

Rotation Matrices

Linear Algebra

Winter 2022 - Dan Calderone

Rotation Matrices

Special Orthogonal Group: $\text{SO}(n)$ Lie group

$$R^T R = R R^T = I \quad \dots \text{columns \& rows orthonormal}$$

$$\det(R) = 1 \quad \dots \text{no reflections}$$

$$R \in \text{SO}(2) \subset \mathbb{R}^{2 \times 2}$$

Diagonalization:

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}}$$

right
evecs left
evecs

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$U^* U = U U^* = I$$

\iff

$$R^{-1} = R^T$$

...columns orthonormal

$$R^T R = \begin{bmatrix} - & R_1^T & - \\ \vdots & \ddots & \vdots \\ - & R_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ R_1 & \cdots & R_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} R_1^T R_1 & \cdots & R_1^T R_n \\ \vdots & \ddots & \vdots \\ R_n^T R_1 & \cdots & R_n^T R_1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

...similar for rows

$$R R^T = \begin{bmatrix} - & r_1^T & - \\ \vdots & \ddots & \vdots \\ - & r_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} r_1^T r_1 & \cdots & r_1^T r_n \\ \vdots & \ddots & \vdots \\ r_n^T r_1 & \cdots & r_n^T r_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Rotation Matrices - 2D

Special Orthogonal Group: $\text{SO}(n)$ Lie group

$$R^T R = RR^T = I \quad \dots \text{columns & rows orthonormal}$$

$$\det(R) = 1 \quad \dots \text{no reflections}$$

$$R \in \text{SO}(2) \subset \mathbb{R}^{2 \times 2}$$

Diagonalization:

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}}$$

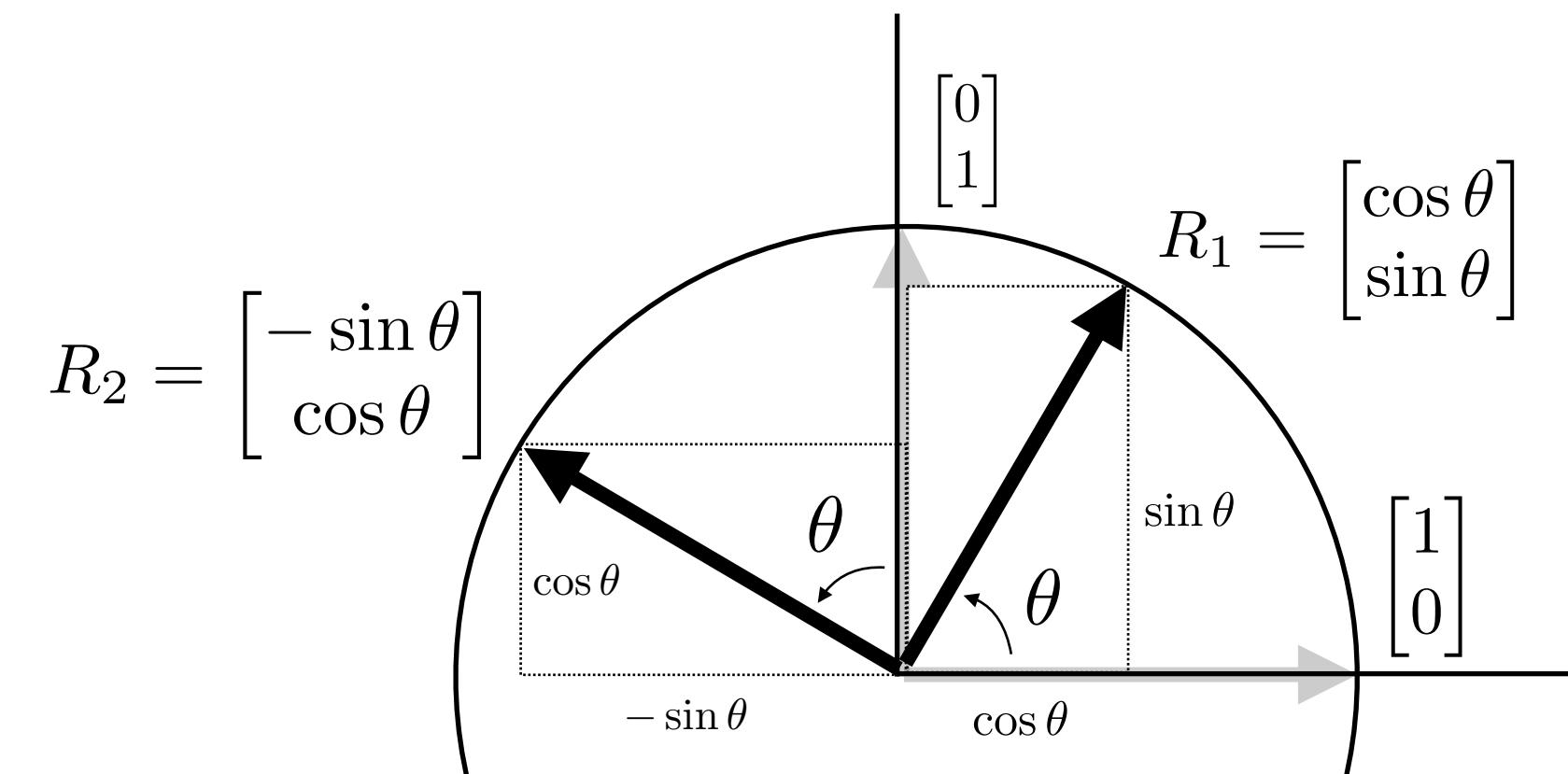
right
evecs

left
