Diagonalization

Square matrix: $A \in \mathbb{R}^{n \times n}$

Assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Right Eigenvectors:

$$V = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \qquad AV = \begin{bmatrix} AV_1 \cdots AV_n \end{bmatrix} = \begin{bmatrix} V_1\lambda_1 \cdots V_n\lambda_n \end{bmatrix} = \begin{bmatrix} V_1 \cdots V_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{D} = VD$$

$$AV = VD$$

$$\Rightarrow \quad A = VDV^{-1}$$

Left Eigenvectors:

$$W = \begin{bmatrix} -W_1^*- \\ \vdots \\ -W_n^*- \end{bmatrix} \qquad WA = \begin{bmatrix} -W_1^*A- \\ \vdots \\ -W_n^*A- \end{bmatrix} = \begin{bmatrix} -\lambda_1 W_1^*- \\ \vdots \\ -\lambda_n W_n^*- \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{D} \begin{bmatrix} -W_1^*- \\ \vdots \\ -W_n^*- \end{bmatrix} = DW$$

$$WA = DW$$

$$\Rightarrow \quad A = W^{-1}DW$$

Assuming V & W are chosen with compatible orderings and lengths of columns/rows… $V^{-1} = W$

Diagonalization

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

$$V^{-1}V = \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} W_1^* V_1 & \cdots & W_1^* V_n \\ \vdots & & \vdots \\ W_n^* V_1 & \cdots & W_n^* V_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

…from off diagonal terms $W_j^* V_i = 0 \quad j \neq i$

V_i orthogonal to all other W_j

…from diagonal terms $W_i^* V_i = 1$

V_i, W_i can be scaled so that $W_i^* V_i = 1$

Diagonalization - Similarity Transform

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

A is similar to a diagonal matrix

…transform into eigenvector coordinates

$$x = Vx' \qquad y = Vy'$$

$$\begin{aligned} y &= Ax \\ Vy' &= AVx' \\ y' &= V^{-1}AVx' \\ y' &= V^{-1}VDV^{-1}Vx' \\ y' &= Dx' \end{aligned}$$

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1' \\ \vdots \\ \lambda_n x_n' \end{bmatrix}$$

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values
(on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication Ax

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}$$

transforming into eigen-vector coords

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication Ax

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}}}_{\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}}$$

transforming into eigen-vector coords

Scaling each coord by eigenvalue

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Interpretation of Matrix Multiplication Ax

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{}$$

$\begin{bmatrix} W_1^* x \\ \vdots \\ W_n^* x \end{bmatrix}$ transforming into eigen-vector coords

$\begin{bmatrix} \lambda_1 W_1^* x \\ \vdots \\ \lambda_n W_n^* x \end{bmatrix}$ Scaling each coord by eigenvalue

$V_1 \lambda_1 W_1^* x + \cdots + V_n \lambda_n W_n^* x$ Transforming back into regular coordinates

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

If x is an eigenvector…

Interpretation of Matrix Multiplication AV_i

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}$$

Orthogonal to all other left eigenvectors

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

If x is an eigenvector…

Interpretation of Matrix Multiplication AV_i

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ Orthogonal to all other left eigenvectors}}}_{\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \text{ Scaled by specific eigenvalue}}$$

Diagonalization - Matrix Multiplication

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\textbf{Eigen-values} \text{ (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} =$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

If x is an eigenvector…

Interpretation of Matrix Multiplication AV_i

$$\underbrace{\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}}_{\begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix}}}_{\lambda_i V_i}$$

Orthogonal to all other left eigenvectors

Scaled by specific eigenvalue

Select out that specific eigenvector

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

$$\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Shuffling eigenvalues and eigenvectors

Order of sum does not matter…

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

Diagonalization

$$A = \begin{bmatrix} \; V \; \end{bmatrix} \begin{bmatrix} \; D \; \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \; A \; \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \; P \; \end{bmatrix} \begin{bmatrix} \; P^T \; \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \; P \; \end{bmatrix} \begin{bmatrix} \; P^T \; \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Shuffling eigenvalues and eigenvectors

$$\begin{bmatrix} \; A \; \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \boxed{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}\begin{bmatrix} P \end{bmatrix}} \boxed{\begin{bmatrix} P^T \end{bmatrix}\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} P \end{bmatrix}} \boxed{\begin{bmatrix} P^T \end{bmatrix}\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

Shuffling eigenvalues and eigenvectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_n & \cdots & V_1 \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_n & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_1 \end{bmatrix} \begin{bmatrix} - & W_n^* & - \\ & \vdots & \\ - & W_1^* & - \end{bmatrix}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

Permutation Matrix $P \in \mathbb{R}^{n\times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $P^T P = I$

Shuffling eigenvalues and eigenvectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization (non-unique) case 1: ordering

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Permutation Matrix $P \in \mathbb{R}^{n \times n}$

Shuffle columns (or rows) of identity…

Ex. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P^T P = I$

Diagonalization

$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big]$$

$$\Big[\ A\ \Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} =$$

$$\boxed{\begin{bmatrix} | & & | \\ V_n & \cdots & V_1 \\ | & & | \end{bmatrix}} \boxed{\begin{bmatrix} \lambda_n & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_1 \end{bmatrix}} \boxed{\begin{bmatrix} - & W_n^* & - \\ & \vdots & \\ - & W_1^* & - \end{bmatrix}}$$

Shuffling eigenvalues and eigenvectors

$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \big[\lambda_i\big] \big[- \quad W_i^* \quad -\big]$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

diagonal matrices commute…

$$A = \left[\; V \; \right] \left[\; D \; \right] \left[\; V^{-1} \right]$$

$$\left[\; A \; \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\textbf{Eigen-values} \text{ (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 \frac{\gamma_1}{\gamma_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \frac{\gamma_n}{\gamma_n} \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Scaling eigenvectors

$$\left[\; A \; \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i \right] \left[- \quad W_i^* \quad - \right]$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

diagonal matrices commute…

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \gamma_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{\gamma_n} \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Scaling eigenvectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Sum of rank-1 matrices

Dyadic Expansion

← Order of sum does not matter…

Diagonalization (non-unique) case 2: scaling

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

diagonal matrices commute…

$$A = \Big[\ V\ \Big]\Big[\ D\ \Big]\Big[\ V^{-1}\Big]$$

$$\Big[\ A\ \Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\textbf{Eigen-values}\text{ (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}} = \underbrace{\begin{bmatrix} | & & | \\ V_1\gamma_1 & \cdots & V_n\gamma_n \\ | & & | \end{bmatrix}}_{V'} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} - & \frac{1}{\gamma_1}W_1^* & - \\ & \vdots & \\ - & \frac{1}{\gamma_n}W_n^* & - \end{bmatrix}}_{V'^{-1}}$$

Scaling eigenvectors

$$\Big[\ A\ \Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \Big[\lambda_i\Big] \Big[- \quad W_i^* \quad -\Big]$$

Sum of rank-1 matrices

Dyadic Expansion

Order of sum does not matter…

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & V_2 \\ | & | \end{matrix}} & \boxed{\begin{matrix} | & | \\ V_3 & V_4 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} \gamma_2 e^{-i\phi_2} & 0 \\ 0 & \gamma_2 e^{i\phi_2} \end{matrix}} & \cdots & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \end{matrix} & \begin{matrix} \vdots & \vdots \end{matrix} & \ddots & \vdots \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & W_2^* & - \end{matrix}} \\ \boxed{\begin{matrix} - & W_3^* & - \\ - & W_4^* & - \end{matrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & V_2 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & W_2^* & - \end{matrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \big[\; V \;\big]\big[\; D \;\big]\big[\; V^{-1} \big]$$

$$\big[\; A \;\big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\textbf{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}}$$

$$\big[\; A \;\big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \big[\lambda_i\big] \big[- \quad W_i^* \quad -\big]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} | & | & & | \\ V_1 & \bar{V}_1 & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1}\frac{z}{z} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1}\frac{z'}{z'} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \qquad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Note: *may differ by any complex scalars* $z, z' \in \mathbb{C}$

