Matrices in $\mathbb{R}^{2\times 2}$

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***Abstract*—This tutorial paper gives basic visualizations for real matrices in $\mathbb{R}^{2\times 2}$ wih a primary focus on column geometry. Basic notation and basic column and row geometry are given followed by visualizations of several basic types of matrices. Matrix multiplication is discussed. Transposes and inverses are discussed with a focus on symmetric/skew-symmetric matrices and visualization. Similarity transforms and eigenvalue decompositions are discussed. Explicit algebraic characterization of eigenvalues/eigenvectors along with thorough visualizations. Special attention is then given to complex eigenvalues/eigenvectors including a discussion of rotation matrices. Symmetric and definite matrices are discussed in the context of quadratic forms. Grammians (shape matrices) are discussed; polar decompositions are derived and discussed, and the singular value decomposition is discussed in detail with analogies drawn from complex numbers. The final section of the paper details linear vector fields. The spectral mapping theorem and the matrix exponential are discussed along with stability criteria in continuous and discrete time.**

I. Introduction

Real matrices in $\mathbb{R}^{2\times 2}$ show up in every corner of modern matheematics. Beyond being useful in their own right, they also provide foundational examples and intuition for studying invertible linear maps (square matrices) in general. In this paper, we give many detailed visualizations of basic structural results about $\mathbb{R}^{2\times 2}$. Of course, it cannot cover all facets of 2×2 matrix structure, but it is meant to be thorough.

The initial section covers basic notation as well as gives basic details of column and row geometry and the image of sets under 2×2 linear transformations. Our analysis in this paper will focus primarily on column geometry as it is the most natural but row geometry will be discussed as well at several points. Matrix multiplication is discussed briefly as well. We next detail the structure of several basic classes of square 2×2 matrix structures including diagonal, upper/lower triangular, symmetric/skew-symmetric, rotations/reflections, and nilpotent structures. This section is meant to give a general flavor and build basic spatial intuition.

The next section discusses the geometry of the matrix transpose, ie. the geometry of the rows relative to the columns. While algebraically immediate, this geometry is actually fairly subtle. Particular attention is given to the symmetric and skew-symmetric portions of the matrix. Inverses are then discussed. Whereas transpose are algebraically simple and geometrically complicated, inverses have the opposite flavor (geometrically simple but algebraically complicated).

We then turn our attention to the rich subject of similarity transformations and eigenvalue decompositions. Similarity transforms are visualized with specific attention given to orthonormal similarity transforms (similarity transforms that are also congruent). The characteristic polynomial and it's relation to eigenvalues and left/right eigenvectors is thoroughly visualized and discussed. Formulas for eigenvalues, eigenvectors, and diagonalizations are given. Special attention is given to the complex eigenvalue case and pseudo-diagonal/rotational forms of complex eigenvalue decompositions. We also include a discussion of repeated eigenvalues, Jordan form, and nilpotent matrices. We then present in detail how eigenvalues and both left/right eigenvectors relate to the column geometry of a matrix in the both the real and complex eigenvalues cases. These particular visualizations are detailed and extensive. The complex case is then expanded further to detail it's rotational structure and specific attention is given to true rotation/reflection matrices. We also give specific attention to skew-symmetric matrices as real matrices with purely imaginary eigenvalues. Finally, we conclude the initial eigen-decomposition discussion with a brief discussion of the spectral mapping theorem. We next turn our attention to symmetric matrices in the context of quadratic forms. Quadratic form surfaces and their relationships to symmetric matrix eigenstructure is discussed. Positive-definite, negative-definite, and indefinite matrices are discussed.

The next section of the paper contains a detailed discussion of matrix shapes including the polar decomposition and singular value decomposition. The two Gramian matrices and, more importantly, their square roots, are discussed as the primary two definitions of matrix's positive definite ”shape”. From there we derive and visualize the polar decomposition in both contexts. Finally, we use the eigen-structure of the Gramian matrices to give the singular value decomposition (the classical construction) and give visualizations. Detailed connections between each of these decompositions as well as the sym/skew-sym decomposition are given as a thorough discussion of analogies with complex numbers in their Cartesian and polar form.

