Eigenvectors & Eigenvalues:

Linear Algebra

Major sources:

Winter 2022 - Dan Calderone

Eigenvectors & Eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$

Eigenvalue/Eigenvector Problem

A transforms $\mathbb{R}^n$ ***…which directions stay unchanged?*** → **Eigenvectors**

…within those directions…

…how much do vectors get stretched → **Eigenvalues**

Eigenvector Equation

$$Ax = x\lambda$$

Eigenvector $x \in \mathbb{C}^n$ **Eigenvalue** $\lambda \in \mathbb{C}$

Spans of eigenvectors (& generalized eigenvectors) are called **A-invariant subspaces**

Eigenvalues:

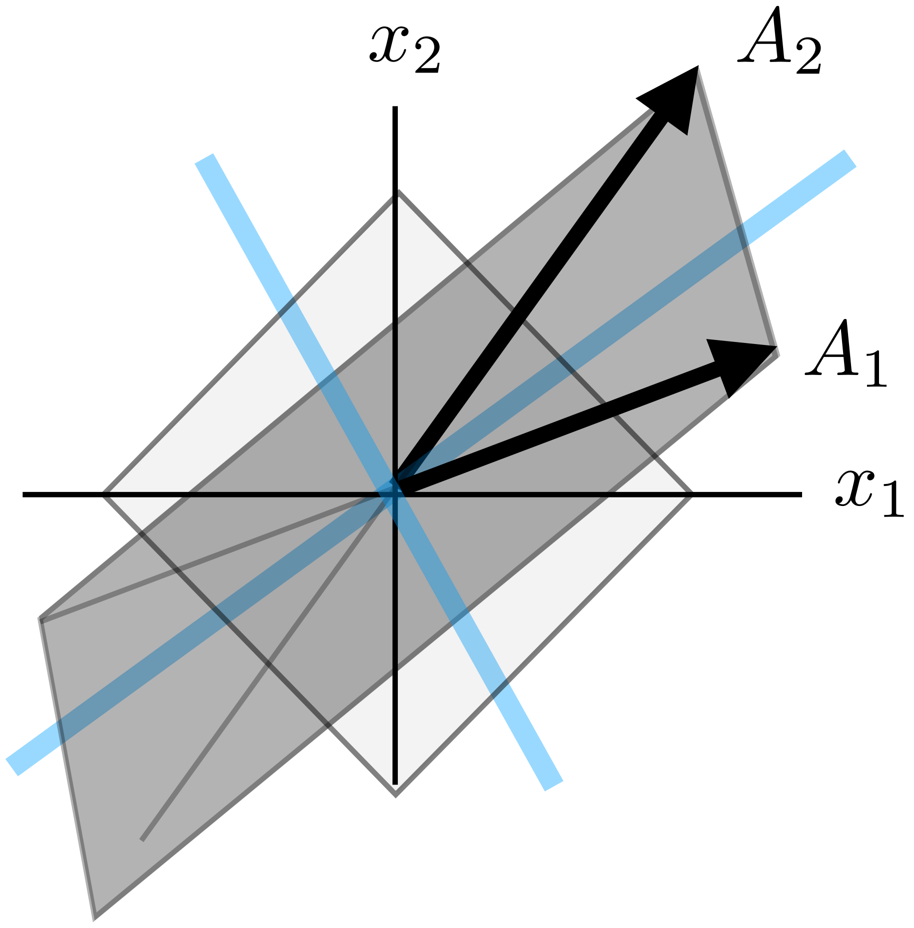
Fundamental property of matrices
Do **not** change with coordinate/similarity transformations

Eigenvectors:

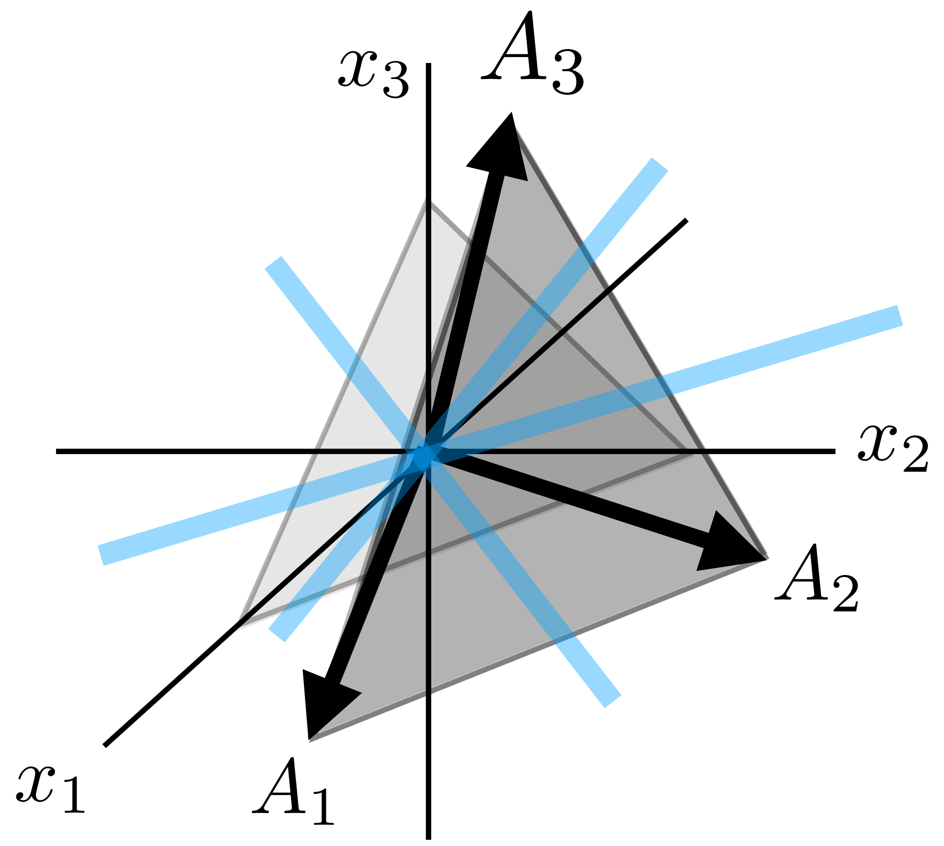
…coordinate dependent (do change with coordinate/similarity transformations)

Picture Examples:

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} \in \mathbb{R}^{2\times 2}$$



$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} \in \mathbb{R}^{3\times 3}$$



Eigenvector/Eigenvalue equation

Square matrix: $A \in \mathbb{R}^{n \times n}$

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0 \qquad v \in \mathcal{N}(A - \lambda I)$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0 \qquad w^* \in \mathcal{N}^L(A - \lambda I) = 0$$

For any eigenvalue, right and left eigenvectors come in pairs
since $A - \lambda I$ drops row and column rank at the same time

Eigenvectors exist only for values of s where $A - sI$ drops rank…

…how to characterize…. ➡ $sI - A$ drops rank only when $\det(sI - A) = 0$

Characteristic Polynomial

$\text{char}_A(s) = \det(sI - A)$ **n-th order polynomial** ➡ **n roots**

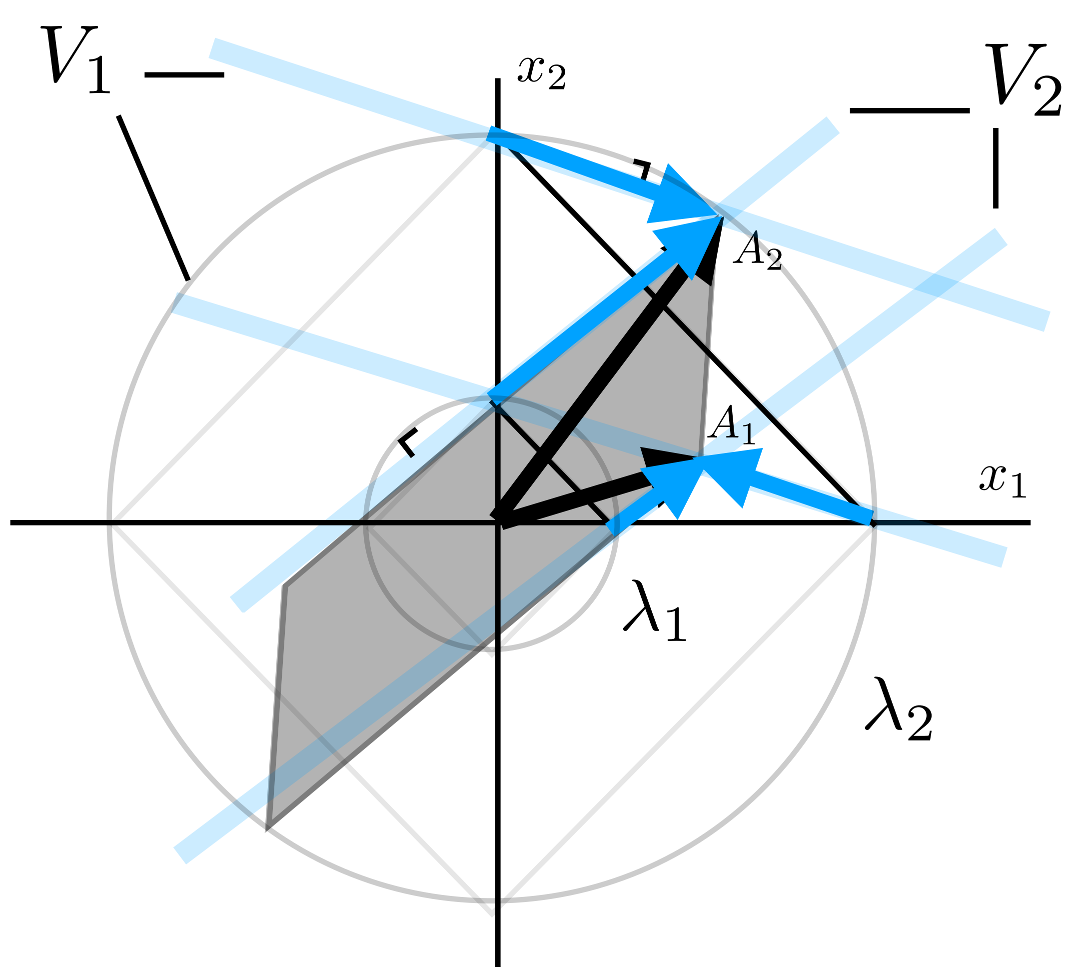
Roots are eigenvalues: λ solution to $\text{char}_A(s) = 0$

Fundamental
Theorem of Algebra

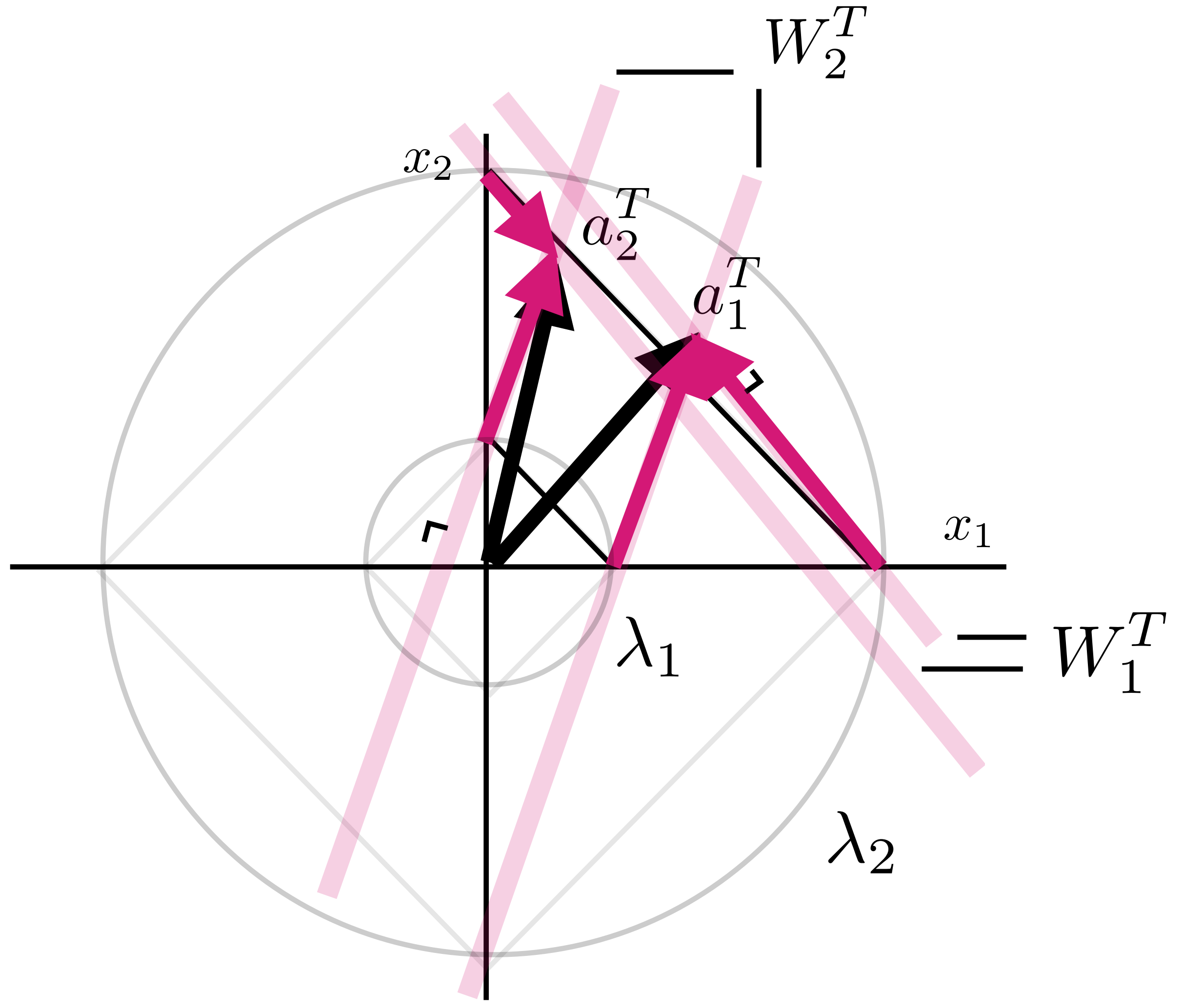
Picture Examples:

$\mathbb{R}^{2\times 2}$

COLUMN GEOMETRY



ROW GEOMETRY



(see below)

Eigenvector/Eigenvalue equation

Square matrix: $A \in \mathbb{R}^{n \times n}$

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0 \qquad v \in \mathcal{N}(A - \lambda I)$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0 \qquad w^* \in \mathcal{N}^L(A - \lambda I) = 0$$

For any eigenvalue, right and left eigenvectors come in pairs since $A - \lambda I$ drops row and column rank at the same time

Eigenvectors exist only for values of s where $A - sI$ drops rank…

…how to characterize…. ➡ $sI - A$ drops rank only when $\det(sI - A) = 0$

Characteristic Polynomial

$\mathrm{char}_A(s) = \det(sI - A)$ **n-th order polynomial** ➡ **n roots**

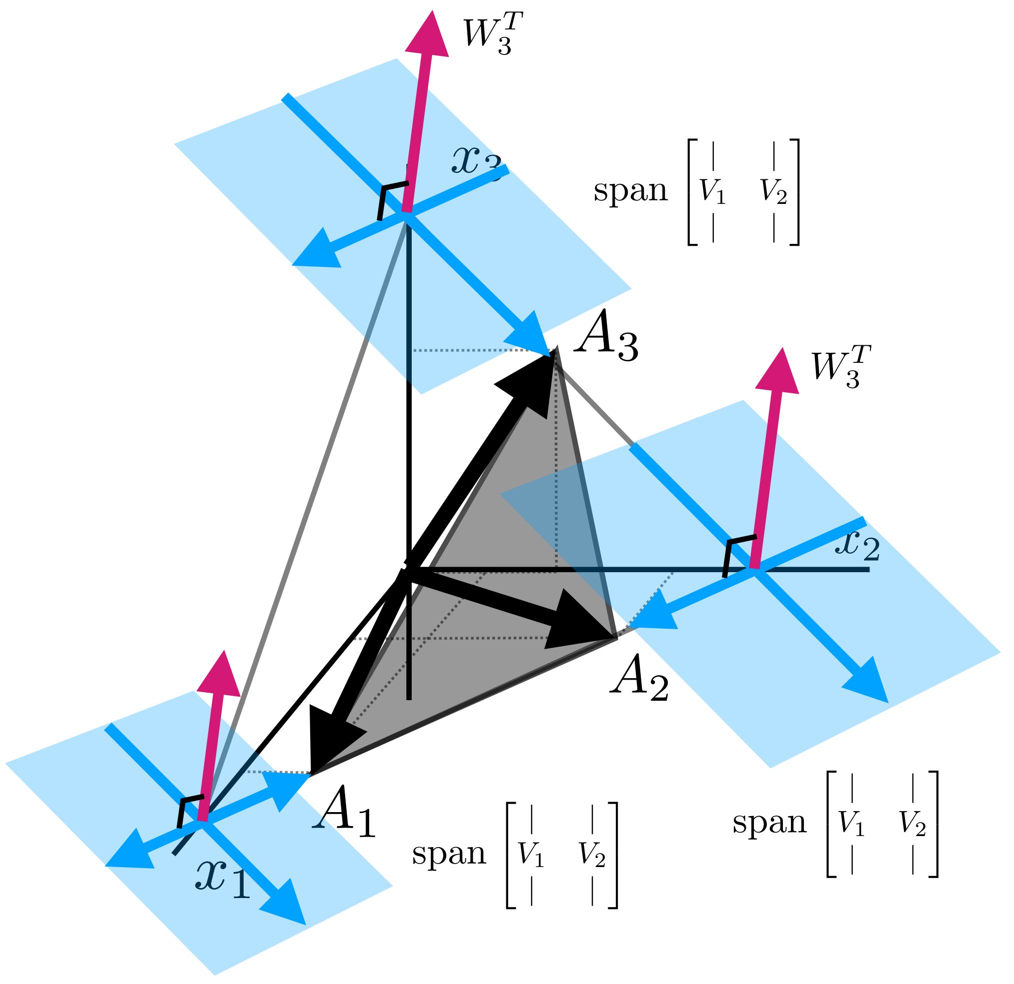
Roots are eigenvalues: λ solution to $\mathrm{char}_A(s) = 0$

Fundamental Theorem of Algebra

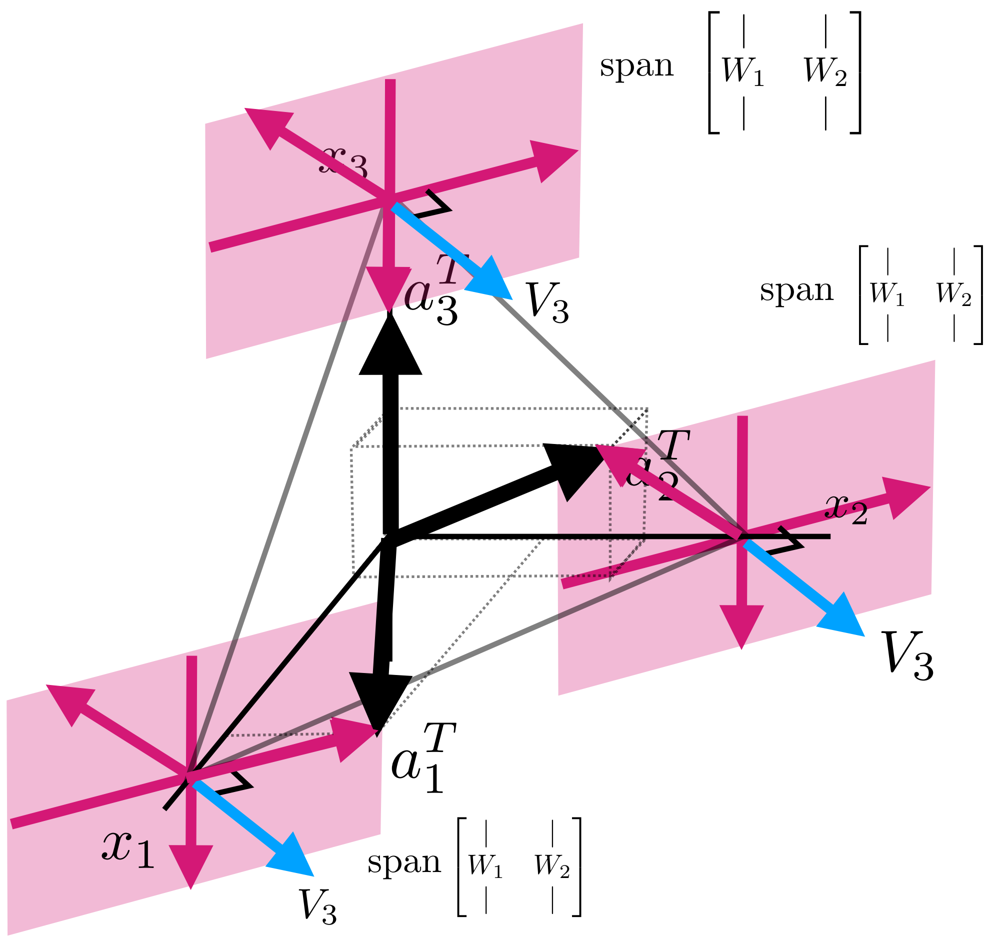
Picture Examples:

$\mathbb{R}^{3\times3}$

COLUMN GEOMETRY



ROW GEOMETRY



(see below)

Eigenvector/Eigenvalue Picture

Eigenvectors exist only for values of s where $A - sI$ drops rank…

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0$$

Characteristic Polynomial

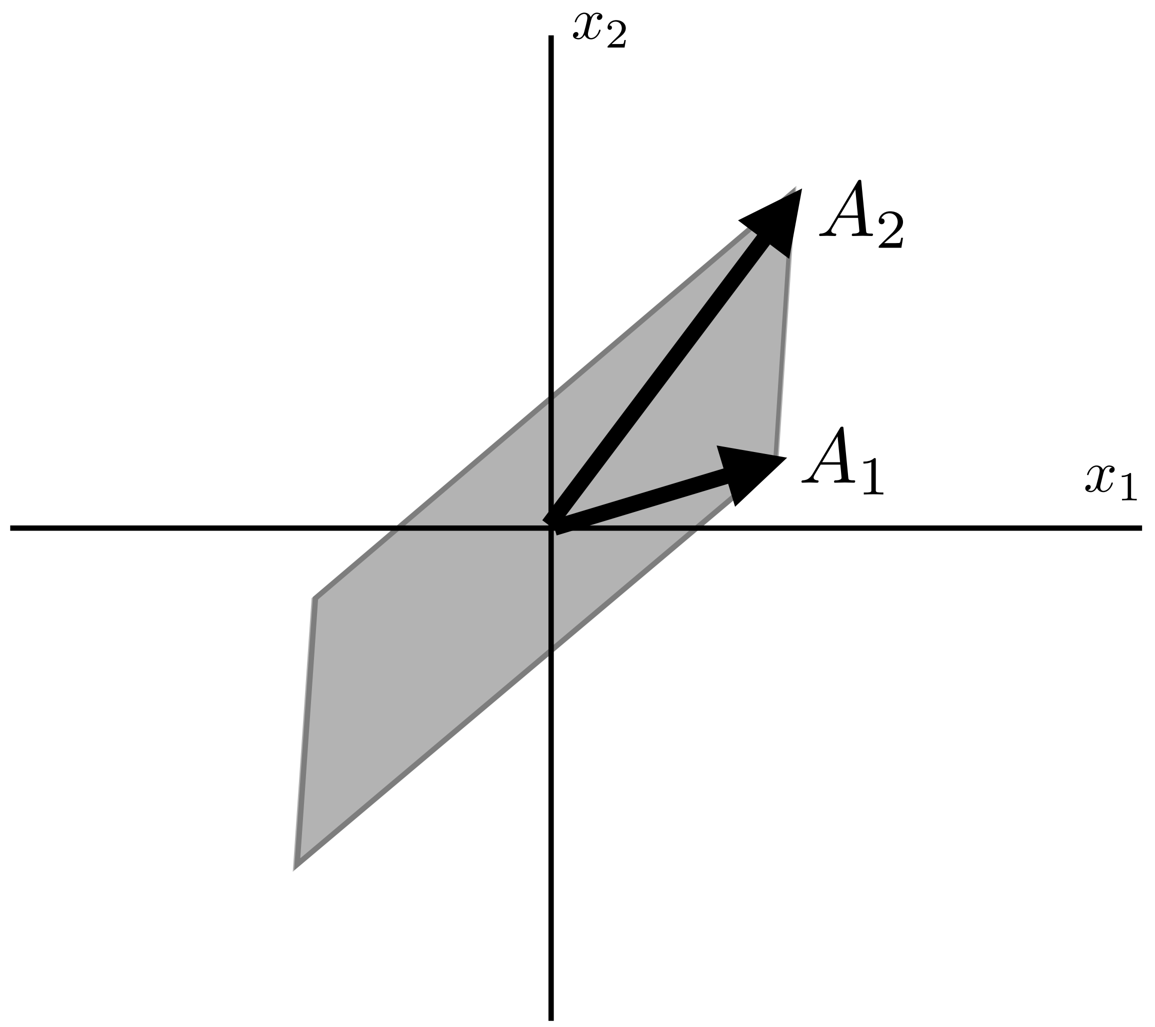
$sI - A$ drops rank only when

$$\mathrm{char}_A(s) = \det(sI - A) = 0$$

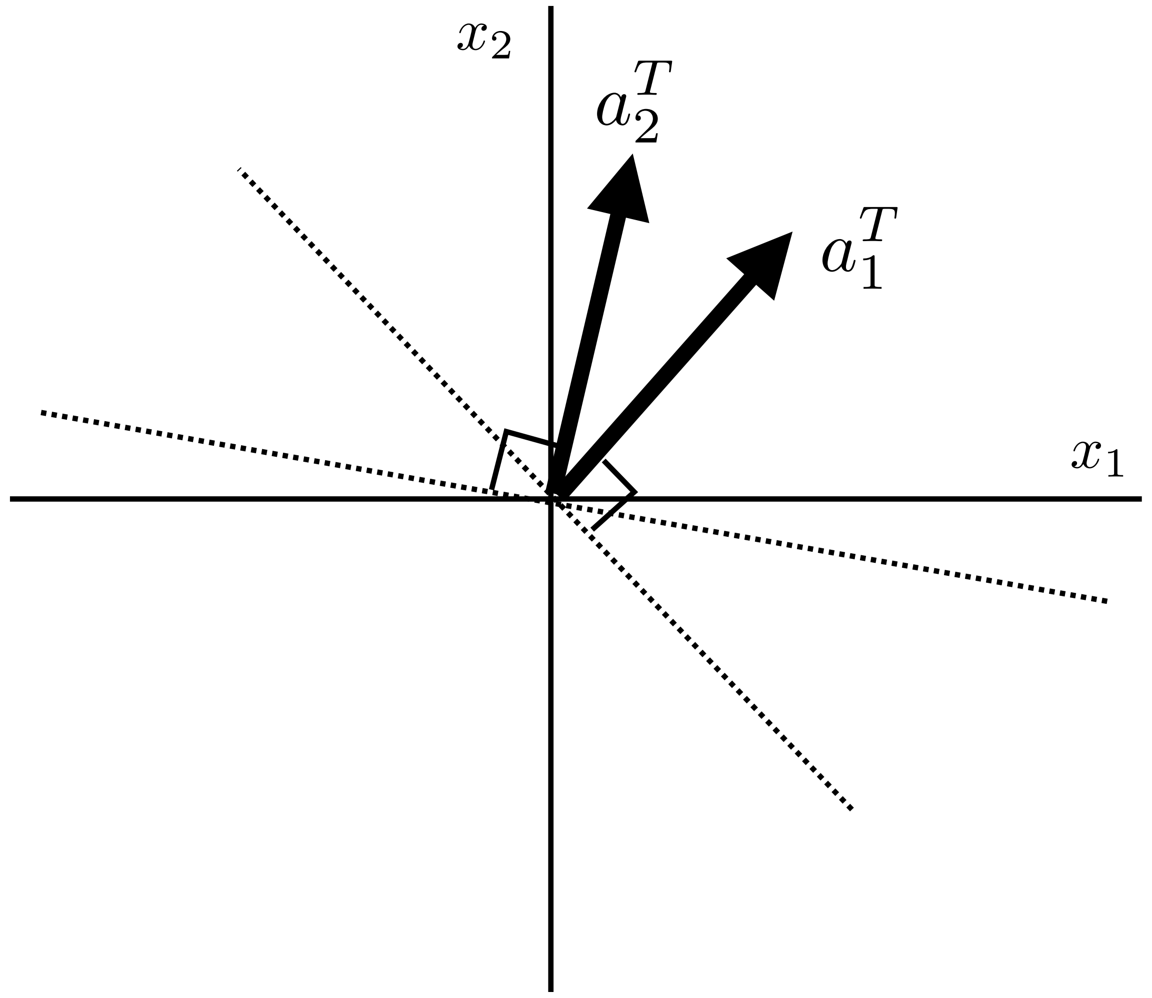
$$A \in \mathbb{R}^{2\times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T- \\ -a_2^T- \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$

COLUMN GEOMETRY



ROW GEOMETRY



Eigenvector/Eigenvalue Picture

Eigenvectors exist only for values of s where $A - sI$ drops rank...

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0$$

Characteristic Polynomial

$sI - A$ drops rank only when

$$\mathrm{char}_A(s) = \det(sI - A) = 0$$

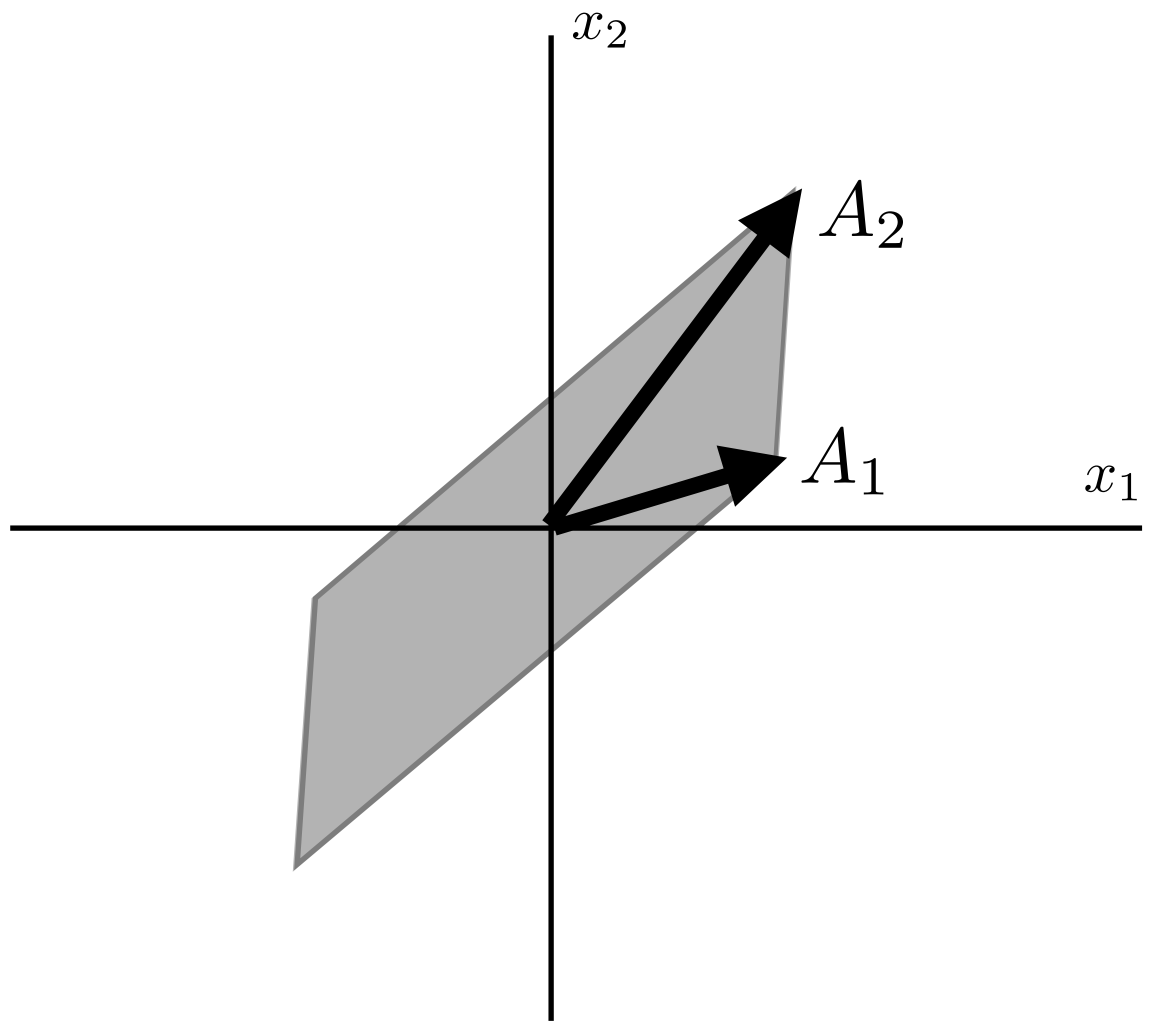
$$A \in \mathbb{R}^{2\times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T- \\ -a_2^T- \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$

COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$

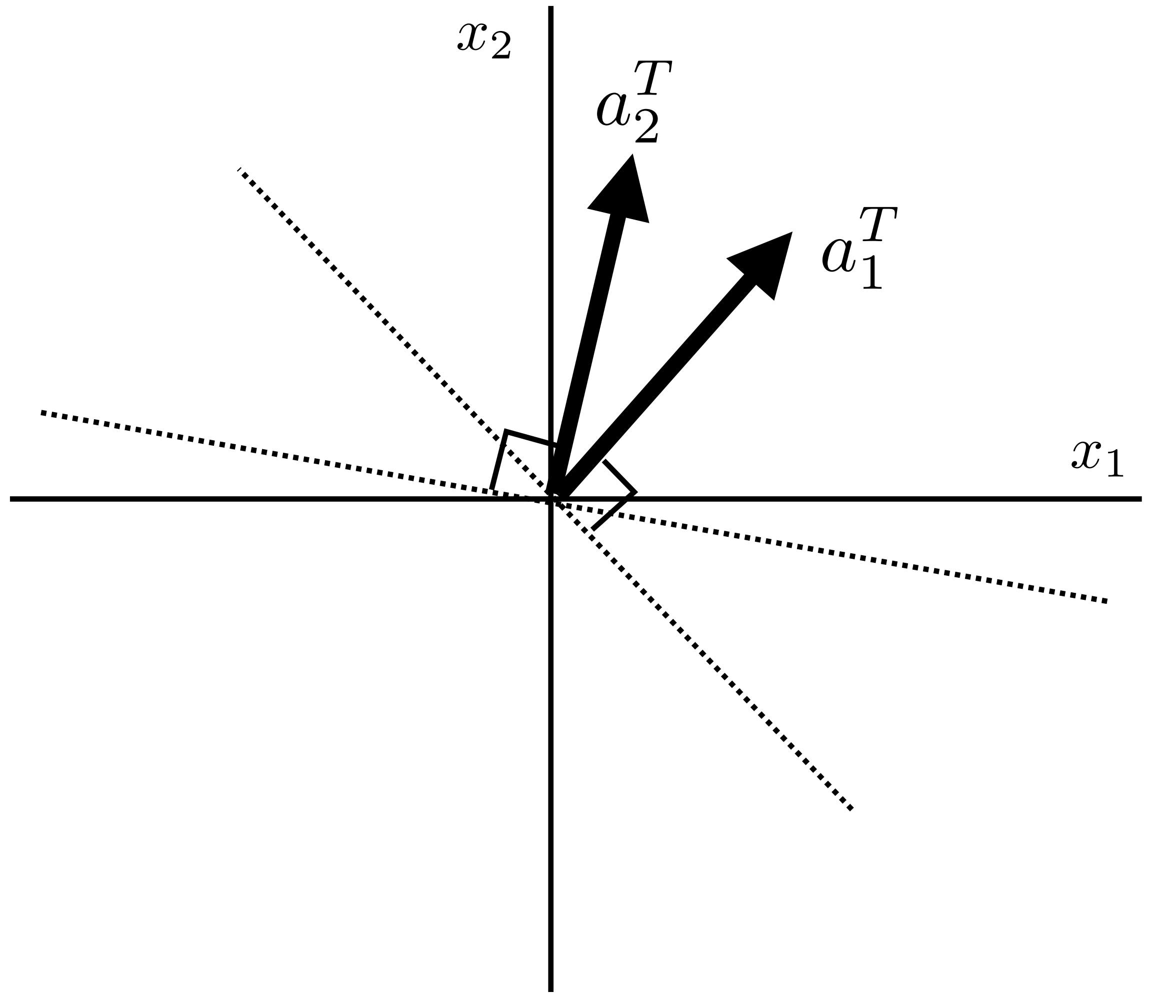
drops rank for eigenvalue



ROW GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$

drops rank for eigenvalue



Eigenvector/Eigenvalue Picture

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of s where $A - sI$ drops rank...

Characteristic Polynomial

$sI - A$ drops rank only when

$$\mathrm{char}_A(s) = \det(sI - A) = 0$$

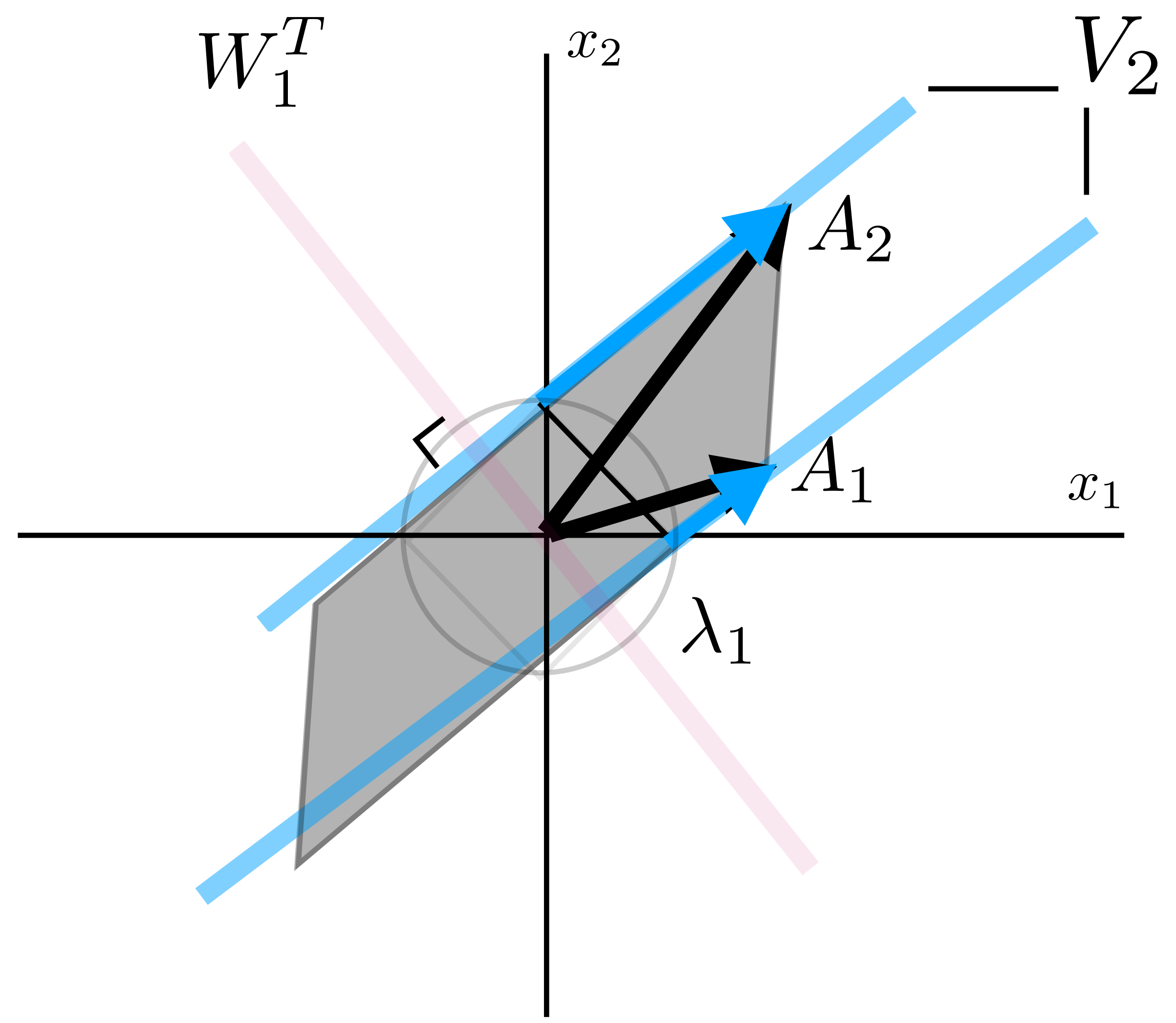
$$A \in \mathbb{R}^{2\times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T- \\ -a_2^T- \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$

COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda_1 I \end{bmatrix}$$

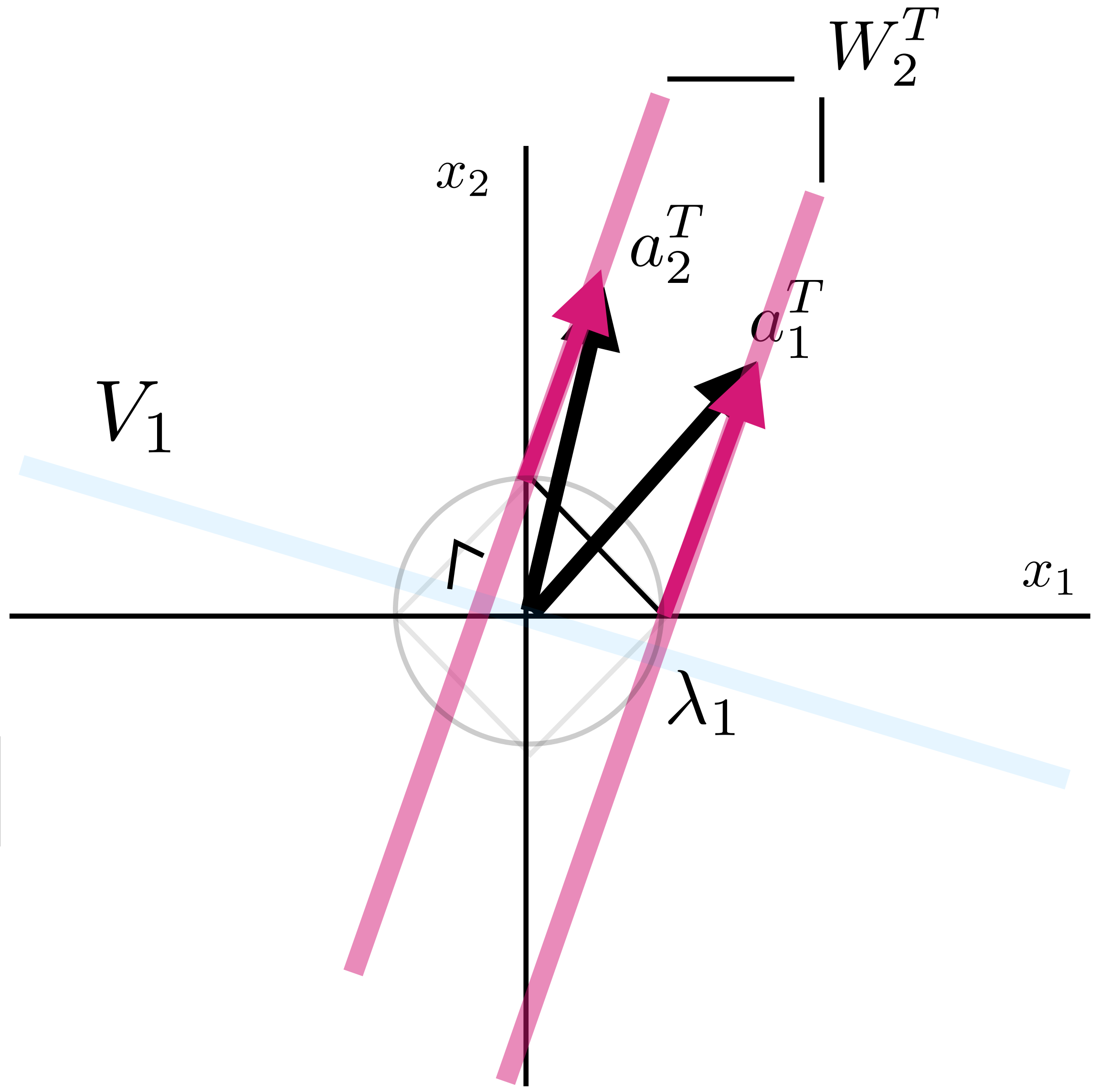
$$\begin{bmatrix} | \\ V_2 \\ | \end{bmatrix} \begin{bmatrix} \lambda_2 - \lambda_1 \end{bmatrix} \begin{bmatrix} - & W_2^T & - \end{bmatrix}$$



ROW GEOMETRY

$$\begin{bmatrix} A - \lambda_1 I \end{bmatrix}$$

$$\begin{bmatrix} | \\ V_2 \\ | \end{bmatrix} \begin{bmatrix} \lambda_2 - \lambda_1 \end{bmatrix} \begin{bmatrix} - & W_2^T & - \end{bmatrix}$$



Eigenvector/Eigenvalue Picture

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^*A = w^*\lambda \qquad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of s where $A - sI$ drops rank...

Characteristic Polynomial

$sI - A$ drops rank only when

$$\text{char}_A(s) = \det(sI - A) = 0$$

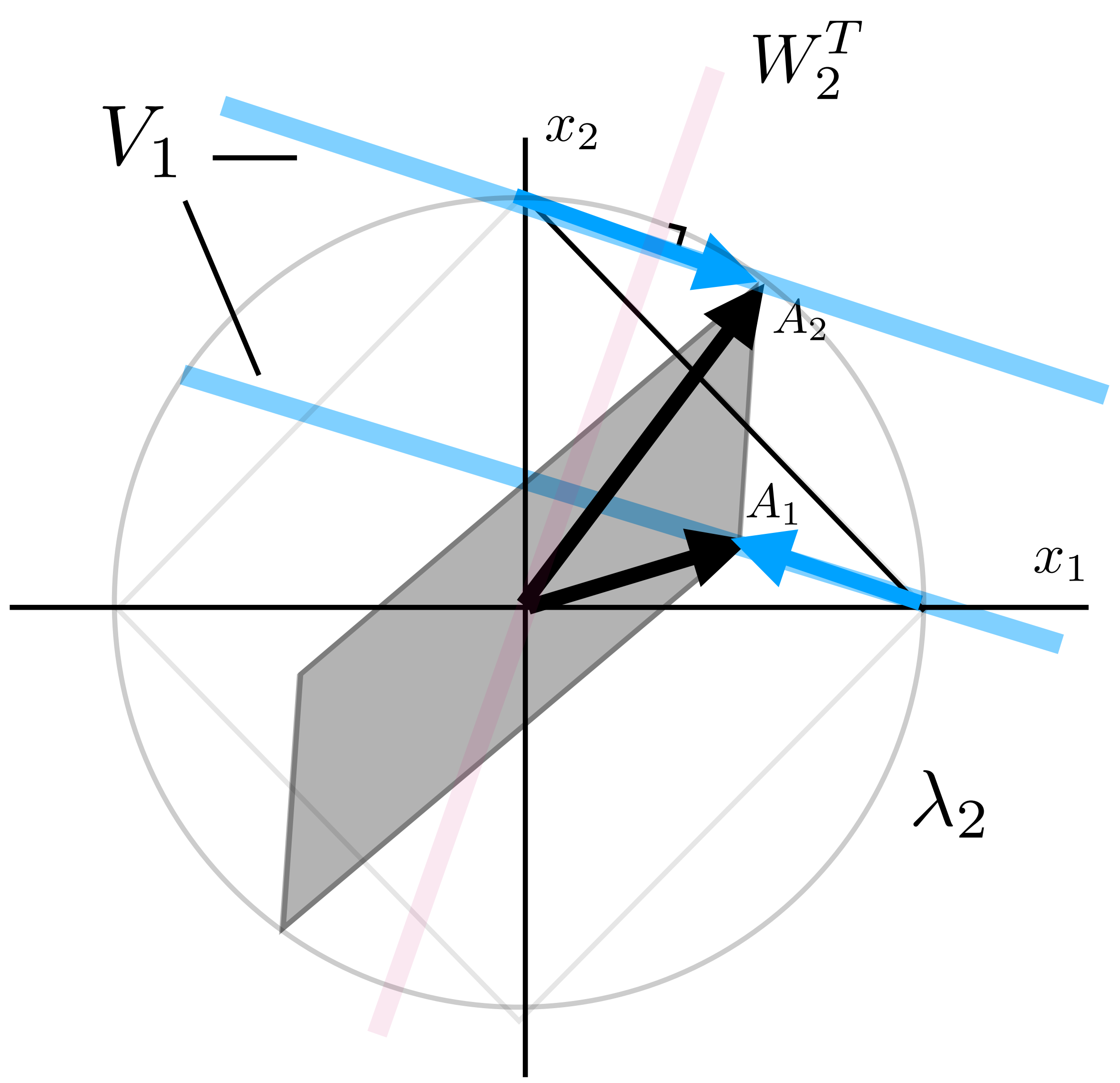
$$A \in \mathbb{R}^{2\times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T- \\ -a_2^T- \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$

COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda_2 I \end{bmatrix}$$

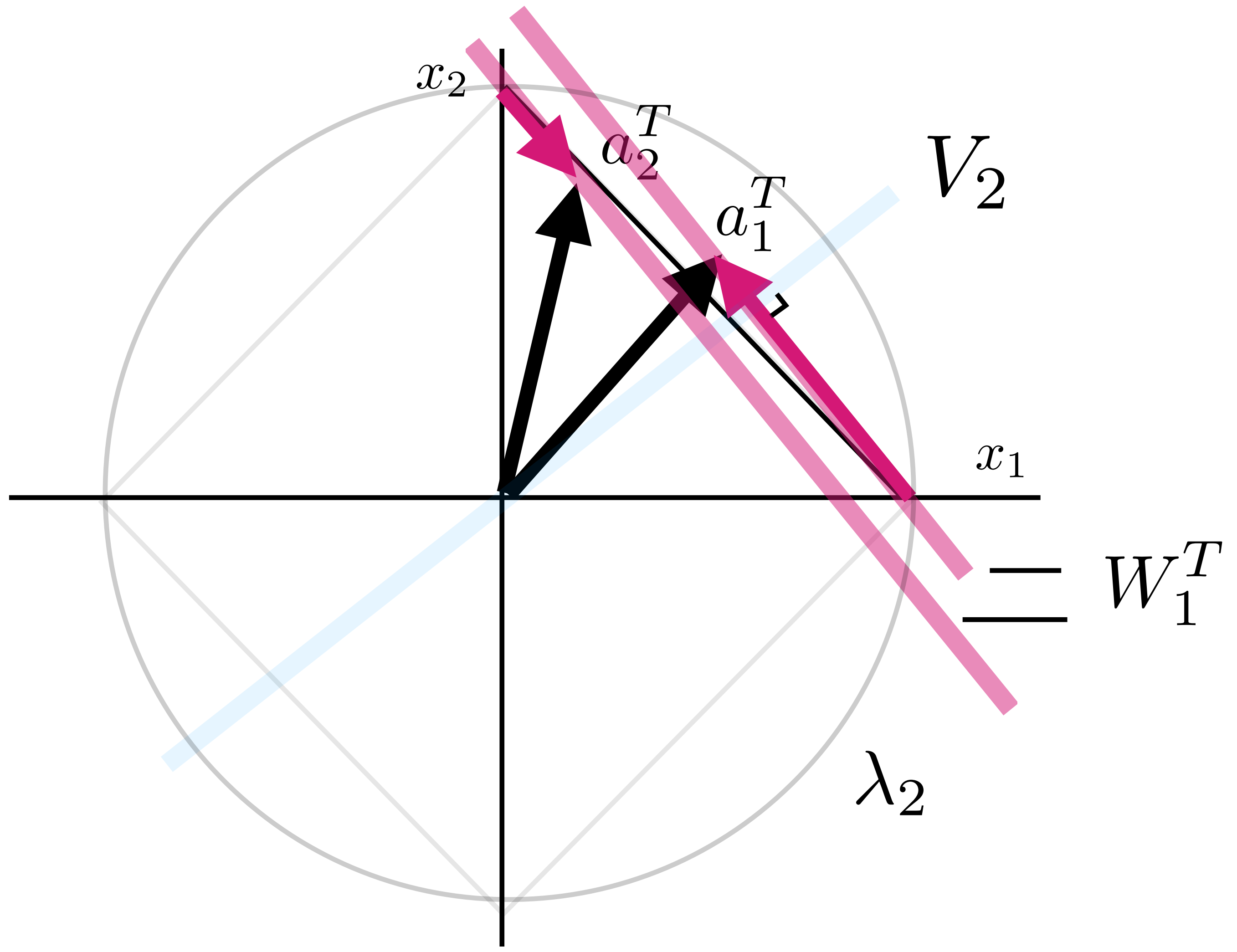
$$\begin{bmatrix} | \\ V_1 \\ | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \end{bmatrix}$$



ROW GEOMETRY

$$\begin{bmatrix} A - \lambda_2 I \end{bmatrix}$$

$$\begin{bmatrix} | \\ V_1 \\ | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \end{bmatrix}$$



Eigenvector/Eigenvalue Picture

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of s where $A - sI$ drops rank...

Characteristic Polynomial

$sI - A$ drops rank only when

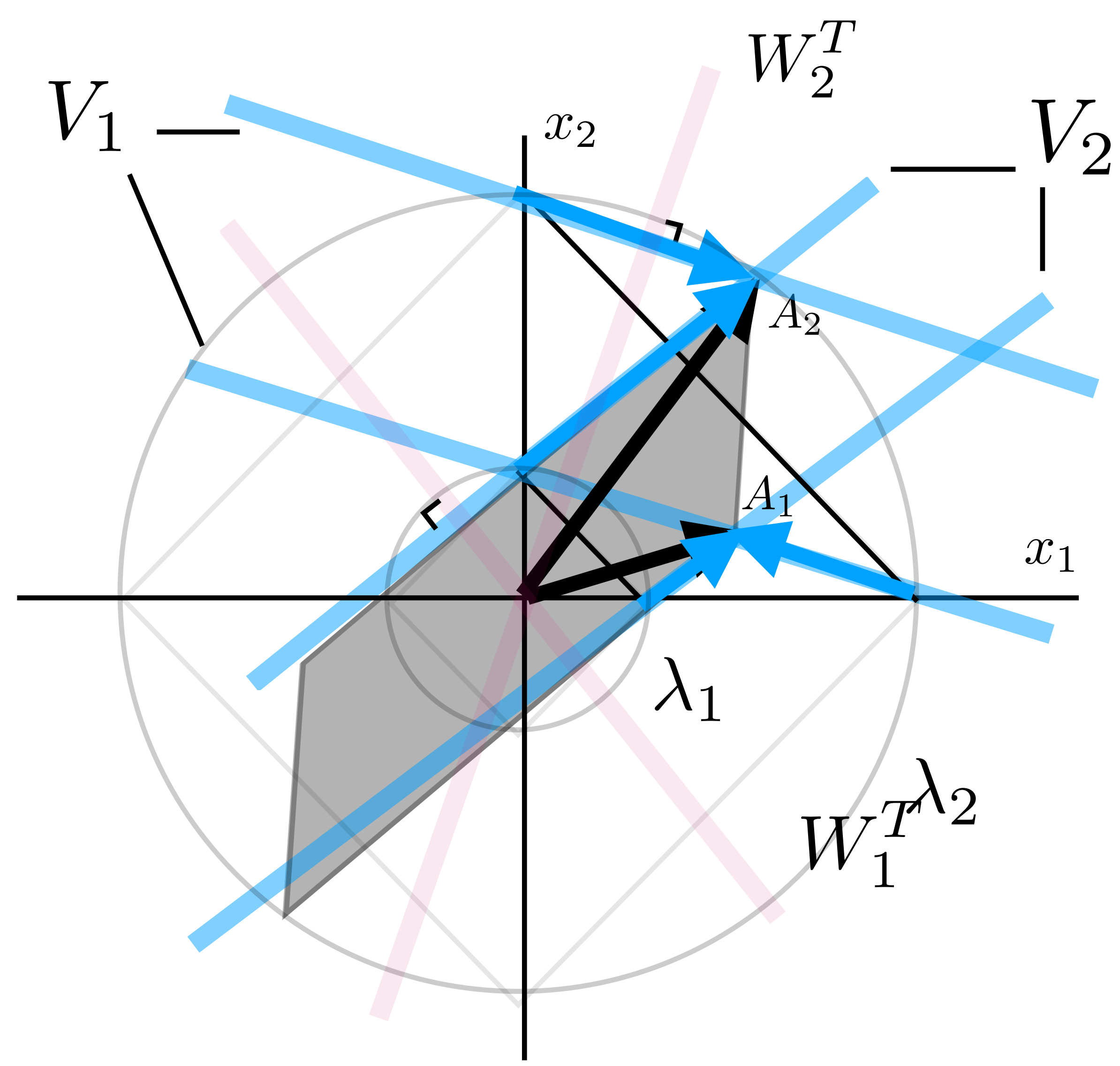
$$\text{char}_A(s) = \det(sI - A) = 0$$

$$A \in \mathbb{R}^{2\times 2}$$

$$A = \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_1^T- \\ -a_2^T- \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$

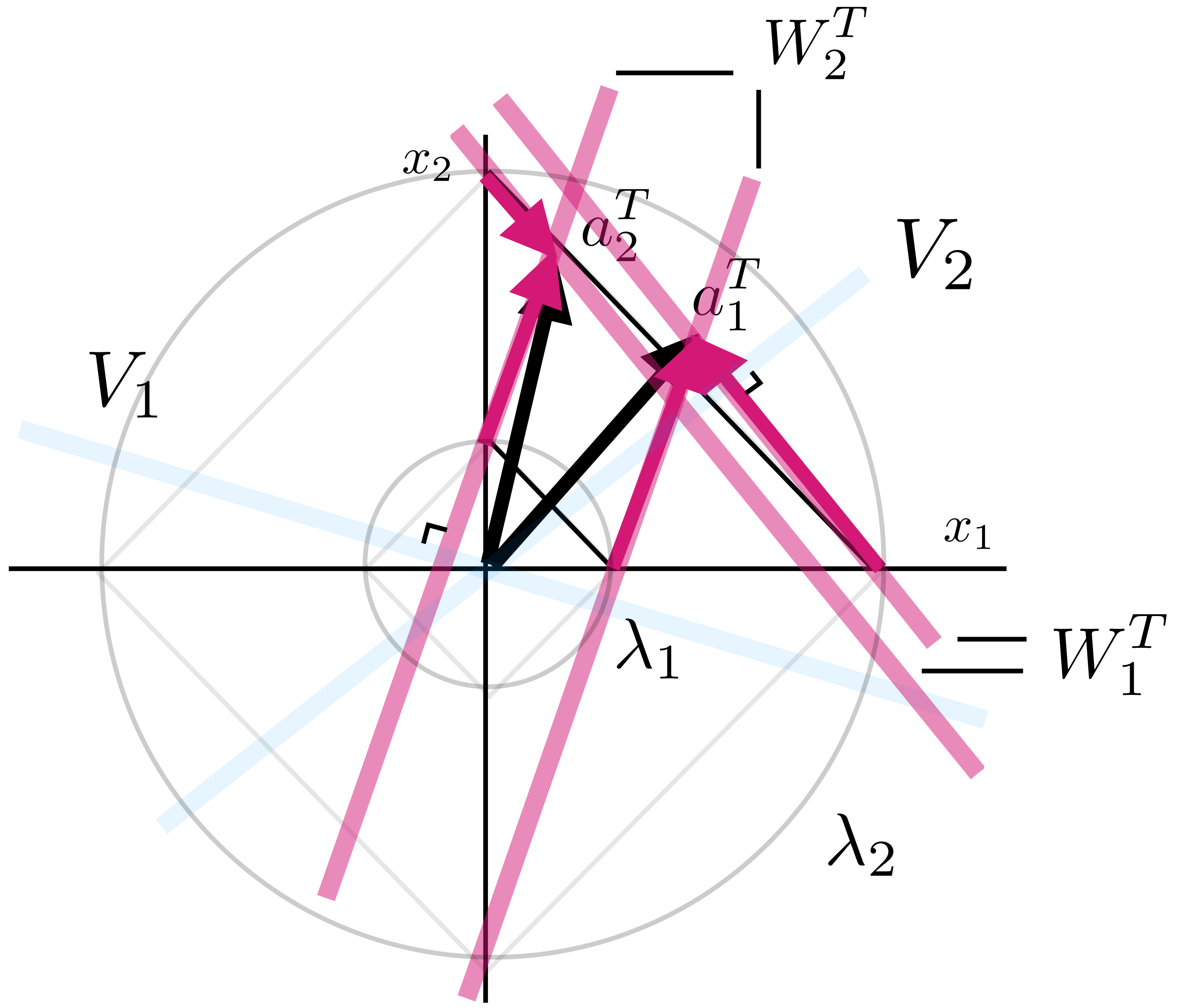
COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$



ROW GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$



Eigenvector/Eigenvalue Picture

Eigenvectors exist only for values of s where $A - sI$ drops rank...