evecs

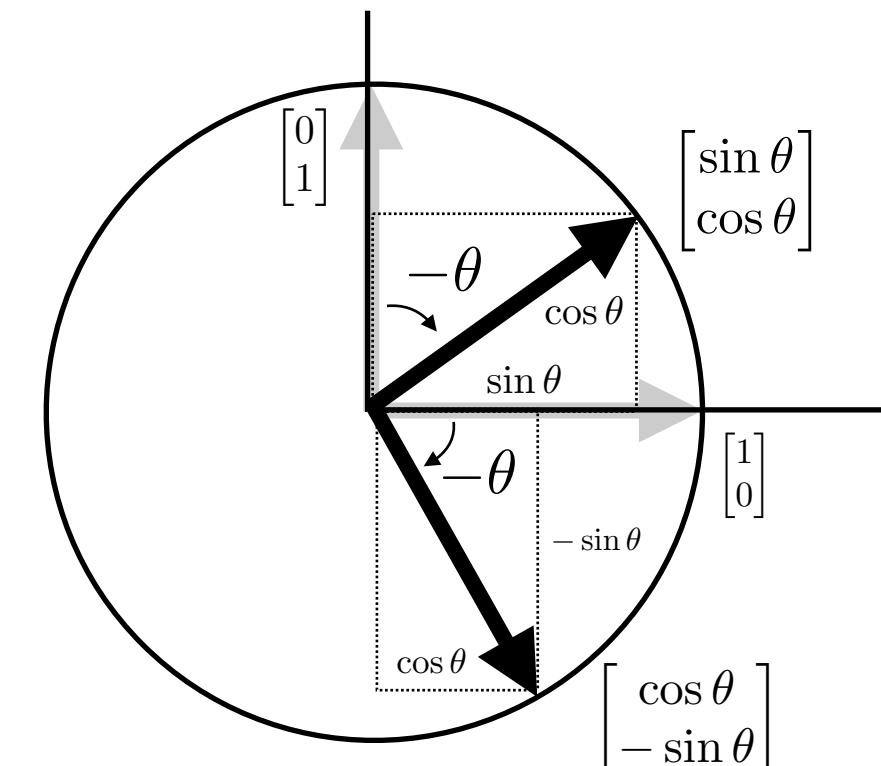
$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$U^* U = U U^* = I$$

