# Quadratic Forms, Definite Matrices, Congruence Transformations

# Linear Algebra:

**Major Contributions: John Simpson-Porco** 

Winter 2022 - Dan Calderone

Quadratic Form:  $f(x) = x^T Q x$   $Q \in \mathbb{R}^{n \times n}$   $Q = Q^T$ 

Definiteness:	Short	Notation	Definition		Analogy	Eigenvalues
Positive definite:	PD	$Q \succ 0$	$x^T Q x > 0$	$\forall x  x \neq 0$	positive orthant	$\lambda_i > 0  \lambda_i \in eig(Q)$
Positive semi-definite	PSD	$Q \succeq 0$	$x^T Q x \ge 0$	$\forall x$	positive orthant w/ boundary	$\lambda_i \ge 0  \lambda_i \in eig(Q)$
Negative-definite	ND	$Q \prec 0$	$x^T Q x < 0$	$\forall x  x \neq 0$	negative orthant	$\lambda_i < 0  \lambda_i \in eig(Q)$
Negative semi-definite	NSD	$Q \preceq 0$	$x^T Q x \le 0$	$\forall x$	negative orthant w/ boundary	$\lambda_i \le 0  \lambda_i \in eig(Q)$
Indefinite:			$x^T Q x > 0$	some $x$	the rest of the space	
			$x^T Q x < 0$	some $x$		

Note: not a useful definition for general matrices

... condition only says something about the symmetric part of Q

Symmetric/Skew-symmetric Decomposition

$$Q = \frac{1}{2} \underbrace{\left(Q + Q^T\right)}_{\text{symmetric}} + \frac{1}{2} \underbrace{\left(Q - Q^T\right)}_{\text{skew-sym}}$$

$$x^{T}Qx = \frac{1}{2}x^{T}\left(Q + Q^{T}\right)x + \frac{1}{2}x^{T}\left(Q - Q^{T}\right)x$$

$$= \frac{1}{2}x^{T}\left(Q + Q^{T}\right)x + \frac{1}{2}x^{T}Qx - \frac{1}{2}x^{T}Q^{T}x$$
...transpose
$$= \frac{1}{2}x^{T}\left(Q + Q^{T}\right)x + \underbrace{\frac{1}{2}x^{T}Qx - \frac{1}{2}x^{T}Qx}_{=0}$$

$$= \frac{1}{2}x^{T}\left(Q + Q^{T}\right)x$$
...only the symmetric part matters

### **Eigenvalue condition proof:**

...consider eigenvector coordinates

$$x = Vx'$$

since V is invertible... 
$$\forall x \iff \forall x'$$

$$x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i}x_{i}'^{2}$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x^{\prime} \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

**Quadratic Form:** 

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

**Definiteness:** 

### **Eigenvalues**

 $\lambda_i > 0 \quad \lambda_i \in eig(Q)$ 

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$ 

 $\lambda_i < 0 \quad \lambda_i \in eig(Q)$ 

### **Eigenvalue condition proof:**

Positive definite:

PSD

ND

$$Q \succ 0$$

 $Q \succeq 0$ 

 $Q \prec 0$ 

 $x^T Q x \ge 0 \quad \forall \ x$ 

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

...the rest of the space

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

$$x = Vx'$$

...consider eigenvector coordinates

$$\forall x \iff \forall x'$$

$$x^T Q x = x V D V^T x = x'^T D x' = \sum_{i} \lambda_i x_i'^2$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x^{\prime} \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Negative semi-definite

Negative-definite

Positive semi-definite

$$Q \le 0 \qquad x^T Q x \le 0 \qquad \forall \ x$$

$$\leq 0$$

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$ 

$$x^T Q x < 0$$
 some

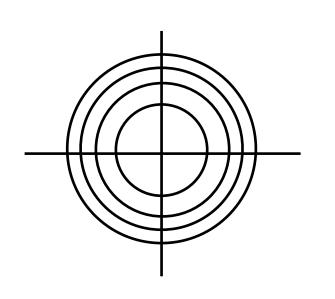
 $x^T Q x > 0$ some x

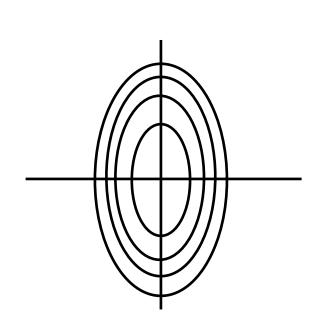
$$x^T Q x < 0$$
 some  $x$ 

 $Q \succ 0$ Surfaces:

$$Q = I$$

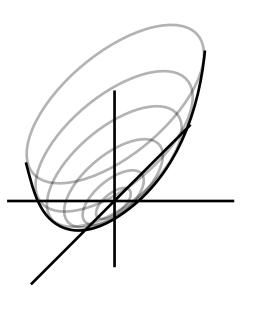
Indefinite:

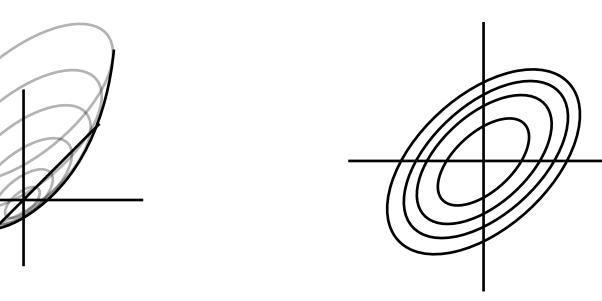


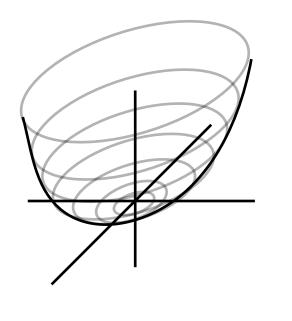


level sets

diagonal







surface

level sets

general

surface

level sets

surface

**Quadratic Form:** 

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

**Definiteness:** 

### **Analogy**

### **Eigenvalues**

## **Eigenvalue condition proof:**

since V is

invertible...

Positive definite:

Negative-definite

$$Q \succ 0$$

 $x^T Q x > 0$ 

 $x^T Q x < 0$ 

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

$$\lambda_i > 0 \quad \lambda_i \in eig(Q)$$

$$x = Vx'$$

...consider eigenvector coordinates

Negative semi-definite

ND

$$Q \succeq 0$$

 $Q \prec 0$ 

$$x^T Q x \ge 0 \quad \forall \ x$$

$$x^T Q x < 0 \quad \forall \ x \quad x \neq 0$$

...negative orthant

...positive orthant

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$ 

$$\lambda_i < 0 \quad \lambda_i \in eig(\zeta_i)$$

$$Q \leq 0$$
  $x^T$ 

$$x^T Q x \le 0 \qquad \forall \ x$$

some x

some x

...the rest of the space

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

# $x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i} x_{i}'^{2}$

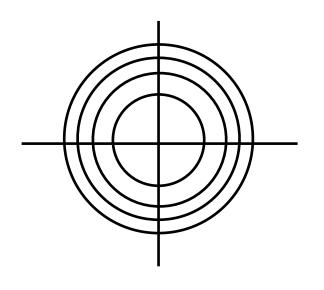
$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

 $\forall x \iff \forall x'$ 

**Surfaces:**  $Q \prec 0$ 

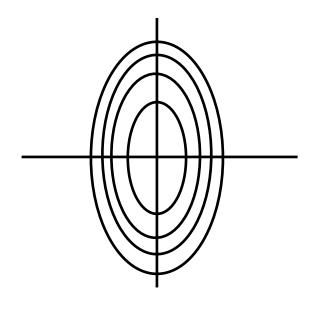
$$Q = I$$



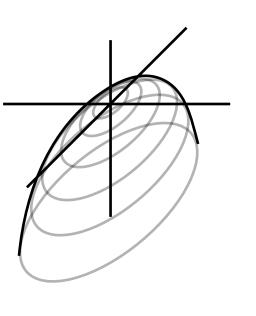
level sets

surface

diagonal

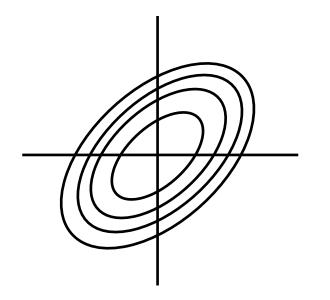




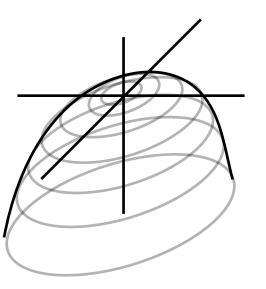


surface

general



level sets



surface

**Quadratic Form:** 

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

 $x^T Q x > 0 \quad \forall \ x \quad x \neq 0$ 

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$ 

$$Q = Q^T$$

**Definiteness:** 

**Definition Notation** 

 $x^T Q x \ge 0 \quad \forall \ x$ 

Positive definite:

Negative-definite

Positive semi-definite

Negative semi-definite

PD

 $Q \succ 0$ 

PSD

 $Q \succeq 0$ 

 $Q \prec 0$ 

ND

NSD

 $Q \leq 0$ 

 $x^T Q x \le 0 \quad \forall x$ 

 $x^T Q x > 0$ 

 $x^T Q x < 0$ some x **Analogy** 

...positive orthant

...positive orthant w/ boundary

...negative orthant

...negative orthant w/ boundary

...the rest of the space

**Eigenvalues** 

 $\lambda_i > 0 \quad \lambda_i \in eig(Q)$ 

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$ 

 $\lambda_i < 0 \quad \lambda_i \in eig(Q)$ 

 $\lambda_i \leq 0 \quad \lambda_i \in eig(Q)$ 

**Eigenvalue condition proof:** 

...consider eigenvector coordinates

$$x = Vx'$$

since V is invertible...

 $\forall x \iff \forall x'$ 

 $x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i} x_{i}'^{2}$ 

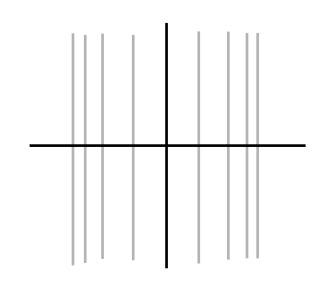
$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

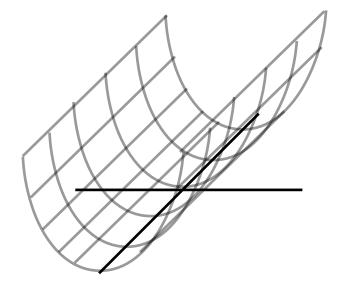
$$x \neq 0$$

Surfaces:  $Q \succeq 0$ 

Indefinite:

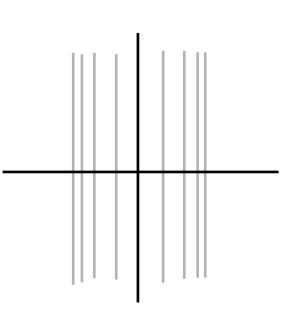
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



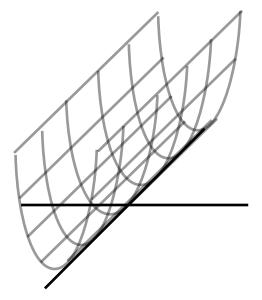


 $Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$ 

some x

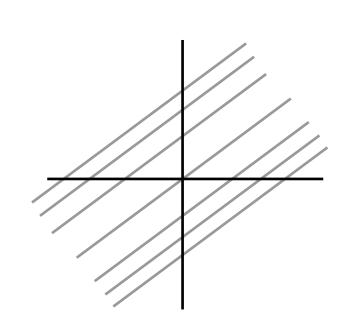


diagonal

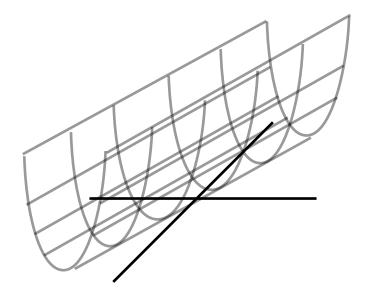


surface

 $Q = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} V^T = \lambda_1 v_1 v_1^T$ 







surface

level sets

surface

level sets

**Quadratic Form:** 

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

**Definiteness:** 

OI	IUI	

### **Eigenvalues**

### **Eigenvalue condition proof:**

...consider eigenvector coordinates

$$Q \succ 0$$

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

$$x \quad x \neq$$

$$\lambda_i > 0 \quad \lambda_i \in eig(Q)$$

$$x = Vx'$$

Negative semi-definite

$$Q \succeq 0$$

$$x^T Q x \ge 0 \quad \forall \ x$$

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$ 

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$ 

invertible... 
$$\forall x \iff \forall x'$$

since V is

NSD

$$Q \prec 0$$

 $Q \leq 0$ 

$$x^T Q x \le 0 \qquad \forall \ x$$

w/ boundary

...the rest of the space

...negative orthant

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

$$x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i} x_{i}'^{2}$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Indefinite:

$$x^T Q x > 0$$

some 
$$x$$

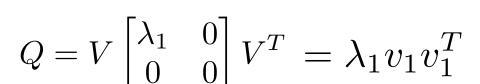
 $Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$ 

$$Qx < 0$$
 some  $x$ 

 $x^T Q x < 0$ 

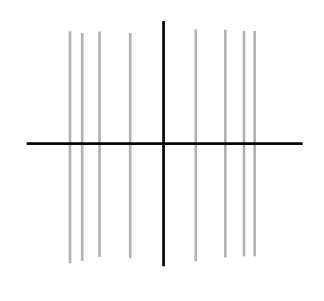
some x

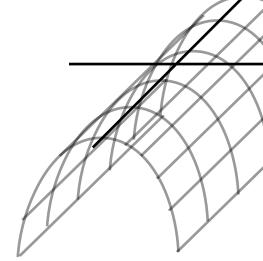
diagonal

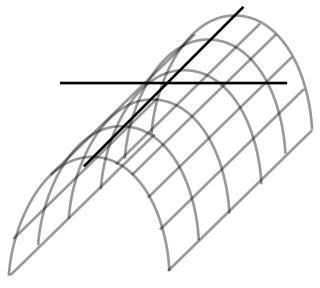


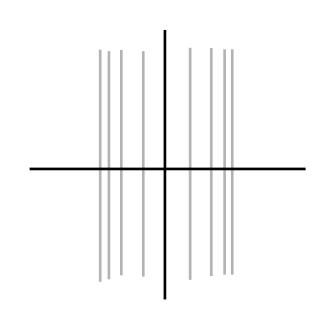
**Surfaces:** 

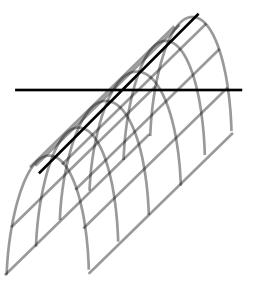
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

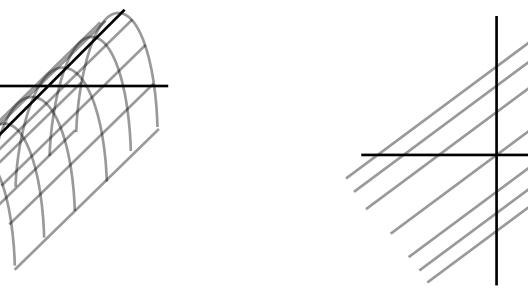


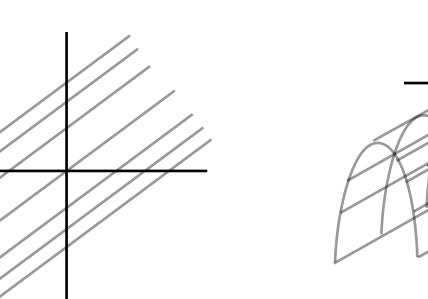


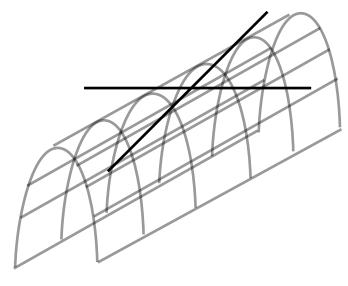












surface level sets

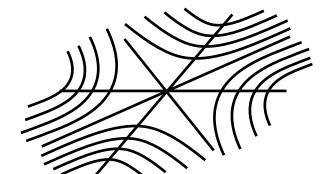
 $Q \leq 0$ 

level sets

surface

level sets

surface



**Quadratic Form:** 

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

**Definiteness:** 

Short Notation	
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**Definition** 

$$Q = Q^T$$

$$Q \succ 0$$

 $Q \succeq 0$ 

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

$$x \neq$$

### **Eigenvalues**

$$\lambda_i > 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

Positive definite:

Negative-definite

Positive semi-definite

Negative semi-definite

NSD

$$Q \prec 0$$

 $Q \leq 0$ 

 $x^T Q x \ge 0 \quad \forall \ x$ 

$$\forall x$$

 $x^T Q x \le 0 \qquad \forall \ x$ 

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$ 

 $x^T Q x > 0$ some x $x^T Q x < 0$ some x

**Analogy** 

...positive orthant

...positive orthant w/ boundary

...negative orthant

...negative orthant w/ boundary

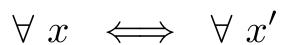
...the rest of the space

**Eigenvalue condition proof:** 

...consider eigenvector coordinates

$$x = Vx'$$

since V is invertible...



