External-Cost Continuous-Type Wardrop Equilibria in Routing Games

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***Abstract*—We propose a bi-criterion Wardrop equilibrium which we call an *external cost continuous type Wardrop equilibrium* for nonatomic routing games where agents incur some type dependent cost to have access to different sets of available routes. Rather than being anonymous, the population is described by a distribution over the type parameter (that can be supported on any compact interval of the real line). At equilibria, no member of the population can improve their type dependent cost either by switching sets of routes or by switching routes within their chosen set. From this equilibrium condition, we derive how the population mass will divide up among the various routing options. We then formulate a potential function optimization program for finding the equilibrium mass distribution. This work revisits the cost-vs-time equilibria of Leurent [1] and Marcotte [2] while specifically allowing the type parameter to be positive or negative. Applications include modeling the value of information in routing, the effect of privacy concern on congestion, and how commuters make tradeoffs between different forms of transportation.**

I. INTRODUCTION

Nonatomic routing games [3]–[5] are a standard of transportation modeling, providing a way to capture the game theoretic decisions each driver faces while still providing a manageable way to compute the overall population behavior. While the classical routing game models just considered travel time, there has been interest in extending these models to incorporate other factors that agents consider along with travel time.

Multi-criteria routing equilibria were considered early on by Dial [6] extending the work of Quandt [7] and Schneider [8]. These early formulations ignored congestion effects which were added by Dafermos [9] in the deterministic case. Since then, multi-criteria models have divided up the population into either finite classes of users each with distinct preferences or have represented the population by some continuous distribution (infinite classes of users). Nagurney and Dong [10], Li and Chen [11], and Yang [12] all consider finite classes of users focusing mainly on variational inequality approaches.

Leurent [1], [13] was one of the earliest to represent the population's value for time vs. money as a continuous distribution over some positive interval and formulate an optimization problem for finding the equilibrium. Dial [14], [15] considered a more general scenario where both travel time and monetary cost can depend on congestion and framed the problem as a variational inequality. Marcotte, et al. [2], [16], [17] generalized the work of Leurent and Dial presenting a general variational inequality formulation. Much of the strength of these formulations has been methods to turn infinite dimensional variational inequalities into finite dimensional problems [2], [18]. Our work follows in the mode of these models. Much of the focus since then has been using these models to devise tolling schemes [19]–[23].

In our formulation, population members consider some external factor along with travel time. We assume an arbitrary distribution over a parameter θ that represents how much different members of the population value this external factor. We model the whole set of routes as being divided into subsets $\{\mathcal{R}_o\}_{o\in\mathcal{O}}$ that each come with an external cost α_o. Drivers in the population select the subset of routes they want to use and and then their individual route within that subset. Their total cost is their travel time plus $\alpha_o\theta$ for their particular subset o and their personal tradeoff parameter θ. We present the appropriate equilibrium definition for agents who consider this cost; and then from this definition, we derive how the population will divide itself up among the various transportation options. We also give a potential function and the appropriate optimization problem that can be used to compute the population mass distribution associated with the equilibrium.

Our approach revisits the formulations of Leurent and Marcotte. It is a cleaner version of Leurent's optimization formulation [1] and a subcase of Marcotte's variational inequality formulation [2] where the external factor cost does not depend on traffic flow. We present our own simple proofs of the form of the equilibrium and equivalence between the minimum of our potential function optimization problem and the equilibrium. A main distinction between our presentation and these previous works is that Leurent, Dial, and Marcotte assume the distribution over the external cost parameter (which they consider to be the monetary value of time) to be supported on the non-negative reals ($\theta \in [0, \infty)$ in our notation). Our proofs highlight the fact that this is not necessary. In addition, the external costs α_o can be positive or negative. We highlight

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why this might be useful in the application section.

This bi-criterion equilibrium has many different applications. Along with standard tolling problems, one such application is providing a general framework for computing the value of routing information. Another interesting example is understanding how drivers' concern for their location privacy will shift the traffic equilibrium.

We can also use this framework to compare different modes of transportation where the parameter θ represents the population's preference for one form over the other. Our work differs from previous work using a continuous distribution to represent transportation mode preferences [24] in that we take advantage of the fact that θ and each α_o can be either negative or positive in order to model the fact that some members of the population may prefer different travel modes even when the travel time is equal. We go into further detail in Section IV.

The rest of the paper is organized as follows. In Section II, we present our new equilibrium concept and derive some of its properties. In Section III, we give a potential function and the corresponding optimization problem for computing the equilibrium. In Section IV, we give several applications where this framework could be applied. In Section V, we conclude and comment on future work.

II. External cost continuous type Wardrop equilibrium

A. Setup

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a graph with nodes and edges and let $\mathcal{R}$ be the set of all routes through a network from an origin node to a destination node. Let $\{\mathcal{R}_o\}_{o\in\mathcal{O}}$ be a collection of subsets of routes. Each subset of routes, $\mathcal{R}_o$ has a price, α_o that users have to pay in order to use those particular routes.