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^*A = w^*\lambda \qquad w^*(A - \lambda I) = 0$$

Characteristic Polynomial

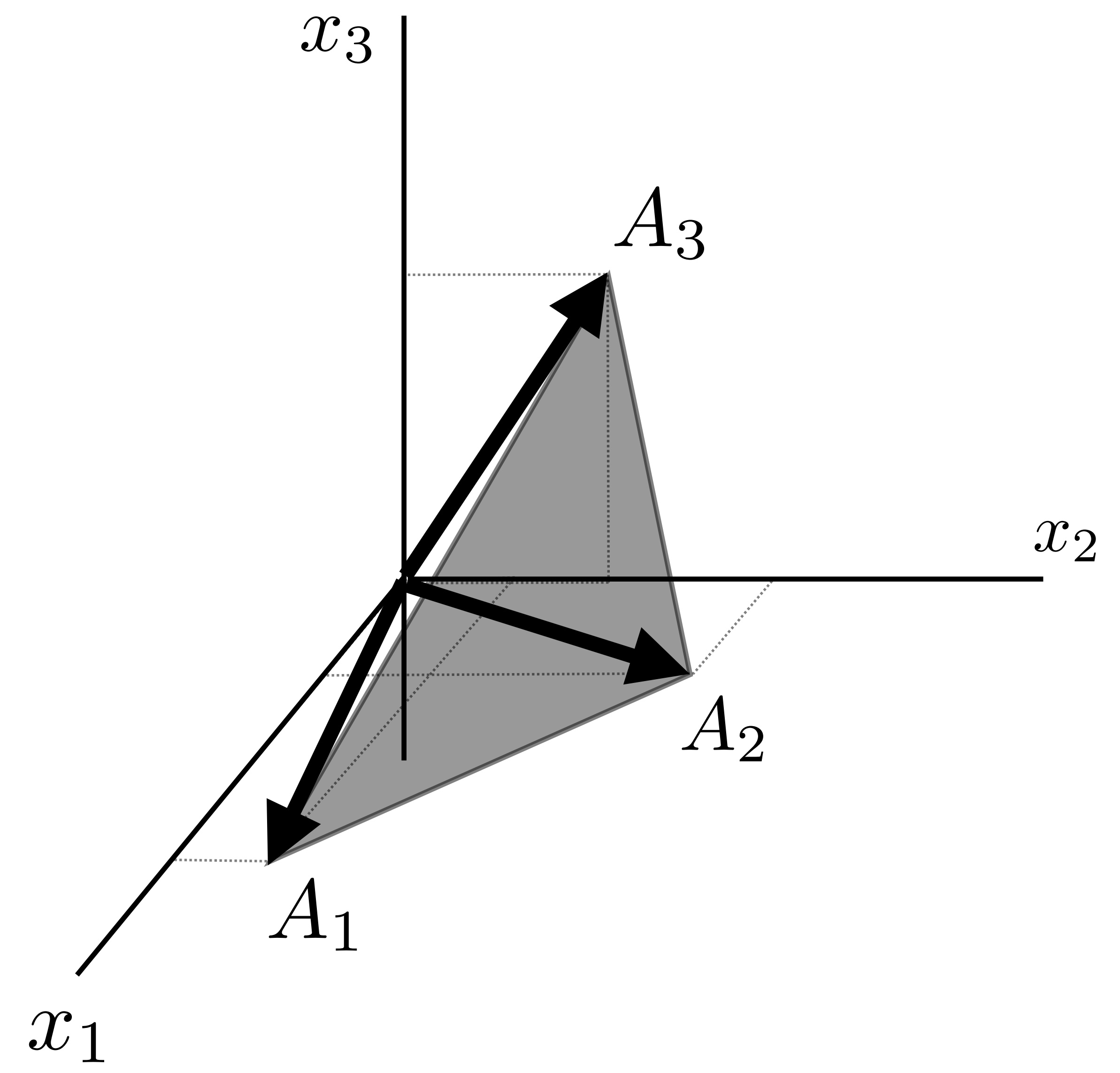
$sI - A$ drops rank only when

$$\mathrm{char}_A(s) = \det(sI - A) = 0$$

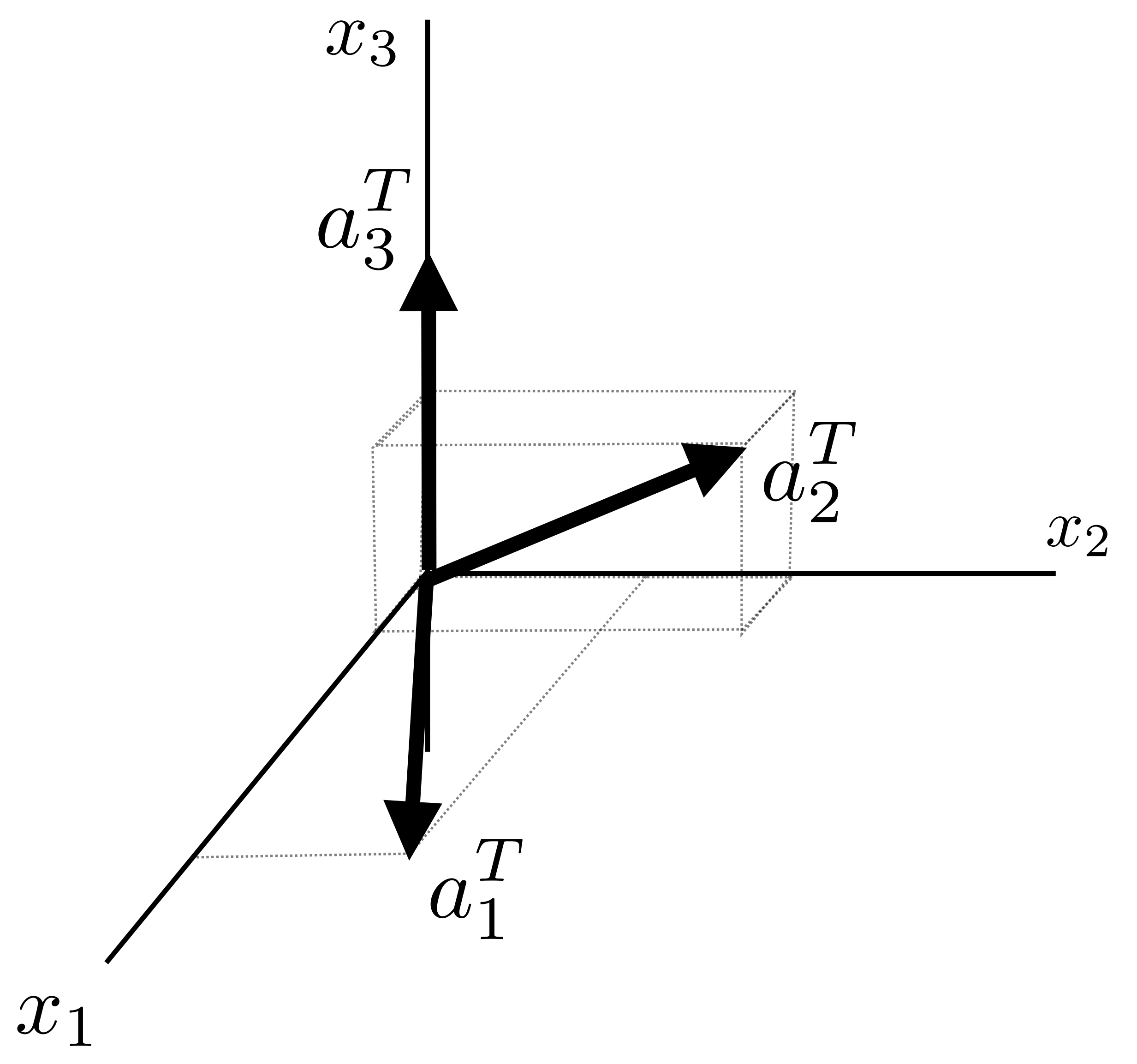
$$A \in \mathbb{R}^{3\times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \\ - & W_3^T & - \end{bmatrix}$$

COLUMN GEOMETRY



ROW GEOMETRY



Eigenvector/Eigenvalue Picture

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of s where $A - sI$ drops rank…

Characteristic Polynomial

$sI - A$ drops rank only when

$$\mathrm{char}_A(s) = \det(sI - A) = 0$$

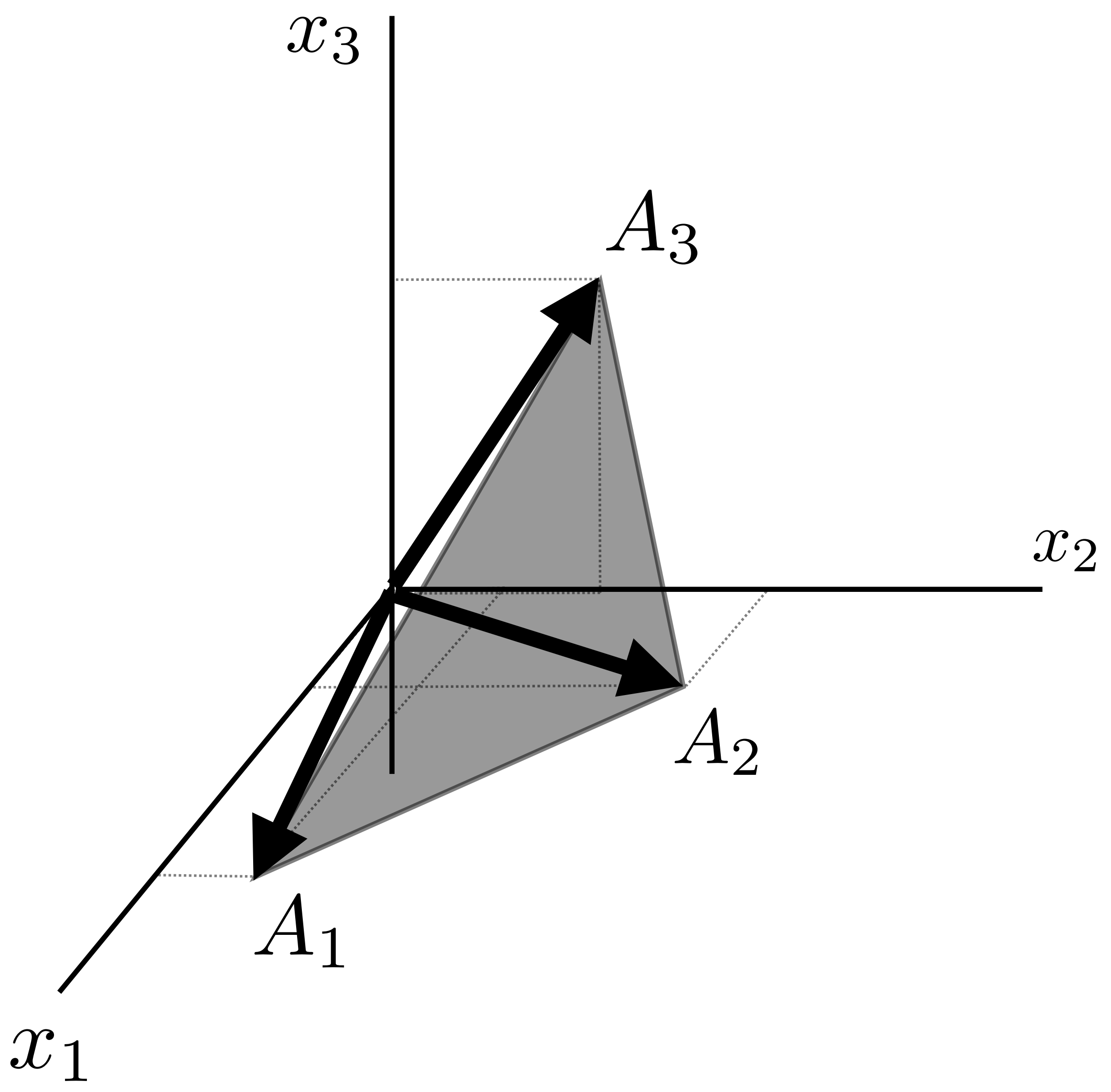
$$A \in \mathbb{R}^{3\times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \\ - & W_3^T & - \end{bmatrix}$$

COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$

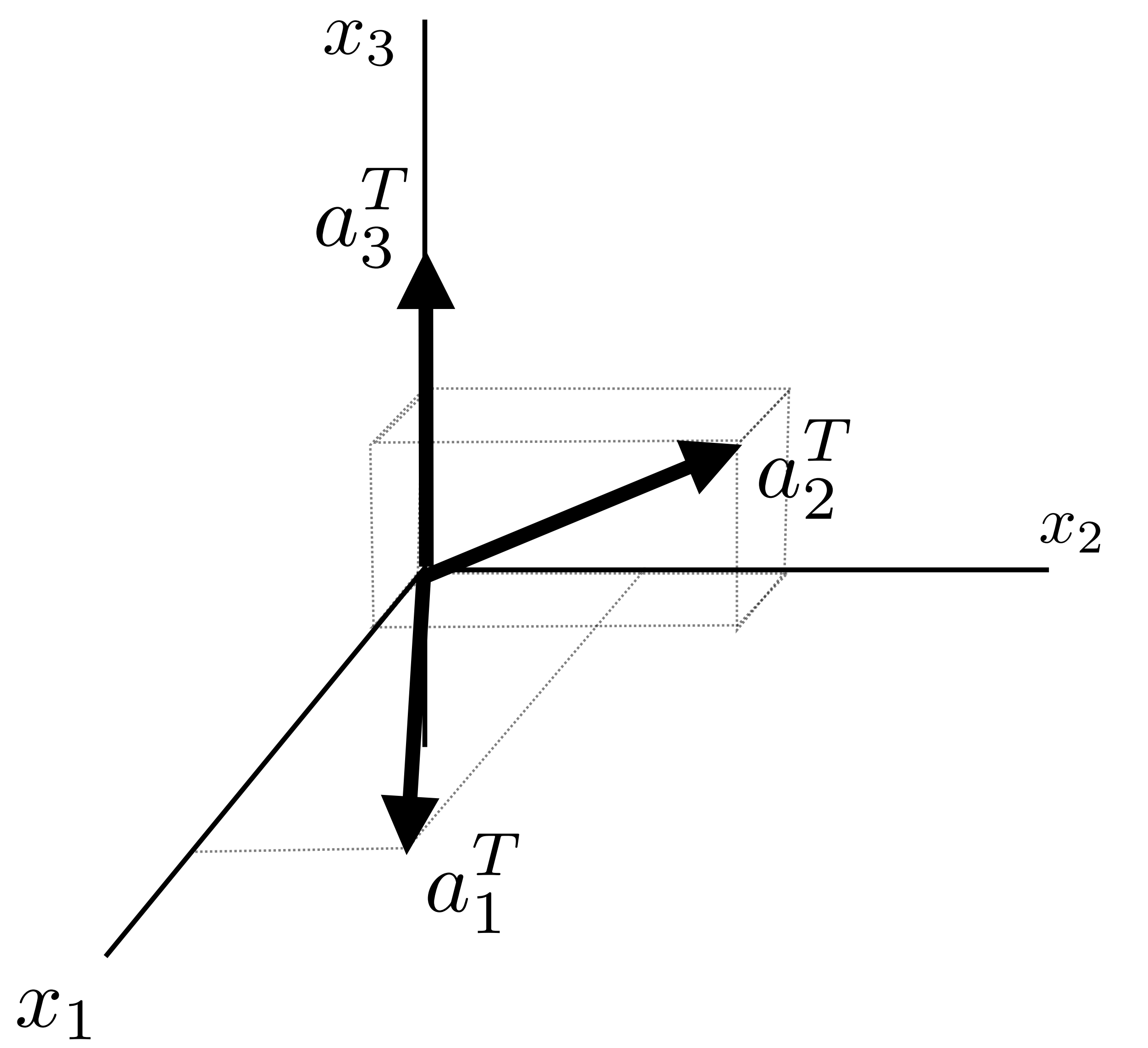
drops rank for eigenvalue



ROW GEOMETRY

$$\begin{bmatrix} A - \lambda I \end{bmatrix}$$

drops rank for eigenvalue



Eigenvector/Eigenvalue Picture

For any eigenvalue $\lambda \in \mathbb{C}$

Eigenvectors exist only for values of s where $A - sI$ drops rank...

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^*A = w^*\lambda \qquad w^*(A - \lambda I) = 0$$

Characteristic Polynomial

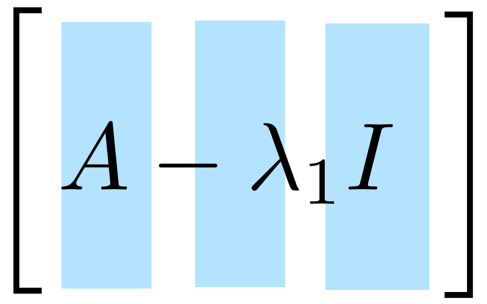
$sI - A$ drops rank only when

$$\mathrm{char}_A(s) = \det(sI - A) = 0$$

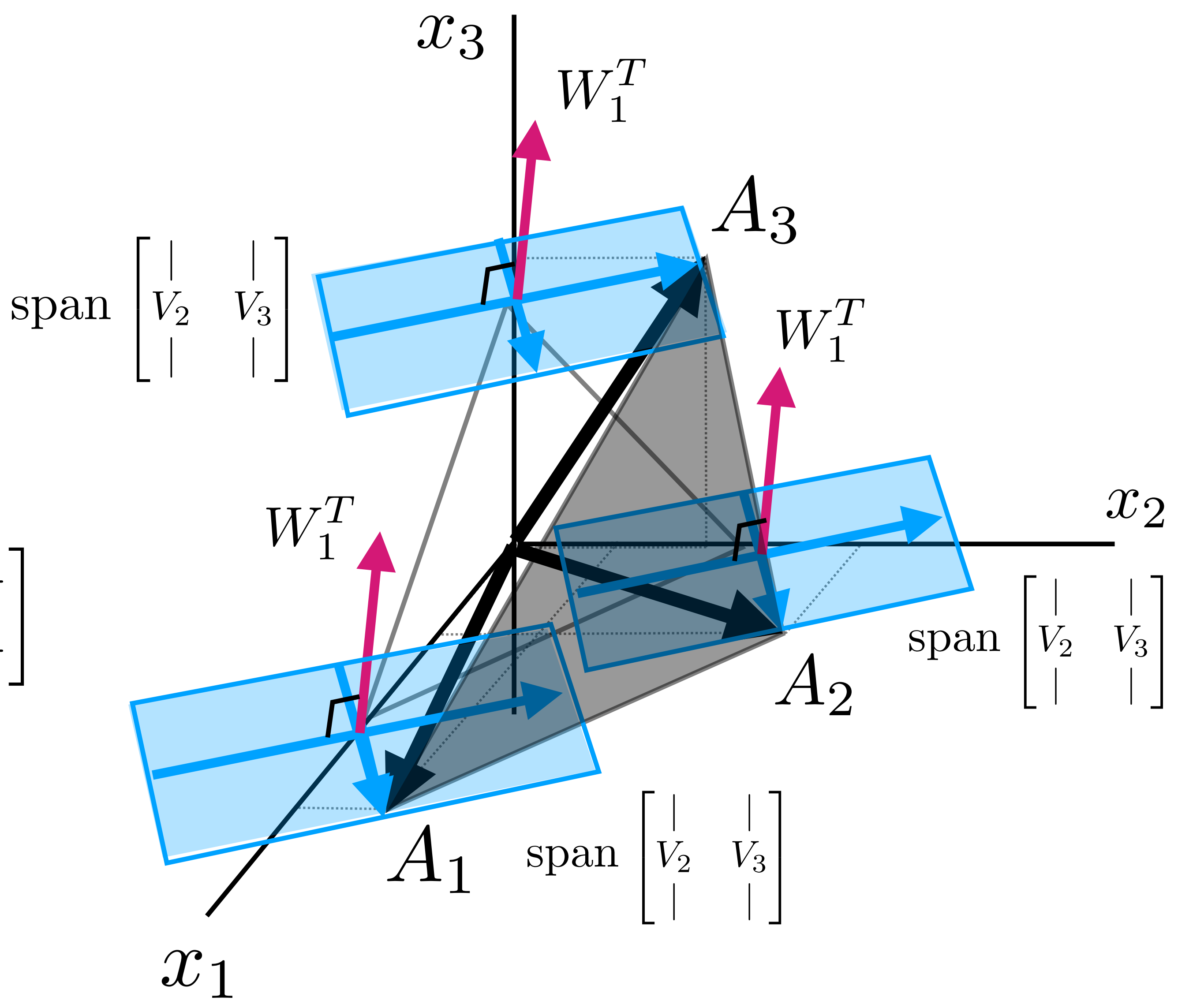
$$A \in \mathbb{R}^{3\times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \\ - & W_3^T & - \end{bmatrix}$$