$$R = \begin{bmatrix} R_1 & R_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



$$R^{-1} = R^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$



Rotation Matrices - 3D

Special Orthogonal Group: $\text{SO}(n)$ Lie group

$$R^T R = R R^T = I \quad \dots \text{columns & rows orthonormal}$$

$$\det(R) = 1 \quad \dots \text{no reflections}$$

$$R \in \text{SO}(3) \subset \mathbb{R}^{3 \times 3}$$

Diagonalization:

$$R = \begin{bmatrix} | & | & | \\ \frac{1}{\sqrt{2}}(v_1 + iv_2) & \frac{1}{\sqrt{2}}(v_1 - iv_2) & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & \frac{1}{\sqrt{2}}(v_1^T - iv_2^T) & - \\ - & \frac{1}{\sqrt{2}}(v_1^T + iv_2^T) & - \\ v_3^T & - \end{bmatrix}$$

U right eigenvectors

U^* left eigenvectors

unitary:

$$U^* U = U U^* = I$$

Rotation plane Rotation axis

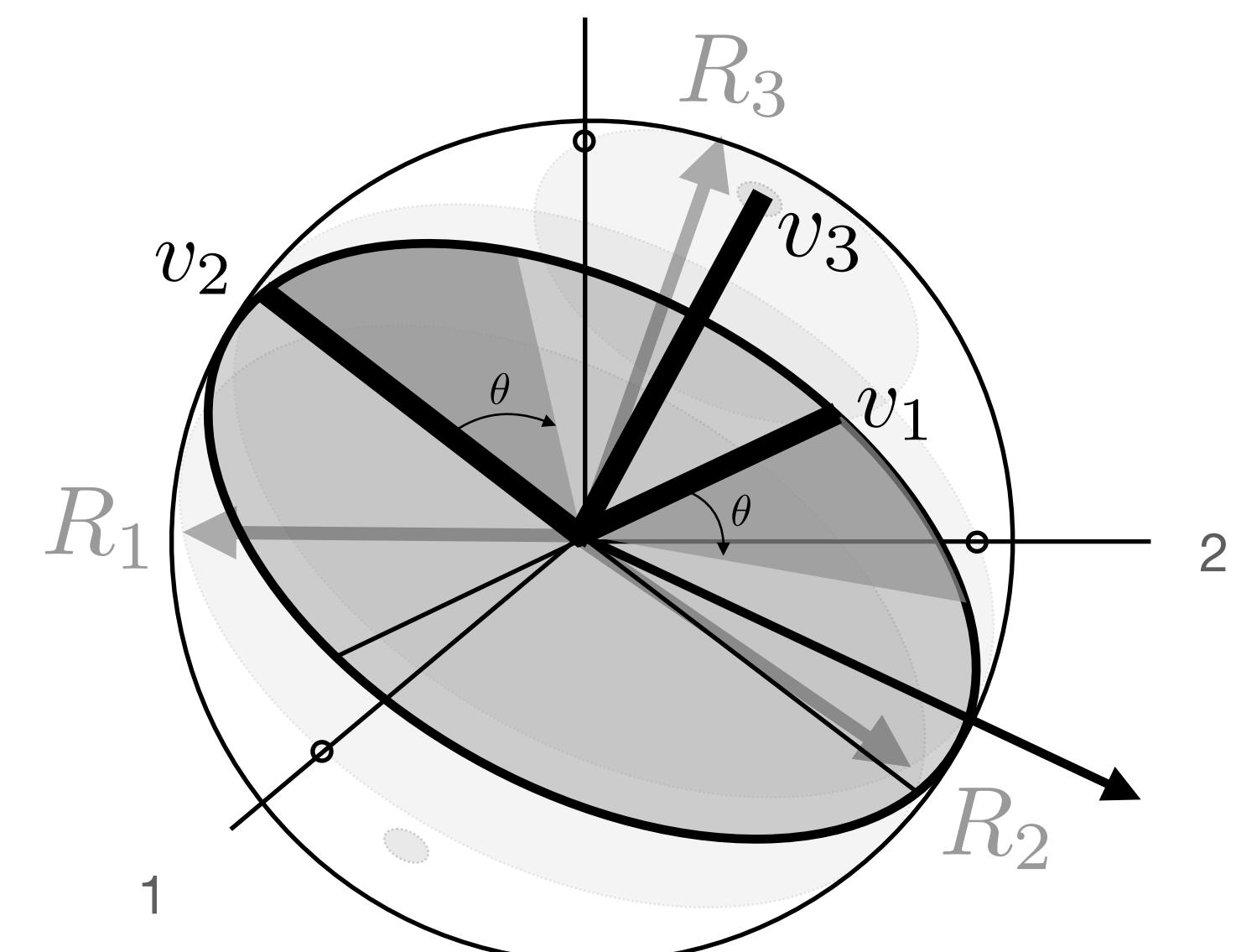
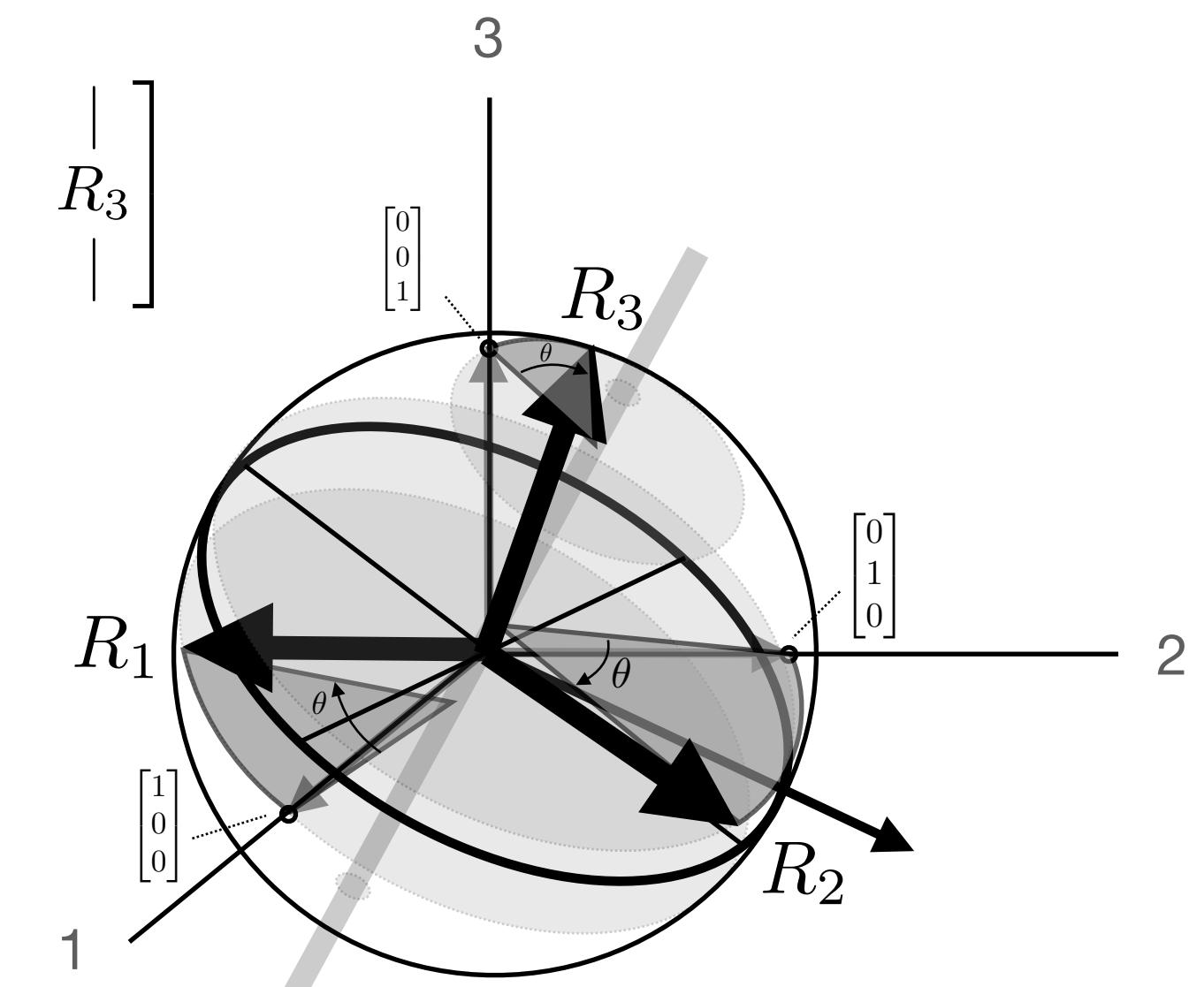
$$R = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & v_3^T & - \end{bmatrix}$$

