

Diagonalization

Square matrix: $A \in \mathbb{R}^{n \times n}$

Assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Right Eigenvectors:

$$V = \begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix} \quad AV = \begin{bmatrix} AV_1 \dots AV_n \end{bmatrix} = \begin{bmatrix} V_1 \lambda_1 \dots V_n \lambda_n \end{bmatrix} = \begin{bmatrix} V_1 \dots V_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_D = VD \quad \rightarrow \quad AV = VD$$

Left Eigenvectors:

$$W = \begin{bmatrix} -W_1^* - \\ \vdots \\ -W_n^* - \end{bmatrix} \quad WA = \begin{bmatrix} -W_1^* A - \\ \vdots \\ -W_n^* A - \end{bmatrix} = \underbrace{\begin{bmatrix} -\lambda_1 W_1^* - \\ \vdots \\ -\lambda_n W_n^* - \end{bmatrix}}_D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* - \\ \vdots \\ -W_n^* - \end{bmatrix} = DW \quad \rightarrow \quad WA = DW$$

$$W^{-1} = W \quad A = W^{-1}DW$$

Assuming V & W are chosen with compatible orderings and lengths of columns/rows...

$$V^{-1} = W$$

Diagonalization

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Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices

Dyadic Expansion

$$\begin{aligned} V^{-1}V &= \begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} W_1^*V_1 & \cdots & W_1^*V_n \\ \vdots & & \vdots \\ W_n^*V_1 & \cdots & W_n^*V_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \end{aligned}$$

...from off diagonal terms $W_j^*V_i = 0 \quad j \neq i$

V_i orthogonal to all other W_j

...from diagonal terms

V_i, W_i

$W_i^*V_i = 1$

can be scaled
so that $W_i^*V_i = 1$

Diagonalization - Similarity Transform

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$ A is similar to a diagonal matrix

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices
Dyadic Expansion

$$\begin{bmatrix} y'_1 \\ \vdots \\ y'_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x'_1 \\ \vdots \\ \lambda_n x'_n \end{bmatrix}$$

$$x = Vx' \quad y = Vy'$$

$$y = Ax$$

$$Vy' = AVx'$$

$$y' = V^{-1}AVx'$$

$$y' = V^{-1}VDV^{-1}Vx'$$

$$y' = Dx'$$

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Interpretation of
Matrix Multiplication

$$[A][x] = \begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} [x]$$

$\underbrace{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$ transforming into eigen-vector coords

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} | & | \\ W_1^* & \dots & W_n^* \\ | & | \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} | & & \\ -W_i^* & - & \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication

Ax

$$\begin{bmatrix} | & | \\ A & x \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} | & | \\ -W_1^* & \dots & -W_n^* \\ | & | \end{bmatrix} \begin{bmatrix} | & | \\ W_1^*x & \dots & W_n^*x \\ | & | \end{bmatrix}$$

transforming into eigen-vector coords

$\overbrace{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}$ Scaling each coord by eigenvalue

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices
Dyadic Expansion

Interpretation of Matrix Multiplication

Ax

$$\begin{bmatrix} A \\ | \\ x \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$\underbrace{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$ transforming into eigen-vector coords

$\underbrace{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}$ Scaling each coord by eigenvalue

$V_1 \lambda_1 W_1^* x + \dots + V_n \lambda_n W_n^* x$ Transforming back into regular coordinates

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **If x is an eigenvector...**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & | \\ V_1 & \cdots & V_n \\ | & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices

Dyadic Expansion

Interpretation of
Matrix Multiplication

AV_i

$$[A][x] = \begin{bmatrix} | & | \\ V_1 & \cdots & V_n \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} \underbrace{\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}$$

Orthogonal to all other left eigenvectors

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **If x is an eigenvector...**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} | & | \\ -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} | & | \\ -W_i^* & - \\ | & | \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication

AV_i

$$\begin{bmatrix} | & | \\ A & x \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} | & | \\ -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} \begin{bmatrix} | & | \\ x \\ | & | \end{bmatrix}$$

$\underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}}$ Orthogonal to all other left eigenvectors

$\underbrace{\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ 0 \end{bmatrix}}$ Scaled by specific eigenvalue

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots** **If x is an eigenvector...**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors **Eigen-values (on diagonal)** **Left eigen-vectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices
Dyadic Expansion

Interpretation of Matrix Multiplication AV_i

$$\begin{bmatrix} A \\ x \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

Orthogonal to all other left eigenvectors

Scaled by specific eigenvalue

Select out that specific eigenvector

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = [V] [D] [V^{-1}]$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Shuffling eigenvalues and eigenvectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic
Expansion

$$[A] = \sum_i \underbrace{\begin{bmatrix} | \\ V_i \\ | \end{bmatrix}}_{\text{Dyadic Expansion}} [\lambda_i] [- W_i^* -]$$

Shuffling eigenvalues and eigenvectors

Order of sum does not matter...



Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = [V] [D] [V^{-1}]$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} [P] \underbrace{\begin{bmatrix} P \\ P^T \end{bmatrix}}_{\text{Shuffling eigenvalues and eigenvectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} [P] \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

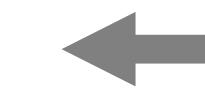
Sum of
rank-1
matrices

Dyadic Expansion

Shuffling eigenvalues and eigenvectors

Order of sum does not matter...

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = [V] [D] [V^{-1}]$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

**Shuffling eigenvalues
and eigenvectors**

Order of sum does not matter...

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$



Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = [V] [D] [V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

**Shuffling eigenvalues
and eigenvectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [- & W_i^* & -]$$



Order of sum does not matter...

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Diagonalization

Shuffle columns (or rows) of identity...

$$A = [V] [D] [V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Sum of
rank-1
matrices

Dyadic Expansion

**Shuffling eigenvalues
and eigenvectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [- & W_i^* & -]$$



Order of sum does not matter...

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

diagonal matrices
commute...

$$A = [V] [D] [V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 \frac{\gamma_1}{\gamma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \frac{\gamma_n}{\gamma_n} \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ \vdots & \vdots & \vdots \\ - & W_n^* & - \end{bmatrix}$$

**Right
eigen-
vectors**

**Eigen-
values
(on diagonal)**

**Left
eigen-
vectors**

**Scaling
eigenvectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices
**Dyadic
Expansion**



Order of sum does not matter...

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

diagonal matrices
commute...

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Scaling eigenvectors}} \underbrace{\begin{bmatrix} \gamma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \gamma_n \end{bmatrix}}_{\text{Sum of rank-1 matrices}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Diagonal matrix}} \underbrace{\begin{bmatrix} \frac{1}{\gamma_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{\gamma_n} \end{bmatrix}}_{\text{Diagonal matrix}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

Scaling eigenvectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [-W_i^* -]$$

Sum of
rank-1
matrices
Dyadic Expansion



Order of sum does not matter...

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

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Diagonalization

diagonal matrices
commute...

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} | & & | \\ V_1\gamma_1 & \cdots & V_n\gamma_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -\frac{1}{\gamma_1}W_1^* & - \\ \vdots & \vdots \\ -\frac{1}{\gamma_n}W_n^* & - \end{bmatrix}}_{\text{Scaling eigenvectors}}$$

$$V'$$

$$V'^{-1}$$

Scaling
eigenvectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Sum of
rank-1
matrices
**Dyadic
Expansion**



Order of sum does not matter...

