

LTI & LTV Systems

Linear System Theory

Major sources:

Winter 2022 - Dan Calderone

CLTI System - Autonomous

LTI = linear time invariant

LTI Scalar ODE $\lambda \in \mathbb{C}$ $x \in \mathbb{R}$

$$\dot{x} = \lambda x \quad x(t_0) = x_0$$

Solution: $x(t) = e^{\lambda(t-t_0)}x_0$

assume (WLOG) $t_0 = 0$

...for $\lambda = a + bi$

$$e^{\lambda t} = e^{(a+bi)t} = e^{at}e^{ibt}$$

Stability:

Exponentially Stable

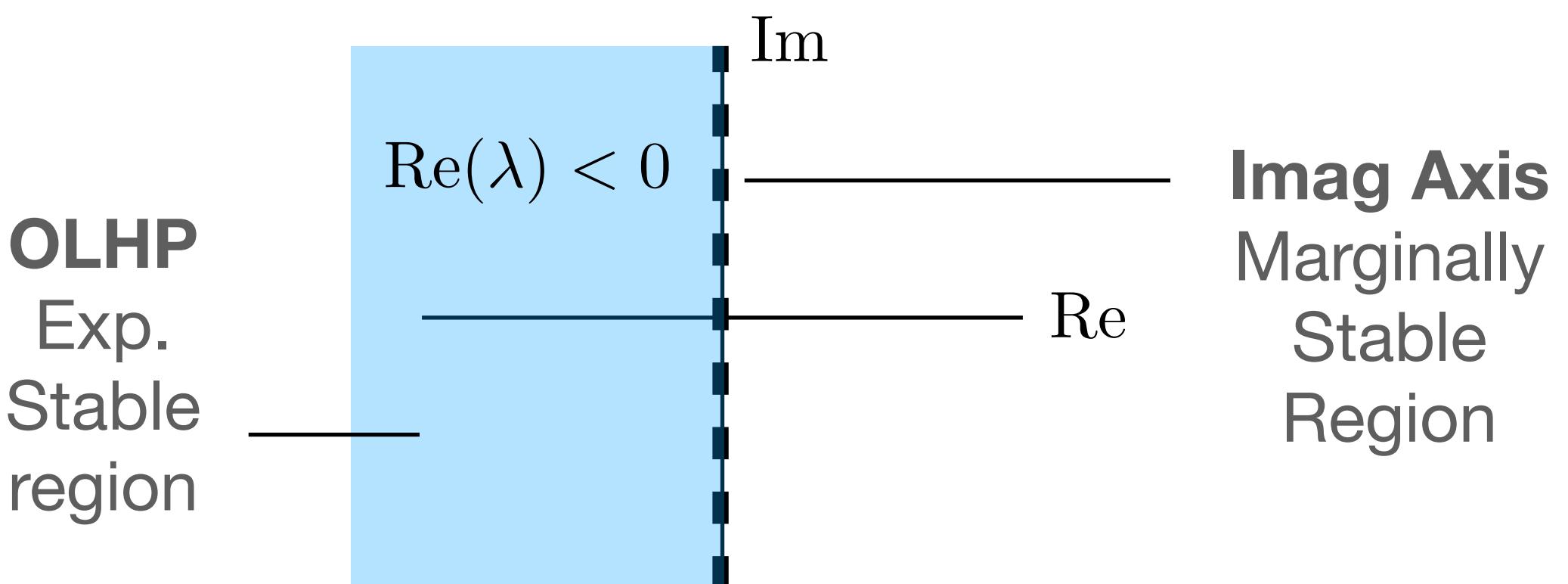
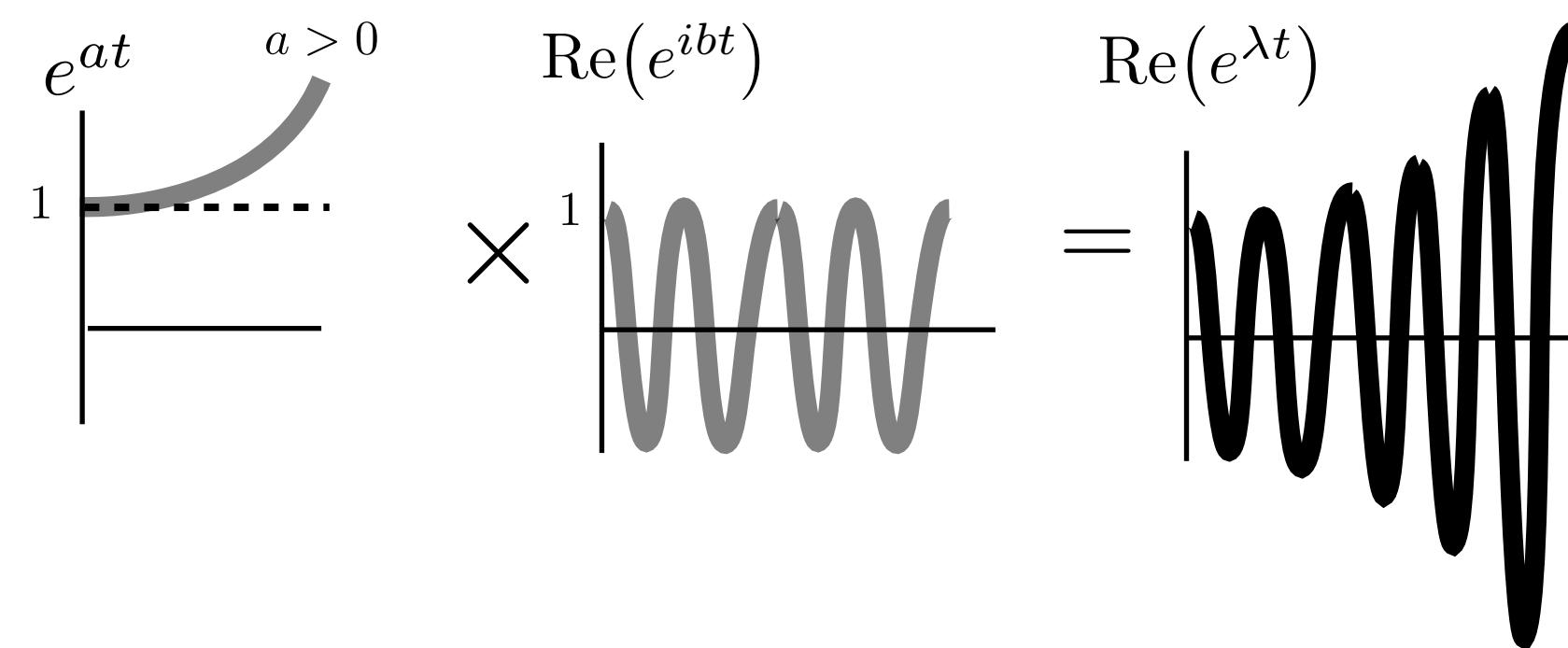
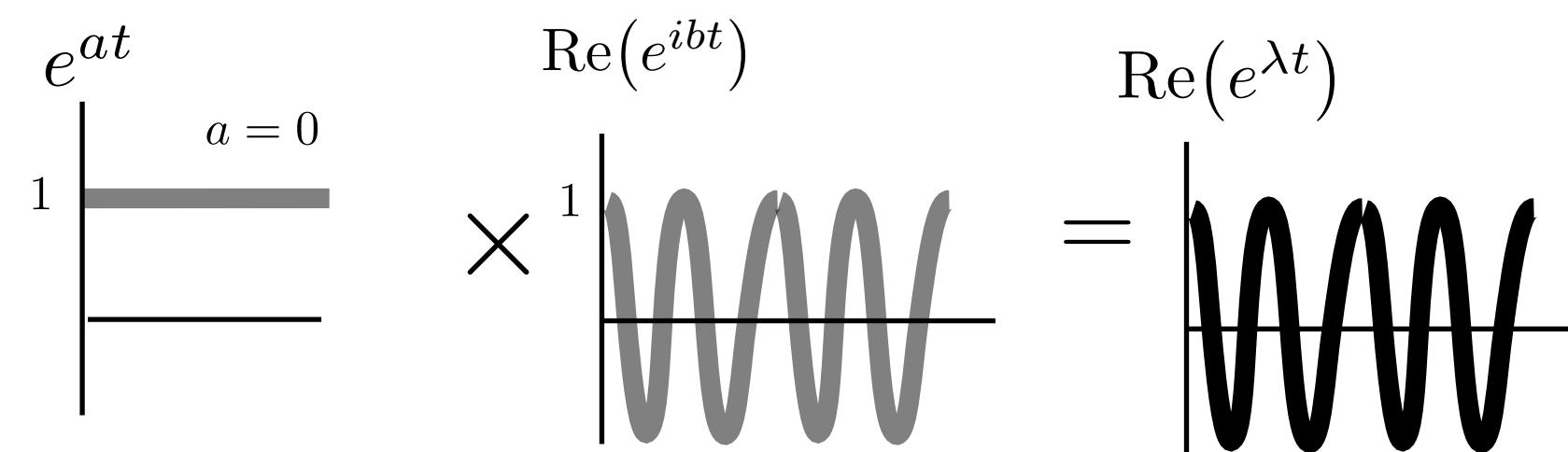
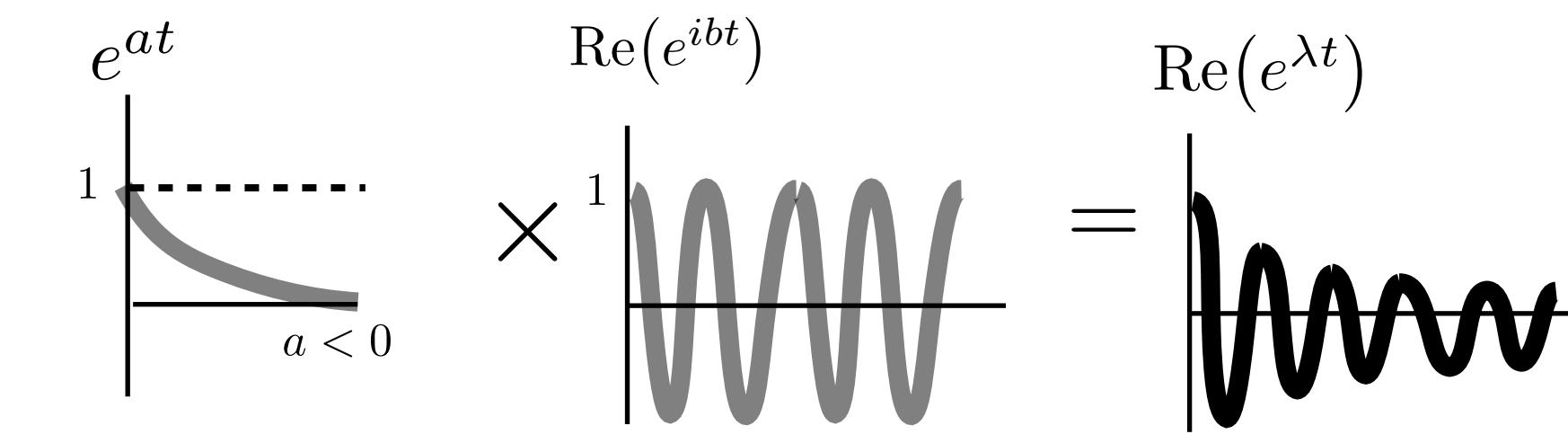
$$\text{Re}(\lambda) = a < 0$$