orthonormal: V

$$V^T V = V V^T = I$$

$$V^T$$

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$



Rotation Matrices - 3D

Special Orthogonal Group: $\text{SO}(n)$ Lie group

$$R^T R = R R^T = I \quad \dots \text{columns & rows orthonormal}$$

$$\det(R) = 1 \quad \dots \text{no reflections}$$

$$R \in \text{SO}(3) \subset \mathbb{R}^{3 \times 3}$$

Diagonalization:

$$R = \begin{bmatrix} | & | & | \\ \frac{1}{\sqrt{2}}(v_1 + iv_2) & \frac{1}{\sqrt{2}}(v_1 - iv_2) & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}}(v_1^T - iv_2^T) & - \\ -\frac{1}{\sqrt{2}}(v_1^T + iv_2^T) & - \\ v_3^T & - \end{bmatrix}$$

U right eigenvectors

U^* left eigenvectors

unitary:

$$U^* U = U U^* = I$$

Rotation plane Rotation axis

$$R = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -v_1^T & - \\ -v_2^T & - \\ -v_3^T & - \end{bmatrix}$$

orthonormal: V

$$V^T V = V V^T = I$$

$$V^T$$

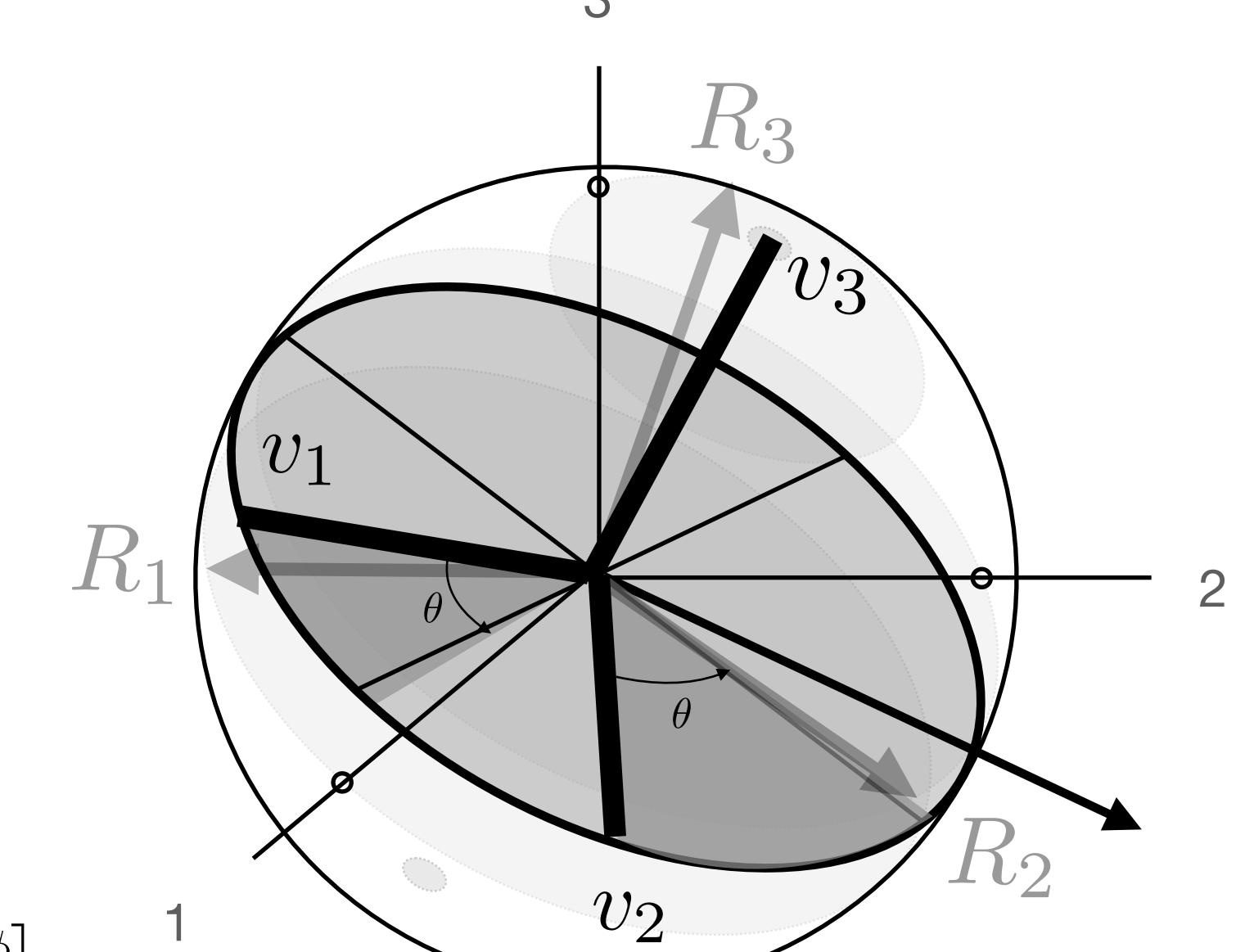
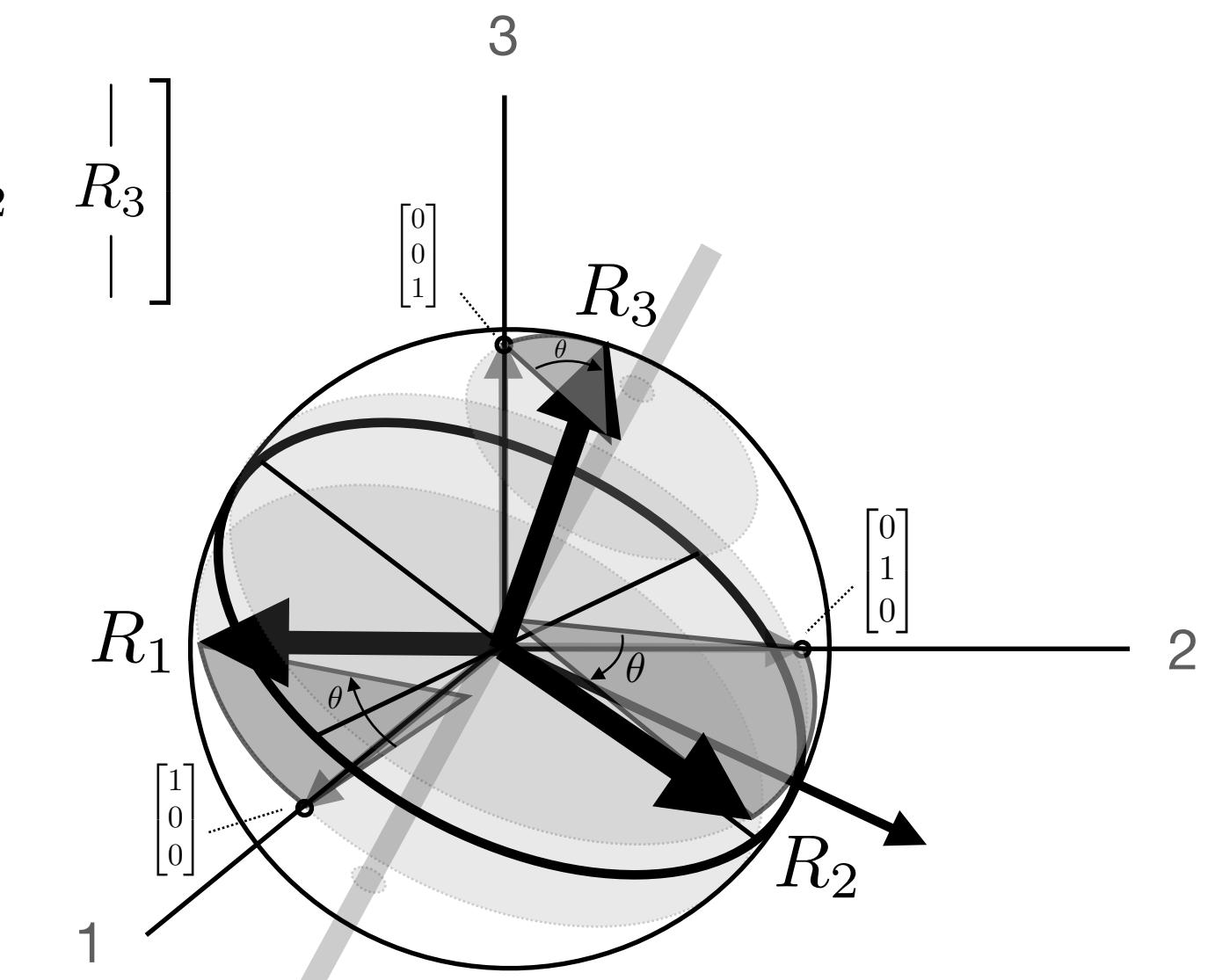
Note: can change
by a phase shift ϕ v_1, v_2