$$x^T Q x = x V D V^T x = x'^T D x' = \sum_{i} \lambda_i x_i'^2$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x^{\prime} \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

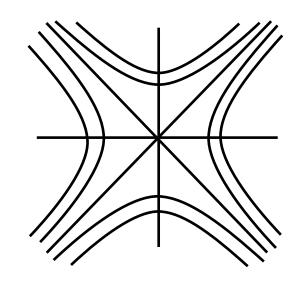
**Surfaces:** 

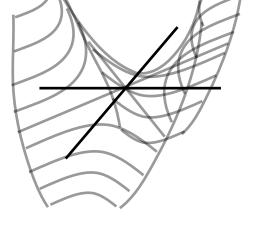
Indefinite:

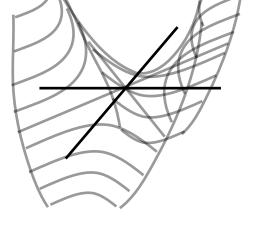


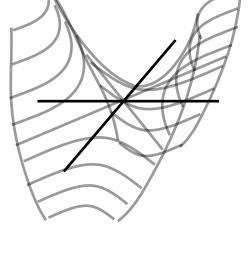
indefinite

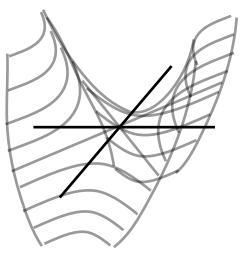
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



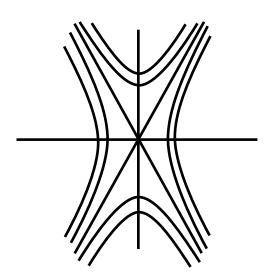




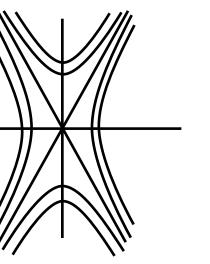


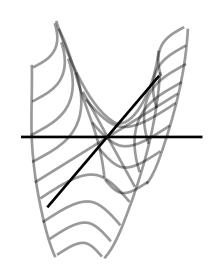




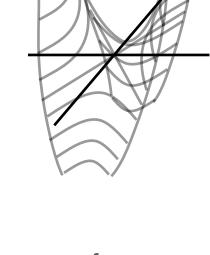


 $Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ 

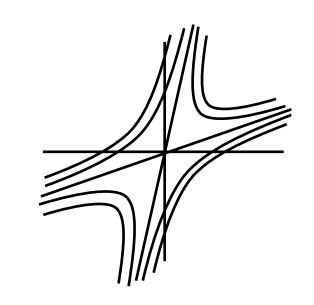




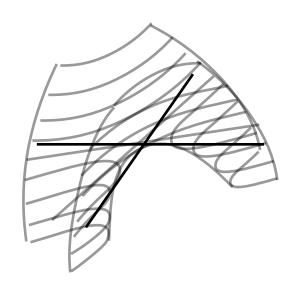
diagonal



$$Q = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^T$$



general



surface level sets

level sets

surface

level sets

surface

**Quadratic Form:** 

$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

$$Q \in \mathbb{R}^{n \times n}$$

$$Q = Q^T$$

**Definiteness:** 

### **Analogy**

### **Eigenvalue condition proof:**

Positive definite:

PSD

$$Q \succ 0$$

 $Q \succeq 0$ 

 $Q \prec 0$ 

$$x^T Q x > 0 \quad \forall \ x \quad x \neq 0$$

 $x^T Q x \ge 0 \quad \forall \ x$ 

 $x^T Q x > 0$ 

 $x^T Q x < 0$ 

$$x \neq 0$$

### ...positive orthant w/ boundary

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

**Eigenvalues** 

 $\lambda_i > 0 \quad \lambda_i \in eig(Q)$ 

 $\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$ 

 $\lambda_i \leq 0 \quad \lambda_i \in eig(Q)$ 

$$\lambda_i \in \mathrm{elg}(Q)$$

x = Vx'

