Farka’s Lemma

Convex Optimization

Dan Calderone

Farka's Lemma Let $H \in \mathbb{R}^{m \times n}$ and $r \in \mathbb{R}^n$.

Then exactly one of the following two assertions is true.

1. There exists $v \in \mathbb{R}^m$ such that $v^\top H = r^\top$ and $v \geq 0$
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Core Intuition: Square Case

"each row of H is orthogonal to the (other) columns of H inverse"

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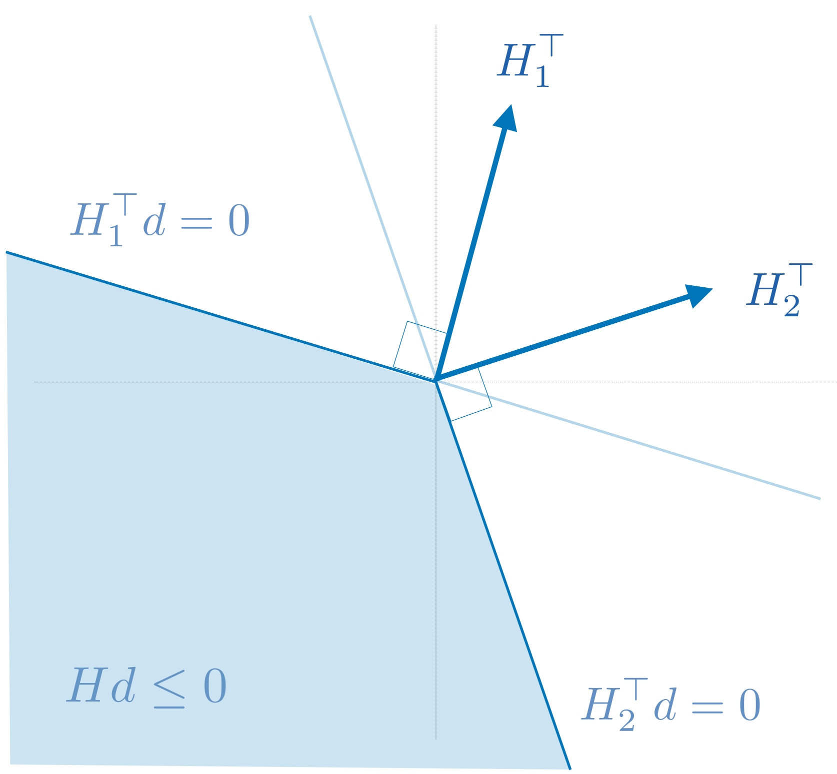

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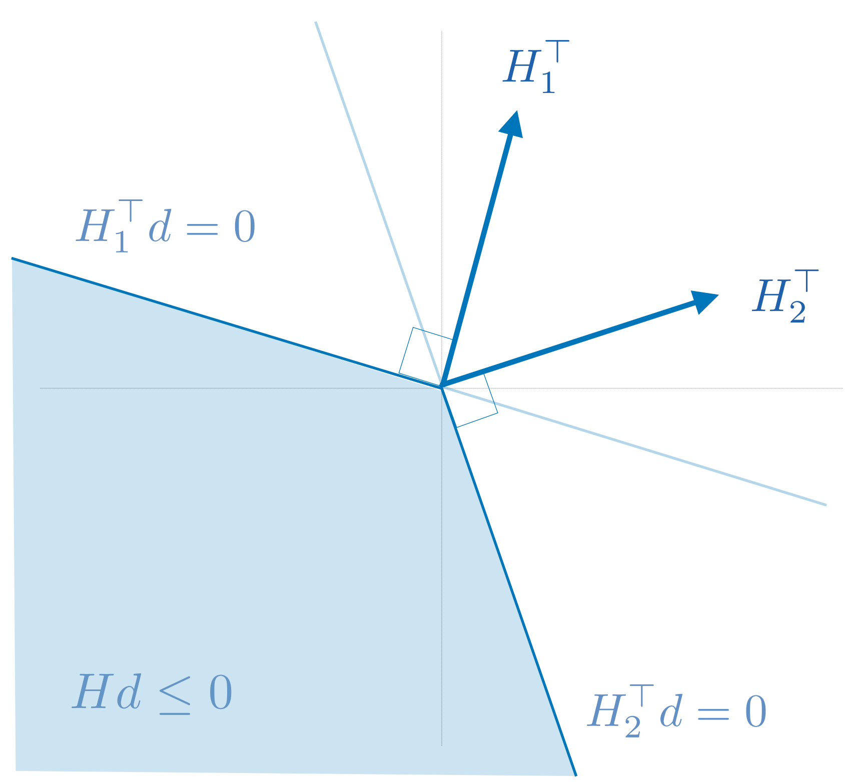

