Quadratic Forms, Definite Matrices, Congruence Transformations

Linear Algebra:

Major Contributions: John Simpson-Porco

Winter 2022 - Dan Calderone

Definite (Symmetric) Matrices

Quadratic Form: $f(x) = x^TQx \qquad Q \in \mathbb{R}^{n\times n} \quad Q = Q^T$

Definiteness:	Short	Notation	Definition			Analogy	Eigenvalues
Positive definite:	PD	$Q \succ 0$	$x^TQx > 0$	$\forall\ x$	$x \neq 0$	…positive orthant	$\lambda_i > 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Positive semi-definite	PSD	$Q \succeq 0$	$x^TQx \geq 0$	$\forall\ x$		…positive orthant w/ boundary	$\lambda_i \geq 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Negative-definite	ND	$Q \prec 0$	$x^TQx < 0$	$\forall\ x$	$x \neq 0$	…negative orthant	$\lambda_i < 0 \quad \lambda_i \in \mathrm{eig}(Q)$
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Indefinite:			$x^TQx > 0$ $x^TQx < 0$	some x some x		…the rest of the space	

Eigenvalue condition proof:

…consider eigenvector coordinates

$$x = Vx'$$

since V is invertible… $\quad \forall\ x \iff \forall\ x'$

$$x^TQx = xVDV^Tx = x'^TDx' = \sum_i \lambda_i x_i'^2$$

$$\sum_i \lambda_i x_i'^2 > 0 \quad \underset{x\neq 0}{\forall x'} \iff \lambda_i > 0 \quad \forall\ \lambda_i \in \mathrm{eig}(Q)$$

Note: not a useful definition for general matrices

… condition only says something about the symmetric part of Q

Symmetric/Skew-symmetric Decomposition

$$Q = \tfrac{1}{2}\underbrace{\left(Q + Q^T\right)}_{\text{symmetric}} + \tfrac{1}{2}\underbrace{\left(Q - Q^T\right)}_{\text{skew-sym}}$$

$$\begin{aligned}
x^TQx &= \tfrac{1}{2}x^T\left(Q+Q^T\right)x + \tfrac{1}{2}x^T\left(Q-Q^T\right)x \\
&= \tfrac{1}{2}x^T\left(Q+Q^T\right)x + \tfrac{1}{2}x^TQx - \underbrace{\tfrac{1}{2}x^TQ^Tx}_{\text{...transpose}} \\
&= \tfrac{1}{2}x^T\left(Q+Q^T\right)x + \underbrace{\tfrac{1}{2}x^TQx - \tfrac{1}{2}x^TQx}_{=0} \\
&= \tfrac{1}{2}x^T\left(Q+Q^T\right)x
\end{aligned}$$

…only the symmetric part matters

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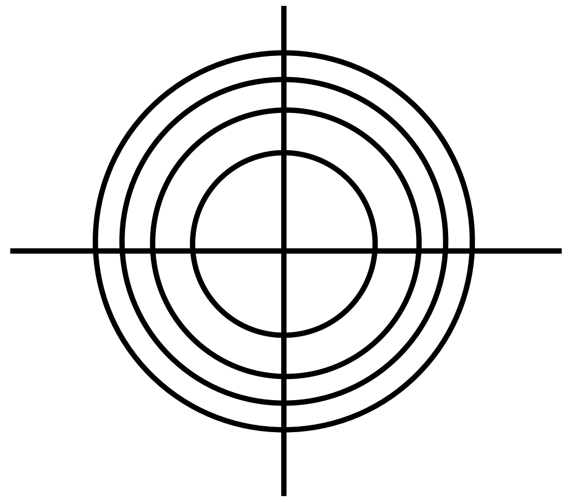
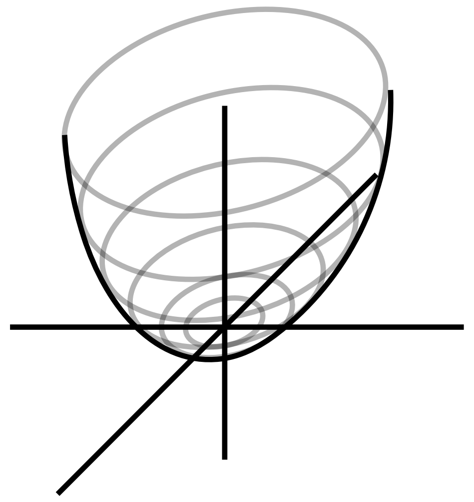
Surfaces: $Q \succ 0$

$Q = I$



level sets



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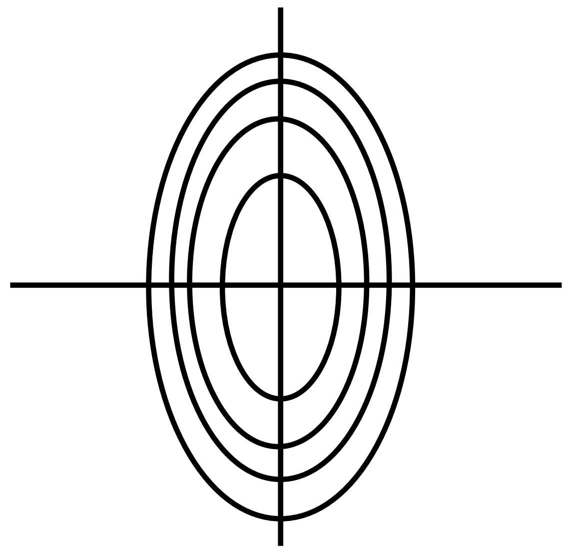
Q diagonal



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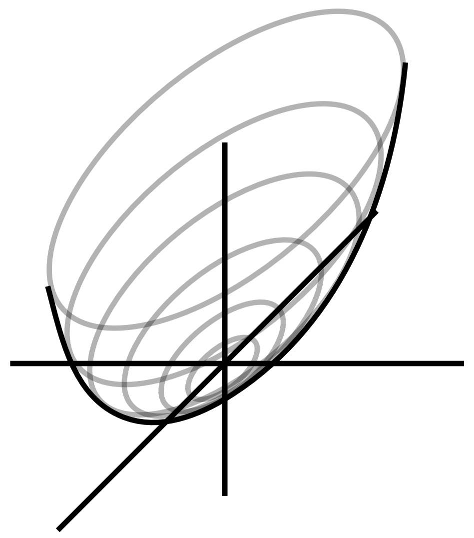

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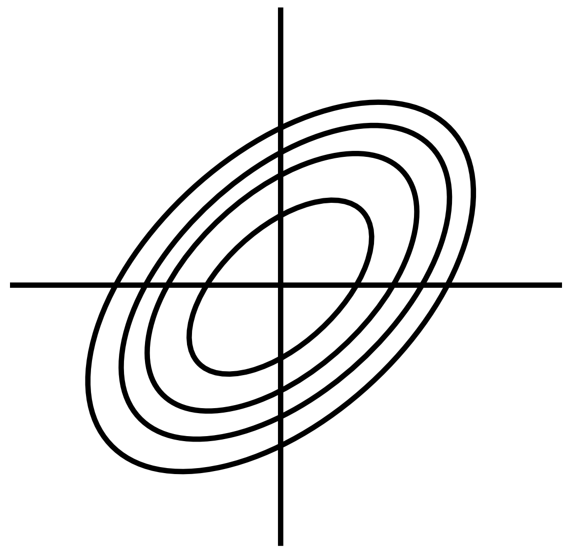