Marginally Stable

$$\text{Re}(\lambda) = a = 0$$

Unstable

$$\text{Re}(\lambda) = a > 0$$



DLTI System - Autonomous

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$$\dot{x} = \lambda x \quad x(t_0) = x_0$$

...evolution over a time step Δt

$$x(t + \Delta t) = e^{\lambda(t + \Delta t - t)} x(t) = e^{\lambda \Delta t} x(t)$$

...discrete time index $x[k] = x(k\Delta t + t_0) = x(t)$

LTI Discrete Update Eqn $\lambda_\Delta = e^{\lambda \Delta t} = e^{a \Delta t} e^{ib \Delta t}$

$$x[k+1] = \lambda_\Delta x[k] \quad x[0] = x_0$$

Stability:

Exponentially Stable

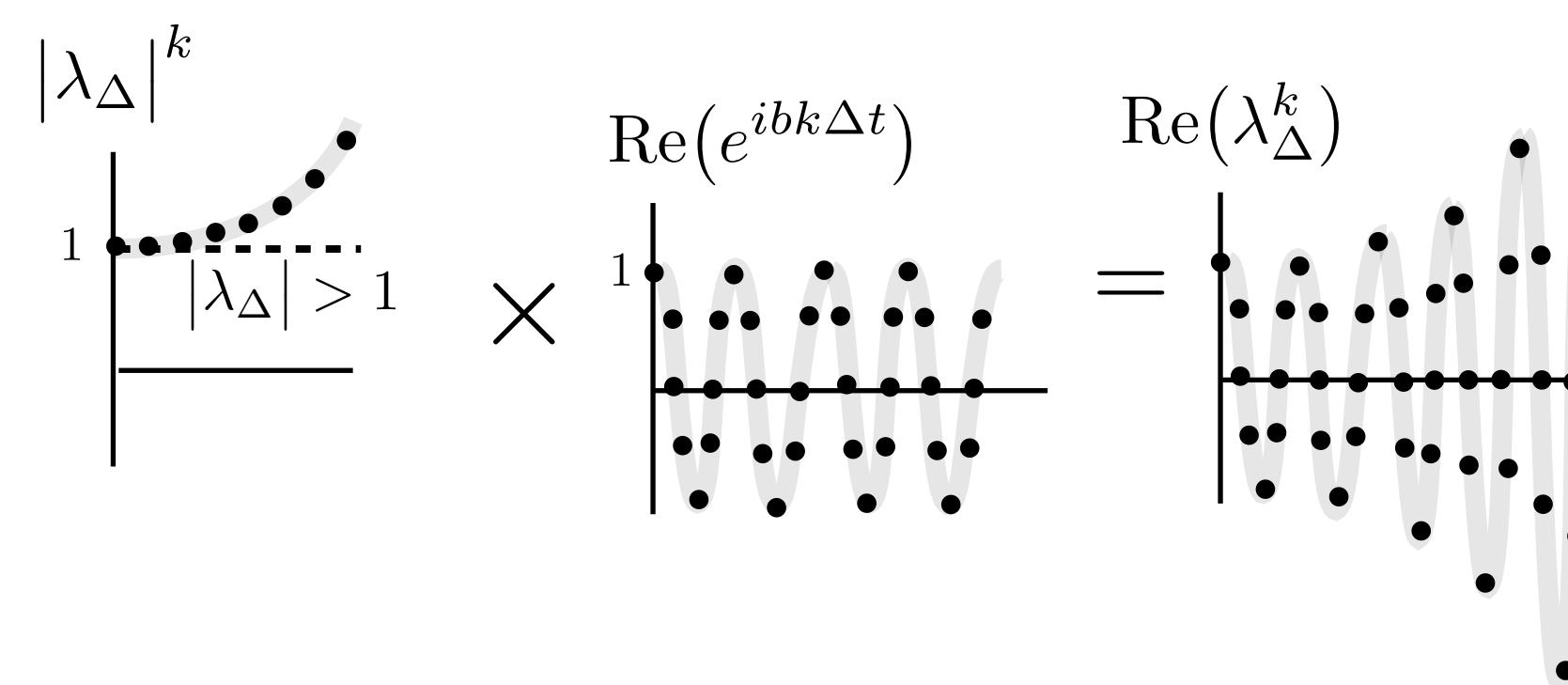
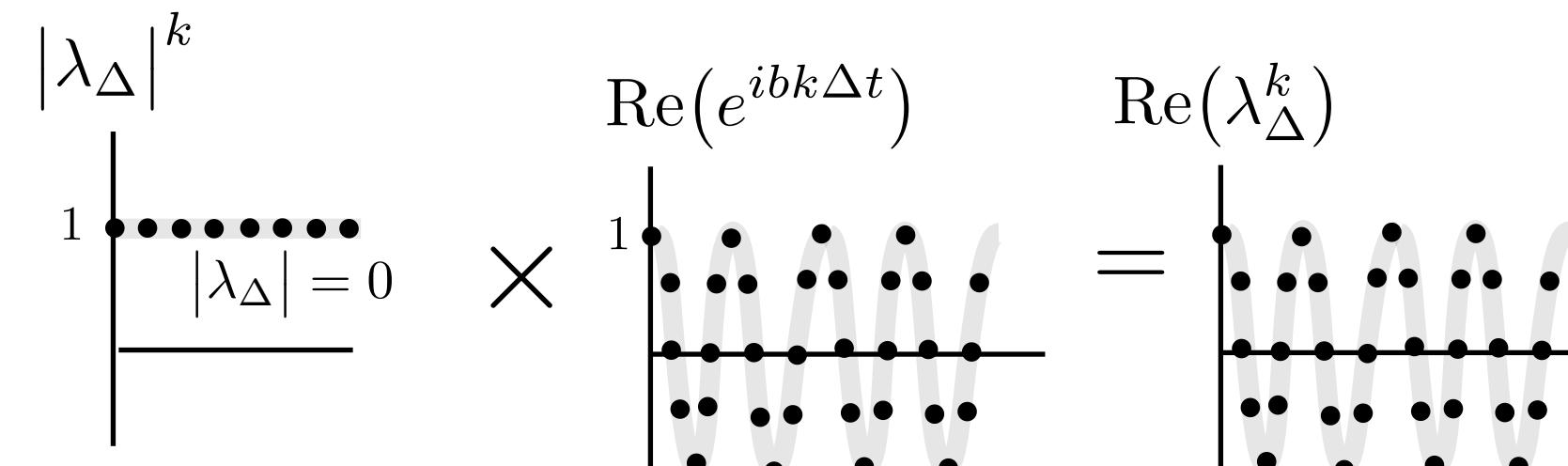
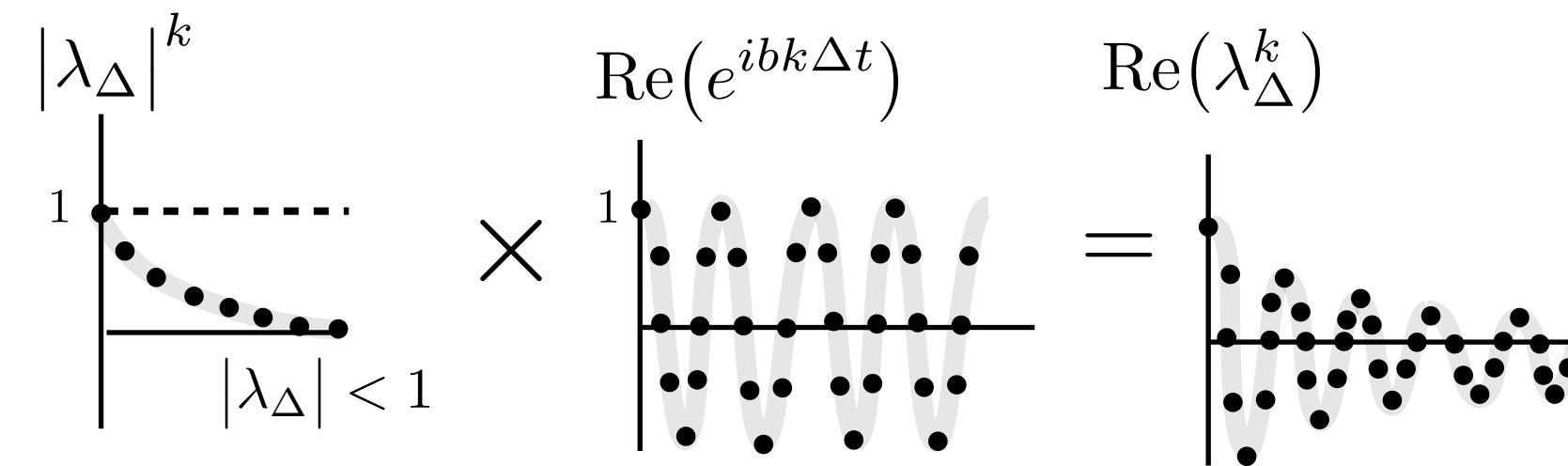
$$|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a \Delta t}| < 1$$