Remark 1: We will assume a single origin-destination pair but as in the classic routing game case, the analysis extends to multiple populations.

Without loss of generality, we will assume that the route subsets are ordered by price,

$$\alpha_{o-1} > \alpha_o \qquad \forall o \in \mathcal{O} \tag{1}$$

Once they pay to use a certain set of routes, drivers then play a standard routing game on those routes (though their congestion costs may depend on members of the population who choose a different set of routes.) We note that we do not assume any specific structure on the subsets $\{\mathcal{R}_o\}_{o\in\mathcal{O}}$. We also note that it is not a restriction to assume that α_{o-1} is strictly greater than α_o. If we have two groups, $\mathcal{R}_o$ and $\mathcal{R}_{o'}$ such that $\alpha_o = \alpha_{o'}$, we can just consider $\mathcal{R}_o \cup \mathcal{R}_{o'}$ as one group.

As previously mentioned, each member of the population has some type θ that represents their tradeoff between time and the external factor, i.e. how much external cost they are willing to incur for access to quicker routes through the network. We assume we are given some population distribution for this parameter, $dF(\theta)$. A sample distribution is illustrated in Figure 1. $\bar{\theta}$ is the maximum tradeoff parameter for anyone in the population and $\underline{\theta}$ is the minimum tradeoff parameter.

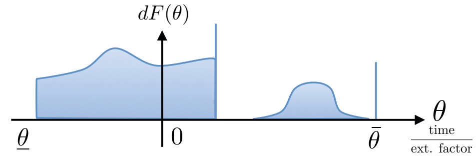

Fig. 1. Example population distribution of time-money tradeoff.

The overall population's decision to pay for each set of routes is encoded by a vector valued indicator function $I : [\underline{\theta}, \bar{\theta}] \to \Delta_{|\mathcal{O}|}$ where $\Delta_{|\mathcal{O}|}$ is the simplex of dimension $|\mathcal{O}|$. The oth element $I_o(\theta)$ represents the fraction of users with type θ that select subset $\mathcal{R}_o$. Here, we have implicitly made the assumption that every member of the population chooses some option. We can compute the total mass of users that pay for routes in $\mathcal{R}_o$ as

$$m_o(I_o) = \int_{[\underline{\theta},\bar{\theta}]} I_o(\theta) dF(\theta) \tag{2}$$

This mass is then divided up over the various routes in $\mathcal{R}_o$. Let $z_o \in \mathbb{R}_+^{|\mathcal{R}_o|}$ be the vector of masses assigned to each of these routes. We have that

$$\sum_{r\in\mathcal{R}_o} (z_o)_r = m_o \tag{3}$$

We will also use $z = (z_o)_{o\in\mathcal{O}}$ as a short hand for the entire set of mass distributions. The total mass is given by $m = \sum_o m_o$.

Given $I(\theta) = (I_o(\theta))_{o\in\mathcal{O}}$ and the corresponding mass distributions $z = (z_o)_{o\in\mathcal{O}}$, we can compute the total flow on each edge of the network $x \in \mathbb{R}_+^{|\mathcal{E}|}$ as

$$x = \sum_o \mathbf{E}_{\mathcal{R}_o} z_o \tag{4}$$

where $\mathbf{E}_{\mathcal{R}_o} \in \{0,1\}^{|\mathcal{E}|\times|\mathcal{R}_o|}$ is an indicator (or *routing*) matrix for the edges in each route in $\mathcal{R}_o$. Each edge has an associated *latency* function $l_e(x_e)$ that is a positive, increasing function of congestion. Latency for a whole route $\ell_r(z)$ is the sum of the edge latencies along that route, i.e. $\ell_r(z) = \sum_{e\in r} l_e(x_e)$.

B. Equilibrium

We now define our Wardrop equilibrium concept.

Definition 1: An *external cost continuous type Wardrop equilibrium* is a set of measurable functions $I = (I_o)_{o\in\mathcal{O}} : [\underline{\theta}, \bar{\theta}] \to \Delta_{|\mathcal{O}|}$ and corresponding mass distributions $z = (z_o)_{o\in\mathcal{O}}$ satisfying Equations (2) and (3) that satisfies the following. For any $\theta \in [\underline{\theta}, \bar{\theta}]$ and $o \in \mathcal{O}$ such that $I_o(\theta) > 0$ and any $r \in \mathcal{R}_o$ such that $(z_o)_r > 0$

$$\ell_r(z) + \alpha_o\theta \le \ell_{r'}(z) + \alpha_{o'}\theta \tag{5}$$

for any $o' \in \mathcal{O}$ and $r' \in \mathcal{R}_{o'}$.

Intuitively, this states that any population member of type θ who pays for $\mathcal{R}_o$ and drives a specific route could not have done better by selecting a different subset of routes and/or a different route. We note that within a specific subset of routes this reduces to the standard Wardrop equilibrium condition.