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} W_1^* & - \\ \vdots & \\ W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix}}_{\text{Red box}} \underbrace{\begin{bmatrix} | & | \\ V_3 & V_4 \\ | & | \end{bmatrix}}_{\text{Blue box}} \dots$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\underbrace{\begin{bmatrix} | & & | \\ V_n & & | \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & 0 & 0 & \dots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & 0 & 0 & \dots & 0 \\ 0 & 0 & \gamma_2 e^{-i\phi_2} & 0 & \dots & 0 \\ 0 & 0 & 0 & \gamma_2 e^{i\phi_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} | & & | \\ W_1^* & & | \\ | & & | \end{bmatrix}}_{\text{Red box}} \underbrace{\begin{bmatrix} | & & | \\ W_2^* & & | \\ | & & | \end{bmatrix}}_{\text{Blue box}} \dots \underbrace{\begin{bmatrix} | & & | \\ W_3^* & & | \\ | & & | \end{bmatrix}}_{\text{Red box}} \underbrace{\begin{bmatrix} | & & | \\ W_4^* & & | \\ | & & | \end{bmatrix}}_{\text{Blue box}} \dots \underbrace{\begin{bmatrix} | & & | \\ W_n^* & & | \\ | & & | \end{bmatrix}}_{\text{Red box}}$$

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} | & & | \\ -W_i^* & & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}} \boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \dots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}} \boxed{\begin{bmatrix} -W_1^* & - \\ -W_2^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \quad W_1^*, \bar{W}_1^*$
can be conjugate pairs

$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \dots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}} \boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: V_1, \bar{V}_1 W_1^*, \bar{W}_1^*
can be conjugate pairs

$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} \frac{z}{z'} & 0 & \dots & 0 \\ 0 & \gamma_1 e^{i\phi_1} \frac{z'}{z} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Note: may differ by any complex scalars $z, z' \in \mathbb{C}$
...with both magnitude and phase shifts

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 z & \bar{V}_1 z' \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: V_1, \bar{V}_1 W_1^*, \bar{W}_1^*
can be conjugate pairs

$$\boxed{\begin{bmatrix} | & | \\ V_n & | \\ | & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \dots \begin{bmatrix} -\frac{1}{z} W_1^* & - \\ -\frac{1}{z'} \bar{W}_1^* & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Note: may differ by any complex scalars $z, z' \in \mathbb{C}$
...with both magnitude and phase shifts

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \dots \boxed{\begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \quad W_1^*, \bar{W}_1^*$
can be conjugate pairs

$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}} \dots \boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} =$$

$v_1, v'_1 \in \mathbb{R}^n$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \dots \boxed{\begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \quad W_1^*, \bar{W}_1^*$
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$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}} \dots \boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix}}$$

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$$\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_U$$

$v_1, v'_1 \in \mathbb{R}^n$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \dots \boxed{\begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \quad W_1^*, \bar{W}_1^*$
can be conjugate pairs

$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}} \dots \boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_{U} \frac{1}{\sqrt{2}}$$

Real Imag
(scaled by $\sqrt{2}$)

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \dots \boxed{\begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: V_1, \bar{V}_1 W_1^*, \bar{W}_1^*
can be conjugate pairs

$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}} \dots \boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} = \boxed{\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_U \frac{1}{\sqrt{2}}$$

similarly...

$$\boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \end{bmatrix}} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}}_{U^*} \boxed{\begin{bmatrix} -w_1^\top & - \\ -w_1'^\top & - \end{bmatrix}}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \dots \boxed{\begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \quad W_1^*, \bar{W}_1^*$
can be conjugate pairs

$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}} \dots \boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_U$$

Note: $U \in \mathbb{C}^{2 \times 2}$
unitary

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \dots \boxed{\begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \quad W_1^*, \bar{W}_1^*$
can be conjugate pairs

$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}} \dots \boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} = \boxed{\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_{U} \frac{1}{\sqrt{2}}$$

Note: $U \in \mathbb{C}^{2 \times 2}$ $U^*U = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = I$
unitary

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \dots \boxed{\begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: V_1, \bar{V}_1 W_1^*, \bar{W}_1^*
can be conjugate pairs

$$\boxed{\begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}} \dots \boxed{\begin{bmatrix} -W_1^* & - \\ -\bar{W}_1^* & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} = \boxed{\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_{U} \frac{1}{\sqrt{2}}$$

Also: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: V_1, \bar{V}_1 W_1^*, \bar{W}_1^*
can be conjugate pairs

$$\boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ \vdots & \\ -\bar{W}_1^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} = \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_{U} \frac{1}{\sqrt{2}}$$

$$\text{Also: } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: V_1, \bar{V}_1 W_1^*, \bar{W}_1^*
can be conjugate pairs

$$\boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \dots \begin{bmatrix} | & | \\ V_n & \bar{V}_n \\ | & | \end{bmatrix} \begin{bmatrix} \gamma_n e^{-i\phi_n} & 0 \\ 0 & \gamma_n e^{i\phi_n} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -W_1^* & - \\ \vdots & \\ -\bar{W}_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \boxed{\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix}} = \boxed{\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_U \frac{1}{\sqrt{2}}$$

$$\text{Also: } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = \gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} | & & | \\ W_1^* & \dots & W_n^* \\ | & & | \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & & | \\ V_1 & \dots & \bar{V}_1 \\ | & & | \end{bmatrix}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} | & & | \\ -W_i^* & \dots & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\begin{array}{c} U \\ \downarrow \\ \begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} | & & | \\ W_1^* & \dots & W_n^* \\ | & & | \end{bmatrix} \\ \boxed{\begin{bmatrix} | & & | \\ V_1 & \dots & \bar{V}_1 \\ | & & | \end{bmatrix}} \dots \boxed{\begin{bmatrix} | & & | \\ V_n & \dots & \bar{V}_n \\ | & & | \end{bmatrix}} \end{array}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \begin{bmatrix} | & & | \\ V_1 & \dots & \bar{V}_1 \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \dots & v'_1 \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_U \frac{1}{\sqrt{2}}$$

$$\text{Also: } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = \gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}}_{\text{Real part}} \dots \underbrace{\begin{bmatrix} | & | \\ V_n & | \\ | & | \end{bmatrix}}_{\text{Complex conjugate pairs}} \underbrace{\begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}}_{\text{Diagonal matrix}} \dots \underbrace{\begin{bmatrix} -w_1^\top & - \\ -w'_1^\top & - \\ \vdots & \vdots \\ \bar{W}_n^* & - \end{bmatrix}}_{\text{Complex conjugate pairs}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$v_1, v'_1 \in \mathbb{R}^n$

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \quad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_U \frac{1}{\sqrt{2}}$$

$$\text{Also: } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = \gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V][D][V^{-1}]$$

$$\begin{bmatrix} A \\ & \vdots & & \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}}_{V'} \cdots \underbrace{\begin{bmatrix} | & | \\ v_n & v'_n \\ | & | \end{bmatrix}}_{V_n} \underbrace{\begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}}_{\gamma} \underbrace{\begin{bmatrix} \cdots & 0 \\ \cdots & 0 \\ \ddots & \ddots \\ \cdots & \lambda_n \end{bmatrix}}_{\text{Diagonal Matrix}} \underbrace{\begin{bmatrix} - & w_1^\top & - \\ - & w'_1^\top & - \\ \vdots & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}}_{V'^{-1}}$$

$V' \in \mathbb{R}^{n \times n}$

$$\begin{bmatrix} A \\ & \vdots & & \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\underbrace{\begin{bmatrix} | & & | \\ v_1 & & v'_1 \\ | & & | \end{bmatrix}}_{V' \in \mathbb{R}^{n \times n}} \cdots \underbrace{\begin{bmatrix} | & & | \\ V_n & & | \\ | & & | \end{bmatrix}}_{V'^{-1}} \underbrace{\begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}}_{\boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}}} \cdots \underbrace{\begin{bmatrix} -w_1^\top & - \\ -w'_1^\top & - \\ \vdots & \\ -\bar{W}_n^* & - \end{bmatrix}}_{\boxed{V'^{-1}}}$$

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V][D][V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\underbrace{\begin{bmatrix} | & & | \\ v_1 & & v'_1 \\ | & & | \end{bmatrix}}_{V' \in \mathbb{R}^{n \times n}} \cdots \underbrace{\begin{bmatrix} | & & | \\ V_n & & | \end{bmatrix}}_{V' \in \mathbb{R}^{n \times n}} \underbrace{\begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}}_{D' \in \mathbb{R}^{n \times n}} \cdots \underbrace{\begin{bmatrix} -w_1^\top & - \\ -w'_1^\top & - \\ \vdots & \\ -\bar{W}_n^* & - \end{bmatrix}}_{V'^{-1}}$$

$V' \in \mathbb{R}^{n \times n}$

$D' \in \mathbb{R}^{n \times n}$

V'^{-1}

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Real Expansion...

