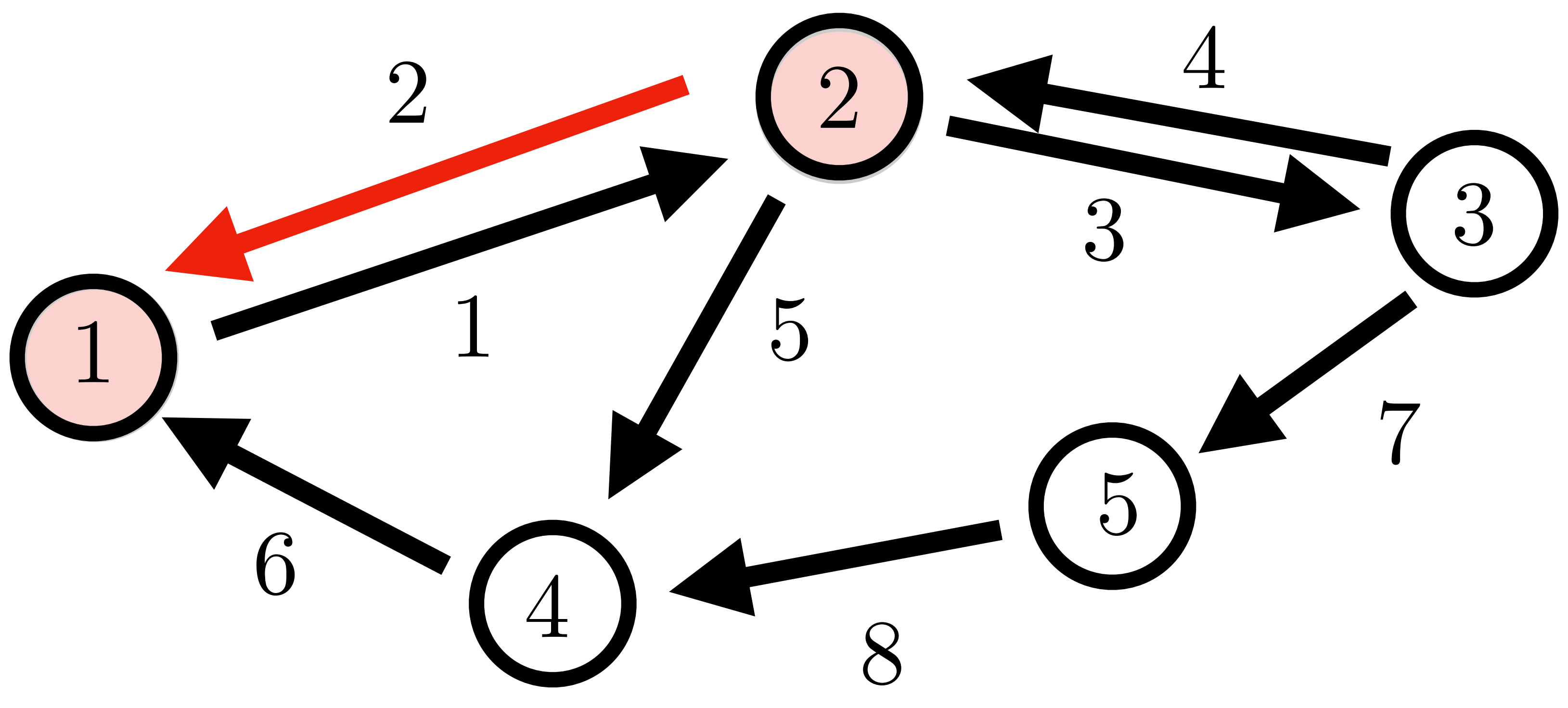

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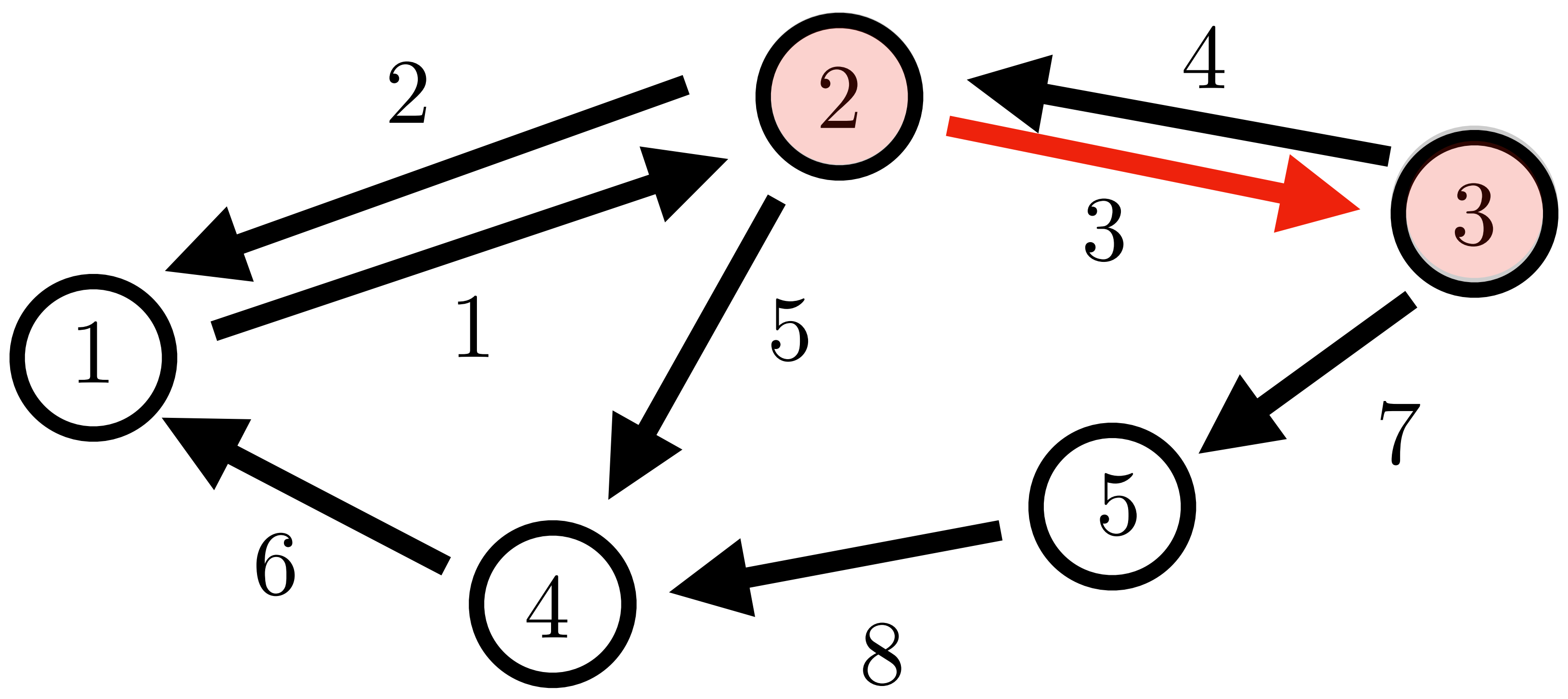

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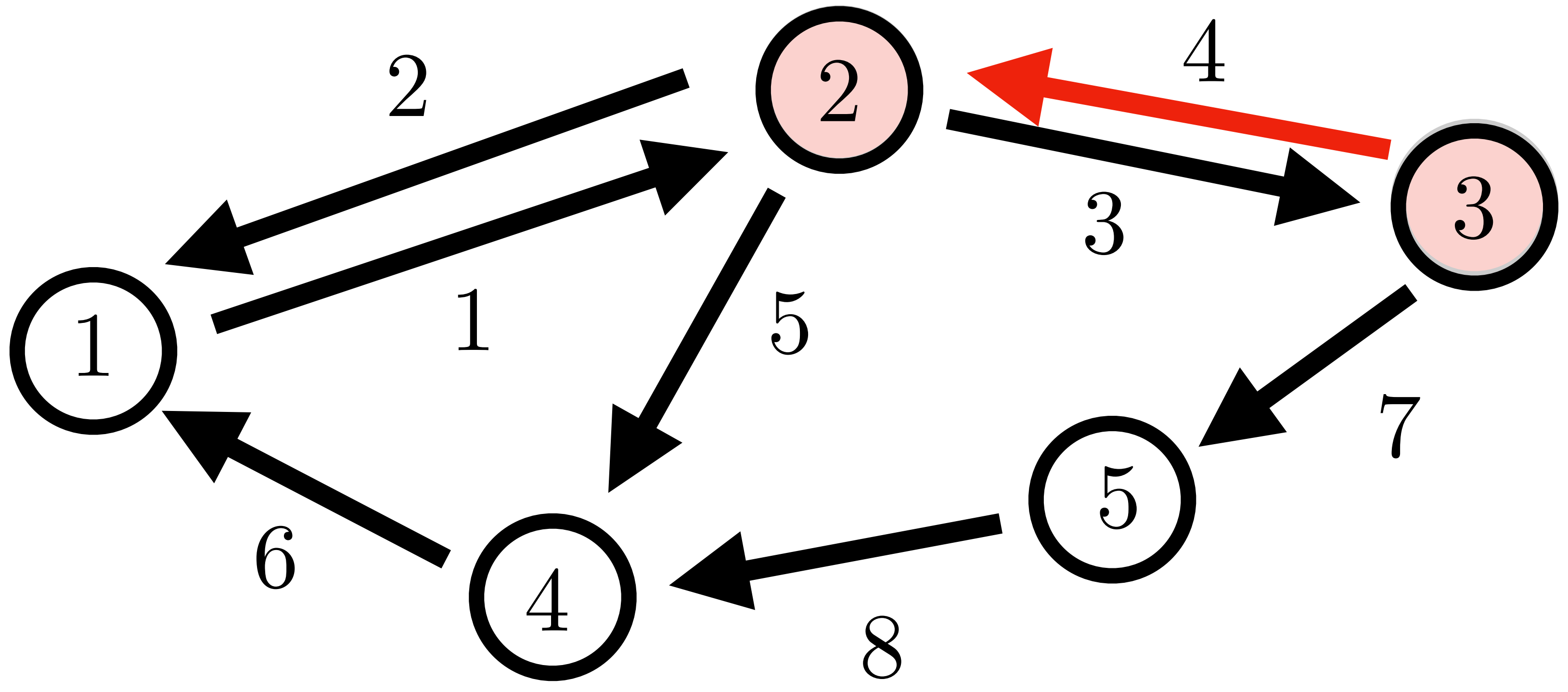

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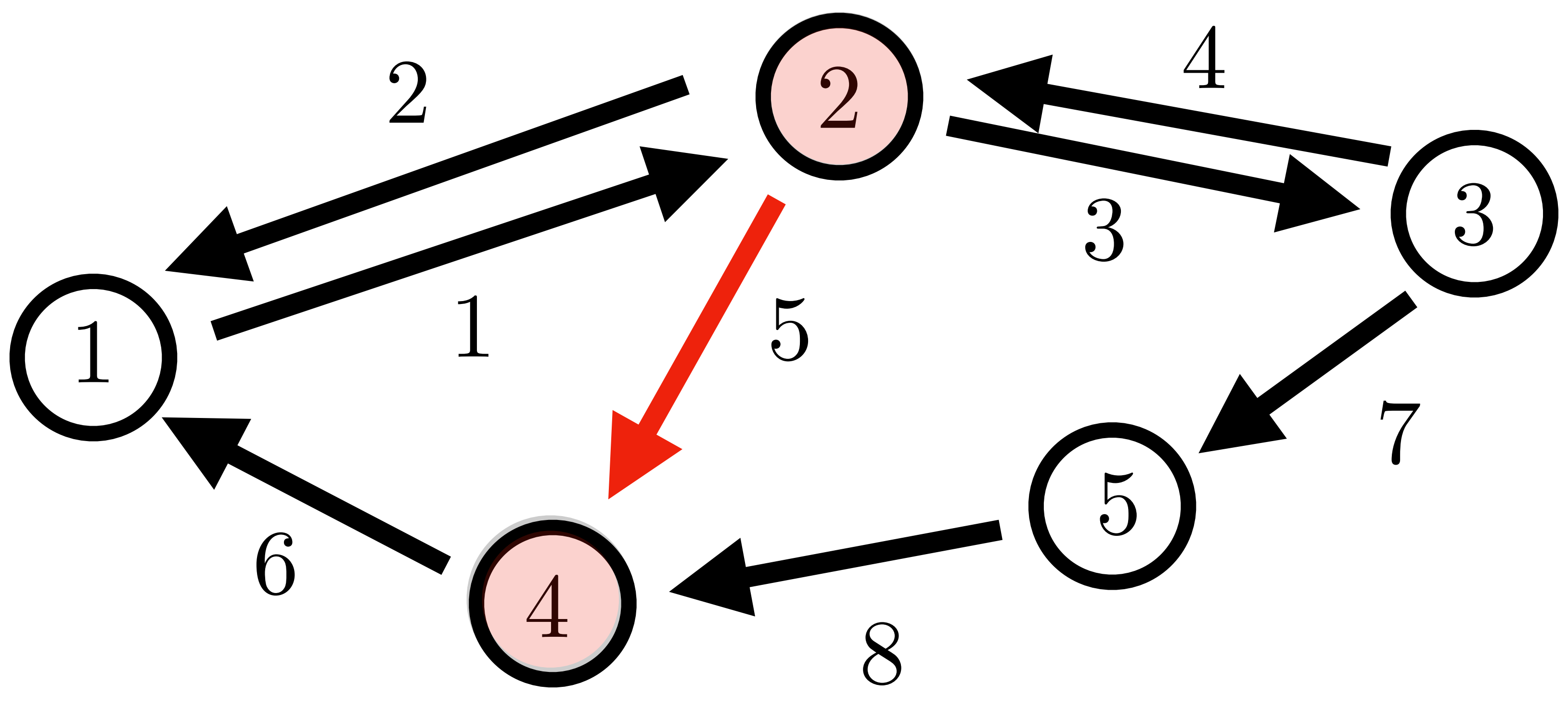

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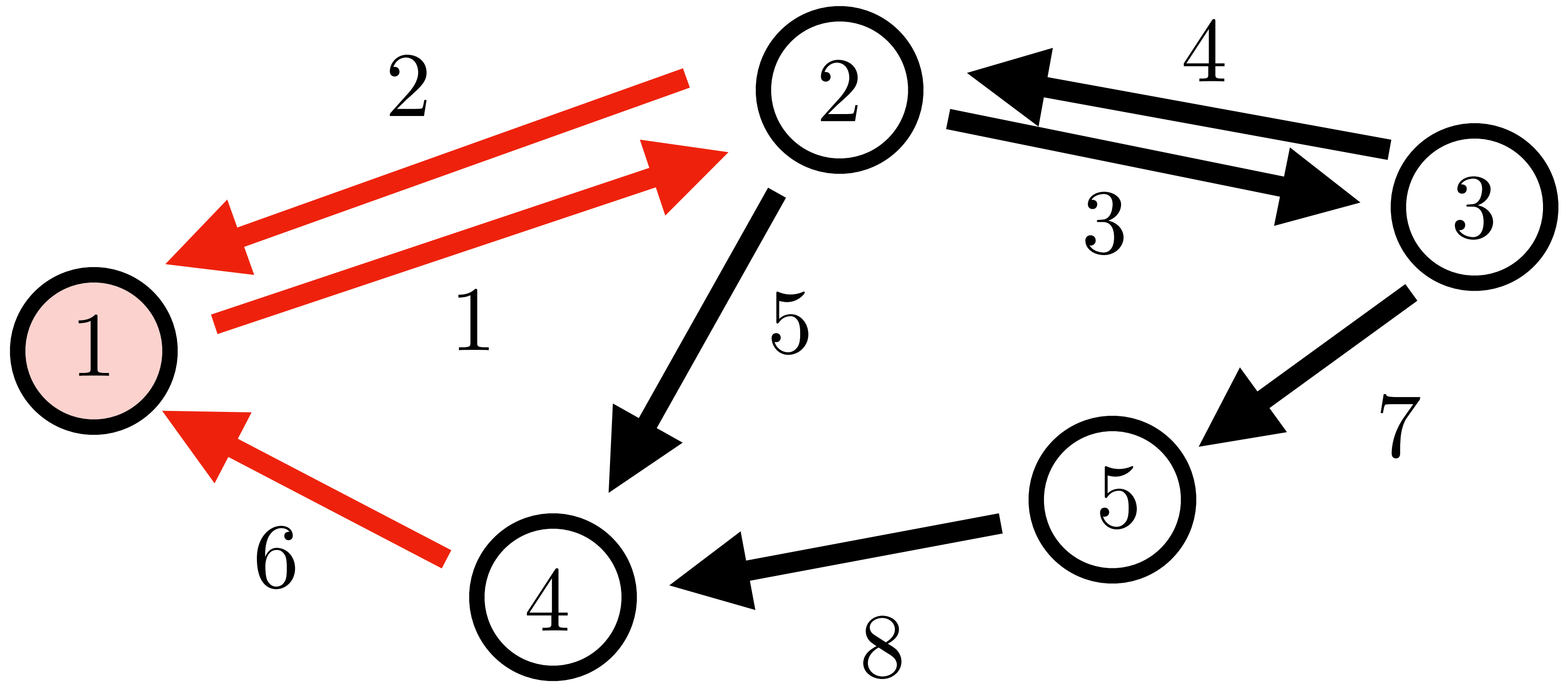

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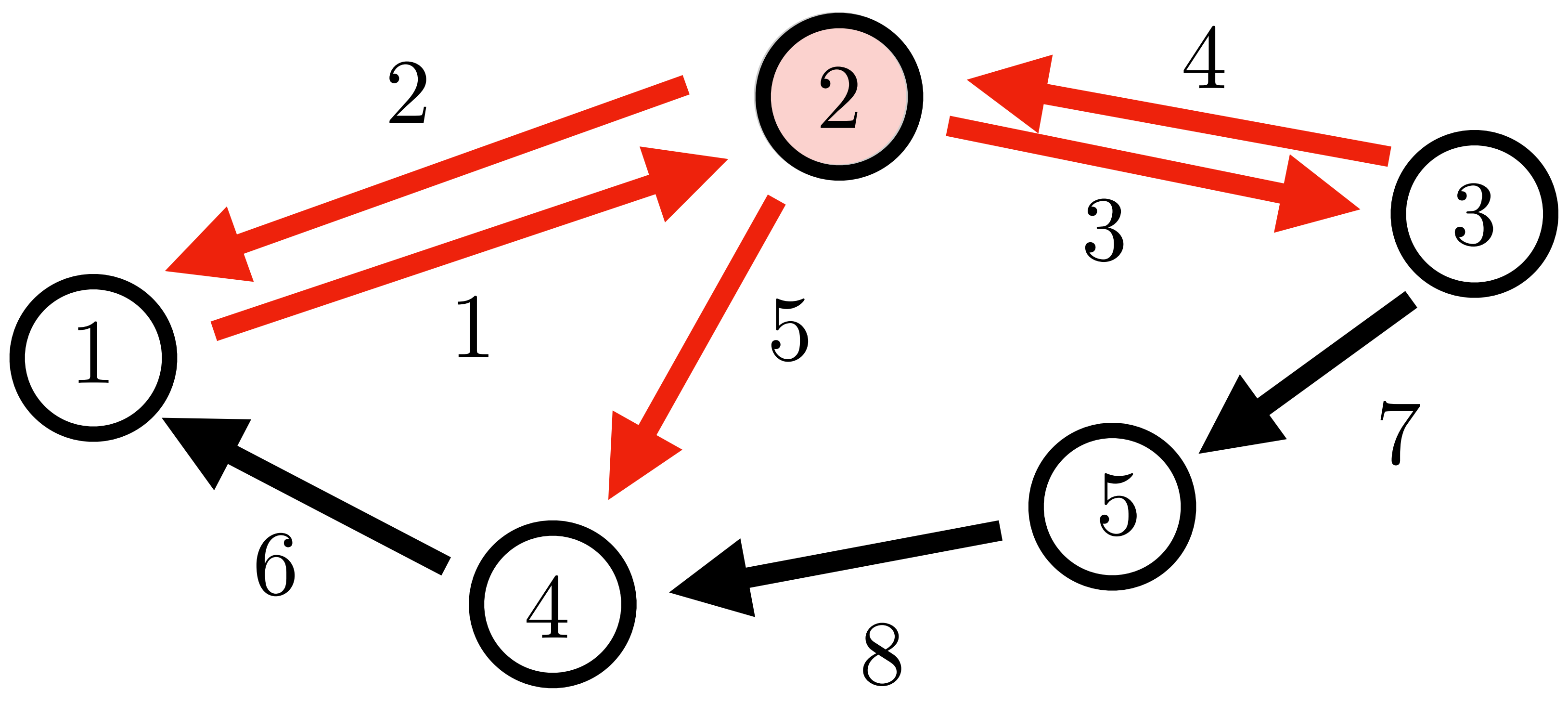

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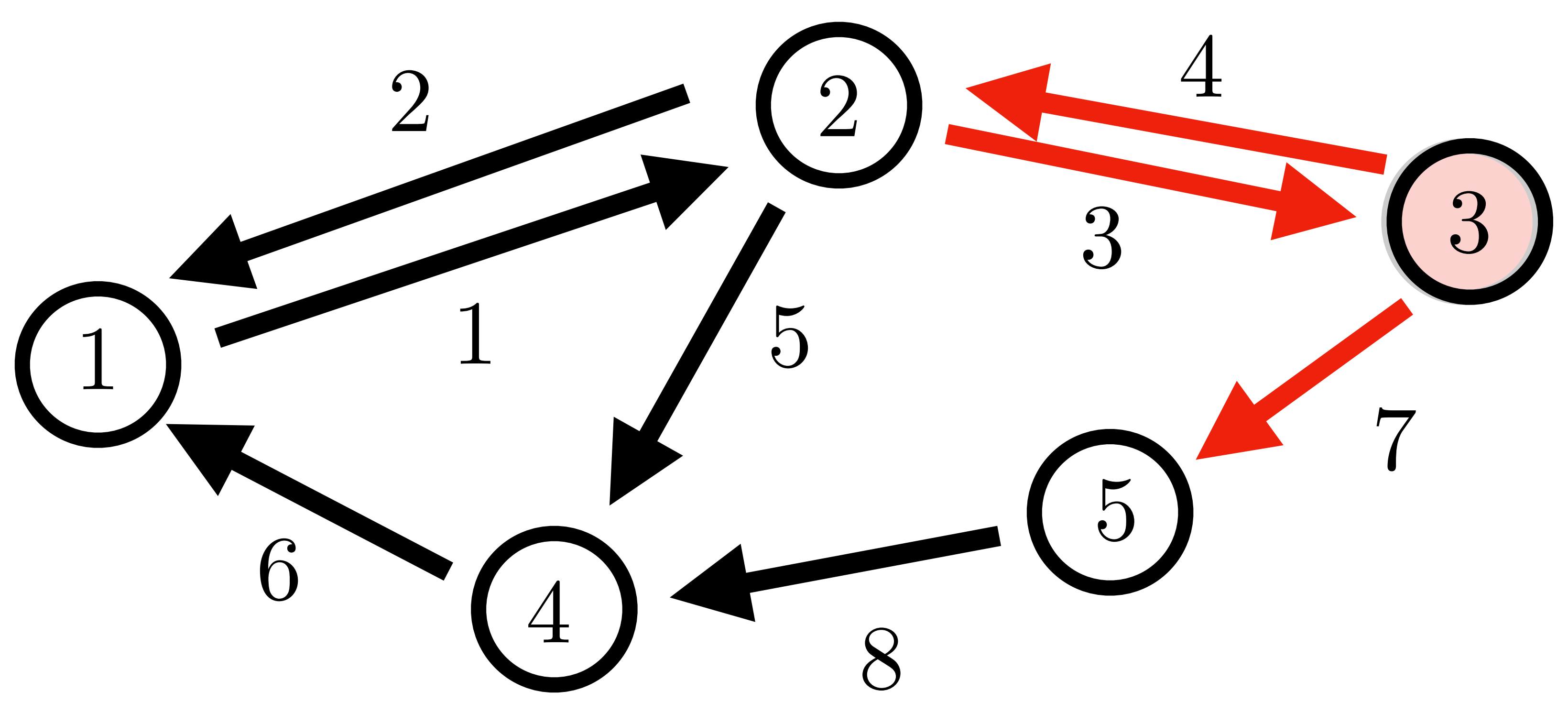

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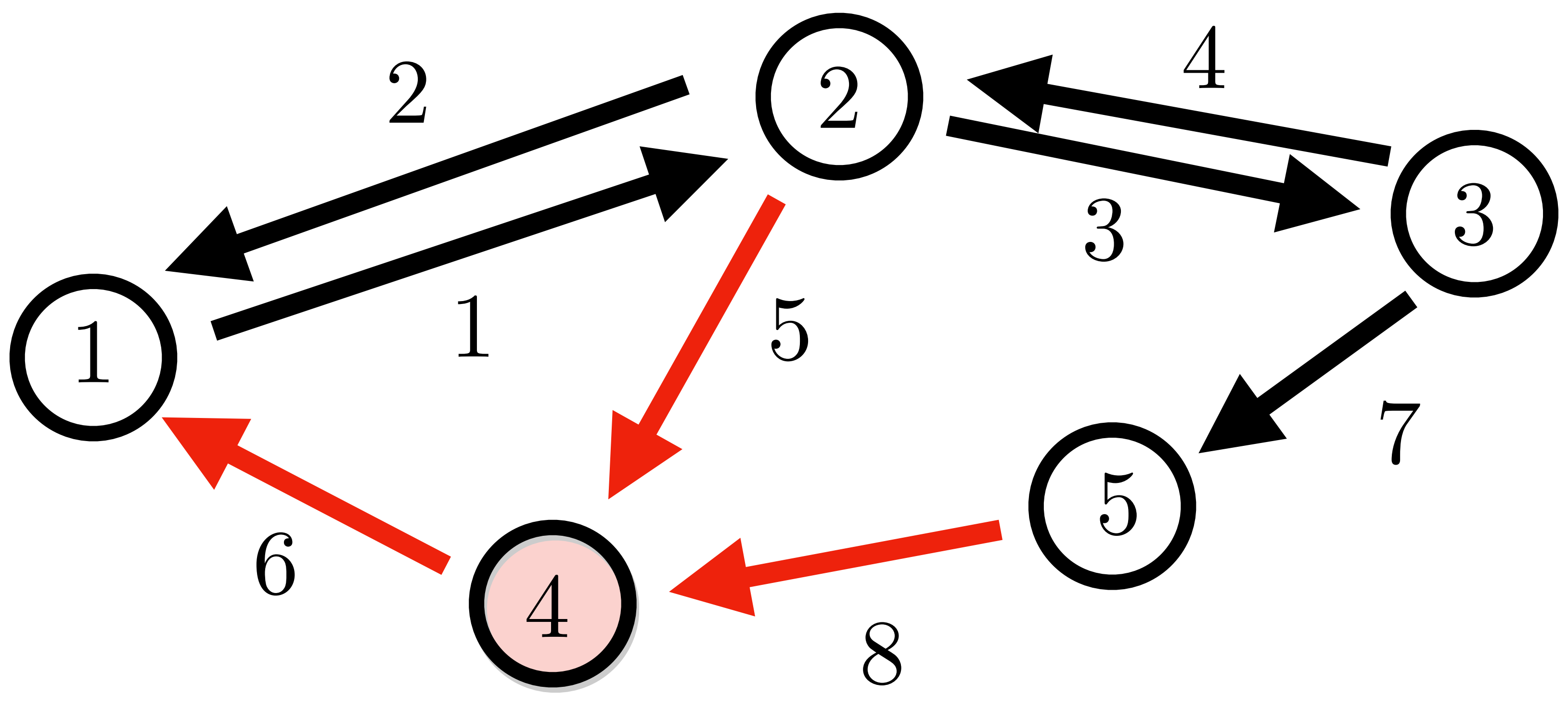

