Quadratic Forms, Definite Matrices, Congruence Transformations

Linear Algebra:

Winter 2022 - Dan Calderone

Definite (Symmetric) Matrices

Quadratic Form: $f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$

Definiteness:	Short	Notation	Definition	Analogy	Eigenvalues
Positive definite:	PD	$Q \succ 0$	$x^T Q x > 0 \quad \forall\, x \quad x \neq 0$	…positive orthant	$\lambda_i > 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Positive semi-definite	PSD	$Q \succeq 0$	$x^T Q x \geq 0 \quad \forall\, x$	…positive orthant w/ boundary	$\lambda_i \geq 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Negative-definite	ND	$Q \prec 0$	$x^T Q x < 0 \quad \forall\, x \quad x \neq 0$	…negative orthant	$\lambda_i < 0 \quad \lambda_i \in \mathrm{eig}(Q)$
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Indefinite:			$x^T Q x > 0 \quad \text{some } x$ $x^T Q x < 0 \quad \text{some } x$	…the rest of the space	

Eigenvalue condition proof:

…consider eigenvector coordinates

$$x = V x'$$

since V is invertible… $\quad \forall\, x \iff \forall\, x'$

$$x^T Q x = x V D V^T x = x'^T D x' = \sum_i \lambda_i {x'_i}^2$$

$$\sum_i \lambda_i {x'_i}^2 > 0 \quad \underset{x \neq 0}{\forall x'} \iff \lambda_i > 0 \quad \forall\, \lambda_i \in \mathrm{eig}(Q)$$

Note: not a useful definition for general matrices

… condition only says something about the symmetric part of Q

Symmetric/Skew-symmetric Decomposition

$$Q = \underbrace{\tfrac{1}{2}\left(Q + Q^T\right)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}\left(Q - Q^T\right)}_{\text{skew-sym}}$$

$$\begin{aligned}
x^T Q x &= \tfrac{1}{2} x^T \left(Q + Q^T\right) x + \tfrac{1}{2} x^T \left(Q - Q^T\right) x \\
&= \tfrac{1}{2} x^T \left(Q + Q^T\right) x + \tfrac{1}{2} x^T Q x - \underbrace{\tfrac{1}{2} x^T Q^T x}_{\text{...transpose}} \\
&= \tfrac{1}{2} x^T \left(Q + Q^T\right) x + \underbrace{\tfrac{1}{2} x^T Q x - \tfrac{1}{2} x^T Q x}_{=0} \\
&= \tfrac{1}{2} x^T \left(Q + Q^T\right) x
\end{aligned}$$

…only the symmetric part matters

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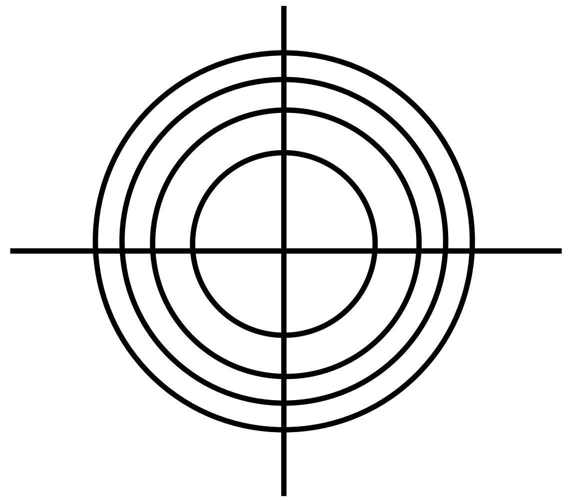
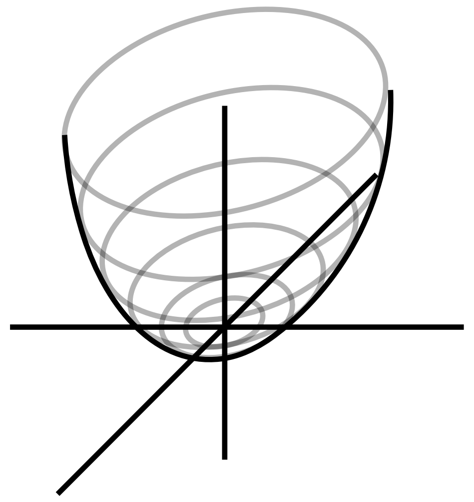
Surfaces: $Q \succ 0$

$Q = I$

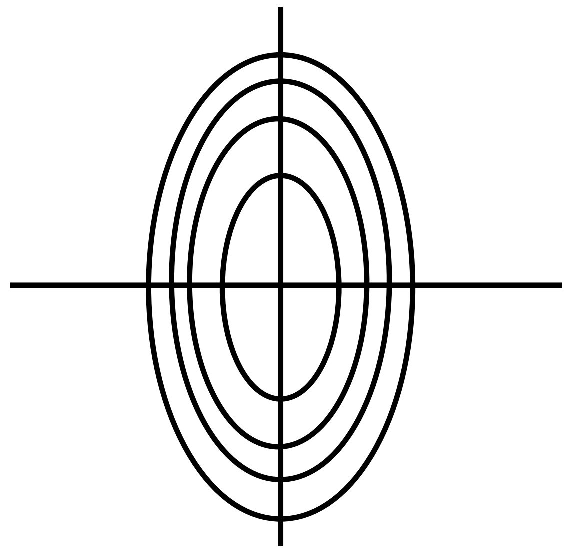


level sets



surface

Q diagonal



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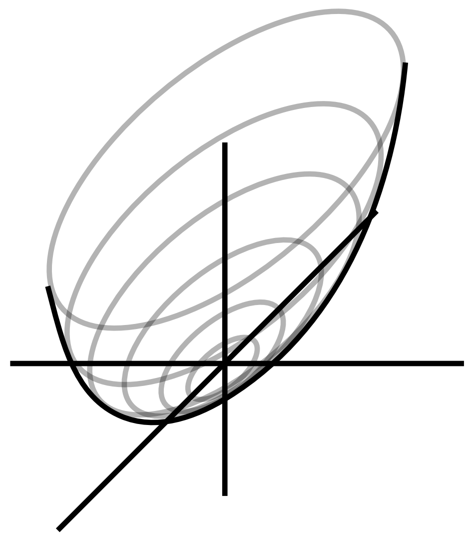

