Transition Kernels, Markov Chains Markov Decision Processes

Algebraic Graph Theory

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Sarah Li
Yue Yu
Shahriar Talebi

Spring 2022 - Dan Calderone

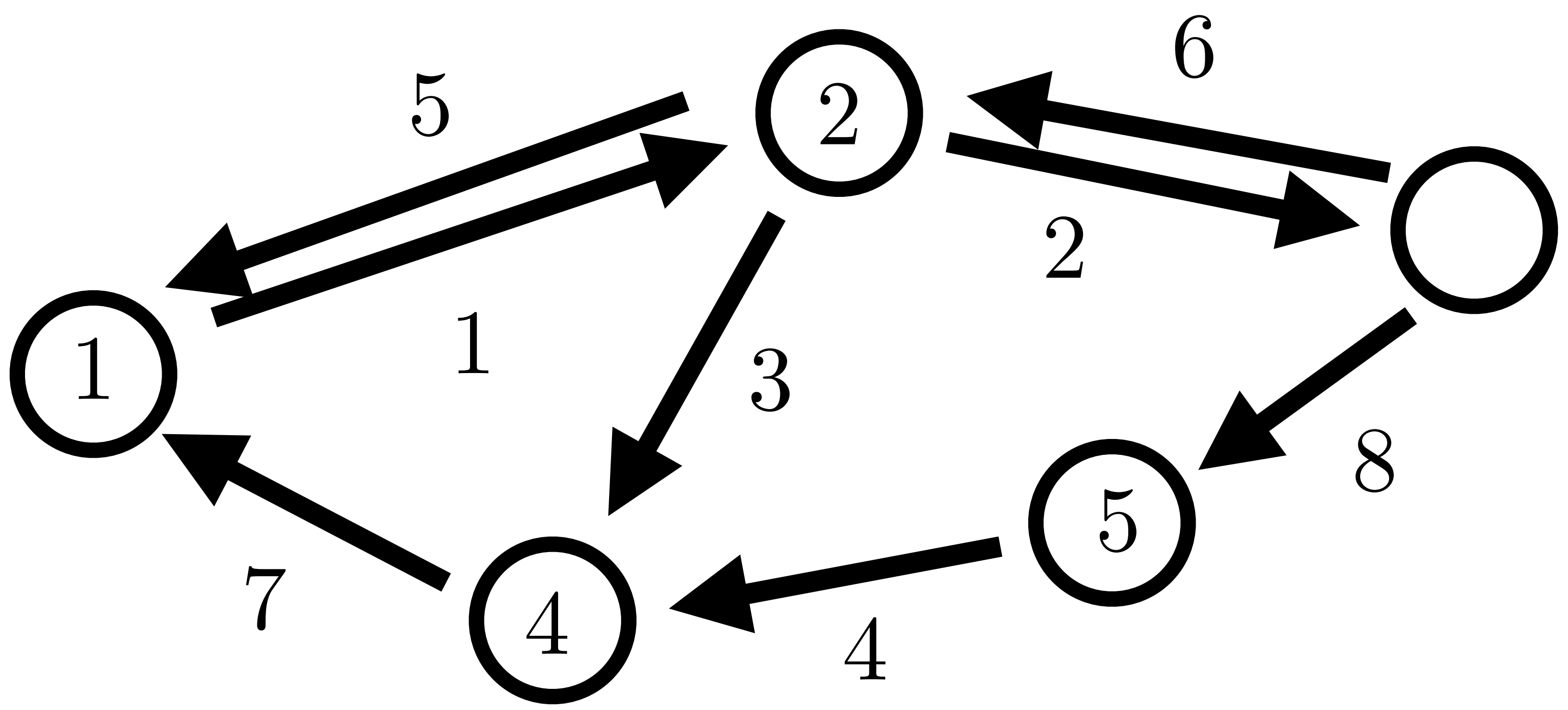
Probabilistic Transitions

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrices $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad E = E_{\text{in}} - E_{\text{out}}$

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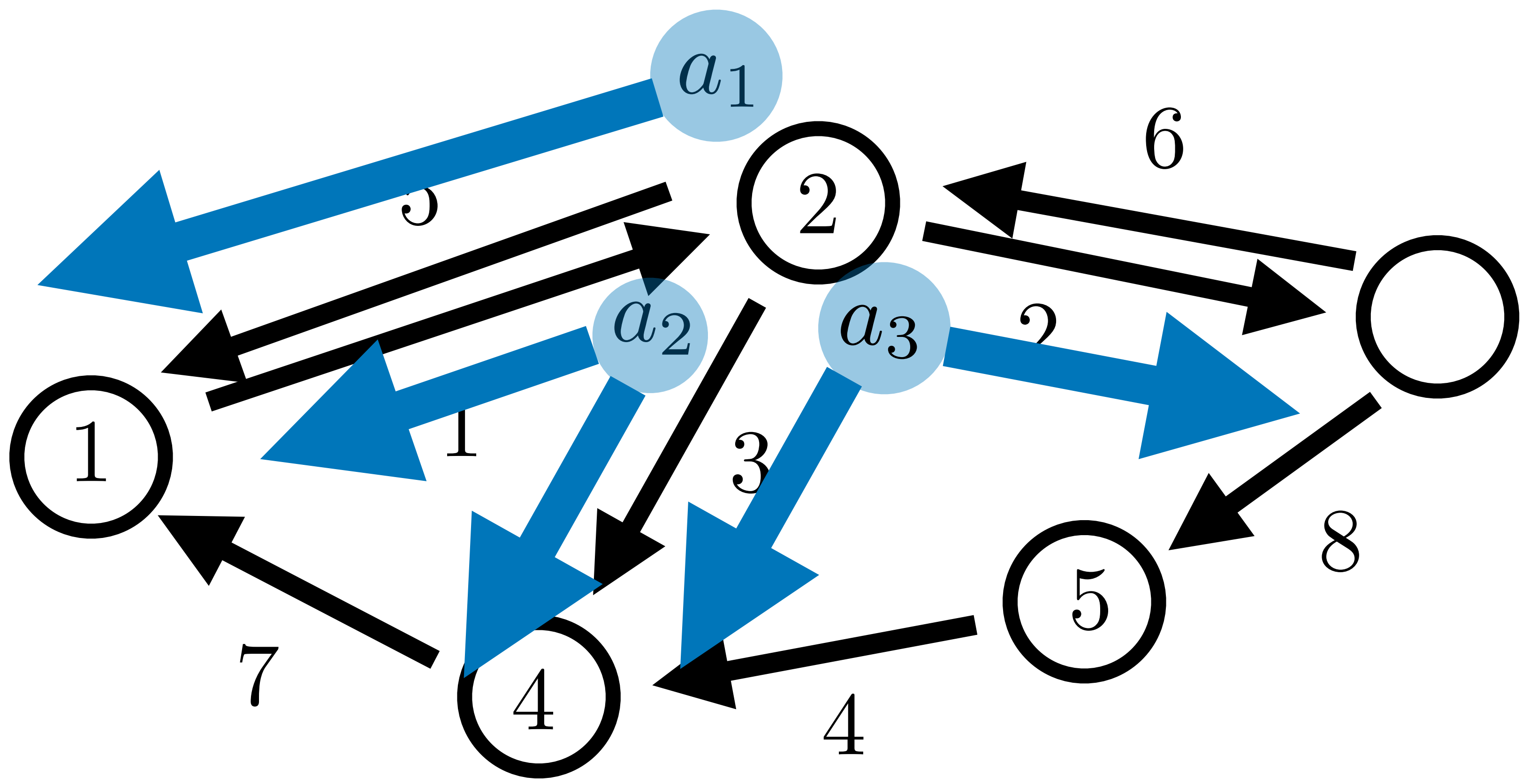

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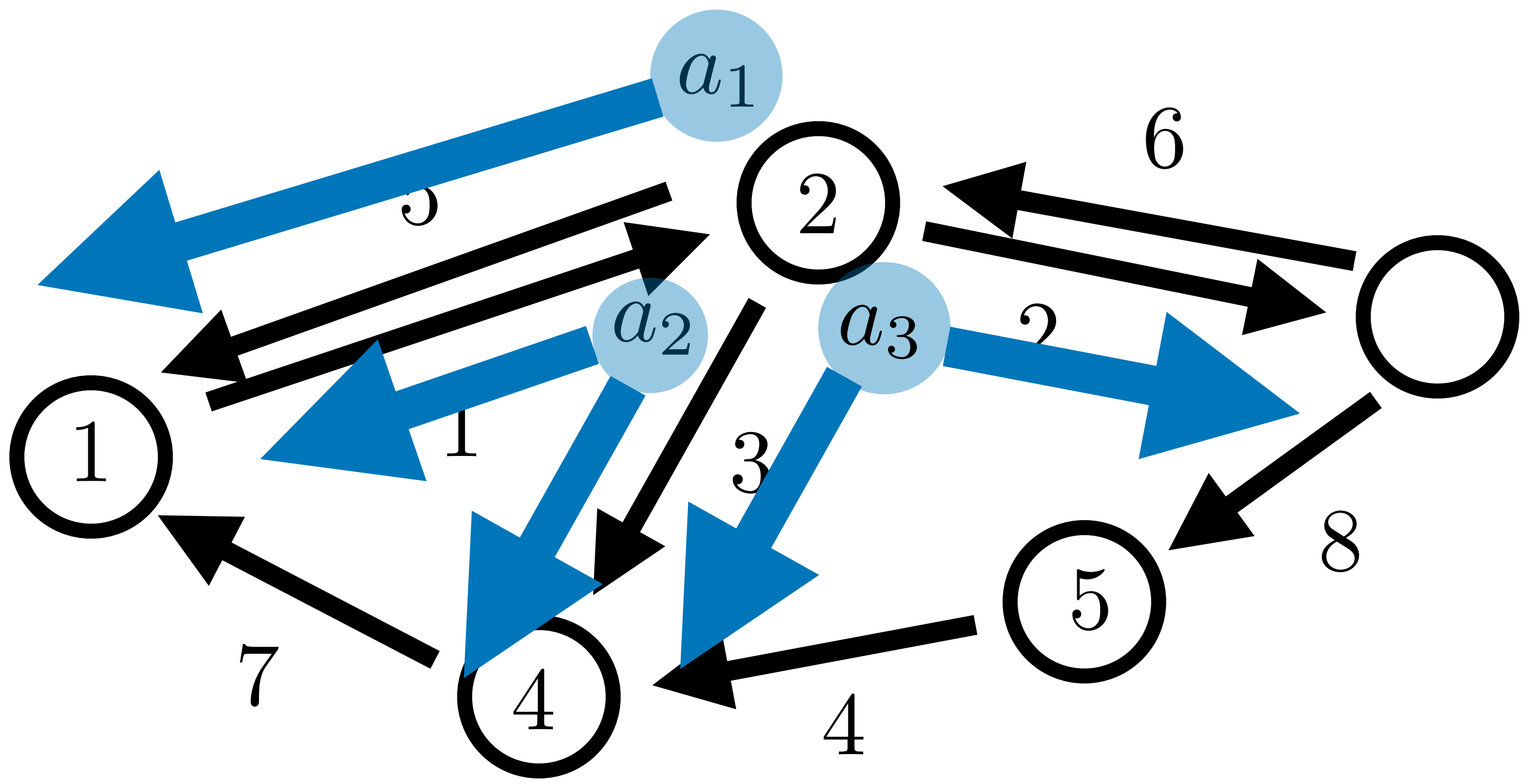

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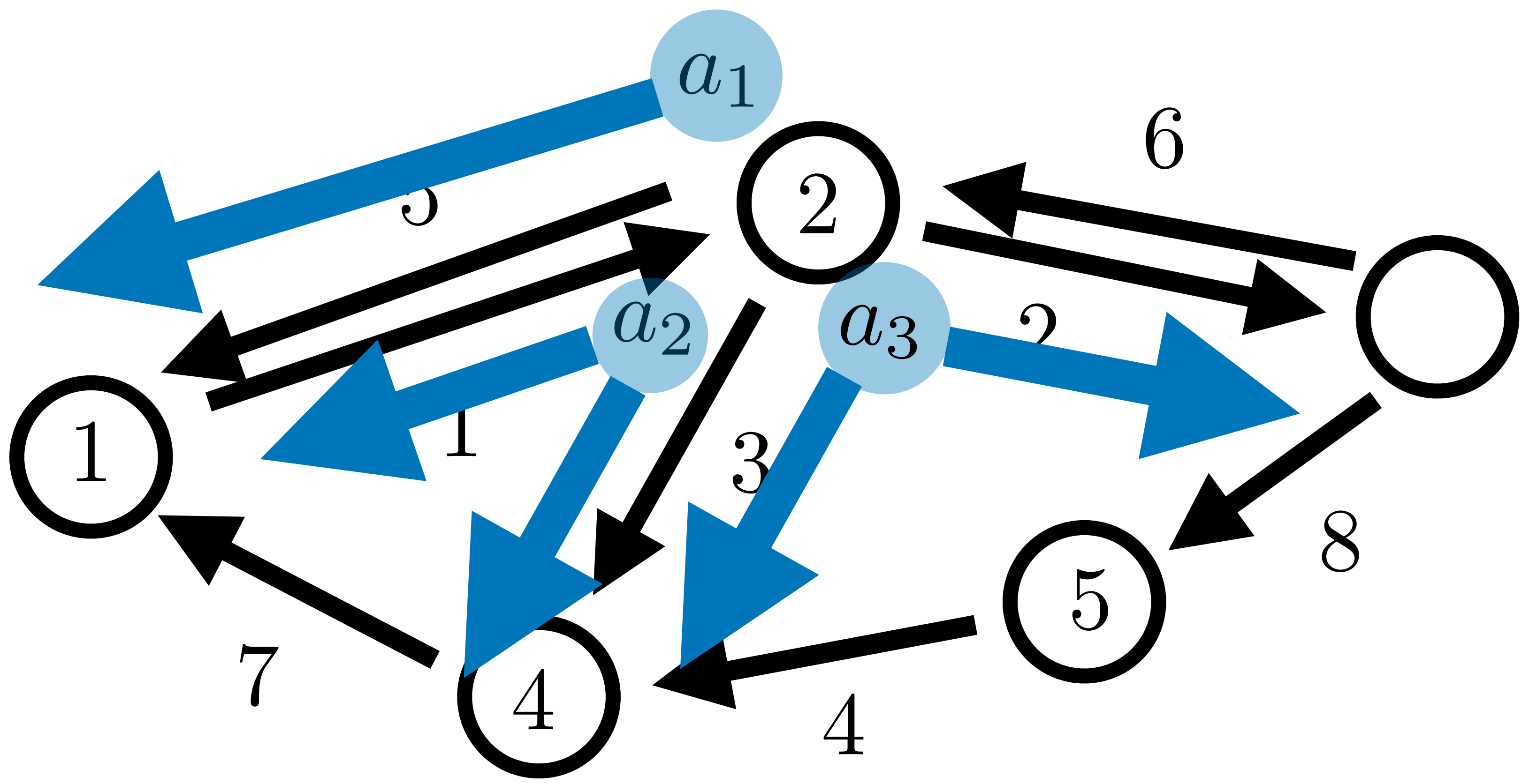

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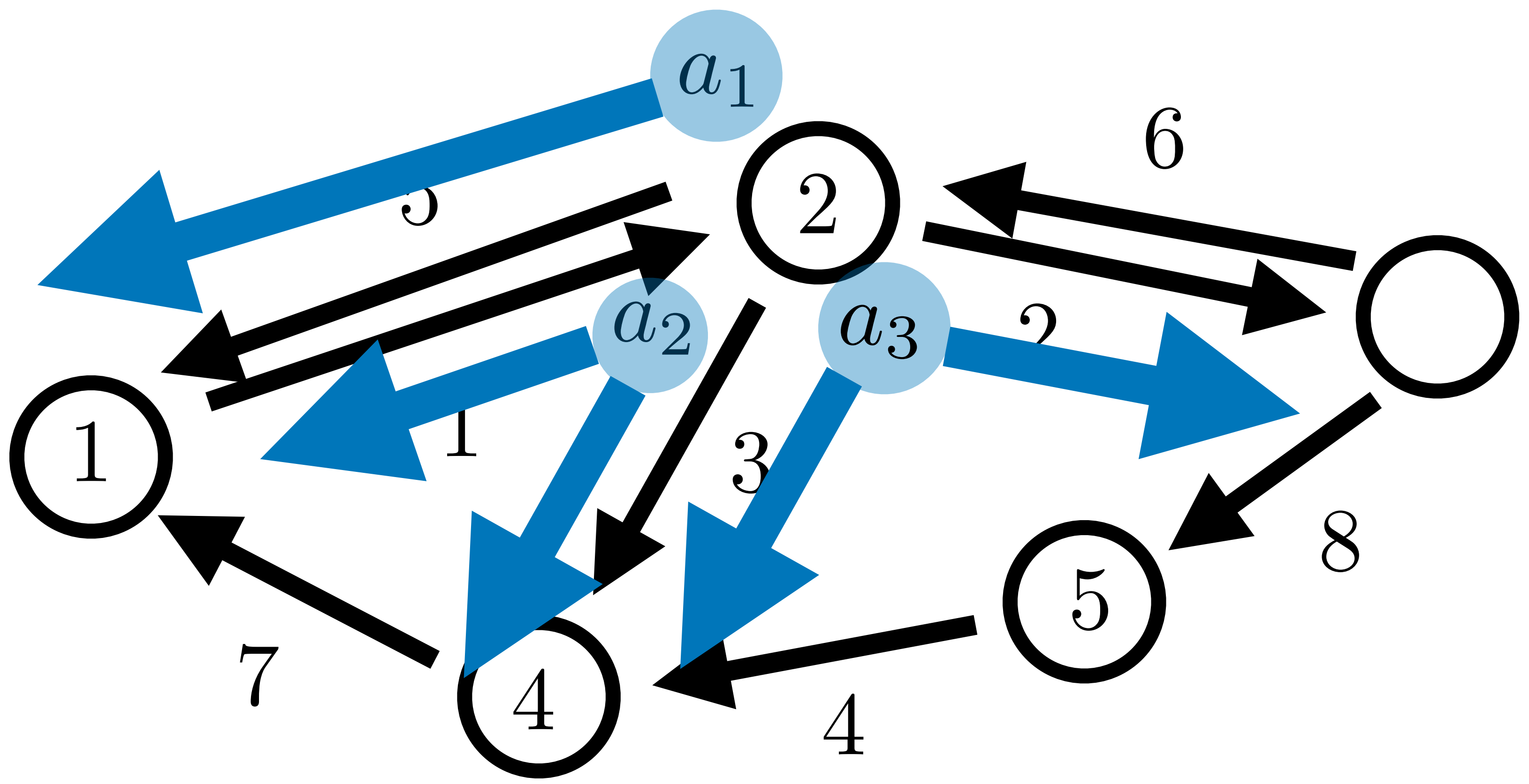