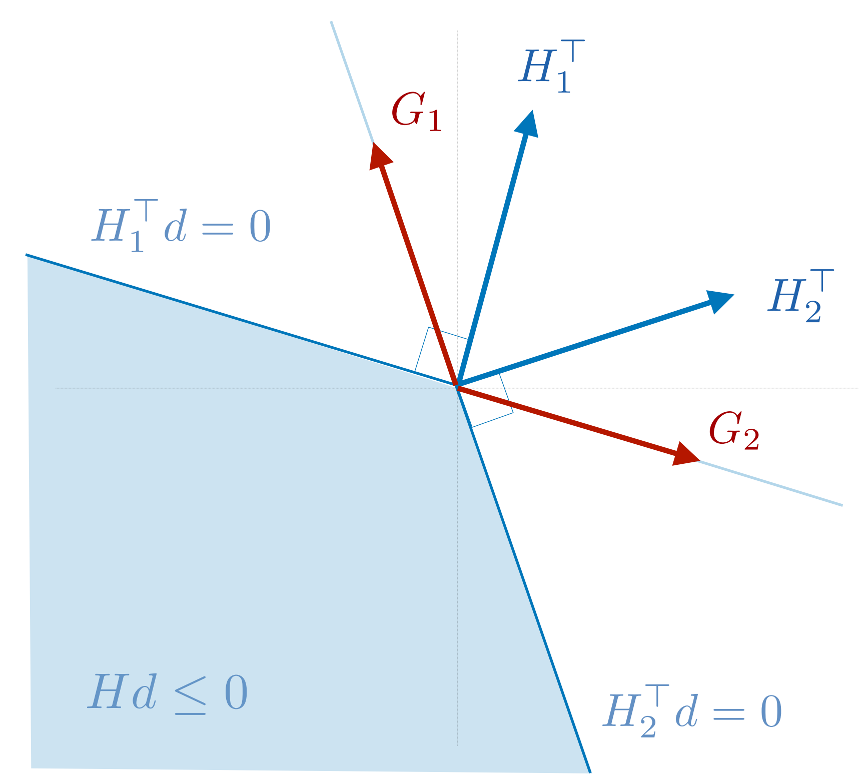

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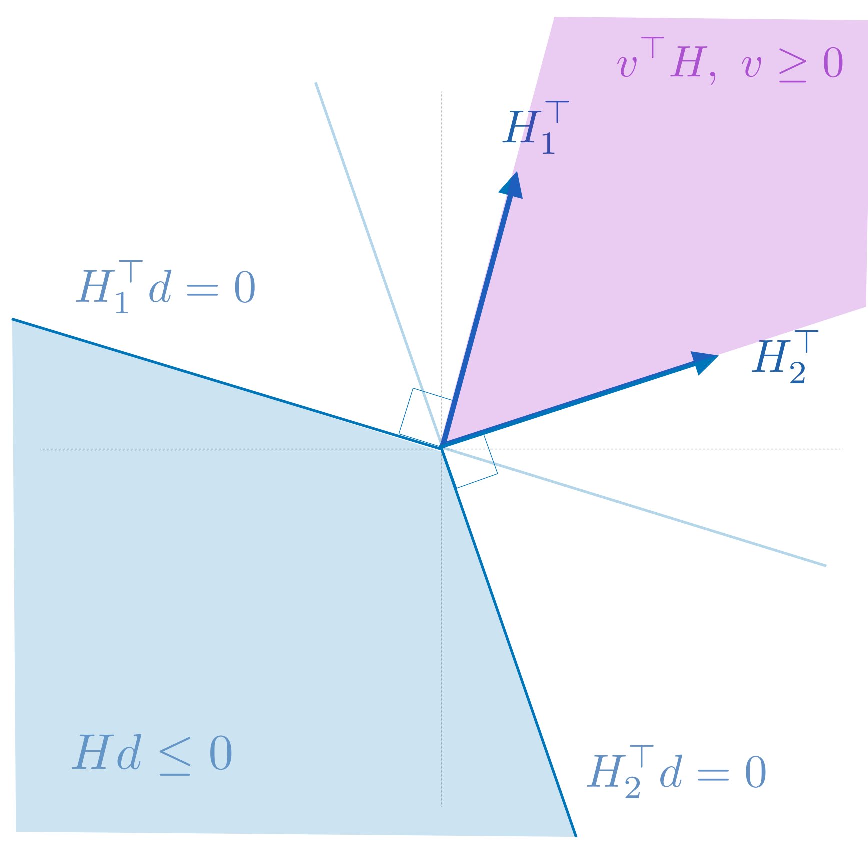