We also note that like in the traditional routing game, the equilibrium condition only places restrictions on strategies with positive mass ($m_o > 0$); however, Equation (2) requires that $I_o(\theta) = 0$ almost everywhere whenever $m_o = 0$.

We now deduce several properties about the choice functions $I = (I_o)_{o\in\mathcal{O}}$.

Lemma 1: Assume an ordering on $\{\alpha_o\}_{o\in\mathcal{O}}$ such that

$$\alpha_1 > \cdots > \alpha_{|\mathcal{O}|} \tag{6}$$

Suppose I and z form an external cost continuous type Wardrop equilibrium. Then there exists $\{\bar{\theta}_o\}_{o\in\mathcal{O}}$ such that

$$\underline{\theta} \leq \bar{\theta}_1 \leq \cdots \leq \bar{\theta}_{|\mathcal{O}|} \leq \bar{\theta} \tag{7}$$

and $\{\underline{\gamma}_o\}_{o\in\mathcal{O}} \in [0,1]$ and $\{\bar{\gamma}_o\}_{o\in\mathcal{O}} \in [0,1]$ such that I satisfies

$$I_o(\theta) = \begin{cases} \underline{\gamma}_o & ; \text{ if } \theta = \bar{\theta}_{o-1} \\ 1 & ; \text{ if } \bar{\theta}_{o-1} < \theta < \bar{\theta}_o \\ \bar{\gamma}_o & ; \text{ if } \theta = \bar{\theta}_o \\ 0 & ; \text{ otherwise} \end{cases} \tag{8}$$

almost everywhere (where we define $\bar{\theta}_0 = \underline{\theta}$).
The set $(I_o)_{o\in\mathcal{O}}$ is illustrated in Figure 2. The set $\{\bar{\theta}_o\}_{o\in\mathcal{O}}$ are the critical points detailed by Marcotte [2], [18].

Proof 1: For every o such that $m_o > 0$, define

$$\underline{\theta}_o = \inf_{\theta}\{\theta : I_o(\theta) > 0\} \qquad \bar{\theta}_o = \sup_{\theta}\{\theta : I_o(\theta) > 0\} \tag{9}$$

First, we show that for any two options o and o' such that $\alpha_{o'} > \alpha_o$ and both $m_o > 0$ and $m_{o'} > 0$, then $\bar{\theta}_{o'} \leq \underline{\theta}_o$. Assume not, i.e. that $\underline{\theta}_o < \bar{\theta}_{o'}$. Select $\theta, \theta' \in [\underline{\theta}_o, \bar{\theta}_{o'}]$ such that $\theta < \theta'$, $I_o(\theta) > 0$, and $I_{o'}(\theta') > 0$. Select routes $r \in \mathcal{R}_o$ and $r' \in \mathcal{R}_{o'}$ such that both $(z_o)_r > 0$ and $(z_{o'})_{r'} > 0$. Applying (5) at θ and θ' respectively gives.

$$(\alpha_{o'} - \alpha_o)\theta \geq \ell_r - \ell_{r'} \geq (\alpha_{o'} - \alpha_o)\theta' \tag{10}$$

Since $\alpha_{o'} - \alpha_o > 0$, it follows that $\theta \geq \theta'$ which is a contradiction. Thus we have that $\bar{\theta}_{o'} \leq \underline{\theta}_o$ for any options with positive mass. For any option o with $m_o = 0$, setting $\bar{\theta}_o = \bar{\theta}_{o-1}$ and $\underline{\gamma}_o = \bar{\gamma}_o = 0$ will satisfy Equations (2) and (7). Finally, since $I(\theta)$ maps to the simplex, we have that $I_o(\theta) = 1$ almost everywhere for $\theta \in (\bar{\theta}_{o-1}, \bar{\theta}_o)$.

We note that given the form of $(I_o)_{o\in\mathcal{O}}$ expounded in Lemma 1, we can compute $\{\bar{\theta}_o\}_{o\in\mathcal{O}}$ along with the appropriate values of $\{\underline{\gamma}_o\}_{o\in\mathcal{O}}$ and $\{\bar{\gamma}_o\}_{o\in\mathcal{O}}$ (to satisfy Equation (2)) using the cumulative distribution function of $dF(\theta)$. Let $\text{CDF} : \theta \mapsto m$ be the cumulative distribution function. Define a function $\Theta : m \mapsto \theta$ as $\Theta(\cdot) = \text{CDF}^{-1}(\cdot)$. We can then compute what $\bar{\theta}_o$ must be from the masses $\{m_o\}_{o\in\mathcal{O}}$.

$$\bar{\theta}_o = \Theta\Big(\textstyle\sum_{i\leq o} m_i\Big) \tag{11}$$

This inverse cumulative distribution is illustrated in Figure 3.

