Graph Structures & Matrices

Algebraic Graph Theory

Acknowledgements: Mehran Mesbahi
Mathias Colbert Russelson,
Sarah Li
Shahriar Talebi

DATES: 3/30/22
4/4/22

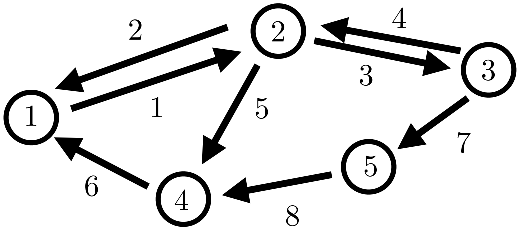
Spring 2022 - Dan Calderone

Graphs

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



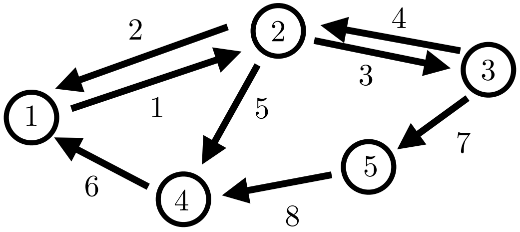
Graphs

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

edge e is "incident" to v and v'



Undirected Graphs

$$e = (v, v')$$

Neighborhoods: set of "adjacent" nodes

$$\mathcal{N}_v = \{v' \in \mathcal{V} \mid e = (v, v') \in \mathcal{E}\}$$

degree of vertex $\quad d_v = |\mathcal{N}_v|$

Directed Graphs

$e = (v, v')$ edge e from v to v'

Neighborhoods: set of "adjacent" nodes

$$\mathcal{N}_v^{\text{out}} = \{v' \in \mathcal{V} \mid e = (v, v') \in \mathcal{E}\}$$

$$\mathcal{N}_v^{\text{in}} = \{v' \in \mathcal{V} \mid e = (v', v) \in \mathcal{E}\}$$

$$\mathcal{N}_v = \mathcal{N}_v^{\text{in}} \cup \mathcal{N}_v^{\text{out}}$$

out-degree $\quad d_v^{\text{in}} = |\mathcal{N}_v^{\text{in}}|$

in-degree $\quad d_v^{\text{out}} = |\mathcal{N}_v^{\text{out}}|$

degree $\quad d_v = d_v^{\text{in}} + d_v^{\text{out}}$

Automorphism of Graph

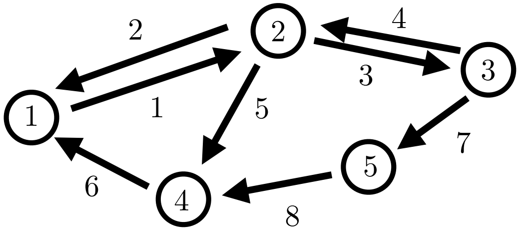
"Relabeling of nodes and edges
that maintains graph structure"

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

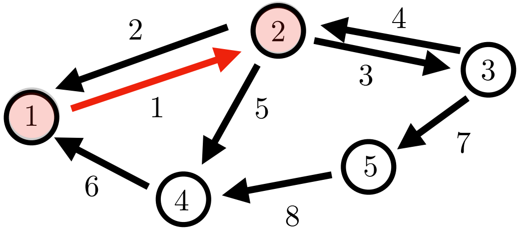
$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{array}{ccc} & \longleftarrow \text{edges} \longrightarrow & \\ D = & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} & \updownarrow \text{vertices} \end{array}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{cc} \longleftarrow \text{edges} \longrightarrow & \\ D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} & \updownarrow \text{vertices} \end{array}$$

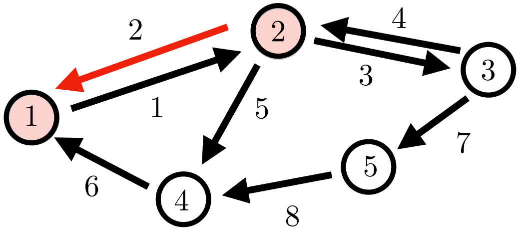
$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, D \,\right]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\left[\, D_{\text{out}} \,\right]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\left[\, D_{\text{in}} \,\right]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

edges

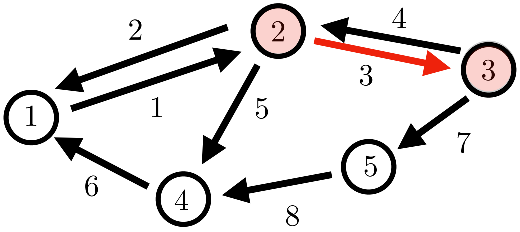
$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

vertices

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

edges

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

vertices

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

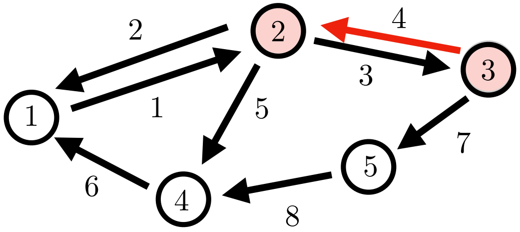
$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

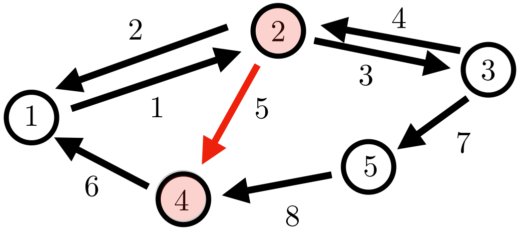
$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{array}{cc} & \longleftarrow \text{edges} \longrightarrow \\ D = & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \updownarrow \text{vertices} \end{array}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

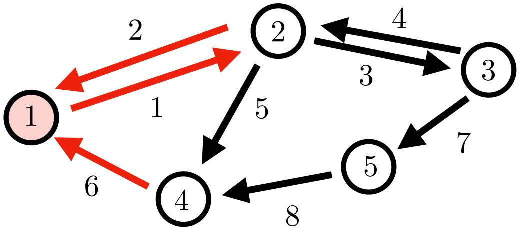
$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{array}{cc} & \longleftarrow \text{edges} \longrightarrow \\ D = & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \updownarrow \text{vertices} \end{array}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

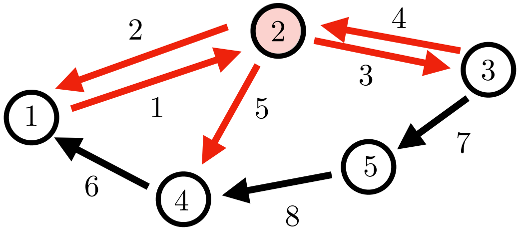
$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{array}{cc} & \longleftarrow \text{edges} \longrightarrow \\ D = & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \updownarrow \text{vertices} \end{array}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\big[\, D \,\big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\big[\, D_{\text{out}} \,\big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\big[\, D_{\text{in}} \,\big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

← edges →

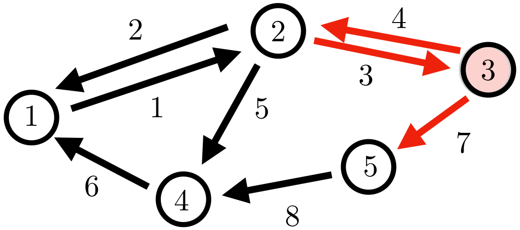
$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

↑ vertices ↓

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

← edges →

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \updownarrow \text{ vertices}$$

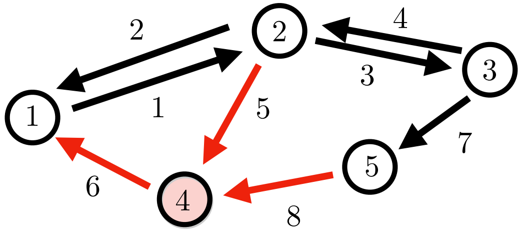
$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{array}{cc} & \longleftarrow \text{edges} \longrightarrow \\ D = & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \updownarrow \text{vertices}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

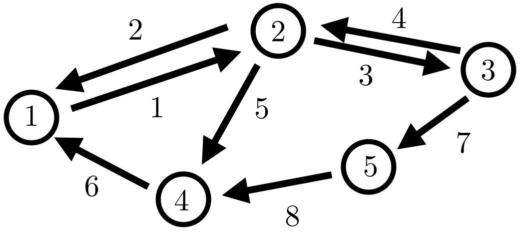
$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$\longleftarrow$ edges $\longrightarrow$

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$\updownarrow$ vertices

…relabeling nodes **rearrange rows**

…relabeling edges **rearrange columns**

Algebraically: multiply by permutation matrices

P, P' permutation matrices

New Incidence Matrix $\quad D' = PDP'$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad AP = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_2 & A_1 & A_3 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & a_3^T & - \end{bmatrix} = \begin{bmatrix} - & a_2^T & - \\ - & a_1^T & - \\ - & a_3^T & - \end{bmatrix}$$

Review Block:

$$Ax = \begin{bmatrix} A_1 \cdots A_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A_1 x_1 + \cdots + A_n x_n$$

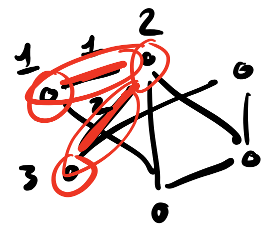
$$Ax = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

$$AB = A \begin{bmatrix} B_1 \cdots B_k \end{bmatrix} = \begin{bmatrix} AB_1 \cdots AB_k \end{bmatrix}$$

Incidences: $D \Longrightarrow D'$ new incidence matrix of subgraph

take subset of nodes/edges

$$D' = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}}_{|V|} \; 3 \quad \overbrace{\begin{bmatrix} & & \\ & D & \\ & & \end{bmatrix}}^{|E|} \quad \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}}^{2}$$



$D' \in \mathbb{R}^{3 \times 2}$

Incidence Matrix - Domain

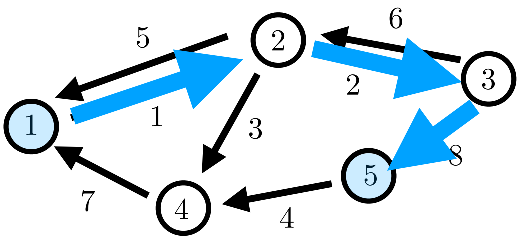
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

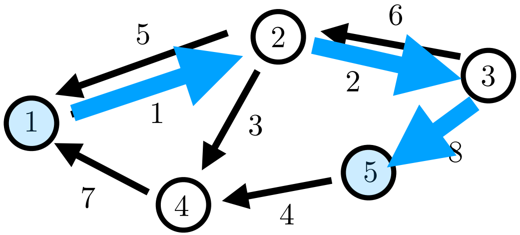
Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Incidence Matrix - Domain

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Examples

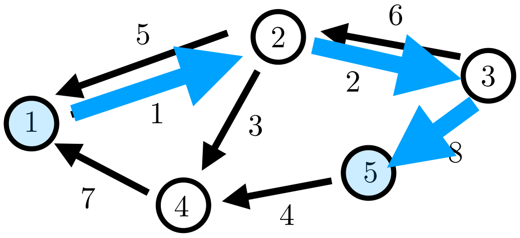
...fluid flow

...traffic flow (transportation networks, commodity flow)