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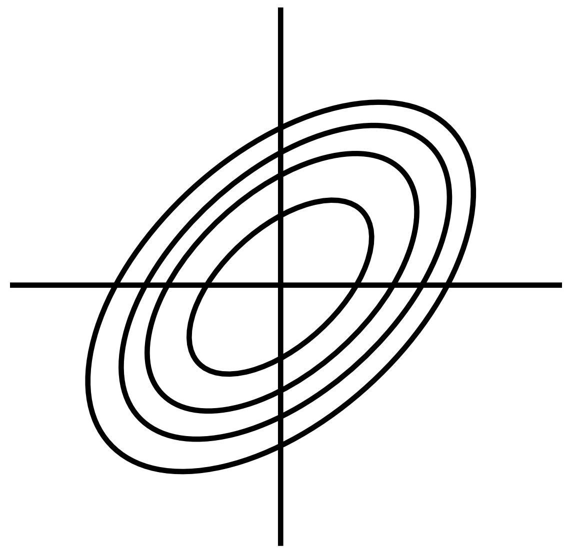

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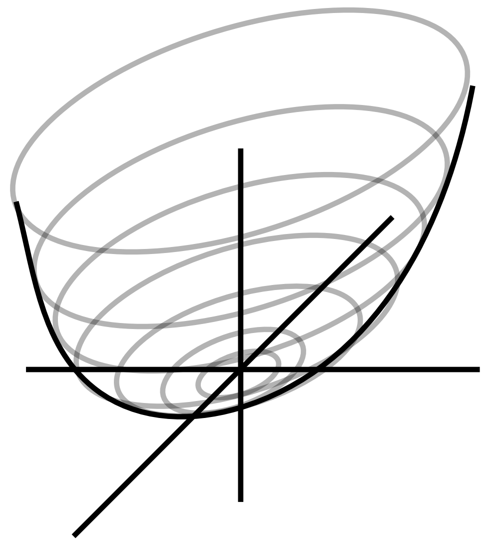

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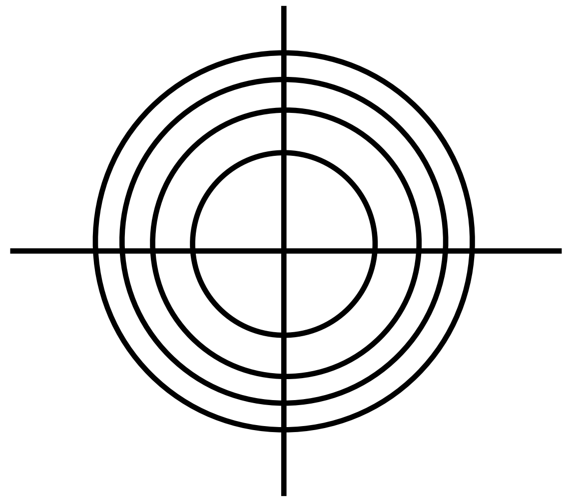

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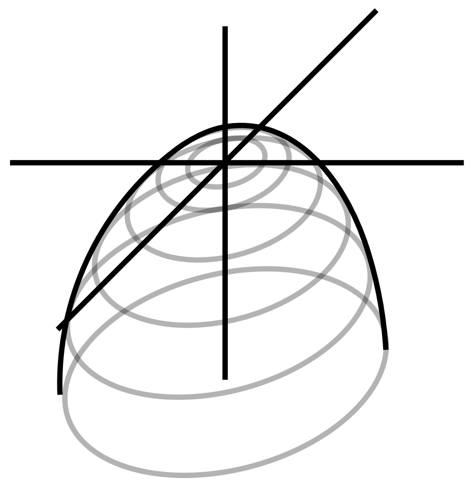

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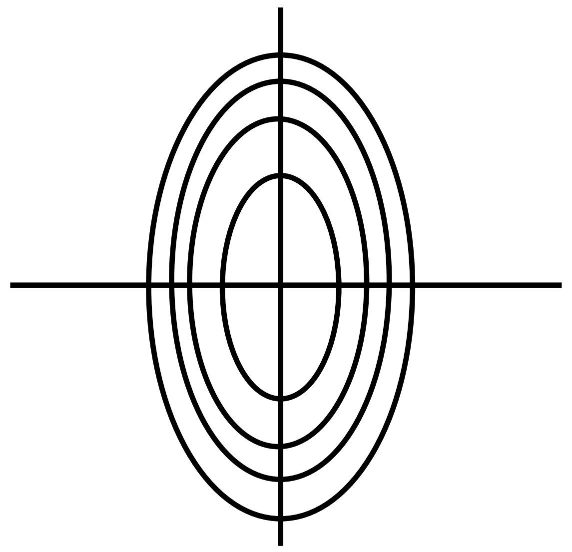

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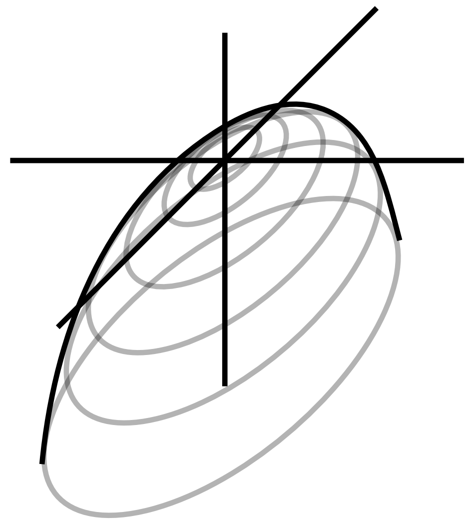

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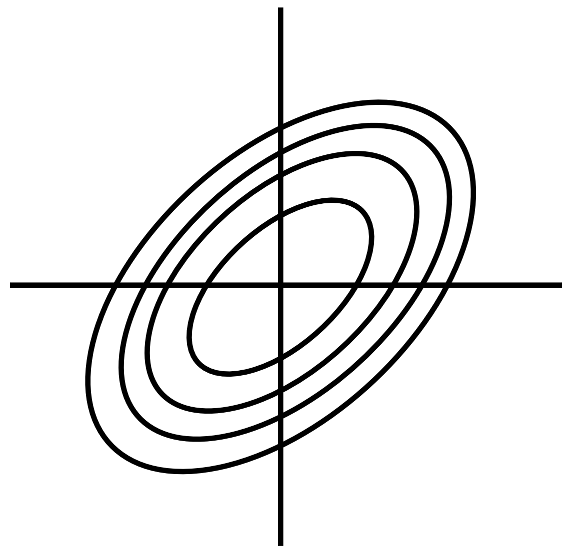