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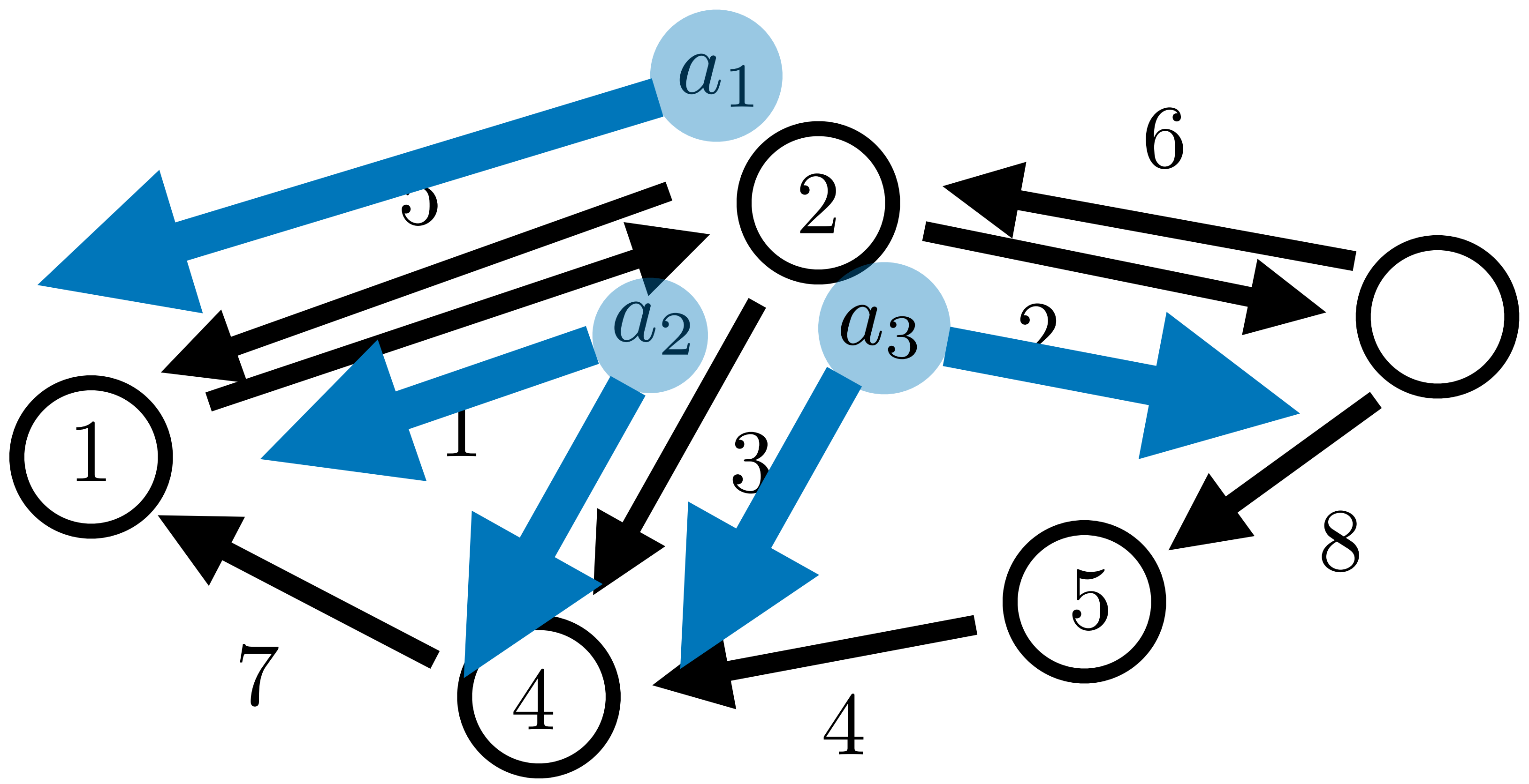

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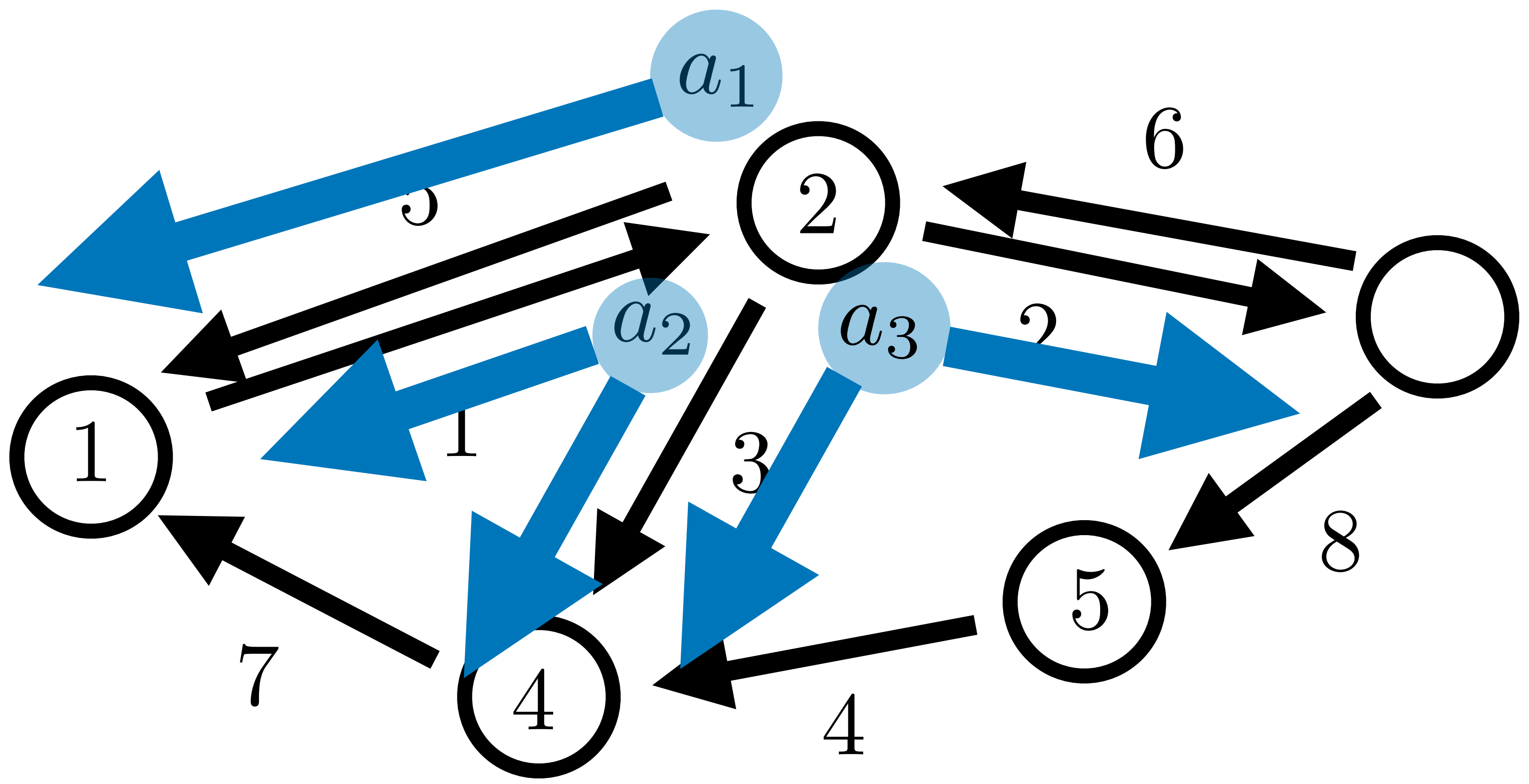

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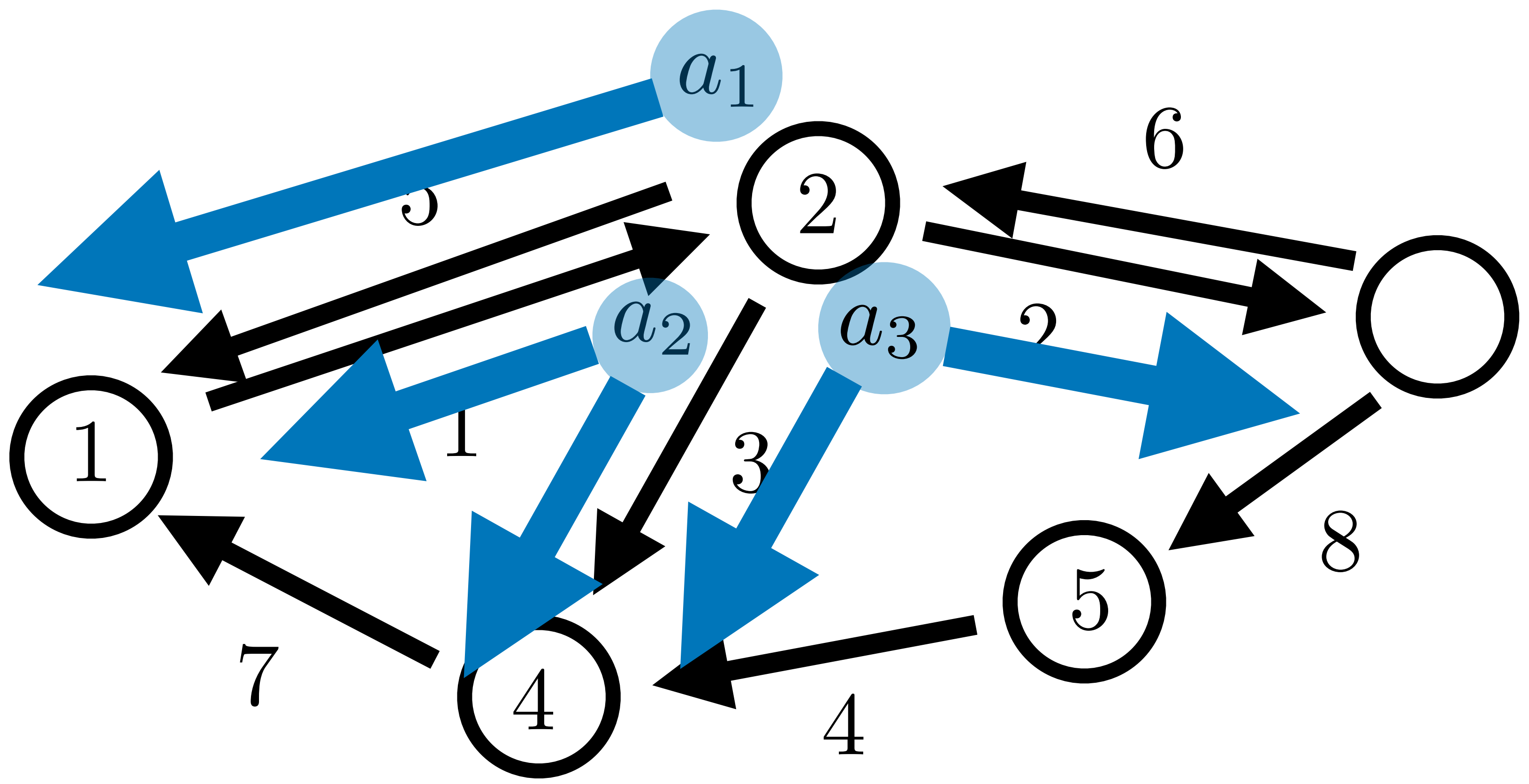

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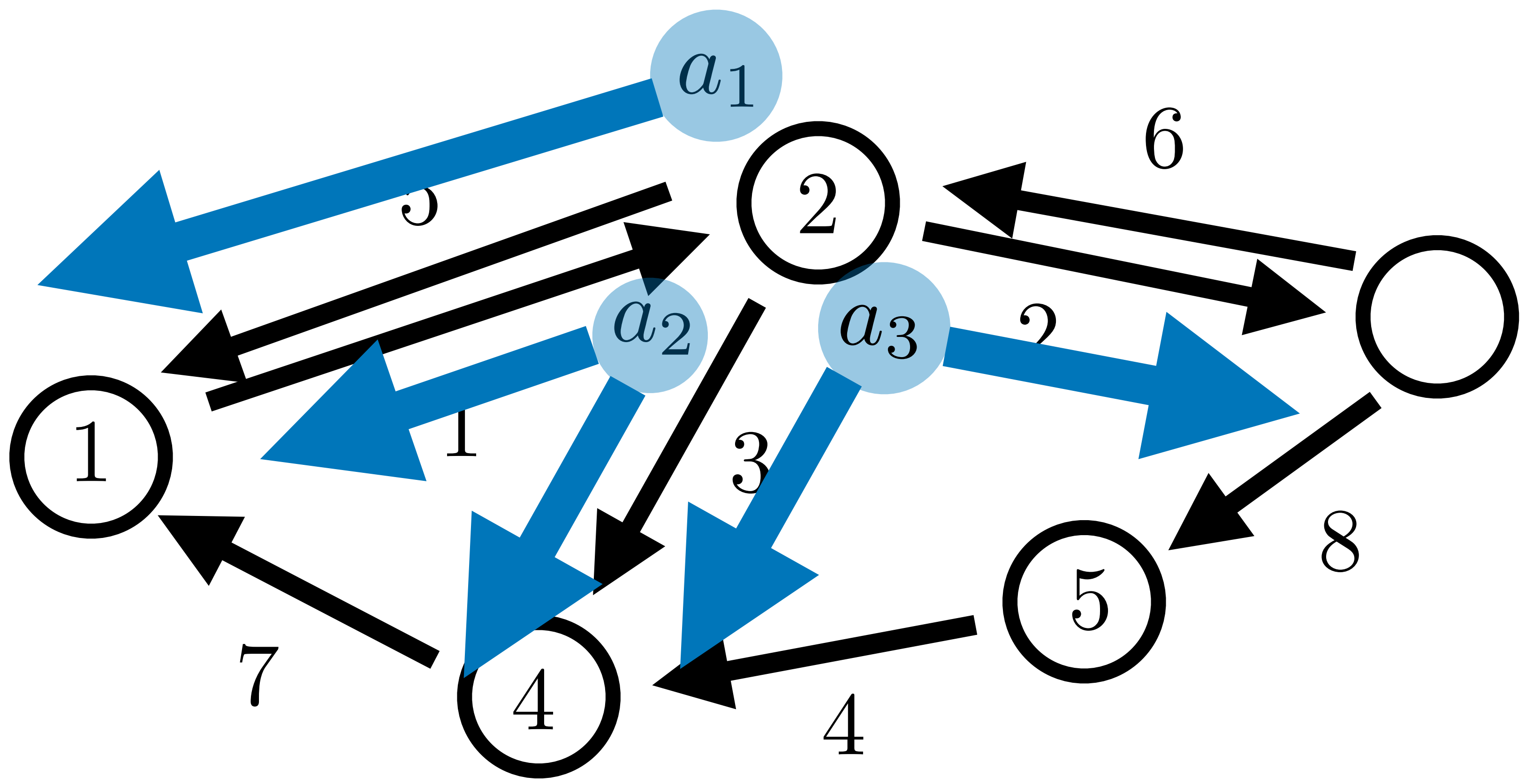

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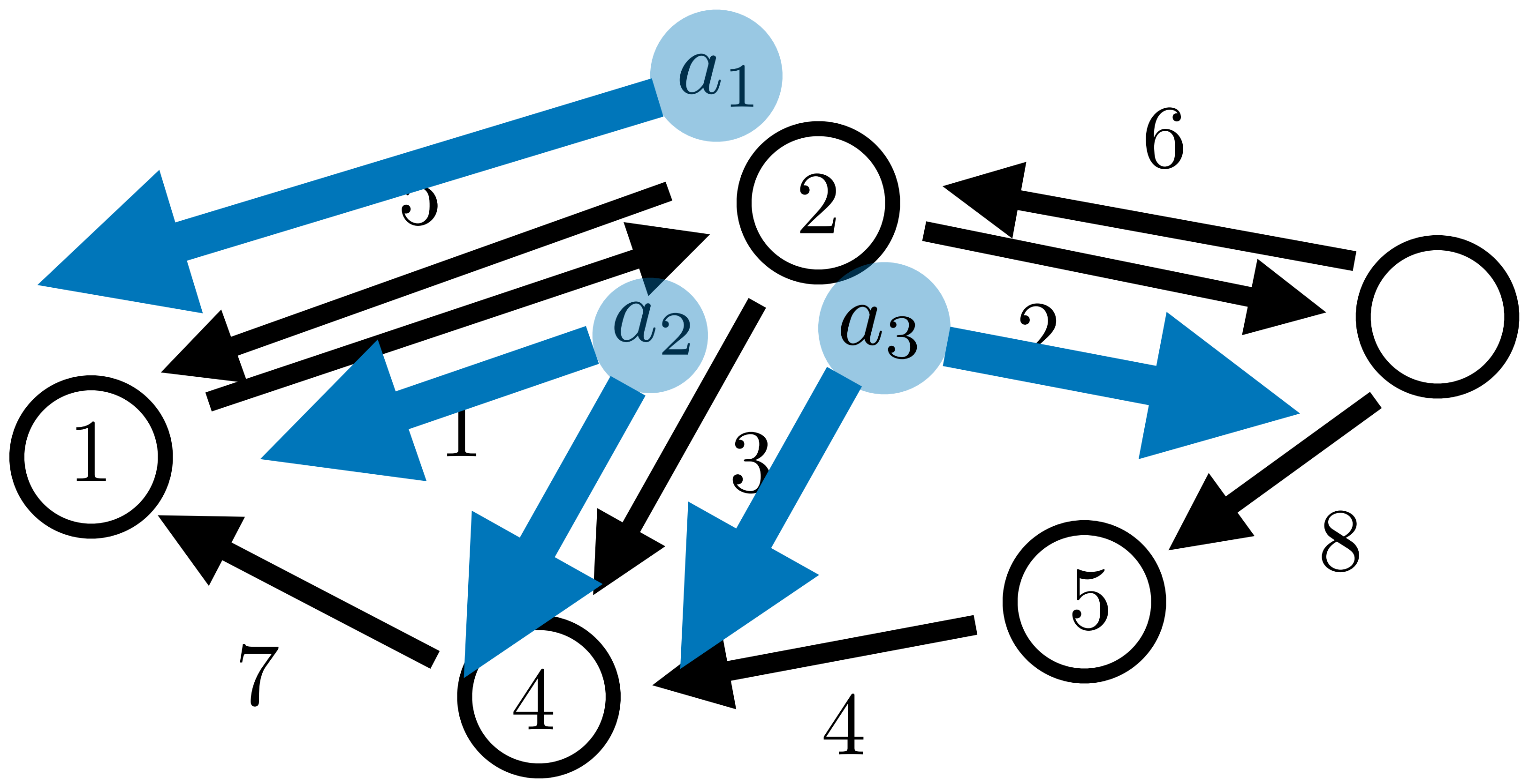
$$E_{\mathcal{A}} = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} \quad P = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} \quad W = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

Policy



$$E_{\mathcal{A}} = E_{\text{out}} W$$
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Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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Incidence Matrices

$E \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$ $E = E_{\text{in}} - E_{\text{out}}$

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Policy

$\Pi \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{S}|}$

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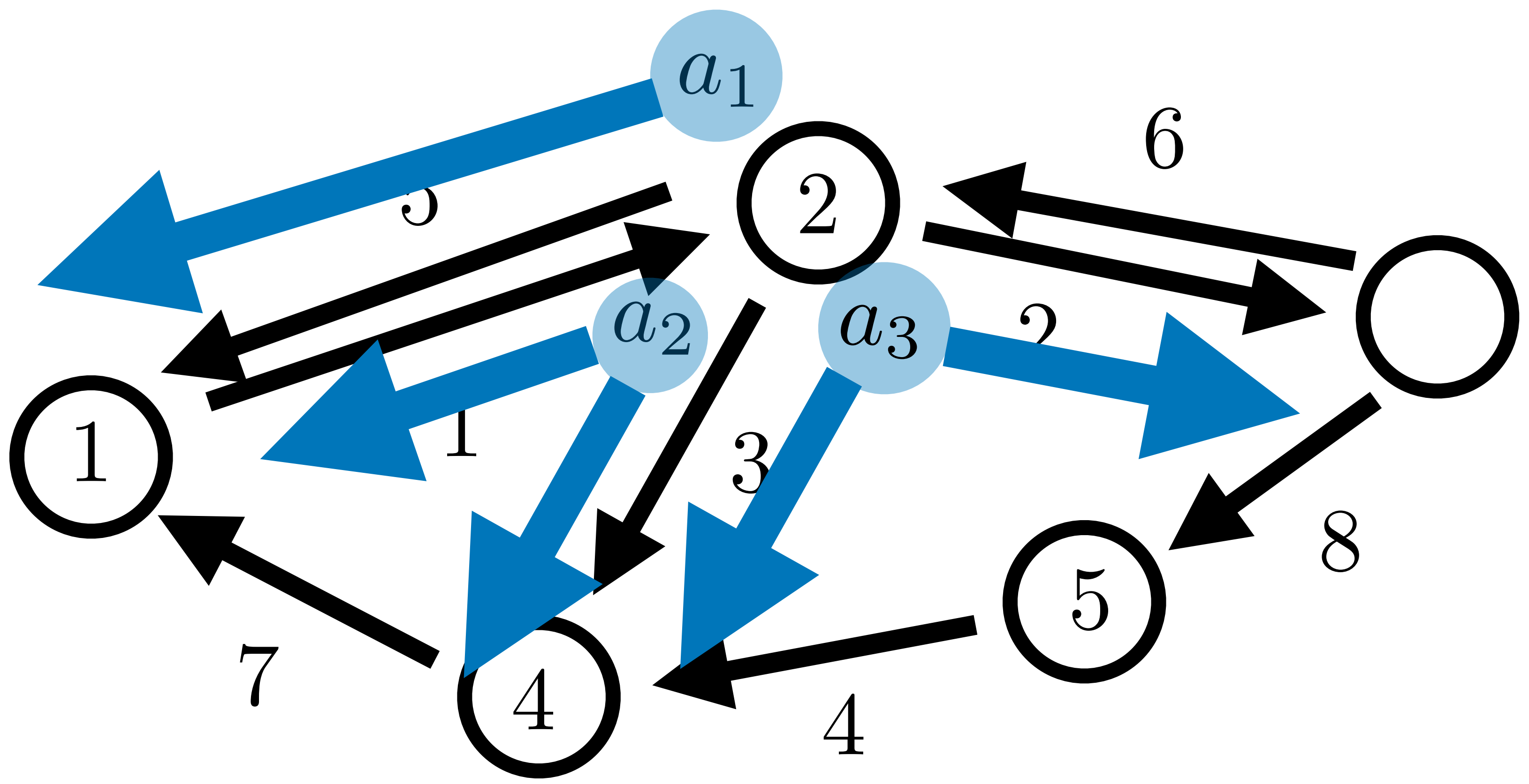

