

Eigenvectors & Eigenvalues:

Linear Algebra

Major sources:

Winter 2022 - Dan Calderone

Eigenvectors & Eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$

Eigenvalue/Eigenvector Problem

A transforms \mathbb{R}^n *...which directions stay unchanged?* \rightarrow **Eigenvectors**
 ...within those directions...
 ...how much do vectors get stretched \rightarrow **Eigenvalues**

Eigenvector Equation

$$Ax = x\lambda \quad \text{Eigenvector } x \in \mathbb{C}^n \quad \text{Eigenvalue } \lambda \in \mathbb{C}$$

Spans of eigenvectors (& generalized eigenvectors) are called **A-invariant subspaces**

Eigenvalues:

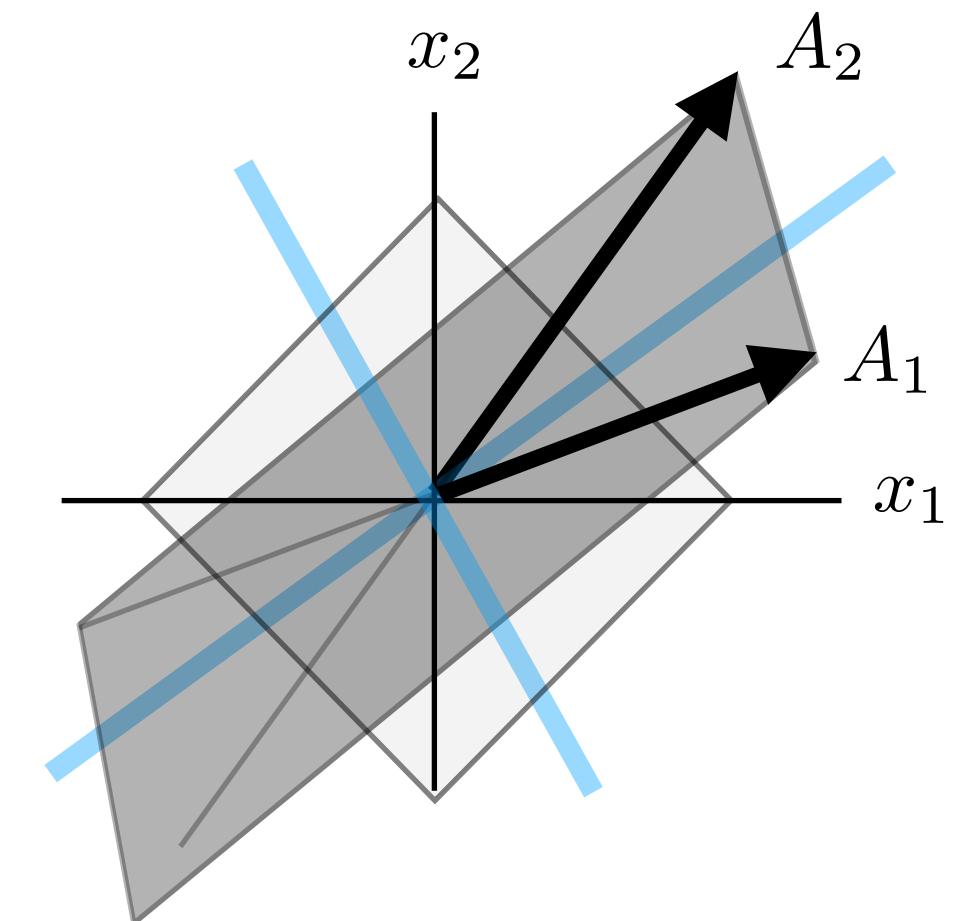
Fundamental property of matrices
Do **not** change with coordinate/similarity transformations

Eigenvectors:

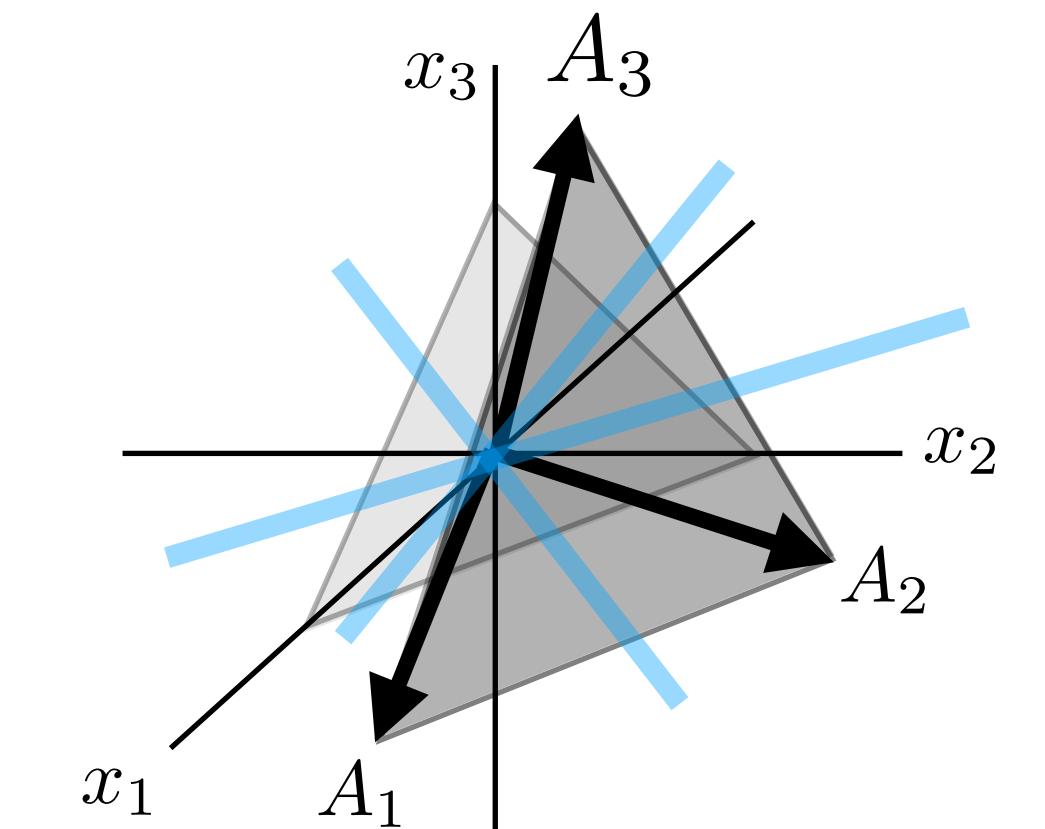
...coordinate dependent (do change with coordinate/similarity transformations)

Picture Examples:

$$A = \begin{bmatrix} & \\ A_1 & A_2 \\ & \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$



$$A = \begin{bmatrix} & & \\ A_1 & A_2 & A_3 \\ & & \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



Eigenvector/Eigenvalue equation

Square matrix: $A \in \mathbb{R}^{n \times n}$

For any eigenvalue $\lambda \in \mathbb{C}$

Right Eigenvector: $v \in \mathbb{C}^n$

$$Av = v\lambda$$

$$(A - \lambda I)v = 0$$

$$v \in \mathcal{N}(A - \lambda I)$$

Left Eigenvectors: $w \in \mathbb{C}^n$

$$w^* A = w^* \lambda$$

$$w^*(A - \lambda I) = 0$$

$$w^* \in \mathcal{N}^L(A - \lambda I) = 0$$

For any eigenvalue, right and left eigenvectors come in pairs since $A - \lambda I$ drops row and column rank at the same time

Eigenvectors exist only for values of s where $A - sI$ drops rank...

...how to characterize.... \rightarrow

$sI - A$ drops rank only when $\det(sI - A) = 0$

Characteristic Polynomial

$$\text{char}_A(s) = \det(sI - A)$$

n-th order polynomial



n roots

Roots are eigenvalues:

λ solution to $\text{char}_A(s) = 0$

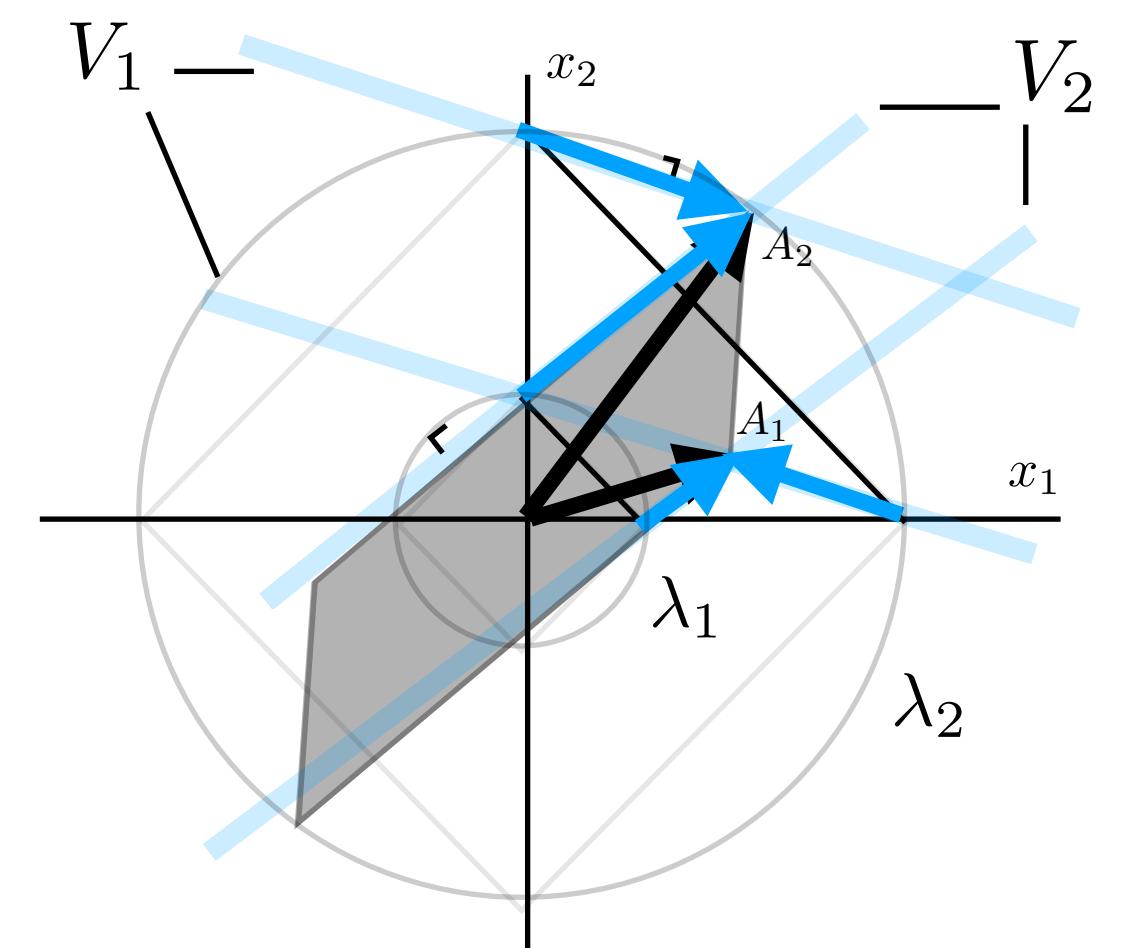
Fundamental
Theorem of Algebra

(see below)

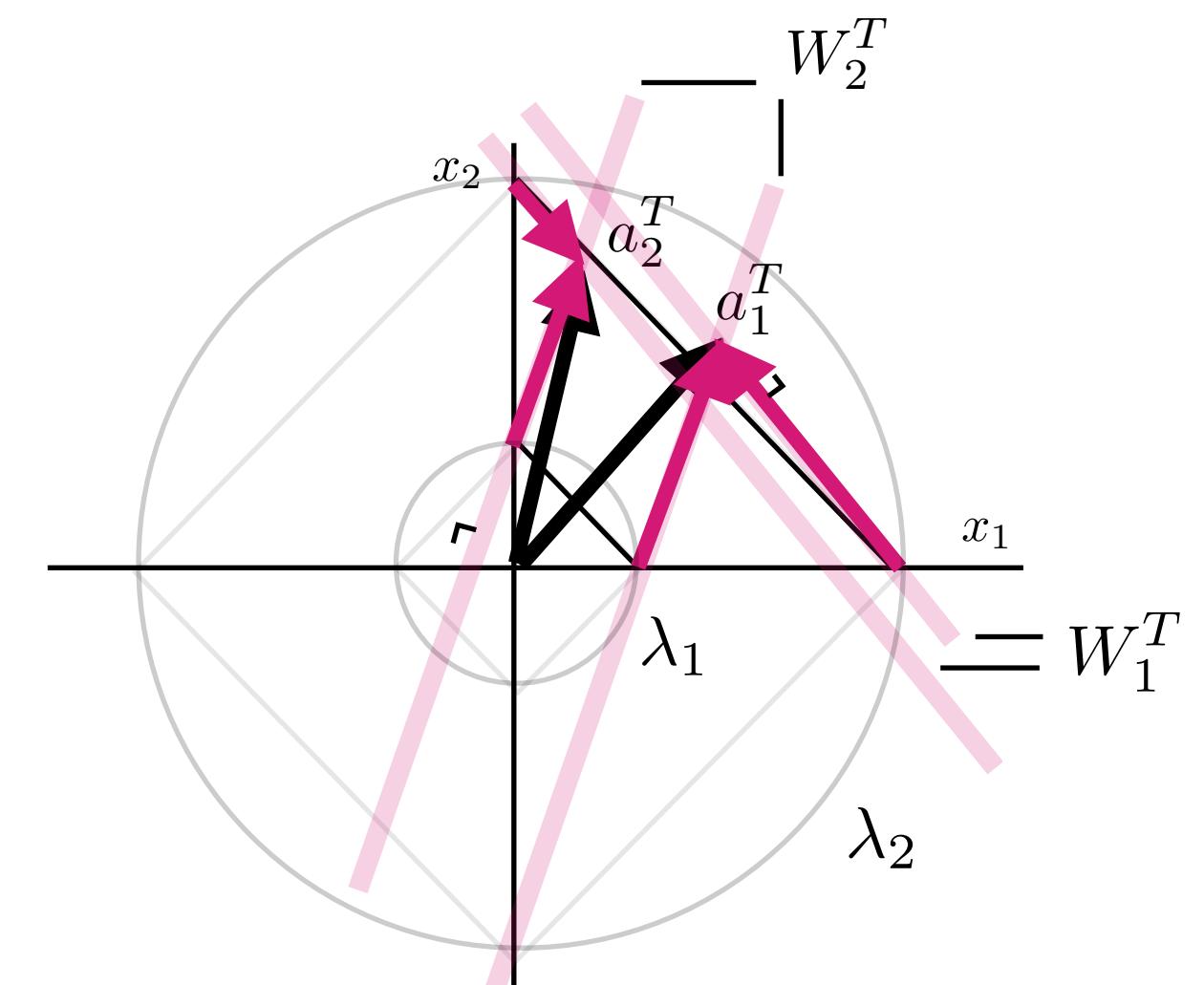
Picture
Examples:

