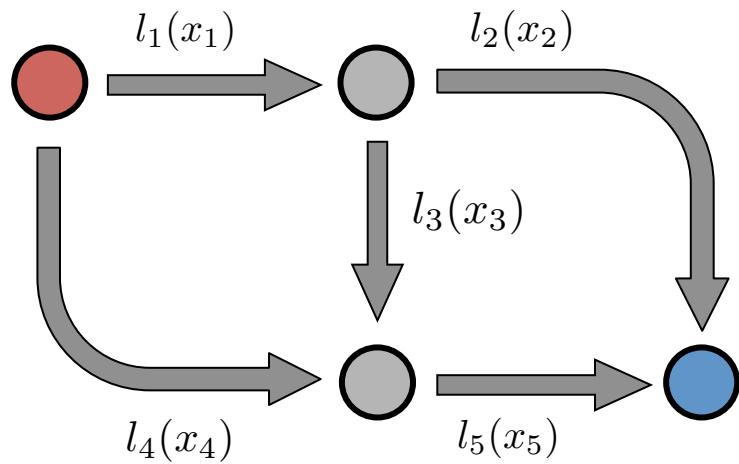


Queue-Routing Game

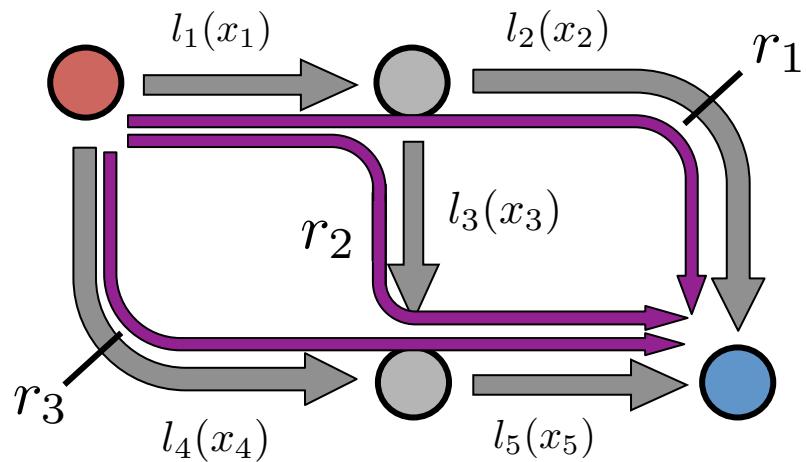
Dan, Eric, Lillian

UC Berkeley
April 8th, 2016

Routing Game Tutorial



Edge & Route Flows



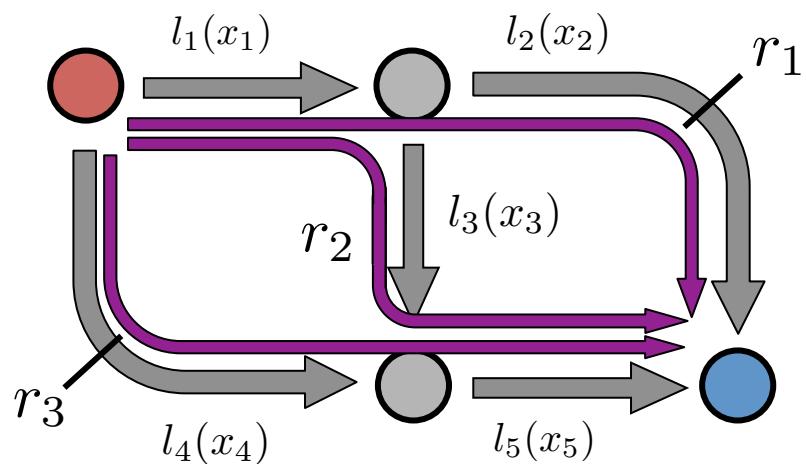
Paths

1	1	0
1	0	0
0	1	0
0	0	1
0	1	1

Edges

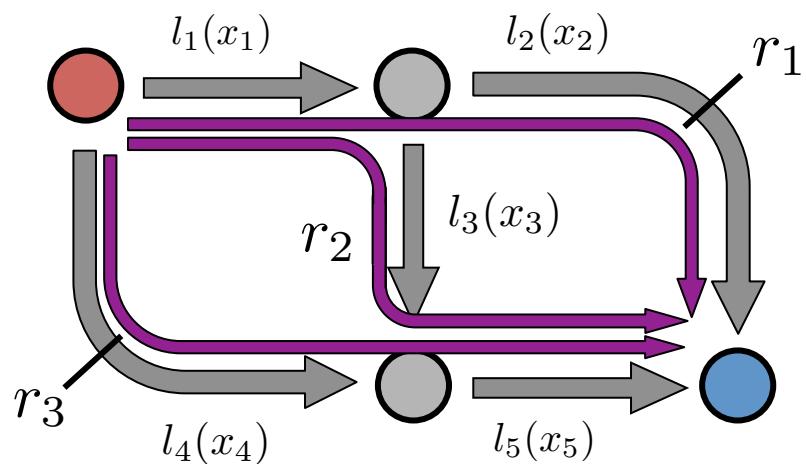
Routing Matrix \mathbb{R}

Edge & Route Flows



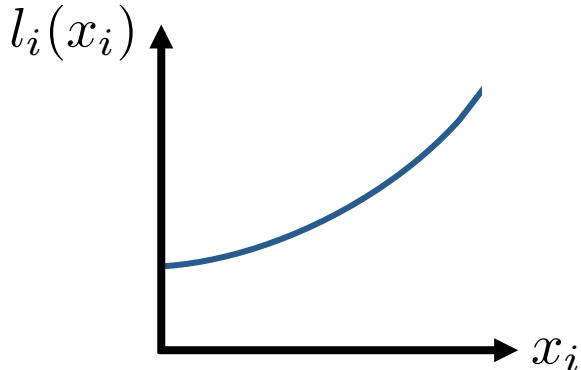
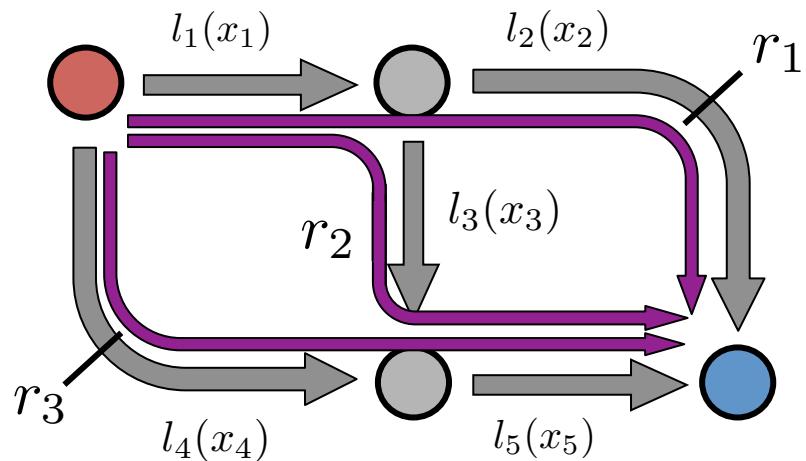
$$\begin{matrix} \text{Routes} \\ \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \right] = \text{Edges} \times \text{Routing Matrix } \mathbb{R} \times \text{Route Flows} \\ \left[\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \right] \times \left[\begin{matrix} x_1^R \\ x_2^R \\ x_3^R \end{matrix} \right] \end{matrix}$$

Edge & Route Flows



$$\begin{array}{c} \text{Routes} \\ \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \right] = \underbrace{\text{Edges} \quad \left[\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \right]}_{\text{Routing Matrix } \mathbb{R}} \times \left[\begin{matrix} x_1^R \\ x_2^R \\ x_3^R \end{matrix} \right] \\ \text{Edge Flows} \\ \text{Route Flows} \\ x = \mathbb{R} \times x^R \end{array}$$

Edge & Route Latencies

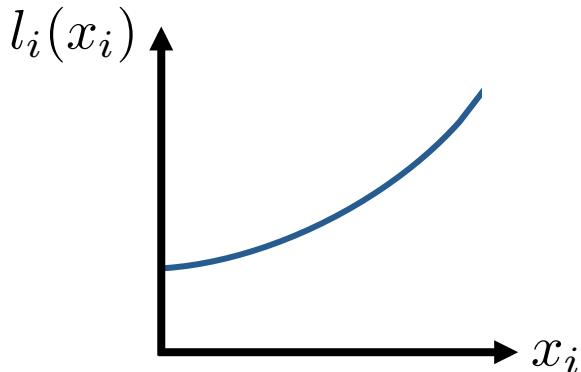
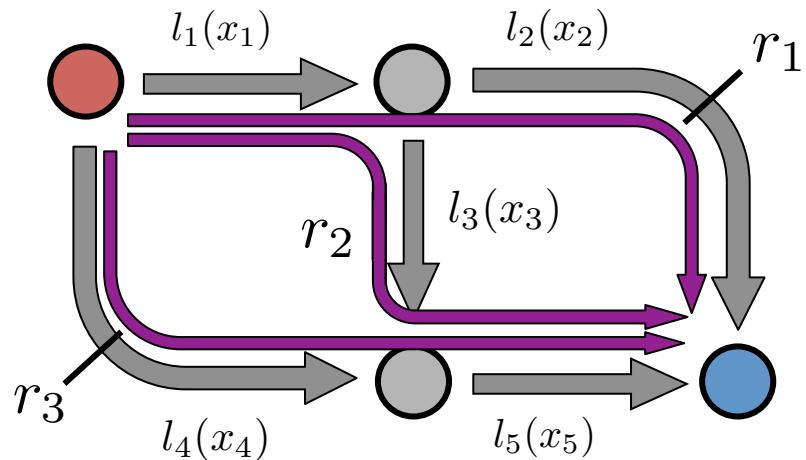


$$\begin{array}{c}
 \text{Routes} \\
 \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \right] = \underbrace{\text{Edges} \left[\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \right] \times \text{Routing Matrix } \mathbb{R}}_{x = \mathbb{R} \times x^R} \times \left[\begin{matrix} x_1^R \\ x_2^R \\ x_3^R \end{matrix} \right] \\
 \text{Route Flows}
 \end{array}$$

Latencies

$$\begin{array}{ll}
 \text{Edge} & l(x) = \left[\begin{matrix} l_1 & l_2 & l_3 & l_4 & l_5 \end{matrix} \right] \\
 \text{Path} & l^R(x) = \left[\begin{matrix} l_1^R & l_2^R & l_3^R \end{matrix} \right]
 \end{array}$$

Edge & Route Latencies



$$\begin{array}{c}
 \text{Routes} \\
 \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \right] = \underbrace{\text{Edges} \quad \left[\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \right]}_{\text{Routing Matrix } \mathbb{R}} \times \left[\begin{matrix} x_1^R \\ x_2^R \\ x_3^R \end{matrix} \right] \\
 \text{Edge Flows} \\
 \text{Route Flows}
 \end{array}$$

$$x = \mathbb{R} \times x^R$$

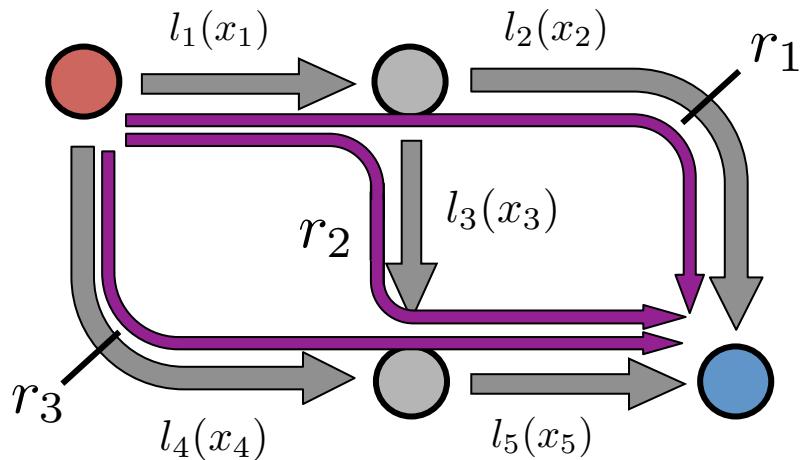
Latencies

$$\text{Edge} \quad l(x) = \left[\begin{matrix} l_1 & l_2 & l_3 & l_4 & l_5 \end{matrix} \right]$$

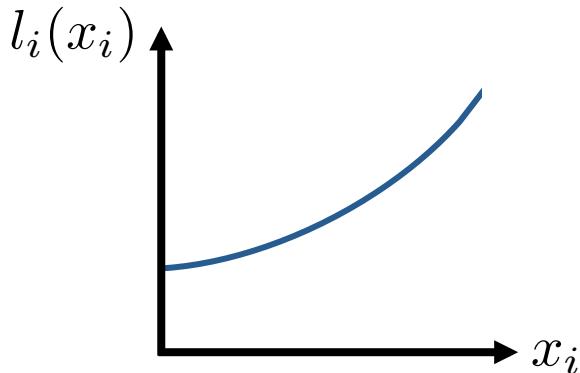
$$\text{Path} \quad l^R(x) = \left[\begin{matrix} l_1^R & l_2^R & l_3^R \end{matrix} \right]$$

$$l \times \mathbb{R} = l^R$$

Potential Function



Potential Function $P(x) = \sum_e \int_0^{x_e} l_e(u) du$



$$\begin{matrix} & \text{Routes} \\ \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \right] & = \underbrace{\text{Edges} \times \text{Routing Matrix } \mathbb{R}}_{\text{Edge Flows}} \times \left[\begin{matrix} x_1^R \\ x_2^R \\ x_3^R \end{matrix} \right] \\ & \text{Route Flows} \end{matrix}$$

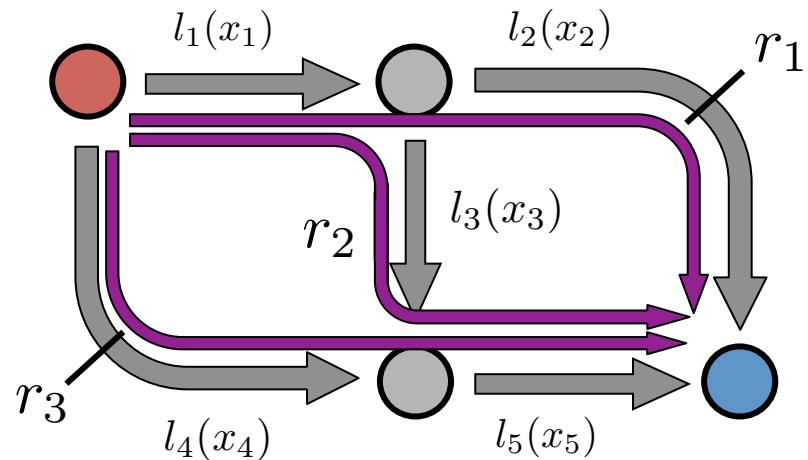
$$x = \mathbb{R} \times x^R$$

Latencies

$$\begin{array}{ll} \text{Edge} & l(x) = [l_1 \ l_2 \ l_3 \ l_4 \ l_5] \\ \text{Path} & l^R(x) = [l_1^R \ l_2^R \ l_3^R] \end{array}$$

$$l \times \mathbb{R} = l^R$$

Potential Function



Potential Function $P(x) = \sum_e \int_0^{x_e} l_e(u) du$

$$\nabla_x P(x) = l(x)$$

Routes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{matrix} \text{Edges} \\ \text{Routing Matrix } \mathbb{R} \end{matrix}}_{\text{Latencies}} \times \begin{bmatrix} x_1^R \\ x_2^R \\ x_3^R \end{bmatrix}$$

Route Flows

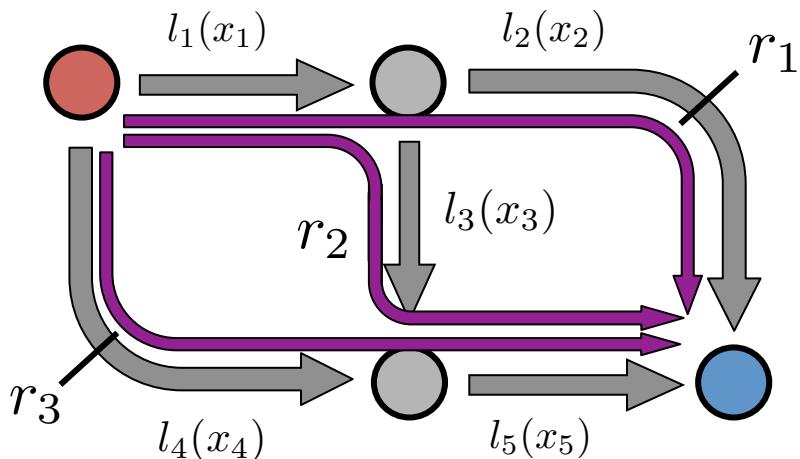
$$x = \mathbb{R} \times x^R$$

Edge $l(x) = [l_1 \ l_2 \ l_3 \ l_4 \ l_5]$

Path $l^R(x) = [l_1^R \ l_2^R \ l_3^R]$

$$l \times \mathbb{R} = l^R$$

Routing Game Formulations



Routes		Edges
$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$		$\mathbb{R} =$
Flows		$x = \mathbb{R} \times x^R$
Latencies		$l \times \mathbb{R} = l^R$

Potential Function $P(x) = \sum_e \int_0^{x_e} l_e(u) du$

$$\begin{array}{ll} \min_{x^R} & P(x) \\ \text{s.t. } & x^R \geq 0, \quad x = \mathbb{R}x^R, \\ & \sum_r x_r^R = s \end{array}$$

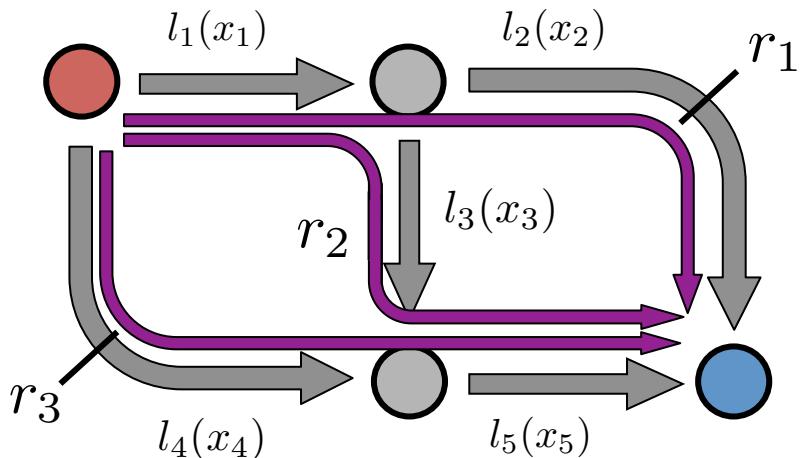
Path Formulation

OR

$$\begin{array}{ll} \min_x & P(x) \\ \text{s.t. } & x \geq 0, \\ & Gx = S \end{array}$$

Edge Formulation

Routing Game Formulations



Potential Function $P(x) = \sum_e \int_0^{x_e} l_e(u) du$

$$\begin{aligned} \min_{x^R} \quad & P(x) \\ \text{s.t.} \quad & x^R \geq 0, \quad x = \mathbb{R}x^R, \\ & \sum_r x_r^R = s \end{aligned}$$

Path Formulation

OR

$$\begin{aligned} \min_x \quad & P(x) \\ \text{s.t.} \quad & x \geq 0, \\ & Gx = S \end{aligned}$$

Edge Formulation

Routing Matrix

$$\mathbb{R} = \begin{matrix} \text{Routes} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \text{Edges} \end{matrix}$$

Flows

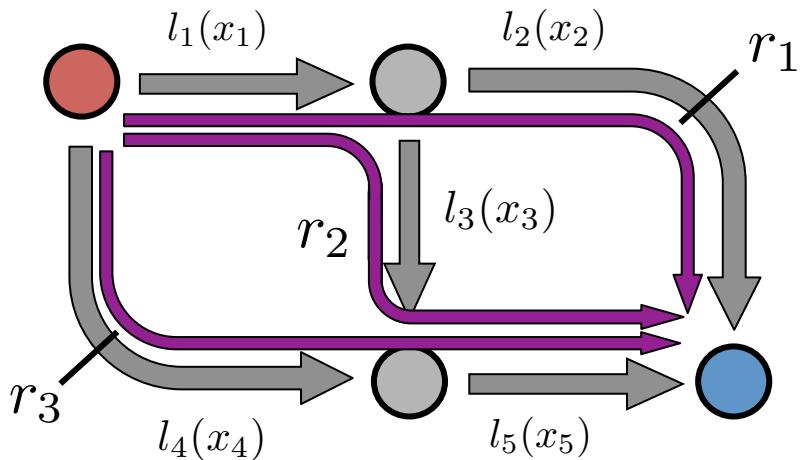
$$x = \mathbb{R} \times x^R$$

Latencies

$$l \times \mathbb{R} = l^R$$

KEY ARGUMENTS

Routing Game Formulations



Potential Function $P(x) = \sum_e \int_0^{x_e} l_e(u) du$

$$\begin{aligned} \min_{x^R} \quad & P(x) \\ \text{s.t.} \quad & x^R \geq 0, \quad x = \mathbb{R}x^R, \\ & \sum_r x_r^R = s \end{aligned}$$

Path Formulation

OR

$$\begin{aligned} \min_x \quad & P(x) \\ \text{s.t.} \quad & x \geq 0, \\ & Gx = S \end{aligned}$$

Edge Formulation

Routing Matrix

$$\mathbb{R} = \begin{matrix} \text{Routes} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \text{Edges} \end{matrix}$$

Flows

$$x = \mathbb{R} \times x^R$$

Latencies

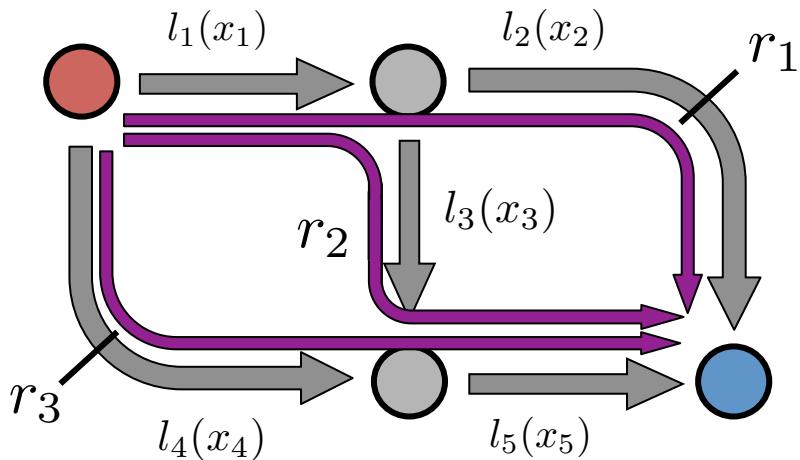
$$l \times \mathbb{R} = l^R$$

KEY ARGUMENTS

First Order Optimality

$$\underbrace{\nabla P(x)}_{l(x)} + \lambda \cdot \text{CONST} - \underbrace{\mu}_{\geq 0} = 0$$

Routing Game Formulations



Potential Function $P(x) = \sum_e \int_0^{x_e} l_e(u) du$

$$\begin{aligned} \min_{x^R} \quad & P(x) \\ \text{s.t.} \quad & x^R \geq 0, \quad x = \mathbb{R}x^R, \\ & \sum_r x_r^R = s \end{aligned}$$

Path Formulation

OR

$$\begin{aligned} \min_x \quad & P(x) \\ \text{s.t.} \quad & x \geq 0, \\ & Gx = S \end{aligned}$$

Edge Formulation

Routing Matrix

$$\mathbb{R} = \begin{matrix} \text{Routes} \\ \hline \text{Edges} \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Flows

$$x = \mathbb{R} \times x^R$$

Latencies

$$l \times \mathbb{R} = l^R$$

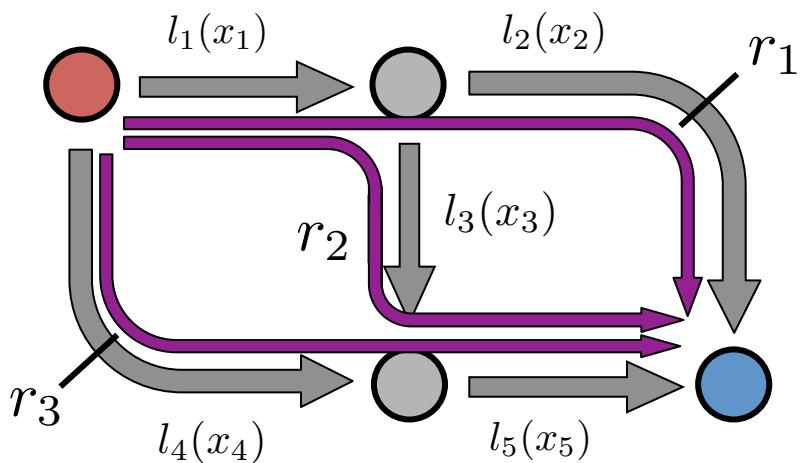
KEY ARGUMENTS

First Order Optimality

$$\underbrace{\nabla P(x)}_{l(x)} + \lambda \cdot \text{CONST} - \underbrace{\mu}_{\geq 0} = 0$$

Wardrop Equilibrium

Wardrop Equilibrium



Potential Function $P(x) = \sum_e \int_0^{x_e} l_e(u) du$

$$\begin{array}{ll} \min_{x^R} & P(x) \\ \text{s.t. } & x^R \geq 0, \quad x = \mathbb{R}x^R, \\ & \sum_r x_r^R = s \end{array}$$

Path Formulation

OR

$$\begin{array}{ll} \min_{x^R} & P(x) \\ \text{s.t. } & x \geq 0, \\ & Gx = S \end{array}$$

Edge Formulation

Routing Matrix

$$\mathbb{R} = \begin{matrix} \text{Routes} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \text{Edges} \end{matrix}$$

Flows

$$x = \mathbb{R} \times x^R$$

Latencies

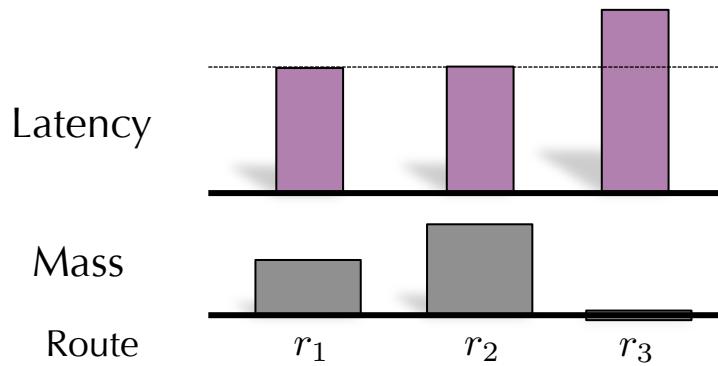
$$l \times \mathbb{R} = l^R$$

KEY ARGUMENTS

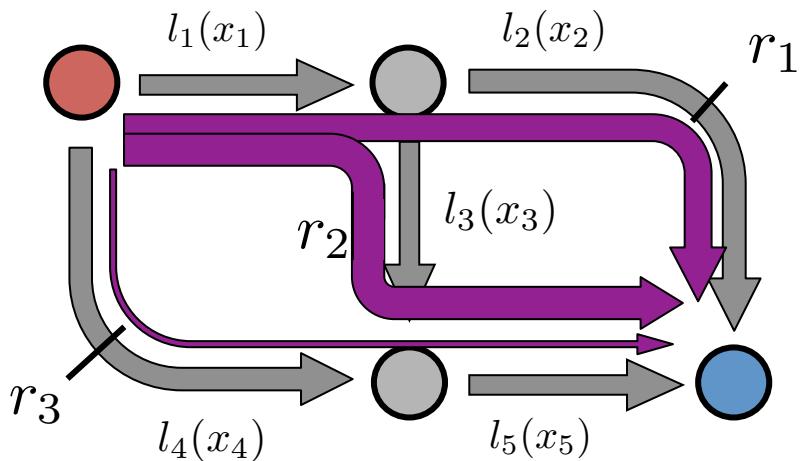
First Order Optimality

$$\underbrace{\nabla P(x)}_{l(x)} + \lambda \cdot \text{CONST} - \underbrace{\mu}_{\geq 0} = 0$$

Wardrop Equilibrium



Path Formulation



Path Formulation

$$\min_{x^R} \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x^R \geq 0, \quad x = \mathbb{R}x^R,$$

$$\sum_r x_r^R = s$$

Routing Matrix

$$\mathbb{R} = \begin{matrix} & \text{Routes} \\ & [1 \ 1 \ 0] \\ & [1 \ 0 \ 0] \\ & [0 \ 1 \ 0] \\ & [0 \ 0 \ 1] \\ & [0 \ 1 \ 1] \end{matrix}$$

Edges

Flows

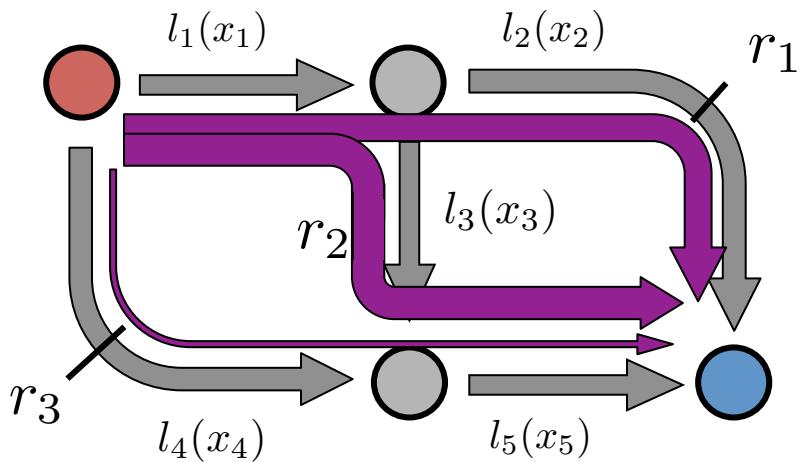
$$x = \mathbb{R} \times x^R$$

Latencies

$$l \times \mathbb{R} = l^R$$

First Order Optimality...

