

Dual Programs: LP

$$\begin{array}{ll} \max_x & r^\top x \\ \text{s.t.} & \begin{array}{l} Ax = b \quad \lambda \\ Cx \geq d \quad \mu \end{array} \end{array} \quad \begin{array}{c} \text{Primal} \\ \text{Program} \end{array}$$

Dual Programs: LP

$$\begin{array}{ll} \max_x & r^\top x \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: LP

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$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

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$$\min_{\lambda, \mu \geq 0} \max_x (r^\top + \lambda^\top A + \mu^\top C)x - \lambda^\top b - \mu^\top d$$

Dual Programs: LP

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 \max_x & r^\top x \\
 \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}
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$$\min_{\lambda, \mu \geq 0} \max_x \underbrace{(r^\top + \lambda^\top A + \mu^\top C)x}_{\text{must be 0 for inner}} - \lambda^\top b - \mu^\top d$$

problem to be bounded

Dual Programs: LP

$$\begin{array}{ll}
 \max_x & r^\top x \\
 \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu}
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$$\begin{aligned}
 \min_{\lambda, \mu \geq 0} \quad & \max_x \quad \underbrace{(r^\top + \lambda^\top A + \mu^\top C)x}_{\text{must be 0 for inner}} - \lambda^\top b - \mu^\top d \\
 & \quad \text{problem to be bounded}
 \end{aligned}$$

$$\begin{array}{ll}
 \min_{\lambda, \mu} & -\lambda^\top b - \mu^\top d \\
 \text{s.t.} & r^\top + \lambda^\top A + \mu^\top C = 0, \quad \mu \geq 0
 \end{array}
 \quad \text{Dual Program}$$

Dual Programs: QP

$$\max_x \quad \frac{1}{2}x^\top Qx + r^\top x$$

Primal Program

s.t. $Ax = b \quad \lambda$ $Cx \geq d \quad \mu$

Note: $Q = Q^\top \prec 0$

Dual Programs: QP

$$\begin{array}{ll} \max_x & \frac{1}{2}x^\top Qx + r^\top x \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \begin{array}{l} \text{Primal Program} \\ \text{Note: } Q = Q^\top \prec 0 \end{array}$$

$$\mathcal{L}(x, \lambda, \mu) = \frac{1}{2}x^\top Qx + r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d)$$

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$$\min_{\lambda, \mu \geq 0} \max_x \frac{1}{2}x^\top Qx + (r^\top + \lambda^\top A + \mu^\top C)x - \lambda^\top b - \mu^\top d$$

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$$\min_{\lambda, \mu \geq 0} \max_x \underbrace{\frac{1}{2}x^\top Qx + (r^\top + \lambda^\top A + \mu^\top C)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d$$

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Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$,

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maximize explicitly...

$$\text{Define } \xi^\top = r^\top + \lambda^\top A + \mu^\top C, \quad \frac{\partial}{\partial x} \left(\frac{1}{2}x^\top Qx + \xi^\top x \right) = 0$$

Dual Programs: QP

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maximize explicitly...

$$\text{Define } \xi^\top = r^\top + \lambda^\top A + \mu^\top C, \quad \frac{\partial}{\partial x} \left(\frac{1}{2}x^\top Qx + \xi^\top x \right) = 0 \quad \Rightarrow \quad x = -Q^{-1}\xi$$

Dual Programs: QP

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maximize explicitly...

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$, $\frac{\partial}{\partial x} \left(\frac{1}{2}x^\top Qx + \xi^\top x \right) = 0 \Rightarrow x = -Q^{-1}\xi$

Plug in x ...