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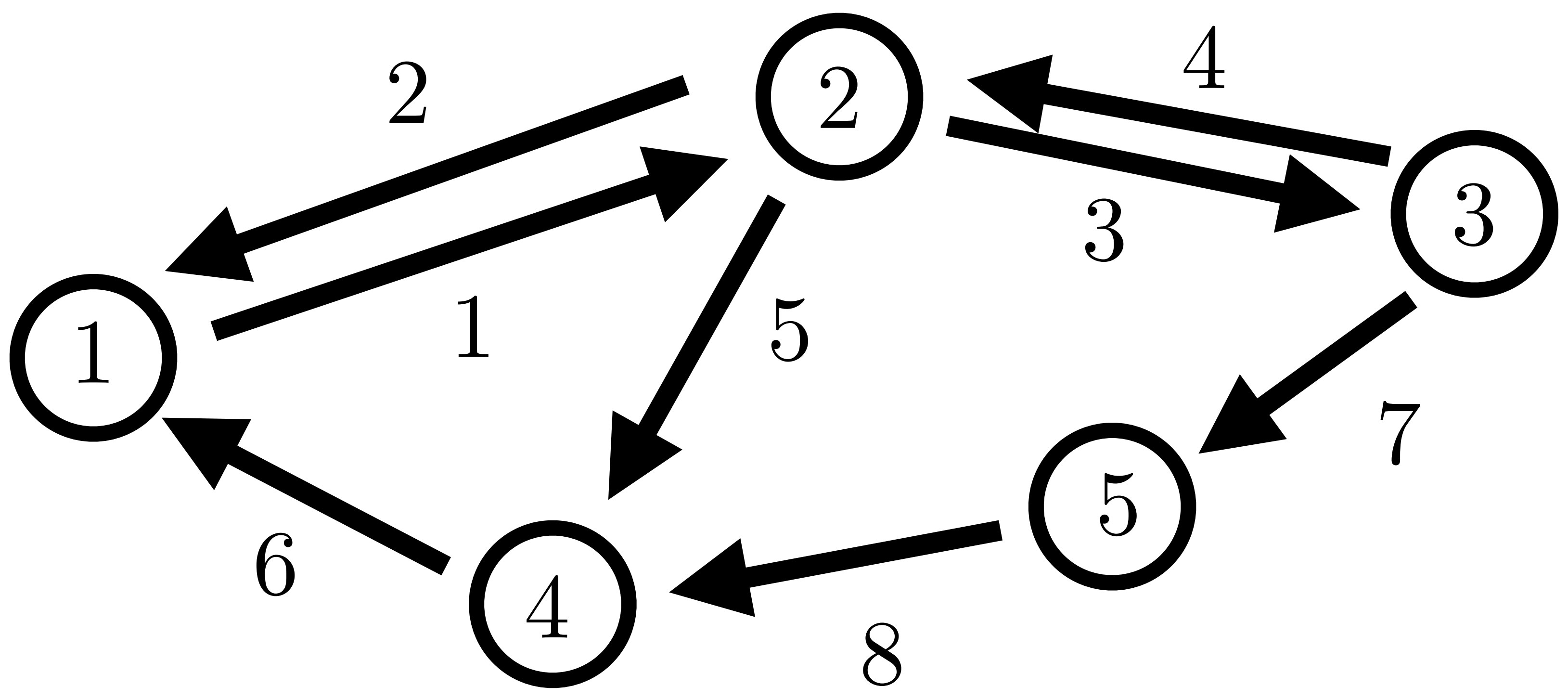

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Left Nullspace $\qquad \mathbf{1}^T E = 0$

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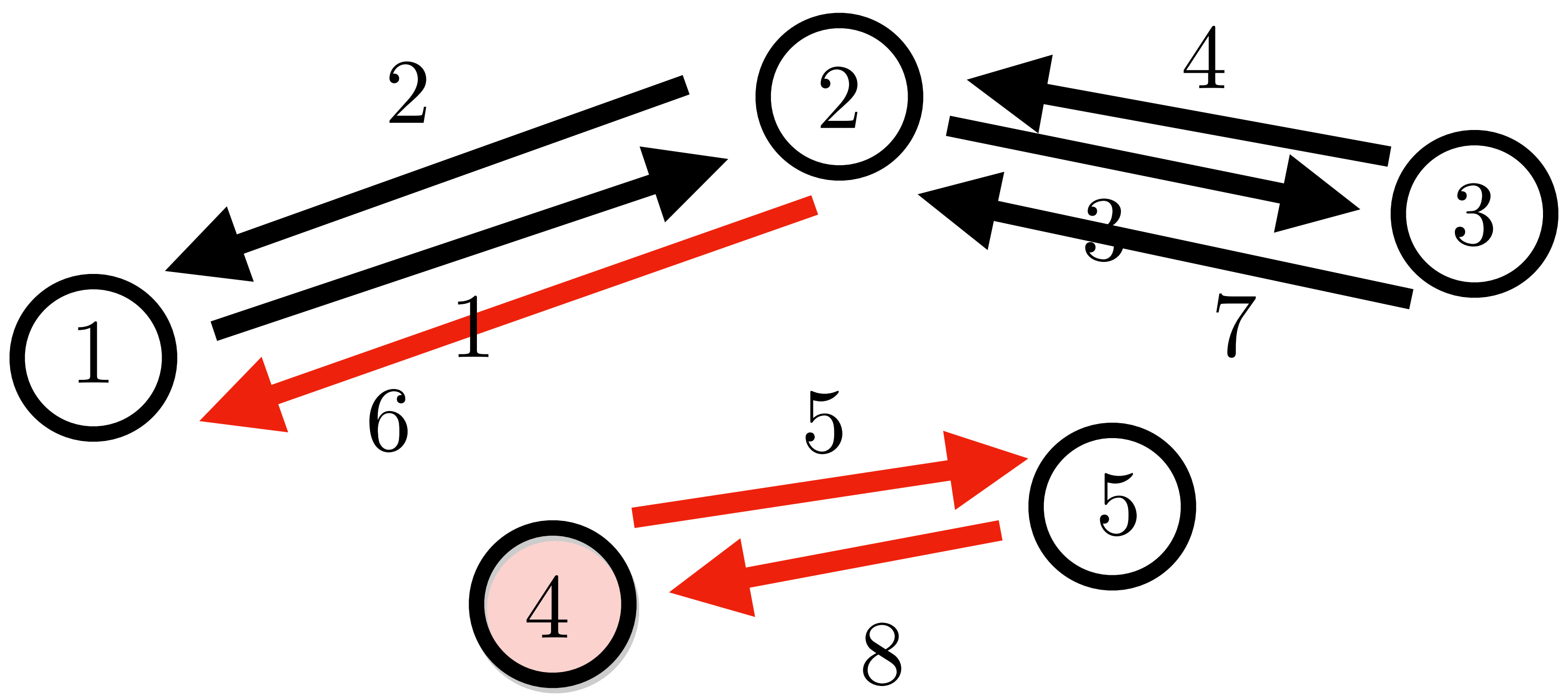

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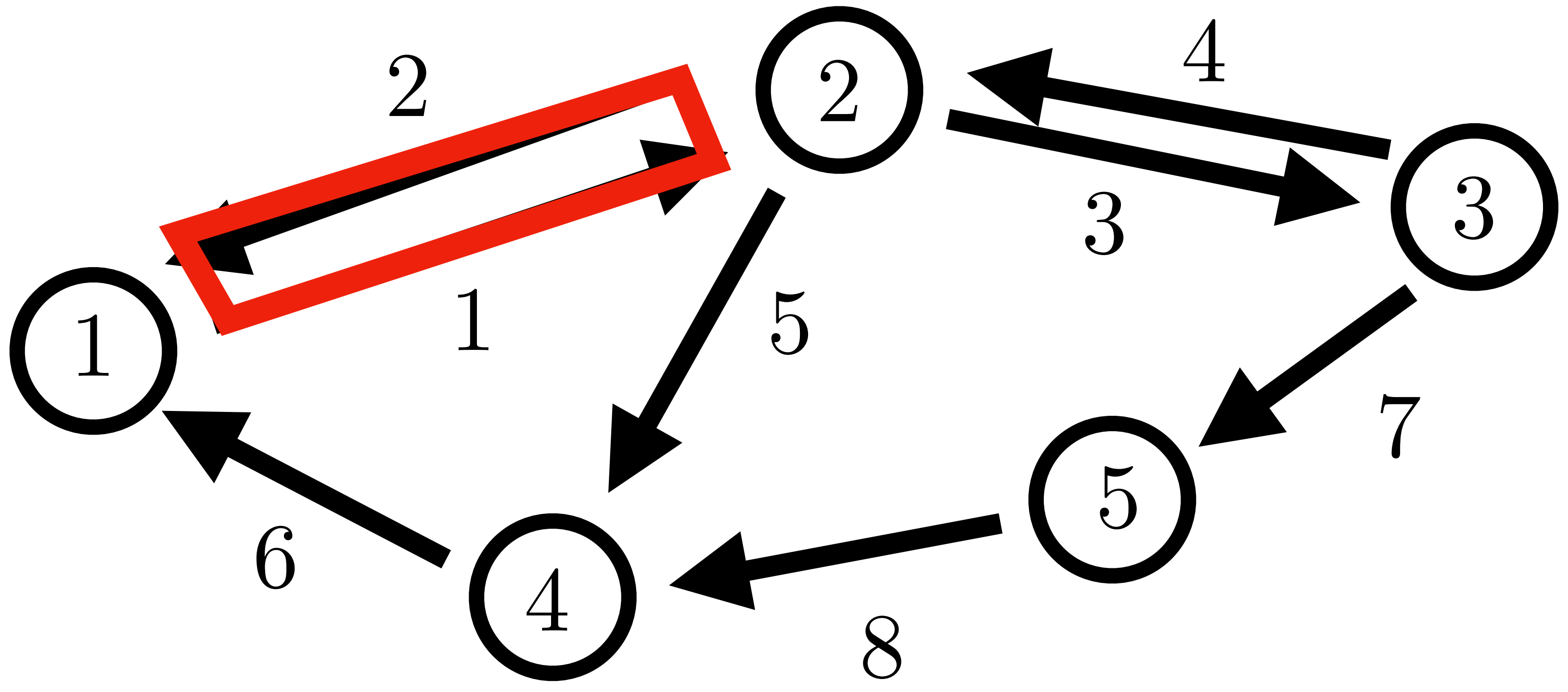
Left Nullspace (General)

$$\begin{bmatrix} \mathbf{1}^T & 0 & \cdots & 0 \\ 0 & \mathbf{1}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}^T \end{bmatrix} \left[\; E \; \right] = \mathbf{0}$$

num = dim

Incidence Matrix - Nullspaces



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ea. equation:

Conservation of flow at ea. node

edge mass flows $\longrightarrow$

Cycle indicator (basis) matrix:

$$\left[\, C \,\right]_{ec} = \begin{cases} 1 & ;\ \text{if } e \text{ flows with cycle } c \\ -1 & ;\ \text{if } e \text{ flows against cycle } c \\ 0 & ;\ e \text{ not in cycle } c \end{cases}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

Sign indicates if cycle goes with or against edge direction

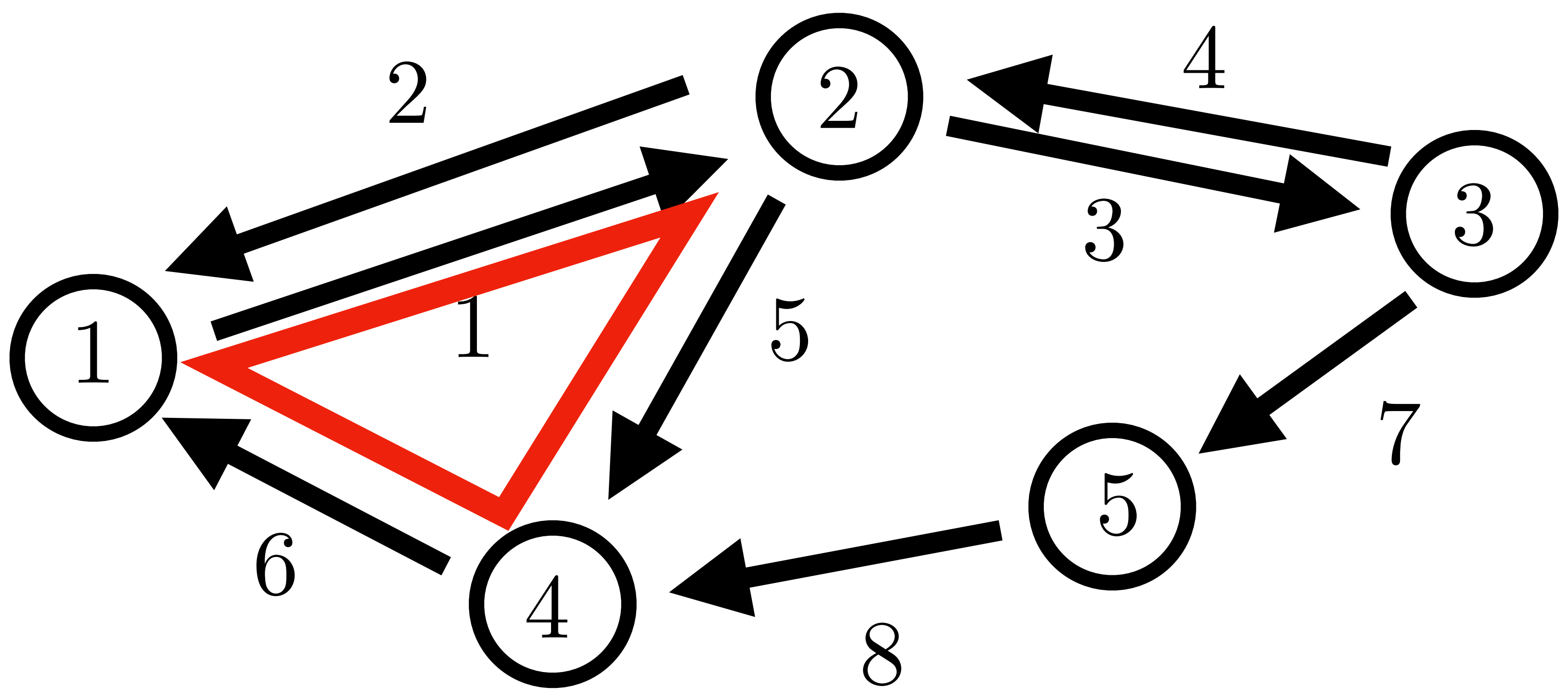
Right Nullspace

$$EC = 0$$

Cycle space $\mathcal{C}$ $\quad \dim \mathcal{C} = |\mathcal{E}| - |\mathcal{V}| + 1$ $\quad$ Ex: $8 - 5 + 1 = 4$

Columns = basis for $\mathcal{C}$

Incidence Matrix - Nullspaces



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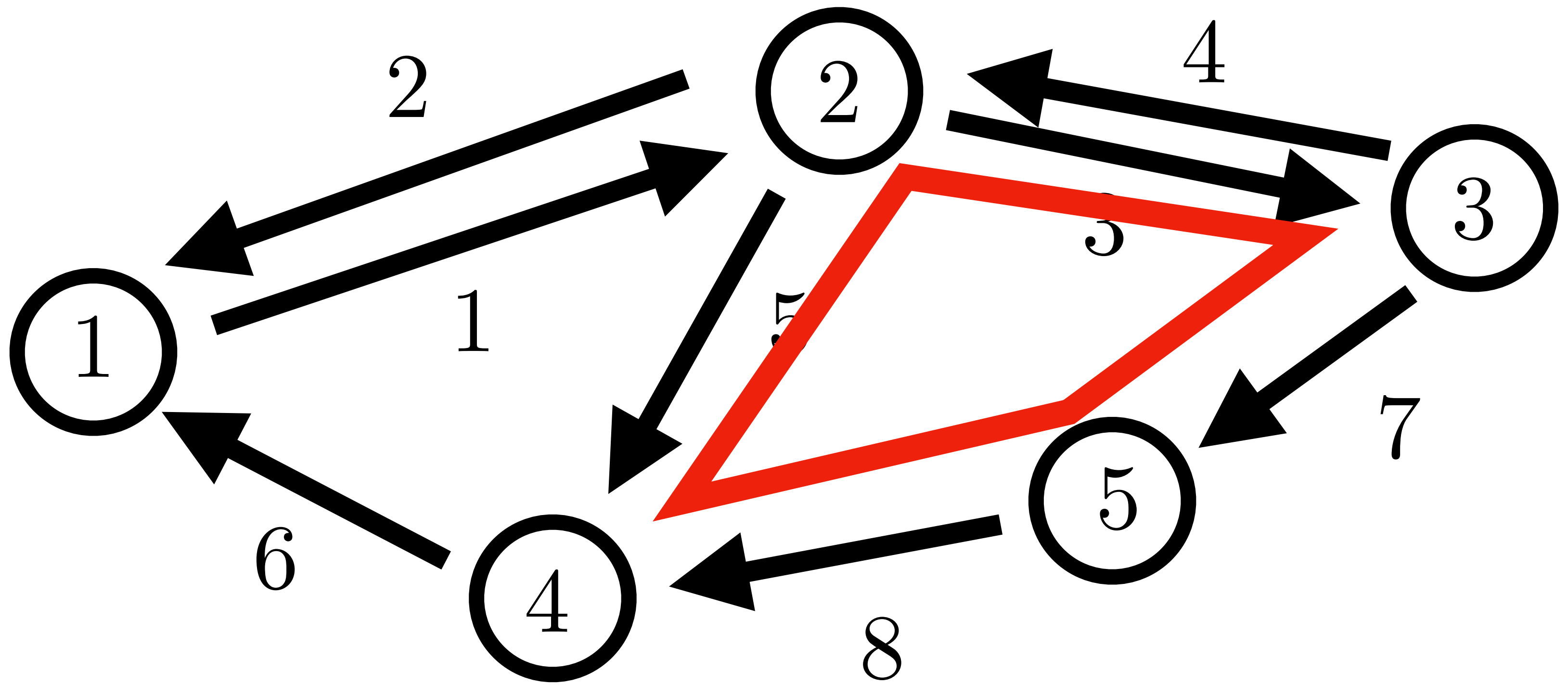

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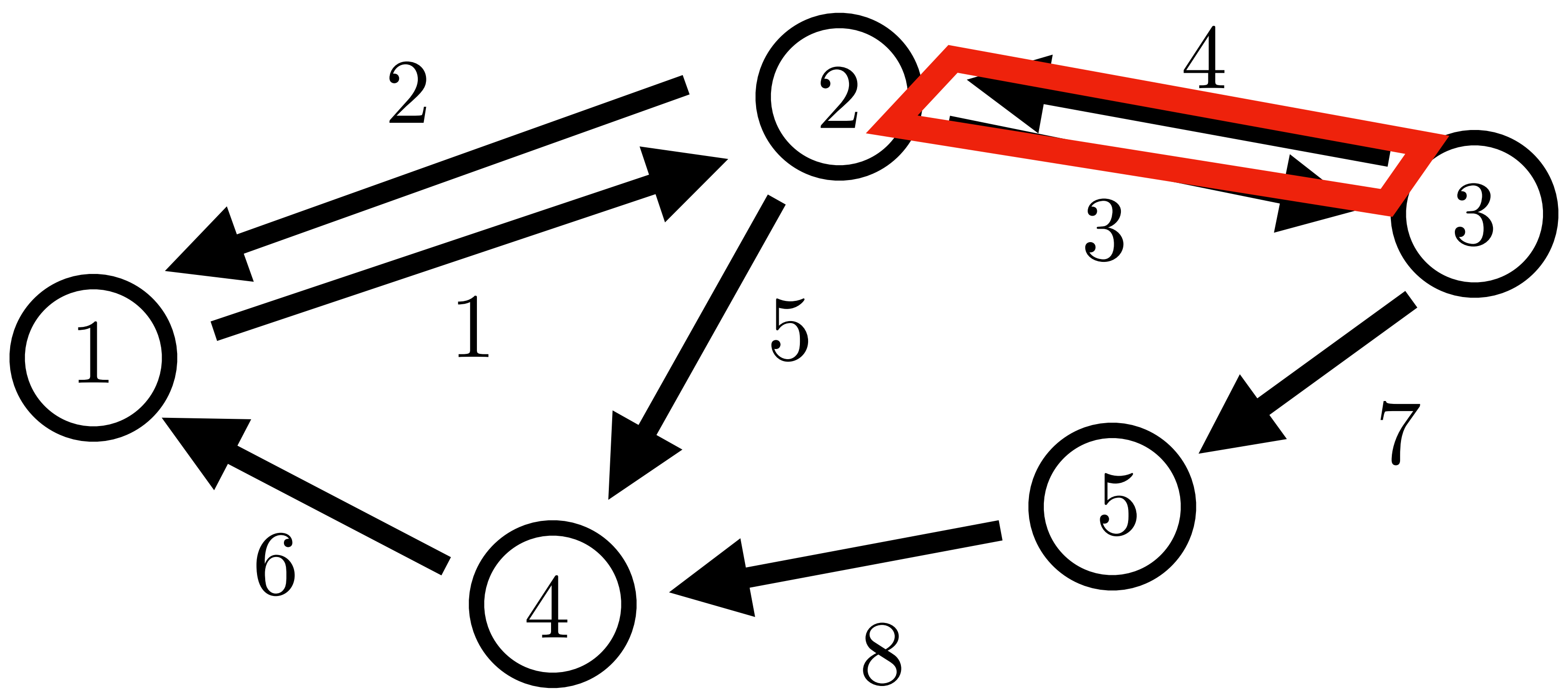

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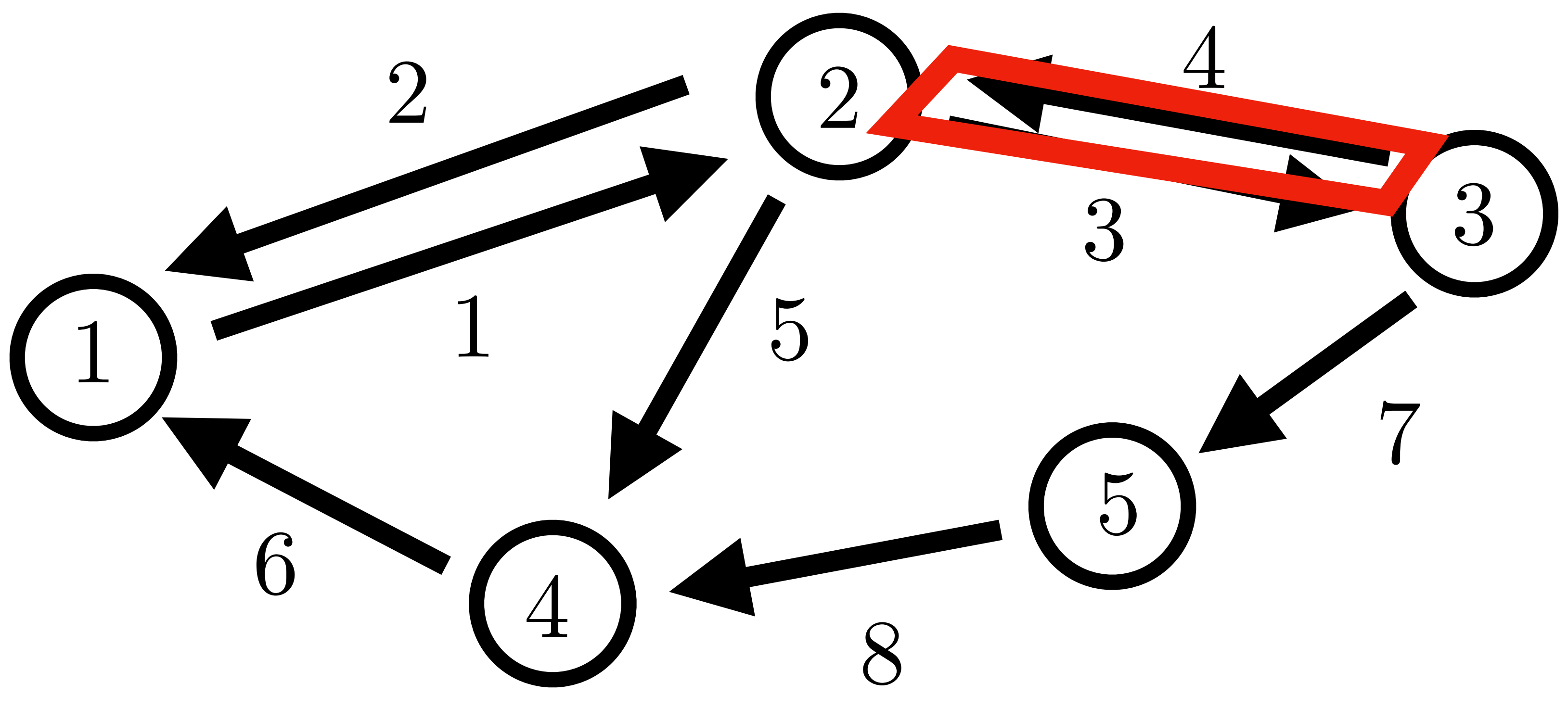

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$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

edge mass flows $\longrightarrow$

ea. equation:

Conservation of flow at ea. node

Subspace Constraint

$$Ex = 0 \quad \Rightarrow \quad x = Cz$$

Cycle space $\mathcal{C}$ $\quad \dim \mathcal{C} = |\mathcal{E}| - |\mathcal{V}| + 1$ $\quad$ Ex: $8 - 5 + 1 = 4$

Cycle indicator (basis) matrix:

$$\big[\, C \,\big]_{ec} = \begin{cases} 1 & ;\ \text{if } e \text{ flows with cycle } c \\ -1 & ;\ \text{if } e \text{ flows against cycle } c \\ 0 & ;\ e \text{ not in cycle } c \end{cases}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\longleftarrow$ Sign indicates if cycle goes with or against edge direction

Columns $=$ basis for $\mathcal{C}$

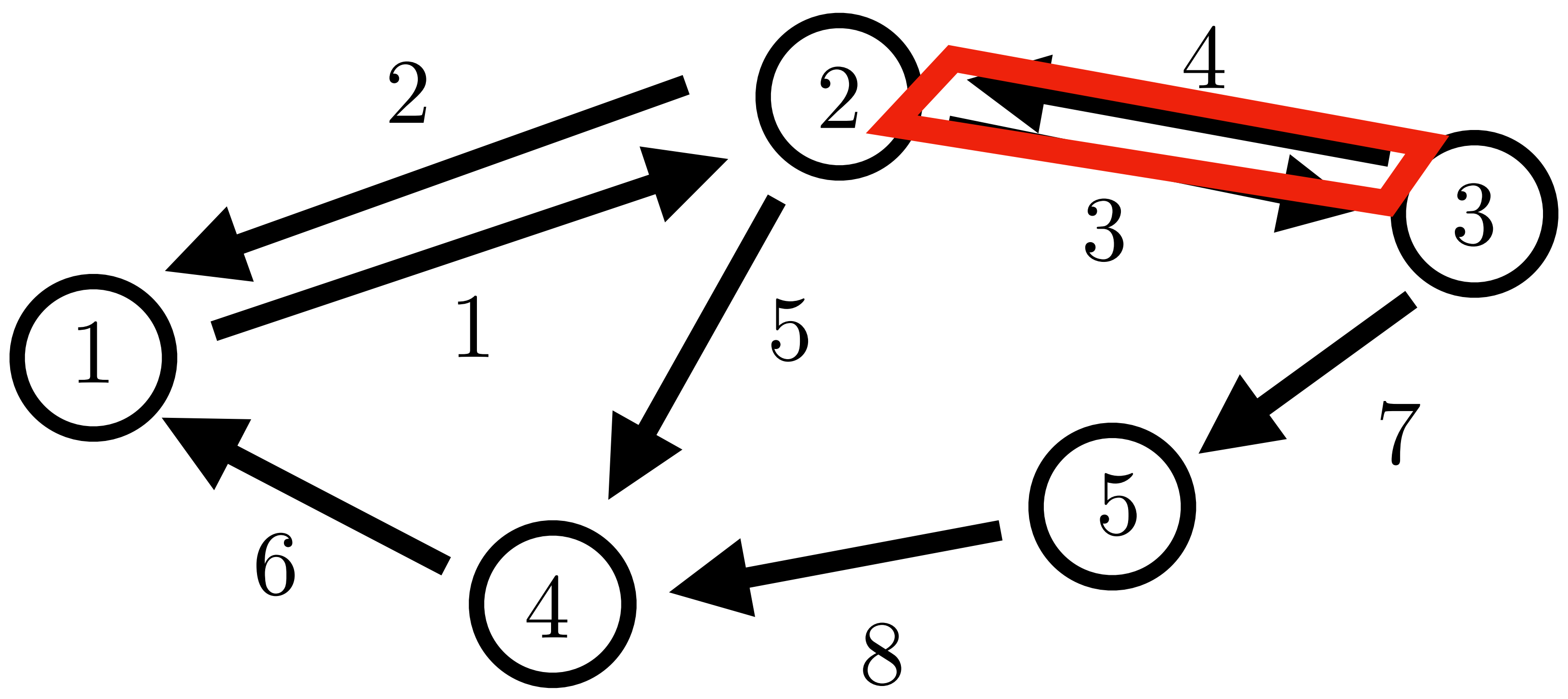
Incidence Matrix - Nullspaces

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows $\longrightarrow$

Cycle indicator (basis) matrix:

$$\left[\, C \,\right]_{ec} = \begin{cases} 1 & ; \text{ if } e \text{ flows with cycle } c \\ -1 & ; \text{ if } e \text{ flows against cycle } c \\ 0 & ; \ e \text{ not in cycle } c \end{cases}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