Marginally Stable

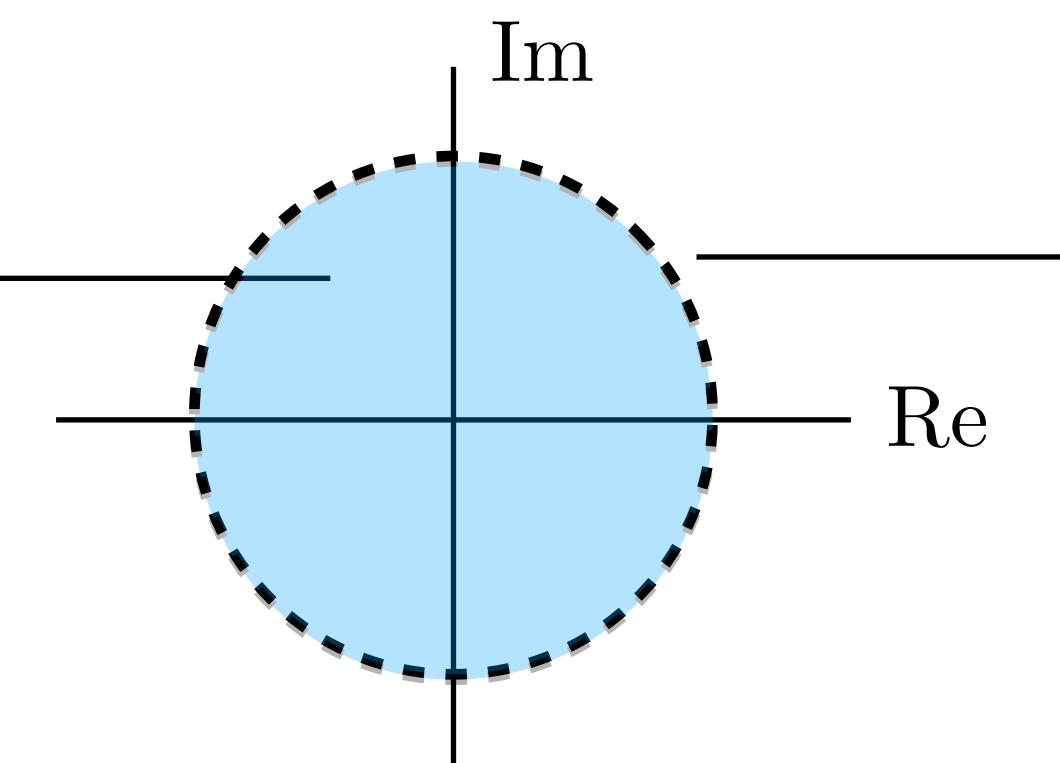
$$|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a \Delta t}| = 1$$

Unstable

$$|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a \Delta t}| > 1$$



Unit Circle Interior
Exp.
Stable
region



Unit Circle Boundary
Marginally
Stable
Region

LTI System - CT vs. DT Stability

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$$\dot{x} = \lambda x \quad x(t_0) = x_0$$

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$$\operatorname{Re}(\lambda) = a > 0$$

LTI Discrete Update Eqn

$$x[k+1] = \lambda_\Delta x[k]$$

$$\lambda_\Delta = e^{\lambda \Delta t} = e^{a \Delta t} e^{ib \Delta t}$$

$$x[0] = x_0$$

Stability:

Exponentially Stable

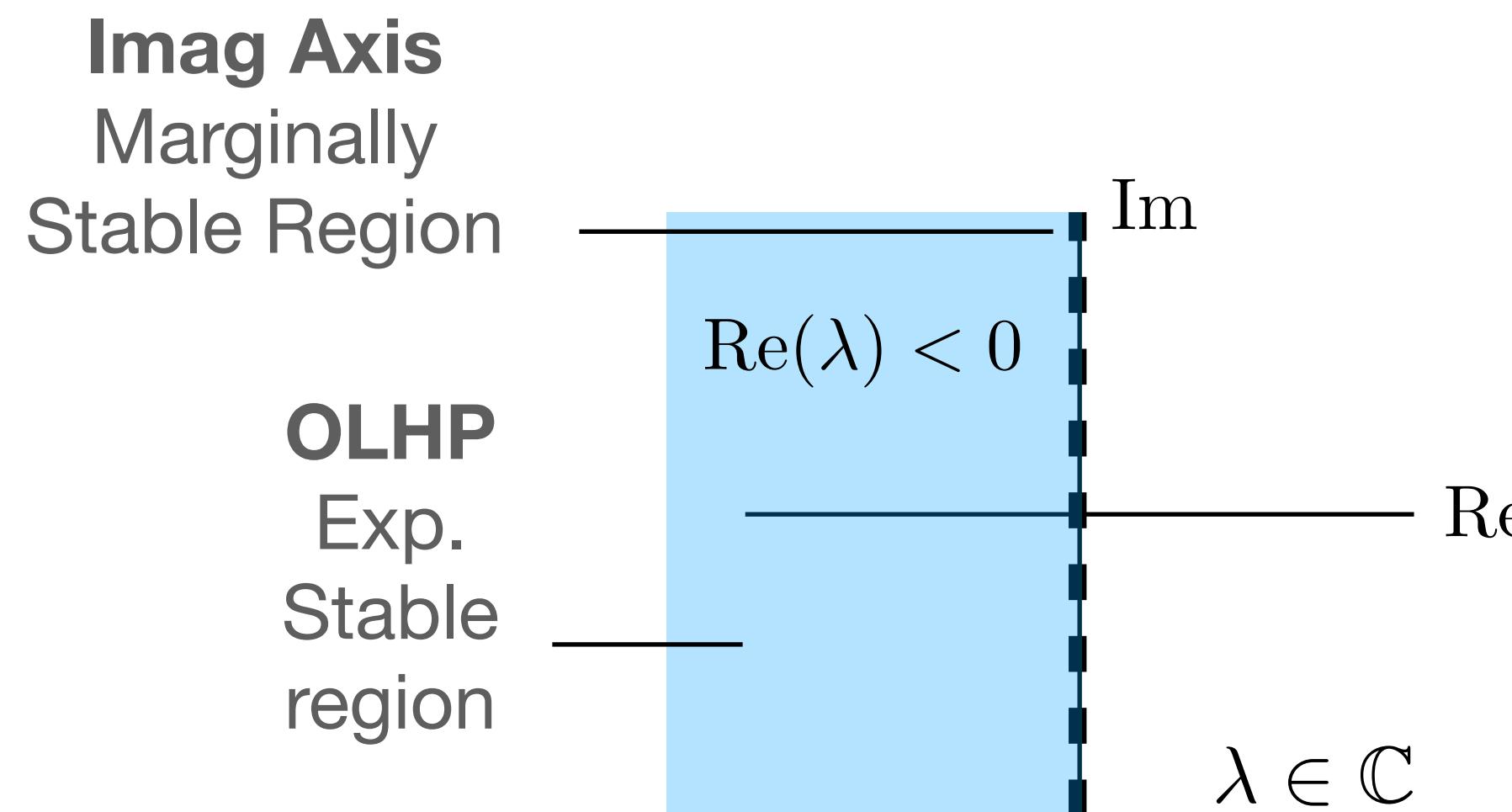
$$|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a \Delta t}| < 1$$