…with both magnitude and phase shifts

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 z & \bar{V}_1 z' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & \frac{1}{z} W_1^* & - \\ - & \frac{1}{z'} \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

Note: *may differ by any complex scalars* $z, z' \in \mathbb{C}$

…with both magnitude and phase shifts

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = [\ V\ ][\ D\ ][\ V^{-1}]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [- \quad W_i^* \quad -]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} =$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors — **Eigen-values** (on diagonal) — **Left eigen-vectors**

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [- \quad W_i^* \quad -]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Real — Imag

(scaled by $\sqrt{2}$)

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} \; V \; \end{bmatrix}\begin{bmatrix} \; D \; \end{bmatrix}\begin{bmatrix} \; V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \; A \; \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\textbf{Eigen-values} \text{ (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}}$$

$$\begin{bmatrix} \; A \; \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} | & | & & | \\ V_1 & \bar{V}_1 & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

similarly…

$$\begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}}_{U^*} \begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Note: $U \in \mathbb{C}^{2\times 2}$

unitary

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = [\ V\ ][\ D\ ][\ V^{-1}]$$

$$[\ A\ ] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-values} \\ \text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$[\ A\ ] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] [- \quad W_i^* \quad -]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1$ $W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} | & | & & | \\ V_1 & \bar{V}_1 & \cdots & V_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \gamma_1 e^{-i\phi_1} & 0 & \cdots & 0 \\ 0 & \gamma_1 e^{i\phi_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \\ & \vdots & \\ - & \bar{W}_n^* & - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Note: $U \in \mathbb{C}^{2\times 2}$ $U^* U = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = I$

unitary

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\frac{1}{\sqrt{2}}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

Eigenvectors: $V_1, \bar{V}_1 \qquad W_1^*, \bar{W}_1^*$

can be conjugate pairs

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\frac{1}{\sqrt{2}} = \gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\ V\ \right]\left[\ D\ \right]\left[\ V^{-1}\right]$$

$$\left[\ A\ \right] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\left[\ A\ \right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} [\lambda_i] \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \gamma_1 e^{-i\phi_1} & 0 \\ 0 & \gamma_1 e^{i\phi_1} \end{matrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & W_1^* & - \\ - & \bar{W}_1^* & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

$U \quad U^* \quad U \quad U^*$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n \qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\frac{1}{\sqrt{2}} = \gamma\begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \begin{matrix} \cdots & 0 \\ \cdots & 0 \end{matrix} \\ \begin{matrix} \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} \ddots & \vdots \\ \cdots & \lambda_n \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix}$$

where: $V_1, \bar{V}_1 \in \mathbb{C}^n$ $\qquad v_1, v_1' \in \mathbb{R}^n$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \end{bmatrix} \quad \bar{V}_1 = \begin{bmatrix} V_{11}^* \\ V_{21}^* \\ \vdots \\ V_{n1}^* \end{bmatrix} \qquad \begin{bmatrix} | & | \\ V_1 & \bar{V}_1 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U}$$

Also: $$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \gamma e^{-i\phi} & 0 \\ 0 & \gamma e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\frac{1}{\sqrt{2}} = \gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

$$A = \begin{bmatrix} & V & \end{bmatrix}\begin{bmatrix} & D & \end{bmatrix}\begin{bmatrix} & V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} & A & \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}} = \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}}_{V' \in \mathbb{R}^{n\times n}} \begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad\quad \vdots & \ddots & \vdots \\ 0 \quad\quad 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & {w_1'}^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}}_{V'^{-1}}$$

$$\begin{bmatrix} & A & \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\substack{\text{Right}\\\text{eigen-}\\\text{vectors}}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{Eigen-}\\\text{values}\\\text{(on diagonal)}}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\substack{\text{Left}\\\text{eigen-}\\\text{vectors}}} = \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}}_{V' \in \mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \cdots & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} \vdots & \quad & \vdots \\ 0 & \quad & 0 \end{matrix} & \ddots & \begin{matrix} \vdots \\ \lambda_n \end{matrix} \end{bmatrix}}_{D' \in \mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}}_{V'^{-1}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

$$= \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}}_{V' \in \mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} \boxed{\gamma \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}} & \cdots & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} \vdots & \quad \vdots \end{matrix} & \ddots & \vdots \\ \begin{matrix} 0 & \quad 0 \end{matrix} & \cdots & \lambda_n \end{bmatrix}}_{D' \in \mathbb{R}^{n\times n}} \underbrace{\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix}}_{V'^{-1}}$$

Real Expansion…

Block diagonal…

Pseudo-diagonalization

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

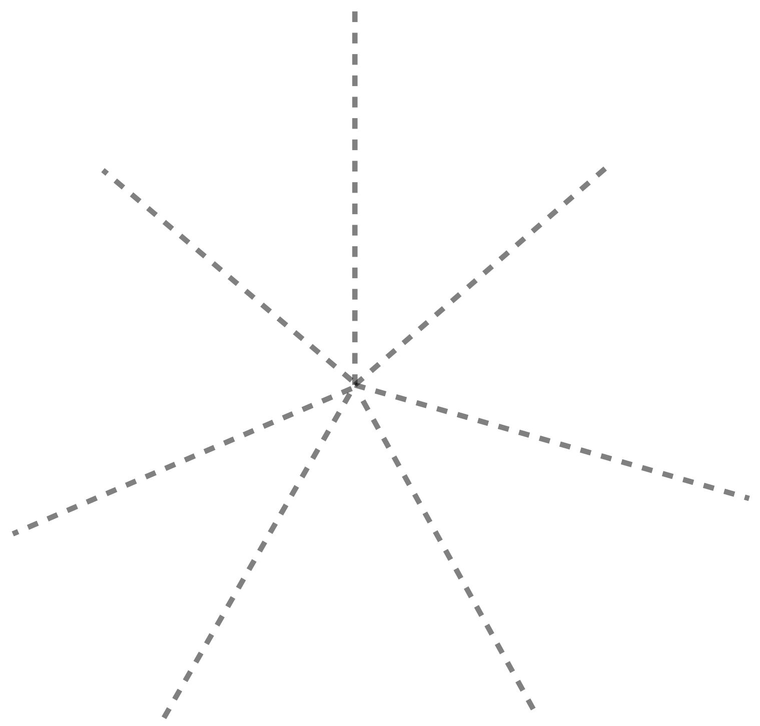
Eigen-values (on diagonal)

Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$\mathbb{R}^n$



Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

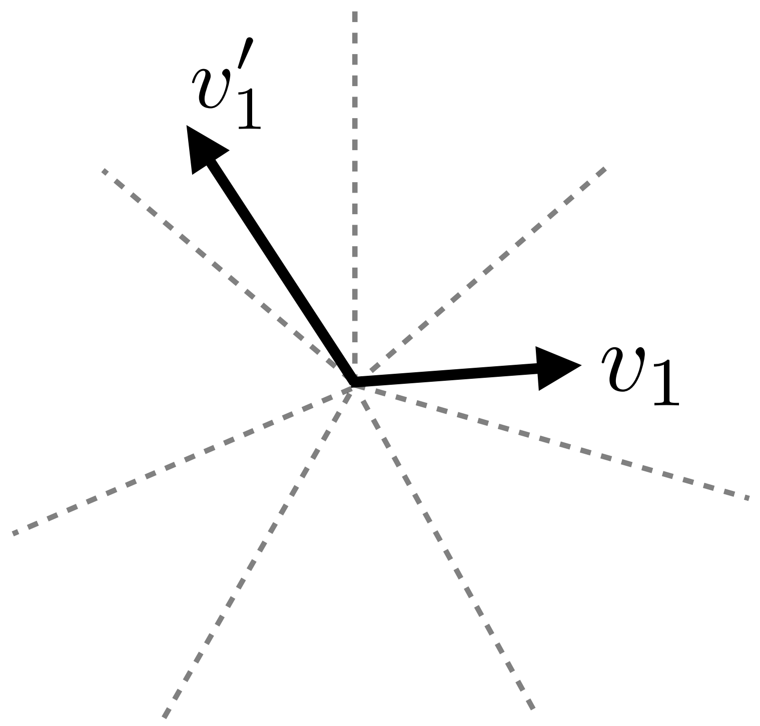
$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$