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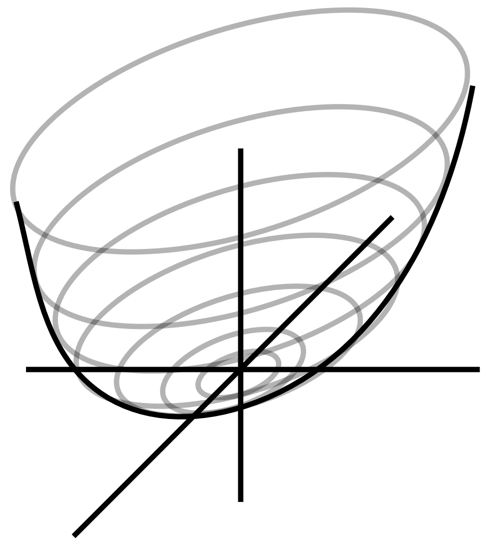

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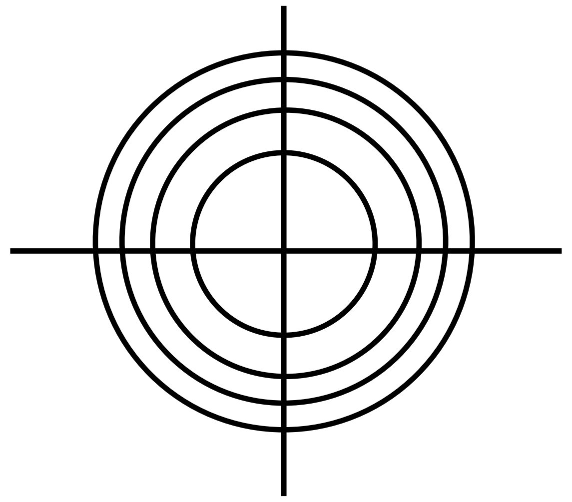

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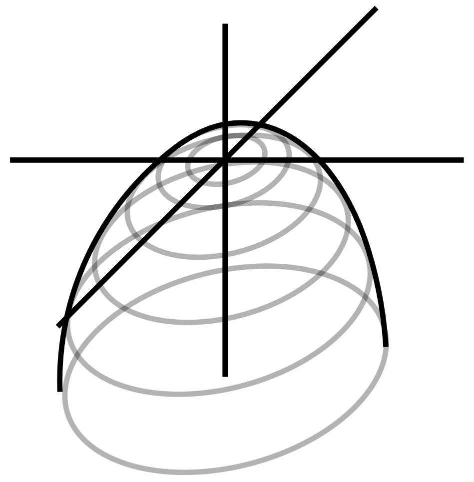

surface

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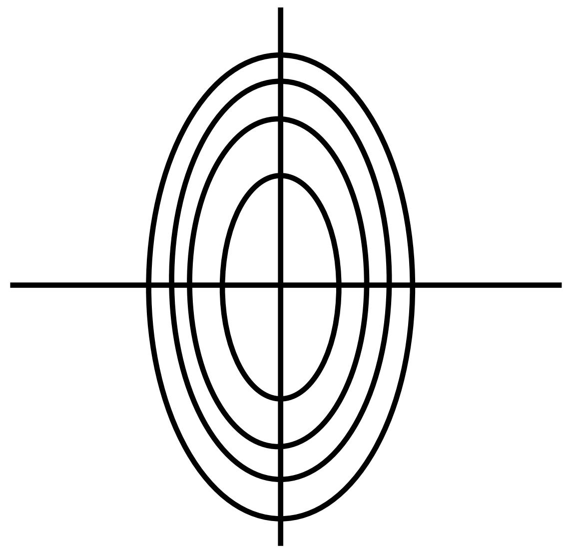

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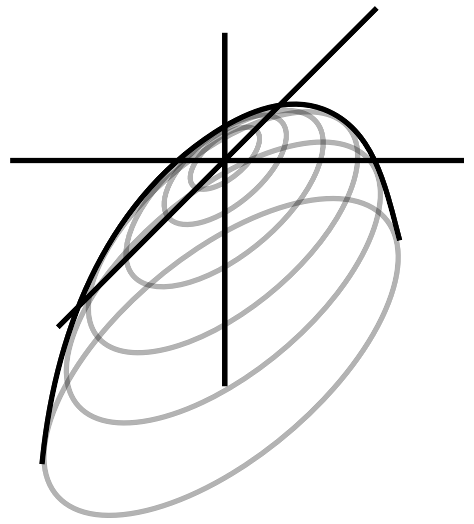

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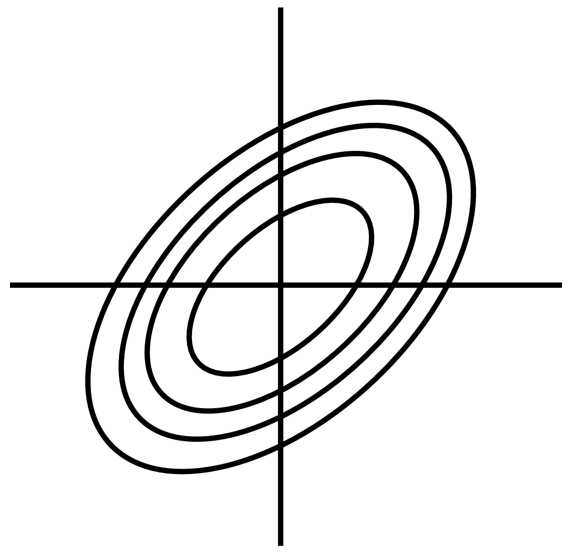

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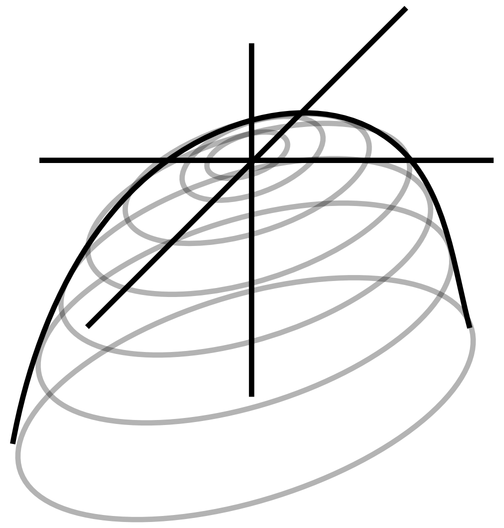

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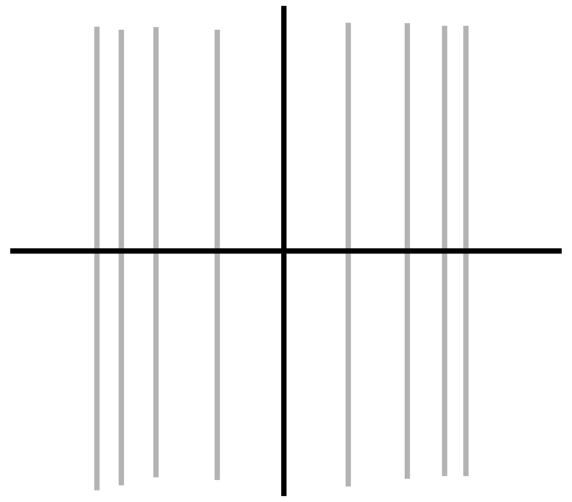