Sign indicates if cycle goes with or against edge direction

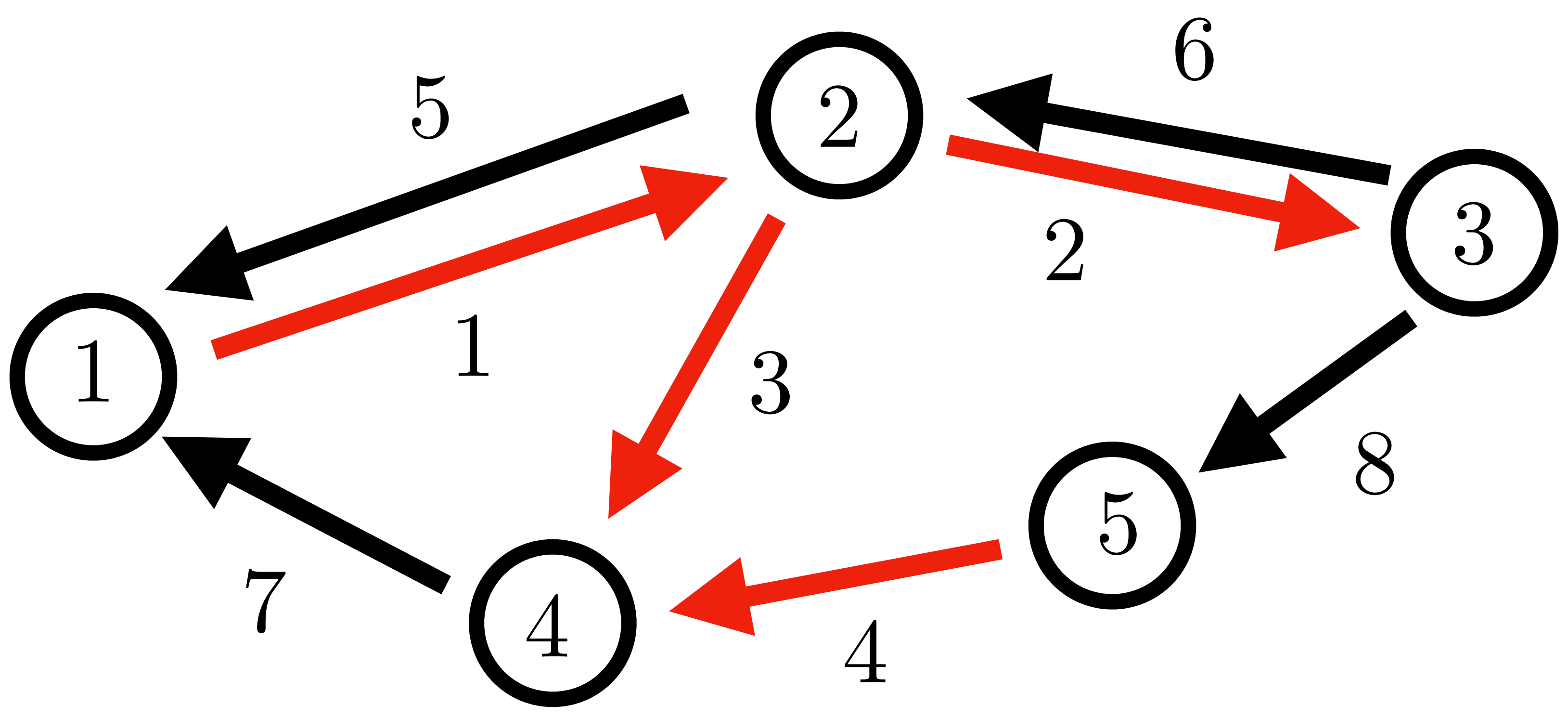
Columns = basis for $\mathcal{C}$

Affine Constraint $\quad Ex = S \;\Rightarrow\; x = \bar{x} + Cz$

Specific Solution $\quad$ Cyclic Flow

Min Norm Solution $\quad \bar{x} = E^T (E E^T)^\dagger S$

Incidence Matrix - Geometry



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\big[\, E \,\big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Affine Constraint

$$Ex = S \quad \Rightarrow \quad x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

Min Norm Solution

$$\bar{x} = E^T (EE^T)^{\dagger} S$$

Spanning Tree Construction:

$$E = \begin{bmatrix} E' & E'' \end{bmatrix} = \begin{bmatrix} E' & E'M \end{bmatrix} = E' \begin{bmatrix} I & M \end{bmatrix}$$

$$|\mathcal{V}| \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

E'

$|\mathcal{V}| - 1$

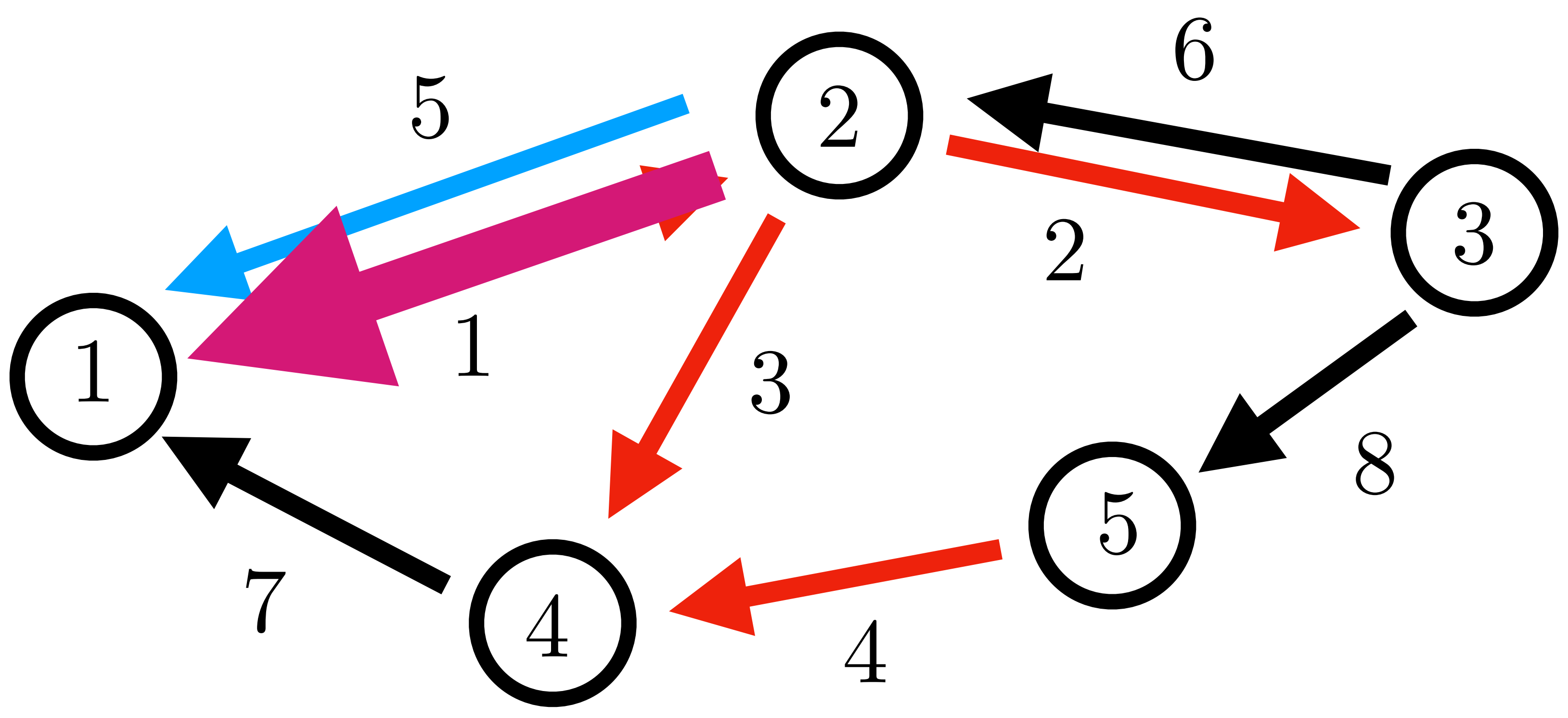
Incidence Matrix - Geometry

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} E \end{bmatrix}_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Spanning Tree Construction:

$$E = \begin{bmatrix} E' & E'' \end{bmatrix} = \begin{bmatrix} E' & E'M \end{bmatrix} = E' \begin{bmatrix} I & M \end{bmatrix}$$

$$|\mathcal{V}| \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

E'

$|\mathcal{V}| - 1$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

M

Affine Constraint

$$Ex = S \quad \Rightarrow \quad x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

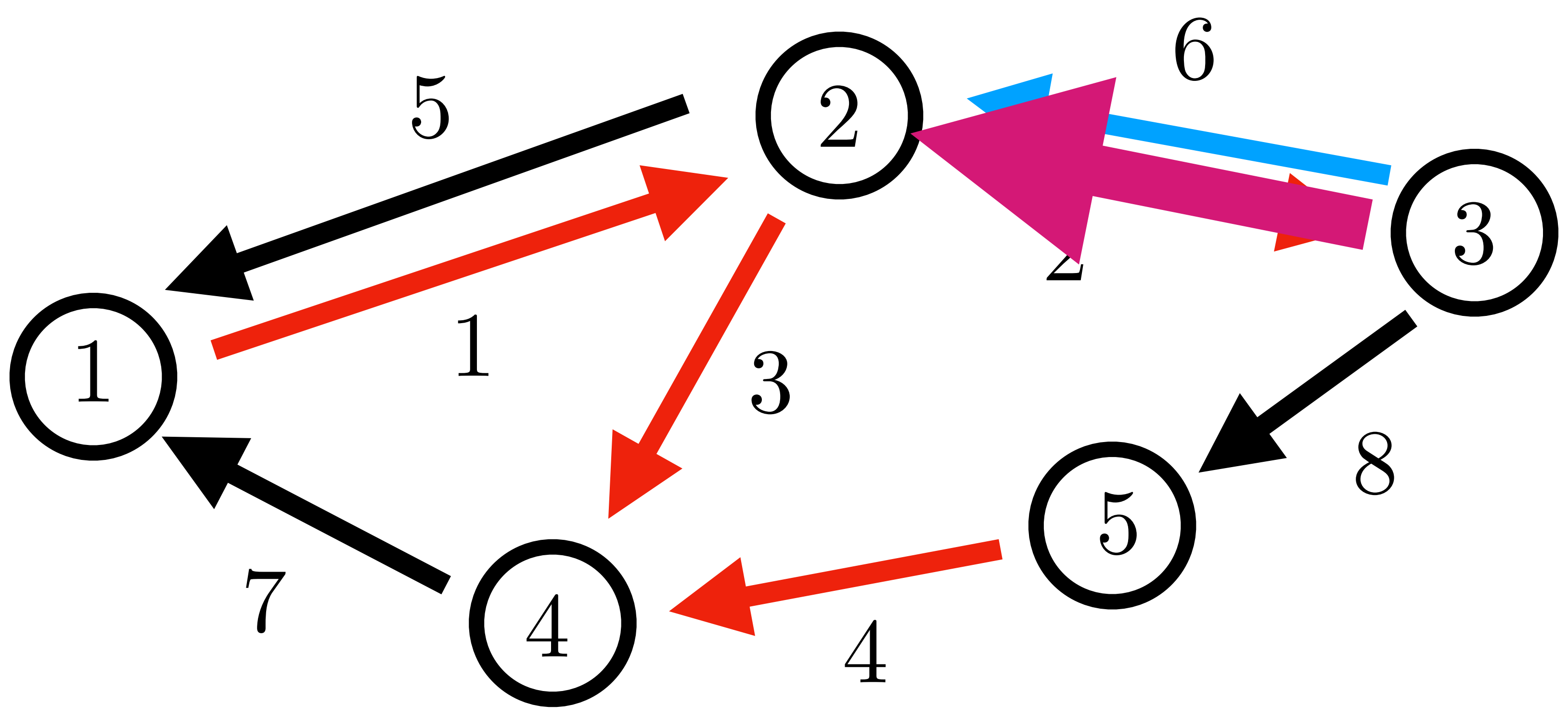
Incidence Matrix - Geometry

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} E \end{bmatrix}_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Spanning Tree Construction:

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$$|\mathcal{V}| \; \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

E'

$|\mathcal{V}| - 1$

Affine Constraint

$$Ex = S \quad \Rightarrow \quad x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

M

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$

Incidence Matrix - Geometry

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} E \end{bmatrix}_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

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Specific Solution $\quad$ Cyclic Flow

Min Norm Solution

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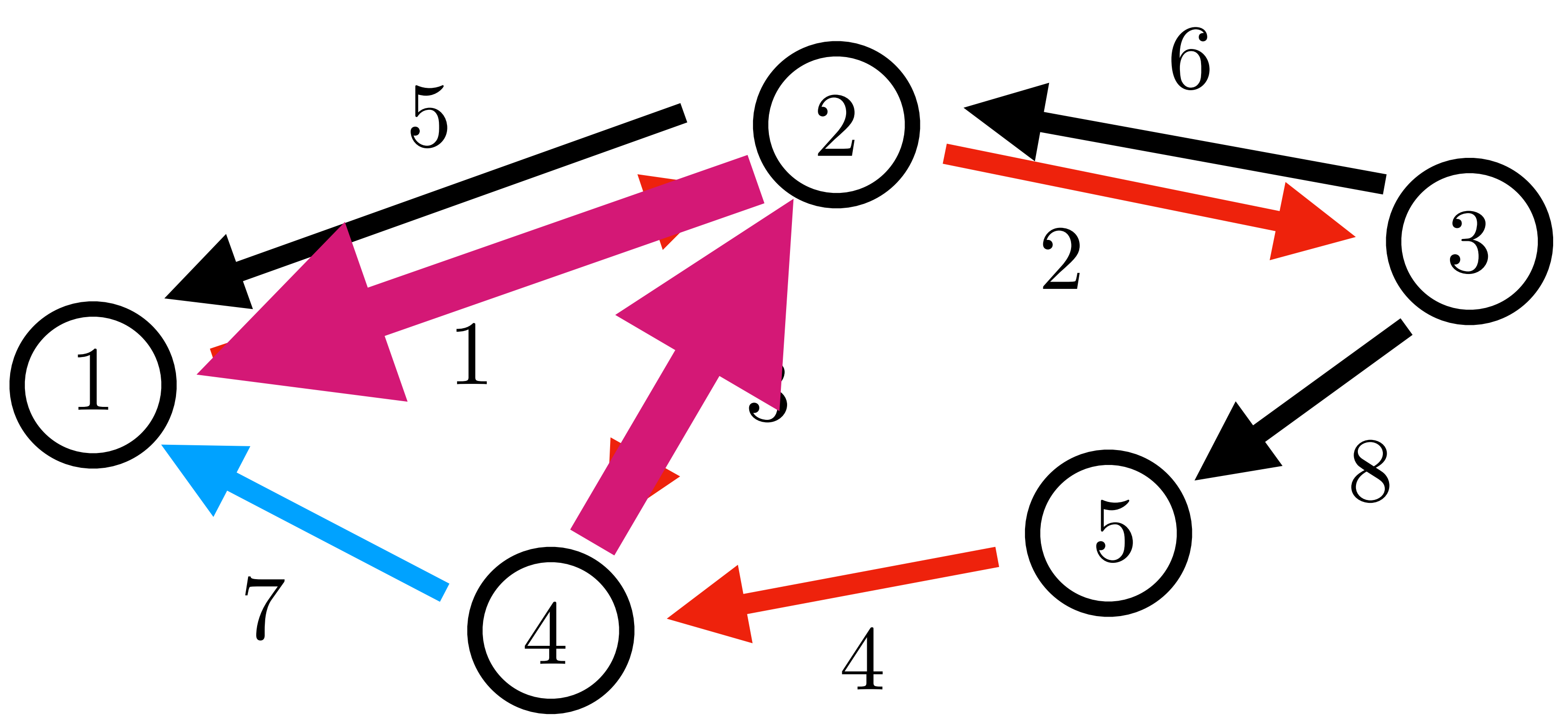

Spanning Tree Construction:

$$E = \begin{bmatrix} E' & E'' \end{bmatrix} = \begin{bmatrix} E' & E'M \end{bmatrix} = E' \begin{bmatrix} I & M \end{bmatrix}$$

$$|\mathcal{V}| \; \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

E' $\quad |\mathcal{V}| - 1$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

M

Incidence Matrix - Geometry

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

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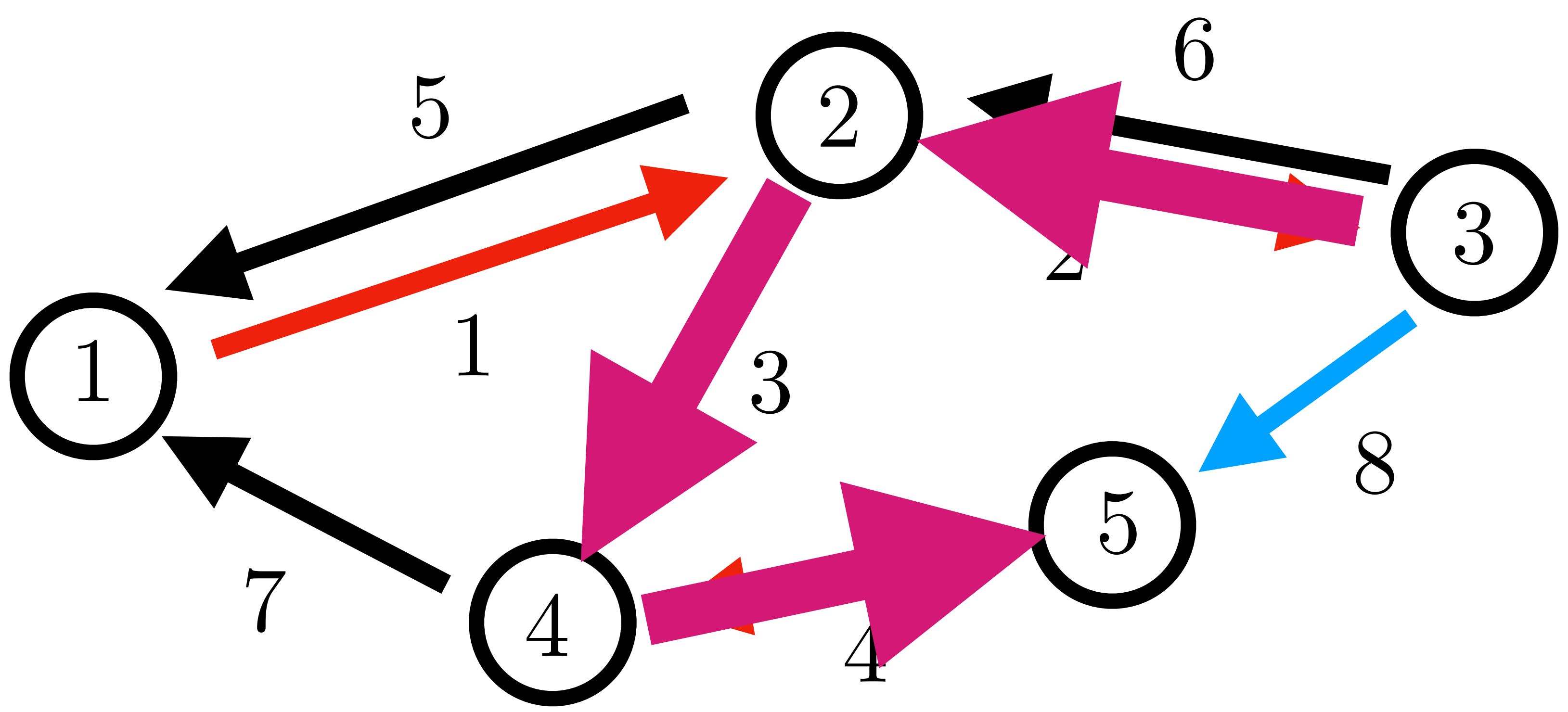

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

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$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Spanning Tree Construction:

$$E = \begin{bmatrix} E' & E'' \end{bmatrix} = \begin{bmatrix} E' & E'M \end{bmatrix} = E' \begin{bmatrix} I & M \end{bmatrix}$$

$$|\mathcal{V}| \quad \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

E'

$|\mathcal{V}| - 1$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

M

Affine Constraint

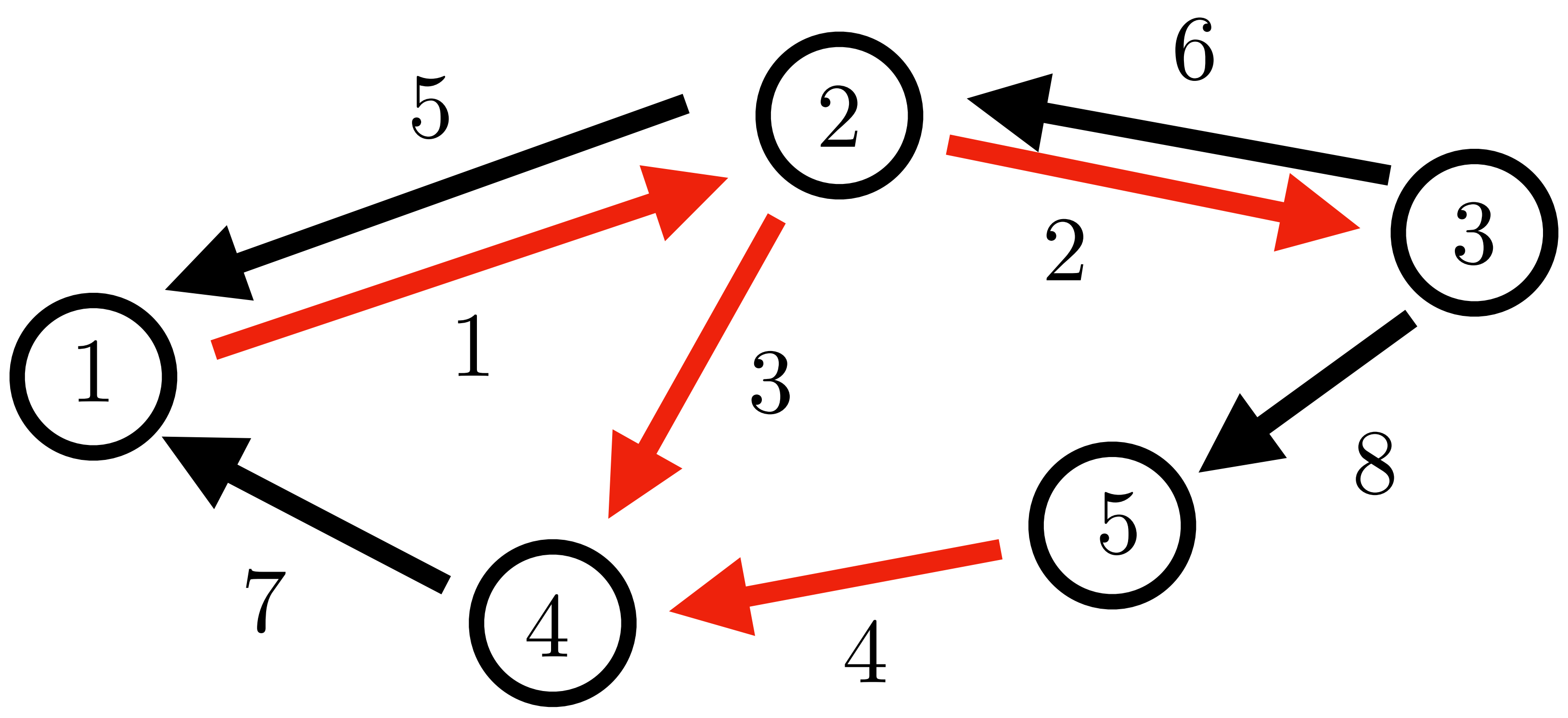
$$Ex = S \quad \Rightarrow \quad x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Incidence Matrix - Geometry



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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… directed or undirected

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$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

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$$E = \left[\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right]$$

Affine Constraint

$$Ex = S \quad \Rightarrow \quad x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Incidence Matrix - Geometry

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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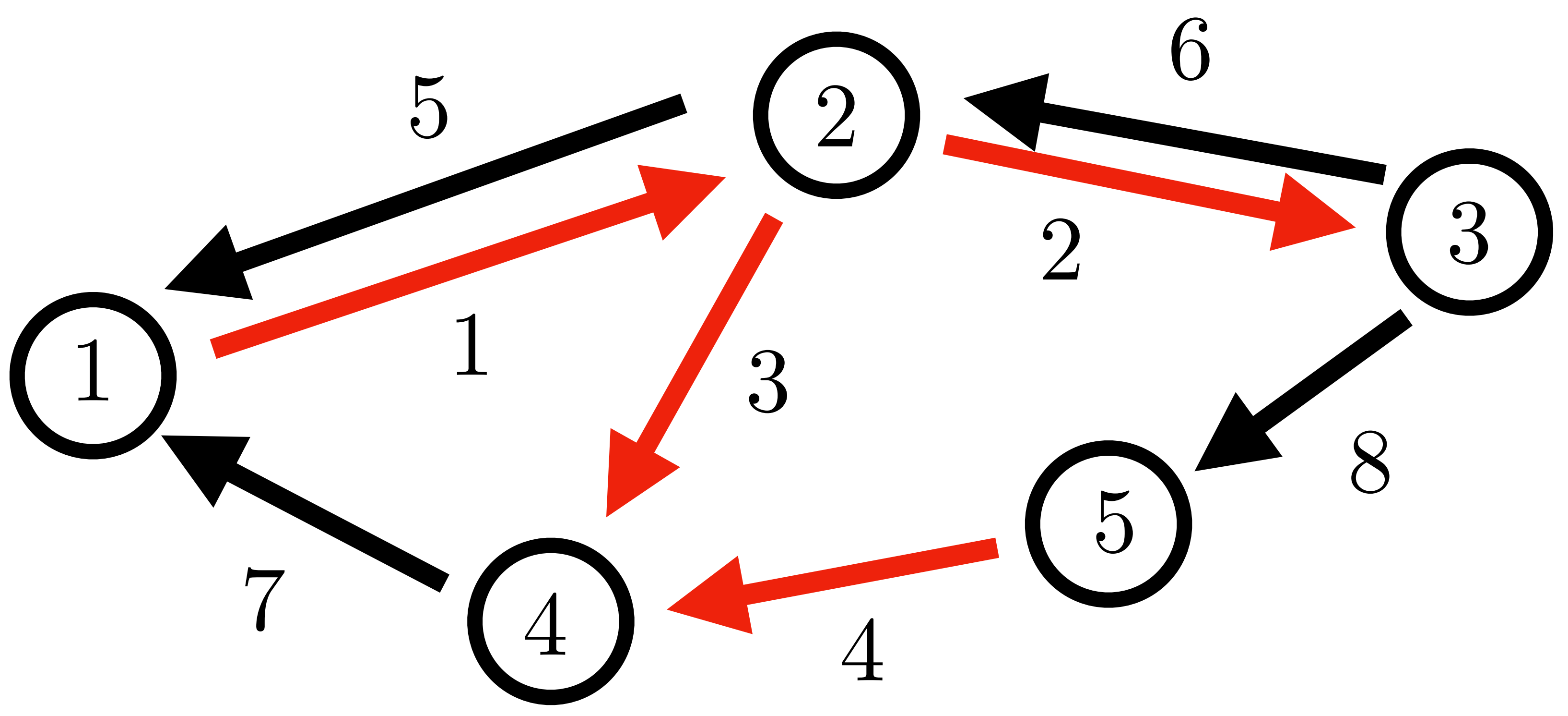

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} E \end{bmatrix}_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Spanning Tree Construction:

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Affine Constraint

$$Ex = S \quad \Rightarrow \quad x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

