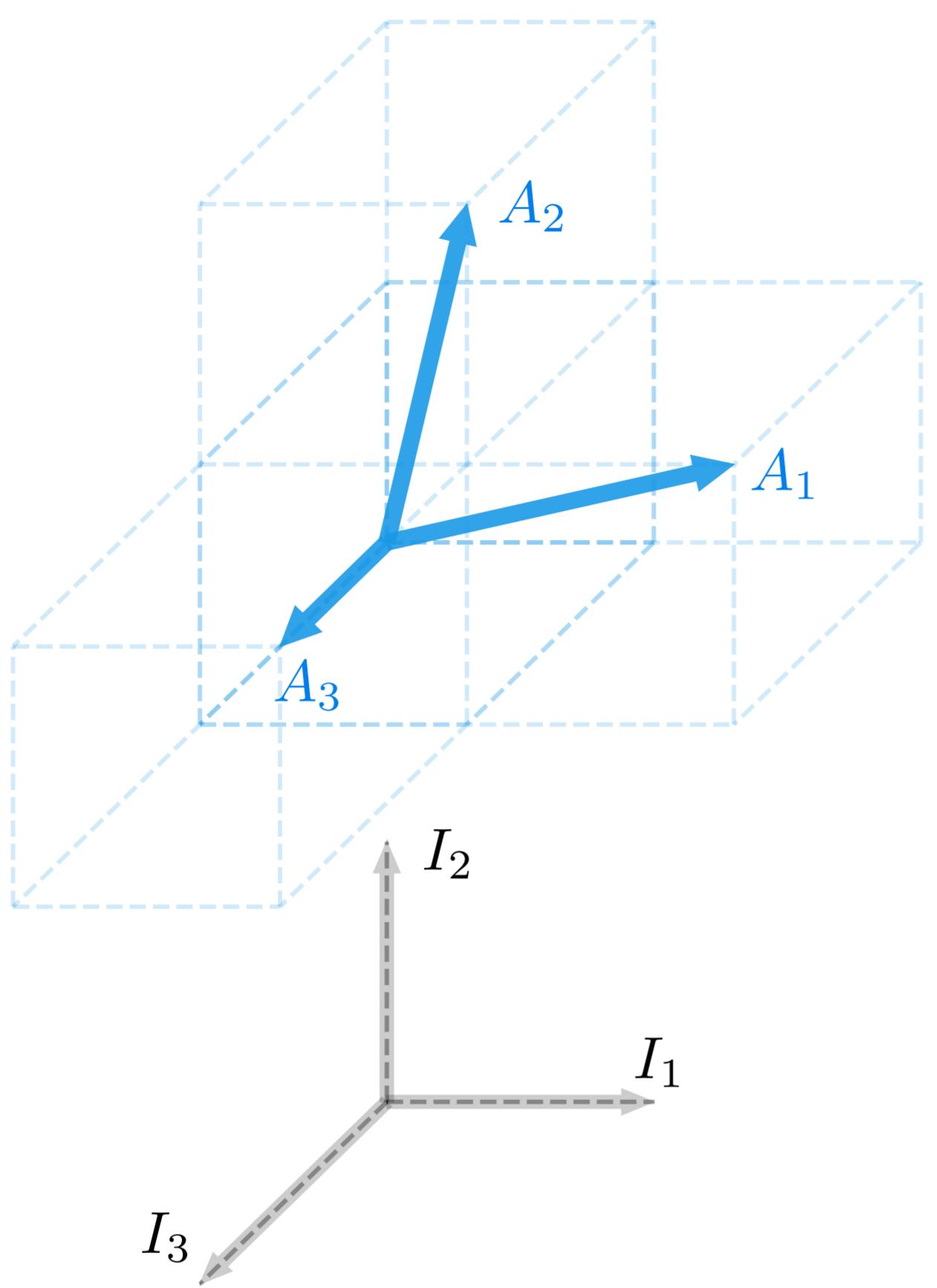


Gaussian Elimination: Elementary Column Operations (column geometry)

Linear Algebra

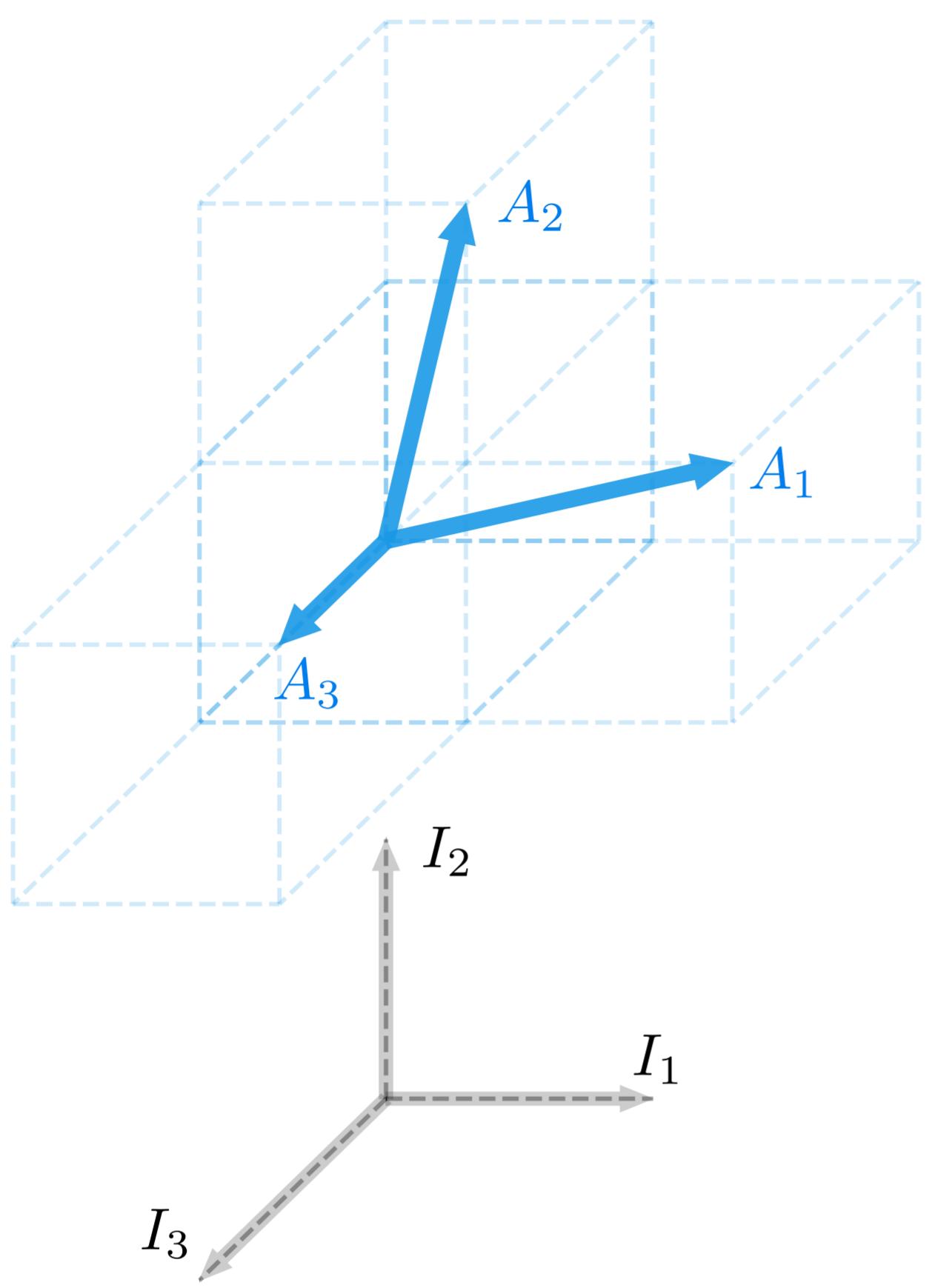
Major sources:

Winter 2022 - Dan Calderone

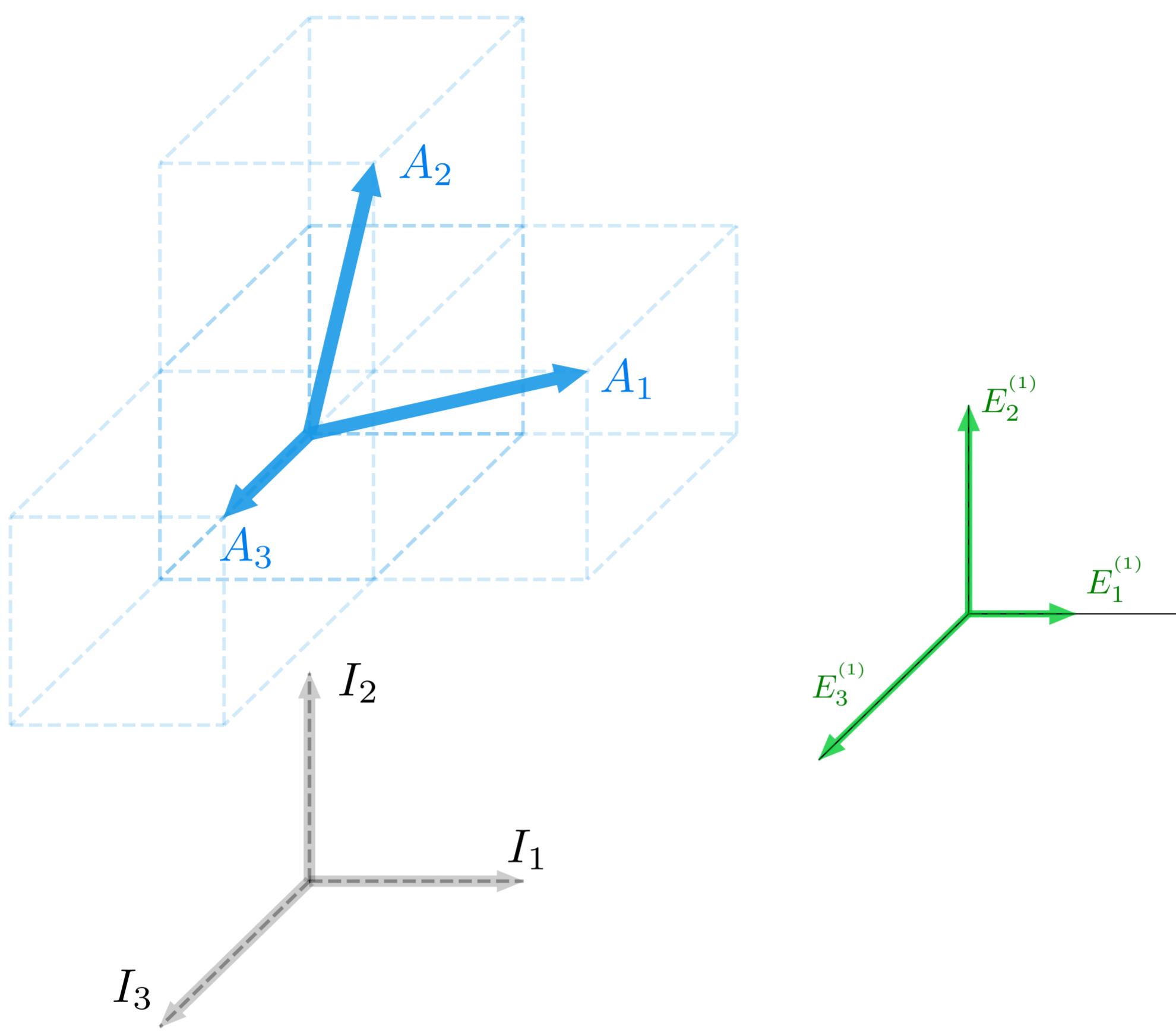


$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & \\ 1 & 2 & 1 & \\ 1 & 1 & 2 & \\ \hline 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array} \right]$$