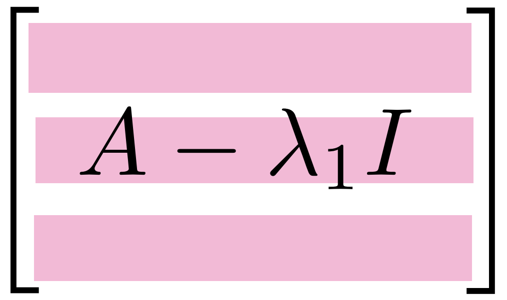
COLUMN GEOMETRY



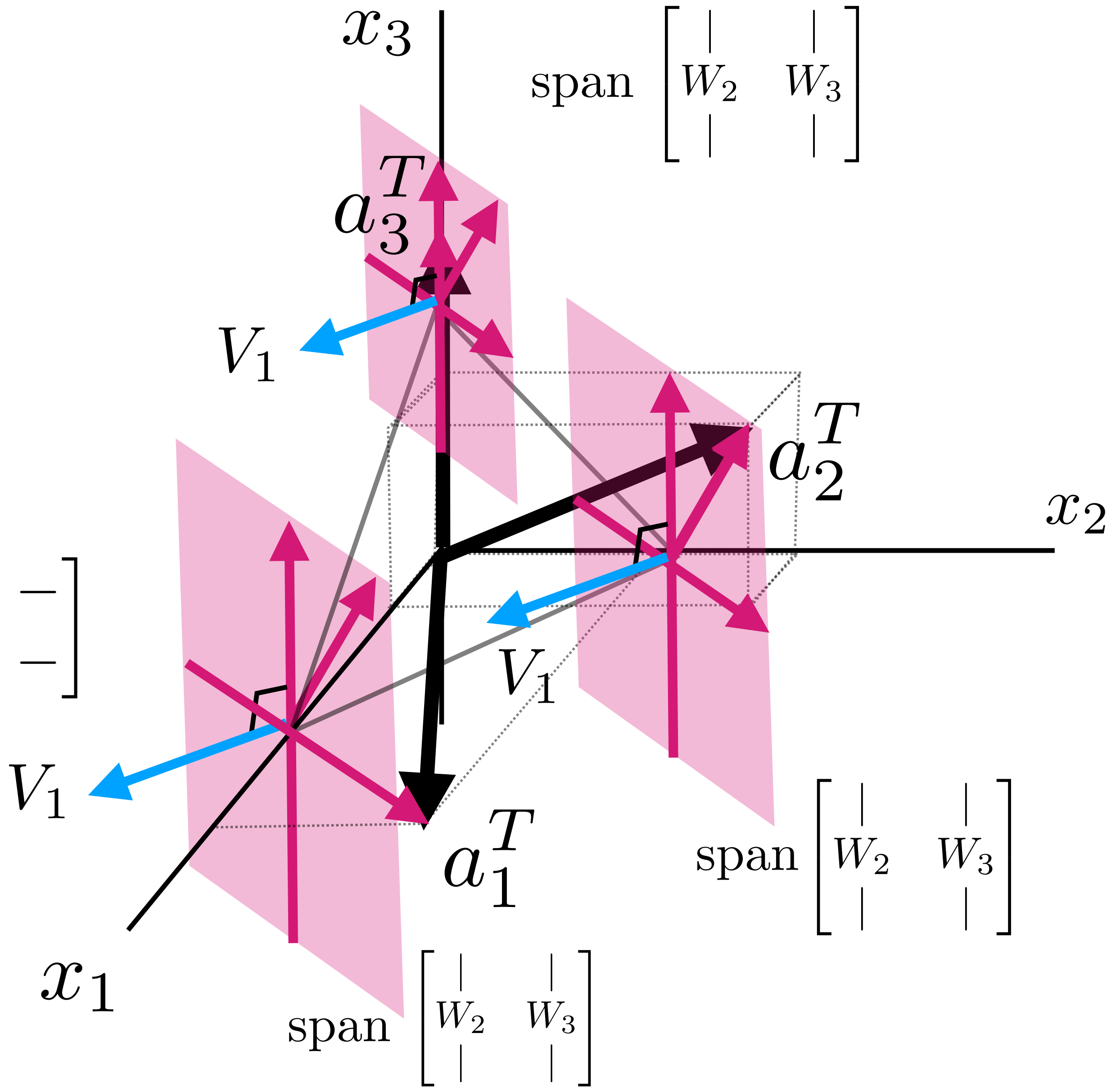
$$\begin{bmatrix} | & | \\ V_2 & V_3 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_2 - \lambda_1 & 0 \\ 0 & \lambda_3 - \lambda_1 \end{bmatrix} \begin{bmatrix} - & W_2^T & - \\ - & W_3^T & - \end{bmatrix}$$



ROW GEOMETRY



$$\begin{bmatrix} | & | \\ V_2 & V_3 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_2 - \lambda_1 & 0 \\ 0 & \lambda_3 - \lambda_1 \end{bmatrix} \begin{bmatrix} - & W_2^T & - \\ - & W_3^T & - \end{bmatrix}$$



Eigenvector/Eigenvalue Picture

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of s where $A - sI$ drops rank...

Characteristic Polynomial

$sI - A$ drops rank only when

$$\text{char}_A(s) = \det(sI - A) = 0$$

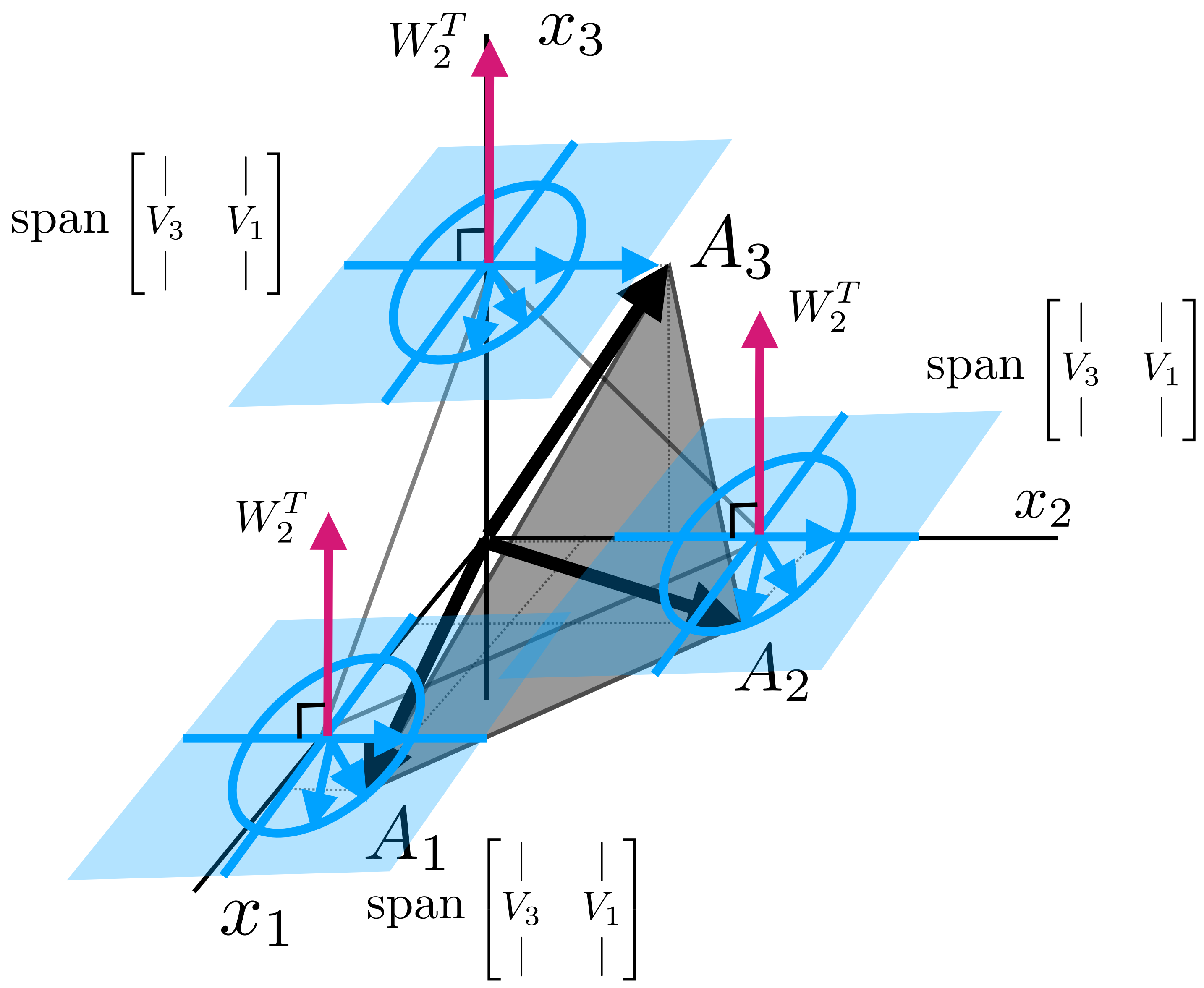
$$A \in \mathbb{R}^{3\times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \\ - & W_3^T & - \end{bmatrix}$$

COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda_2 I \end{bmatrix}$$

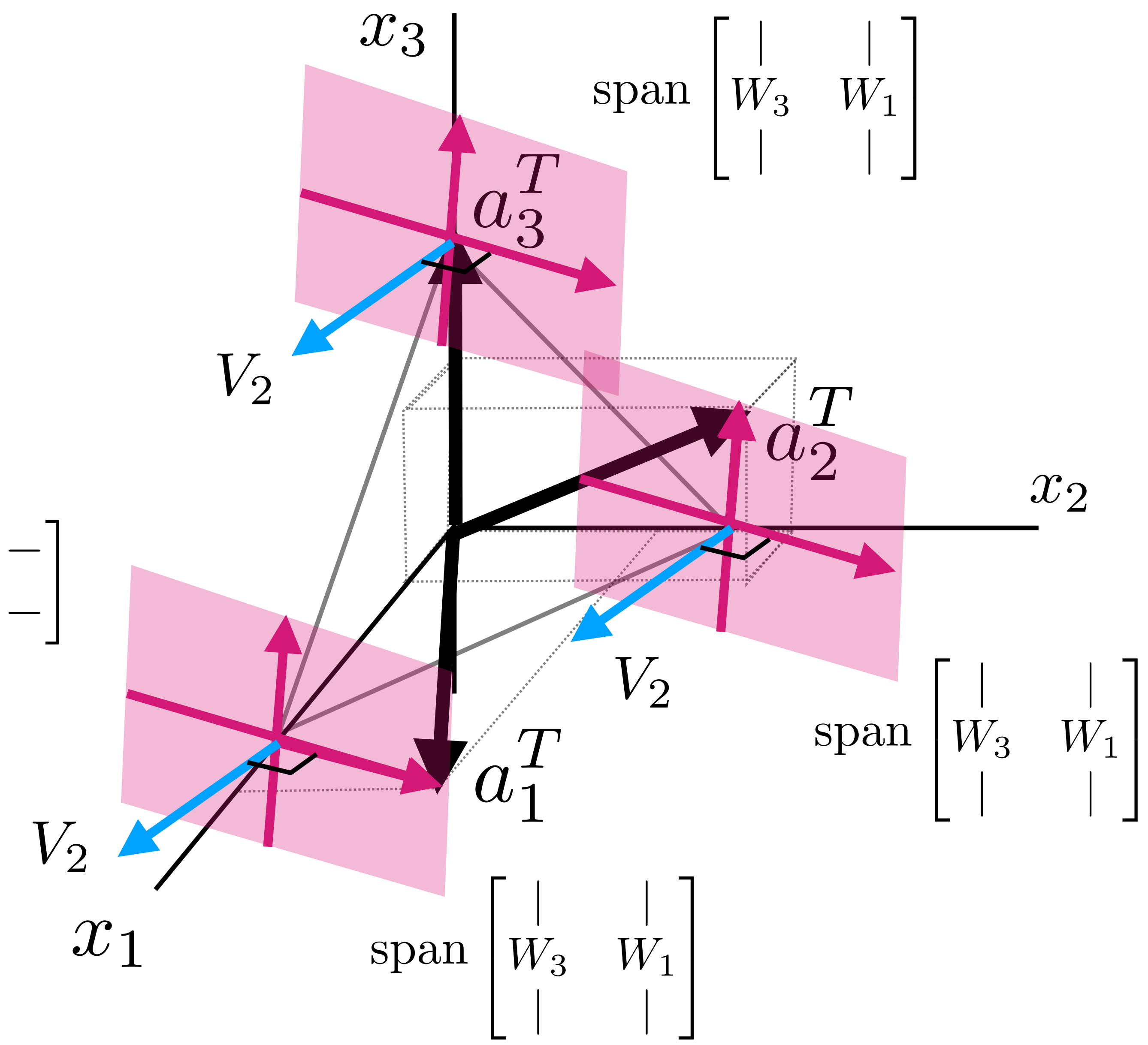
$$\begin{bmatrix} | & | \\ V_1 & V_3 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & \lambda_3 - \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_3^T & - \end{bmatrix}$$



ROW GEOMETRY

$$\begin{bmatrix} A - \lambda_2 I \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & V_3 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & \lambda_3 - \lambda_2 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_3^T & - \end{bmatrix}$$



Eigenvector/Eigenvalue Picture

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda \qquad (A - \lambda I)v = 0$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda \qquad w^*(A - \lambda I) = 0$$

Eigenvectors exist only for values of s where $A - sI$ drops rank...

Characteristic Polynomial

$sI - A$ drops rank only when

$$\mathrm{char}_A(s) = \det(sI - A) = 0$$

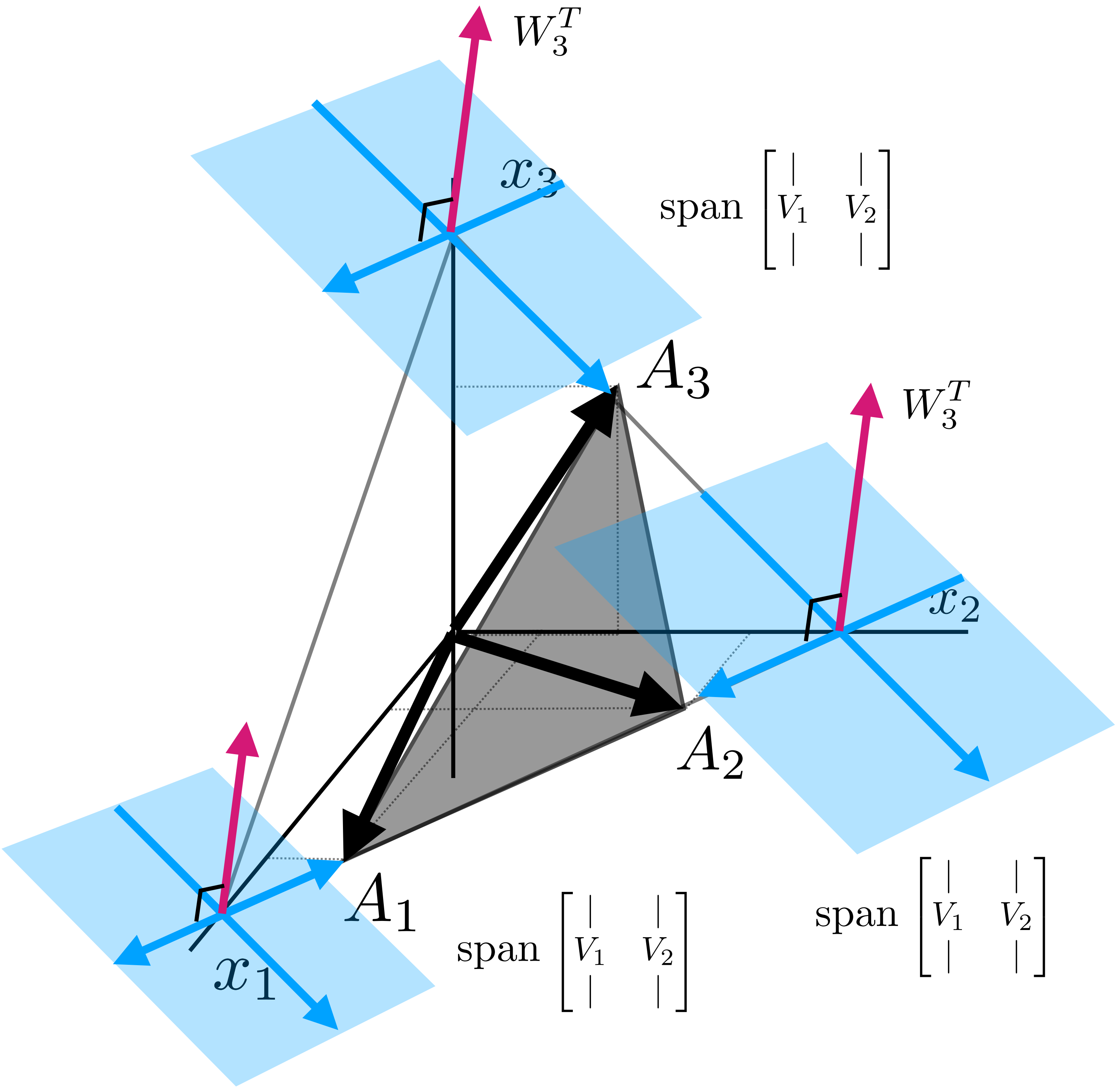
$$A \in \mathbb{R}^{3\times 3}$$

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & a_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \\ - & W_3^T & - \end{bmatrix}$$

COLUMN GEOMETRY

$$\begin{bmatrix} A - \lambda_3 I \end{bmatrix}$$

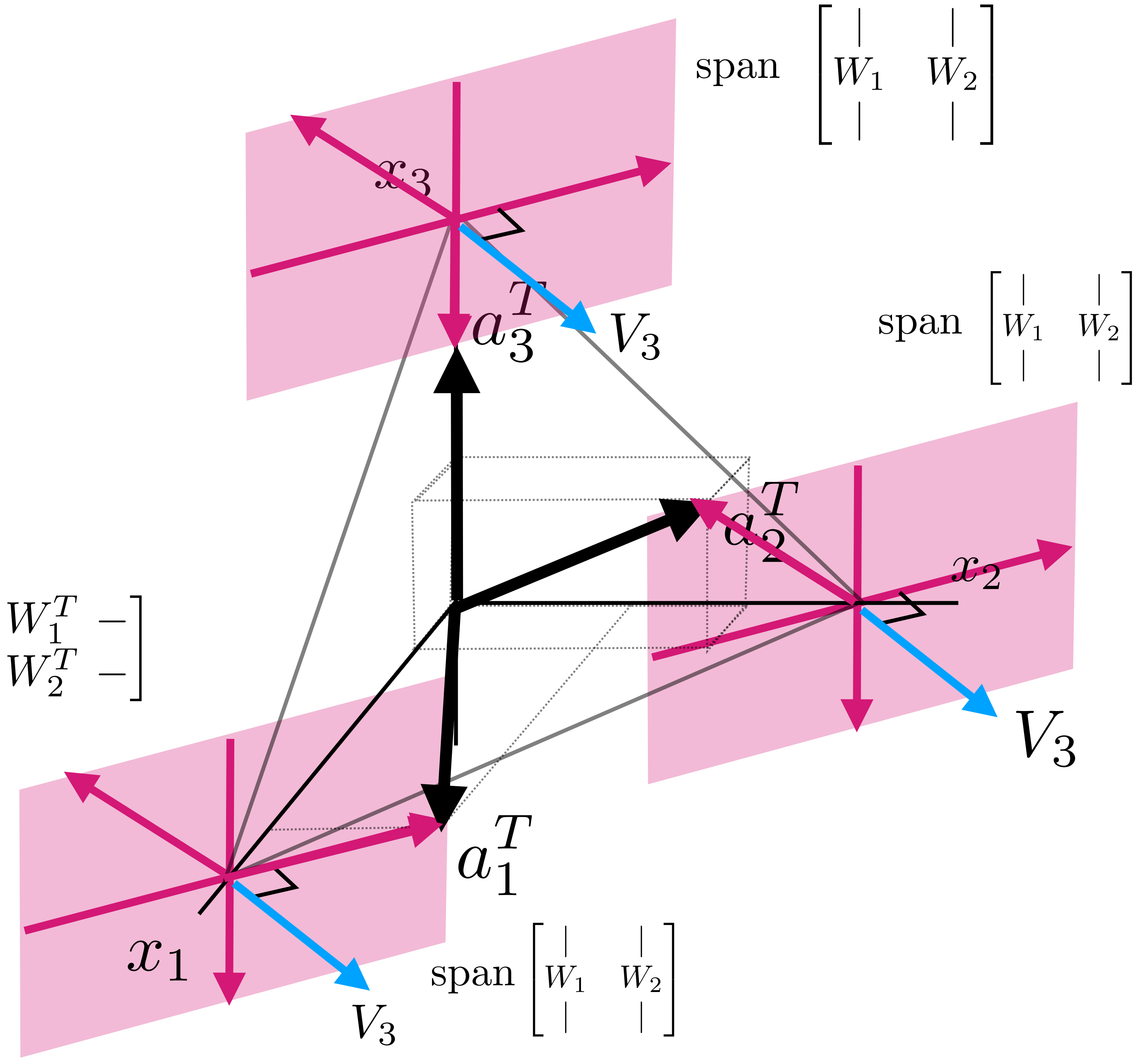
$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_3 & 0 \\ 0 & \lambda_2 - \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$



ROW GEOMETRY

$$\begin{bmatrix} A - \lambda_3 I \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_3 & 0 \\ 0 & \lambda_2 - \lambda_3 \end{bmatrix} \begin{bmatrix} - & W_1^T & - \\ - & W_2^T & - \end{bmatrix}$$