Path Formulation



Path Formulation

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Flows

$$x = \mathbb{R} \times x^R$$

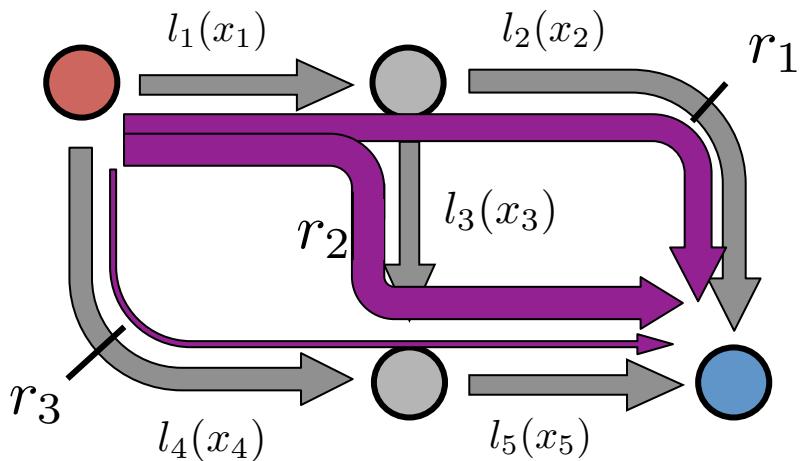
Latencies

$$l \times \mathbb{R} = l^R$$

First Order Optimality...

Gradient... $\nabla_{x^R} P = l(x)\mathbb{R} = l^R(x)$

Path Formulation



Path Formulation

$$\min_{x^R} \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

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Flows

$$x = \mathbb{R} \times x^R$$

Latencies

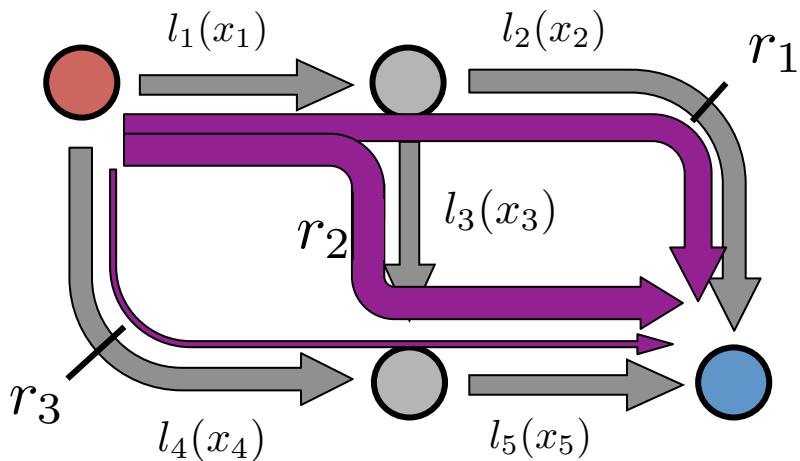
$$l \times \mathbb{R} = l^R$$

First Order Optimality...

Gradient... $\nabla_{x^R} P = l(x)\mathbb{R} = l^R(x)$

$$\mathcal{L}(x, \lambda, \mu) = P(x) - \lambda(\mathbf{1}^T x^R - s) - \mu^T x^R$$

Path Formulation



Path Formulation

$$\min_{x^R} \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x^R \geq 0, \quad x = \mathbb{R}x^R,$$

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Edges

Flows

$$x = \mathbb{R} \times x^R$$

Latencies

$$l \times \mathbb{R} = l^R$$

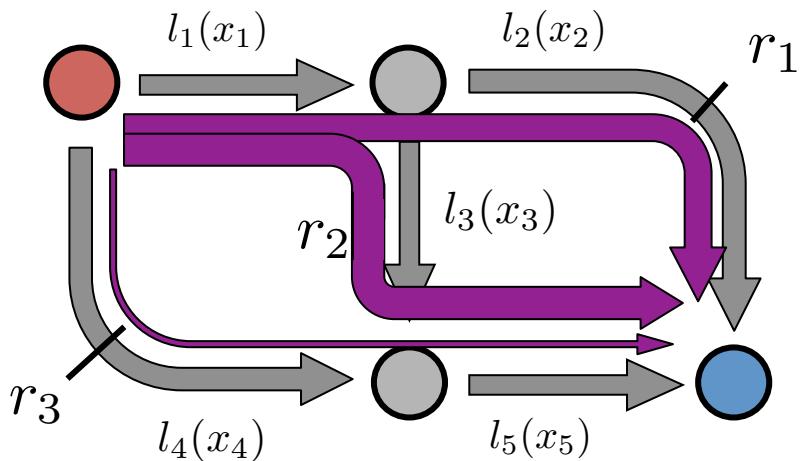
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Gradient... $\nabla_{x^R} P = l(x)\mathbb{R} = l^R(x)$

$$\mathcal{L}(x, \lambda, \mu) = P(x) - \lambda(\mathbf{1}^T x^R - s) - \mu^T x^R$$

$$\frac{\partial \mathcal{L}}{\partial x^R} : \quad l^R(x) = \lambda \mathbf{1}^T + \mu^T$$

Path Formulation



Path Formulation

$$\min_{x^R} \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

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Routing Matrix

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Flows

$$x = \mathbb{R} \times x^R$$

Latencies

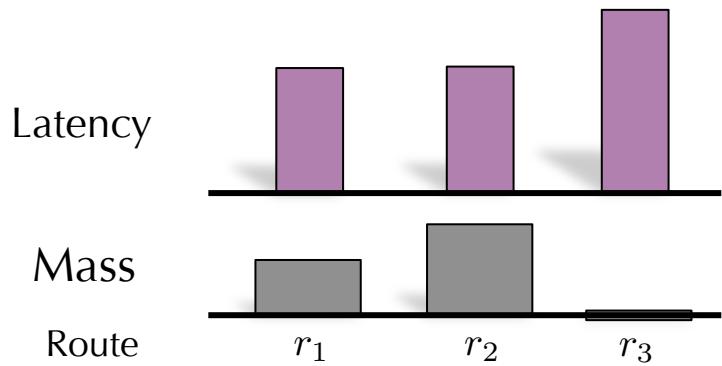
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First Order Optimality...

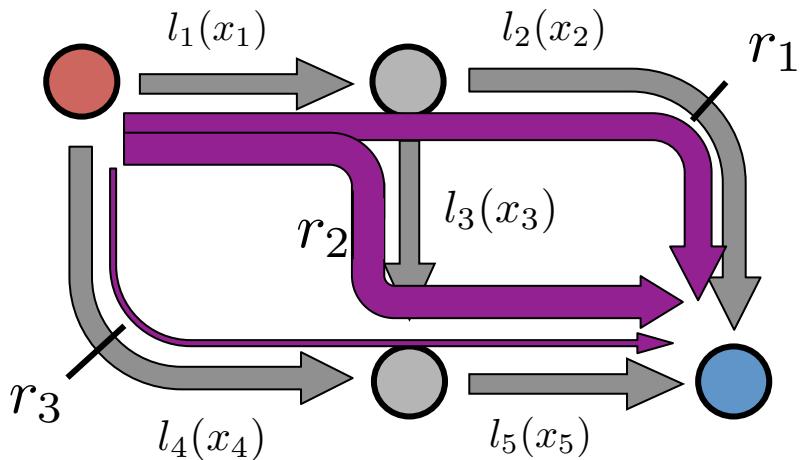
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Path Formulation



Path Formulation

$$\min_{x^R} \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x^R \geq 0, \quad x = \mathbb{R}x^R,$$

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Routing Matrix $\mathbb{R} = \begin{matrix} & \text{Routes} \\ & [1 \ 1 \ 0] \\ & [1 \ 0 \ 0] \\ & [0 \ 1 \ 0] \\ & [0 \ 0 \ 1] \\ & [0 \ 1 \ 1] \end{matrix}$

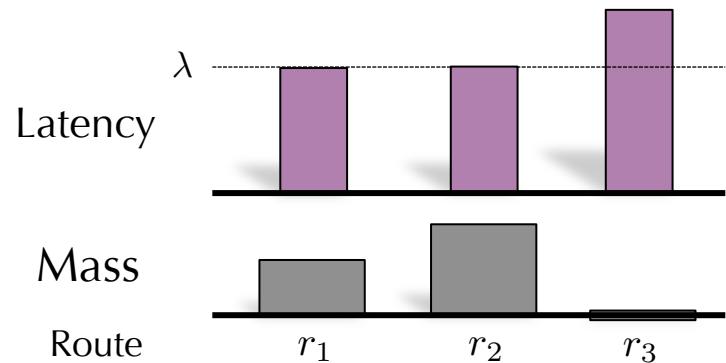
Flows $x = \mathbb{R} \times x^R$
Latencies $l \times \mathbb{R} = l^R$

First Order Optimality...

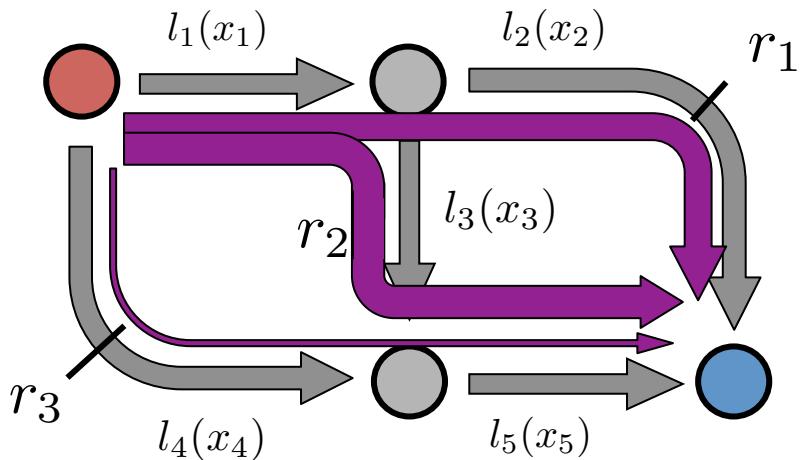
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$$\mathcal{L}(x, \lambda, \mu) = P(x) - \lambda(\mathbf{1}^T x^R - s) - \mu^T x^R$$

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Path Formulation



Path Formulation

$$\min_{x^R} \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x^R \geq 0, \quad x = \mathbb{R}x^R,$$

$$\sum_r x_r^R = s$$

Routing Matrix $\mathbb{R} = \begin{matrix} & \text{Routes} \\ & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \text{Edges} & \end{matrix}$

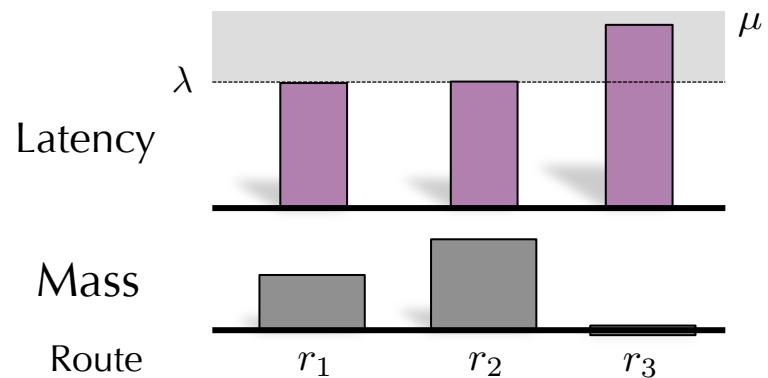
Flows $x = \mathbb{R} \times x^R$
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First Order Optimality...

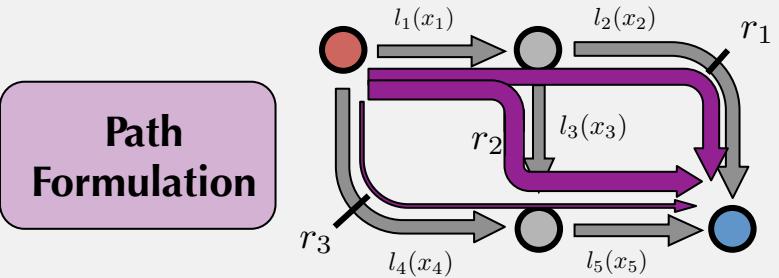
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Path Formulation

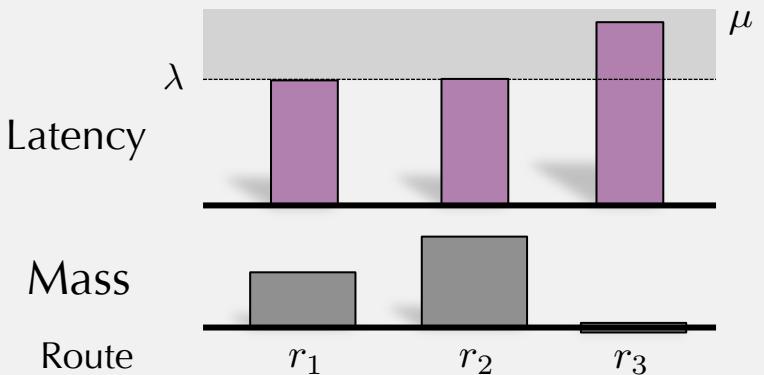


$$\min_{x^R} P(x) \text{ s.t. } x^R \geq 0, x = \mathbb{R}x^R, \sum_r x_r^R = s$$

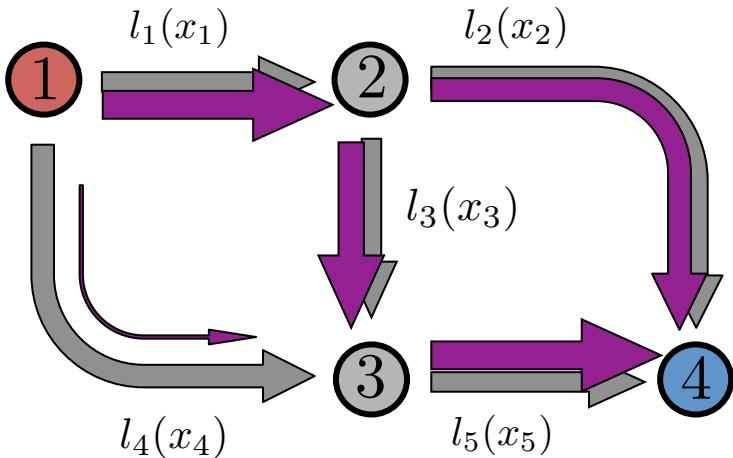
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Path Formulation

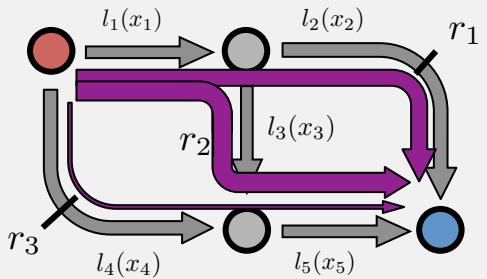


Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

Path Formulation

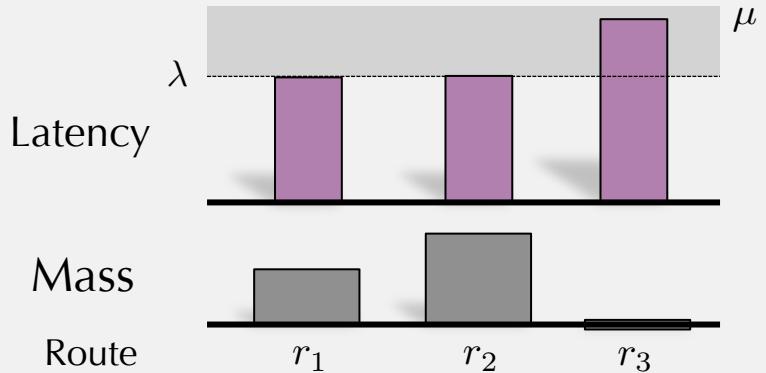


$$\min_{x^R} P(x) \text{ s.t. } x^R \geq 0, x = \mathbb{R}x^R, \sum_r x_r^R = s$$

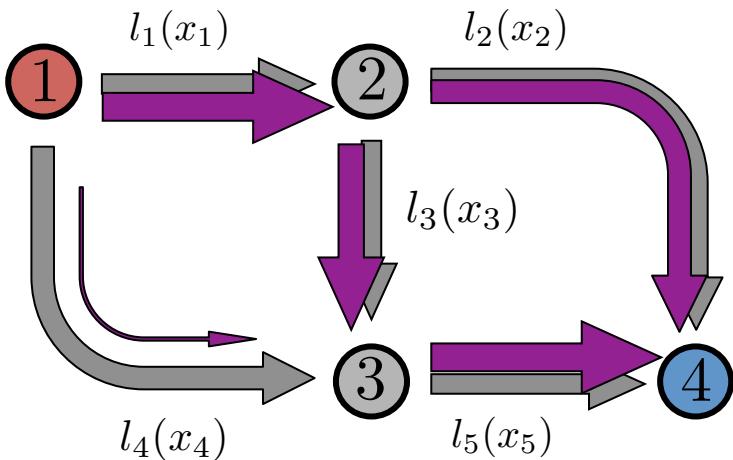
$$\textbf{Gradient...} \quad \nabla_{x^R} P = l(x)\mathbb{R} = l^R(x)$$

$$\mathcal{L}(x, \lambda, \mu) = P(x) - \lambda(\mathbf{1}^T x^R - s) - \mu^T x^R$$

$$\frac{\partial \mathcal{L}}{\partial x^R} : \quad l^R(x) = \lambda \mathbf{1}^T + \mu^T$$



Path Formulation



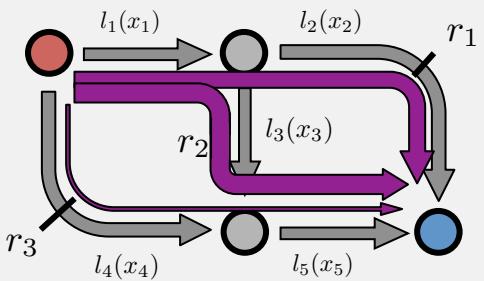
Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

$$G \left\{ \begin{array}{c} \text{edges} \\ \text{nodes} \end{array} \right. \left[\begin{matrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{matrix} \right] \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \right] = \left[\begin{matrix} s \\ 0 \\ 0 \\ 0 \\ -s \end{matrix} \right] \right\} S$$

Path Formulation

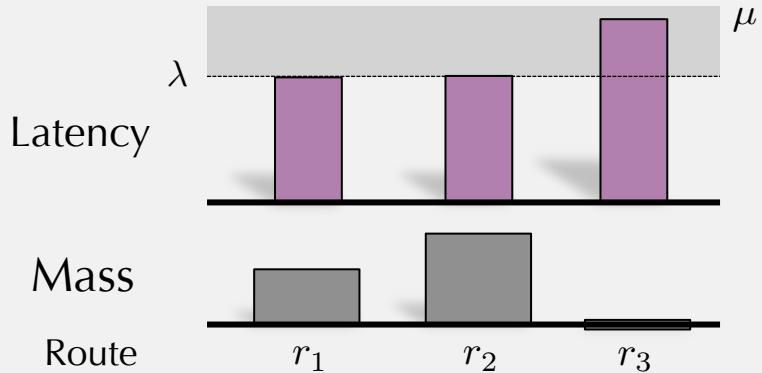


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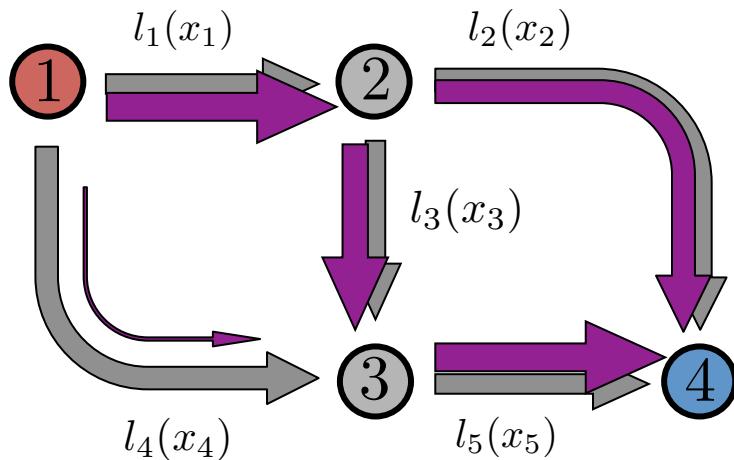
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Edge Formulation



First Order Optimality...

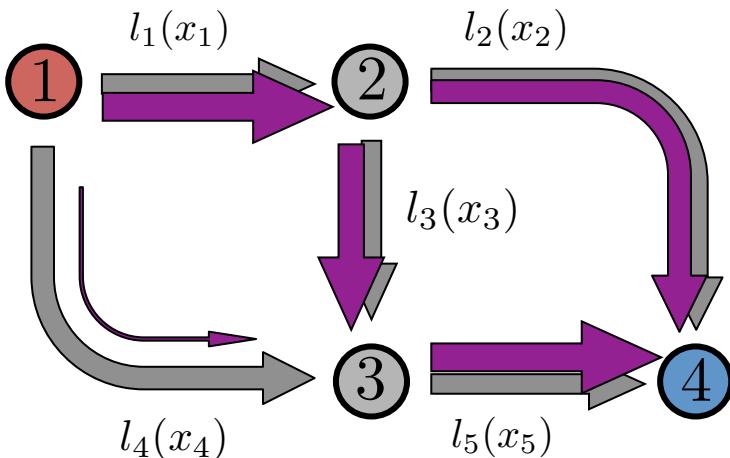
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Edge Formulation



First Order Optimality...

Gradient... $\nabla_x P = l(x)$

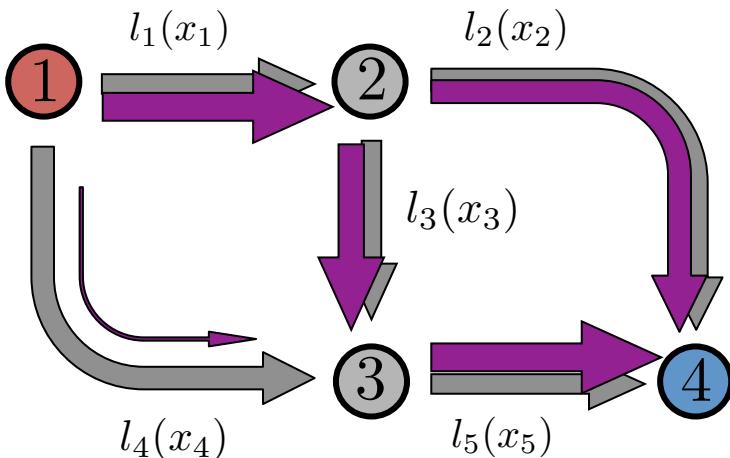
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$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

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Edge Formulation



First Order Optimality...

Gradient... $\nabla_x P = l(x)$

$$\mathcal{L}(x, \pi, \nu) = P(x) - \pi^T(Gx - S) - \nu^T x$$

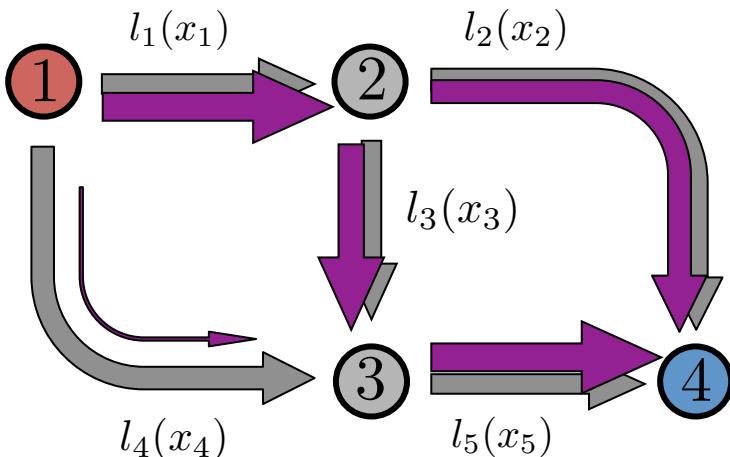
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Edge Formulation



Edge Formulation

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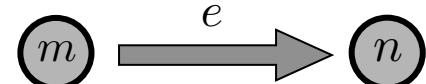
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First Order Optimality...