Dual Programs: QP

$$\begin{array}{ll} \max_x & \frac{1}{2}x^\top Qx + r^\top x \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \begin{array}{l} \text{Primal Program} \\ \text{Note: } Q = Q^\top \prec 0 \end{array}$$

$$\mathcal{L}(x, \lambda, \mu) = \frac{1}{2}x^\top Qx + r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d)$$

$$\begin{array}{ll} \max_x & \min_{\lambda, \mu \geq 0} \quad \frac{1}{2}x^\top Qx + r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) \\ \min_{\lambda, \mu \geq 0} & \max_x \quad \underbrace{\frac{1}{2}x^\top Qx + (r^\top + \lambda^\top A + \mu^\top C)x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d \end{array}$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C$, $\frac{\partial}{\partial x} \left(\frac{1}{2}x^\top Qx + \xi^\top x \right) = 0 \Rightarrow x = -Q^{-1}\xi$

Plug in x ...

$$\begin{array}{ll} \min_{\xi, \lambda, \mu} & -\frac{1}{2}\xi^\top Q^{-1}\xi - \lambda^\top b - \mu^\top d \end{array} \quad \begin{array}{l} \text{Dual Program} \end{array}$$

$$\text{s.t.} \quad \xi^\top = r^\top + \lambda^\top A + \mu^\top C, \quad \mu \geq 0 \quad \begin{array}{l} \text{Note: } -Q^{-1} = -Q^{-\top} \succ 0 \end{array}$$

Dual Programs: SOCP (lin objective)

$$\begin{array}{ll} \max_x & r^\top x \\ \text{s.t.} & \begin{array}{l} Ax = b \quad \lambda \\ Cx \geq d \quad \mu \\ \|E_i x + e_i\|_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \end{array} \end{array} \quad \begin{array}{l} \text{Primal Program} \\ \text{Dual Program} \end{array}$$

Dual Programs: SOCP (lin objective)

$$\begin{array}{ll} \max_{x, y_i} & r^\top x \\ \text{s.t.} & \begin{array}{l} Ax = b \quad \lambda \\ Cx \geq d \quad \mu \\ \|y_i\|_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i \end{array} \\ & y_i = E_i x + b_i \quad i \in \mathcal{I} \quad \theta_i \end{array}$$

Primal Program

Dual Programs: SOCP (lin objective)

$$\max_{x, y_i} \quad r^\top x$$

Primal Program

s.t.

$$Ax = b \quad \lambda$$

$$Cx \geq d \quad \mu$$

$$\|y_i\|_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$$

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$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = -r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i(g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top(E_i x + b_i - y_i)$$

Dual Programs: SOCP (lin objective)

	$\max_{x, y_i} r^\top x$	Primal Program
	s.t.	$Ax = b \quad \lambda$ $Cx \geq d \quad \mu$ $\ y_i\ _2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$
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$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = -r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i(g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top(E_i x + e_i - y_i)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} -r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i(g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top(E_i x + e_i - y_i)$$

Dual Programs: SOCP (lin objective)

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$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} r^\top x + \lambda^\top(Ax - b) + \mu^\top(Cx - d) + \sum_i \nu_i(g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top(E_i x + e_i - y_i)$$

$$\begin{aligned} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} & \left(r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] \right) x - \lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i] \\ & + \sum_i -(\nu_i \|y_i\|_2 + \theta_i^\top y_i) \end{aligned}$$

Dual Programs: SOCP (lin objective)

$$\max_{x, y_i} \quad r^\top x$$

s.t.

$$Ax = b$$

$$Cx \geq d$$

$$\|y_i\|_2 \leq g_i^\top x + h_i$$

Primal Program

$$y_i = E_i x + b_i \quad i \in \mathcal{I}$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] \right) x}_{\text{must be 0 for inner problem to be bounded}} - \lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i]$$

$$+ \sum_i \underbrace{- (\nu_i \|y_i\|_2 + \theta_i^\top y_i)}_{\text{must be bounded above...}}$$

Dual Programs: SOCP (lin objective)

$$\max_{x, y_i} \quad r^\top x$$

s.t.

$$Ax = b \quad \lambda$$

$$Cx \geq d \quad \mu$$

Primal Program

$$\|y_i\|_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$$

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$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