The final section of this paper details the structure of linear vector fields (ordinary differential equations) in the linear time invariant case. Basic solutions in the form of the matrix exponential in continuous and discrete time are given. The relationship between eigen-structure and trajectories is detailed. Stability criteria in both continous and discrete time are given in terms of eigenvalues as well as various parametric tests for stability. Some of these are classical results while others are somewhat novel.

Remark 1. *The primary section missing from this paper is perhaps one focusing on matrix commutators. The authors hope to add this section at some point in the future.*

A. *Prerequisites and Follow-ups*

This paper can be read on it's own without much difficulty; however it does assume a familiarity with the notation and vector visualization techniques presented in the following monographs.

- Vector visualizations
- Column geometry

This paper is also meant to be part one of a three part series. The second paper expands many of these results/visualizations to real matrices in $\mathbb{R}^{3\times3}$; the third paper discusses extensions to general matrices in $\mathbb{R}^{n\times n}$ with visualizations given in $\mathbb{R}^{4\times4}$. This last paper is (of course) far less thorough since the space $\mathbb{R}^{n\times n}$ is a vast mathematical landscape that has never been fully explored. Any "thorough" discussion would have to include countless specific types of matrices. The visualizations in this last paper are also only meant to be experimental and to give a flavor for how an ambitious student of visualization might seek to extend the the visualization techniques in the first two papers to higher dimensional geometries. As such, they should only be viewed in parallel to the first two monographs. The authors also take no responsibility for any confusion that may result from viewing them. The dissatisfied reader is always heartily encouraged to make improvements or re-fall in love with pure algebraic insight. [1]

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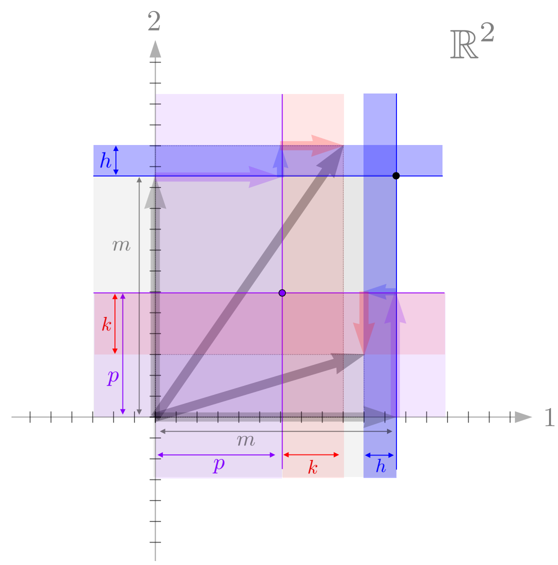
II. Basic Column Geometry

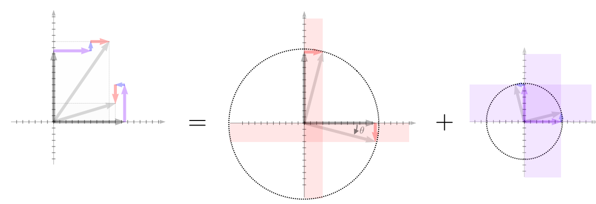
The columns of a matrix $A \in \mathbb{R}^{m\times n}$ are vectors in the co-domain of the linear map

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$

Each individual column $A_j \in \mathbb{R}^m$ tells where the jth standard basis vector (in the domain) gets mapped under the transformation. Explicitly $AI_j = A_j$. We can see where a vector $x \in \mathbb{R}^n$ in the domain gets mapped by breaking up x into a linear combination of standard basis vectors (ie. $x = I_1x_1 + \cdots + I_nx_n$), transforming each standard basis vector to the appropriate column, and then recombining. Algebraically, this is given by

$$\begin{aligned} Ax &= A\big(I_1x_1 + \cdots + I_nx_n\big) \\ &= AI_1x_1 + \cdots + AI_n \\ &= A_1x_1 + \cdots + A_nx_n \end{aligned}$$