We can also compute the values of $\{\underline{\gamma}_o\}_{o\in\mathcal{O}}$ and $\{\bar{\gamma}_o\}_{o\in\mathcal{O}}$. Whenever $m_o = 0$, then $\underline{\gamma}_o = \bar{\gamma}_o = 0$. Whenever $m_o > 0$ and $\bar{\theta}_{o-1} = \bar{\theta}_o$, then $\underline{\gamma}_o = \bar{\gamma}_o = m_o / F(\{\bar{\theta}_o\})$. The remaining values are then computed inductively. Let $1, \ldots, o'$ be the set of options such that $\underline{\theta} = \bar{\theta}_o$ for $o \leq o'$. We can compute $\underline{\gamma}_{o'+1} = 1 - \sum_{i=1}^{o'} \bar{\gamma}_i$ and we can use $m_{o'+1} = \underline{\gamma}_{o'+1} F(\{\bar{\theta}_{o'}\}) + F((\bar{\theta}_{o'}, \bar{\theta}_{o'+1})) + \bar{\gamma}_{o'+1} F(\{\bar{\theta}_{o'+1}\})$ to solve for $\bar{\gamma}_{o'+1}$. We can repeat this procedure with $\bar{\theta}_{o'+1}$ instead of $\underline{\theta}$ and the rest of the $\gamma's$ follow by induction.

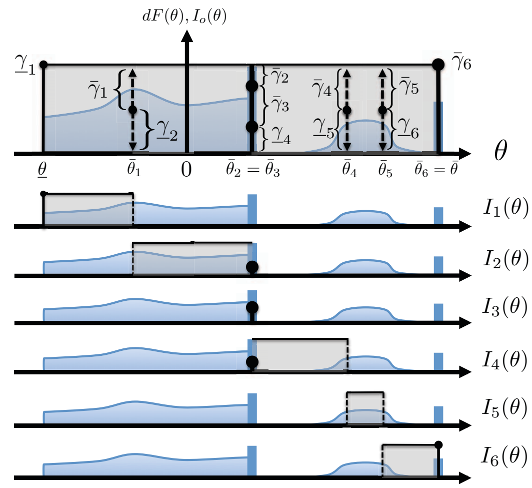

Fig. 2. Illustration of a possible set of indicator functions, $(I_o)_{o\in\mathcal{O}}$, at equilibrium for the distribution in Figure 1.

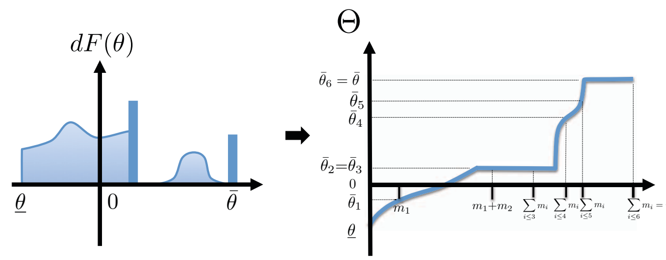

Fig. 3. Inverse cummulative distribution function, $\Theta(m)$, for the distribution in Figure 1.

We note that whenever $\bar{\theta}_o$ falls on a set of measure zero (as is the case with $\bar{\theta}_1$, $\bar{\theta}_4$, and $\bar{\theta}_5$ in Figure 2), there is some ambiguity in the choice of $\bar{\gamma}_o$ and $\underline{\gamma}_{o+1}$. In this case, the choice of $\bar{\gamma}_o$ does not affect the equilibrium mass distribution.

III. Equilibrium Computation

A. Potential Function

Lemma 1 indicates that given an equilibrium mass distribution $\{z_o\}_{o\in\mathcal{O}}$, we can determine the choice functions $(I_o(\theta))_{o\in\mathcal{O}}$ of the form (8) using Equations (11) and solving for the gammas inductively as discussed above. These arguments allow us to solve for the equilibrium mass directly. We define the appropriate potential function and show that the KKT necessary conditions for minimizing this function with respect to the appropriate constraints gives a mass distribution satisfying the external cost continuous type Wardrop equilibrium conditions.

The potential function is given by

$$F(z) = \sum_e \int_0^{x_e} l_e(u)\ du + \sum_i \int_{\sum_{o<i} m_o}^{\sum_{o\leq i} m_o} \alpha_i \Theta(u)\ du \tag{12}$$

We note that we can write the second summation in (12) as

$$\sum_{i=1}^{|\mathcal{O}|-1} \int_0^{\sum_{o\leq i} m_o} (\alpha_i - \alpha_{i+1})\Theta(u)\ du + \int_0^m \alpha_{|\mathcal{O}|}\Theta(u)\ du \tag{13}$$

and on the set defined by the conservation of mass constraint, the last term is constant.

In this form, we can see that the potential function is convex on the set defined by the conservation of mass constraint. This follows from the fact that $\Theta(\cdot)$ is increasing for any distribution $dF(\theta)$ as illustrated in Figure 3 and $\alpha_i - \alpha_{i+1} > 0$ for any set of α_i's that satisfy the ordering convention (1). Note that we can write $F(z)$ as a function of z only since $\{m_o\}_{o\in\mathcal{O}}$ and x are both functions of z.