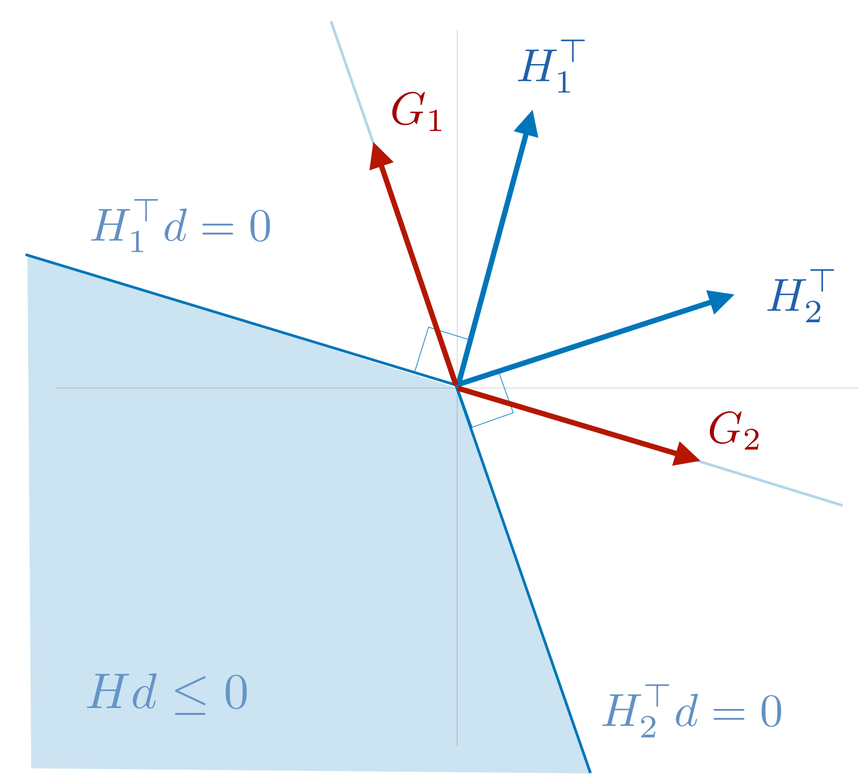

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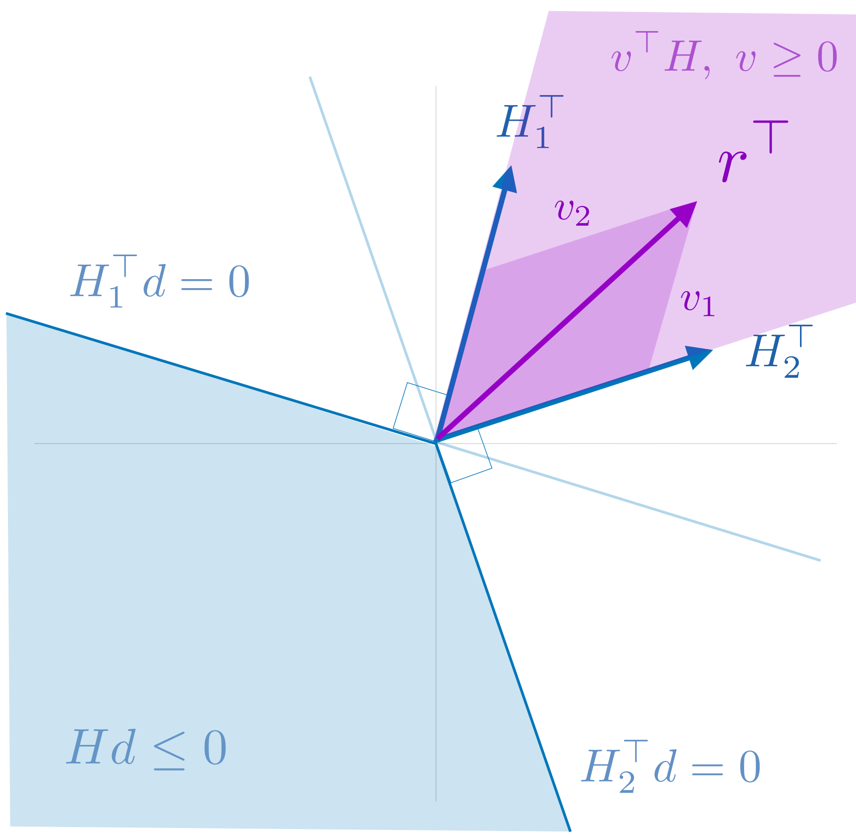