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Policy

probability of actions conditioned on state

$$\Pi = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}$$

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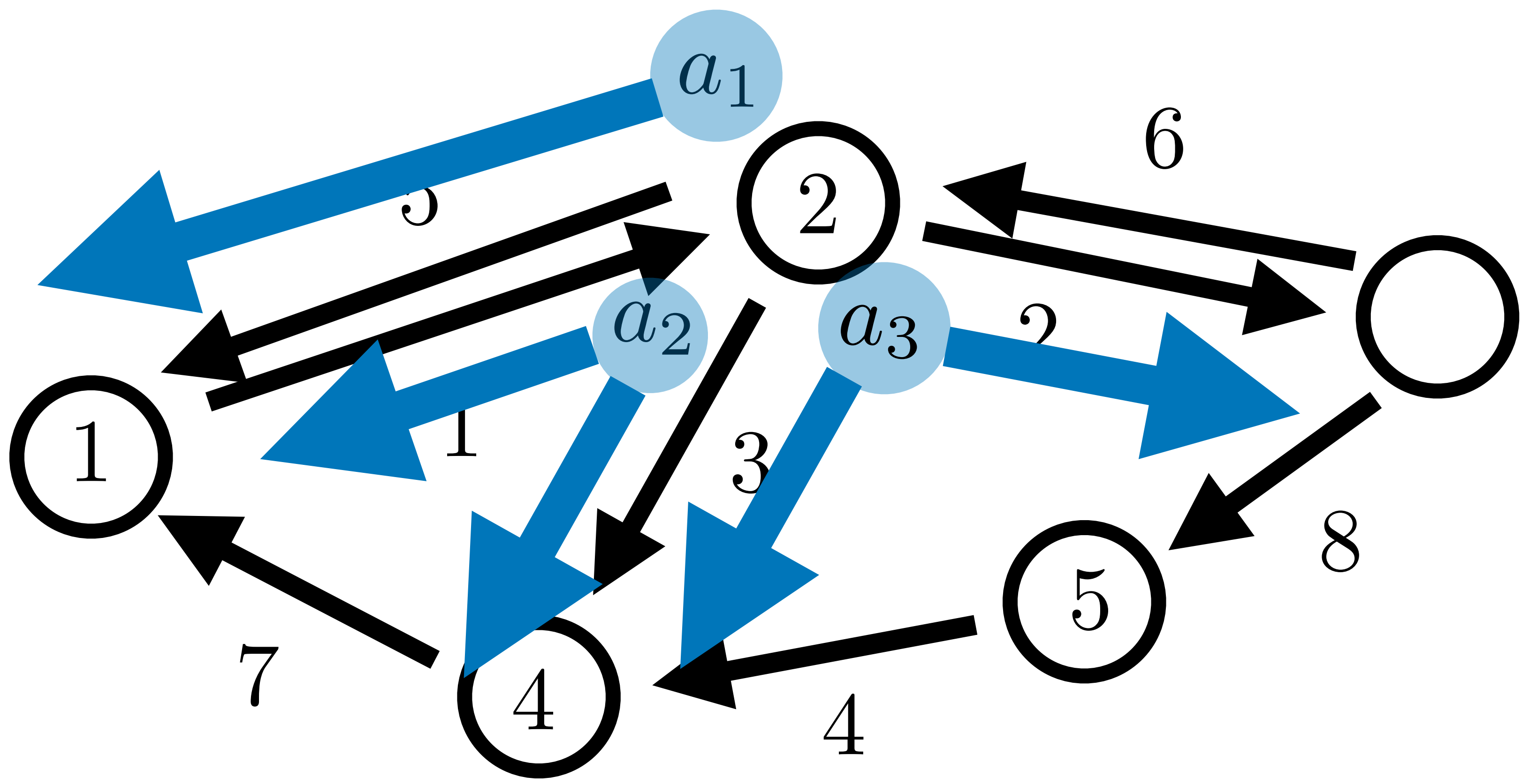

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Policy

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Policy

$$\Pi = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}$$

Markov Matrix

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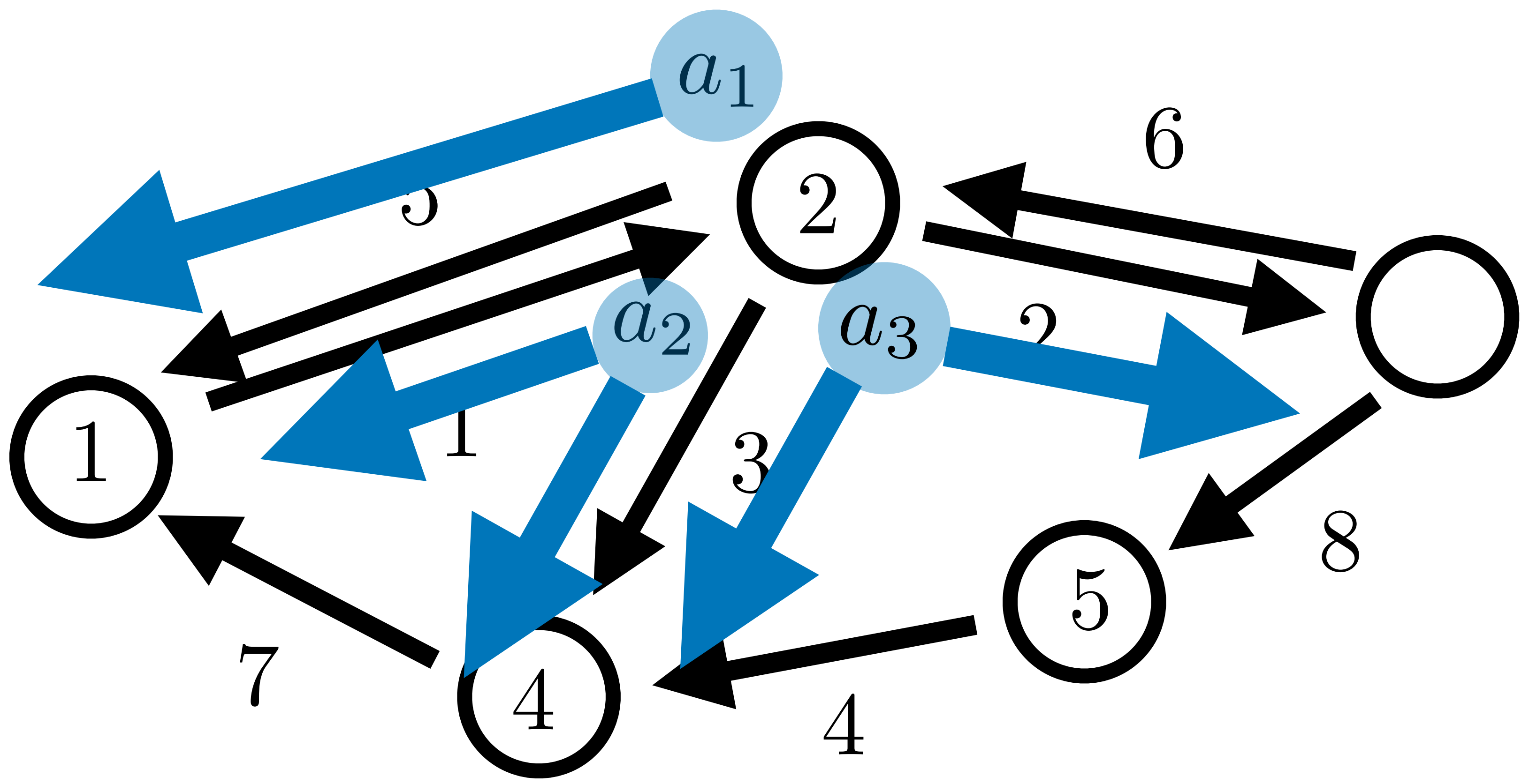
Policy



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Policy

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$$\Pi = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}$$

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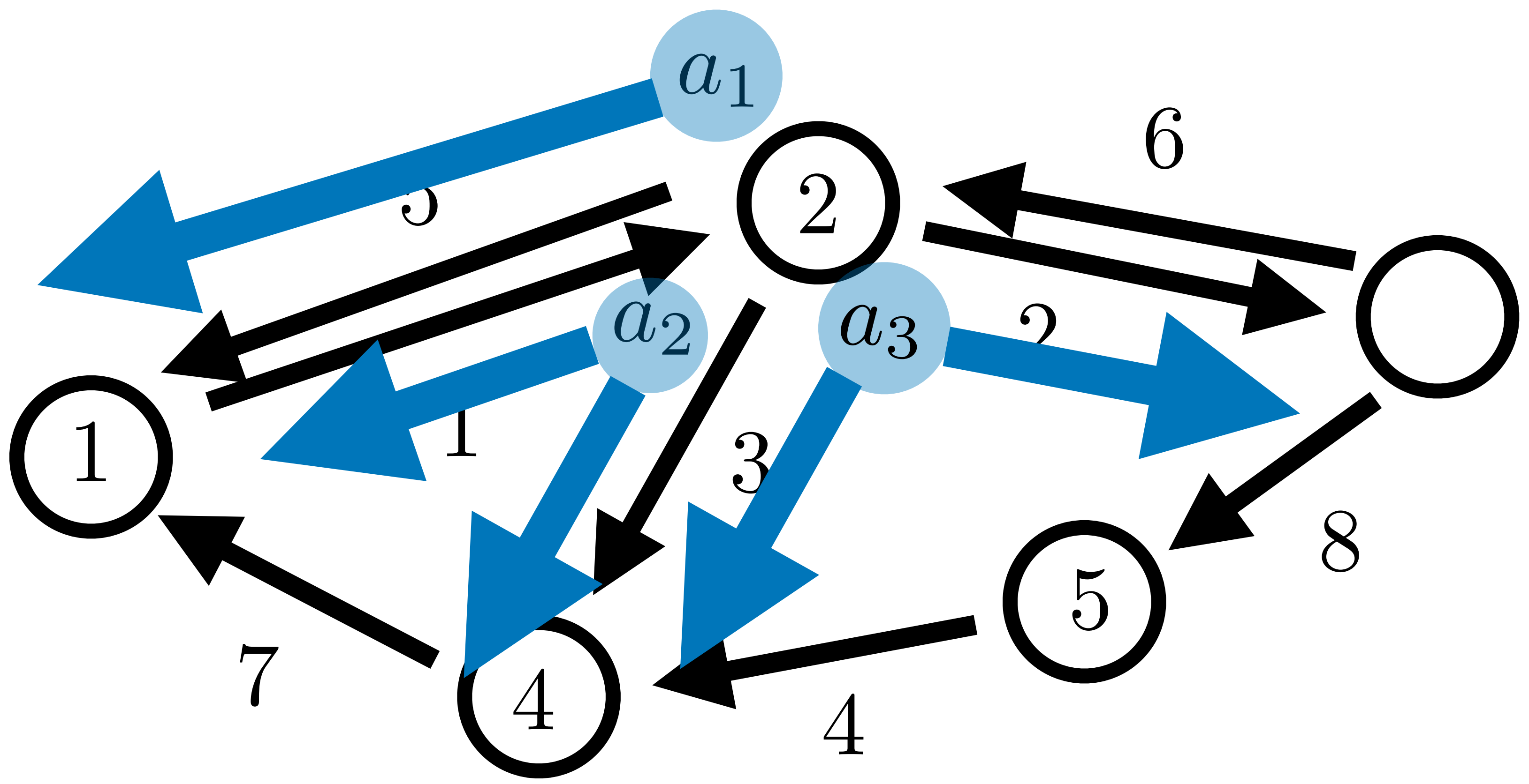
also… $N = \Pi P \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$

Policy



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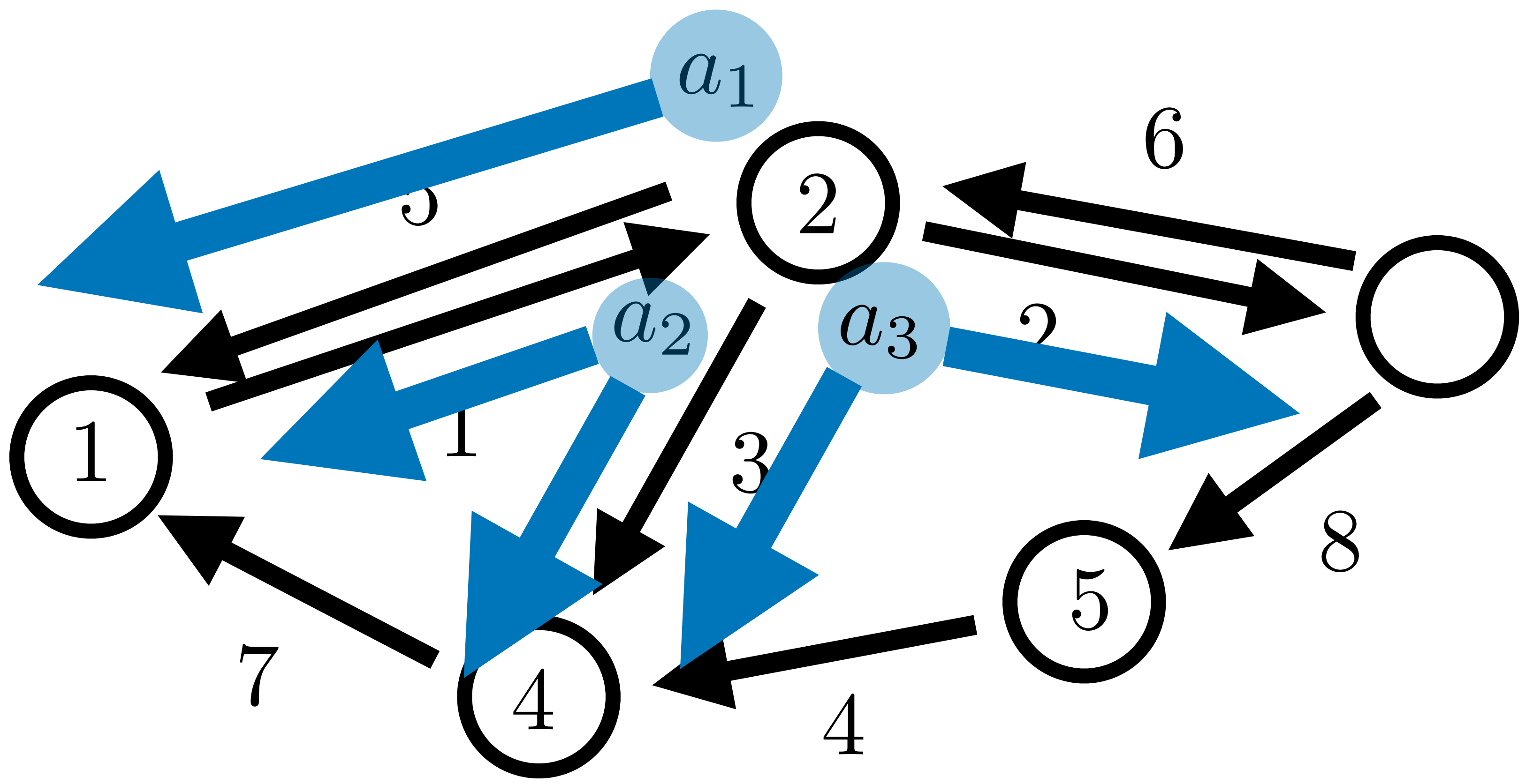

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$$\left[\, E_{\mathcal{A}} - P \,\right] x = \left[\, E_{\text{out}} - E_{\text{in}} \,\right] W x = S$$

$$E_{\mathcal{A}} = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} \quad P = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} \quad W = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \quad M = P\Pi = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}$$

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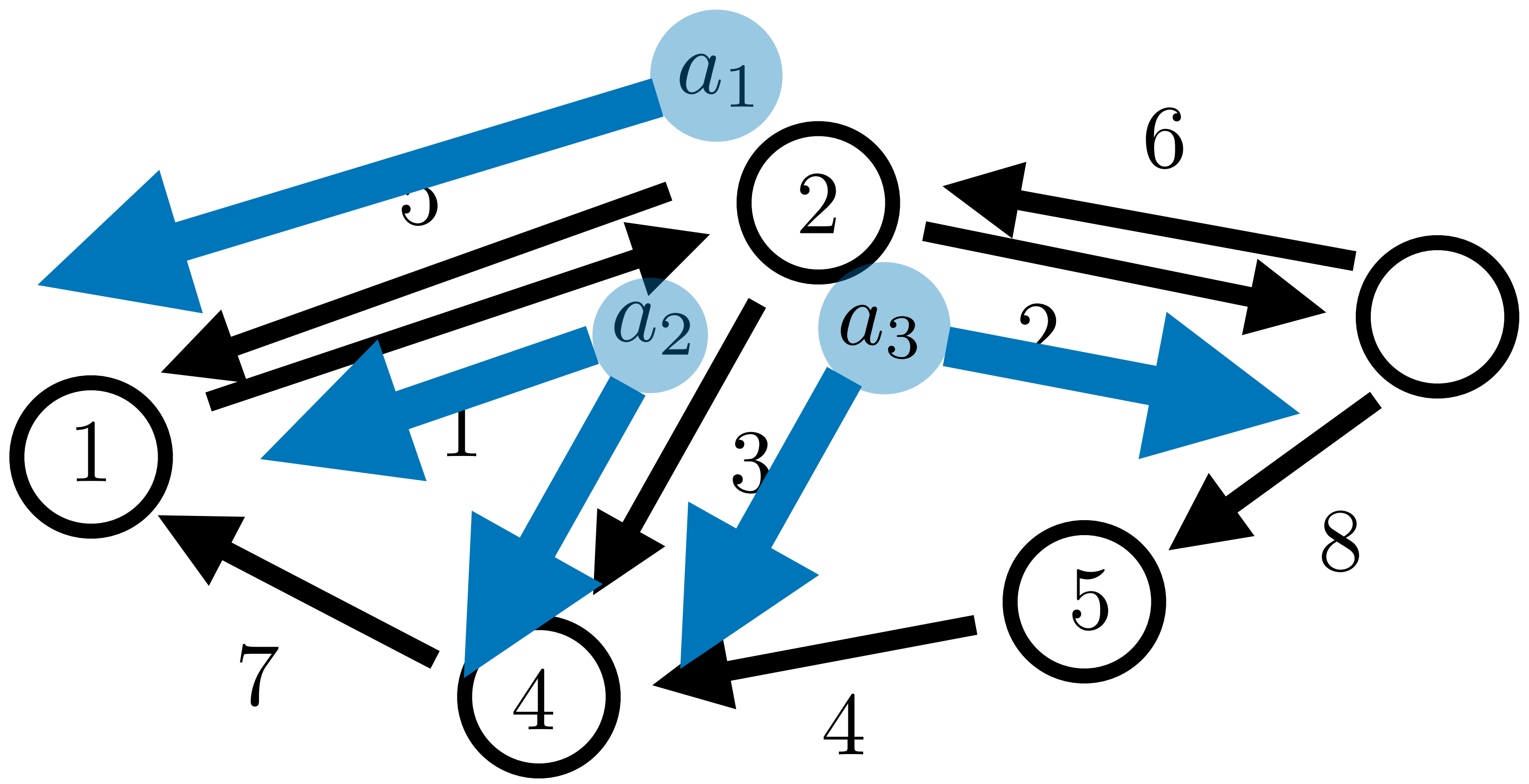
Column Stochastic

positive & sum to 1

$\mathbf{1}^T E_{\text{out}} = \mathbf{1}^T$ $\mathbf{1}^T E_{\mathcal{A}} = \mathbf{1}^T$ $\mathbf{1}^T W = \mathbf{1}^T$ $\mathbf{1}^T \Pi = \mathbf{1}^T$

$\mathbf{1}^T E_{\text{in}} = \mathbf{1}^T$ $\mathbf{1}^T P = \mathbf{1}^T$ $\mathbf{1}^T M = \mathbf{1}^T$