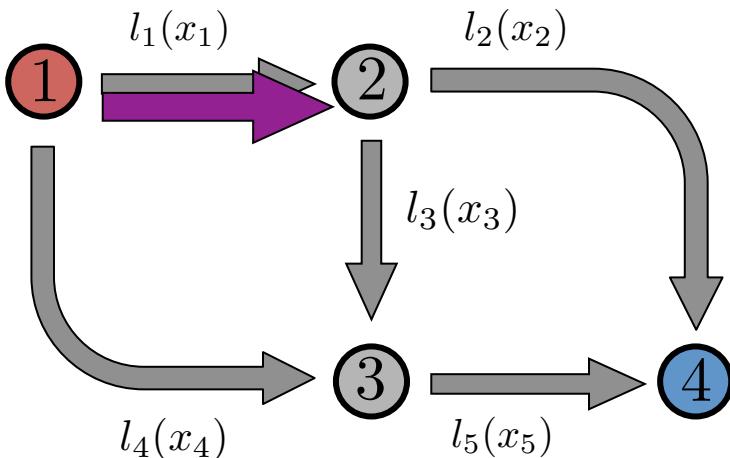
Gradient... $\nabla_x P = l(x)$

$$\mathcal{L}(x, \pi, \nu) = P(x) - \pi^T(Gx - S) - \nu^T x$$

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Edge Formulation



Edge Formulation

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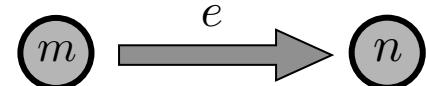
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First Order Optimality...

Gradient... $\nabla_x P = l(x)$

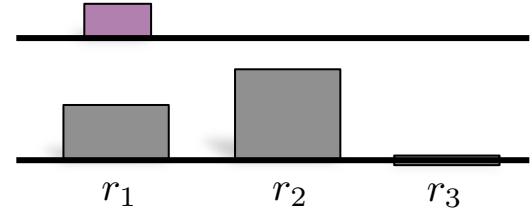
$$\mathcal{L}(x, \pi, \nu) = P(x) - \pi^T(Gx - S) - \nu^T x$$

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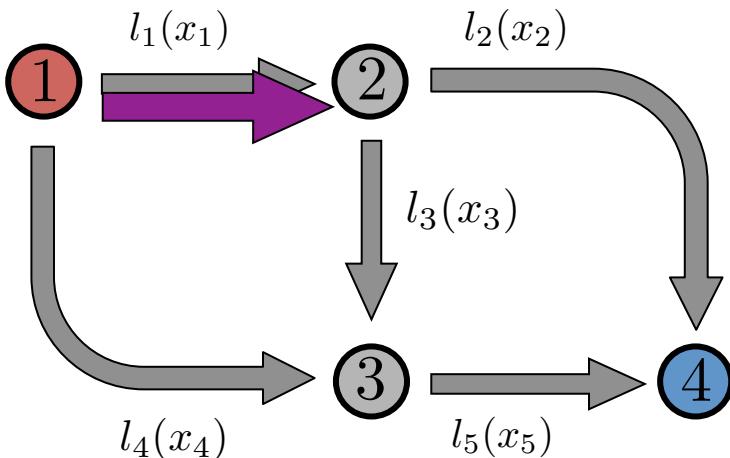
$$r_1 : \quad l_1 = \pi_2 - \pi_1 + \nu_1$$

Latency



Mass
Route

Edge Formulation



Edge Formulation

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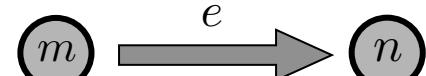
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First Order Optimality...

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$$\mathcal{L}(x, \pi, \nu) = P(x) - \pi^T(Gx - S) - \nu^T x$$

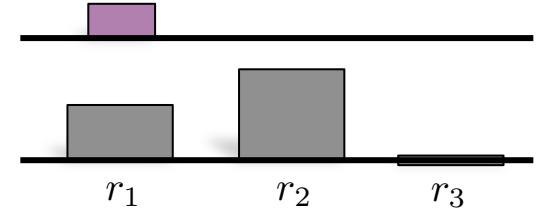
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$$r_1 :$$

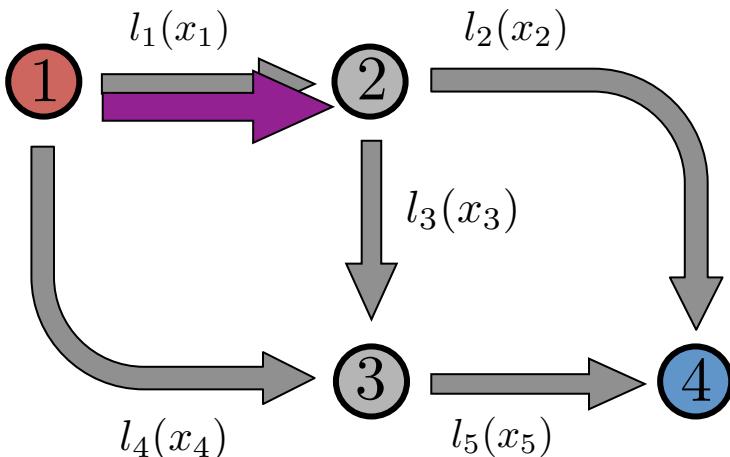
$$l_1 = \pi_2 - \pi_1 + \cancel{\nu_1}^0$$

Latency



Mass
Route

Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

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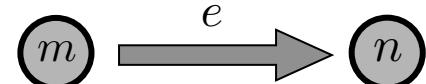
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First Order Optimality...

Gradient... $\nabla_x P = l(x)$

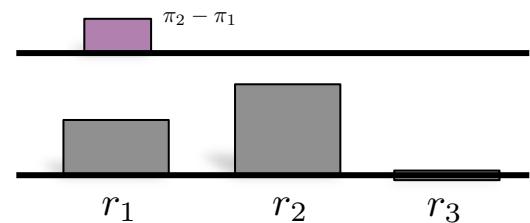
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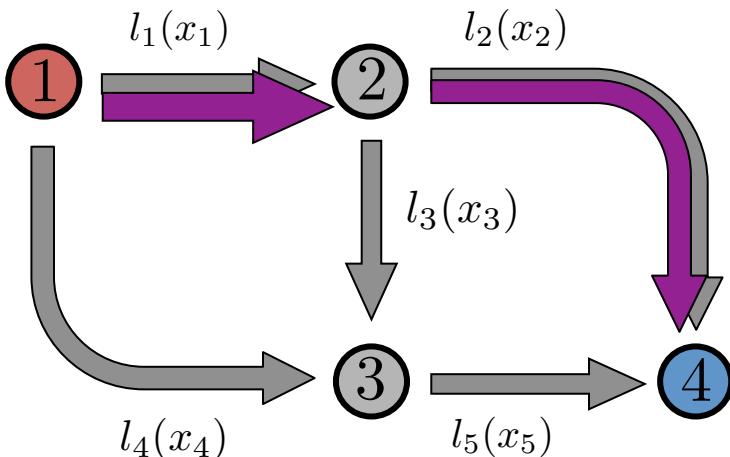
Latency



Mass

Route

Edge Formulation



Edge Formulation

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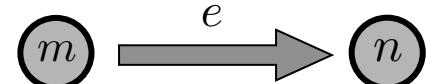
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First Order Optimality...

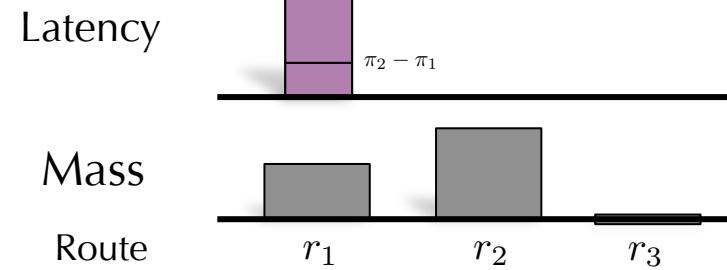
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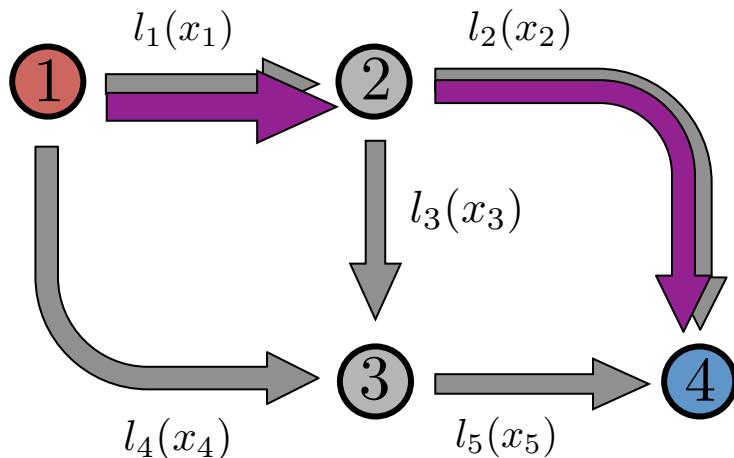
$$\frac{\partial \mathcal{L}}{\partial x_e} : l_e(x_e) = (\pi_n - \pi_m) + \nu_e$$



$$r_1 : \quad l_2 + l_1 = \pi_4 - \pi_2 + \pi_2 - \pi_1 + \nu_2$$



Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

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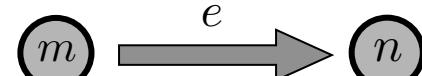
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First Order Optimality...

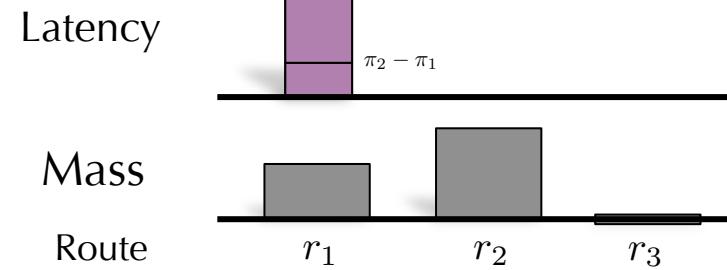
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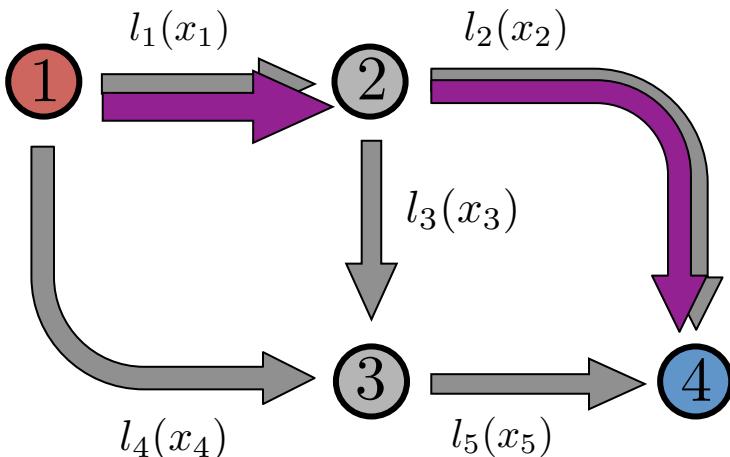
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$$r_1 : \quad l_2 + l_1 = \pi_4 - \cancel{\pi_2} + \cancel{\pi_2}^0 - \pi_1 + \cancel{\nu_2}^0$$



Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

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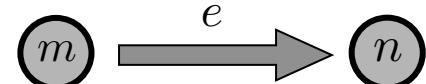
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First Order Optimality...

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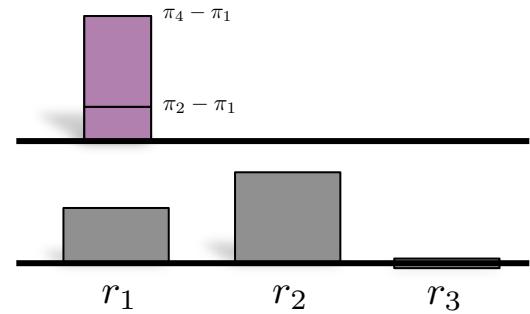
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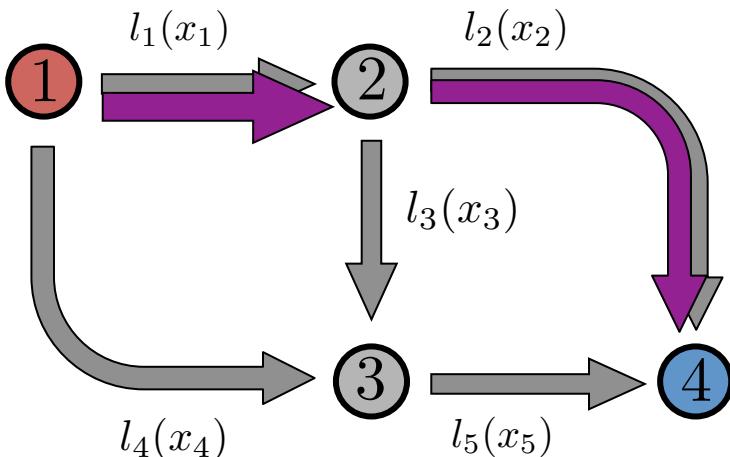


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Latency



Edge Formulation



Edge Formulation

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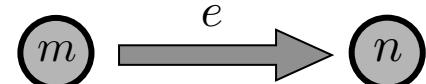
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First Order Optimality...

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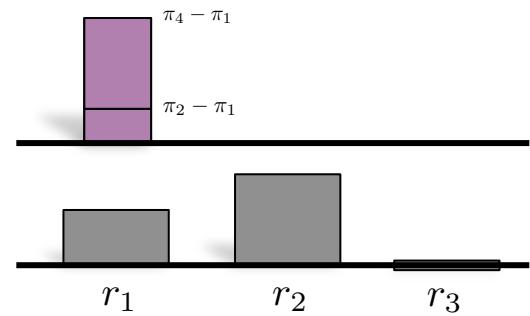
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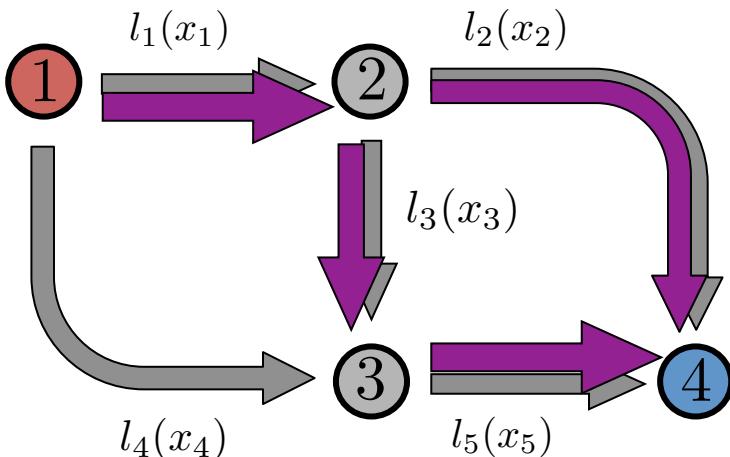
Latency



Mass

Route

Edge Formulation



Edge Formulation

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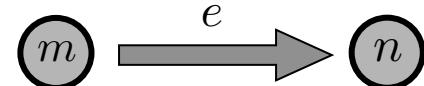
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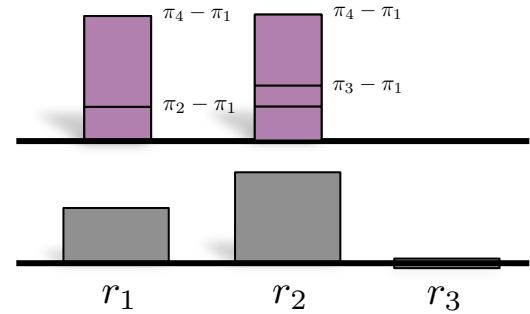
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$$r_1 : \quad l_1^R = \pi_4 - \pi_1$$

$$r_2 : l_1 + l_3 + l_5 = \pi_4 - \pi_1 + \nu_1 + \nu_3 + \nu_5$$

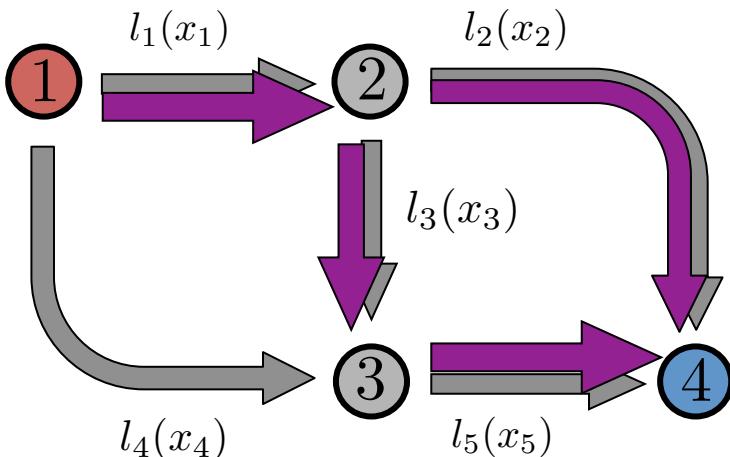
Latency



Mass

Route

Edge Formulation



Edge Formulation

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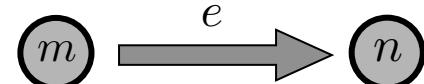
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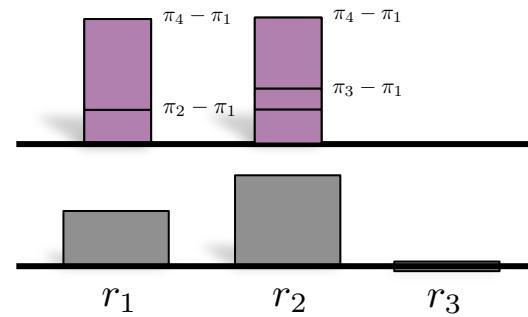
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$$r_1 : \quad l_1^R = \pi_4 - \pi_1$$

$$r_2 : l_1 + l_3 + l_5 = \pi_4 - \pi_1 + \cancel{\nu_1} + \cancel{\nu_3} + \cancel{\nu_5} + 0$$

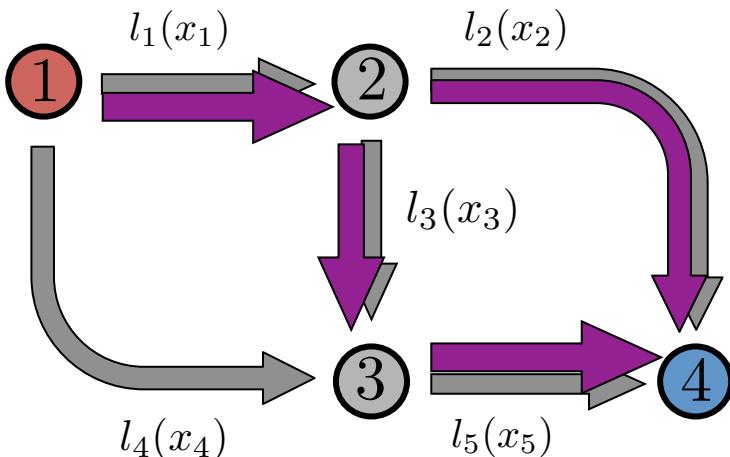
Latency



Mass

Route

Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

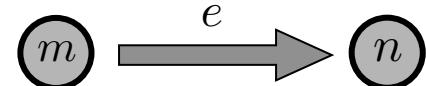
$$G \left\{ \begin{array}{c} \text{edges} \\ \text{nodes} \end{array} \right. \left[\begin{matrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{matrix} \right] \left[\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \right] = \left[\begin{matrix} s \\ 0 \\ 0 \\ 0 \\ -s \end{matrix} \right] \right\} S$$

First Order Optimality...

Gradient... $\nabla_x P = l(x)$

$$\mathcal{L}(x, \pi, \nu) = P(x) - \pi^T (Gx - S) - \nu^T x$$

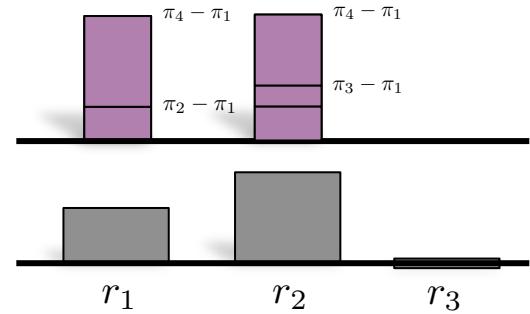
$$\frac{\partial \mathcal{L}}{\partial x_e} : l_e(x_e) = (\pi_n - \pi_m) + \nu_e$$



$$r_1 : \quad l_1^R = \pi_4 - \pi_1$$

$$r_2 : \quad l_2^R = \pi_4 - \pi_1$$

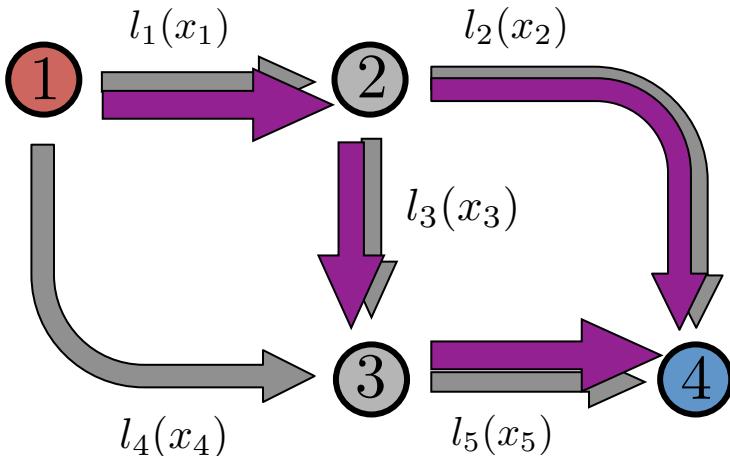
Latency



Mass

Route

Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

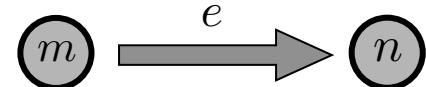
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First Order Optimality...

Gradient... $\nabla_x P = l(x)$

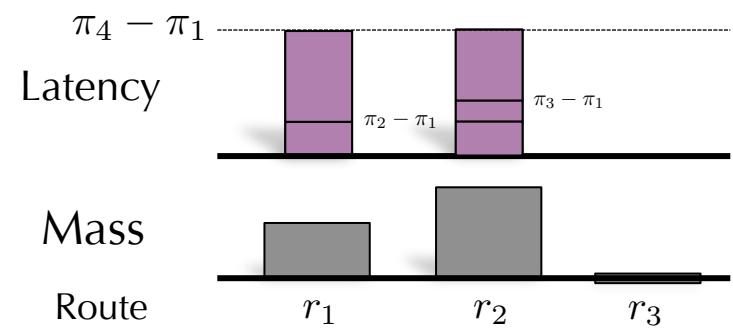
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$$\frac{\partial \mathcal{L}}{\partial x_e} : l_e(x_e) = (\pi_n - \pi_m) + \nu_e$$

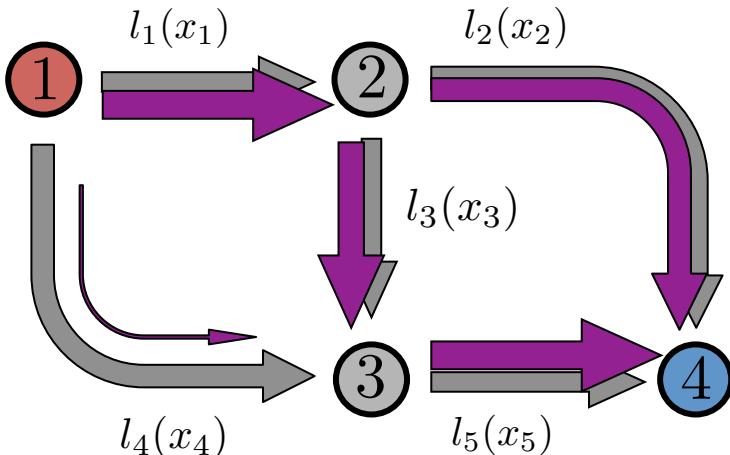


$$r_1 : \quad l_1^R = \pi_4 - \pi_1$$

$$r_2 : \quad l_2^R = \pi_4 - \pi_1$$



Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

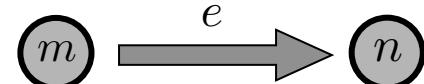
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First Order Optimality...

Gradient... $\nabla_x P = l(x)$

$$\mathcal{L}(x, \pi, \nu) = P(x) - \pi^T(Gx - S) - \nu^T x$$

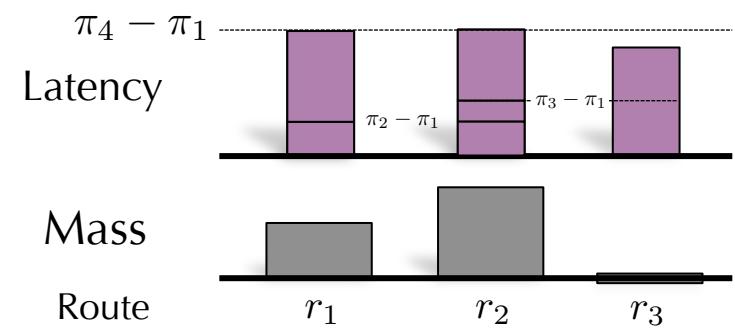
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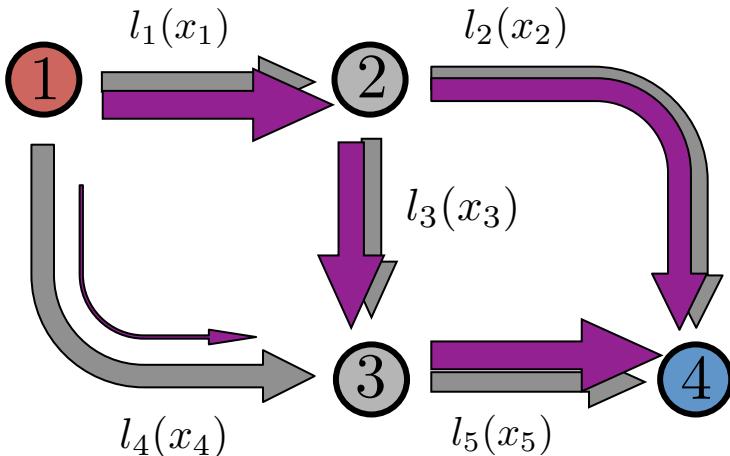
$$r_1 : \quad l_1^R = \pi_4 - \pi_1$$

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$$r_3 : \quad l_4 = \pi_3 - \pi_1 + \nu_4$$



Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

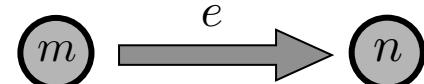
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First Order Optimality...