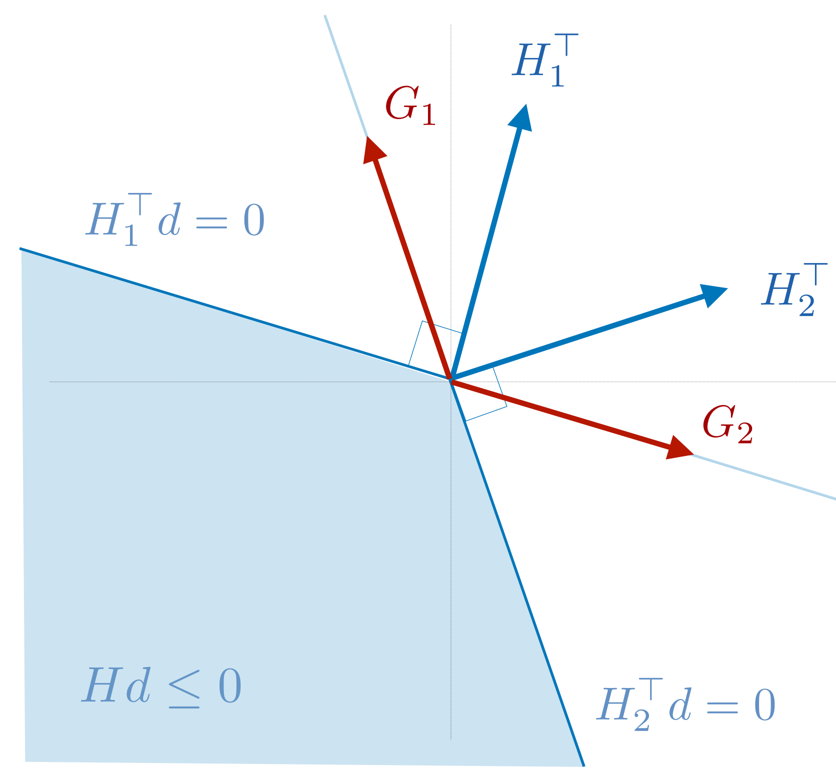

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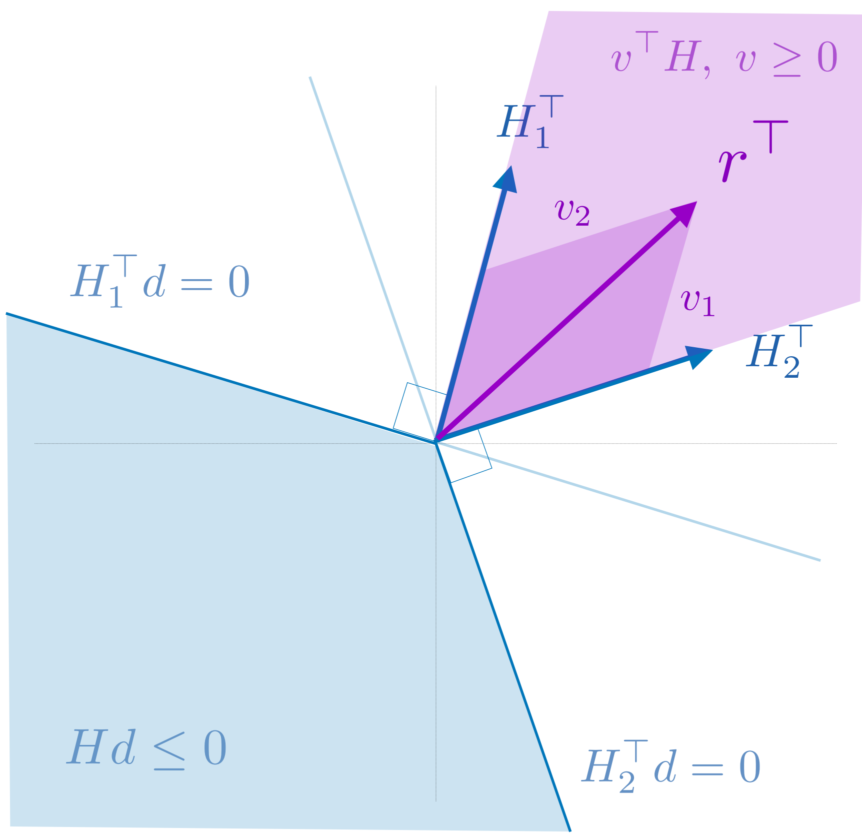
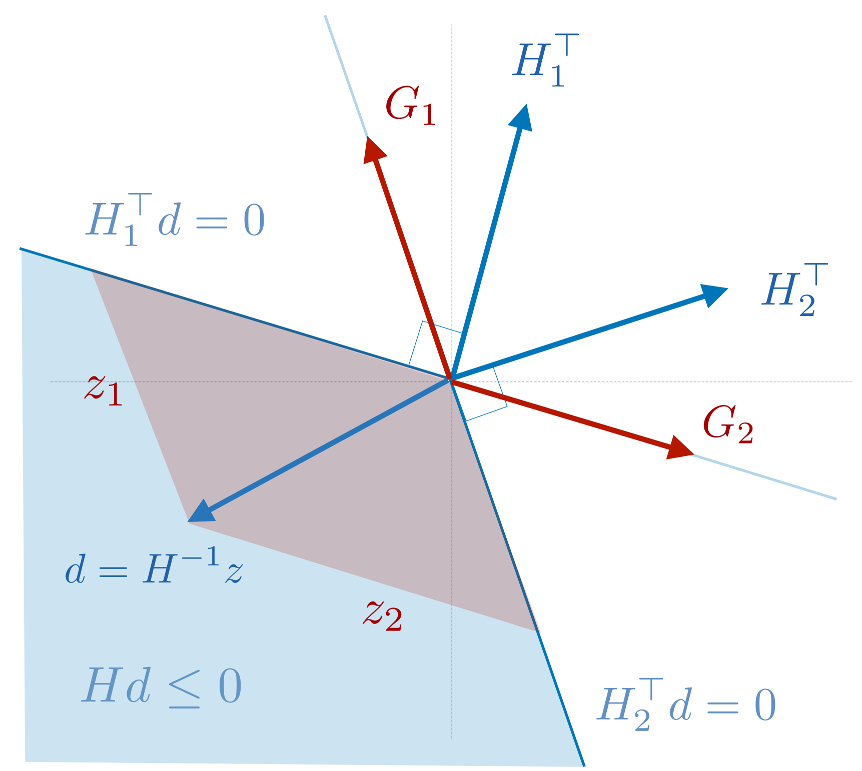
For square, invertible H $\quad \exists\, d \in \mathbb{R}^n$ s.t. $Hd \leq 0 \iff d = H^{-1}z$ for $z \leq 0$





