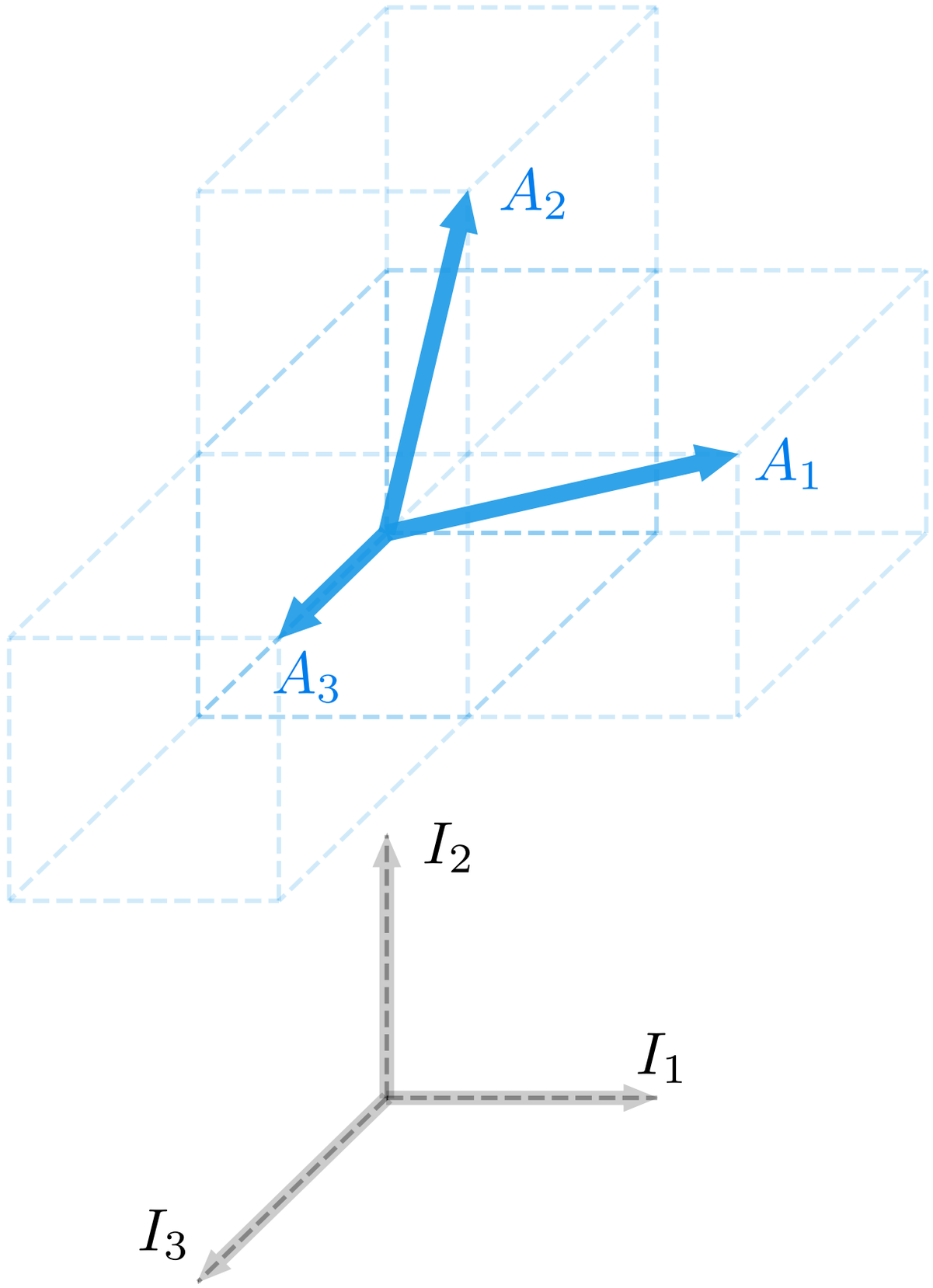
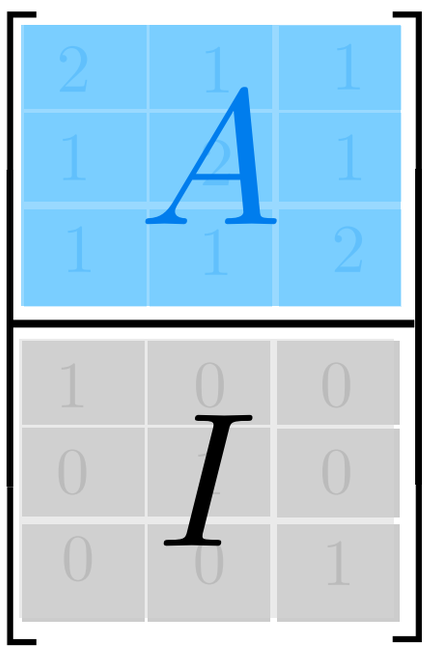
# Gaussian Elimination:
## Elementary Column Operations (column geometry)

# Linear Algebra

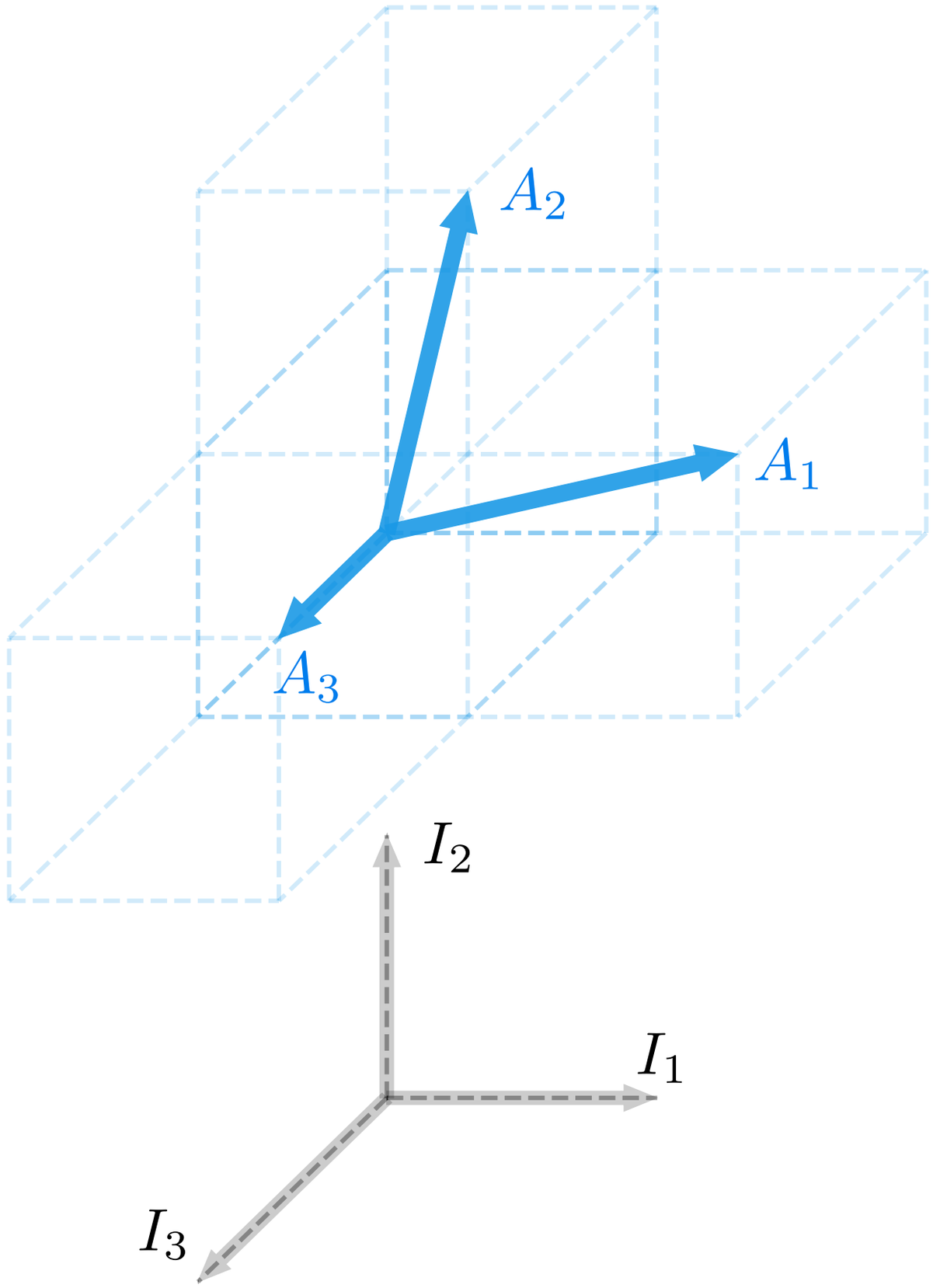
**Major sources:**

**Winter 2022 - Dan Calderone**



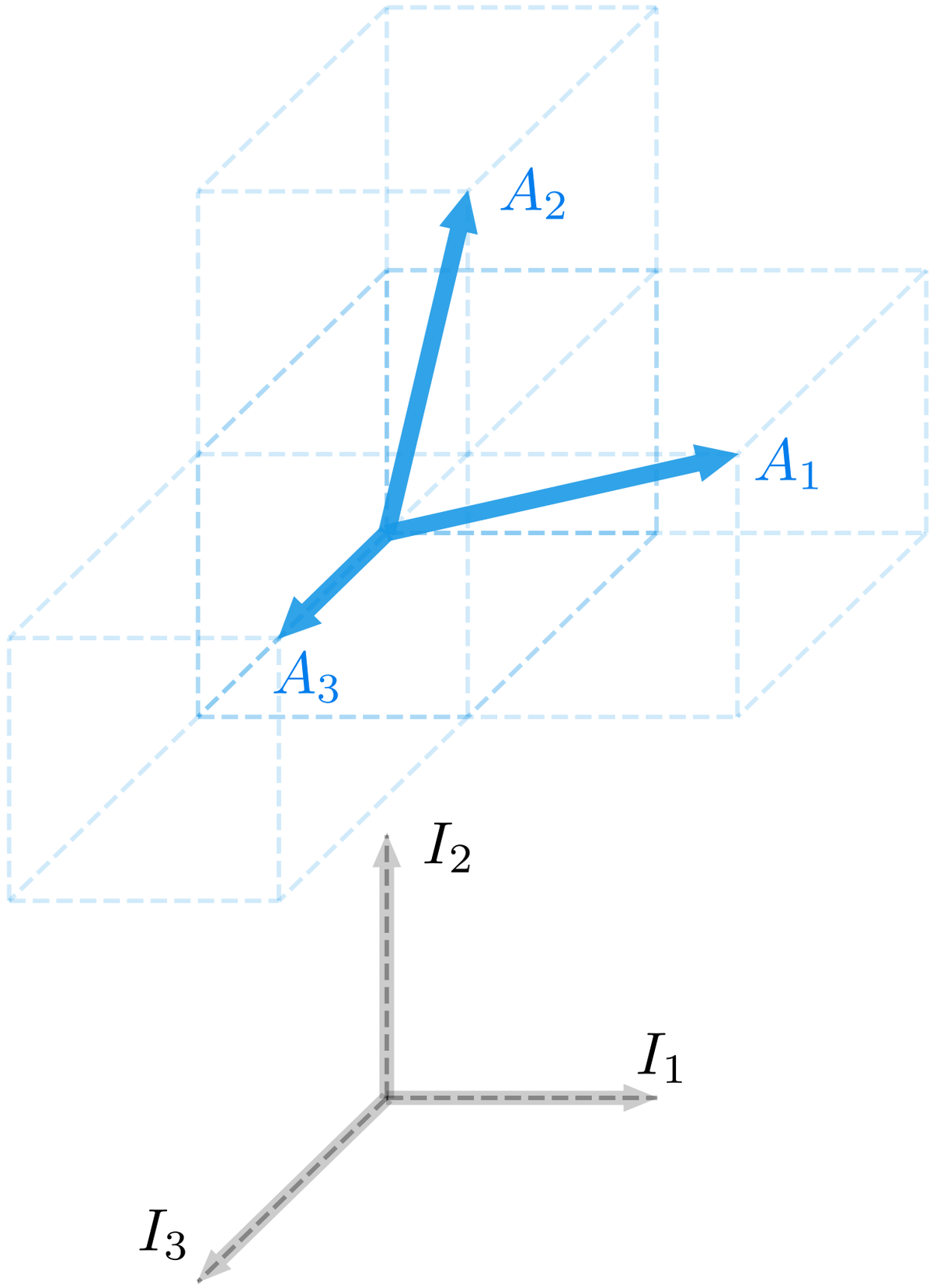



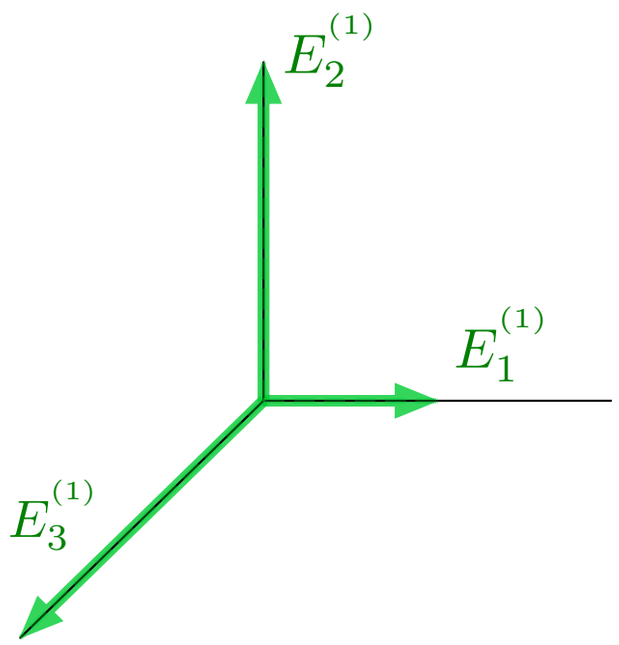

**Augmented System**




$$
\begin{array}{c}
A \\
\\
I
\end{array}
\left[\begin{array}{ccc}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2 \\
\hline
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

**Augmented System**



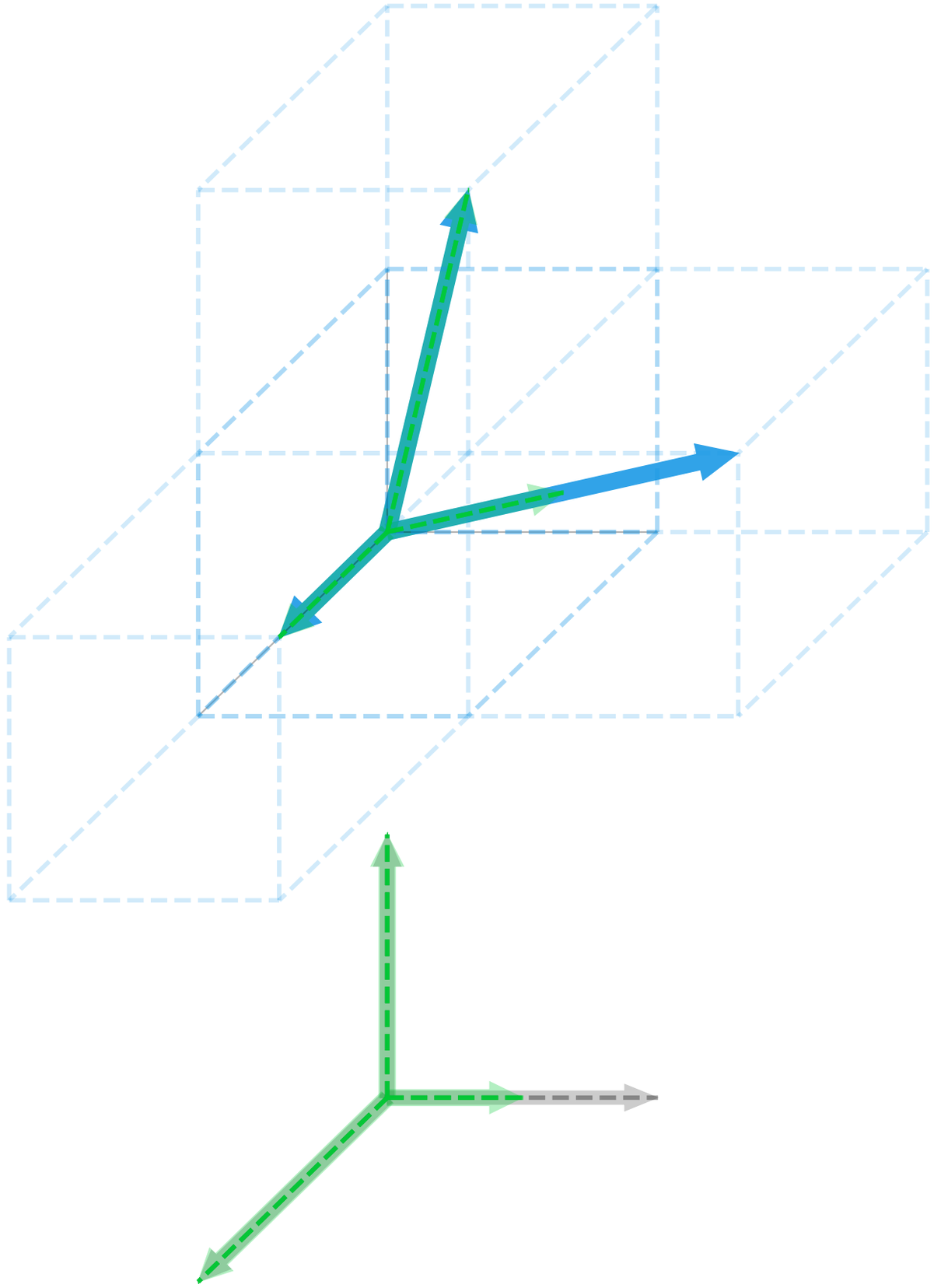

$$A \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$I \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