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$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] \right) x}_{\text{must be 0 for inner problem to be bounded}} - \lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i]$$

$$+ \sum_i \underbrace{- (\nu_i \|y_i\|_2 + \theta_i^\top y_i)}_{\text{must be bounded above...}}$$

$$\sup_{y_i} - (\nu_i \|y_i\|_2 + \theta_i^\top y_i)$$

Dual Programs: SOCP (lin objective)

$$\max_{x, y_i} \quad r^\top x$$

s.t.

$$Ax = b$$

$$Cx \geq d$$

$$\|y_i\|_2 \leq g_i^\top x + h_i$$

Primal Program

$$y_i = E_i x + b_i \quad i \in \mathcal{I}$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

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$$+ \sum_i \underbrace{- (\nu_i \|y_i\|_2 + \theta_i^\top y_i)}_{\text{must be bounded above...}}$$

$$\begin{aligned} & \sup_{y_i} - (\nu_i \|y_i\|_2 + \theta_i^\top y_i) \\ &= \begin{cases} 0 & ; \|\theta_i\|_2 \leq \nu_i \\ \infty & ; \text{otherwise} \end{cases} \end{aligned}$$

Dual Programs: SOCP (lin objective)

$$\max_{x, y_i} \quad r^\top x$$

s.t.

$$Ax = b \quad \lambda$$

$$Cx \geq d \quad \mu$$

Primal Program

$$\|y_i\|_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$$

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$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] \right) x}_{\text{must be 0 for inner problem to be bounded}} - \lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i] + \sum_i \underbrace{- (\nu_i \|y_i\|_2 + \theta_i^\top y_i)}_{\text{must be bounded above...}}$$

$$\text{Suppose } \|\theta_i\|_2 \leq \nu_i \quad (\nu_i \geq 0)$$

$$\begin{aligned} -\theta_i^\top y_i &\leq \nu_i \|y_i\|_2 \quad \forall y_i \\ -(\nu_i \|y_i\|_2 + \theta_i^\top y_i) &\leq 0 \quad \forall y_i \end{aligned}$$

$$\sup_{y_i} - (\nu_i \|y_i\|_2 + \theta_i^\top y_i)$$

$$= \begin{cases} 0 & ; \quad \|\theta_i\|_2 \leq \nu_i \\ \infty & ; \quad \text{otherwise} \end{cases}$$

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$$\max_{x, y_i} \quad r^\top x$$

s.t.

$$Ax = b$$

$$Cx \geq d$$

$$\|y_i\|_2 \leq g_i^\top x + h_i$$

Primal Program

$$y_i = E_i x + b_i \quad i \in \mathcal{I}$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] \right) x}_{\text{must be 0 for inner problem to be bounded}} - \lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i]$$

$$+ \sum_i \underbrace{- (\nu_i \|y_i\|_2 + \theta_i^\top y_i)}$$

must be bounded above...

$$\text{Suppose } \|\theta_i\|_2 > \nu_i \quad (\nu_i \geq 0)$$

$$\text{take } y_i = -s\theta_i \quad s \in \mathbb{R}_+$$

$$-(\nu_i \|y_i\|_2 + \theta_i^\top y_i) = -s(\nu_i \|\theta_i\|_2 - \|\theta_i\|_2^2) > 0$$

$$\begin{aligned} \sup_{y_i} - (\nu_i \|y_i\|_2 + \theta_i^\top y_i) \\ = \begin{cases} 0 & ; \|\theta_i\|_2 \leq \nu_i \\ \infty & ; \text{otherwise} \end{cases} \end{aligned}$$

Dual Programs: SOCP (lin objective)

	$\max_{x, y_i} r^\top x$	Primal Program
	s.t.	$Ax = b \quad \lambda$ $Cx \geq d \quad \mu$ $\ y_i\ _2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i$
		$y_i = E_i x + b_i \quad i \in \mathcal{I} \quad \theta_i$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\left(r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] \right) x}_{\text{must be 0 for inner problem to be bounded}} - \lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i] + \sum_i \underbrace{- (\nu_i \|y_i\|_2 + \theta_i^\top y_i)}_{\text{must be bounded above...}}$$