$$\mathbb{R}^{2 \times 2}$$

COLUMN GEOMETRY



ROW GEOMETRY



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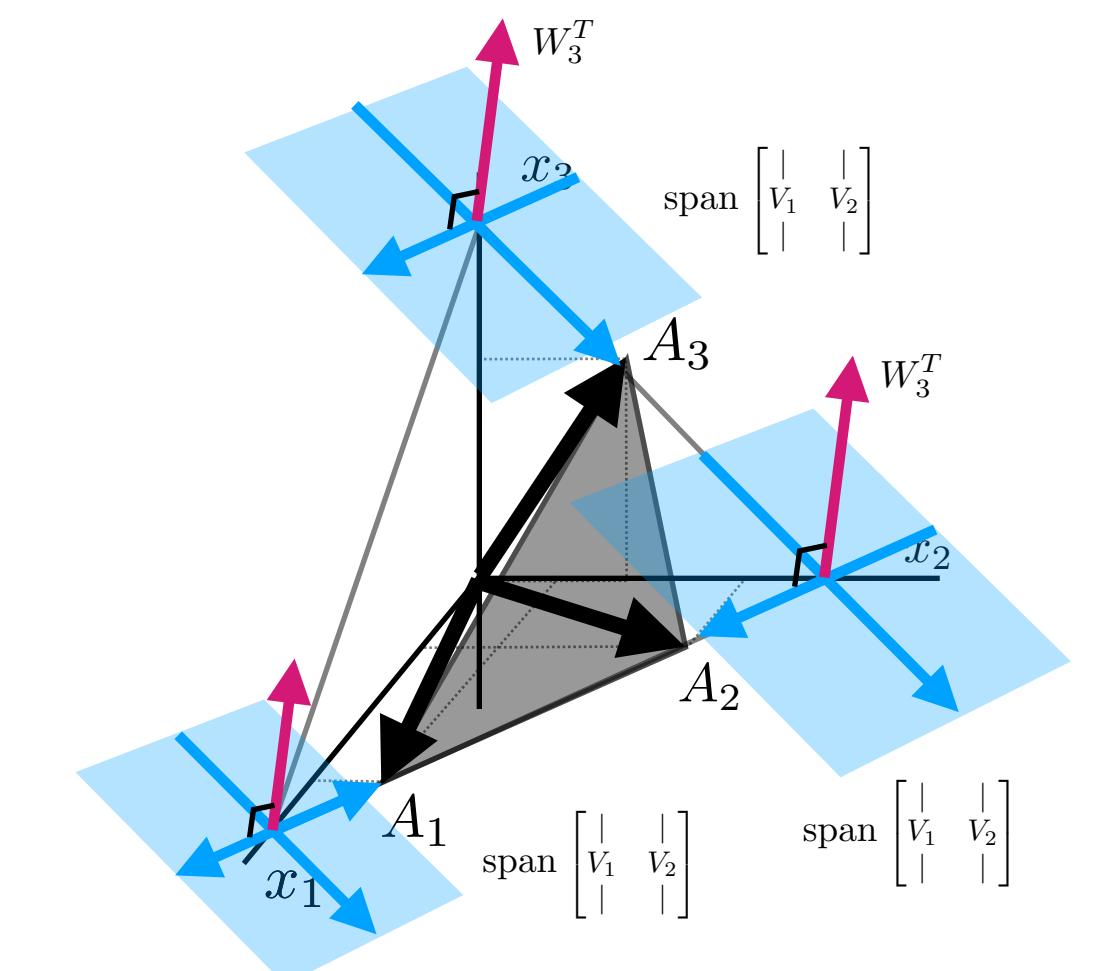
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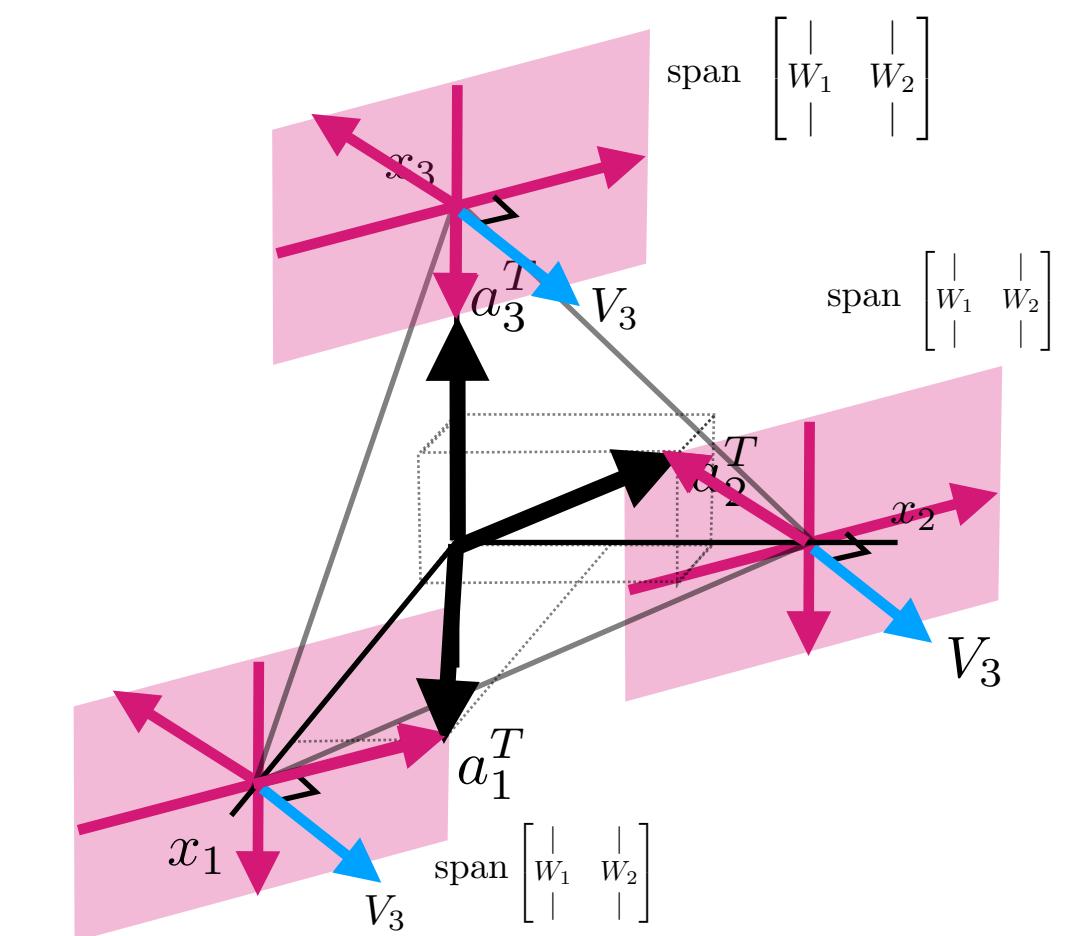
$$\lambda \text{ solution to } \text{char}_A(s) = 0$$

Fundamental
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COLUMN GEOMETRY



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(see below)

Eigenvector/Eigenvalue Picture

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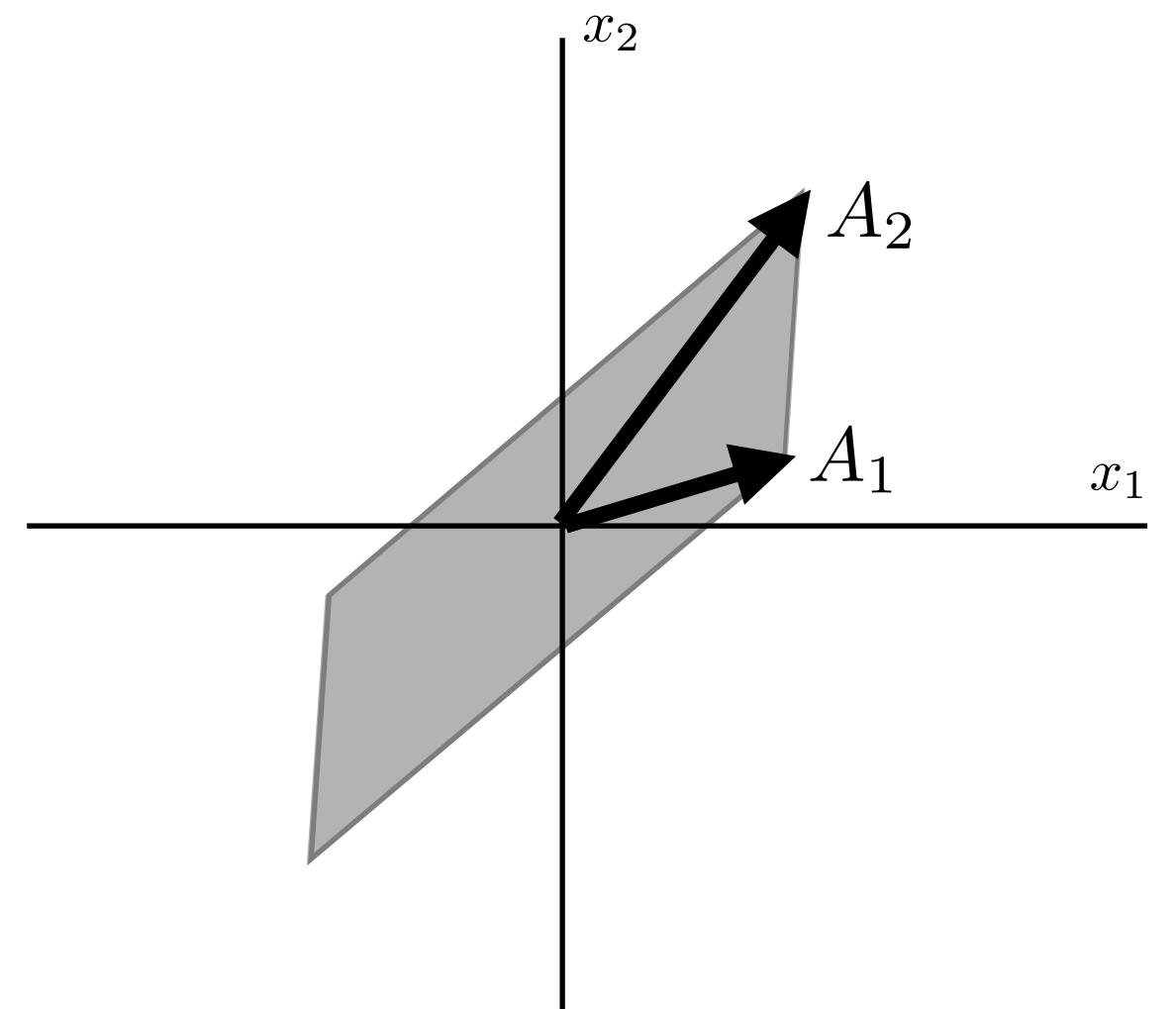
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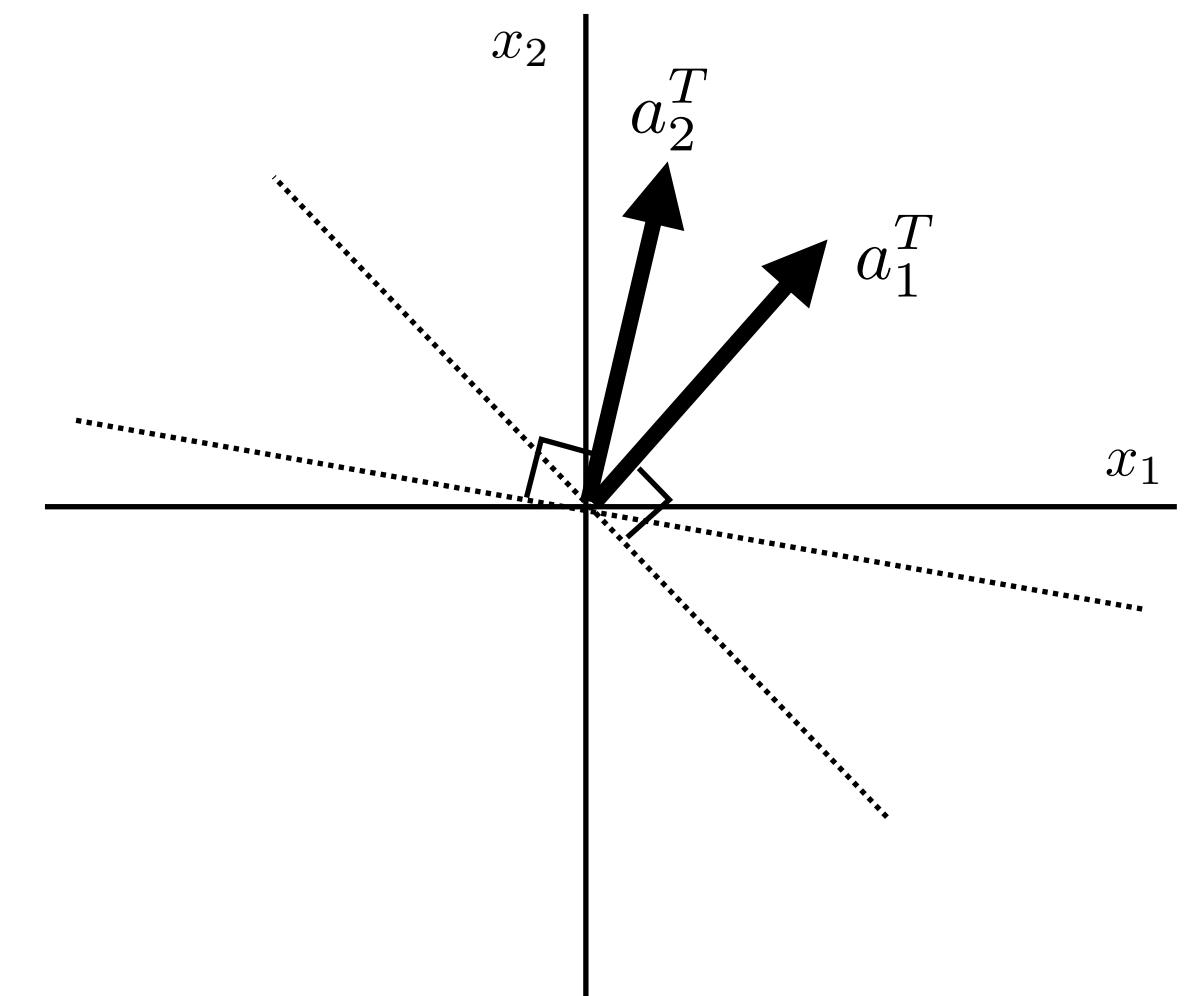
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**COLUMN
GEOMETRY**