Remark 2: This potential function is closely related to Leurent's objective function, Equation (8) in [1], and is equivalent to the potential function given in Equation (54) in [2] and Equation (40) in [18] when θ is supported only on $\mathbb{R}_+$.

B. Optimization Formulation

We can write the following optimization problem for finding the equilibrium.

Theorem 1: Let $z = (z_o)_{o\in\mathcal{O}}$ be a mass distribution that solves the following optimization problem and $I = (I_o)_{o\in\mathcal{O}}$ be a set of choice functions with form determined by (8), (11), and the inductive procedure outlined above.

$$\min_z \quad F(z) \tag{14a}$$

$$\text{s.t.} \quad m = \sum_o \sum_{r\in\mathcal{R}_o} (z_o)_r, \quad z_o \geq 0, \qquad \forall o \in \mathcal{O} \tag{14b}$$

$$m_o = \sum_{r\in\mathcal{R}_o} (z_o)_r, \quad x = \sum_{o\in\mathcal{O}} \mathbf{E}_{\mathcal{R}_o} z_o \quad \forall o \in \mathcal{O} \tag{14c}$$

It follows that $z = (z_o)_{o\in\mathcal{O}}$ and $I = (I_o)_{o\in\mathcal{O}}$ are an external cost continuous type Wardrop equilibrium.

Proof 2: The Lagrangian is given by

$$\mathcal{L} = F(z) - \lambda\Big(m - \sum_o \sum_{r\in\mathcal{R}_o} (z_o)_r\Big) - \sum_o \mu_o^T z_o \tag{15}$$

where $\lambda \in \mathbb{R}$ and $\mu_o \in \mathbb{R}_+^{|\mathcal{R}_o|}$ and where we have substituted in (14c). For a given $r \in \mathcal{R}_o$, the first order optimality conditions give

$$\ell_r + \sum_{o\leq i} \alpha_i \bar{\theta}_i - \sum_{o<i} \alpha_i \bar{\theta}_{i-1} = \lambda + (\mu_o)_r \tag{16}$$

where $(\mu_o)_r \geq 0$ with equality achieved whenever $(z_o)_r > 0$ by complementary slackness. We can rewrite Equation (16) in two different ways

$$\ell_r + \sum_{o\leq i} (\alpha_i - \alpha_{i+1})\bar{\theta}_i = \lambda + (\mu_o)_r \tag{17a}$$

$$\ell_r + \sum_{o<i} (\alpha_{i-1} - \alpha_i)\bar{\theta}_{i-1} = \lambda + (\mu_o)_r \tag{17b}$$

We now consider a specific choice function $I_o(\theta)$ and a route $r \in \mathcal{R}_o$ such that $(z_o)_r > 0$. By Lemma 1, we simply need to show that (5) is satisfied for any $\theta \in [\bar{\theta}_{o-1}, \bar{\theta}_o]$. Take any $\theta \in [\bar{\theta}_{o-1}, \bar{\theta}_o]$ and two subsets $o, o' \in \mathcal{O}$ such that $o' > o$ and corresponding routes $r \in \mathcal{R}_o$ and $r' \in \mathcal{R}_{o'}$ such that $(z_o)_r > 0$. From (17a), we have that

$$\ell_r - \ell_{r'} + \sum_{o\leq i<o'} (\alpha_i - \alpha_{i+1})\bar{\theta}_i = (\mu_o)_r - (\mu_{o'})_{r'} \tag{18a}$$

$$\ell_r - \ell_{r'} + \sum_{o\leq i<o'} (\alpha_i - \alpha_{i+1})\theta \leq (\mu_o)_r - (\mu_{o'})_{r'} \tag{18b}$$

$$\ell_r - \ell_{r'} + (\alpha_o - \alpha_{o'})\theta \leq (\mu_o)_r - (\mu_{o'})_{r'} \tag{18c}$$

where, in Equation (18b), we have used that $\alpha_i - \alpha_{i+1} > 0$ for all i and $\theta \leq \bar{\theta}_i$ for all $i \geq o$. Since $(z_o)_r > 0$ (and thus by complementary slackness $(\mu_o)_r = 0$) and $(\mu_{o'})_{r'} \geq 0$, it follows that

$$\ell_r + \alpha_o\theta \leq \ell_{r'} + \alpha_{o'}\theta \tag{19}$$

Now if $o' < o$, we have that

$$\ell_{r'} - \ell_r + \sum_{o'<i\leq o} (\alpha_{i-1} - \alpha_i)\bar{\theta}_{i-1} = (\mu_o)_{r'} - (\mu_{o'})_r \tag{20a}$$

$$\ell_{r'} - \ell_r + \sum_{o'<i\leq o} (\alpha_{i-1} - \alpha_i)\theta \geq (\mu_o)_{r'} - (\mu_{o'})_r \tag{20b}$$

$$\ell_{r'} - \ell_r + (\alpha_{o'} - \alpha_o)\theta \geq (\mu_o)_{r'} - (\mu_{o'})_r \tag{20c}$$

since $\alpha_{i-1} - \alpha_i > 0$ for all i and $\theta \geq \bar{\theta}_{i-1}$ for $i \leq o$. This yields

$$\ell_r + \alpha_o\theta \leq \ell_{r'} + \alpha_{o'}\theta \tag{21}$$

again by complementary slackness which proves the result. Figure 4 gives a graphical illustration of the equilibrium condition for the sample distribution in Figure 1.