**Augmented
System**

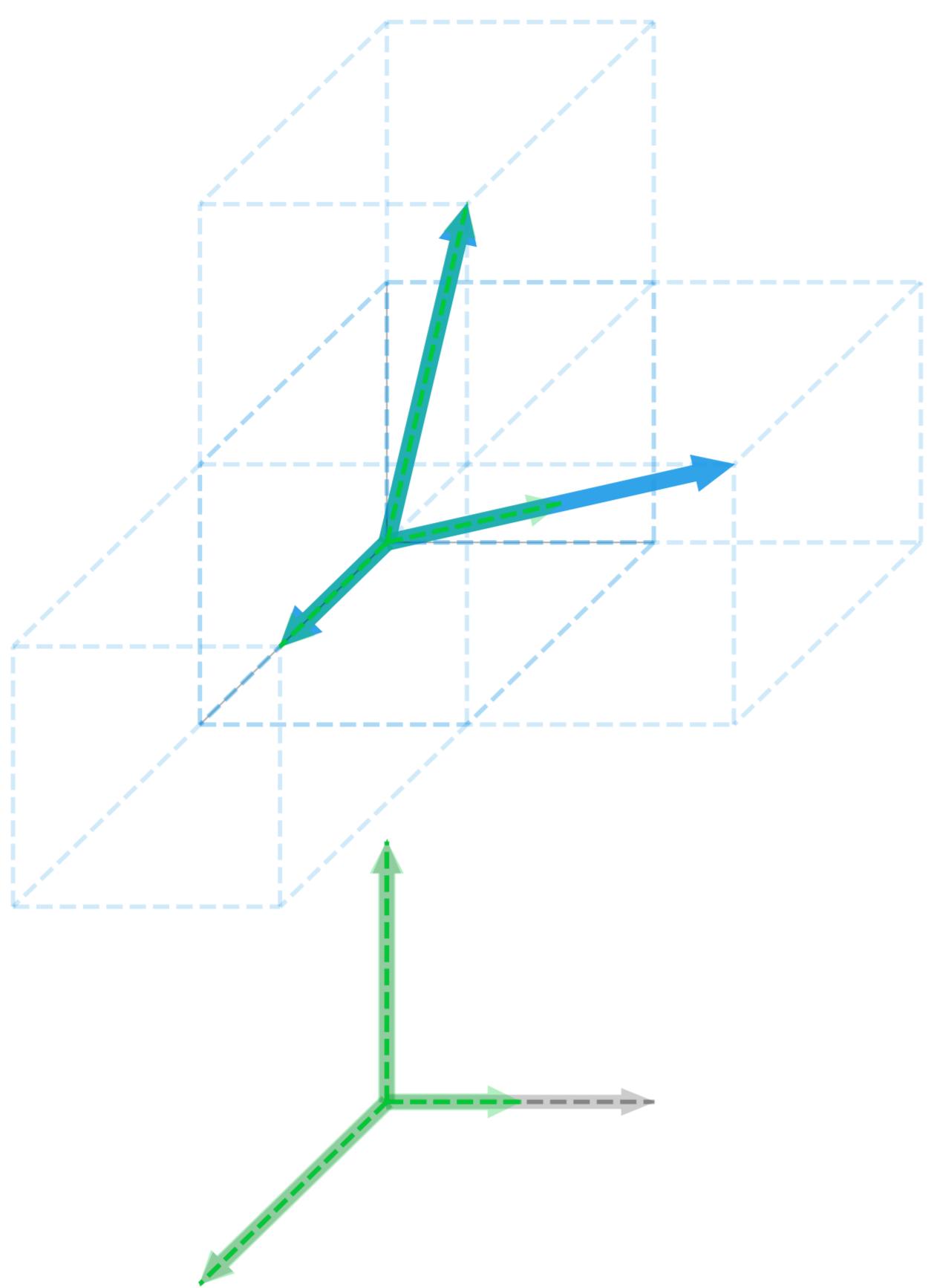


$$\begin{array}{c}
 \begin{bmatrix} A & | & I \end{bmatrix} = \\
 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{array}
 \quad \text{Augmented System}$$

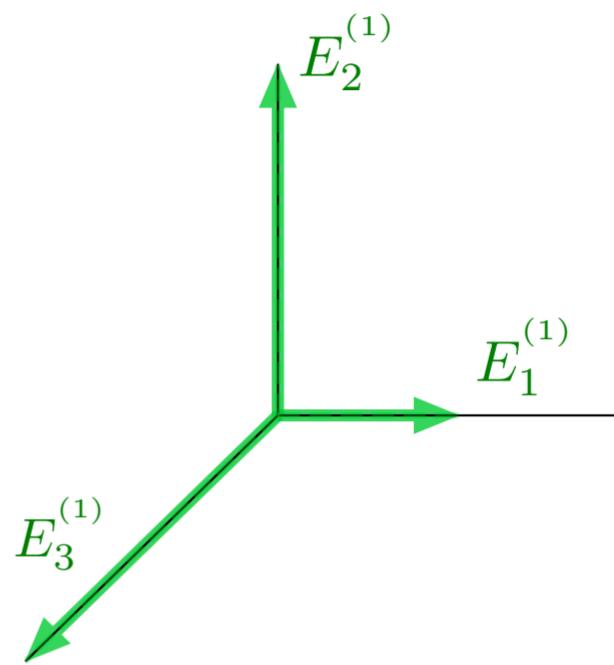


$$\begin{array}{c}
 A \\
 \times \\
 I
 \end{array}
 \quad
 \begin{bmatrix}
 2 & 1 & 1 \\
 1 & 2 & 1 \\
 1 & 1 & 2
 \end{bmatrix}
 \quad
 \begin{bmatrix}
 1/2 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
 \end{bmatrix}
 \quad
 E^{(1)}$$

Below the matrices, there is a multiplication symbol (\times) indicating that the matrix A is multiplied by the matrix $E^{(1)}$. The matrix I is positioned between the two matrices.

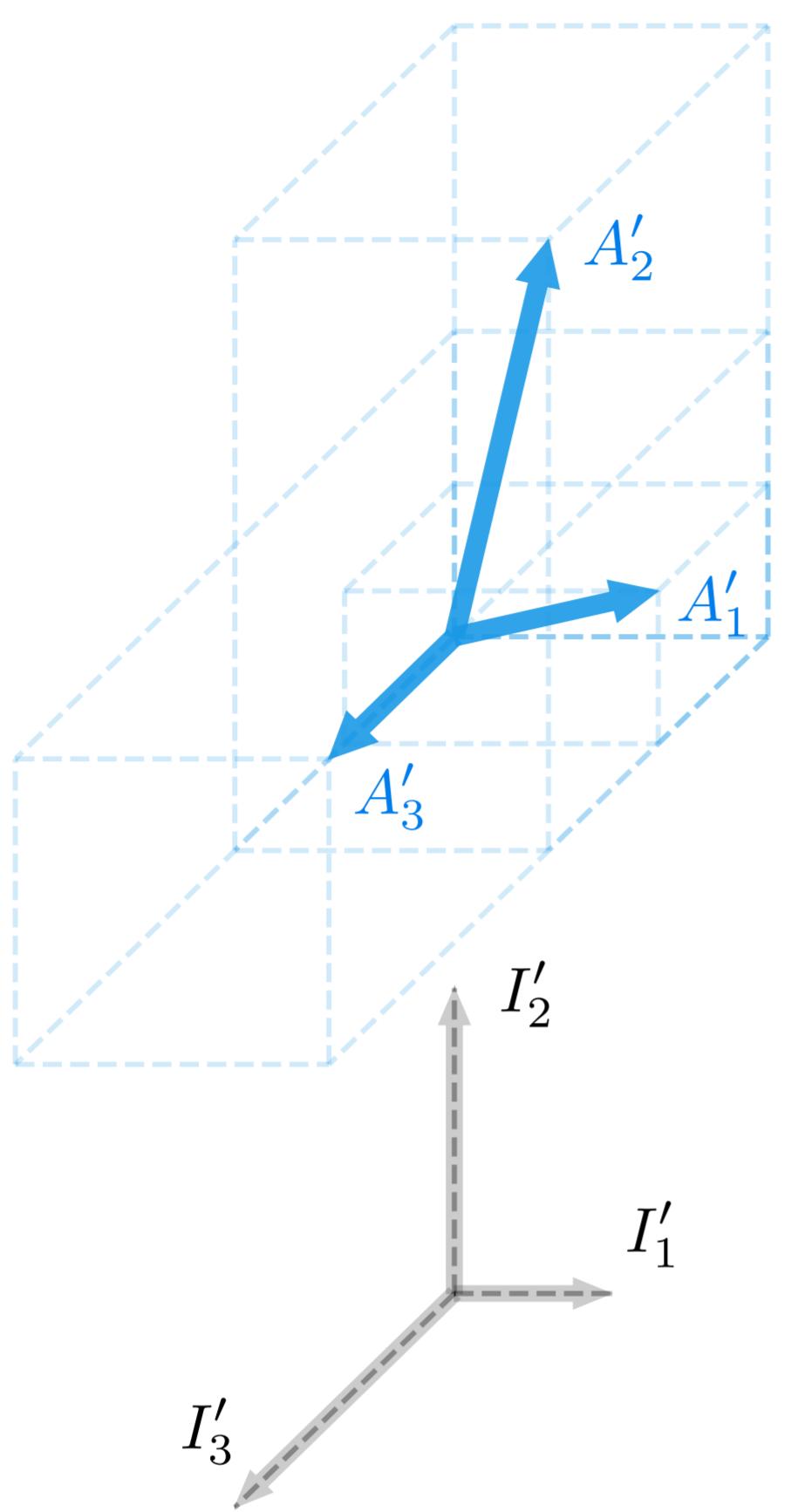


Column-reduction:
operation 1...