...data flow

...current

Incidence Matrix - Domain



Graph: Vertices $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ Edges $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

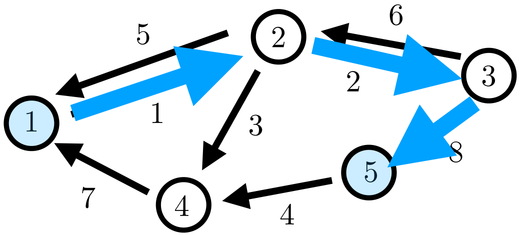
Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes

Examples

...fluid flow
...traffic flow
...data flow
...current

Incidence Matrix - Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

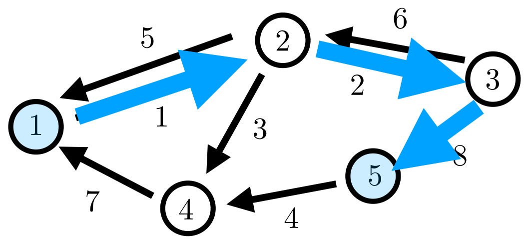
Non-conserved flow $\quad S = Dx \quad$ **Edge flow vector**

$$S = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

Incidence Matrix - Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

Non-conserved flow $\quad S = Dx \quad$ **Edge flow vector**

$$x = \boxed{\bar{x}} + Cz$$

Specific Solution $\quad$ Cyclic Flow

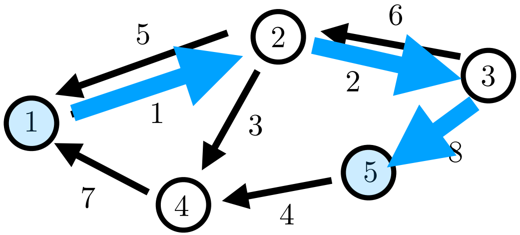
$$S = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}}_{x}$$

Incidence Matrix - Domain

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \boxed{\bar{x}} + Cz$$

Specific Solution

Cyclic Flow

$$\mathbb{1}^T D = 0$$

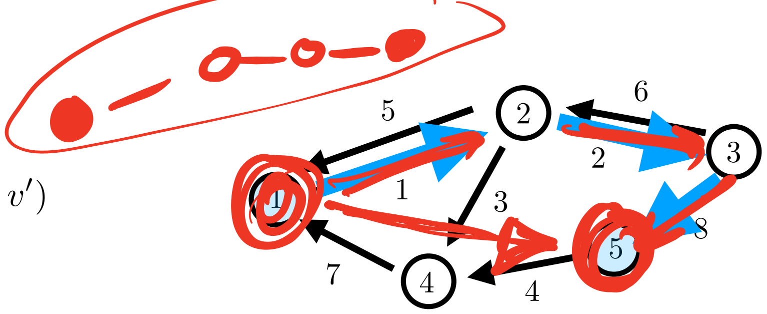
$$S = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

edge 1

edge 2

edge 8

Incidence Matrix - Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes

Non-conserved flow $\quad S = Dx \quad$ **Edge flow vector**

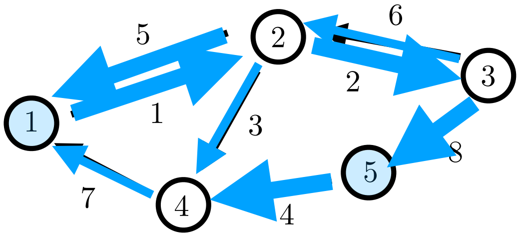
$$x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

$$S = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Col 1 $\quad$ Col 2 $\quad$ Col 8

Incidence Matrix - Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$S = \begin{bmatrix} -1 \\ 0.5 \\ -0.3 \\ -0.2 \\ 1 \end{bmatrix}$$

$$x = \bar{x} + Cz$$

Specific Solution

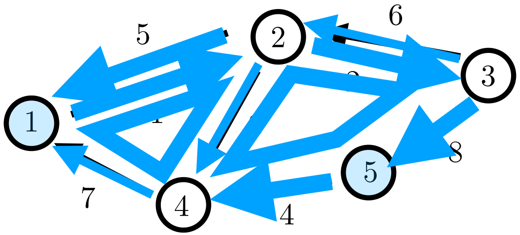
Cyclic Flow

Incidence Matrix - Domain

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

Non-conserved flow

$$S = Dx$$

Edge flow vector

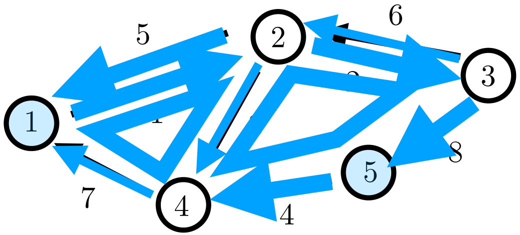
$$S = \begin{bmatrix} -1 \\ 0.5 \\ -0.3 \\ -0.2 \\ 1 \end{bmatrix}$$

$$x = \boxed{\bar{x} + Cz}$$

Specific Solution

Cyclic Flow

Incidence Matrix - Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

Non-conserved flow $\quad S = Dx \quad$ **Edge flow vector**

$$x = \boxed{\bar{x} + Cz}$$

Specific Solution $\quad$ Cyclic Flow

Minimum Norm Solution: $x = D^T (DD^T)^\dagger S$

… no component of x in nullspace, ie. no cycle flows

Moore Penrose Pseudoinverse

… gives the minimum norm/least squares solution

… to be an exact solution S needs to be in range of D (conservation of flow in & out of network)

Incidence Matrix - Co-Domain

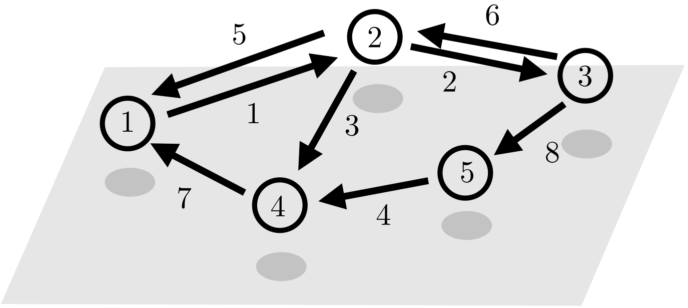
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

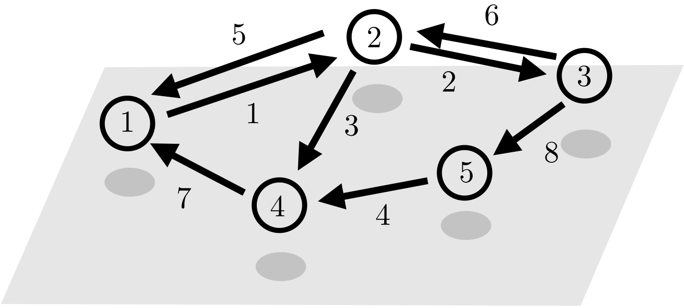
Cyclic Flow

Incidence Matrix - Co-Domain

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Non-conserved flow $\quad S = Dx \quad$ **Edge flow vector**

$$x = \bar{x} + Cz$$

Specific Solution (for $\bar{x}$) — Cyclic Flow (for Cz)

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

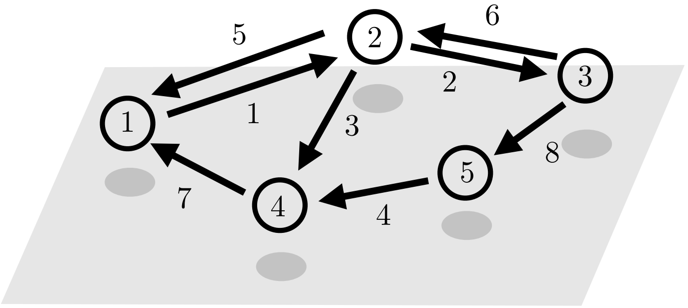
$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

Examples **...gravitational potential**

...voltage

...cost-to-go

Incidence Matrix - Co-Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ **...tension/difference on edges**

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

Non-conserved flow $$S = Dx$$ **Edge flow vector**

$$x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

voltages on nodes

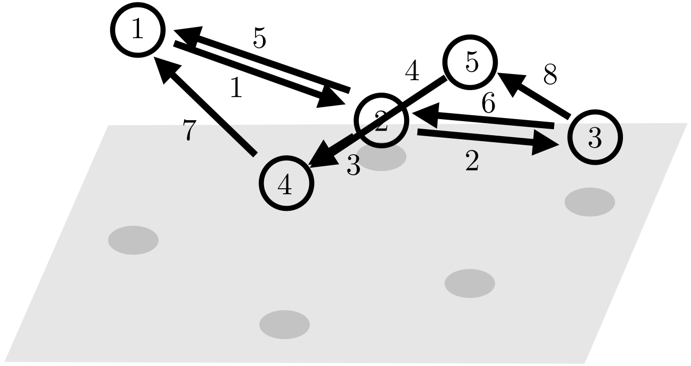
$$w^T D = \tau^T$$

voltage drops on edges

Value function $\quad$ **Edge tension**

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} w_1 - w_2 & w_2 - w_3 & w_3 - w_1 \end{bmatrix}$$

Incidence Matrix - Co-Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\qquad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ **...tension/difference on edges**

Non-conserved flow $\qquad$ **Edge flow vector**

$$S = Dx$$

$$x = \bar{x} + Cz$$

Specific Solution $\qquad$ Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

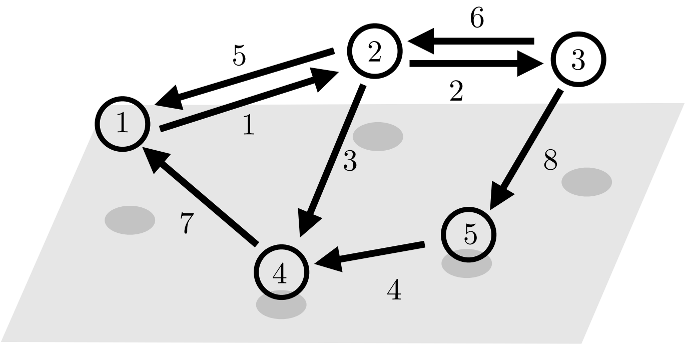
$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

$$w^T D = \tau^T$$

Value function $\qquad$ **Edge tension**

$$\left[w^T D\right]_e = w_i - w_{i'} = \tau_e$$

Incidence Matrix - Co-Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ **...tension/difference on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

$$w^T D = \tau^T$$

Value function

Edge tension

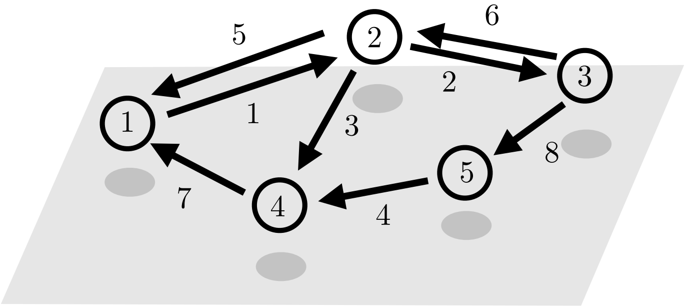
$$\left[w^T D\right]_e = w_i - w_{i'} = \tau_e$$

