Dual Programs: LP

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Primal Program

Dual Programs: LP

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Primal Program

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

Dual Programs: LP

$$\max_{x} \quad r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Primal Program

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: LP

$$\max_{x} \quad r^\top x$$
$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Primal Program

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_{x} \quad (r^\top + \lambda^\top A + \mu^\top C) x - \lambda^\top b - \mu^\top d$$

Dual Programs: LP

Primal Program

$$\max_x \quad r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x \quad \underbrace{(r^\top + \lambda^\top A + \mu^\top C)}_{\substack{\text{must be 0 for inner} \\ \text{problem to be bounded}}} x \quad - \lambda^\top b - \mu^\top d$$

Dual Programs: LP

Primal Program

$$\max_{x} \quad r^\top x$$
$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_{x} \quad \underbrace{(r^\top + \lambda^\top A + \mu^\top C)}_{\substack{\text{must be 0 for inner} \\ \text{problem to be bounded}}} x \; - \lambda^\top b - \mu^\top d$$

Dual Program

$$\min_{\lambda, \mu} \quad -\lambda^\top b - \mu^\top d$$
$$\text{s.t.} \quad r^\top + \lambda^\top A + \mu^\top C = 0 \,, \quad \mu \geq 0$$

Dual Programs: QP

$$\max_x \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Primal Program

Note: $Q = Q^\top \prec 0$

Dual Programs: QP

$$\max_x \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

Primal Program

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \; \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

Dual Programs: QP

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

Primal Program

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

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$$\max_{x} \quad \min_{\lambda, \mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

Dual Programs: QP

Primal Program

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

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$$\min_{\lambda, \mu \geq 0} \; \max_{x} \quad \tfrac{1}{2}x^\top Q x \; + \; \big(r^\top + \lambda^\top A + \mu^\top C\big) x \; - \lambda^\top b - \mu^\top d$$

Dual Programs: QP

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x \ + \ r^\top x$$

Primal Program

$$\text{s.t.} \quad Ax = b \quad \lambda \qquad Cx \geq d \quad \mu$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \ \tfrac{1}{2}x^\top Q x \ + r^\top x \ + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\max_{x} \quad \min_{\lambda, \mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \ + r^\top x \ + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\min_{\lambda, \mu \geq 0} \quad \max_{x} \quad \underbrace{\tfrac{1}{2}x^\top Q x \ + \ \big(r^\top + \lambda^\top A + \mu^\top C\big)x}_{\text{maximize explicitly...}} \ -\lambda^\top b - \mu^\top d$$

Dual Programs: QP

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

Primal Program

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\max_{x} \quad \min_{\lambda,\mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\min_{\lambda,\mu \geq 0} \quad \max_{x} \quad \underbrace{\tfrac{1}{2}x^\top Q x \; + \; \big(r^\top + \lambda^\top A + \mu^\top C\big)\, x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C,$

Dual Programs: QP

Primal Program

$$\max_x \quad \tfrac{1}{2}x^\top Q x \; + \; r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \quad \min_{\lambda, \mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \; + r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \quad \max_x \quad \underbrace{\tfrac{1}{2}x^\top Q x \; + \; (r^\top + \lambda^\top A + \mu^\top C) x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$, $\quad \dfrac{\partial}{\partial x}\left(\tfrac{1}{2}x^\top Q x + \xi^\top x\right) = 0$

Dual Programs: QP

$$\max_x \quad \tfrac{1}{2}x^\top Q x \;+\; r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Primal Program

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x \;+ r^\top x \;+ \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \quad \min_{\lambda, \mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \;+ r^\top x \;+ \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \quad \max_x \quad \underbrace{\tfrac{1}{2}x^\top Q x \;+\; (r^\top + \lambda^\top A + \mu^\top C)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C, \quad \frac{\partial}{\partial x}\left(\tfrac{1}{2}x^\top Q x + \xi^\top x\right) = 0 \quad \Rightarrow \quad x = -Q^{-1}\xi$

Dual Programs: QP

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x \ + \ r^\top x$$

Primal Program

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x \ + r^\top x \ + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\max_{x} \quad \min_{\lambda,\mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \ + r^\top x \ + \lambda^\top \big(Ax - b\big) + \mu^\top \big(Cx - d\big)$$

$$\min_{\lambda,\mu \geq 0} \quad \max_{x} \quad \underbrace{\tfrac{1}{2}x^\top Q x \ + \ \big(r^\top + \lambda^\top A + \mu^\top C\big) x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$, $\quad \frac{\partial}{\partial x}\big(\tfrac{1}{2}x^\top Q x + \xi^\top x\big) = 0 \quad \Rightarrow \quad x = -Q^{-1}\xi$

Plug in x...