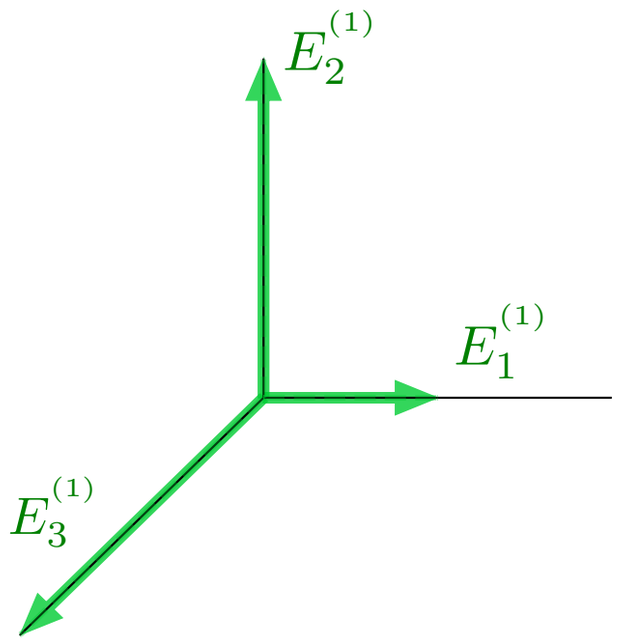
$$\times$$

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E^{(1)}$$

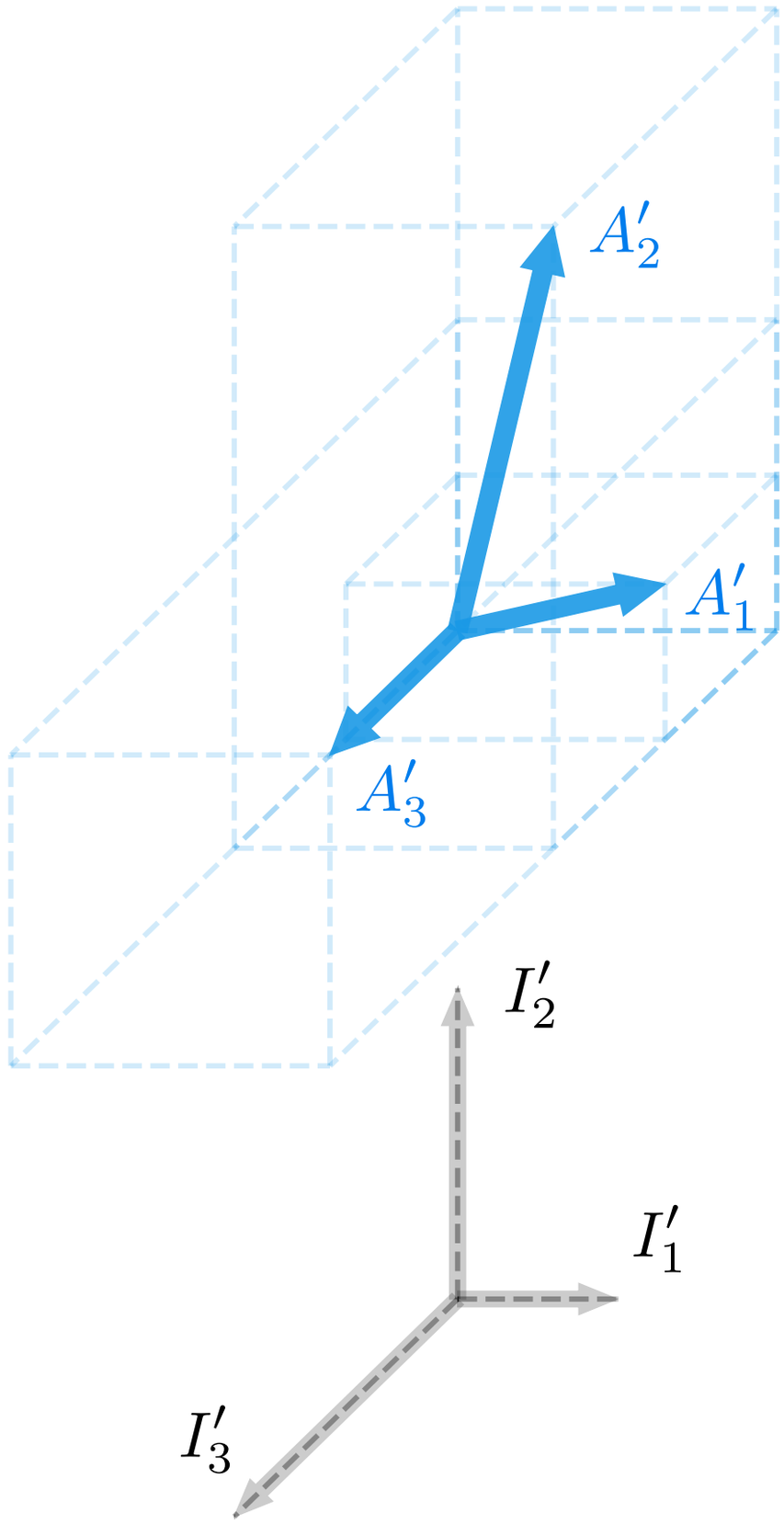


## Column-reduction: operation 1…




$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad E^{(1)} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

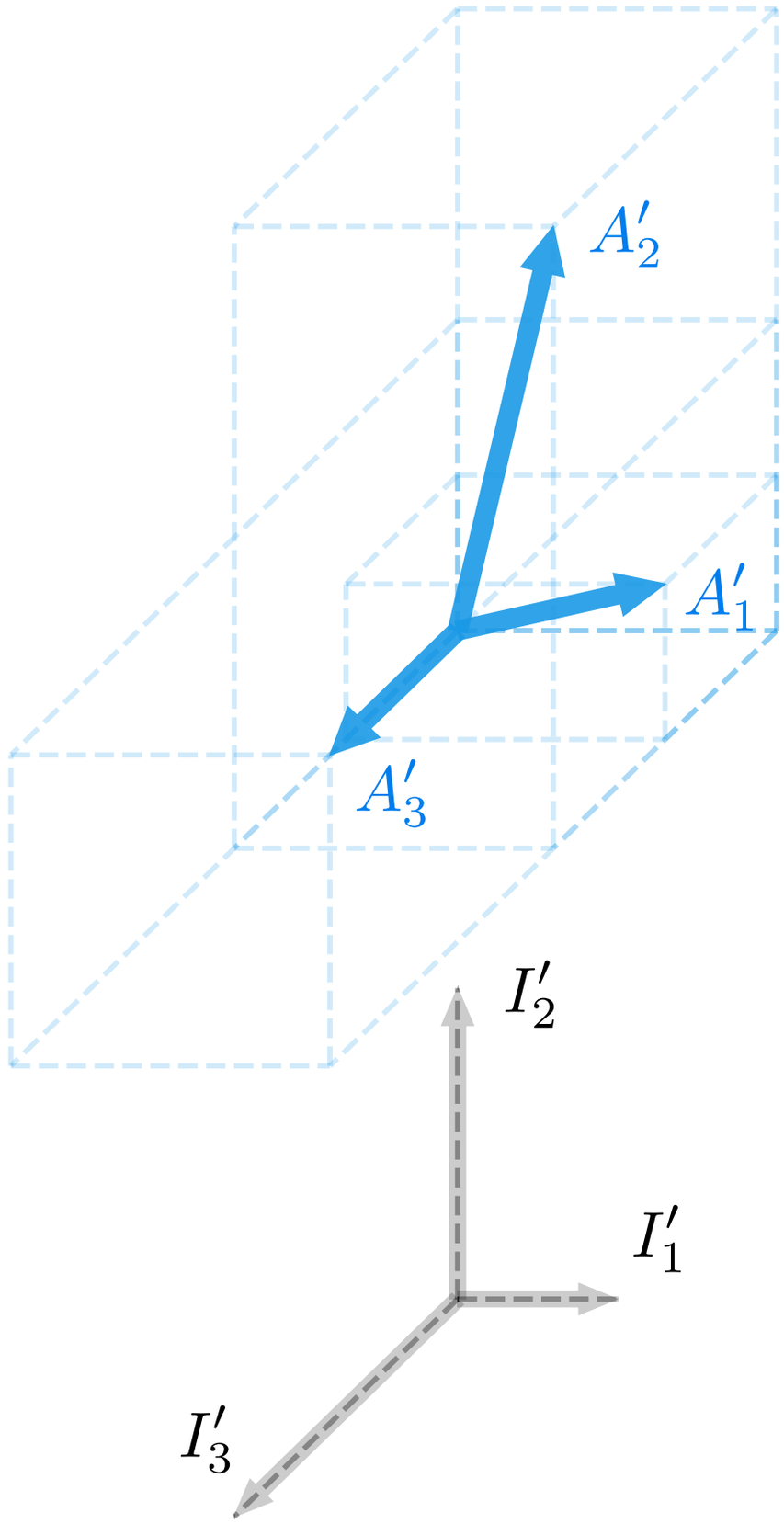


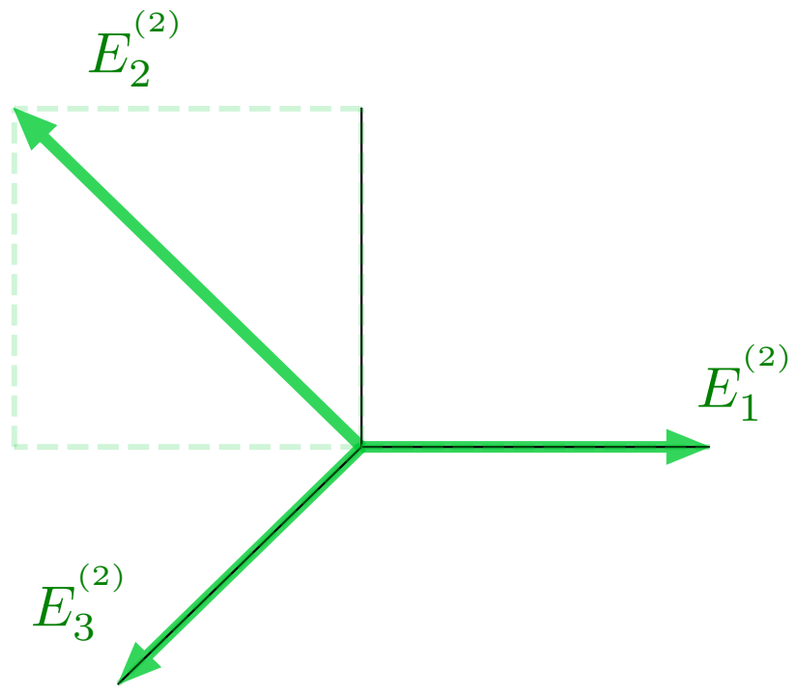

**Column-reduction: operation 1…**

$$\underbrace{AE^{(1)}}_{A'} \qquad \underbrace{IE^{(1)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1/2 & 2 & 1 \\ 1/2 & 1 & 2 \\ \hline 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

## Column-reduction: operation 2…

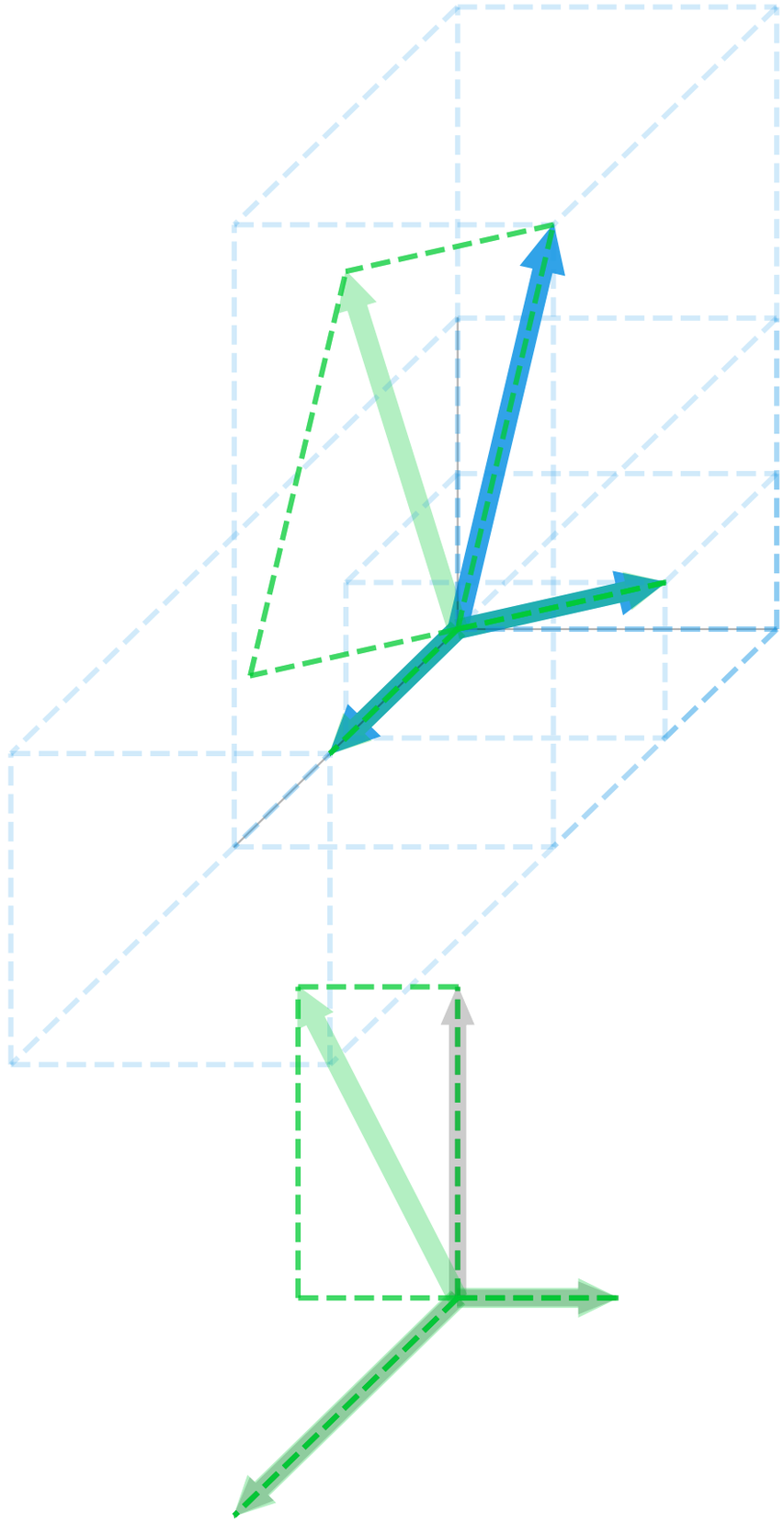


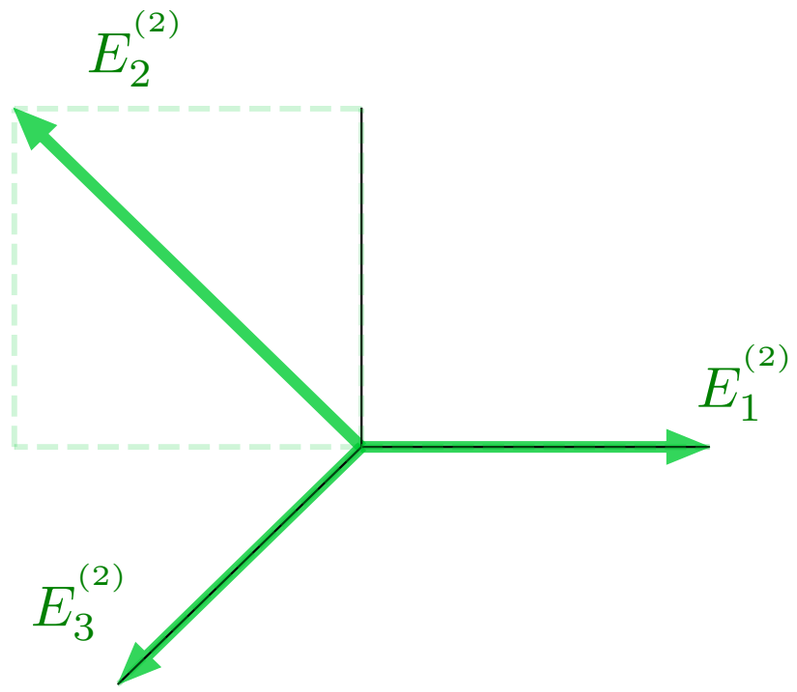

$$\underbrace{AE^{(1)}}_{A'} = \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 2 & 1 \\ 1/2 & 1 & 2 \end{bmatrix}$$

$$\underbrace{IE^{(1)}}_{I'} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \quad \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{(2)}}$$