Marginally Stable

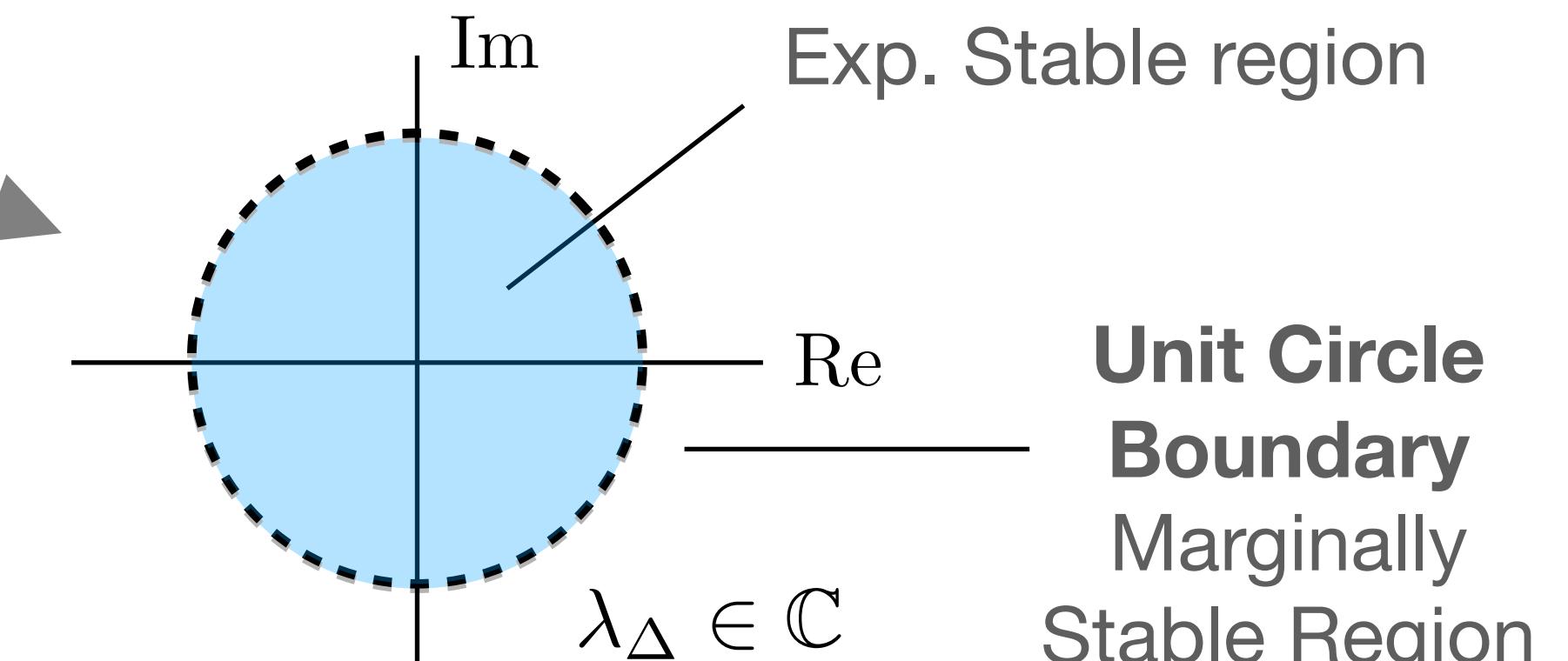
$$|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a \Delta t}| = 1$$

Unstable

$$|\lambda_\Delta| = |e^{\lambda \Delta t}| = |e^{a \Delta t}| > 1$$



$$\lambda_\Delta = e^{\lambda \Delta t}$$



CLTI System - Autonomous

LTI = linear time invariant

LTI Vector ODE $A \in \mathbb{R}^{n \times n}$ $x \in \mathbb{R}^n$

$$\dot{x} = Ax \quad x(t_0) = x_0$$

Solution:

$$x(t) = e^{A(t-t_0)}x_0 \quad \text{matrix exponential}$$

... eigenvalues control decay/expansion
and oscillation for eigensubspaces

For diagonalizable A: $A = VDV^{-1}$

$$x(t) = e^{At}x_0 = Ve^{Dt}V^{-1}x_0$$

\Rightarrow

$$V^{-1}x(t) = e^{Dt}V^{-1}x_0$$

$$x = Vx'$$

$$\Rightarrow x'(t) = e^{Dt}x'_0$$

...in eigenvector coordinates

$$\begin{bmatrix} x'_1(t) \\ \vdots \\ x'_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x'_1(0) \\ \vdots \\ x'_n(0) \end{bmatrix} \Rightarrow$$

$$\begin{aligned} x'_1(t) &= e^{\lambda_1 t}x'_1(0) \\ &\vdots \\ x'_n(t) &= e^{\lambda_n t}x'_n(0) \end{aligned}$$

Example: $A \in \mathbb{R}^{2 \times 2}$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

Eigenvalues

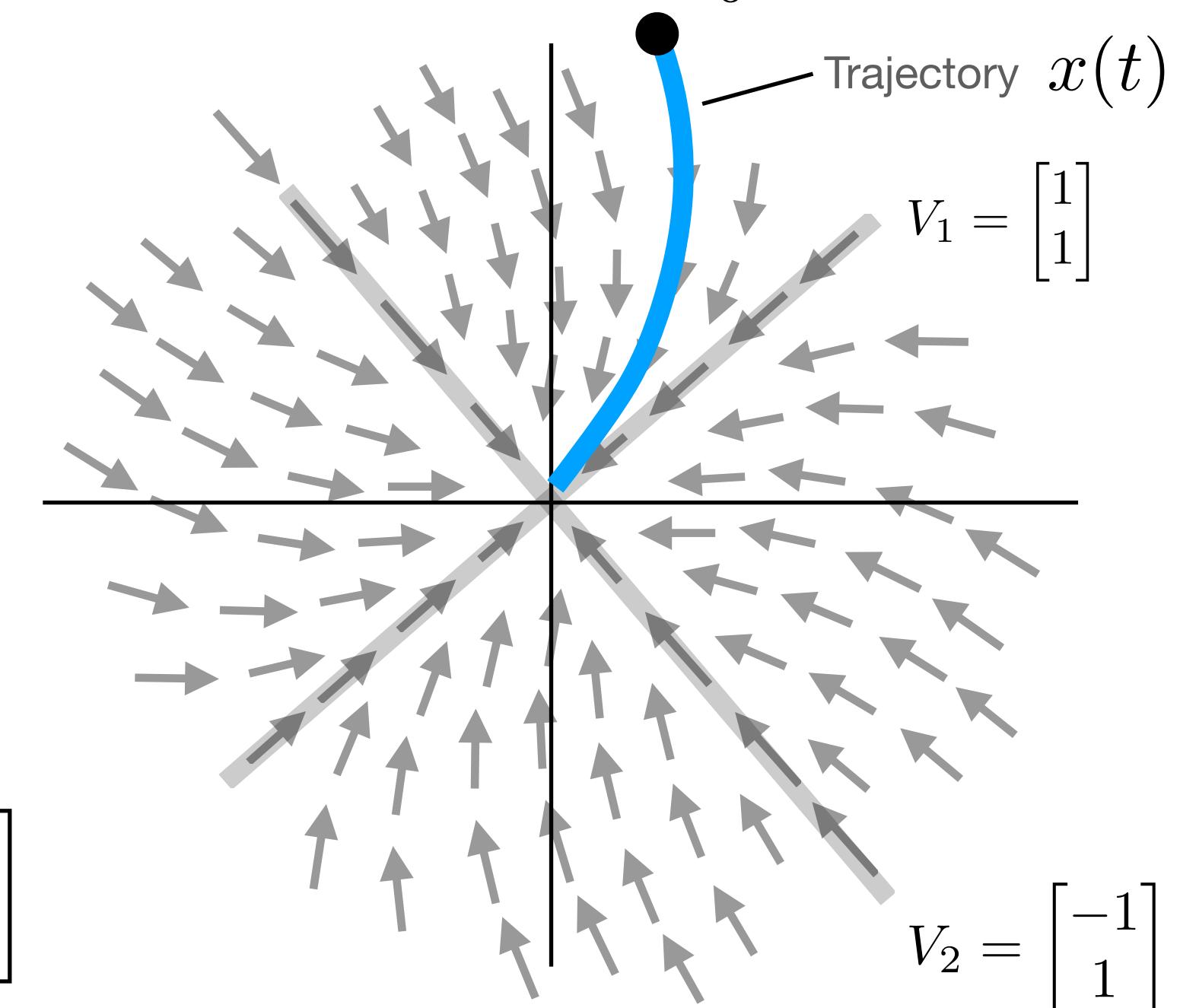
$$\begin{aligned} \lambda_1 &= -2, \\ \lambda_2 &= -4 \end{aligned}$$