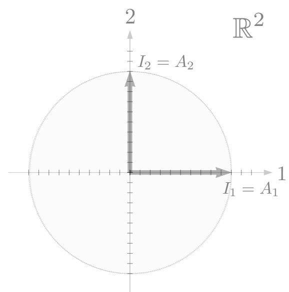




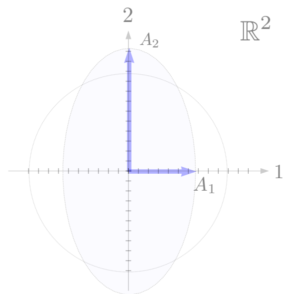
III. Matrix Types

Identity, diagonal

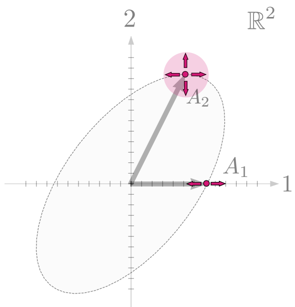
Upper/lower triangular
Symmetric, Skew-symmetric
Rotations/Reflections



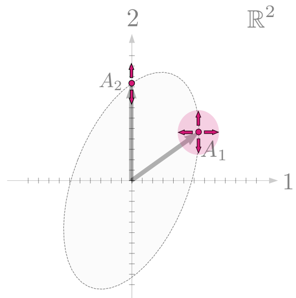
Identity



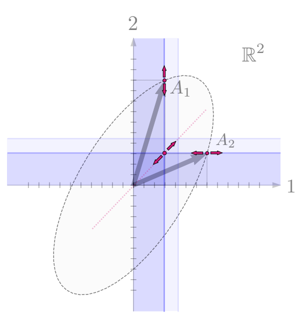
Diagonal



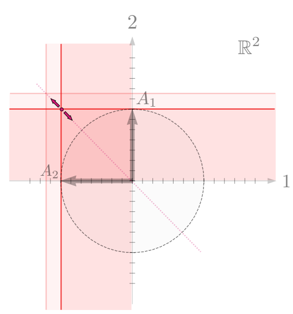
Upper
triangular



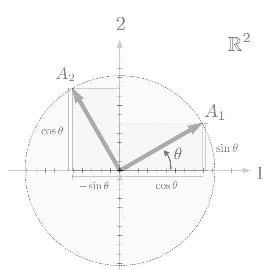
Lower
triangular



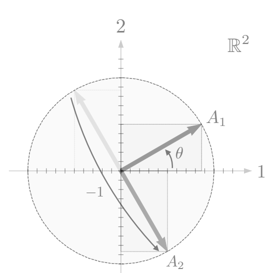
Symmetric



Skew-symmetric



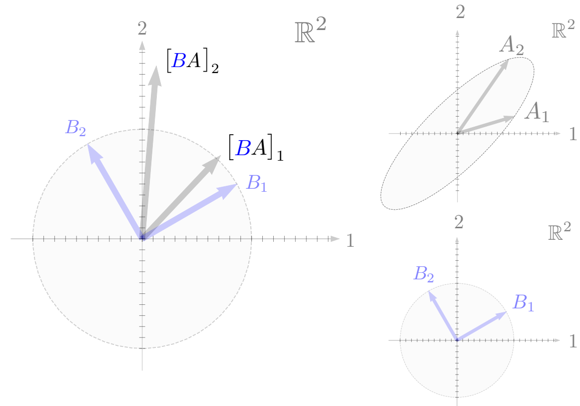
Rotation

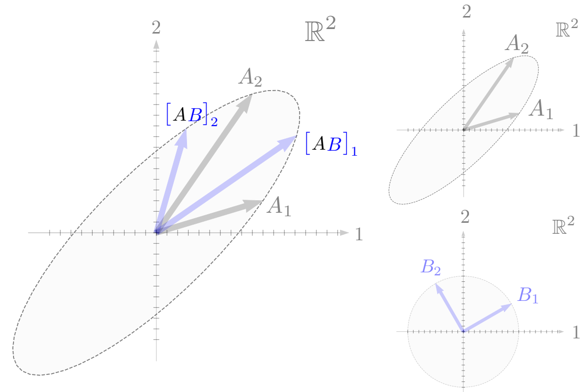


Rotation/Reflection

IV. Matrix Multiplication

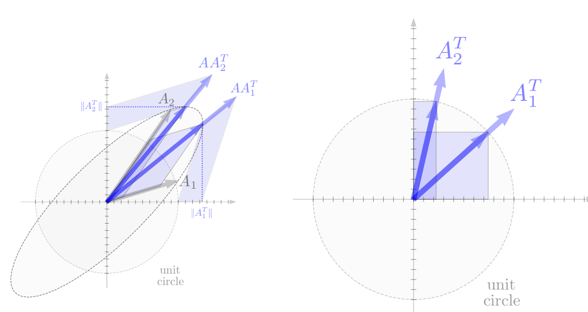
Left multiplication
Right multiplication