$\mathbb{R}^n$

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

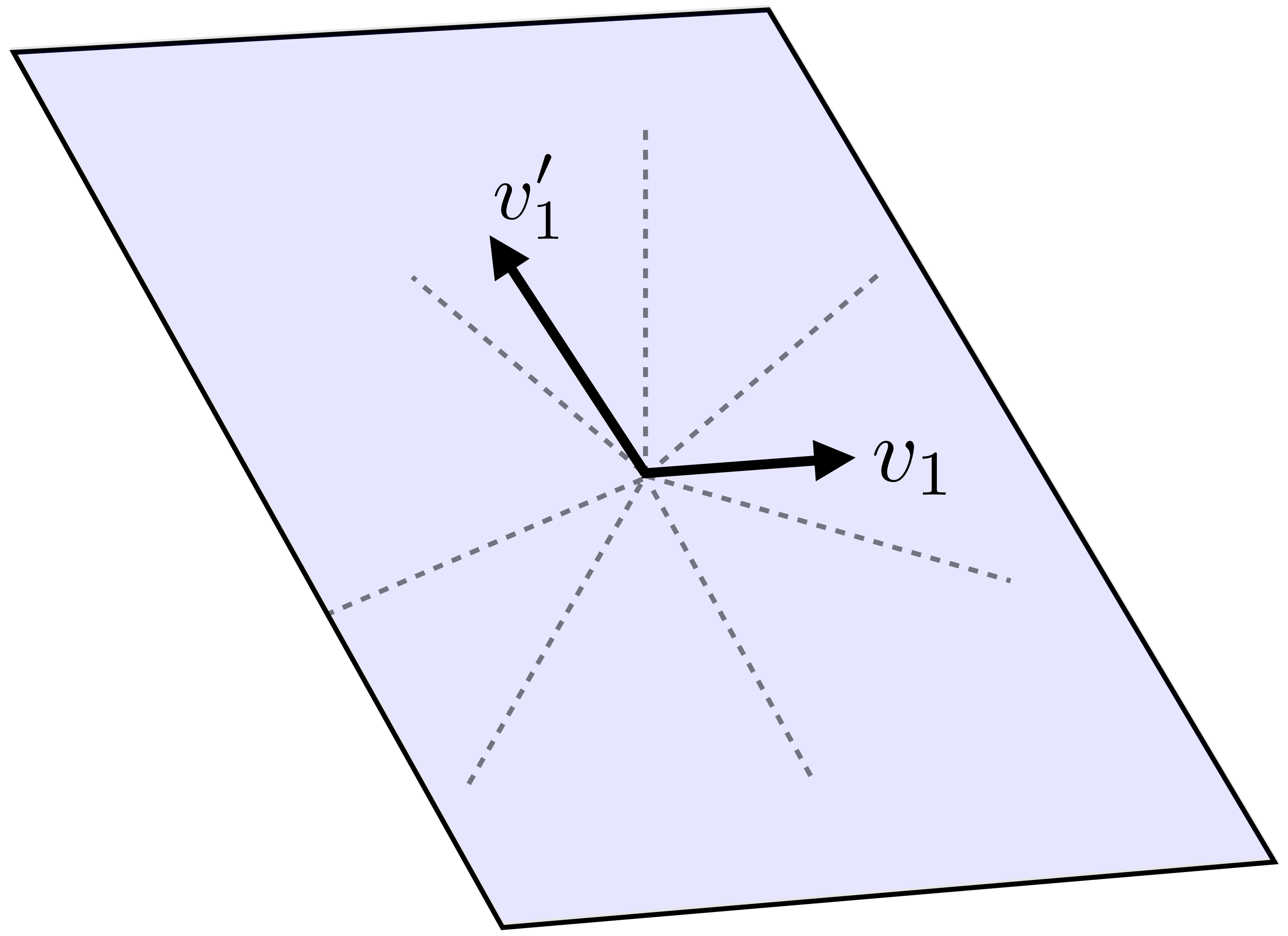
$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}$$



$\mathbb{R}^n$

$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\text{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\text{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} & V & \end{bmatrix}\begin{bmatrix} & D & \end{bmatrix}\begin{bmatrix} & V^{-1} & \end{bmatrix} =$$

$$\begin{bmatrix} & A & \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

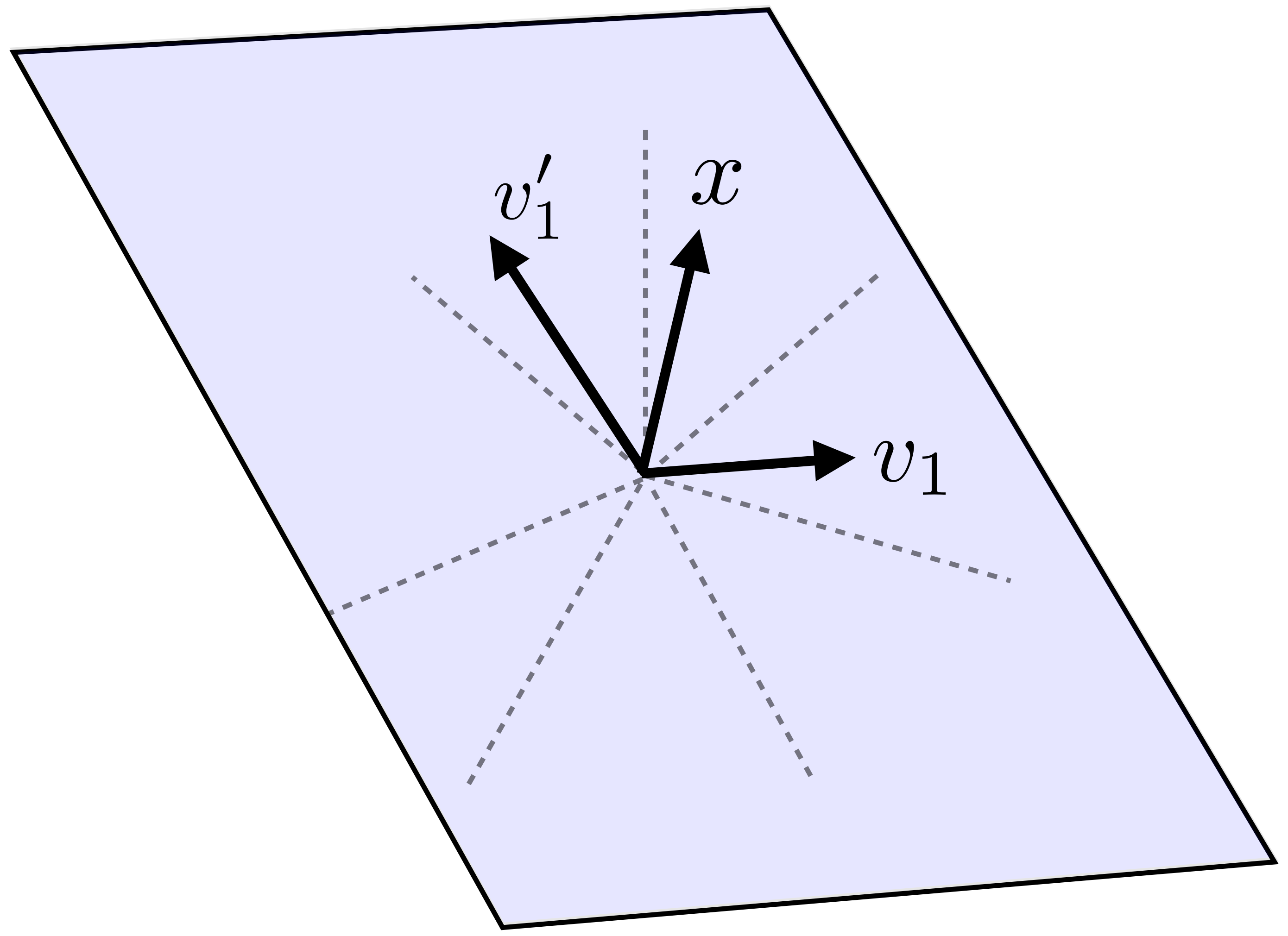
$$\begin{bmatrix} & A & \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$\begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

vector in plane of rotation

$\mathbb{R}^n$



$\text{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} & V & \end{bmatrix}\begin{bmatrix} & D & \end{bmatrix}\begin{bmatrix} & V^{-1} & \end{bmatrix}$$

$$\begin{bmatrix} & A & \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}}\underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}}\underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