Block diagonal...
Pseudo-diagonalization

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V] [D] [V^{-1}] = \underbrace{\begin{bmatrix} | & | \\ V_1 & \cdots & V_n \\ | & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

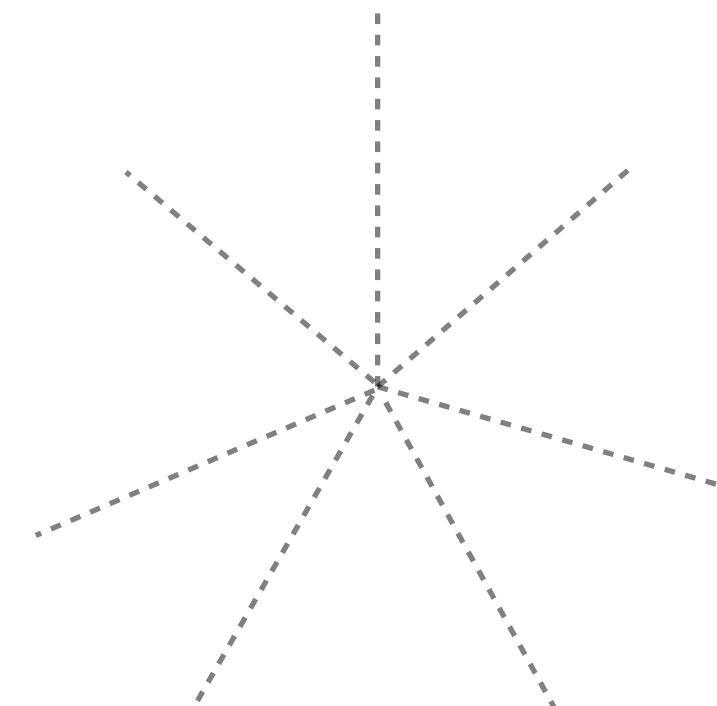
Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [-W_i^* -]$$

\mathbb{R}^n

$$\underbrace{\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}}_{V_1} \cdots \underbrace{\begin{bmatrix} | & | \\ v_n & v'_n \\ | & | \end{bmatrix}}_{V_n} \underbrace{\begin{bmatrix} \gamma R_\phi & & & \\ & \ddots & & \\ & & 0 & 0 \\ & & \vdots & \vdots \\ & & & \lambda_n \end{bmatrix}}_{\text{Diagonal Matrix}} \underbrace{\begin{bmatrix} -w_1^\top & - \\ -w_1'^\top & - \\ \vdots & \\ -\bar{W}_n^* & - \end{bmatrix}}_{W_n^*}$$



Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}] = \underbrace{\begin{bmatrix} | & | \\ V_1 & \cdots & V_n \\ | & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ - & \vdots & - \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

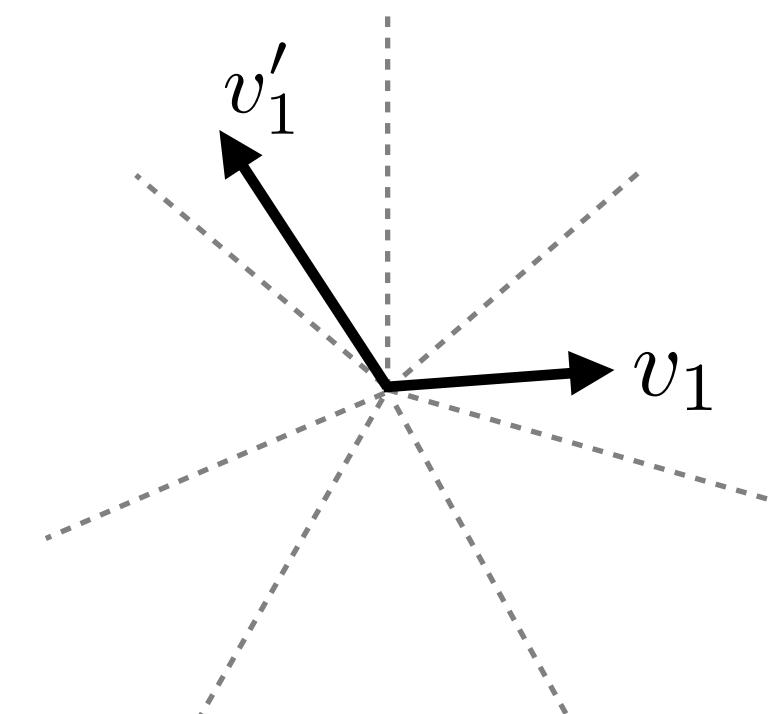
Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix} \quad \mathbb{R}^n$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \cdots \begin{bmatrix} | & | \\ V_n & | \\ | & | \end{bmatrix} \boxed{\gamma R_\phi} \cdots \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \\ - & \vdots & - \end{bmatrix} \quad \lambda_n \begin{bmatrix} - & \bar{W}_n^* & - \end{bmatrix}$$



Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}]$$

$$[A] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \dots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

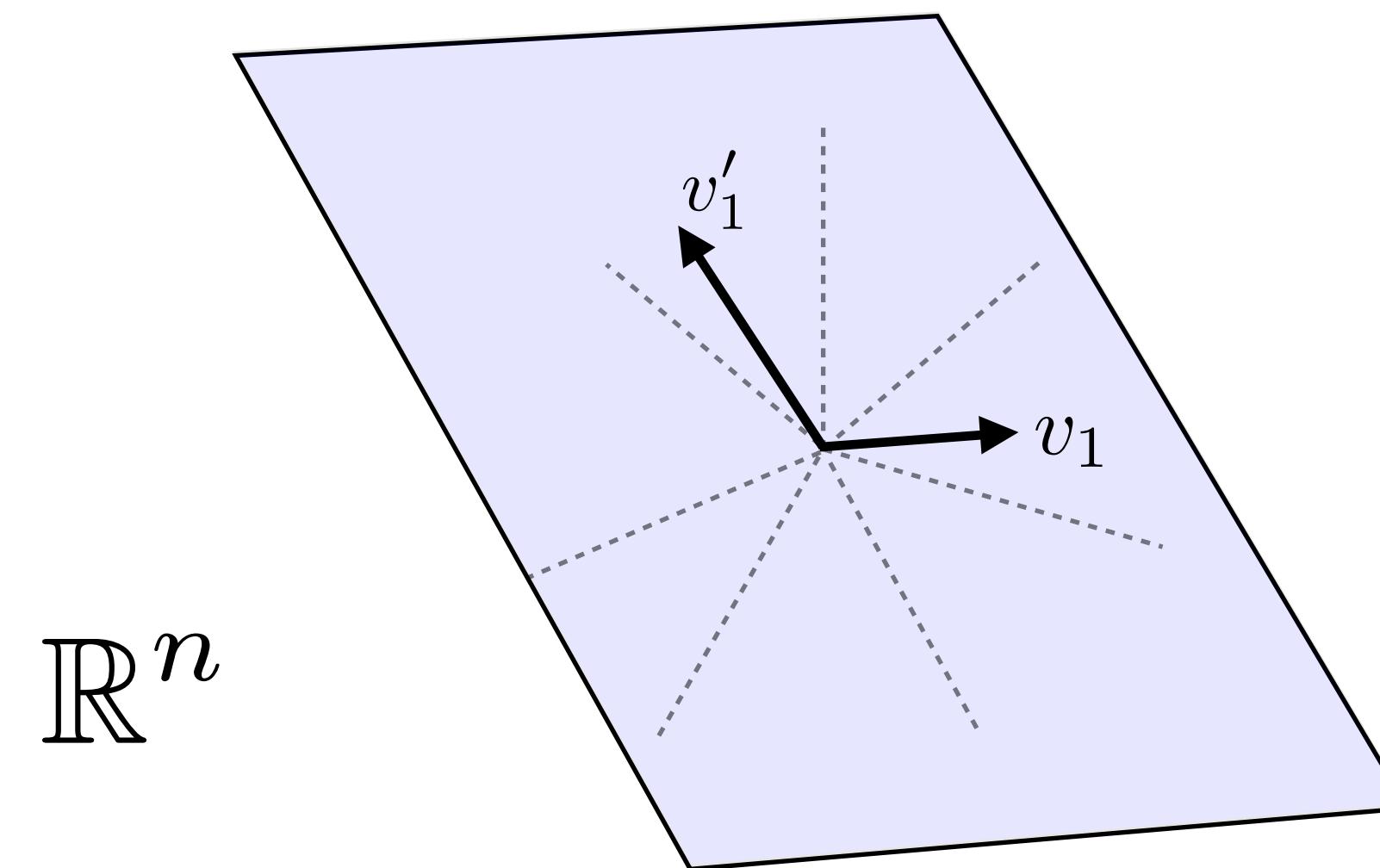
Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ V_n & | \\ | & | \end{bmatrix} \boxed{\gamma R_\phi} \dots \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \dots & \dots \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \dots & \dots \end{bmatrix} \begin{bmatrix} -w_1^\top & - \\ -w'_1^\top & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix}$$



$$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \quad \text{2D}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}]$$

$$[A] = \underbrace{[V_1 \dots V_n]}_{\text{Right eigen-vectors}} \underbrace{[\lambda_1 \dots 0 \dots \lambda_n]}_{\text{Eigen-values (on diagonal)}} \underbrace{[-W_1^* \dots \vdots \dots W_n^*]}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