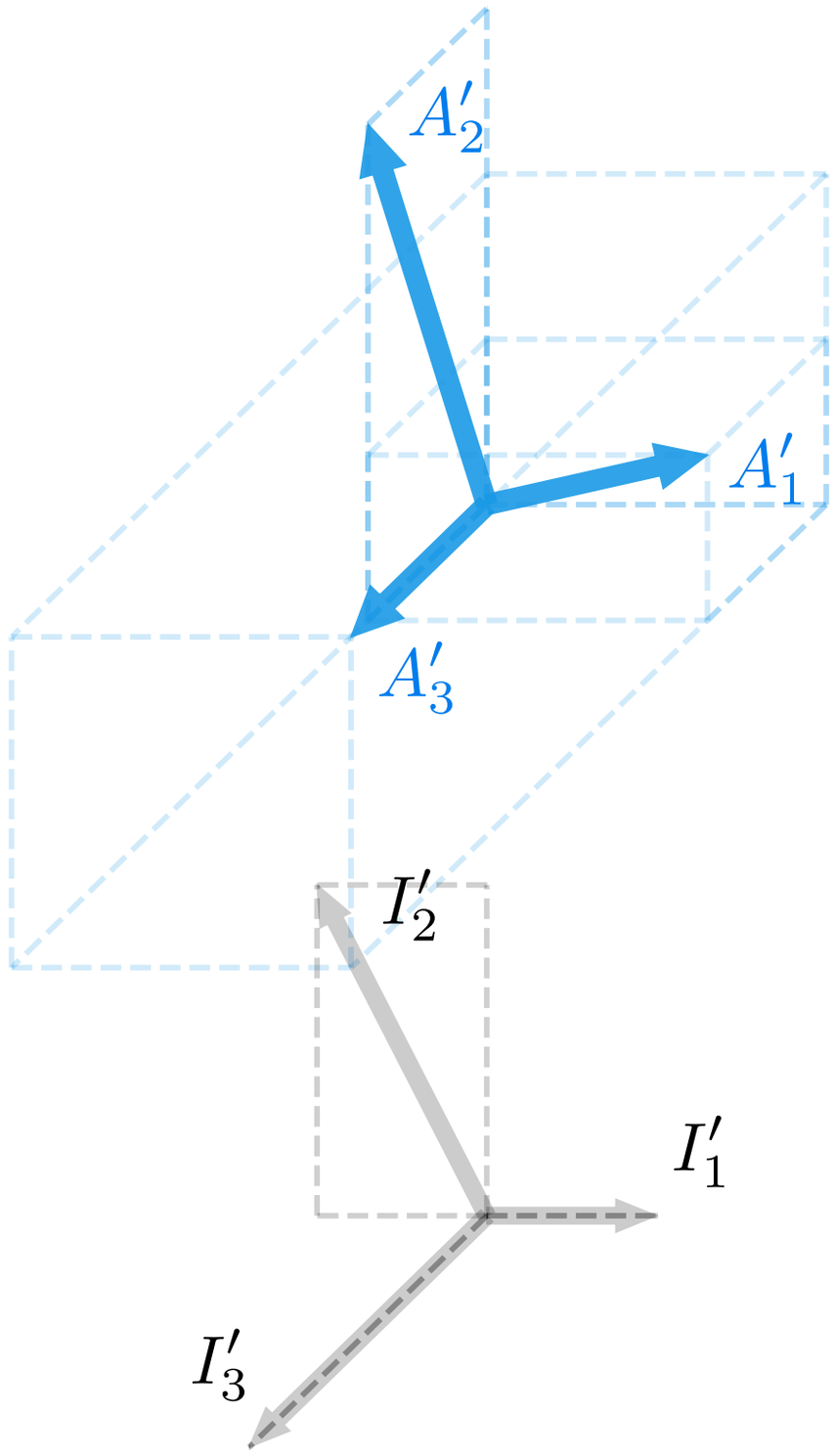


## Column-reduction: operation 2…




$$
\underbrace{AE^{(1)}}_{A'}
\qquad
\underbrace{IE^{(1)}}_{I'}
\qquad
\left[\begin{array}{ccc}
1 & 1 & 1 \\
1/2 & 2 & 1 \\
1/2 & 1 & 2 \\
\hline
1/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
\qquad
\underbrace{\left[\begin{array}{ccc}
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]}_{E^{(2)}}
$$

## Column-reduction: operation 2…

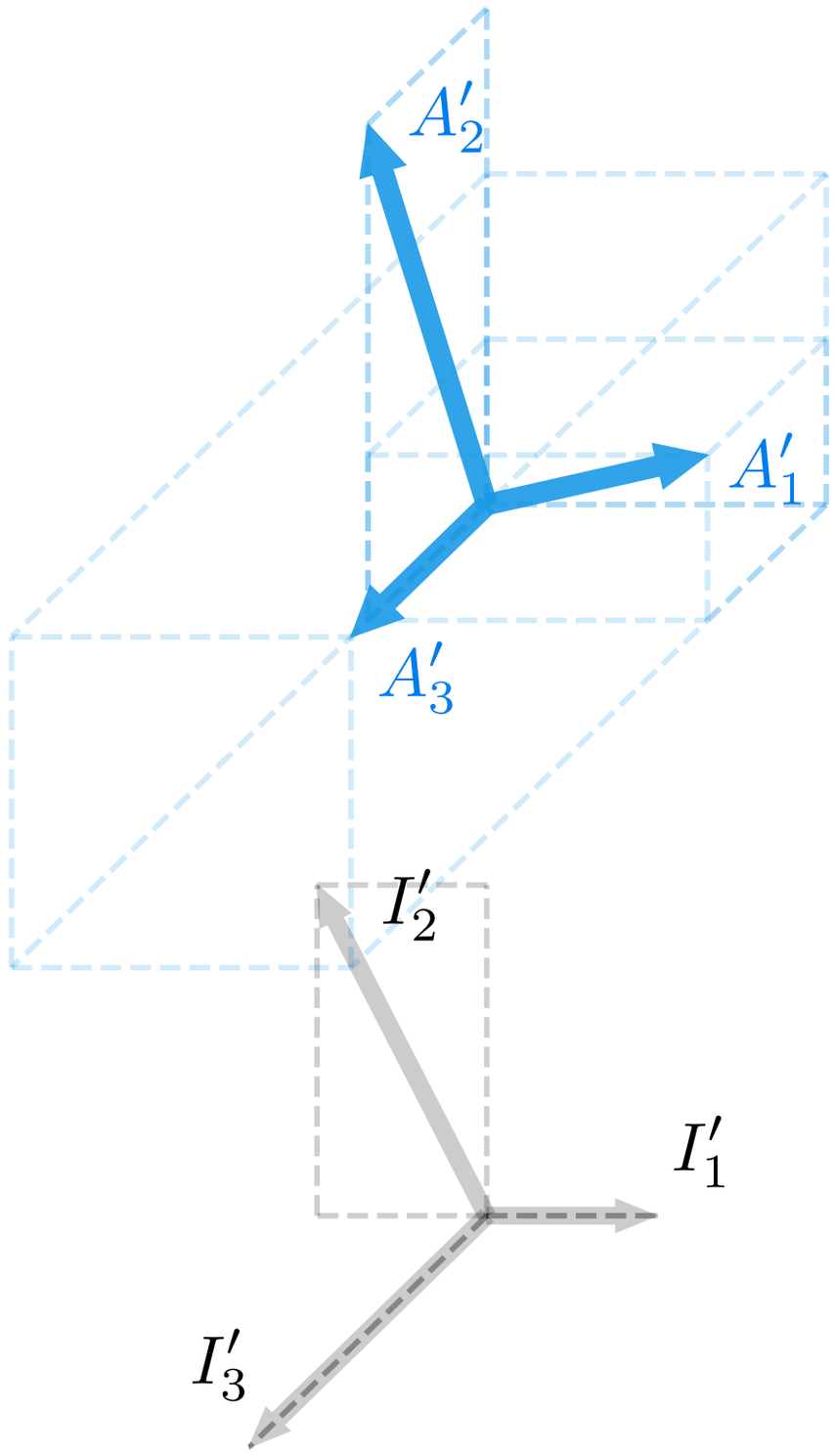


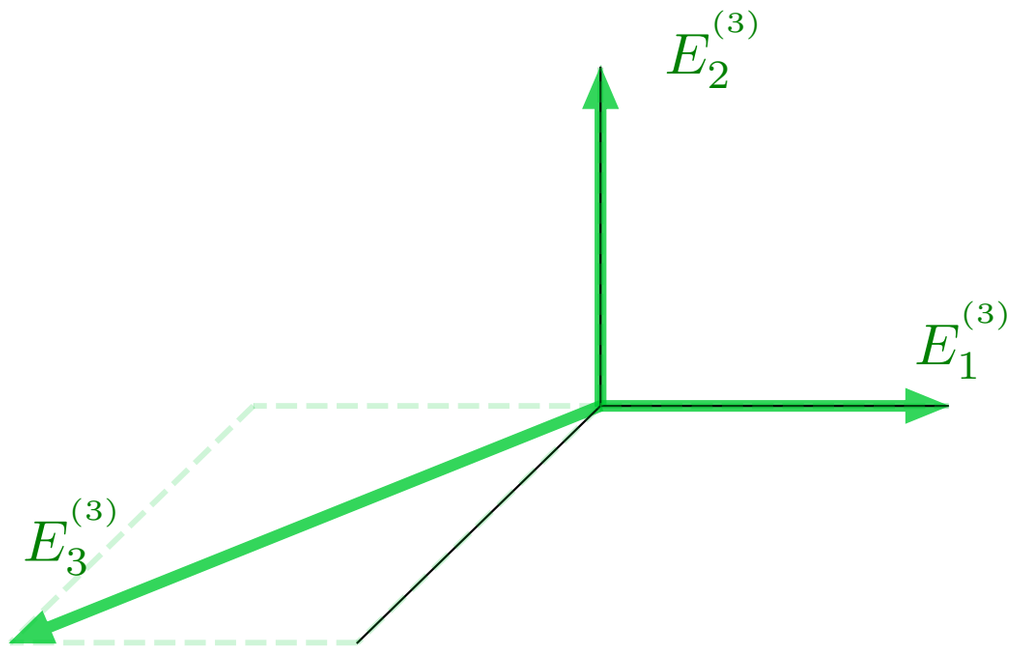

$$\underbrace{A\,E^{(1)}E^{(2)}}_{A'}$$

$$\underbrace{I\,E^{(1)}E^{(2)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 1 \\ 1/2 & 3/2 & 1 \\ 1/2 & 1/2 & 2 \\ \hline 1/2 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

## Column-reduction: operation 3…

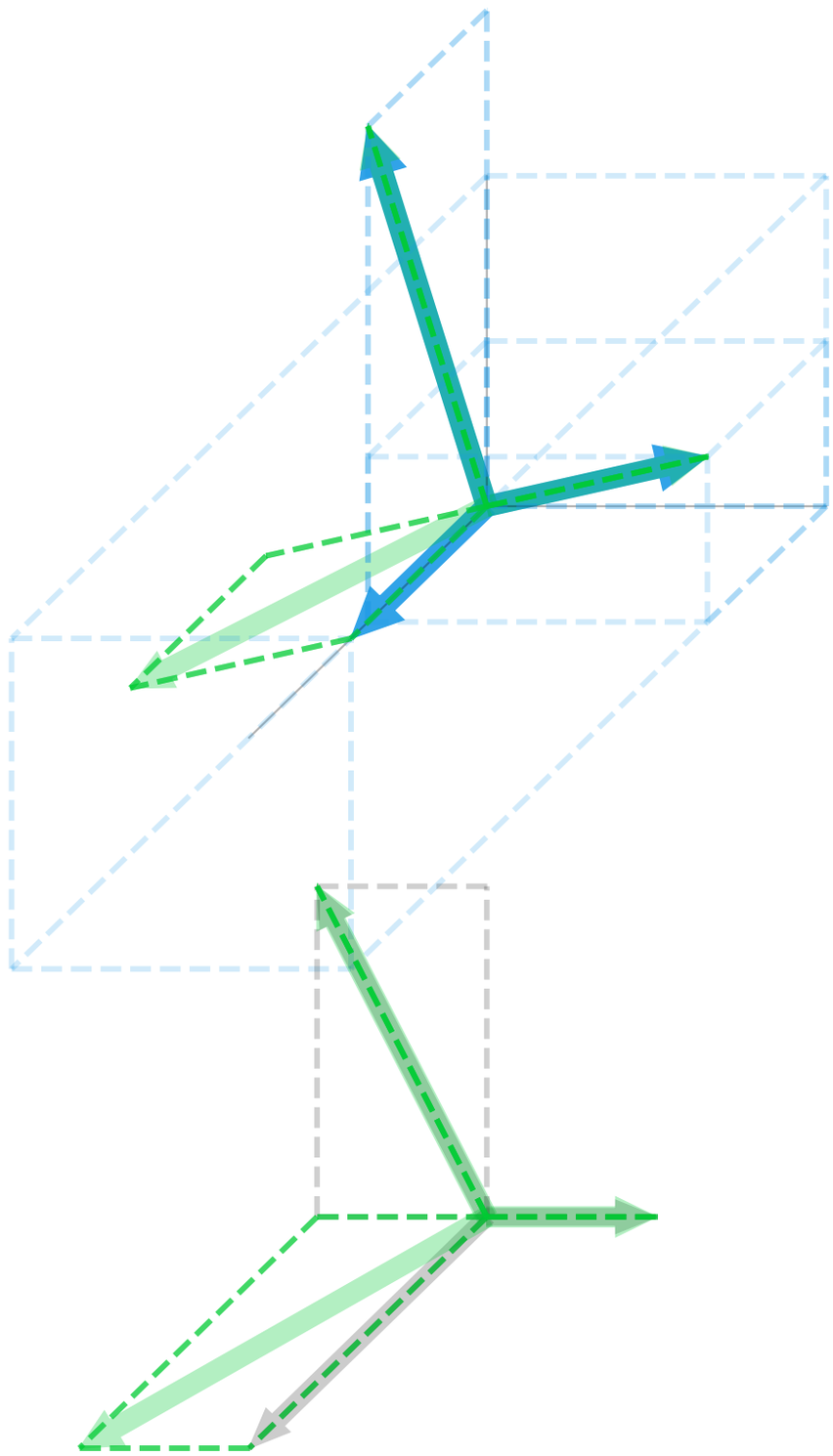


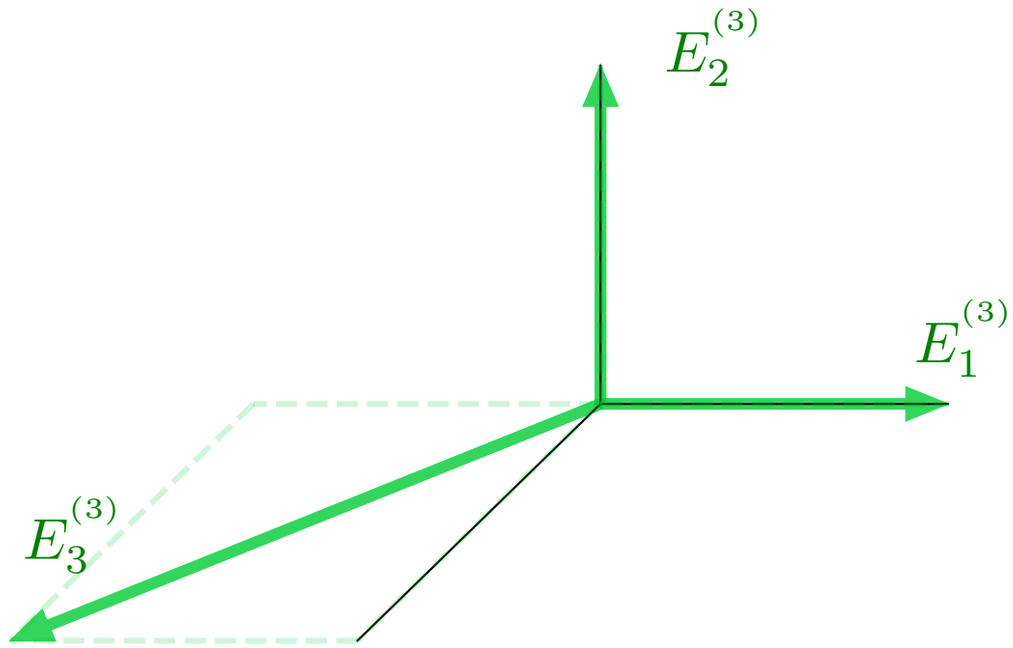




$$\underbrace{AE^{(1)}E^{(2)}}_{A'} \quad \underbrace{IE^{(1)}E^{(2)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 1 \\ 1/2 & 3/2 & 1 \\ 1/2 & 1/2 & 2 \\ \hline 1/2 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \times \underbrace{\left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{E^{(3)}}$$