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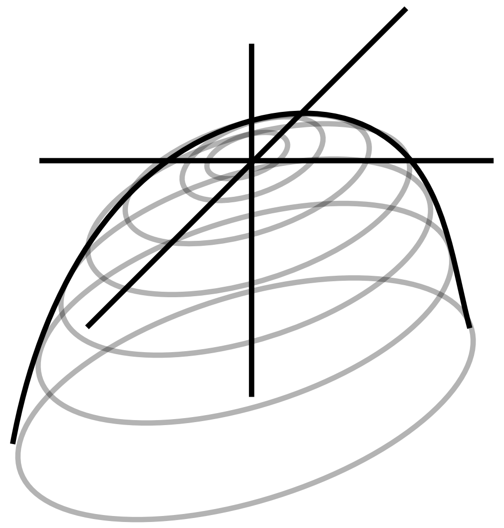

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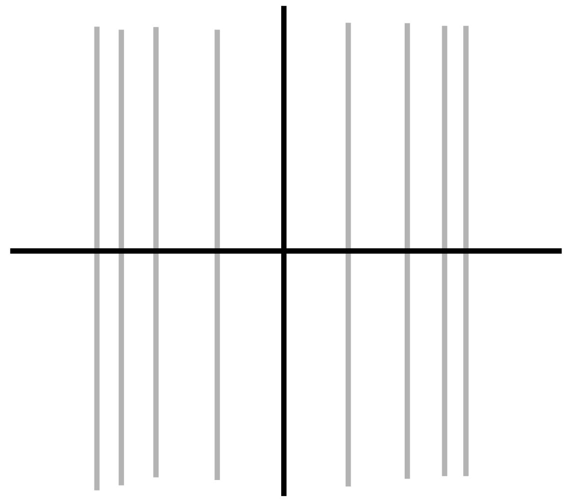

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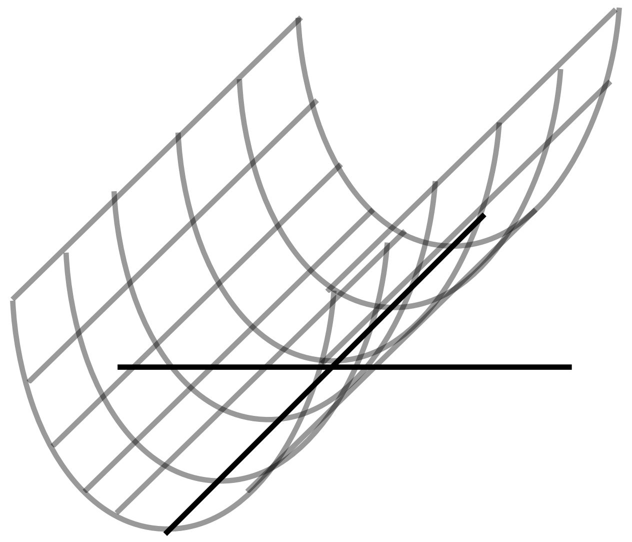
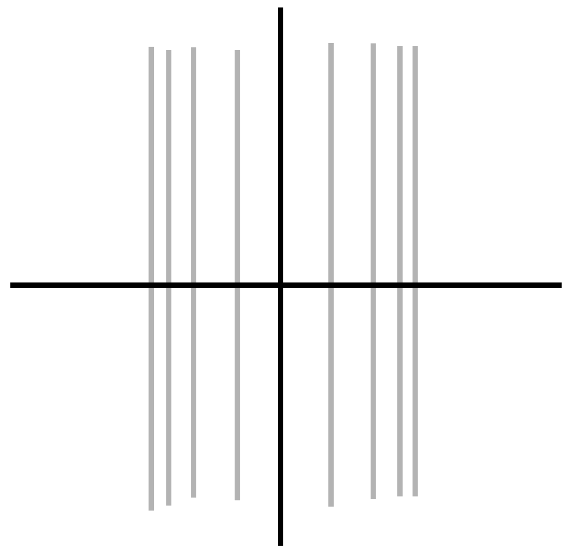


surface

$Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$ diagonal



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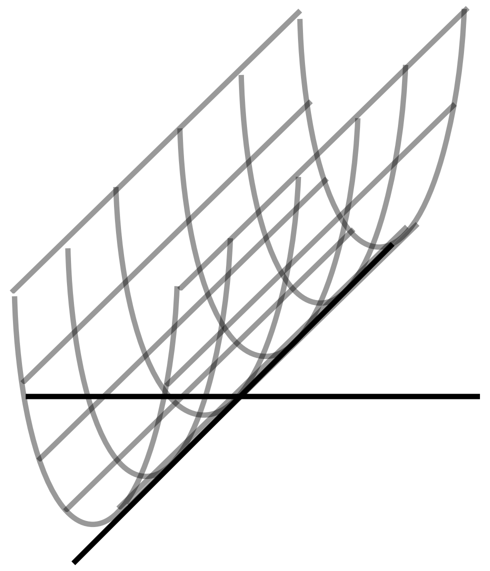

surface

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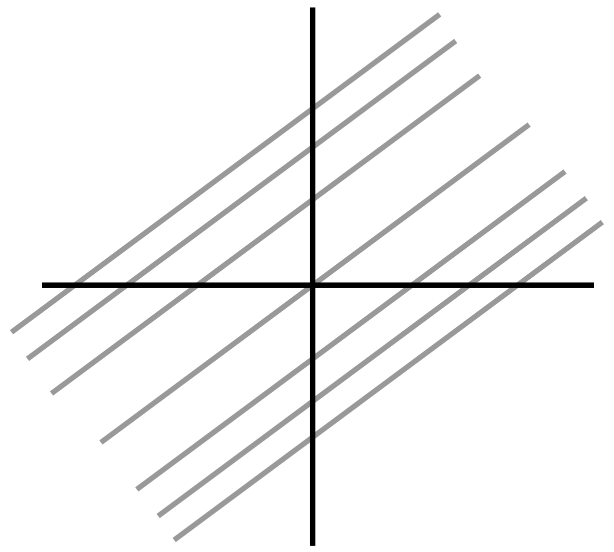

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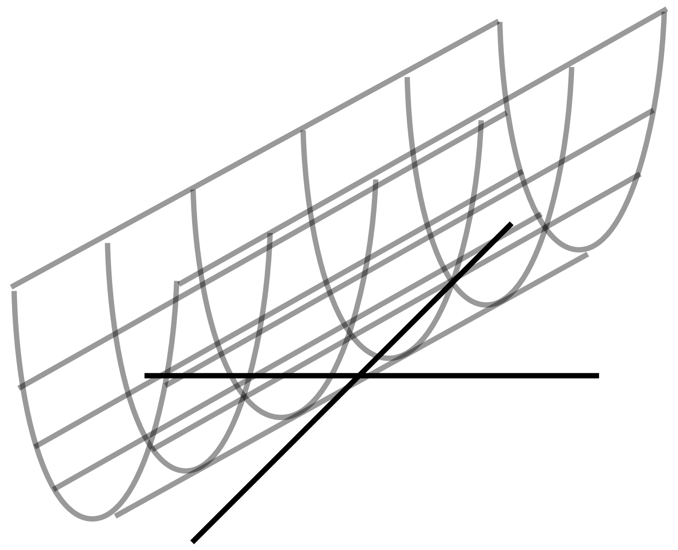