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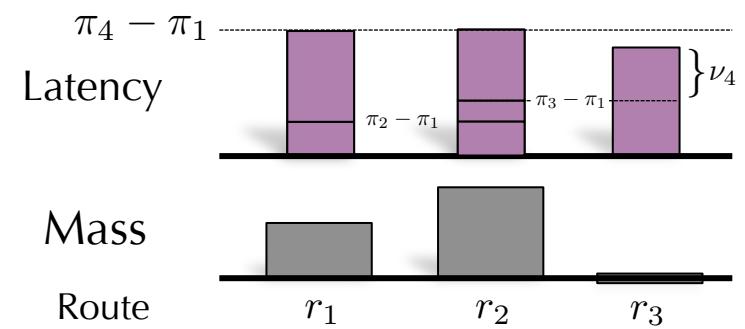
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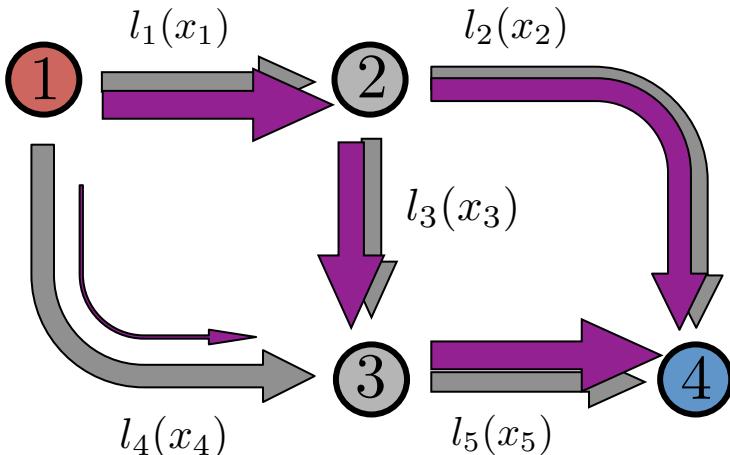
$$r_1 : \quad l_1^R = \pi_4 - \pi_1$$

$$r_2 : \quad l_2^R = \pi_4 - \pi_1$$

$$r_3 : \quad l_4 = \pi_3 - \pi_1 + \nu_4 \neq 0$$



Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

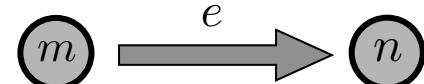
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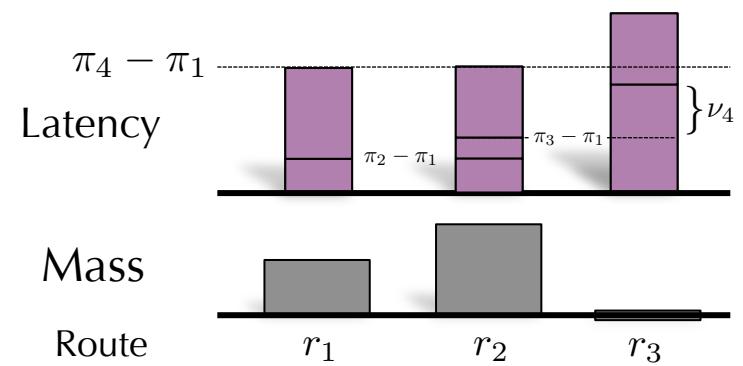
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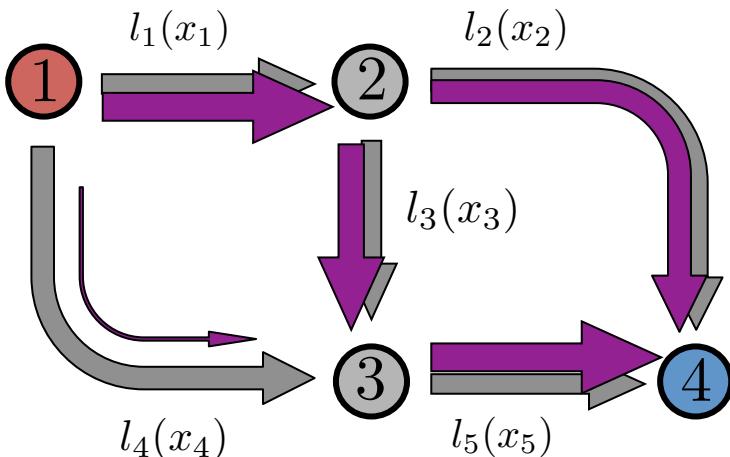
$$r_1 : \quad l_1^R = \pi_4 - \pi_1$$

$$r_2 : \quad l_2^R = \pi_4 - \pi_1$$

$$r_3 : \quad l_5 + l_4 = \pi_4 - \pi_1 + \nu_4 + \cancel{\nu_5} \neq 0$$



Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \ du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

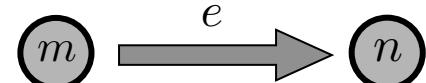
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First Order Optimality...

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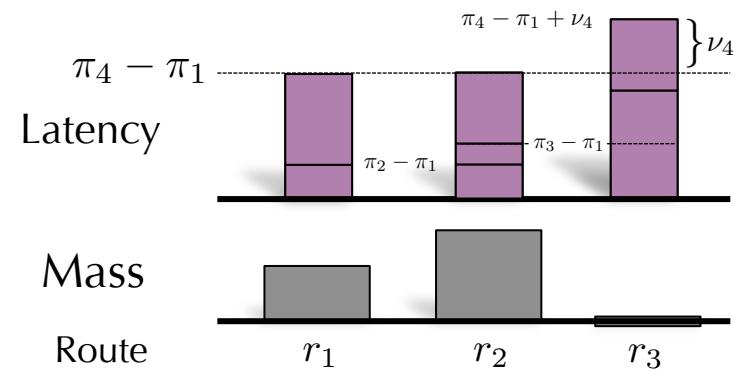
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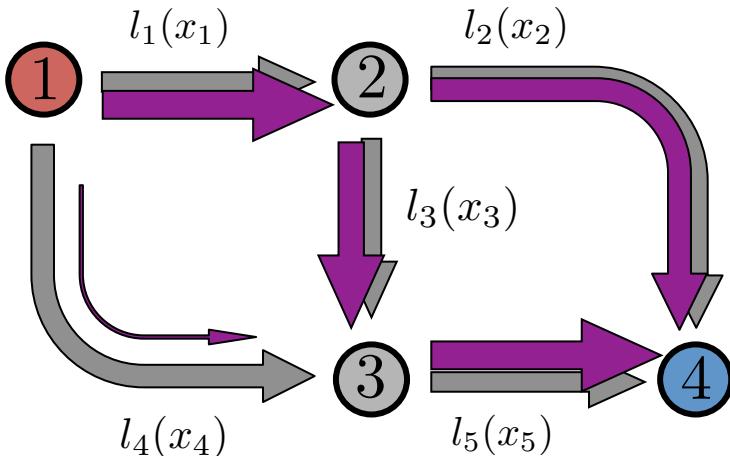
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Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

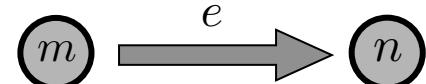
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First Order Optimality...

Gradient... $\nabla_x P = l(x)$

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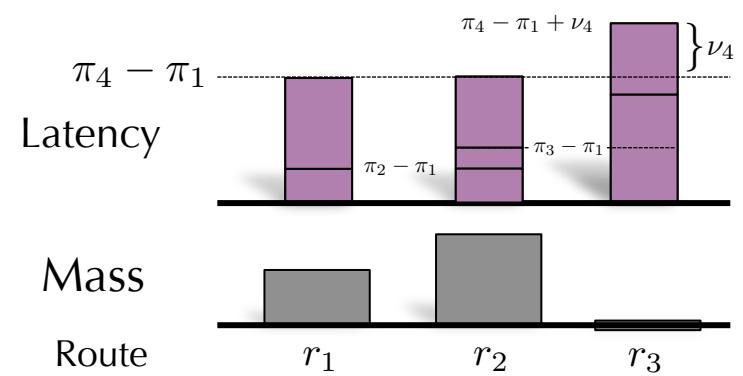
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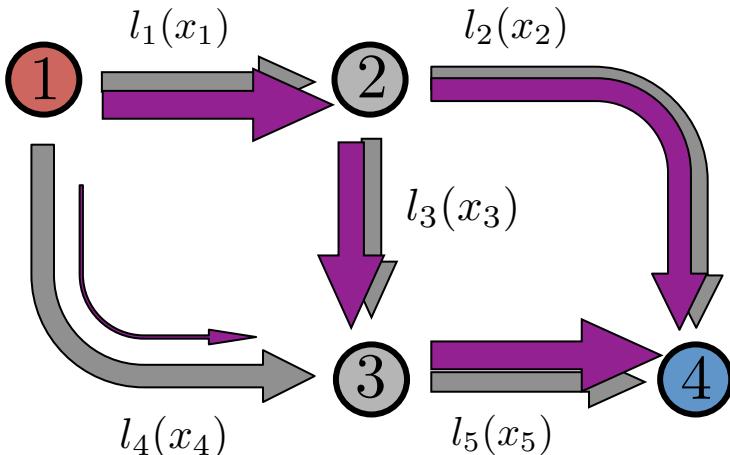
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$$r_3 : \quad l_3^R = \pi_4 - \pi_1 + \nu_4$$



Edge Formulation



Edge Formulation

$$\min_x \quad P(x) = \sum_e \int_0^{x_e} l_e(u) \, du$$

$$\text{s.t.} \quad x \geq 0, \quad Gx = S$$

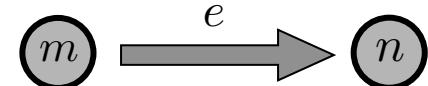
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First Order Optimality...

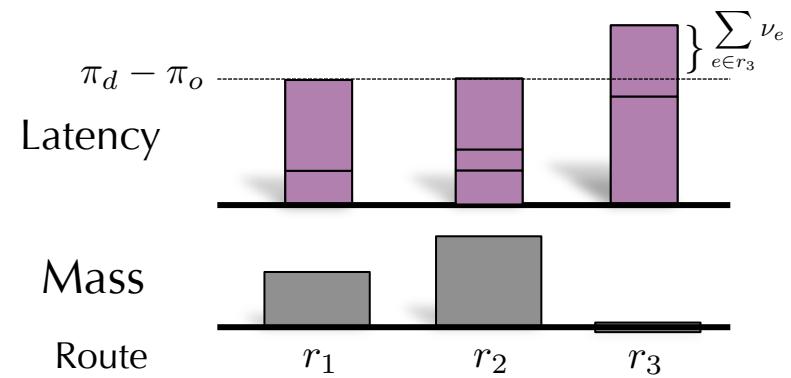
Gradient... $\nabla_x P = l(x)$

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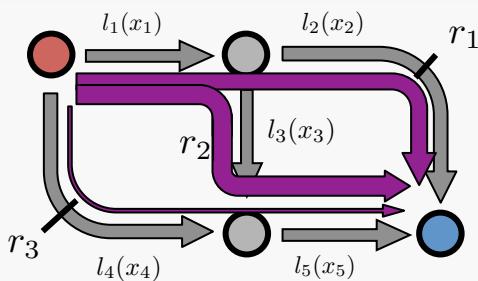


$$\begin{aligned} l_r^R &= \sum_{e \in r} l_e(x_e) \\ &= \pi_d - \pi_o + \sum_{e \in r} \nu_e \end{aligned}$$



Summary

Path Formulation

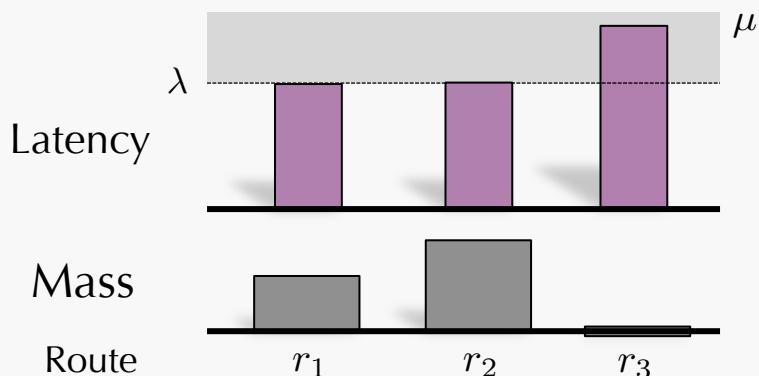


$$\min_{x^R} P(x) \text{ s.t. } x^R \geq 0, x = \mathbb{R}x^R, \sum_r x_r^R = s$$

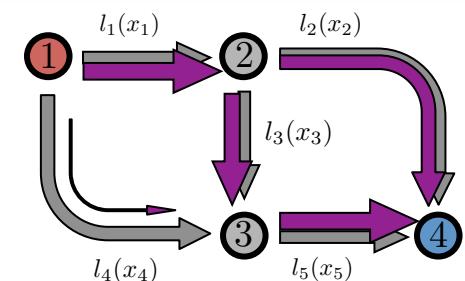
Gradient... $\nabla_{x^R} P = l(x)\mathbb{R} = l^R(x)$

$$\mathcal{L}(x, \lambda, \mu) = P(x) - \lambda(\mathbf{1}^T x^R - s) - \mu^T x^R$$

$$\frac{\partial \mathcal{L}}{\partial x^R} : \quad l^R(x) = \lambda \mathbf{1}^T + \mu^T$$



Edge Formulation

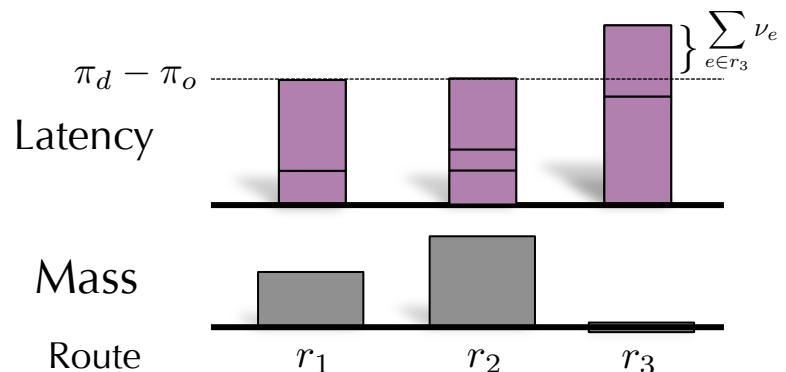


$$\min_x P(x) \text{ s.t. } x \geq 0, \quad Gx = S$$

Gradient... $\nabla_x P = l(x)$

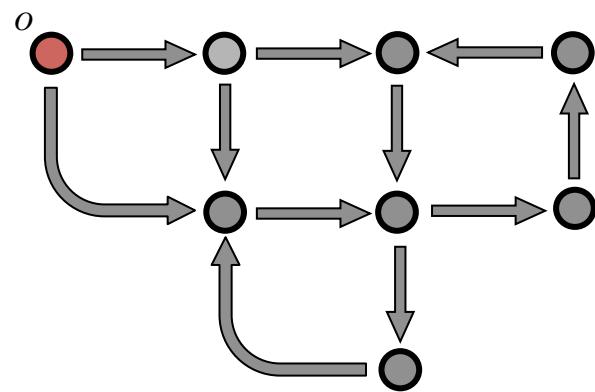
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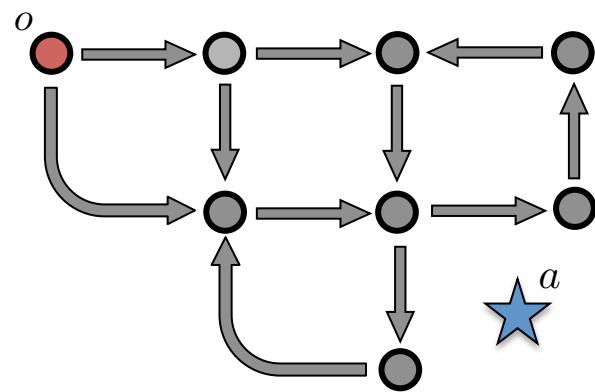


Parking Traffic Problem

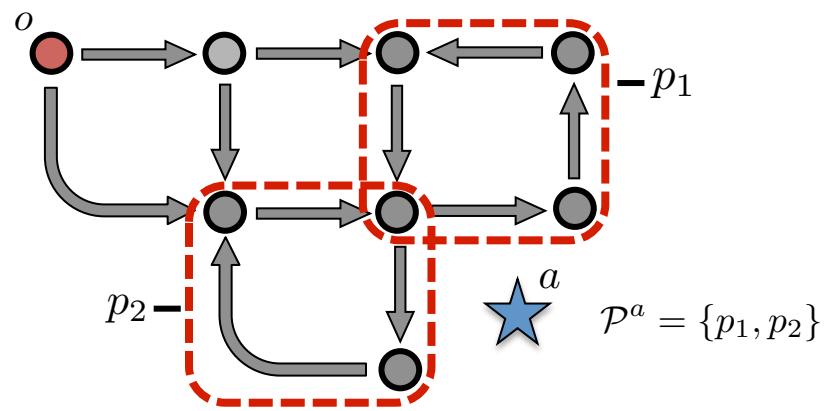
Edge Formulation



Edge Formulation



Edge Formulation



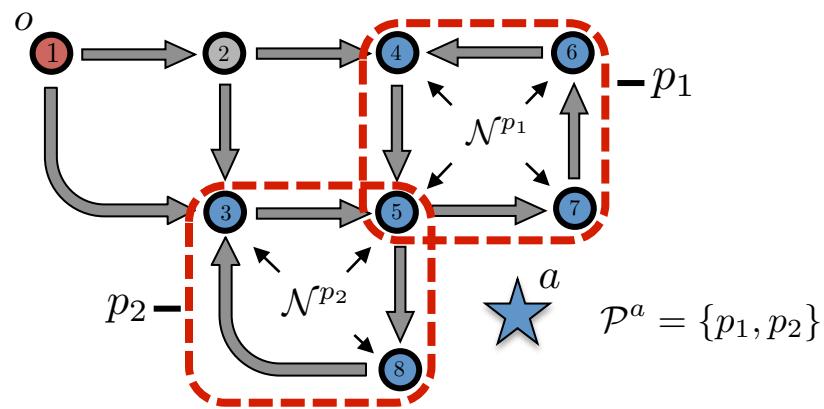
Strategy

Parking Area

Area 1

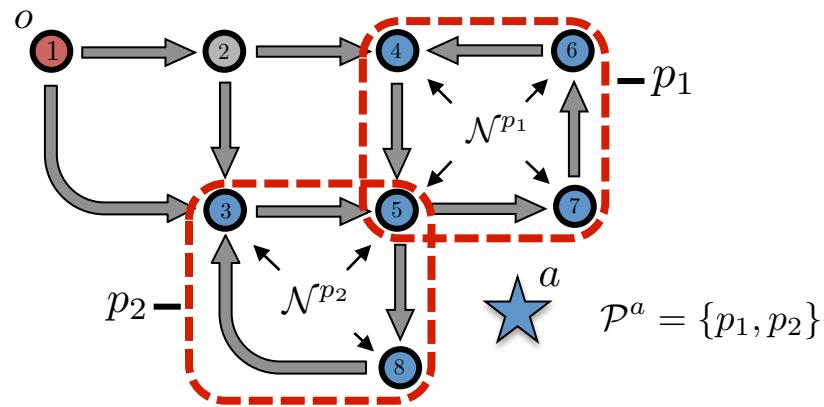
Area 2

Edge Formulation



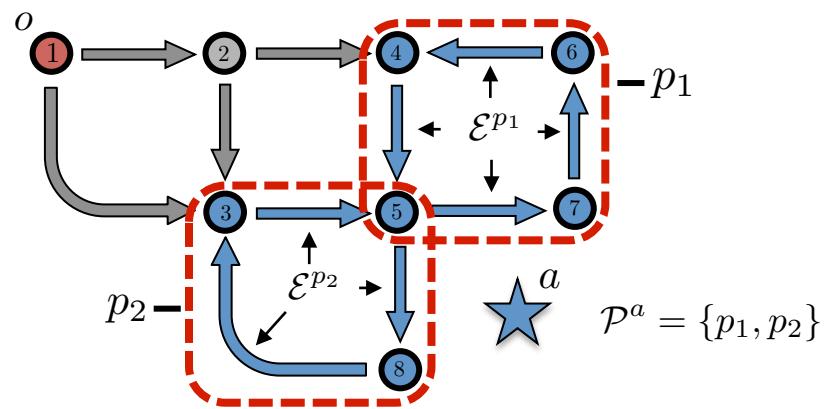
Strategy	Node	4	5	6	7	3	5	8
	Parking Area	Area 1				Area 2		

Edge Formulation



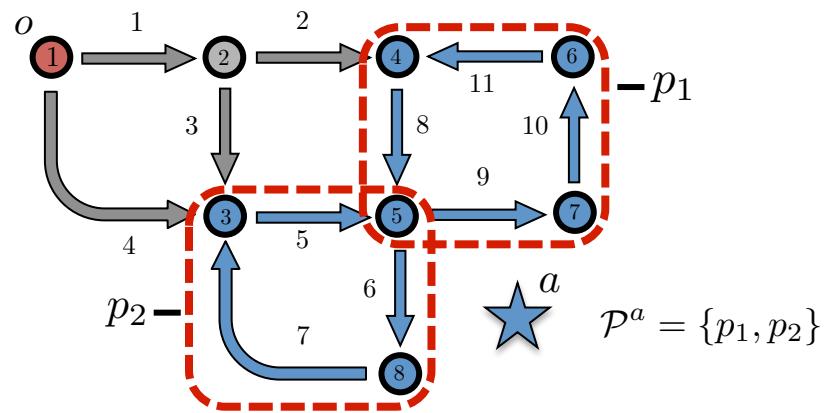
Strategy	Node	4	5	5	3
Parking Area		Area 1		Area 2	

Edge Formulation



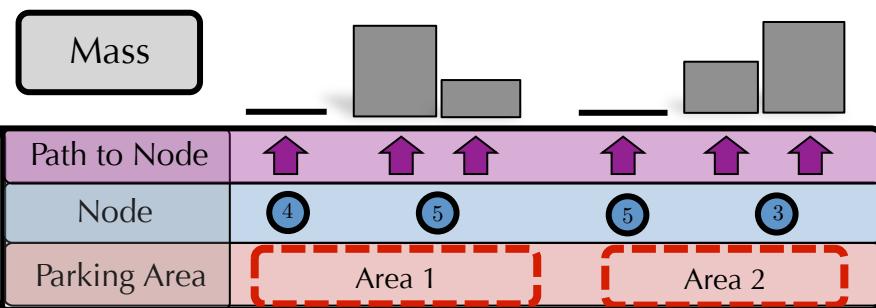
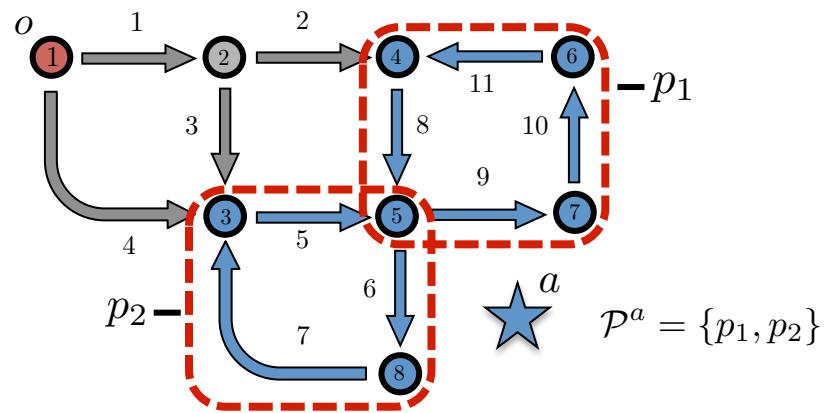
Strategy	Node	4	5	5	3
Strategy	Parking Area	Area 1		Area 2	
Strategy					

Edge Formulation

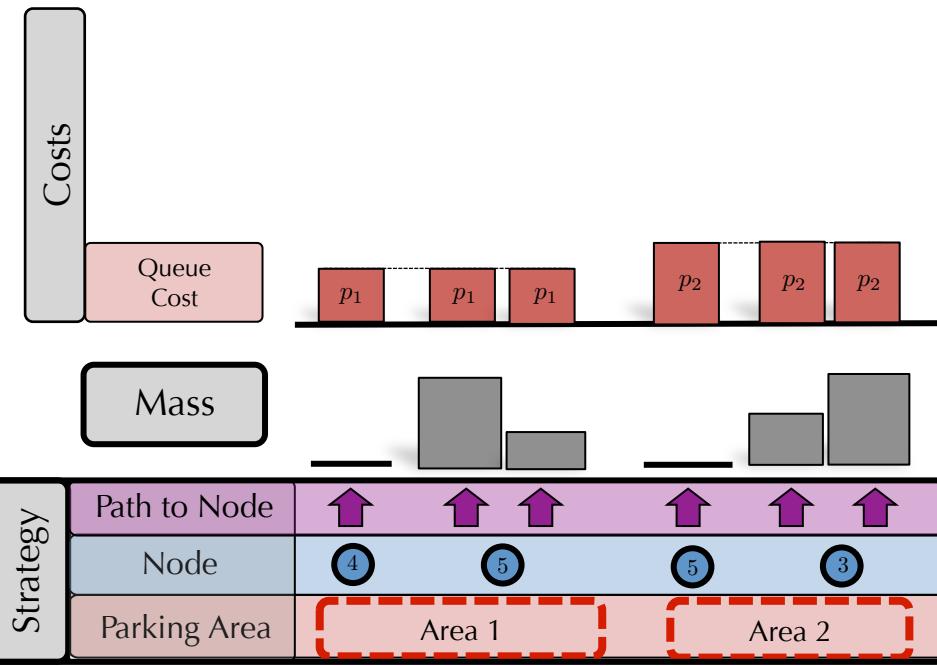
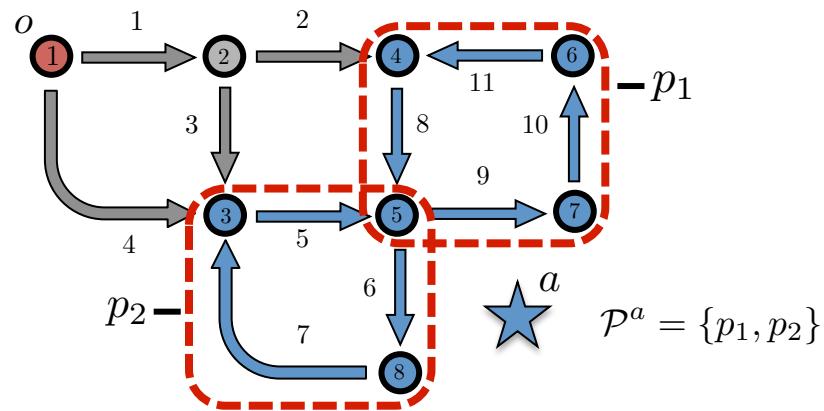