Transition Kernel



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Transition Kernel Laplacian

…specific weighted graph Laplacian

$$L_P = (E_{\mathcal{A}} - P)\,(E_{\mathcal{A}} - P)^T = EWW^TE^T$$

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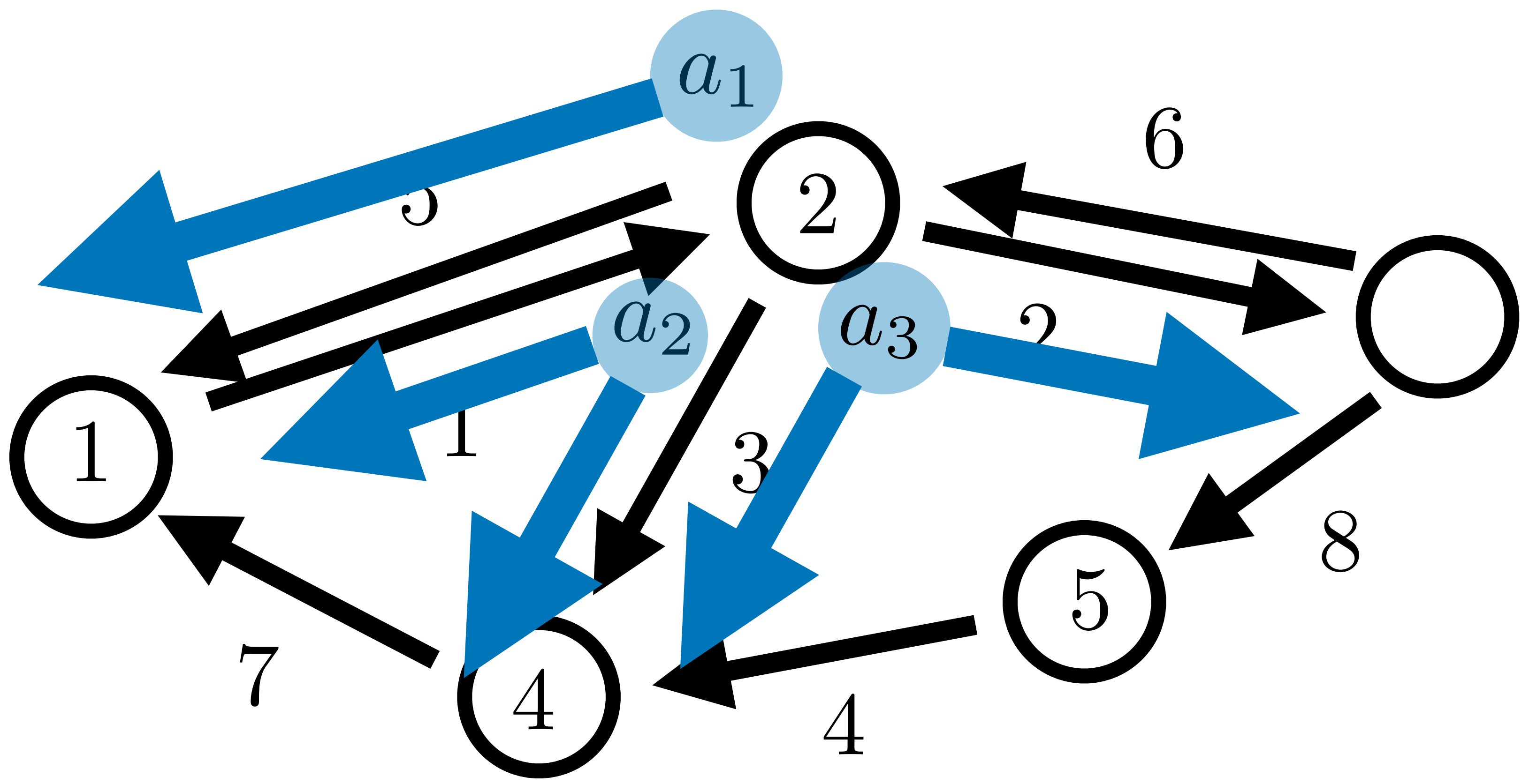

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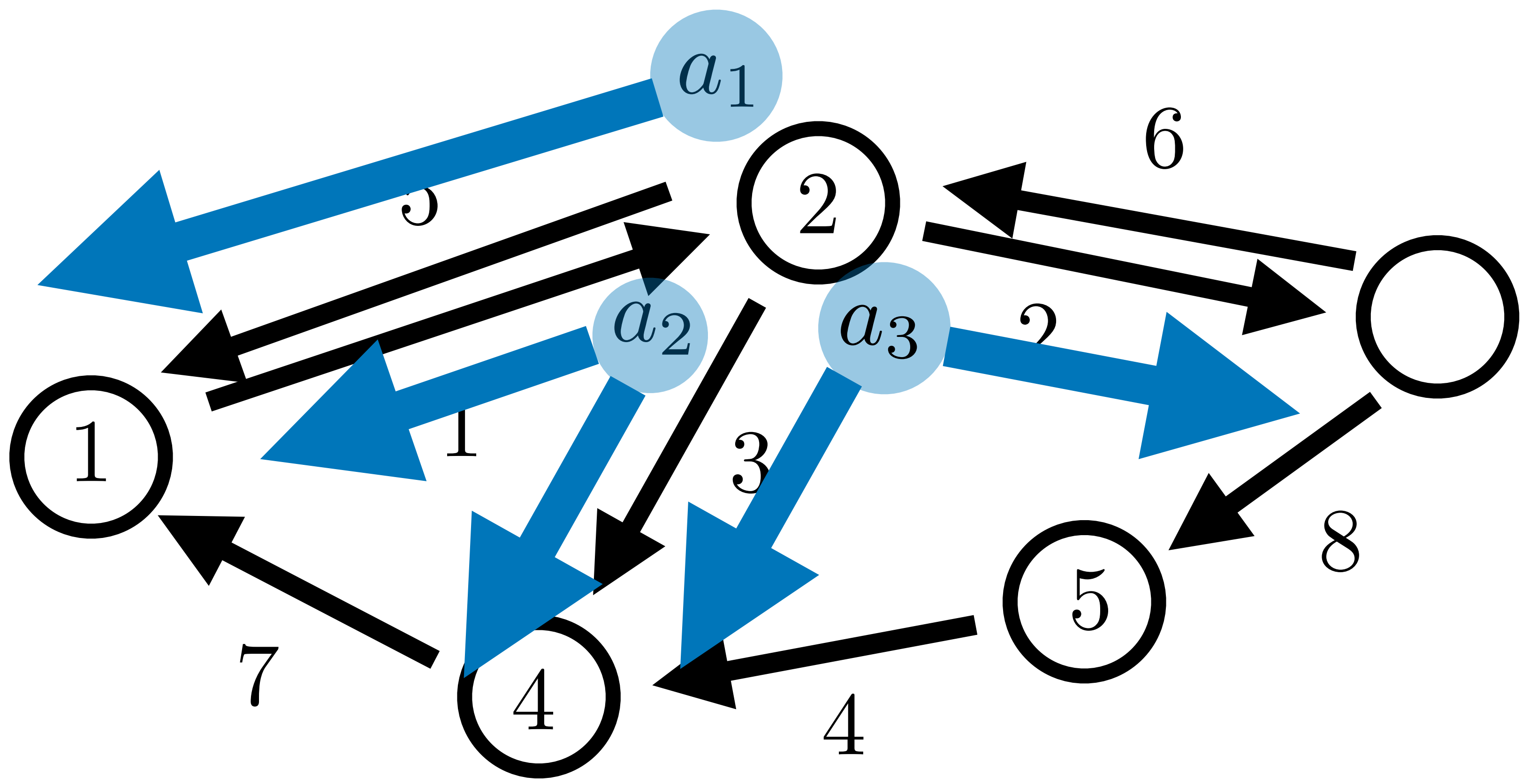

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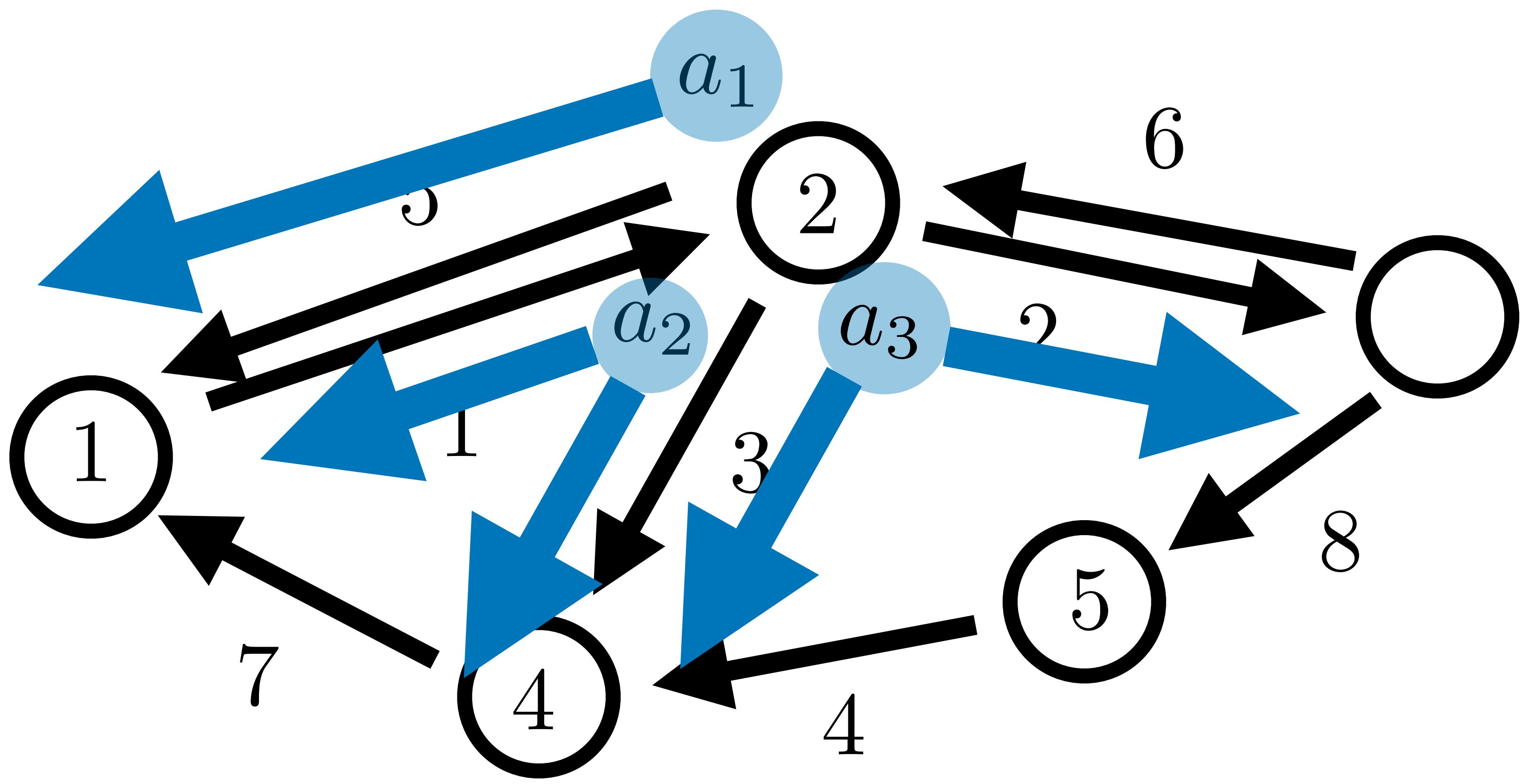

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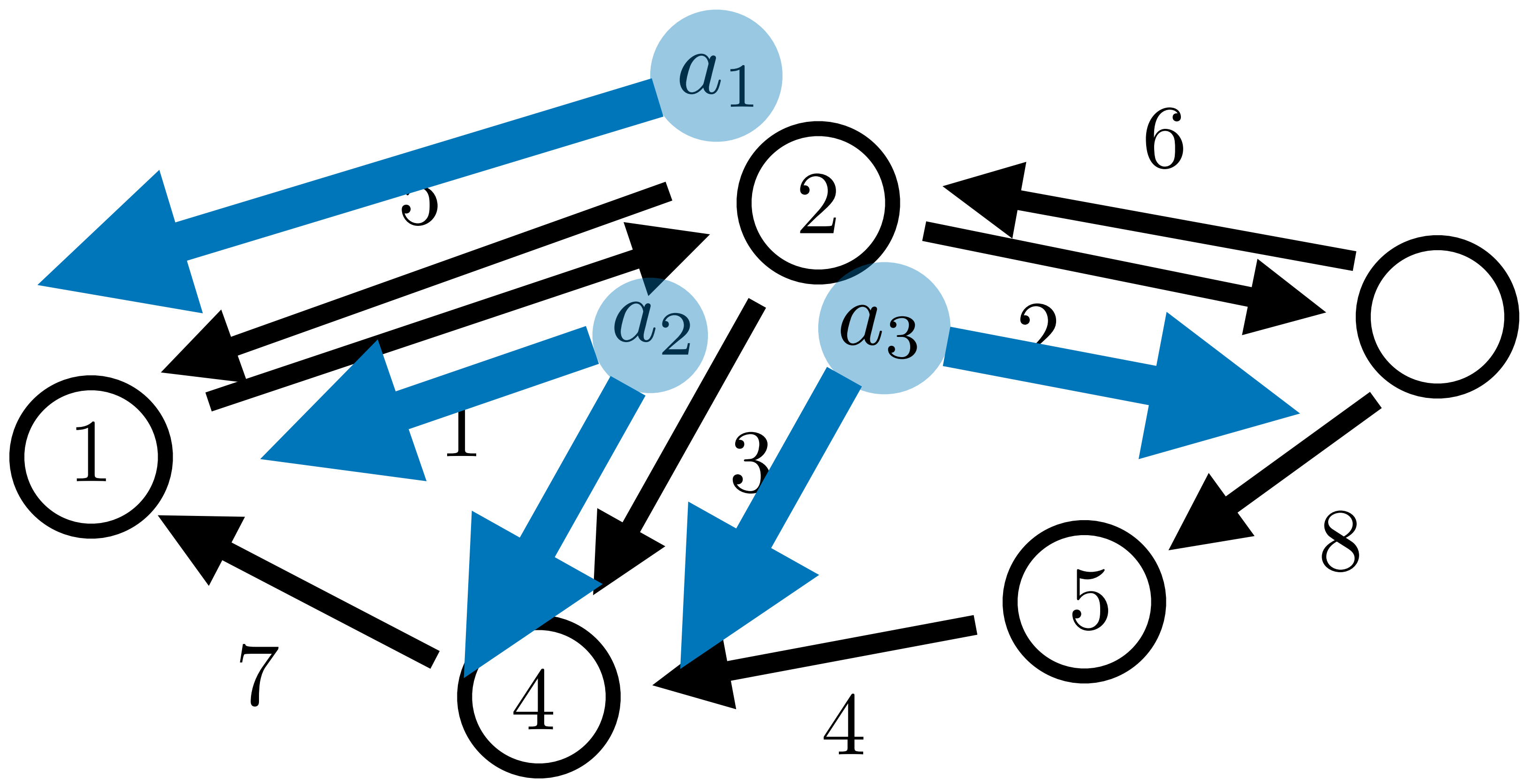

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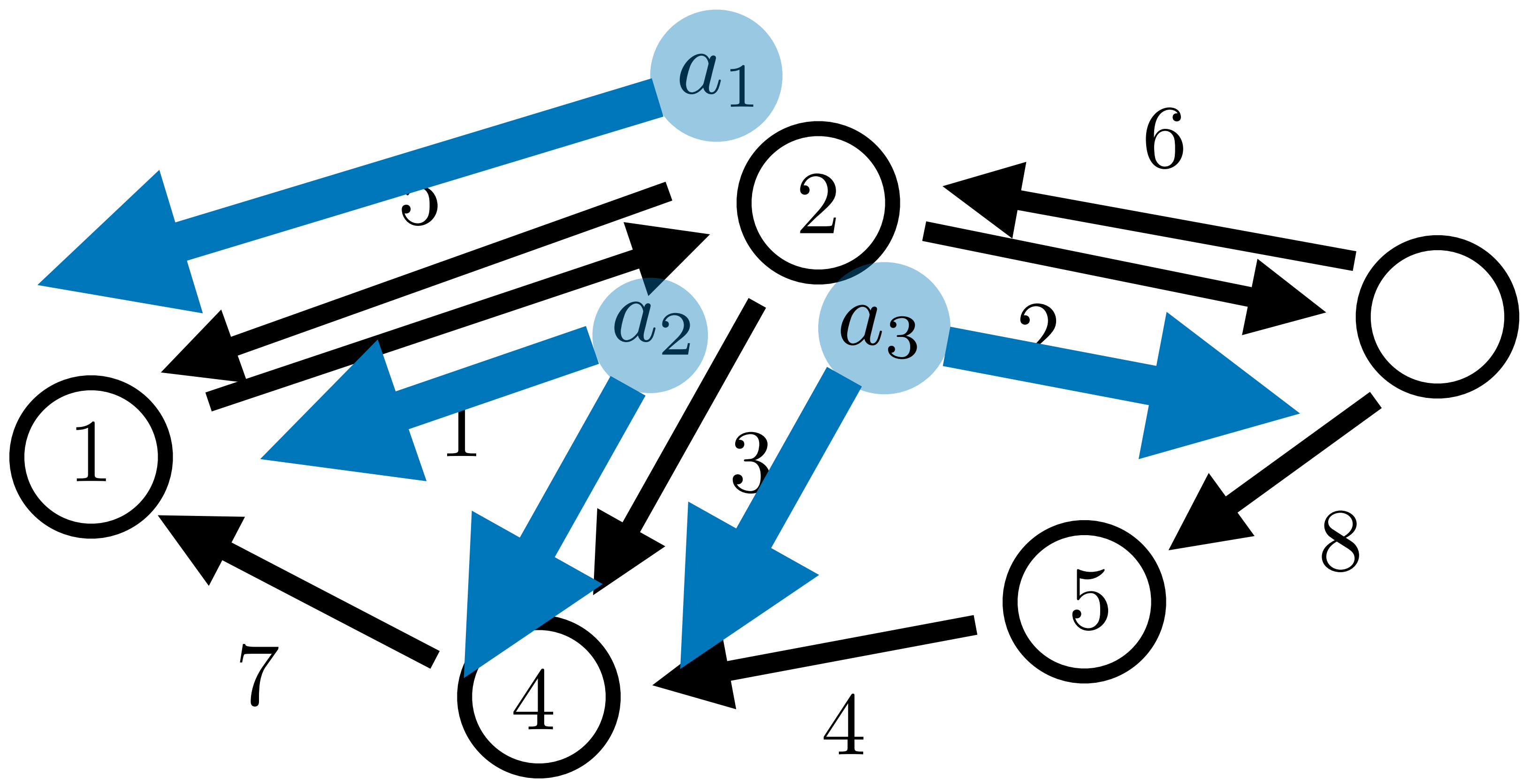
Steady-state (state) distribution $z = Mz$ ...corresponding right eigenvector

$$\left[\,I - M\right]z = \left[E_{\mathcal{A}} - P\right]\Pi z = \left[E_{\mathcal{A}} - P\right]x = 0$$