## Column-reduction: operation 3…

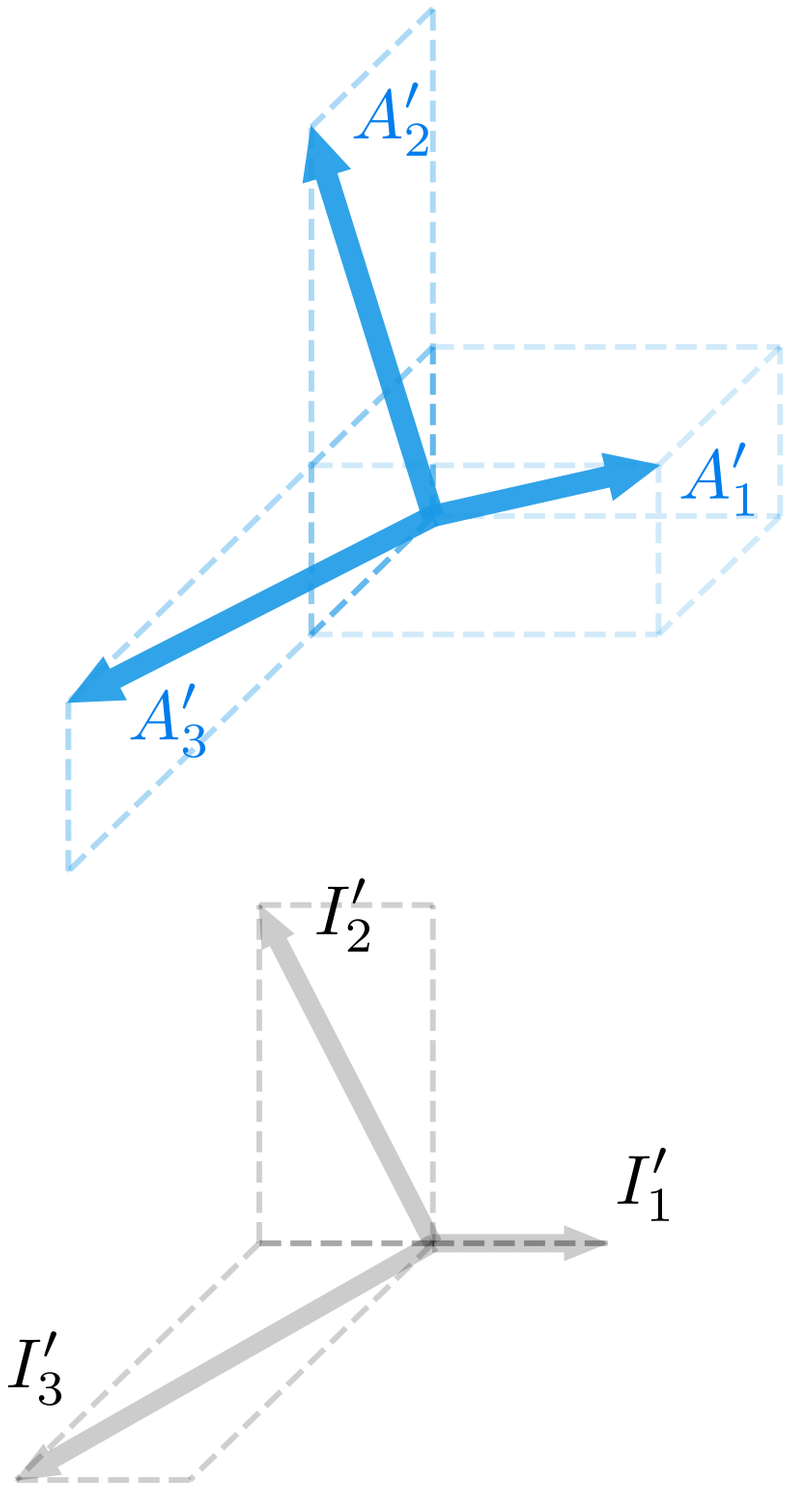





$$\underbrace{A\,E^{(1)}E^{(2)}}_{A'} \qquad \underbrace{I\,E^{(1)}E^{(2)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 1 \\ 1/2 & 3/2 & 1 \\ 1/2 & 1/2 & 2 \\ \hline 1/2 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{(3)}}$$

## Column-reduction: operation 3…



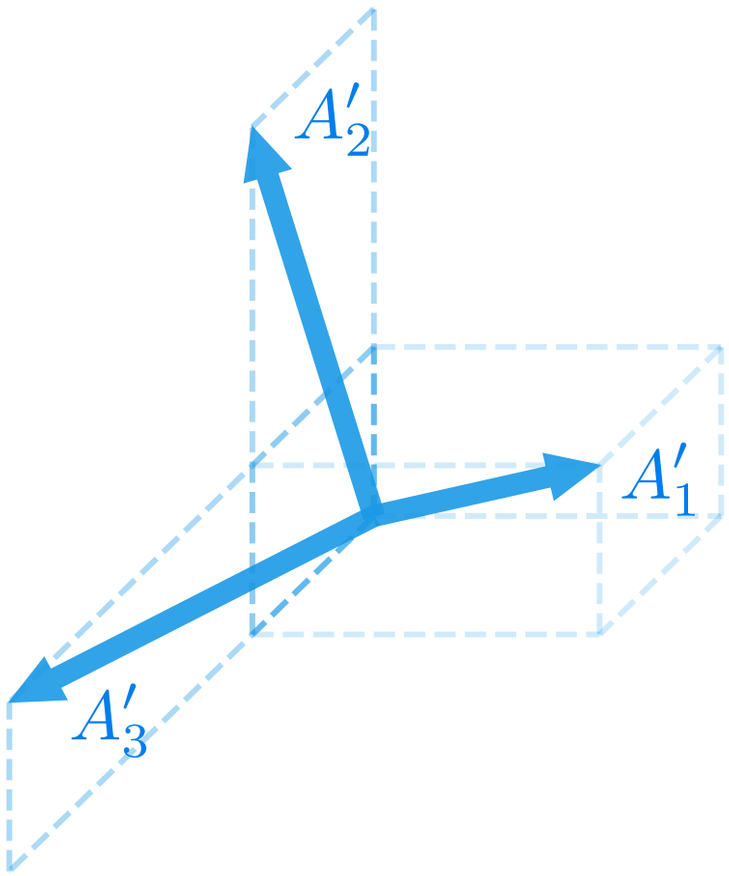

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}}_{A'}$$

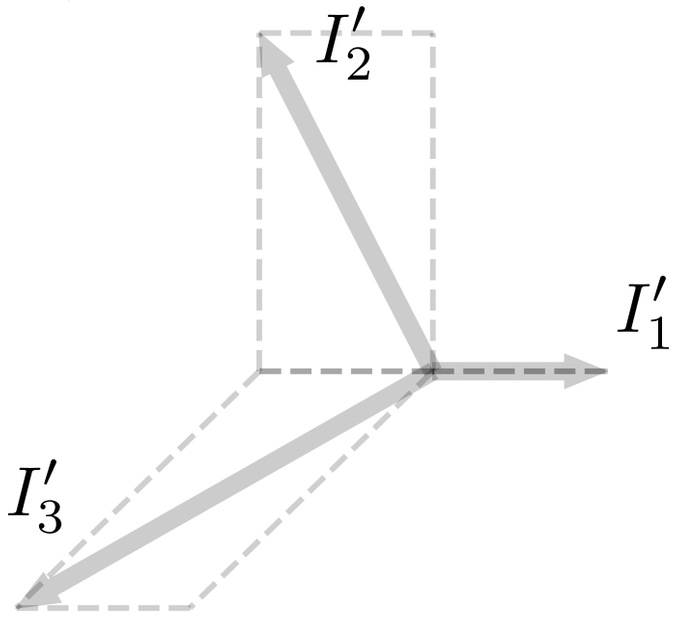
$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}}_{I'}$$

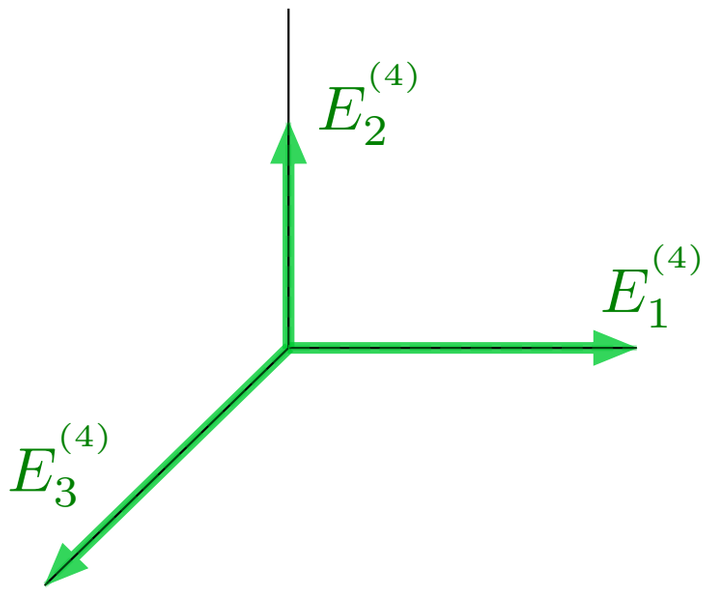
$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & 3/2 & 1/2 \\ 1/2 & 1/2 & 3/2 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1/2 & -1/2 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

## Column-reduction: operation 4…



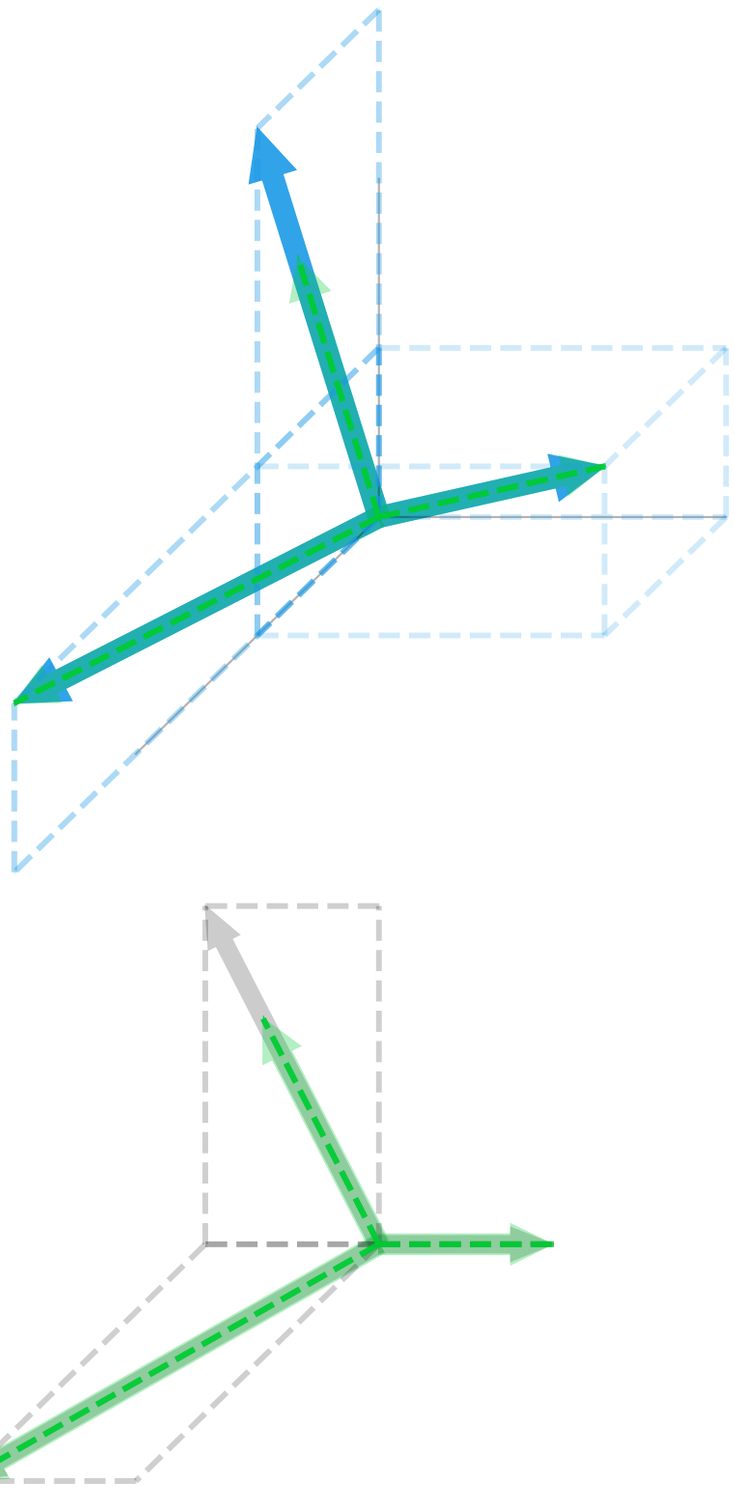



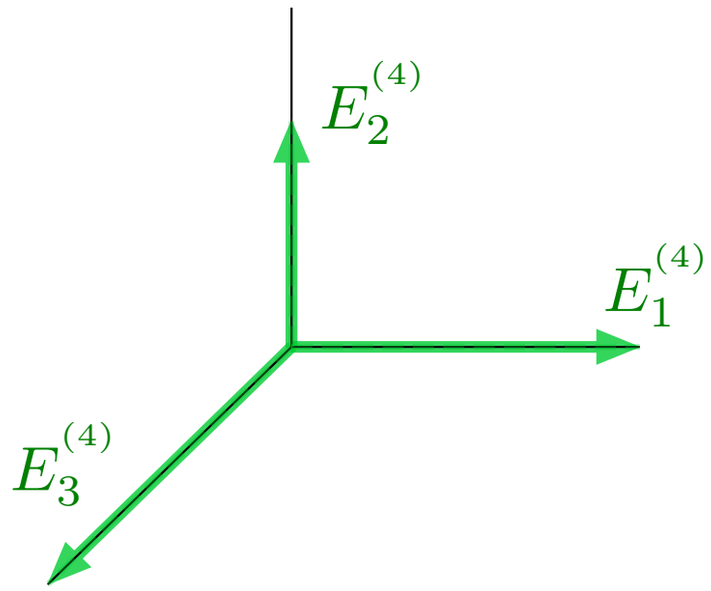

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}}_{A'} \qquad \underbrace{IE^{(1)}E^{(2)}E^{(3)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & 3/2 & 1/2 \\ 1/2 & 1/2 & 3/2 \\ \hline 1/2 & -1/2 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \times \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$E^{(4)}$

## Column-reduction: operation 4…



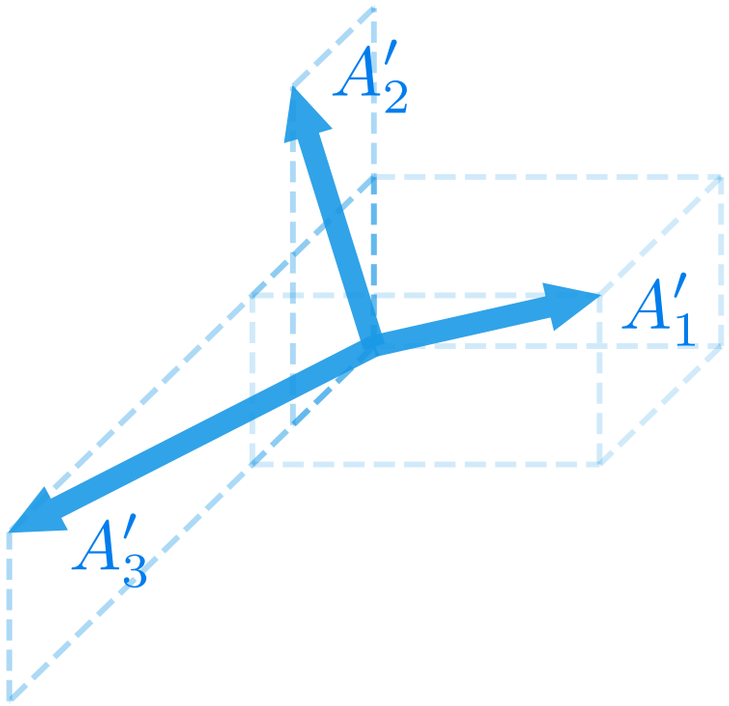


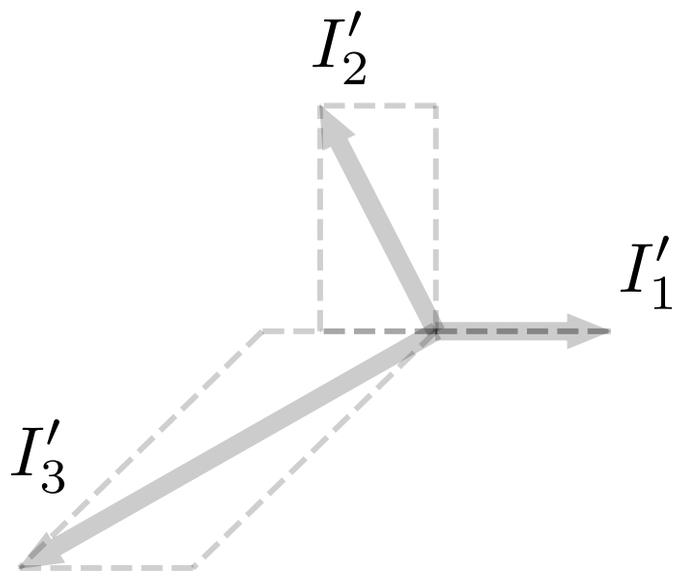

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}}_{A'} \qquad \underbrace{IE^{(1)}E^{(2)}E^{(3)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & 3/2 & 1/2 \\ 1/2 & 1/2 & 3/2 \\ \hline 1/2 & -1/2 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$E^{(4)}$