IV. Applications

A. Classical routing and variable demand

It is straightforward to see that this framework reduces to the classical routing game whenever the population distribution is a delta function at zero. The variable demand routing game can also be thought of as a special case. Consider the route groupings, $\{\mathcal{R}_1, \mathcal{R}_2\} = \{\mathcal{R}, \emptyset\}$ and prices $\{\alpha_1, \alpha_2\} = \{\alpha, 0\}$ with $\alpha > 0$ and $dF(\theta)$ supported on $\mathbb{R}_-$. The latency for taking a route in the empty set (not driving) is considered 0. The equilibrium condition for any route $r \in \mathcal{R}$ such that $z_r > 0$ is given by $\ell_r + \alpha\theta \leq 0$. Drivers with more negative values of θ are okay with longer travel times before they decide not to drive. The potential function is given by

$$F(z) = \sum_e \int_0^{x_e} l_e(u)\ du + \int_0^{m_1} \alpha\Theta(u)\ du \tag{22a}$$

Here the demand curve is given by $d(\cdot) = -\frac{1}{\alpha}\Theta^{-1}(\cdot)$.

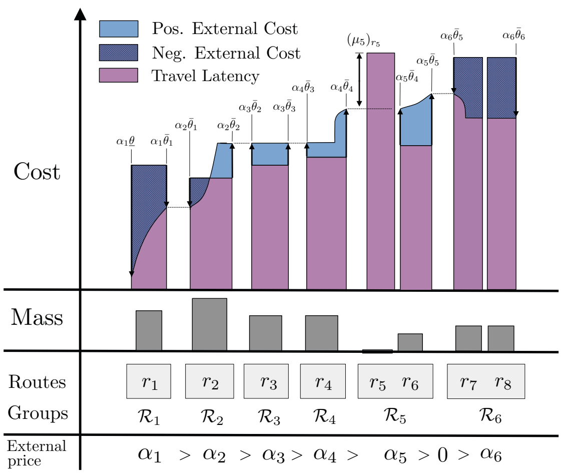

Fig. 4. Visualization of the external cost continuous type Wardrop equilibrium condition for the distribution shown in Figure 1 and the route subset structure given. Purple represents the travel latency for specific routes. Blue represents the perceived external cost which depends on the price of each routing option α_o and individual population members type θ. Note that the perceived external cost can be negative either because θ is negative (as is the case for $\mathcal{R}_1$ in the figure) or because α_o is negative (as is the case for $\mathcal{R}_6$ in the figure). The familiar Wardrop balance condition is preserved at the transitions between various routing options. We note that the distribution of mass on the various routes could be significantly more complicated for more complex route groupings.

B. Traveler information systems market and privacy

One clear application of this framework would be modeling the market for traveler information systems. Various routing apps such as Google maps or Waze each provide users with a group of routes to choose from. Better apps would provide more routing options, shortcuts, etc. The price α_o would be the amount users pay for each service.

Another interesting application of this framework would be to model drivers' interest in their location privacy. Often, users only receive congestion information from services such as Google maps if they allow the navigation service to monitor their location. Any user who "opts out", decides not to share their location and not receive congestion information, takes the nominally shortest route, the shortest route without congestion. Any users that "opt in", decide to share their location information, receive information about all possible routes. The routing groups are $\mathcal{R}_{\text{out}} = \{r_1\}$ where r_1 is the shortest uncongested route and $\mathcal{R}_{\text{in}} = \mathcal{R}$. Members who opt out do not pay any additional cost ($\alpha_{\text{out}} = 0$) and members who opt in pay an additional cost of $\alpha_{\text{in}} > 0$ times θ. The parameter θ here represents each population member's value of privacy versus their value of travel time.

C. Multi-modal routing

This framework also provides a way to study the choice commuters make between different modes of transportation such as taking the subway or driving. We assume there are two commuting options with sets of routes $\mathcal{R}_1$ and $\mathcal{R}_2$.

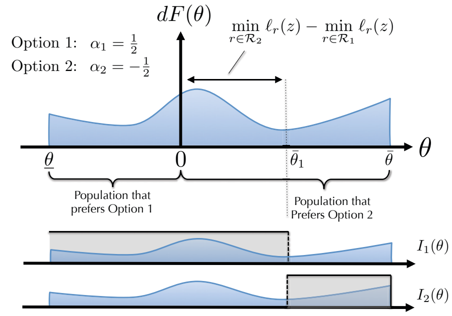

Fig. 5. Preference distribution for two different commuting options.