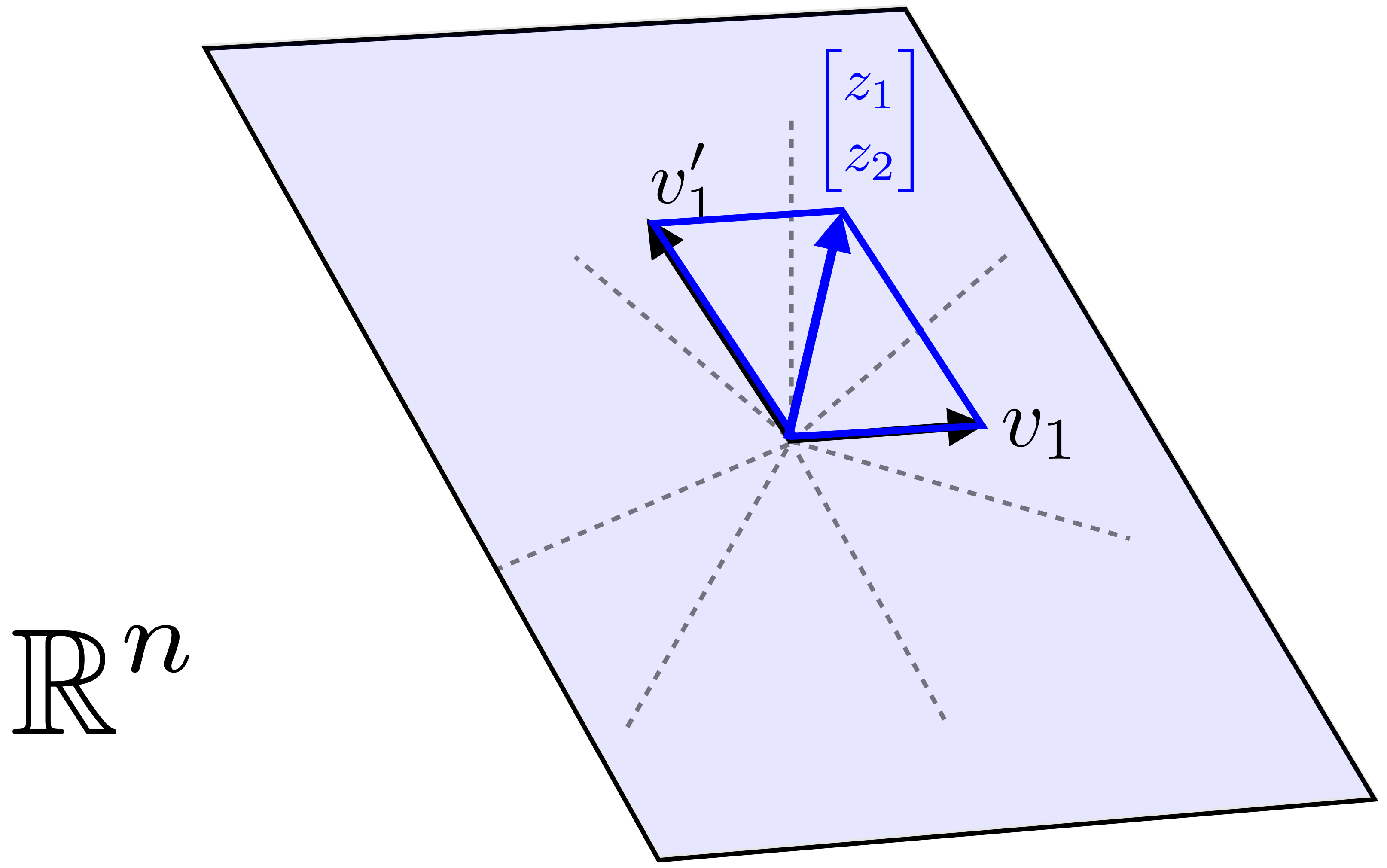
Right eigen-vectors **Eigen-values** (on diagonal) **Left eigen-vectors**

$$\begin{bmatrix} & A & \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix}\lambda_i\end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

vector in plane of rotation



$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

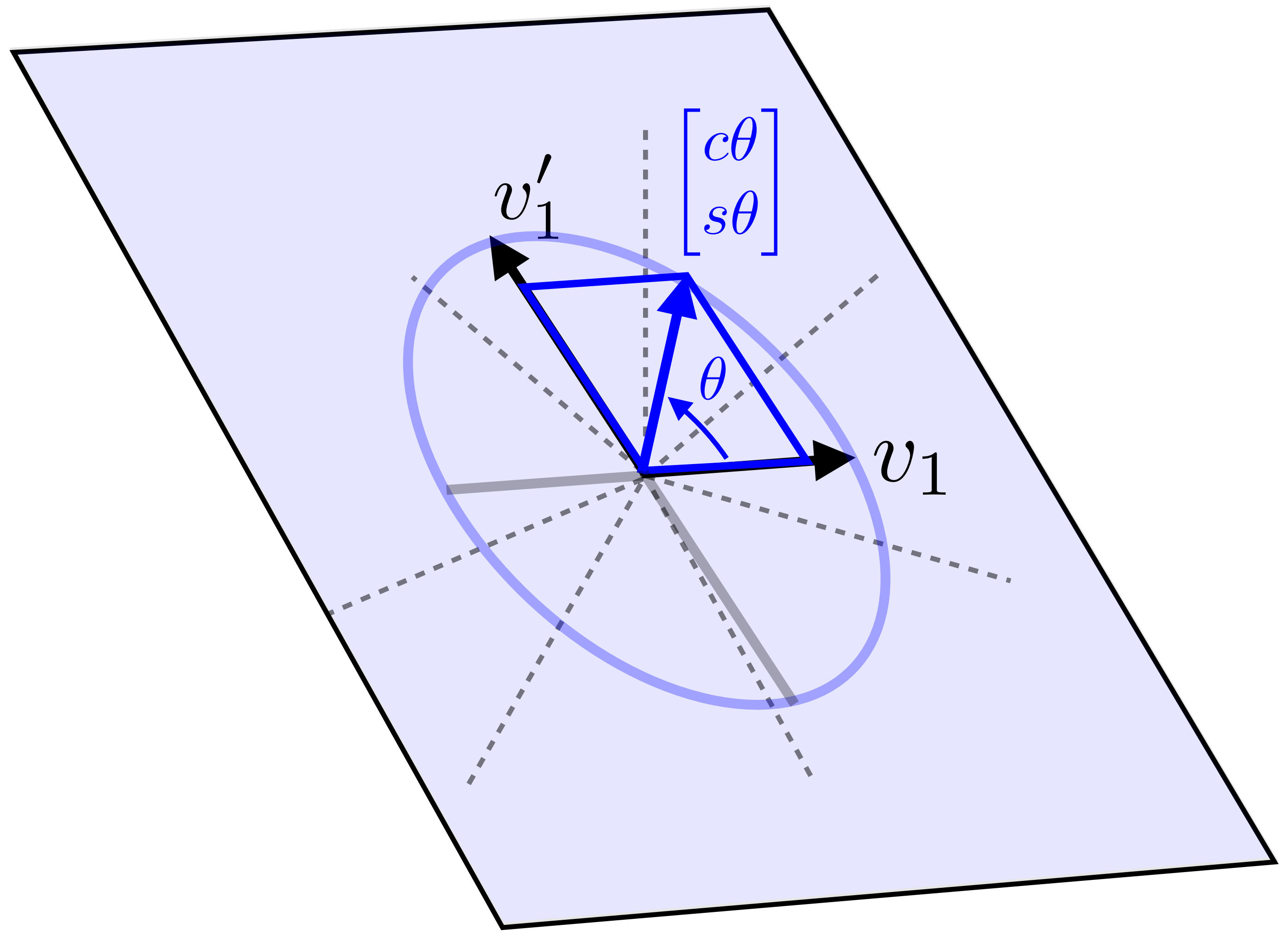
Right eigen-vectors | **Eigen-values** (on diagonal) | **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix}\begin{bmatrix} \lambda_i \end{bmatrix}\begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\gamma \geq 0$