## Column-reduction: operation 4…



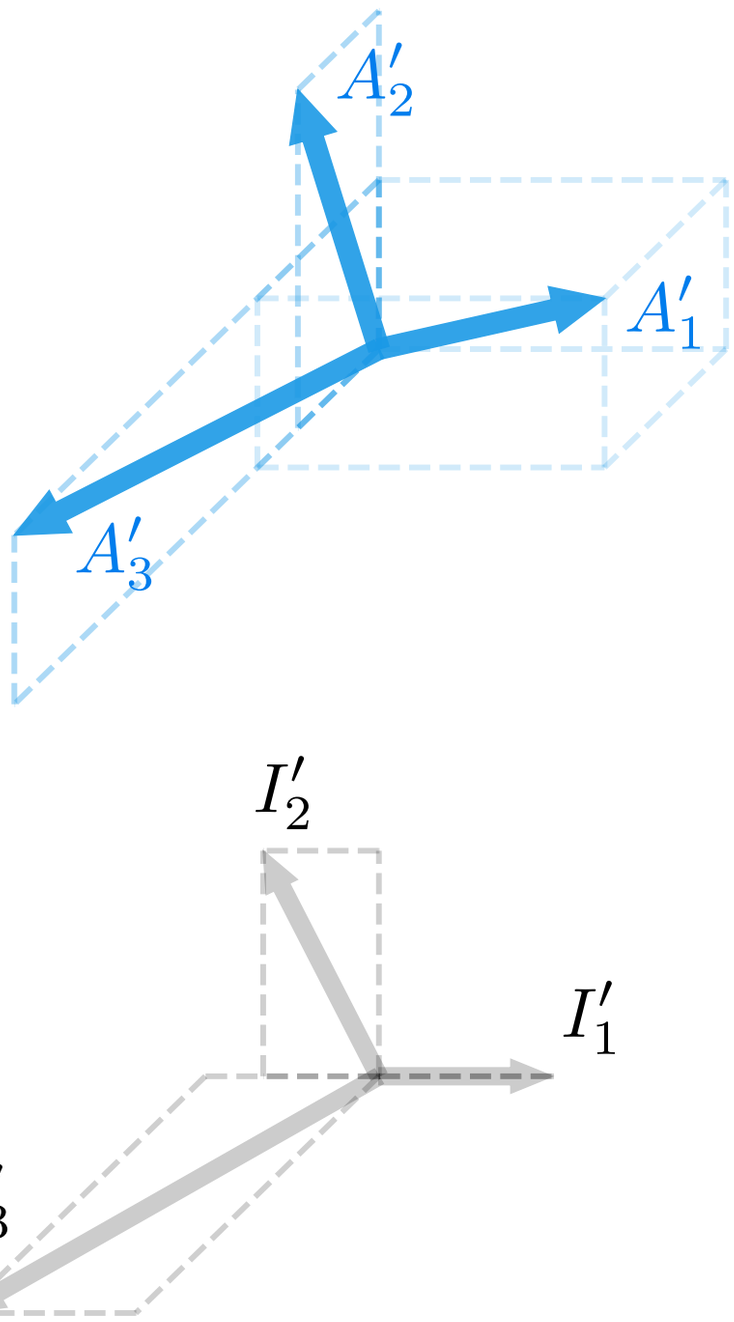



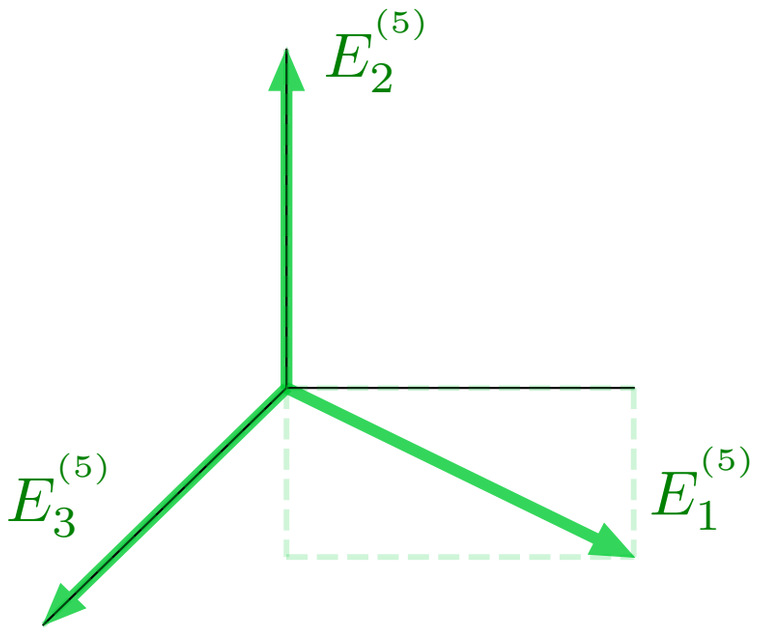

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{A'}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/3 & 3/2 \\ \hline 1/2 & -1/3 & -1/2 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{I'}$$

## Column-reduction: operation 5…



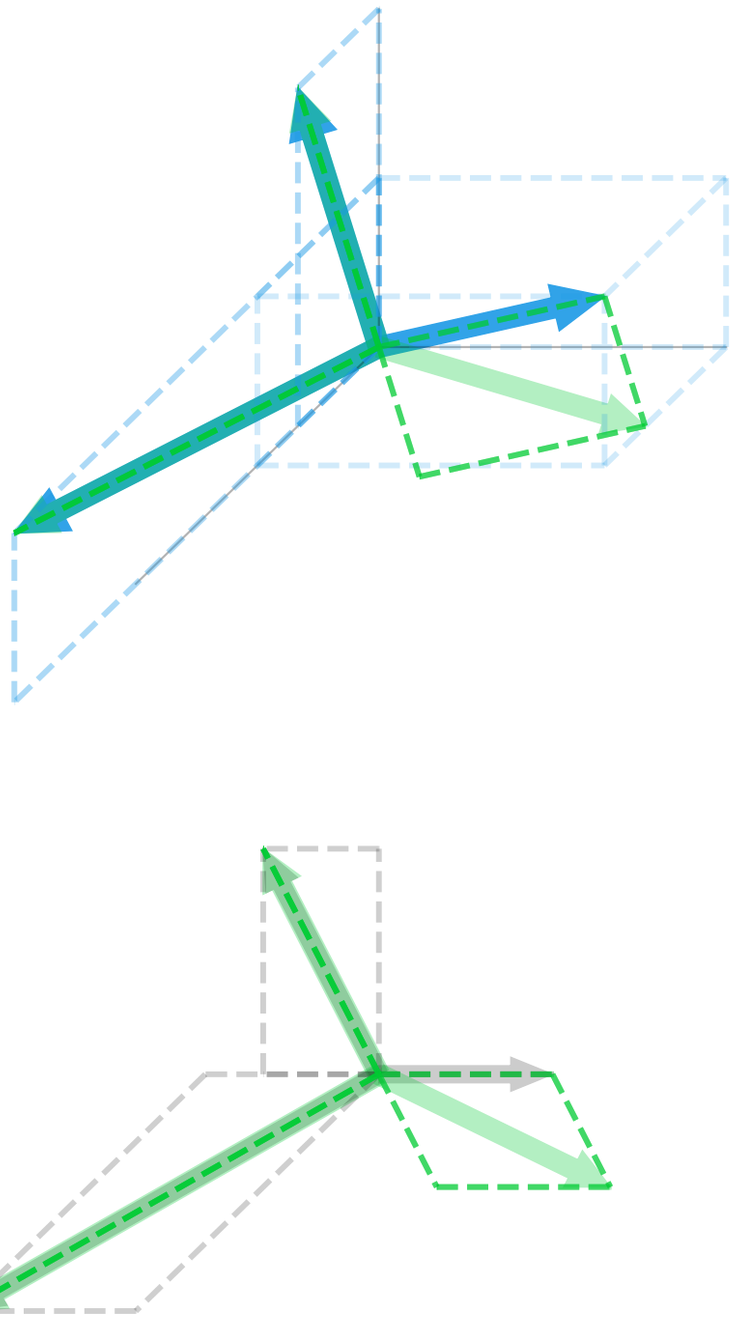



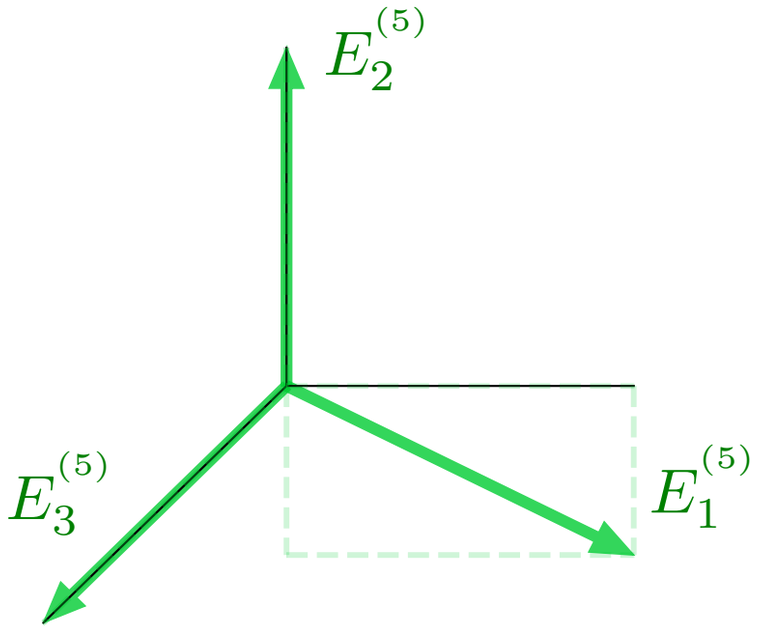

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{A'} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/3 & 3/2 \end{bmatrix}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{I'} \quad \begin{bmatrix} 1/2 & -1/3 & -1/2 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{(5)}}$$

## Column-reduction: operation 5…

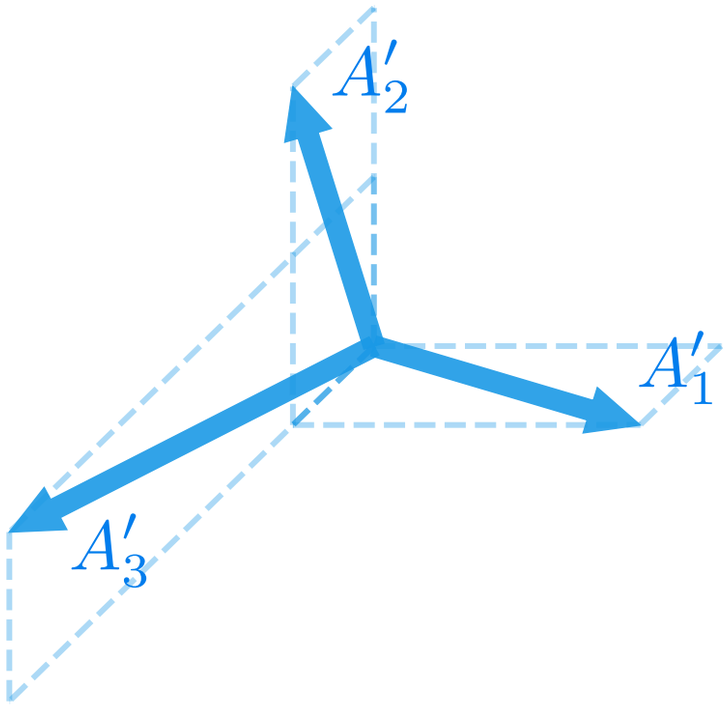


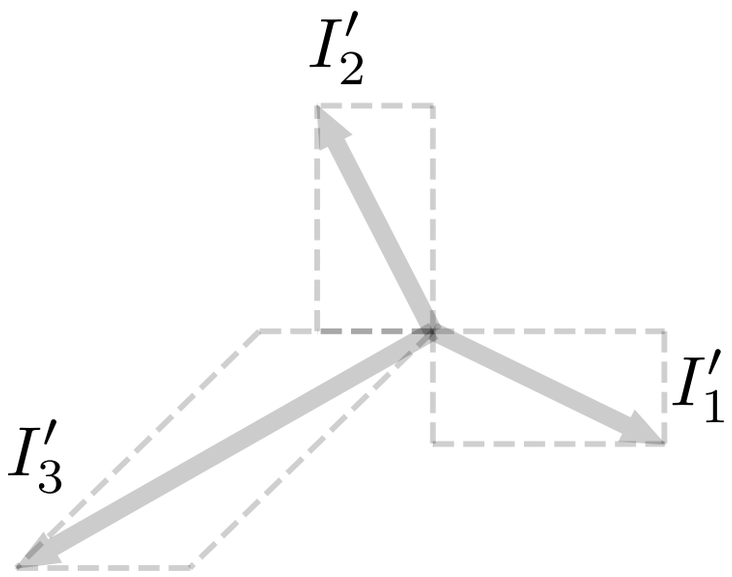



$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{A'} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/3 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E^{(5)}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{I'} \begin{bmatrix} 1/2 & -1/3 & -1/2 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Column-reduction: operation 5…**



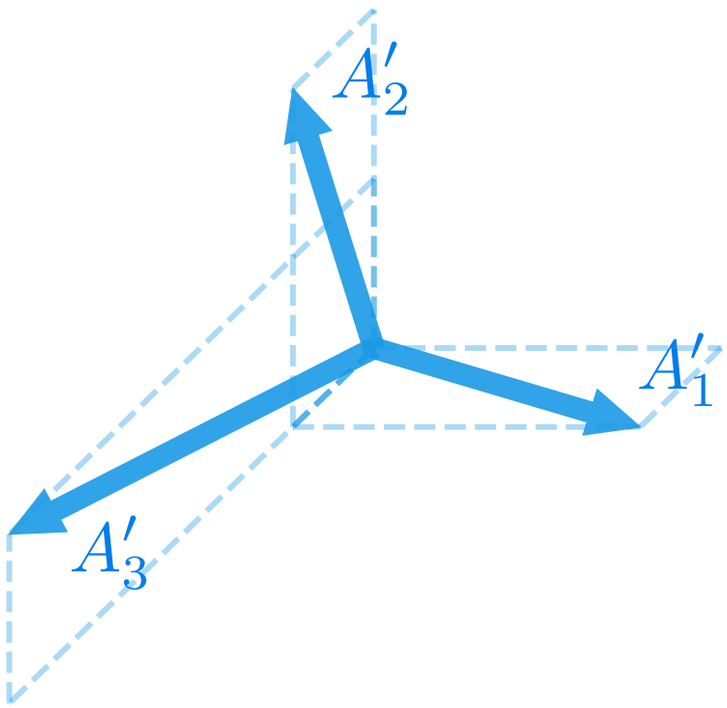



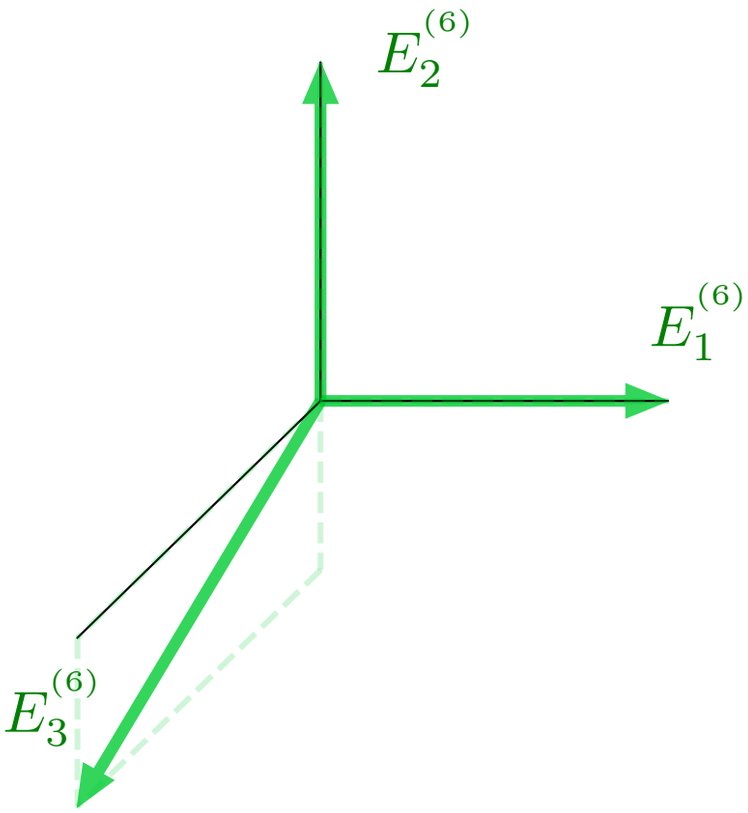

$$\underbrace{A E^{(1)} E^{(2)} E^{(3)} E^{(4)} E^{(5)}}_{A'}$$

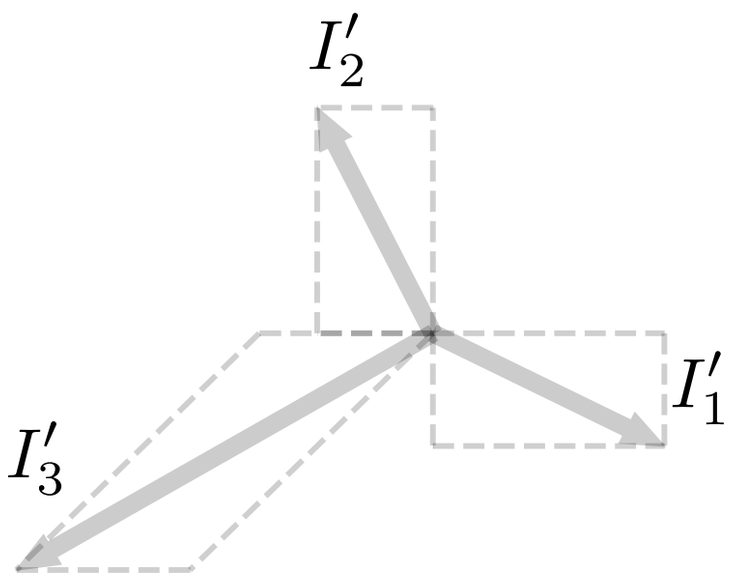
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 1/3 & 1/3 & 3/2 \\ \hline 2/3 & -1/3 & -1/2 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{I E^{(1)} E^{(2)} E^{(3)} E^{(4)} E^{(5)}}_{I'}$$