**ROW
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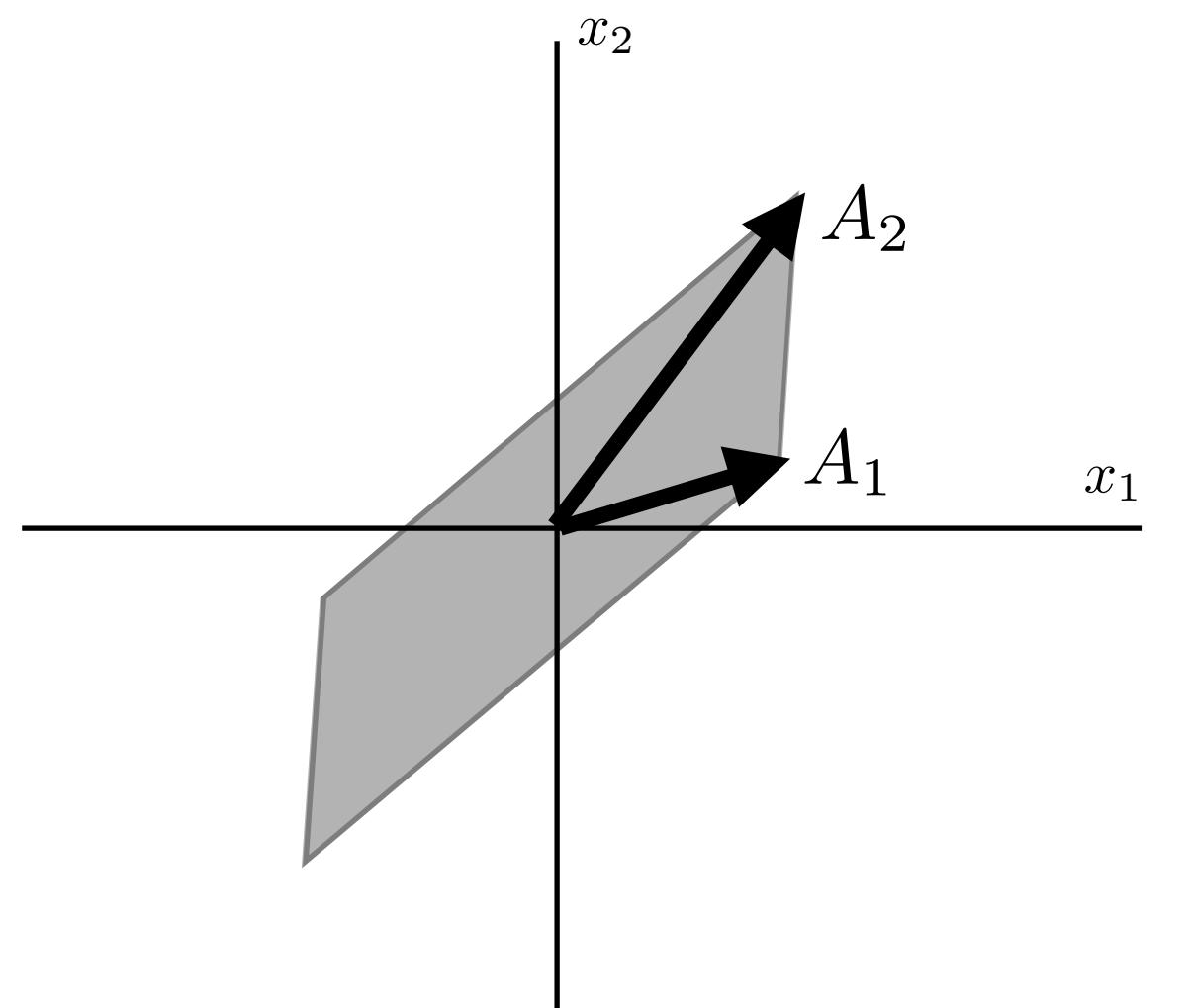
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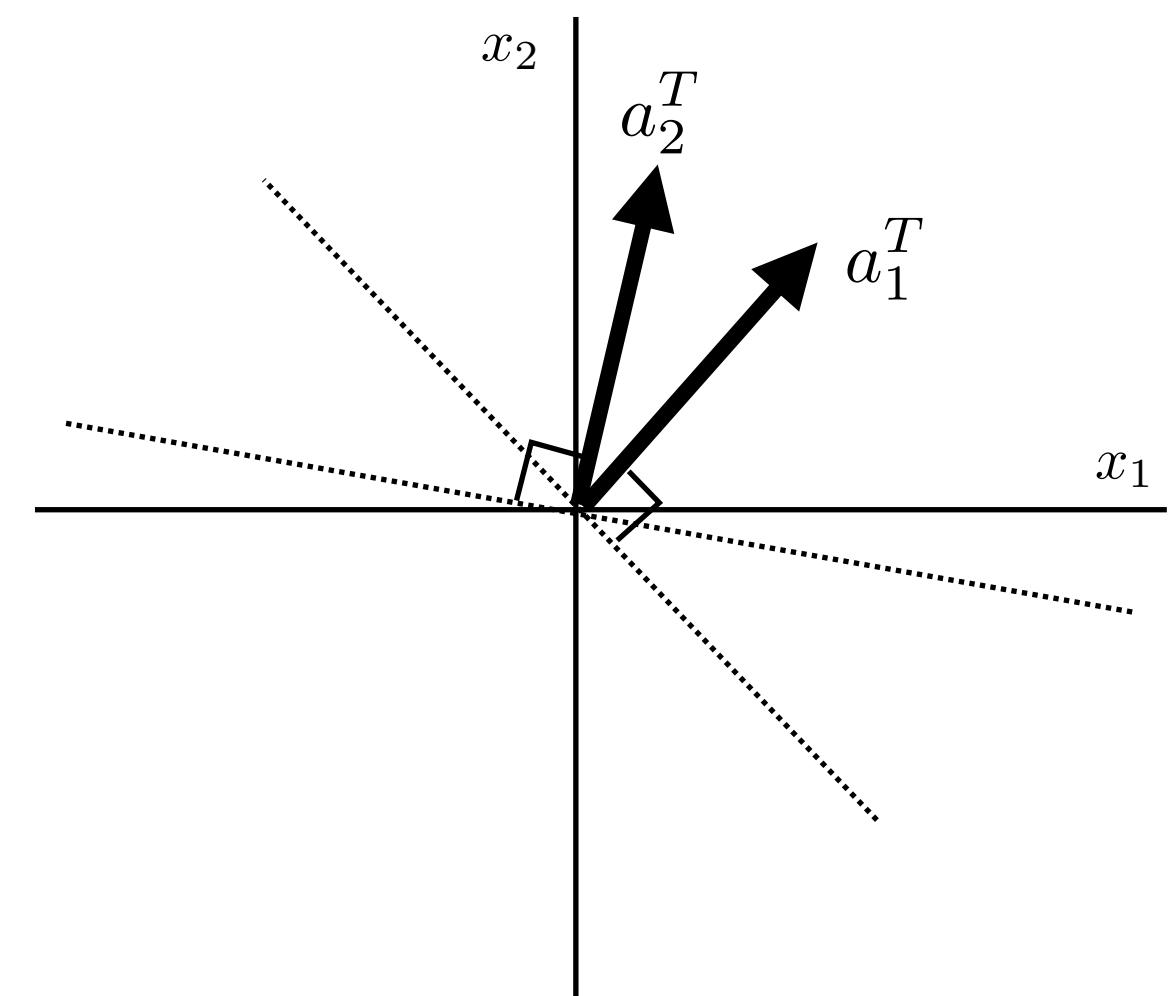
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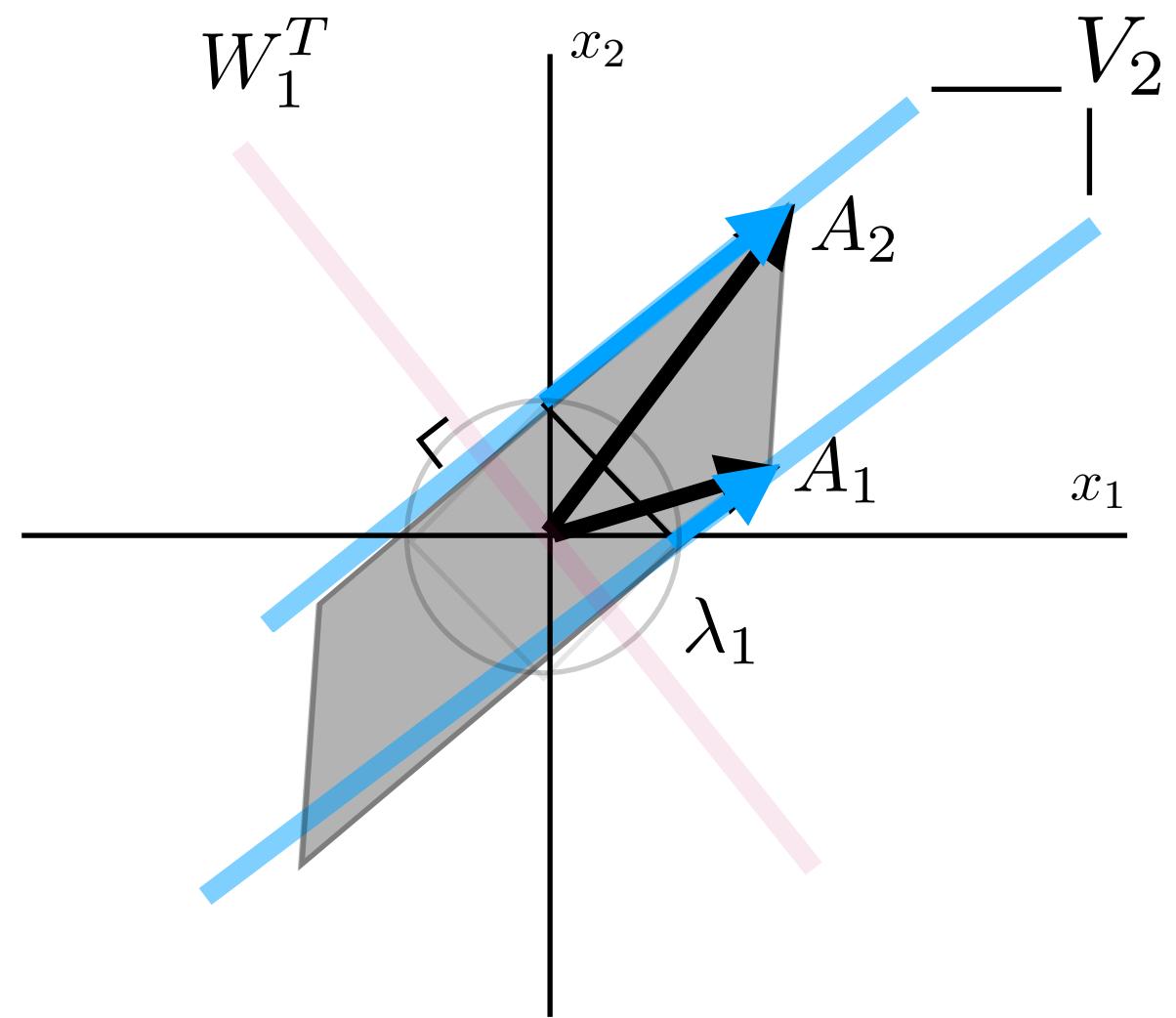
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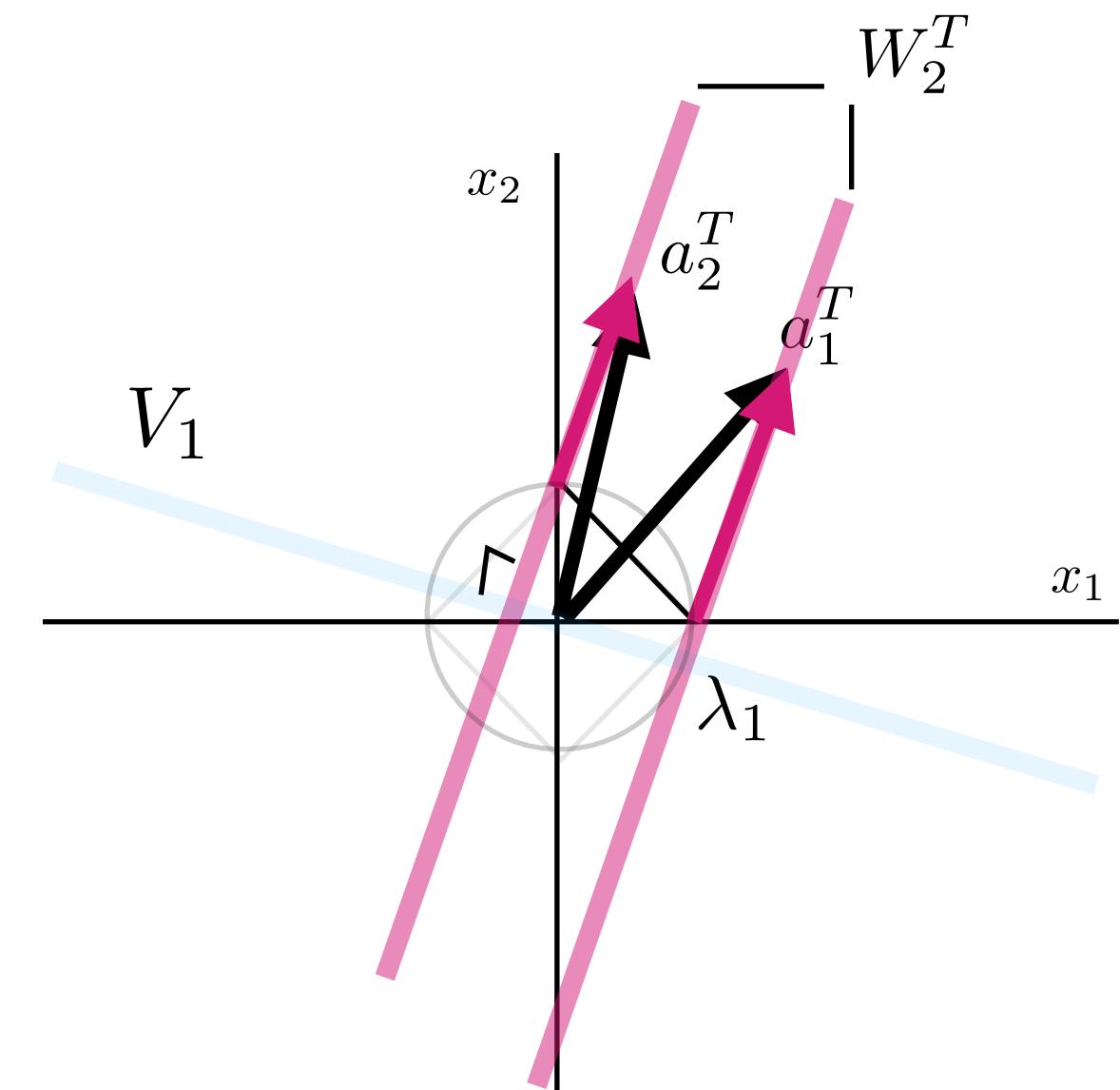
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ROW GEOMETRY