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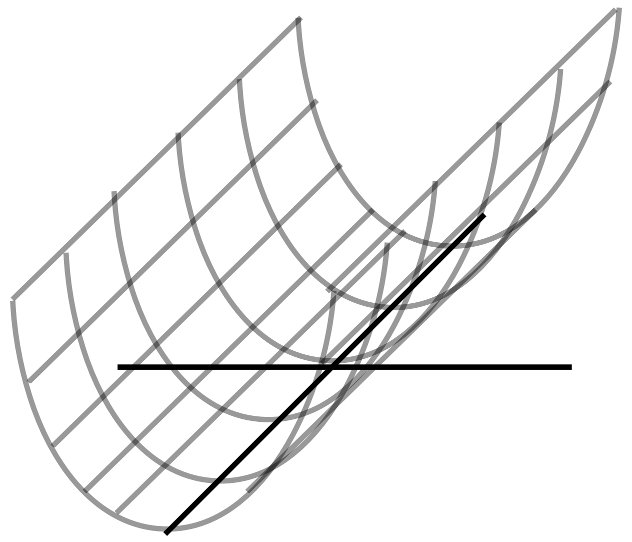
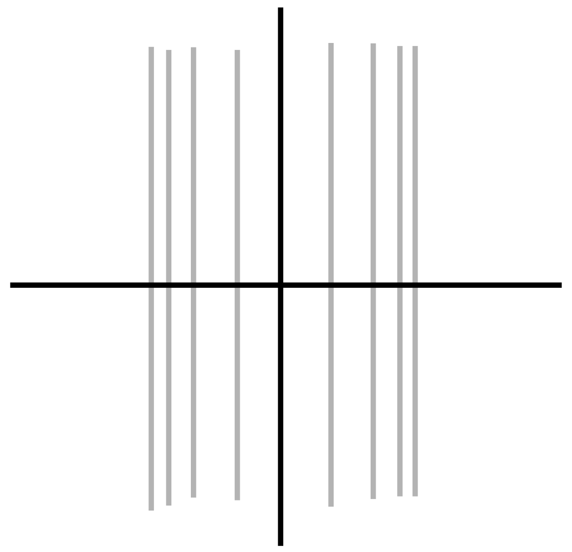


surface

$Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$ diagonal



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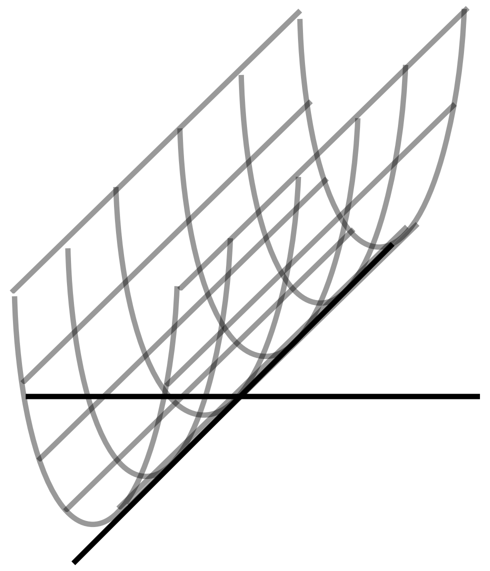

surface

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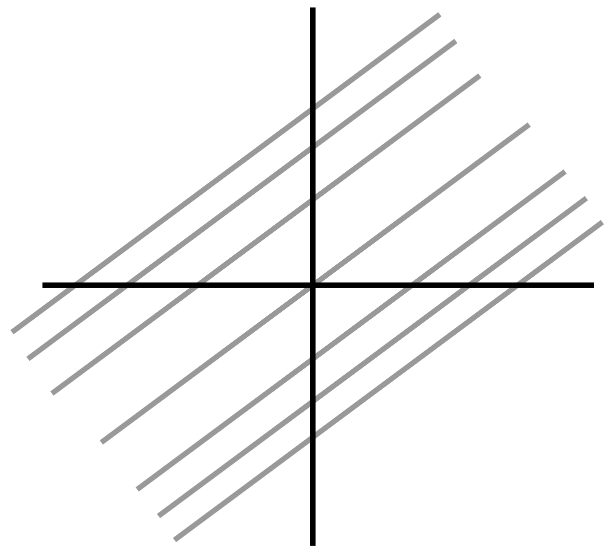

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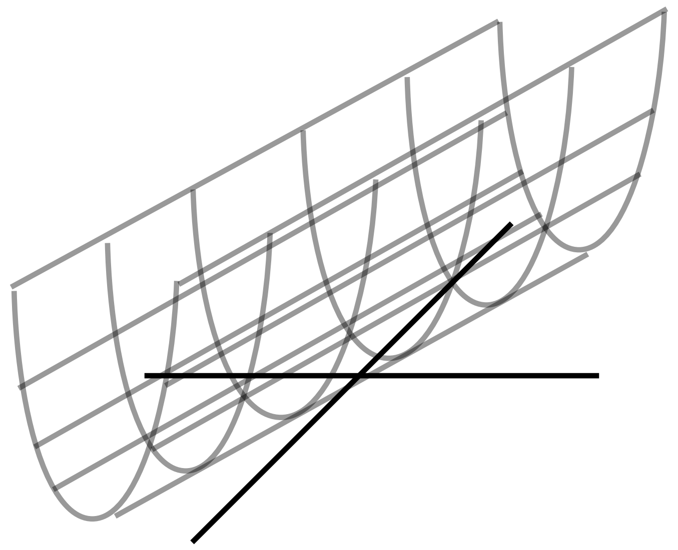

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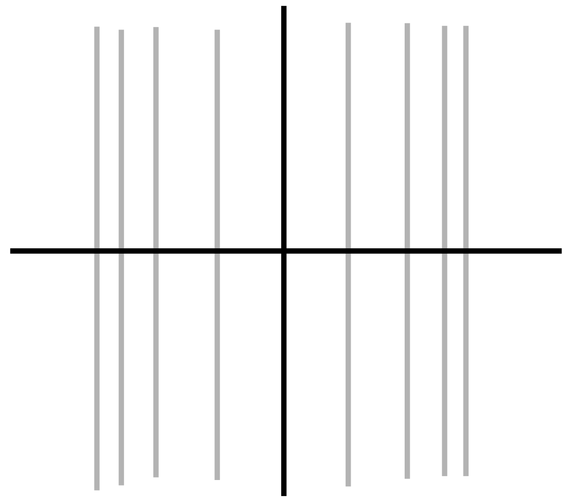

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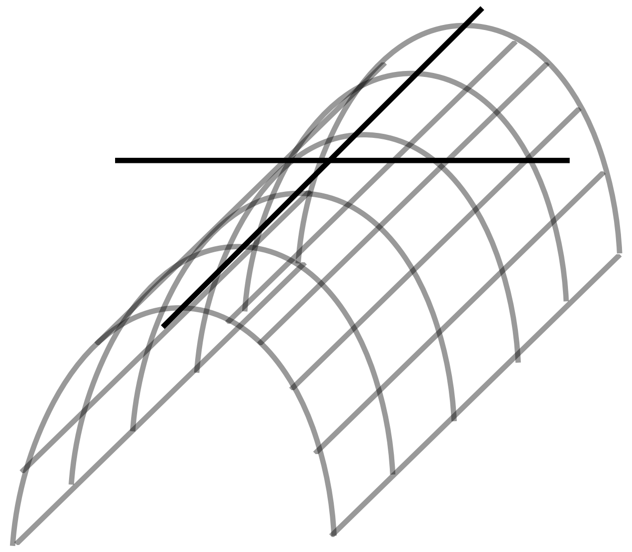

surface

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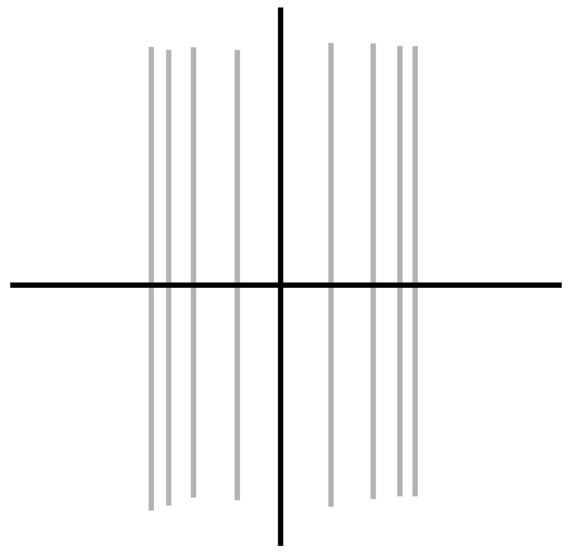