Diagonalization

Square matrix: $A \in \mathbb{R}^{n\times n}$

Assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Right Eigenvectors:

$$V = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \qquad AV = \begin{bmatrix} AV_1 \cdots AV_n \end{bmatrix} = \begin{bmatrix} V_1\lambda_1 \cdots V_n\lambda_n \end{bmatrix} = \begin{bmatrix} V_1 \cdots V_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{D} = VD$$

$$AV = VD$$

$$\Rightarrow \quad A = VDV^{-1}$$

Left Eigenvectors:

$$W = \begin{bmatrix} -W_1^* - \\ \vdots \\ -W_n^* - \end{bmatrix} \qquad WA = \begin{bmatrix} -W_1^* A - \\ \vdots \\ -W_n^* A - \end{bmatrix} = \begin{bmatrix} -\lambda_1 W_1^* - \\ \vdots \\ -\lambda_n W_n^* - \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{D} \begin{bmatrix} -W_1^* - \\ \vdots \\ -W_n^* - \end{bmatrix} = DW$$

$$WA = DW$$

$$\Rightarrow \quad A = W^{-1}DW$$

Assuming V & W are chosen with compatible orderings and lengths of columns/rows…

$$V^{-1} = W$$

Diagonalization

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\; V \;\right]\left[\; D \;\right]\left[\; V^{-1} \right]$$

$$\left[\; A \;\right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\left[\; A \;\right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Sum of rank-1 matrices

Dyadic Expansion

$$V^{-1}V = \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} W_1^*V_1 & \cdots & W_1^*V_n \\ \vdots & & \vdots \\ W_n^*V_1 & \cdots & W_n^*V_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

…from off diagonal terms $W_j^* V_i = 0 \quad j \neq i$

V_i orthogonal to all other W_j

…from diagonal terms $W_i^* V_i = 1$

V_i, W_i can be scaled so that $W_i^* V_i = 1$

Diagonalization - Similarity Transform

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

A is similar to a diagonal matrix

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

…transform into eigenvector coordinates

$$x = Vx' \qquad y = Vy'$$

$$y = Ax$$
$$Vy' = AVx'$$
$$y' = V^{-1}AVx'$$
$$y' = V^{-1}VDV^{-1}Vx'$$
$$y' = Dx'$$

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1' \\ \vdots \\ \lambda_n x_n' \end{bmatrix}$$

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication Ax

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$$

transforming into eigen-vector coords

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication Ax

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}}_{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}$$

transforming into eigen-vector coords

Scaling each coord by eigenvalue

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication Ax

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}}_{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}}_{V_1 \lambda_1 W_1^* x + \cdots + V_n \lambda_n W_n^* x}$$

transforming into eigen-vector coords

Scaling each coord by eigenvalue

Transforming back into regular coordinates

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

If x is an eigenvector…

Interpretation of Matrix Multiplication AV_i

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Orthogonal to all other left eigenvectors}} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

If x is an eigenvector…

Interpretation of Matrix Multiplication AV_i

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}\underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ Orthogonal to all other left eigenvectors}}}_{\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \text{ Scaled by specific eigenvalue}}$$

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

If x is an eigenvector…

Interpretation of Matrix Multiplication AV_i

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}}_{\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix}}}_{\lambda_i V_i}$$

Orthogonal to all other left eigenvectors

Scaled by specific eigenvalue

Select out that specific eigenvector

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} & V & \end{bmatrix}\begin{bmatrix} & D & \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} & A & \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

$$\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Shuffling eigenvalues and eigenvectors

$$\begin{bmatrix} & A & \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

← Order of sum does not matter…

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

Diagonalization

$$A = \begin{bmatrix} \; V \; \end{bmatrix} \begin{bmatrix} \; D \; \end{bmatrix} \begin{bmatrix} \; V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \; A \; \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \; P \; \end{bmatrix} \begin{bmatrix} \; P^T \; \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \; P \; \end{bmatrix} \begin{bmatrix} \; P^T \; \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Shuffling eigenvalues and eigenvectors

$$\begin{bmatrix} \; A \; \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & & & | \\ V_1 & & \cdots & V_n \\ | & & & | \end{bmatrix} \begin{bmatrix} P \end{bmatrix}} \boxed{\begin{bmatrix} P^T \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} P \end{bmatrix}} \boxed{\begin{bmatrix} P^T \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}$$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

Shuffling eigenvalues and eigenvectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

⟵ Order of sum does not matter…

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_n & \cdots & V_1 \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_n & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_1 \end{bmatrix} \begin{bmatrix} - & W_n^* & - \\ & \vdots & \\ - & W_1^* & - \end{bmatrix}$$

Permutation Matrix $P \in \mathbb{R}^{n\times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\qquad P^T P = I$

Shuffling eigenvalues and eigenvectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\; V \;\right]\left[\; D \;\right]\left[\; V^{-1} \right]$$

$$\left[\; A \;\right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\left[\; A \;\right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Sum of rank-1 matrices

Dyadic Expansion

Permutation Matrix $P \in \mathbb{R}^{n\times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\qquad P^T P = I$

$$\begin{bmatrix} | & & | \\ V_n & \cdots & V_1 \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_n & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_1 \end{bmatrix} \begin{bmatrix} - & W_n^* & - \\ & \vdots & \\ - & W_1^* & - \end{bmatrix}$$

Shuffling eigenvalues and eigenvectors

Order of sum does not matter…

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

diagonal matrices commute…

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 \frac{\gamma_1}{\gamma_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \frac{\gamma_n}{\gamma_n} \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

Scaling eigenvectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

← Order of sum does not matter…

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

diagonal matrices commute…

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \gamma_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{\gamma_n} \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

Scaling eigenvectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

⟵ Order of sum does not matter…

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

diagonal matrices commute…

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} | & & | \\ V_1\gamma_1 & \cdots & V_n\gamma_n \\ | & & | \end{bmatrix}}_{V'} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & \frac{1}{\gamma_1}W_1^* & - \\ & \vdots & \\ - & \frac{1}{\gamma_n}W_n^* & - \end{bmatrix}}_{V'^{-1}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

Scaling eigenvectors

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \begin{bmatrix} | & | & | & | & & | \\ V_1 & V_2 & V_3 & V_4 & \cdots & V_n \\ | & | & | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_2 e^{-i\phi_2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \gamma_2 e^{i\phi_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & W_2^* & - \\ - & W_3^* & - \\ - & W_4^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$A = \left[\; V \; \right] \left[\; D \; \right] \left[\; V^{-1} \right]$$

$$\left[\; A \; \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & V_2 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & W_2^* & - \end{matrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix}$$

$$\left[\; A \; \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \big[\ V\ \big]\big[\ D\ \big]\big[\ V^{-1}\big]$$

$$\Big[\ A\ \Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} | & | & & | \\ V_1 & \bar{V}_1 & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\;V\;\right]\left[\;D\;\right]\left[\;V^{-1}\right]$$

$$\left[\;A\;\right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\left[\;A\;\right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} \frac{z}{z} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \frac{z'}{z'} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Note: *may differ by any complex scalars* $z, z' \in \mathbb{C}$

…with both magnitude and phase shifts

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} | & | & & | \\ V_1 z & \bar{V}_1 z' & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & \frac{1}{z} W_1^* & - \\ - & \frac{1}{z'} \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Note: *may differ by any complex scalars* $z, z' \in \mathbb{C}$

…with both magnitude and phase shifts

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors — **Eigen-values** (on diagonal) — **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \;\; \bar{W}_n^* \;\; - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} =$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Real Imag

(scaled by $\sqrt{2}$)

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\; V \;\right]\left[\; D \;\right]\left[\; V^{-1} \right]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

similarly…

$$\begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}}_{U^*} \begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\; V \;\right]\left[\; D \;\right]\left[\; V^{-1} \right]$$

$$\left[\; A \;\right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\left[\; A \;\right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} | & | & & | \\ V_1 & \bar{V}_1 & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Note: $U \in \mathbb{C}^{2\times 2}$

unitary

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \big[\ V\ \big]\big[\ D\ \big]\big[\ V^{-1}\big]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Note: $U \in \mathbb{C}^{2 \times 2}$ $\quad U^* U = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = I$

unitary

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$= \begin{bmatrix} | & | & & | \\ V_1 & \bar{V}_1 & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\quad V \quad \right]\left[\quad D \quad \right]\left[\quad V^{-1} \right]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\frac{1}{\sqrt{2}} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $\quad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $\quad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\frac{1}{\sqrt{2}} = \gamma\begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\; V \;\right]\left[\; D \;\right]\left[\; V^{-1} \right]$$

$$\left[\; A \;\right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\left[\; A \;\right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \overset{U}{\boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}}} & \cdots & \overset{U^*}{\begin{matrix} | \\ V_n \\ | \end{matrix}} \end{bmatrix} \begin{bmatrix} \overset{U}{\boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \overset{U^*}{\begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $\qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = \gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $\qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\frac{1}{\sqrt{2}} = \gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\left[\boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} \quad \cdots \quad \begin{matrix} | \\ V_n \\ | \end{matrix}\right]}_{V' \in \mathbb{R}^{n\times n}} \begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \qquad \vdots & \ddots & \vdots \\ 0 \qquad 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}}_{V'^{-1}}$$

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}}_{V' \in \mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \begin{matrix} \cdots \\ \cdots \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix}}_{D' \in \mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & {w_1'}^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}}_{V'^{-1}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}}_{V' \in \mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix}}_{D' \in \mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}}_{V'^{-1}}$$

Real Expansion…

Block diagonal…

Pseudo-diagonalization

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

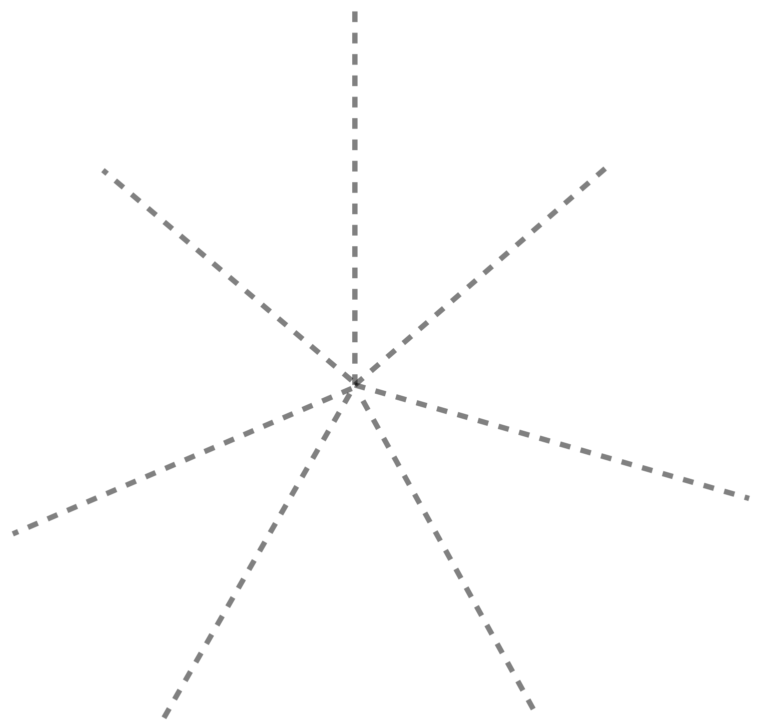
$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & & \cdots & 0 \\ & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

$\mathbb{R}^n$



Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big]$$

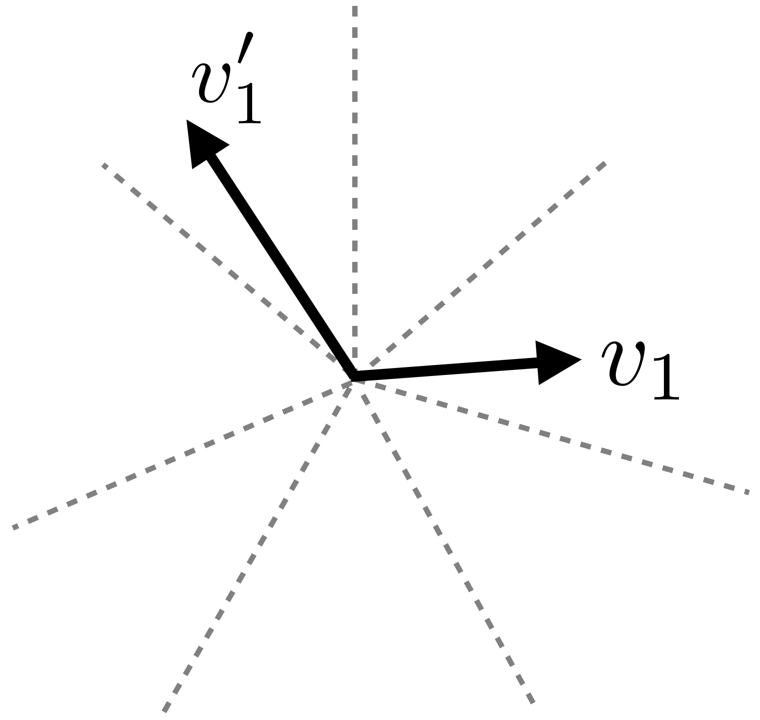
$$\Big[\ A\ \Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}}$$

$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \Big[\lambda_i\Big]\Big[- \quad W_i^* \quad -\Big]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \left[\boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} \ \cdots \ \begin{matrix} | \\ V_n \\ | \end{matrix}\right] \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix}$$

$\mathbb{R}^n$



Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} & V & \end{bmatrix}\begin{bmatrix} & D & \end{bmatrix}\begin{bmatrix} & V^{-1} & \end{bmatrix}$$

$$\begin{bmatrix} & A & \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

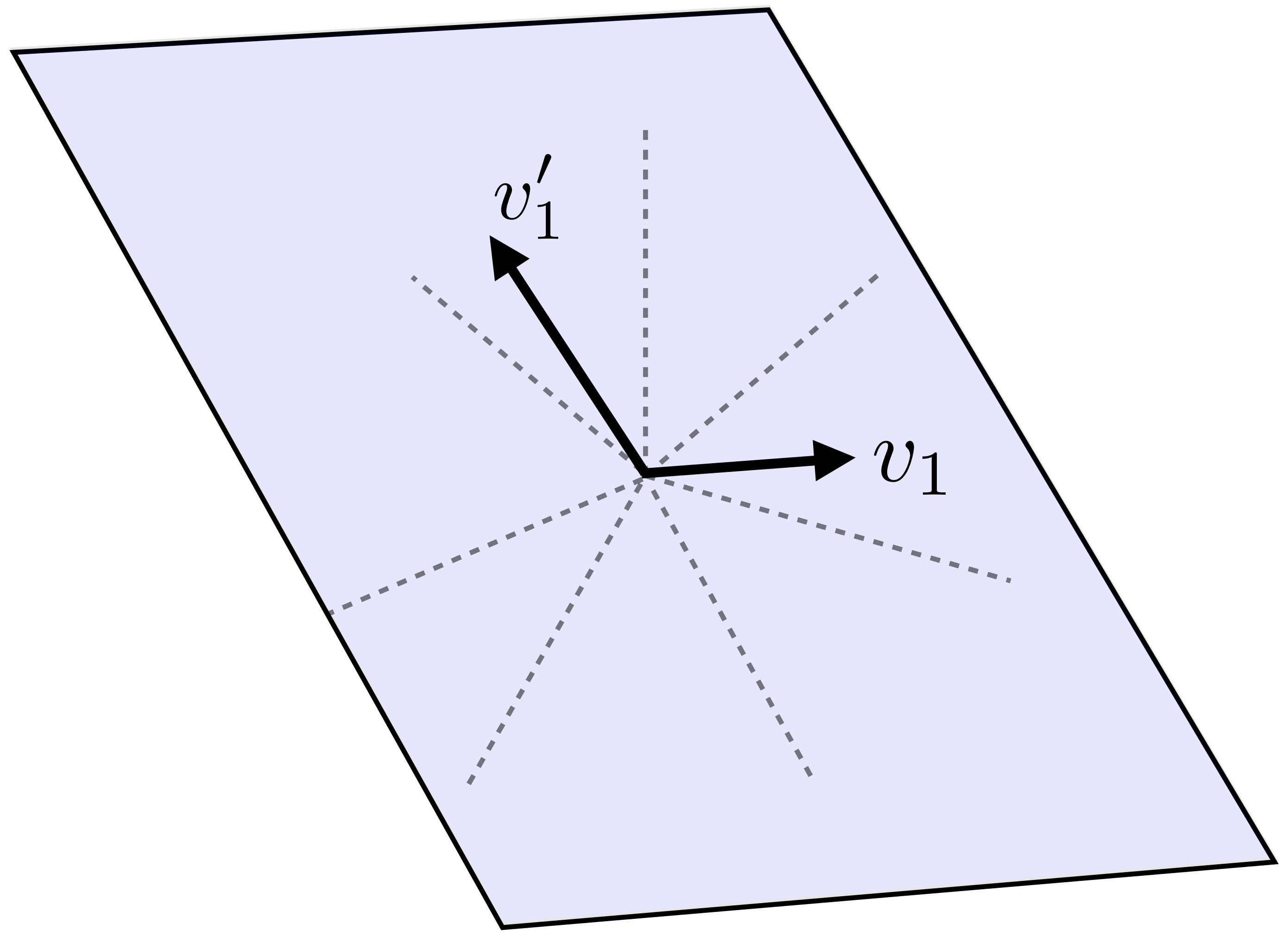
Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} & A & \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

$\mathbb{R}^n$



$$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \quad \text{2D}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big]$$

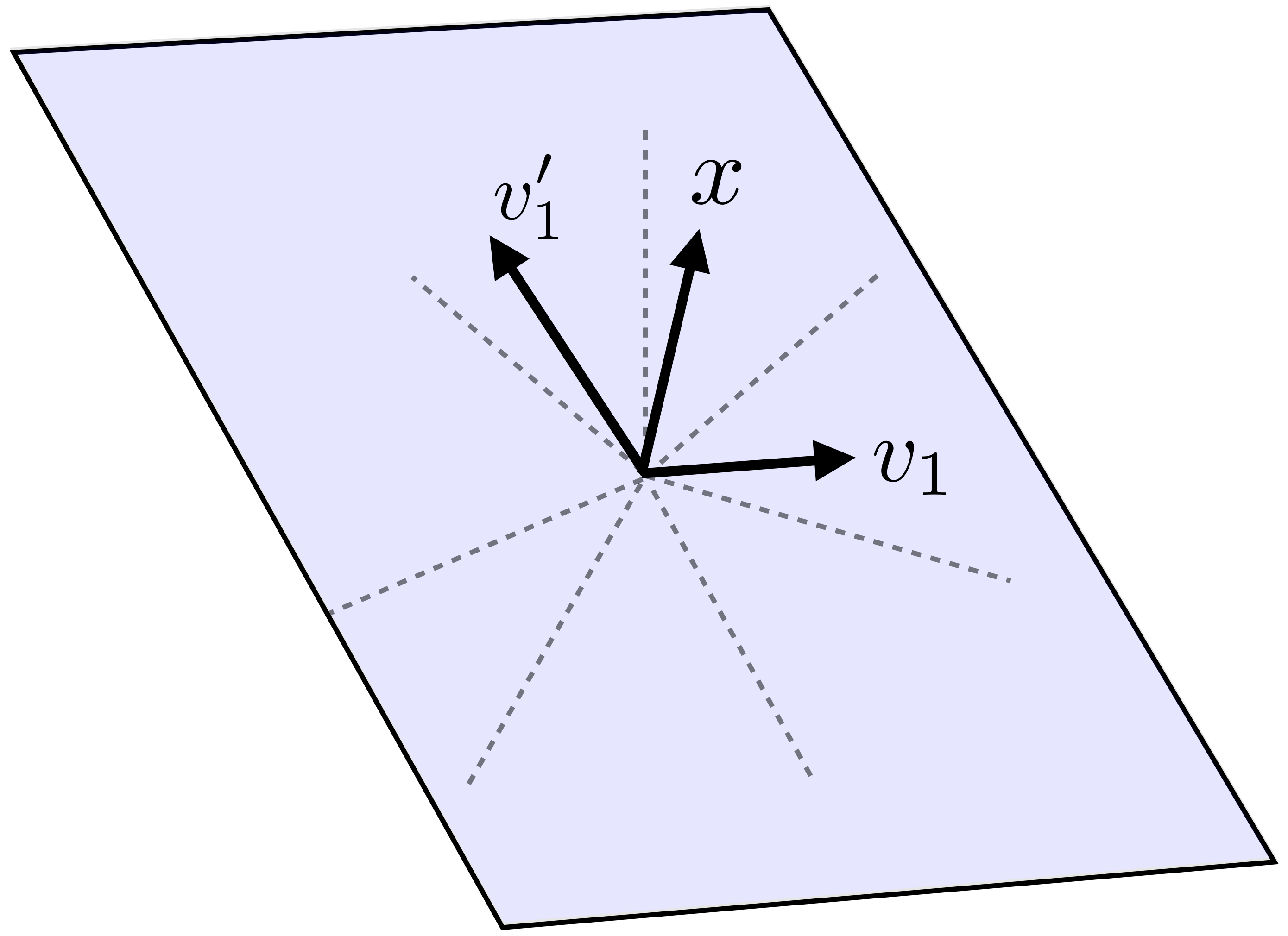
$$\Big[\ A\ \Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \big[\lambda_i\big]\big[- \quad W_i^* \quad -\big]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} \cdots \begin{bmatrix} | \\ V_n \\ | \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