Min Norm Solution $\quad \bar{x} = E^T (E E^T)^{\dagger} S$

Incidence Matrix - Geometry

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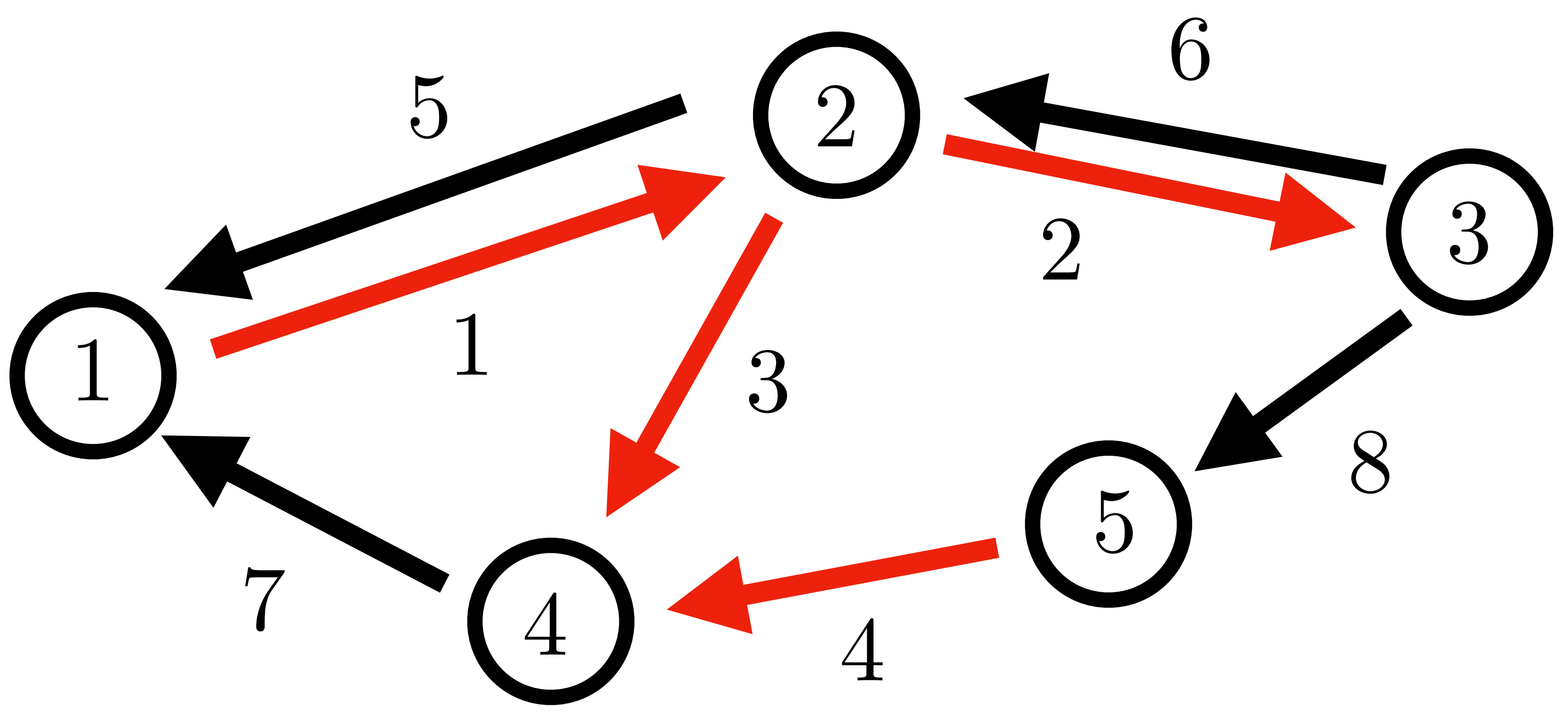

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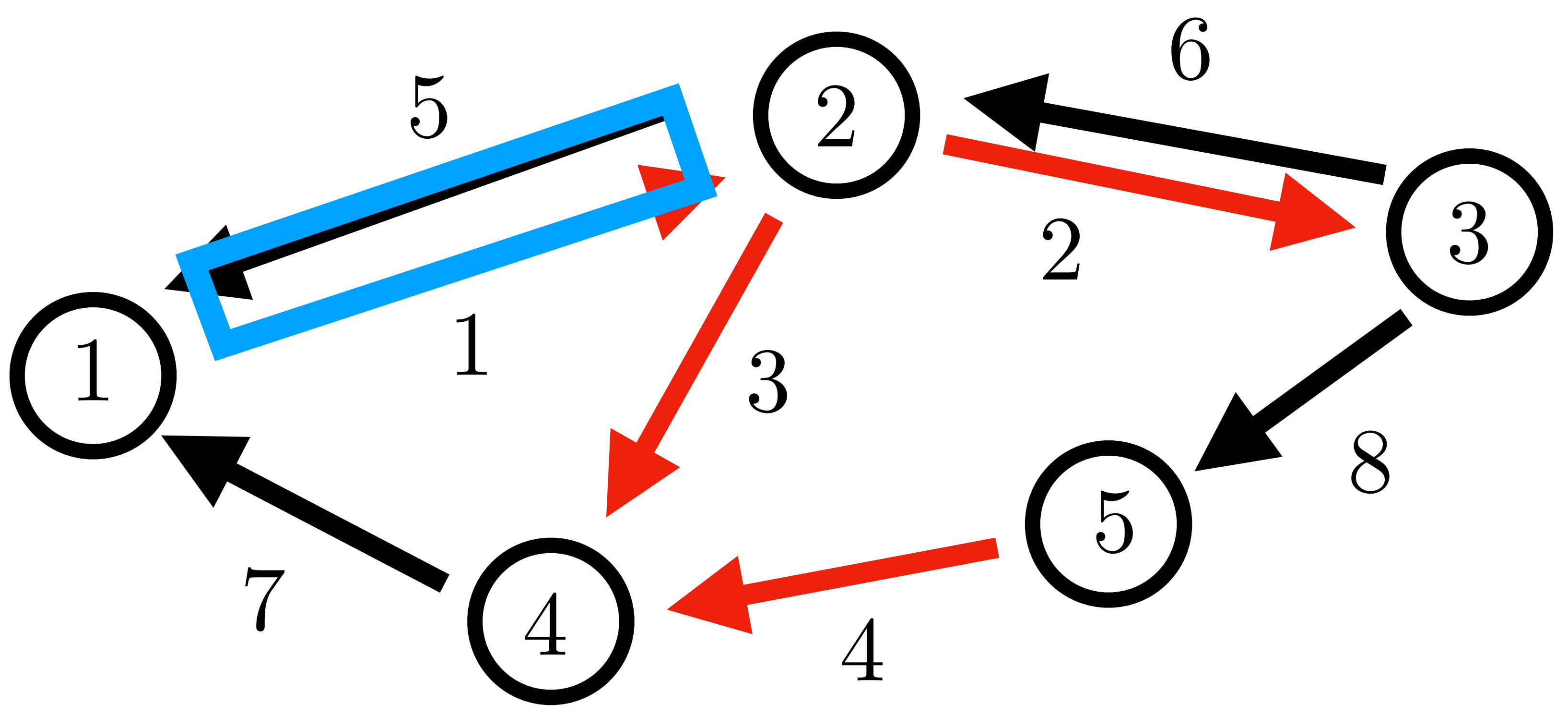
$$C = \begin{bmatrix} M \\ -I \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$EC = \begin{bmatrix} E' & E'' \end{bmatrix} \begin{bmatrix} M \\ -I \end{bmatrix} = E'M - E'' = 0$$

Incidence Matrix - Geometry



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$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

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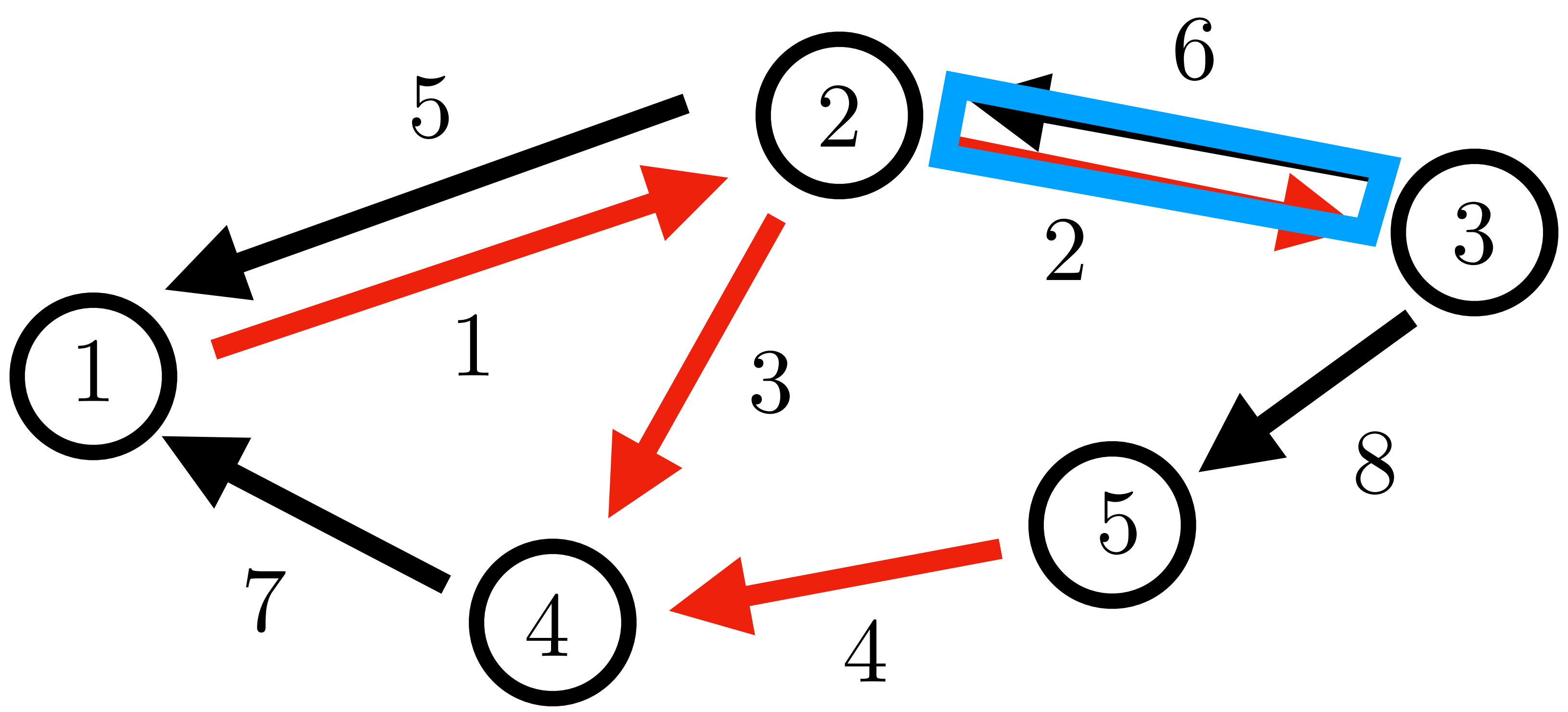
Incidence Matrix - Geometry



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edge mass flows $\longrightarrow$

Affine Constraint $\quad Ex = S \quad \Rightarrow \quad x = \bar{x} + Cz$

Specific Solution $\quad$ Cyclic Flow

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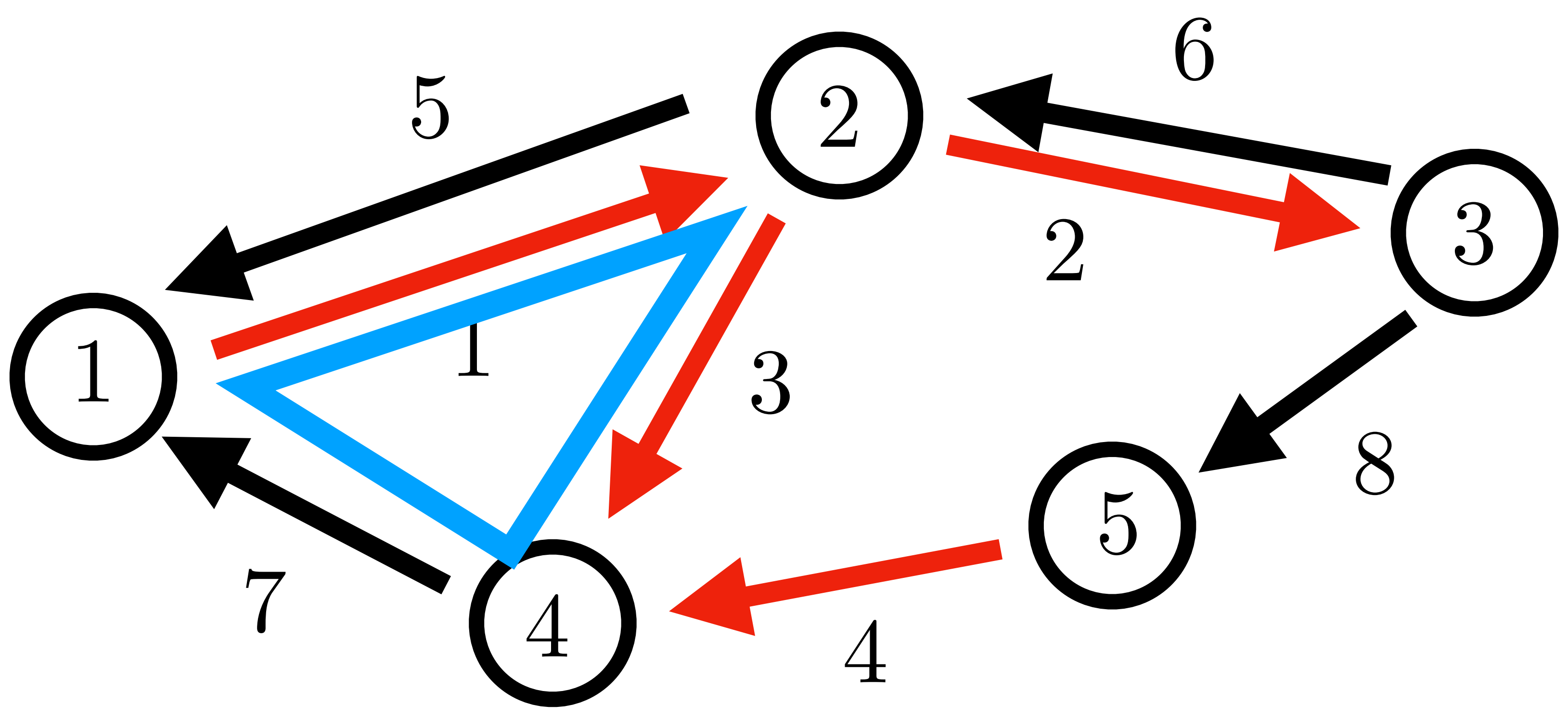

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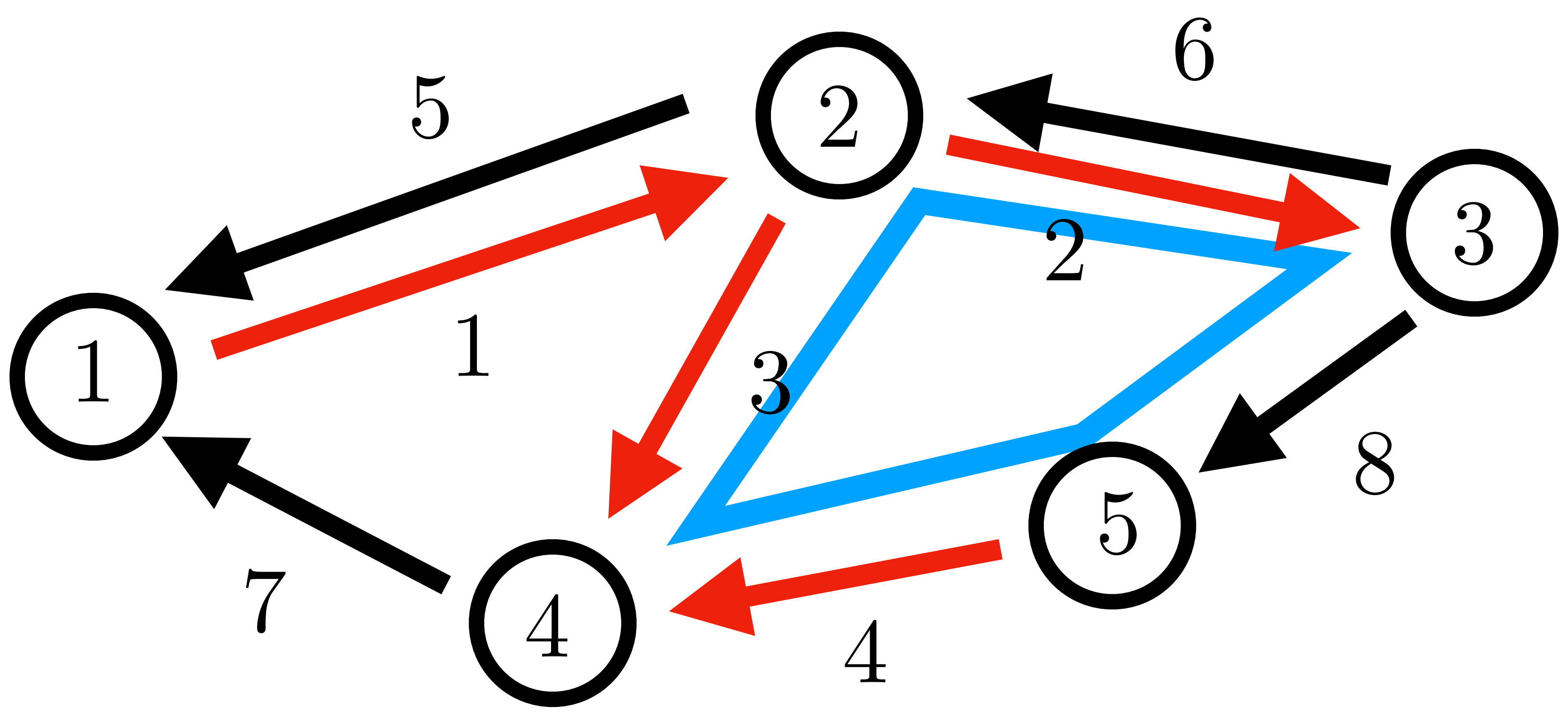

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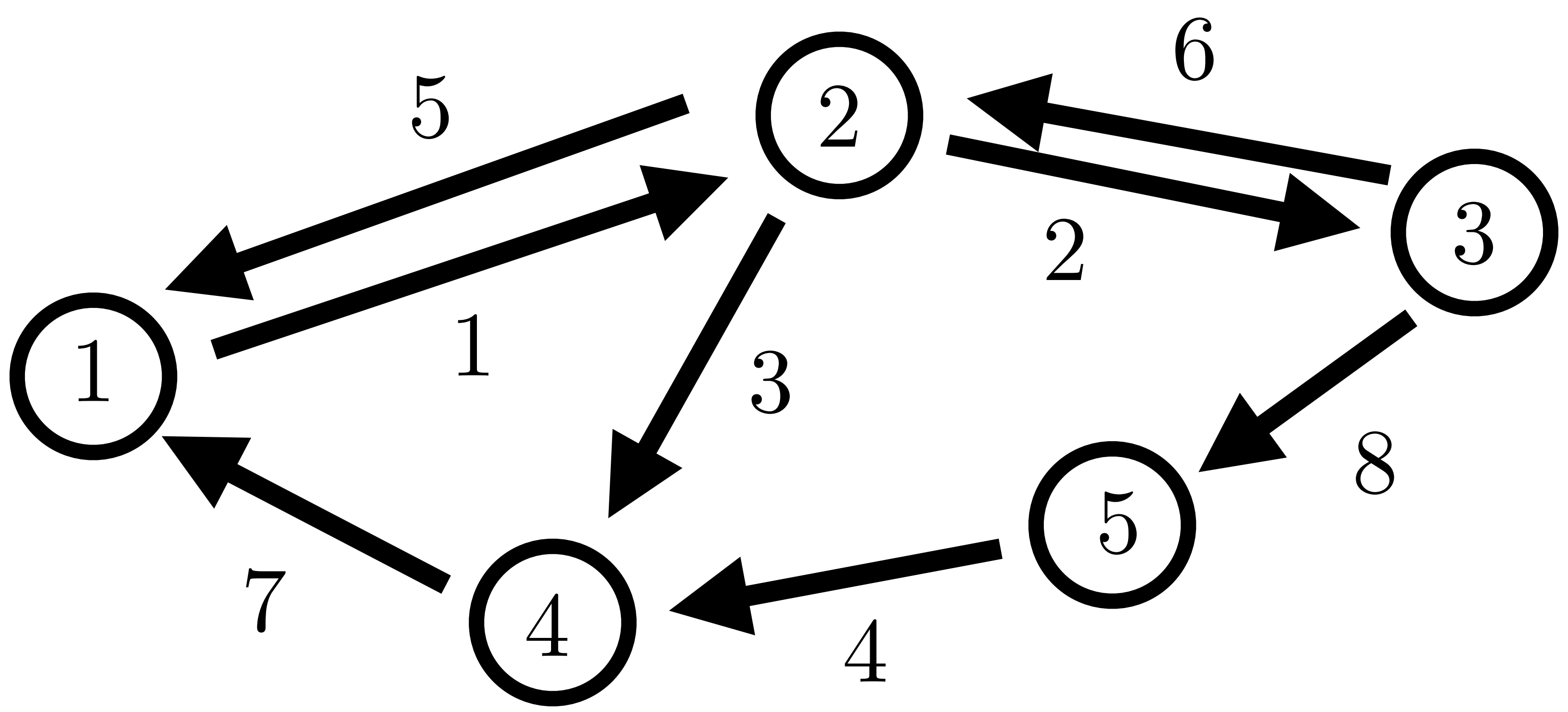

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Domain & Codomain Interpretations

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **Mass flow on edges**

Co-domain: $v \in \mathbb{R}^{|\mathcal{V}|}$ **…source-sink on nodes**

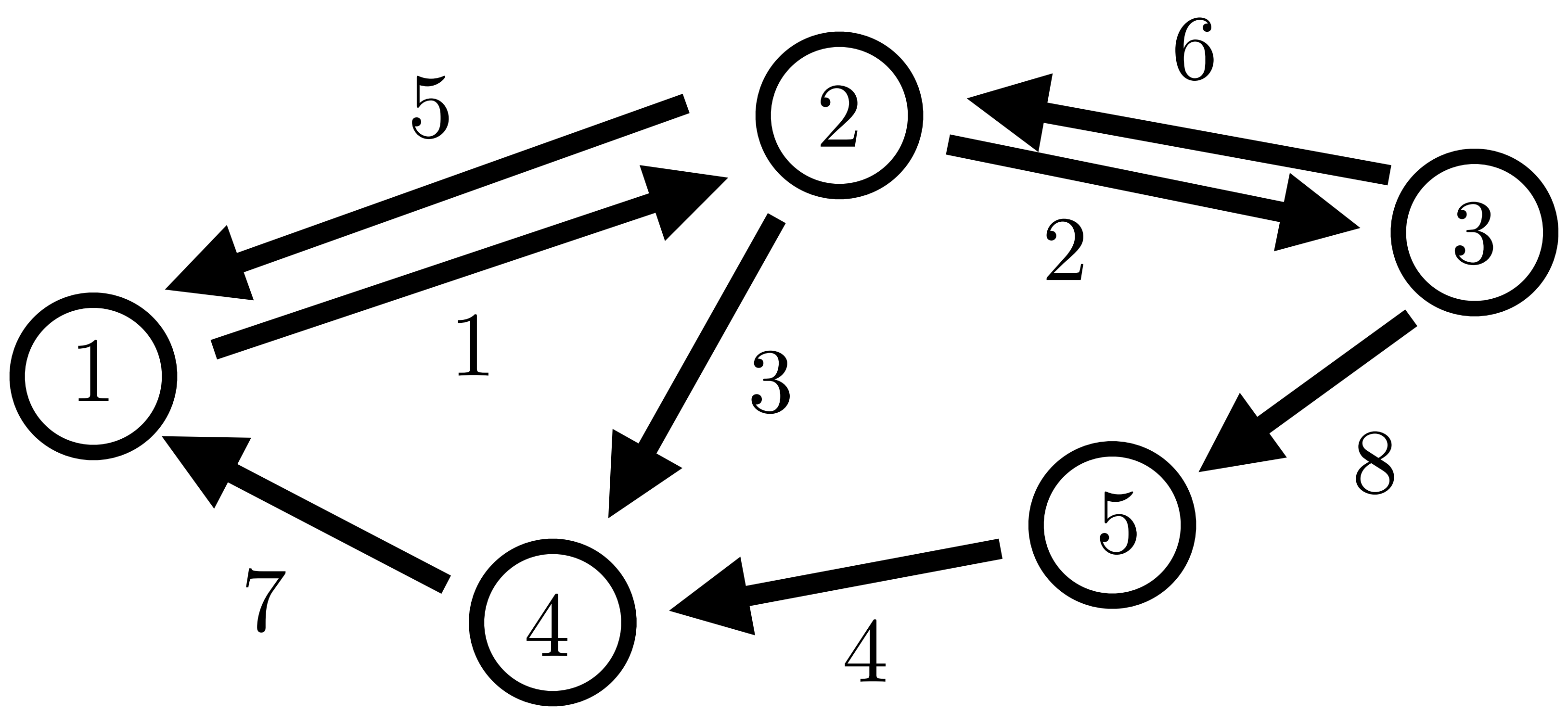
…value function on nodes

value: **Cost-to-go**

Potential value

“Height” gravitational potential

Incidence Matrix - Geometry



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Column Geometry

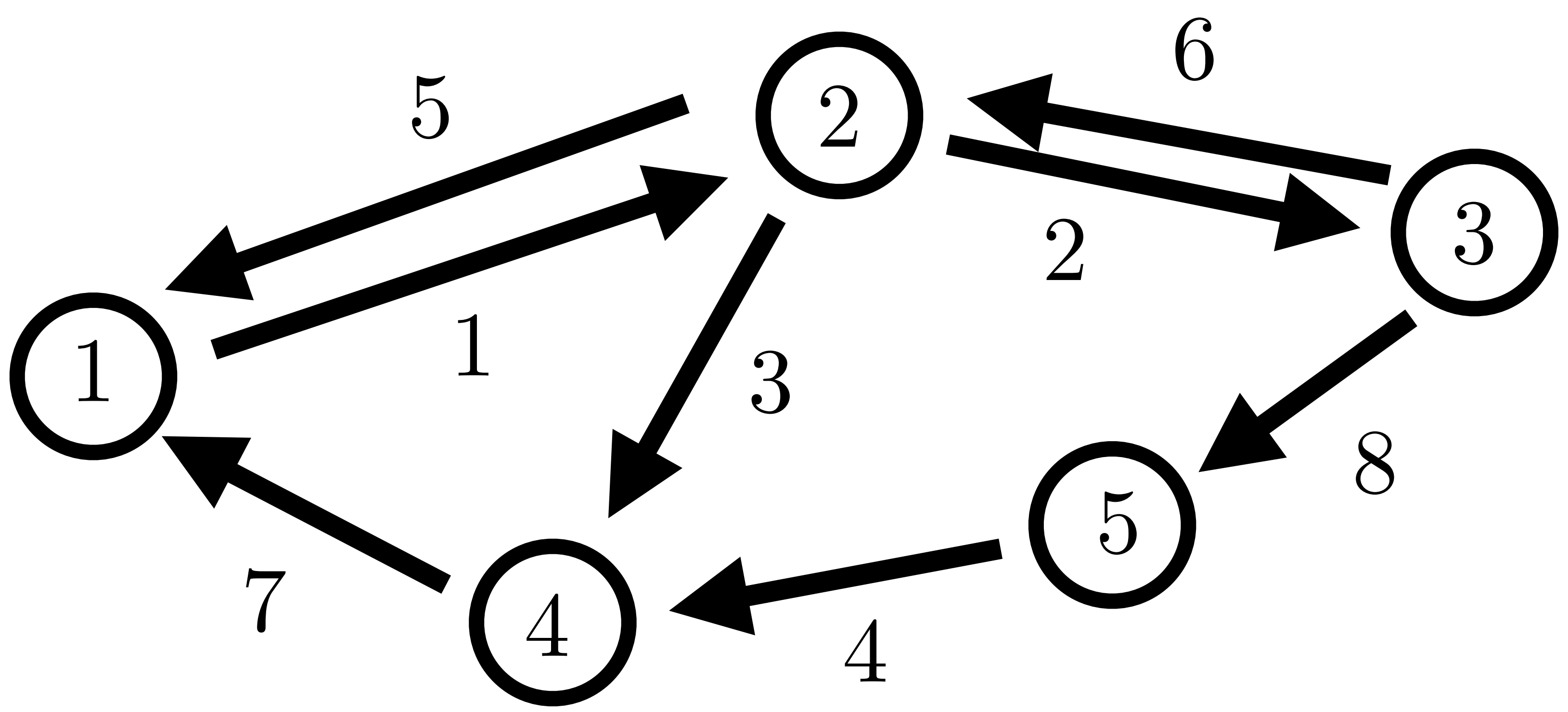
$$S = Ex$$

Non-conserved flow **Edge flow vector**

$$x = \bar{x} + Cz$$

Specific Solution Cyclic Flow

Incidence Matrix - Geometry



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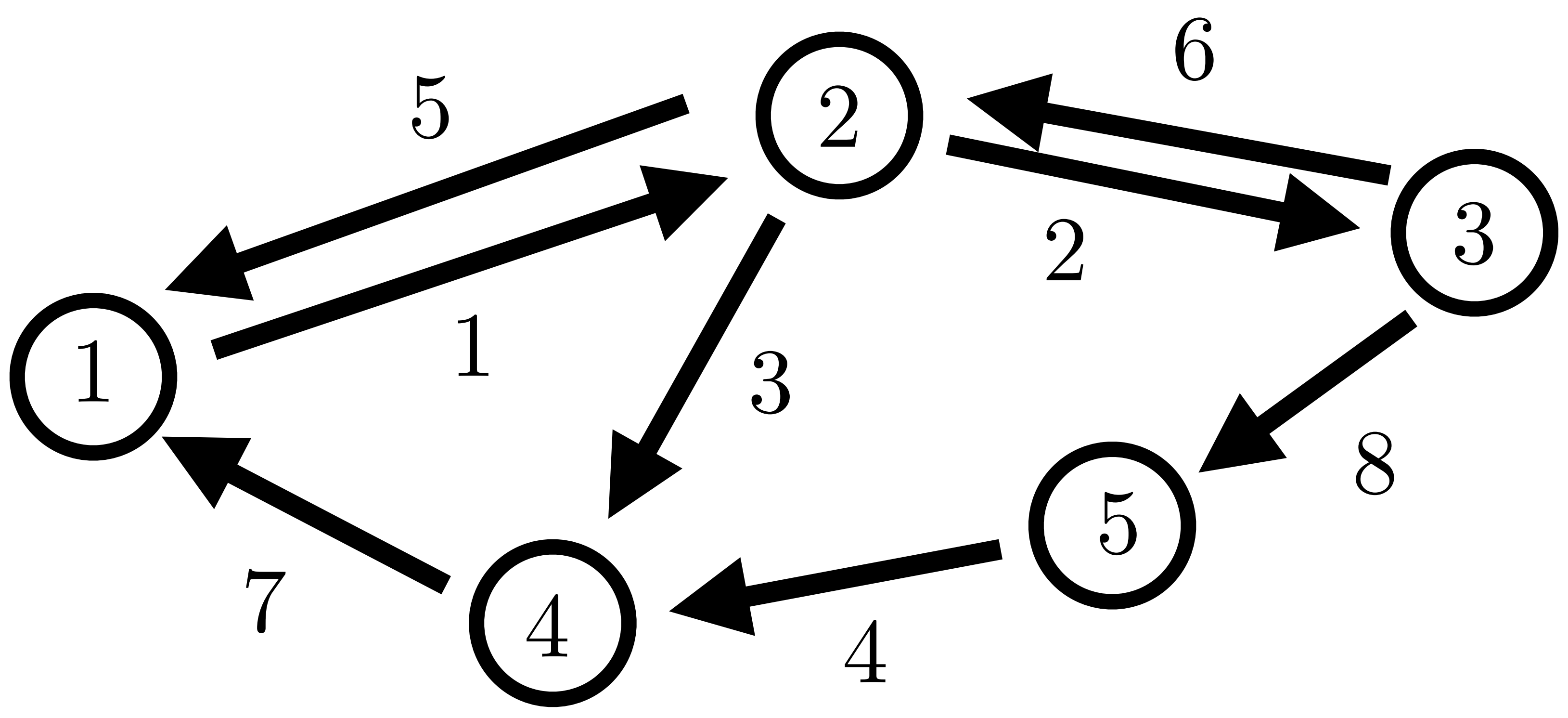
Row Geometry

$$v^T E = \tau^T$$

"Height" of nodes $\qquad$ **Tension in edges**

$$\left[v^T E \right]_e = v_i - v_{i'} = \tau_e$$

Graph Laplacian



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cols... $\qquad$ **rows...**