Steady-state (state-action) distribution

Markov Chains



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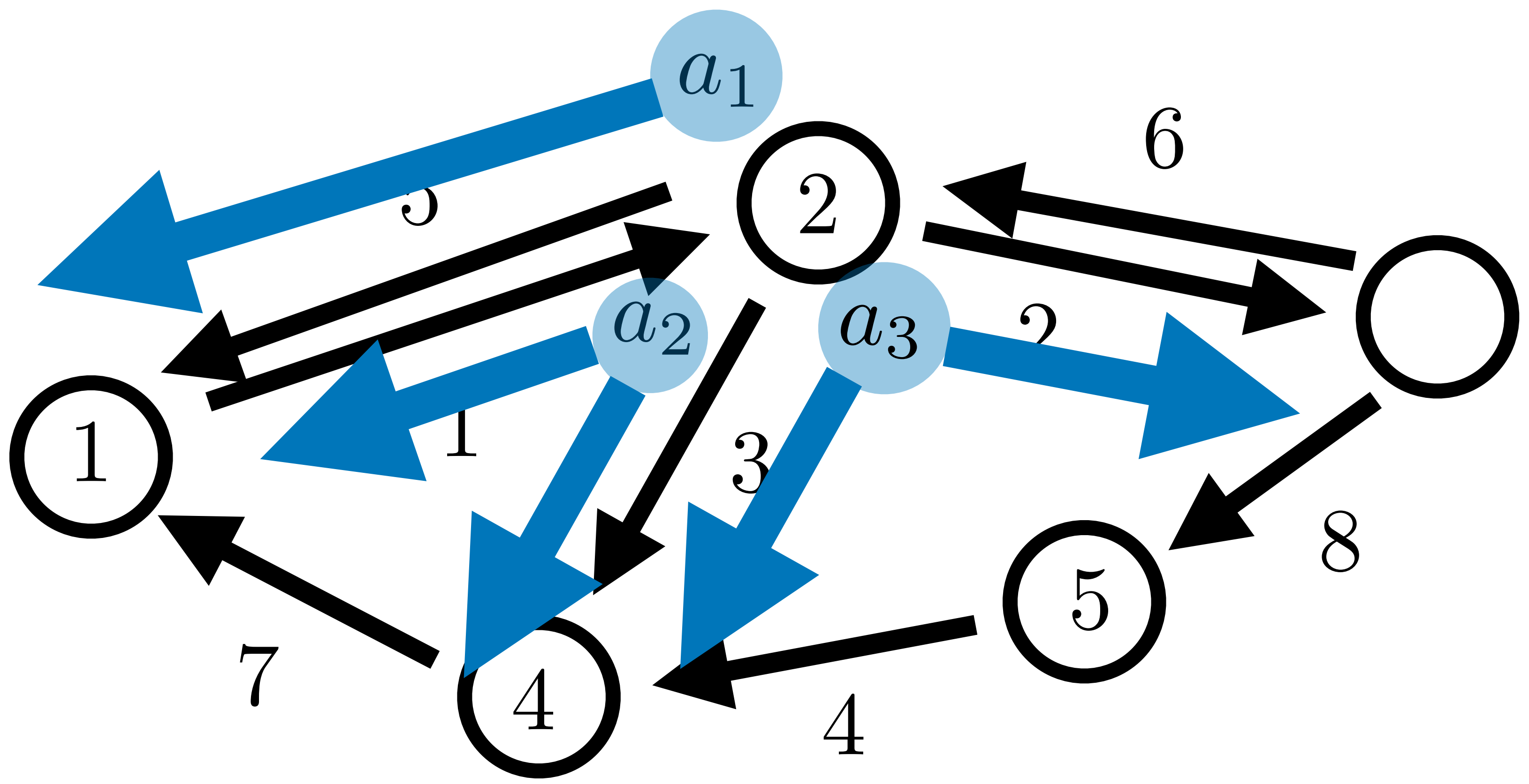

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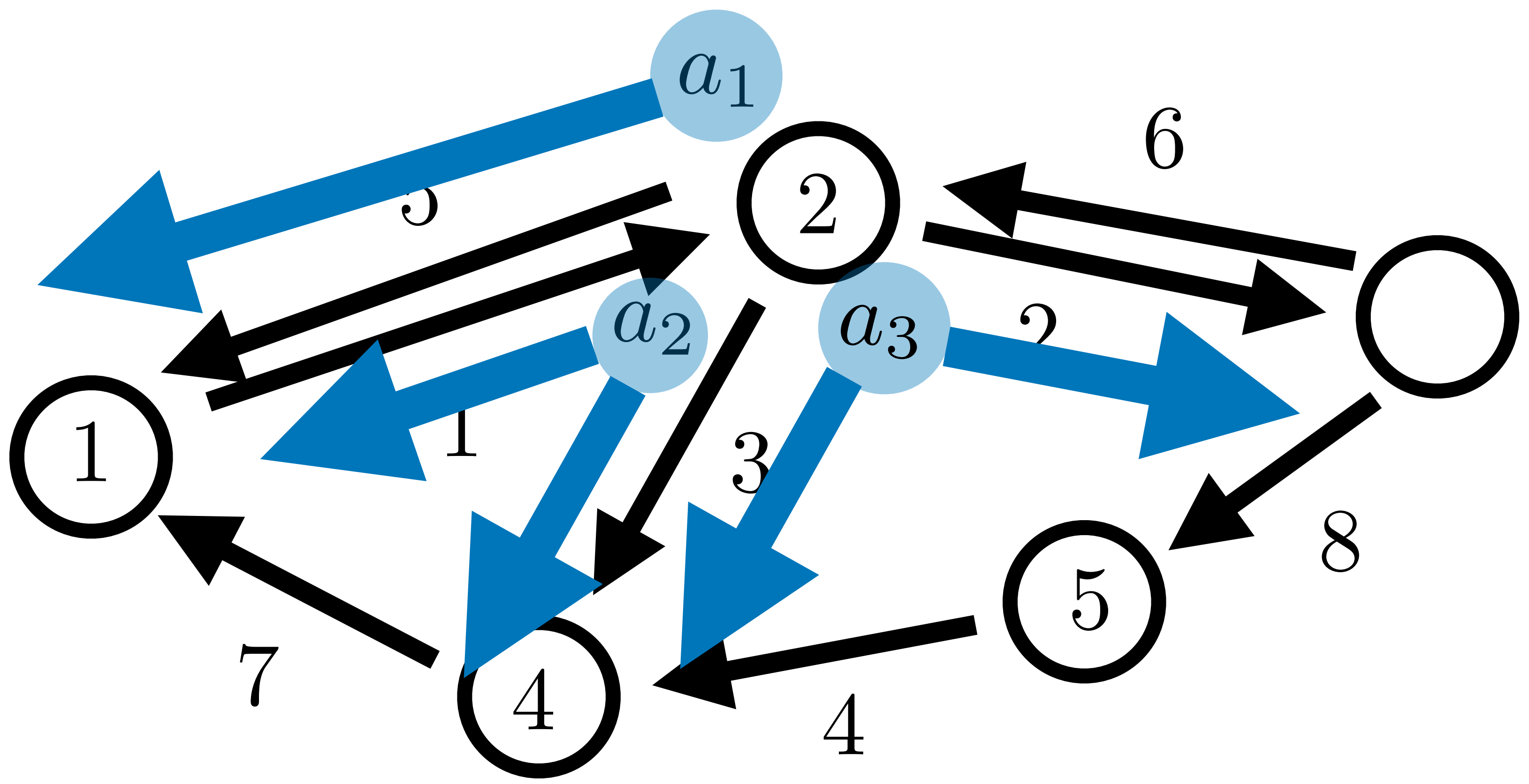

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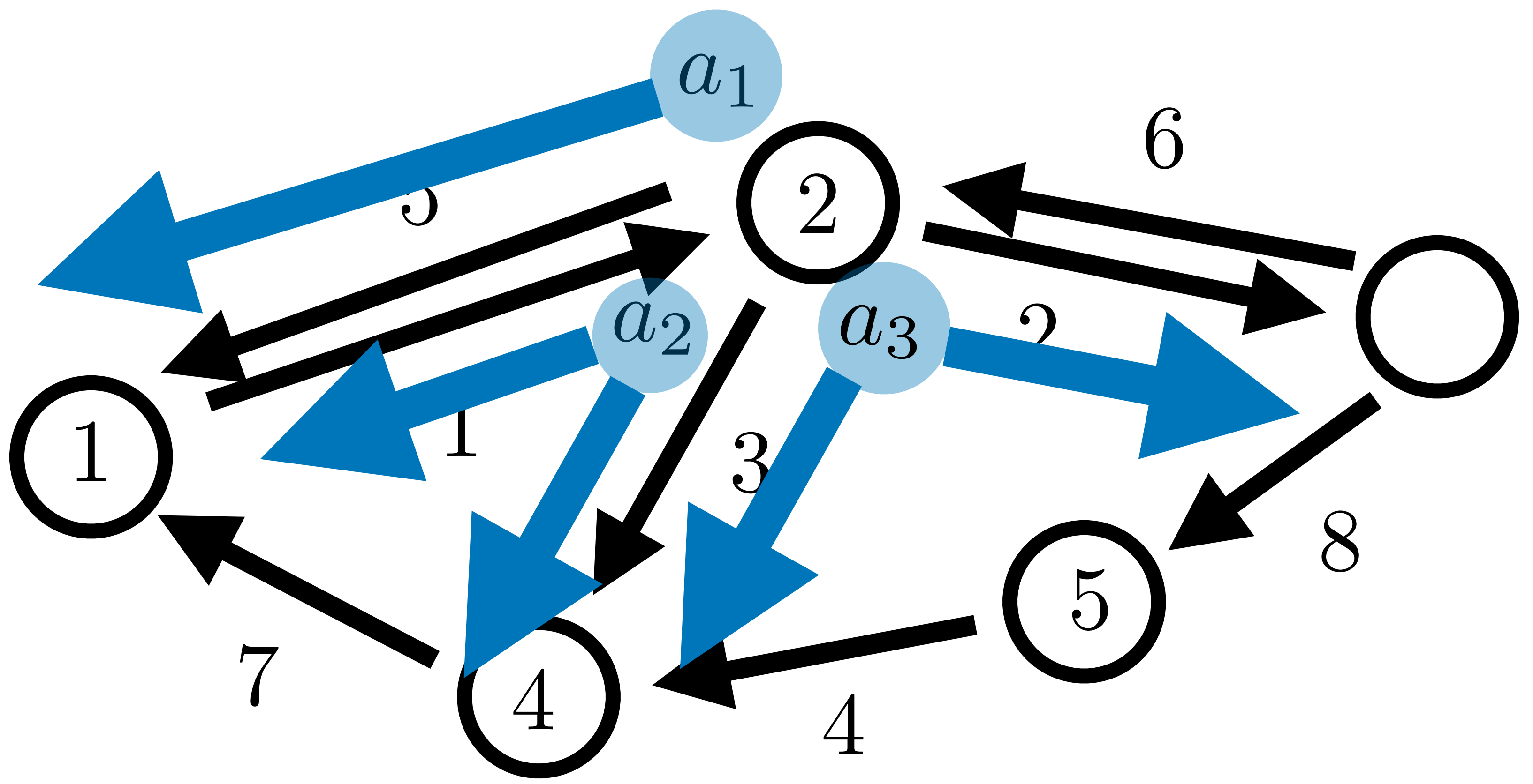

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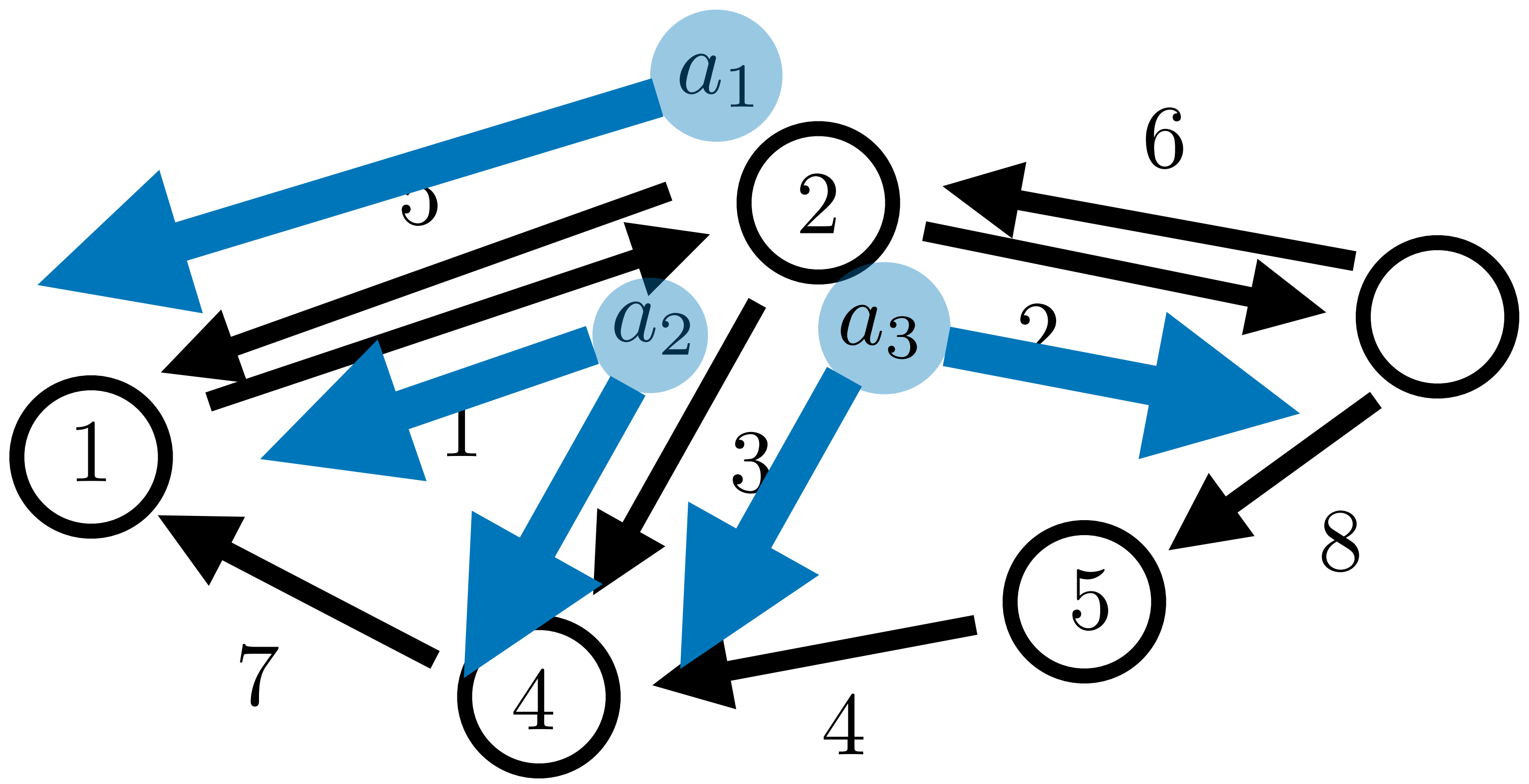

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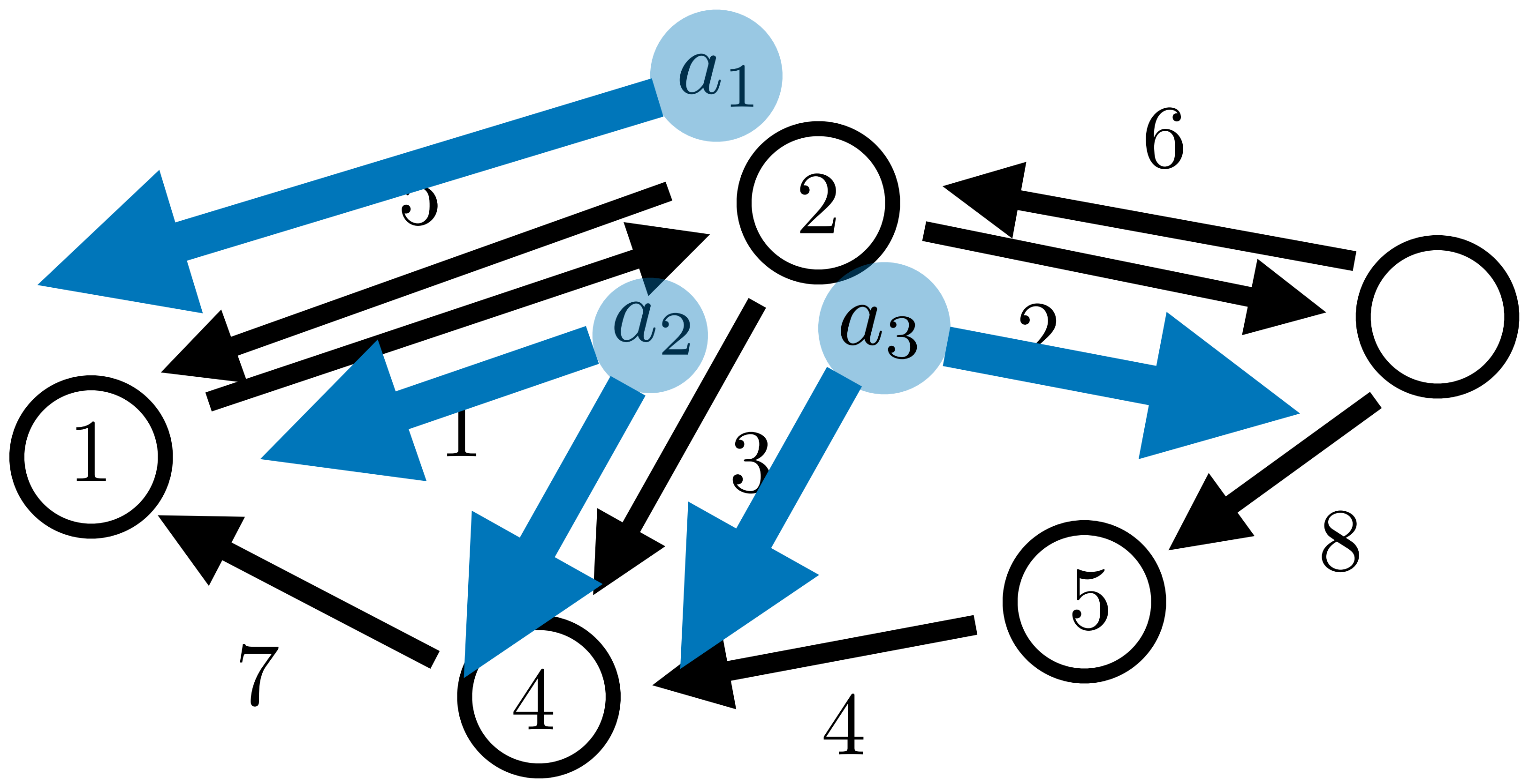

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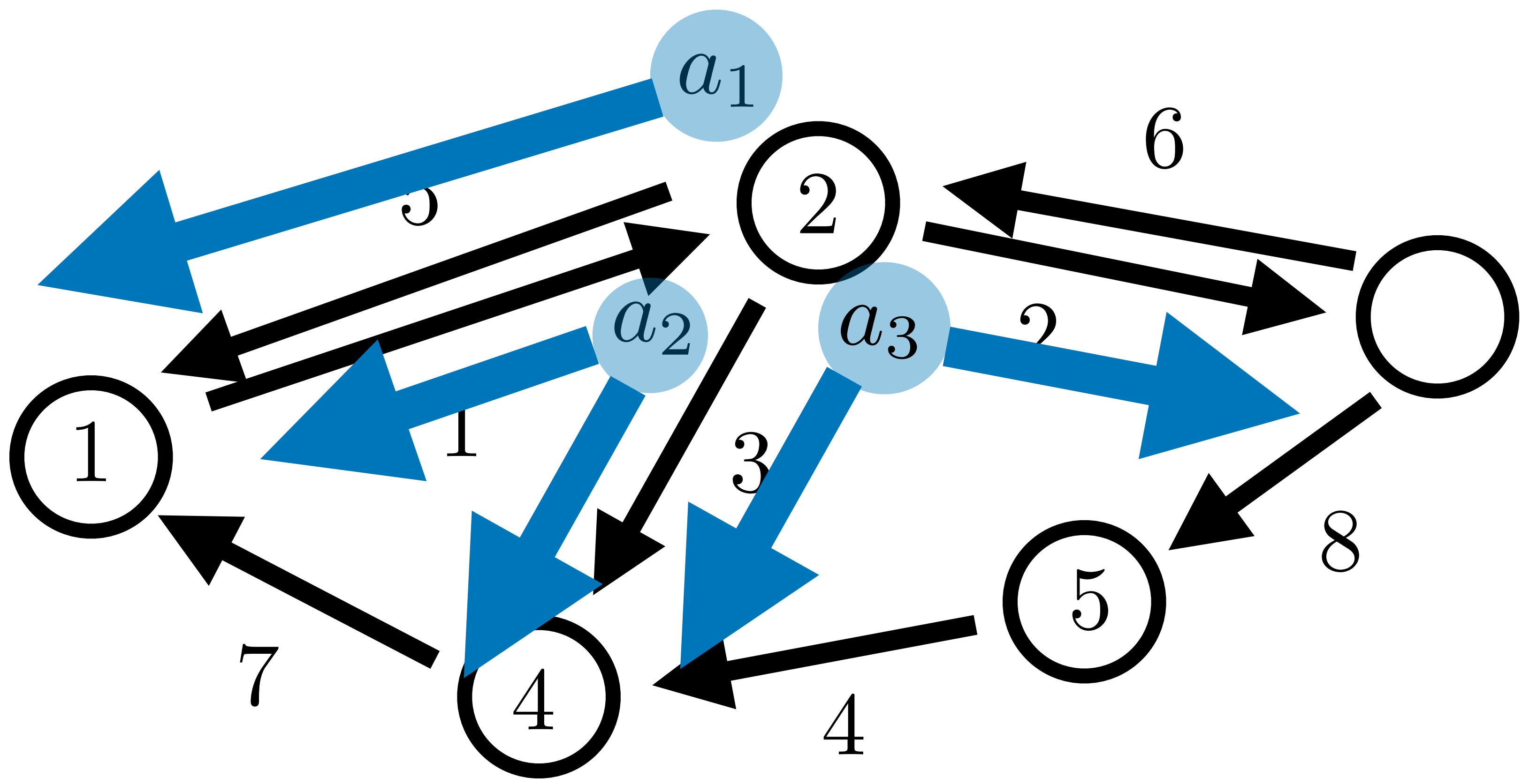