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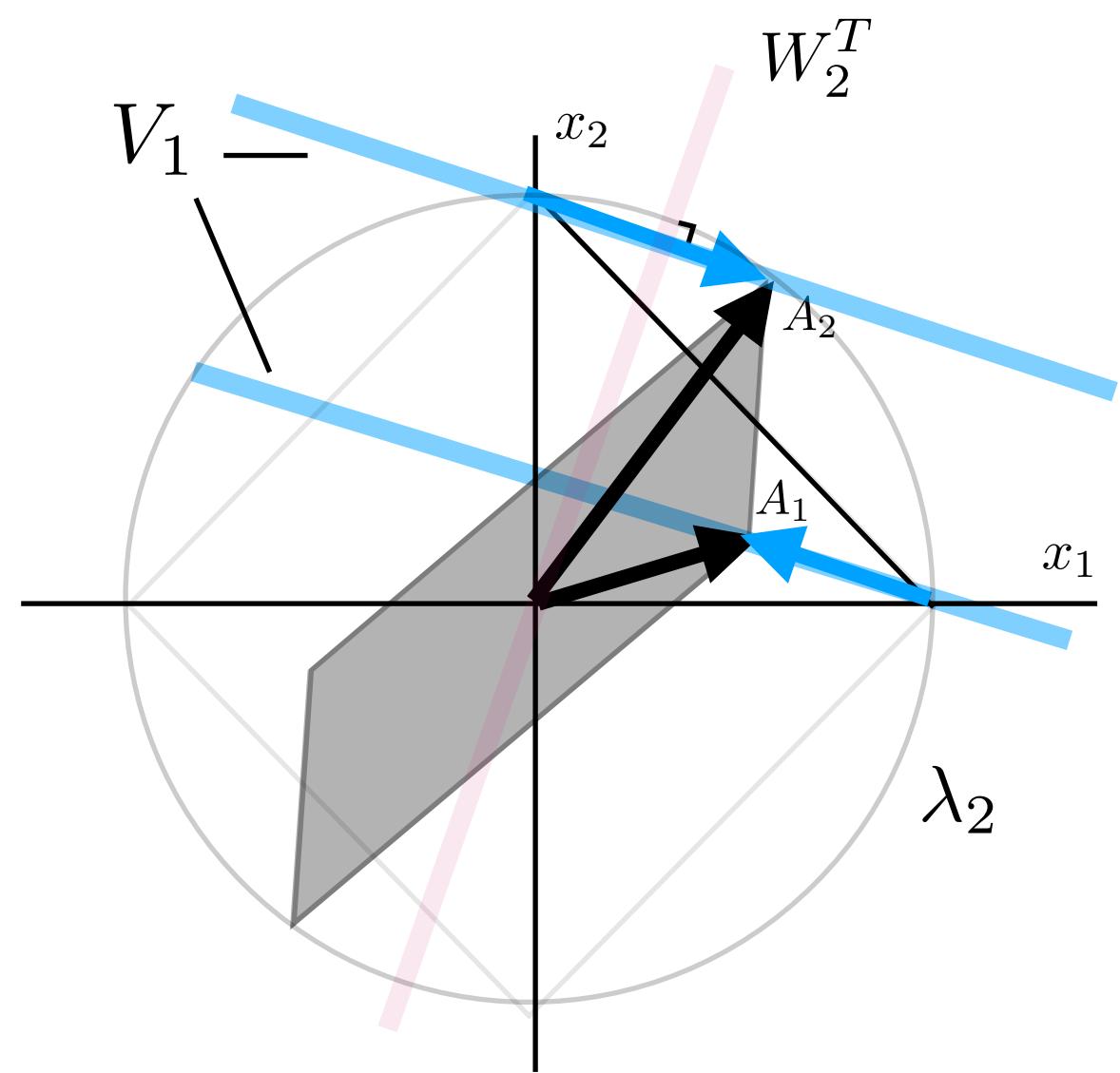
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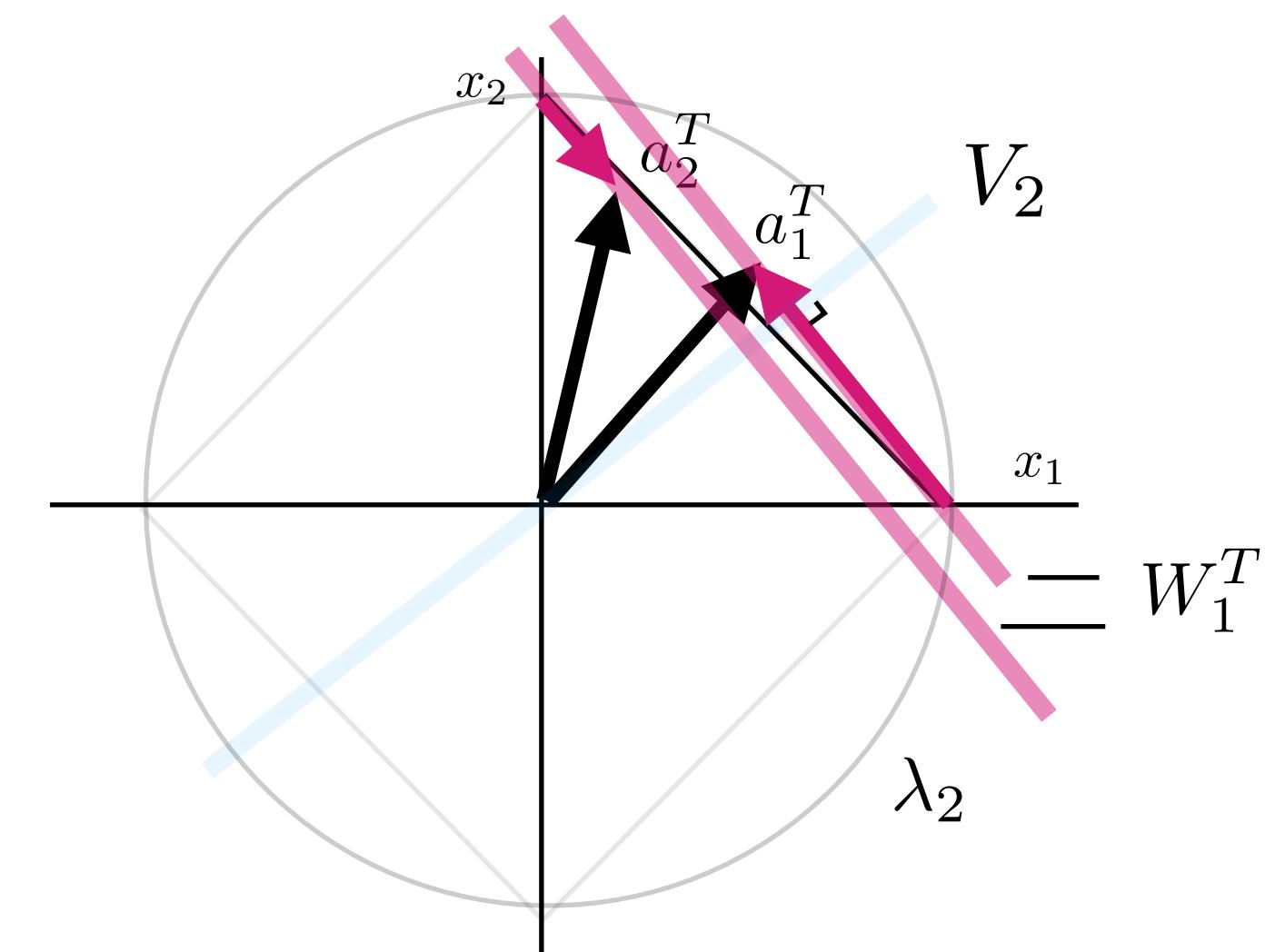
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Right Eigenvector: $v \in \mathbb{C}^n$