vector in plane of rotation



$\mathbb{R}^n$

$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

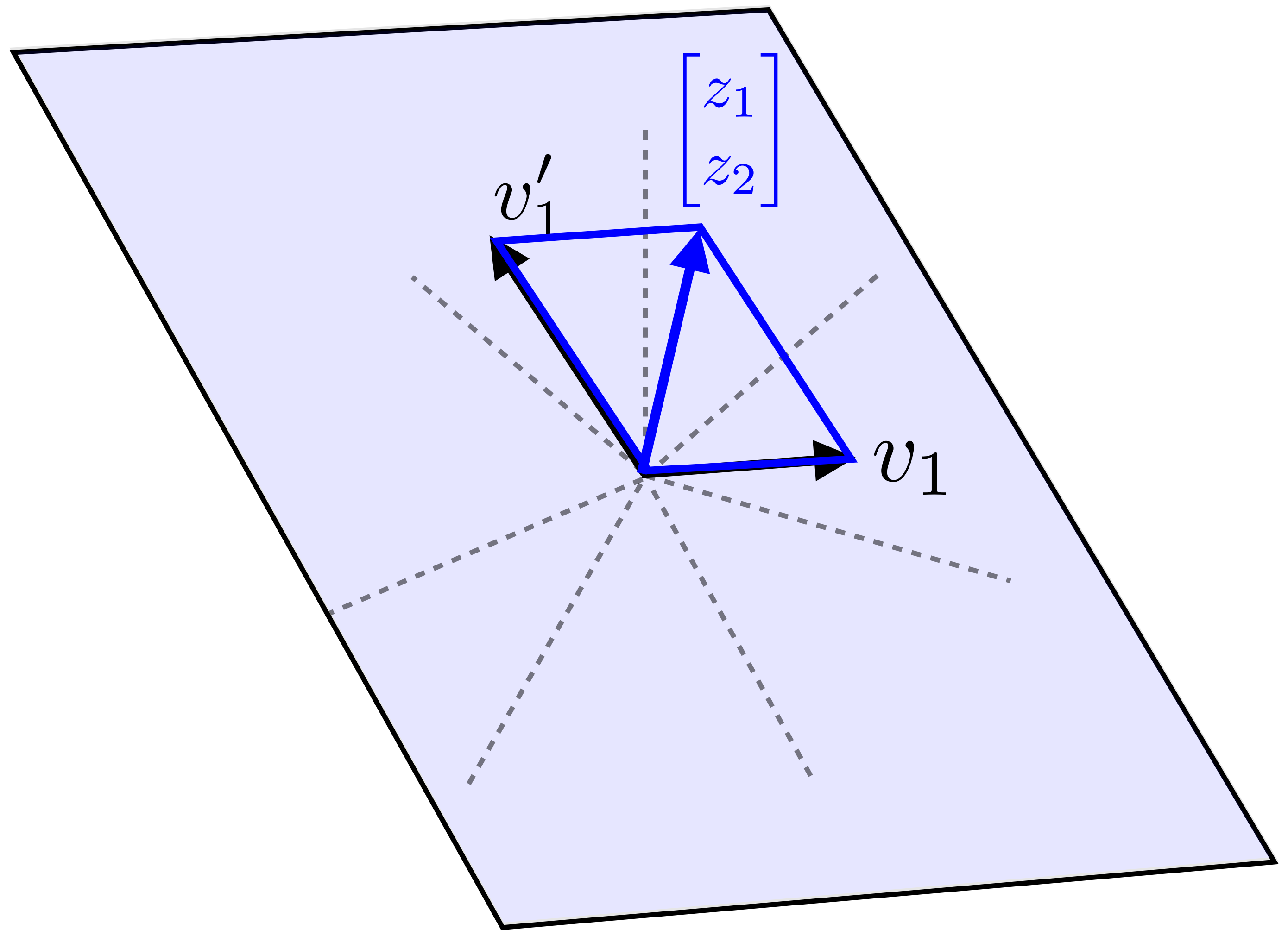
Left eigen-vectors

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [- \quad W_i^* \quad -]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & & \cdots & 0 \\ & & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

vector in plane of rotation



$\mathbb{R}^n$

$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

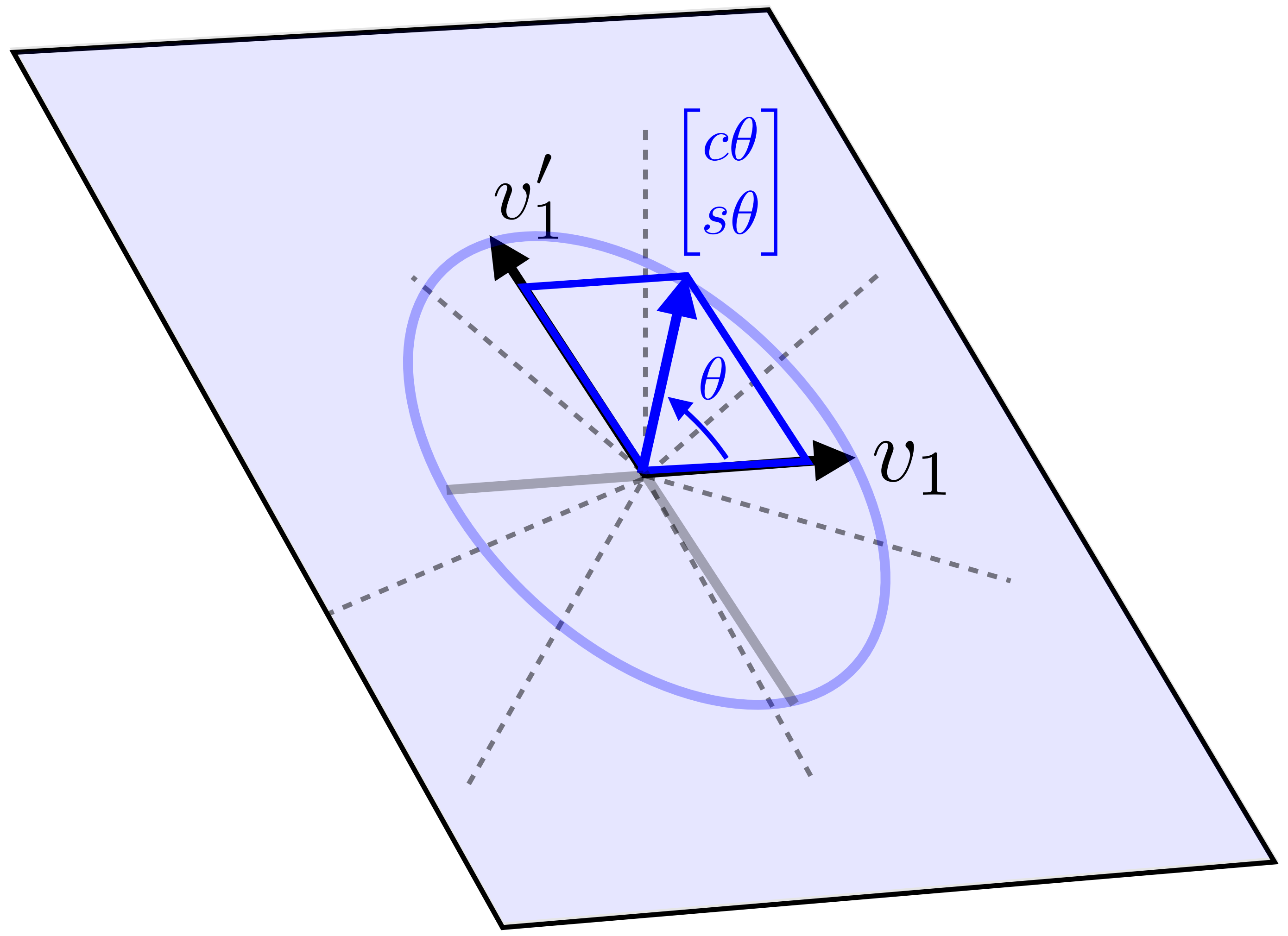
$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} c\theta \\ s\theta \end{bmatrix}^{|z|_2}$$

vector in plane of rotation

$\mathbb{R}^n$



$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

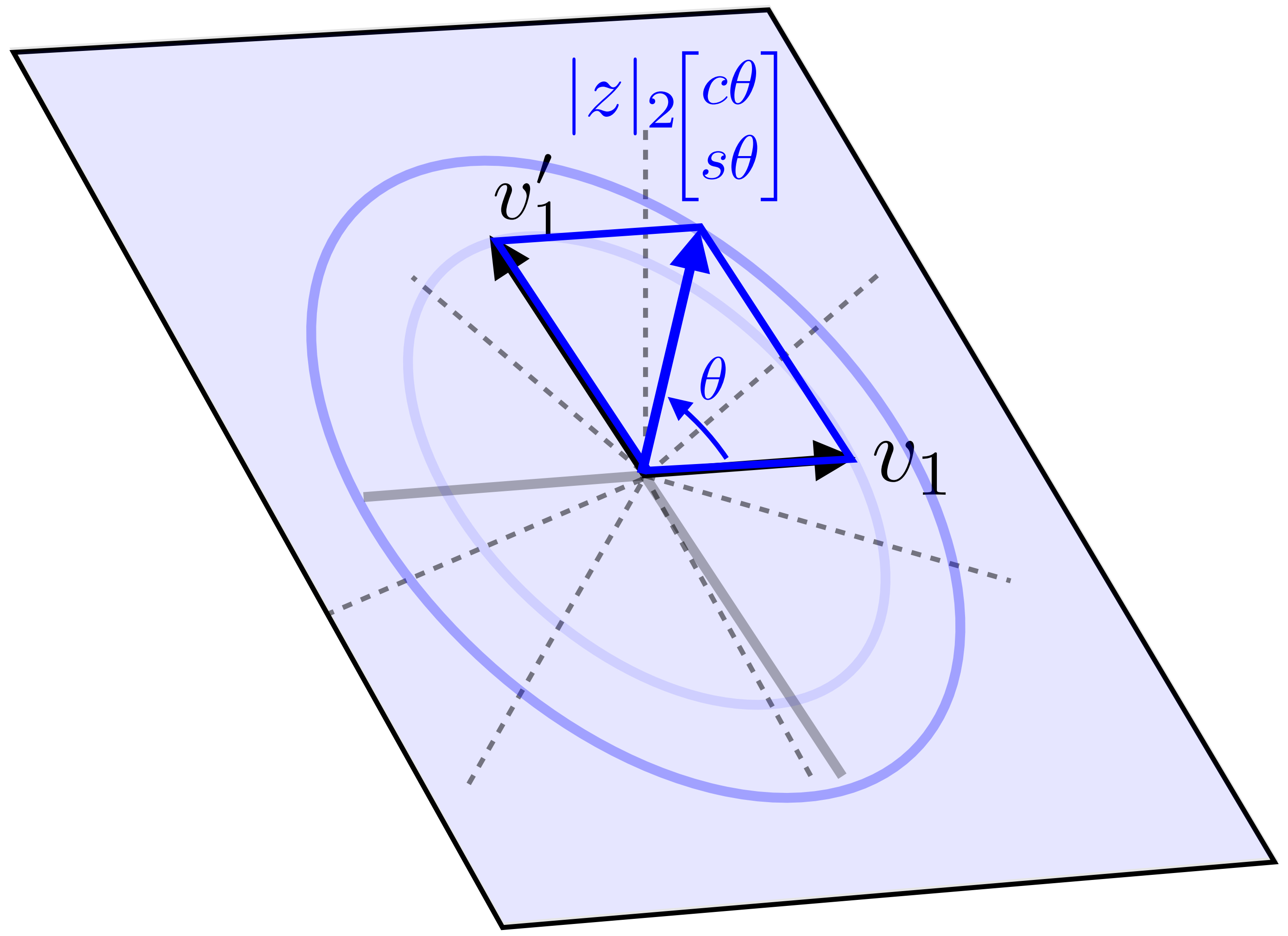
Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} c\theta \\ s\theta \end{bmatrix} |z|_2$$

$\mathbb{R}^n$



$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

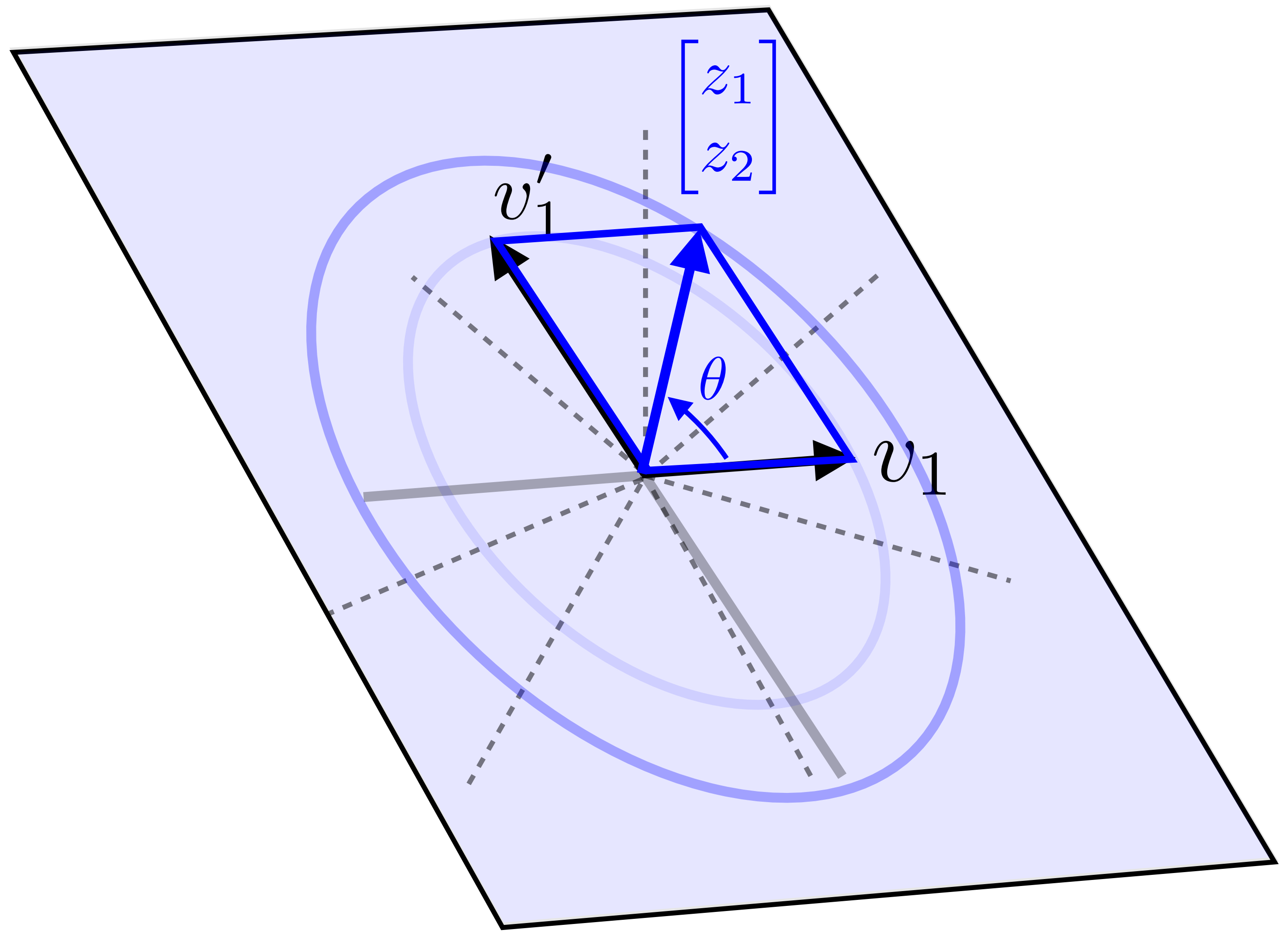
$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad\quad \vdots & \ddots & \vdots \\ 0 \quad\quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



$\mathbb{R}^n$

$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big] =$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

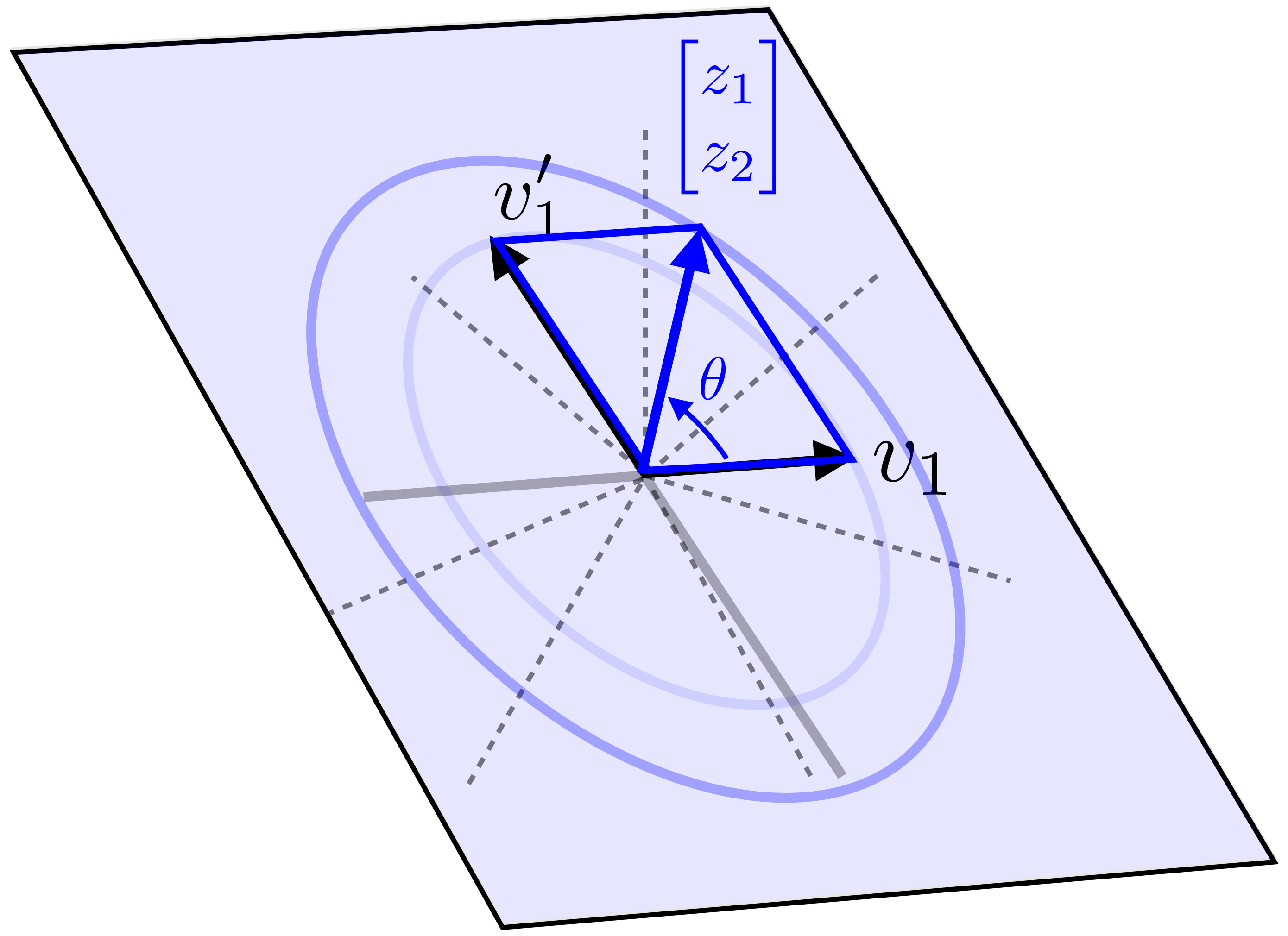
Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} z_1 \\ z_2 \\ 0 \\ \vdots \end{bmatrix}}$$