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$P \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{A}|}$

$W \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{A}|}$

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Policy

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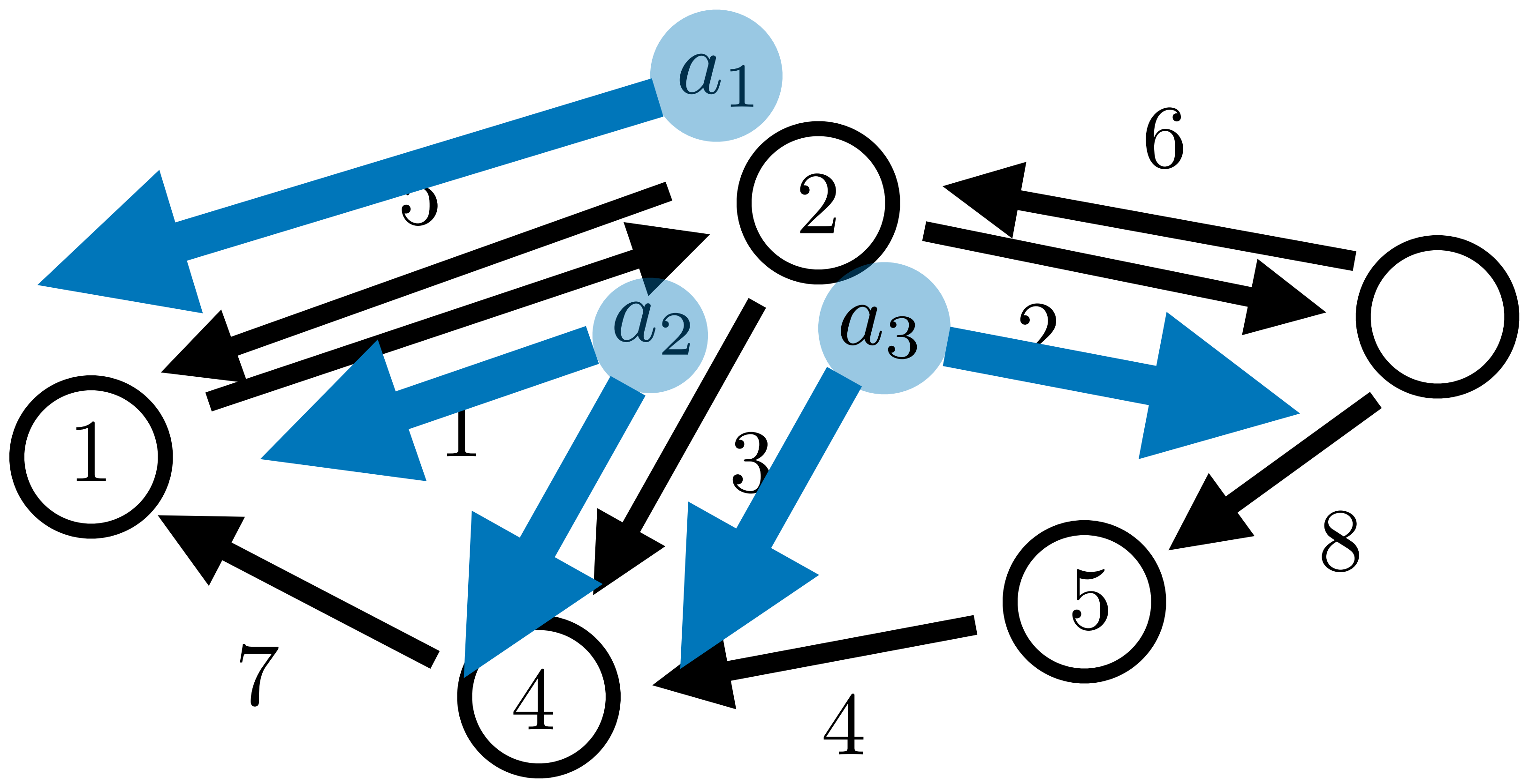

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Steady state constraint with discount

$$\left[\, E_{\mathcal{A}} - \gamma P \,\right] x = \left[\, E_{\text{out}} - \gamma E_{\text{in}} \,\right] W x = (1-\gamma) z_0$$

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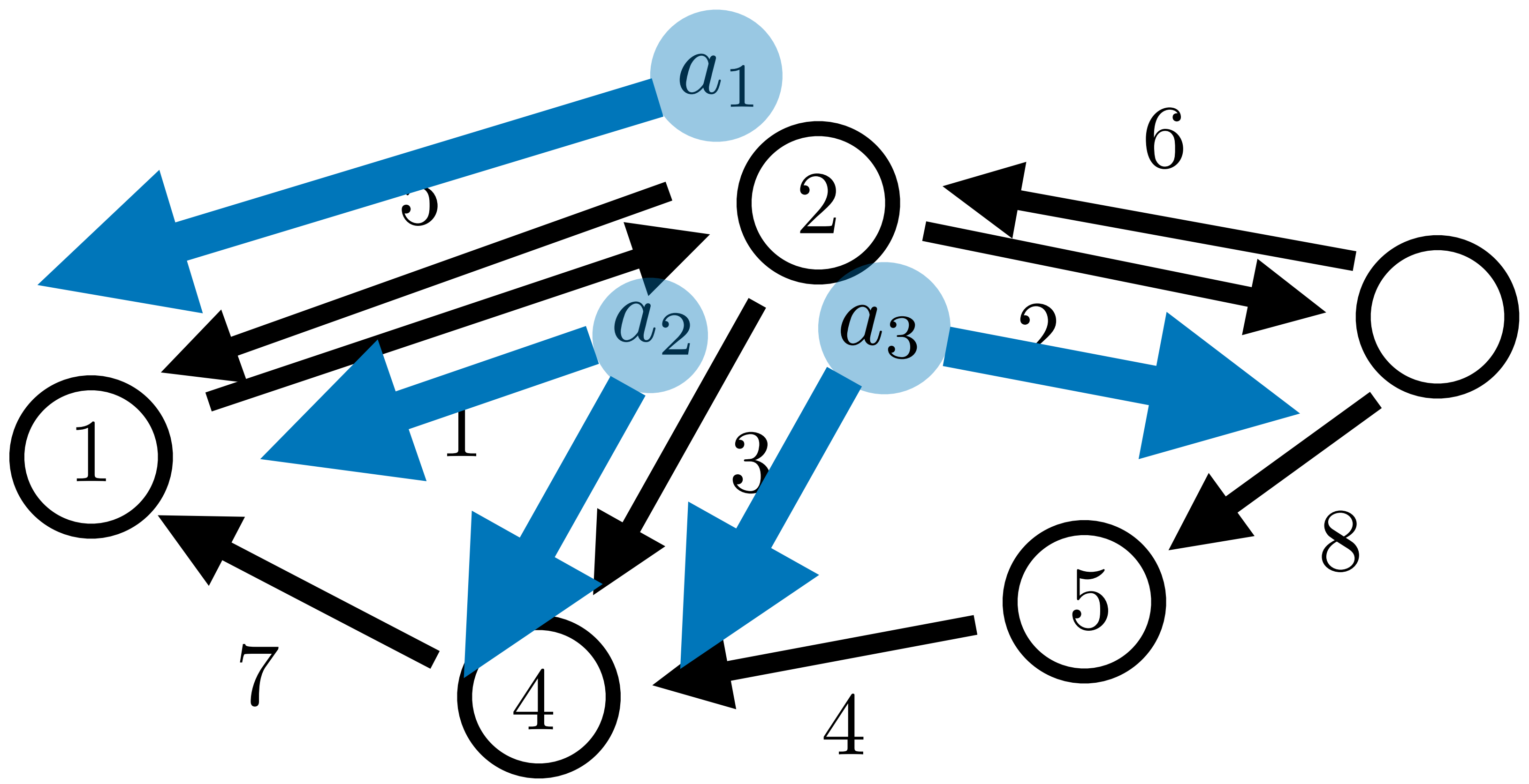

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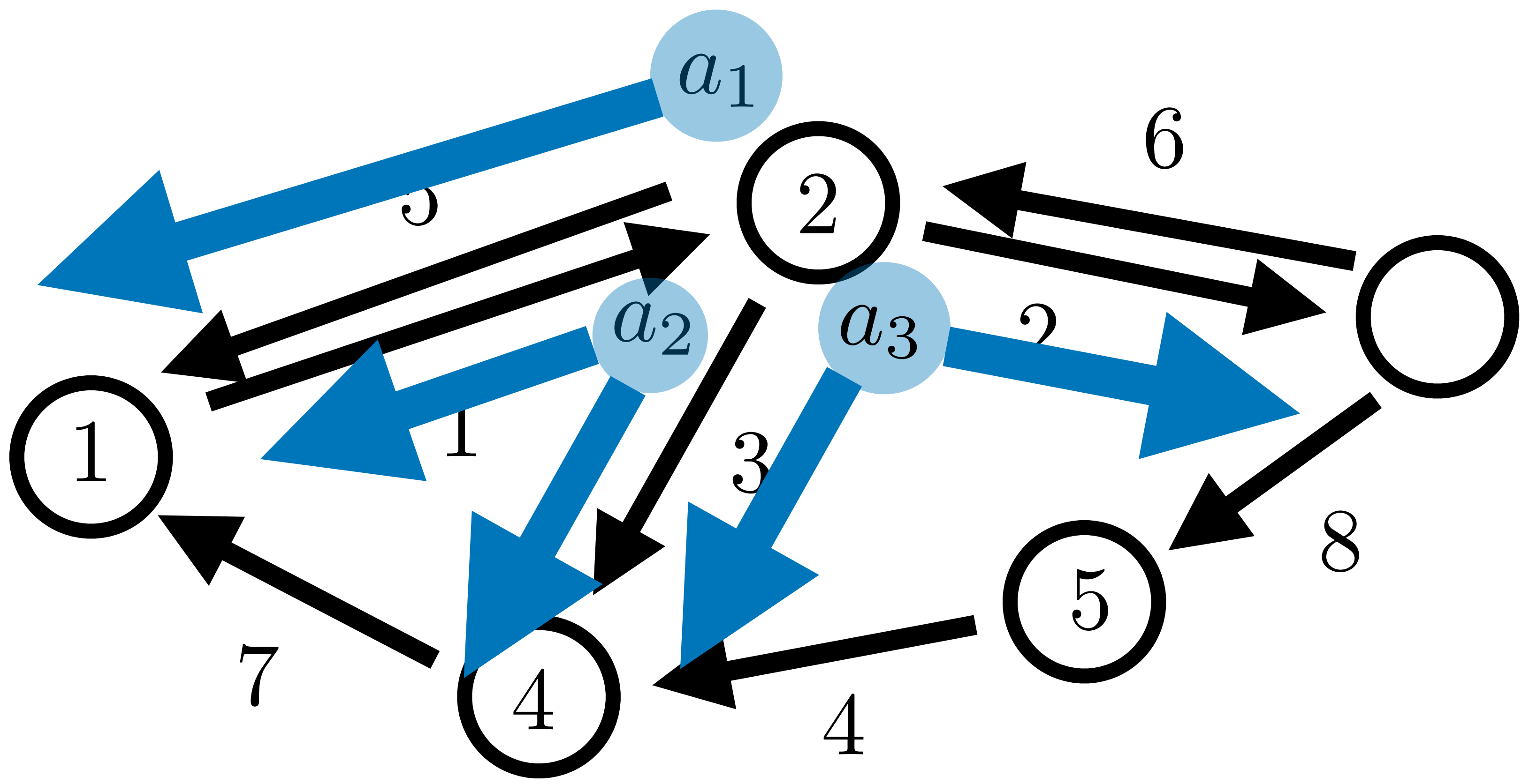

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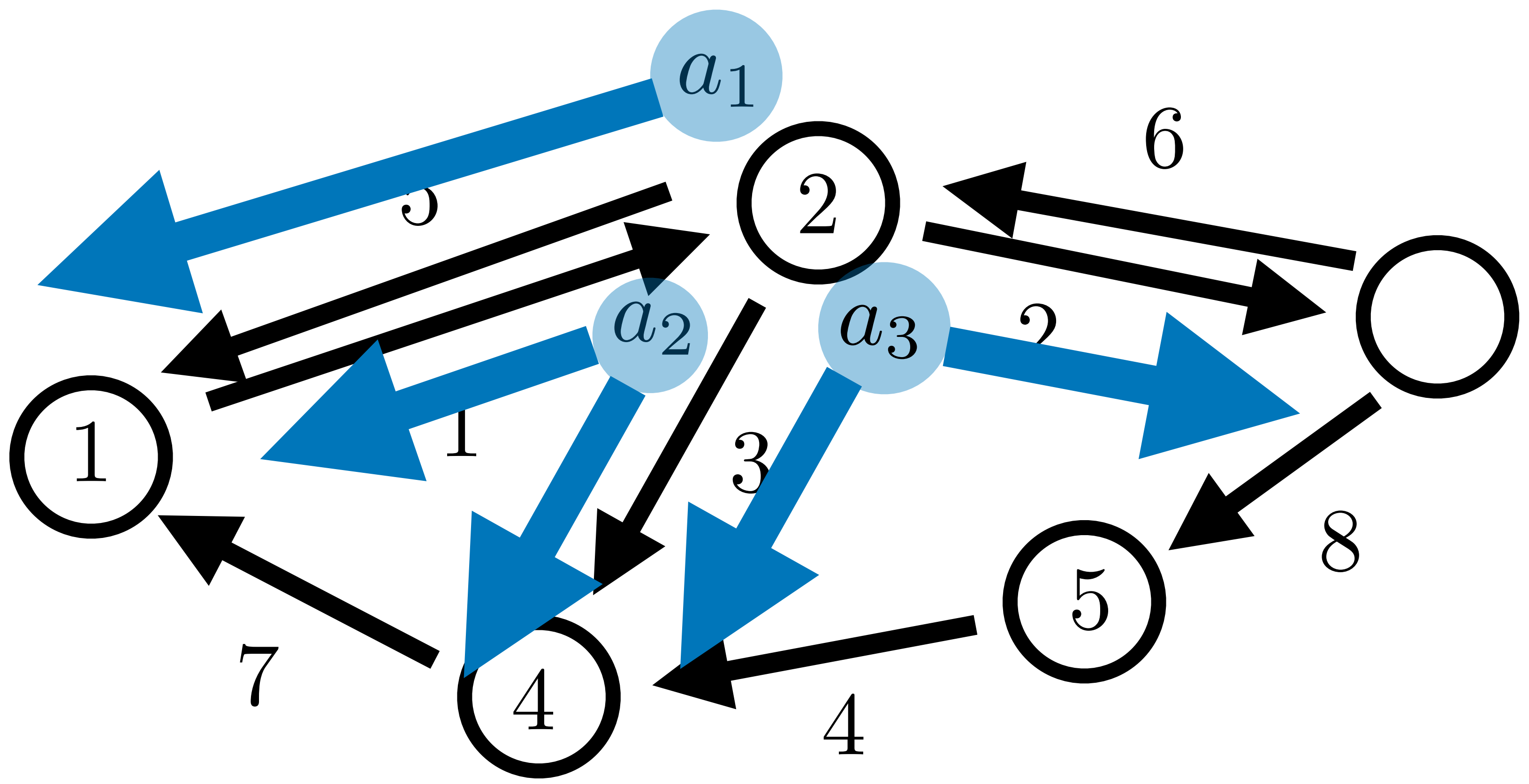

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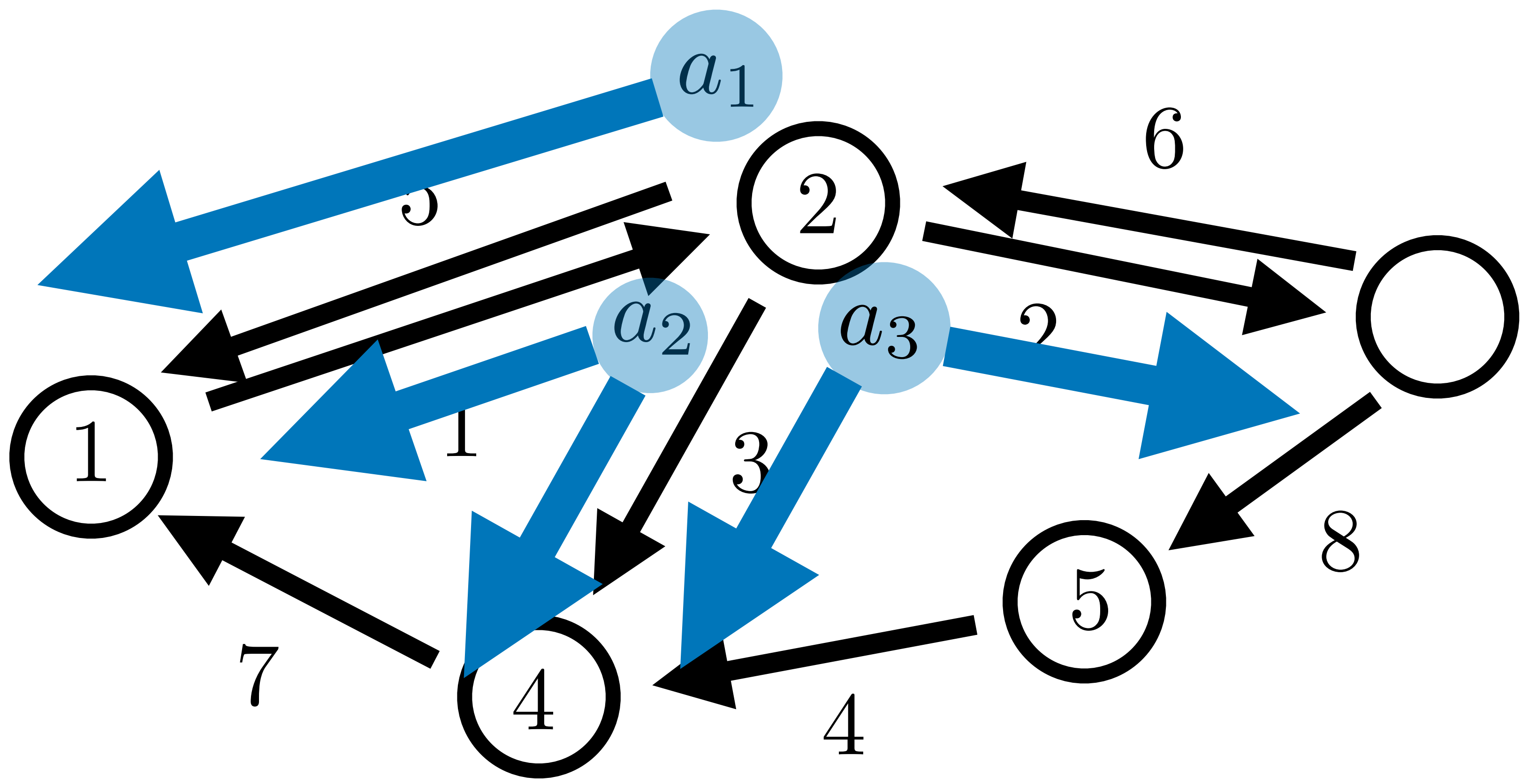
Markov Decision Processes