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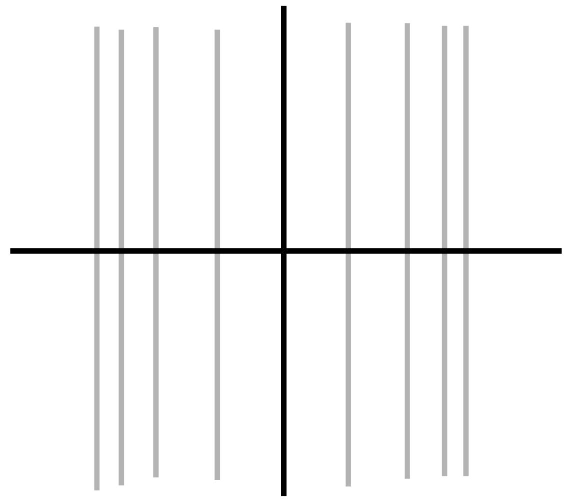

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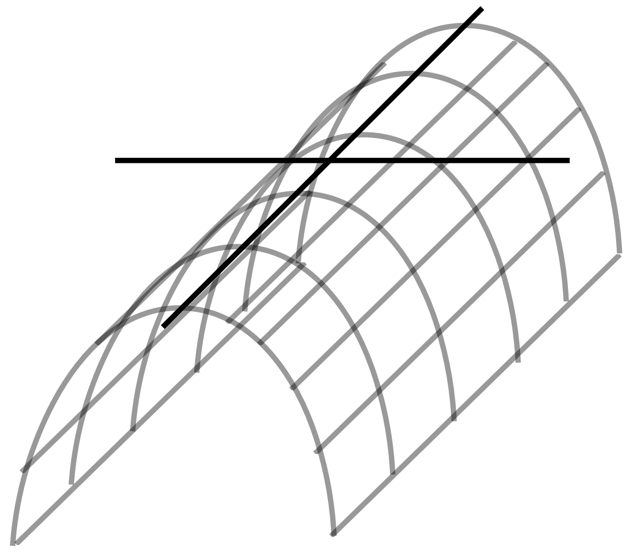
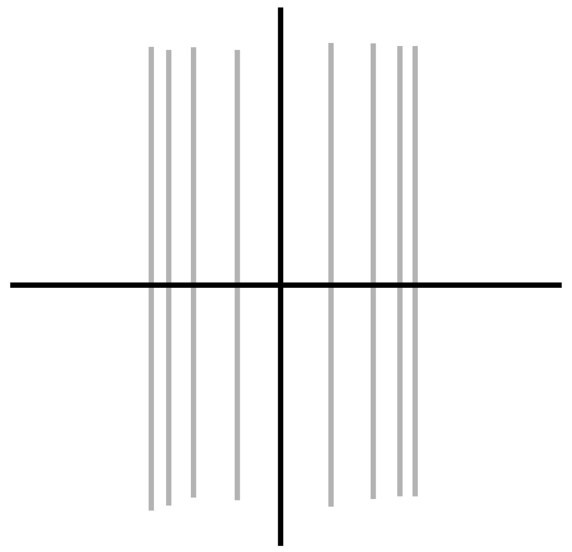


surface

$Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$ diagonal



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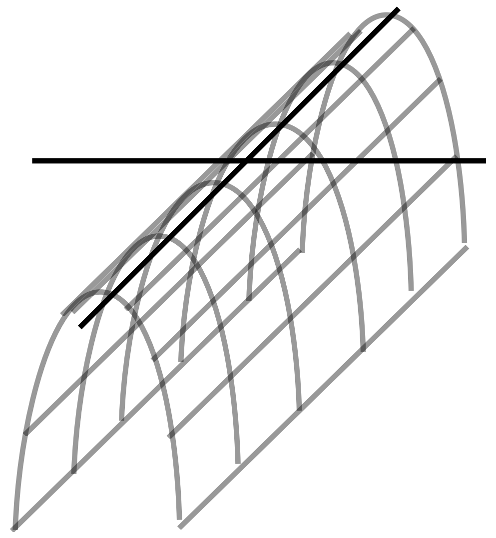

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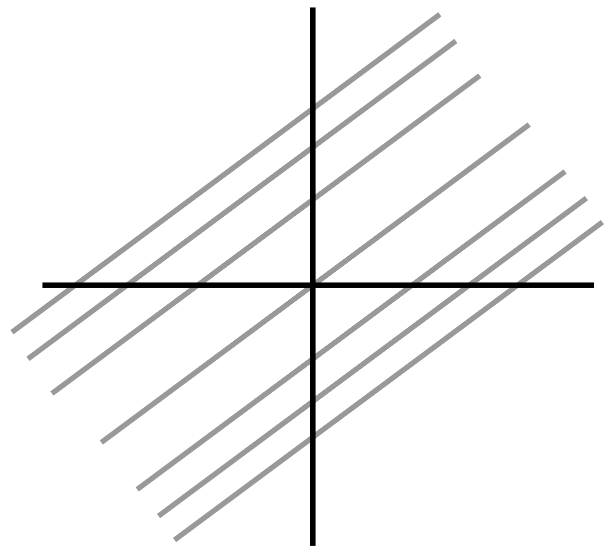