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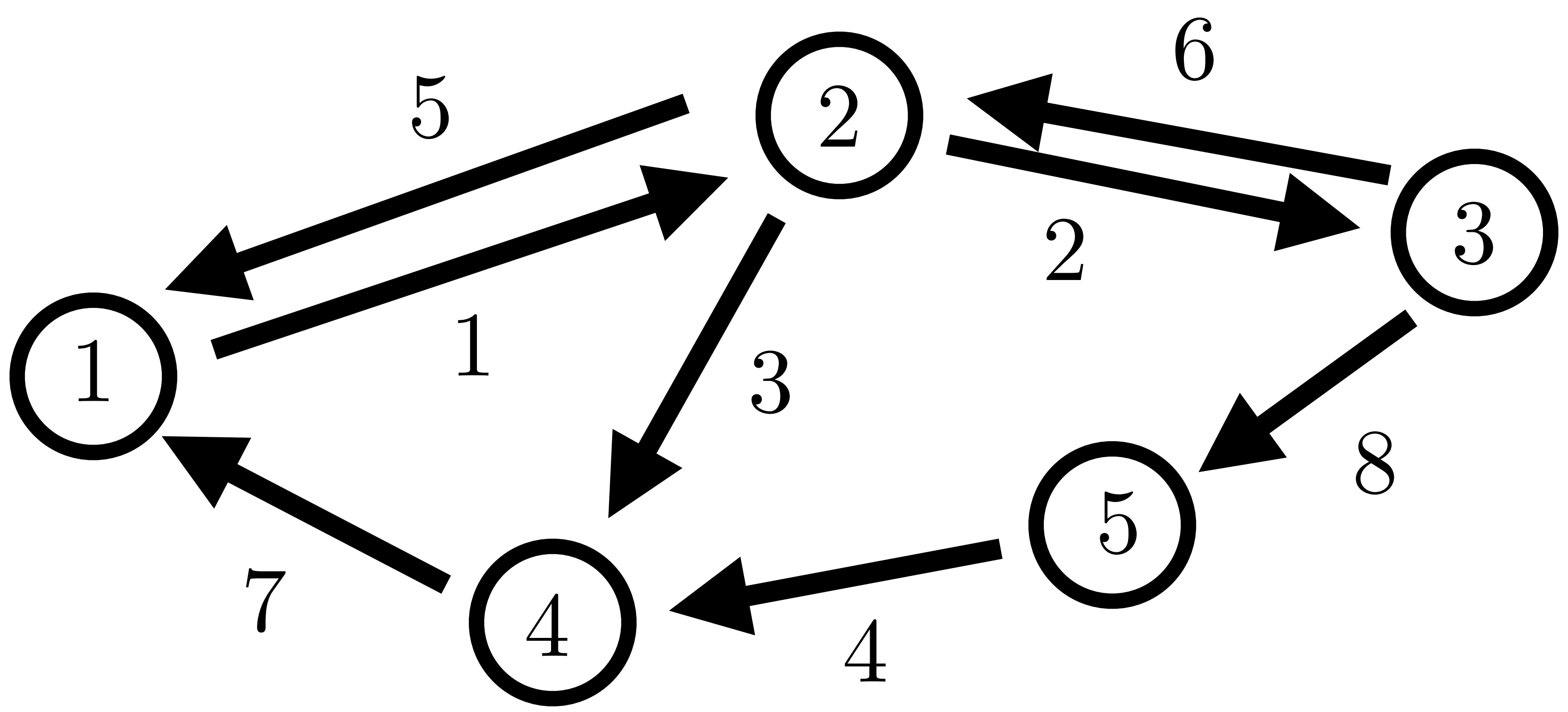

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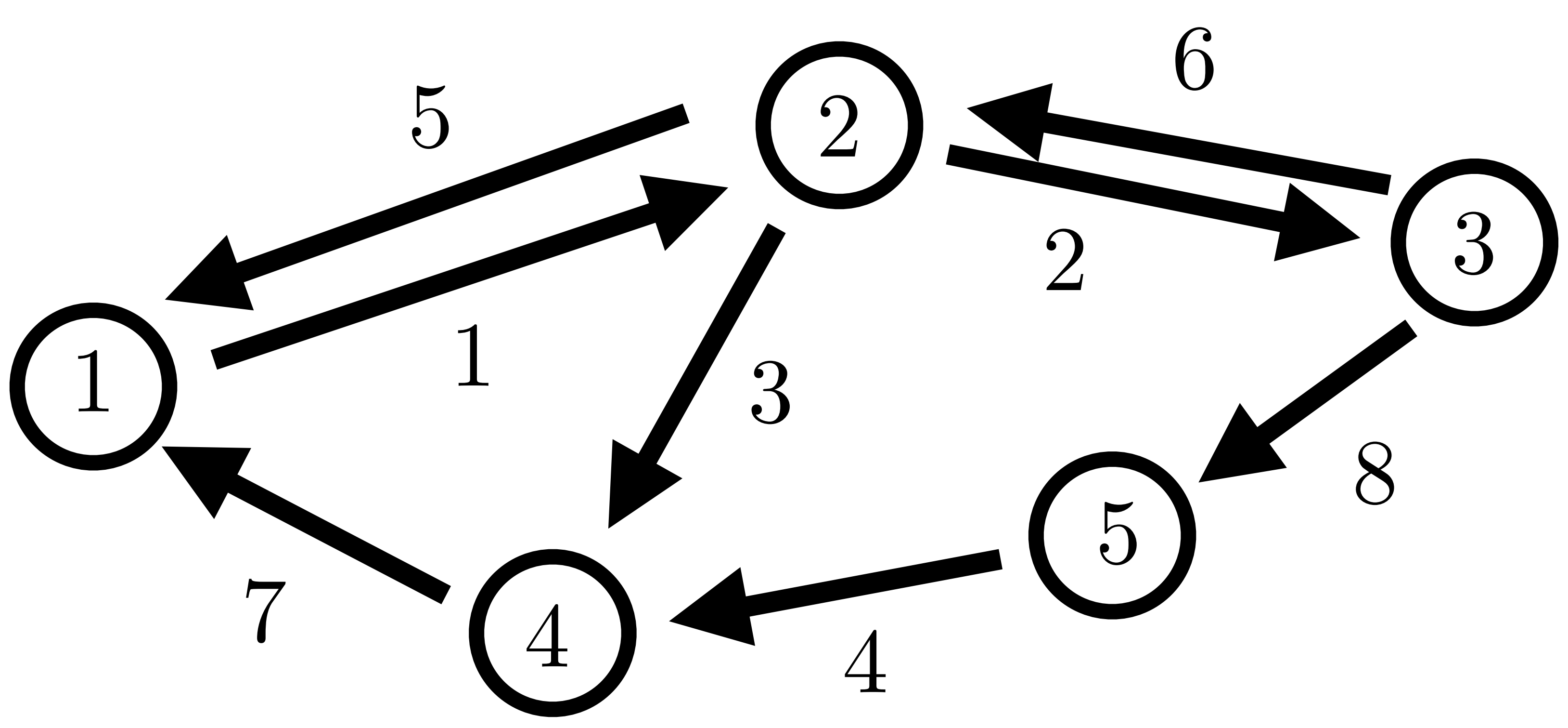
cols... **rows...**

$$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix}$$

Inner products of columns

“Relative geometry of columns”

Graph Laplacian



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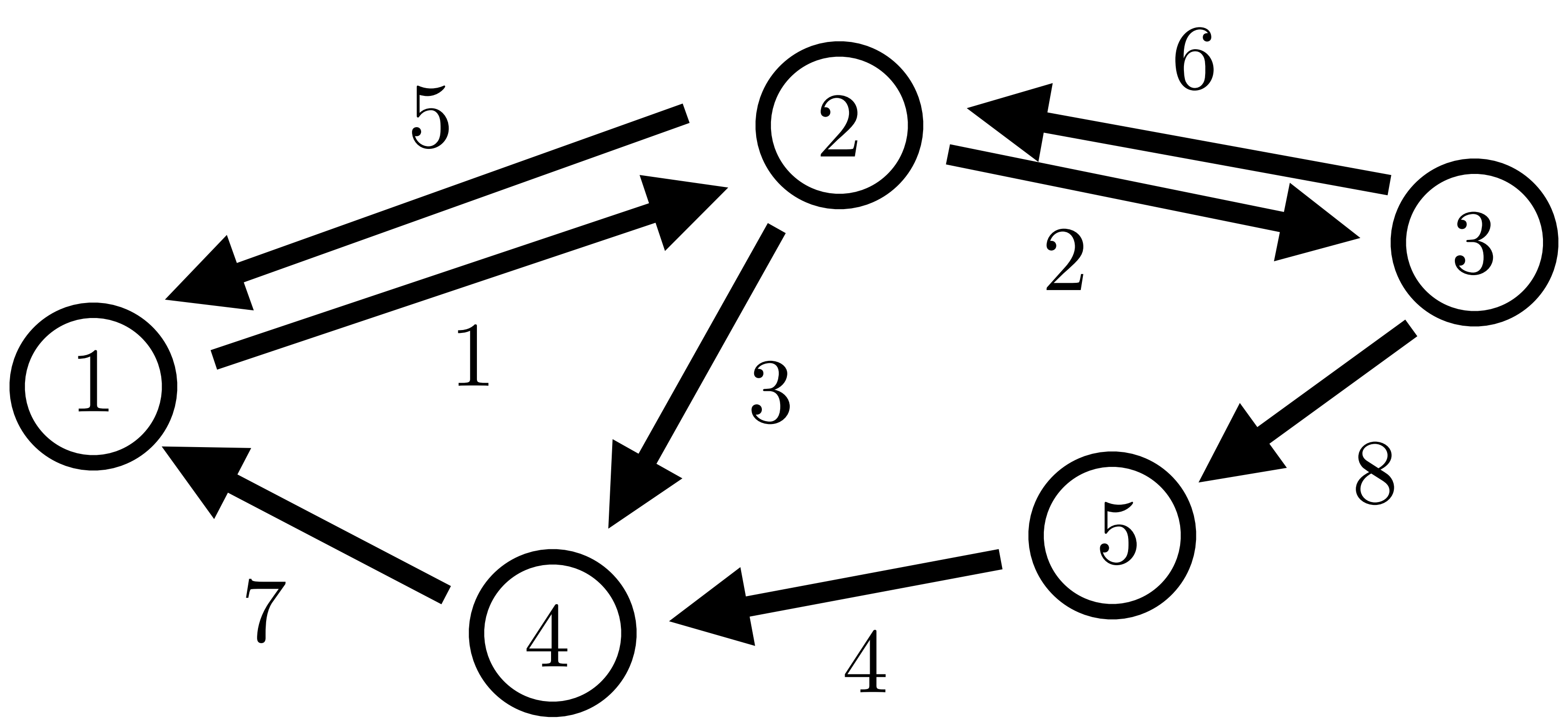
cols… $\qquad$ **rows…**

$$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix}$$

Inner products of rows

“Relative geometry of rows”

Graph Laplacian



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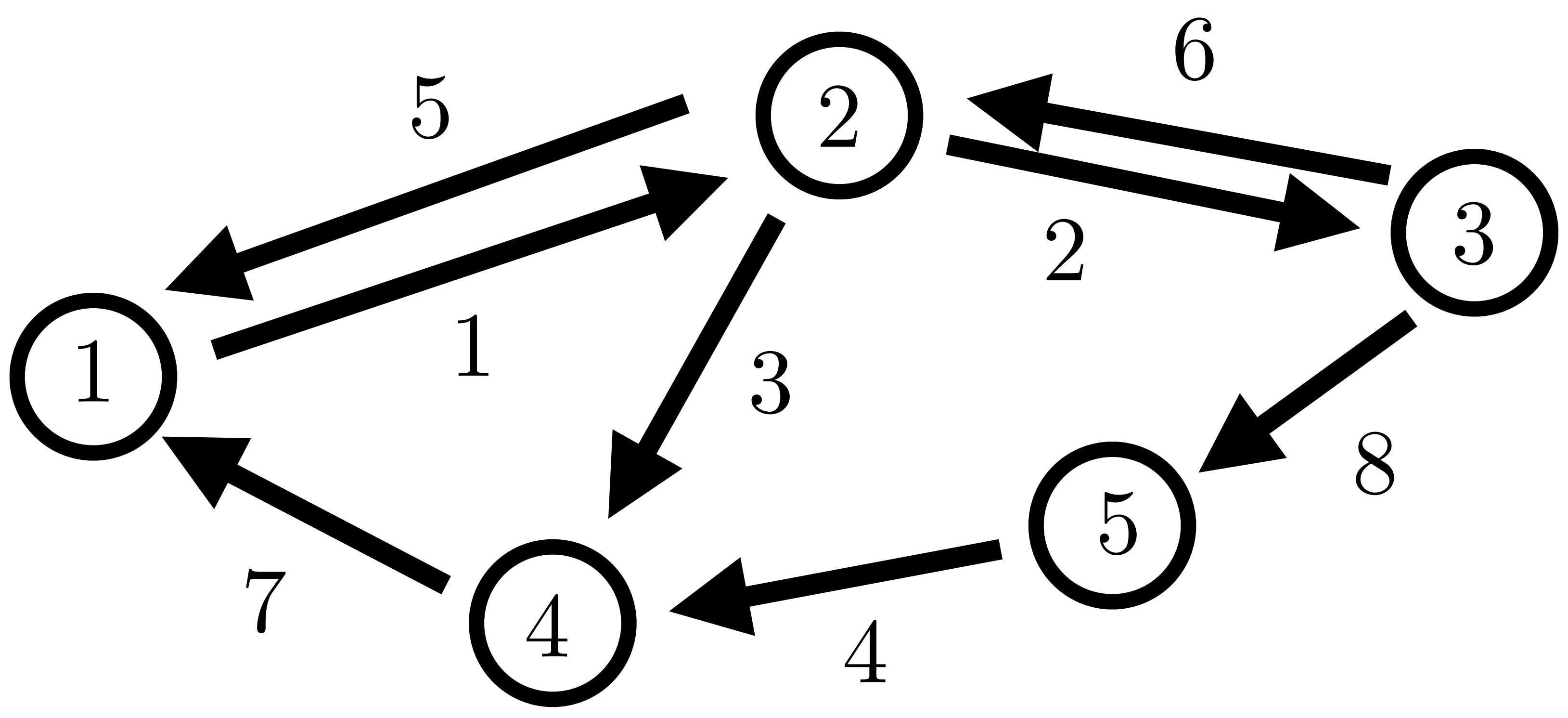
cols… $\qquad$ **rows…**

RA $\quad$ rotate columns of A…

relative geometry stays the same.

$$(AR)^T (RA) = A^T R^T R A = A^T A$$

Graph Laplacian



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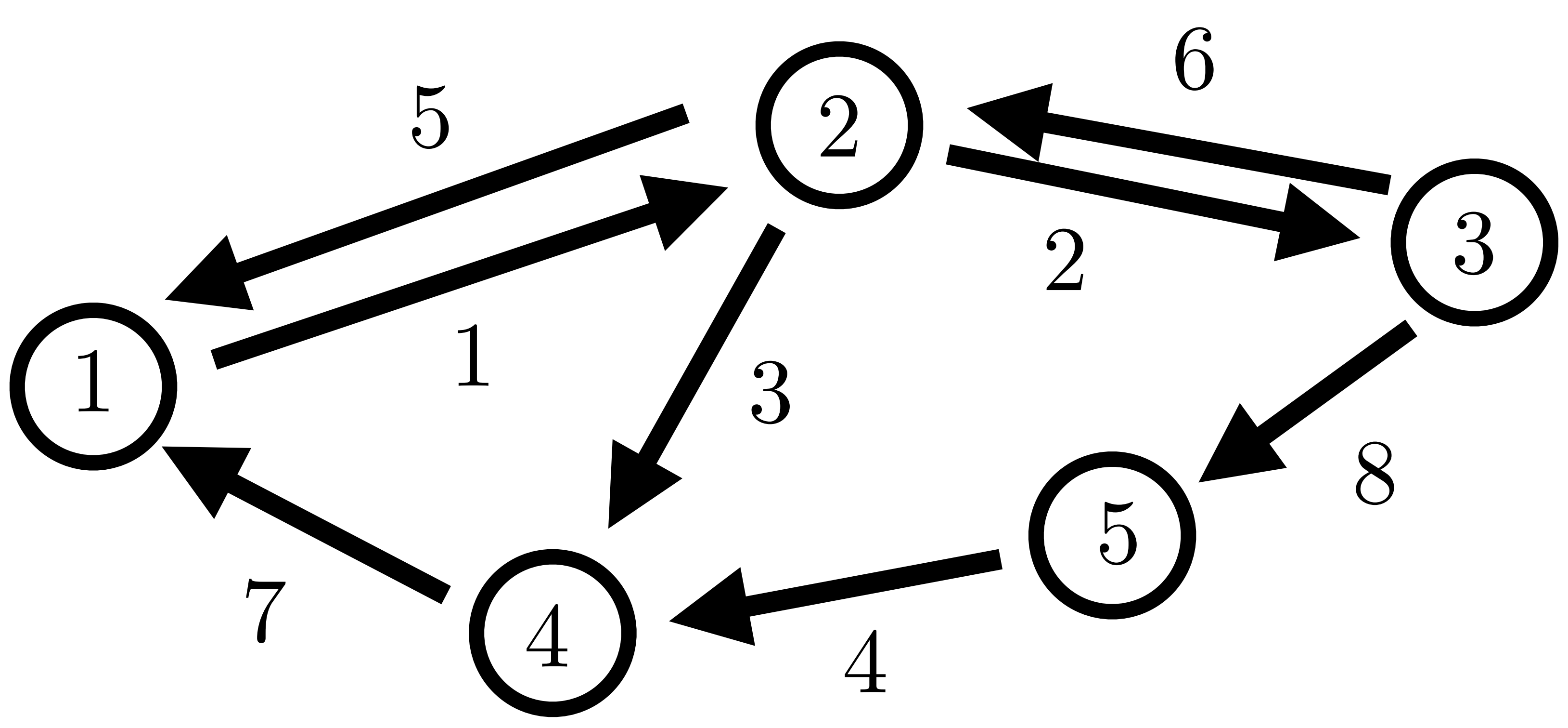
cols… **rows…**

$A^T A \quad$ "Shape" of the columns of A

Graph Laplacian



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edge mass flows $\longrightarrow$

Affine Constraint $\quad Ex = S$

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^{\dagger} S$

Shape Matrices

…recall $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

cols… **rows…**

$A^T A$ ~~“Shape” of the columns of A~~

$(A^T A)^{1/2}$ “Shape” of the columns of A

More accurate

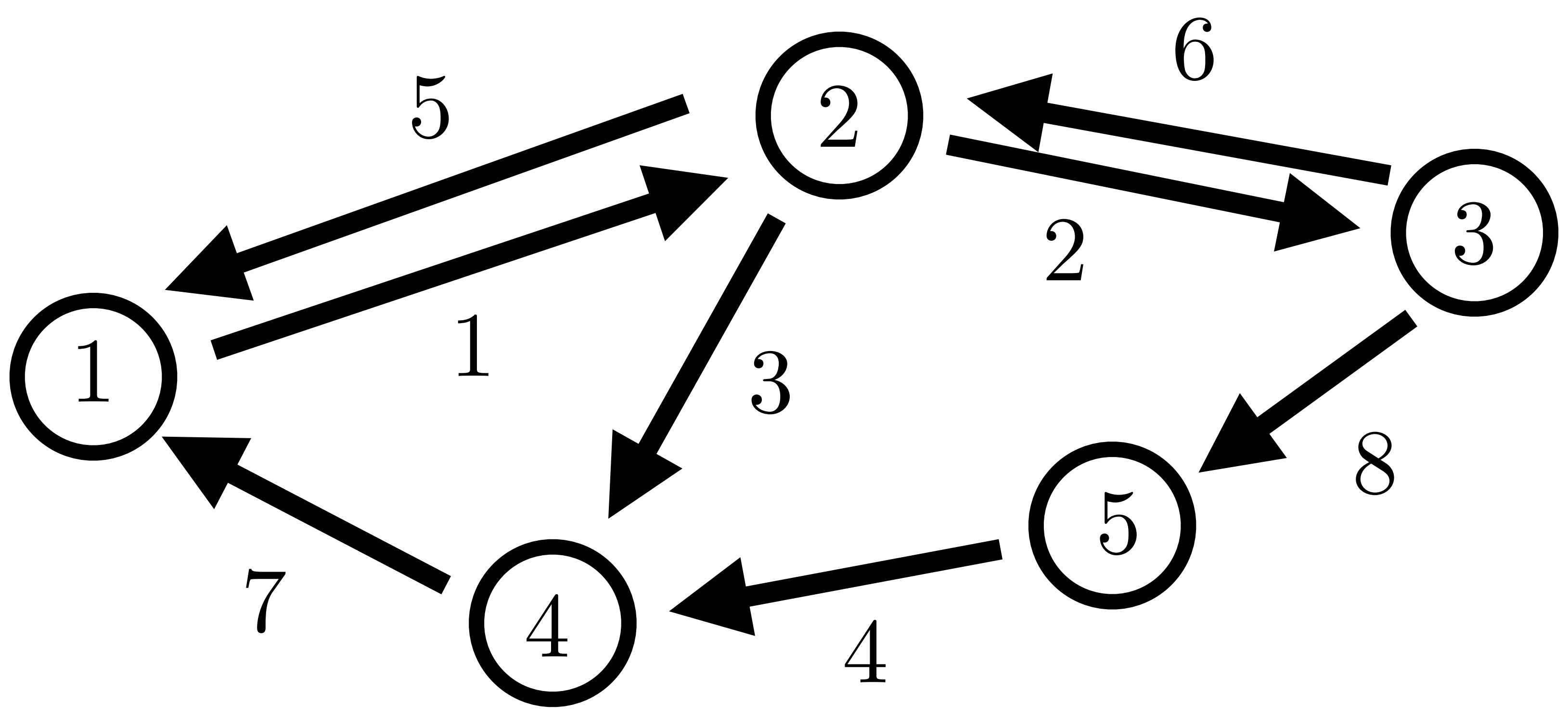
Graph Laplacian

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



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$(A^T A)^{1/2}$ “Shape” of the columns of A

$(AA^T)^{1/2}$ “Shape” of rows of A

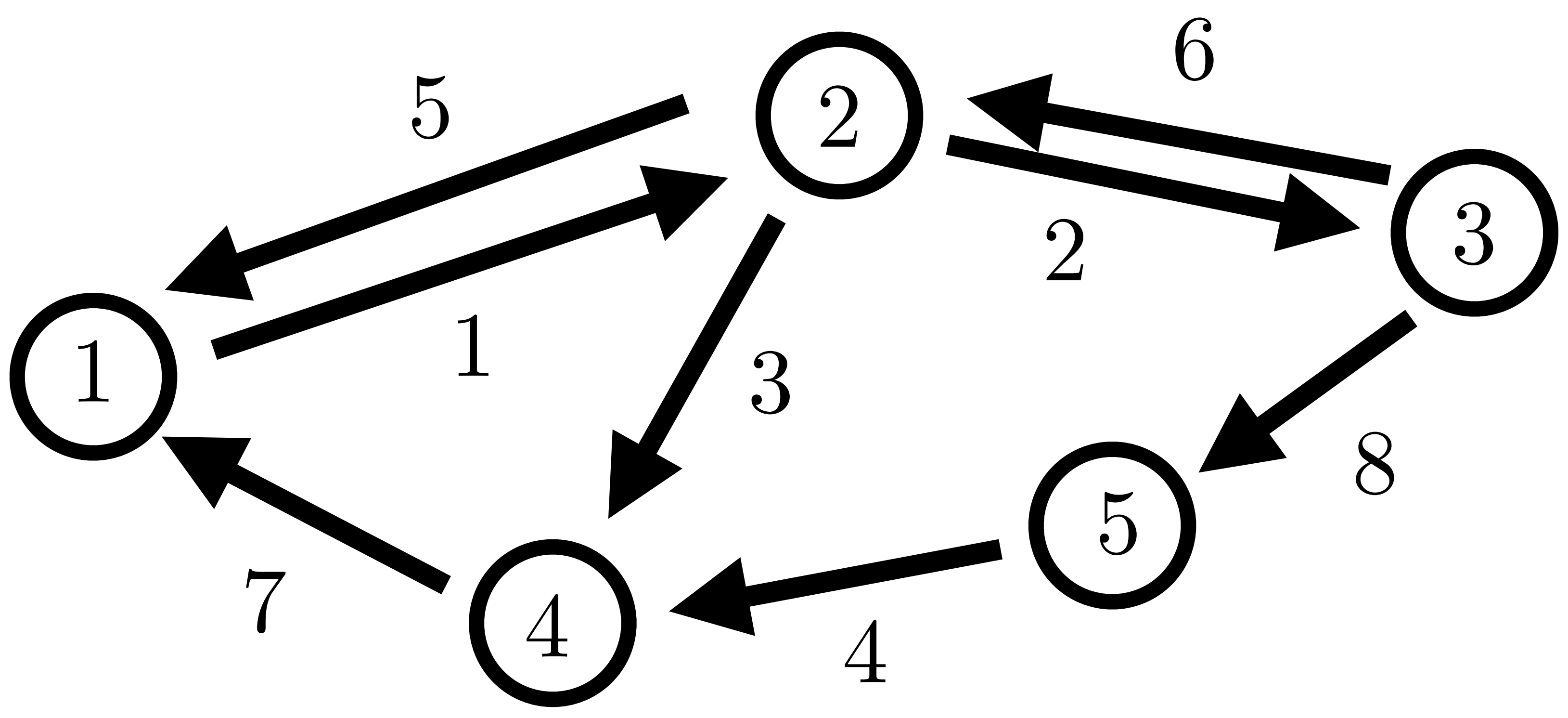
Analogy: $\quad z \in \mathbb{C} \qquad |z| = \sqrt{z^* z} \qquad z = |z| e^{i\phi}$