Incidence Matrix - Co-Domain

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ **...tension/difference on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

$$w^T D = \tau^T$$

Value function $\quad$ **Edge tension**

$$\left[w^T D\right]_e = w_i - w_{i'} = \tau_e$$

Incidence Matrix - Co-Domain

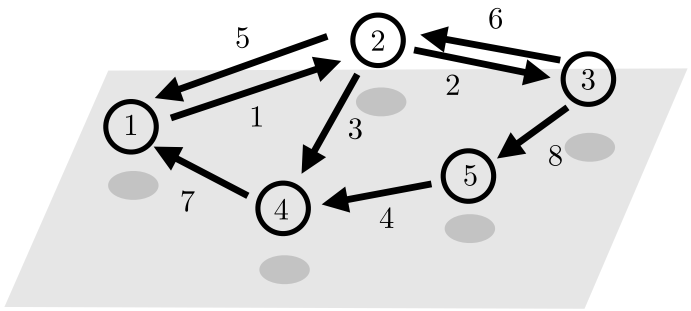
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ ...tension/difference on edges

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes

$w \in \mathbb{R}^{|\mathcal{V}|}$ ...value function on nodes

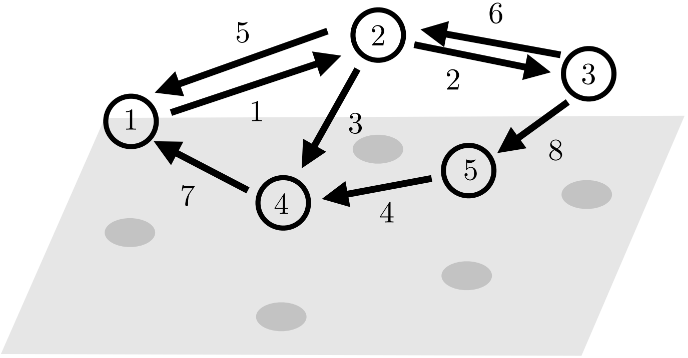
$$w^T D = \tau^T$$

Value function $\quad$ **Edge tension**

$$(w^T + \mathbf{1}^T)D = \tau^T$$

Constant shift (doesn't change tension)

Incidence Matrix - Co-Domain



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ **...tension/difference on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

$$w^T D = \tau^T$$

Value function **Edge tension**

$$(w^T + \mathbf{1}^T)D = \tau^T$$

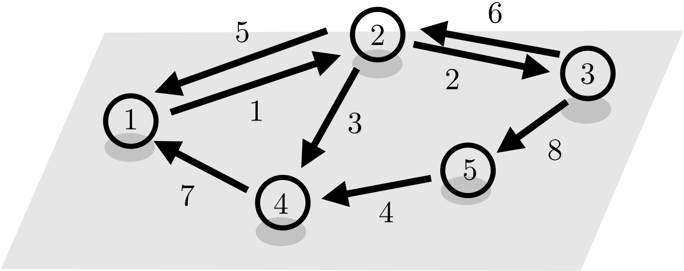
Constant shift (doesn't change tension)

Incidence Matrix - Co-Domain

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ **...tension/difference on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

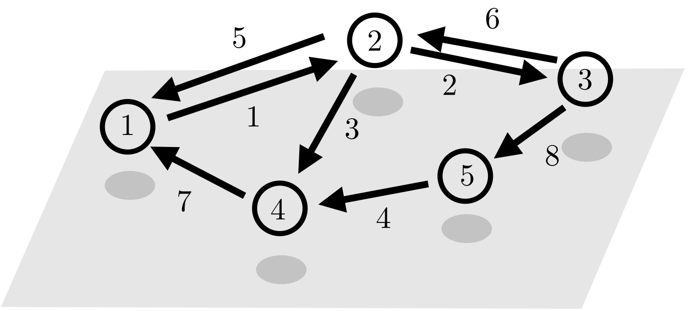
$$w^T D = \tau^T$$

Value function **Edge tension**

$$(w^T + \mathbf{1}^T)D = \tau^T$$

Constant shift (doesn't change tension)

Incidence Matrix - Co-Domain



Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ **...tension/difference on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

$w \in \mathbb{R}^{|\mathcal{V}|}$ **...value function on nodes**

$$w^T D = \tau^T$$

Value function **Edge tension**

$$(w^T + \mathbf{1}^T)D = \tau^T$$

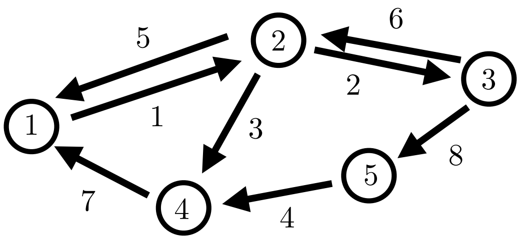
Constant shift (doesn't change tension)

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Fundamental Thm of Linear Algebra

$$A \in \mathbb{R}^{m \times n}$$

Co-Domain

Range A

dim = k

$\oplus^{\perp}$

Nullspace A^T

dim = m-k

"Span of the columns"

"Orthogonal to columns"

rank A = k

$$y \quad \begin{bmatrix} A \end{bmatrix} \quad x$$

Rank-nullity

$$\mathrm{rank}(A) + \mathrm{null}(A) = n$$

"Span of the rows"

"Orthogonal to rows"

Domain

Range A^T

dim = k

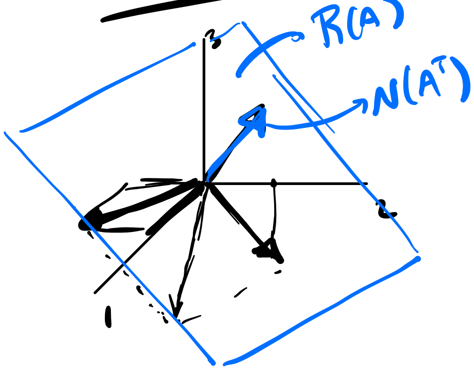
$\oplus^{\perp}$

Nullspace A

dim = n-k

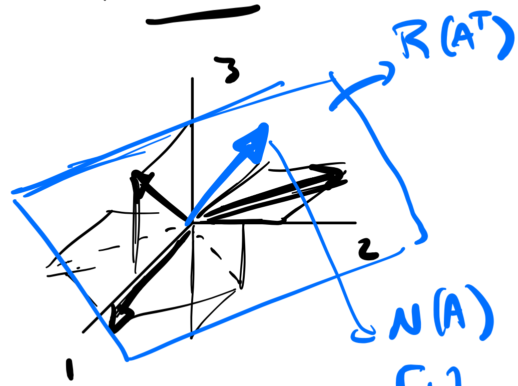
$A \in \mathbb{R}^{3\times 3}$ $\quad$ $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} =$

CODOMAIN



A

DOMAIN



$\text{span} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$y^T A = 0$

$y = [1\ 1\ 1]$ $\quad$ $y^T [A_1\, A_2\, A_3] = 0$

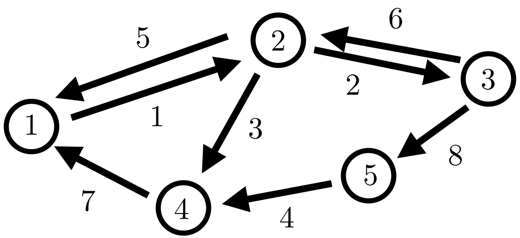
$[y^T A_1\ \ y^T A_2\ \ y^T A_3] = [0\ 0\ 0]$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Structure Overview

Co-Domain

Range D

dim = rk D

$\oplus^{\perp}$

Nullspace D^T

$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ $\boxed{\text{Spanning Tree (Forest)}}$

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ $\boxed{\text{Constant vectors}}$

$$\begin{bmatrix} \quad D \quad \end{bmatrix}$$

?

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Domain

Range D^T

dim = rk D

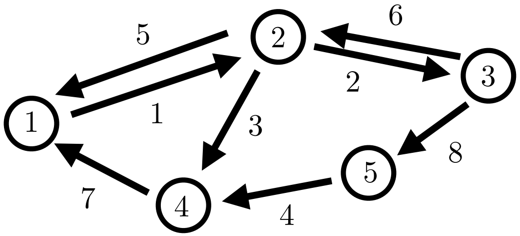
$\oplus^{\perp}$

Nullspace D

$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Cycles $\boxed{\text{Basis } \begin{bmatrix} | \\ C \\ | \end{bmatrix}}$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ **rank** D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Structure Overview

Co-Domain

Range D
dim = rk D

$\oplus^{\perp}$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

$$\begin{bmatrix} & D & \end{bmatrix}$$

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = rk D

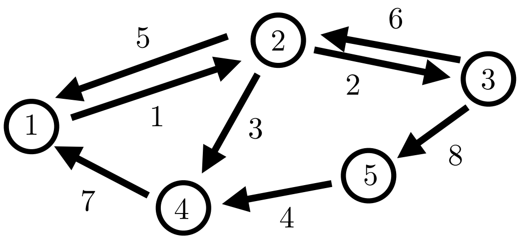
$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Structure Overview

Co-Domain

$\boxed{\text{Range } D \quad \text{dim = rk } D}$

$\oplus^{\perp}$

$\boxed{\text{Nullspace } D^T \quad \textbf{dim} = |\mathcal{V}| - \textbf{rk}D}$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

$$\begin{bmatrix} & D & \end{bmatrix}$$

$$D = \begin{bmatrix} | & | \\ T & \mathbf{1} \\ | & | \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & M \\ - & C^T & - \end{bmatrix}$$

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

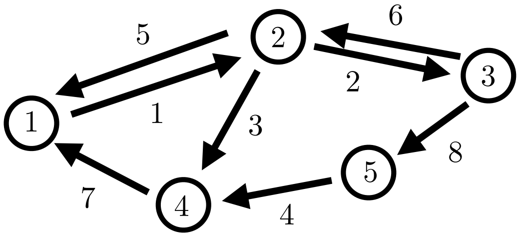
$\boxed{\text{Range } D^T \quad \text{dim = rk } D}$

$\oplus^{\perp}$

Basis Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

$\boxed{\text{Nullspace } D \quad \textbf{dim} = |\mathcal{E}| - \textbf{rk}D}$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Structure Overview

Co-Domain

Range D

dim = rk D

$\oplus^{\perp}$

Nullspace D^T

$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk} D$

Basis

$\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis

$\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

$$\begin{bmatrix} & D & \end{bmatrix}$$

$$D = \begin{bmatrix} | & | \\ T & \mathbf{1} \\ | & | \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & M \\ M^T & I \end{bmatrix}$$

Domain

Basis

$\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T

dim = rk D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D

$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk} D$

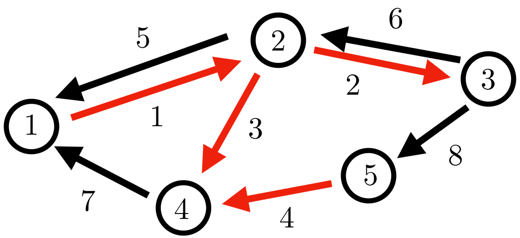
Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