True | **False**

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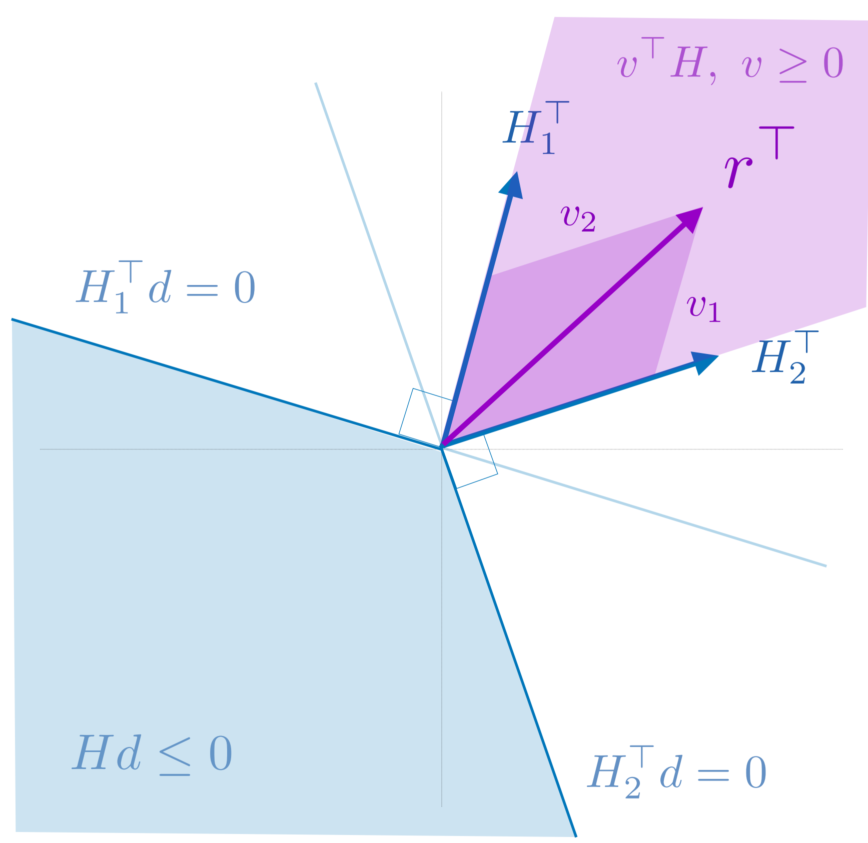
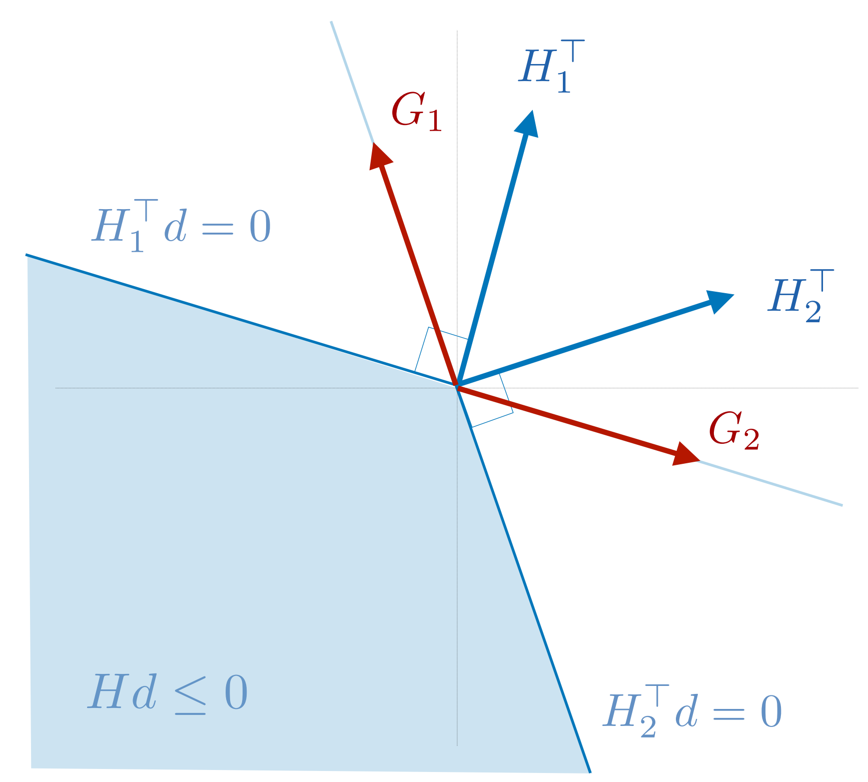