Strategy	Path to Node	↑	↑	↑	↑	↑	↑
Node	4	5	5	3			
Parking Area		Area 1			Area 2		

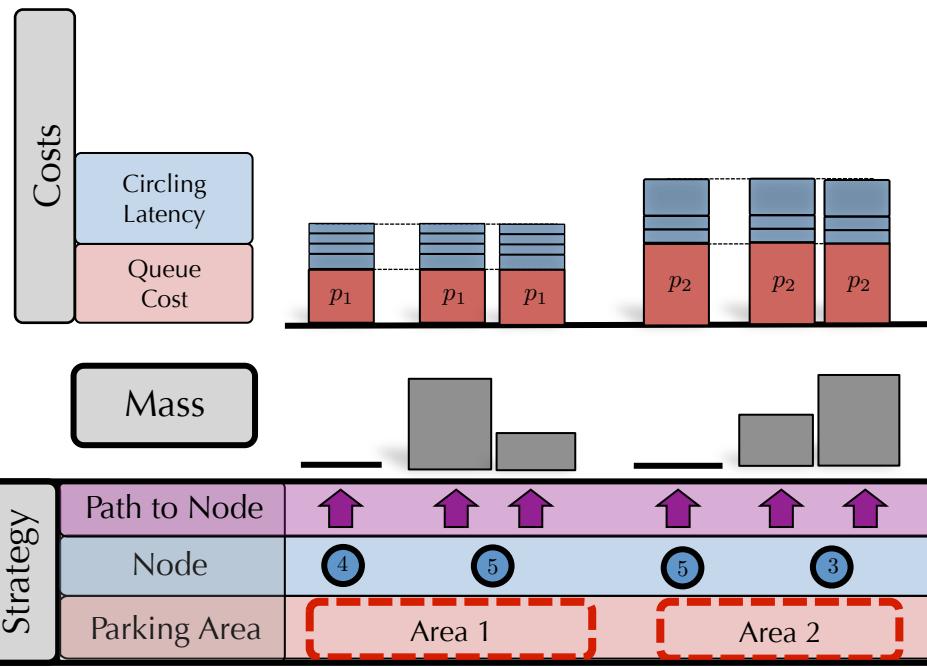
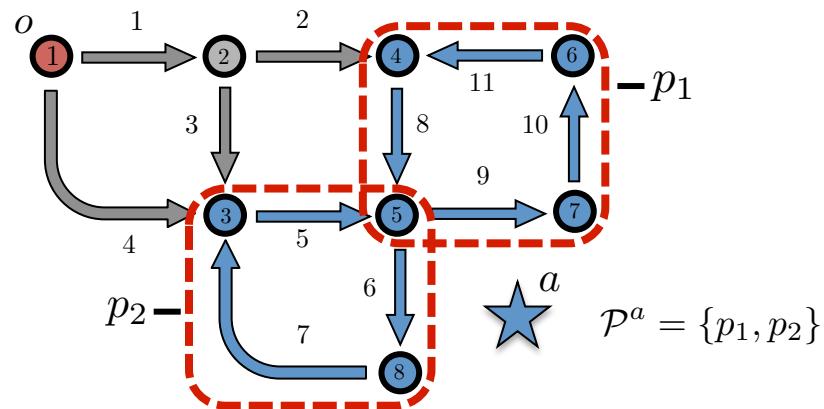
Edge Formulation



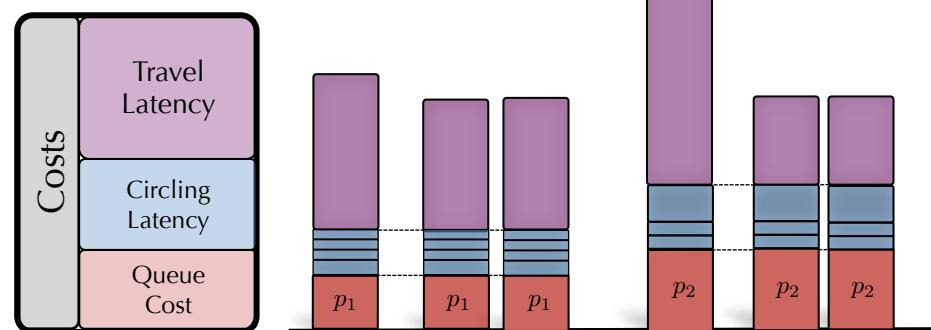
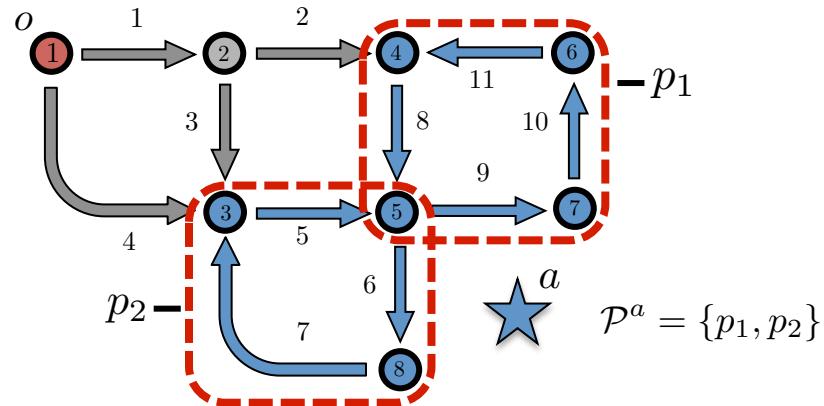
Edge Formulation



Edge Formulation



Edge Formulation

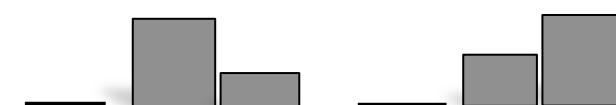
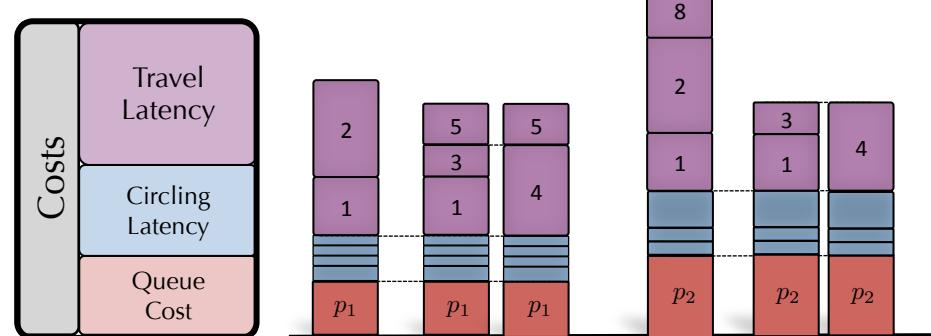
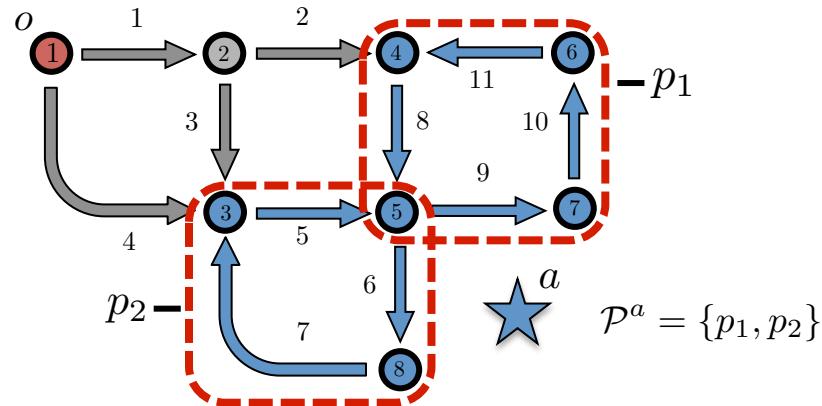


Mass



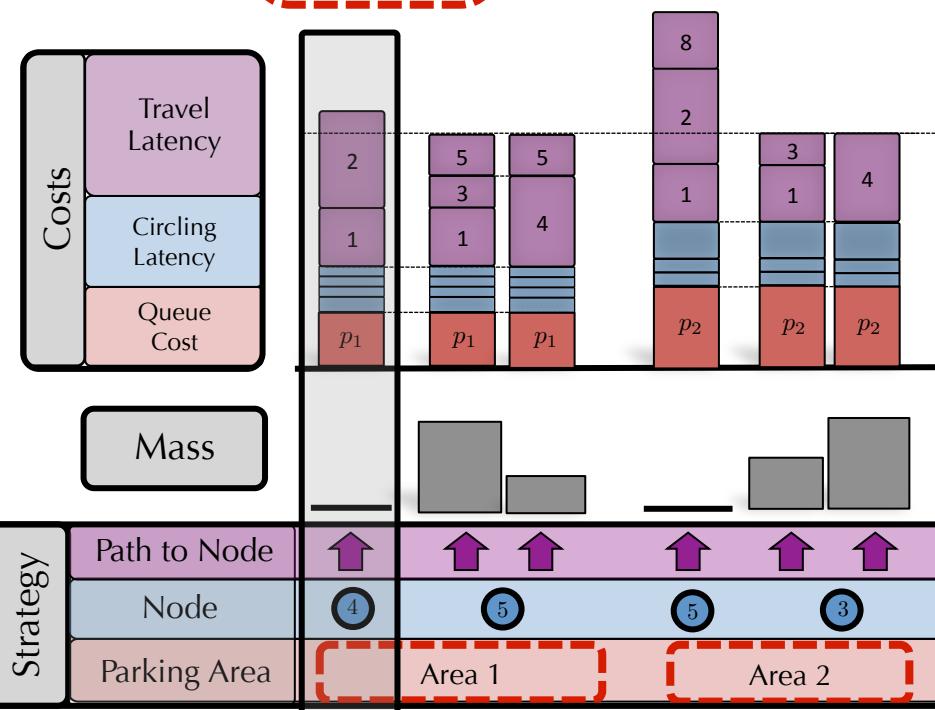
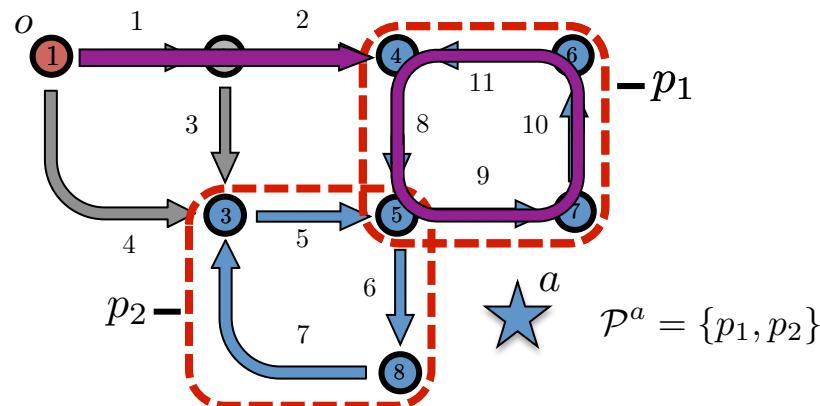
Strategy	Path to Node	Node	Parking Area
			Area 1 Area 2

Edge Formulation

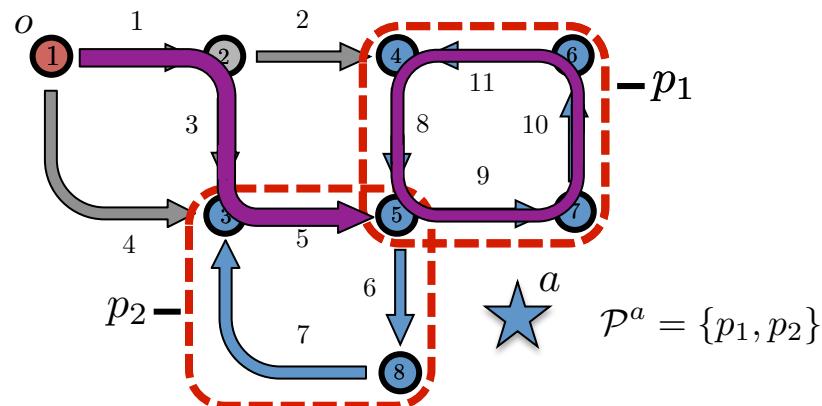


Strategy	Path to Node	Node	Parking Area
Path to Node	↑ ↑ ↑ ↑ ↑		
Node	4 5 5 3		Area 1 Area 2
Parking Area			

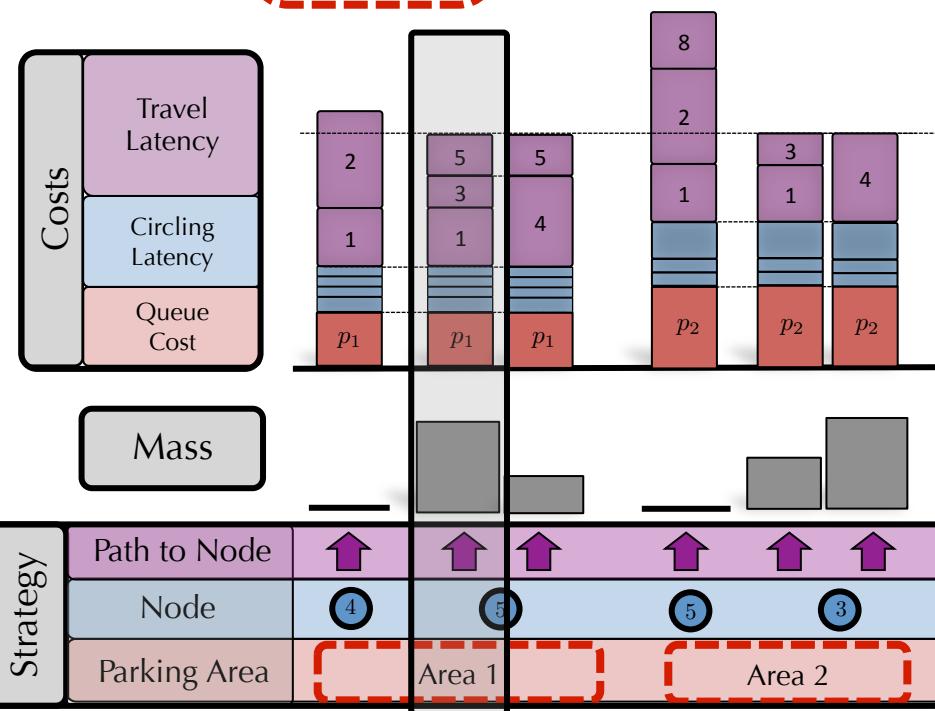
Edge Formulation



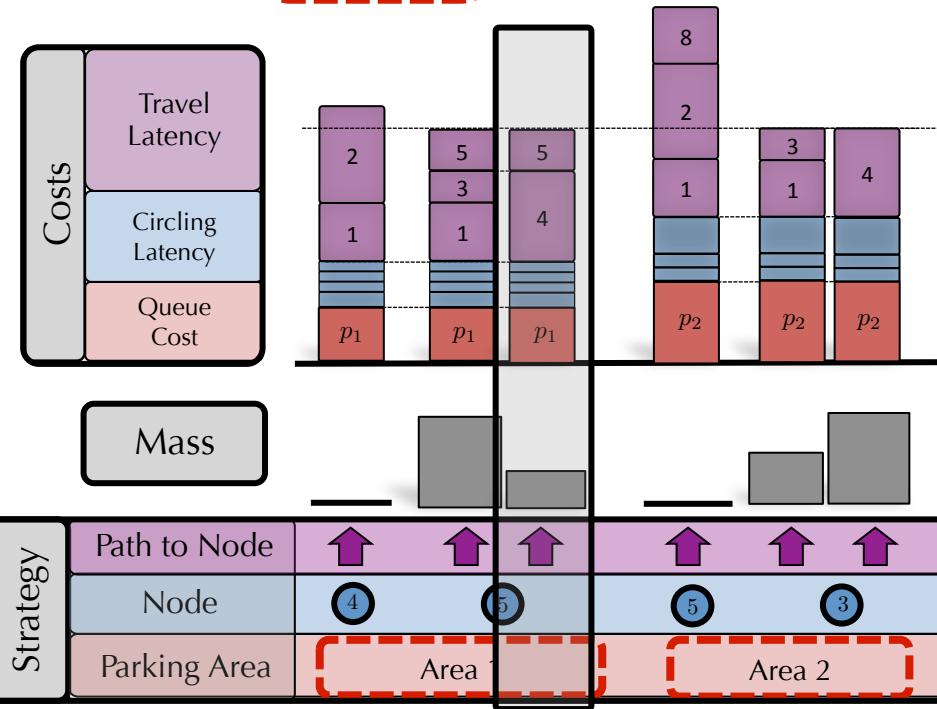
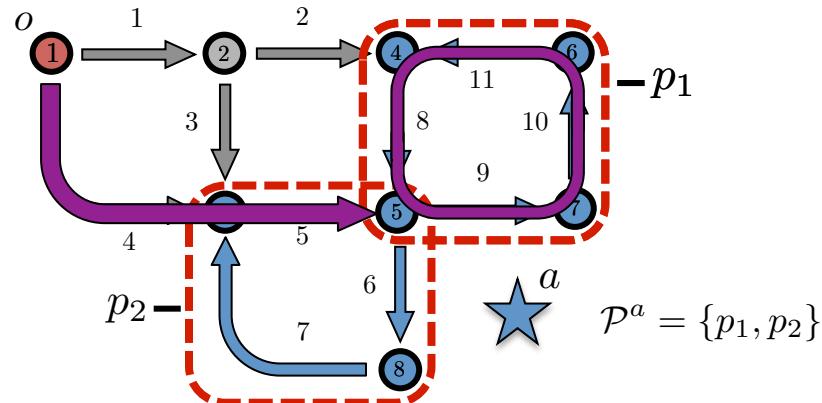
Edge Formulation



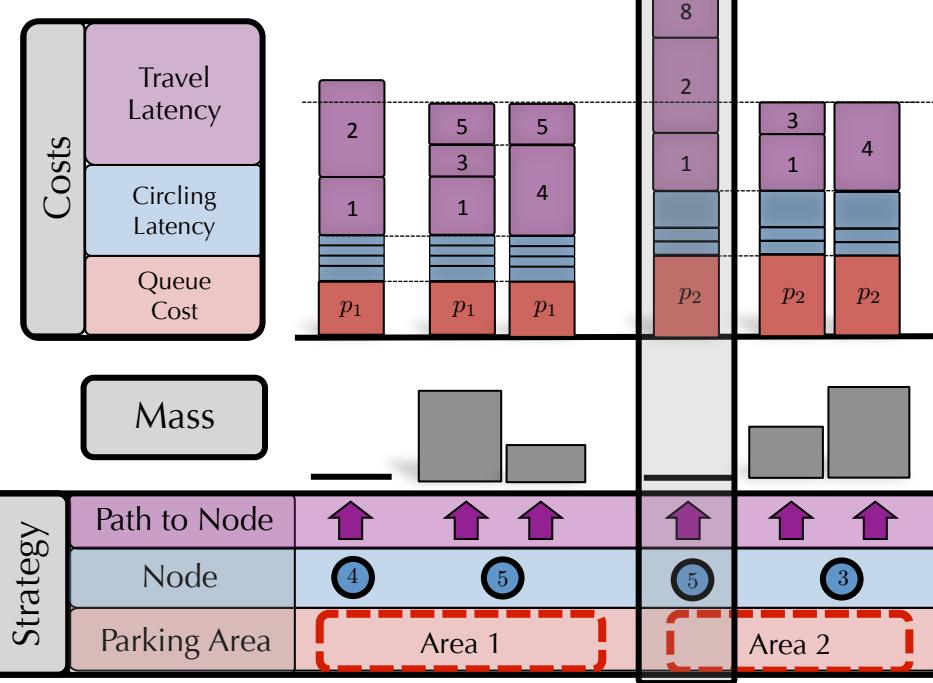
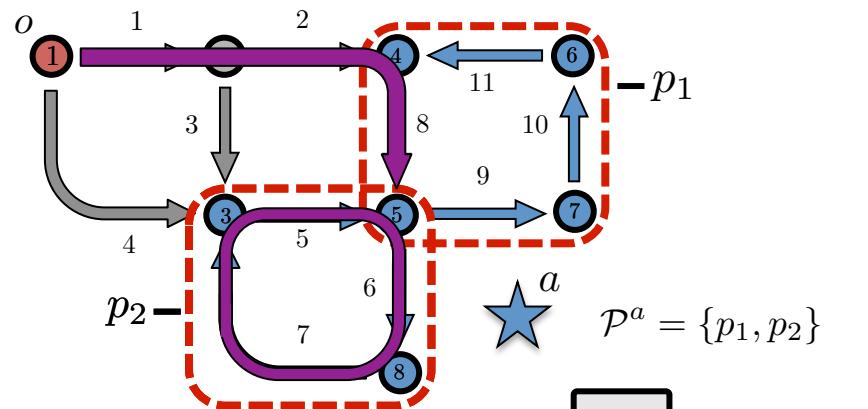
$$\mathcal{P}^a = \{p_1, p_2\}$$



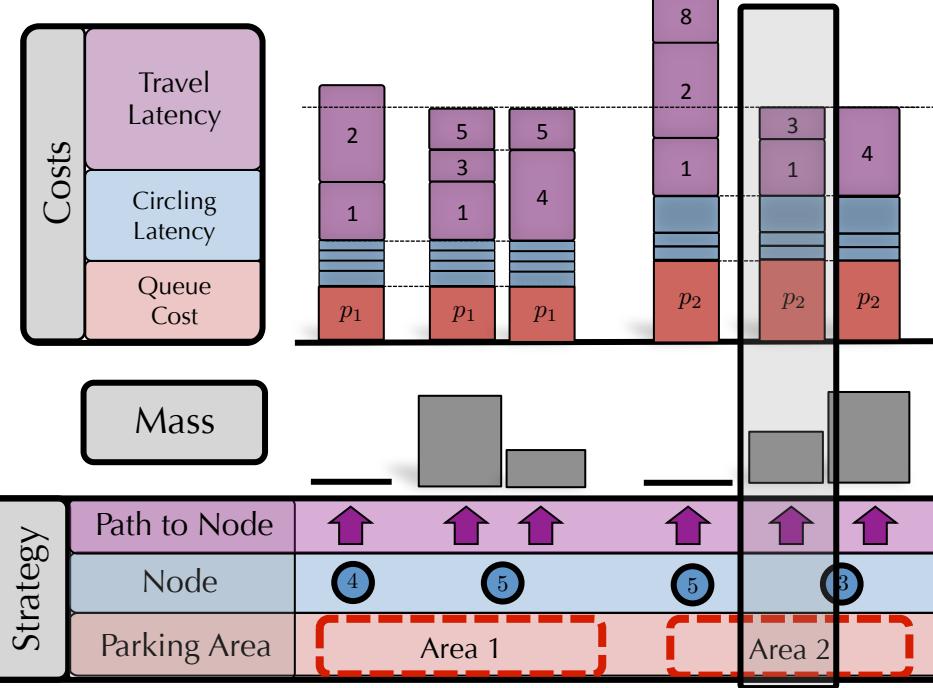
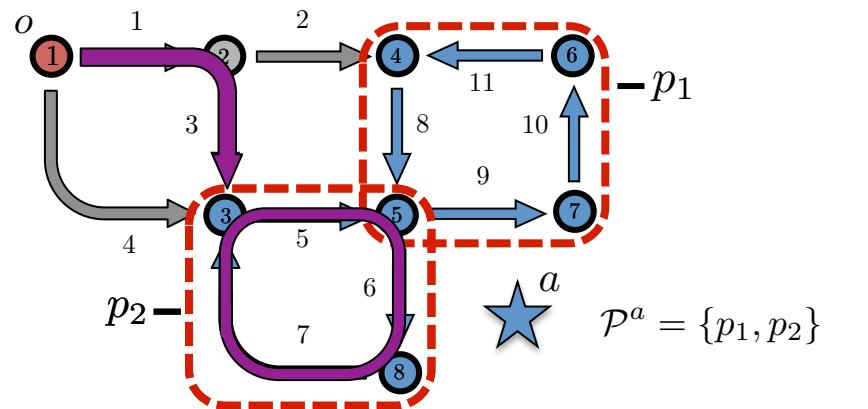
Edge Formulation



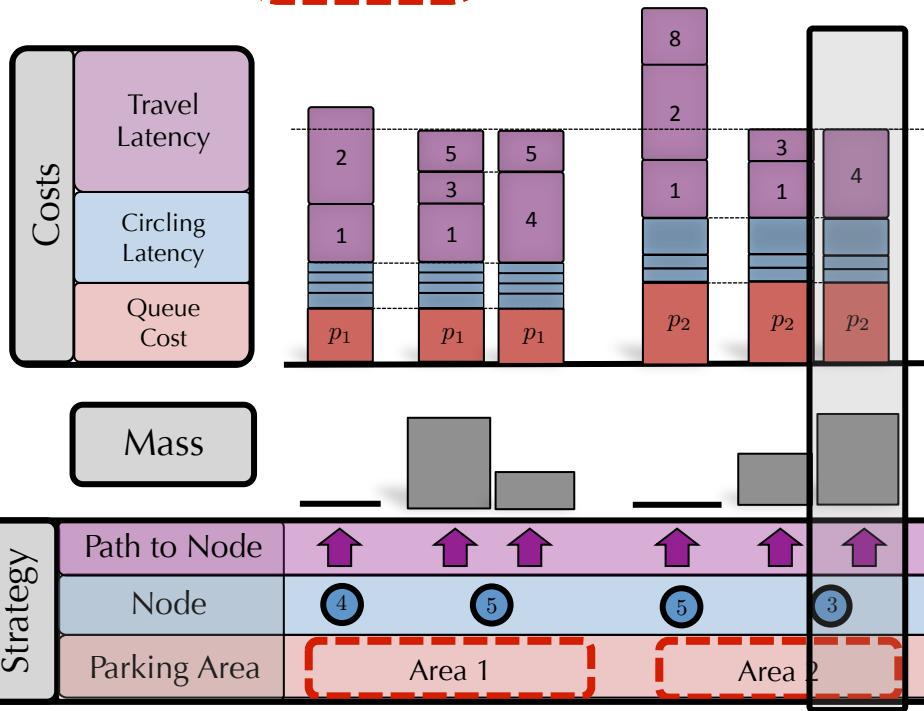
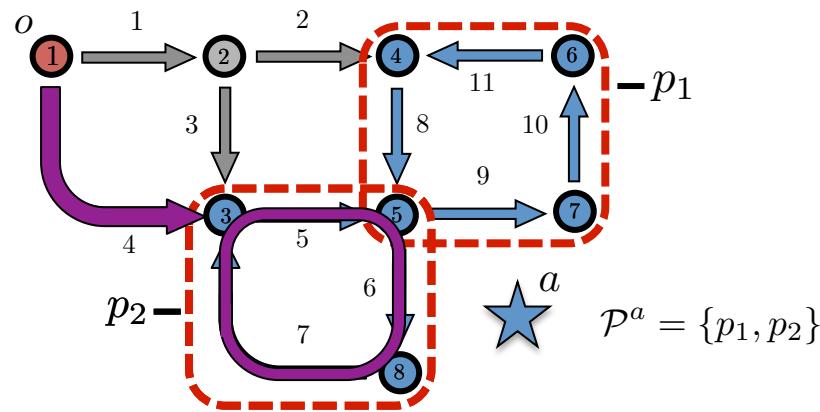
Edge Formulation