...consider eigenvector coordinates

$$\forall x \iff \forall x'$$

$$x^{T}Qx = xVDV^{T}x = x'^{T}Dx' = \sum_{i} \lambda_{i} x_{i}'^{2}$$

$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Negative-definite

Negative semi-definite

Positive semi-definite

NSD

$$Q \leq 0$$
  $x^T Q$ 

$$x^T Q x \le 0 \qquad \forall \ x$$

 $x^T Q x < 0 \quad \forall \ x \quad x \neq 0$ 

some x

some x

...negative orthant w/ boundary

...the rest of the space

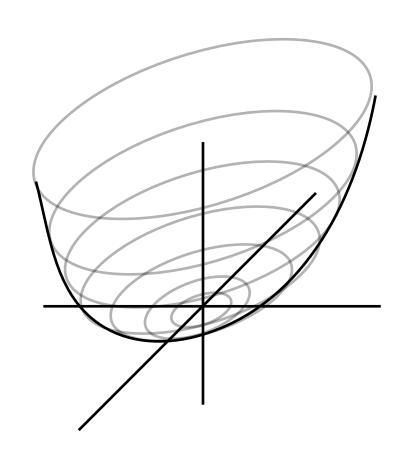
$$T = \begin{bmatrix} | & | \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$Q = VDV^T = \begin{bmatrix} \begin{vmatrix} & & \\ v_1 & v_2 \\ & & \end{vmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$||v_i||_2 =$$

Surfaces: 
$$Q \succ 0$$

Indefinite:



 $f(x) = x^T Q x = 1$ 

$$f\left(\frac{1}{\sqrt{\lambda_{1}}}v_{1}\right) = \frac{1}{\sqrt{\lambda_{1}}}v_{1}^{T}Qv_{1}\frac{1}{\sqrt{\lambda_{1}}}$$

$$= \frac{1}{\sqrt{\lambda_{1}}}\begin{bmatrix} 1 \\ v_{1} \\ 1 \end{bmatrix}^{T}\begin{bmatrix} 1 & 1 \\ v_{1} & v_{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{bmatrix}\begin{bmatrix} - & v_{1}^{T} & - \\ - & v_{2}^{T} & - \end{bmatrix}\begin{bmatrix} 1 \\ v_{1} \\ 1 \end{bmatrix}\frac{1}{\sqrt{\lambda_{1}}}$$

$$= \frac{1}{\sqrt{\lambda_{1}}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{T}\begin{bmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{\sqrt{\lambda_{1}}} = \frac{\lambda_{1}}{(\sqrt{\lambda_{1}})^{2}} = 1$$

surface

level sets

**Quadratic Form:** 

ND

 $f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$ 

Definiteness:	Short	Notation
Positive definite:	PD	$Q \succ 0$
Positive semi-definite	PSD	$0 \leq 0$

$$\begin{array}{cc} \mathsf{PSD} & Q \succeq 0 \\ \mathsf{ND} & Q \prec 0 \end{array}$$

Negative semi-definite NSD 
$$Q \leq 0$$

Indefinite:

Negative-definite

### **Definition** Notation

$$x^{T}Qx > 0 \quad \forall x \quad x \neq 0$$
$$x^{T}Qx \ge 0 \quad \forall x$$

$$x^T Q x < 0 \quad \forall \ x \quad x \neq 0$$

$$x^T Q x \le 0 \qquad \forall \ x$$

$$x^T Qx > 0$$
 some  $x$ 

$$x^T Q x < 0$$
 some  $x$ 

### **Analogy**

...positive orthant

...positive orthant w/ boundary

...negative orthant

...negative orthant w/ boundary

...the rest of the space

### **Eigenvalues**

$$\lambda_i > 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i \ge 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i < 0 \quad \lambda_i \in eig(Q)$$

$$\lambda_i \le 0 \quad \lambda_i \in eig(Q)$$

### **Eigenvalue condition proof:**

...consider eigenvector coordinates

$$x = Vx'$$

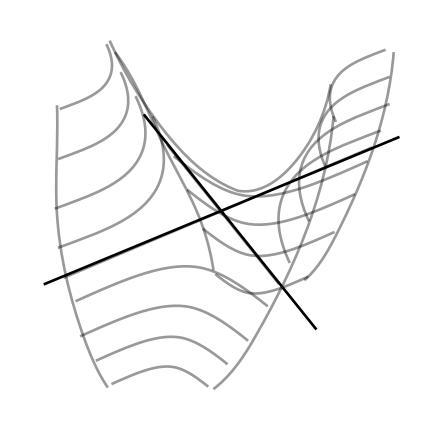
since V is  $\forall x \iff \forall x'$ invertible...