## Column-reduction: operation 6…



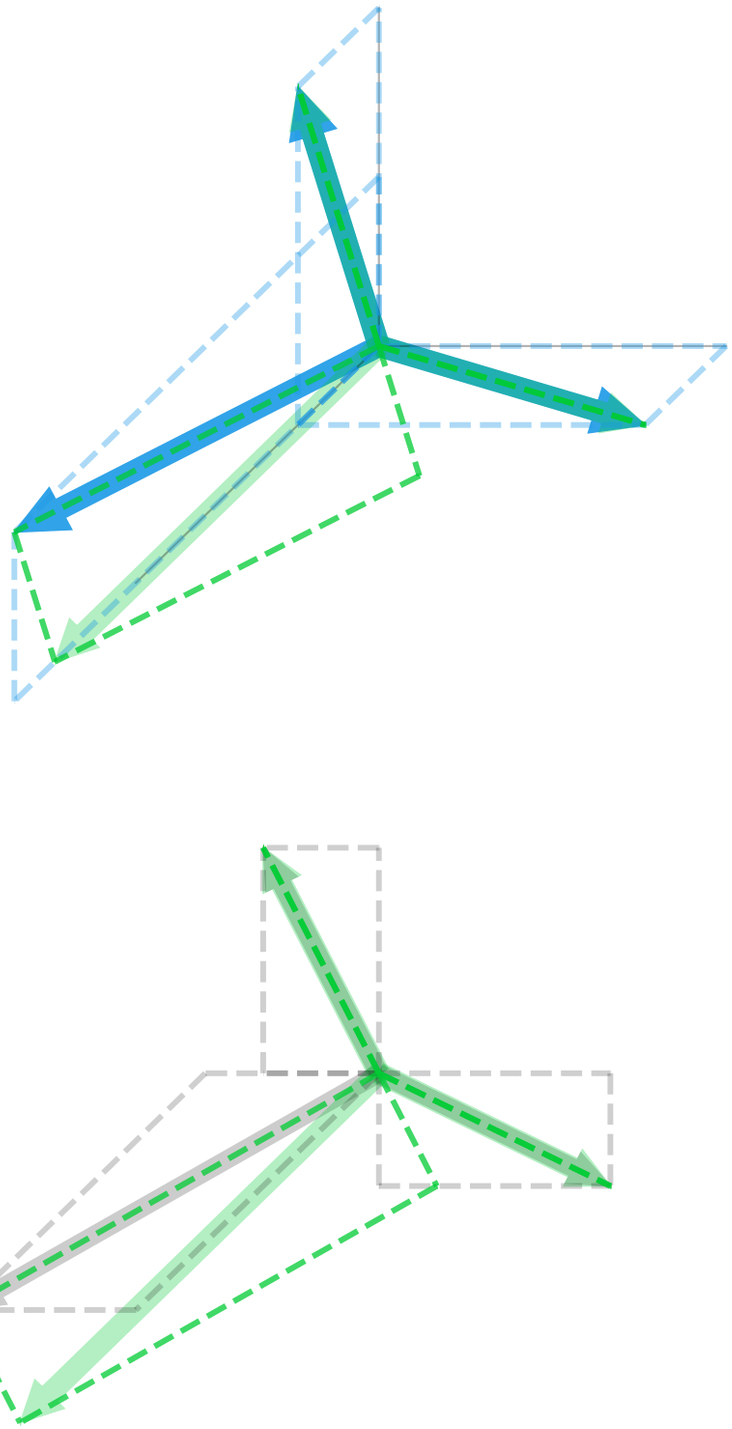



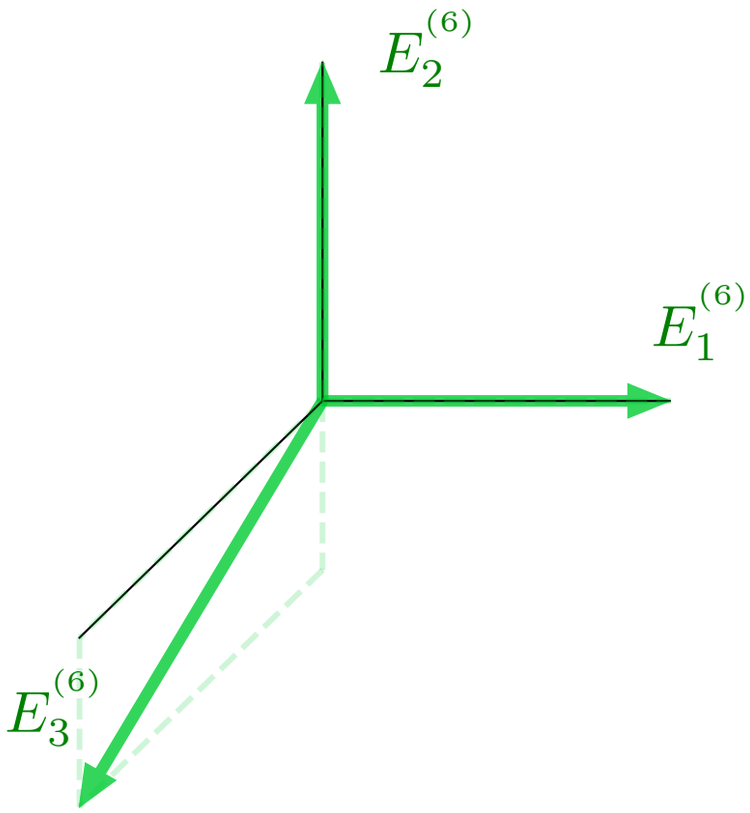

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}}_{A'}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 1/3 & 1/3 & 3/2 \\ \hline 2/3 & -1/3 & -1/2 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}}_{I'} \qquad E^{(6)}$$

## Column-reduction: operation 6…






$$AE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)} = A'$$

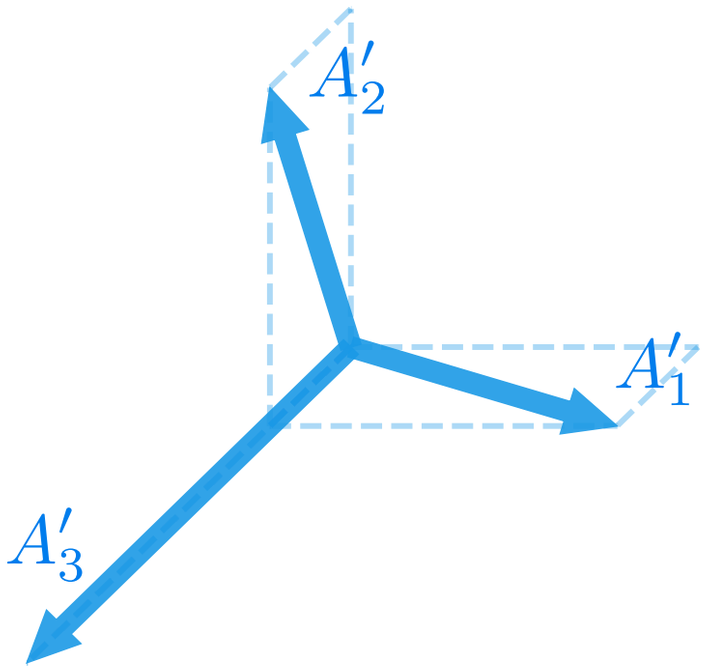
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 1/3 & 1/3 & 3/2 \end{bmatrix}$$

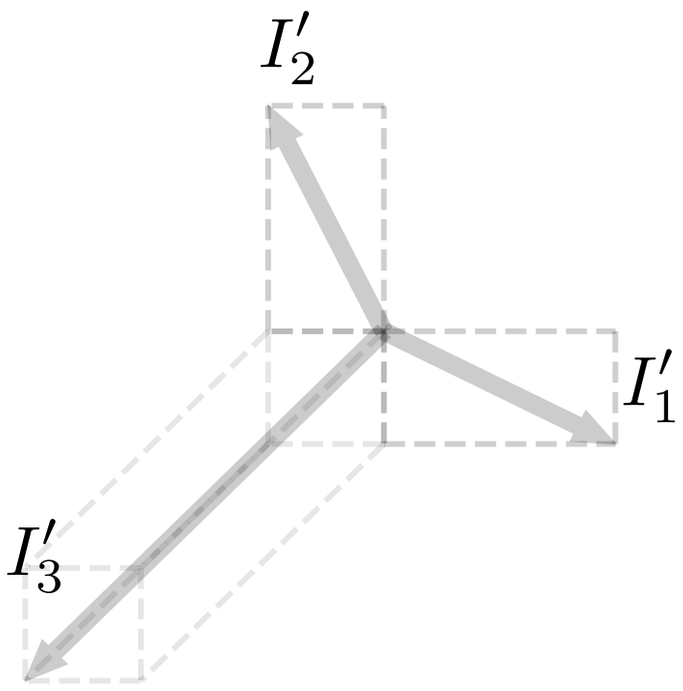
$$IE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)} = I'$$

$$\begin{bmatrix} 2/3 & -1/3 & -1/2 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} = E^{(6)}$$

**Column-reduction: operation 6…**

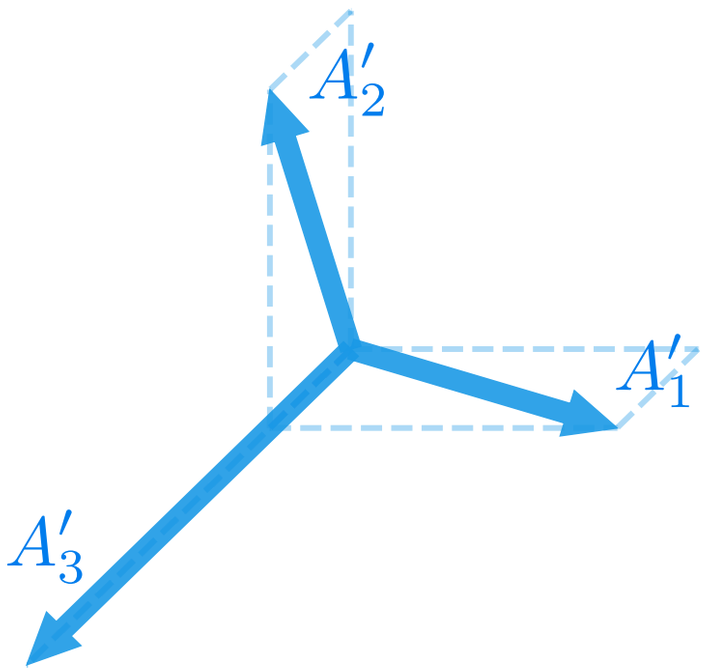


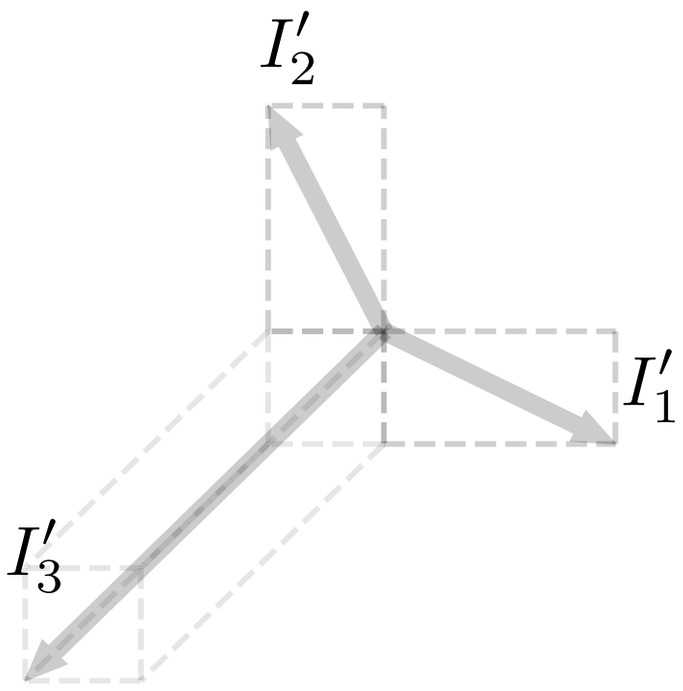



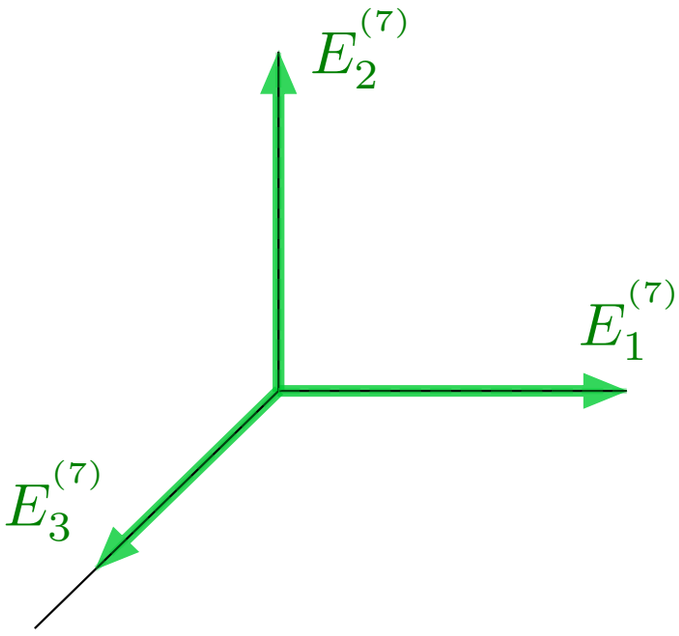

$$\underbrace{A E^{(1)} E^{(2)} E^{(3)} E^{(4)} E^{(5)} E^{(6)}}_{A'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 4/3 \end{bmatrix}$$

$$\underbrace{I E^{(1)} E^{(2)} E^{(3)} E^{(4)} E^{(5)} E^{(6)}}_{I'} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

## Column-reduction: operation 7…



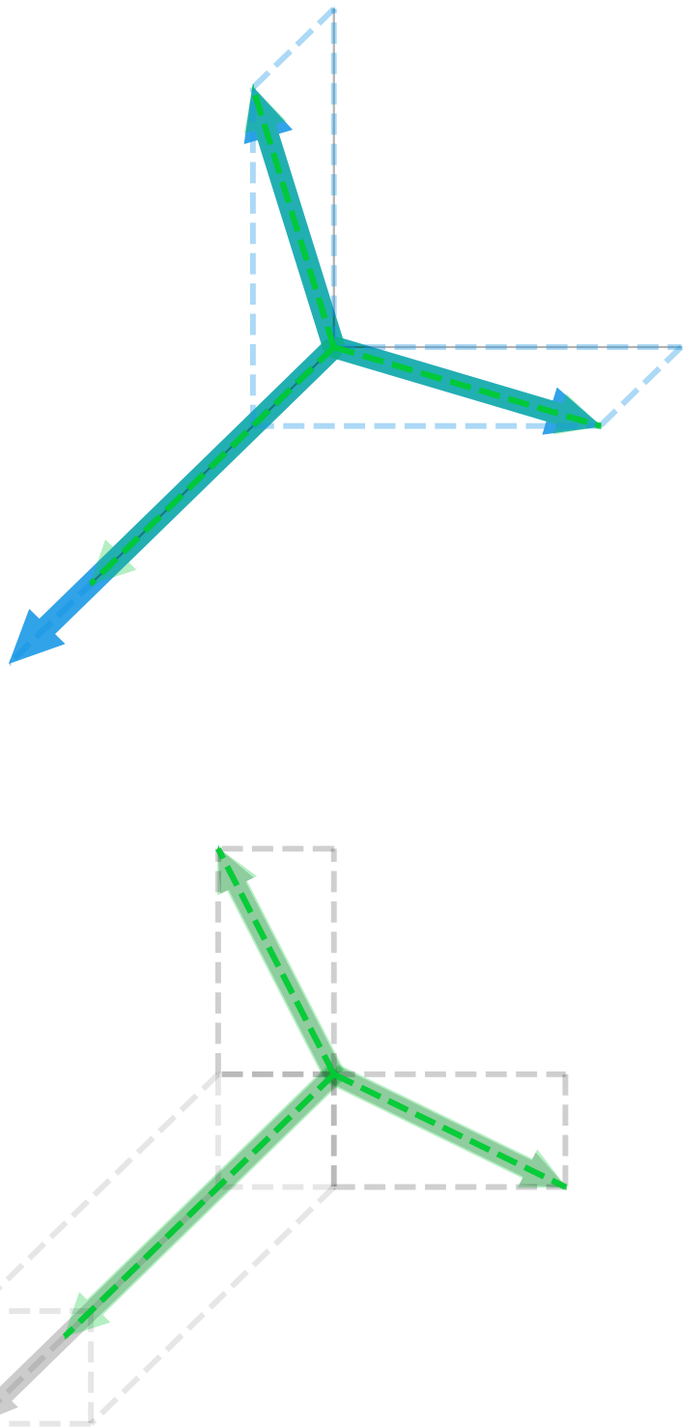



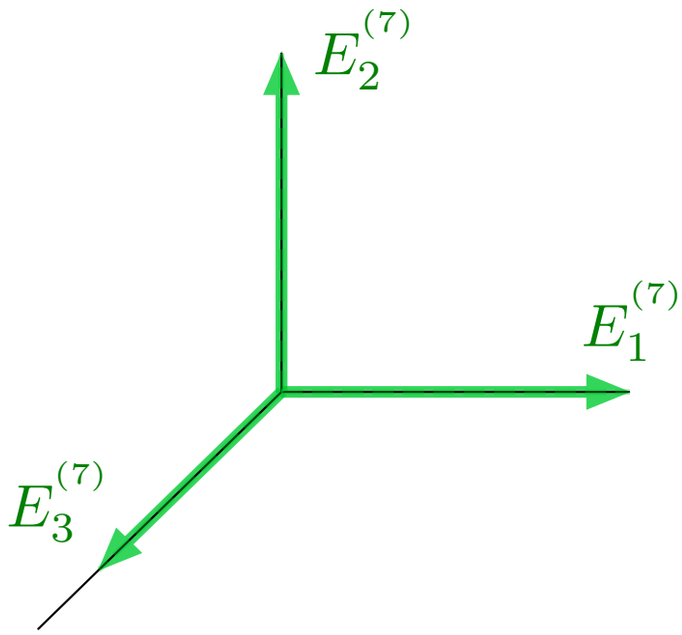

$$A E^{(1)} E^{(2)} E^{(3)} E^{(4)} E^{(5)} E^{(6)} = A'$$

$$I E^{(1)} E^{(2)} E^{(3)} E^{(4)} E^{(5)} E^{(6)} = I'$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 4/3 \\ \hline 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{bmatrix} = E^{(7)}$$