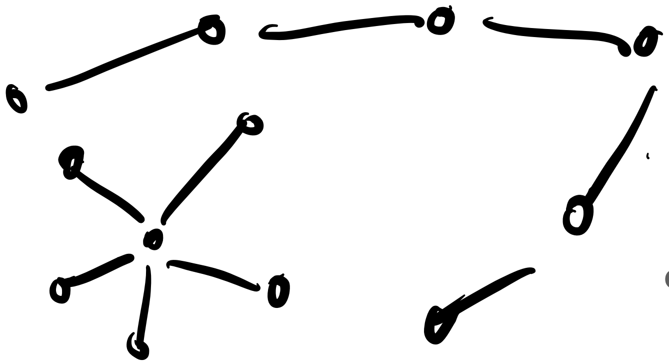
Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$



$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Spanning Tree Construction



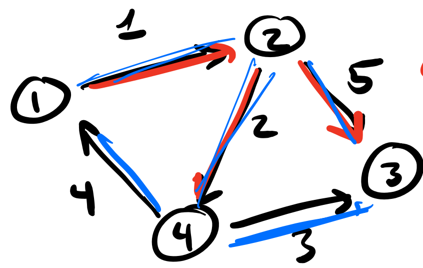
Co-Domain

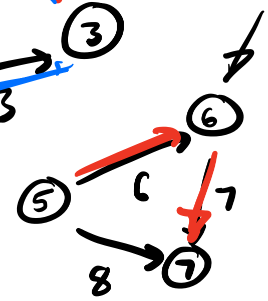
Co-Domain	Basis	
Range D dim = D	$\begin{bmatrix} \mid \\ T \\ \mid \end{bmatrix}$	Spanning Tree (Forest)
$\oplus^{\perp}$		
Nullspace D^T $\dim = \lvert\mathcal{V}\rvert - \mathbf{rk}D$	$\begin{bmatrix} \mid \\ \mathbb{1} \\ \mid \end{bmatrix}$	Constant vectors

Domain

	Basis	Domain
	$\begin{bmatrix} I \\ M^T \end{bmatrix}$	Range D^T dim = D
		$\oplus^{\perp}$
Cycles	$\begin{bmatrix} M \\ -I \end{bmatrix}$	Nullspace D $\dim = \lvert\mathcal{E}\rvert - \mathbf{rk}D$

$$
\begin{array}{cc}
 & 8 \\
7 & D=\begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$





$$D = [T]\,[I \;\; M] = [T \;\; TM]$$

n_c = num of connected components

$n_c = 2$

$$
\begin{array}{cc}
 & \begin{array}{c} 5 \\ |V| - n_c \end{array} \\
7 \;\; |V| & \begin{bmatrix}
-1 & 0 & 0 & 0 & 0\\
1 & -1 & -1 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
\quad
\begin{array}{cc}
 & 8 \\
5 & \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

edge1 edge2 edge5

edges of spanning tree (forest)

Assumption: first edges are spanning tree

$$D = T\,[I \;\; M]\,P$$

$$L = DD^T = T[I \;\; M]\begin{bmatrix} I \\ M^T \end{bmatrix} T^T = T(I + MM^T)T^T$$

$$L = DWD^T$$

$$(I + MM^T)$$

$W =$ [diagonal]

$W \succ 0$

$$L = TWT^T \qquad W = I + MM^T$$

PSD

CODOMAIN

Range(D)

basis : T

1st connctd

2nd connctd

$$\begin{bmatrix} D \end{bmatrix}$$

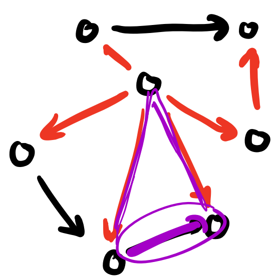
$$\downarrow$$

$$T\begin{bmatrix} I & M \end{bmatrix}$$

Nullspace D^T

basis $\bar{\mathbb{1}} = \begin{bmatrix} \mathbb{1} & & \\ & \mathbb{1} & \\ & & \ddots \ \mathbb{1} \end{bmatrix}$

$\mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$



$$\bar{\mathbb{1}}^T D = 0$$

$$\begin{bmatrix} \mathbb{1}^T & \cdots & \mathbb{1}^T \end{bmatrix} \begin{bmatrix} D & 0 & 0 \\ & D & \\ & & D \end{bmatrix}$$

$$\downarrow$$

$$D = \begin{bmatrix} T & | & TM \end{bmatrix}$$

$$\downarrow$$

$$TM_1$$

DOMAIN

Range (D^T)

basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

"cuts of graph"?

nullspace (D)

basis: $\begin{bmatrix} M \\ -I \end{bmatrix}$

cycles of graph

"edge flows"

$$M = \begin{bmatrix} M_1 & \cdots & M_k \end{bmatrix}$$

Incidence Matrix

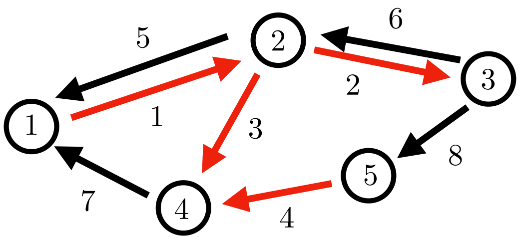
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Spanning Tree Construction

Co-Domain

$\boxed{\text{Range } D \quad \text{dim} = D}$

$\oplus^{\perp}$

$\boxed{\text{Nullspace } D^T \quad \mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D}$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Spanning Tree (Forest)

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$ $\boxed{\text{Range } D^T \quad \text{dim} = D}$

$\oplus^{\perp}$

Basis

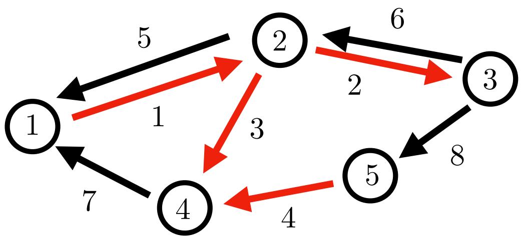
Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$ $\boxed{\text{Nullspace } D \quad \mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D}$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Spanning Tree Construction

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathrm{dim} = |\mathcal{V}| - \mathrm{rk} D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

$$D = \begin{bmatrix} T & TM \end{bmatrix}$$

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Spanning Tree (Forest)

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
$\mathrm{dim} = |\mathcal{E}| - \mathrm{rk} D$

Incidence Matrix

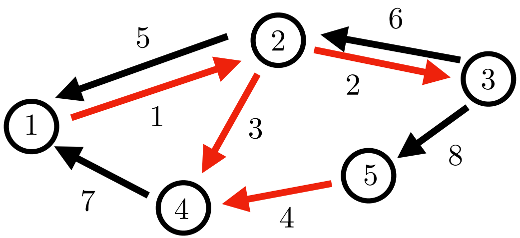
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Spanning Tree (Forest)

Co-Domain

Range D $\quad$ dim = D

$\oplus^{\perp}$

Nullspace D^T $\quad$ $\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T $\quad$ dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D $\quad$ $\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

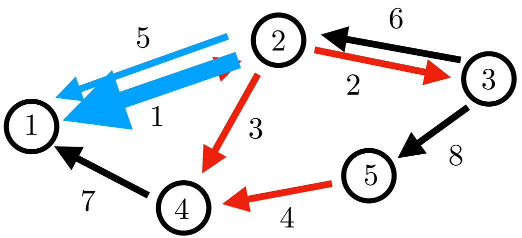
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Co-Domain

Range D
dim = D

$\oplus \perp$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus \perp$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

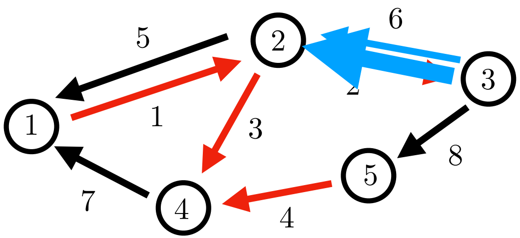
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Co-Domain

Range D $\quad$ dim = D

$\oplus \perp$

Nullspace D^T $\quad$ $\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T $\quad$ dim = D

$\oplus \perp$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

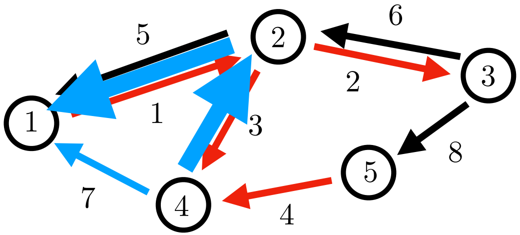
Nullspace D $\quad$ $\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Co-Domain

Range D dim = D

$\oplus \perp$

Nullspace D^T $\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T dim = D

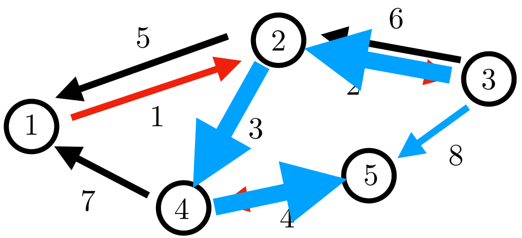
$\oplus \perp$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D $\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Co-Domain

Range D
dim = D

$\oplus \perp$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

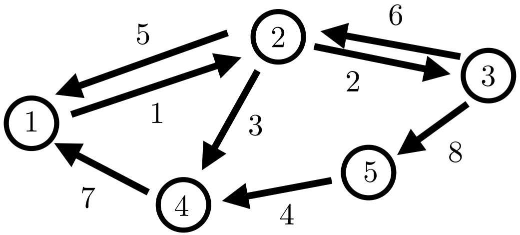
$\oplus \perp$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{vertices}$$

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

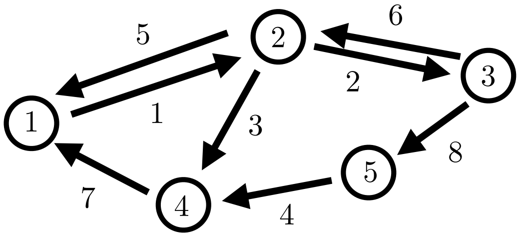
$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

$\boxed{\text{Nullspace } D \quad \mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D}$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \updownarrow \text{ vertices}$$

$\longleftarrow$ edges $\longrightarrow$

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} M \\ -I \end{bmatrix}$$

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ **Constant vectors**

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

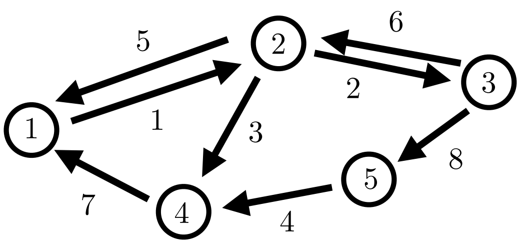
Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ **rank** D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} M \\ -I \end{bmatrix}$$

$$DC = T \begin{bmatrix} I & M \end{bmatrix} \begin{bmatrix} M \\ -I \end{bmatrix} = T(M - M) = 0$$

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ **Constant vectors**

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

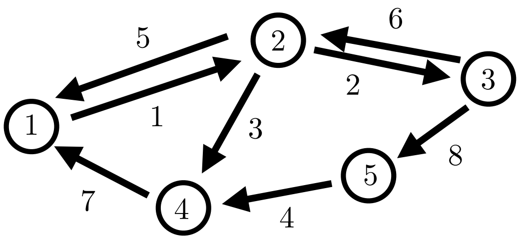
$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x is cycle flow $\quad x = Cz$