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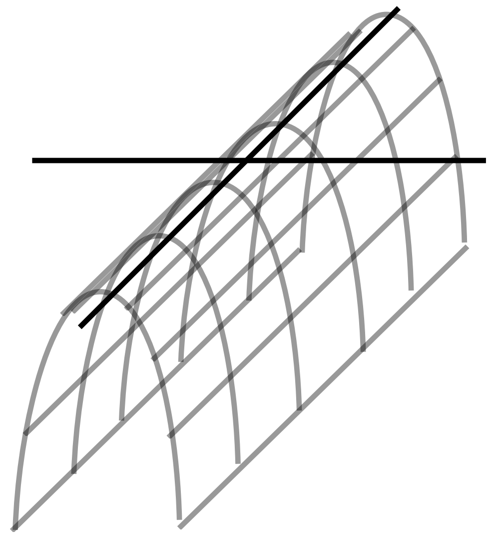

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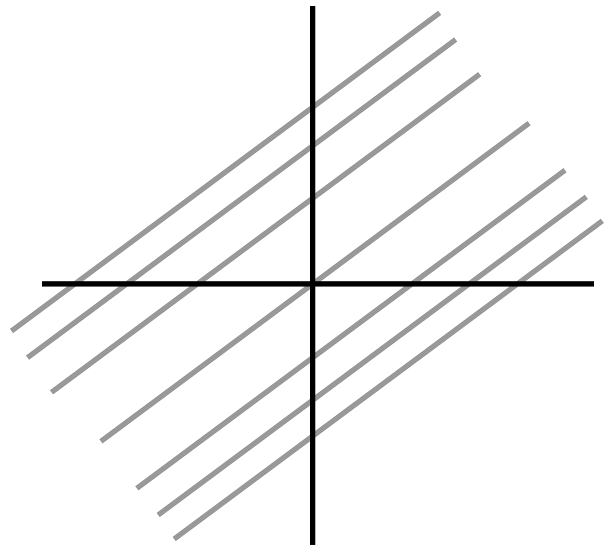

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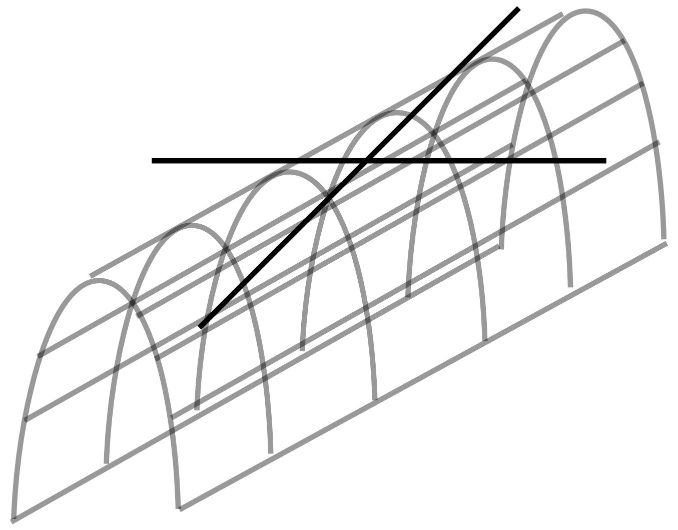
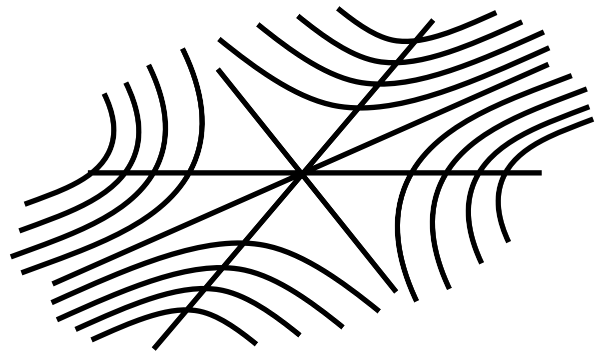


surface

Definite (Symmetric) Matrices



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Surfaces: Q indefinite

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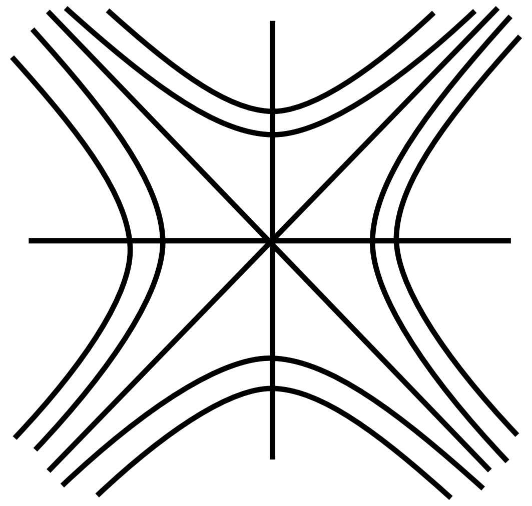

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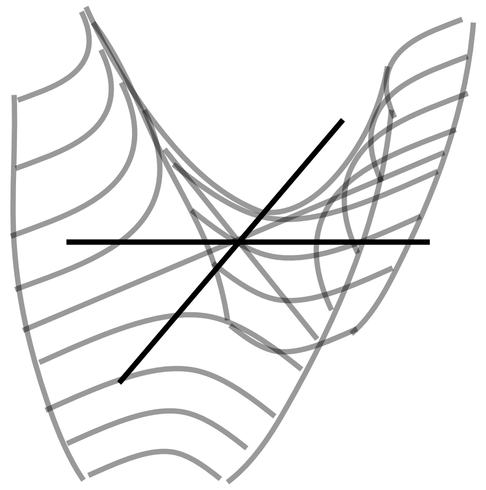

surface

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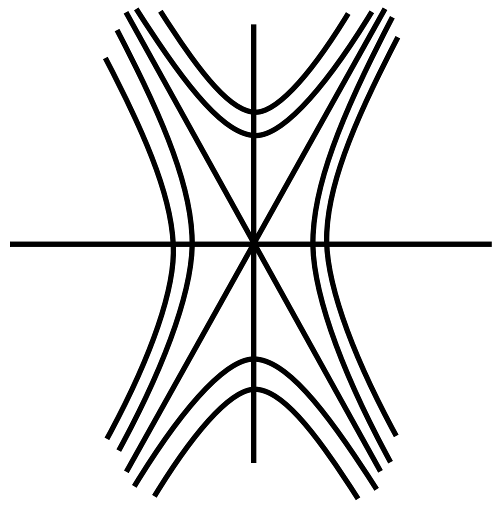

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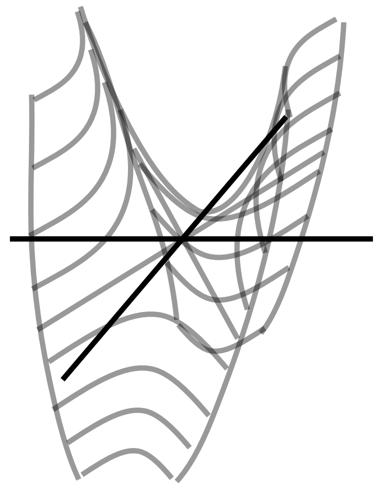