$$\begin{bmatrix} \boxed{\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix}\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} \boxed{\begin{bmatrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{bmatrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}\begin{bmatrix} c\theta \\ s\theta \end{bmatrix}^{|z|_2}$$

vector in plane of rotation



$\mathbb{R}^n$

$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \left[\; V \;\right]\left[\; D \;\right]\left[\; V^{-1} \right]$$

$$\left[\; A \;\right] = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}$$

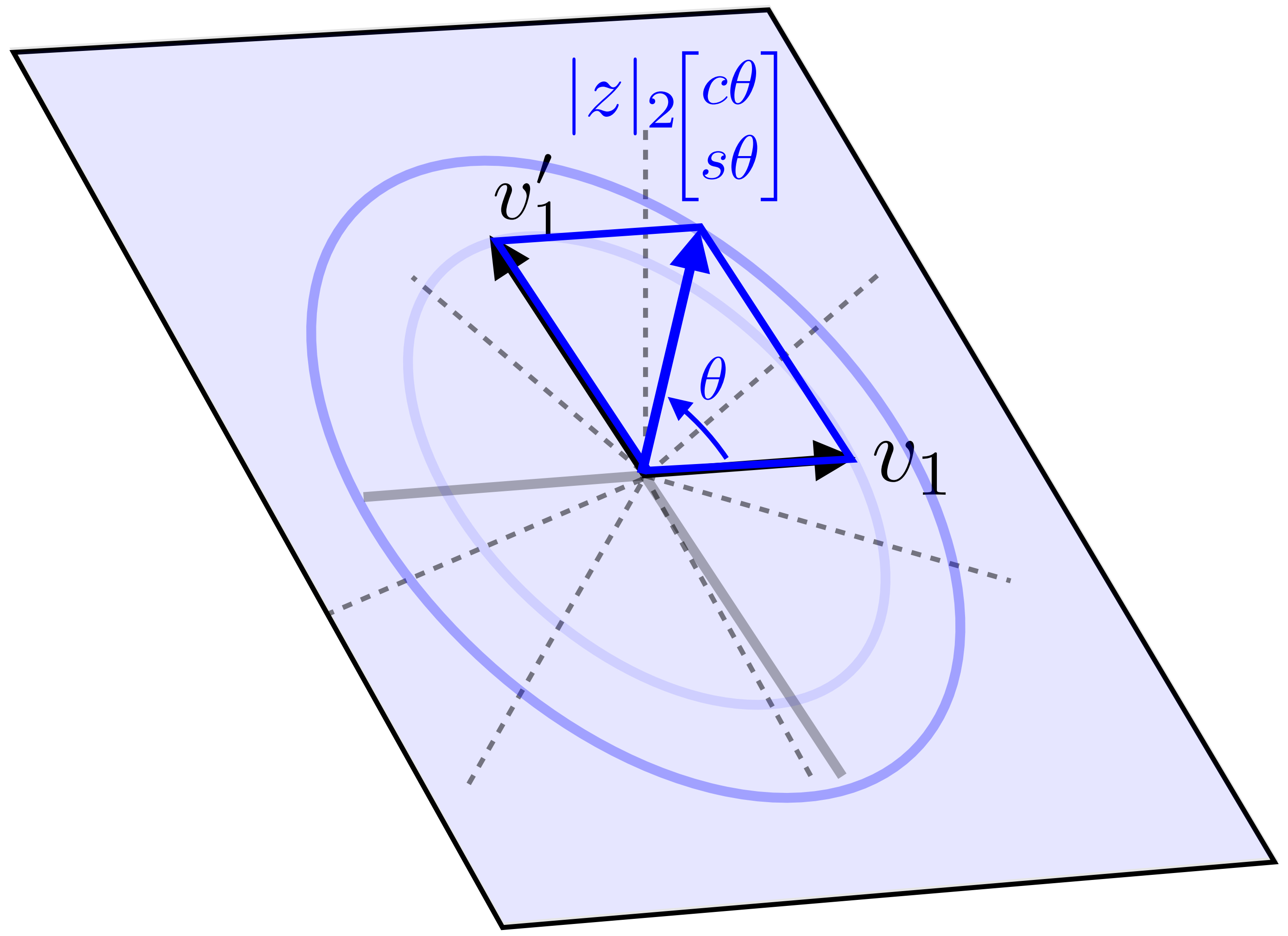
Right eigen-vectors | **Eigen-values** (on diagonal) | **Left eigen-vectors**

$$\left[\; A \;\right] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \left[\lambda_i\right] \left[- \quad W_i^* \quad -\right]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \left[\boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} \;\cdots\; \begin{matrix} | \\ V_n \\ | \end{matrix}\right] \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad W_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} c\theta \\ s\theta \end{bmatrix} |z|_2$$

$\mathbb{R}^n$



$\mathrm{span} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} & V & \end{bmatrix}\begin{bmatrix} & D & \end{bmatrix}\begin{bmatrix} & V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} & A & \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