Cycle indicator matrix

$$[\, C \,]_{ec} = \begin{cases} 1 & ;\ \text{if } e \text{ flows with cycle } c \\ -1 & ;\ \text{if } e \text{ flows against cycle } c \\ 0 & ;\ e \text{ not in cycle } c \end{cases}$$

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

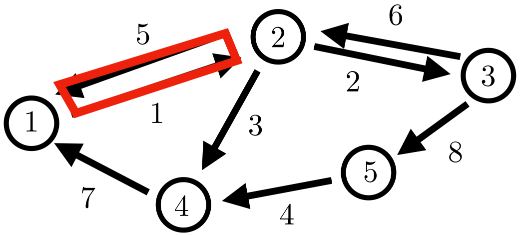
Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ **rank** D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Co-Domain

	Basis	
Range D dim = D	$\begin{bmatrix} \vert \\ T \\ \vert \end{bmatrix}$	**Spanning Tree (Forest)**
$\oplus^{\perp}$		
Nullspace D^T $\mathbf{dim} = \vert\mathcal{V}\vert - \mathbf{rk}D$	$\begin{bmatrix} \vert \\ \mathbf{1} \\ \vert \end{bmatrix}$	**Constant vectors**

Cycle indicator matrix

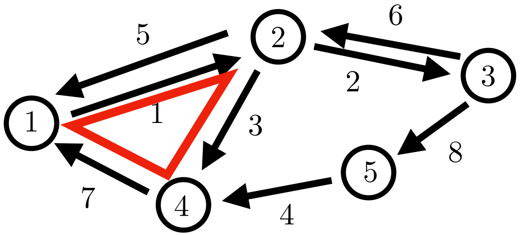
$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\longleftarrow$ Sign indicates if cycle goes with or against edge direction

Domain

	Basis	
	$\begin{bmatrix} I \\ M^T \end{bmatrix}$	Range D^T dim = D
		$\oplus^{\perp}$
Cycles	$\begin{bmatrix} \vert \\ C \\ \vert \end{bmatrix}$	Nullspace D $\mathbf{dim} = \vert\mathcal{E}\vert - \mathbf{rk}D$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{vertices}$$

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\leftarrow$ Sign indicates if cycle goes with or against edge direction

Co-Domain

Range D, dim = D	**Basis** $\begin{bmatrix} \vert \\ T \\ \vert \end{bmatrix}$	**Spanning Tree (Forest)**
$\oplus^{\perp}$		
Nullspace D^T, $\mathbf{dim} = \vert\mathcal{V}\vert - \mathbf{rk}D$	**Basis** $\begin{bmatrix} \vert \\ \mathbf{1} \\ \vert \end{bmatrix}$	**Constant vectors**

Domain

	Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$	Range D^T, dim = D
		$\oplus^{\perp}$
Cycles	**Basis** $\begin{bmatrix} \vert \\ C \\ \vert \end{bmatrix}$	Nullspace D, $\mathbf{dim} = \vert\mathcal{E}\vert - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

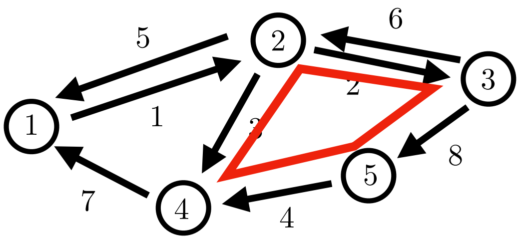
Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges



Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

Sign indicates if cycle goes with or against edge direction

Co-Domain

Range D dim = D

$\oplus^{\perp}$

Nullspace D^T $\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ **Constant vectors**

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T dim = D

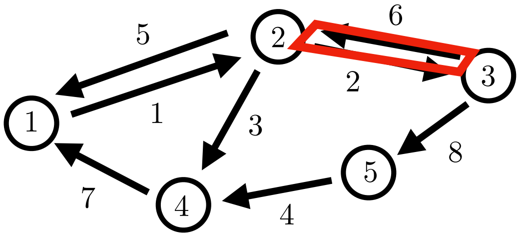
$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Nullspace D $\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\leftarrow$ Sign indicates if cycle goes with or against edge direction

Co-Domain

Range D dim = D

$\oplus^{\perp}$

Nullspace D^T $\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$ Range D^T dim = D

$\oplus^{\perp}$

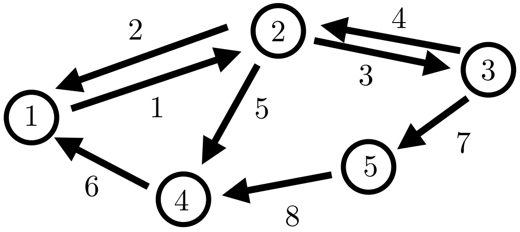
Basis Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$ Nullspace D $\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ **rank** $D = |\mathcal{V}| - n_c$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{0}$$

Left Nullspace

$$D = T[I\ M]$$

$$\mathbf{1}^T D = 0$$

Co-Domain

Range D, dim = D

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

$\oplus^\perp$

Nullspace D^T, $\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T, dim = D

$\oplus^\perp$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

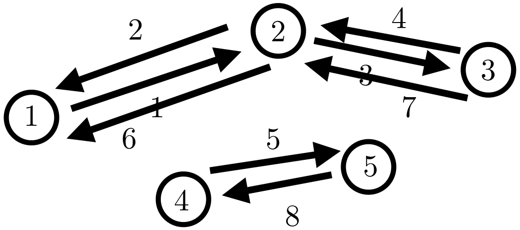
Nullspace D, $\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} = \mathbf{0}$$

Left Nullspace

Co-Domain

Basis

Range D dim = D

$\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

$\oplus^{\perp}$

Basis

Nullspace D^T $\mathrm{dim} = |\mathcal{V}| - \mathrm{rk}D$

$\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

$$\mathbf{1}^T D = 0$$

$\mathbb{1}^T D = 0$

Domain

Basis

$\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T dim = D

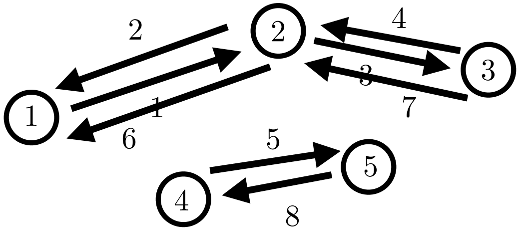
$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D $\mathrm{dim} = |\mathcal{E}| - \mathrm{rk}D$

Incidence Matrix



Graph: Vertices $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ Edges $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ rank D = $|\mathcal{V}| - n_c$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} = \mathbf{0}$$

Left Nullspace

$\mathbb{1}^T = [1 \cdots 1]$

$$\underbrace{\begin{bmatrix} \mathbf{1}^T & 0 & \cdots & 0 \\ 0 & \mathbf{1}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}^T \end{bmatrix}}_{\bar{\mathbf{1}}^T} \begin{bmatrix} D \end{bmatrix} = \mathbf{0}$$

dim = num connected components

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\dim = |\mathcal{V}| - \mathrm{rk} D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \bar{\mathbf{1}} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

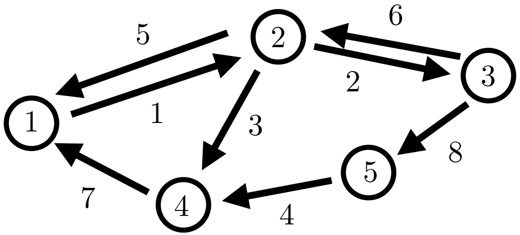
$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
$\dim = |\mathcal{E}| - \mathrm{rk} D$

Graph Laplacians



Review: Shape Matrices

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Graph Laplacians

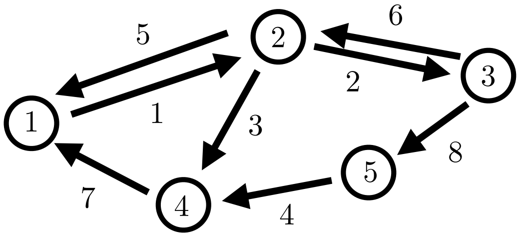
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Review: Shape Matrices

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$

cols…

$$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix}$$

Inner products of columns

"Relative geometry of columns"

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

rows…

$$A A^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix}$$

Inner products of rows

"Relative geometry of rows"

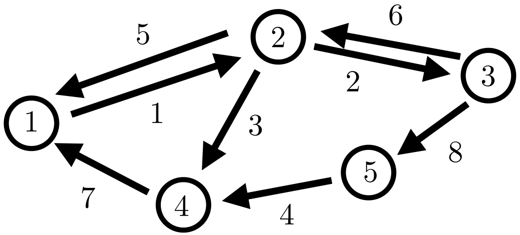
Graph Laplacians

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$

cols...

$$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix}$$

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

rows...

$$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix}$$

RA rotate columns of A…

….relative geometry stays the same.

$$(RA)^T (RA) = A^T R^T R A = A^T A$$

AR rotate rows of A…

….relative geometry stays the same.

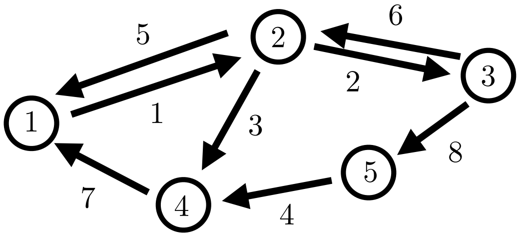
$$(AR)(AR)^T = A R R^T A^T = A A^T$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

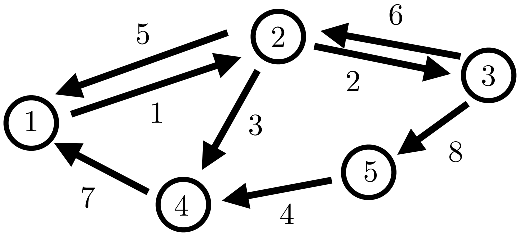
$A^T A$ "Shape" of the columns of A

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Graph Laplacians



Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

$A^T A$ ~~“Shape” of the columns of A~~

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

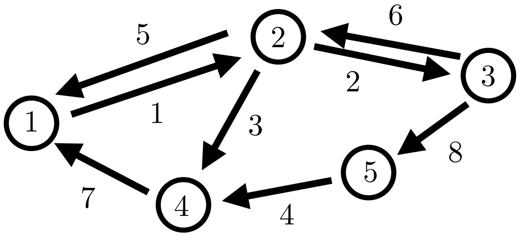
$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of the columns of A

More Accurate

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

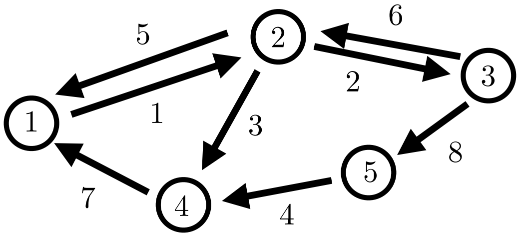
$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