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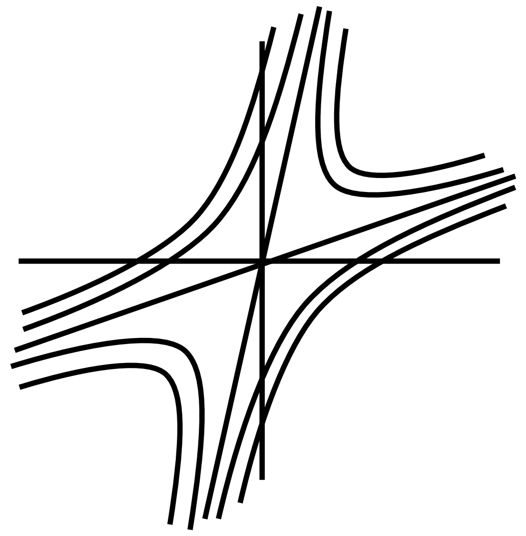

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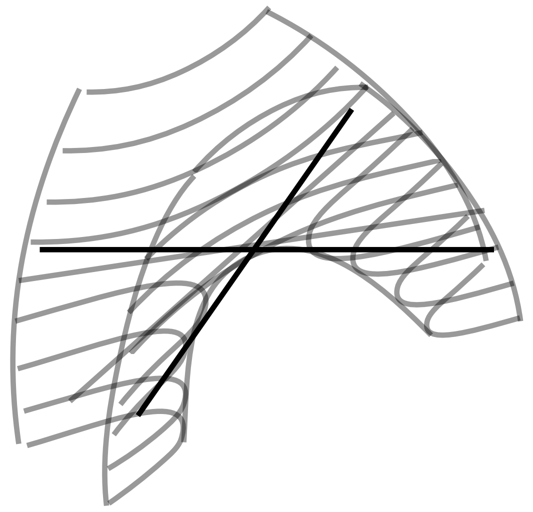

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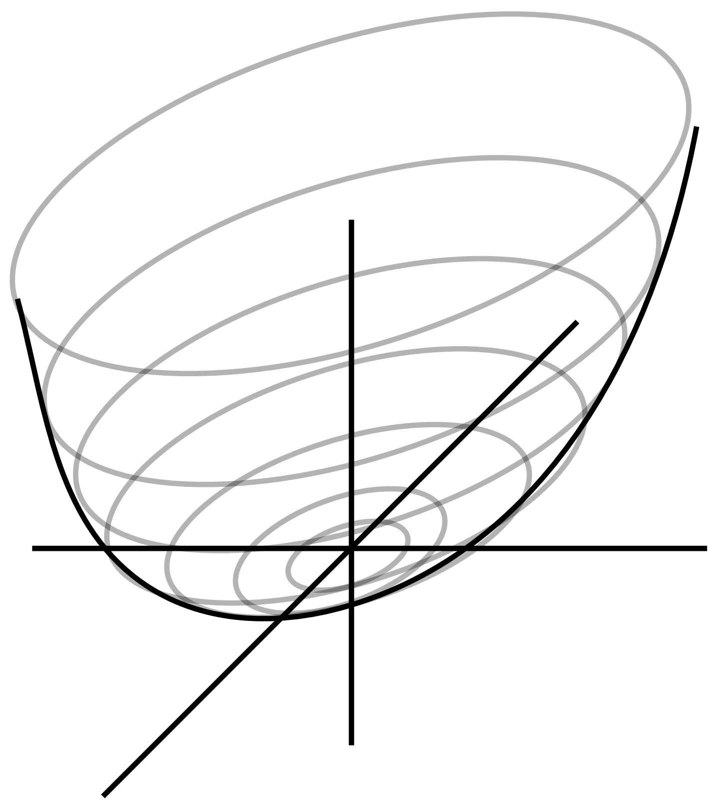

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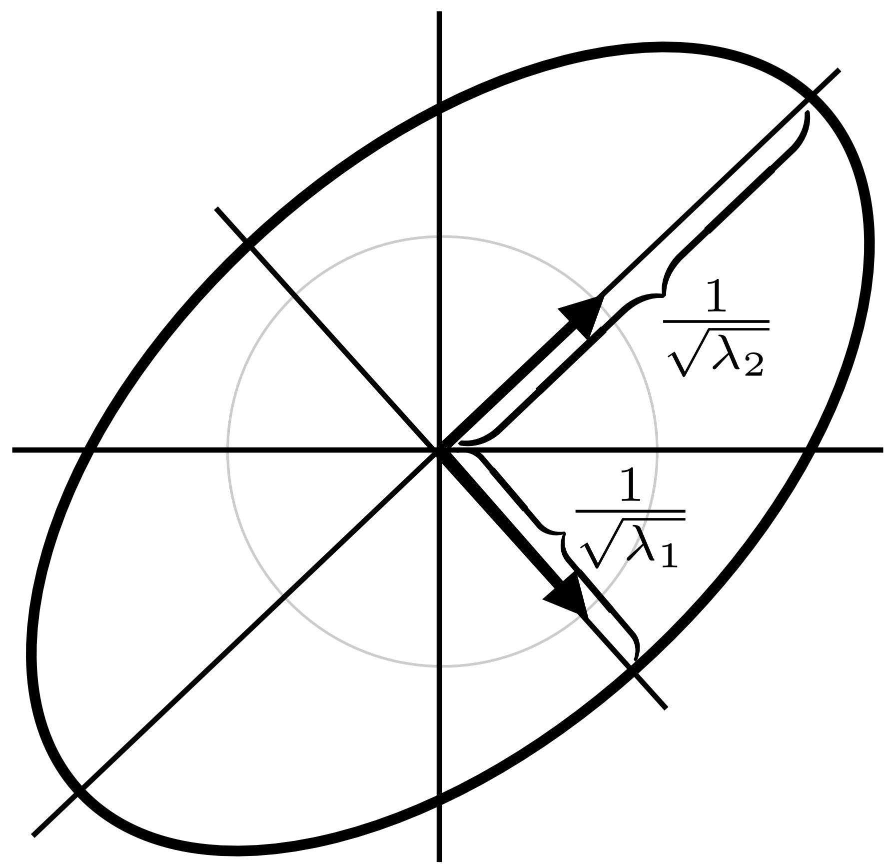
$f(x) = x^T Q x = 1$



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$$Q = V D V^T = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$f\left(\frac{1}{\sqrt{\lambda_1}} v_1\right) = \frac{1}{\sqrt{\lambda_1}} v_1^T Q v_1 \frac{1}{\sqrt{\lambda_1}}$$

$$= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix}^T \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \frac{1}{\sqrt{\lambda_1}}$$

$$= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{\lambda_1}} = \frac{\lambda_1}{(\sqrt{\lambda_1})^2} = 1$$

Definite (Symmetric) Matrices

Quadratic Form: $f(x) = x^T Q x \quad Q \in \mathbb{R}^{n\times n} \quad Q = Q^T$

Definiteness:	Short	Notation	Definition			Analogy	Eigenvalues
Positive definite:	PD	$Q \succ 0$	$x^TQx > 0$	$\forall\ x$	$x \neq 0$	…positive orthant	$\lambda_i > 0 \quad \lambda_i \in \text{eig}(Q)$
Positive semi-definite	PSD	$Q \succeq 0$	$x^TQx \geq 0$	$\forall\ x$		…positive orthant w/ boundary	$\lambda_i \geq 0 \quad \lambda_i \in \text{eig}(Q)$
Negative-definite	ND	$Q \prec 0$	$x^TQx < 0$	$\forall\ x$	$x \neq 0$	…negative orthant	$\lambda_i < 0 \quad \lambda_i \in \text{eig}(Q)$
Negative semi-definite	NSD	$Q \preceq 0$	$x^TQx \leq 0$	$\forall\ x$		…negative orthant w/ boundary	$\lambda_i \leq 0 \quad \lambda_i \in \text{eig}(Q)$
Indefinite:			$x^TQx > 0$	some x		…the rest of the space	
			$x^TQx < 0$	some x			