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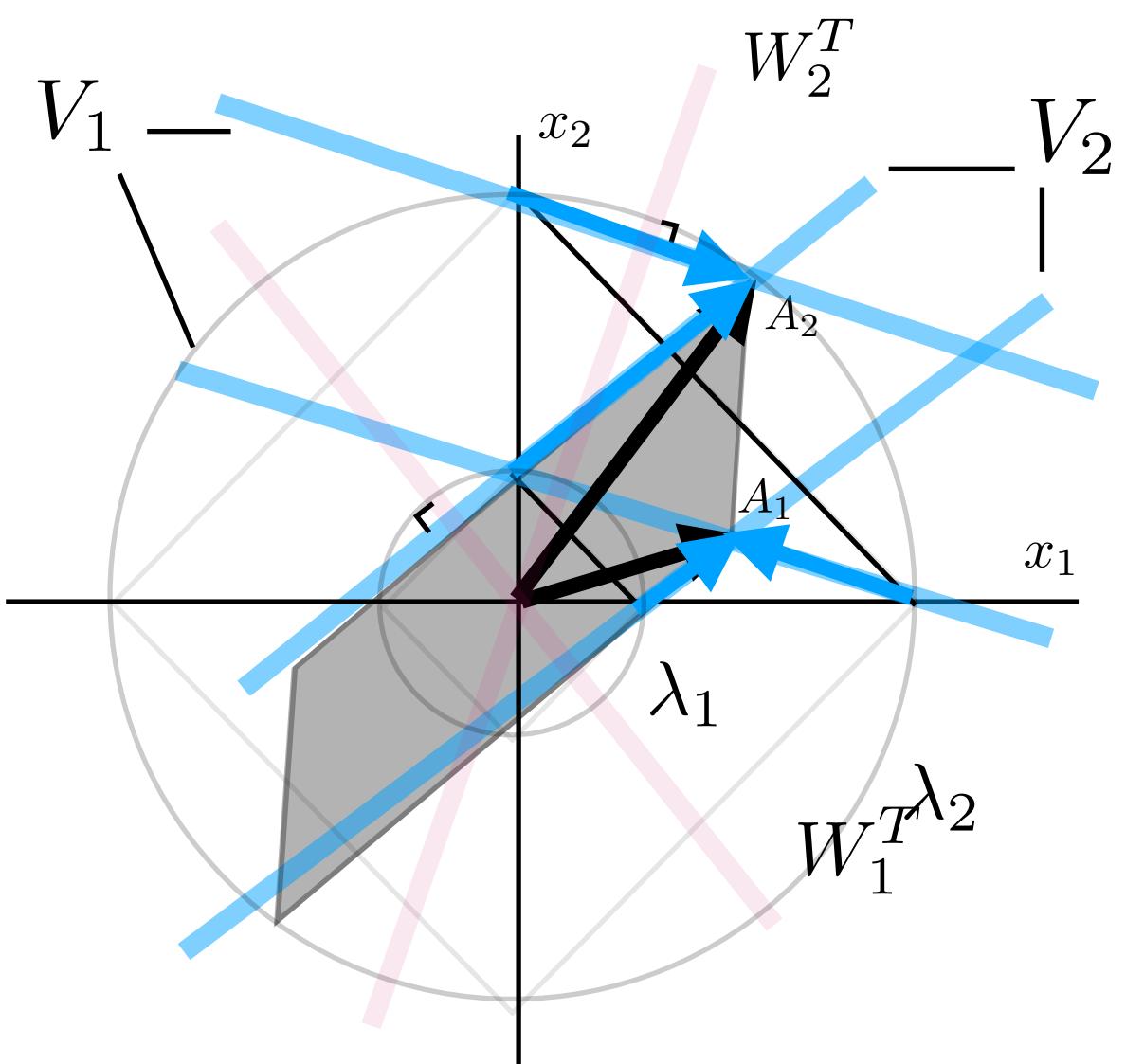
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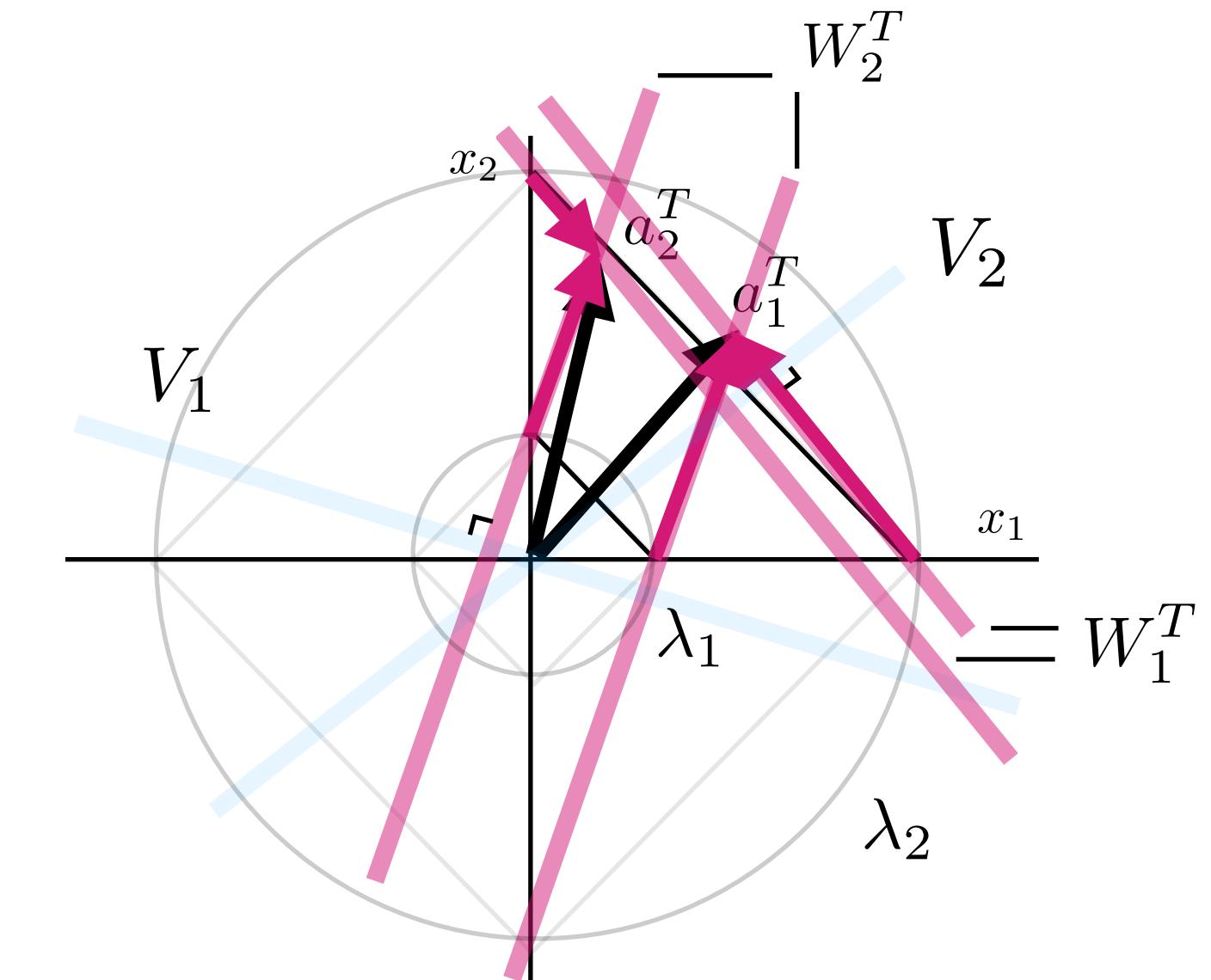
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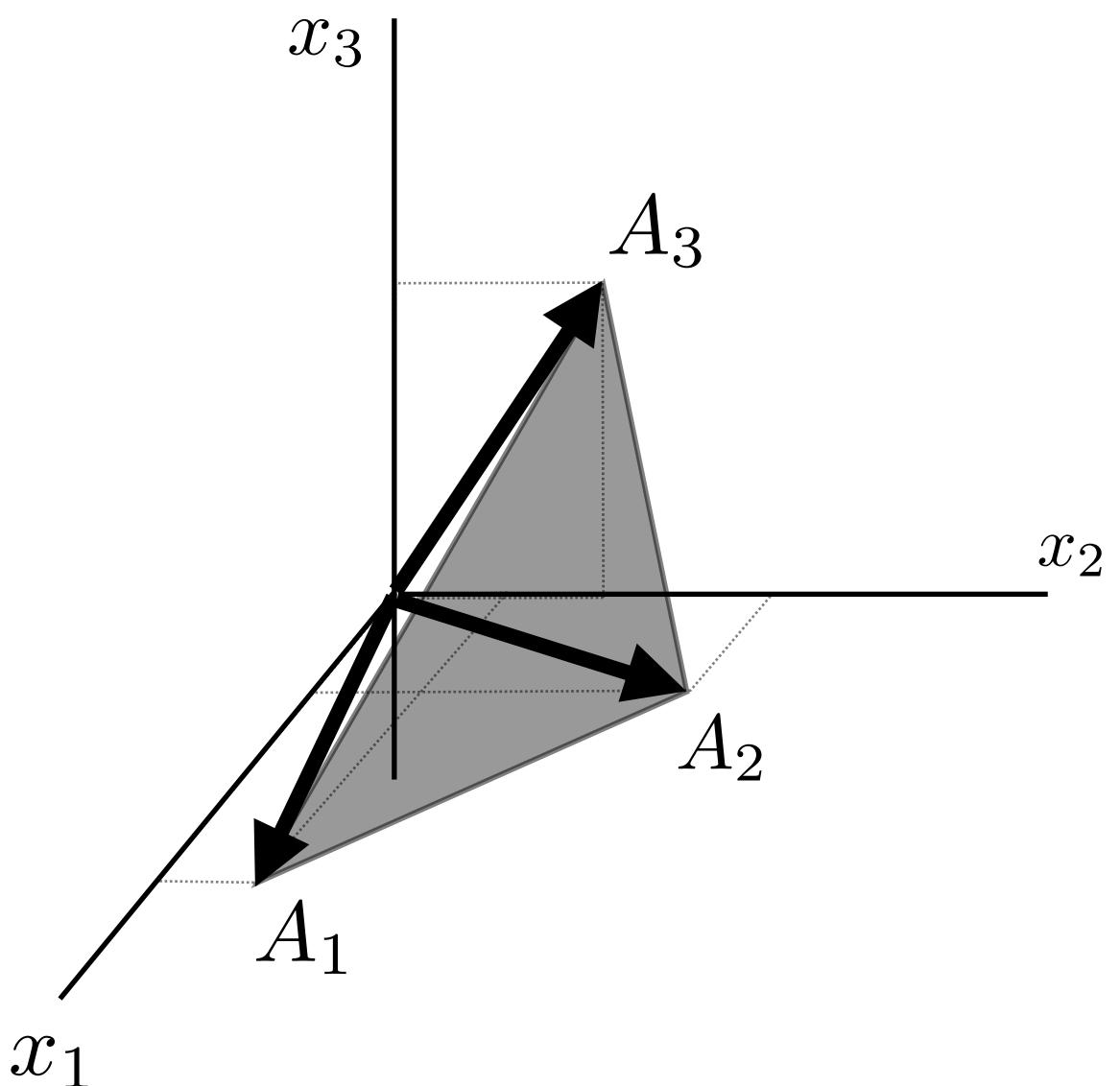
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$$sI - A \quad \text{drops rank only when} \\ \text{char}_A(s) = \det(sI - A) = 0$$