Eigenvectors

$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Quiver Plot & Trajectory:

Initial Conditions x_0



DLTI System - Autonomous

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...evolution over a time step Δt

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...discrete time index $x[k] = x(k\Delta t + t_0) = x(t)$

LTI Discrete Update Eqn $A_\Delta = e^{A\Delta t}$

$$x[k+1] = A_\Delta x[k] \quad x[0] = x_0$$

Solution: $x[k] = A_\Delta^k x_0$

if diagonalizable $x[k] = A_\Delta^k x_0 = V D_\Delta^k V^{-1} x_0 \quad D_\Delta = e^{D\Delta t}$

$$\Rightarrow V^{-1} x[k] = D_\Delta^k V^{-1} x_0 \quad x = Vx'$$

$$\Rightarrow x'[k] = D_\Delta^k x'_0$$

...in eigenvector coordinates

$$\begin{bmatrix} x_1[k] \\ \vdots \\ x_n[k] \end{bmatrix} = \begin{bmatrix} \lambda_{1\Delta}^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n\Delta}^k \end{bmatrix} \begin{bmatrix} x_1[0] \\ \vdots \\ x_n[0] \end{bmatrix}$$

$$\lambda_{1\Delta} = e^{\lambda_1 \Delta t} \quad \dots \quad \lambda_{n\Delta} = e^{\lambda_n \Delta t}$$

Example: $A \in \mathbb{R}^{2 \times 2}$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \quad \lambda_1 = -2, \quad \lambda_2 = -4$$

Time step $\Delta t = 0.01$

$$A_\Delta = \begin{bmatrix} 0.9705 & 0.0097 \\ 0.0097 & 0.9705 \end{bmatrix}$$

Eigenvalues

$$\lambda_{1\Delta} = 0.9802$$

$$\lambda_{2\Delta} = 0.9608$$

Eigenvectors

$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

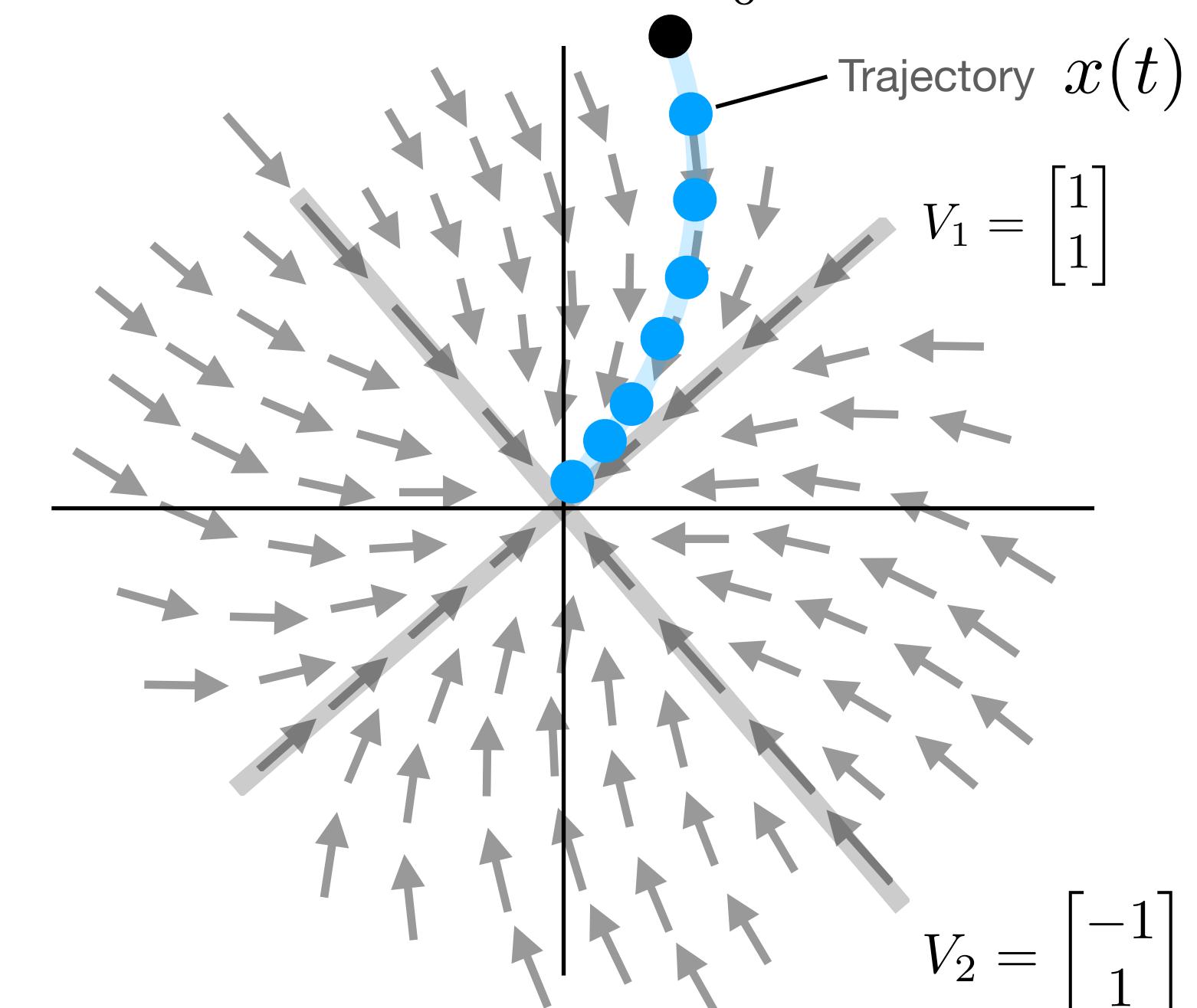
$$x'_1[k] = \lambda_{1\Delta}^k x'_1[0]$$

\vdots

$$x'_n[k] = \lambda_{n\Delta}^k x'_n[0]$$

Quiver Plot & Trajectory:

Initial Conditions x_0



CLTI System - with controls

LTI = linear time invariant

LTI Vector ODE

$$A = \mathbb{R}^{n \times n} \quad x \in \mathbb{R}^n$$

$$\dot{x} = Ax + Bu \quad x(t_0) = x_0$$

Solution:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$$

autonomous
“drift” term
affect of all
control inputs

...effect of control input at time $\tau \in [t_0, t]$
on state at time t $x(t)$

$$\text{Effect of all inputs added up} = \int_{t_0}^t e^{A(t-\tau)} B u(\tau) \quad | \quad | \quad |$$

propagated forward from time τ to t . | input at τ | input matrix B

...in eigenvector coordinates

$$\begin{bmatrix} x'_1(t) \\ \vdots \\ x'_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x'_1(0) \\ \vdots \\ x'_n(0) \end{bmatrix} =$$

$$\begin{aligned} x'_1(t) &= e^{\lambda_1 t} x'_1(0) \\ &\vdots \\ x'_n(t) &= e^{\lambda_n t} x'_n(0) \end{aligned}$$