Polar Decomposition “column version”

$$A = A(A^T A)^{-1/2} \cdot (A^T A)^{1/2}$$

Rotation $\qquad$ **PSD “shape"**

Graph Laplacian



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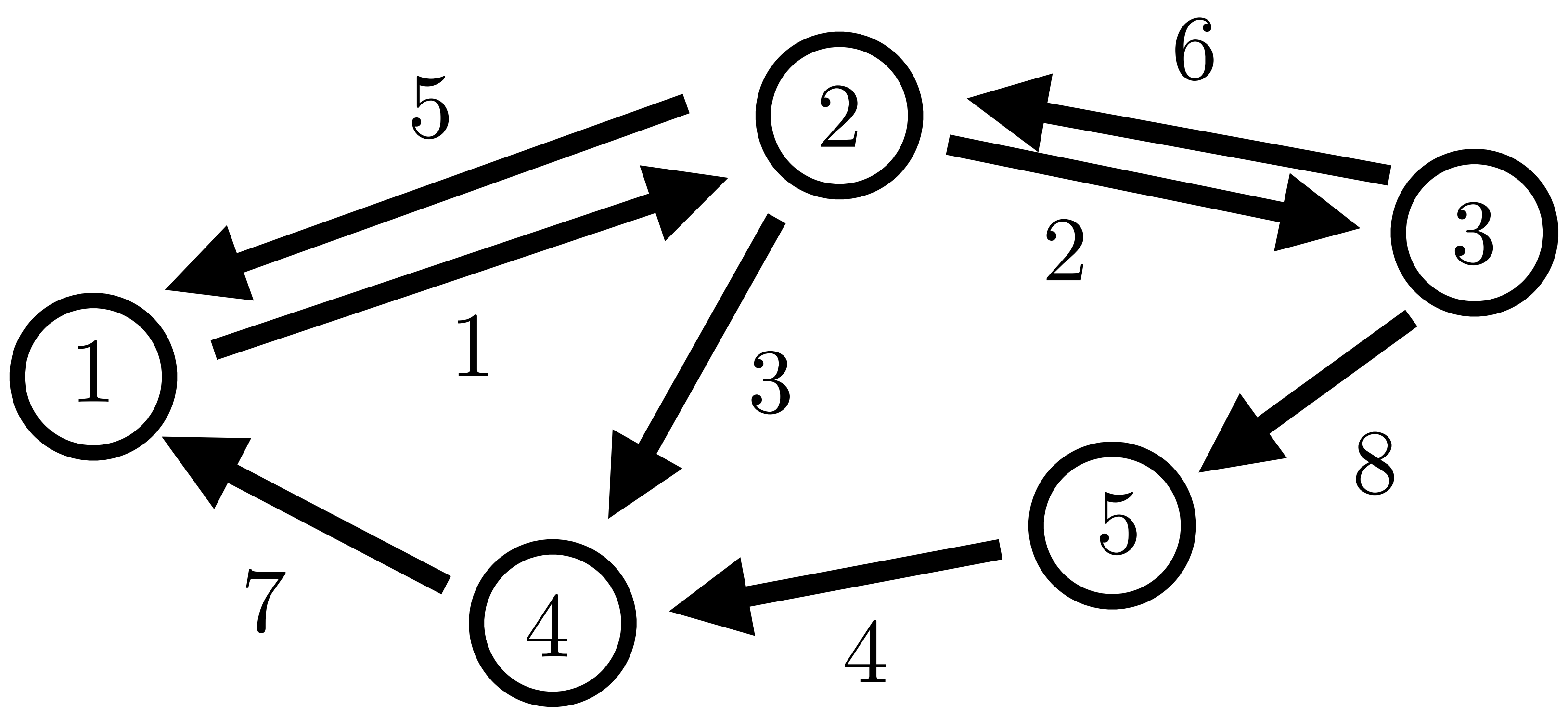
PSD "shape" **Rotation**

Graph Laplacian



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EVD of shapes $\quad (A^TA)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$

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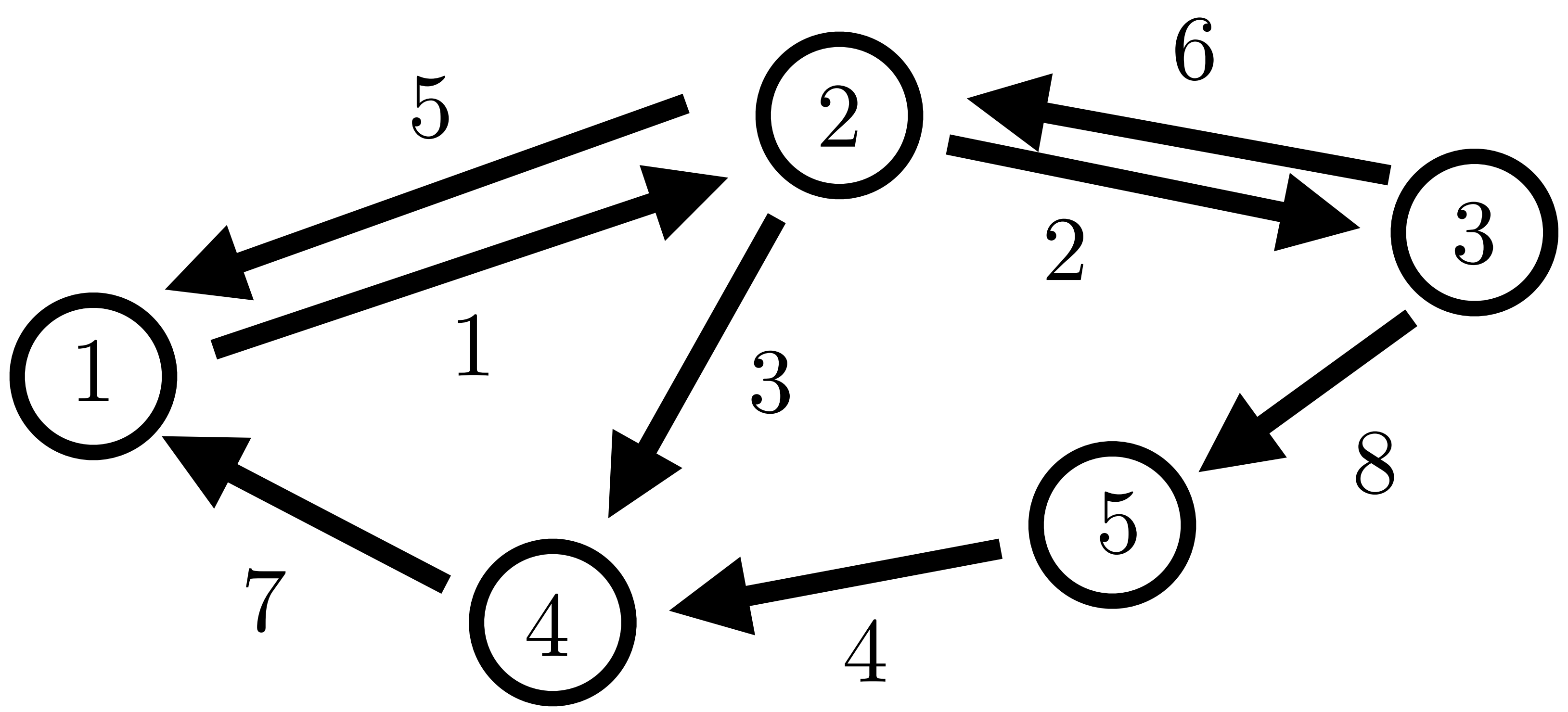

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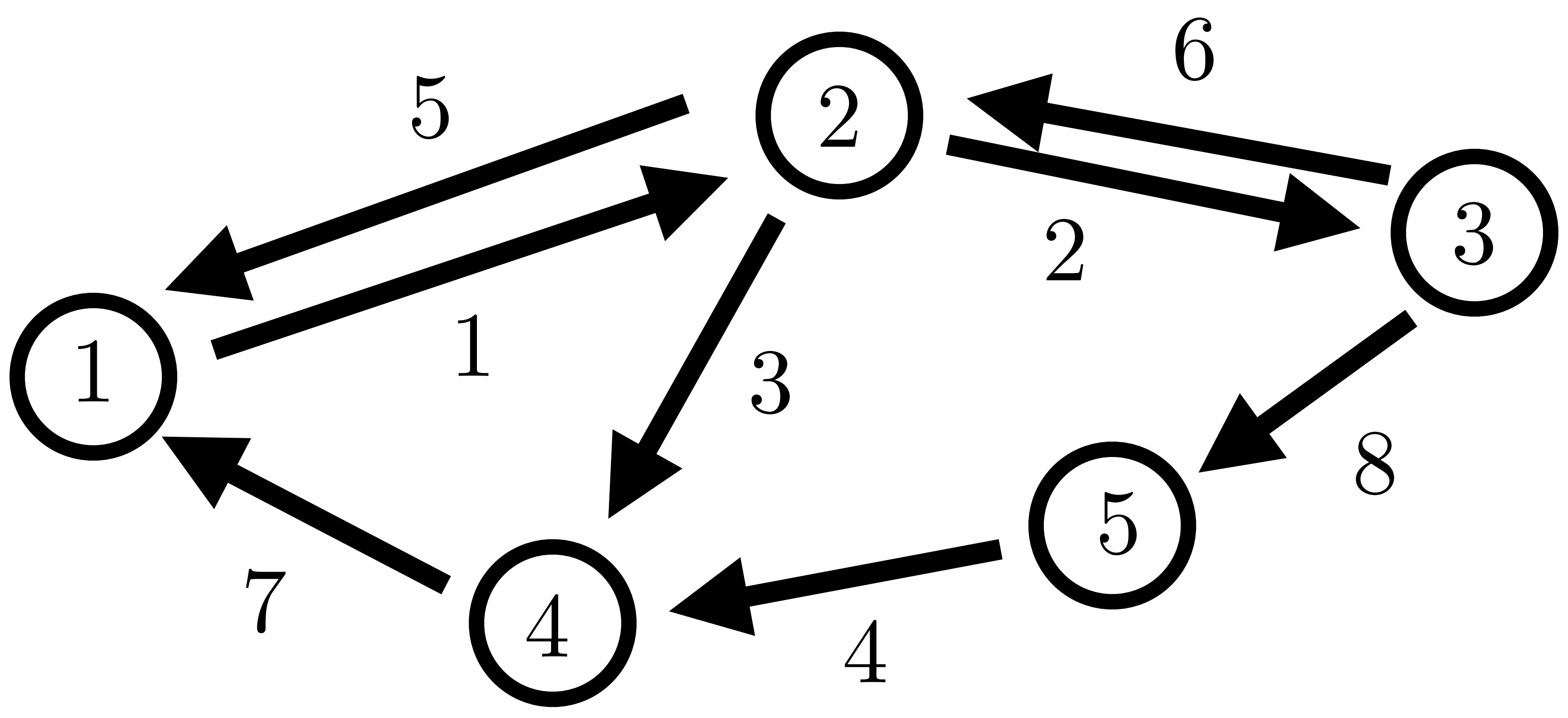
Polar Decomposition "row version" $\quad A = \underbrace{U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T}_{\text{PSD "shape"}} \cdot \underbrace{U V^T}_{\text{Rotation}}$

Graph Laplacian



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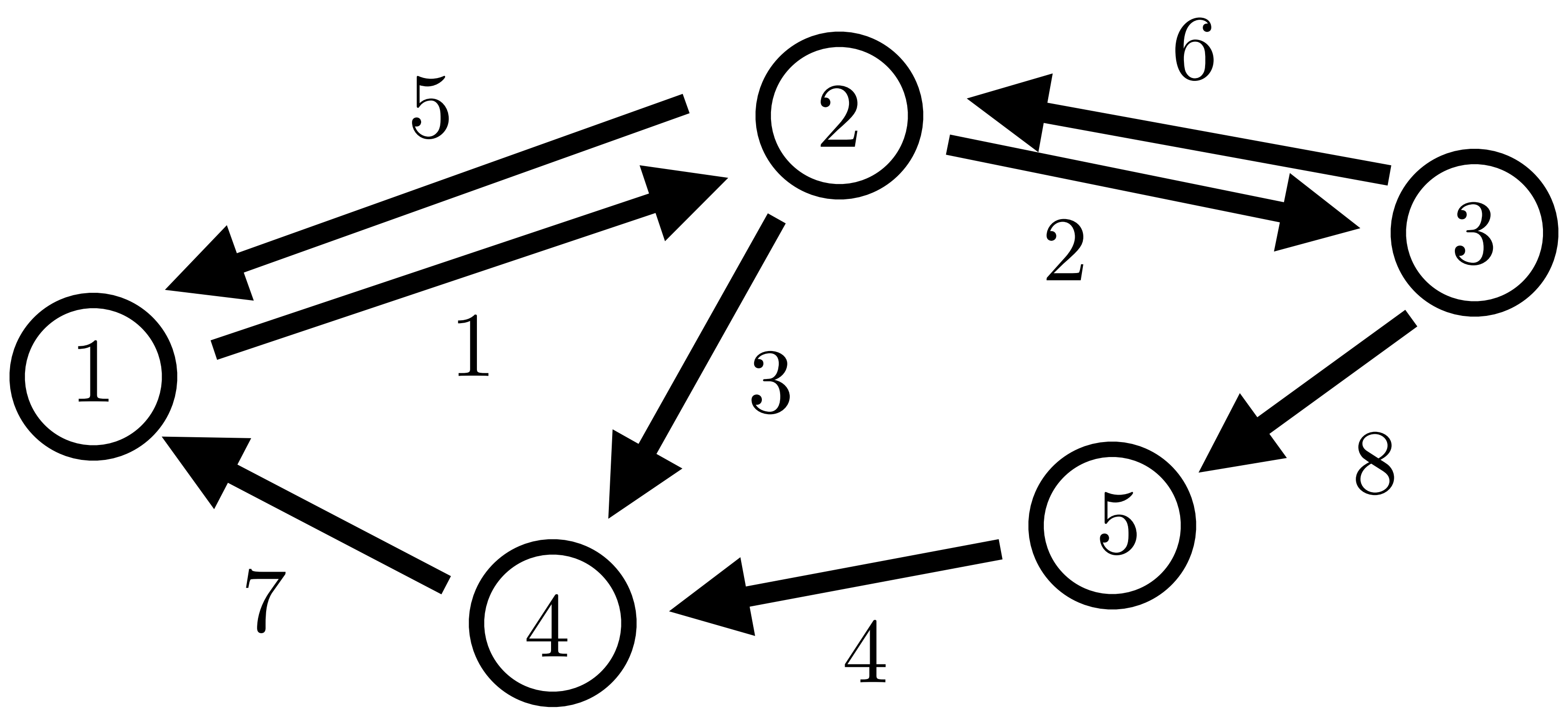
$$A = \underbrace{UV^T}_{\text{Rotation}} \cdot \underbrace{V\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^T}_{\text{PSD “shape"}}$$

Graph Laplacian



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Singular Value Decomposition $\quad A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Graph Laplacian

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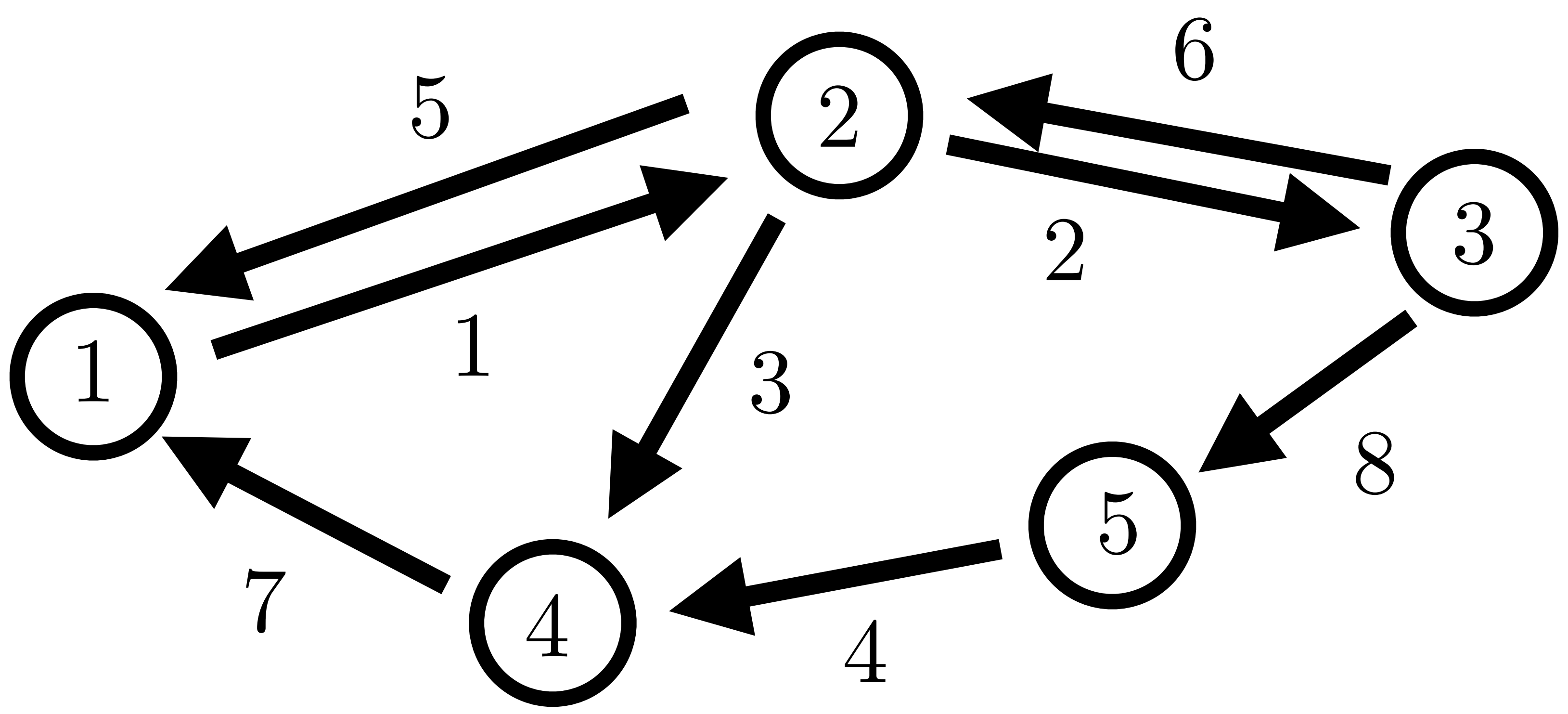

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Singular Value Decomposition

$$U' = AV'\Sigma^{-1}$$

$$V'^T = \Sigma^{-1} U'^T A$$

for singular vectors w/ non-zero values

Graph Laplacian

Graph:

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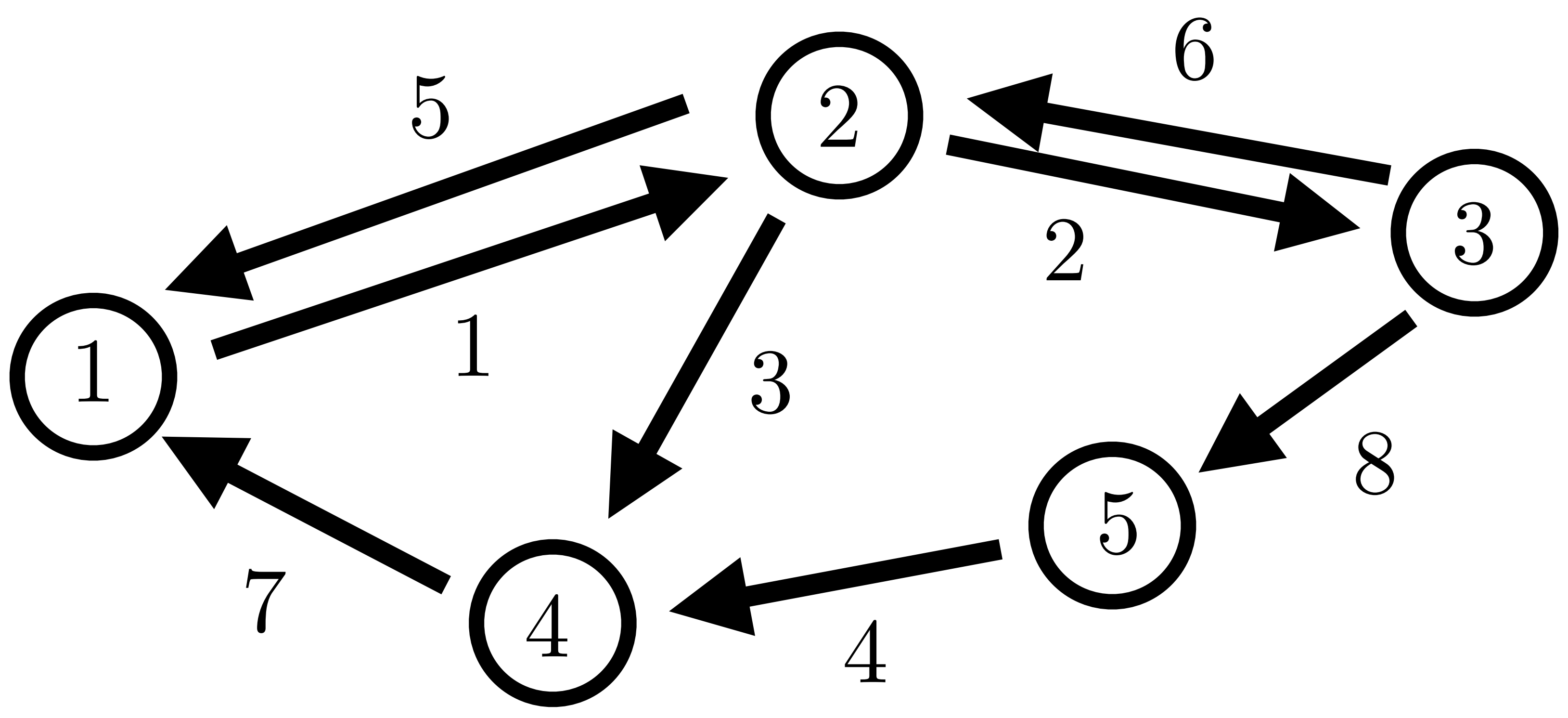

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Graph Laplacian "shape" matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

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SVD of incidence matrix

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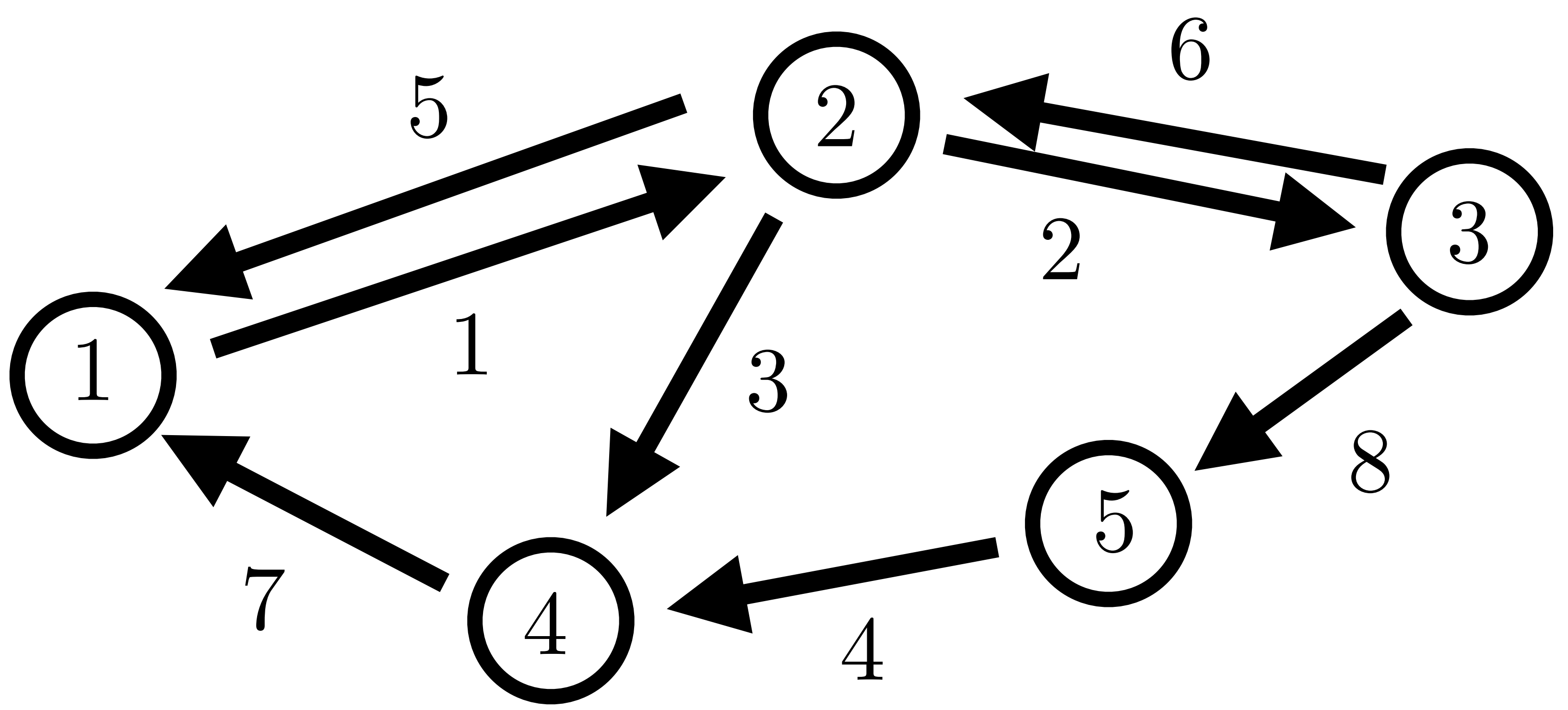

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$$\bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix}$$

Laplacian eigenvectors

(Incidence matrix left singular vectors)