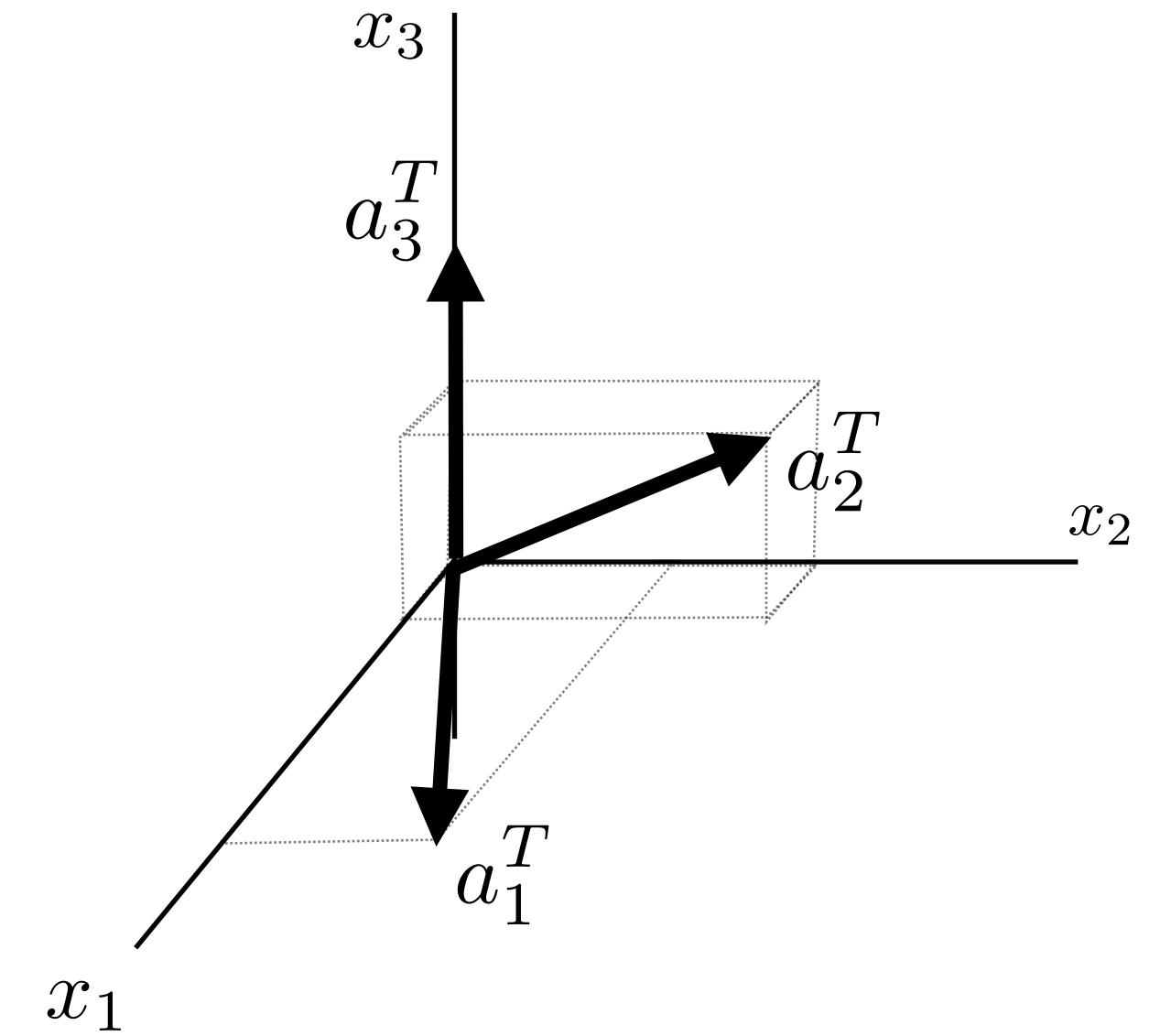
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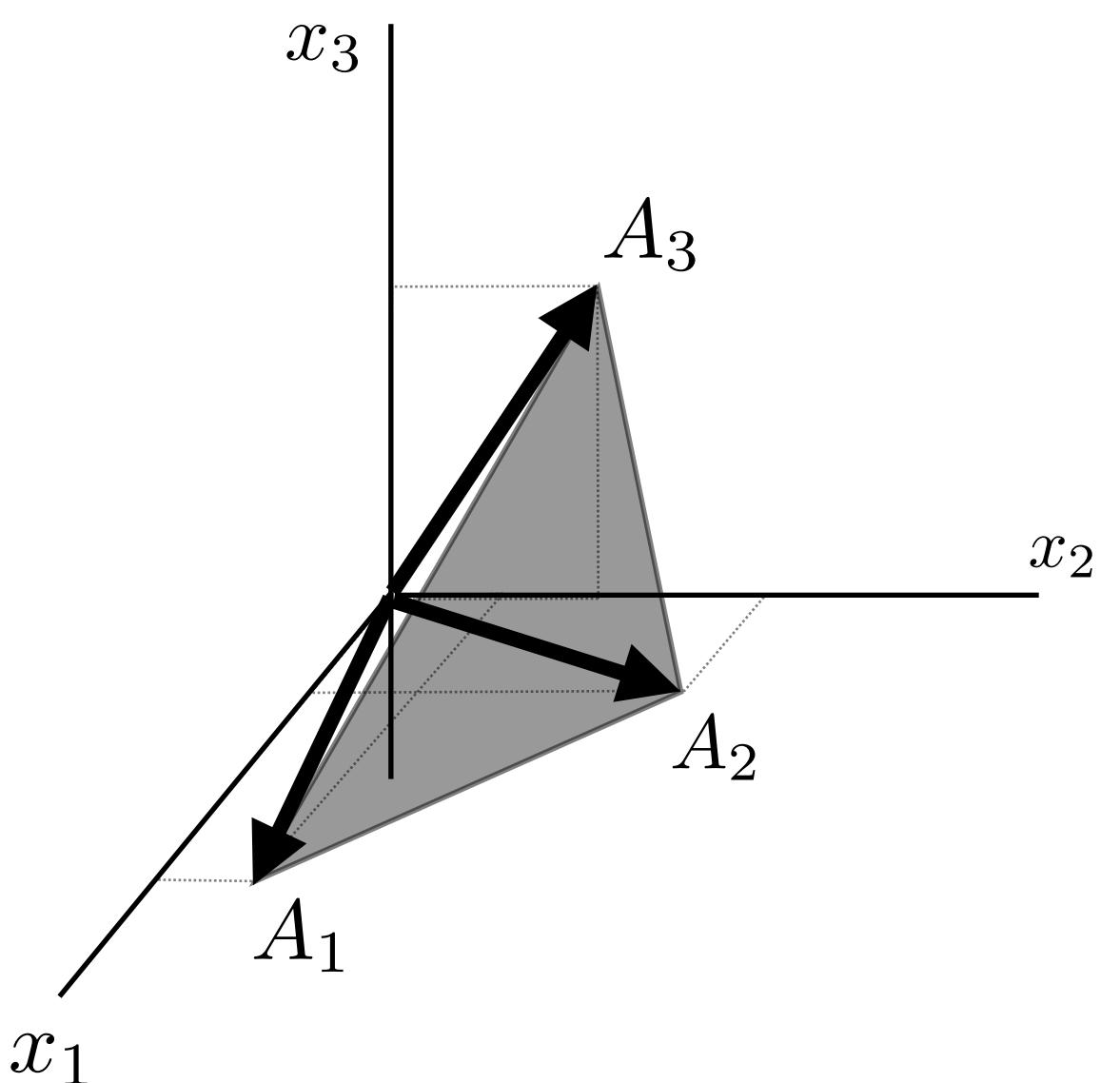
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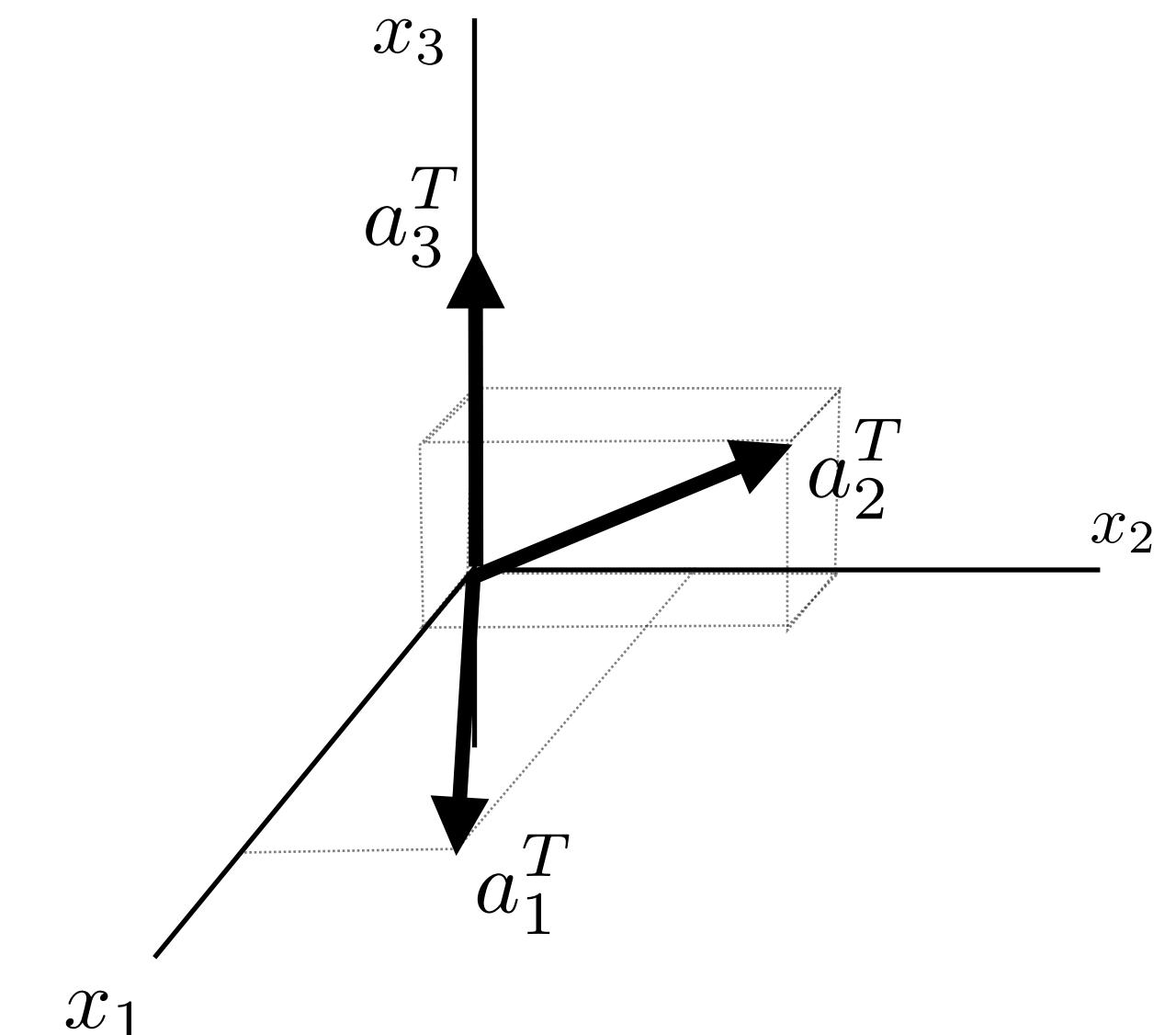
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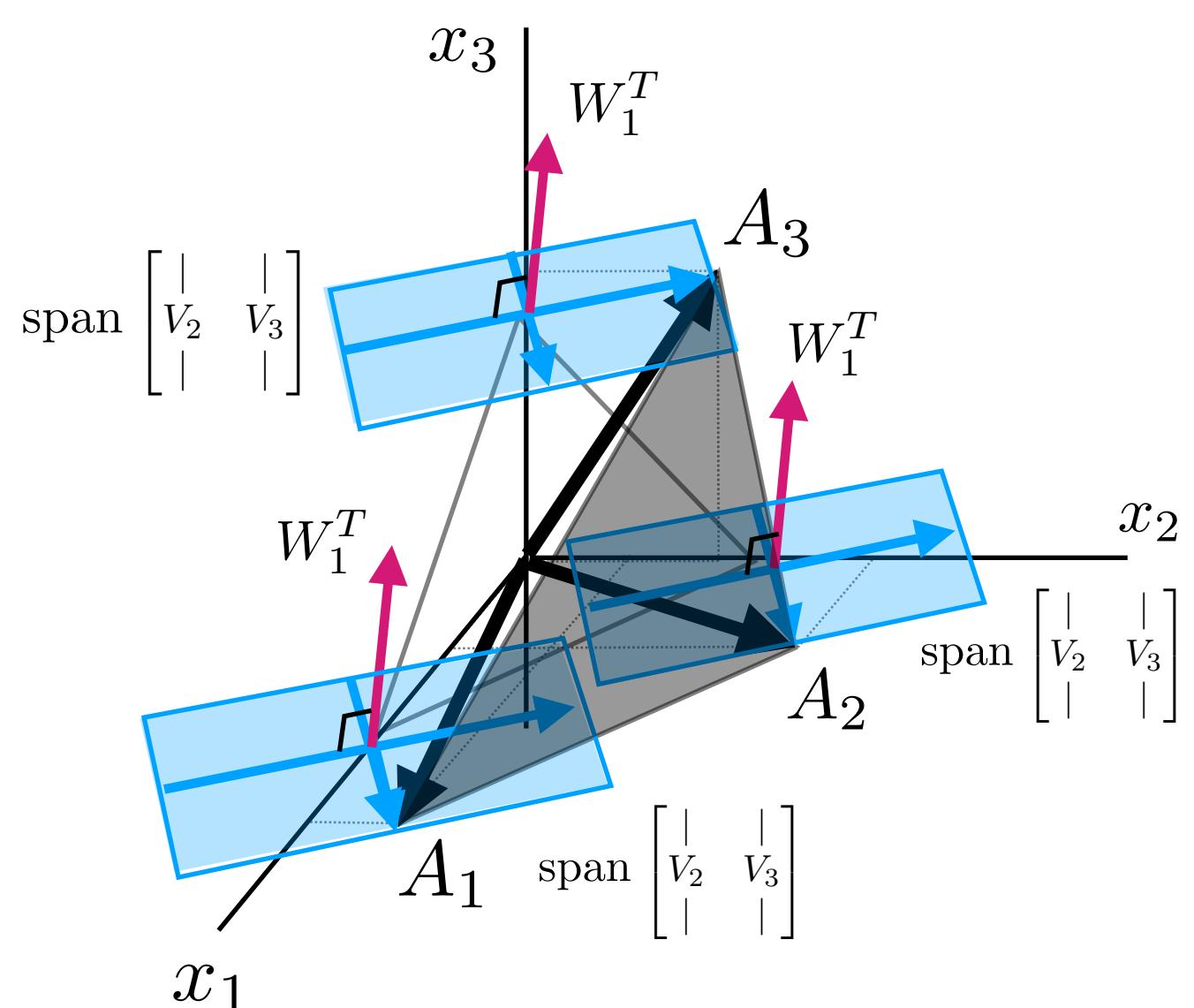
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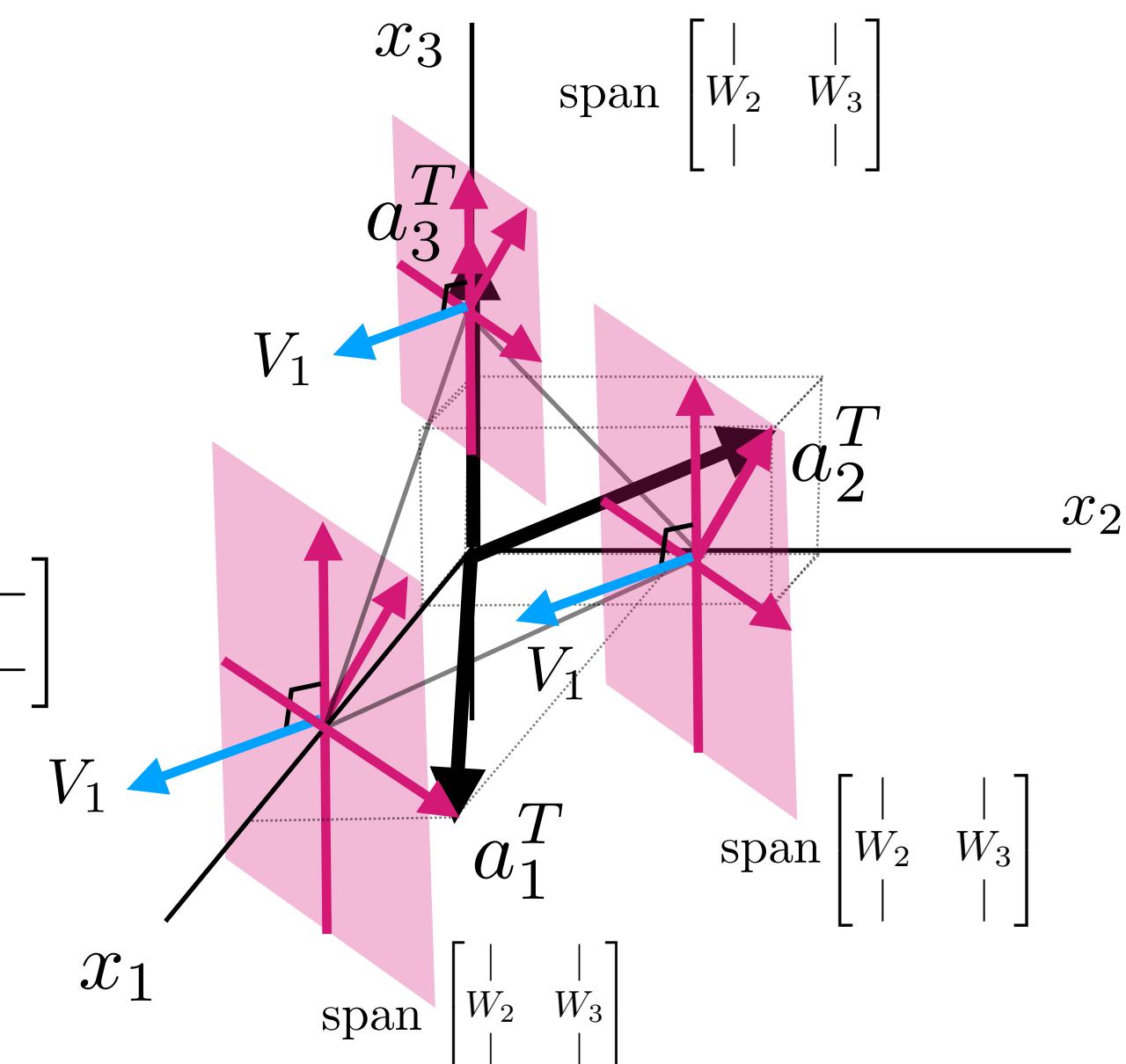
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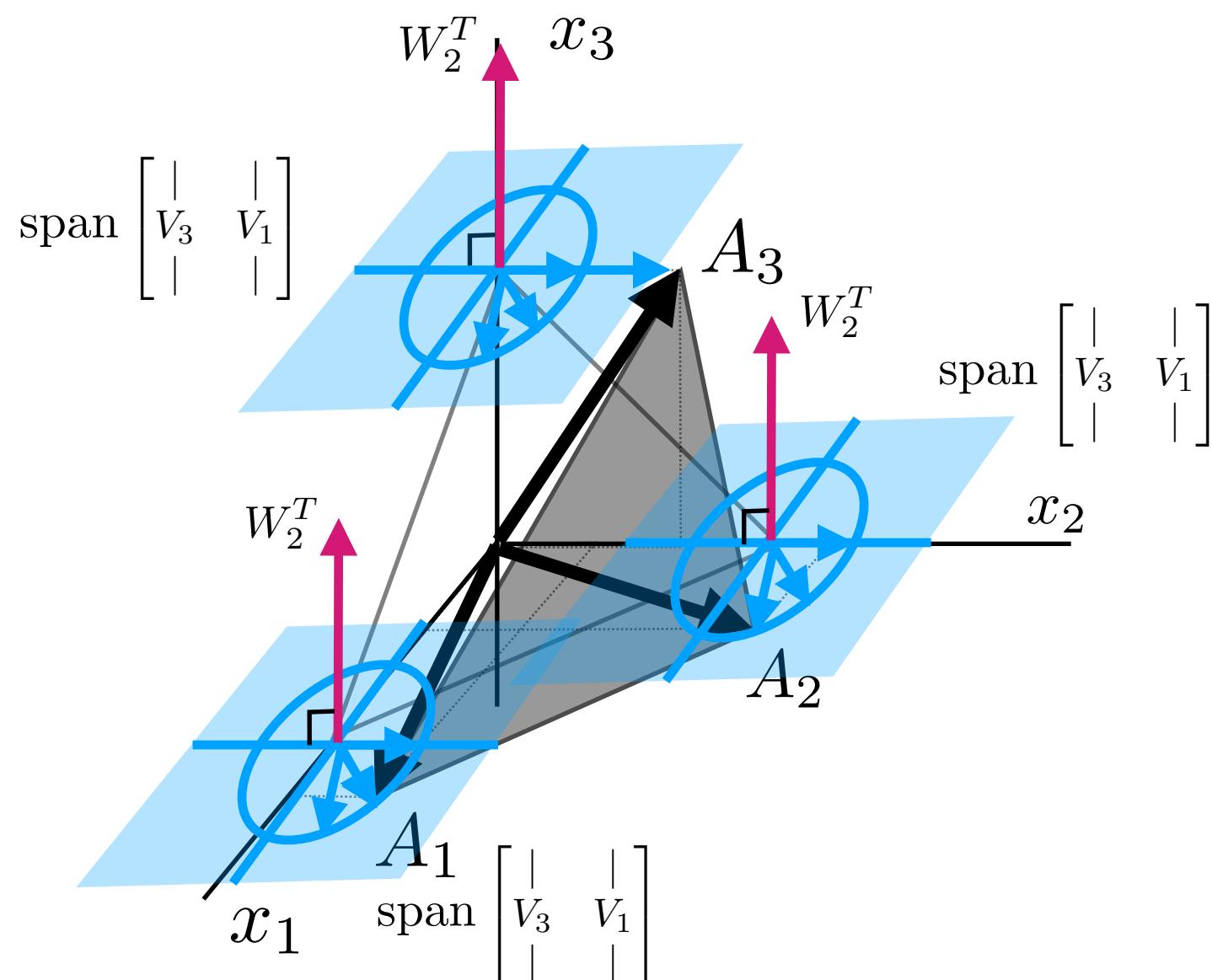
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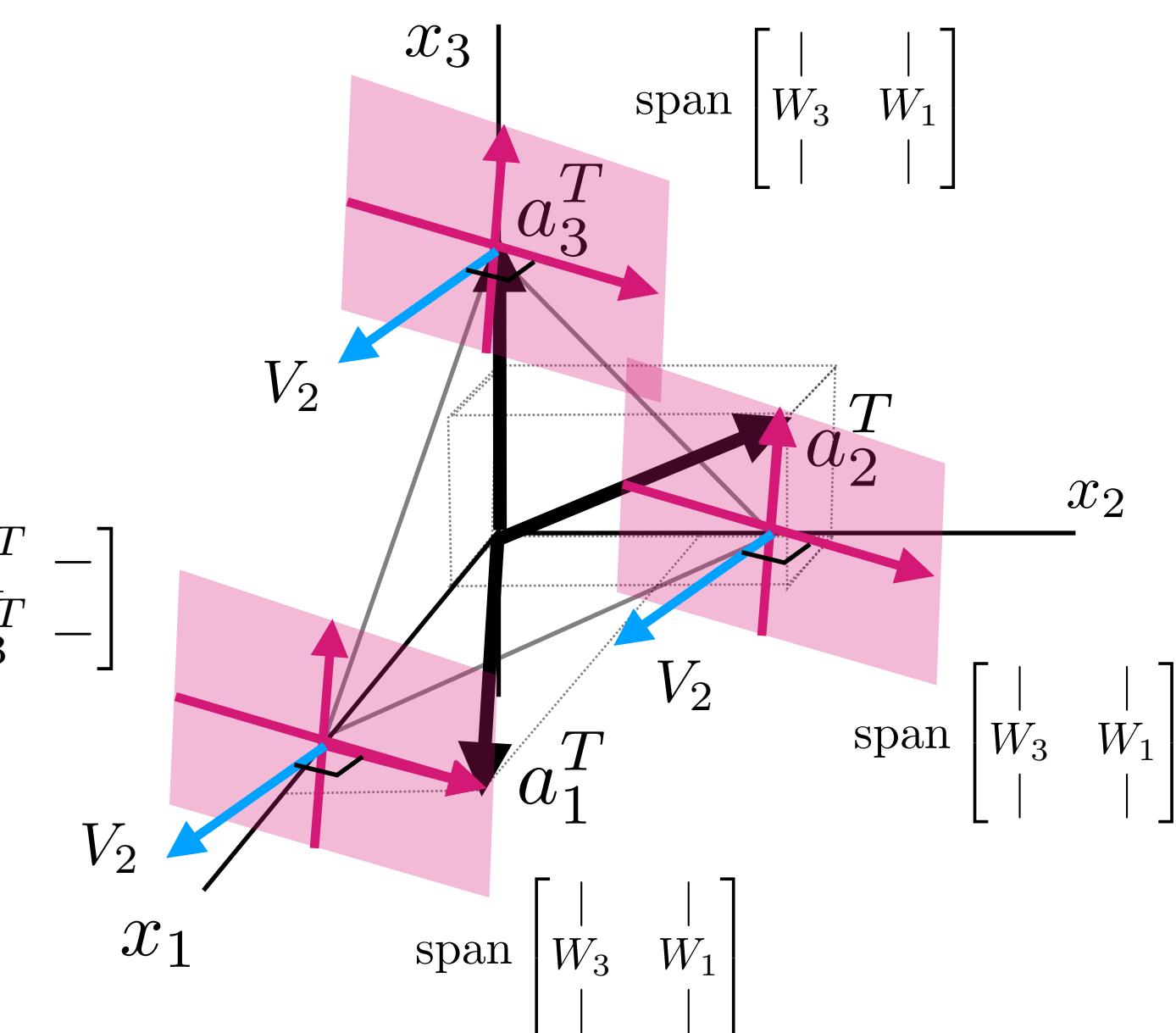
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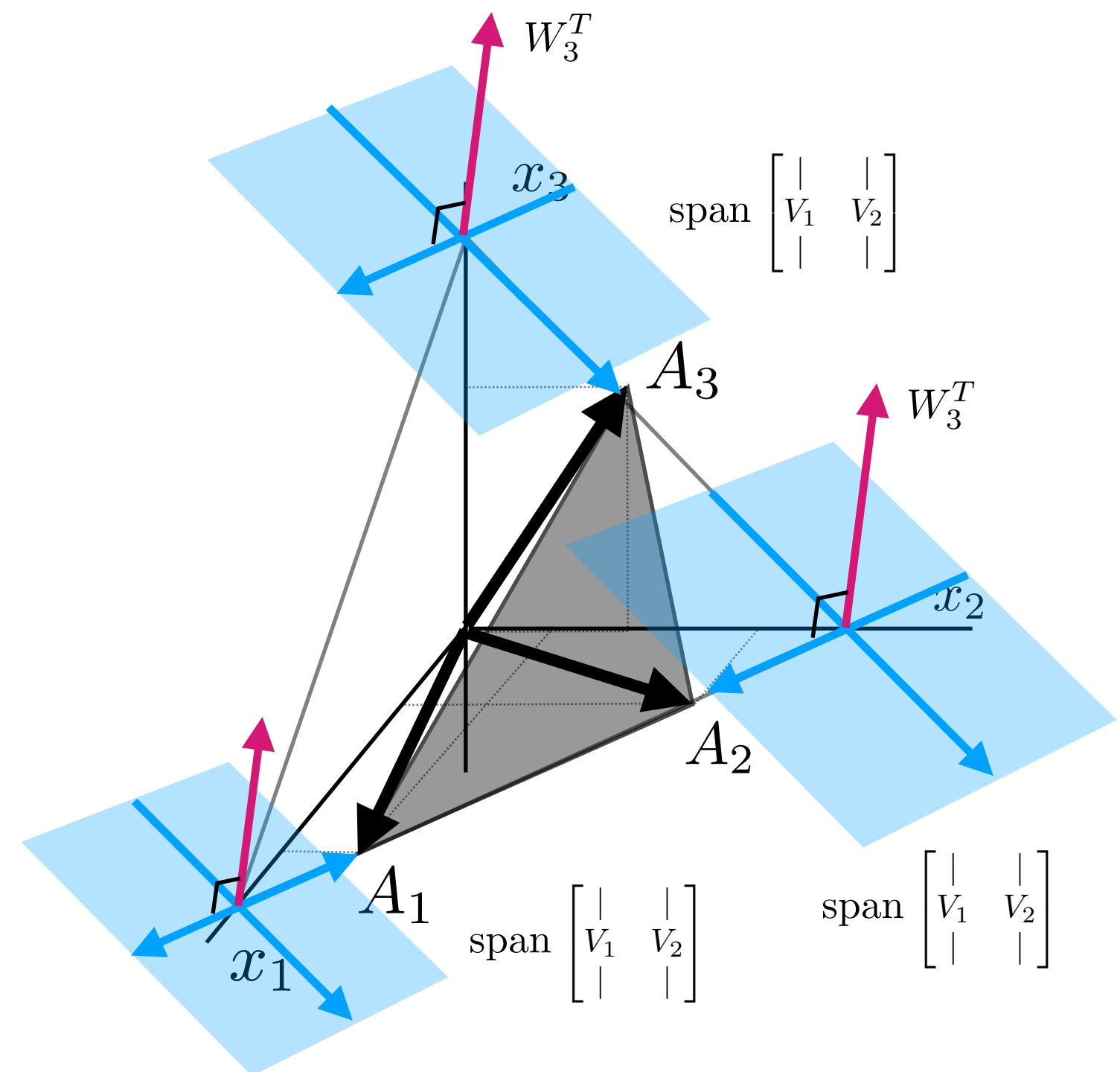
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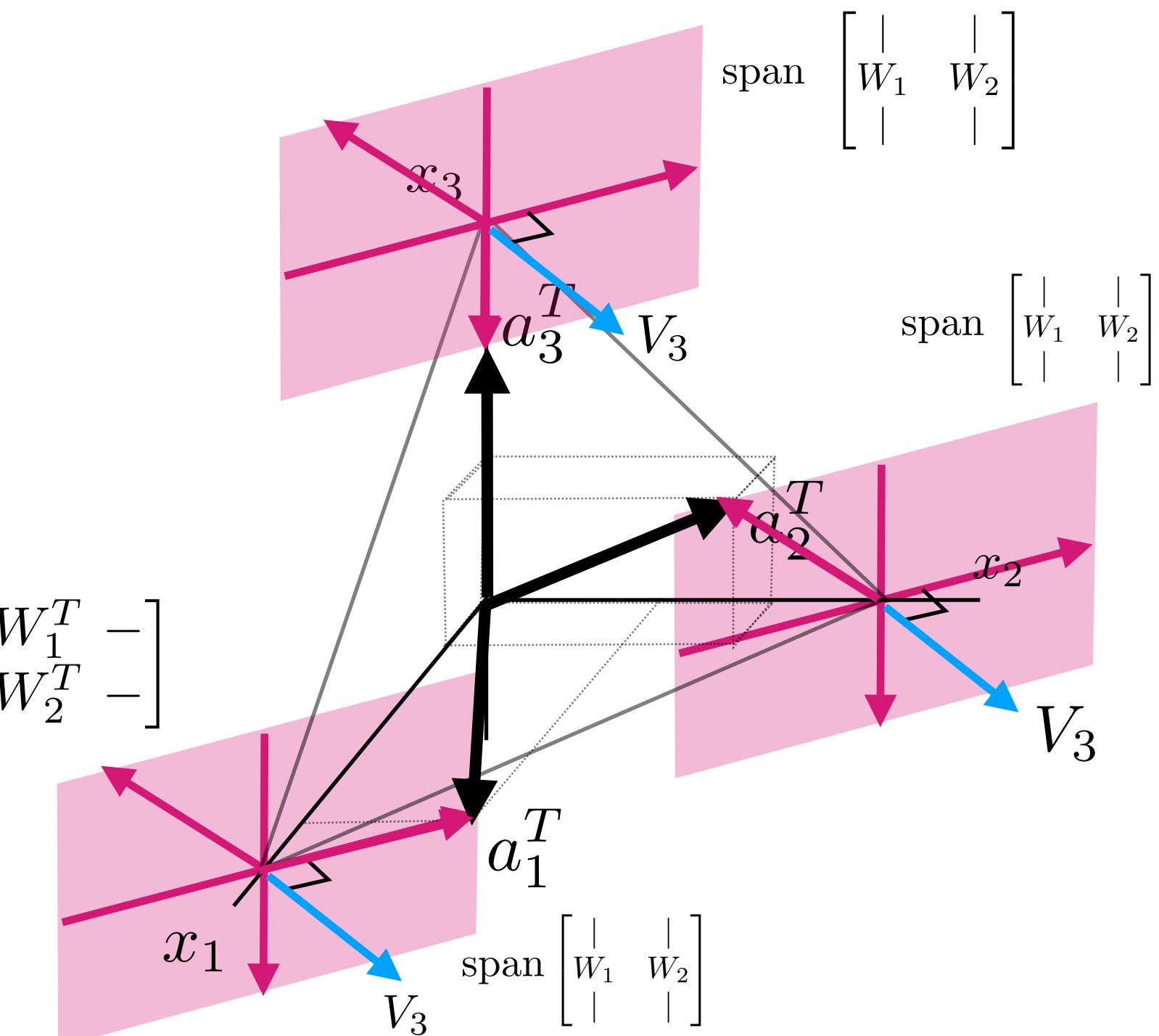
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Diagonalization