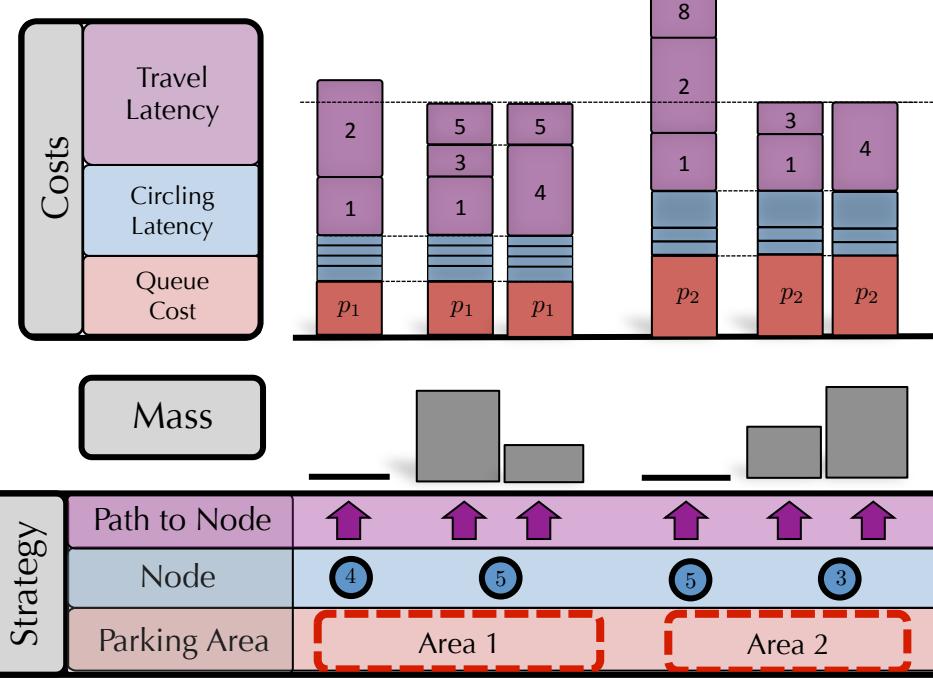
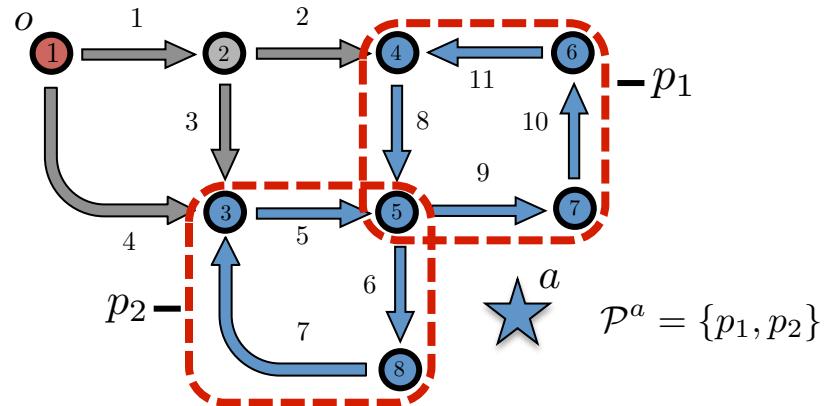
Edge Formulation



Edge Formulation



Edge Formulation

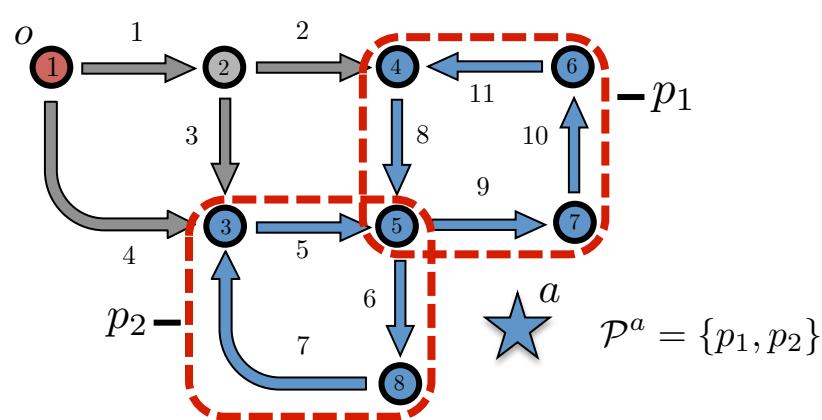


Equilibrium Condition

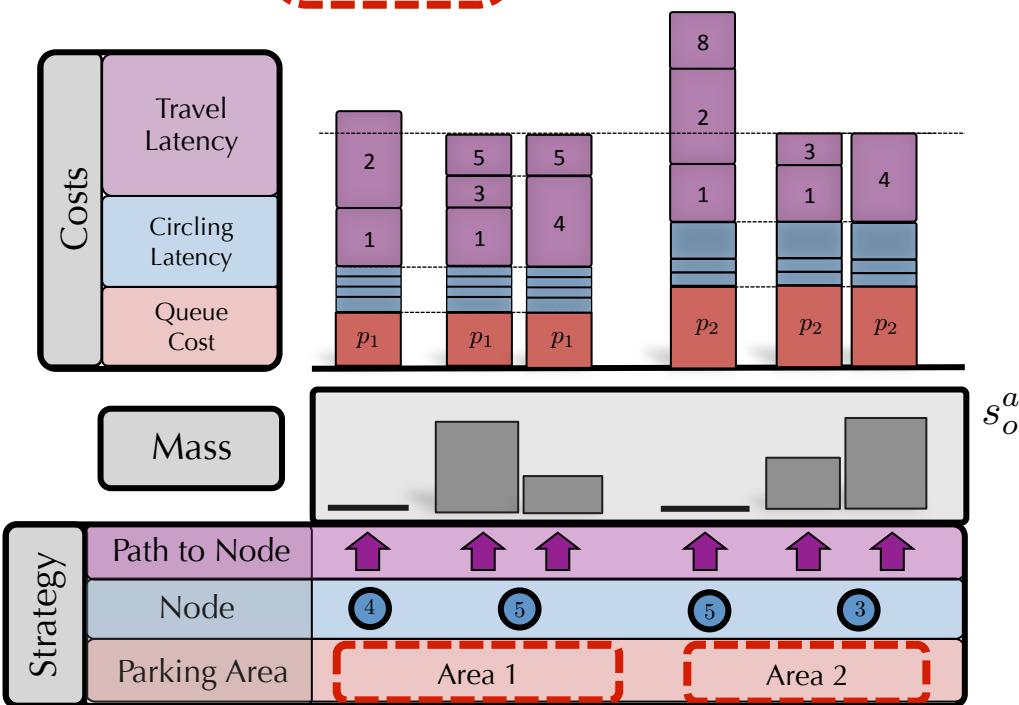


Queue-Routing Wardrop Equilibrium

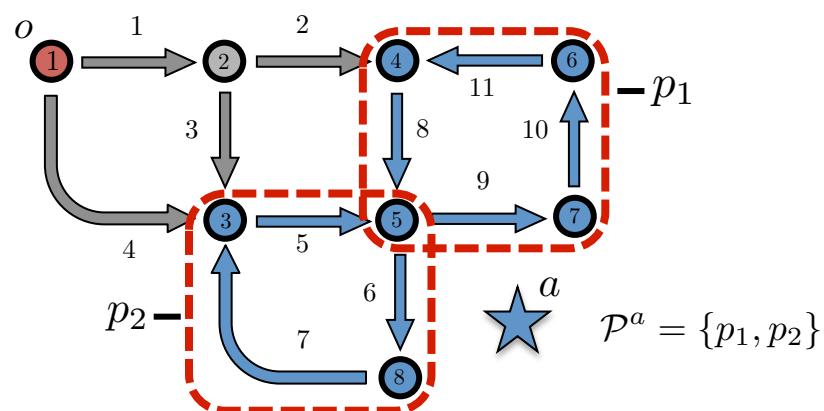
Edge Formulation



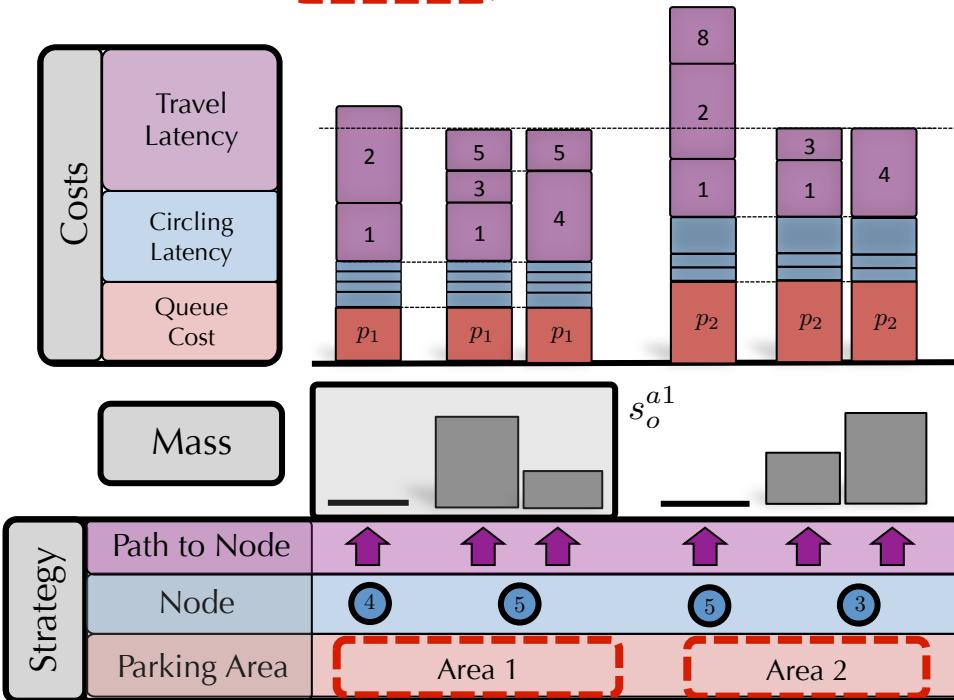
s_o^a : Traffic from origin o to attraction a



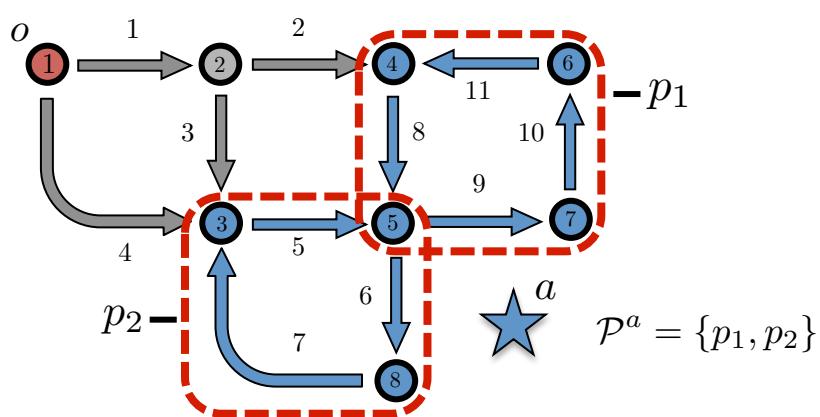
Edge Formulation



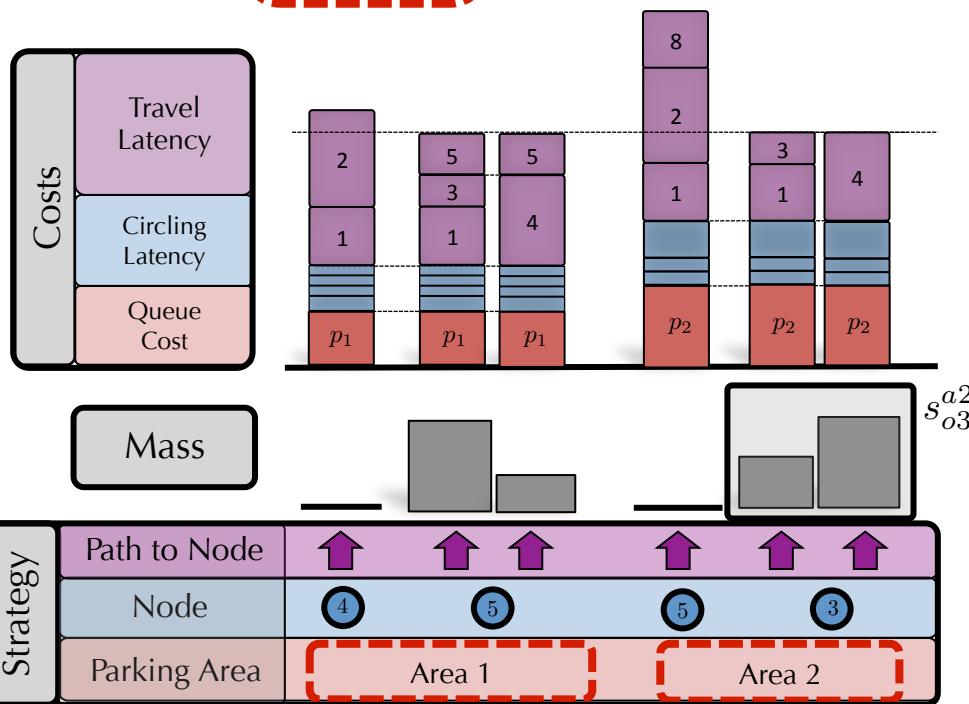
s_o^a : Traffic from origin o to attraction a
 s_o^{ap} : ...parking in area p $s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap}$



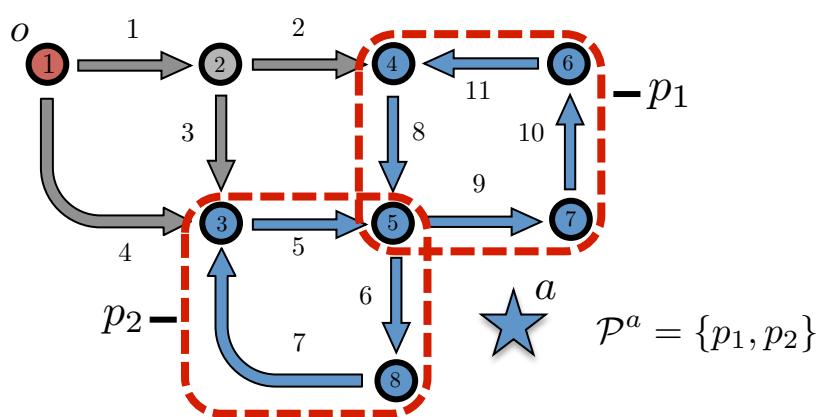
Edge Formulation



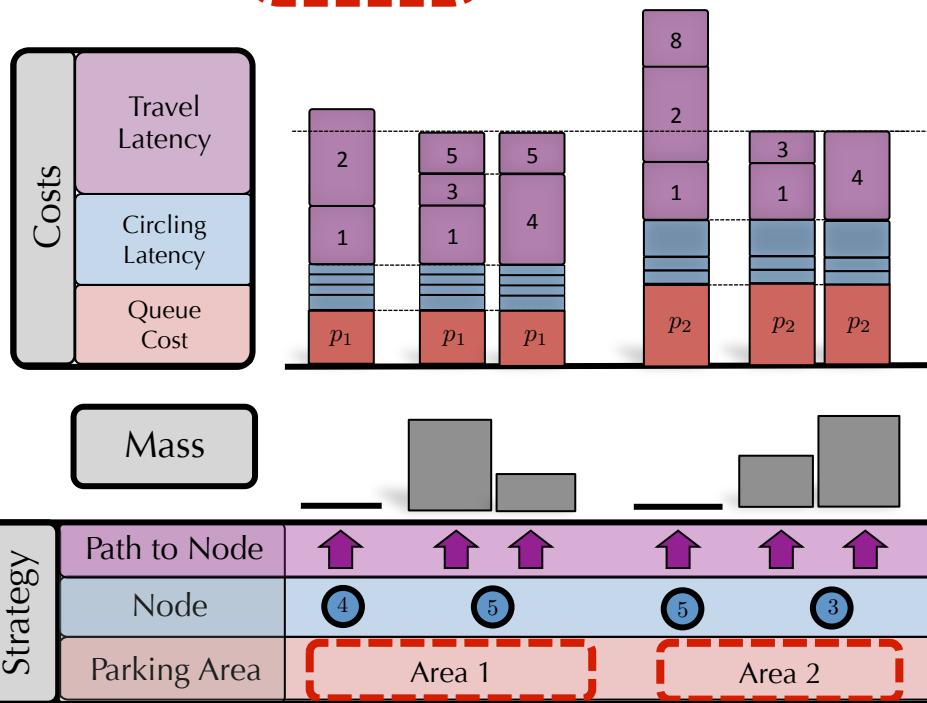
- s_o^a : Traffic from origin o to attraction a
- s_o^{ap} : ...parking in area p $s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap}$
- s_{od}^{ap} : ...entering thru node d $s_o^{ap} = \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$



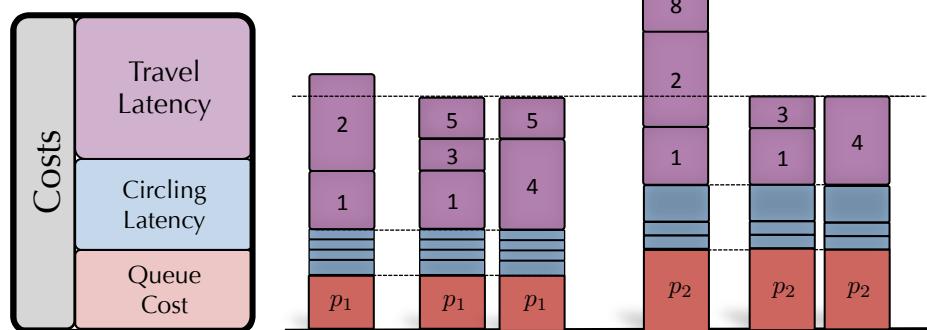
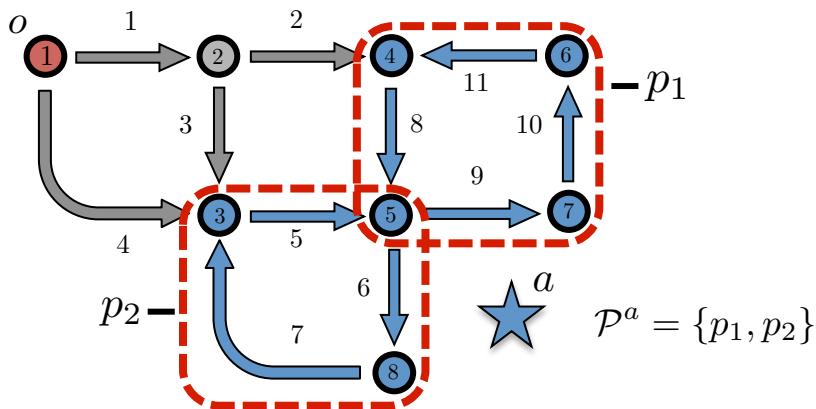
Edge Formulation



- s_o^a : Traffic from origin o to attraction a
- s_o^{ap} : ...parking in area p $s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap}$
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- s^p : Total traffic in area p $s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$



Edge Formulation



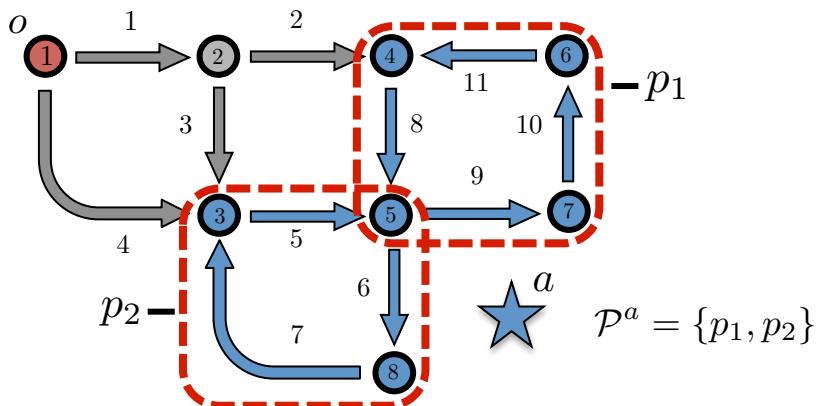
Mass



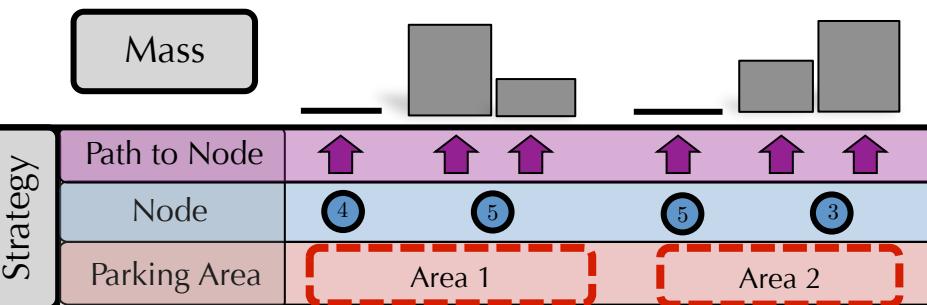
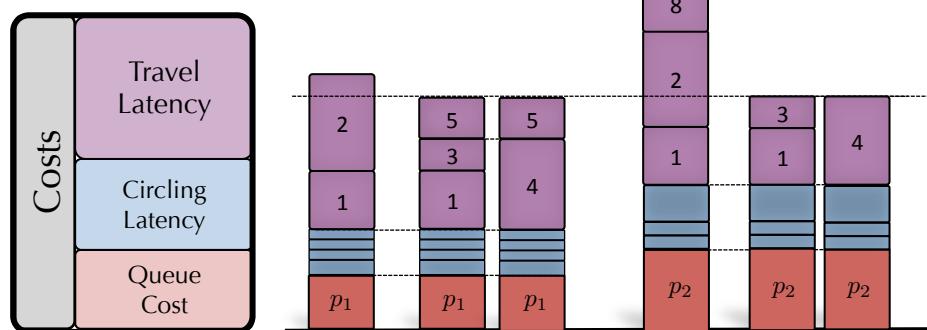
Strategy	Path to Node	Node	Parking Area
Path to Node	↑ ↑ ↑ ↑ ↑	4 5 5 3	Area 1 Area 2
Node			
Parking Area			

$$\begin{aligned}
 s_o^a &: \text{Traffic from origin } o \text{ to attraction } a \\
 s_o^{ap} &: \dots \text{parking in area } p \quad s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap} \\
 s_{od}^{ap} &: \dots \text{entering thru node } d \quad s_o^{ap} = \sum_{d \in \mathcal{N}^p} s_{od}^{ap} \\
 s^p &: \text{Total traffic in area } p \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap} \\
 \hline
 x_{od}^{ap} &: \text{Edge flows from population } s_{od}^{ap} \\
 Gx_{od}^{ap} = S_{od}^{ap} & \quad (S_{od}^{ap})_i = \begin{cases} s_{od}^{ap} & ; \text{ if } i = o \\ -s_{od}^{ap} & ; \text{ if } i = d \\ 0 & ; \text{ otherwise} \end{cases}
 \end{aligned}$$

Edge Formulation



$$\mathcal{P}^a = \{p_1, p_2\}$$



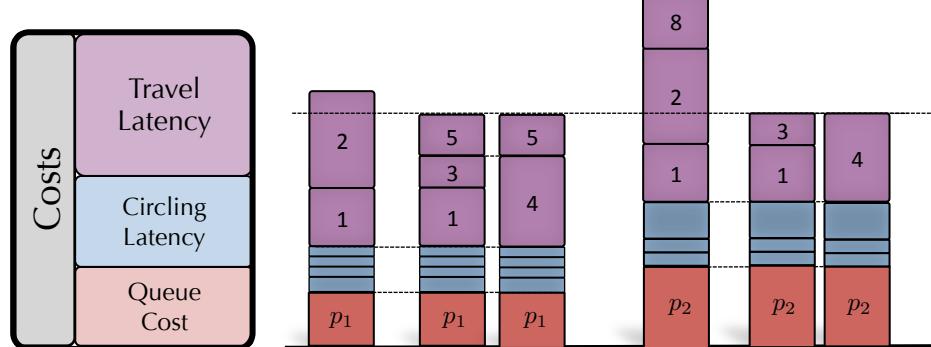
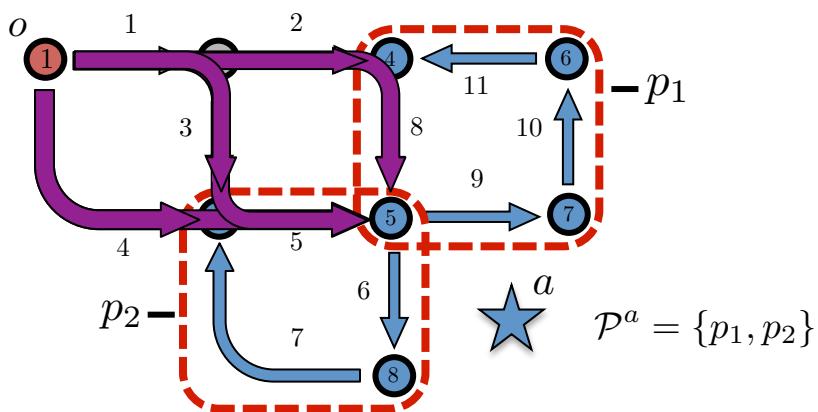
s_o^a	: Traffic from origin o to attraction a
s_o^{ap}	: ...parking in area p
s_{od}^{ap}	: ...entering thru node d
s^p	: Total traffic in area p

s_o^{ap}	: Edge flows from population s_{od}^{ap}
$Gx_{od}^{ap} = S_{od}^{ap}$	$(S_{od}^{ap})_i = \begin{cases} s_{od}^{ap} & ; \text{ if } i = o \\ -s_{od}^{ap} & ; \text{ if } i = d \\ 0 & ; \text{ otherwise} \end{cases}$

x : Total edge flows

$$x = \underbrace{\sum_{o,a,p,d} x_{od}^{ap}}_{\text{Traveling Flows}} + \underbrace{\sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p s_o^{ap}}_{\text{Circling Flows}}$$

Edge Formulation



Mass



Strategy	Path to Node	Node	Parking Area
Path to Node	↑ ↑ ↑ ↑ ↑ ↑		
Node	4 5 5 3		Area 1 Area 2
Parking Area			

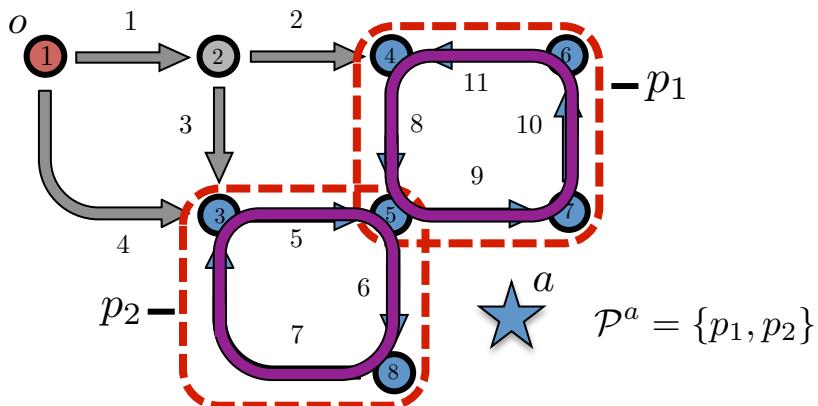
$$\begin{aligned}
 s_o^a &: \text{Traffic from origin } o \text{ to attraction } a \\
 s_o^{ap} &: \dots \text{parking in area } p \quad s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap} \\
 s_{od}^{ap} &: \dots \text{entering thru node } d \quad s_o^{ap} = \sum_{d \in \mathcal{N}^p} s_{od}^{ap} \\
 s^p &: \text{Total traffic in area } p \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}
 \end{aligned}$$

$$\begin{aligned}
 x_{od}^{ap} &: \text{Edge flows from population } s_{od}^{ap} \\
 Gx_{od}^{ap} = S_{od}^{ap} & \quad (S_{od}^{ap})_i = \begin{cases} s_{od}^{ap} & ; \text{ if } i = o \\ -s_{od}^{ap} & ; \text{ if } i = d \\ 0 & ; \text{ otherwise} \end{cases}
 \end{aligned}$$

x : Total edge flows

$$x = \underbrace{\sum_{o,a,p,d} x_{od}^{ap}}_{\text{Traveling Flows}} + \underbrace{\sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p s_o^{ap}}_{\text{Circling Flows}}$$

Edge Formulation



The diagram illustrates the cost and mass distribution of parking strategies across two areas, Area 1 and Area 2.