## Column-reduction: operation 7…



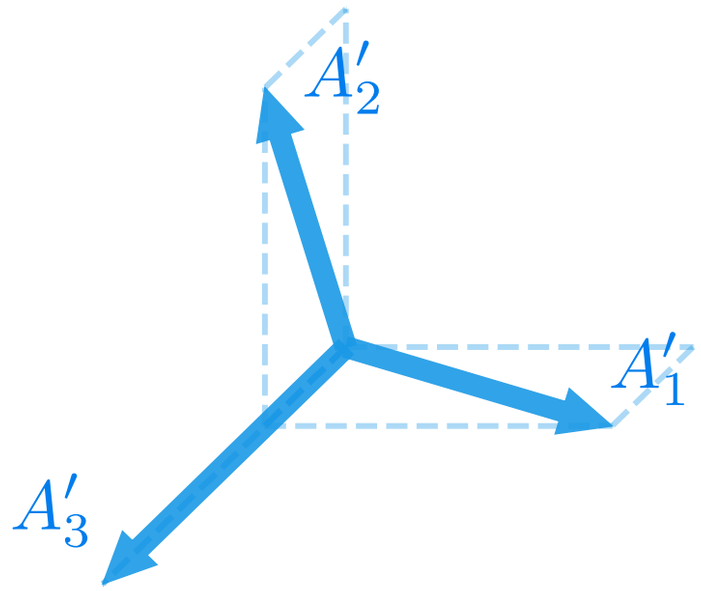


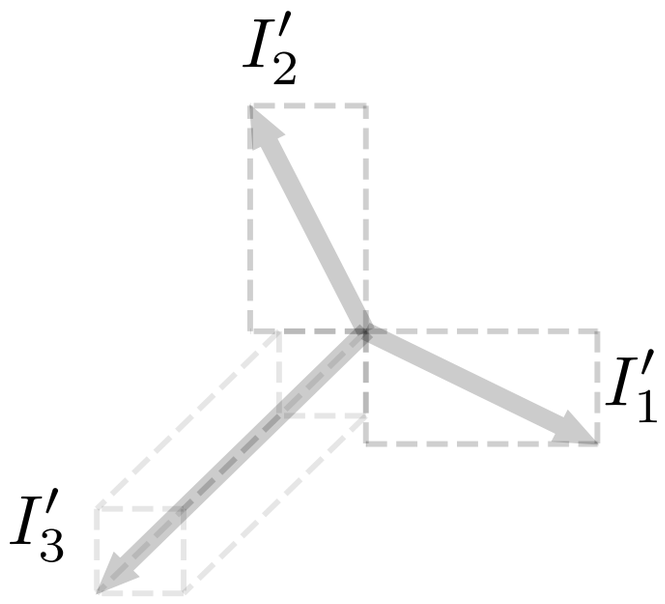

$$AE^{(1)}E^{(2)}E^{(3)}\underbrace{E^{(4)}E^{(5)}E^{(6)}}_{A'} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 4/3 \end{array}\right]$$

$$IE^{(1)}E^{(2)}E^{(3)}\underbrace{E^{(4)}E^{(5)}E^{(6)}}_{I'} = \left[\begin{array}{ccc} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ 0 & 0 & 1 \end{array}\right]$$

$$E^{(7)} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array}\right]$$

## Column-reduction: operation 7…

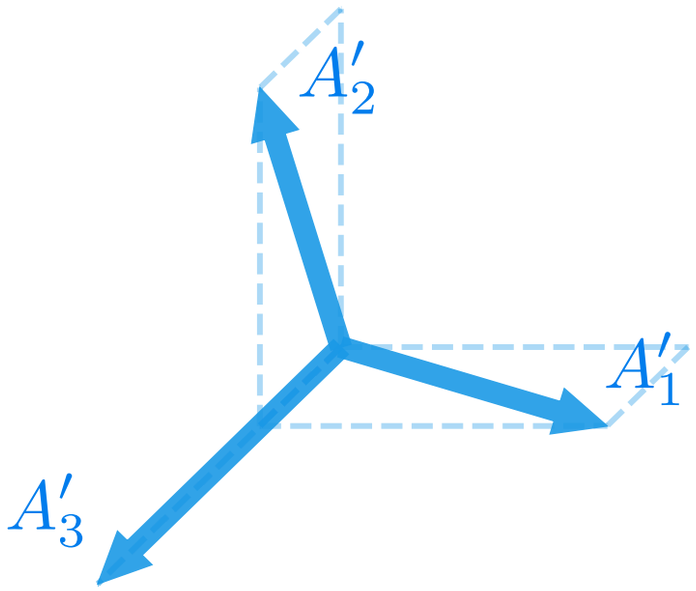


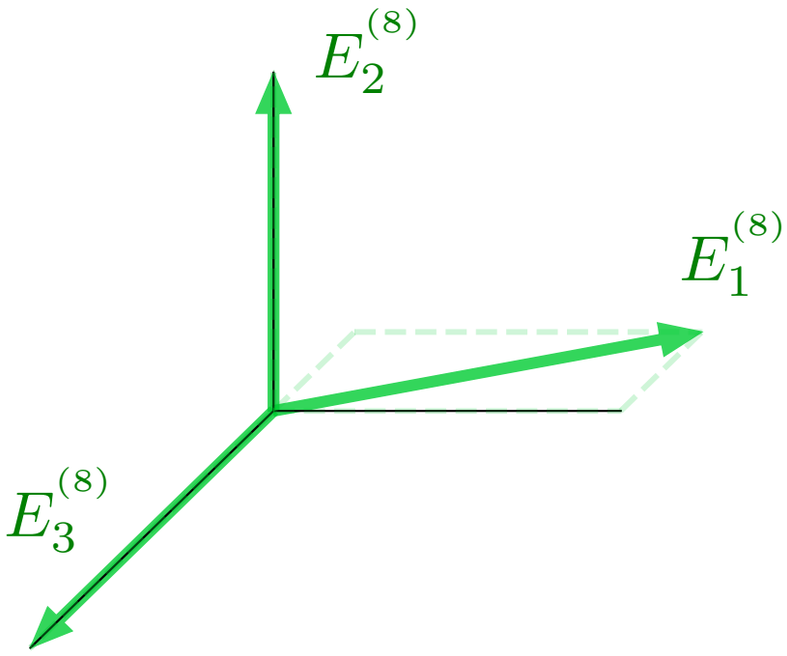



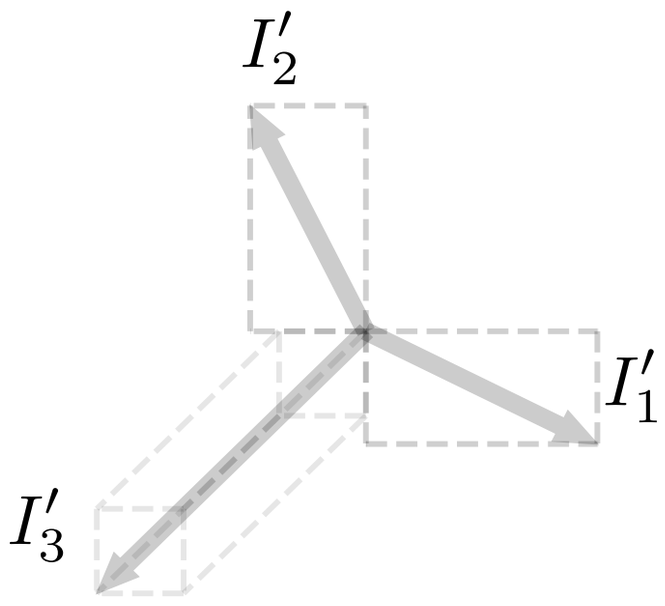

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{A'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1 \end{bmatrix}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{I'} = \begin{bmatrix} 2/3 & -1/3 & -1/4 \\ -1/3 & 2/3 & -1/4 \\ 0 & 0 & 3/4 \end{bmatrix}$$

## Column-reduction: operation 8…




$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{A'}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1 \end{bmatrix}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{I'}$$

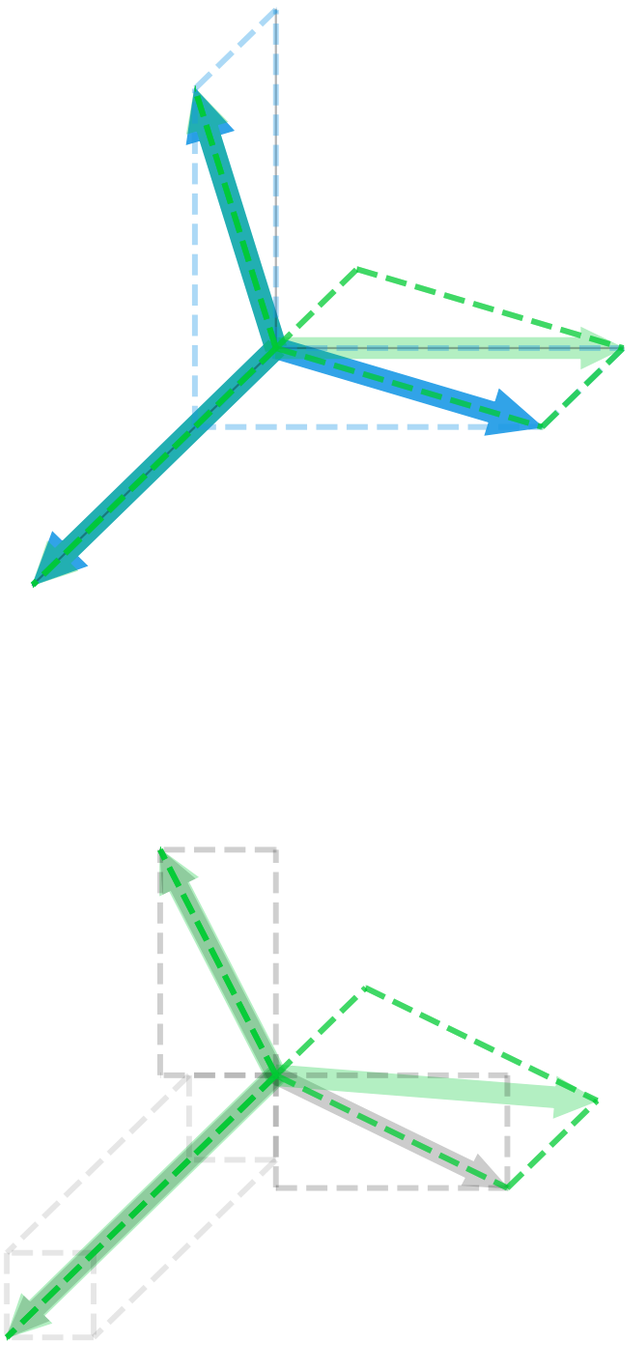
$$\begin{bmatrix} 2/3 & -1/3 & -1/4 \\ -1/3 & 2/3 & -1/4 \\ 0 & 0 & 3/4 \end{bmatrix}$$

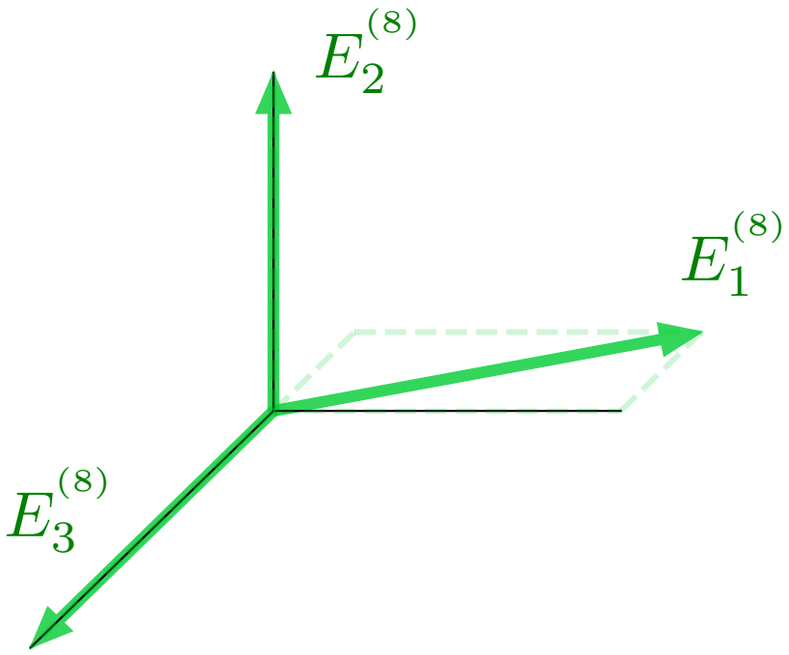
$$\times$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

$E^{(8)}$

## Column-reduction: operation 8…






$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{A'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1 \end{array}\right]$$

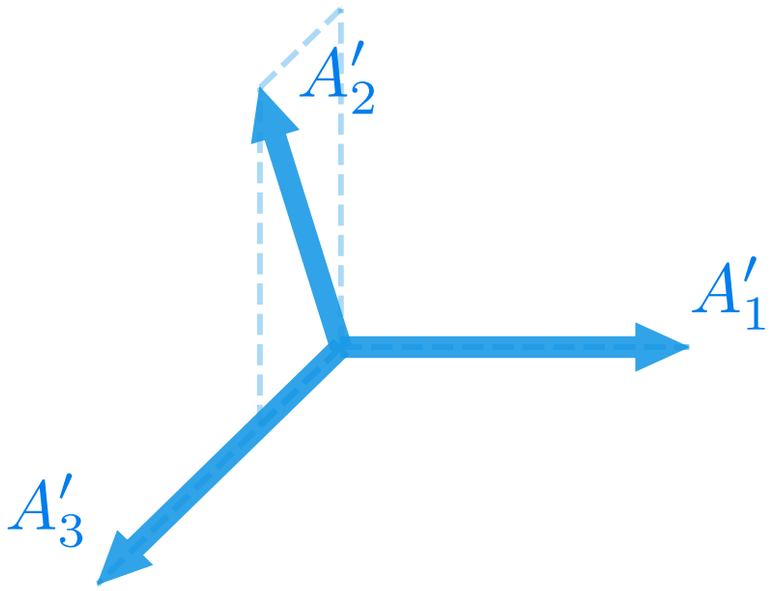
$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{array}\right]$$

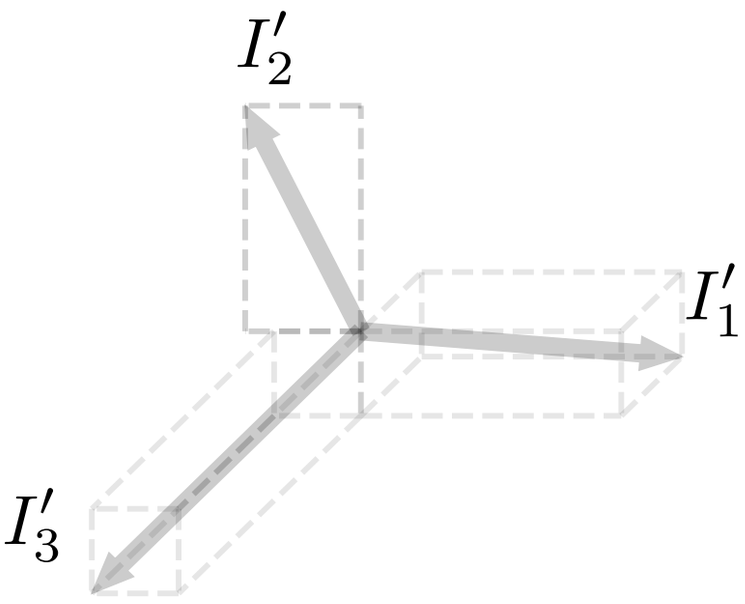
$$E^{(8)}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{I'}$$

$$\left[\begin{array}{ccc} 2/3 & -1/3 & -1/4 \\ -1/3 & 2/3 & -1/4 \\ 0 & 0 & 3/4 \end{array}\right]$$