Square matrix: $A \in \mathbb{R}^{n \times n}$

Assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Right Eigenvectors:

$$V = \begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix} \quad AV = \begin{bmatrix} AV_1 \dots AV_n \end{bmatrix} = \begin{bmatrix} V_1 \lambda_1 \dots V_n \lambda_n \end{bmatrix} = \begin{bmatrix} V_1 \dots V_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_D = VD \quad \rightarrow \quad AV = VD$$

Left Eigenvectors:

$$W = \begin{bmatrix} -W_1^* - \\ \vdots \\ -W_n^* - \end{bmatrix} \quad WA = \begin{bmatrix} -W_1^* A - \\ \vdots \\ -W_n^* A - \end{bmatrix} = \begin{bmatrix} -\lambda_1 W_1^* - \\ \vdots \\ -\lambda_n W_n^* - \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_D \begin{bmatrix} -W_1^* - \\ \vdots \\ -W_n^* - \end{bmatrix} = DW \quad \rightarrow \quad WA = DW$$

Assuming V & W are chosen with compatible orderings and lengths of columns/rows...

$$V^{-1} = W$$

Diagonalization

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = V D V^{-1}$$

$$\underbrace{\begin{bmatrix} A \\ | \\ V_1 & \cdots & V_n \\ | \end{bmatrix}}_{\text{Right eigen-vectors}} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{D}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\underbrace{\begin{bmatrix} A \\ | \\ V_i \\ | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{[\lambda_i]}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_i^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \\ | \\ V_i \\ | \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices

Dyadic
Expansion

$$\begin{aligned} V^{-1}V &= \begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} W_1^* V_1 & \cdots & W_1^* V_n \\ \vdots & & \vdots \\ W_n^* V_1 & \cdots & W_n^* V_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \end{aligned}$$

...from off diagonal terms $W_j^* V_i = 0 \quad j \neq i$

V_i orthogonal to all other W_j

...from diagonal terms

V_i, W_i

$W_i^* V_i = 1$

can be scaled
so that $W_i^* V_i = 1$

Diagonalization - Similarity Transform

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$ A is similar to a diagonal matrix