Case 1.





Case 1.

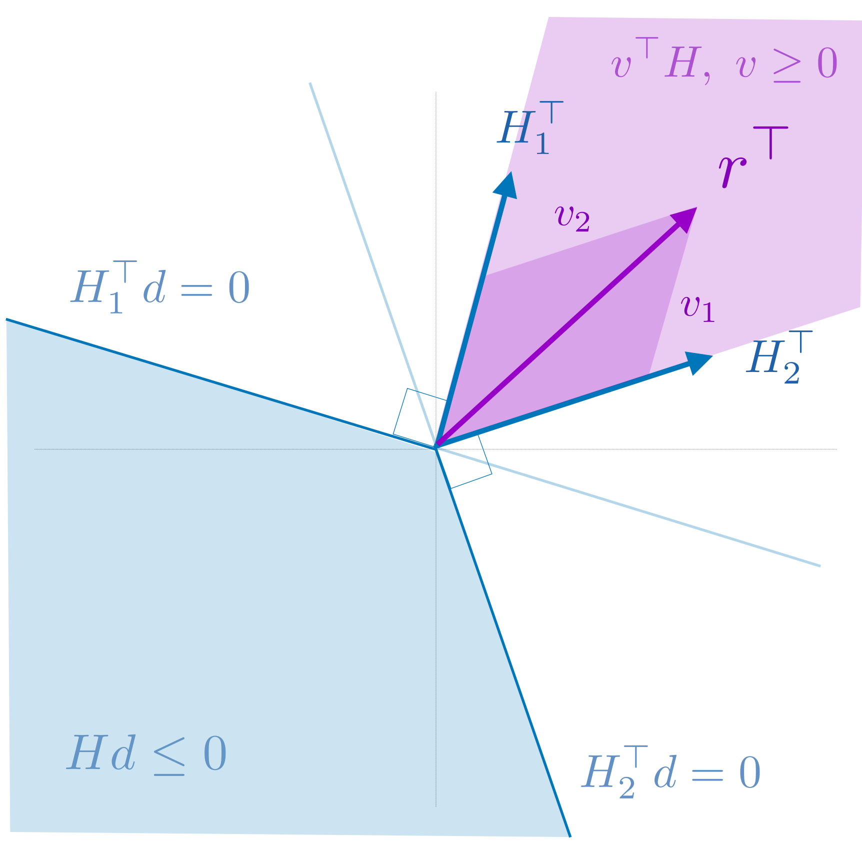
Farka's Lemma Let $H \in \mathbb{R}^{m \times n}$ and $r \in \mathbb{R}^n$.