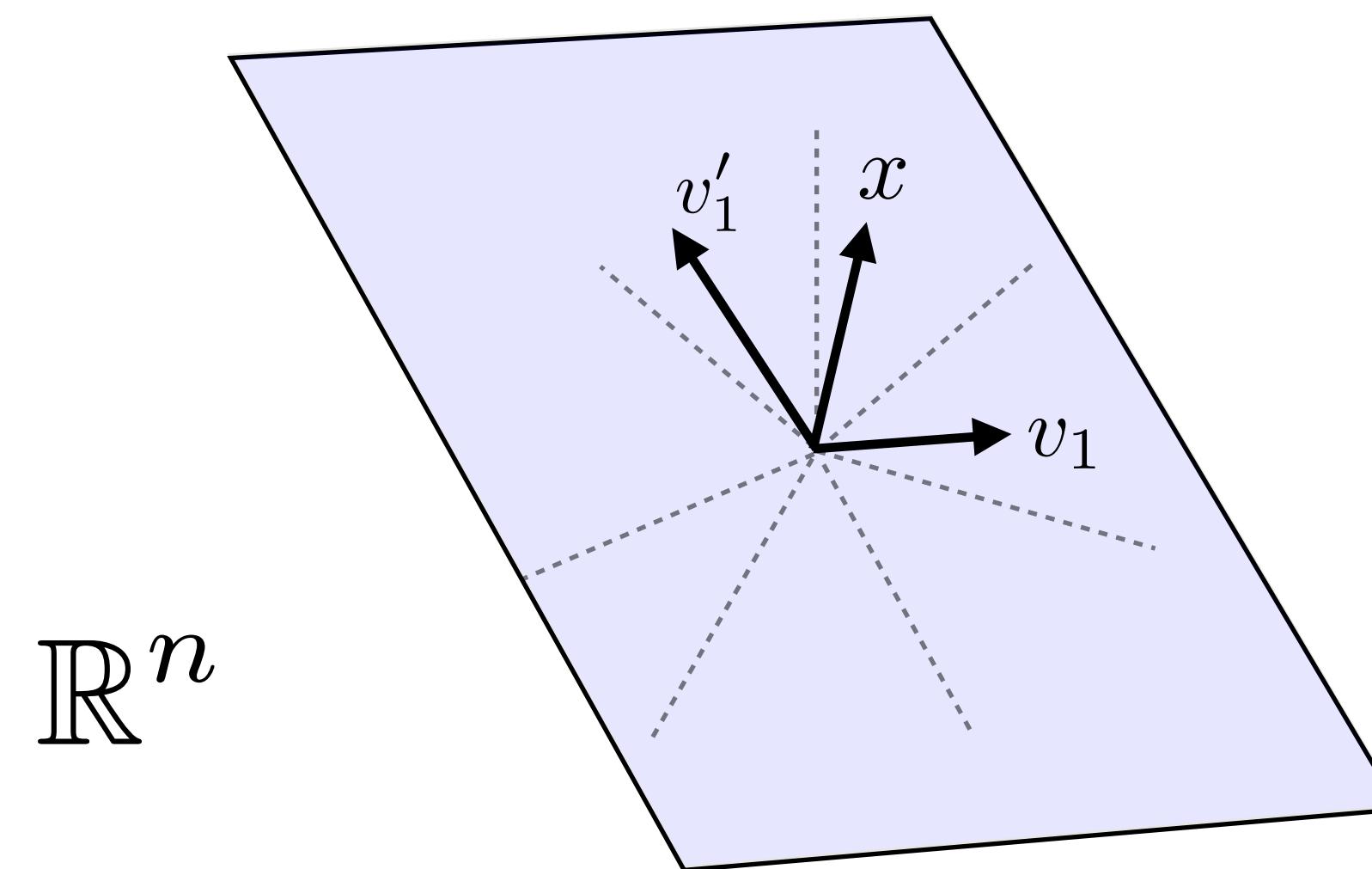
Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} | & -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$[A] = \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ V_n & | \\ | & | \end{bmatrix} \boxed{\gamma R_\phi} \begin{bmatrix} \dots & 0 \\ \dots & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \dots & \dots \\ \lambda_n & \end{bmatrix} \begin{bmatrix} | & -W_1^* & - \\ | & -W_1'^\top & - \\ | & \vdots & | \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

vector in plane of rotation



$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}]$$

$$[A] = \underbrace{[V_1 \dots V_n]}_{\text{Right eigen-vectors}} \underbrace{[\lambda_1 \dots 0 \dots \lambda_n]}_{\text{Eigen-values (on diagonal)}} \underbrace{[-W_1^* \dots \vdots \dots W_n^*]}_{\text{Left eigen-vectors}}$$

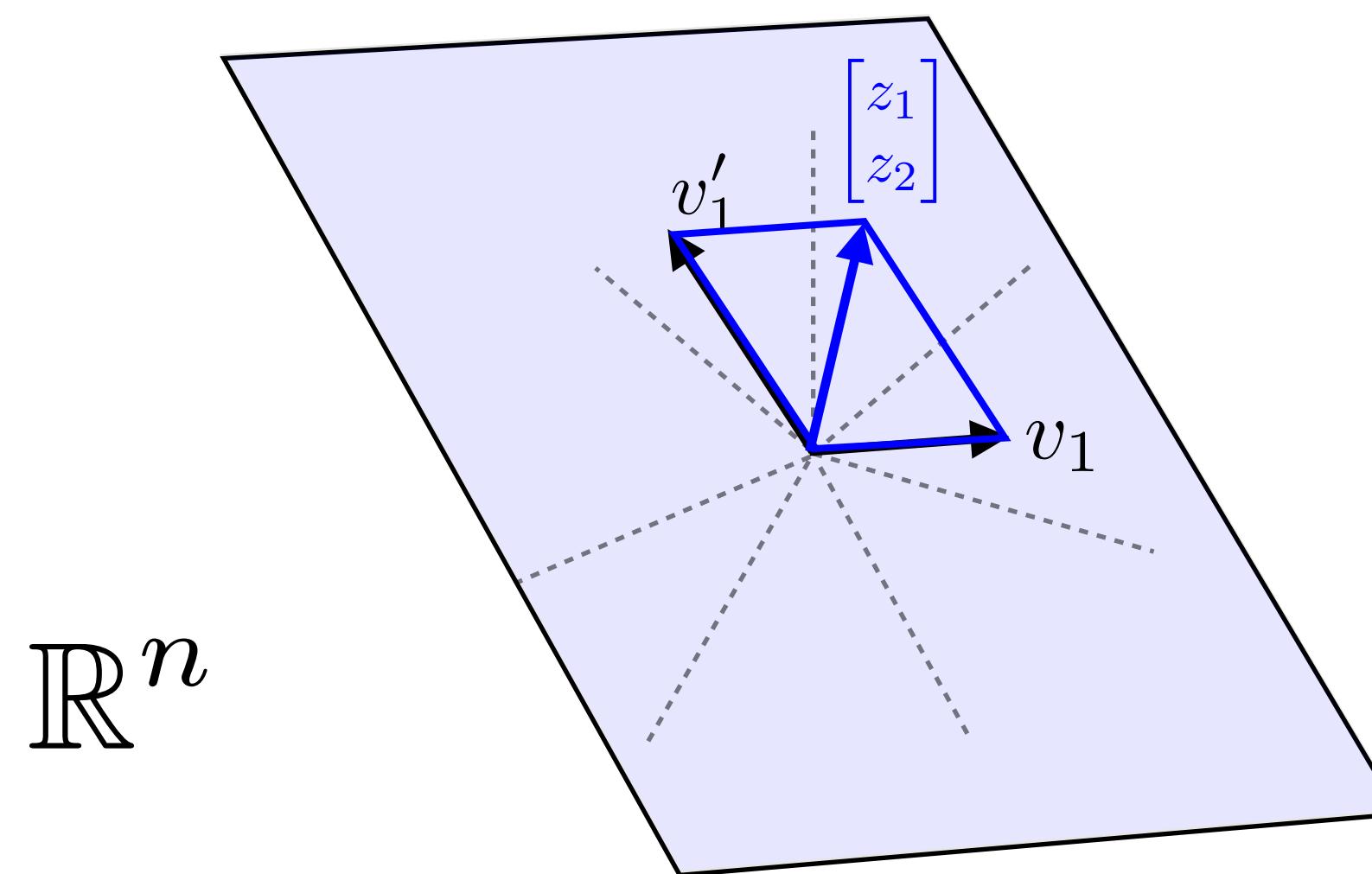
Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [-W_i^* \dots]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$[A] = \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ v_n & v'_n \\ | & | \end{bmatrix} \boxed{\gamma R_\phi} \begin{bmatrix} \dots & 0 \\ \vdots & 0 \\ 0 & 0 \\ \ddots & \ddots \\ \dots & \lambda_n \end{bmatrix} \begin{bmatrix} | & | \\ -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix} \begin{bmatrix} | & | \\ z_1 & z_2 \\ | & | \end{bmatrix}$$