Dual Program

$$\begin{aligned} \min_{\substack{\lambda, \mu \\ \nu_i, \theta_i}} & -\lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i] \\ \text{s.t.} & r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] = 0, \quad \mu \geq 0, \\ & \|\theta_i\|_2 \leq \nu_i, \quad \nu_i \geq 0 \quad i \in \mathcal{I} \end{aligned} \quad \begin{aligned} \sup_{y_i} & - (\nu_i \|y_i\|_2 + \theta_i^\top y_i) \\ & = \begin{cases} 0 & ; \quad \|\theta_i\|_2 \leq \nu_i \\ \infty & ; \quad \text{otherwise} \end{cases} \end{aligned}$$

Dual Programs: SOCP (quad objective)

Primal Program

$$\begin{aligned} \max_{x, y_i} & \frac{1}{2} x^\top Q x + r^\top x \\ \text{s.t.} & Ax = b \quad \lambda \\ & Cx \geq d \quad \mu \\ & \|y_i\|_2 \leq g_i^\top x + h_i \quad i \in \mathcal{I} \quad \nu_i \\ & y_i = E_i x + b_i \quad i \in \mathcal{I} \quad \theta_i \end{aligned}$$

$$\mathcal{L}(x, y_i, \lambda, \mu, \nu_i, \theta_i) = \frac{1}{2} x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\max_{x, y_i} \min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \frac{1}{2} x^\top Q x + r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d) + \sum_i \nu_i (g_i^\top x + h_i - \|y_i\|_2) + \sum_i \theta_i^\top (E_i x + e_i - y_i)$$

$$\min_{\substack{\lambda, \mu \geq 0 \\ \nu_i \geq 0, \theta_i}} \max_{x, y_i} \underbrace{\frac{1}{2} x^\top Q x + \left(r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] \right) x}_{\text{maximize explicitly...}} - \lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i] + \sum_i \underbrace{- (\nu_i \|y_i\|_2 + \theta_i^\top y_i)}_{\text{must be bounded above...}}$$

Dual Program

$$\begin{aligned} \min_{\substack{\lambda, \mu, \xi \\ \nu_i, \theta_i}} & -\frac{1}{2} \xi^\top Q^{-1} \xi - \lambda^\top b - \mu^\top d + \sum_i [\nu_i h_i + \theta_i^\top e_i] \\ \text{s.t.} & r^\top + \lambda^\top A + \mu^\top C + \sum_i [\nu_i g_i^\top + \theta_i^\top E_i] = \xi^\top, \quad \mu \geq 0, \\ & \|\theta_i\|_2 \leq \nu_i, \quad \nu_i \geq 0 \quad i \in \mathcal{I} \end{aligned} \quad \begin{aligned} & \sup_{y_i} - (\nu_i \|y_i\|_2 + \theta_i^\top y_i) \\ & = \begin{cases} 0 & ; \quad \|\theta_i\|_2 \leq \nu_i \\ \infty & ; \quad \text{otherwise} \end{cases} \end{aligned}$$

Dual Programs: SDP (lin objective)

$$\max_X \quad \langle R, X \rangle$$

Primal Program

$$\text{s.t.} \quad \langle A_i, X \rangle = b_i \quad i \in \mathcal{I} \quad \cancel{\lambda_i}$$

$$\langle C_j, X \rangle \geq d_j \quad j \in \mathcal{J} \quad \cancel{\mu_j}$$

$$X = X^\top \succeq 0 \quad \cancel{U}$$

Dual Programs: SDP (lin objective)

$$\begin{array}{ll} \max_X & \text{Tr}(R^\top X) \\ \text{s.t.} & \begin{array}{l} \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \end{array} \bigcup \lambda_i \\ & \begin{array}{l} \text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \end{array} \bigcup \mu_j \\ & X = X^\top \succeq 0 \bigcup U \end{array}$$