Costs:

- Travel Latency:** Represented by purple bars. In Area 1, there are 2, 5, and 5 units respectively. In Area 2, there are 2, 3, and 4 units respectively.
- Circling Latency:** Represented by blue bars. In Area 1, there are 1, 1, and 4 units respectively. In Area 2, there are 1, 1, and 4 units respectively.
- Queue Cost:** Represented by orange-red bars. In Area 1, there are p_1 , p_1 , and p_1 units respectively. In Area 2, there are p_2 , p_2 , and p_2 units respectively.

Mass:

Gray bars representing mass are shown above the parking areas. The total mass in Area 1 is approximately 1.5 times that of Area 2.

Strategy:

Path to Node	↑	↑	↑	↑	↑	↑
Node	4	5	5	5	3	
Parking Area	Area 1			Area 2		

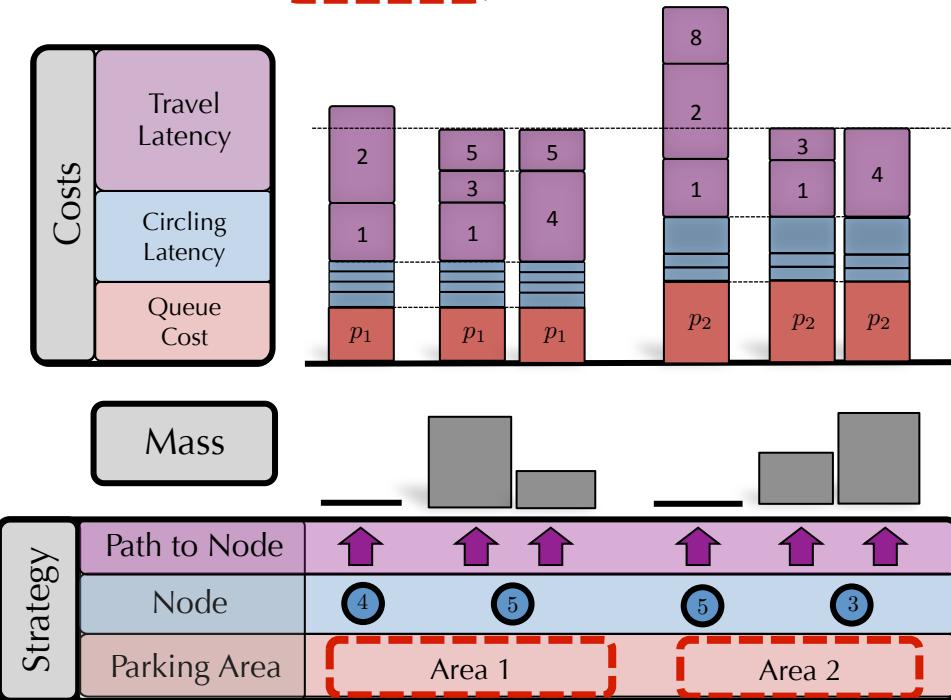
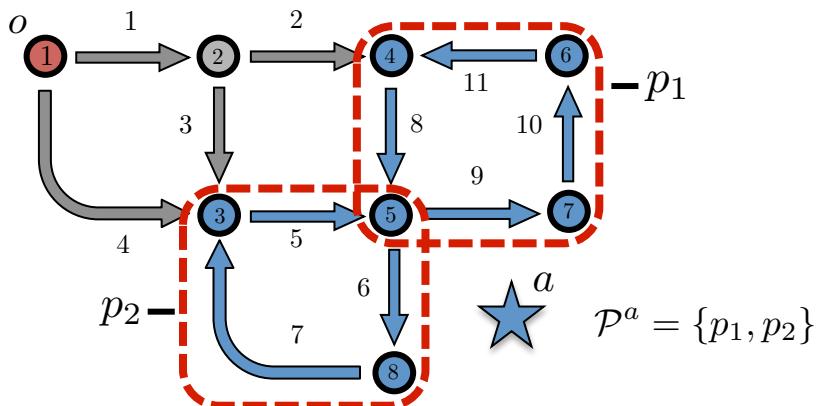
s_o^a :	Traffic from origin o to attraction a	
s_o^{ap} :	...parking in area p	$s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap}$
s_{od}^{ap} :	...entering thru node d	$s_o^{ap} = \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$
s^p :	Total traffic in area p	$s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$
<hr/>		
x_{od}^{ap} :	Edge flows from population	s_{od}^{ap}

$$Gx_{od}^{ap} = S_{od}^{ap} \quad (S_{od}^{ap})_i = \begin{cases} s_{od}^{ap} & ; \text{ if } i = o \\ -s_{od}^{ap} & ; \text{ if } i = d \\ 0 & ; \text{ otherwise} \end{cases}$$

x : Total edge flows

$$x = \underbrace{\sum_{o,a,p,d} x_{od}^{ap}}_{\text{Traveling Flows}} + \underbrace{\sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p s_o^{ap}}_{\text{Circling Flows}}$$

Edge Formulation



$$\begin{aligned}
 s_o^a &: \text{Traffic from origin } o \text{ to attraction } a \\
 s_o^{ap} &: \dots \text{parking in area } p \quad s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap} \\
 s_{od}^{ap} &: \dots \text{entering thru node } d \quad s_o^{ap} = \sum_{d \in \mathcal{N}^p} s_{od}^{ap} \\
 s^p &: \text{Total traffic in area } p \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}
 \end{aligned}$$

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 x_{od}^{ap} &: \text{Edge flows from population } s_{od}^{ap} \\
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 \end{aligned}$$

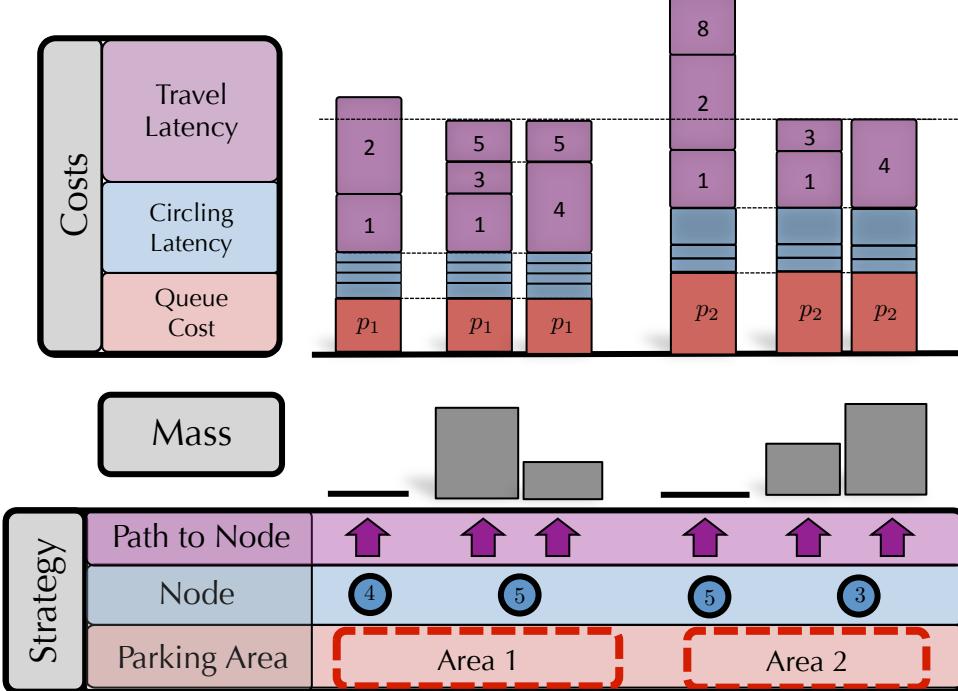
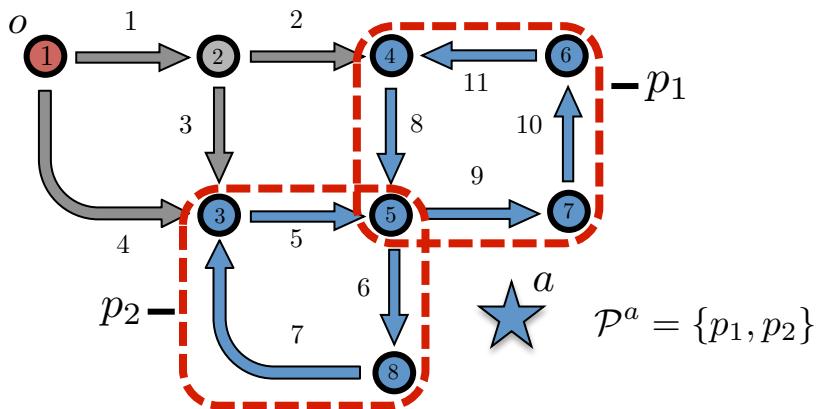
x : Total edge flows

$$x = \underbrace{\sum_{o,a,p,d} x_{od}^{ap}}_{\text{Traveling Flows}} + \underbrace{\sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p s_o^{ap}}_{\text{Circling Flows}}$$

Potential Function

$$P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du$$

Edge Formulation



$$\begin{aligned}
 s_o^a &: \text{Traffic from origin } o \text{ to attraction } a \\
 s_o^{ap} &: \dots \text{parking in area } p \quad s_o^a = \sum_{p \in \mathcal{P}^a} s_o^{ap} \\
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 s^p &: \text{Total traffic in area } p \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}
 \end{aligned}$$

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 x_{od}^{ap} &: \text{Edge flows from population } s_{od}^{ap} \\
 Gx_{od}^{ap} &= S_{od}^{ap} \quad (S_{od}^{ap})_i = \begin{cases} s_{od}^{ap} & ; \text{ if } i = o \\ -s_{od}^{ap} & ; \text{ if } i = d \\ 0 & ; \text{ otherwise} \end{cases}
 \end{aligned}$$

x : Total edge flows

$$x = \underbrace{\sum_{o,a,p,d} x_{od}^{ap}}_{\text{Traveling Flows}} + \underbrace{\sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p s_o^{ap}}_{\text{Circling Flows}}$$

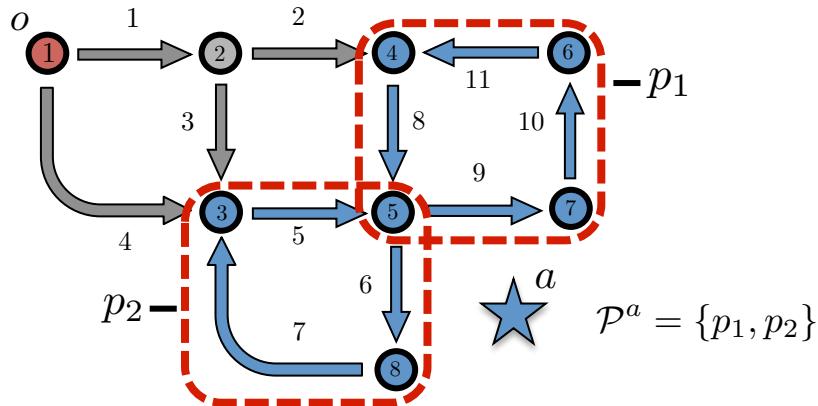
Optimization Problem

$$\min_{x_{od}^{ap}, s_{od}^{ap}} P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du$$

$$\text{s.t. } x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p$$

$$s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p$$

Edge Formulation



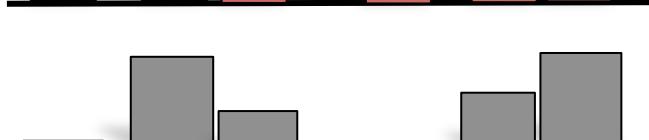
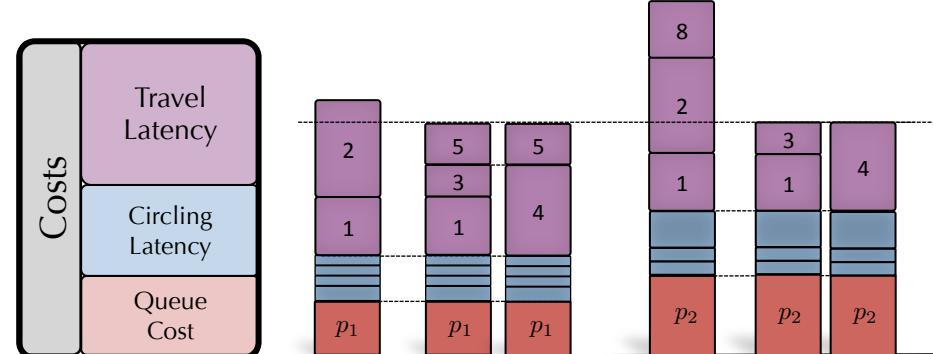
$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap}, s_{od}^{ap}} P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du$$

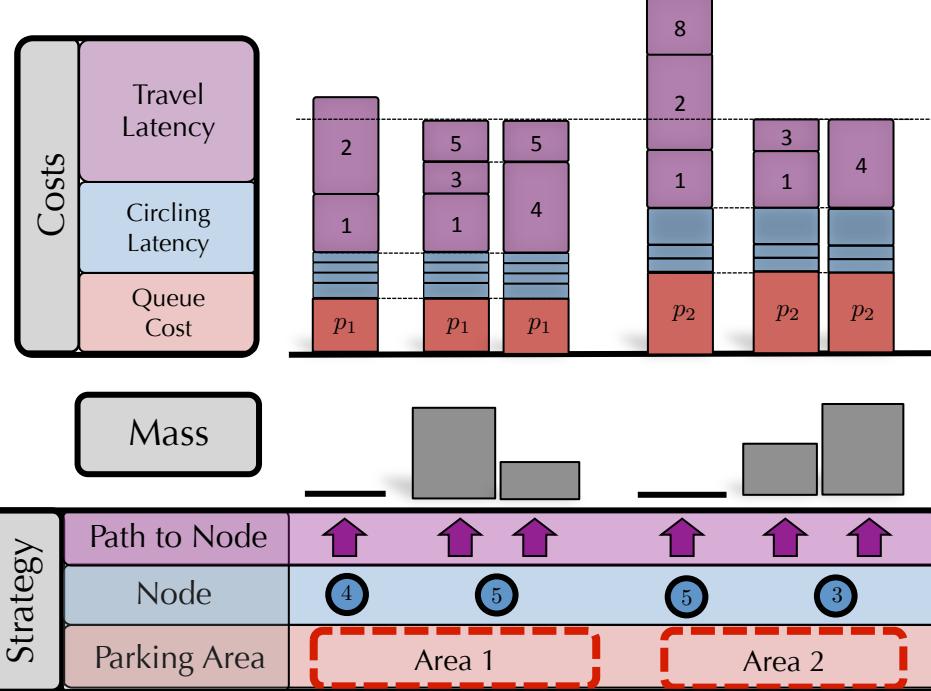
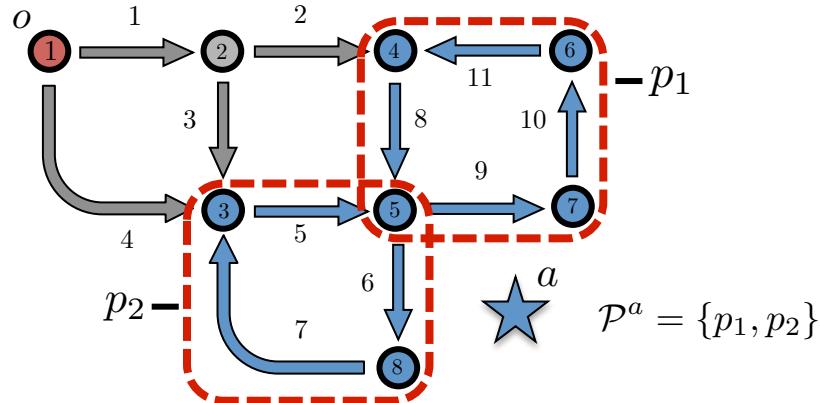
$$\text{s.t.} \quad x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p$$

$$s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p$$



Strategy	Path to Node	↑	↑	↑	↑	↑
	Node	4	5	5	3	
	Parking Area	Area 1				Area 2

Edge Formulation



$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap}, s_{od}^{ap}} P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du$$

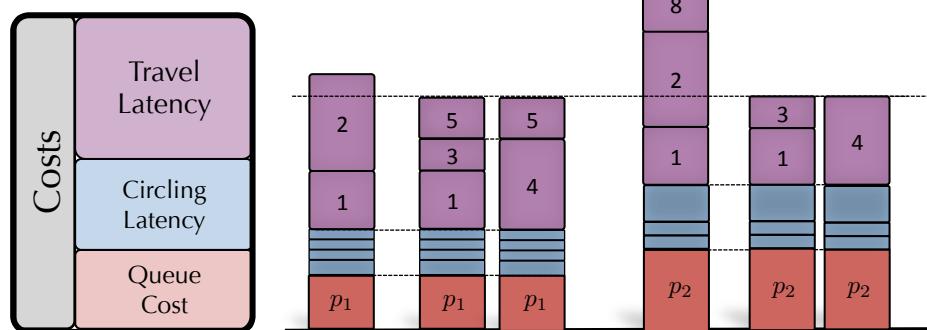
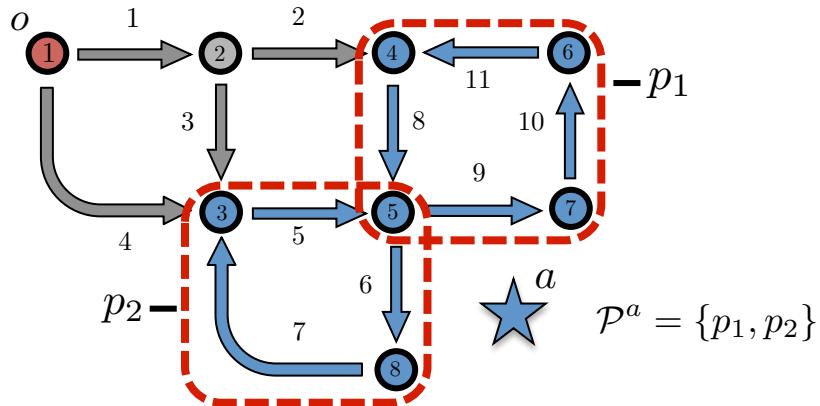
$$\text{s.t.} \quad x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p$$

$$s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

Travel Latency

Edge Formulation



Mass



Strategy	Path to Node	Node	Parking Area
Path to Node	↑ ↑ ↑ ↑ ↑ ↑		
Node	4 5 5 3		Area 1 Area 2
Parking Area			

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

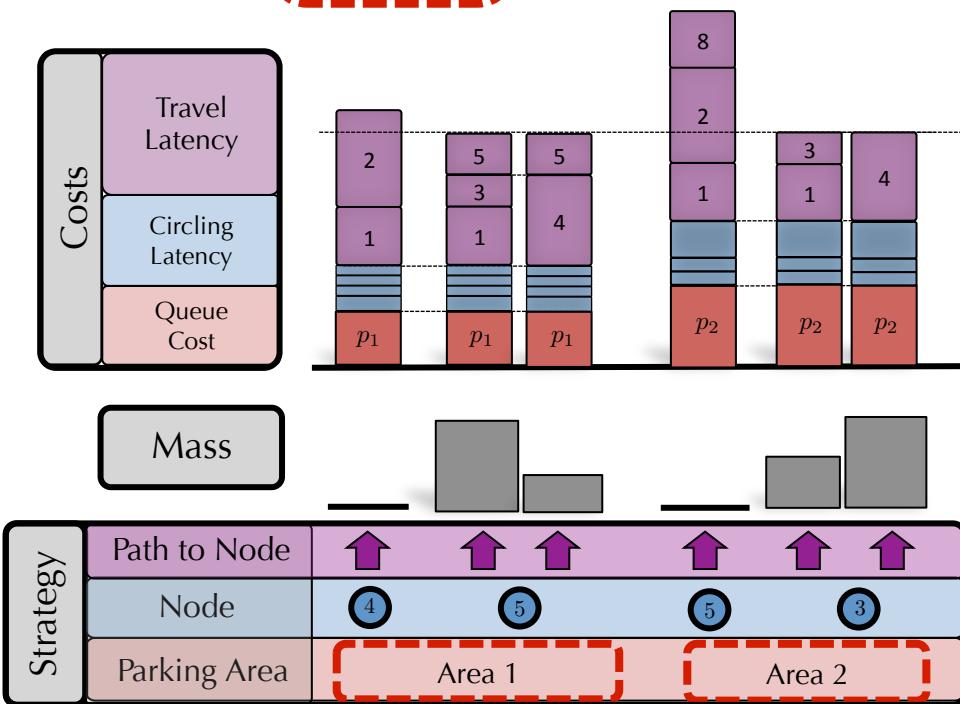
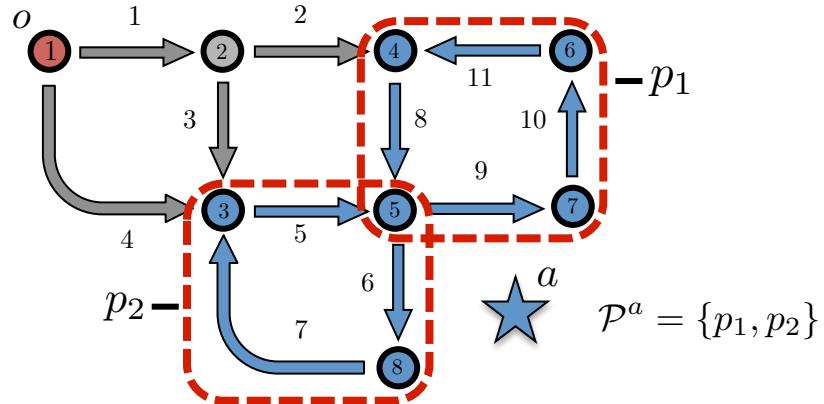
$$\begin{aligned} \min_{x_{od}^{ap}, s_{od}^{ap}} \quad & P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du \\ \text{s.t.} \quad & x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p \\ & s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p \end{aligned}$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\begin{aligned} \nabla_{s_{od}^{ap}} P = & \frac{1}{|\mathcal{E}^p|} l(x) \mathbf{E}^p + \\ & -R_p + C_p s^p \end{aligned}$$

Travel Latency
Circling Latency
Queue Cost

Edge Formulation



$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\begin{aligned} \min_{x_{od}^{ap}, s_{od}^{ap}} \quad & P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du \\ \text{s.t.} \quad & x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p \\ & s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p \end{aligned}$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

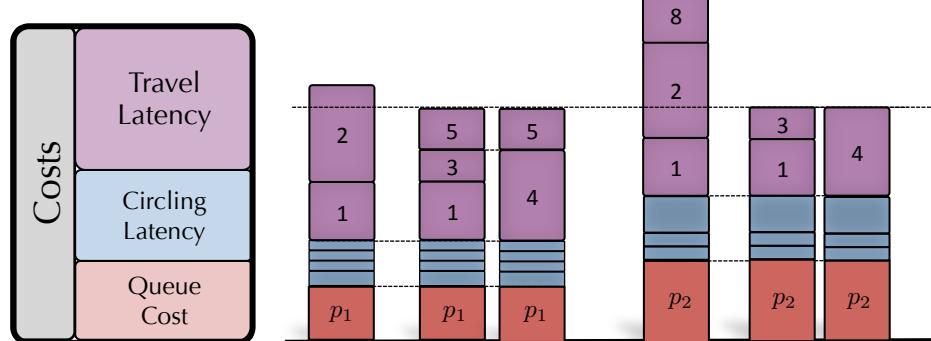
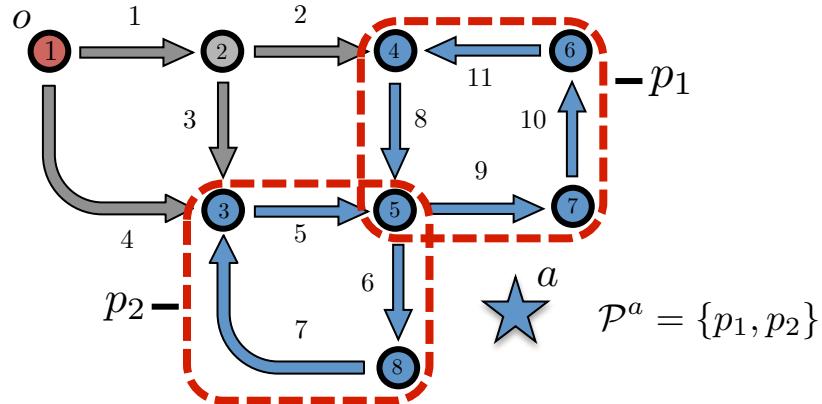
$$\begin{aligned} \nabla_{s_{od}^{ap}} P = & \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + \\ & -R_p + C_p s^p \end{aligned}$$

First Order Optimality...

$$\begin{aligned} \mathcal{L}(x, s, \pi, \lambda, \nu, \mu) = & P(x) + \\ & - \sum_{o,d,a,p} (\pi_{od}^{ap})^T (Gx_{od}^{ap} - S_{od}^{ap}) - \sum_{o,d,a,p} (\nu_{od}^{ap})^T x_{od}^{ap} \\ & - \sum_{o,a} \lambda_o^a [\sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} - s_o^a] - \sum_{o,d,a,p} (\mu_{od}^{ap})^T s_{od}^{ap} \end{aligned}$$

Travel Latency
Circling Latency
Queue Cost

Edge Formulation



Mass



Strategy	Path to Node	Node	Parking Area
Path to Node	4	5	Area 1
Node	5	3	Area 2
Parking Area			

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap}, s_{od}^{ap}} P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du$$

$$\text{s.t.} \quad x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p$$

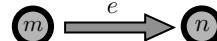
$$s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + \\ -R_p + C_p s^p$$

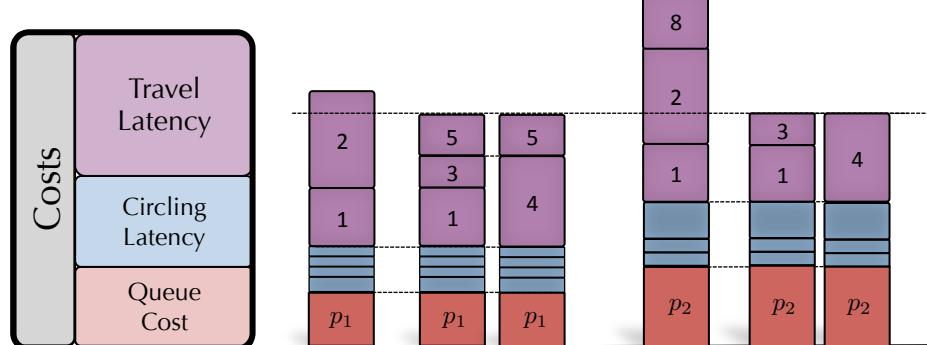
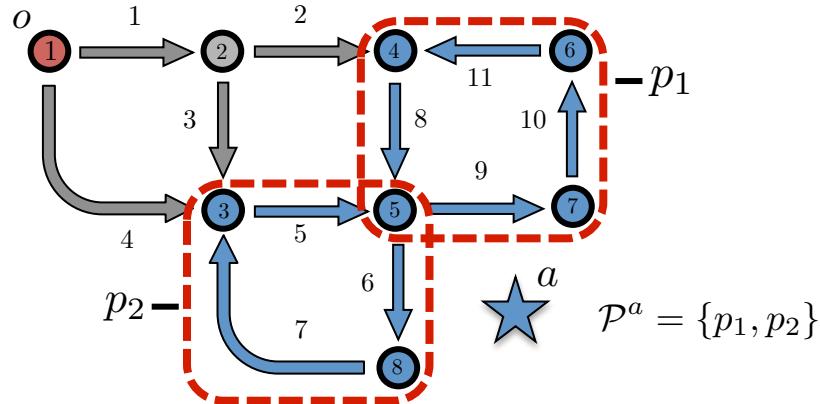
First Order Optimality...