Diagonalization

$$A = V D V^{-1}$$

$$\left[\begin{array}{c|c|c|c} A & | & | & | \\ \hline V_1 & \dots & V_n & | \\ | & & | & | \end{array} \right] = \underbrace{\left[\begin{array}{c|c|c|c} | & & | & | \\ \hline V_1 & \dots & V_n & | \\ | & & | & | \end{array} \right]}_{\text{Right eigen-vectors}} \underbrace{\left[\begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \underbrace{\left[\begin{array}{ccc} -W_1^* & - & | \\ | & \vdots & | \\ -W_n^* & - & | \end{array} \right]}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\left[\begin{array}{c|c|c|c} A & | & | & | \\ \hline V_i & & & | \\ | & & & | \end{array} \right] = \sum_i \left[\begin{array}{c|c|c|c} | & & | & | \\ \hline V_i & & & | \\ | & & & | \end{array} \right] \left[\begin{array}{c} \lambda_i \\ | \\ | \end{array} \right] \left[\begin{array}{ccc} -W_i^* & - & | \\ | & \vdots & | \\ -W_i^* & - & | \end{array} \right]$$

Sum of
rank-1
matrices
Dyadic Expansion

$$\left[\begin{array}{c} y'_1 \\ \vdots \\ y'_n \end{array} \right] = \left[\begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right] \left[\begin{array}{c} x'_1 \\ \vdots \\ x'_n \end{array} \right] = \left[\begin{array}{c} \lambda_1 x'_1 \\ \vdots \\ \lambda_n x'_n \end{array} \right]$$

$$x = Vx' \quad y = Vy'$$

$$y = Ax$$

$$Vy' = AVx'$$

$$y' = V^{-1}AVx'$$

$$y' = V^{-1}VDV^{-1}Vx'$$

$$y' = Dx'$$

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Interpretation of
Matrix Multiplication

Ax

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

transforming into eigen-vector coords

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Dyadic Expansion

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices

Dyadic Expansion

Interpretation of
Matrix Multiplication

Ax

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$\underbrace{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$ transforming into eigen-vector coords

$\underbrace{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}$ Scaling each coord by eigenvalue

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices
**Dyadic
Expansion**

Interpretation of
Matrix Multiplication

Ax

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ \vdots & \ddots & \vdots \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$\underbrace{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$ transforming into eigen-vector coords

$\underbrace{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}$ Scaling each coord by eigenvalue

$V_1 \lambda_1 W_1^* x + \cdots + V_n \lambda_n W_n^* x$ Transforming back into regular coordinates

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **If x is an eigenvector...**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices

Dyadic Expansion

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

Orthogonal to all other left eigenvectors

AV_i

Interpretation of
Matrix Multiplication

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **If x is an eigenvector...**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices

Dyadic Expansion

Interpretation of
Matrix Multiplication

AV_i

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

Orthogonal to all other left eigenvectors

Scaled by specific eigenvalue

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **If x is an eigenvector...**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors **Eigen-values (on diagonal)** **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication

AV_i

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

Orthogonal to all other left eigenvectors

Scaled by specific eigenvalue

$$\underbrace{\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix}}_{\lambda_i V_i} \quad \underbrace{\begin{bmatrix} | \\ \lambda_i \\ | \end{bmatrix}}_{\text{Select out that specific eigenvector}}$$

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

$$\left[\begin{array}{c} A \\ | \\ V_1 \dots V_n \\ | \end{array} \right] = \left[\begin{array}{ccc} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{array} \right] \underbrace{\left[\begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \left[\begin{array}{ccc} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{array} \right] =$$

$$\left[\begin{array}{ccc} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{array} \right] \underbrace{\left[\begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right]}_{\text{Shuffling eigenvalues and eigenvectors}} \left[\begin{array}{ccc} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{array} \right]$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

$$\left[\begin{array}{c} A \\ | \\ V_i \\ | \end{array} \right] = \sum_i \left[\begin{array}{c} | \\ V_i \\ | \end{array} \right] [\lambda_i] \left[\begin{array}{ccc} - & W_i^* & - \end{array} \right]$$



Order of sum does not matter...

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

$$\left[\begin{array}{c} A \\ | \\ V_1 \quad \cdots \quad V_n \\ | \end{array} \right] = \underbrace{\left[\begin{array}{ccc} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right]}_{\text{Right eigen-vectors}} \underbrace{\left[\begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \underbrace{\left[\begin{array}{ccc} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{array} \right]}_{\text{Left eigen-vectors}} = \left[\begin{array}{ccc} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right] \left[\begin{array}{c} P \\ | \\ P^T \end{array} \right] \left[\begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right] \left[\begin{array}{c} P \\ | \\ P^T \end{array} \right] \left[\begin{array}{ccc} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{array} \right]$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

Shuffling eigenvalues and eigenvectors

$$\left[\begin{array}{c} A \\ | \\ V_i \\ | \end{array} \right] = \sum_i \left[\begin{array}{c} | \\ V_i \\ | \end{array} \right] [\lambda_i] \left[\begin{array}{ccc} - & W_i^* & - \end{array} \right]$$



Order of sum does not matter...

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

$$\left[\begin{array}{c} A \\ | \\ V_1 & \cdots & V_n \\ | \end{array} \right] = \left[\begin{array}{c|c|c} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right] \underbrace{\left[\begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \underbrace{\left[\begin{array}{ccc} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{array} \right]}_{\text{Left eigen-vectors}} = \left[\begin{array}{c|c|c} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right] \left[\begin{array}{c} P \\ | \\ P^T \end{array} \right] \left[\begin{array}{ccc} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right] \left[\begin{array}{c} P \\ | \\ P^T \end{array} \right] \left[\begin{array}{ccc} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{array} \right]$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Shuffling eigenvalues and eigenvectors

$$\left[\begin{array}{c} A \\ | \\ V_i \\ | \end{array} \right] = \sum_i \left[\begin{array}{c} | \\ V_i \\ | \end{array} \right] [\lambda_i] \left[\begin{array}{ccc} -W_i^* & - \end{array} \right]$$

Sum of rank-1 matrices
Dyadic Expansion



Order of sum does not matter...

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

**Shuffling eigenvalues
and eigenvectors**

$$\begin{bmatrix} A \\ | \\ V_i \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



Order of sum does not matter...

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

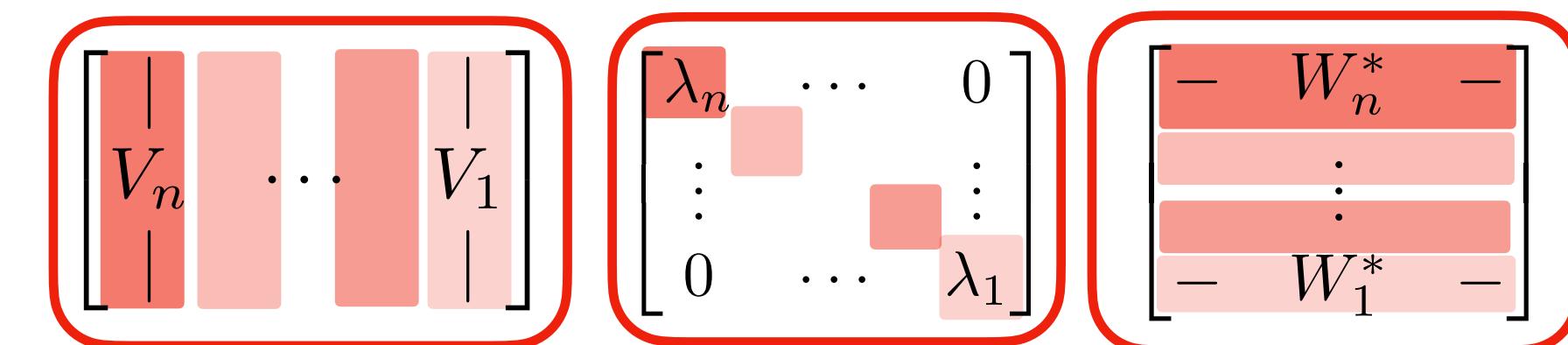
Diagonalization

Shuffle columns (or rows) of identity...

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$



Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

**Shuffling eigenvalues
and eigenvectors**

$$\begin{bmatrix} A \\ | \\ V_i \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



Order of sum does not matter...

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

diagonal matrices
commute...

$$A = V D V^{-1}$$

$$\begin{bmatrix} A \\ | \\ V_1 & \cdots & V_n \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 \frac{\gamma_1}{\gamma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \frac{\gamma_n}{\gamma_n} \end{bmatrix}}_{\text{Scaling eigenvectors}} \begin{bmatrix} - & W_1^* & - \\ | & \vdots & | \\ - & W_n^* & - \end{bmatrix}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Scaling eigenvectors

$$\begin{bmatrix} A \\ | \\ V_i \\ | \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices
Dyadic Expansion



Order of sum does not matter...

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Diagonalization

diagonal matrices
commute...

$$A = V D V^{-1}$$

$$\left[\begin{array}{c|c} A & \\ \hline V_1 & \dots & V_n \end{array} \right] = \underbrace{\left[\begin{array}{c|c} | & | \\ V_1 & \dots & V_n \\ | & & | \end{array} \right]}_{\text{Right eigen-vectors}} \underbrace{\left[\begin{array}{ccc} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{array} \right]}_{\text{Eigen-values (on diagonal)}} \underbrace{\left[\begin{array}{c|c} | & | \\ W_1^* & \dots & W_n^* \\ | & & | \end{array} \right]}_{\text{Left eigen-vectors}} =$$

$$\left[\begin{array}{c|c} | & | \\ V_1 & \dots & V_n \\ | & & | \end{array} \right] \left[\begin{array}{ccc} \gamma_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \gamma_n \end{array} \right] \left[\begin{array}{ccc} \frac{1}{\gamma_1} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \frac{1}{\gamma_n} \end{array} \right] \left[\begin{array}{c|c} | & | \\ W_1^* & \dots & W_n^* \\ | & & | \end{array} \right]$$

**Right
eigen-
vectors**

**Eigen-
values
(on diagonal)**

**Left
eigen-
vectors**

**Scaling
eigenvectors**

$$\left[\begin{array}{c|c} A & \\ \hline V_i \end{array} \right] = \sum_i \left[\begin{array}{c|c} | & | \\ V_i & | \end{array} \right] [\lambda_i] \left[\begin{array}{c|c} | & | \\ W_i^* & | \end{array} \right]$$

Sum of
rank-1
matrices
**Dyadic
Expansion**



Order of sum does not matter...

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

diagonal matrices
commute...

$$A = V D V^{-1}$$

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$$\begin{bmatrix} | & & | \\ V_1\gamma_1 & \cdots & V_n\gamma_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & \frac{1}{\gamma_1}W_1^* & - \\ | & \vdots & | \\ - & \frac{1}{\gamma_n}W_n^* & - \end{bmatrix}}_{\text{Scaling eigenvectors}}$$

$$V'$$

$$V'^{-1}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

$$\begin{bmatrix} A \\ | \\ A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



Order of sum does not matter...

Spectral Mapping Theorem

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization $A = VDV^{-1}$ $A^k = VD^kV^{-1}$

Powers of A
$$\begin{aligned} A^k &= VDV^{-1} \times VDV^{-1} \times \dots \times VDV^{-1} \\ &= VDV^{-1}VDV^{-1} \dots VDV^{-1} \\ &= VD^kV^{-1} \end{aligned} \quad = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

Polynomials of A

polynomial $\Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \dots + \alpha_1 s + \alpha_0 1$ $\Psi(A) = V\Psi(D)V^{-1}$

plugging in A...

$$\begin{aligned} \Psi(A) &= \alpha_k A^k + \alpha_{k-1} A^{k-1} + \alpha_{k-2} A^{k-2} + \dots + \alpha_1 A + \alpha_0 I \\ &= \alpha_k VD^kV^{-1} + \alpha_{k-1} VD^{k-1}V^{-1} + \alpha_{k-2} VD^{k-2}V^{-1} + \dots + \alpha_1 VDV^{-1} + \alpha_0 VV^{-1} \\ &= V(\alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \dots + \alpha_1 D + \alpha_0 I)V^{-1} \end{aligned} \quad = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \Psi(\lambda_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Psi(\lambda_n) \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

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Polynomials of A

polynomial $\Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \dots + \alpha_1 s + \alpha_0 1$

$$\Psi(A) = V(\alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \dots + \alpha_1 D + \alpha_0 I)V^{-1}$$

Spectral Mapping Theorem for $f(s)$ analytic

$$\lambda \in \text{eig}(A) \rightarrow f(\lambda) \in \text{eig}(f(A))$$

$A, f(A)$ have the same eigenvectors

$$\begin{aligned} \Psi(A) &= V\Psi(D)V^{-1} \\ &= \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \Psi(\lambda_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Psi(\lambda_n) \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix} \end{aligned}$$

Spectral Mapping Theorem

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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$$\text{polynomial} \quad \Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \dots + \alpha_1 s + \alpha_0 1$$

$$\Psi(A) = V \left(\alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \dots + \alpha_1 D + \alpha_0 I \right) V^{-1}$$

Spectral Mapping Theorem for $f(s)$ analytic

$$\lambda \in \text{eig}(A) \rightarrow f(\lambda) \in \text{eig}(f(A))$$

$A, f(A)$ have the same eigenvectors

$$= \begin{bmatrix} V \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

Specific Useful Case: Matrix Exponential

$$\begin{aligned}e^{At} &= I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k\end{aligned}$$

$$\textbf{Derivative: } \frac{d}{dt} (e^{At}) = Ae^{At}$$

- can see from polynomial definition
- related to definition of e

Spectral Mapping Theorem

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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$$\text{polynomial} \quad \Psi(s) = \alpha_k s^k + \alpha_{k-1} s^{k-1} + \alpha_{k-2} s^{k-2} + \dots + \alpha_1 s + \alpha_0 1$$

$$\Psi(A) = V(\alpha_k D^k + \alpha_{k-1} D^{k-1} + \alpha_{k-2} D^{k-2} + \dots + \alpha_1 D + \alpha_0 I)V^{-1}$$

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$$= \begin{bmatrix} V \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

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$$\begin{aligned}e^{At} &= Ve^{Dt}V^{-1} \\ &= \begin{bmatrix} V \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}\end{aligned}$$