Primal Program

Dual Programs: SDP (lin objective)

$$\max_X \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t. } \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

$$\max_X \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t. } \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

$$\max_X \quad \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t.} \quad \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \quad \Leftarrow \quad \min_{U \succeq 0} \text{Tr}(U^\top X)$$

Dual Programs: SDP (lin objective)

$$\max_{X} \quad \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t. } \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; \quad X = X^\top \succeq 0 \\ -\infty & ; \quad \text{otherwise} \end{cases}$$

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$	Primal Program
s.t. Tr($A_i^\top X$) = b_i $i \in \mathcal{I}$ Tr($C_j^\top X$) ≥ d_j $j \in \mathcal{J}$ $X = X^\top \succeq 0$	λ_{i} μ_{j} U

$$\mathcal{L}(X, \lambda_i, \mu_j, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; \quad X = X^\top \succeq 0 \\ -\infty & ; \text{ otherwise} \end{cases}$$

Suppose $X \succeq 0$ ($X = X^\top$)

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$	Primal Program
s.t. Tr($A_i^\top X$) = b_i $i \in \mathcal{I}$ λ_i Tr($C_j^\top X$) ≥ d_j $j \in \mathcal{J}$ μ_j $X = X^\top \succeq 0$ U	

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; \quad X = X^\top \succeq 0 \\ -\infty & ; \text{ otherwise} \end{cases}$$

$$\text{Suppose } X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix} \text{exists} \\ \text{unique} \end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$	Primal Program
s.t. Tr($A_i^\top X$) = b_i $i \in \mathcal{I}$ λ_i Tr($C_j^\top X$) ≥ d_j $j \in \mathcal{J}$ μ_j $X = X^\top \succeq 0$ U	

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; \quad X = X^\top \succeq 0 \\ -\infty & ; \text{ otherwise} \end{cases}$$

$$\text{Suppose } X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix} \text{exists} \\ \text{unique} \end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \text{Tr}(U^\top X) = \text{Tr}(U^\top X^{1/2} X^{\top/2}) = \text{Tr}(X^{\top/2} U^\top X^{1/2})$$

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$	Primal Program
s.t. Tr($A_i^\top X$) = b_i $i \in \mathcal{I}$ λ_i Tr($C_j^\top X$) ≥ d_j $j \in \mathcal{J}$ μ_j $X = X^\top \succeq 0$ U	

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; \quad X = X^\top \succeq 0 \\ -\infty & ; \text{ otherwise} \end{cases}$$

$$\text{Suppose } X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix} \text{exists} \\ \text{unique} \end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \text{Tr}(U^\top X) = \text{Tr}(U^\top X^{1/2} X^{\top/2}) = \underbrace{\text{Tr}(X^{\top/2} U^\top X^{1/2})}_{\text{congruent to } U}$$

Dual Programs: SDP (lin objective)

$$\max_X \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t. } \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; X = X^\top \succeq 0 \\ -\infty & ; \text{otherwise} \end{cases}$$

$$\text{Suppose } X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix} \text{exists} \\ \text{unique} \end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \text{Tr}(U^\top X) = \text{Tr}(U^\top X^{1/2} X^{\top/2}) = \underbrace{\text{Tr}(X^{\top/2} U^\top X^{1/2})}_{\text{congruent to } U}$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

Dual Programs: SDP (lin objective)

$$\max_X \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t. } \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; X = X^\top \succeq 0 \\ -\infty & ; \text{otherwise} \end{cases}$$

$$\text{Suppose } X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix} \text{exists} \\ \text{unique} \end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \text{Tr}(U^\top X) = \text{Tr}(U^\top X^{1/2} X^{\top/2}) = \underbrace{\text{Tr}(X^{\top/2} U^\top X^{1/2})}_{\text{congruent to } U}$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

$$\text{Tr} = \text{sum eigs} \Rightarrow \iff \text{Tr}(X^{\top/2} U^\top X^{1/2}) \geq 0$$