Dual Programs: QP

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x \;+\; r^\top x$$

Primal Program

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x, \lambda, \mu) = \tfrac{1}{2}x^\top Q x \;+ r^\top x \;+ \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \quad \min_{\lambda,\mu \geq 0} \quad \tfrac{1}{2}x^\top Q x \;+ r^\top x \;+ \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda,\mu \geq 0} \quad \max_{x} \quad \underbrace{\tfrac{1}{2}x^\top Q x \;+\; (r^\top + \lambda^\top A + \mu^\top C)\, x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$, $\quad \frac{\partial}{\partial x}\left(\tfrac{1}{2}x^\top Q x + \xi^\top x\right) = 0 \quad \Rightarrow \quad x = -Q^{-1}\xi$

Plug in x...

$$\min_{\xi,\lambda,\mu} \quad -\tfrac{1}{2}\xi^\top Q^{-1}\xi \;\; -\lambda^\top b - \mu^\top d$$

Dual Program

$$\text{s.t.} \quad \xi^\top = r^\top + \lambda^\top A + \mu^\top C\,, \quad \mu \geq 0$$

Note: $-Q^{-1} = -Q^{-\top} \succ 0$

Dual Programs: SOCP (lin objective)

$$\max_x \quad r^\top x$$

Primal Program

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \quad \boxed{||E_i x + e_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i}$$

Dual Programs: SOCP (lin objective)

$$\max_{x, y_i} \quad r^\top x$$

Primal Program

$$\text{s.t.} \quad Ax = b \quad \lambda \qquad Cx \geq d \quad \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$$

$$y_i = E_i x + b_i \quad i \in \mathcal{I} \quad \theta_i$$

Dual Programs: SOCP (lin objective)

$$\max_{x, y_i} \quad r^\top x$$

Primal Program

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \;\; i \in \mathcal{I} \;\; \nu_i$$

$$y_i = E_i x + b_i \;\; i \in \mathcal{I} \;\; \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

Dual Programs: SOCP (lin objective)

$$\max_{x, y_i} \quad r^\top x$$

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$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \quad \boxed{||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i}$$

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$$\max_{x, y_i} \; \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

Dual Programs: SOCP (lin objective)

Primal Program

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$$\max_{x, y_i} \; \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \; \max_{x, y_i} \quad \left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right) x \quad - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i - \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x,y_i} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \;\; i \in \mathcal{I} \;\; \nu_i$$

$$y_i = E_i x + b_i \;\; i \in \mathcal{I} \;\; \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x,y_i} \min_{\substack{\lambda,\mu \geq 0 \\ \nu_i \geq 0, \theta_i}} r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda,\mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x,y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right)}_{\text{must be 0 for inner problem to be bounded}} x \; - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \quad \lambda \qquad Cx \geq d \quad \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$$

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$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x, y_i} \; \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \; \max_{x, y_i} \quad \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right)}_{\text{must be 0 for inner problem to be bounded}} x \quad -\lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

$$\sup_{y_i} - \quad \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x,y_i} \quad r^\top x$$