$$\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \Leftarrow \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix}$$

...since

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} = \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\phi & s\phi \\ -s\phi & c\phi \end{bmatrix}$$

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$



Skew symmetric matrices

Special orthogonal Lie algebra $\text{so}(n)$ Lie algebra

$$K = -K^T$$

...skew symmetric matrix

$$k \in \mathbb{R}^3 \quad \|k\|_2 = 1$$

...axis of rotation

$$K = \hat{k} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

...hat operator

$$R = e^{K\theta}$$

...matrix exponential

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

$$K\theta = \begin{bmatrix} \frac{1}{\sqrt{2}}(v_1 + iv_2) & \frac{1}{\sqrt{2}}(v_1 - iv_2) & | \\ | & | & k \end{bmatrix}$$

right eigenvectors

Diagonalization:

$$\begin{bmatrix} | & | & | \\ -i\theta & 0 & 0 \\ 0 & i\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & \frac{1}{\sqrt{2}}(v_1^T - iv_2^T) & - \\ - & \frac{1}{\sqrt{2}}(v_1^T + iv_2^T) & - \\ - & k^T & - \end{bmatrix}$$

left eigenvectors

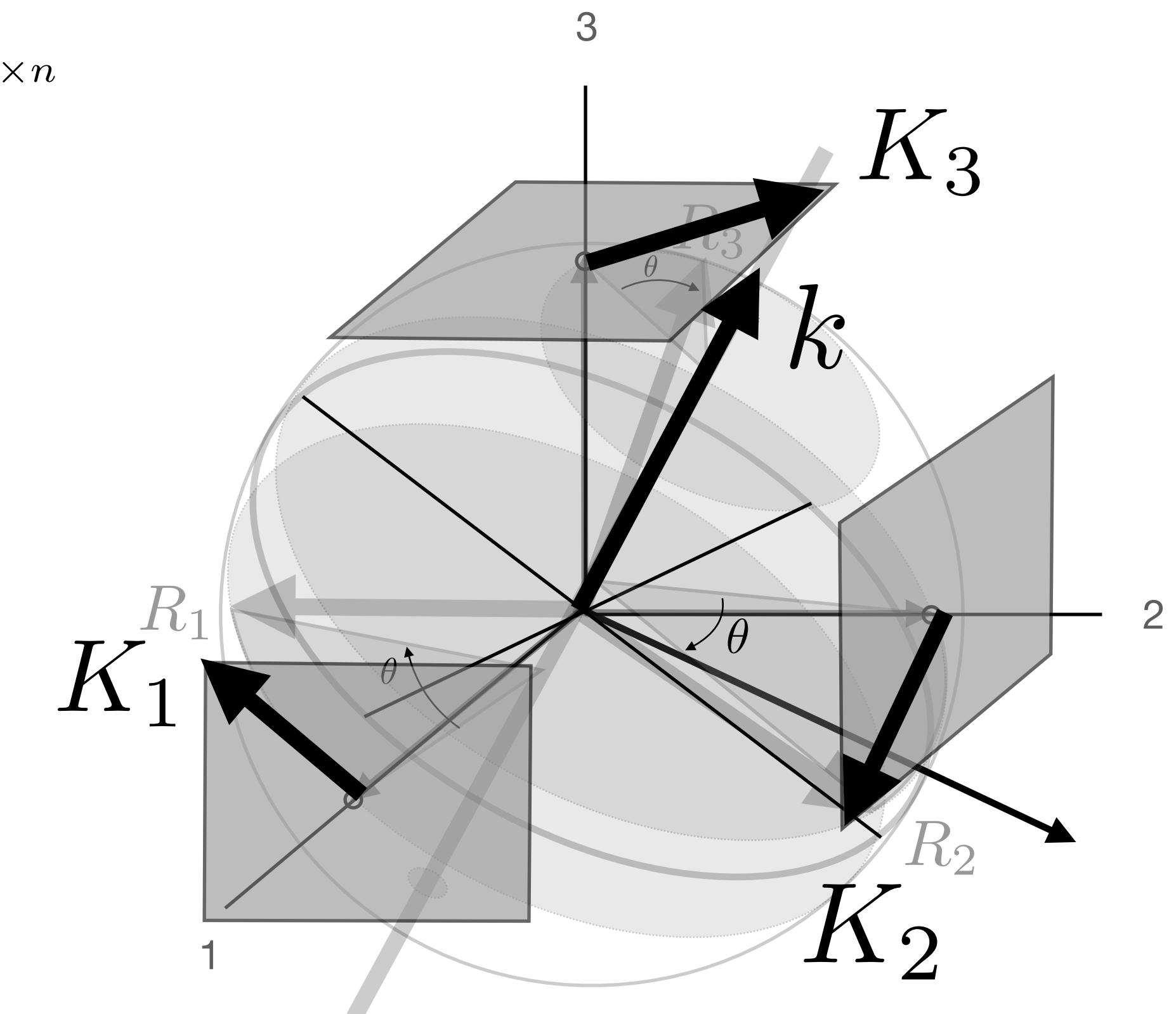
Rotation plane Rotation axis

$$K\theta = \underbrace{\begin{bmatrix} | & | & | \\ v_1 & v_2 & k \\ | & | & | \end{bmatrix}}_{\text{orthonormal: } V} \begin{bmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & k^T & - \end{bmatrix}$$

orthonormal: V

$$V^T V = V V^T = I$$

$$V^T$$



$$K = \begin{bmatrix} | & | & | \\ K_1 & K_2 & K_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$