KKT Matrix Conditioning

Convex Optimization

Major sources: Govind Chari Skye Mceowen

Spring 2023 - Dan Calderone

Quadratic Optimization & KKT System

$$\min_{x} \quad \frac{1}{2}x^{\top}Qx + c^{\top}x$$

Primal Variables: $x \in \mathbb{R}^n$

 $Q \in \mathbb{R}^{n \times n} \qquad Q = Q^{\top} \succ 0 \qquad c \in \mathbb{R}^n$

s.t.
$$Ax = b$$

Dual Variables: $v \in \mathbb{R}^m$

 $A \in \mathbb{R}^{m \times n}$

 $b \in \mathbb{R}^m$

$$\mathcal{L}(x,v) = \frac{1}{2}x^{\mathsf{T}}Qx + c^{\mathsf{T}}x + v^{\mathsf{T}}(Ax - b)$$

fat (m < n),

full row rank (rank = m)

KKT Conditions:

1. Stationarity:

 $x^{\top}Q + c^{\top} + v^{\top}A = 0$

2. Feasibility:

$$Ax - b = 0$$

KKT System:

$$\begin{bmatrix} Q & A^{\top} \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

KKT Matrix

Solution:

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q & A^{\top} \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} -c \\ b \end{bmatrix}$$

KKT Solutions

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2. Feasibility: Ax - b = 0

KKT Matrix

Solution:

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q^{-1} - Q^{-1}A^{\top} (AQ^{-1}A^{\top})^{-1} AQ^{-1} & Q^{-1}A^{\top} (AQ^{-1}A^{\top})^{-1} \\ (AQ^{-1}A^{\top})^{-1} AQ^{-1} & -(AQ^{-1}A^{\top})^{-1} \end{bmatrix} \begin{bmatrix} -c \\ b \end{bmatrix}$$

...using block matrix inversion (or just directly verify).

KKT Solutions

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KKT Matrix

Solution:

...projection matrix

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} Q^{-1} \begin{bmatrix} I - A^{\top} (AQ^{-1}A^{\top})^{-1} AQ^{-1} \\ (AQ^{-1}A^{\top})^{-1} AQ^{-1} \end{bmatrix}$$

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KKT Matrix

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full row rank (rank = m)

KKT Conditions:

Lagrangian:

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KKT System:

$$\begin{bmatrix} Q & A^{\top} \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

Solution:

$$x = Q^{-1}A^{\top} (AQ^{-1}A^{\top})^{-1} (AQ^{-1}c + b) - Q^{-1}c$$

$$v = -(AQ^{-1}A^{\top})^{-1}(AQ^{-1}c + b)$$

KKT Matrix Properties

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KKT Matrix

Matrix Properties:

$$M = \begin{bmatrix} Q & A^{\top} \\ A & 0 \end{bmatrix}$$

$$\operatorname{spec}(M) = \rho_+ \sqcup \rho_-$$

Cardinality

Positive:
$$ho_+ = \{\lambda_1, \ldots, \lambda_n\} > 0$$

$$|\rho_+| = n$$

$$\rho_{-} = \{\lambda_{n+1}, \dots, \lambda_{n+m}\} < 0 \qquad |\rho_{-}| = m$$

$$|\rho_-|=m$$

indefinite

*many conditionings are possible

Coordinate Transforms

$$\min_{x} \quad \frac{1}{2}x'^{\top}P^{\top}QPx' + c^{\top}Px'$$

s.t.
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Lagrangian:
$$\mathcal{L}(x',v') = \tfrac{1}{2}x'^\top P^\top Q P x' + c^\top P x' + v'^\top W^\top (A P x' - b)$$

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KKT System:

$$\begin{bmatrix} P^{\top}QP & P^{\top}A^{\top}W \\ W^{\top}AP & 0 \end{bmatrix} \begin{bmatrix} x' \\ v' \end{bmatrix} = \begin{bmatrix} -P^{\top}c \\ W^{\top}b \end{bmatrix}$$

Conditioning:

$$M \Rightarrow \begin{bmatrix} P^{\top}QP & P^{\top}A^{\top}W \\ W^{\top}AP & 0 \end{bmatrix} = \begin{bmatrix} P^{\top} & 0 \\ 0 & W^{\top} \end{bmatrix} \begin{bmatrix} Q & A^{\top} \\ A & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & W \end{bmatrix}$$

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$$M \Rightarrow \begin{bmatrix} P^\top Q P & P^\top A^\top W \\ W^\top A P & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & \Sigma \\ 0 & I & 0 \\ \Sigma & 0 & 0 \end{bmatrix} \xrightarrow{\text{Permute...}} \begin{bmatrix} I & \Sigma & 0 \\ \Sigma & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

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Subblocks:

$$\begin{bmatrix} 1 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} \qquad s(s-1) - \sigma_j^2 = 0$$

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$$\times (n - m) \times m \times m$$

$$\downarrow \qquad \qquad \downarrow$$

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