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$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \ + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\max_{x,y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \ + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x,y_i} \quad \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right)}_{\text{must be 0 for inner problem to be bounded}} x \quad -\lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

$$\sup_{y_i} - \ \left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)$$

$$= \begin{cases} 0 & ; \ ||\theta_i||_2 \leq \nu_i \\ \infty & ; \ \text{otherwise} \end{cases}$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \;\; i \in \mathcal{I} \;\; \nu_i$$

$$y_i = E_i x + b_i \;\; i \in \mathcal{I} \;\; \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x, y_i} \; \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \; \max_{x, y_i} \quad \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right) x}_{\text{must be 0 for inner problem to be bounded}} \; - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{-\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

$$\text{Suppose} \quad ||\theta_i||_2 \leq \nu_i \qquad (\nu_i \geq 0)$$

$$-\theta_i^\top y_i \leq \nu_i ||y_i||_2 \qquad \forall y_i$$

$$-\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right) \leq 0 \qquad \forall y_i$$

$$\sup_{y_i} -\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)$$

$$= \begin{cases} 0 & ; \; ||\theta_i||_2 \leq \nu_i \\ \infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \quad \boxed{||y_i||_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i}$$

$$\boxed{y_i = E_i x + b_i \quad i \in \mathcal{I} \quad \theta_i}$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \;+\; \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2 \right) \;+\; \sum_i \theta_i^\top \left(E_i x + e_i - y_i \right)$$

$$\max_{x, y_i} \quad \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \;+\; \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2 \right) \;+\; \sum_i \theta_i^\top \left(E_i x + e_i - y_i \right)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad \max_{x, y_i} \quad \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i \right] \right) x}_{\text{must be 0 for inner problem to be bounded}} \quad - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i \right]$$

$$+ \sum_i \underbrace{- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i \right)}_{\text{must be bounded above...}}$$

$$\text{Suppose} \quad ||\theta_i||_2 > \nu_i \qquad (\nu_i \geq 0)$$

$$\text{take} \;\; y_i = -s\theta_i \qquad s \in \mathbb{R}_+$$

$$- \left(\nu_i ||y_i||_2 + \theta_i^\top y_i \right) = - \; s\left(\nu_i ||\theta_i||_2 - ||\theta_i||_2^2 \right) > 0$$

$$\sup_{y_i} - \left(\nu_i ||y_i||_2 + \theta_i^\top y_i \right)$$

$$= \begin{cases} 0 & ; \; ||\theta_i||_2 \leq \nu_i \\ \infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SOCP (lin objective)

Primal Program

$$\max_{x, y_i} \quad r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu \qquad \|y_i\|_2 \leq g_i^\top x + h_i \;\; i \in \mathcal{I} \;\; \nu_i$$

$$y_i = E_i x + b_i \;\; i \in \mathcal{I} \;\; \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - \|y_i\|_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\max_{x, y_i} \; \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \quad r^\top x \; + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - \|y_i\|_2\right) + \sum_i \theta_i^\top \left(E_i x + e_i - y_i\right)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \; \max_{x, y_i} \quad \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\right)}_{\text{must be 0 for inner problem to be bounded}} x \quad - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right] + \sum_i \underbrace{- \left(\nu_i \|y_i\|_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

Dual Program

$$\min_{\substack{\lambda, \mu \\ \nu_i, \theta_i}} \quad -\lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$\text{s.t.} \quad r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right] = 0, \quad \mu \geq 0,$$

$$\|\theta_i\|_2 \leq \nu_i, \;\; \nu_i \geq 0 \quad i \in \mathcal{I}$$

$$\sup_{y_i} - \left(\nu_i \|y_i\|_2 + \theta_i^\top y_i\right) = \begin{cases} 0 & ; \;\; \|\theta_i\|_2 \leq \nu_i \\ \infty & ; \;\; \text{otherwise} \end{cases}$$

Dual Programs: SOCP (quad objective)

Primal Program

$$\max_{x,y_i} \boxed{\tfrac{1}{2}x^\top Q x} + r^\top x$$

$$\text{s.t.} \quad Ax = b \;\; \lambda \qquad Cx \geq d \;\; \mu \qquad ||y_i||_2 \leq g_i^\top x + h_i \;\; i \in \mathcal{I} \;\; \nu_i$$

$$y_i = E_i x + b_i \;\; i \in \mathcal{I} \;\; \theta_i$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \boxed{\tfrac{1}{2}x^\top Q x} + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x,y_i} \min_{\substack{\lambda,\mu\geq 0\\ \nu_i \geq 0, \theta_i}} \boxed{\tfrac{1}{2}x^\top Q x} + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i \left(g_i^\top x + h_i - ||y_i||_2\right) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda,\mu\geq 0\\ \nu_i \geq 0, \theta_i}} \max_{x,y_i} \underbrace{\boxed{\tfrac{1}{2}x^\top Q x} + \Big(r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right]\Big)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$+ \sum_i \underbrace{-\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right)}_{\text{must be bounded above...}}$$