↑
vector in
plane of
rotation



\mathbb{R}^n

$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}]$$

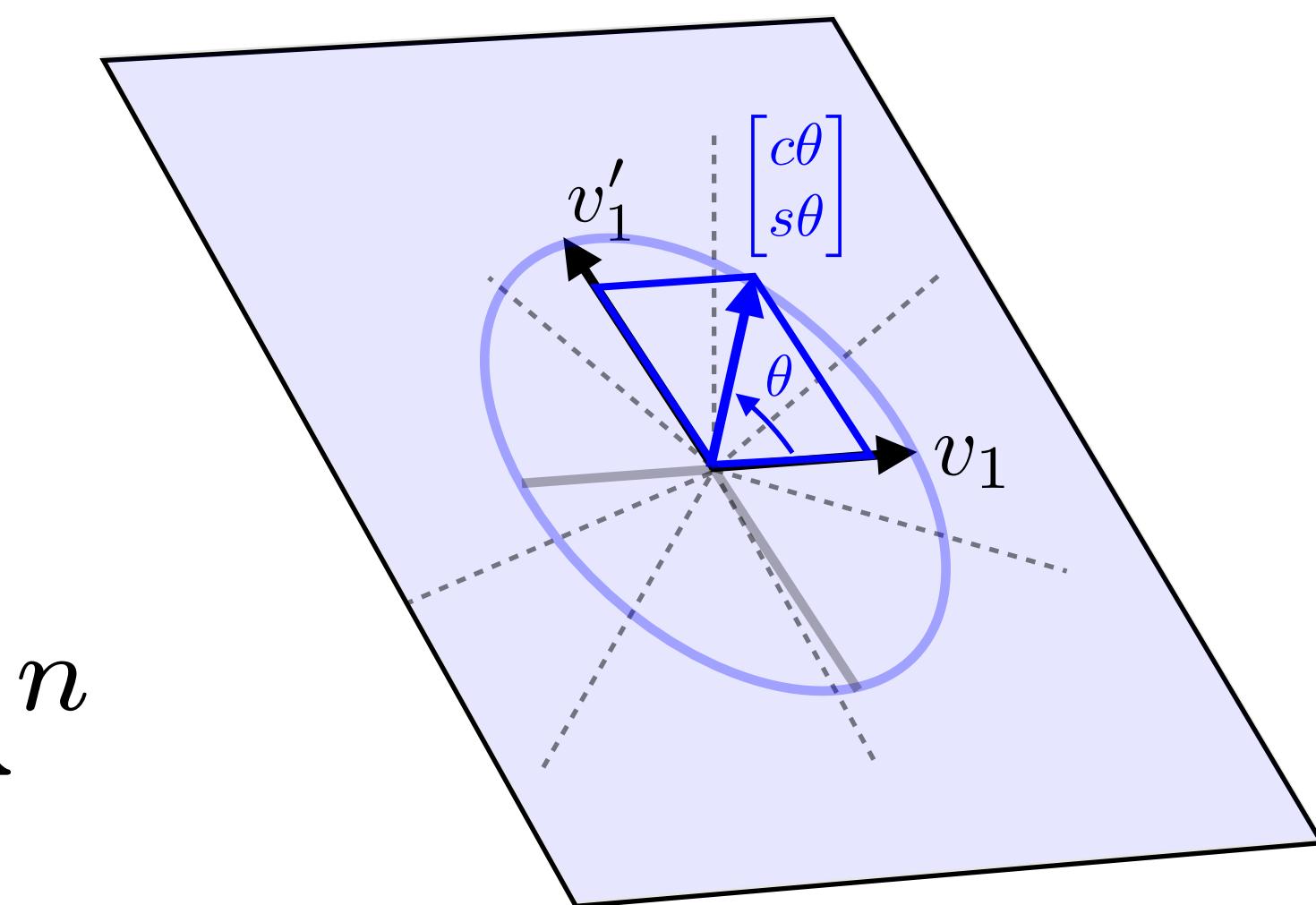
$$[A] = \underbrace{[V_1 \dots V_n]}_{\text{Right eigen-vectors}} \underbrace{[\lambda_1 \dots 0 \dots \lambda_n]}_{\text{Eigen-values (on diagonal)}} \underbrace{[-W_1^* \dots \vdots \dots W_n^*]}_{\text{Left eigen-vectors}}$$

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} | & -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ v_n & v'_n \\ | & | \end{bmatrix} \begin{bmatrix} \gamma R_\phi & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \lambda_n \\ & & & -\bar{W}_n^* & - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \begin{bmatrix} c\theta & |z|_2 \\ s\theta & \end{bmatrix}$$

↑
vector in
plane of
rotation



$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}]$$

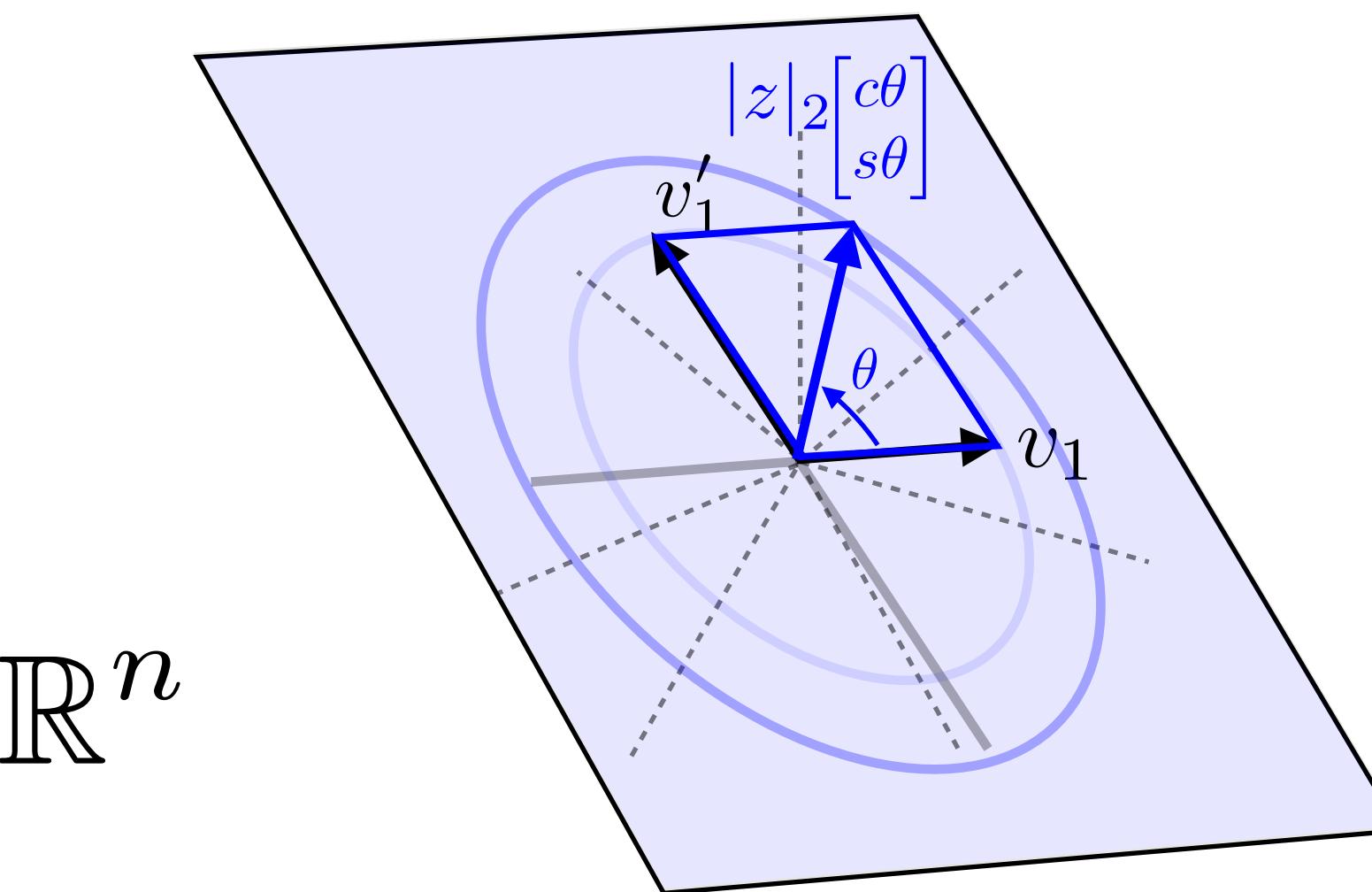
$$[A] = \underbrace{[V_1 \dots V_n]}_{\text{Right eigen-vectors}} \underbrace{[\lambda_1 \dots 0 \dots \lambda_n]}_{\text{Eigen-values (on diagonal)}} \underbrace{[-W_1^* \dots \vdots \dots W_n^*]}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [-W_i^* \dots]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ v_n & v'_n \\ | & | \end{bmatrix} \boxed{\gamma R_\phi} \dots \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ \lambda_n & \lambda_n \end{bmatrix} \begin{bmatrix} -w_1^\top & - \\ -w_1'^\top & - \\ \vdots & \vdots \\ -\bar{W}_n^* & - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} [c\theta \quad |z|_2 \quad s\theta]$$