$$x^T Q x = x V D V^T x = x'^T D x' = \sum_{i} \lambda_i x_i'^2$$

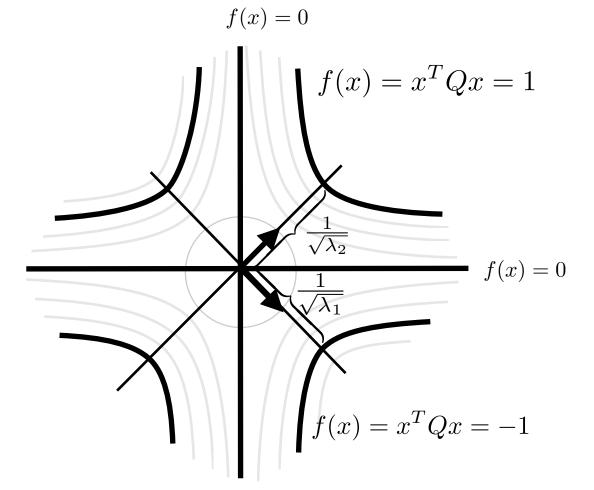
$$\sum_{i} \lambda_{i} x_{i}^{\prime 2} > 0 \quad \forall x' \iff \lambda_{i} > 0 \quad \forall \lambda_{i} \in eig(Q)$$

$$x \neq 0$$

Surfaces: 
$$Q$$
 indefinite



surface



level sets

$$Q = VDV^{T} = \begin{bmatrix} \begin{vmatrix} & & | \\ v_1 & v_2 \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$f\left(\frac{1}{\sqrt{\lambda_{1}}}v_{1}\right) = \frac{1}{\sqrt{\lambda_{1}}}v_{1}^{T}Qv_{1}\frac{1}{\sqrt{\lambda_{1}}}$$

$$= \frac{1}{\sqrt{\lambda_{1}}}\begin{bmatrix} \begin{vmatrix} 1 \\ v_{1} \end{vmatrix} \end{bmatrix}^{T}\begin{bmatrix} \begin{vmatrix} 1 \\ v_{1} \end{vmatrix} v_{2} \\ \begin{vmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{bmatrix}\begin{bmatrix} - v_{1}^{T} & - \\ - v_{2}^{T} & - \end{bmatrix}\begin{bmatrix} \begin{vmatrix} 1 \\ v_{1} \end{vmatrix} \frac{1}{\sqrt{\lambda_{1}}}$$

$$= \frac{1}{\sqrt{\lambda_{1}}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{T}\begin{bmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{\lambda_{1}}} = \frac{\lambda_{1}}{(\sqrt{\lambda_{1}})^{2}} = 1$$

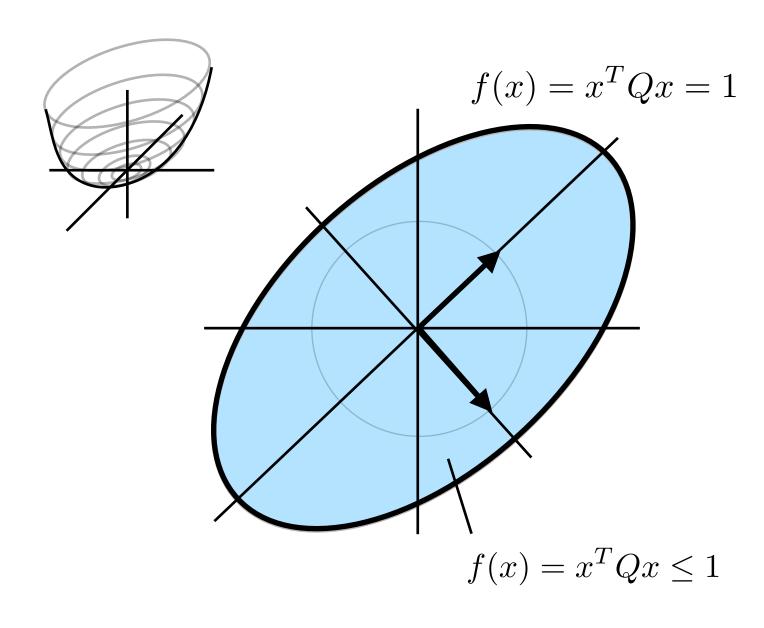
# **Quadratic Form - Level Sets**

Quadratic Form:

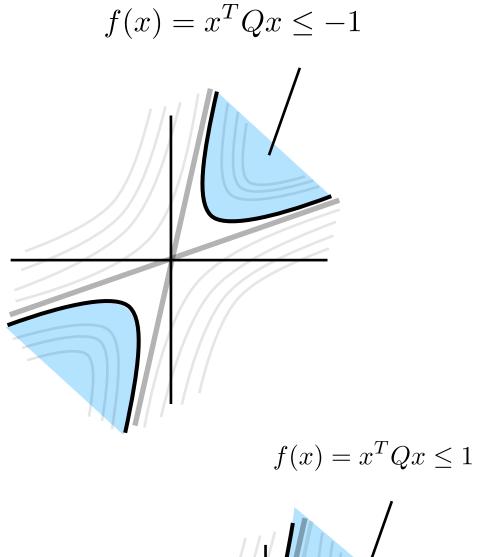
$$f(x) = x^T Q x$$

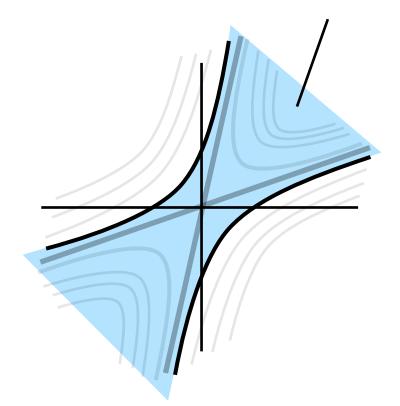
$$f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$$

### **Ellipsoids**

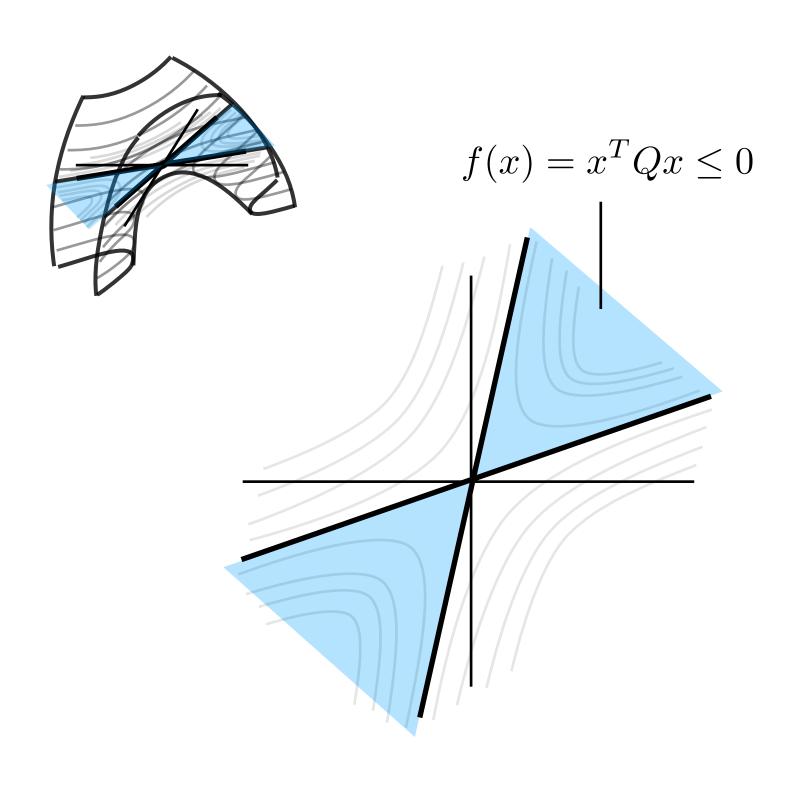


### **Hyperboloids**





### Cones



# **Quadratic Form - Level Sets**

$$f(x) = x^T Q x$$

Quadratic Form: 
$$f(x) = x^T Q x$$
  $Q \in \mathbb{R}^{n \times n}$   $Q = Q^T$ 

$$\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix} \qquad b \in [\underline{\mathbf{b}}, \ \overline{\mathbf{b}}]$$

$$c \in [\underline{\mathbf{c}}, \ \overline{\mathbf{c}}]$$

Cones