Depending on the commuting options, these route groups may be disjoint and the congestion effects may be different for each. For example, if commuters choose between driving or taking the subway, the subway routes will be separate from the driving routes and have different congestion effects. Congestion on driving routes will primarily result in increased travel time. Congestion on the subway routes could result in greater inconvenience, more crowded platforms, less available seats, etc.

The population distribution $dF(\theta)$ is supported from the $\underline{\theta} \leq 0$ to $\bar{\theta} \geq 0$. If we set $\alpha_1 > 0$ and $\alpha_2 < 0$, then $\theta < 0$ models a preference for option 1 and $\theta > 0$ models a preference for option 2. If we set, $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = -\frac{1}{2}$, then at $\bar{\theta}_1$ at equilibrium we have that

$$\min_{r\in\mathcal{R}_1} \ell_r(z) + \frac{1}{2}\bar{\theta}_1 = \min_{r\in\mathcal{R}_2} \ell_r(z) - \frac{1}{2}\bar{\theta}_1 \tag{23a}$$

$$\Rightarrow \qquad \bar{\theta}_1 = \min_{r\in\mathcal{R}_2} \ell_r(z) - \min_{r\in\mathcal{R}_1} \ell_r(z) \tag{23b}$$

Here, the value of θ for each member of the population indicates how much faster option 1 must be than option 2 before they will switch to option 1. We illustrate a sample preference distribution for two options in Figure 5.

We note that this framework can be used to model two different commuting options; however, more than two is problematic. Since θ is one dimensional, it can represent relative preference between two options; however, it would only make sense to compare three or more commuting options if there was a clear preference ordering for these options that *all* population members agreed upon which is unlikely.

The two commuting options in this framework can be varied and it is not required that the two sets of routes be disjoint. For example, we might seek to compare the demand for driving vs the demand for taxis. If the two options share routes, however, it is important that mass from either group have the same effect on congestion on the shared routes in order for the game to be a potential game.

To further illustrate the use of the model in this way, we consider a simple parallel network with two commuting options each with two routes (shown in Figure 6a). We assume a total mass of $M = 8$ and $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = -\frac{1}{2}$. The population distribution is uniform from $\underline{\theta} = -4$ to $\bar{\theta} = 4$.

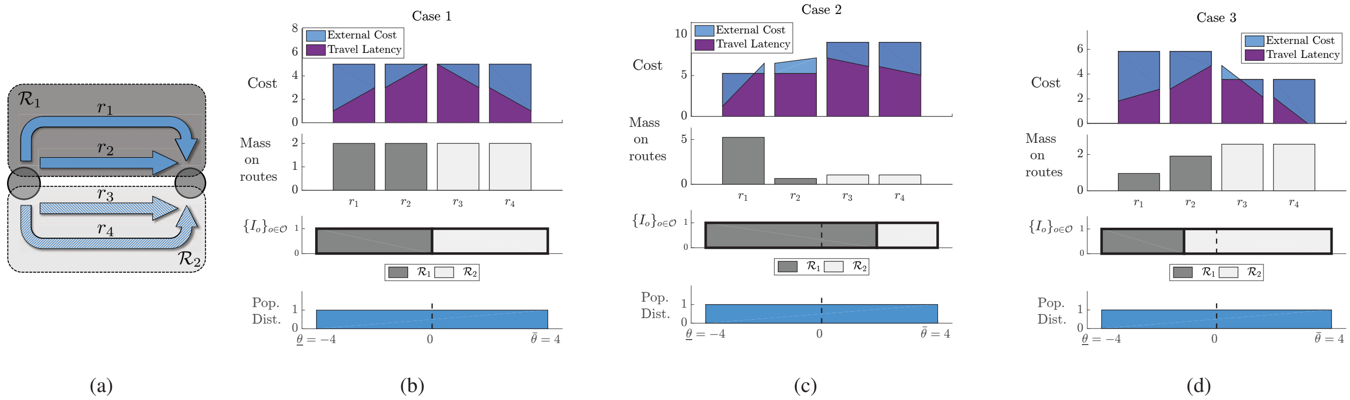

Fig. 6. Equilibrium and balance conditions for the commuting preference example for the simple network shown in (a). (b) Case 1: the two routing options are symmetric so the population mass chooses their preferred mode of transportation and the population mass is split around $\theta = 0$. (c) Case 2: option 2 becomes more congested so some mass that prefers option 2 switches to option 1. (d) Case 3: option 1 becomes more congested so some mass switches from option 2 to option 1. Note that the effective external-cost $\alpha_o\theta$ is negative in some cases.

We consider three different sets of latencies of the form $\ell_r(z_r) = a_r z_r + b_r$ with coefficients shown in the table below.

	a_r				b_r			
Routes	r_1	r_2	r_3	r_4	r_1	r_2	r_3	r_4
Case 1	2	2	2	2	1	1	1	1
Case 2	1	8	1	1	0	0	8	8
Case 3	4	2	1	1	2	2	1	1

The resulting equilibria and balance conditions are illustrated in Figures 6b, 6c, and 6d.