Dual Programs: SDP (lin objective)

$$\max_X \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t. } \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; X = X^\top \succeq 0 \\ -\infty & ; \text{otherwise} \end{cases}$$

$$\text{Suppose } X \succeq 0 \quad (X = X^\top) \quad \Rightarrow \quad \begin{matrix} \text{exists} \\ \text{unique} \end{matrix} \quad X^{1/2} = X^{\top/2} \succeq 0$$

$$\Rightarrow \text{Tr}(U^\top X) = \text{Tr}(U^\top X^{1/2} X^{\top/2}) = \underbrace{\text{Tr}(X^{\top/2} U^\top X^{1/2})}_{\text{congruent to } U} \geq 0$$

$$U \succeq 0 \iff X^{\top/2} U^\top X^{1/2} \succeq 0$$

$$\text{Tr} = \text{sum eigs} \Rightarrow \iff \text{Tr}(X^{\top/2} U^\top X^{1/2}) \geq 0$$

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$	Primal Program
s.t. Tr($A_i^\top X$) = b_i $i \in \mathcal{I}$ Tr($C_j^\top X$) ≥ d_j $j \in \mathcal{J}$ $X = X^\top \succeq 0$	λ_{i} μ_{j} U

$$\mathcal{L}(X, \lambda_i, \mu_j, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; \quad X = X^\top \succeq 0 \\ -\infty & ; \quad \text{otherwise} \end{cases}$$

Suppose $X \not\succeq 0$ ($X = X^\top$)

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$	Primal Program
s.t. Tr($A_i^\top X$) = b_i $i \in \mathcal{I}$ λ_i Tr($C_j^\top X$) ≥ d_j $j \in \mathcal{J}$ μ_j $X = X^\top \succeq 0$ U	

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; \quad X = X^\top \succeq 0 \\ -\infty & ; \quad \text{otherwise} \end{cases}$$

Suppose $X \not\succeq 0$ ($X = X^\top$) \Rightarrow exists neg. (real) eval with evec u

Dual Programs: SDP (lin objective)

$$\max_{X} \quad \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t. } \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J} \quad \mu_j$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; X = X^\top \succeq 0 \\ -\infty & ; \text{otherwise} \end{cases}$$

Suppose $X \not\succeq 0$ ($X = X^\top$) \Rightarrow exists neg. (real) eval with evec u

take $U = suu^\top \succeq 0$

$$s \in \mathbb{R}_+$$

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$	Primal Program
s.t. Tr($A_i^\top X$) = b_i $i \in \mathcal{I}$ Tr($C_j^\top X$) ≥ d_j $j \in \mathcal{J}$ $X = X^\top \succeq 0$	U

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\max_X \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$X \succeq 0 \iff \min_{U \succeq 0} \text{Tr}(U^\top X) = \begin{cases} 0 & ; \quad X = X^\top \succeq 0 \\ -\infty & ; \quad \text{otherwise} \end{cases}$$

Suppose $X \not\succeq 0$ ($X = X^\top$) \Rightarrow exists neg. (real) eval with evec u

$$\text{take } U = suu^\top \succeq 0 \Rightarrow \text{Tr}(U^\top X) = s \text{Tr}(u^\top Xu) < 0$$

$$s \in \mathbb{R}_+$$

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$		Primal Program
s.t. Tr($A_i^\top X$) = b_i $i \in \mathcal{I}$ λ_i Tr($C_j^\top X$) ≥ d_j $j \in \mathcal{J}$ μ_j $X = X^\top \succeq 0$ U		

$$\mathcal{L}(X, \lambda_i, \mu_j, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\begin{aligned} & \max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X) \\ & \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \max_X \underbrace{\text{Tr}\left(\left[R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top\right] X\right)}_{\text{must be 0 for inner problem to be bounded}} - \sum_i \lambda_i b_i - \sum_j \mu_j d_j \end{aligned}$$