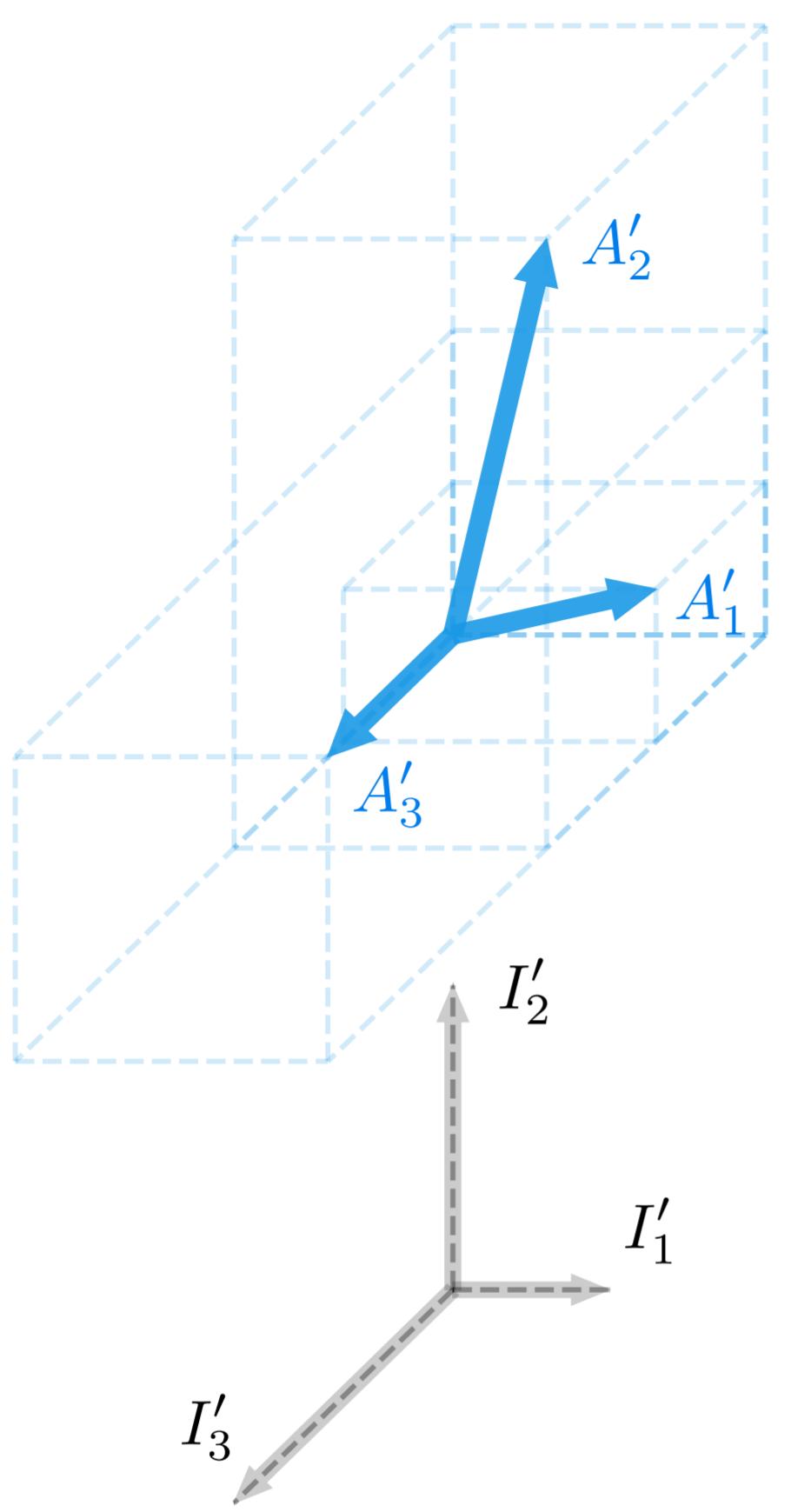
$$\begin{array}{c}
 A \\
 \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array} \right] \left[\begin{array}{ccc} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \\
 I \\
 \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]
 \end{array}$$

$E^{(1)}$

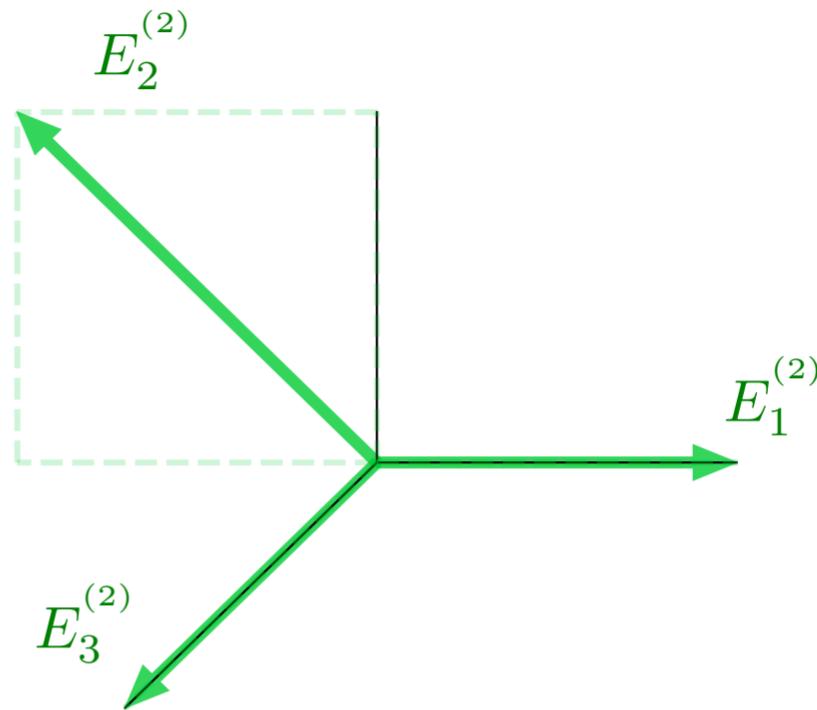


Column-reduction:
operation 1...

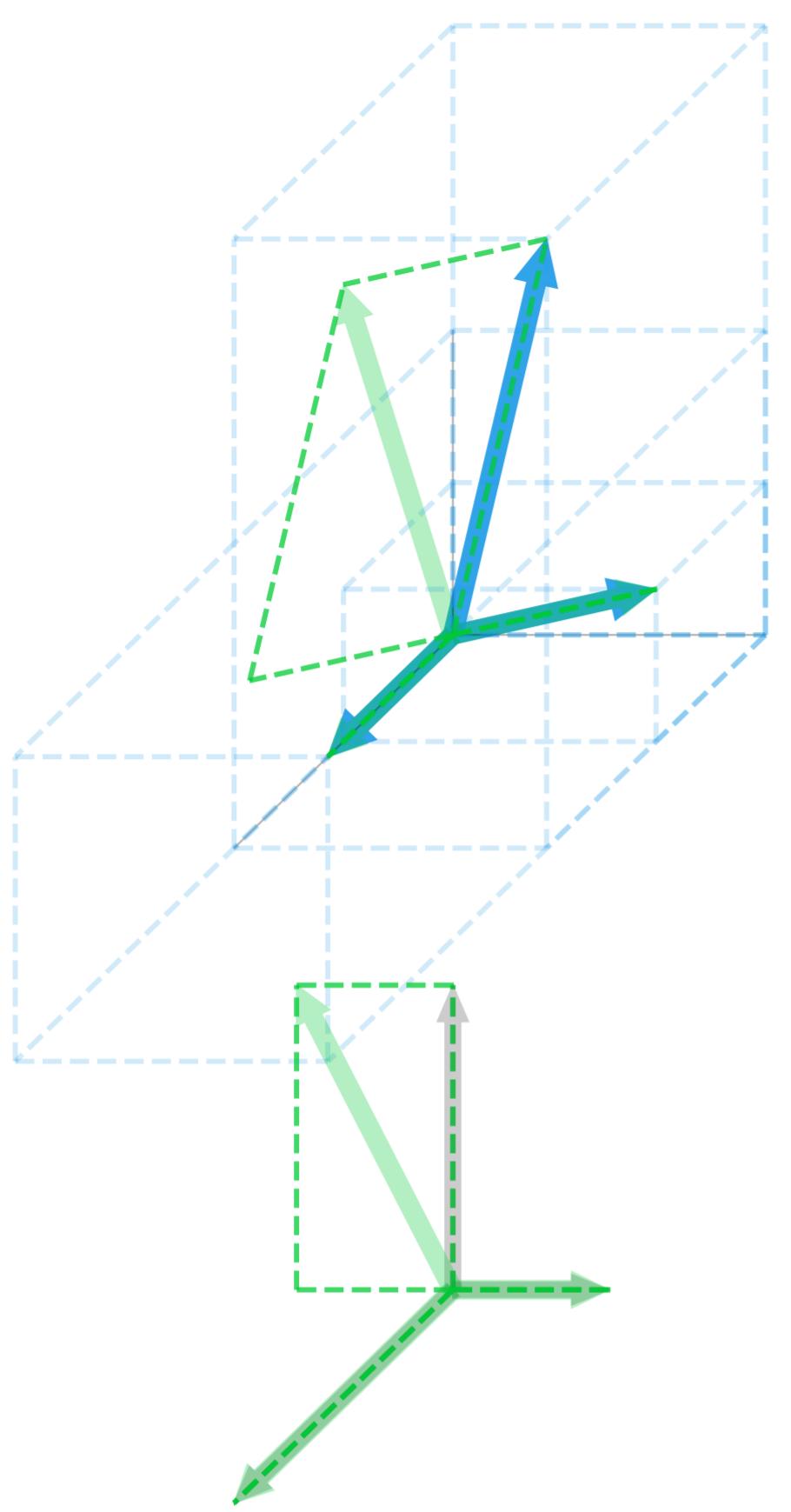
$$\begin{array}{c}
 \underbrace{AE^{(1)}}_{A'} \\
 \hline
 \underbrace{IE^{(1)}}_{I'}
 \end{array}
 \quad
 \begin{bmatrix}
 1 & 1 & 1 \\
 1/2 & 2 & 1 \\
 1/2 & 1 & 2 \\
 \hline
 1/2 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
 \end{bmatrix}$$



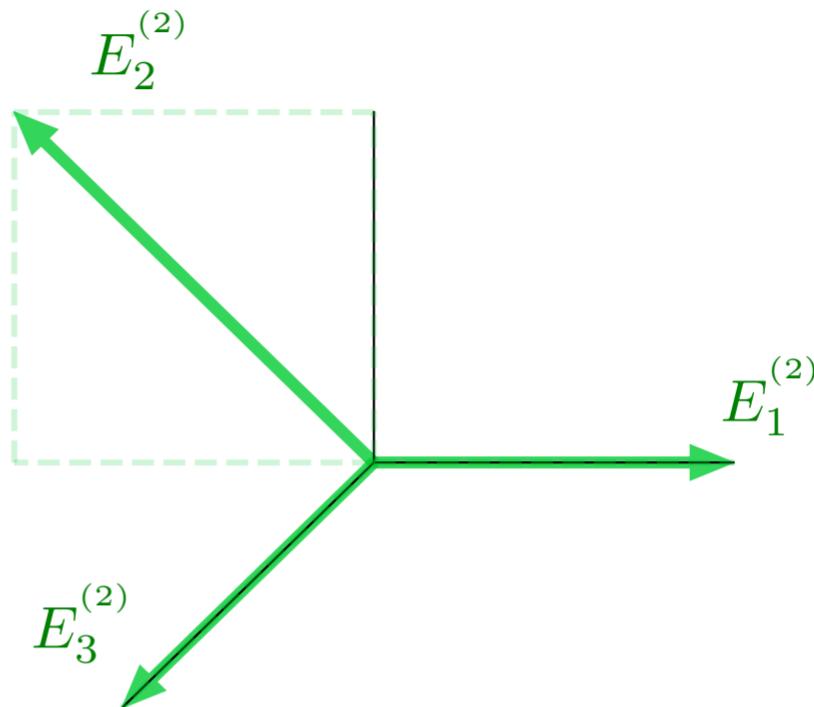
Column-reduction:
operation 2...



$$\begin{array}{c}
 \underbrace{AE^{(1)}}_{A'} \\
 \times \\
 \underbrace{IE^{(1)}}_{I'}
 \end{array}
 \quad \times \quad
 \begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 2 & 1 \\ 1/2 & 1 & 2 \end{bmatrix} \quad \times \quad
 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E^{(2)}$$