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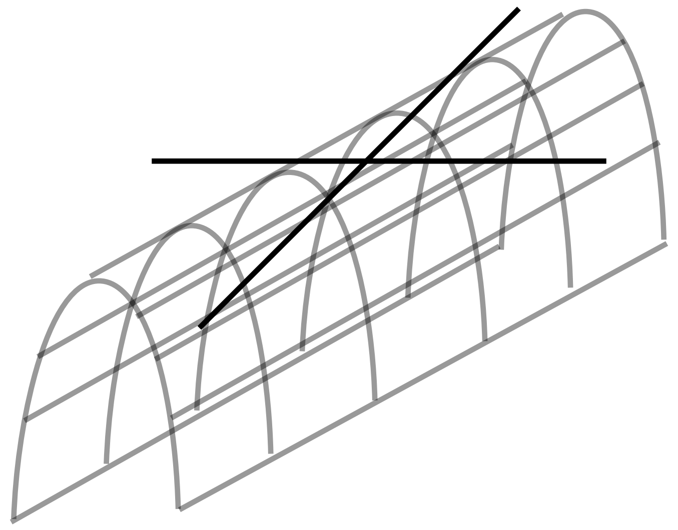
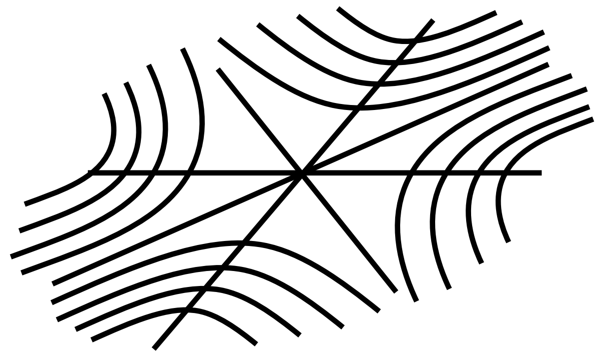


surface

Definite (Symmetric) Matrices



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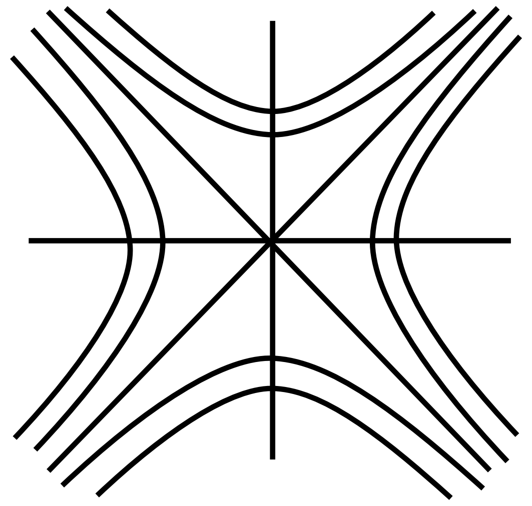
$x \neq 0$

Surfaces: Q indefinite

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



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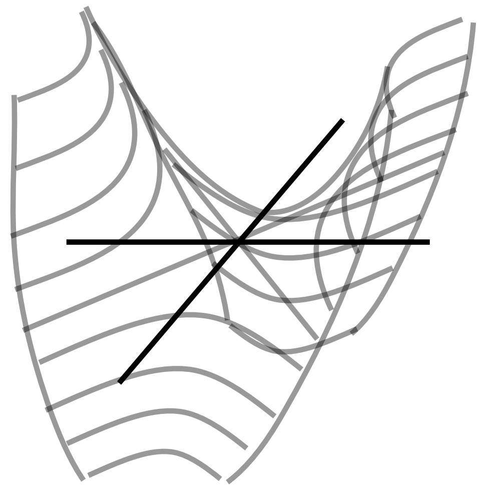

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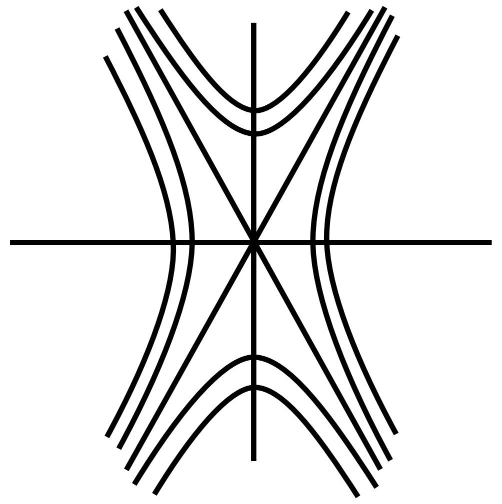
diagonal



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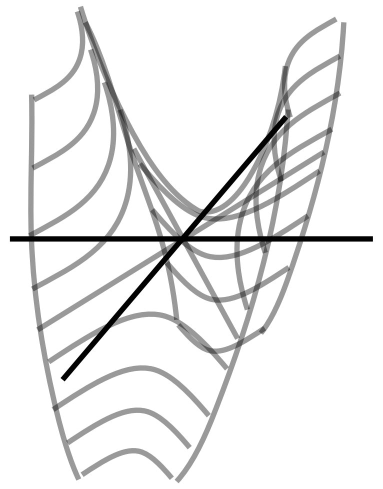

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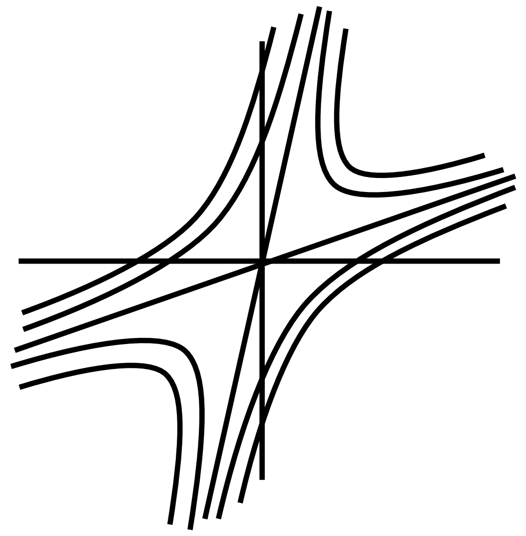
$$Q = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^T$$

general



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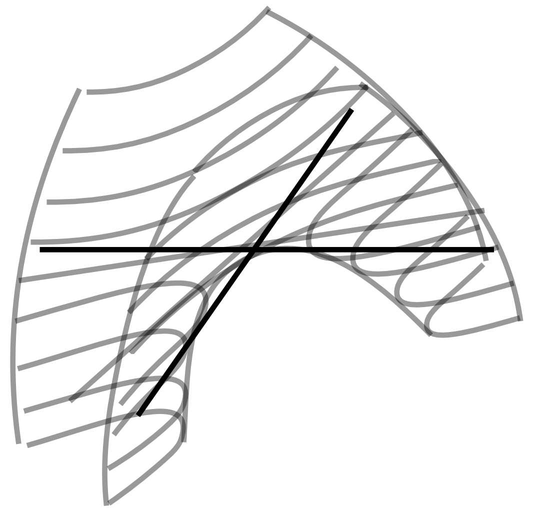