SVD of incidence matrix

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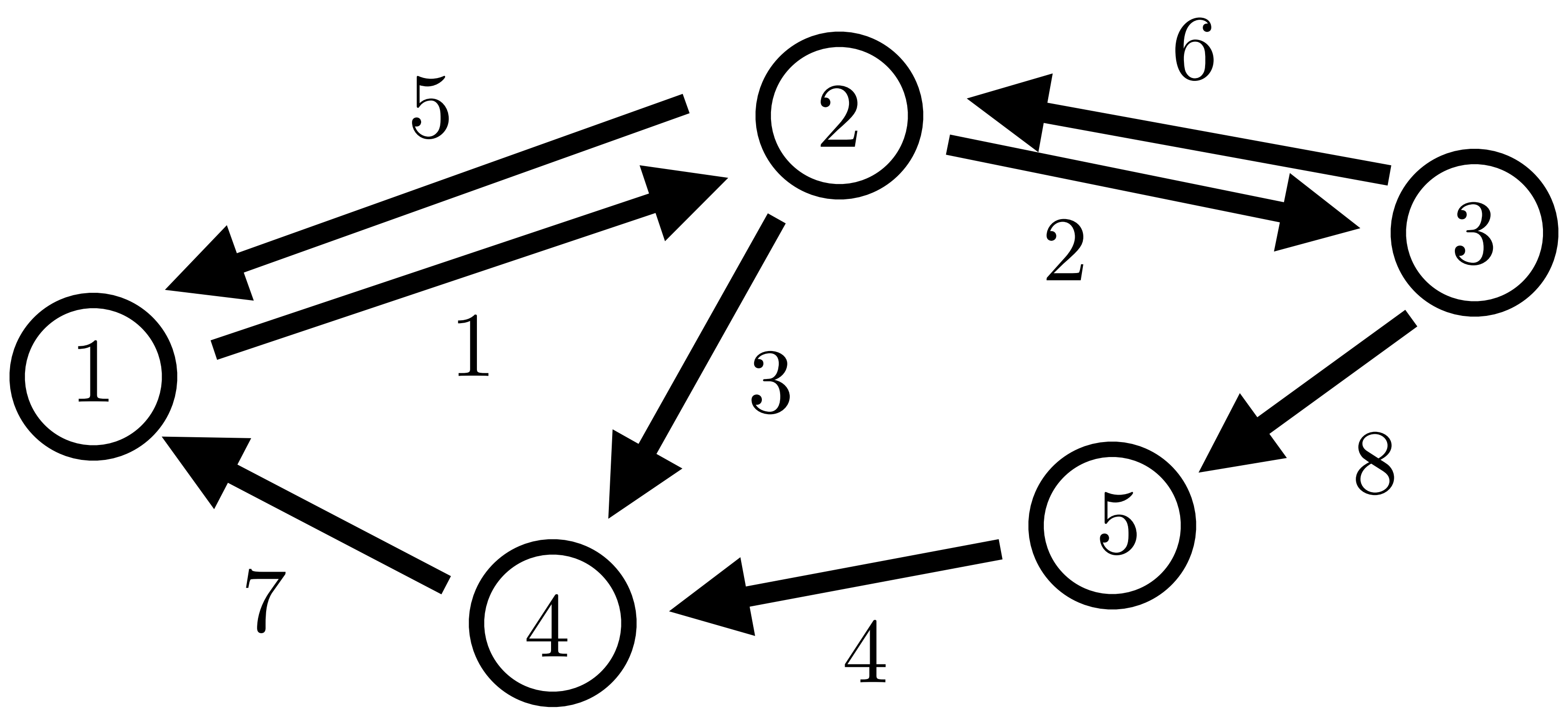
Graph Laplacian



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Action of Laplacian: $\quad Lu = \Big[\ E\ \Big]\Big[\ E^T\ \Big]\begin{bmatrix} | \\ u \\ | \end{bmatrix}$

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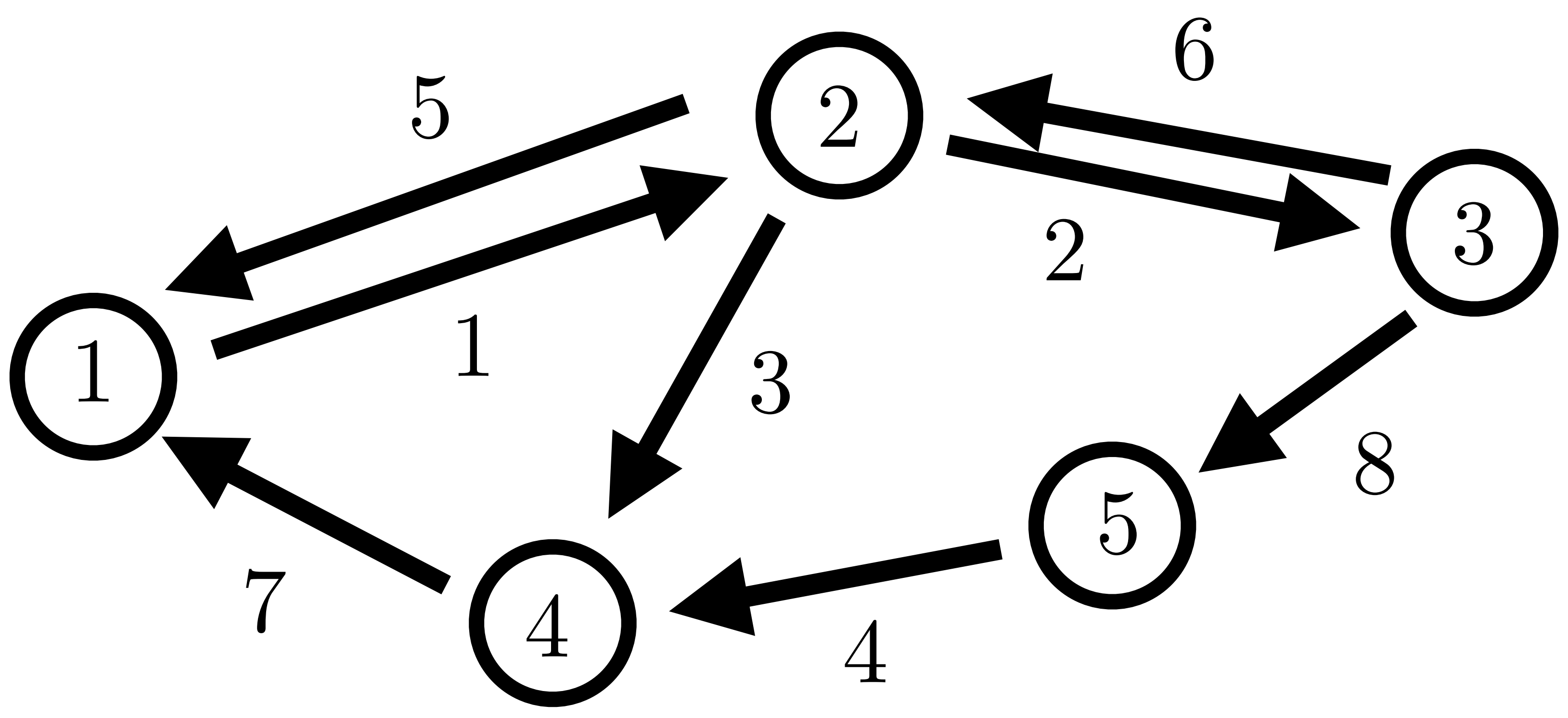

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Graph Laplacian “shape” matrix of rows (squared)

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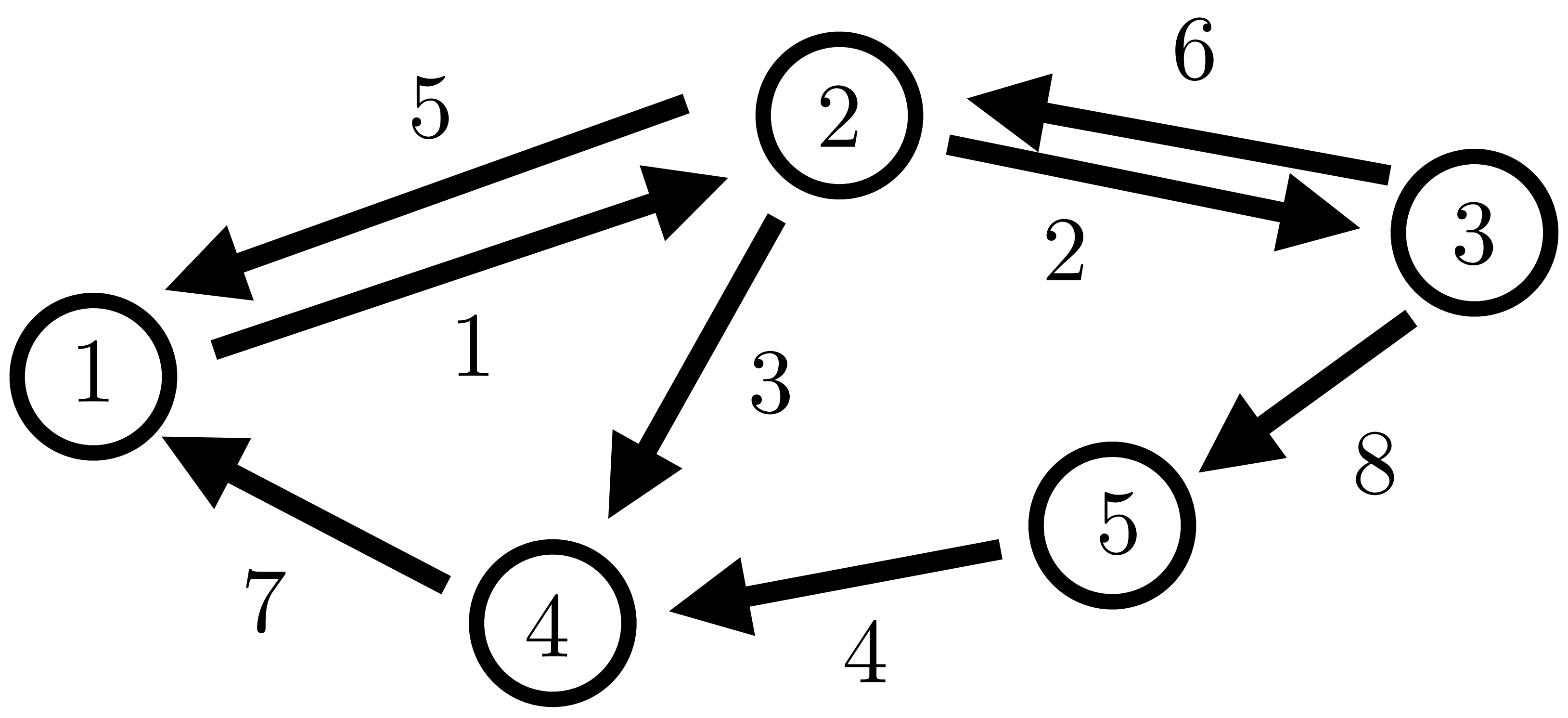
$$= \begin{bmatrix} | & | \\ U' & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & U'^T & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix} \qquad \bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix}$$

Action of Laplacian:

$$Lu = \underbrace{\Big[\, E \,\Big] \underbrace{\Big[\, E^T \,\Big] \begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{…tension created in edges}}}_{\text{… summed resulting tension on nodes}}$$

“heights” of nodes

Graph Laplacian



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\big[\, E \,\big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Affine Constraint $\quad Ex = S$

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Graph Laplacian "shape" matrix of rows (squared)

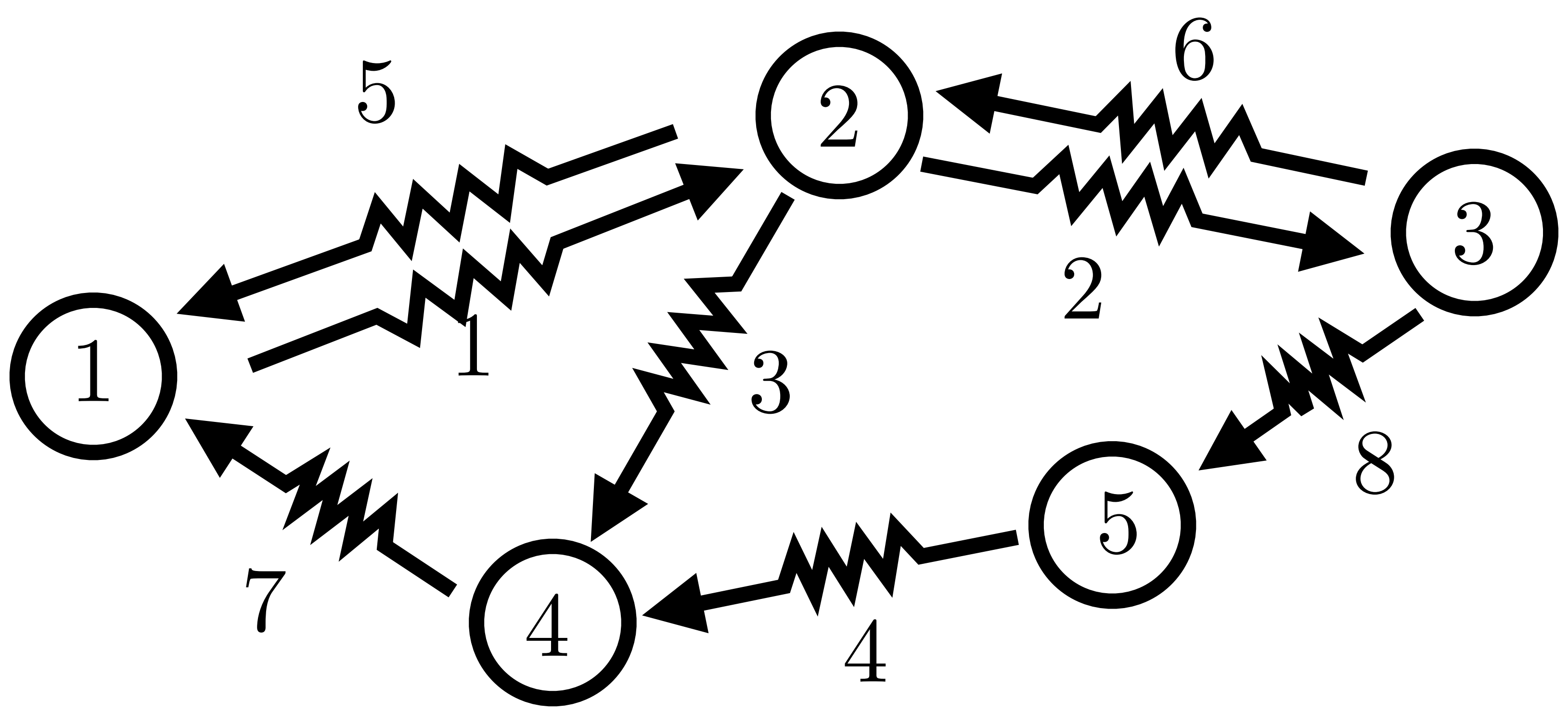
$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= \begin{bmatrix} | & | \\ U' & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & U'^T & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix} \qquad \bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix}$$

Action of Laplacian: $\quad Lu = \underbrace{\big[\, E \,\big] \big[\, E^T \,\big] \begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{… summed tension from node height differences}}$ "heights" of nodes

Linear ODE $\quad \dot{u} = -Lu \quad$ *Eigenvectors are oscillation modes*

Graph Laplacian



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\big[\, E \,\big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Affine Constraint $\quad Ex = S$

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Weighted Graph Laplacian "shape" matrix of rows (squared)

$$L = EWE^T = U_W \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} U_W^T$$

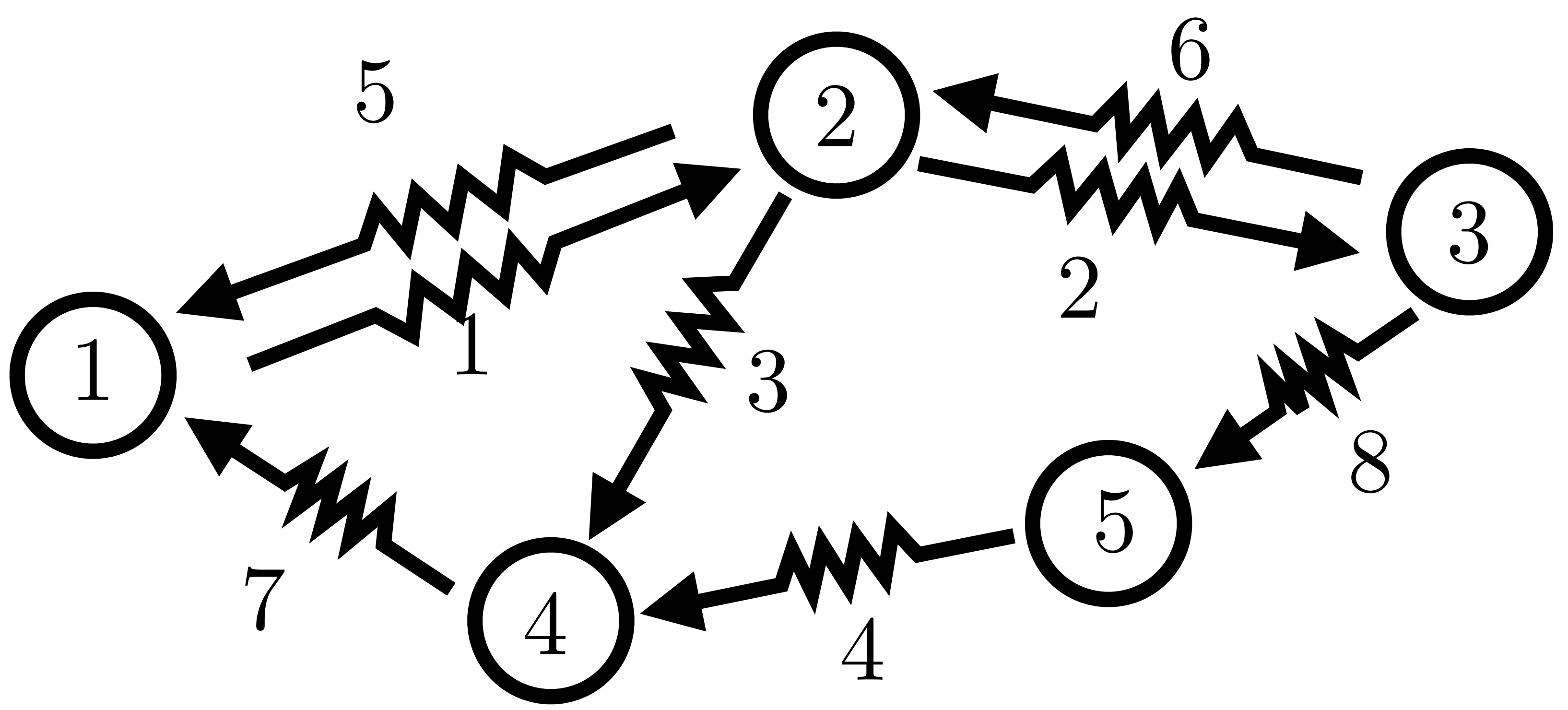
$$= \begin{bmatrix} | & | \\ U'_W & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & U'^T_W & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix}^T \quad \bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix}$$

Action of Laplacian:

$$Lu = \underbrace{\big[\, E \,\big] \underbrace{\big[\, W \,\big] \big[\, E^T \,\big] \begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{...tension created in edges scaled by weights}}}_{\text{... summed resulting tension on nodes}}$$

"heights" of nodes

Graph Laplacian



Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} E \end{bmatrix}_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Affine Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$

Weighted Graph Laplacian "shape" matrix of rows (squared)

$$L = EWE^T = U_W \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} U_W^T$$

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Action of Laplacian:

$$Lu = \underbrace{\begin{bmatrix} E \end{bmatrix} \begin{bmatrix} W \end{bmatrix} \begin{bmatrix} E^T \end{bmatrix}}_{\text{… summed scaled tension from node height differences}} \begin{bmatrix} | \\ u \\ | \end{bmatrix}$$

"heights" of nodes

Linear ODE $\quad \dot{u} = -Lu$

Eigenvectors are oscillation modes

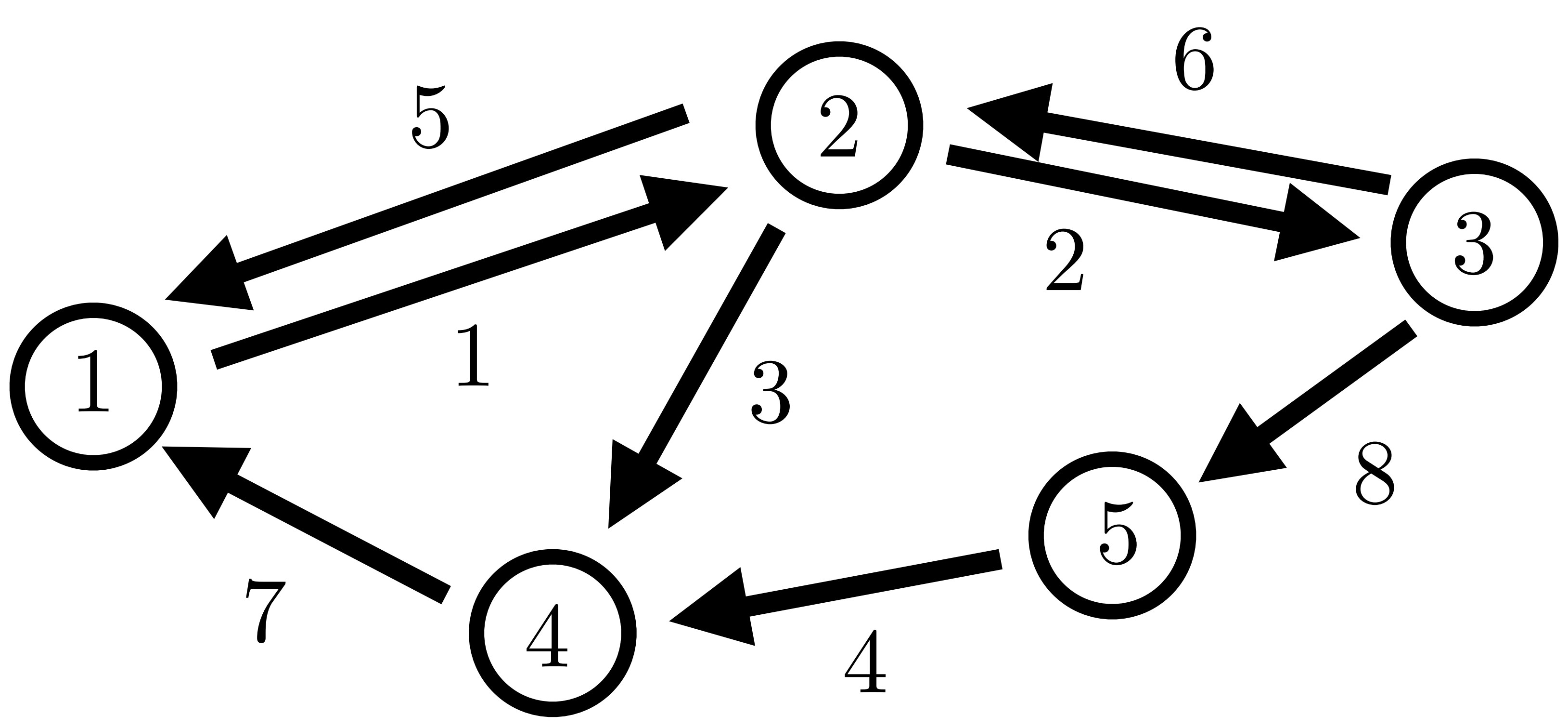
Graph Laplacian

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[E \Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows $\longrightarrow$

Affine Constraint $\quad Ex = S$

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Graph Laplacian "shape" matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= \begin{bmatrix} 3 & -2 & 0 & -1 & 0 \\ -2 & 5 & -2 & -1 & 0 \\ 0 & -2 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Graph Laplacian

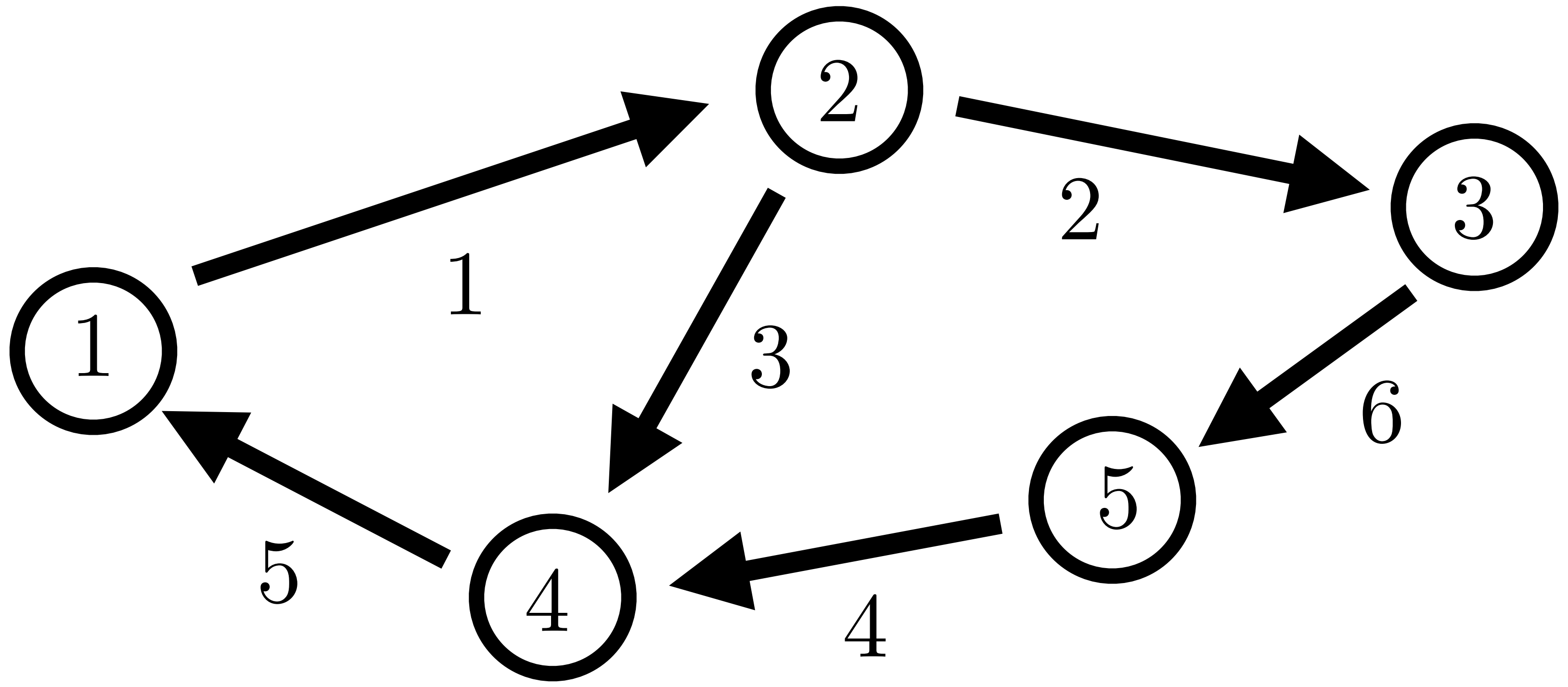
Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} E \end{bmatrix}_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows

Affine Constraint $\quad Ex = S$

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Graph Laplacian "shape" matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Independent of edge direction

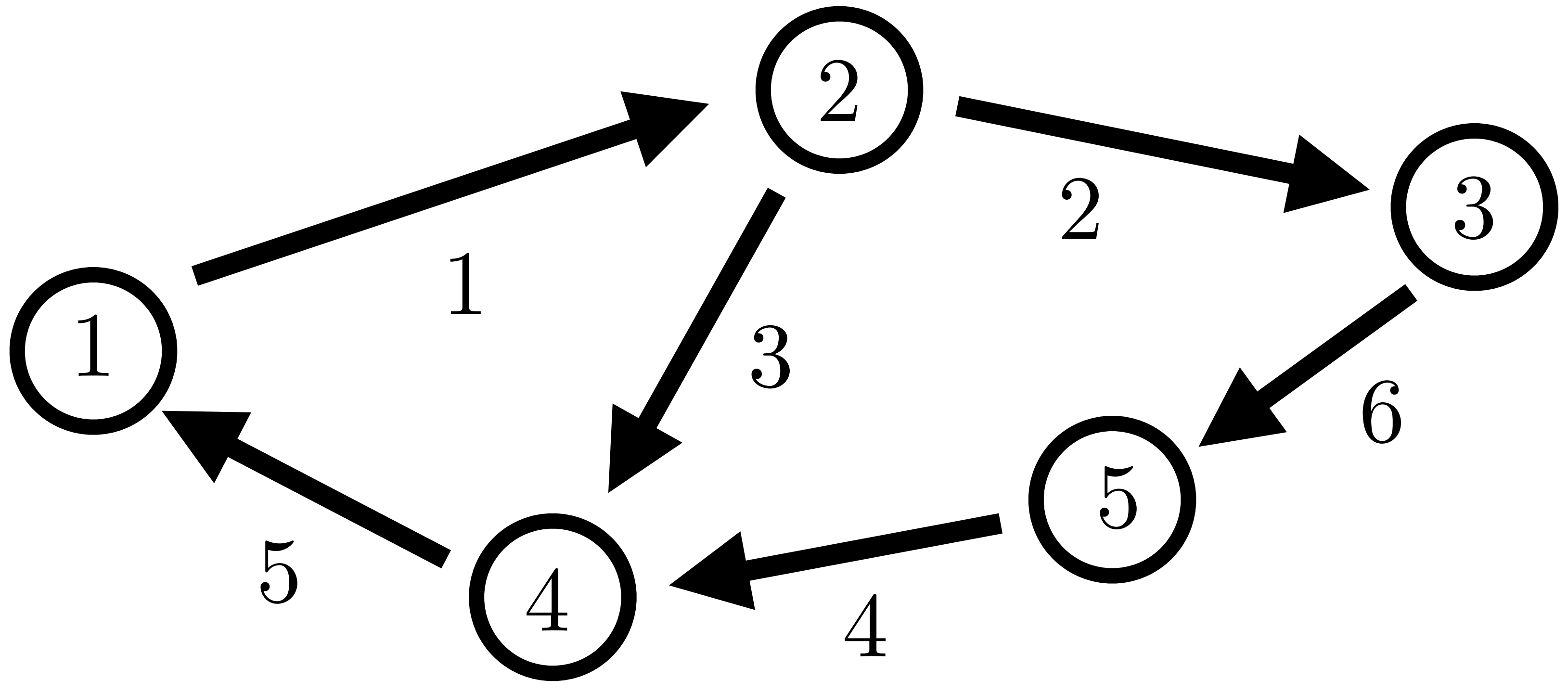
Degree & Adjacency Matrices

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\qquad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows

Affine Constraint $\qquad Ex = S$

Min Norm Solution $\qquad \bar{x} = E^T (EE^T)^{\dagger} S$

Graph Laplacian "shape" matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

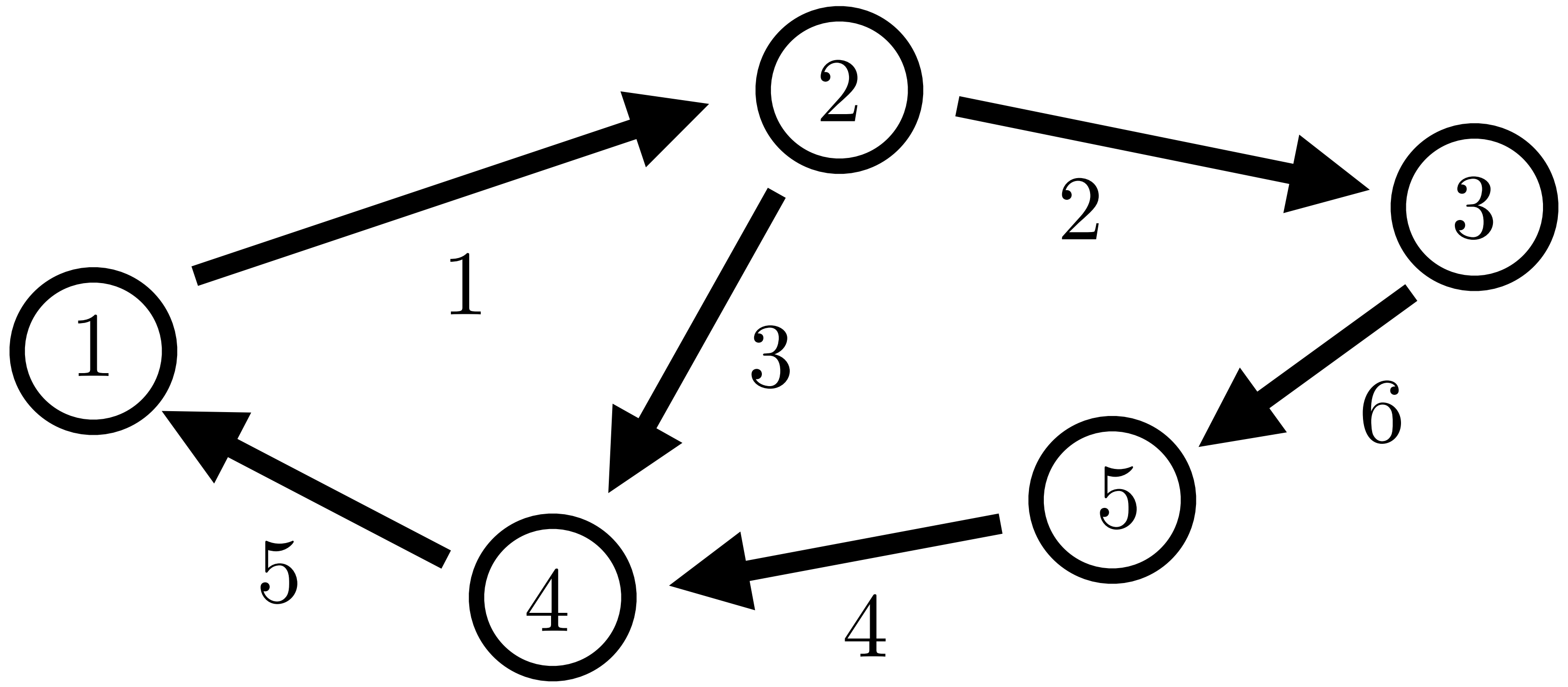
Independent of edge direction

$$L = D - A$$

Degree Matrix $D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Adjacency Matrix $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

Degree & Adjacency Matrices



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows

Affine Constraint $\quad Ex = S$

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Graph Laplacian "shape" matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Independent of edge direction

$$L = D - A$$

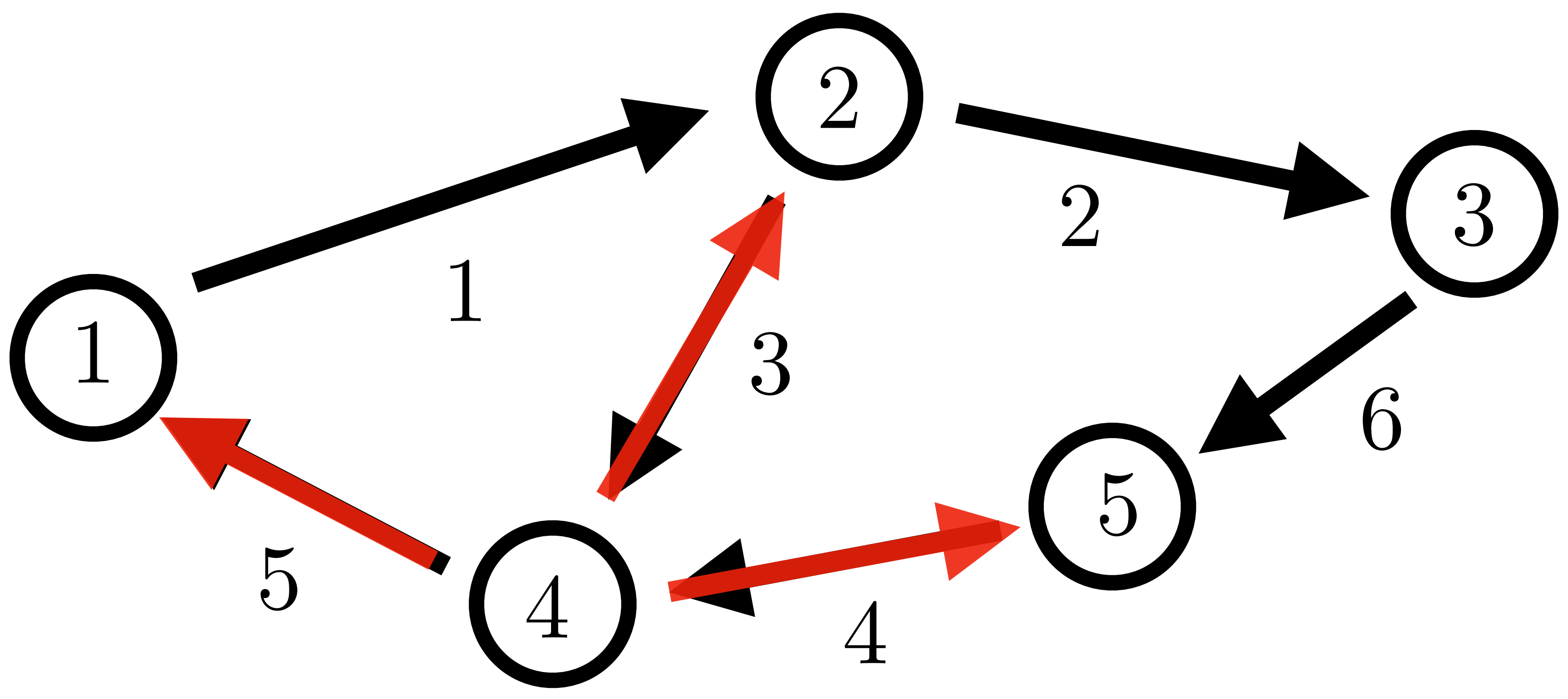
Degree Matrix $\quad \left[\, D \,\right]_{vv} = |\mathcal{N}_v|$

diagonal

\# edges (in & out)

Adjacency Matrix $\quad \left[\, A \,\right]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\,E\,\right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows

Affine Constraint $\quad Ex = S$

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Graph Laplacian “shape” matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

Degree Matrix diagonal $\left[\,D\,\right]_{vv} = |\mathcal{N}_v|$

Adjacency Matrix $\left[\,A\,\right]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Adjacency Matrix

From Node 4

Edges to Nodes 1,2, & 5

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

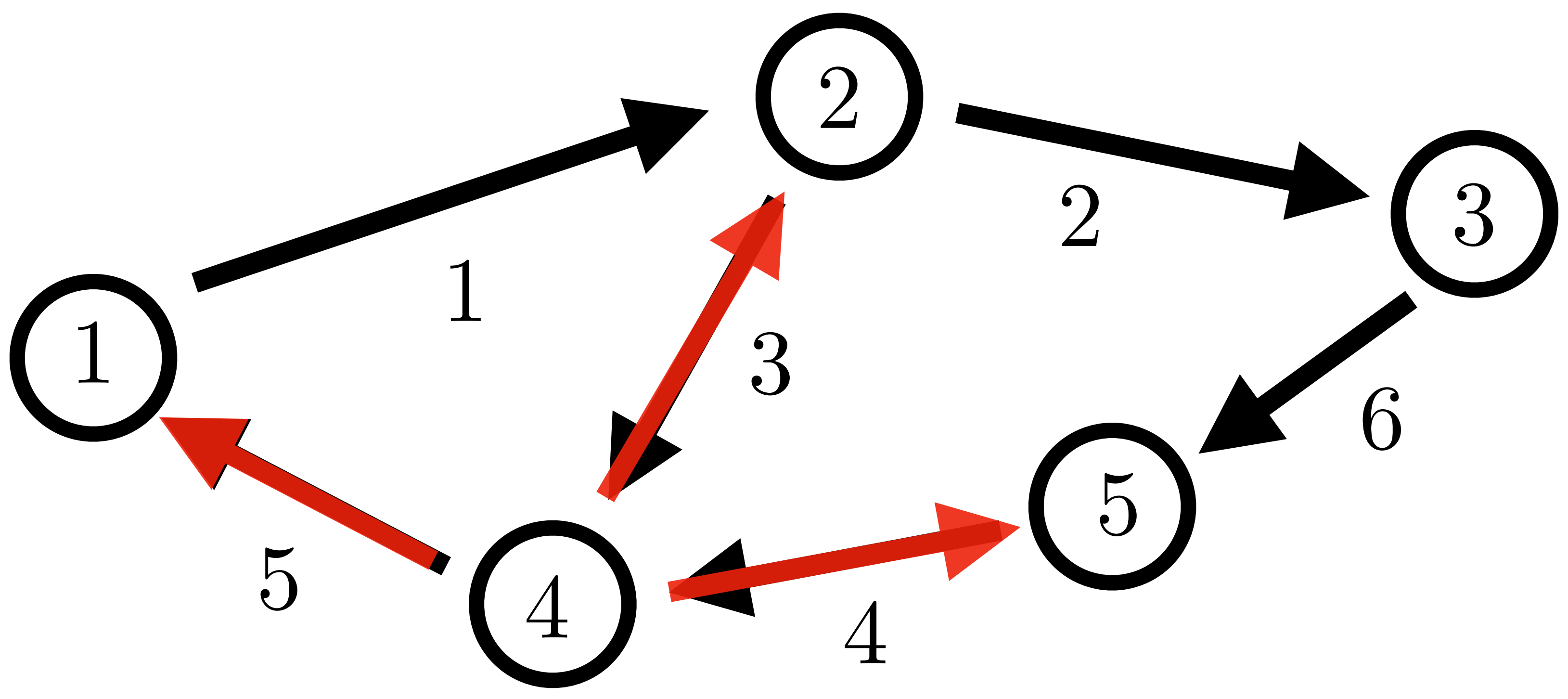
Adjacency Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows

Affine Constraint $\qquad Ex = S$

Min Norm Solution $\qquad \bar{x} = E^T (EE^T)^\dagger S$

Graph Laplacian “shape” matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

Degree Matrix — diagonal $\quad \left[\, D \,\right]_{vv} = |\mathcal{N}_v|$

Adjacency Matrix $\quad \left[\, A \,\right]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

Start @ node 4

Edges to Nodes 1,2, & 5

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

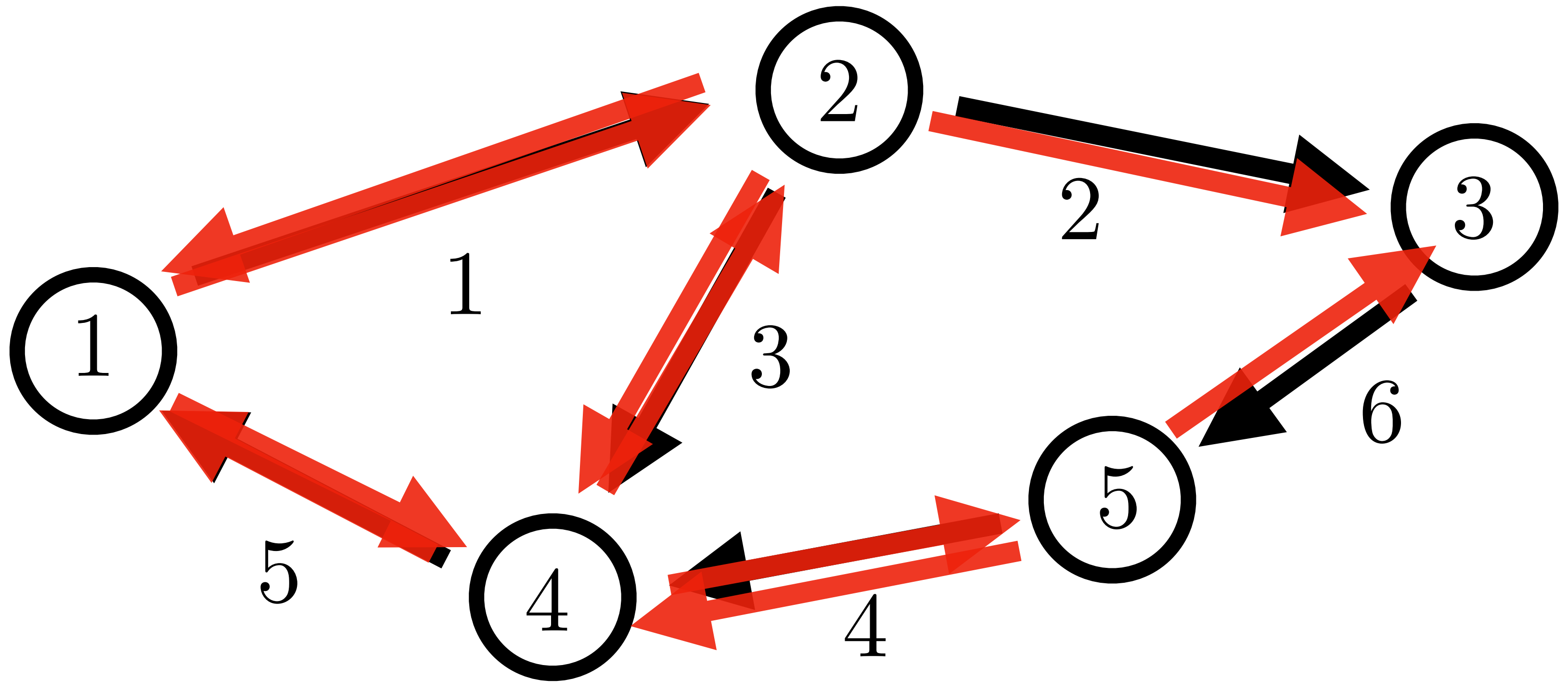
Adjacency Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows

Affine Constraint $\qquad Ex = S$

Min Norm Solution $\qquad \bar{x} = E^T (EE^T)^{\dagger} S$

Graph Laplacian "shape" matrix of rows (squared)

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$$L = D - A$$

Degree Matrix diagonal $\left[\, D \,\right]_{vv} = |\mathcal{N}_v|$

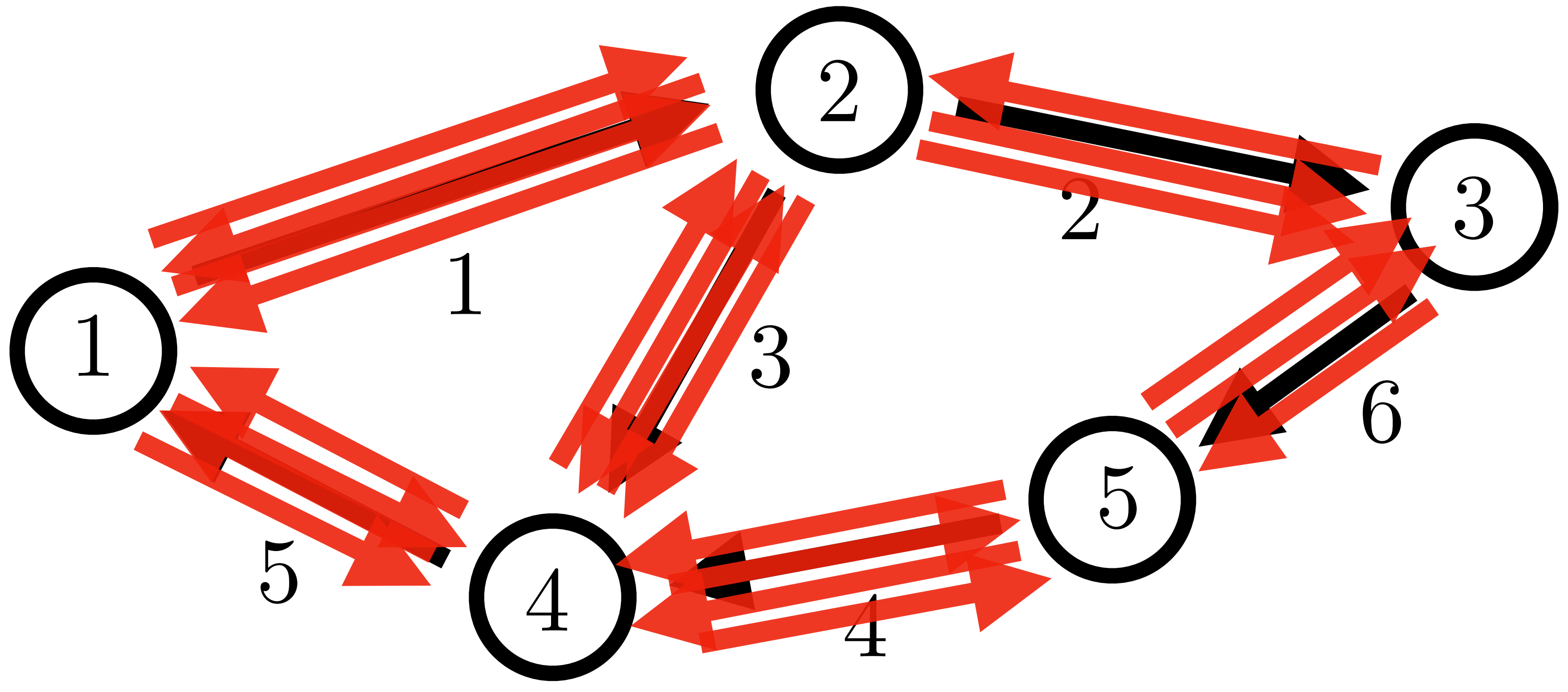
Adjacency Matrix $\left[\, A \,\right]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

Thru Node 1 $\quad$ Thru Node 2 $\quad$ Thru Node 5 $\quad$ Start @ node 4

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Graph Laplacian



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows

Affine Constraint $\quad Ex = S$

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Graph Laplacian "shape" matrix of rows (squared)

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Degree Matrix diagonal $\left[\, D \,\right]_{vv} = |\mathcal{N}_v|$

Adjacency Matrix $\left[\, A \,\right]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

Thru Node 1 Thru Node 2 Thru Node 5 Start @ node 4

3-step paths from node 4 to node 1
3-step paths from node 4 to node 2
⋮
3-step paths from node 4 to node 5

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

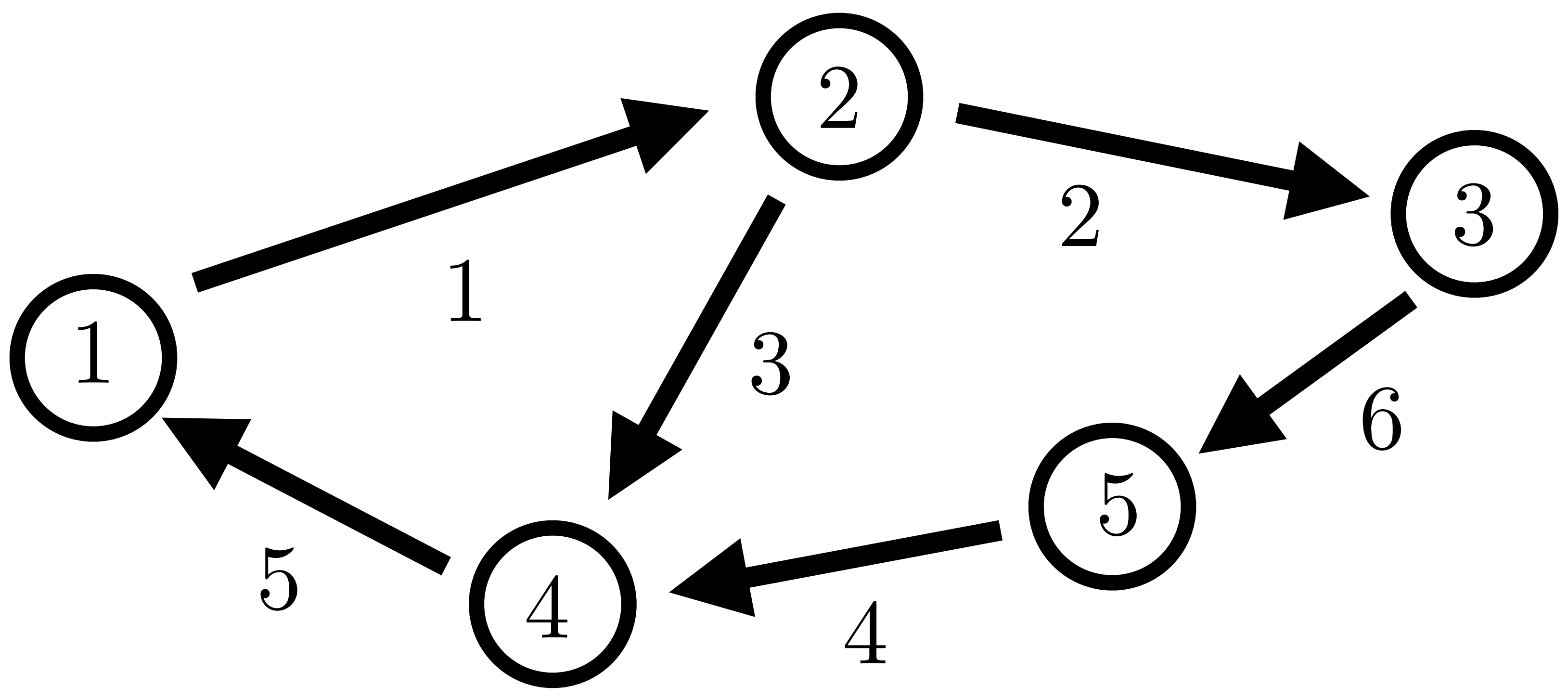
Adjacency Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected



Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows

Affine Constraint $\quad Ex = S$

Min Norm Solution $\quad \bar{x} = E^T (EE^T)^\dagger S$

Graph Laplacian "shape" matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

Degree Matrix diagonal $\left[\, D \,\right]_{vv} = |\mathcal{N}_v|$

Adjacency Matrix $\left[\, A \,\right]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

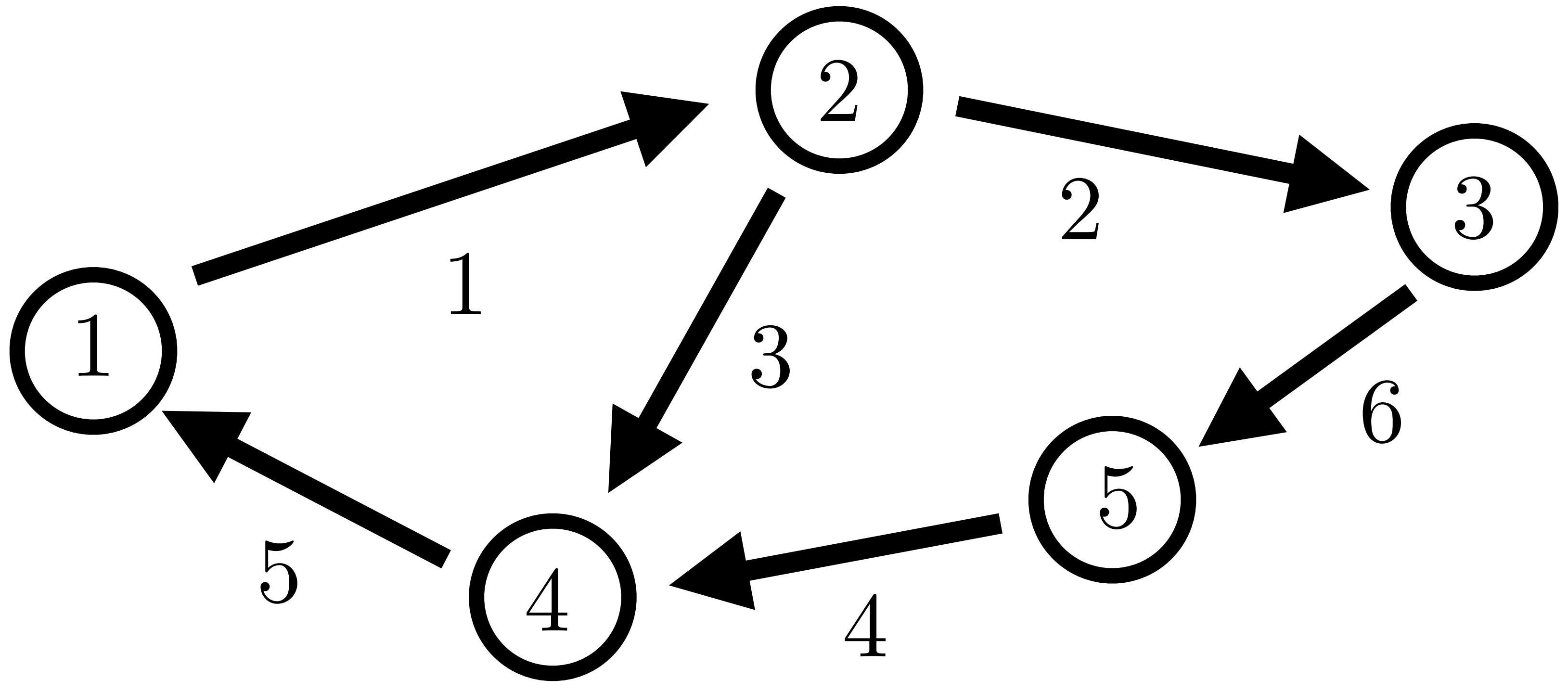
$$\left[\, A^k \,\right]_{vv'}$$

 # k-step paths from node v to node v'

Thru Node 1 $\quad$ Thru Node 2 $\quad$ Thru Node 5 $\quad$ Start @ node 4

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

In/out Incidence Matrices



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

… directed or undirected

Incidence Matrix: $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

In/out Incidence Matrices:

$$\left[\, E_{\text{out}} \,\right]_{ve} = \begin{cases} 1; & \text{if } e \text{ out of } v \\ 0; & \text{otherwise} \end{cases}$$

$$\left[\, E_{\text{in}} \,\right]_{ve} = \begin{cases} 1; & \text{if } e \text{ into } v \\ 0; & \text{otherwise} \end{cases}$$

$$E = E_{\text{in}} - E_{\text{out}}$$

Graph Laplacian "shape" matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

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Powers of Adjacency

$$\left[\, A^k \,\right]_{vv'}$$

 # k-step paths from node v to node v'

Thru Node 1 Thru Node 2 Thru Node 5 Start @ node 4

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$