V. Conclusion and Future Work

In this paper, we have a presented a bi-criterion routing game equilibrium where the population's preference for an external factor is captured by an arbitrary distribution over a parameter θ. Our work differs from previous formulations in that θ can be negative. We derive properties of the equilibrium, present an optimization problem for computation, and examine several application domains including privacy and multi-modal routing. One interesting direction for future work is expanding the framework to more than one external factor, i.e. a multi-dimensional θ parameter.

References

[1] F. Leurent, "Cost versus time equilibrium over a network," *European Journal of Operational Research*, vol. 71, no. 2, pp. 205–221, 1993.

[2] P. Marcotte, "Reformulations of a bicriterion equilibrium model," in *Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods*. Springer, 1998, pp. 269–291.

[3] J. G. Wardrop, "Road paper. some theoretical aspects of road traffic research." in *ICE Proceedings: Engineering Divisions*, vol. 1, no. 3. Thomas Telford, 1952, pp. 325–362.

[4] M. Beckmann, C. McGuire, and C. B. Winsten, "Studies in the economics of transportation," Tech. Rep., 1956.

[5] S. C. Dafermos and F. T. Sparrow, "The traffic assignment problem for a general network," *Journal of Research of the National Bureau of Standards, Series B*, vol. 73, no. 2, pp. 91–118, 1969.

[6] R. B. Dial, "A model and algorithm for multicriteria route-mode choice," *Transportation Research Part B: Methodological*, vol. 13, no. 4, pp. 311–316, 1979.

[7] R. E. Quandt, "A probabilistic abstract mode model," *Studies in Travel Demand VIII, Mathematica, Inc., Princeton, New Jersey*, pp. 127–149, 1967.

[8] M. Schneider, "Access and land development," *Highway Research Board Special Report*, no. 97, 1968.

[9] S. Dafermos, "The general multimodal network equilibrium problem with elastic demand," *Networks*, vol. 12, no. 1, pp. 57–72, 1982.

[10] A. Nagurney, "A multiclass, multicriteria traffic network equilibrium model," *Mathematical and Computer Modelling*, vol. 32, no. 3-4, pp. 393–411, 2000.

[11] S. jie LI and G. ya CHEN, "Multiclass, two-criteria traffic network equilibrium models and vector variational inequalities," *Systems Engineering - Theory Practice*, vol. 28, no. 1, pp. 141 – 145, 2008.

[12] H. Yang and H.-J. Huang, "The multi-class, multi-criteria traffic network equilibrium and systems optimum problem," *Transportation Research Part B: Methodological*, vol. 38, no. 1, pp. 1 – 15, 2004.

[13] F. Leurent, "The theory and practice of a dual criteria assignment model with a continuously distributed value-of-time," in *ISTTT*. Pergamon, 1996, pp. pp–455.

[14] R. B. Dial, "Bicriterion traffic assignment: basic theory and elementary algorithms," *Transportation Science*, vol. 30, no. 2, pp. 93–111, 1996.

[15] ——, "Bicriterion traffic assignment: efficient algorithms plus examples," *Transportation Research Part B: Methodological*, vol. 31, no. 5, pp. 357–379, 1997.

[16] P. Marcotte and D. Zhu, "An efficient algorithm for a bicriterion traffic assignment problem," in *Advanced methods in transportation analysis*. Springer, 1996, pp. 63–73.

[17] P. Marcotte and D. L. Zhu, "Equilibria with infinitely many differentiated classes of customers," 1997.

[18] P. Marcotte and M. Patriksson, "Traffic equilibrium," *Handbooks in Operations Research and Management Science*, vol. 14, pp. 623–713, 2007.

[19] H. Yang and X. Zhang, "Multiclass network toll design problem with social and spatial equity constraints," *Journal of Transportation Engineering*, vol. 128, no. 5, pp. 420–428, 2002.

[20] M. Florian, "Network equilibrium models for analyzing toll highways," in *Mathematical and Computational Models for Congestion Charging*. Springer, 2006, pp. 105–115.

[21] C.-C. Lu, X. Zhou, and H. Mahmassani, "Variable toll pricing and heterogeneous users: Model and solution algorithm for bicriterion dynamic traffic assignment problem," *Transportation Research Record: Journal of the Transportation Research Board*, no. 1964, pp. 19–26, 2006.

[22] P. Marcotte and D. Zhu, "Existence and computation of optimal tolls in multiclass network equilibrium problems," *Operations Research Letters*, vol. 37, no. 3, pp. 211–214, 2009.

[23] W.-X. Wu and H.-J. Huang, "Finding anonymous tolls to realize target flow pattern in networks with continuously distributed value of time," *Transportation Research Part B: Methodological*, vol. 65, pp. 31–46, 2014.

[24] H. Yang, H. Y. Kong, and Q. Meng, "Value-of-time distributions and competitive bus services," *Transportation Research Part E: Logistics and Transportation Review*, vol. 37, no. 6, pp. 411–424, 2001.