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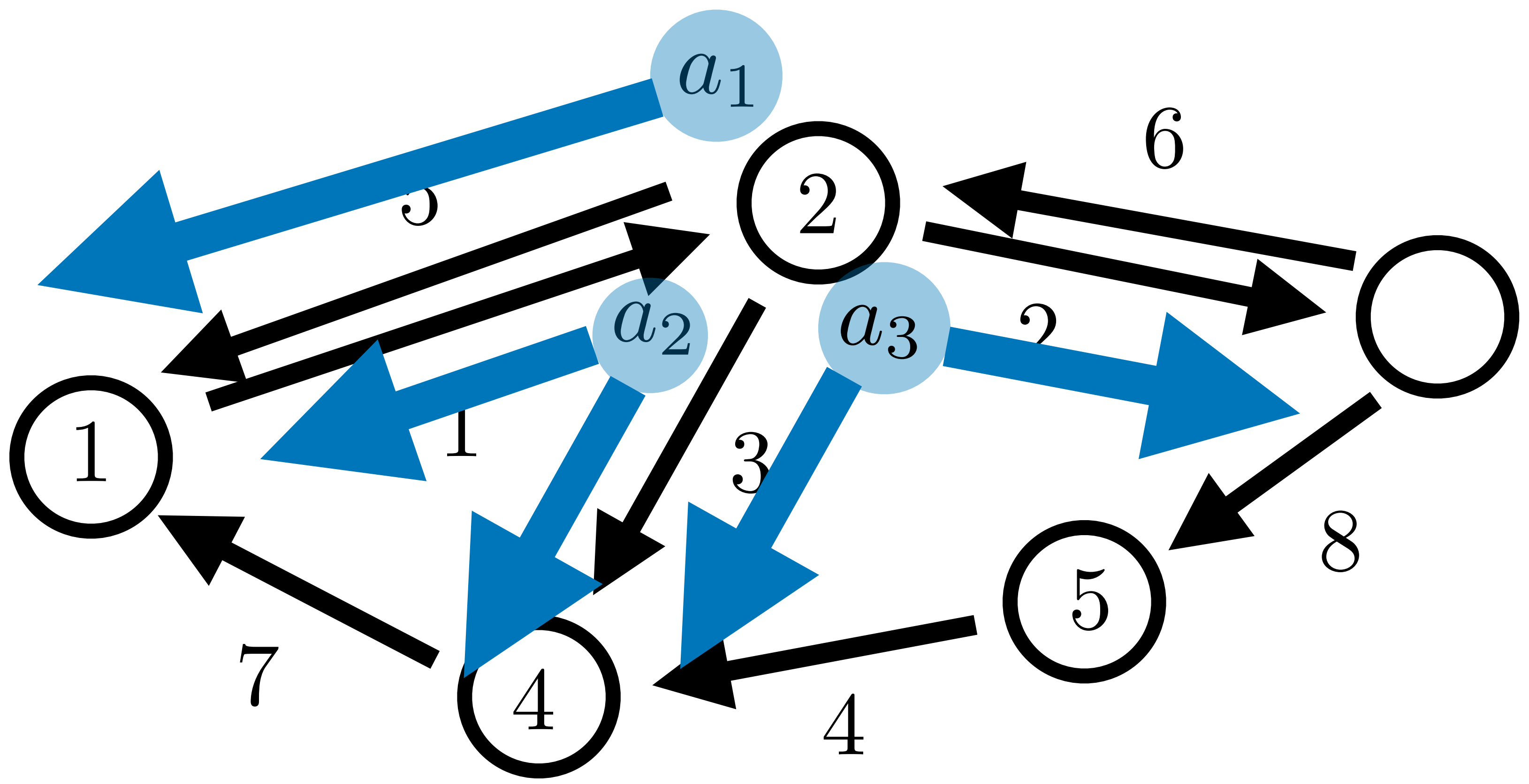
Markov Decision Processes



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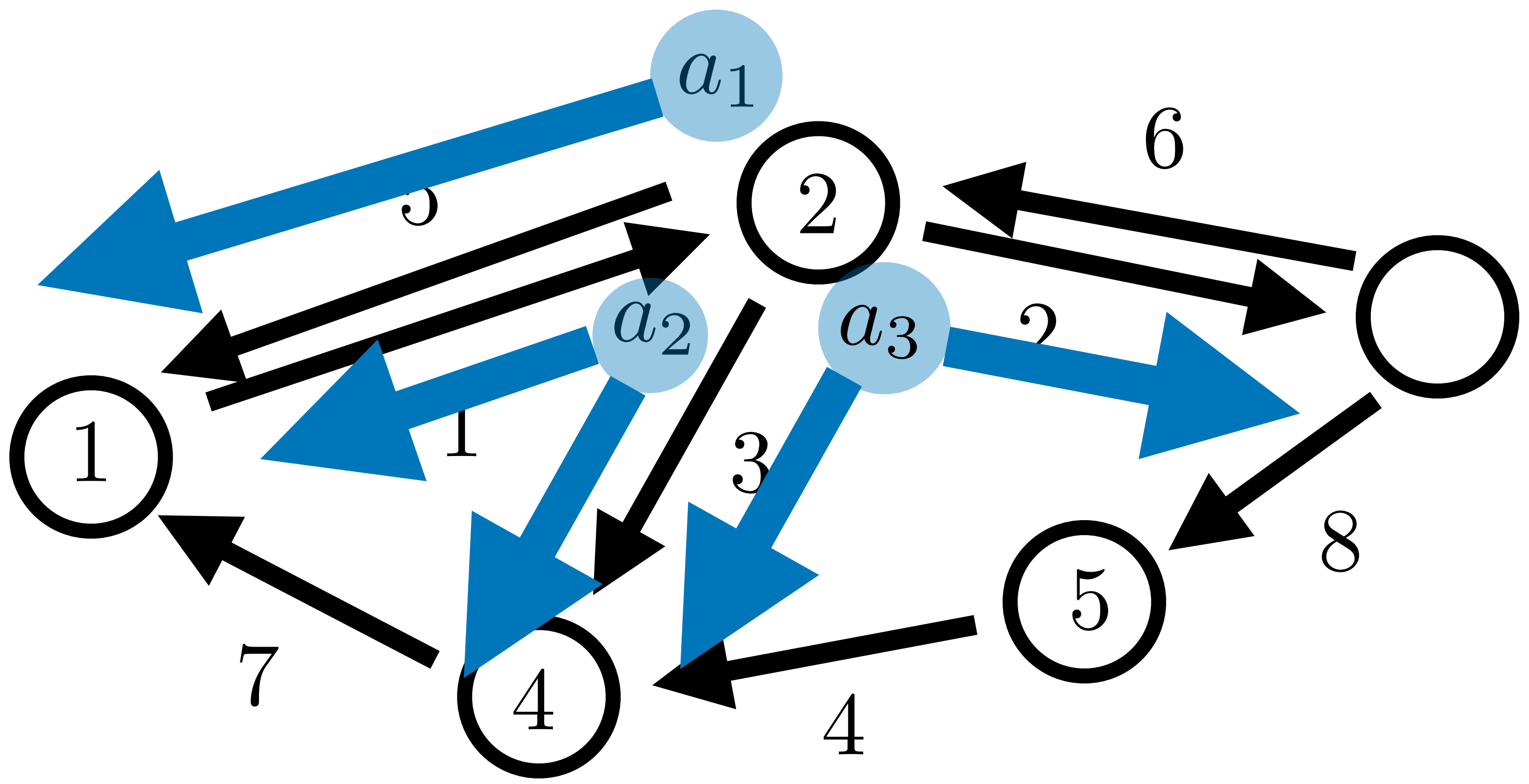
$$\mathbf{1}^T x = 1, \quad \lambda \in \mathbb{R} \qquad x \geq 0 \quad \mu \in \mathbb{R}^{|\mathcal{A}|}_+$$

Markov Decision Processes



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$$\max_x \quad r^T x$$

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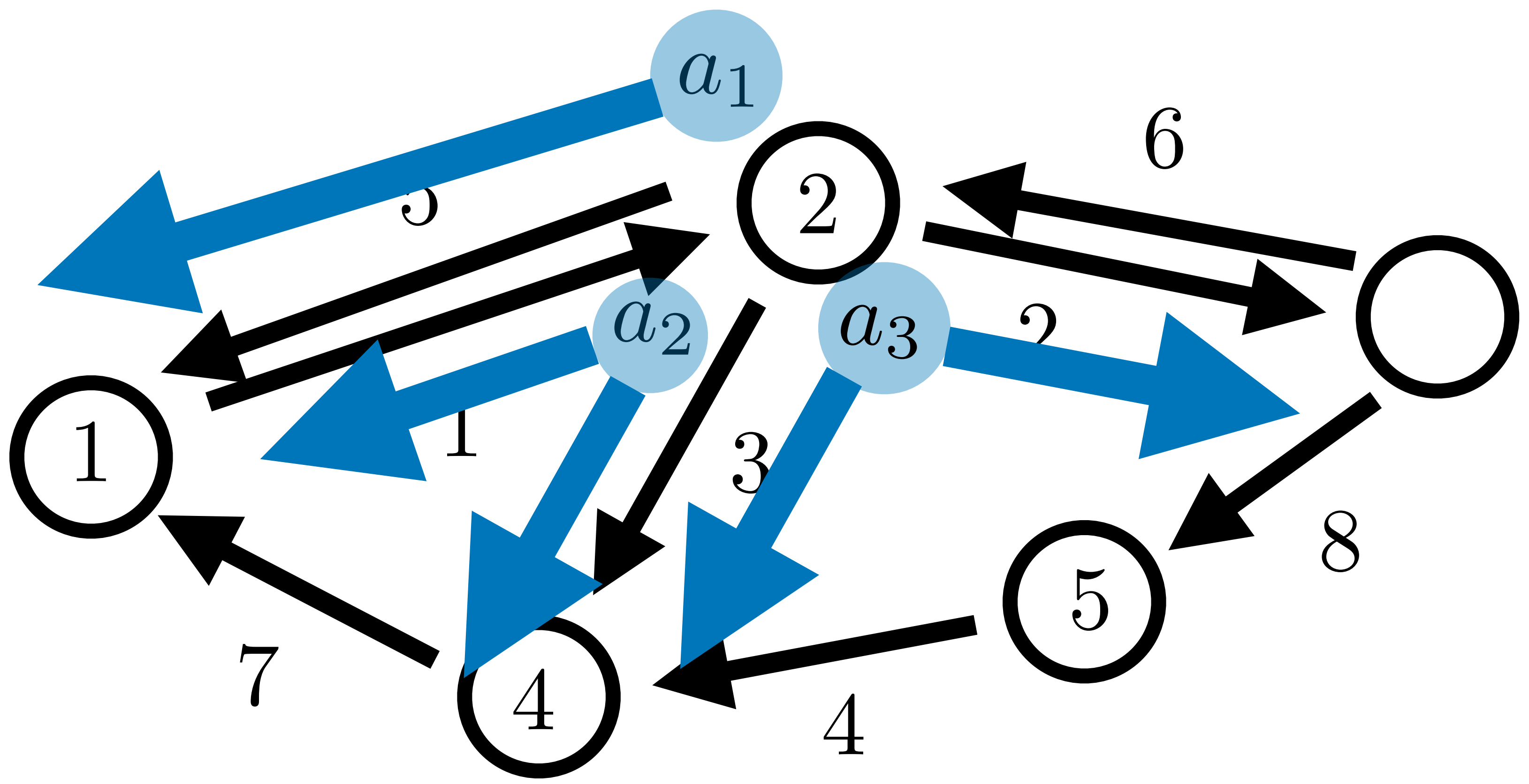
$\lambda \in \mathbb{R}$ **optimal average reward**

Markov Decision Processes



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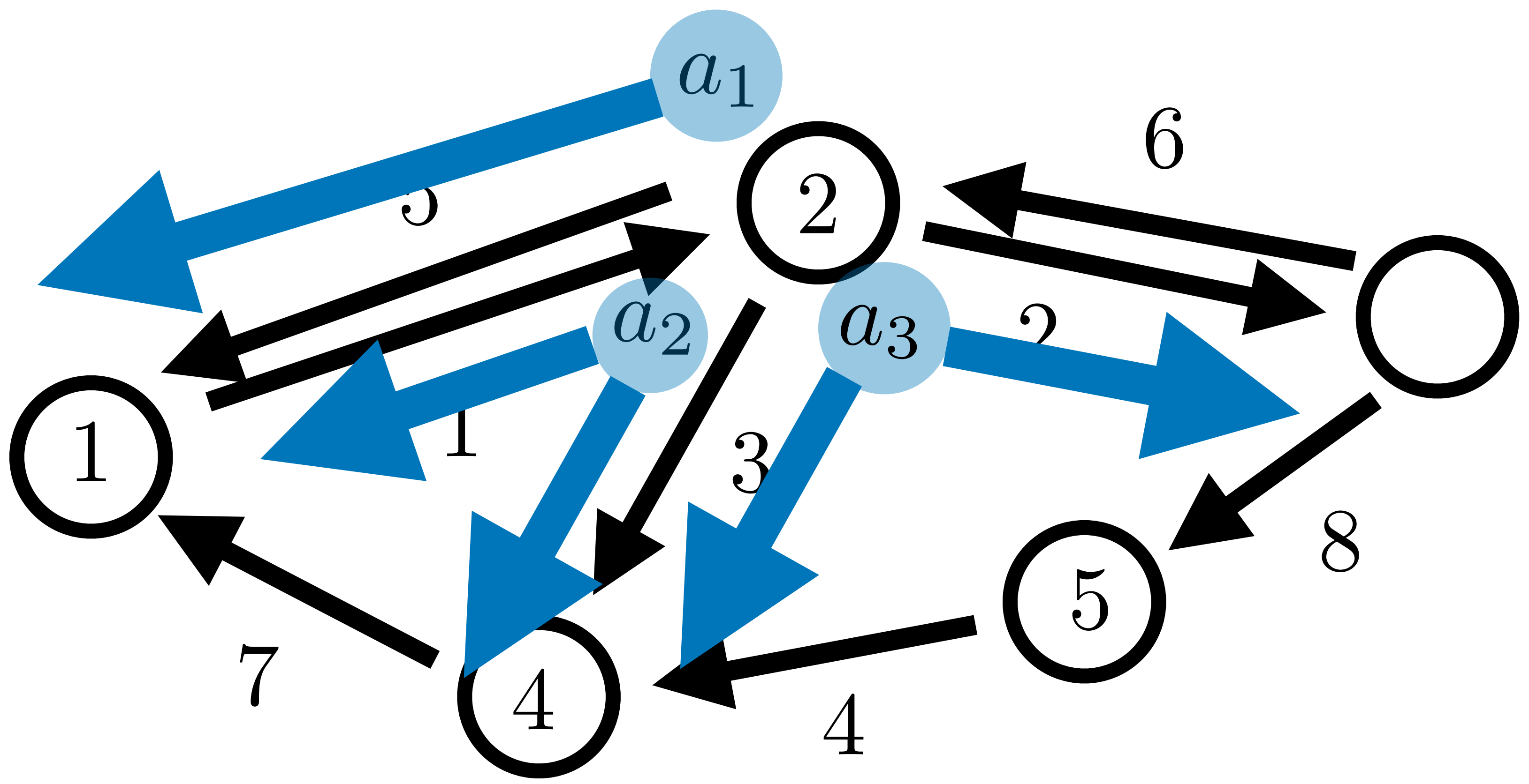

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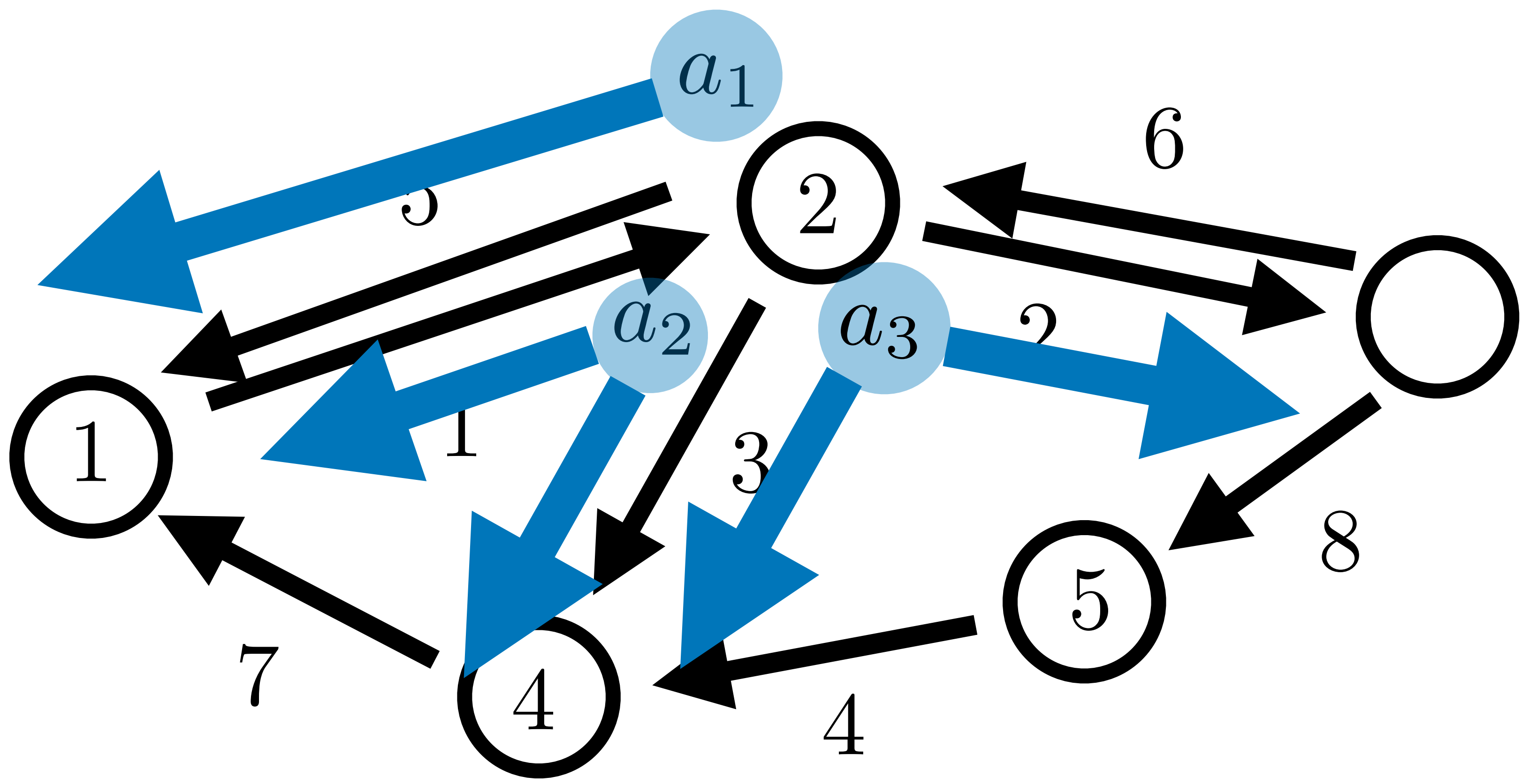
Markov Decision Processes



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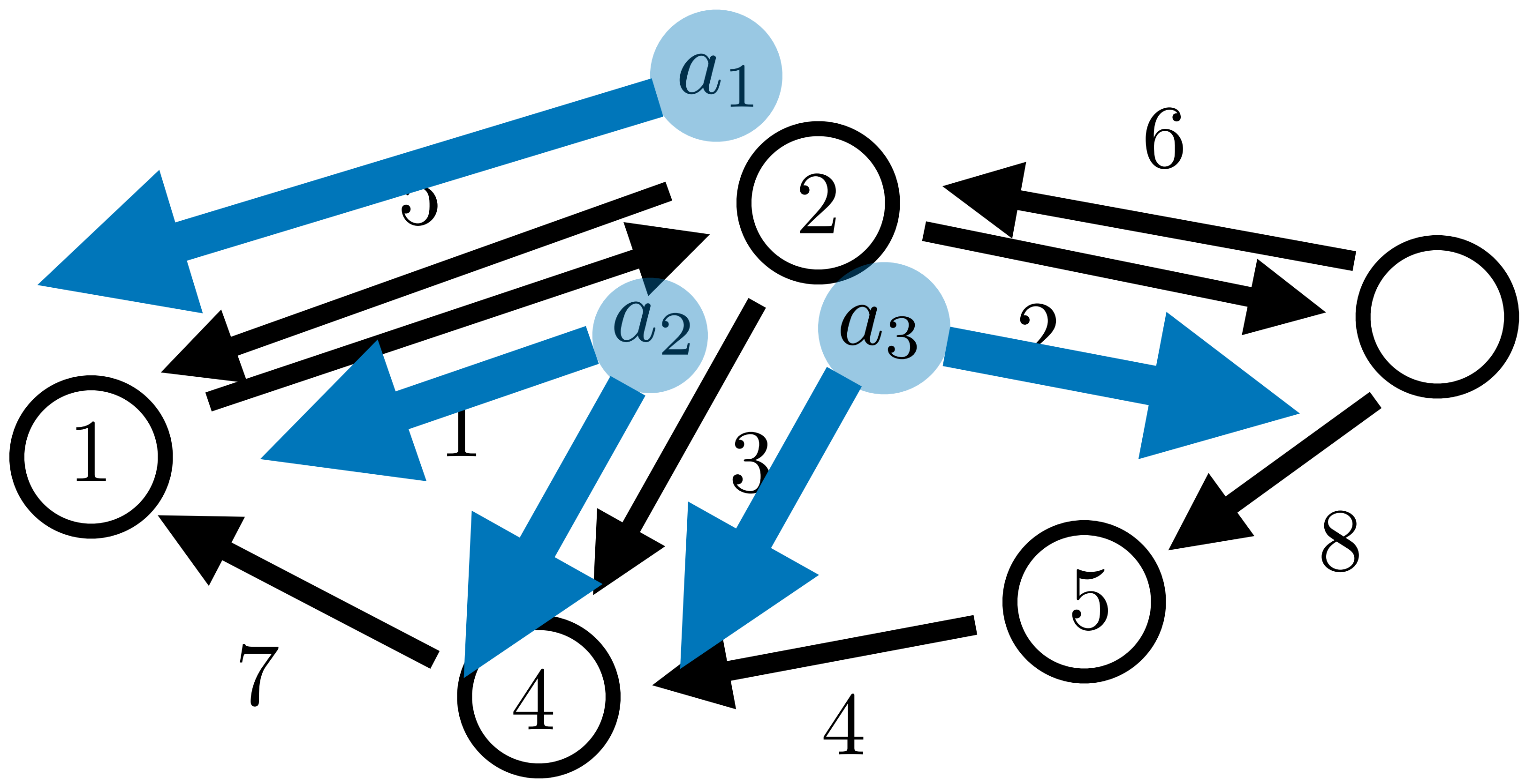
Complementary slackness $\mu^T x = 0$