Quiver Plot & Trajectory:

Example: $A \in \mathbb{R}^{2 \times 2}$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues

$$\lambda_1 = 1 \quad \lambda_2 = 2$$

Eigenvectors

$$\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

CLTI System - with controls

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autonomous
“drift” term affect of all
control inputs

Checking solution:

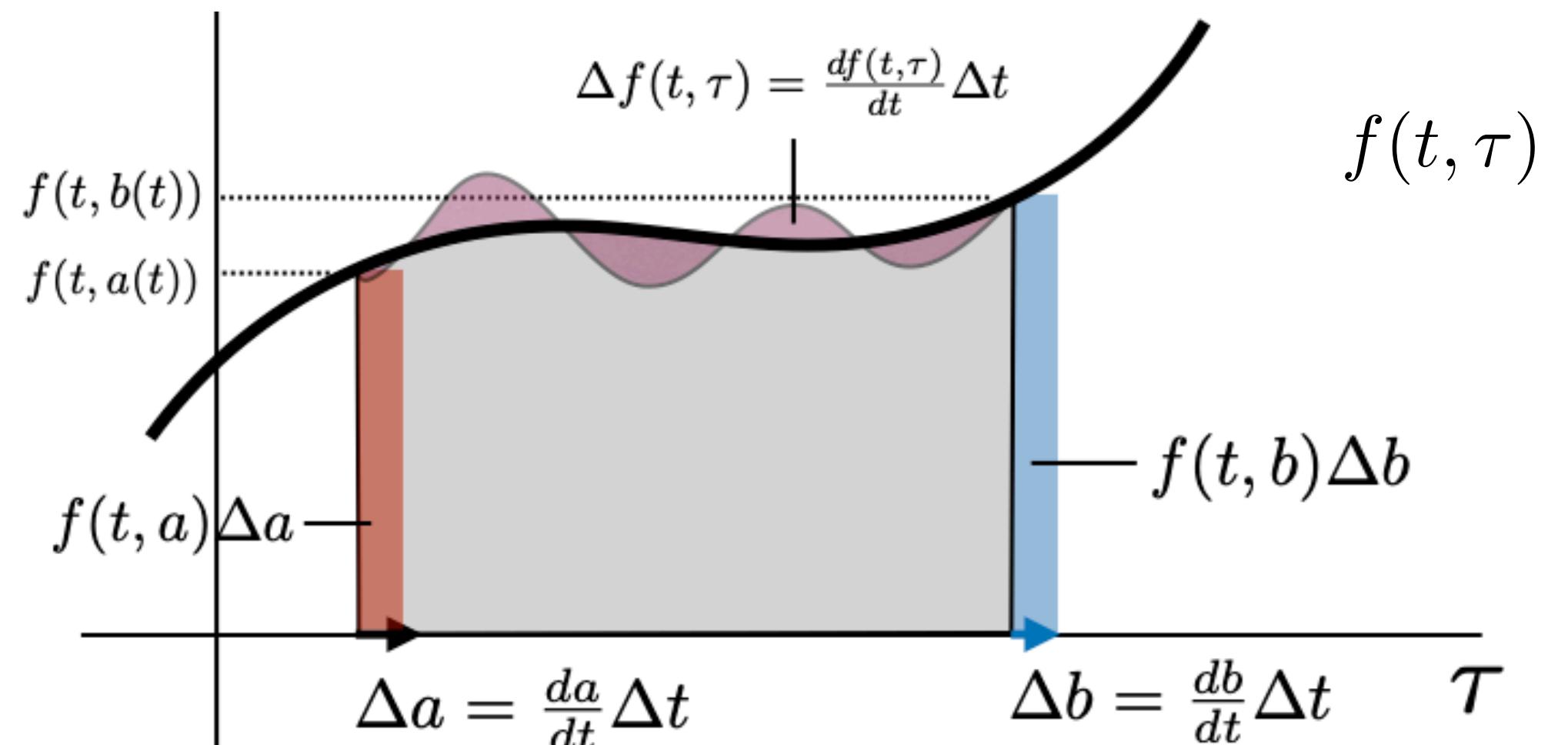
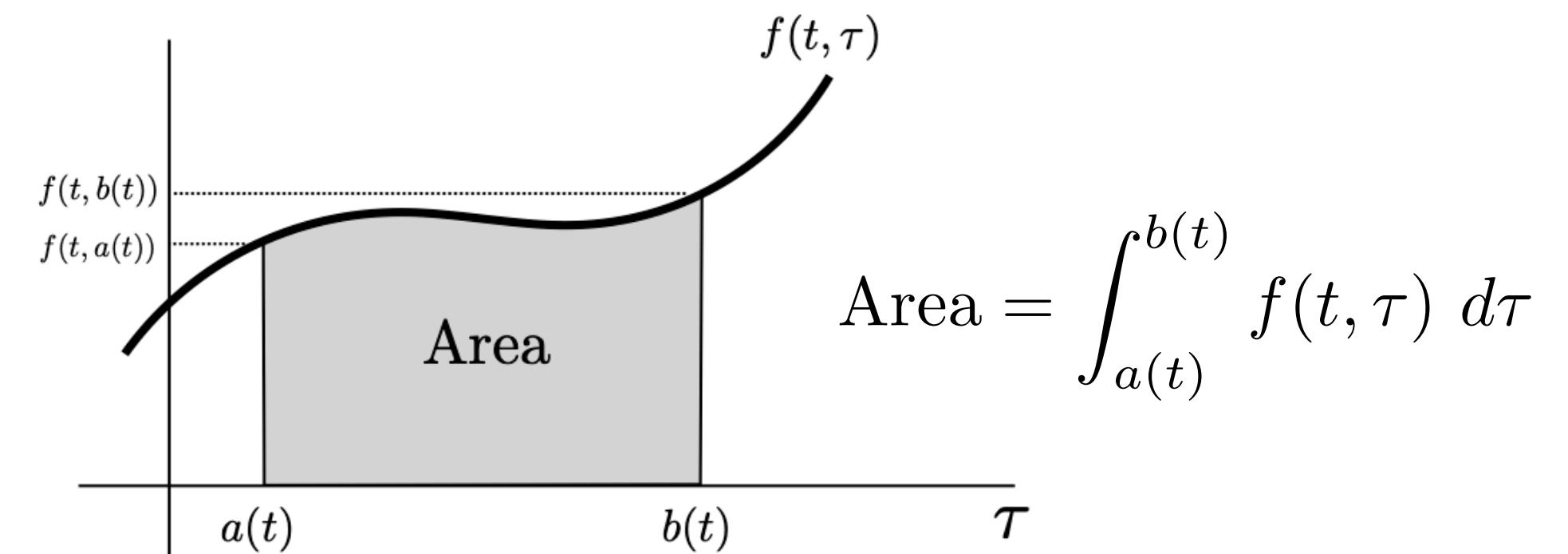
1. Derivative

$$\frac{dx}{dt} = Ae^{A(t-t_0)}x_0 + A \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau + e^{A(t-t_0)}Bu(t)$$