$(A^TA)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

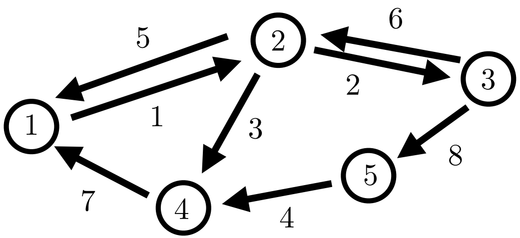
$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

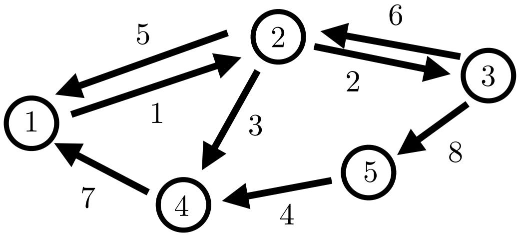
Analogy: $z \in \mathbb{C}$ $\quad |z| = \sqrt{z^* z}$ $\quad z = |z| e^{i\phi}$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

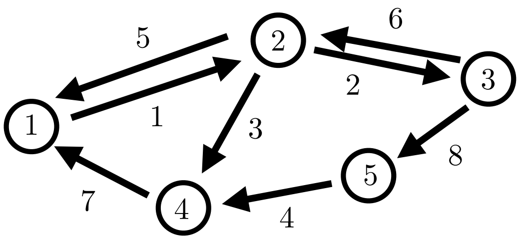
Analogy: $z \in \mathbb{C}$ $\quad |z| = \sqrt{z^* z}$ $\quad z = |z| e^{i\phi}$

Polar Decomposition

$$A = \underbrace{A(A^T A)^{-1/2}}_{\text{Rotation}} \cdot \underbrace{(A^T A)^{1/2}}_{\text{PSD "shape"}}$$

"Column version"

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

Analogy: $z \in \mathbb{C} \quad |z| = \sqrt{z^* z} \quad z = |z| e^{i\phi}$

Polar Decomposition

$$A = A(A^T A)^{-1/2} \cdot (A^T A)^{1/2}$$

Rotation PSD "shape"

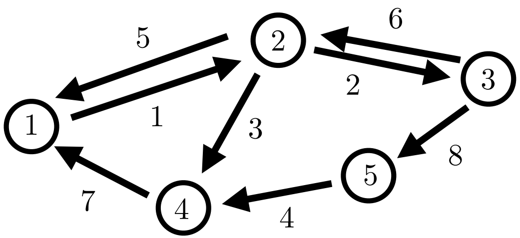
"Column version"

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD "shape" Rotation

"Row version"

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

Analogy: $z \in \mathbb{C} \quad |z| = \sqrt{z^* z} \quad z = |z| e^{i\phi}$

Polar Decomposition

$$A = A(A^T A)^{-1/2} \cdot (A^T A)^{1/2}$$

Rotation $\quad$ PSD "shape" $\quad$ **"Column version"**

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD "shape" $\quad$ Rotation $\quad$ **"Row version"**

Checking rotation…

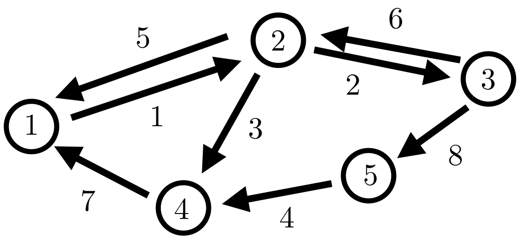
$$(A^T A)^{-1/2} A^T A (A^T A)^{-1/2} = I$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \quad \textbf{cols...}$$

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \quad \textbf{rows...}$$

Review: Sym/PSD Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes $\quad (A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \quad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$

Nullspace $\quad$ Null space A = Null space $A^T A$ $\quad$ Null space A^T = Null space AA^T

Rank $\quad \mathrm{rank}(A) = \mathrm{rank}(A^T) = \mathrm{rank}(A^T A) = \mathrm{rank}(AA^T)$

Symmetric matrix $\quad S \in \mathbb{R}^{n \times n}$ has orthonormal eigenvectors

Positive semi-definite $\quad x^T S x \geq 0 \quad \forall x \iff \lambda_i \geq 0 \quad \lambda_i \in \mathrm{eig}(S)$

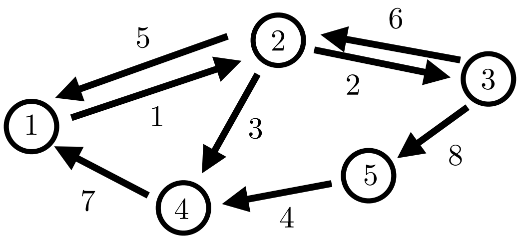
$S \succeq 0 \quad A^T A,\ AA^T,\ (A^T A)^{1/2},\ (AA^T)^{1/2}$ all PSD

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\qquad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Polar Decomposition

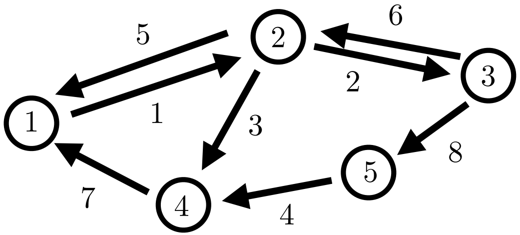
$$A = \underbrace{A(A^T A)^{-1/2}}_{\text{Rotation}} \cdot \underbrace{(A^T A)^{1/2}}_{\text{PSD "shape"}}$$

"Column version"

$$A = \underbrace{(AA^T)^{1/2}}_{\text{PSD "shape"}} \cdot \underbrace{(AA^T)^{-1/2} A}_{\text{Rotation}}$$

"Row version"

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

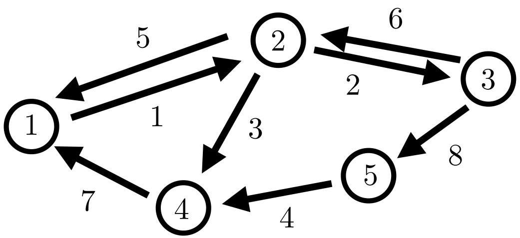
$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Polar Decomposition

$$A = \underbrace{UV^T}_{\text{Rotation}} \cdot \underbrace{V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T}_{\text{PSD "shape"}}$$ **"Column version"**

$$A = \underbrace{U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T}_{\text{PSD "shape"}} \cdot \underbrace{UV^T}_{\text{Rotation}}$$ **"Row version"**

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Singular Value Decomposition

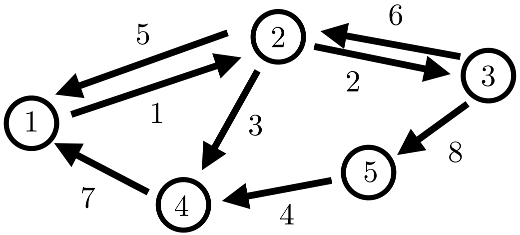
$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \quad \textbf{cols...}$$

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \quad \textbf{rows...}$$

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\qquad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

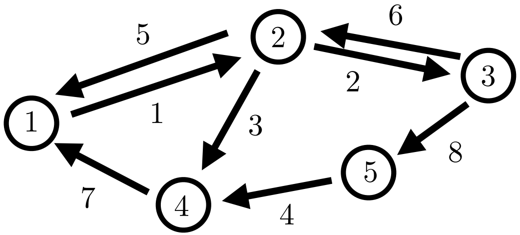
$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Singular Value Decomposition

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$= \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix}$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Review: Shape Matrices

$(A^TA)^{1/2}$ "Shape" of columns $\qquad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

$$(A^TA)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Singular Value Decomposition

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$= \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & {V'}^T & - \\ - & {V''}^T & - \end{bmatrix}$$

$$U' = AV'\Sigma^{-1} \qquad {V'}^T = \Sigma^{-1}{U'}^T A$$

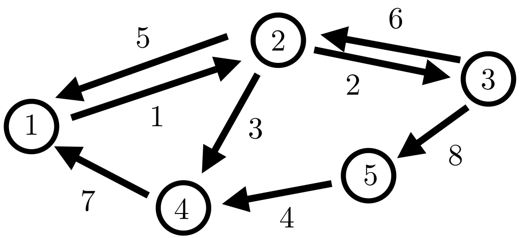
for singular vectors w/ non-zero values

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Laplacian $L = DD^T$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

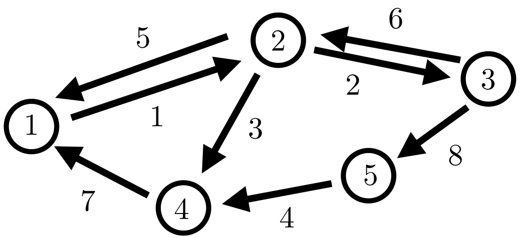
$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



$$\dot{u} = -Lu$$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Laplacian $L = DD^T$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Action:

$$Lu = \underbrace{\begin{bmatrix} D \end{bmatrix} \underbrace{\begin{bmatrix} D^T \end{bmatrix} \begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{…tension created in edges}}}_{\text{… summed resulting tension on nodes}}$$

"heights" of nodes

$$L = \Delta - A$$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$3u_1 - u_2 - u_5 - u_7$

$-(u_1 - u_2) - (u_1 - u_3) - (u_1 - u_7) =$

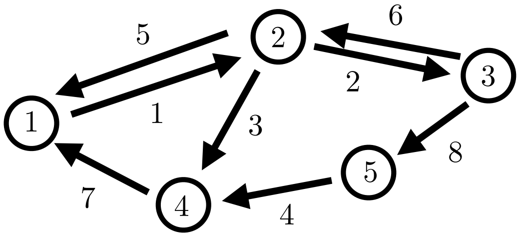
$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ & \vdots & & \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $L = DD^T$

Action: $$Lu = \underbrace{[\ D\ ]\underbrace{[\ D^T\ ]\begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{...tension created in edges}}}_{\text{... summed resulting tension on nodes}}$$

"heights" of nodes

Linear ODE $\dot{u} = -Lu$

Eigenvectors are oscillation modes

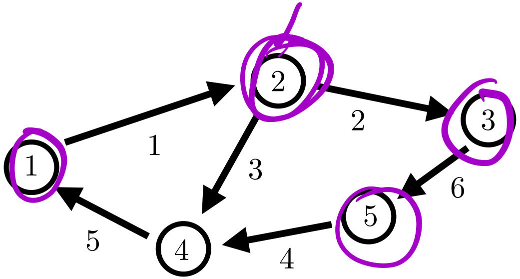
"Vibration modes" of a graph

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Laplacian $L = DD^T$

$$L = DD^T = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

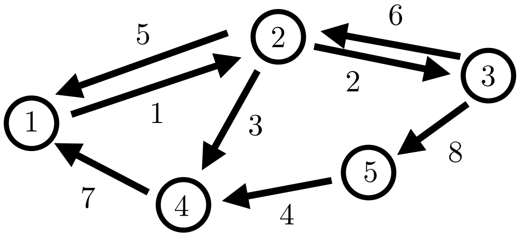
Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T$