**Column-reduction: operation 8…**



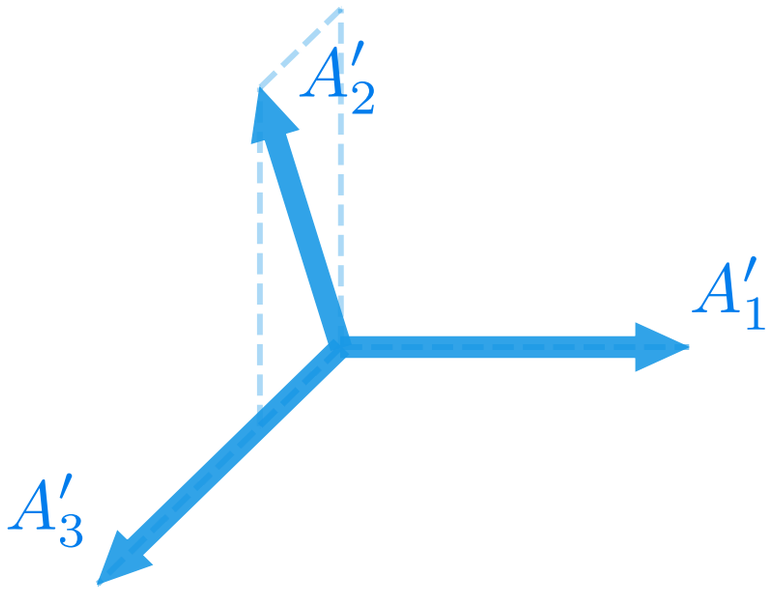



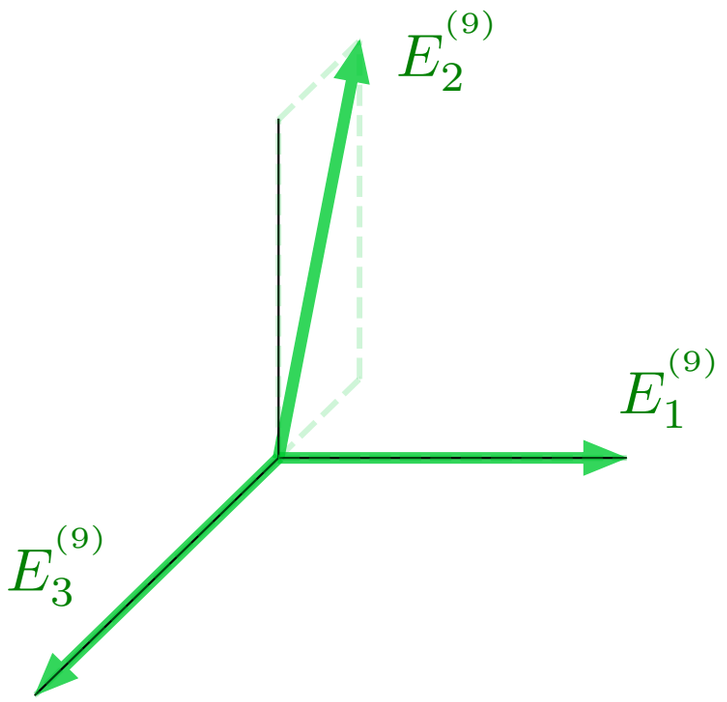

$$A\underbrace{E^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}E^{(8)}}_{A'} = A'$$

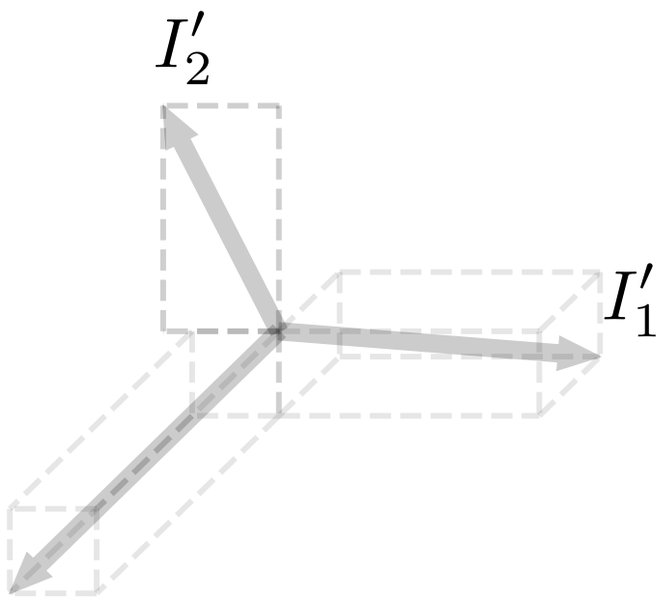
$$I\underbrace{E^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}E^{(8)}}_{I'} = I'$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \\ \hline 3/4 & -1/3 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & 0 & 3/4 \end{array}\right]$$

## Column-reduction: operation 9…



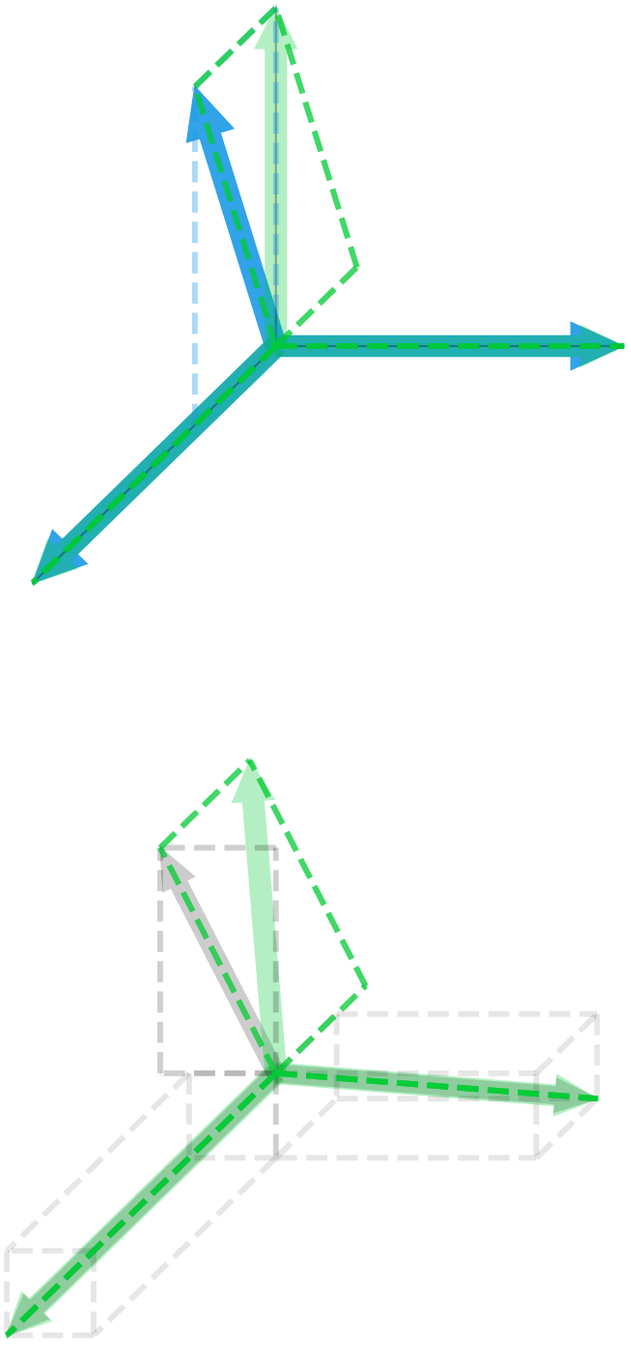



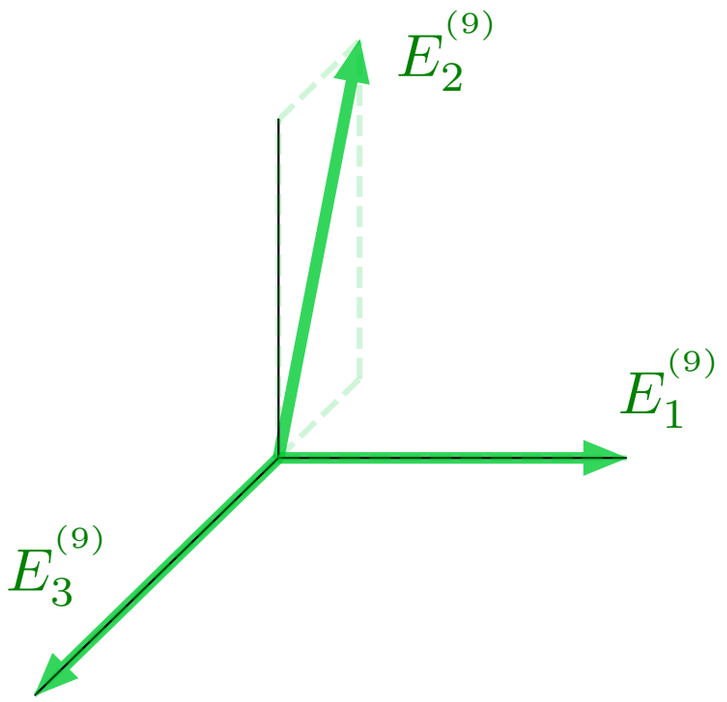

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}E^{(8)}}_{A'}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}E^{(8)}}_{I'}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \\ \hline 3/4 & -1/3 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & 0 & 3/4 \end{bmatrix} \times \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix}}_{E^{(9)}}$$

## Column-reduction: operation 9…





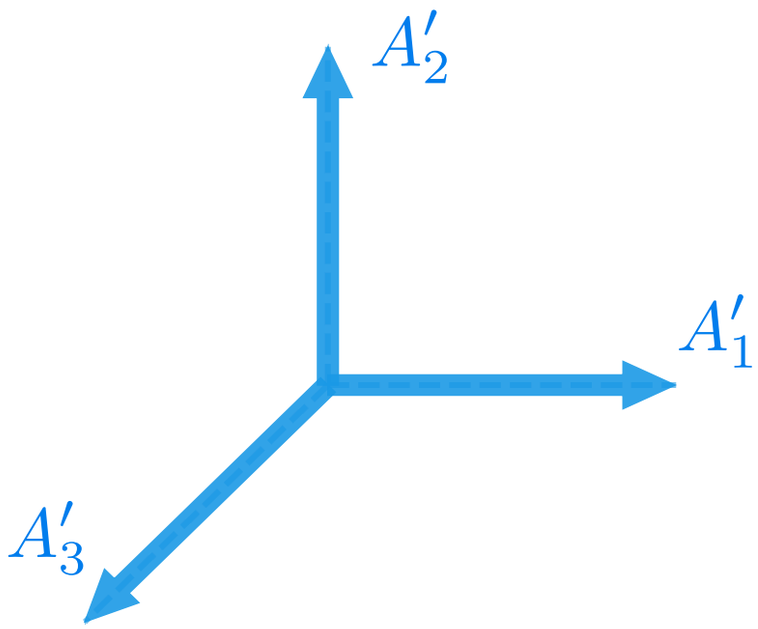

$$A E^{(1)} E^{(2)} E^{(3)} E^{(4)} E^{(5)} E^{(6)} E^{(7)} E^{(8)} = A'$$

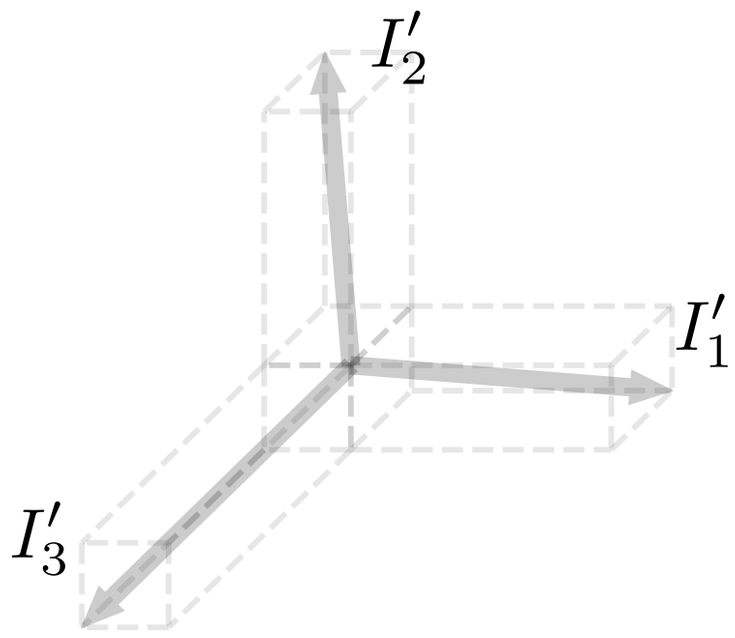
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/3 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & 0 & 3/4 \end{bmatrix}$$

$E^{(9)}$

$$I E^{(1)} E^{(2)} E^{(3)} E^{(4)} E^{(5)} E^{(6)} E^{(7)} E^{(8)} = I'$$

**Column-reduction: operation 9…**



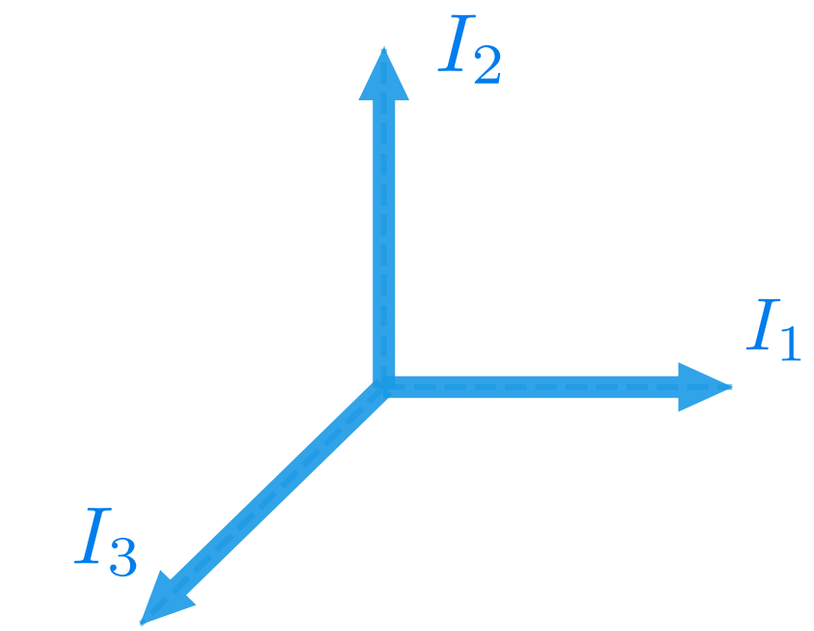



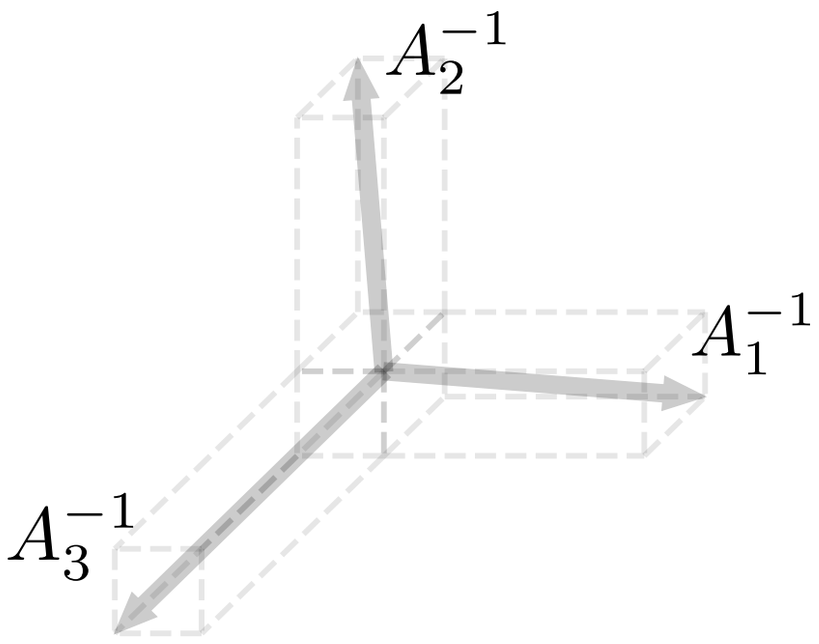

$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}E^{(8)}E^{(9)}}_{A'}$$

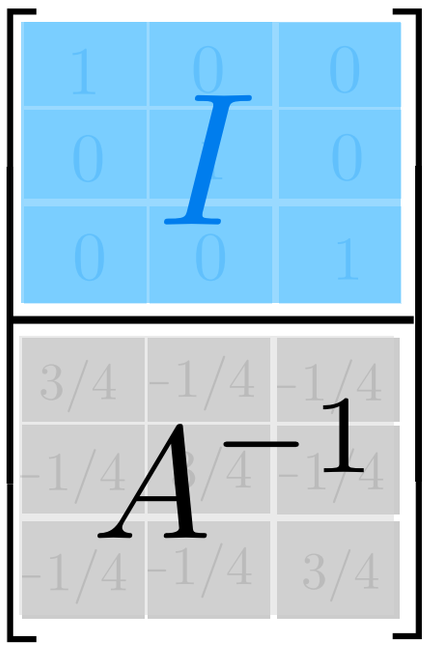
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}E^{(5)}E^{(6)}E^{(7)}E^{(8)}E^{(9)}}_{I'}$$

**final system…**




**final system…**