Dual Program

$$\min_{\substack{\lambda,\mu,\xi\\ \nu_i,\theta_i}} \boxed{-\tfrac{1}{2}\xi^\top Q^{-1}\xi} - \lambda^\top b - \mu^\top d + \sum_i \left[\nu_i h_i + \theta_i^\top e_i\right]$$

$$\text{s.t.} \quad r^\top + \lambda^\top A + \mu^\top C + \sum_i \left[\nu_i g_i^\top + \theta_i^\top E_i\right] = \boxed{\xi^\top,} \quad \mu \geq 0,$$

$$||\theta_i||_2 \leq \nu_i, \;\; \nu_i \geq 0 \quad i \in \mathcal{I}$$

$$\sup_{y_i} -\left(\nu_i ||y_i||_2 + \theta_i^\top y_i\right) = \begin{cases} 0 & ; \; ||\theta_i||_2 \leq \nu_i \\ \infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \langle R, X \rangle$$

$$\text{s.t.} \quad \boxed{\langle A_i, X \rangle = b_i \;\; i \in \mathcal{I} \;\; \lambda_i} \quad \boxed{\langle C_j, X \rangle \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j} \quad \boxed{X = X^\top \succeq 0 \;\; U}$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \; \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \; \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ -\infty & ; \; \text{otherwise} \end{cases}$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} \boxed{0 \quad ; \; X = X^\top \succeq 0} \\ -\infty \quad ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top)$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} \boxed{0 \quad ; \; X = X^\top \succeq 0} \\ -\infty \quad ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \underset{\text{unique}}{\text{exists}} \quad X^{1/2} = X^{\top/2} \succeq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} \boxed{0 \quad ; \; X = X^\top \succeq 0} \\ -\infty \quad ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \substack{\text{exists} \\ \text{unique}} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(X^{\top/2} U^\top X^{1/2})$$

Dual Programs: SDP (lin objective)

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

Primal Program

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} \boxed{0 \quad ; \; X = X^\top \succeq 0} \\ -\infty \quad ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix}\text{exists} \\ \text{unique}\end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(\underbrace{X^{\top/2} U^\top X^{1/2}}_{\text{congruent to U}})$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \;\; i \in \mathcal{I} \;\; \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \;\; j \in \mathcal{J} \;\; \mu_j \qquad X = X^\top \succeq 0 \;\; U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} \boxed{0 \quad ; \; X = X^\top \succeq 0} \\ -\infty \quad ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \underset{\text{unique}}{\text{exists}} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(\underbrace{X^{\top/2} U^\top X^{1/2}}_{\text{congruent to U}})$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

Dual Programs: SDP (lin objective)

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

Primal Program

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} \boxed{0 \quad ; \; X = X^\top \succeq 0} \\ -\infty \quad ; \; \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix}\text{exists} \\ \text{unique}\end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(\underbrace{X^{\top/2} U^\top X^{1/2}}_{\text{congruent to U}})$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

$$\mathrm{Tr} = \text{sum eigs} \quad \Rightarrow \qquad \iff \mathrm{Tr}(X^{\top/2} U^\top X^{1/2}) \geq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \ \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \ \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \ \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \ \mathrm{Tr}(U^\top X) \ = \begin{cases} \boxed{0 \qquad ; \ X = X^\top \succeq 0} \\ -\infty \quad ; \ \text{otherwise} \end{cases}$$

$$\text{Suppose} \quad X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \ \begin{matrix}\text{exists} \\ \text{unique}\end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \quad \mathrm{Tr}(U^\top X) = \mathrm{Tr}(U^\top X^{1/2} X^{\top/2}) = \mathrm{Tr}(\underbrace{X^{\top/2} U^\top X^{1/2}}_{\text{congruent to U}}) \geq 0$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

$$\mathrm{Tr} = \text{sum eigs} \ \Rightarrow \qquad \iff \mathrm{Tr}(X^{\top/2} U^\top X^{1/2}) \geq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \ X = X^\top \succeq 0 \\ \boxed{-\infty \quad ; \ \text{otherwise}} \end{cases}$$