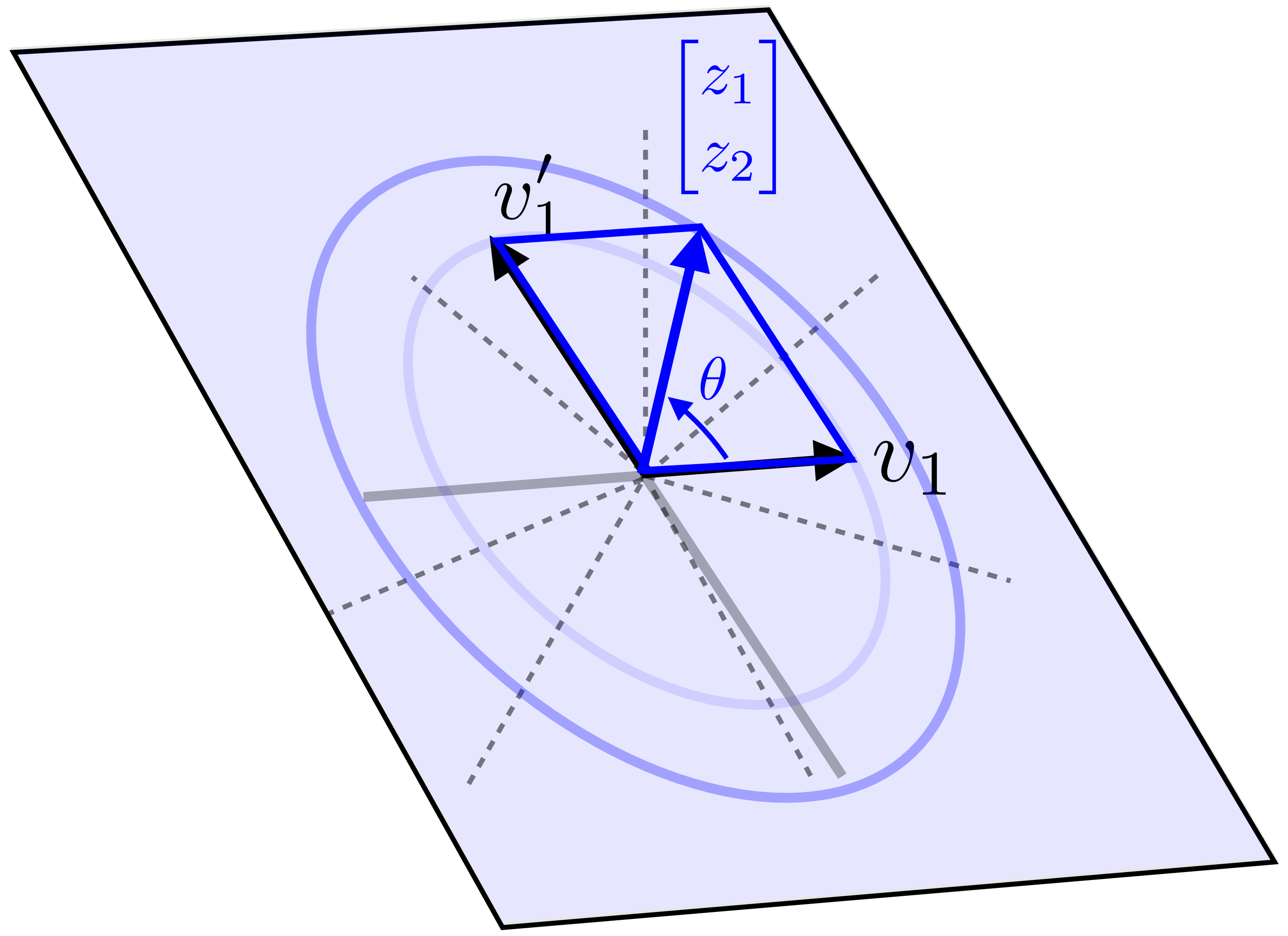
Left eigen-vectors

$$\begin{bmatrix} & A & \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$\mathbb{R}^n$



$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

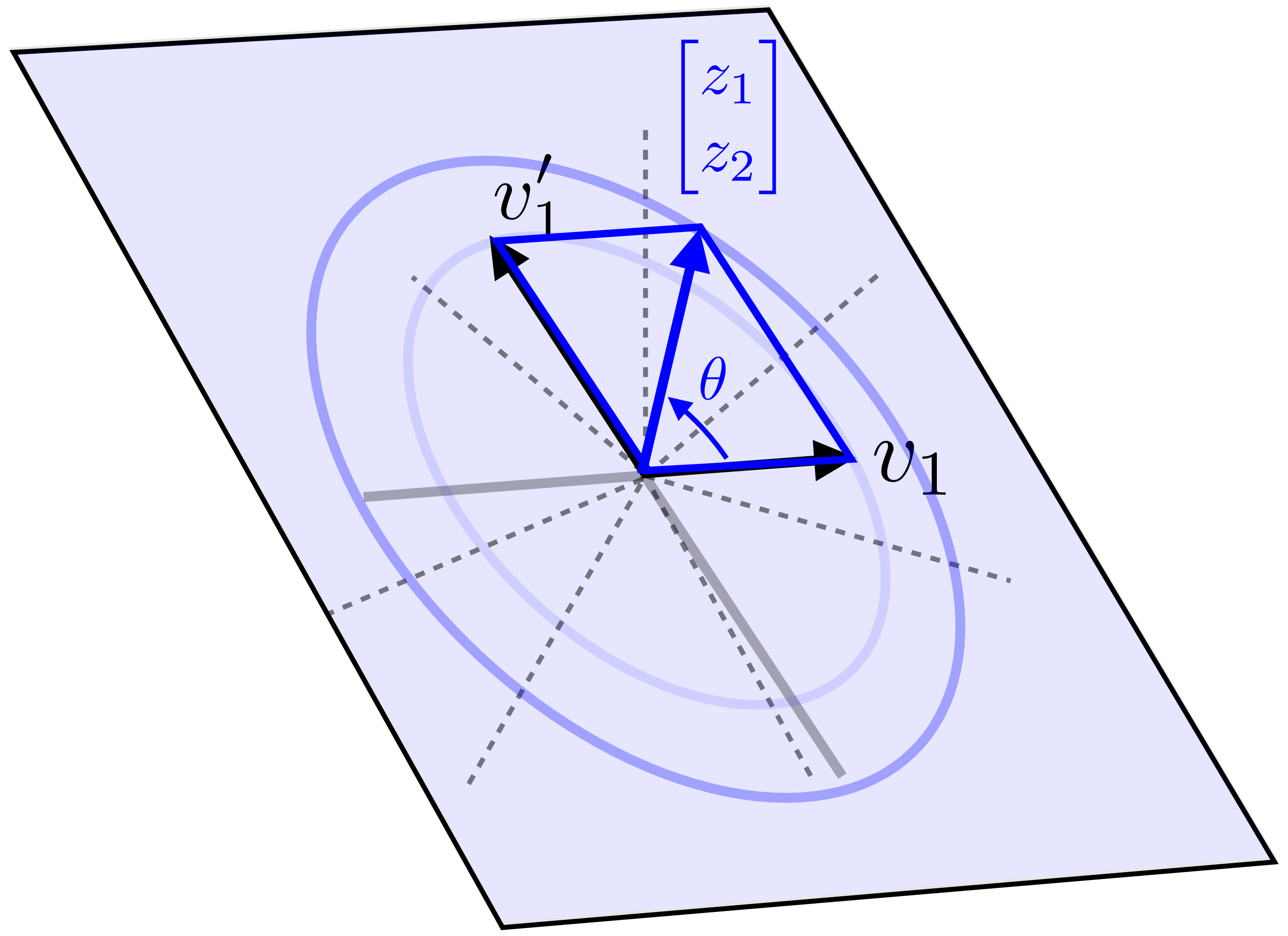
$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors | **Eigen-values** (on diagonal) | **Left eigen-vectors**

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} z_1 \\ z_2 \\ 0 \\ \vdots \end{bmatrix}}$$



$\mathbb{R}^n$

$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \Big[\;V\;\Big]\Big[\;D\;\Big]\Big[\;V^{-1}\Big]$$

$$\Big[\;A\;\Big] = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\textbf{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\textbf{Eigen-values} \text{ (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\textbf{Left eigen-vectors}}$$

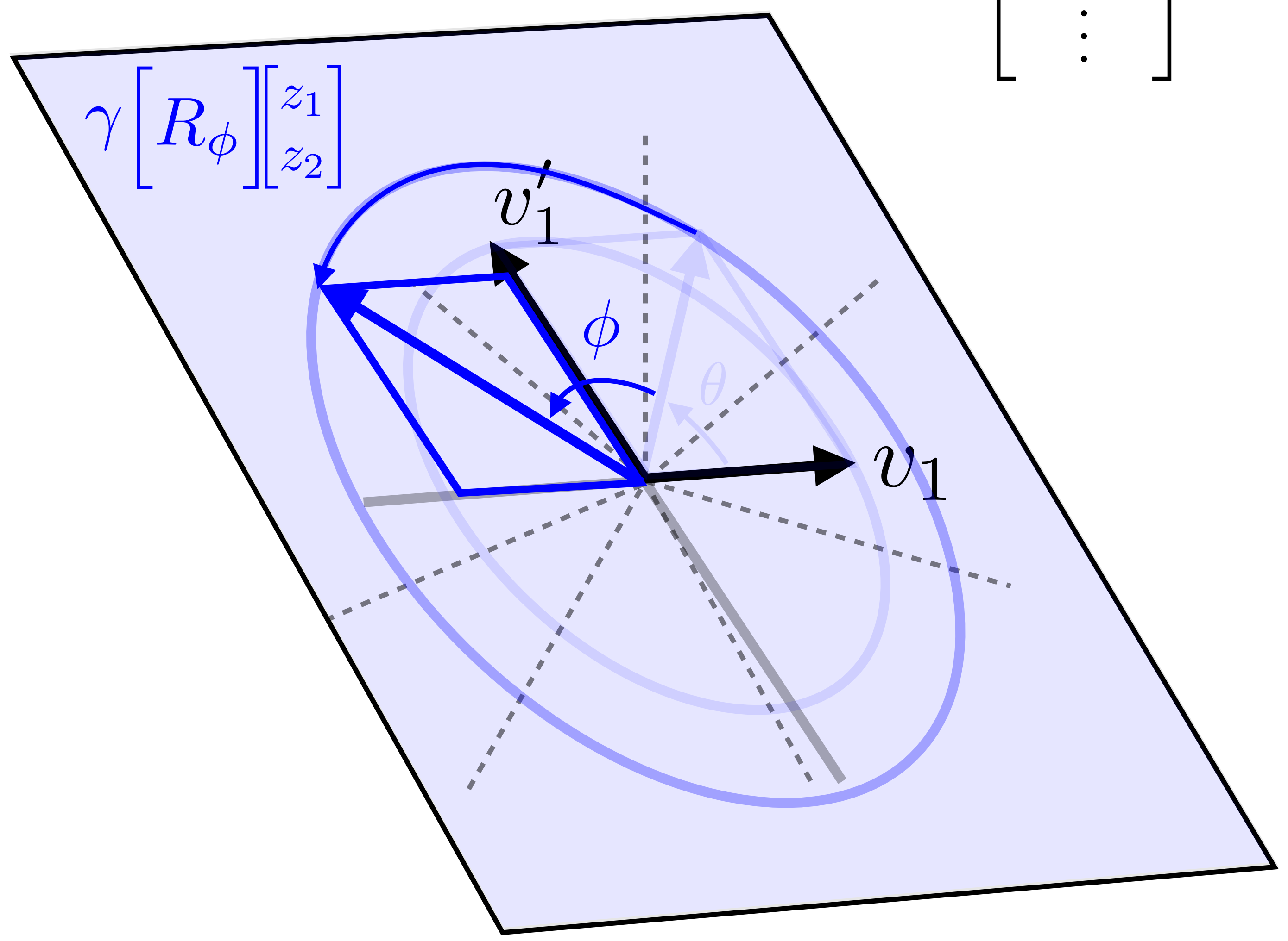
$$\Big[\;A\;\Big] = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \Big[\lambda_i\Big]\Big[- \quad W_i^* \quad -\Big]$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma = 1$