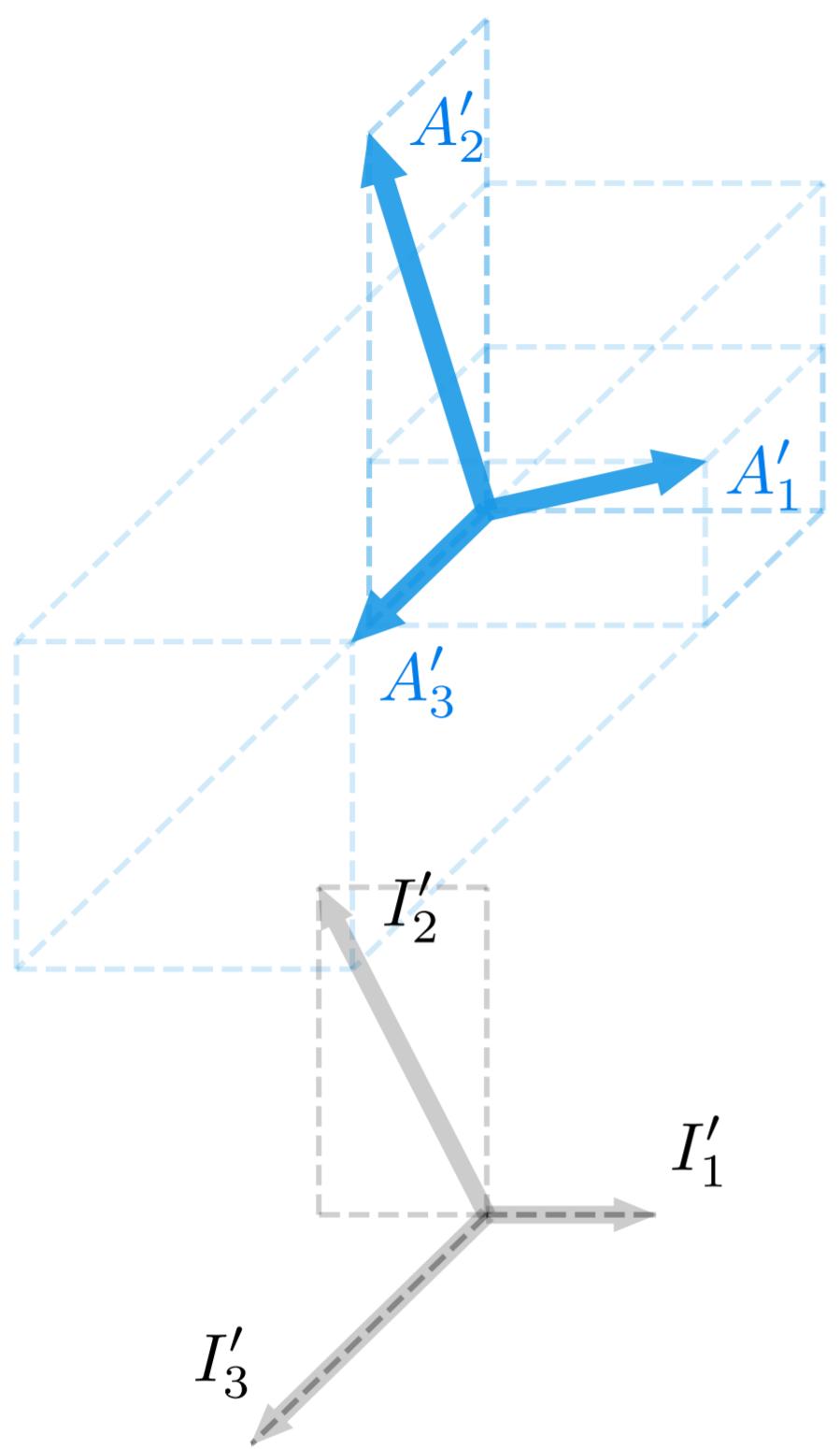


Column-reduction:
operation 2...



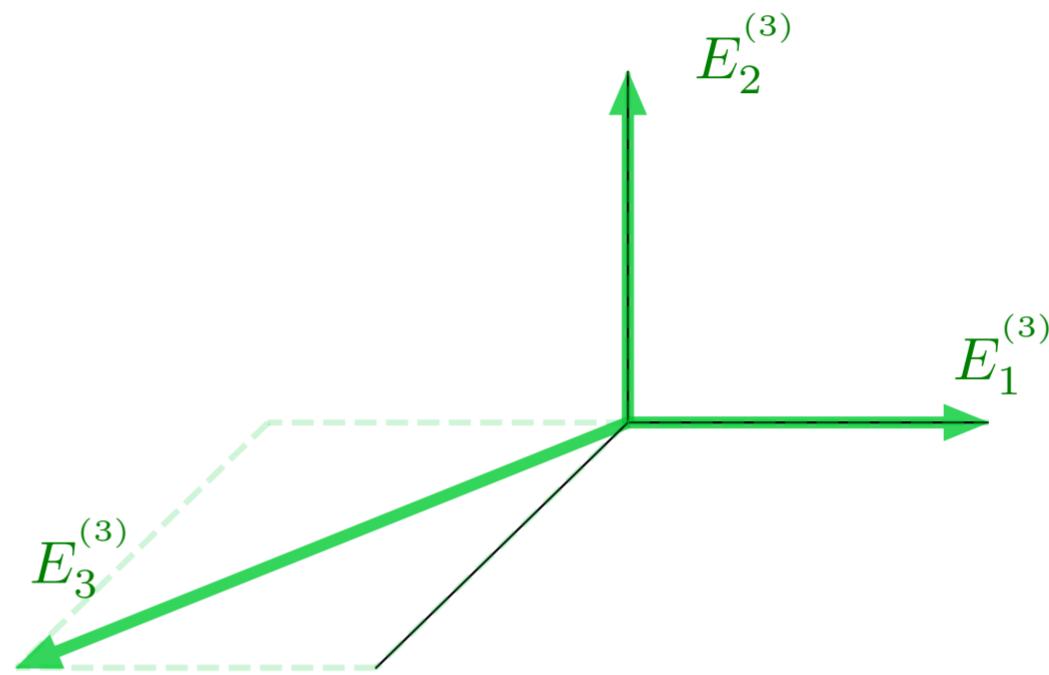
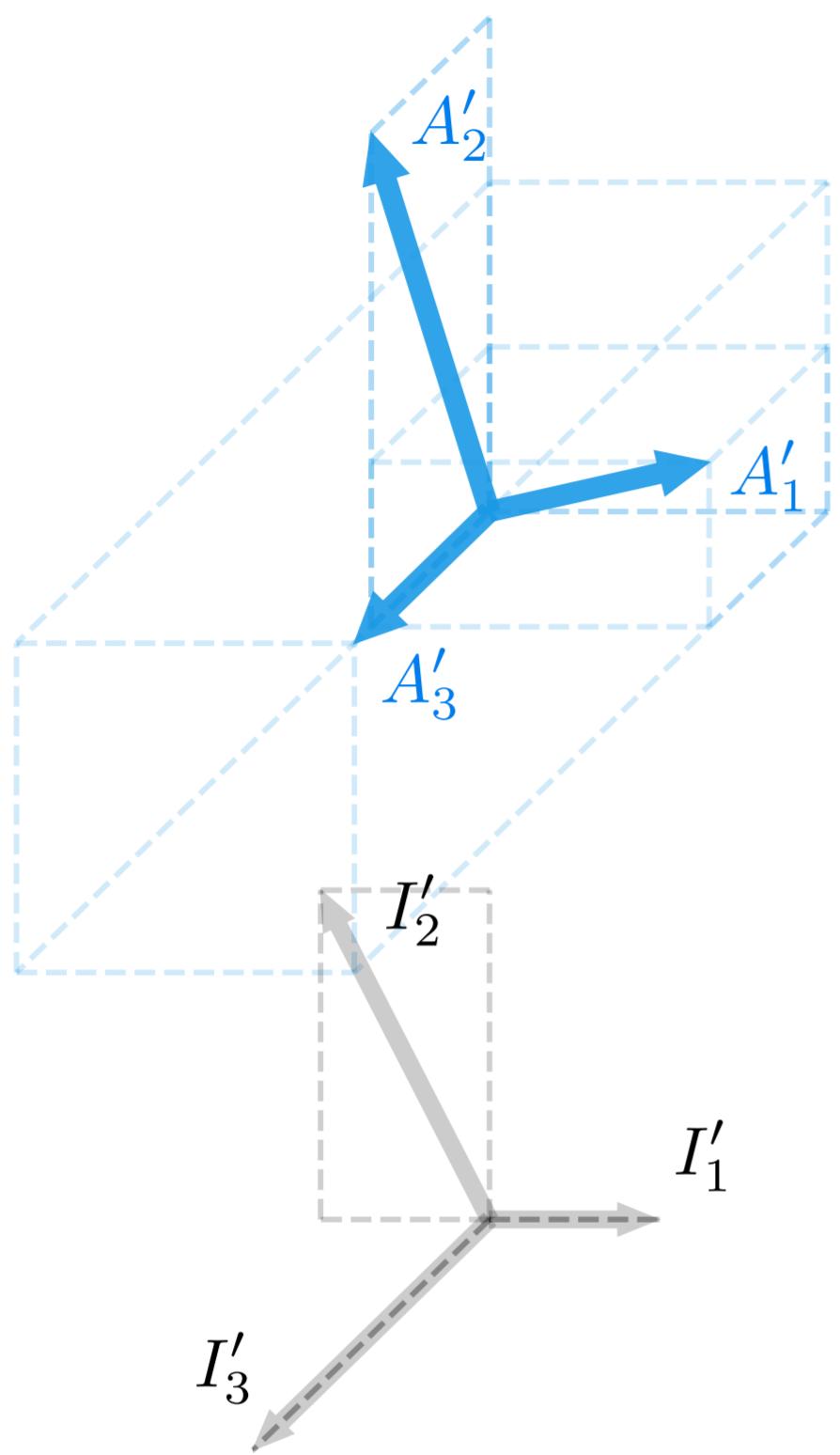
$$\begin{array}{c}
 \underbrace{AE^{(1)}}_{A'} \\
 \hline
 \underbrace{IE^{(1)}}_{I'} \\
 \end{array}
 \left[\begin{array}{ccc|c}
 1 & 1 & 1 & 1 \\
 1/2 & 2 & 1 & 1 \\
 1/2 & 1 & 2 & 1 \\
 \hline
 1/2 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 \\
 0 & 0 & 1 & 1 \\
 \end{array} \right]
 \left[\begin{array}{ccc}
 1 & -1 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
 \end{array} \right] = E^{(2)}$$

Column-reduction: operation 2...