Suppose $\quad X \not\succeq 0 \quad (X = X^\top)$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \ X = X^\top \succeq 0 \\ \boxed{-\infty \quad ; \ \text{otherwise}} \end{cases}$$

Suppose $\quad X \not\succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad$ exists neg. (real) eval with evec u

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i\Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j\Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i\Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j\Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ \boxed{-\infty \quad ; \; \text{otherwise}} \end{cases}$$

$$\text{Suppose} \quad X \not\succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \text{exists neg. (real) eval with evec } u$$

$$\text{take} \quad U = suu^\top \succeq 0$$

$$s \in \mathbb{R}_+$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_X \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \mathrm{Tr}(U^\top X) = \begin{cases} 0 & ; \; X = X^\top \succeq 0 \\ \boxed{-\infty \quad ; \; \text{otherwise}} \end{cases}$$

$$\text{Suppose} \quad X \not\succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \text{exists neg. (real) eval with evec } u$$

$$\text{take} \quad \underset{s \in \mathbb{R}_+}{U = suu^\top \succeq 0} \quad \Rightarrow \quad \mathrm{Tr}(U^\top X) = s\mathrm{Tr}(u^\top X u) < 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \quad \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \quad \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \max_{X} \quad \mathrm{Tr}\Bigg(\underbrace{\Big[R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top\Big]}_{\text{must be 0 for inner problem to be bounded}} X\Bigg) - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \boxed{\mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i} \quad \boxed{\mathrm{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j} \quad \boxed{X = X^\top \succeq 0 \quad U}$$

$$\mathcal{L}\big(X, \lambda_i, \mu_i, U\big) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i\Big) + \sum_j \mu_j \Big(\mathrm{Tr}(C_j^\top X) - d_j\Big) + \mathrm{Tr}(U^\top X)$$

$$\min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \max_{X} \mathrm{Tr}\Bigg(\underbrace{\Big[R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top\Big]}_{\text{must be 0 for inner problem to be bounded}} X\Bigg) - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

Dual Program

$$\min_{\lambda_i, \mu_j, U} \quad -\sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

$$\text{s.t.} \quad R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top = 0\,, \quad \mu_j \geq 0 \quad j \in \mathcal{J} \quad U \succeq 0$$

Dual Programs: SDP (lin objective)

Primal Program

$$\max_{X} \quad \mathrm{Tr}(R^\top X)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i \qquad \boxed{X \geq M \quad W} \qquad X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i \Big) + \boxed{\mathrm{Tr}\big(W^\top (X - M)\big)} + \mathrm{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \boxed{W \geq 0} \\ U \succeq 0}} \mathrm{Tr}(R^\top X) + \sum_i \lambda_i \Big(\mathrm{Tr}(A_i^\top X) - b_i \Big) + \boxed{\mathrm{Tr}\big(W^\top (X - M)\big)} + \mathrm{Tr}(U^\top X)$$

$$\min_{\substack{\lambda_i, \boxed{W \geq 0} \\ U \succeq 0}} \max_{X} \mathrm{Tr}\Bigg(\underbrace{\Big[R^\top + \sum_i \lambda_i A_i^\top + \boxed{W^\top} + U^\top \Big] X}_{\text{must be 0 for inner problem to be bounded}} \Bigg) \boxed{- \mathrm{Tr}(W^\top M)} - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

Dual Program

$$\min_{\lambda_i, W, U} \quad \boxed{- \mathrm{Tr}(W^\top M)} - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

$$\text{s.t.} \quad R^\top + \sum_i \lambda_i A_i^\top + \boxed{W^\top} + U^\top = 0 \,, \quad \boxed{W \geq 0} \quad U \succeq 0$$