Dual Programs: SDP (lin objective)

$\max_{X} \quad \text{Tr}(R^\top X)$	Primal Program	
s.t. $\text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I}$ λ_i	$\text{Tr}(C_j^\top X) \geq d_j \quad j \in \mathcal{J}$ μ_j	$X = X^\top \succeq 0$ U

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X)$$

$$\begin{aligned} & \max_{X} \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \sum_j \mu_j (\text{Tr}(C_j^\top X) - d_j) + \text{Tr}(U^\top X) \\ & \min_{\substack{\lambda_i, \mu_j \geq 0 \\ U \succeq 0}} \max_X \underbrace{\text{Tr}\left(\left[R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top\right] X\right)}_{\text{must be 0 for inner problem to be bounded}} - \sum_i \lambda_i b_i - \sum_j \mu_j d_j \end{aligned}$$

$\min_{\lambda_i, \mu_j, U} - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$	Dual Program
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s.t. $R^\top + \sum_i \lambda_i A_i^\top + \sum_j \mu_j C_j^\top + U^\top = 0, \quad \mu_j \geq 0 \quad j \in \mathcal{J} \quad U \succeq 0$	
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Dual Programs: SDP (lin objective)

$$\max_{X} \quad \text{Tr}(R^\top X)$$

Primal Program

$$\text{s.t.} \quad \text{Tr}(A_i^\top X) = b_i \quad i \in \mathcal{I} \quad \lambda_i$$

$$X \geq M \quad W$$

$$X = X^\top \succeq 0 \quad U$$

$$\mathcal{L}(X, \lambda_i, \mu_i, U) = \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \text{Tr}(W^\top (X - M)) + \text{Tr}(U^\top X)$$

$$\max_{X} \min_{\substack{\lambda_i \\ W \geq 0 \\ U \succeq 0}} \text{Tr}(R^\top X) + \sum_i \lambda_i (\text{Tr}(A_i^\top X) - b_i) + \text{Tr}(W^\top (X - M)) + \text{Tr}(U^\top X)$$

$$\min_{\substack{\lambda_i \\ W \geq 0 \\ U \succeq 0}} \max_{X} \text{Tr} \left(\underbrace{\left[R^\top + \sum_i \lambda_i A_i^\top + W^\top + U^\top \right] X}_{-\text{Tr}(W^\top M)} - \sum_i \lambda_i b_i - \sum_j \mu_j d_j \right)$$

must be 0 for inner problem to be bounded

$$\min_{\lambda_i, W, U} \quad \boxed{-\text{Tr}(W^\top M)} - \sum_i \lambda_i b_i - \sum_j \mu_j d_j$$

Dual Program

$$\text{s.t.} \quad R^\top + \sum_i \lambda_i A_i^\top + \boxed{W^\top} + U^\top = 0, \quad \boxed{W \geq 0} \quad U \succeq 0$$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & Ax = b \quad \lambda \\ & Cx \geq d \quad \mu \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d$$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\begin{aligned} \max_x \quad & \min_{\lambda, \mu \geq 0} \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d) \\ \min_{\lambda, \mu \geq 0} \quad & \max_x \quad F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d \end{aligned}$$

$$\text{Define } \xi^\top = -[\lambda^\top A + \mu^\top C]$$

Dual Programs: General Convex Function

$\max_x \quad F(x)$		Primal Program
s.t.	\$Ax = b\$ λ	\$Cx \geq d\$ μ

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \quad \min_{\lambda, \mu \geq 0} \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \quad \max_x \quad F(x) + \underbrace{\left[\lambda^\top A + \mu^\top C \right] x}_{-\lambda^\top b - \mu^\top d}$$