$\mathbb{R}^n$



$\text{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

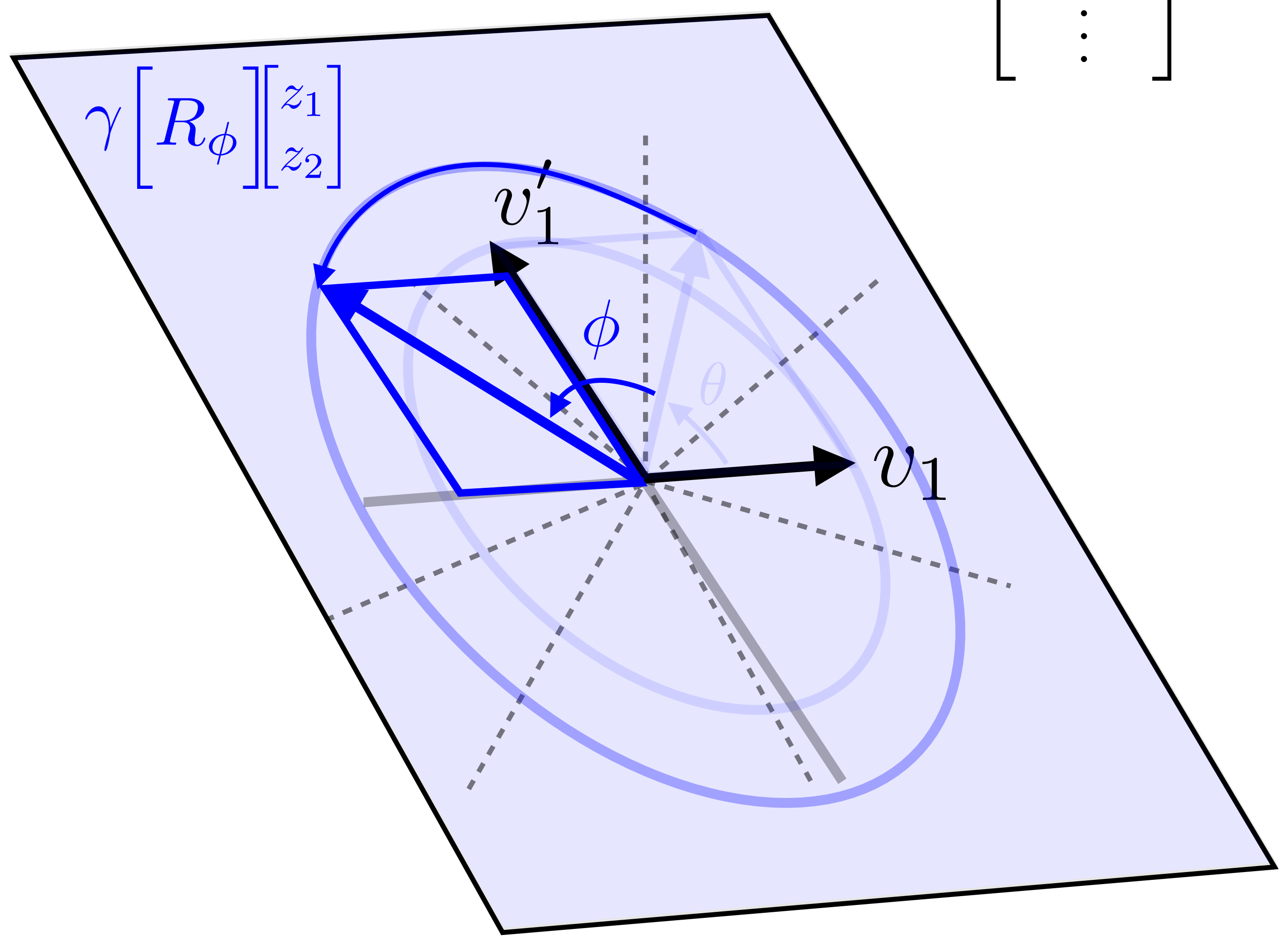
$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma = 1$

$\mathbb{R}^n$



$\text{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\substack{\textbf{Right} \\ \textbf{eigen-} \\ \textbf{vectors}}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-} \\ \textbf{values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\substack{\textbf{Left} \\ \textbf{eigen-} \\ \textbf{vectors}}}$$

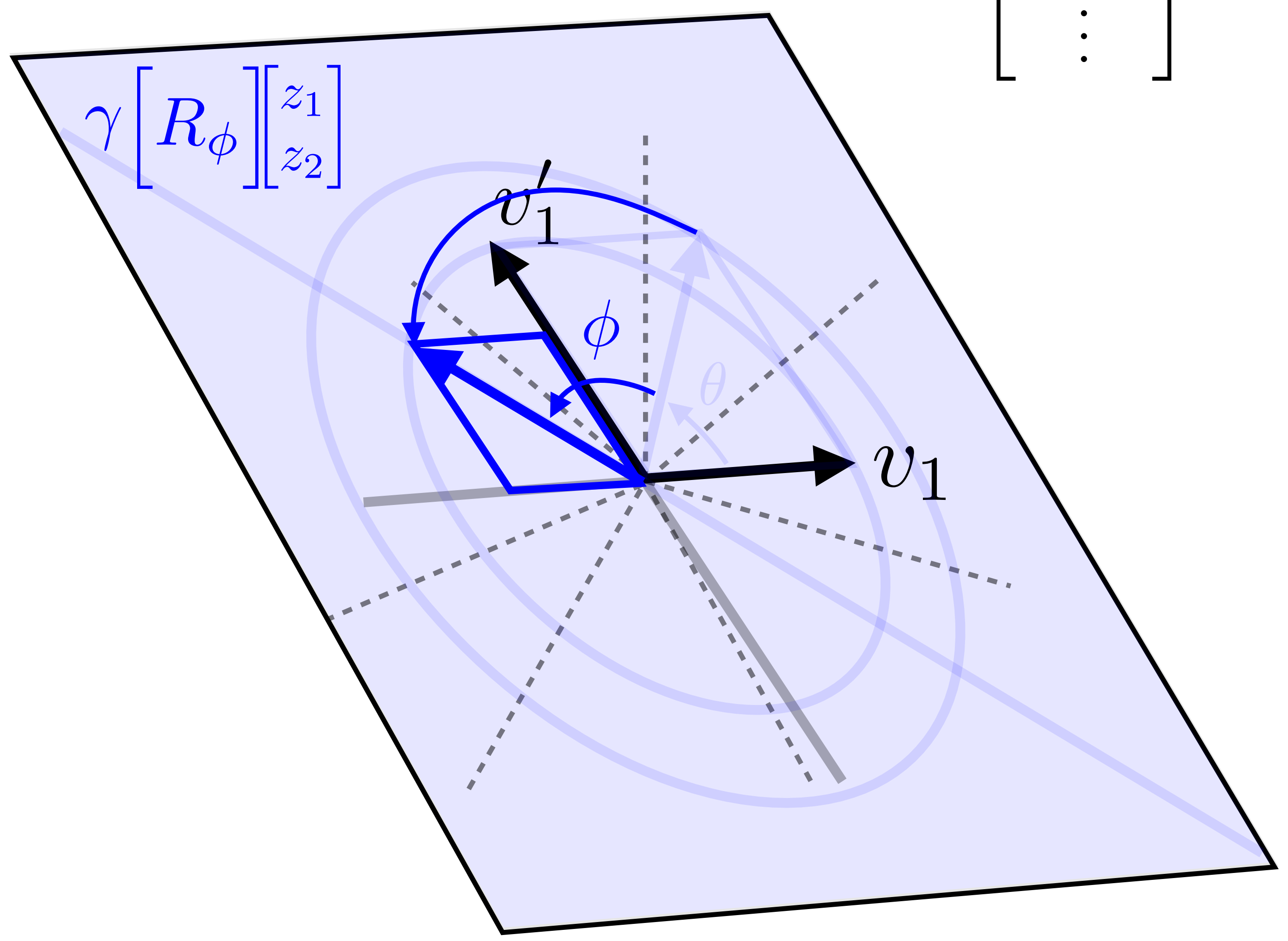
$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} \cdots \begin{bmatrix} | \\ V_n \\ | \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \;\; W_n^* \;\; - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma < 1$

$\mathbb{R}^n$



$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \Big[\; V \;\Big]\Big[\; D \;\Big]\Big[\; V^{-1} \Big]$$

$$\Big[\; A \;\Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

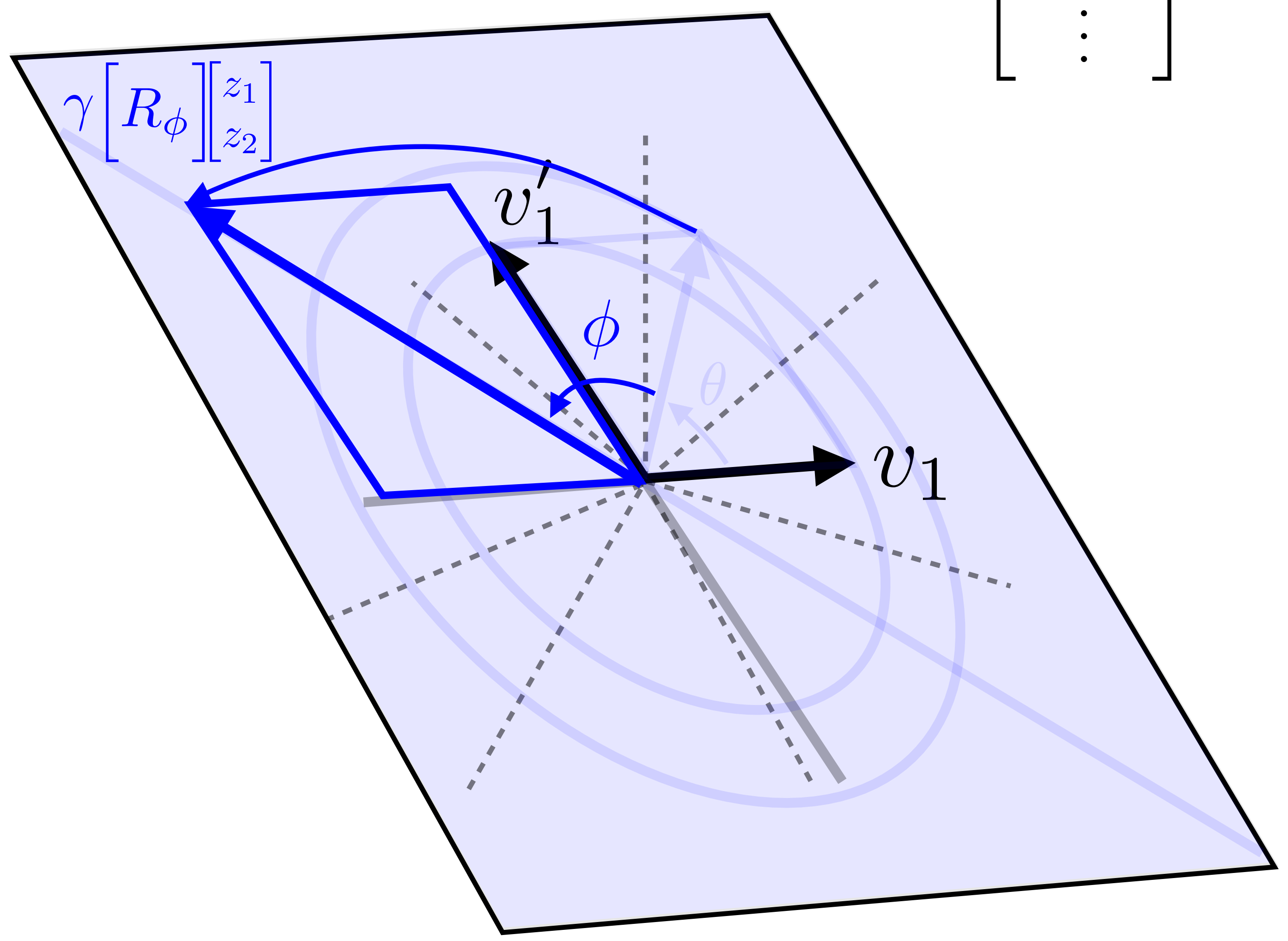
$$\Big[\; A \;\Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \big[\lambda_i\big]\big[- \quad W_i^* \quad -\big]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & {w_1'}^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma > 1$

$\mathbb{R}^n$



$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

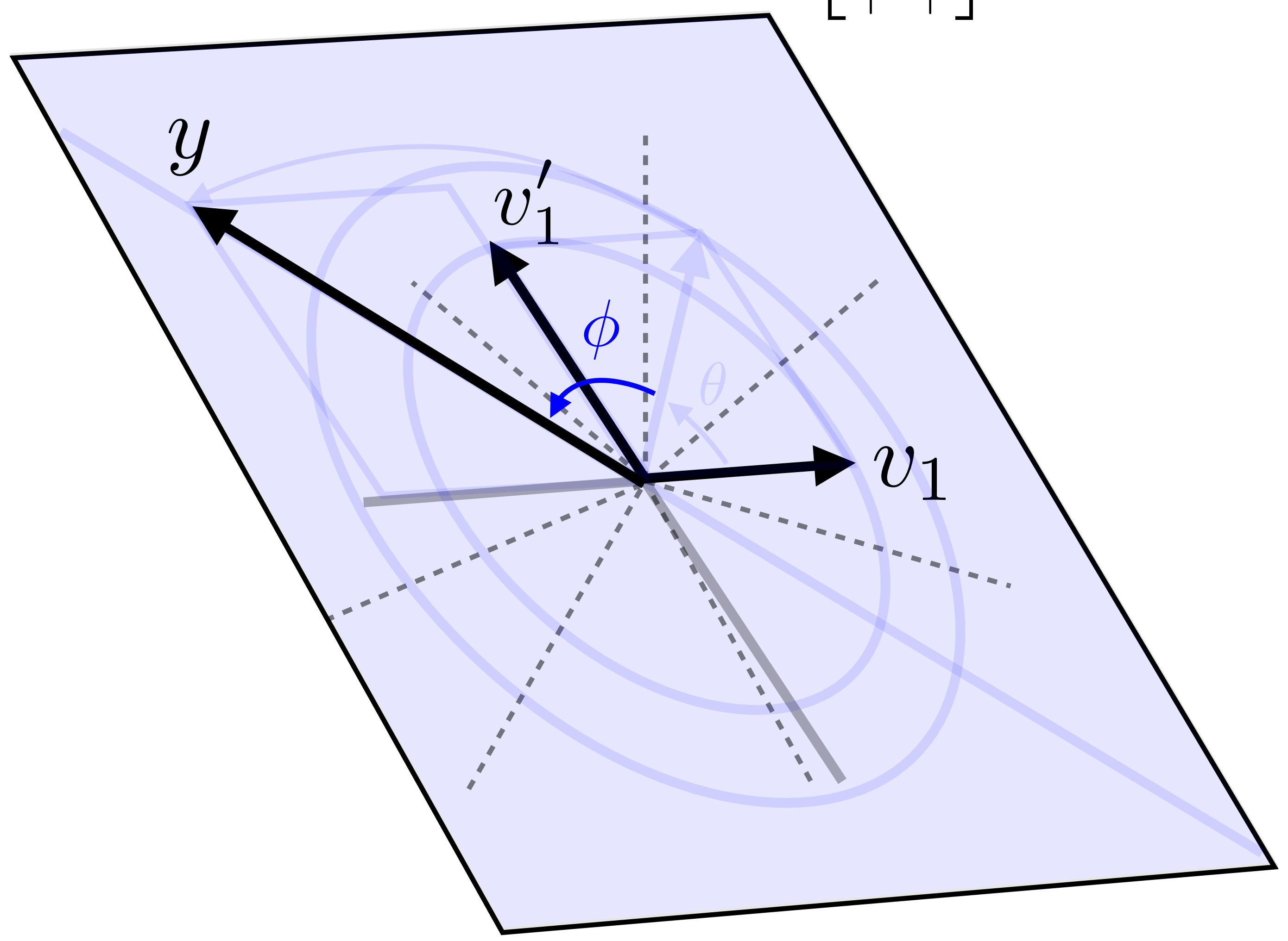
$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \underbrace{\begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{y = \gamma \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} R_\phi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}$$

$\gamma > 1$

$\mathbb{R}^n$



$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Spectral Mapping Theorem

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization $A = VDV^{-1}$

$$A^k = VD^kV^{-1}$$

Powers of A

$$A^k = VDV^{-1} \times VDV^{-1} \times \cdots \times VDV^{-1}$$
$$= VDV^{-1}VDV^{-1} \cdots VDV^{-1}$$
$$= VD^kV^{-1}$$

$$= \begin{bmatrix} & & \\ & V & \\ & & \end{bmatrix} \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} & & \\ & V^{-1} & \\ & & \end{bmatrix}$$

Polynomials of A

polynomial $\Psi(s) = \alpha_k s^k + \alpha_{k-1}s^{k-1} + \alpha_{k-2}s^{k-2} + \cdots + \alpha_1 s + \alpha_0 1$

plugging in A…

$$\Psi(A) = \alpha_k A^k + \alpha_{k-1}A^{k-1} + \alpha_{k-2}A^{k-2} + \cdots + \alpha_1 A + \alpha_0 I$$
$$= \alpha_k VD^kV^{-1} + \alpha_{k-1}VD^{k-1}V^{-1} + \alpha_{k-2}VD^{k-2}V^{-1} + \cdots + \alpha_1 VDV^{-1} + \alpha_0 VV^{-1}$$
$$= V\Big(\alpha_k D^k + \alpha_{k-1}D^{k-1} + \alpha_{k-2}D^{k-2} + \cdots + \alpha_1 D + \alpha_0 I\Big)V^{-1}$$

$$\Psi(A) = V\Psi(D)V^{-1}$$

$$= \begin{bmatrix} & & \\ & V & \\ & & \end{bmatrix} \begin{bmatrix} \Psi(\lambda_1) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \Psi(\lambda_n) \end{bmatrix} \begin{bmatrix} & & \\ & V^{-1} & \\ & & \end{bmatrix}$$

Spectral Mapping Theorem

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization $A = VDV^{-1}$

Powers of A

$$A^k = VDV^{-1} \times VDV^{-1} \times \cdots \times VDV^{-1}$$
$$= VDV^{-1}VDV^{-1} \cdots VDV^{-1}$$
$$= VD^kV^{-1}$$

$$A^k = VD^kV^{-1}$$

$$= \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} & V^{-1} & \end{bmatrix}$$

Polynomials of A

polynomial $\Psi(s) = \alpha_k s^k + \alpha_{k-1}s^{k-1} + \alpha_{k-2}s^{k-2} + \cdots + \alpha_1 s + \alpha_0 1$

$$\Psi(A) = V\Big(\alpha_k D^k + \alpha_{k-1}D^{k-1} + \alpha_{k-2}D^{k-2} + \cdots + \alpha_1 D + \alpha_0 I\Big)V^{-1}$$

$$\Psi(A) = V\Psi(D)V^{-1}$$

$$= \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} \Psi(\lambda_1) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \Psi(\lambda_n) \end{bmatrix} \begin{bmatrix} & V^{-1} & \end{bmatrix}$$

Spectral Mapping Theorem for f(s) analytic

$$\lambda \in \mathrm{eig}(A) \Rightarrow f(\lambda) \in \mathrm{eig}\big(f(A)\big)$$

$A, f(A)$ have the same eigenvectors

Spectral Mapping Theorem

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization $A = VDV^{-1}$

Powers of A

$$A^k = VDV^{-1} \times VDV^{-1} \times \cdots \times VDV^{-1}$$
$$= VDV^{-1}VDV^{-1} \cdots VDV^{-1}$$
$$= VD^kV^{-1}$$

$$A^k = VD^kV^{-1}$$
$$= \Big[\ V\ \Big]\begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix}\Big[\ V^{-1}\ \Big]$$

Polynomials of A

polynomial $\Psi(s) = \alpha_k s^k + \alpha_{k-1}s^{k-1} + \alpha_{k-2}s^{k-2} + \cdots + \alpha_1 s + \alpha_0 1$

$$\Psi(A) = V\Big(\alpha_k D^k + \alpha_{k-1}D^{k-1} + \alpha_{k-2}D^{k-2} + \cdots + \alpha_1 D + \alpha_0 I\Big)V^{-1}$$

Spectral Mapping Theorem for f(s) analytic

$$\lambda \in \mathrm{eig}(A) \Rightarrow f(\lambda) \in \mathrm{eig}\Big(f(A)\Big)$$

$A, f(A)$ have the same eigenvectors

Specific Useful Case: Matrix Exponential

$$e^{At} = I + At + \tfrac{1}{2!}A^2t^2 + \tfrac{1}{3!}A^3t^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \tfrac{1}{k!}A^k t^k$$

Derivative: $\frac{d}{dt}\Big(e^{At}\Big) = Ae^{At}$

- can see from polynomial definition
- related to definition of e

Spectral Mapping Theorem

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization $A = VDV^{-1}$

Powers of A

$$A^k = VDV^{-1} \times VDV^{-1} \times \cdots \times VDV^{-1}$$

$$= VDV^{-1}VDV^{-1} \cdots VDV^{-1}$$

$$= VD^kV^{-1}$$

$$A^k = VD^kV^{-1}$$

$$= \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} & V^{-1} & \end{bmatrix}$$

Polynomials of A

polynomial $\Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \cdots + \alpha_1 s + \alpha_0 1$

$$\Psi(A) = V\left(\alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \cdots + \alpha_1 D + \alpha_0 I\right)V^{-1}$$

Spectral Mapping Theorem for f(s) analytic

$$\lambda \in \mathrm{eig}(A) \Rightarrow f(\lambda) \in \mathrm{eig}\big(f(A)\big)$$

$A, f(A)$ have the same eigenvectors

Specific Useful Case: Matrix Exponential

$$e^{At} = I + At + \tfrac{1}{2!}A^2t^2 + \tfrac{1}{3!}A^3t^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \tfrac{1}{k!} A^k t^k$$

$$e^{At} = Ve^{Dt}V^{-1}$$

$$= \begin{bmatrix} & V & \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} & V^{-1} & \end{bmatrix}$$