\mathbb{R}^n

$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}]$$

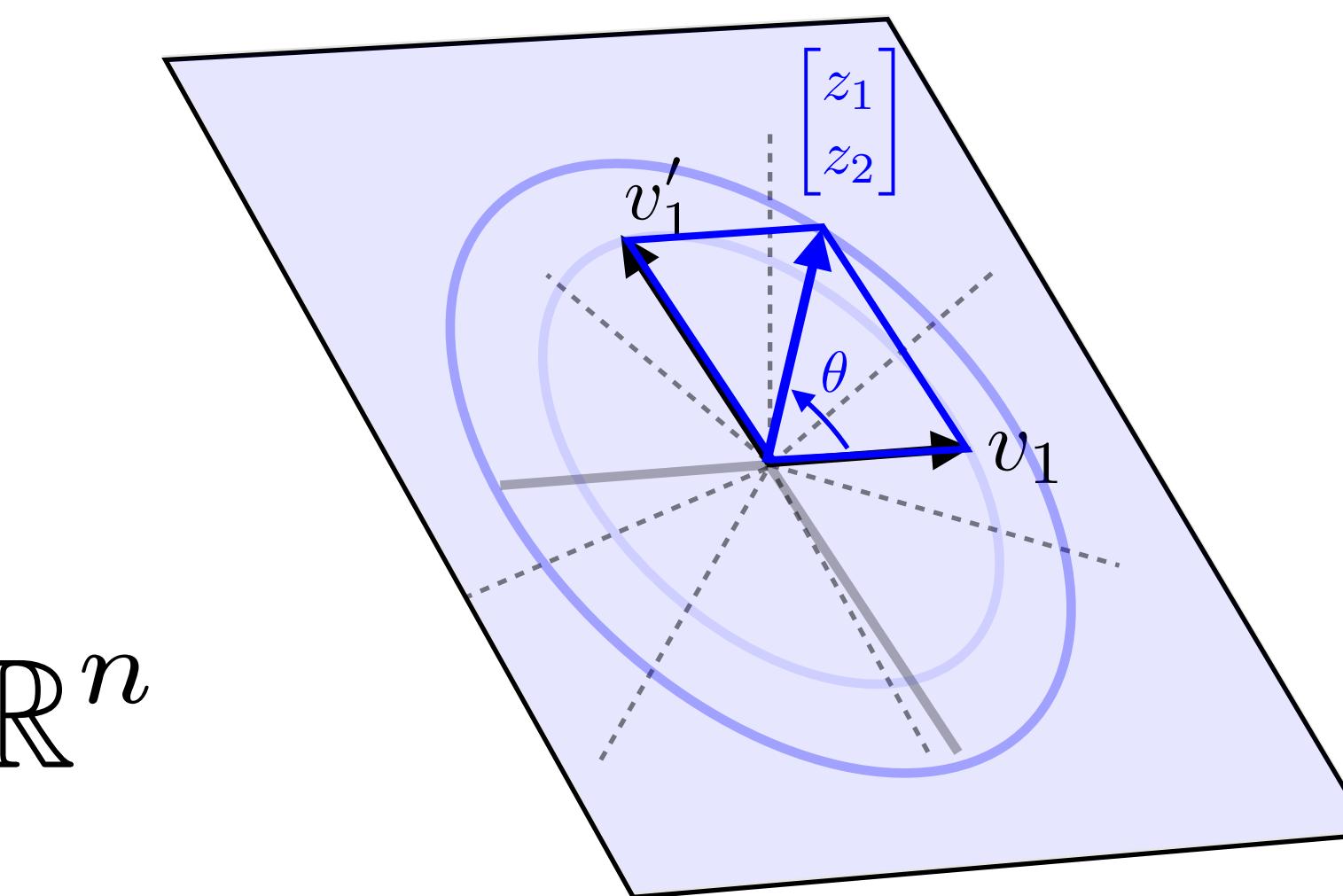
$$[A] = \underbrace{[V_1 \dots V_n]}_{\text{Right eigen-vectors}} \underbrace{[\lambda_1 \dots 0 \dots \lambda_n]}_{\text{Eigen-values (on diagonal)}} \underbrace{[-W_1^* \dots \vdots \dots W_n^*]}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [-W_i^* \dots]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$[A] = \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ v_n & v'_n \\ | & | \end{bmatrix} \boxed{\gamma R_\phi} \begin{bmatrix} \dots & 0 \\ \vdots & 0 \\ 0 & 0 \\ \ddots & \ddots \\ \dots & \lambda_n \end{bmatrix} \begin{bmatrix} | & | \\ -W_1^* & - \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ -W_n^* & - \\ | & | \end{bmatrix} \begin{bmatrix} | & | \\ z_1 & z_2 \\ | & | \end{bmatrix}$$



\mathbb{R}^n

$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}]$$

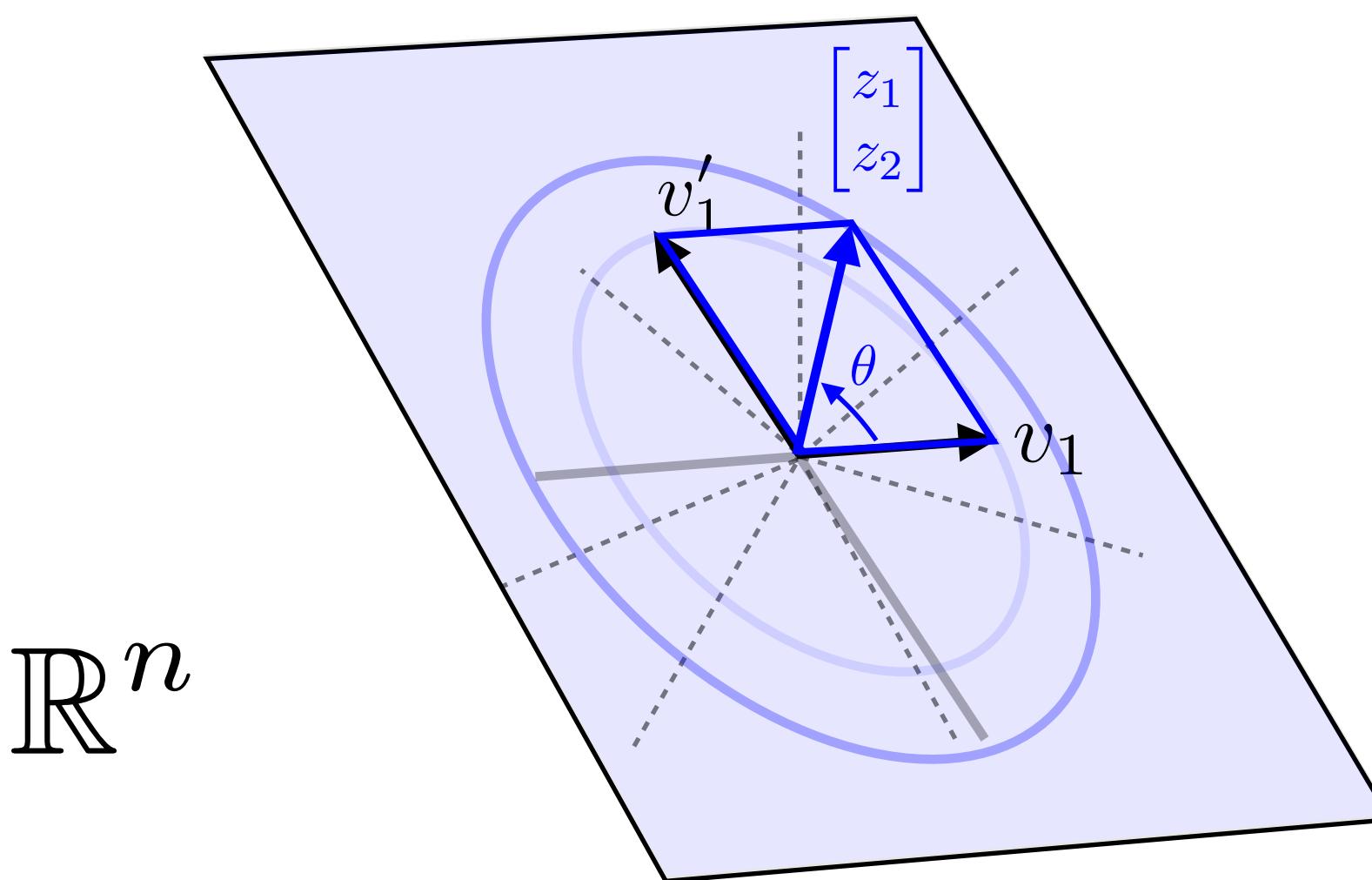
$$[A] = \underbrace{[V_1 \dots V_n]}_{\text{Right eigen-vectors}} \underbrace{[\lambda_1 \dots 0 \dots \lambda_n]}_{\text{Eigen-values (on diagonal)}} \underbrace{[-W_1^* \dots \vdots \dots W_n^*]}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [-W_i^* \dots]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\begin{aligned} & \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ v_n & v'_n \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} \gamma R_\phi & & & \\ & \ddots & & \\ & & 0 & \\ & & 0 & \\ & & \ddots & \\ & & & \ddots & \\ & & & & \lambda_n & \\ & & & & & \bar{W}_n^* & \end{bmatrix}}_{\text{Complex Conjugate Pairs}} \underbrace{\begin{bmatrix} | & | \\ w_1^\top & w_1'^\top \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ w_n^\top & w_n'^\top \\ | & | \end{bmatrix}}_{\text{Complex Conjugate Pairs}} \begin{bmatrix} | \\ z_1 \\ | \\ z_2 \\ | \\ \vdots \end{bmatrix} \\ & \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ 0 \\ \vdots \end{bmatrix}}_{\text{span } \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}} \end{aligned}$$