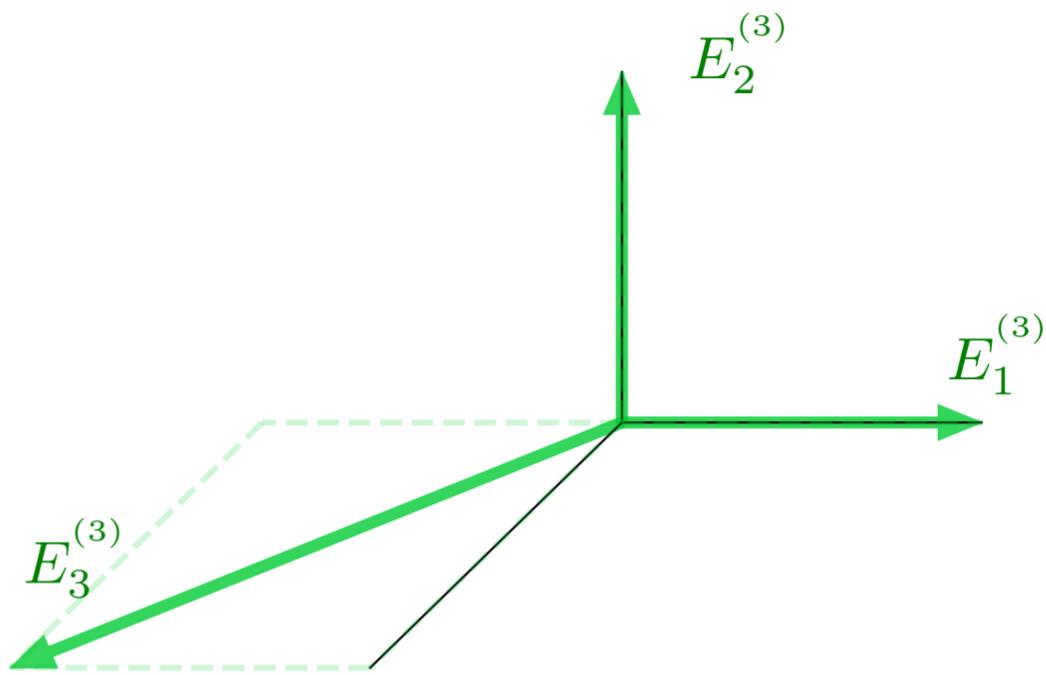
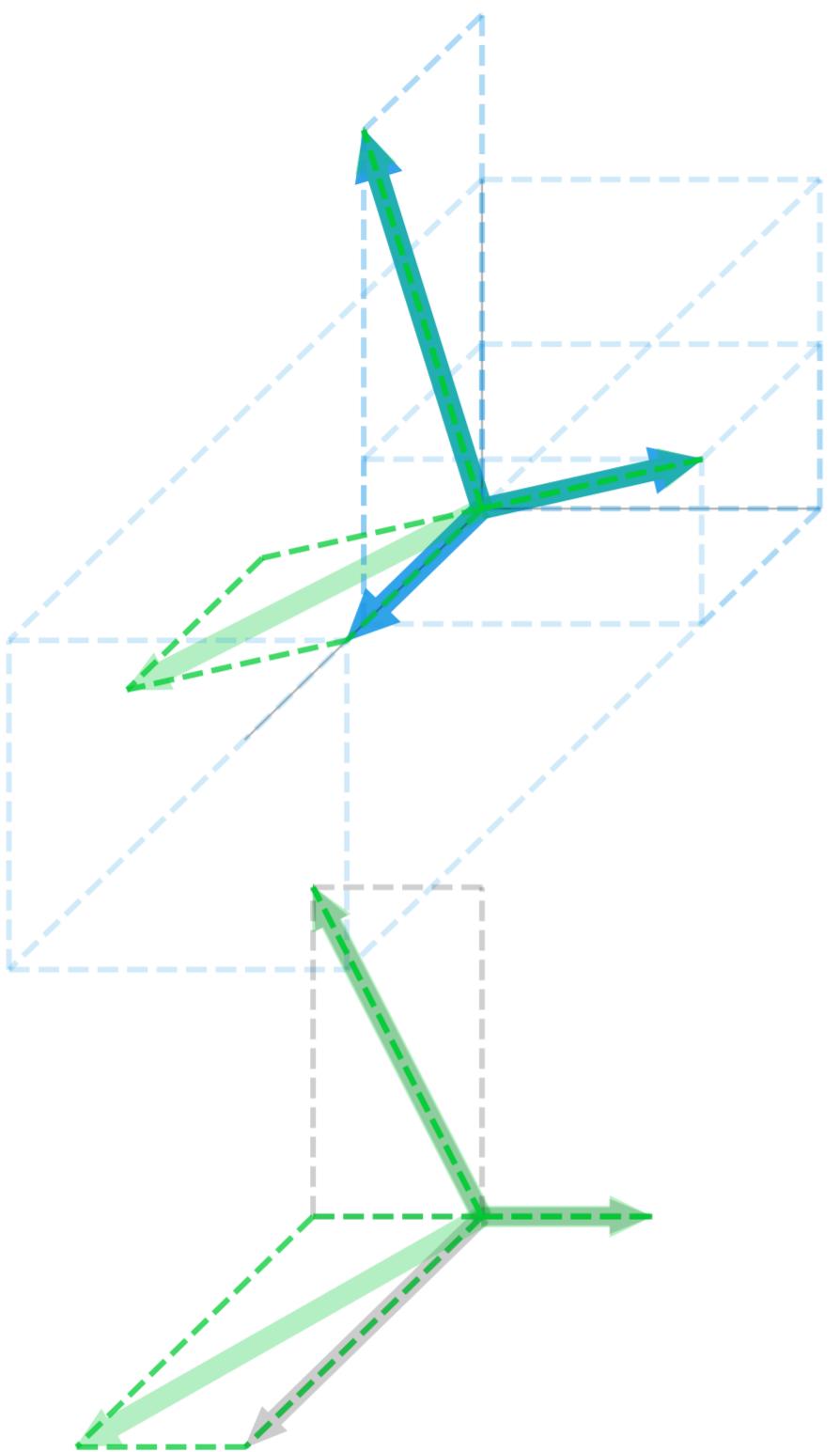
$$\begin{array}{c}
 \underbrace{AE^{(1)}E^{(2)}}_{A'} \\
 \hline
 IE^{(1)}E^{(2)} \\
 \hline
 I'
 \end{array}
 \quad
 \left[\begin{array}{ccc|c}
 1 & 0 & 1 & \\
 1/2 & 3/2 & 1 & \\
 1/2 & 1/2 & 2 & \\
 \hline
 1/2 & -1/2 & 0 & \\
 0 & 1 & 0 & \\
 0 & 0 & 1 &
 \end{array} \right]$$

Column-reduction: operation 3...



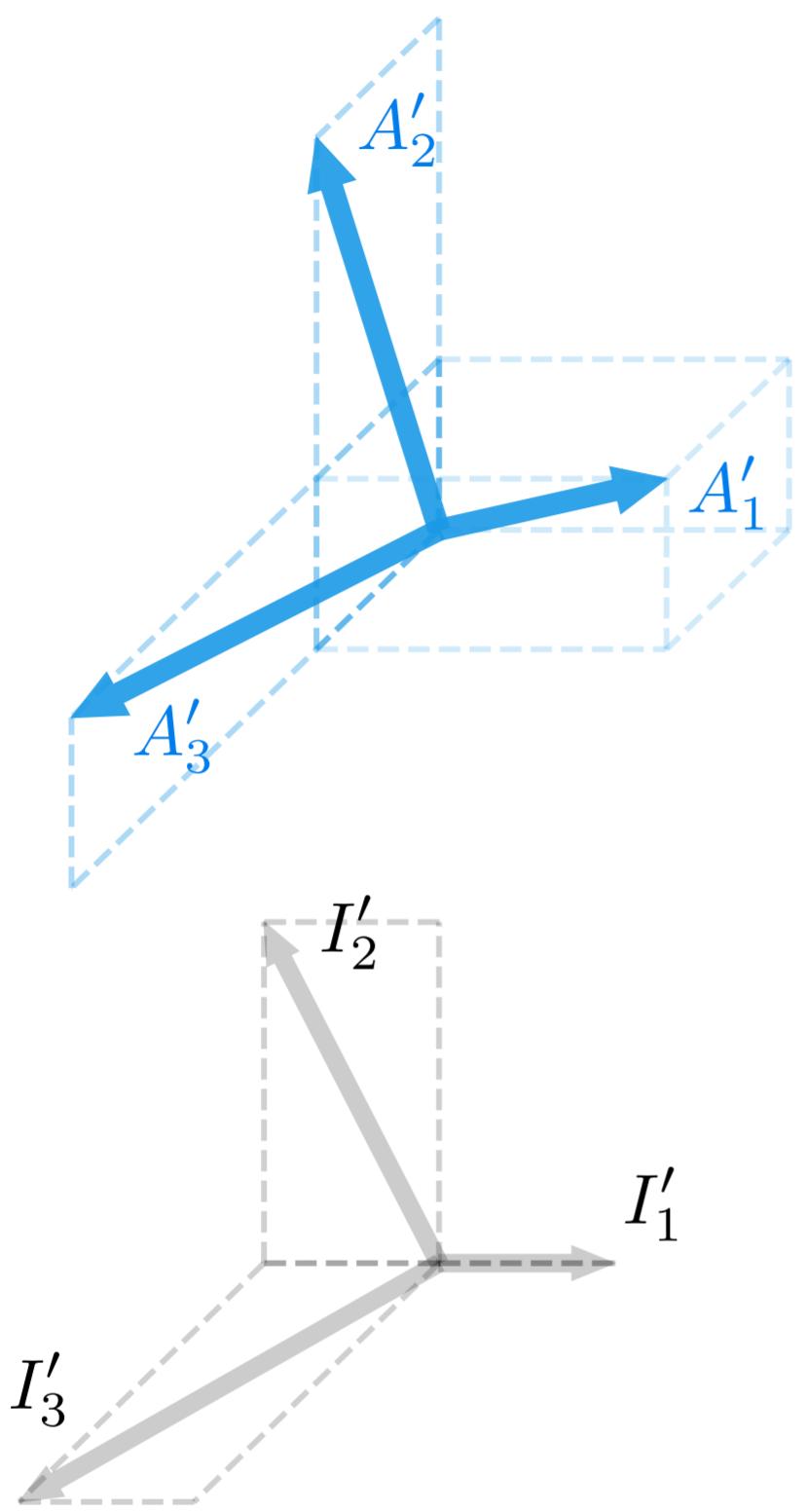
$$\begin{array}{c}
 \underbrace{AE^{(1)}E^{(2)}}_{A'} \\
 \times \\
 \underbrace{IE^{(1)}E^{(2)}}_{I'} \\
 \end{array}
 \quad \times \quad
 \begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 3/2 & 1 \\ 1/2 & 1/2 & 2 \end{bmatrix} \quad \times \quad
 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E^{(3)}$$

Column-reduction: operation 3...