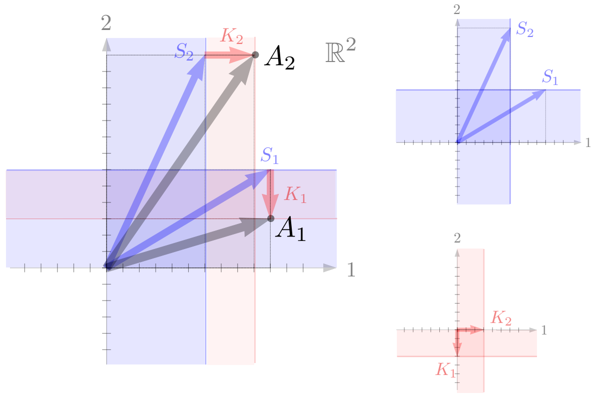


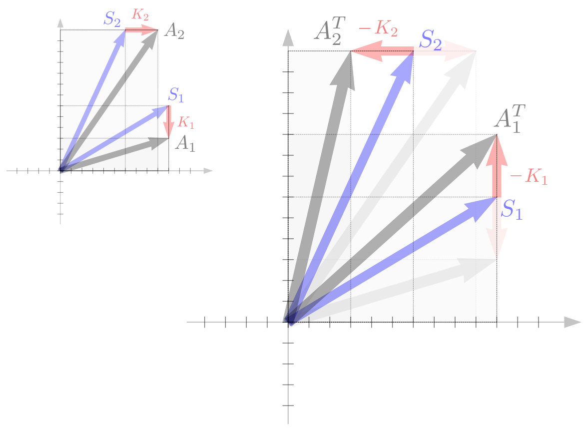


V. Transposes

Rows in parallel
Range of A^T
Sym-skew decomposition







VI. Inverses

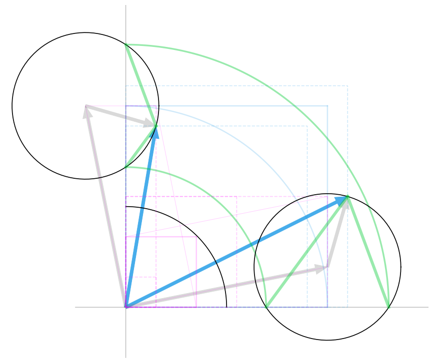
Geometry
Formulas

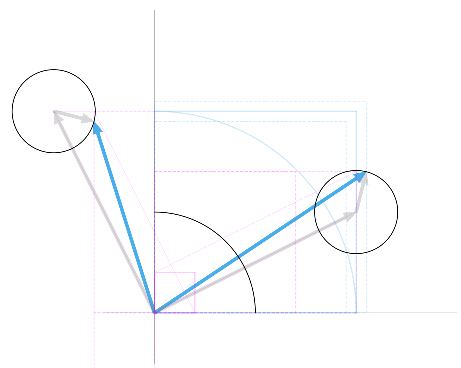
VII. Similarity Transforms

Visualization
Orthonormal transformations

VIII. Eigenvalue Decomposition

Eigenvalues basics
Characteristic polynomial
Algebraic formulas
Real eigenvalues/eigenvectors
Complex eigenvalues/eigenvectors





IX. Complex Eigenvalues

Rotation shape