$\mathbb{R}^n$



$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n \times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots** **GEOMETRY**

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors | **Eigen-values** (on diagonal) | **Left eigen-vectors**

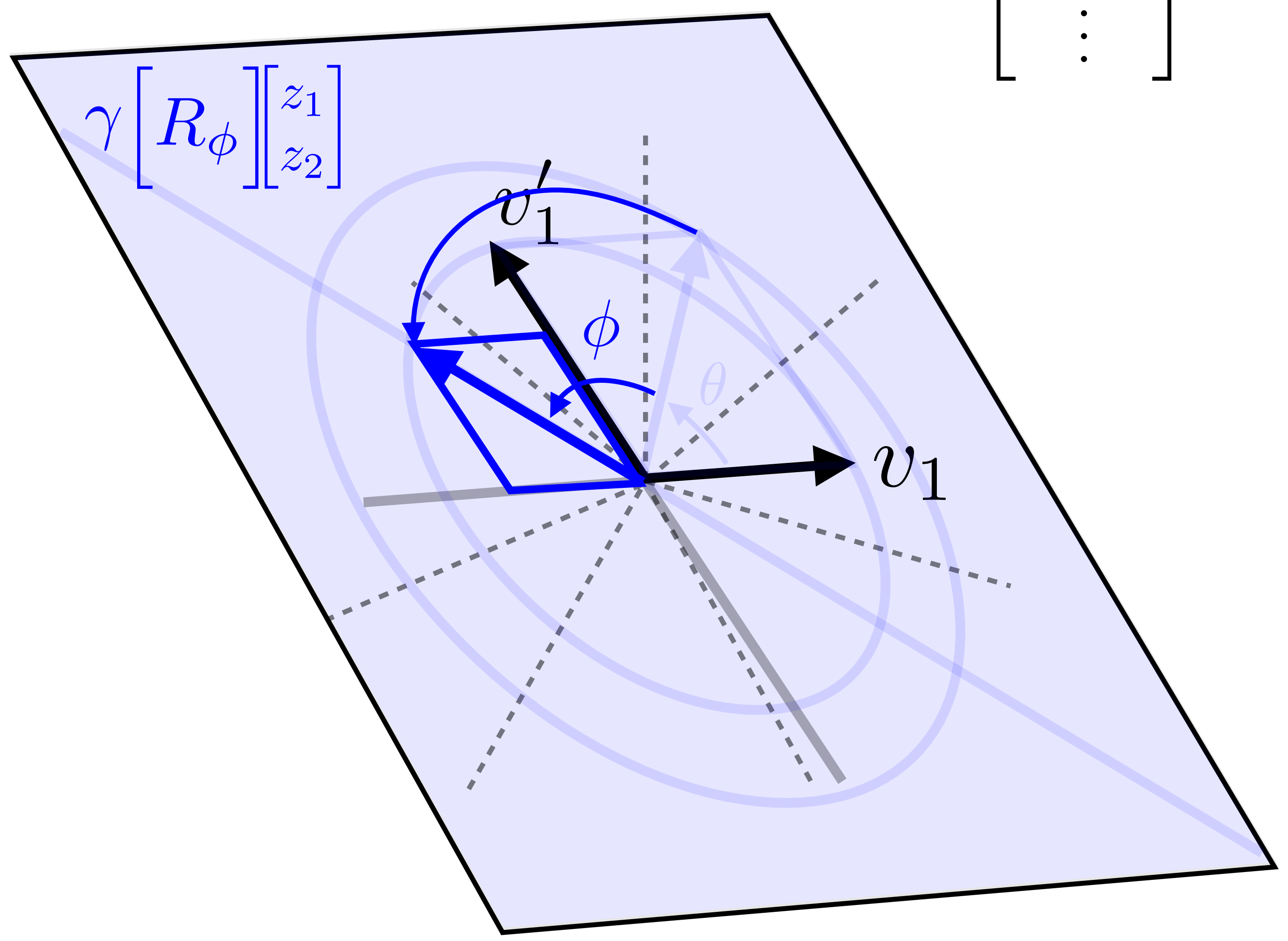
$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

$$\begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma < 1$

$\mathbb{R}^n$



span $\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

Right eigen-vectors

Eigen-values (on diagonal)