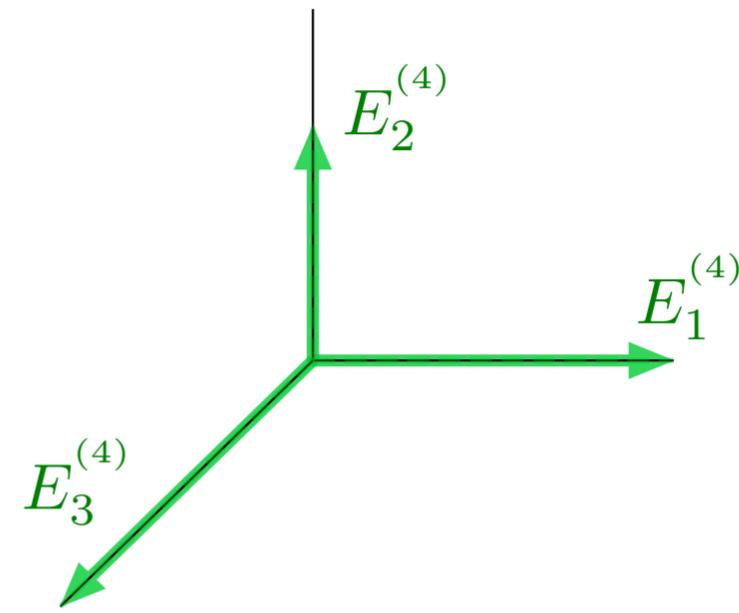
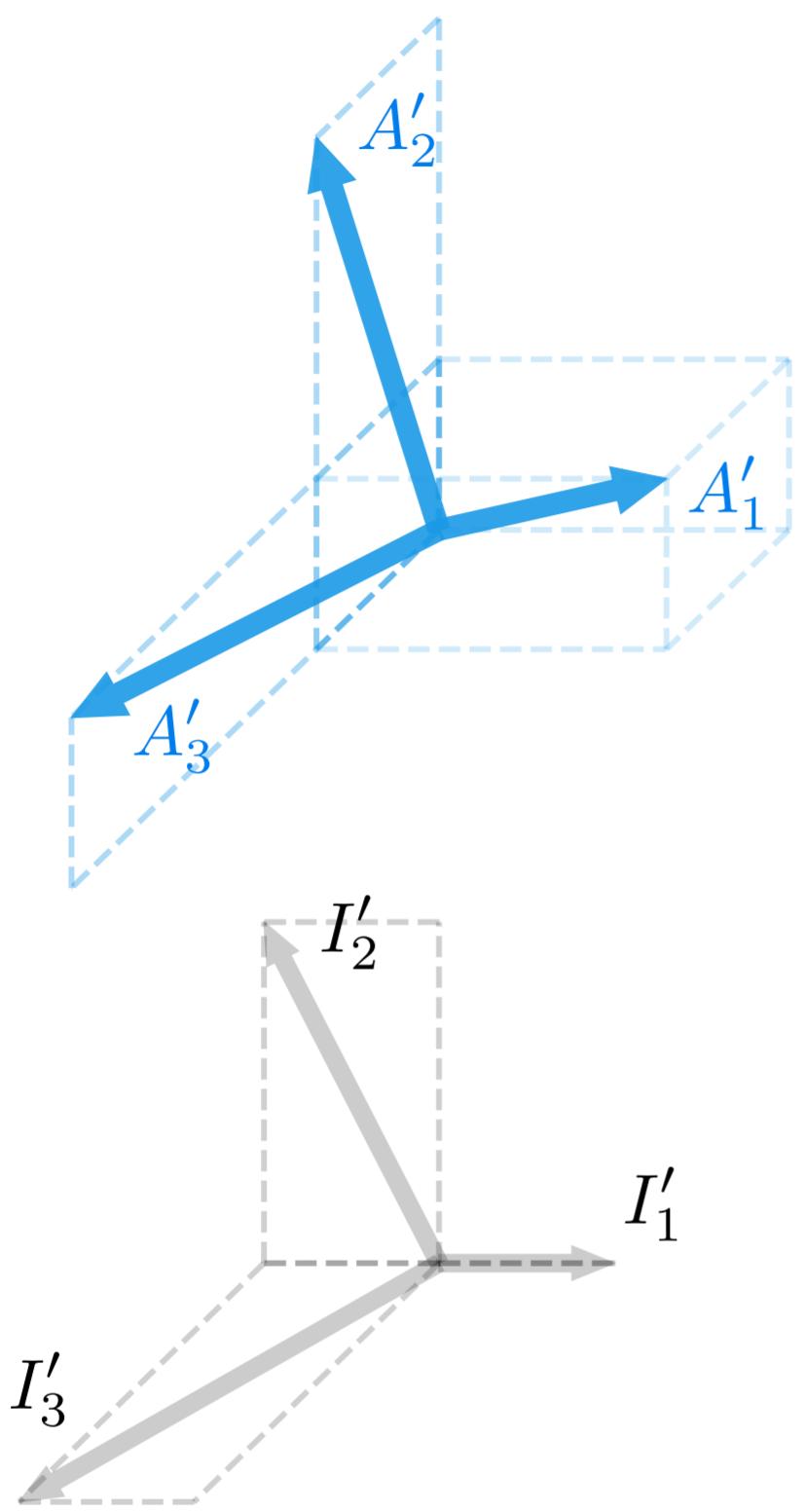


$$\begin{array}{c}
 \underbrace{AE^{(1)}E^{(2)}}_{A'} \\
 \\[10pt]
 \underbrace{IE^{(1)}E^{(2)}}_{I'}
 \end{array}
 \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1/2 & 3/2 & 1 \\ 1/2 & 1/2 & 2 \end{array} \right]
 \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]
 \left[\begin{array}{ccc} 1/2 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]
 E^{(3)}$$

Column-reduction: operation 3...

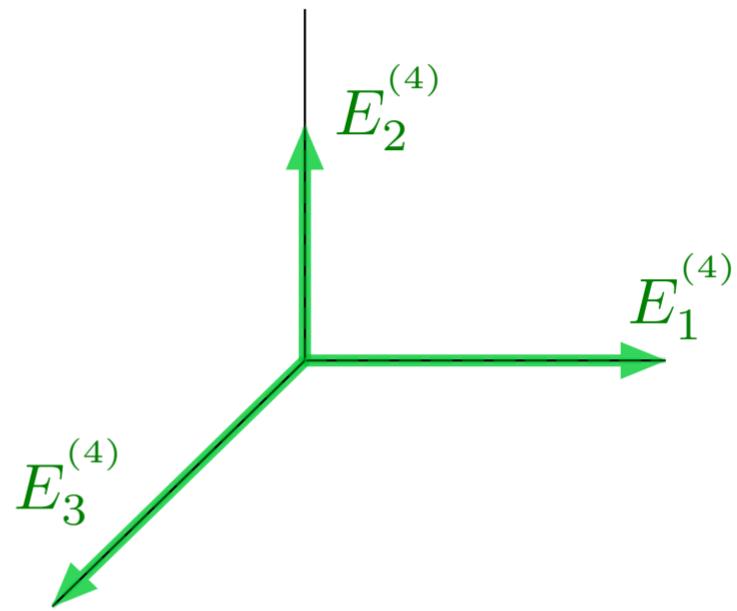
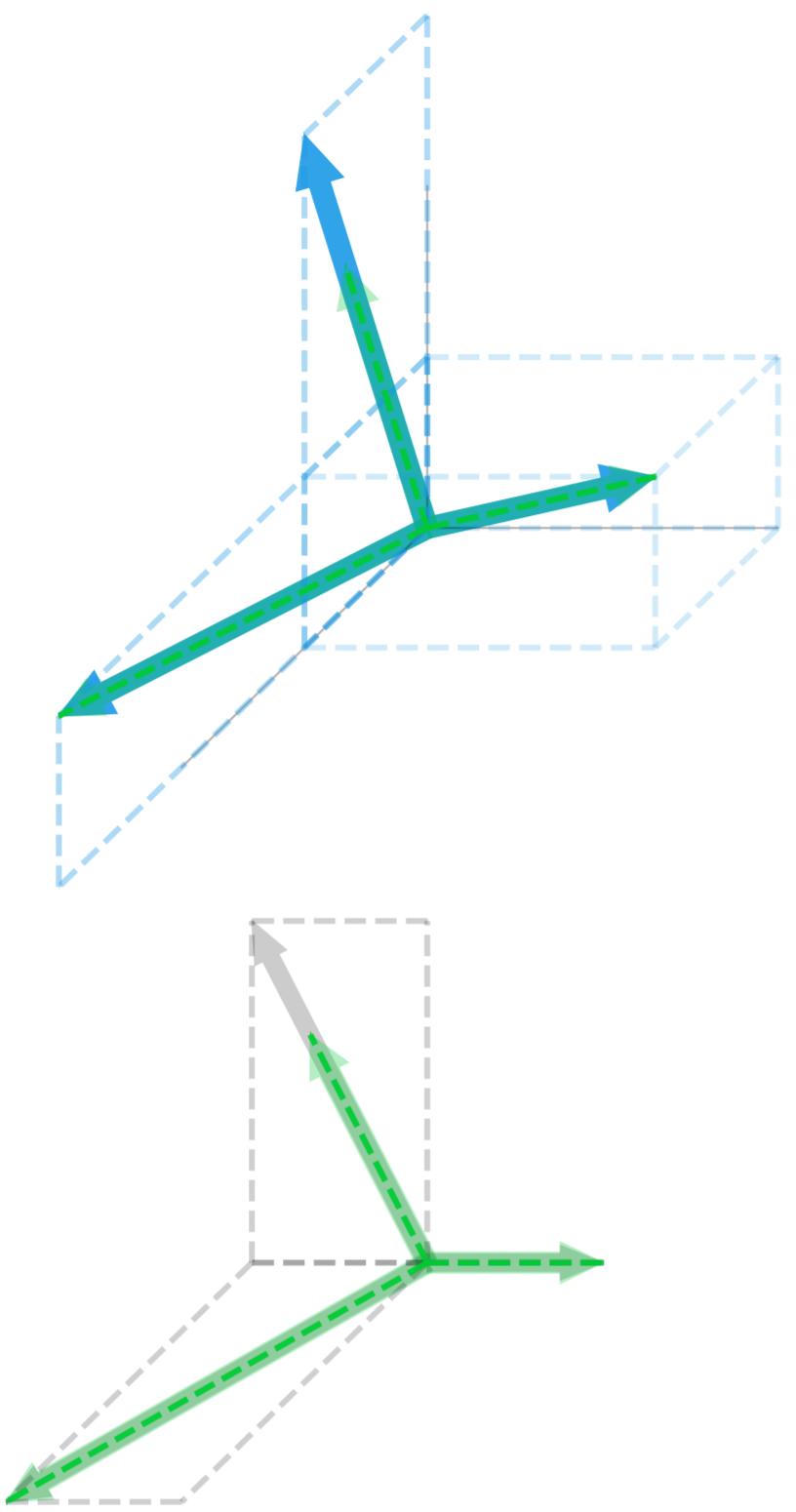


Column-reduction: operation 4...



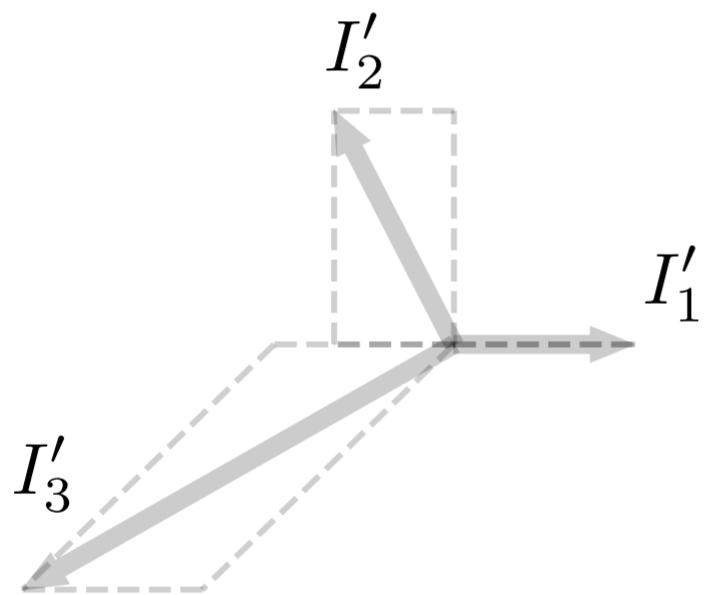
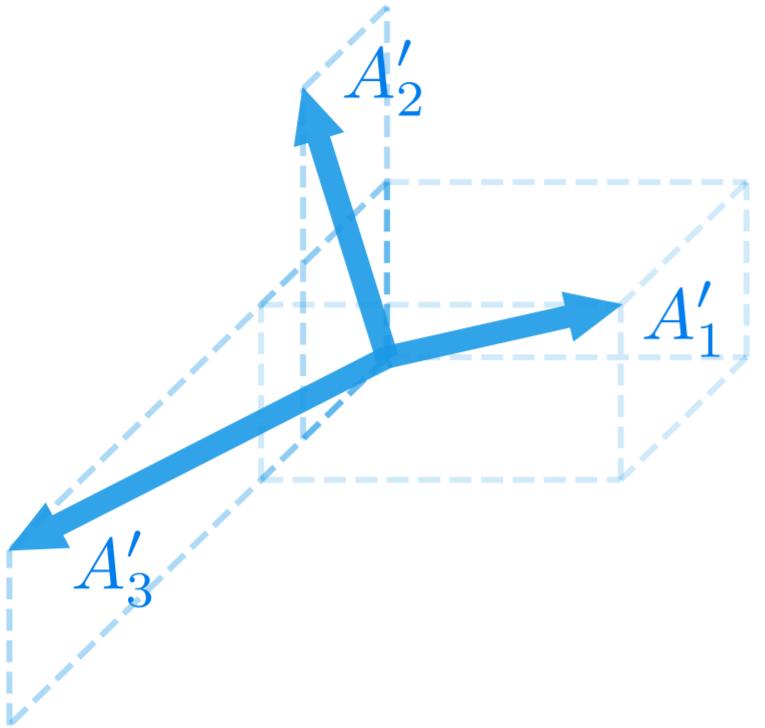
$$\begin{array}{c}
 \underbrace{AE^{(1)}E^{(2)}E^{(3)}}_{A'} \quad \times \quad \underbrace{IE^{(1)}E^{(2)}E^{(3)}}_{I'} \\
 \left[\begin{array}{ccc|ccc}
 1 & 0 & 0 & 1 & 0 & 0 \\
 1/2 & 3/2 & 1/2 & 0 & 2/3 & 0 \\
 1/2 & 1/2 & 3/2 & 0 & 0 & 1 \\
 \hline
 1/2 & -1/2 & -1/2 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0
 \end{array} \right] \quad E^{(4)}
 \end{array}$$

Column-reduction: operation 4...