"No inefficient action is used"

Markov Decision Processes



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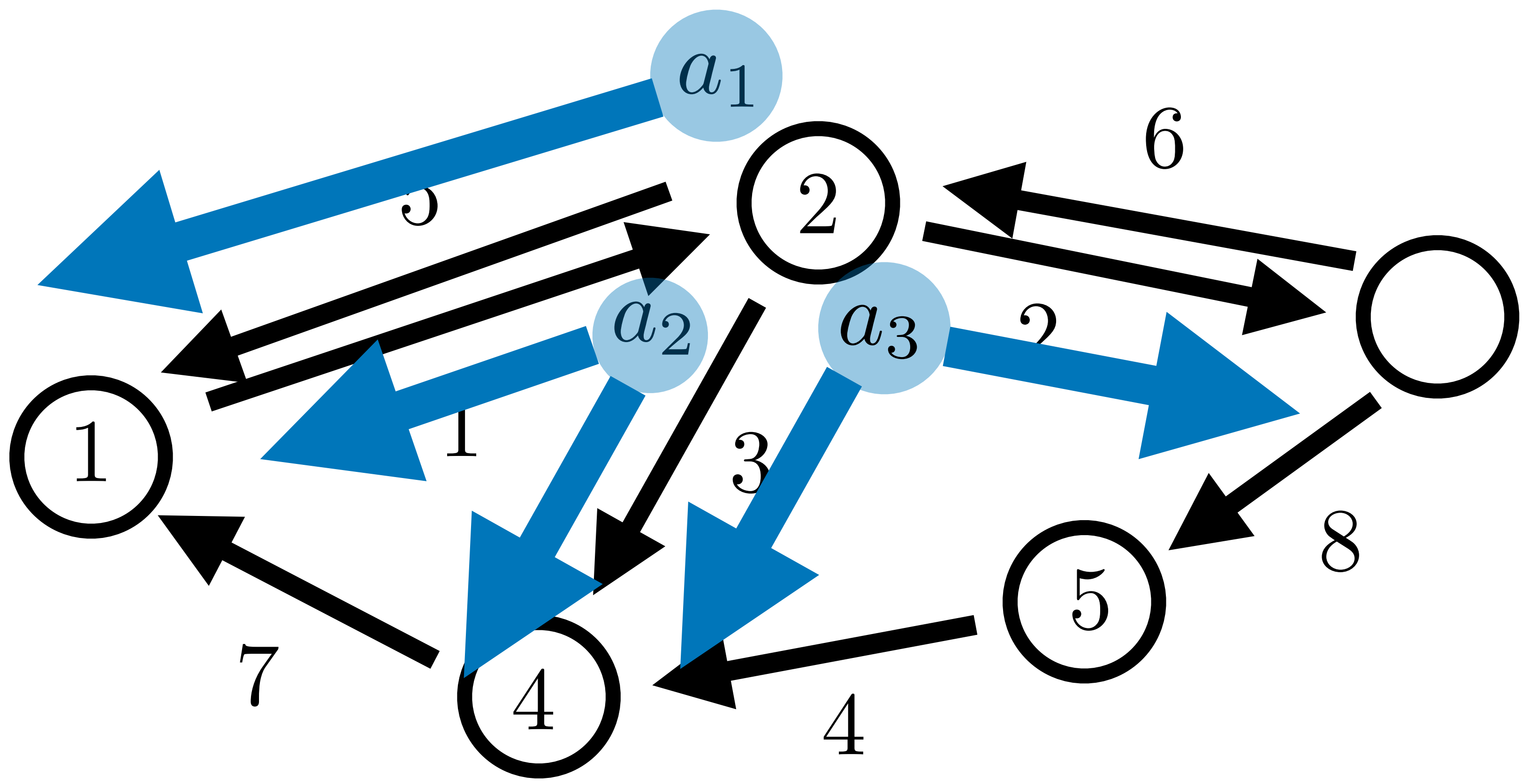
Markov Decision Processes



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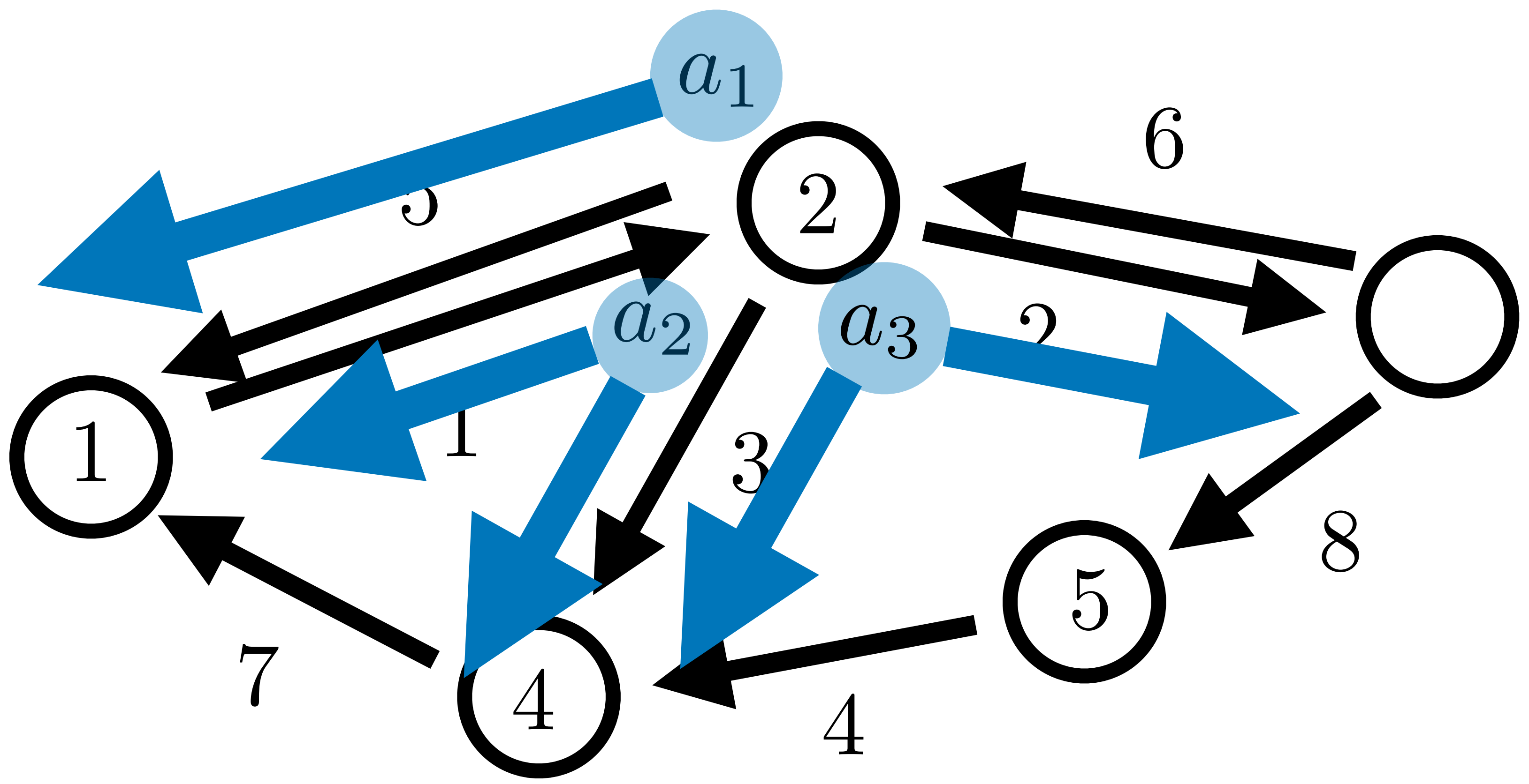

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Optimization of Discounted Rewards

$$\max_x \quad r^T x = \sum_{t=0}^{\infty} \gamma^t r^T x_t$$

$$\text{s.t.} \quad \left[\, E_{\mathcal{A}} - \gamma P \,\right] x = \left[\, E_{\text{out}} - \gamma E_{\text{in}} \,\right] W x = (1-\gamma) z_0 \qquad v \in \mathbb{R}^{|\mathcal{S}|}$$

$$\mathbf{1}^T x = 1, \quad \lambda \in \mathbb{R} \qquad x \geq 0 \quad \mu \in \mathbb{R}_+^{|\mathcal{A}|}$$

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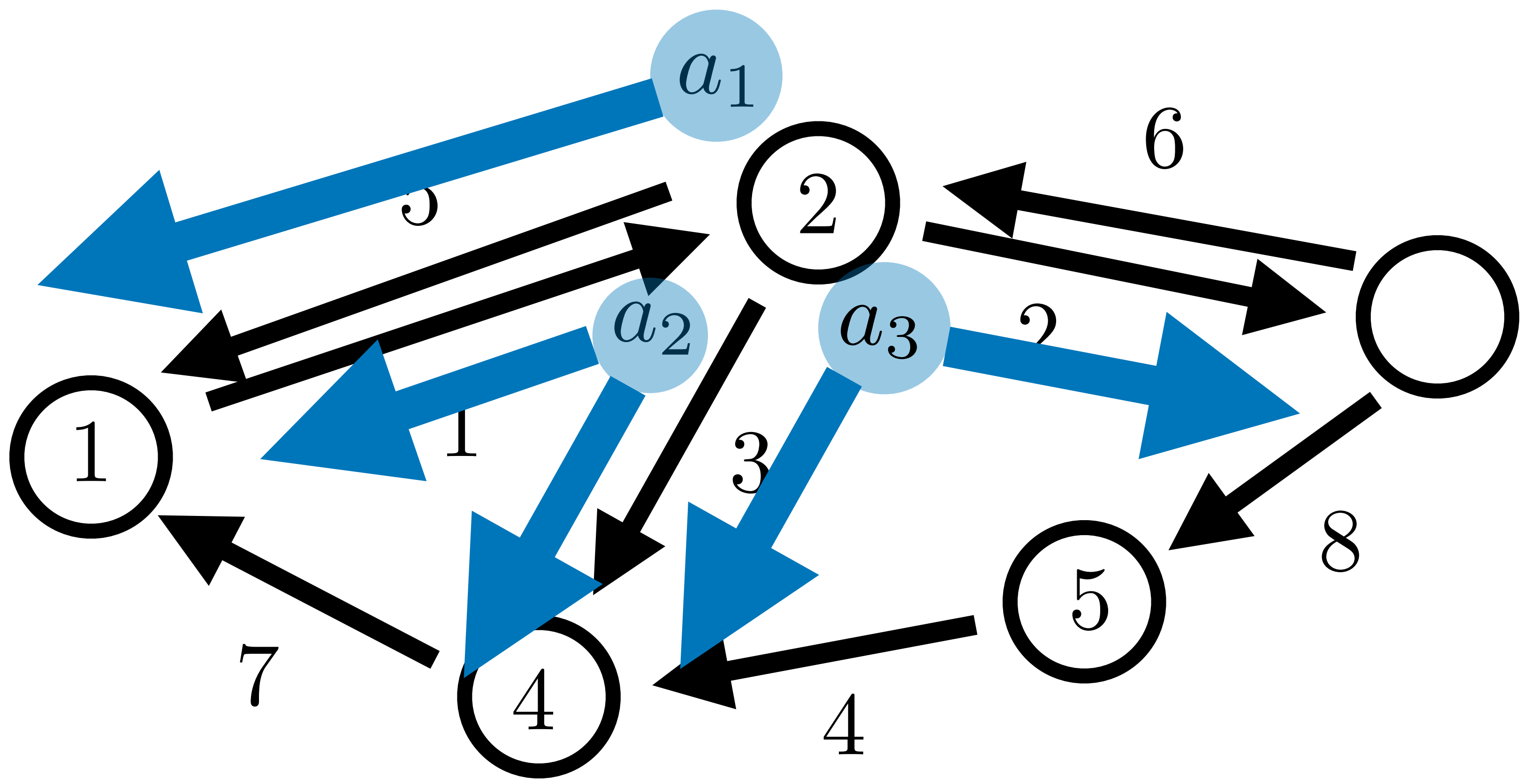

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$P = E_{\text{in}} W$

$M = P\Pi$

$I = E_{\mathcal{A}}\Pi$

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$ **States** $s \in \mathcal{S}$ $\mathcal{V} = \mathcal{S}$

Edges $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrices

$E \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$ $E = E_{\text{in}} - E_{\text{out}}$

$$[\ E\ ]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

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Markov Decision Process

Actions $a \in \mathcal{A}$ total actions $\mathcal{A} = \cup_{s\in\mathcal{S}}\ \mathcal{A}_s$

$a \in \mathcal{A}_s$ actions from ea. state

For each action: $\text{Prob}(s'|s,a)$ Probability of transitioning to state s' from state s

Transition Kernel

$P \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$

$W \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{A}|}$

$$[\ E_{\mathcal{A}}\ ]_{sa} = \begin{cases} 1 & ;\ \text{if } a \in \mathcal{A}_s \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$[\ P\ ]_{s'a} = \begin{cases} \text{Prob}(s'|s,a); & \text{prob. of trans from } s \text{ to } s' \text{ given } a \\ 0; & \text{otherwise} \end{cases}$$

$$[\ W\ ]_{ea} = \begin{cases} \text{Prob}(e|s,a); & \text{prob. of taking edge } e \text{ from } s \text{ given } a \\ 0; & \text{otherwise} \end{cases}$$

Policy

$\Pi \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{S}|}$

$M \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{S}|}$

$$[\ \Pi\ ]_{as} = \begin{cases} \text{Prob}(a|s) & ;\ \text{prob. of taking } a \text{ given being in } s \\ 0 & \quad \text{otherwise} \end{cases}$$

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$y \in \mathbb{R}^{|\mathcal{E}|}$	mass distribution on edges	$y = Wx$
$z \in \mathbb{R}^{|\mathcal{S}|}$	mass distribution on states	$z = E_{\text{out}} y = E_{\mathcal{A}} x$
$r \in \mathbb{R}^{|\mathcal{A}|}$	rewards on state-actions	

Optimization of Discounted Rewards

$$\max_x \quad r^T x \ = \sum_{t=0}^{\infty} \gamma^t r^T x_t$$

$$\text{s.t.} \quad [\ E_{\mathcal{A}} - \gamma P\ ]x = [\ E_{\text{out}} - \gamma E_{\text{in}}\ ]Wx = (1-\gamma) z_0 \qquad v \in \mathbb{R}^{|\mathcal{S}|}$$

$$\cancel{\mathbf{1}^T x = 1, \quad \lambda \in \mathbb{R}} \qquad x \geq 0 \quad \mu \in \mathbb{R}_+^{|\mathcal{A}|}$$

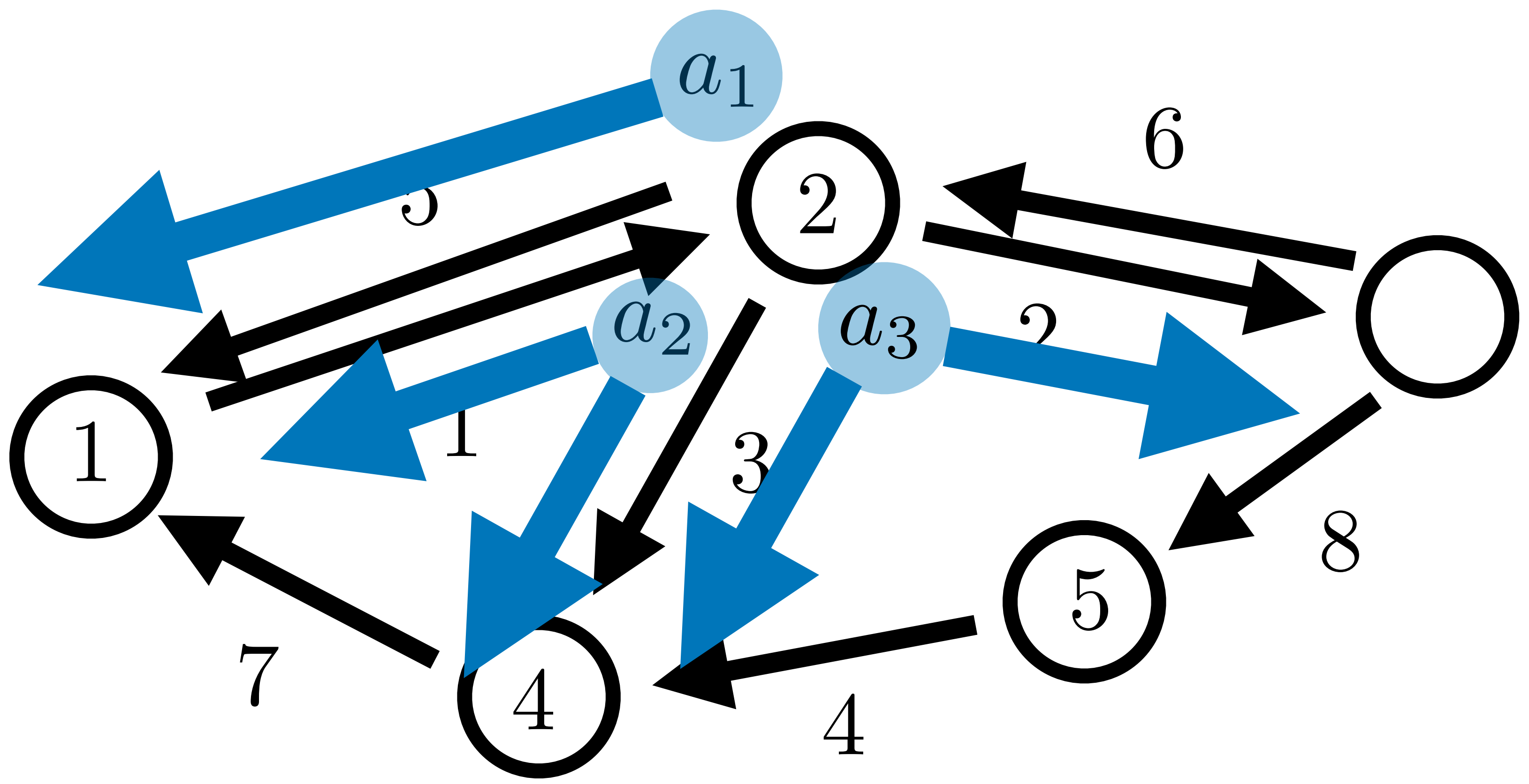
~~$\lambda \in \mathbb{R}$ optimal average reward~~

$v \in \mathbb{R}^{|\mathcal{S}|}$ discounted value on states

$\mu \in \mathbb{R}_+^{|\mathcal{A}|}$ inefficiency of ea. action

Discounted Bellman Equation $v^T E_{\mathcal{A}} = r^T + \gamma v^T P - \mu^T$

Markov Decision Processes



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Discounted Q-value $q^T = v^T P$