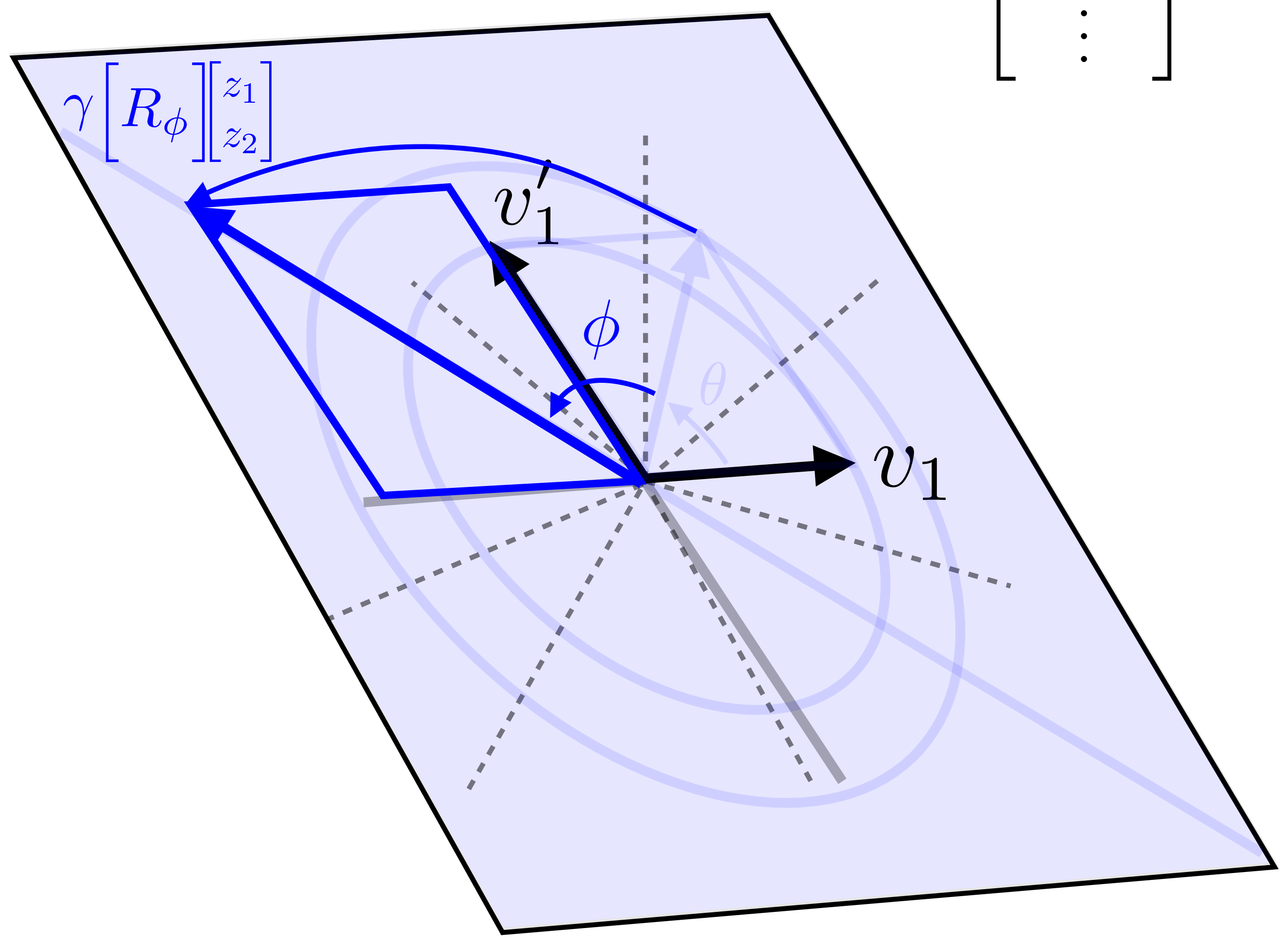
Left eigen-vectors

$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi} \quad \gamma \geq 0$

$$= \begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} \gamma R_\phi z \\ 0 \\ \vdots \end{bmatrix}}$$

$\gamma > 1$



$\mathbb{R}^n$

$\mathrm{span}\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D

Diagonalization: Complex eigenvalues

Square matrix: $A \in \mathbb{R}^{n\times n}$ assume $\mathrm{char}_A(s) = \det(sI - A)$ has **n distinct roots**

GEOMETRY

Eigenvalues: $\mathrm{eig}(A) = \{\lambda_1, \cdots, \lambda_n\}$ with $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Diagonalization

$$A = \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} D \end{bmatrix}\begin{bmatrix} V^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ V_1 & \cdots & V_n \\ | & & | \end{bmatrix}}_{\text{Right eigen-vectors}} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{Eigen-values (on diagonal)}} \underbrace{\begin{bmatrix} - & W_1^* & - \\ & \vdots & \\ - & W_n^* & - \end{bmatrix}}_{\text{Left eigen-vectors}}$$

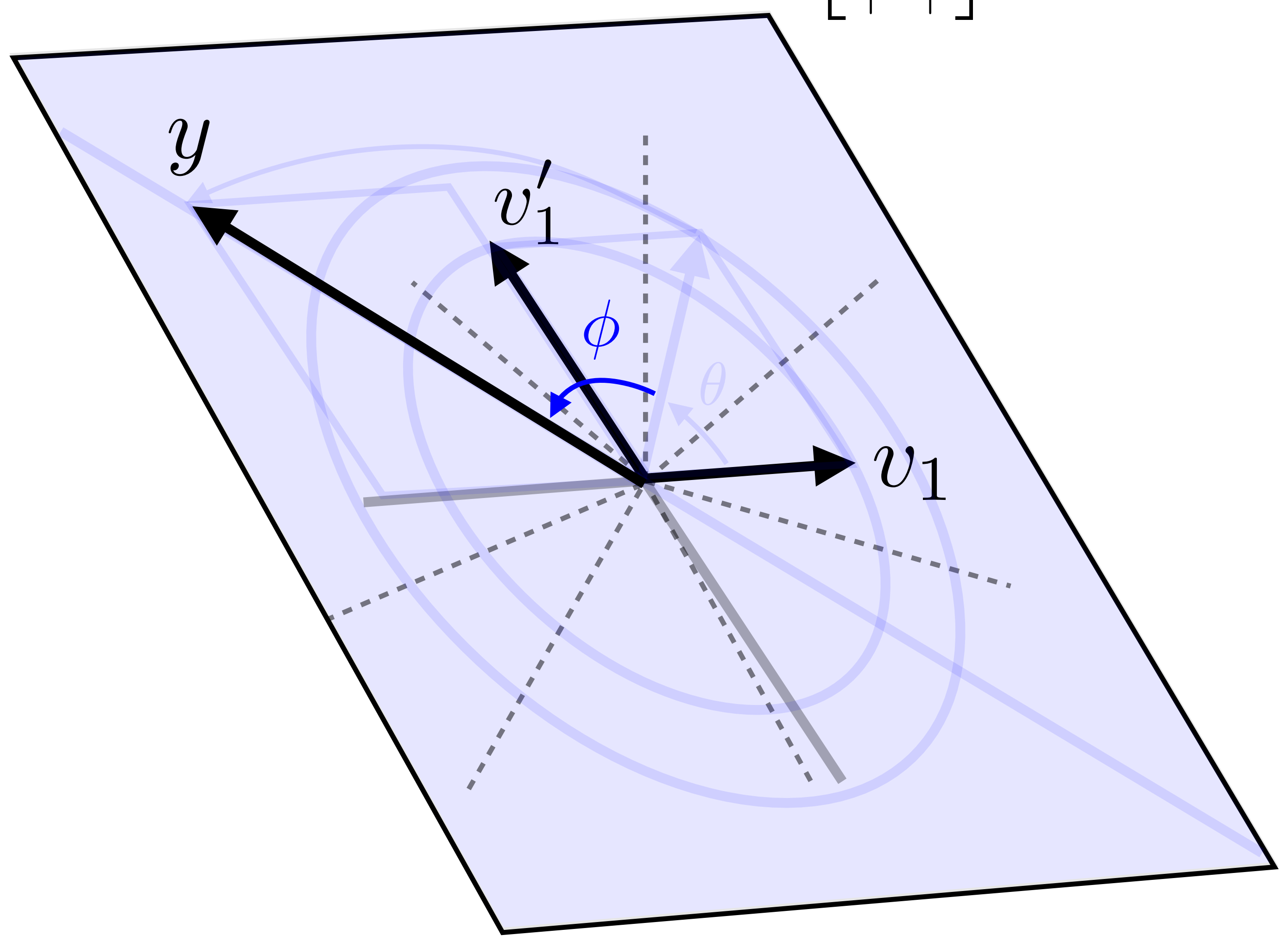
$$\begin{bmatrix} A \end{bmatrix} = \sum_i \begin{bmatrix} | \\ V_i \\ | \end{bmatrix} \begin{bmatrix} \lambda_i \end{bmatrix} \begin{bmatrix} - & W_i^* & - \end{bmatrix}$$

Complex Conjugate Pairs: $\lambda, \lambda^* = a \pm bi = \gamma e^{\pm i\phi}$ $\quad \gamma \geq 0$

$$= \underbrace{\begin{bmatrix} \boxed{\begin{matrix} | & | \\ v_1 & v_1' \\ | & | \end{matrix}} & \cdots & \begin{matrix} | \\ V_n \\ | \end{matrix} \end{bmatrix} \begin{bmatrix} \boxed{\gamma R_\phi} & \cdots & 0 \\ & \cdots & 0 \\ \vdots \quad \vdots & \ddots & \vdots \\ 0 \quad 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} - & w_1^\top & - \\ - & w_1'^\top & - \end{matrix}} \\ \vdots \\ - \quad \bar{W}_n^* \quad - \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{y = \gamma \begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix} \begin{bmatrix} R_\phi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}$$

$\gamma > 1$

$\mathbb{R}^n$



span $\begin{bmatrix} | & | \\ v_1 & v_1' \\ | & | \end{bmatrix}$ 2D