Eigenvalue condition proof:

…consider eigenvector coordinates

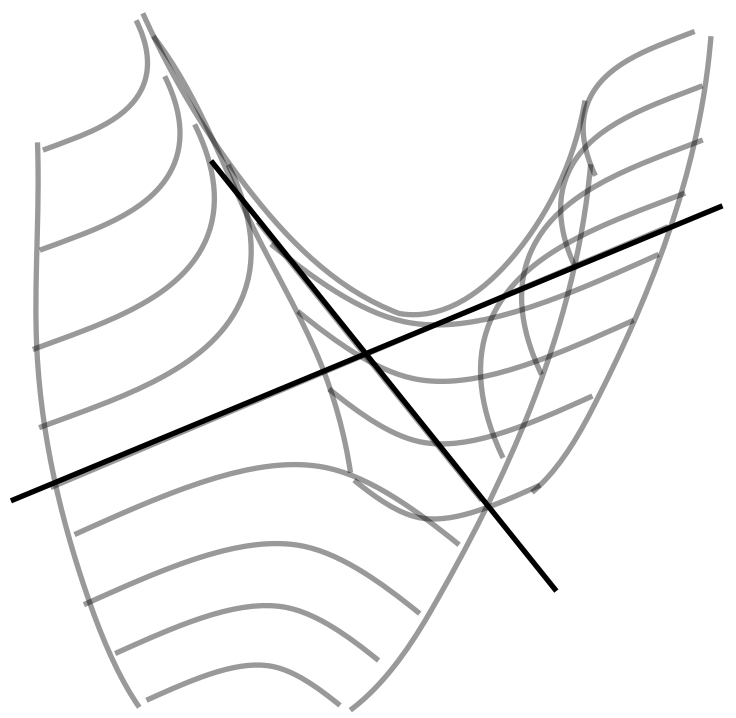
$$x = Vx'$$

since V is invertible… $\quad \forall\ x \iff \forall\ x'$

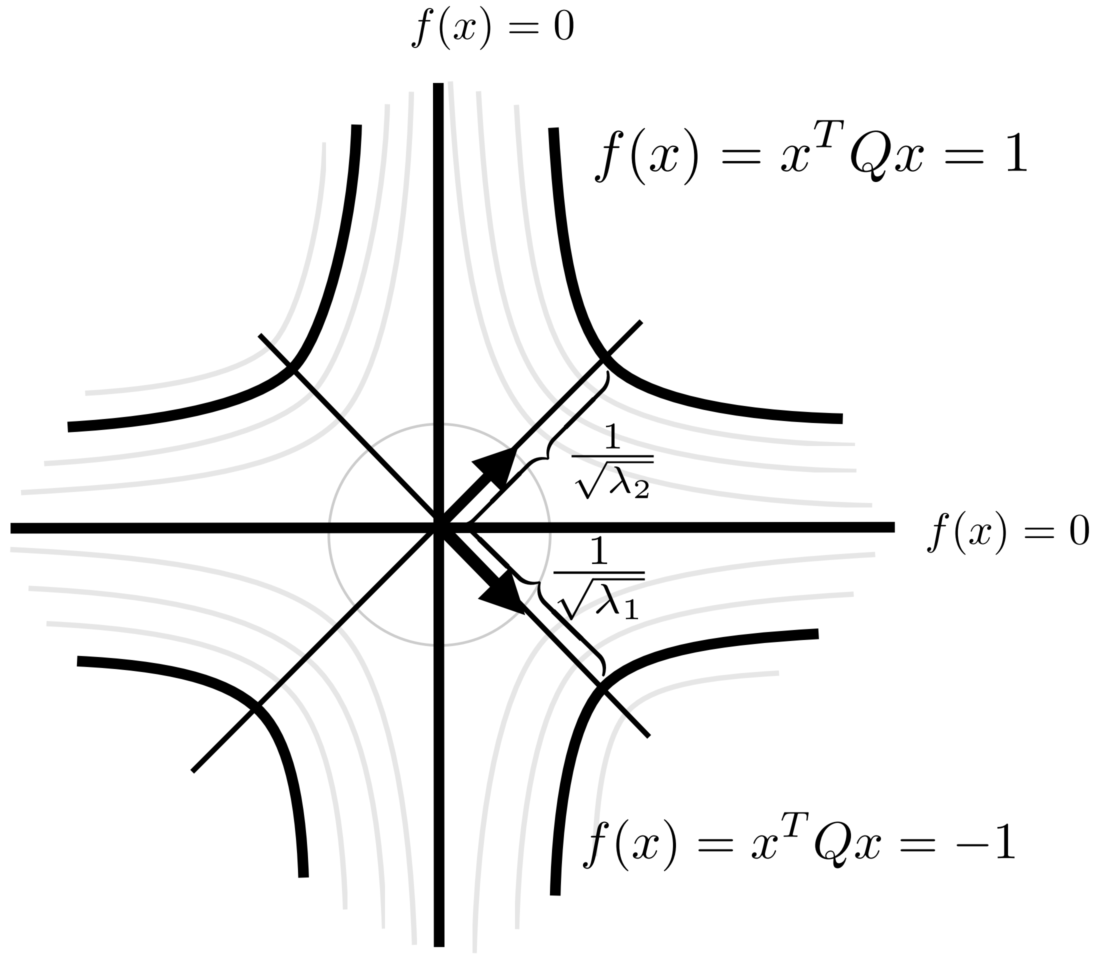
$$x^TQx = xVDV^Tx = x'^TDx' = \sum_i \lambda_i {x'_i}^2$$

$$\sum_i \lambda_i {x'_i}^2 > 0 \quad \underset{x\neq 0}{\forall x'} \iff \lambda_i > 0 \quad \forall\ \lambda_i \in \text{eig}(Q)$$

Surfaces: Q indefinite



surface



level sets

$$Q = VDV^T = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$f\left(\frac{1}{\sqrt{\lambda_1}}v_1\right) = \frac{1}{\sqrt{\lambda_1}}v_1^TQv_1\frac{1}{\sqrt{\lambda_1}}$$