Rotation angle

X. Rotation Matrices

XI. Definite Matrices

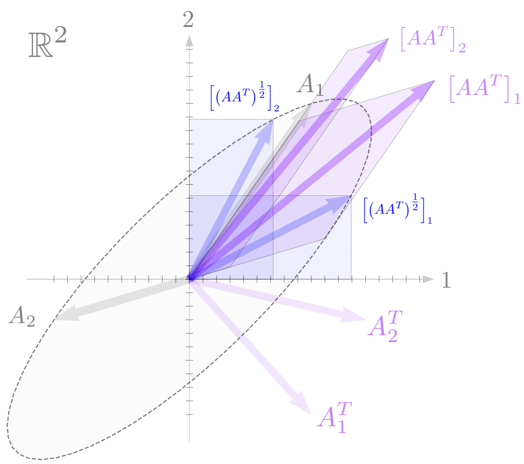
Quadratic forms
Eigenvectors (orthonormal)
Eigenvalues (real)
Positive, Negative, Indefinite

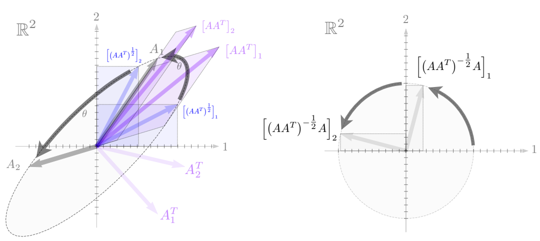
XII. Matrix Shapes

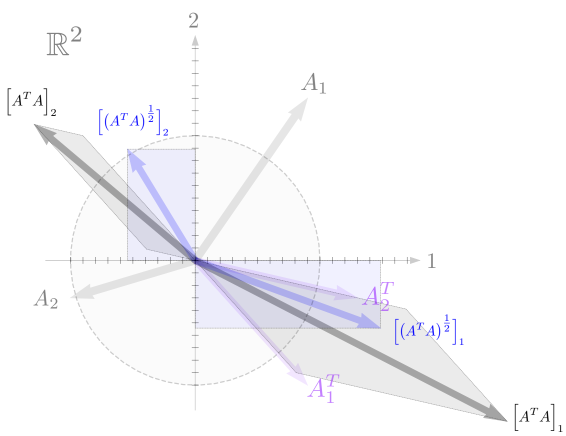
Grammians, rank
Matrix shapes

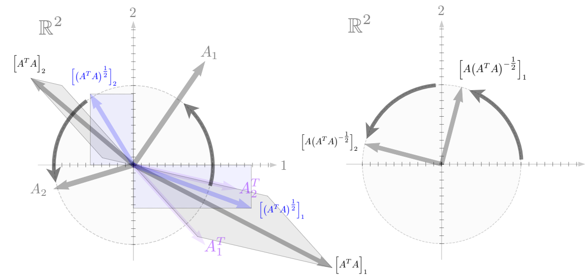
XIII. Polar Decomposition

Form 1
Form 2
Complex number analogy









XIV. Singular Value Decomposition

Forms
Connection to polar decomposition

XV. Linear Vector Fields