$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \quad l_e(x_e) = (\pi_{od}^{ap})_n - (\pi_{od}^{ap})_m + (\nu_{od}^{ap})_e$$



Travel Latency
Circling Latency
Queue Cost

Edge Formulation



Mass



Strategy	Path to Node	Node	Parking Area
Path to Node	↑ ↑ ↑ ↑ ↑		
Node	4 5 5 3		Area 1 Area 2
Parking Area			

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

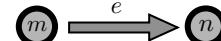
$$\begin{aligned} \min_{x_{od}^{ap}, s_{od}^{ap}} \quad & P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du \\ \text{s.t.} \quad & x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p \\ & s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p \end{aligned}$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\begin{aligned} \nabla_{s_{od}^{ap}} P = & \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + \\ & -R_p + C_p s^p \end{aligned}$$

First Order Optimality...

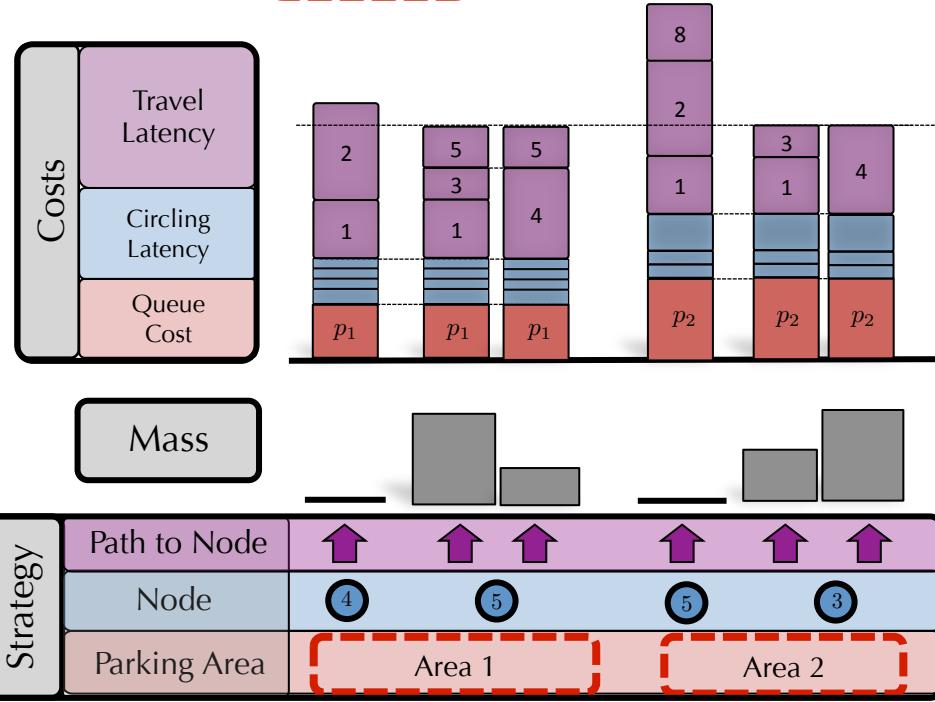
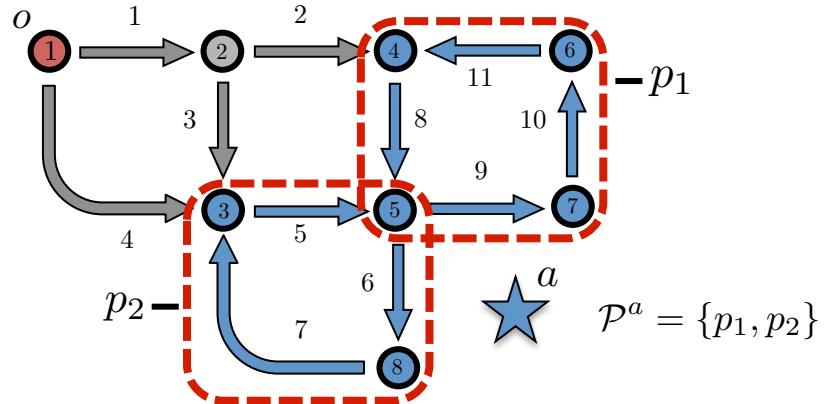
$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \quad l_e(x_e) = (\pi_{od}^{ap})_n - (\pi_{od}^{ap})_m + (\nu_{od}^{ap})_e$$



summing over path from o to d

Travel Latency
Circling Latency
Queue Cost

Edge Formulation



$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\begin{aligned} \min_{x_{od}^{ap}, s_{od}^{ap}} \quad & P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du \\ \text{s.t.} \quad & x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p \\ & s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p \end{aligned}$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\begin{aligned} \nabla_{s_{od}^{ap}} P = & \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + \\ & -R_p + C_p s^p \end{aligned}$$

First Order Optimality...

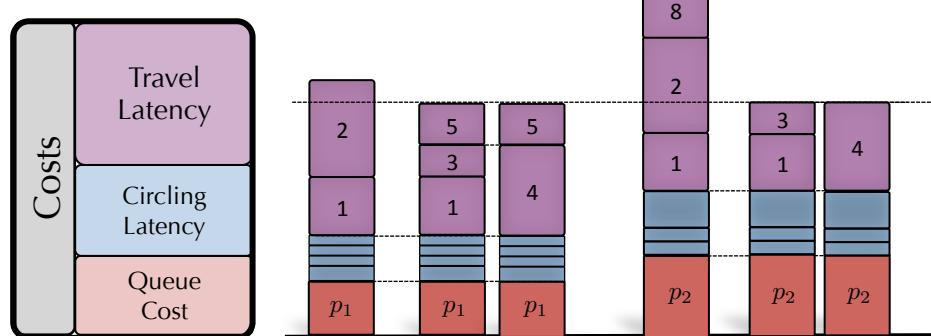
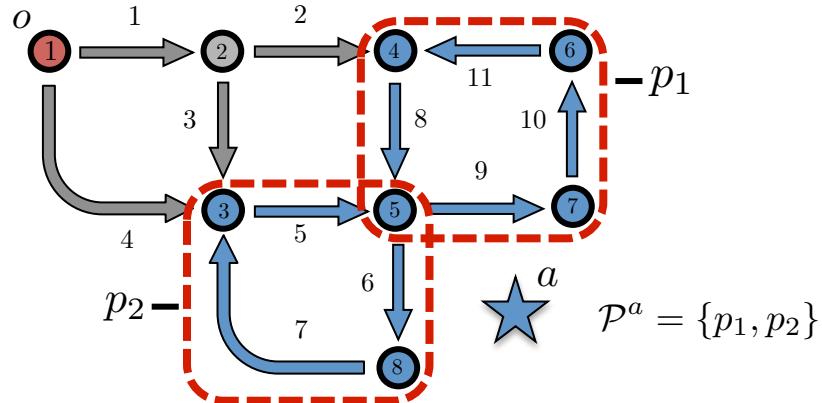
$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \quad l_e(x_e) = (\pi_{od}^{ap})_n - (\pi_{od}^{ap})_m + (\nu_{od}^{ap})_e$$

summing over path from o to d

$$\sum_{e \in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e \in r} (\nu_{od}^{ap})_e$$

Travel Latency
Circling Latency
Queue Cost

Edge Formulation



Mass



Strategy	Path to Node
	Node
	Parking Area
Area 1	
Area 2	

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\begin{aligned} \min_{x_{od}^{ap}, s_{od}^{ap}} \quad & P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du \\ \text{s.t.} \quad & x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p \\ & s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p \end{aligned}$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + -R_p + C_p s^p$$

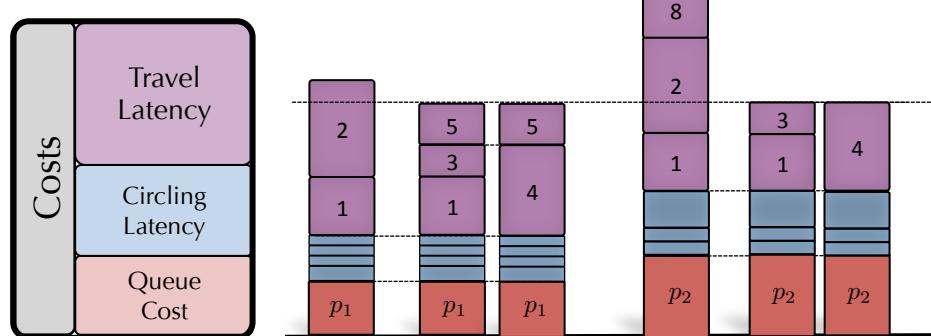
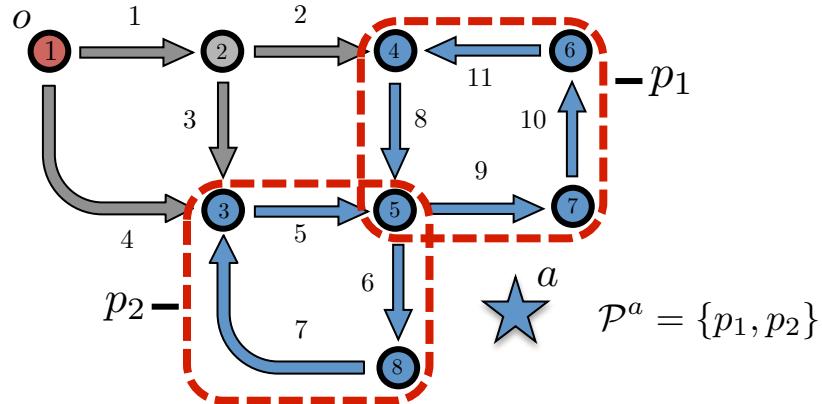
First Order Optimality...

$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \left\{ \sum_{e \in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e \in r} (\nu_{od}^{ap})_e \right\}$$

$$\frac{\partial \mathcal{L}}{\partial s_{od}^{ap}} :$$

Travel Latency
Circling Latency
Queue Cost

Edge Formulation



Mass



Strategy	Path to Node	Node	Parking Area
Path to Node	↑ ↑ ↑ ↑ ↑		
Node	4 5 5 3		Area 1 Area 2
Parking Area			

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\begin{aligned} \min_{x_{od}^{ap}, s_{od}^{ap}} \quad & P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du \\ \text{s.t.} \quad & x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p \\ & s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p \end{aligned}$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^p|} l(x) \mathbf{E}^p + -R_p + C_p s^p$$

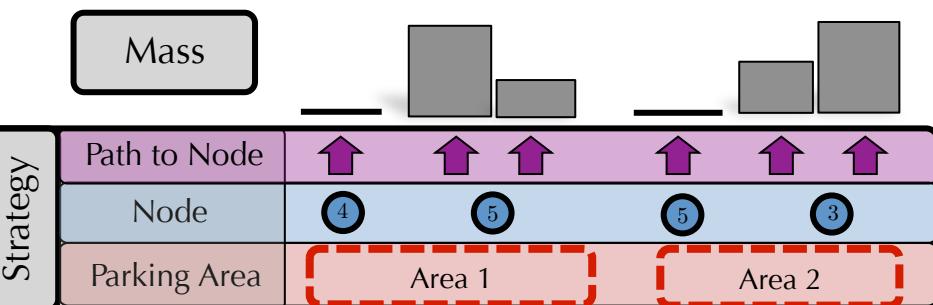
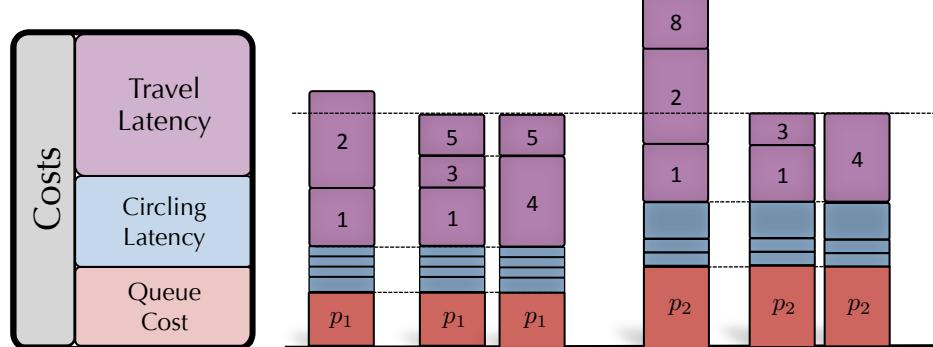
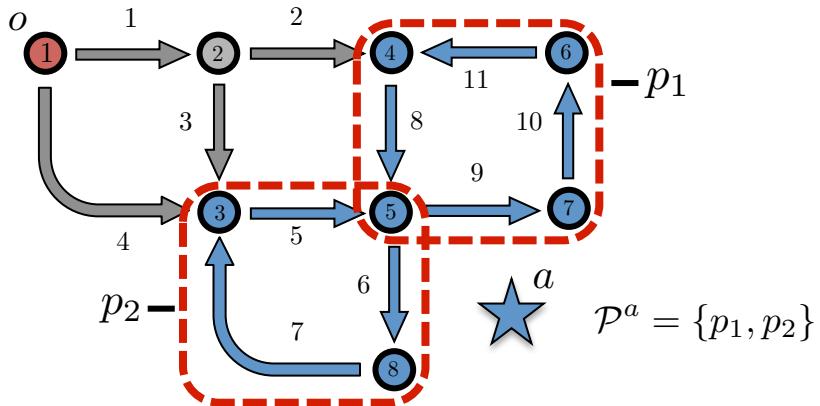
First Order Optimality...

$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \left(\sum_{e \in r} l_e(x_e) \right) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e \in r} (\nu_{od}^{ap})_e$$

$$\frac{\partial \mathcal{L}}{\partial s_{od}^{ap}} : \frac{1}{|\mathcal{E}^p|} l(x) \mathbf{E}^p - R_p + C_p s^p = (\pi_{od}^{ap})_o - (\pi_{od}^{ap})_d + \lambda_o^a + \mu_{od}^{ap}$$

Travel Latency
Circling Latency
Queue Cost

Edge Formulation



$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\begin{aligned} \min_{x_{od}^{ap}, s_{od}^{ap}} \quad & P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du \\ \text{s.t.} \quad & x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p \\ & s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p \end{aligned}$$

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$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + -R_p + C_p s^p$$

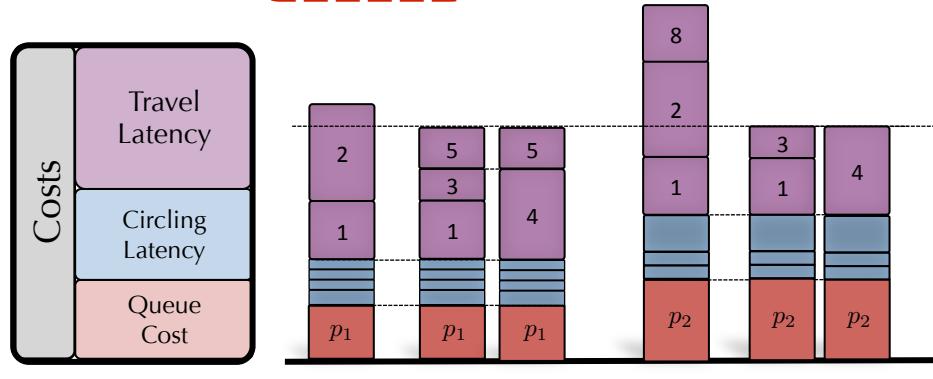
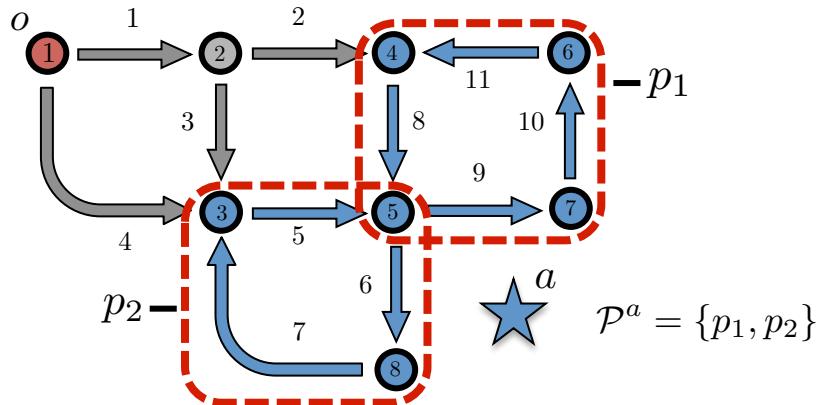
First Order Optimality...

$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \left\{ \sum_{e \in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e \in r} (\nu_{od}^{ap})_e \right\}$$

$$\frac{\partial \mathcal{L}}{\partial s_{od}^{ap}} : \left\{ \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p - R_p + C_p s^p = (\pi_{od}^{ap})_o - (\pi_{od}^{ap})_d + \lambda_o^a + \mu_{od}^{ap} \right\}$$

$$\sum_{e \in r} l_e(x_e) + \frac{1}{|\mathcal{E}^p|} l(x) \mathbf{E}^p - R_p + C_p s^p = \lambda_o^a + \sum_{e \in r} (\nu_{od}^{ap})_e + \mu_{od}^{ap}$$

Edge Formulation



Strategy	Path to Node	Node	Parking Area
Path to Node	↑↑↑↑↑		
Node	4 5 5 3		Area 1 Area 2
Parking Area			

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap}, s_{od}^{ap}} P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du$$

$$\text{s.t.} \quad x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p$$

$$s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + \\ -R_p + C_p s^p$$

First Order Optimality...

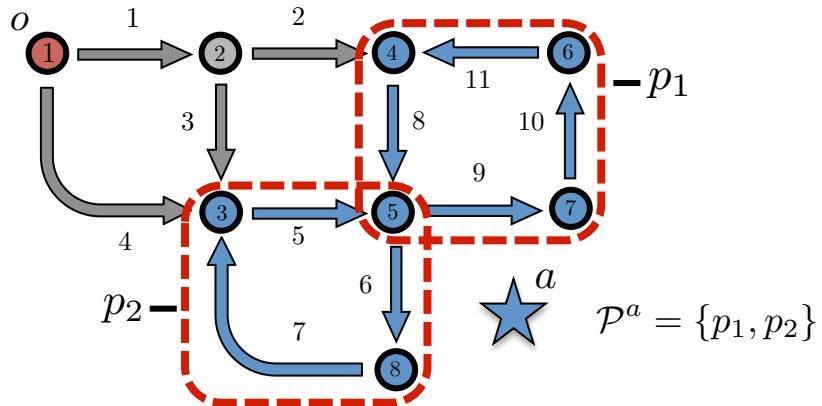
$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \quad \sum_{e \in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e \in r} (\nu_{od}^{ap})_e$$

$$\frac{\partial \mathcal{L}}{\partial s_{od}^{ap}} : \quad \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p - R_p + C_p s^p = (\pi_{od}^{ap})_o - (\pi_{od}^{ap})_d + \lambda_o^a + \mu_{od}^{ap}$$

$$\sum_{e \in r} l_e(x_e) + \frac{1}{|\mathcal{E}^p|} l(x) \mathbf{E}^p - R_p + C_p s^p = \lambda_o^a + \sum_{e \in r} (\nu_{od}^{ap})_e + \mu_{od}^{ap}$$

Traveling Cost
Circling Cost
Queue Cost

Edge Formulation



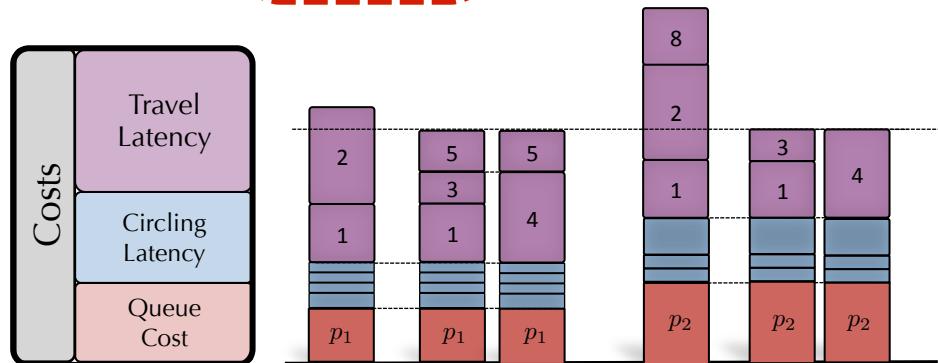
$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap}, s_{od}^{ap}} P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du$$

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Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\lambda_o^a \quad \nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + \\ -R_p + C_p s^p$$

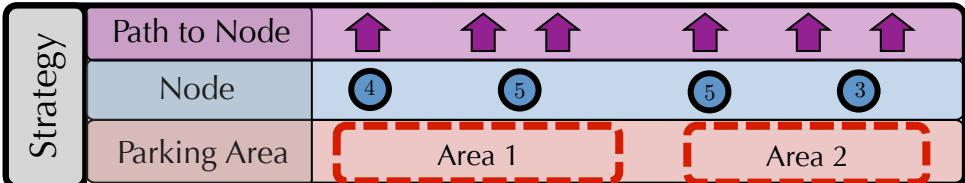
First Order Optimality...

$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \quad \sum_{e \in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e \in r} (\nu_{od}^{ap})_e$$

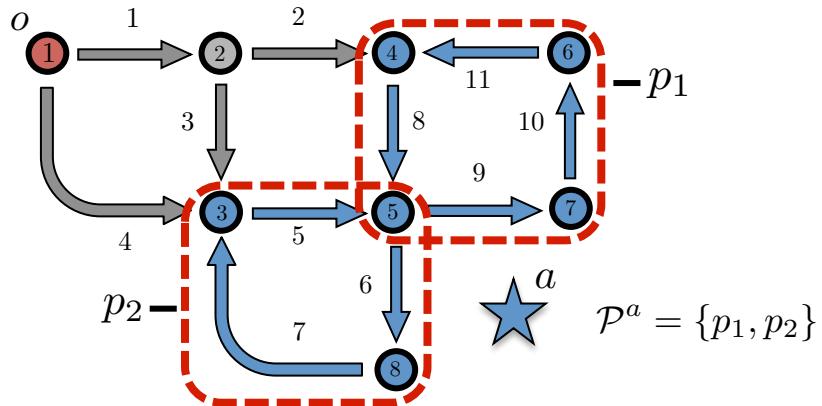
$$\frac{\partial \mathcal{L}}{\partial s_{od}^{ap}} : \quad \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p - R_p + C_p s^p = (\pi_{od}^{ap})_o - (\pi_{od}^{ap})_d + \lambda_o^a + \mu_{od}^{ap}$$

$$\sum_{e \in r} l_e(x_e) + \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p - R_p + C_p s^p = \lambda_o^a + \sum_{e \in r} (\nu_{od}^{ap})_e + \mu_{od}^{ap}$$

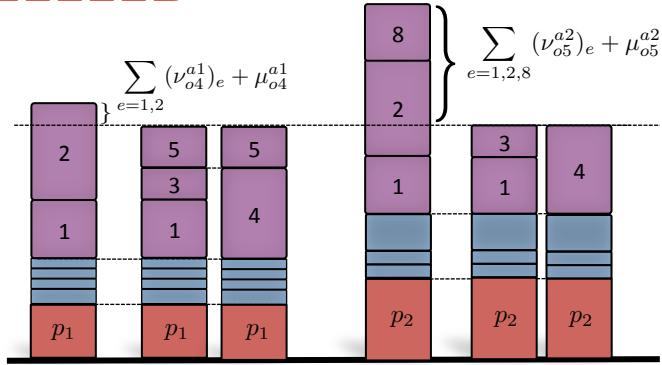
Traveling Cost	Circling Cost	Queue Cost	Total Cost
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Edge Formulation



Costs	Travel Latency
	Circling Latency
	Queue Cost



Mass
Bar chart showing mass distribution along the paths.

Strategy	Path to Node	Node	Parking Area
	↑ ↑ ↑ ↑	4 5 5 3	Area 1 Area 2

$$x = \sum_{o,a,p,d} x_{od}^{ap} + \sum_{o,a,p} \frac{1}{|\mathcal{E}^p|} \mathbf{E}^p \left[\sum_{d \in \mathcal{N}^p} s_{od}^{ap} \right] \quad s^p = \sum_{o,a} \sum_{d \in \mathcal{N}^p} s_{od}^{ap}$$

Optimization Problem

$$\min_{x_{od}^{ap}, s_{od}^{ap}} P(x) = \sum_e \int_0^{x_e} l_e(u) du + \sum_p \int_0^{s^p} -R_p + C_p u du$$

$$\text{s.t.} \quad x_{od}^{ap} \geq 0, \quad Gx_{od}^{ap} = S_{od}^{ap} \quad \forall o, d, a, p$$

$$s_{od}^{ap} \geq 0, \quad \sum_p \sum_{d \in \mathcal{N}^p} s_{od}^{ap} = s_o^a \quad \forall o, d, a, p$$

Gradient... $\nabla_{x_{od}^{ap}} P = l(x)$

$$\nabla_{s_{od}^{ap}} P = \frac{1}{|\mathcal{E}^P|} l(x) \mathbf{E}^p + \\ -R_p + C_p s^p$$

First Order Optimality...

$$\frac{\partial \mathcal{L}}{\partial (x_{od}^{ap})_e} : \quad \sum_{e \in r} l_e(x_e) = (\pi_{od}^{ap})_d - (\pi_{od}^{ap})_o + \sum_{e \in r} (\nu_{od}^{ap})_e$$

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$$\sum_{e \in r} l_e(x_e) + \frac{1}{|\mathcal{E}^p|} l(x) \mathbf{E}^p - R_p + C_p s^p = \lambda_o^a + \sum_{e \in r} (\nu_{od}^{ap})_e + \mu_{od}^{ap}$$

Traveling Cost	Circling Cost	Queue Cost	Total Cost	Extra Cost
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