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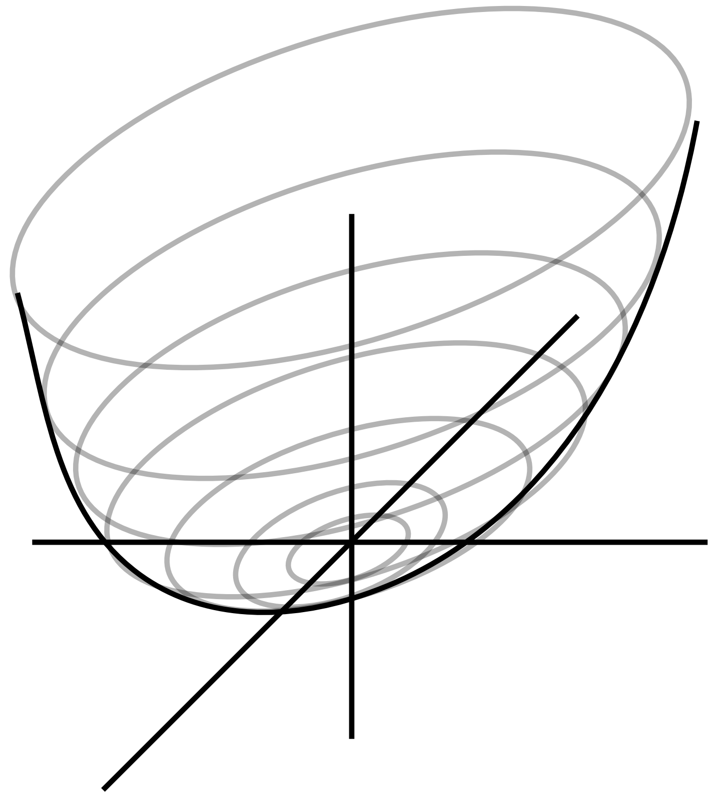

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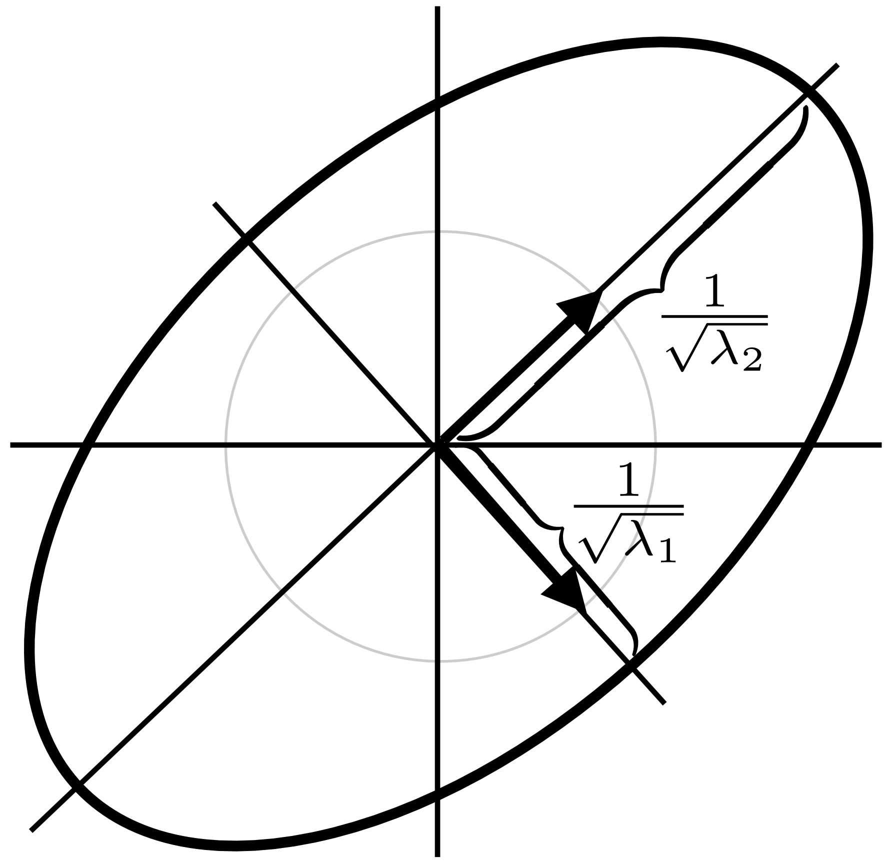
$f(x) = x^TQx = 1$



level sets

$$Q = VDV^T = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$f\left(\frac{1}{\sqrt{\lambda_1}}v_1\right) = \frac{1}{\sqrt{\lambda_1}}v_1^TQv_1\frac{1}{\sqrt{\lambda_1}}$$

$$= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix}^T \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \frac{1}{\sqrt{\lambda_1}}$$

$$= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{\lambda_1}} = \frac{\lambda_1}{(\sqrt{\lambda_1})^2} = 1$$

Definite (Symmetric) Matrices

Quadratic Form: $f(x) = x^TQx \qquad Q \in \mathbb{R}^{n\times n} \quad Q = Q^T$

Definiteness:	Short	Notation	Definition	Analogy	Eigenvalues
Positive definite:	PD	$Q \succ 0$	$x^TQx > 0 \quad \forall\ x \quad x \neq 0$	…positive orthant	$\lambda_i > 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Positive semi-definite	PSD	$Q \succeq 0$	$x^TQx \geq 0 \quad \forall\ x$	…positive orthant w/ boundary	$\lambda_i \geq 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Negative-definite	ND	$Q \prec 0$	$x^TQx < 0 \quad \forall\ x \quad x \neq 0$	…negative orthant	$\lambda_i < 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Negative semi-definite	NSD	$Q \preceq 0$	$x^TQx \leq 0 \quad \forall\ x$	…negative orthant w/ boundary	$\lambda_i \leq 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Indefinite:			$x^TQx > 0$ some x $x^TQx < 0$ some x	…the rest of the space	

Eigenvalue condition proof:

…consider eigenvector coordinates

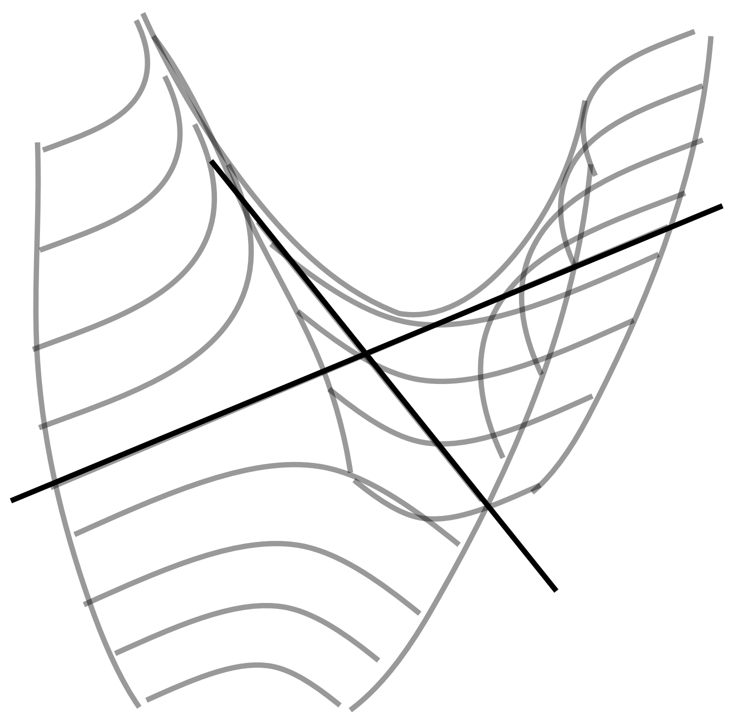
$$x = Vx'$$

since V is invertible… $\quad \forall\ x \iff \forall\ x'$

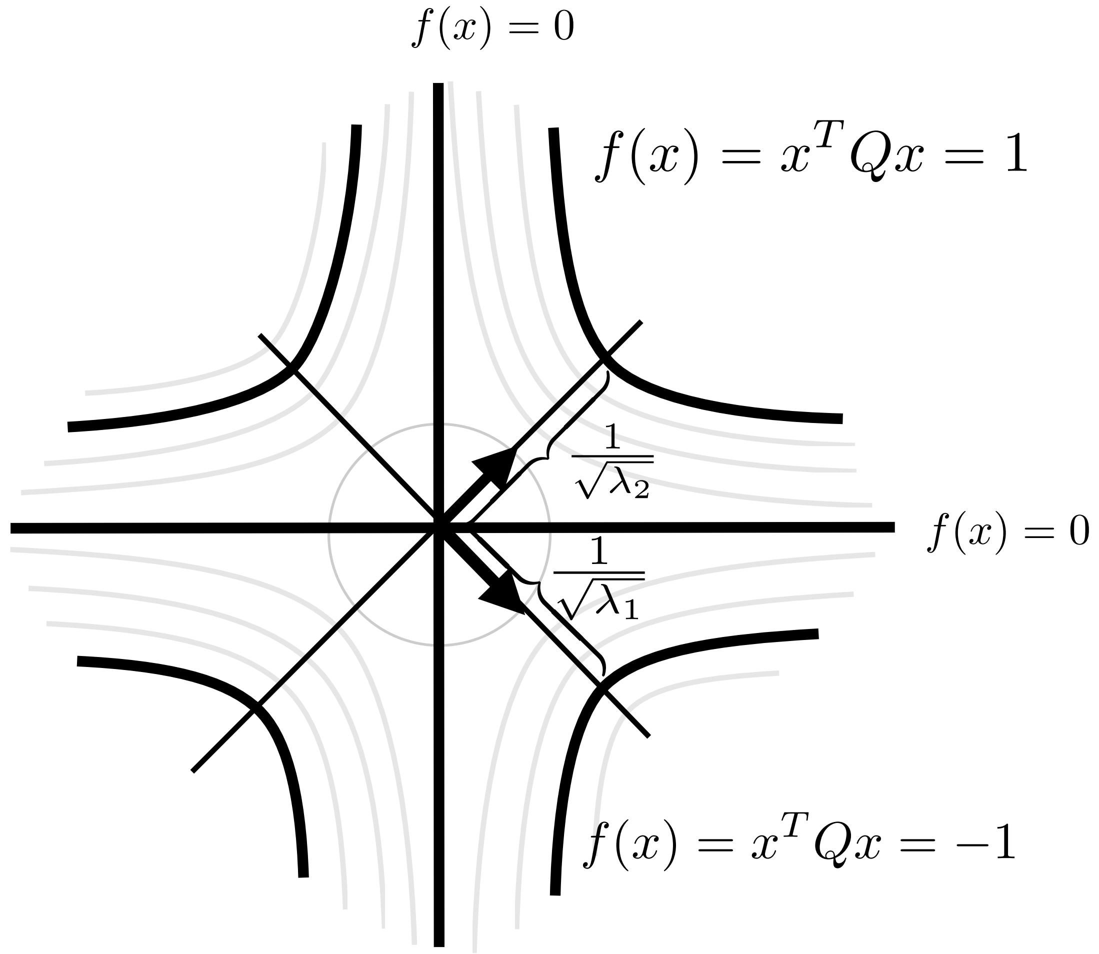
$$x^TQx = xVDV^Tx = x'^TDx' = \sum_i \lambda_i {x'_i}^2$$

$$\sum_i \lambda_i {x'_i}^2 > 0 \quad \underset{x \neq 0}{\forall x'} \iff \lambda_i > 0 \quad \forall\ \lambda_i \in \mathrm{eig}(Q)$$

Surfaces: Q indefinite



surface



level sets

$$Q = VDV^T = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$f\left(\frac{1}{\sqrt{\lambda_1}} v_1\right) = \frac{1}{\sqrt{\lambda_1}} v_1^T Q v_1 \frac{1}{\sqrt{\lambda_1}}$$