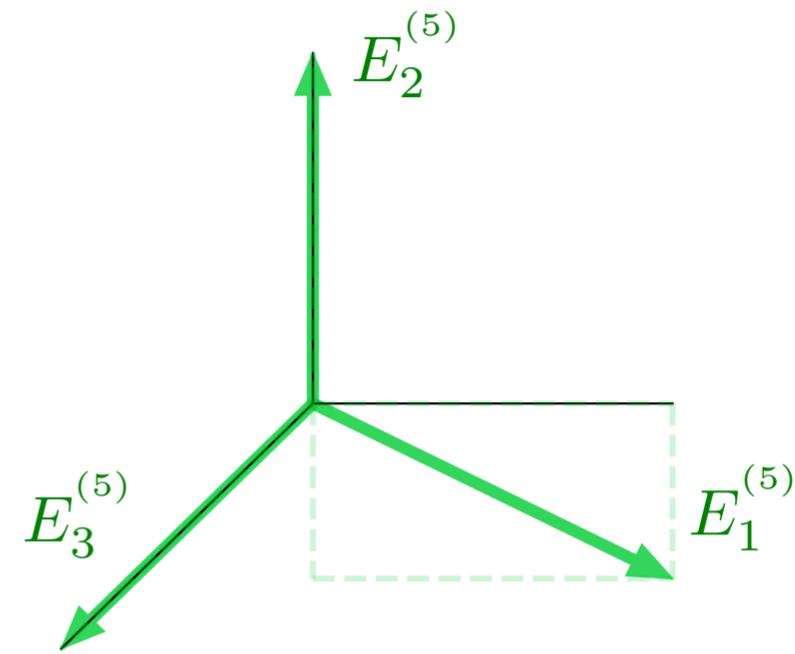
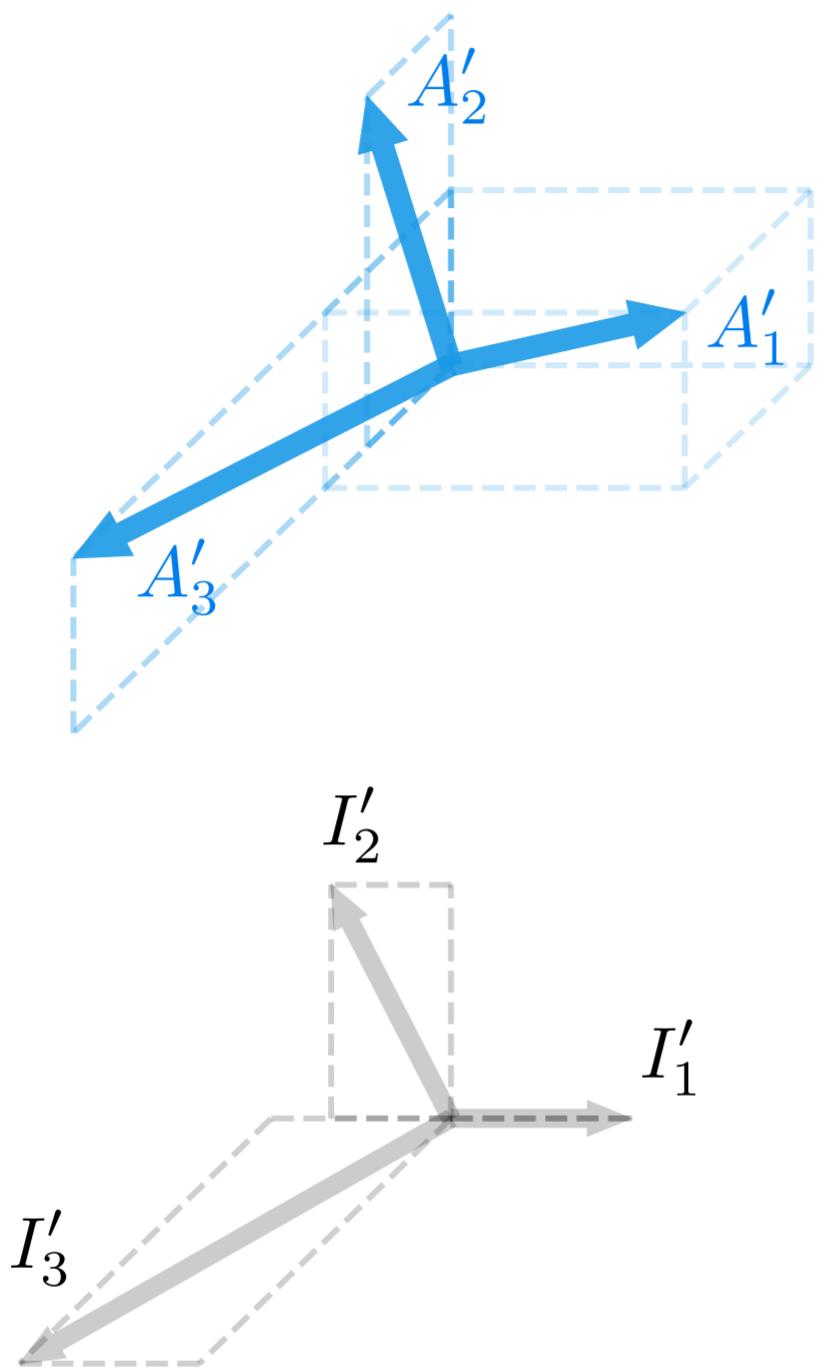
$$\begin{array}{c}
 \underbrace{AE^{(1)}E^{(2)}E^{(3)}}_{A'} \\
 \\
 \underbrace{IE^{(1)}E^{(2)}E^{(3)}}_{I'}
 \end{array}
 \left[\begin{array}{ccc|c}
 1 & 0 & 0 & 1 \\
 1/2 & 3/2 & 1/2 & 0 \\
 1/2 & 1/2 & 3/2 & 0 \\
 \hline
 1/2 & -1/2 & -1/2 & 1/2 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0
 \end{array} \right]
 \left[\begin{array}{ccc|c}
 1 & 0 & 0 & 1 \\
 0 & 2/3 & 0 & 0 \\
 0 & 0 & 1 & 0
 \end{array} \right]
 \quad E^{(4)}$$

Column-reduction: operation 4...



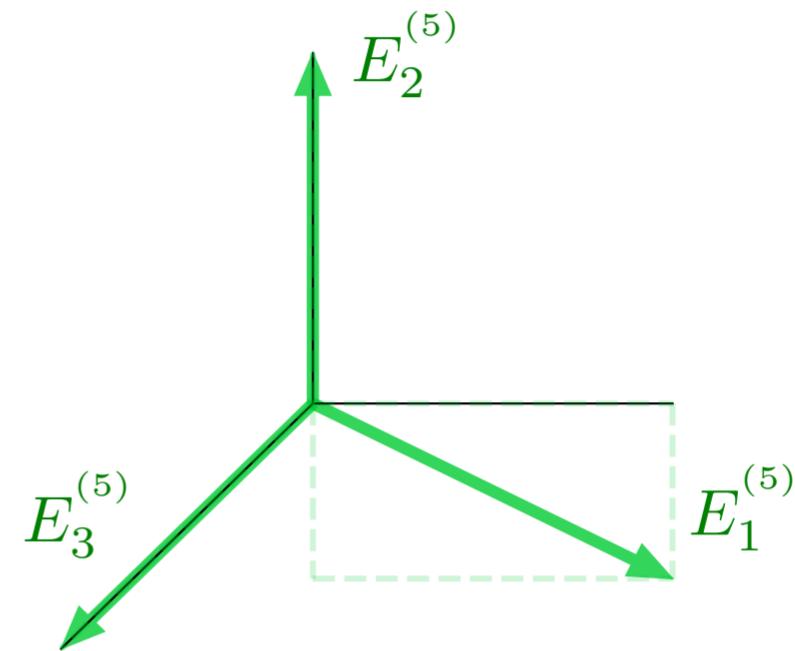
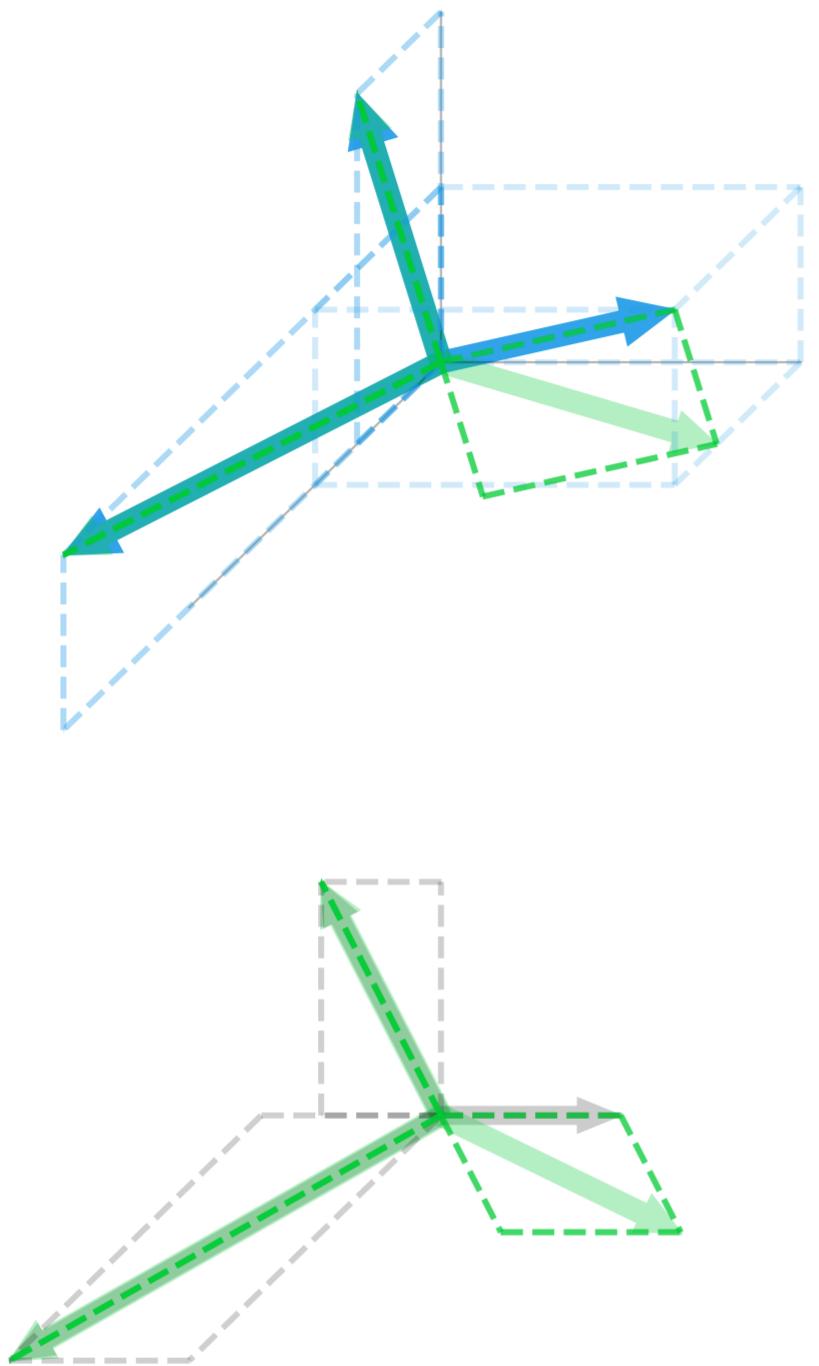
$\underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{A'}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/3 & 3/2 \end{bmatrix}$
$\underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{I'}$	$\begin{bmatrix} 1/2 & -1/3 & -1/2 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Column-reduction: operation 5...