Define $\xi^\top = -[\lambda^\top A + \mu^\top C]$ $F^*(\xi) = \sup_x F(x) - \xi^\top x$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d$$



Define $\xi^\top = -[\lambda^\top A + \mu^\top C]$

$F^*(\xi) = \sup_x F(x) - \xi^\top x$

**Legendre
Transform
(Fenchel
Conjugate)**

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\begin{aligned} \max_x \quad & \min_{\lambda, \mu \geq 0} \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d) \\ \min_{\lambda, \mu \geq 0} \quad & \max_x \quad F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d \end{aligned}$$

Define $\xi^\top = -[\lambda^\top A + \mu^\top C]$

$$F^*(\xi) = \sup_x F(x) - \xi^\top x$$

Legendre
Transform
(Fenchel
Conjugate)

$$\begin{array}{ll} \min_{\xi, \lambda, \mu} & F^*(\xi) - \lambda^\top b - \mu^\top d \\ \text{s.t.} & \xi^\top = -[\lambda^\top A + \mu^\top C], \quad \mu \geq 0 \end{array} \quad \text{Dual Program}$$

Dual Programs: General Convex Function

$\max_x \quad F(x)$		Primal Program
s.t.	\$Ax = b \quad \lambda\$ \$Cx \geq d \quad \mu\$	

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \quad \min_{\lambda, \mu \geq 0} \quad F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \quad \max_x \quad F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -[\lambda^\top A + \mu^\top C]$

$$F^*(\xi) = \sup_x F(x) - \xi^\top x$$

**Legendre
Transform
(Fenchel
Conjugate)**

Example: $F(x) = \frac{1}{2}x^\top Qx$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -[\lambda^\top A + \mu^\top C]$

$$F^*(\xi) = \sup_x F(x) - \xi^\top x$$

Legendre
Transform
(Fenchel
Conjugate)

Example: $F(x) = \frac{1}{2}x^\top Qx$

$$\Rightarrow \frac{d}{dx} (F - \xi^\top x) = 0$$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -[\lambda^\top A + \mu^\top C]$

$$F^*(\xi) = \sup_x F(x) - \xi^\top x$$

Legendre
Transform
(Fenchel
Conjugate)

Example: $F(x) = \frac{1}{2}x^\top Qx$

$$\Rightarrow \frac{d}{dx}(F - \xi^\top x) = 0 \Rightarrow x = Q^{-1}\xi$$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -[\lambda^\top A + \mu^\top C]$

$$F^*(\xi) = \sup_x F(x) - \xi^\top x$$

Legendre
Transform
(Fenchel
Conjugate)

Example: $F(x) = \frac{1}{2}x^\top Qx$

$$\Rightarrow \frac{d}{dx}(F - \xi^\top x) = 0 \Rightarrow x = Q^{-1}\xi \Rightarrow F^*(\xi) = -\frac{1}{2}\xi^\top Q^{-1}\xi$$

Dual Programs: General Convex Function

$$\begin{array}{ll} \max_x & F(x) \\ \text{s.t.} & \boxed{Ax = b \quad \lambda} \quad \boxed{Cx \geq d \quad \mu} \end{array} \quad \text{Primal Program}$$

$$\mathcal{L}(x, \lambda, \mu) = F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_x \min_{\lambda, \mu \geq 0} F(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_x F(x) + [\lambda^\top A + \mu^\top C] x - \lambda^\top b - \mu^\top d$$

Define $\xi^\top = -[\lambda^\top A + \mu^\top C]$

$$F^*(\xi) = \sup_x F(x) - \xi^\top x$$

Legendre
Transform
(Fenchel
Conjugate)

Example: $F(x) = \frac{1}{2}x^\top Qx$

$$\Rightarrow \frac{d}{dx}(F - \xi^\top x) = 0 \Rightarrow x = Q^{-1}\xi \Rightarrow F^*(\xi) = -\frac{1}{2}\xi^\top Q^{-1}\xi$$