$$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \quad \text{2D}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V] [D] [V^{-1}]$$

$$[A] = \left[\begin{array}{ccc|c} | & & & | \\ V_1 & \cdots & V_n \\ | & & | \end{array} \right] \left[\begin{array}{ccc|c} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{array} \right] \left[\begin{array}{ccc|c} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{array} \right]$$

$\brace{ } \quad \brace{ } \quad \brace{ }$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen- vectors

$$[A] = \sum_i [V_i] [\lambda_i] [-W_i^* -]$$

$$\gamma = 1$$

\mathbb{R}^n

The diagram shows a 3D coordinate system with axes x , y , and z . A vector v_1 is shown in the x - y plane. A second vector v'_1 is shown, rotated by an angle ϕ around the z -axis. The angle θ is also indicated between the x -axis and the projection of v_1 onto the x - y plane. The expression $\gamma [R_\phi]^{z_1}_{z_2}$ is displayed at the top left, representing the rotation matrix component from axis z_2 to z_1 .

$$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \quad \text{2D}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = [V] [D] [V^{-1}]$$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Right eigen-vectors

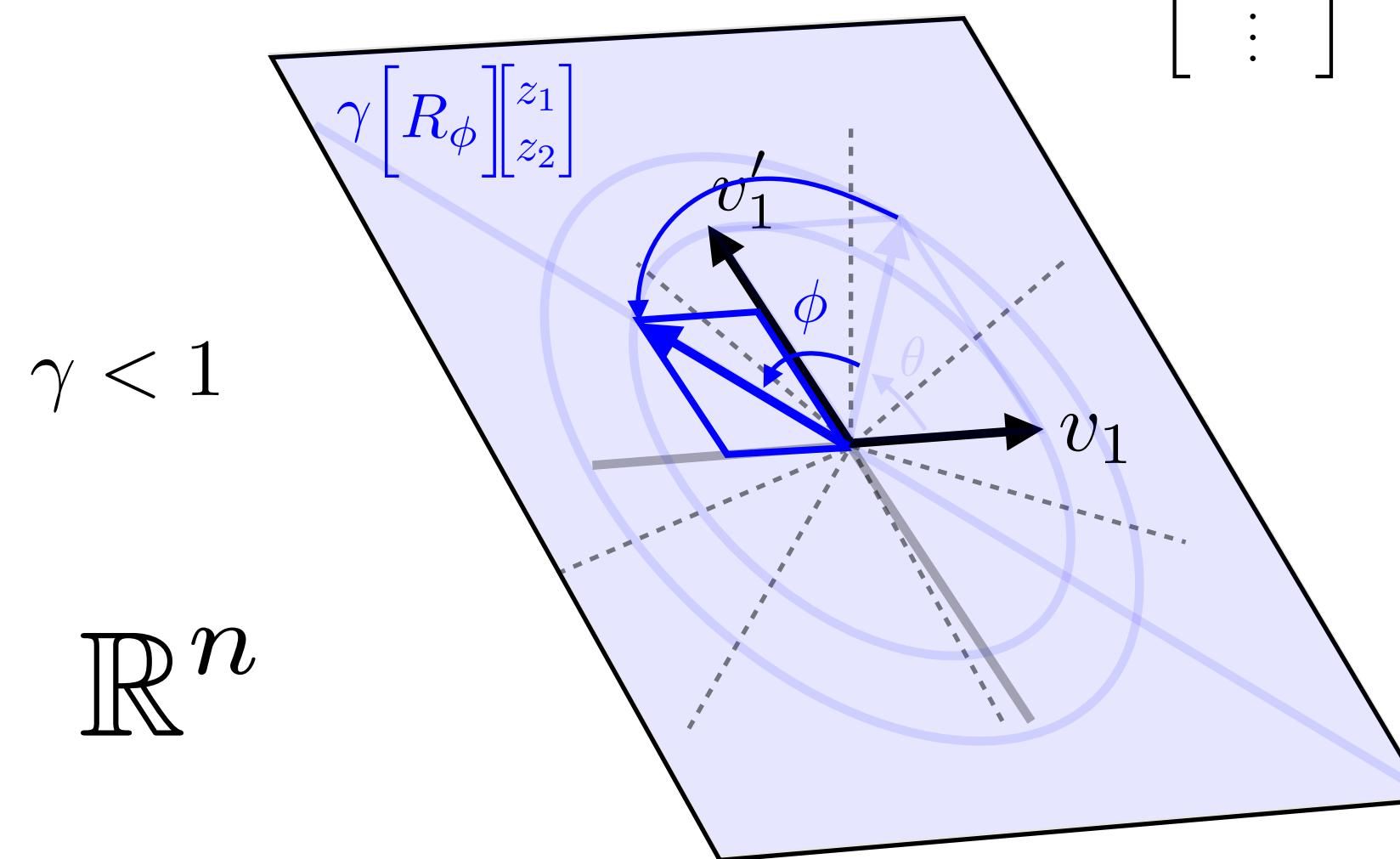
Eigen-values (on diagonal)

Left eigen- vectors

$$[A] = \sum_i [V_i] [\lambda_i] [-W_i^* -]$$

$$\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \cdots \begin{bmatrix} | \\ V_n \\ | \end{bmatrix} \boxed{\gamma R_\phi} \cdots \begin{bmatrix} \cdots & 0 \\ \cdots & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \cdots & \lambda_n \end{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \\ - & & - \end{bmatrix}} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$





$$\gamma < 1$$

\mathbb{R}^n

$$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \quad 2D$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}] = \underbrace{\begin{bmatrix} | & | \\ V_1 & \dots & V_n \\ | & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} -W_1^* & - \\ \vdots & \vdots \\ -W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

