Controllability & Observability

Linear System Theory

Major sources:

Winter 2022 - Dan Calderone

DLTI System - Reachability

LTI Discrete Update Eqn $A_\Delta \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^n$

$$x[k+1] = A_\Delta x[k] + B_\Delta u[k] \qquad x[0] = x_0$$

Discrete Time Matrices

$$A_\Delta = e^{A\Delta t} \qquad B_\Delta = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \; d\tau$$

assuming $u[k]$ constant over Δt

Solutions

$$x[k] = A_\Delta^k x_0 + \sum_{k'=0}^{k-1} A_\Delta^{k-1-k'} B_\Delta u[k']$$

$$= A_\Delta^k x_0 + \underbrace{\begin{bmatrix} A_\Delta^{k-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}}_{U}$$

Reachability/Controllability

…where can you drive the system to?

reachable space = range of G

Reaching a particular state: x_{des}

…solve $x_{\text{des}} - A_\Delta^k x_0 = GU$ for U

Minimum norm solution:

$$U^* = G^T (GG^T)^{-1} \left(x_{\text{des}} - A_\Delta^k x_0 \right)$$
$$= G^T W^{-1} \left(x_{\text{des}} - A_\Delta^k x_0 \right)$$

DT Controllability Grammian: $W = GG^T$

$$W = \sum_{k'=0}^{k-1} A_\Delta^{k'} B_\Delta B_\Delta^T A_\Delta^{k'\,T}$$
$$= \begin{bmatrix} A_\Delta^{k-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} B_\Delta^T A_\Delta^{k-1\,T} \\ \vdots \\ B_\Delta^T A_\Delta^T \\ B_\Delta^T \end{bmatrix}$$
$$= GG^T$$

if G is fat, then W is invertible, if and only if G has full row rank

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by Cayley-Hamilton

$$\mathcal{R}(G) = \mathcal{R}\Big(\begin{bmatrix} A_\Delta^{n-1} B_\Delta & \cdots & A_\Delta B_\Delta & B_\Delta \end{bmatrix}\Big)$$

…since $A_\Delta^{k'} = \beta_{n-1} A_\Delta^{n-1} + \cdots + \beta_1 A_\Delta^1 + \beta_0 I$

for $k' > n-1$

CLTI System - Reachability

LTI Continuous ODE $A = \mathbb{R}^{n \times n}$ $x \in \mathbb{R}^n$

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

Solution:

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau$$

$$= e^{A(t-t_0)}x_0 + \tilde{G}(u[t_0, t])$$

Operator

- infinite-dimensional input $u[t_0, t]$
- n dimensional output

$$\tilde{G}(\,\cdot\,) = \int_{t_0}^{t} e^{A(t-\tau)}B(\cdot)\, d\tau$$

Reachability/Controllability

…where can you drive the system to?

reachable space = range of $\tilde{G}(\,\cdot\,) = \int_{t_0}^{t} e^{A(t-\tau)}B(\cdot)\, d\tau$

Reaching a particular state: x_{des} at time t

…solve $x_{\text{des}} - e^{A(t-t_0)}x_0 = \tilde{G}(u)$ for u

Minimum norm solution:

…works with infinite-dimensional operators too!

…recall **in DT**

$$U^* = G^T W^{-1}\big(x_{\text{des}} - A_\Delta^k x_0\big) \qquad W = \sum_{k'=0}^{k-1} A_\Delta^{k'} B_\Delta B_\Delta^T A_\Delta^{k'^T}$$

in CT **CT Controllability Grammian:**

$$\tilde{W} = \int_{t_0}^{t} e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}\, d\tau \in \mathbb{R}^{n \times n}$$

Solution:

$$u^*(\tau) = B^T e^{A^T(t-\tau)}\tilde{W}^{-1}\big(x_{\text{des}} - e^{A(t-t_0)}x_0\big)$$

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$$y[k] = Cx[k] + Du[k]$$

Discrete Time Matrices

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assuming $u[k]$ constant over Δt

Solutions

$$y[k] = Cx[k] + Du[k] = CA_\Delta^k x_0 + \sum_{k'=0}^{k-1} CA_\Delta^{k-1-k'} B_\Delta u[k'] + Du[k]$$

Observations over time: no controls, with noise $v_{k'} \sim \mathcal{N}(0, \sigma I)$

normal distribution

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^k \end{bmatrix}}_{H} x_0 + \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$$

Observability

...can you estimate the initial state from measurements

unobservable subspace = null space of H

Least Squares Solution

$$x_0 = (H^T H)^{-1} H^T Y$$
$$= X^{-1} H^T Y$$

DT Observability Grammian:

$$X = \sum_{k'=0}^{k} A_\Delta^{k'\,T} C^T C A_\Delta^{k'}$$

$$= \begin{bmatrix} C^T & A_\Delta^T C^T & A_\Delta^{2\;T} C^T & \cdots & A_\Delta^{k\;T} C^T \end{bmatrix} \begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^k \end{bmatrix}$$

$$= H^T H$$

if H is tall, then X is invertible
if and only if H has full col rank

DLTI System - Observability

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Observations over time:

with controls, with noise $v_{k'} \sim \mathcal{N}(0, \sigma I)$

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^k \end{bmatrix}}_{H} x_0 + \underbrace{\begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB_\Delta & D & 0 & \cdots & 0 \\ CA_\Delta B_\Delta & CB_\Delta & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA_\Delta^{k-1}B_\Delta & CA_\Delta^{k-2}B_\Delta & CA_\Delta^{k-3}B_\Delta & \cdots & D \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ \vdots \\ u[k] \end{bmatrix}}_{U} + \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$$

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Least Squares Solution

$$x_0 = (H^T H)^{-1} H^T \left(Y - \mathbf{G}U\right)$$
$$= X^{-1} H^T \left(Y - \mathbf{G}U\right)$$

DT Observability Grammian:

$$X = \sum_{k'=0}^{k} {A_\Delta^{k'}}^T C^T C A_\Delta^{k'}$$

$$= \begin{bmatrix} C^T & A_\Delta^T C^T & {A_\Delta^2}^T C^T & \cdots & {A_\Delta^k}^T C^T \end{bmatrix} \begin{bmatrix} C \\ CA_\Delta \\ CA_\Delta^2 \\ \vdots \\ CA_\Delta^k \end{bmatrix}$$

$$= H^T H$$

if H is tall, then X is invertible
if and only if H has full col rank