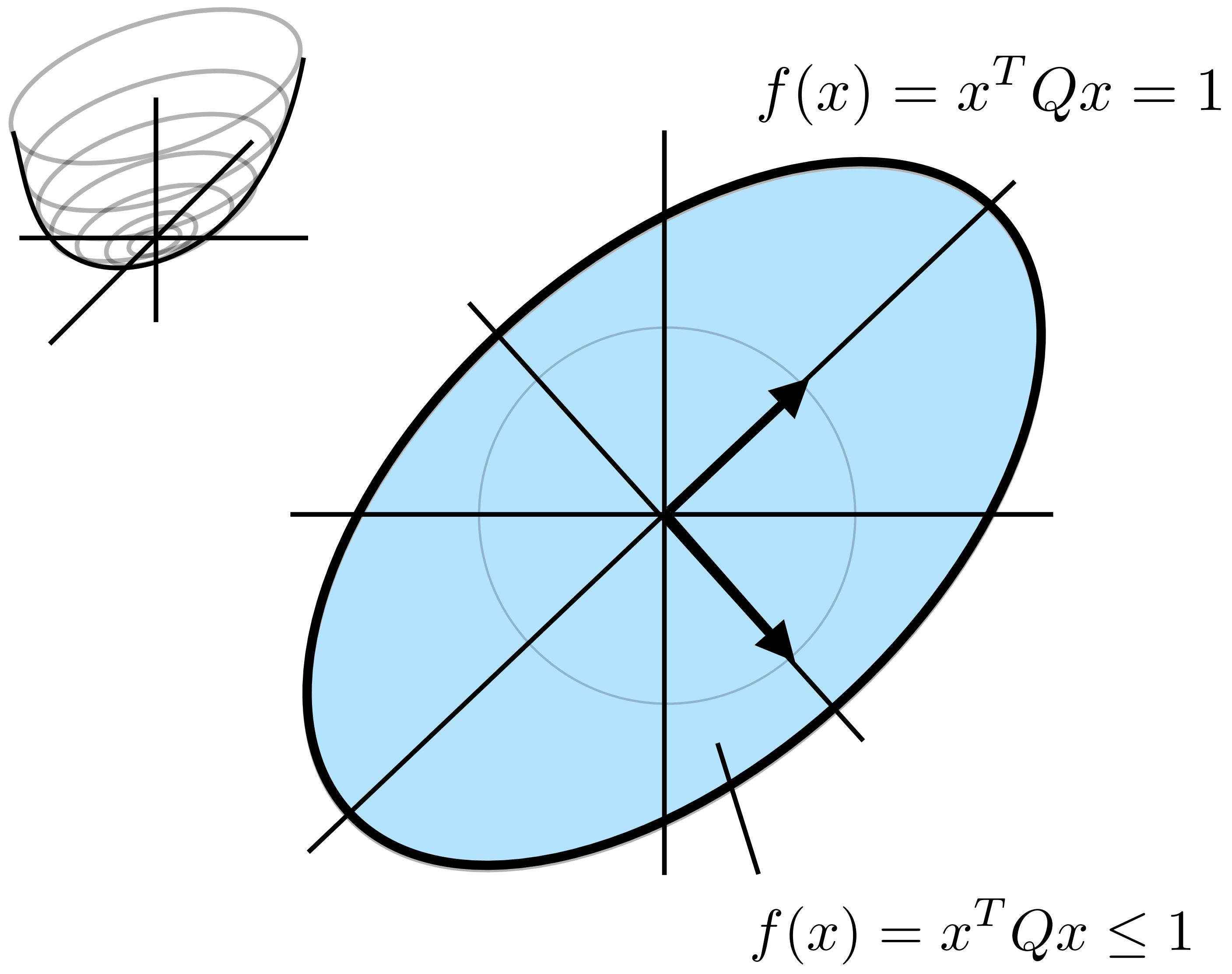
$$= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix}^T \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \frac{1}{\sqrt{\lambda_1}}$$

$$= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{\lambda_1}} = \frac{\lambda_1}{(\sqrt{\lambda_1})^2} = 1$$

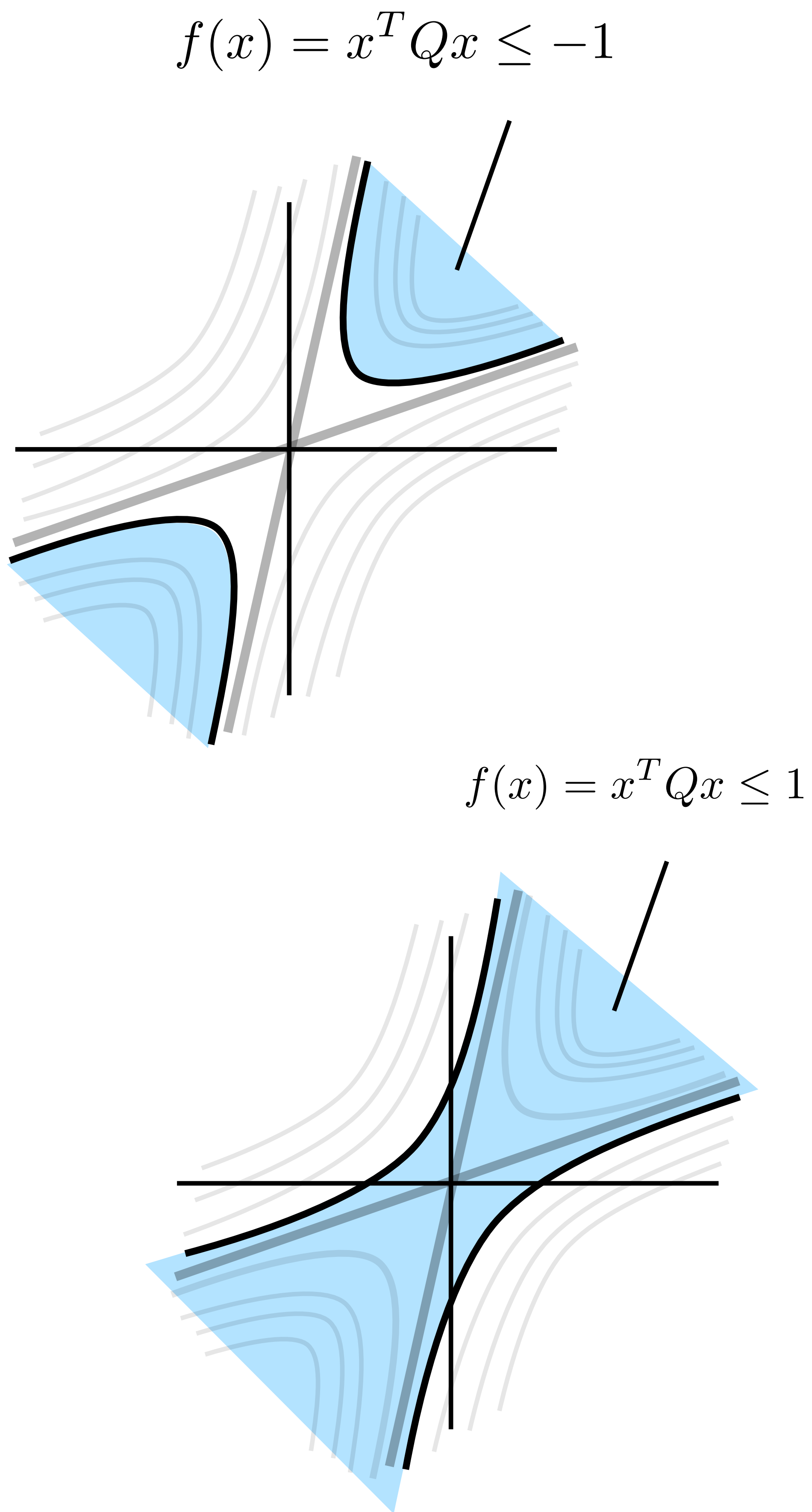
Quadratic Form - Level Sets

Quadratic Form: $f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$

Ellipsoids



Hyperboloids



Cones