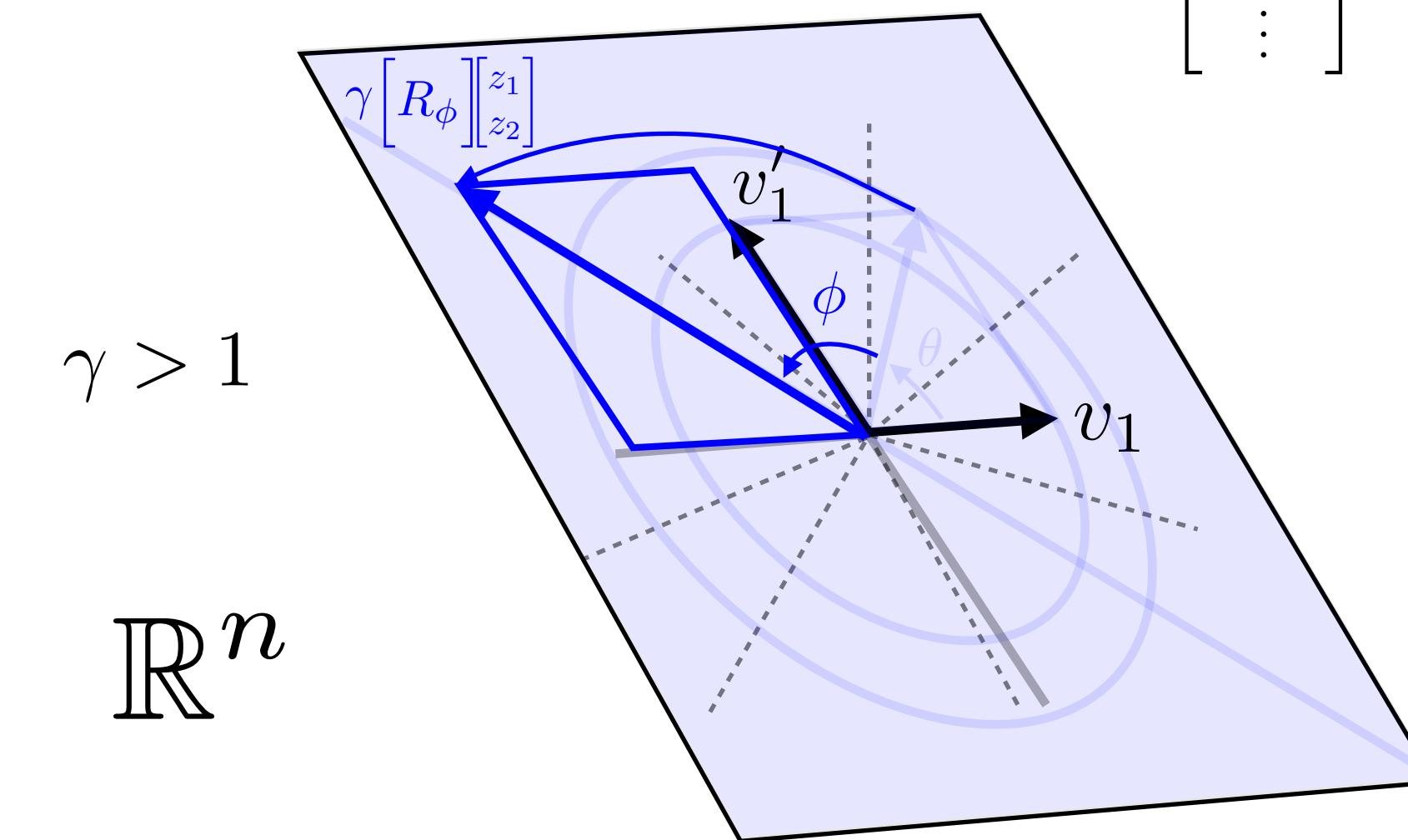
Eigen-values
(on diagonal)

Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\underbrace{\begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \dots \begin{bmatrix} | & | \\ v_n & v'_n \\ | & | \end{bmatrix} \begin{bmatrix} \gamma R_\phi & & & \\ \vdots & \ddots & \ddots & \\ 0 & 0 & \dots & \lambda_n \\ & & \ddots & -\bar{W}_n^* \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\gamma R_\phi z} \begin{bmatrix} | \\ 0 \\ | \end{bmatrix} \dots \begin{bmatrix} | \\ 0 \\ | \end{bmatrix} \begin{bmatrix} w_1^\top & - \\ -w_1'^\top & - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



$\text{span} \begin{bmatrix} | & | \\ v_1 & v'_1 \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$

Diagonalization

$$A = [V] [D] [V^{-1}]$$

$$[A] = \underbrace{[V_1 \dots V_n]}_{\text{Right eigen-vectors}} \underbrace{[\lambda_1 \dots 0 \dots \lambda_n]}_{\text{Eigen-values (on diagonal)}} \underbrace{[-W_1^* \dots \vdots \dots W_n^*]}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values
(on diagonal)

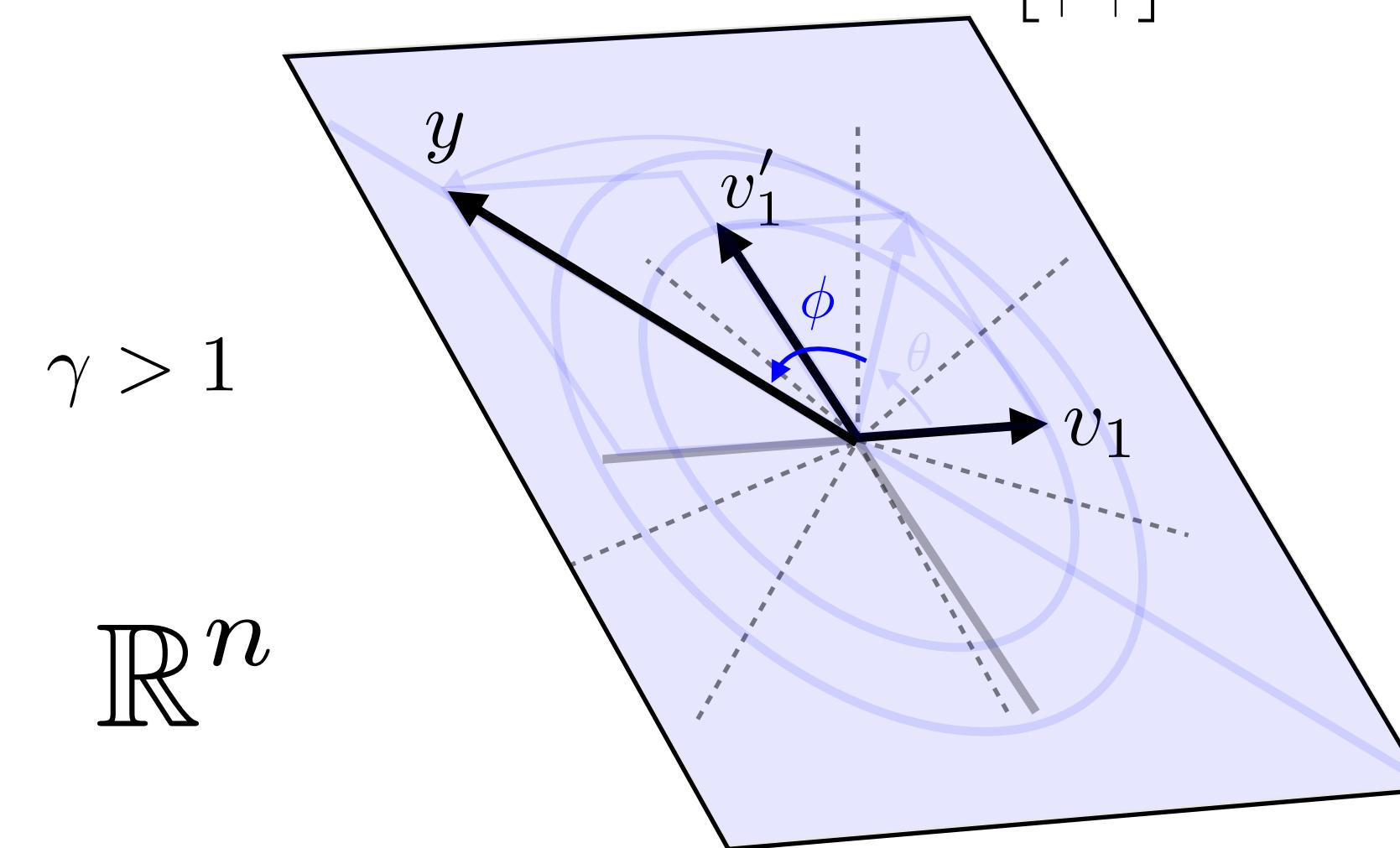
Left eigen-vectors

$$[A] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} | & -W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\underbrace{[V_1 \dots V_n]}_{\text{Right eigen-vectors}} \underbrace{[\gamma R_\phi \dots 0 \dots \bar{W}_n^*]}_{\text{Eigen-values}} \underbrace{[w_1^\top \dots w_n^\top]}_{\text{Left eigen-vectors}}$$

$$y = \gamma \begin{bmatrix} | \\ v_1 v_1' \\ | \end{bmatrix} [R_\phi] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



$\text{span} \begin{bmatrix} | \\ v_1 v_1' \\ | \end{bmatrix}$ 2D