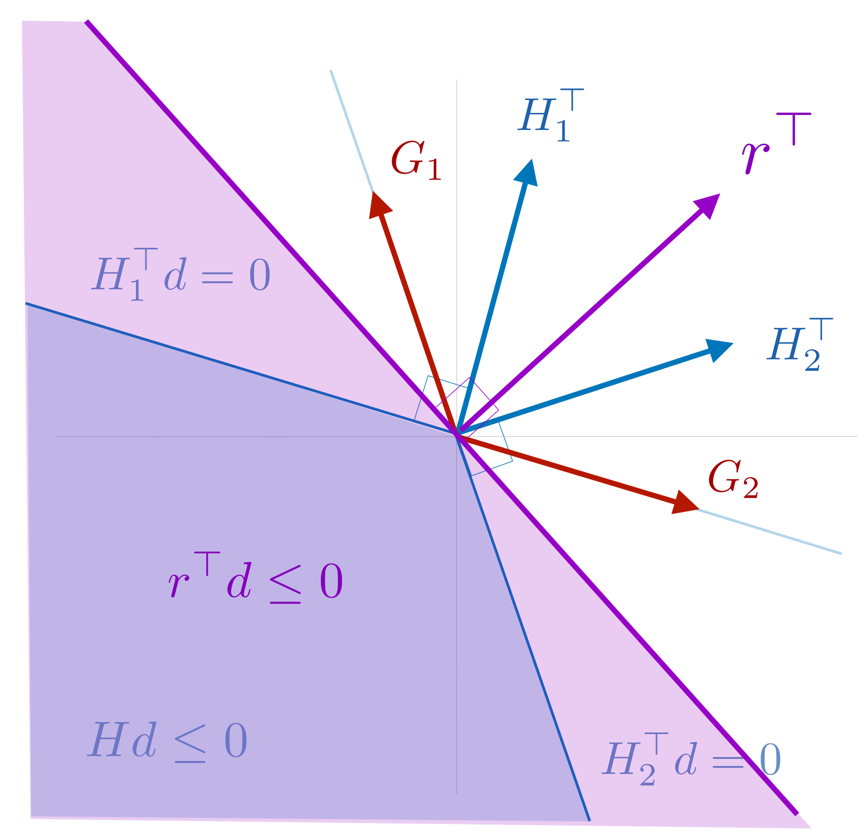
Then exactly one of the following two assertions is true.

True 1. There exists $v \in \mathbb{R}^m$ such that $v^\top H = r^\top$ and $v \geq 0$

False 2. There exists $d \in \mathbb{R}^n$ such that $Hd \leq 0$ and $r^\top d > 0$

$\overset{\text{not}}{\Rightarrow}$ For all $d \in \mathbb{R}^n$ s.t. $Hd \leq 0$, $r^\top d \leq 0$





Case 2.