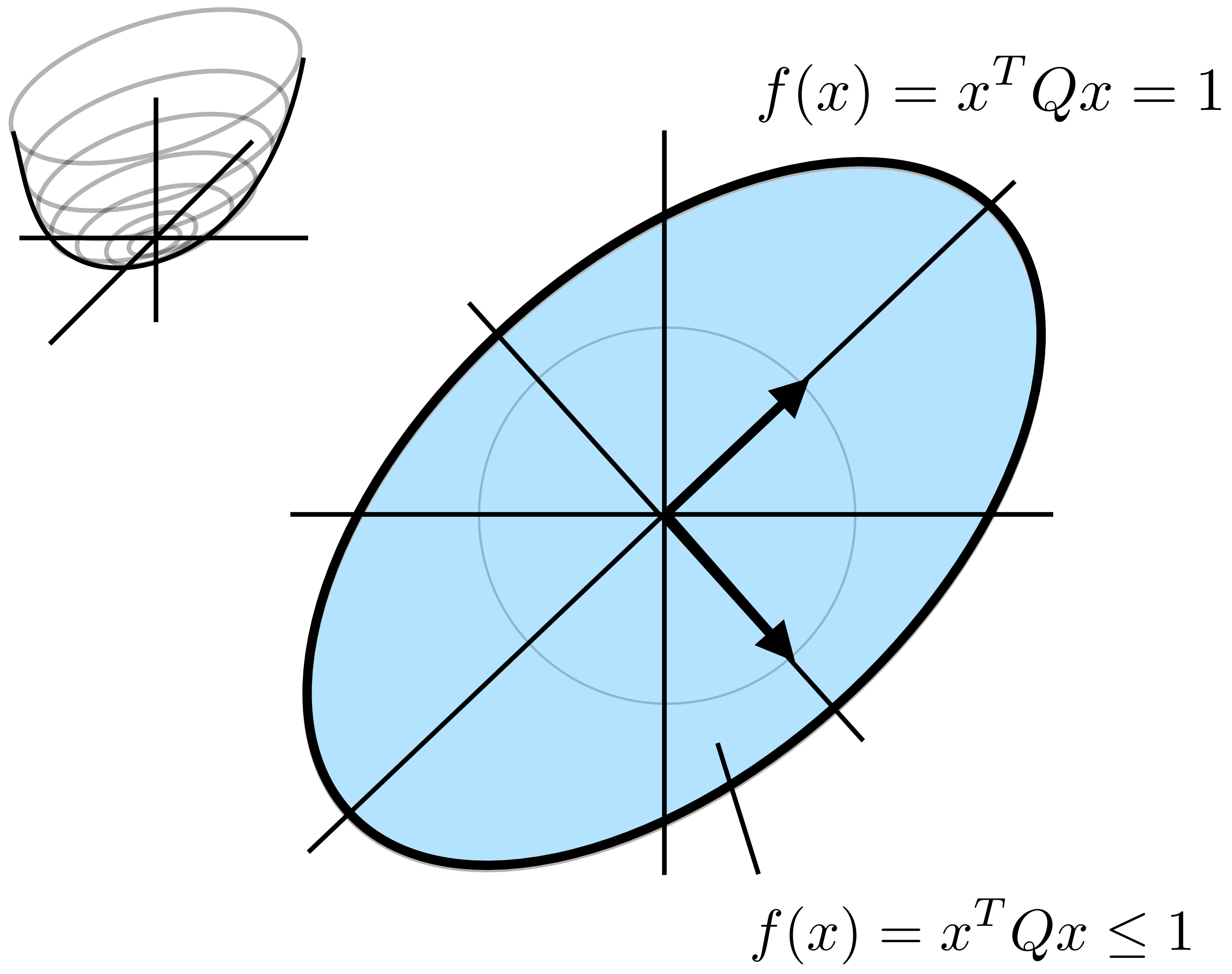
$$= \frac{1}{\sqrt{\lambda_1}}\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix}^T\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix}\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix}\frac{1}{\sqrt{\lambda_1}}$$

$$= \frac{1}{\sqrt{\lambda_1}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{\sqrt{\lambda_1}} = \frac{\lambda_1}{(\sqrt{\lambda_1})^2} = 1$$

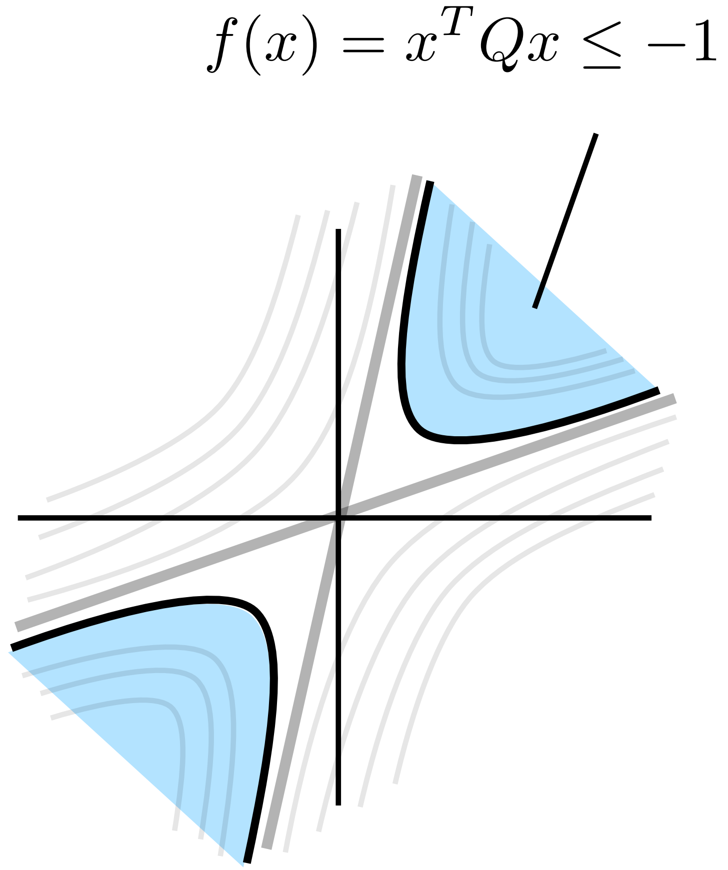
Quadratic Form - Level Sets

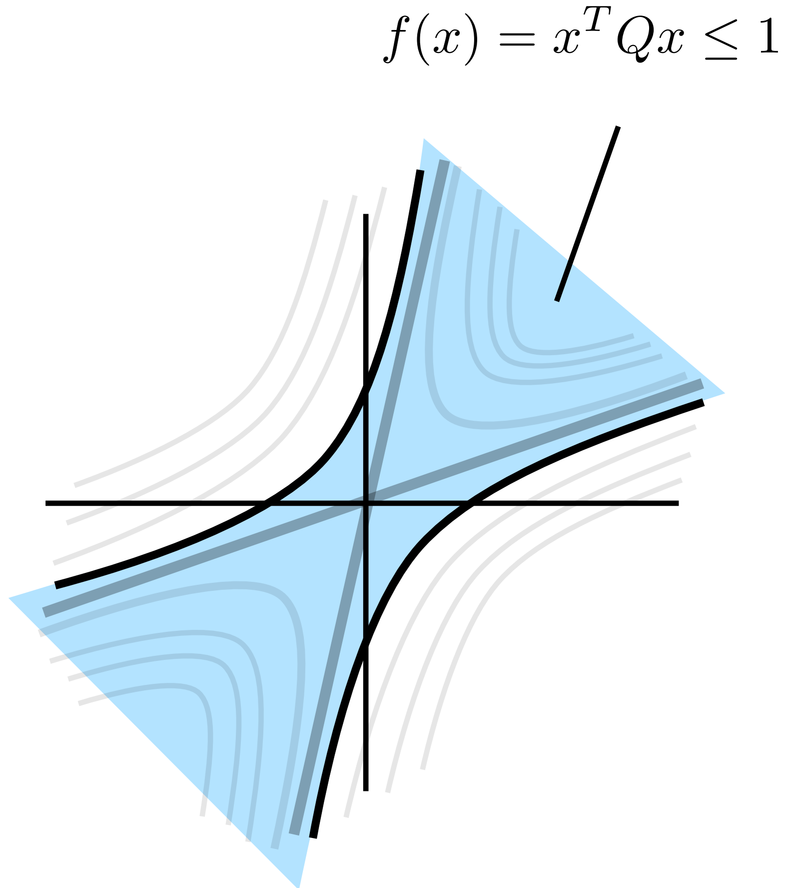
Quadratic Form: $f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$