2. Initial Cond.

$$x(t_0) = e^{A(t_0-t_0)}x_0 + \int_{t_0}^{t_0} e^{A(t_0-\tau)}Bu(\tau) d\tau$$

Leibniz Rule



$$\frac{d\text{Area}}{dt} = f(t, b) \frac{db}{dt} - f(t, a) \frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{df(t, \tau)}{dt} d\tau$$

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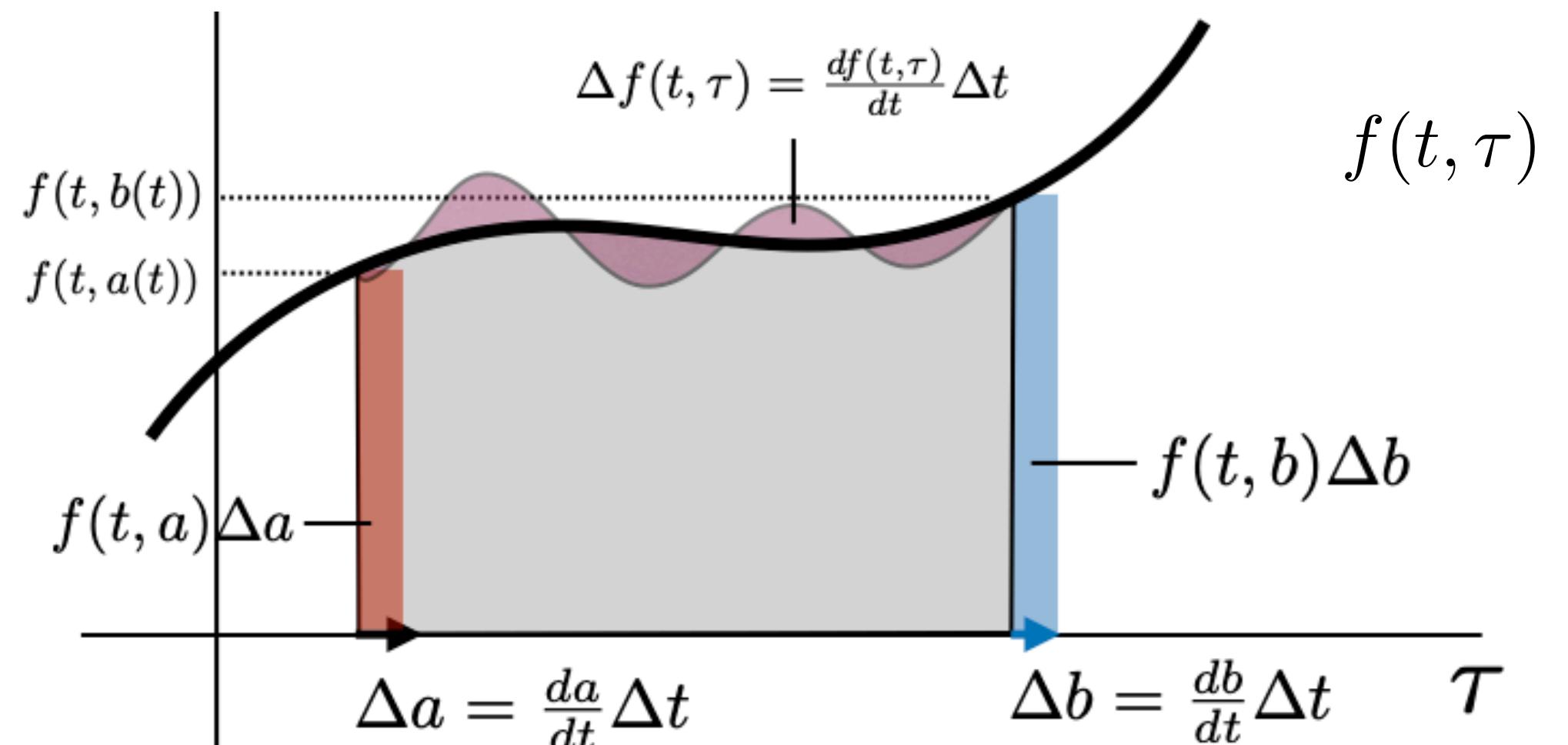
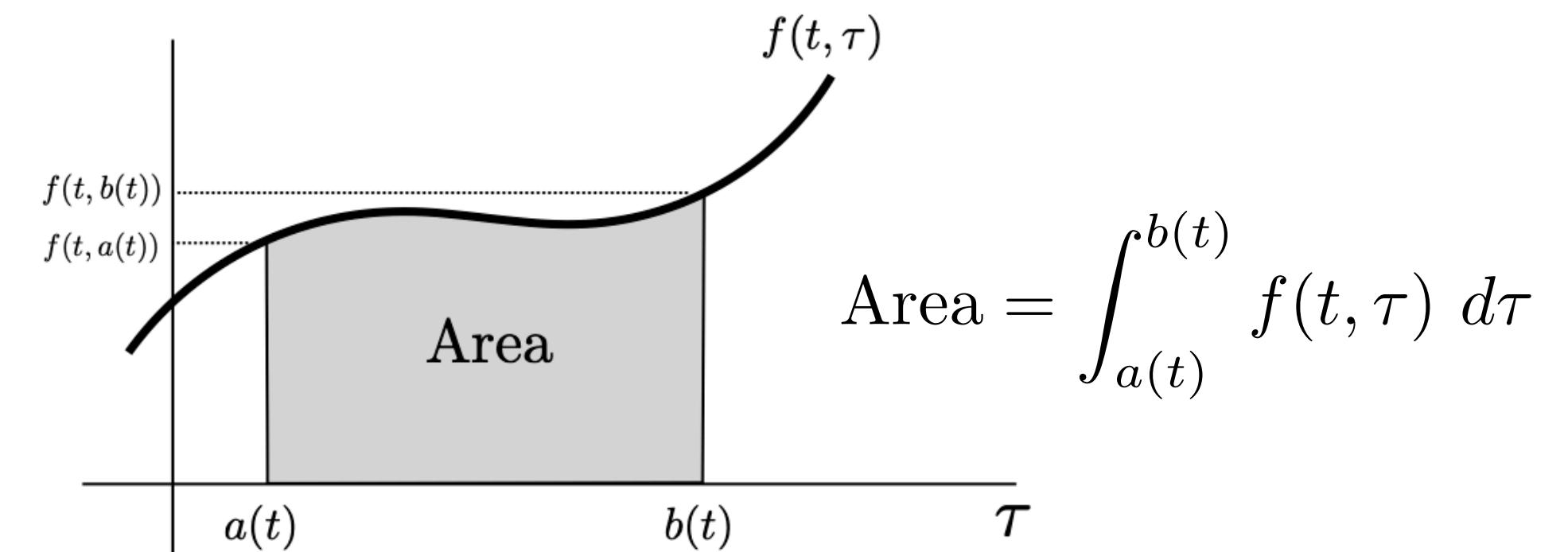
1. Derivative

$$\begin{aligned} \frac{dx}{dt} &= Ae^{A(t-t_0)}x_0 + A \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau + \cancel{e^{A(t-t_0)}Bu(t)}^I \\ &= Ax(t) + Bu(t) \end{aligned}$$

2. Initial Cond.

$$\cancel{x(t_0) = e^{A(t_0-t_0)}x_0}^I + \cancel{\int_{t_0}^{t_0} e^{A(t_0-\tau)}Bu(\tau) d\tau}^0 = x_0$$

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