False | True

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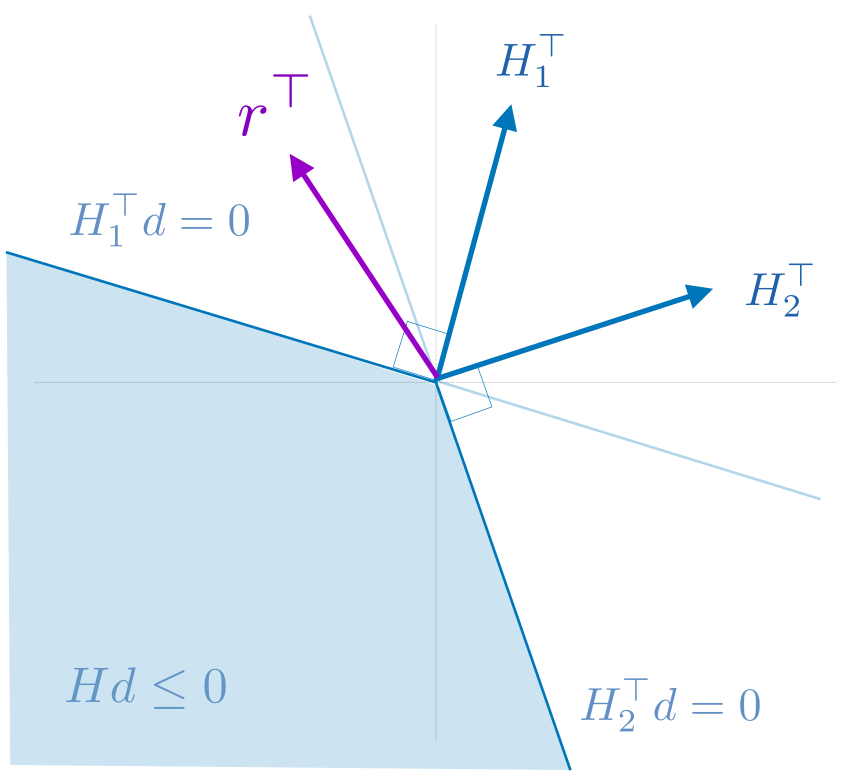

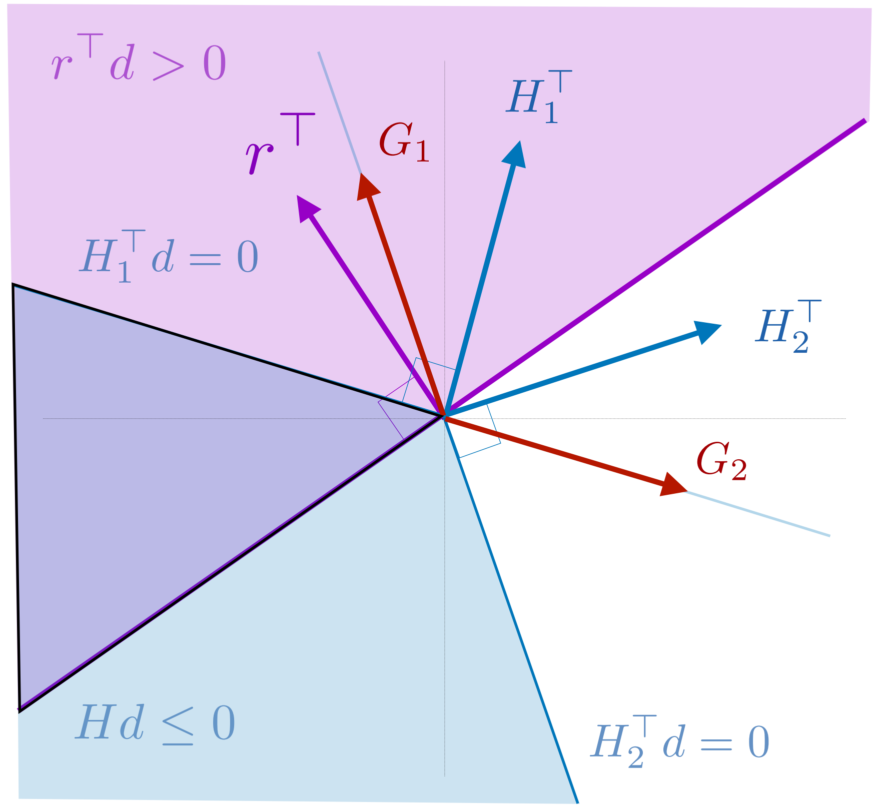

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Then exactly one of the following two assertions is true.

False 1. There exists $v \in \mathbb{R}^m$ such that $v^\top H = r^\top$ and $v \geq 0$

True 2. There exists $d \in \mathbb{R}^n$ such that $Hd \leq 0$ and $r^\top d > 0$

Case 2.

not $\Rightarrow$ For all $v \in \mathbb{R}^m$ s.t. $v^\top H = r^\top, v \not\geq 0$