Ellipsoids

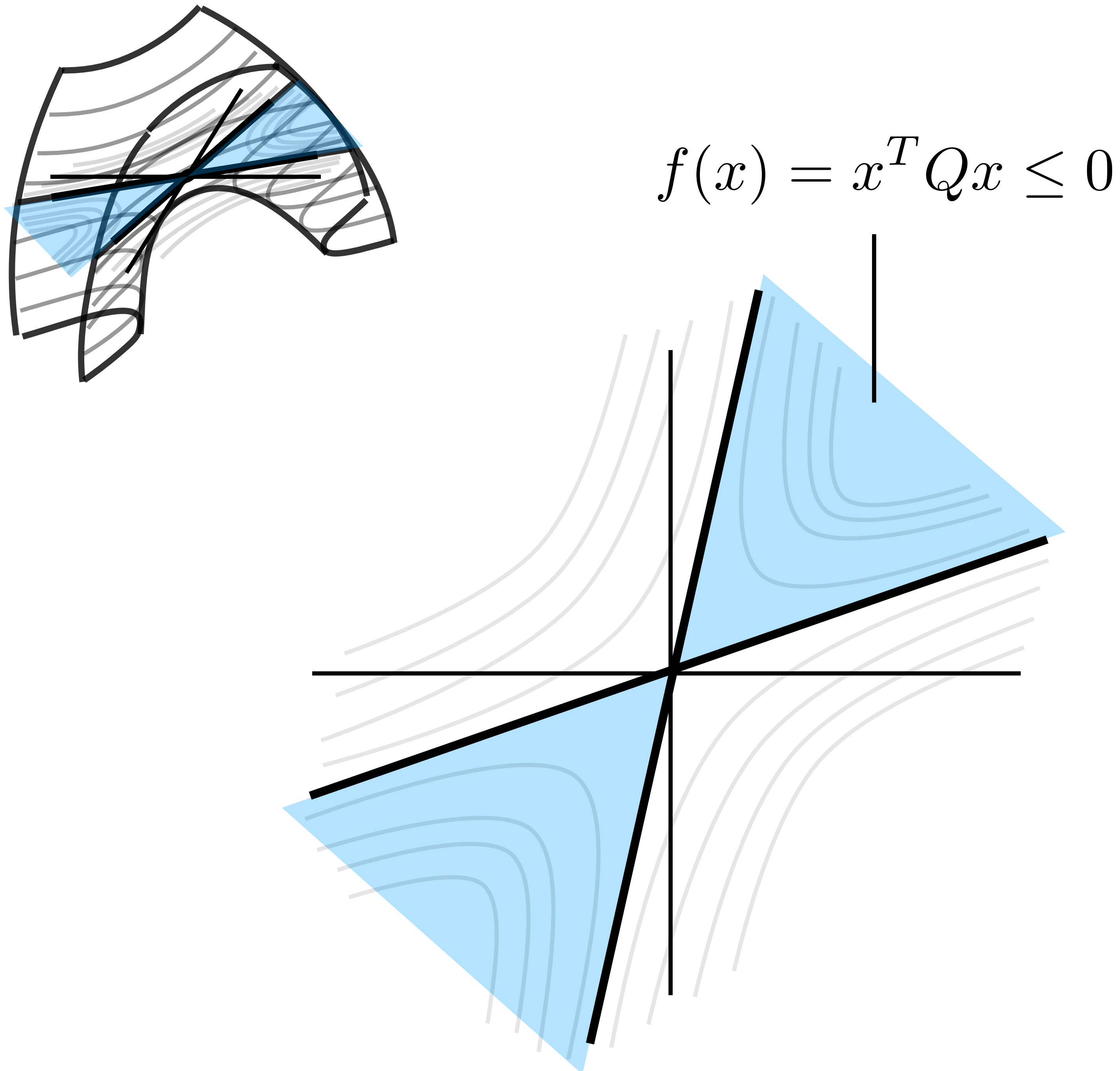


Hyperboloids





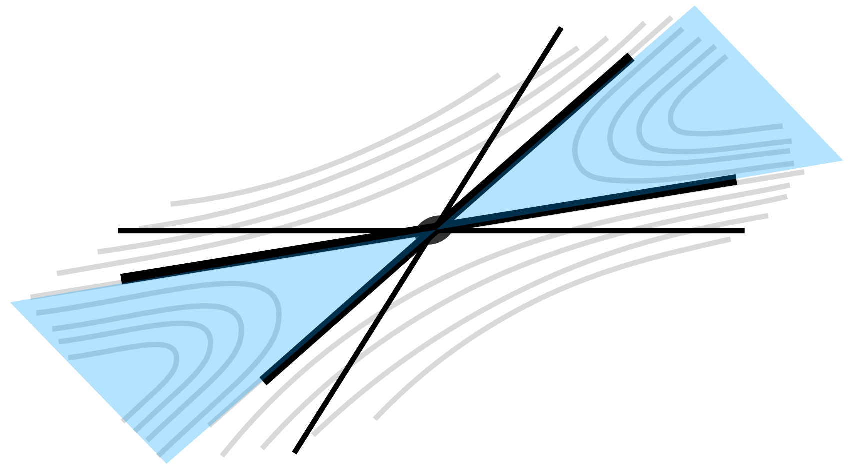
Cones

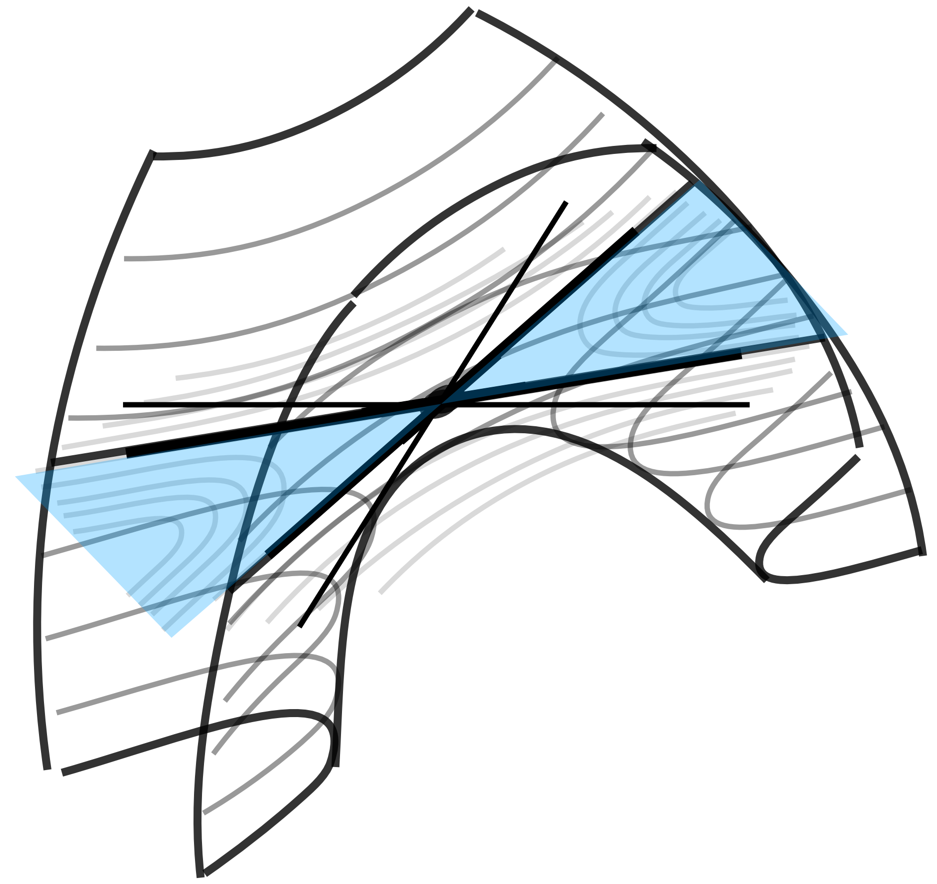


Quadratic Form - Level Sets

Quadratic Form: $f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$

<u>**Cones**</u>





$$\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix} \qquad \begin{matrix} b \in [\underline{\mathrm{b}},\ \bar{\mathrm{b}}] \\ c \in [\underline{\mathrm{c}},\ \bar{\mathrm{c}}] \end{matrix}$$

$\Pi \in \mathbb{S}_n$

$\det(1 - bc)$