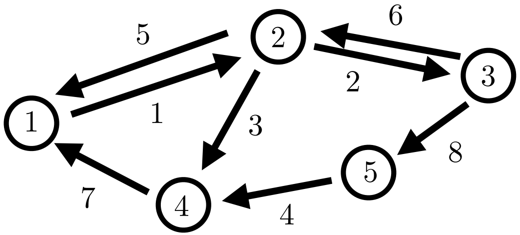
$$L = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} | & | \\ U' & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & U'^T & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix}$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Laplacian $L = DD^T$

$$L = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} | & | \\ \bar{\mathbf{1}} & U' \\ | & | \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma^2 \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U'^T & - \end{bmatrix}$$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

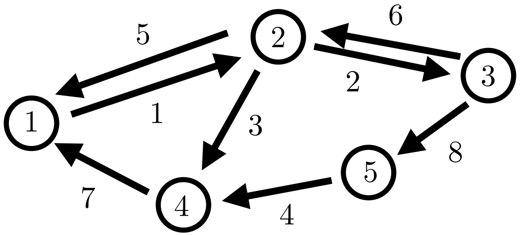
$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

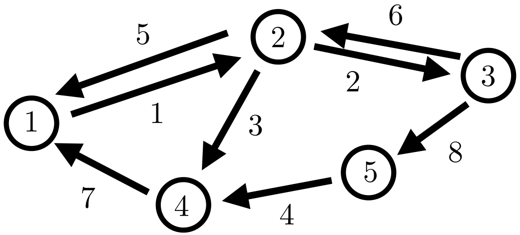
Laplacian $\quad L = DD^T$

$$L = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} | & | \\ \bar{\mathbf{1}} & U' \\ | & | \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma^2 \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U'^T & - \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \bar{\mathbf{1}} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & - & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

Eigenvalues $\quad \underbrace{0 = \cdots = 0}_{\text{num of connected components}} < \lambda_1 \leq \cdots \leq \lambda_n$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T$

$$L = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} | & | \\ \bar{\mathbf{1}} & U' \\ | & | \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma^2 \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U'^T & - \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \bar{\mathbf{1}} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & - & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

Eigenvectors

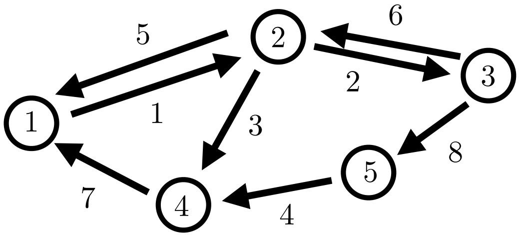
Constant vectors (zero eigenvalues) $\longleftarrow$ $\begin{bmatrix} \boxed{\begin{matrix} | \\ \bar{\mathbf{1}} \\ | \end{matrix}} & \begin{matrix} | \\ U_1 \\ | \end{matrix} & \cdots & \begin{matrix} | \\ U_k \\ | \end{matrix} \end{bmatrix}$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$DD^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0$$

Laplacian $L = DD^T$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

$$L = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} | & | \\ \mathbf{1} & U' \\ | & | \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma^2 \end{bmatrix} \begin{bmatrix} - & \mathbf{1}^T & - \\ - & U'^T & - \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \mathbf{1} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & & \ddots & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \mathbf{1}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Eigenvectors

$$\begin{bmatrix} | & \boxed{\begin{matrix} | & & | \\ U_1 & \cdots & U_k \\ | & & | \end{matrix}} \\ \mathbf{1} & \\ | & \end{bmatrix}$$

Constant vectors (zero eigenvalues) ←

→ Oscillation modes of graph (non-zero eigenvalues)

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

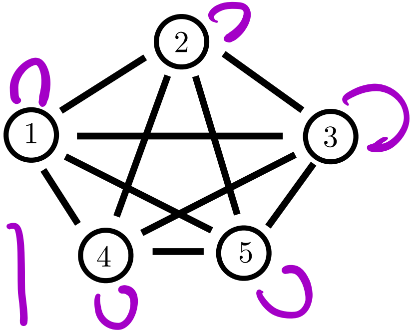
Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$



Laplacian $\quad L = DD^T \qquad$ **Complete Graph**

$$L = \begin{bmatrix} | & | & & | \\ \mathbf{1} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & - & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \mathbf{1}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

Eigenvalues $\quad 0 < |\mathcal{V}| = \cdots = |\mathcal{V}|$

Eigenvectors $\quad \begin{bmatrix} | & | & & | \\ \mathbf{1} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \rightarrow$ Any orthonormal basis vectors perpendicular to $\mathbf{1}$

Proof (sketch) $\quad L = -\mathbf{1}\mathbf{1}^T + |\mathcal{V}|$

$$\underline{L} = M + \alpha I = VSV^{-1} + \alpha VV^{-1}$$
$$= V(S + \alpha I)V^{-1} \quad \Rightarrow \quad \alpha + \lambda \in \text{eig}(L)$$

$\lambda \in \text{eig}(M)$

M: VSV^{-1} → diagonal diagonalization

$V(S+\alpha I)V^{-1}$: diagonalization

$S + \alpha$

$$M = -\mathbb{1}\mathbb{1}^T \qquad \alpha = |v| \text{ or } |v| - 1$$

$$\underline{M} = VSV^{-1} \quad S$$

$$\begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

right evecs — evals — rows are left evecs

$n, 0, \cdots, 0$

$\mathbb{1}$

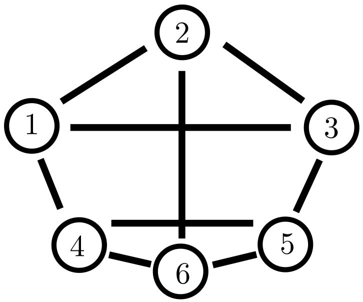
$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad \alpha$$

$$L = M + \alpha I = VSV^{-1} + \alpha VV^{-1} = V(S + \alpha I)V^{-1}$$

VV^{-1}

spectral mapping theorem

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T$ $\qquad$ **d-Regular Graph**

(all nodes have same degree)

$$L = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{1}} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & - & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

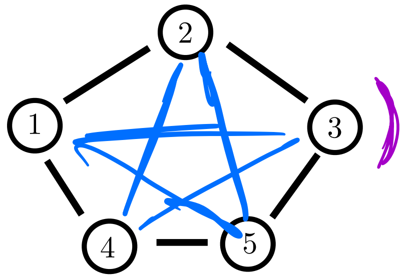
Eigenvalues (same as - adjacency matrix + d)

Eigenvectors (same as adjacency matrix)

see following slides

Proof (sketch) $\quad L = \Delta - A = dI - A$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Lu

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T$

Cycle Graph
(or any circulant graph)

$$L = \begin{bmatrix} | & | & & | \\ \mathbf{1} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & - & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \mathbf{1}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

Eigenvalues (related to DFT)

Eigenvectors discrete Fourier basis vectors

Proof (sketch) Related to theory of circulant/shift matrices

Ask Dan (other materials)

Note: Eigenvectors of L called Graph "Fourier" Transform …. extension of DFT

$c = [c_0 \cdots c_{n-1}]$

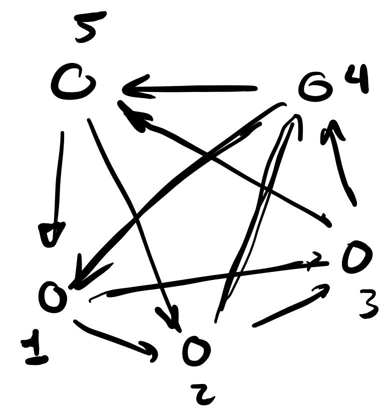
$$C = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & & c_1 \\ & c_0 & c_{n-1} & & c_2 \\ \vdots & & c_0 & & \\ \vdots & & & & \\ c_{n-1} & c_{n-2} & c_{n-3} & & c_0 \end{bmatrix}$$ circulant matrix

(Toeplitz matrix)

$c * x = Cx \rightarrow$ convolution of c & x.

for cycle graph:

$$L = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{bmatrix}$$ is a circulant matrix.



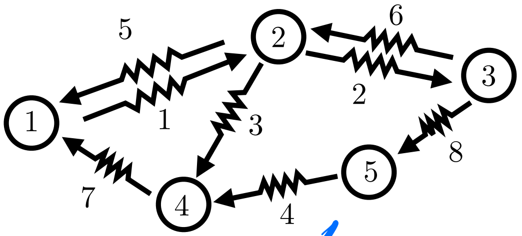
$C \rightarrow$ eigenvectors are DFT basis vectors $\quad F$ — DFT vectors (columns)

$$C = \underbrace{F}_{\text{right evecs}} \underbrace{(dg(F^* c))}_{\text{diagonal is DFT of vector } c} \underbrace{F^*}_{\text{left evecs}}$$

Graph Laplacians

Graph: Vertices $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Weighted Laplacian $\quad L_W = DWD^T$

Edge weights $\quad W_e \geq 0 \qquad W = \mathrm{diag}\big([W_1 \ \cdots \ W_{|\mathcal{E}|}]\big)$

Interpretation: **resistance, travel time/cost**

$$L_W = DWD^T = U_W \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} U_W^T$$

$$= \begin{bmatrix} | & | \\ U'_W & \mathbf{1} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & {U'}_W^T & - \\ - & \mathbf{1}^T & - \end{bmatrix}^T$$

$$
L_w = \begin{array}{c c} & \begin{array}{c c c} & v' & \end{array} \\ v & \begin{bmatrix} \sum_{j \in N_1} w_{1j} & & -w_{vv'} \\ & \ddots & \\ & & \sum_{j \in N_{|V|}} w_{|V|j} \end{bmatrix} \end{array}
$$

$$
= \underbrace{\begin{bmatrix} \sum_{j \in N_1} w_{1j} & & 0 \\ & \ddots & \\ 0 & & \sum_{j \in N_{|V|}} w_{|V|j} \end{bmatrix}}_{\substack{\Delta_w \\ \text{weighted degree} \\ \text{matrix}}} - \underbrace{\begin{array}{c c} & \begin{array}{c} v' \end{array} \\ v & \begin{bmatrix} & & \\ & w_{vv'} & \\ & & \end{bmatrix} \end{array}}_{\substack{A_w \\ \text{weighted adjacency} \\ \text{matrix}}}
$$

For directed graphs:

in-degree Laplacian

$$L_{in} = \Delta_{in} - A$$

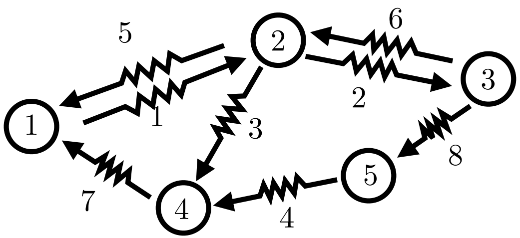
$\rightarrow [\Delta_{in}]_{vv} =$ # of edges coming in to node v

out degree Laplacian

$$L_{out} = \Delta_{out} - A$$

$\rightarrow [\Delta_{out}]_{vv} =$ # of edges going out of node

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

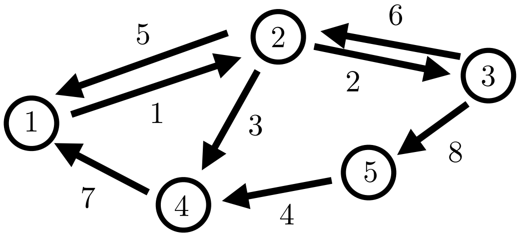
Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Weighted Laplacian $\quad L_W = DWD^T$

Action: $\quad L_W u = \underbrace{\Big[\ D\ \Big] \underbrace{\Big[\ W\ \Big] \Big[\ D^T\ \Big] \begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{...tension created in edges scaled by weights}}}_{\text{... summed resulting tension on nodes}}$ “heights” of nodes

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Edge Laplacian $L_e = D^T D$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

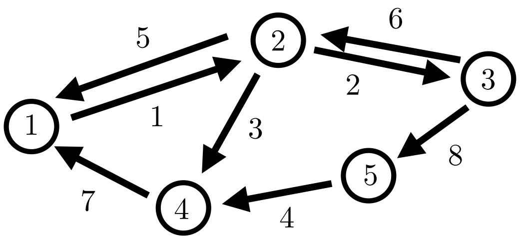
$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Edge Laplacian $\quad L_e = D^T D$

Action:

$$L_e \tau = \underbrace{\left[\; D^T \; \right] \underbrace{\left[\; D \; \right] \begin{bmatrix} | \\ \tau \\ | \end{bmatrix}}_{\text{...summed tension on nodes}}}_{\text{... differential in tension along edges}}$$

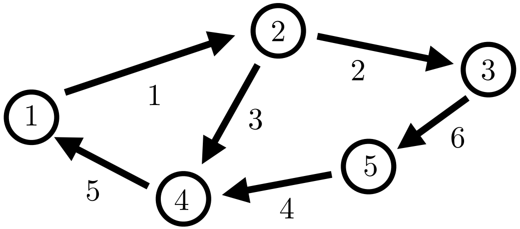
"Tension" in edges

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Laplacian $L = DD^T$

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

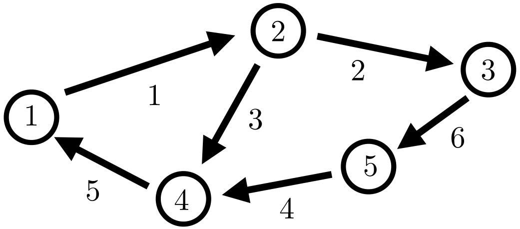
Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Degree & Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = \underline{DD^T}$

Independent of edge direction

nodes

$$L = \boxed{\Delta} - \boxed{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

nodes

Degree Matrix $\quad [\ \Delta\ ]_{vv} = |\mathcal{N}_v|$ **diagonal**

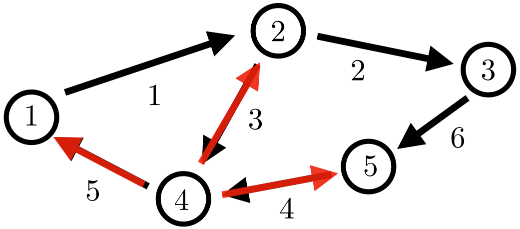
Adjacency Matrix $\quad [\ A\ ]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$

Adjacency Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\,\Delta\,]_{vv} = |\mathcal{N}_v| \quad$ **diagonal**

Adjacency Matrix $\quad [\,A\,]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$

Adjacency Matrix

Start @ node 4

Edges to Nodes 1,2, & 5

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

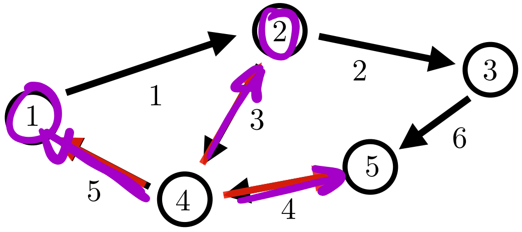
A

Adjacency Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\,\Delta\,]_{vv} = |\mathcal{N}_v| \quad$ **diagonal**

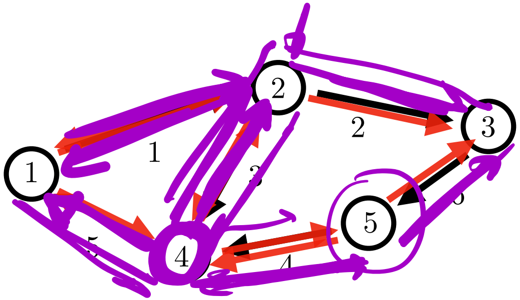
Adjacency Matrix $\quad [\,A\,]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$

Powers of Adjacency

Start @ node 4

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\,\Delta\,]_{vv} = |\mathcal{N}_v|$ **diagonal**

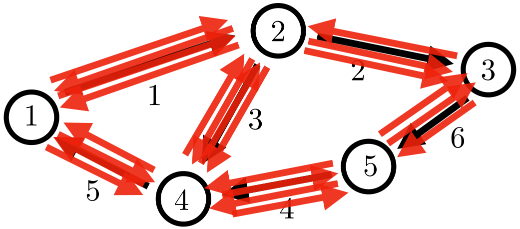
Adjacency Matrix $\quad [\,A\,]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \quad \begin{array}{ccccc} \text{Thru Node 1} & \text{Thru Node 2} & & & \text{Thru Node 5} \\ \end{array} \qquad \begin{array}{ccccc} & & & \text{Start @ node 4} & \\ \end{array}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\,\Delta\,]_{vv} = |\mathcal{N}_v| \quad$ **diagonal**

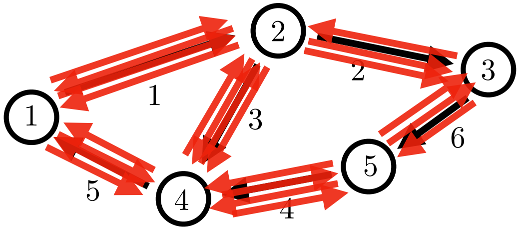
Adjacency Matrix $\quad [\,A\,]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

Thru Node 1 $\quad$ Thru Node 2 $\quad$ Thru Node 5 $\quad$ Start @ node 4

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\,\Delta\,]_{vv} = |\mathcal{N}_v|$ **diagonal**

Adjacency Matrix $\quad [\,A\,]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

Thru Node 1 $\quad$ Thru Node 2 $\quad$ Thru Node 5 $\quad$ Start @ node 4

\# 3-step paths from node 4 to node 1

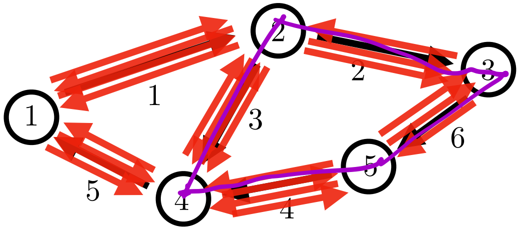
\# 3-step paths from node 4 to node 2

⋮

\# 3-step paths from node 4 to node 5

$$\leftarrow \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\ \Delta\ ]_{vv} = |\mathcal{N}_v| \quad$ **diagonal**

Adjacency Matrix $\quad [\ A\ ]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

$$[\ A^k\ ]_{vv'}$$

 # k-step ~~paths~~ wal from node v to node v'

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \begin{array}{ccccc} \text{Thru Node 1} & \text{Thru Node 2} & & & \text{Thru Node 5} \\ \end{array} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Start @ node 4

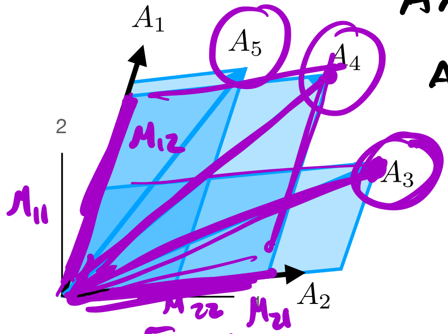
REVIEW: Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} \overline{A'} & A'' \end{bmatrix} = \begin{array}{c} \quad\quad\downarrow \quad\quad\quad\quad\quad 5 \\ 2\begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \end{bmatrix} \end{array}$$

A' Linear independent columns

A'' Linear dependent columns

$$\begin{array}{c} 3 \\ 2\begin{bmatrix} M \\ -I_3 \end{bmatrix} \end{array}$$



$Ax = 0$

$A_1x_1 + \cdots + A_5x_5 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \end{bmatrix}$$

$$A'' = A'B$$

$$A = \begin{bmatrix} A' & A'B \end{bmatrix} = A' \begin{bmatrix} I & B \end{bmatrix}$$

$$\begin{bmatrix} \begin{smallmatrix}|\\ A_3 \\ |\end{smallmatrix} & \begin{smallmatrix}|\\ A_4 \\ |\end{smallmatrix} & \begin{smallmatrix}|\\ A_5 \\ |\end{smallmatrix} \end{bmatrix} = \begin{bmatrix} A_1B_{13} + A_2B_{23} & A_1B_{14} + A_2B_{24} & A_1B_{15} + A_2B_{25} \end{bmatrix}$$

$$\begin{bmatrix} M \\ -I \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Nullspace basis:

$$N = \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$

PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$$

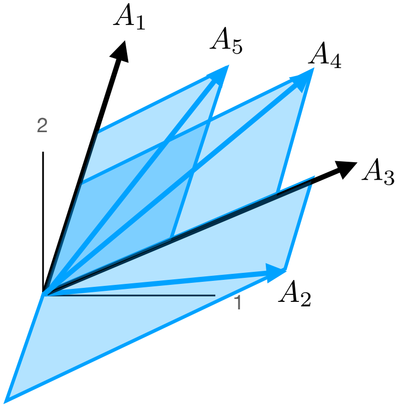
$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

REVIEW: Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns



Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{33} & B_{34} & B_{35} \end{bmatrix}$$

$$A'' = A'B$$

$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_4 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_3 B_{32} & A_1 B_{14} + A_3 B_{34} & A_1 B_{15} + A_3 B_{35} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ B_{33} & B_{34} & B_{35} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$

PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

A' lin. ind.

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \Rightarrow \quad x' + Bx'' = 0$$

$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

REVIEW: Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{12} & B_{13} & B_{15} \\ B_{42} & B_{43} & B_{45} \end{bmatrix}$$

$$A'' = A'B$$

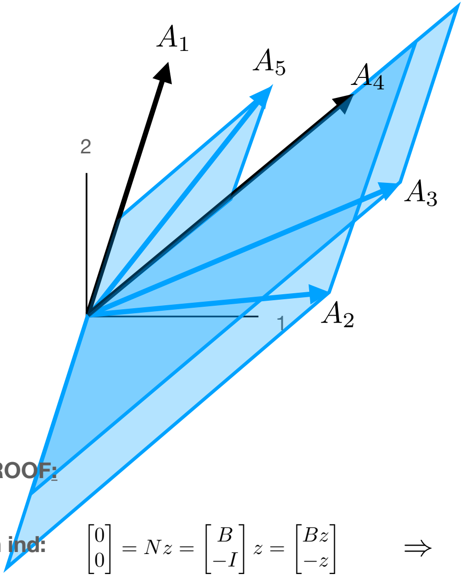
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_3 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_4 B_{42} & A_1 B_{13} + A_4 B_{43} & A_1 B_{15} + A_4 B_{45} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ B_{43} & B_{44} & B_{45} \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$



PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$

$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$