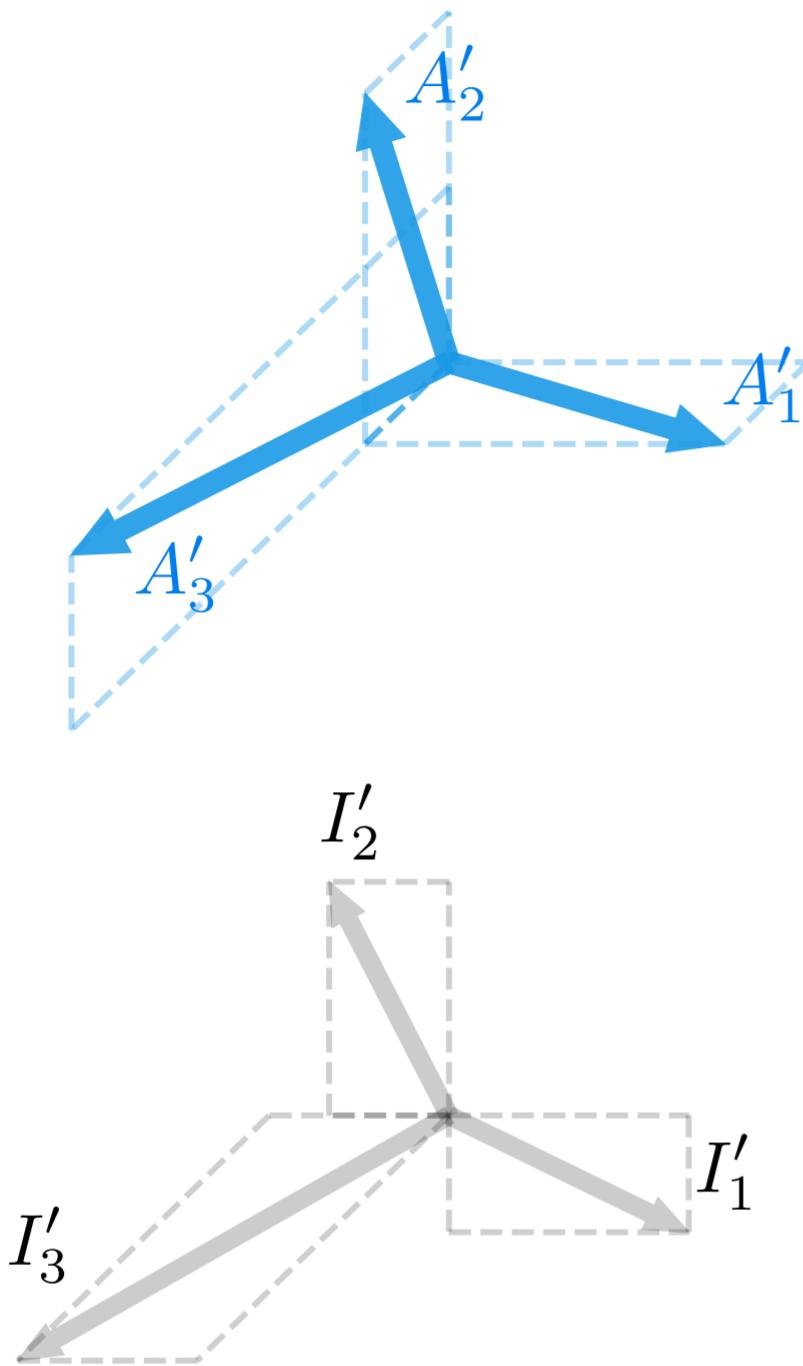
$$\begin{array}{c}
 \underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{A'} \\
 \times \\
 \underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{I'}
 \end{array}
 \quad \times \quad
 \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/3 & 3/2 \\ \hline 1/2 & -1/3 & -1/2 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E^{(5)}$$

Column-reduction: operation 5...



$$\begin{array}{c}
 \underbrace{AE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{A'} \\
 \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/3 & 3/2 \end{array} \right] \quad \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \\
 E^{(5)} \\
 \hline
 \underbrace{IE^{(1)}E^{(2)}E^{(3)}E^{(4)}}_{I'} \\
 \left[\begin{array}{ccc} 1/2 & -1/3 & -1/2 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{array} \right]
 \end{array}$$

Column-reduction: operation 5...

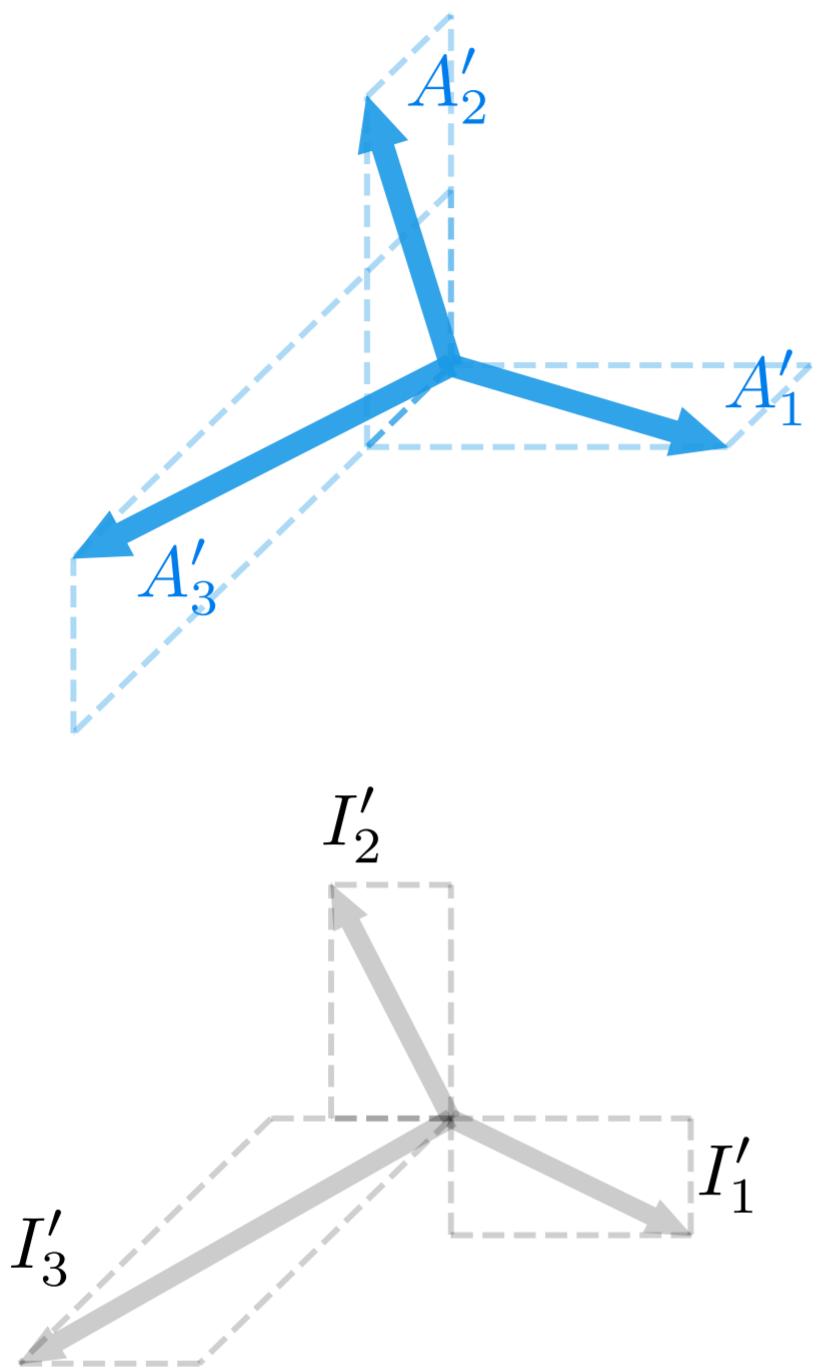


$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}}_{A'} E^{(4)}E^{(5)}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}}_{I'} E^{(4)}E^{(5)}$$

1	0	0
0	1	1/2
1/3	1/3	3/2
2/3	-1/3	-1/2
-1/3	2/3	0
0	0	1

Column-reduction: operation 6...



$$AE \underbrace{E^{(4)} E^{(5)}}_{A'} E^{(1)} E^{(2)} E^{(3)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 1/3 & 1/3 & 3/2 \end{bmatrix}$$

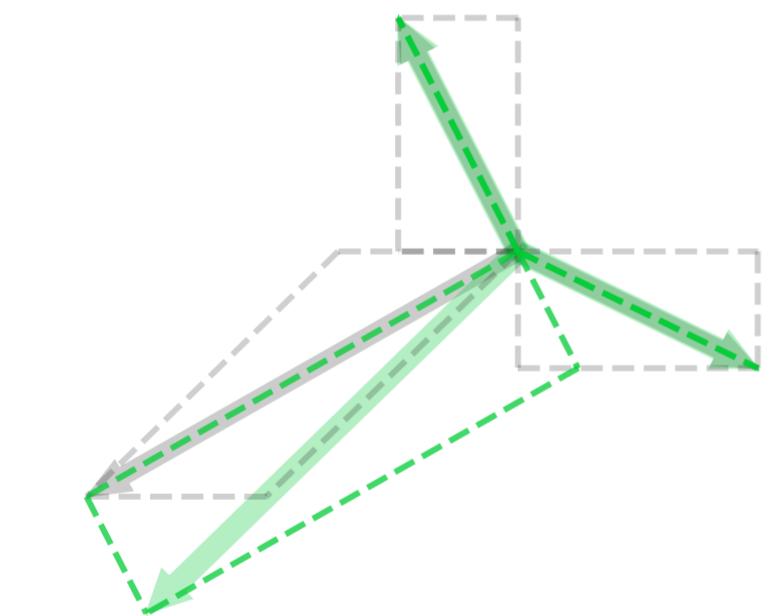
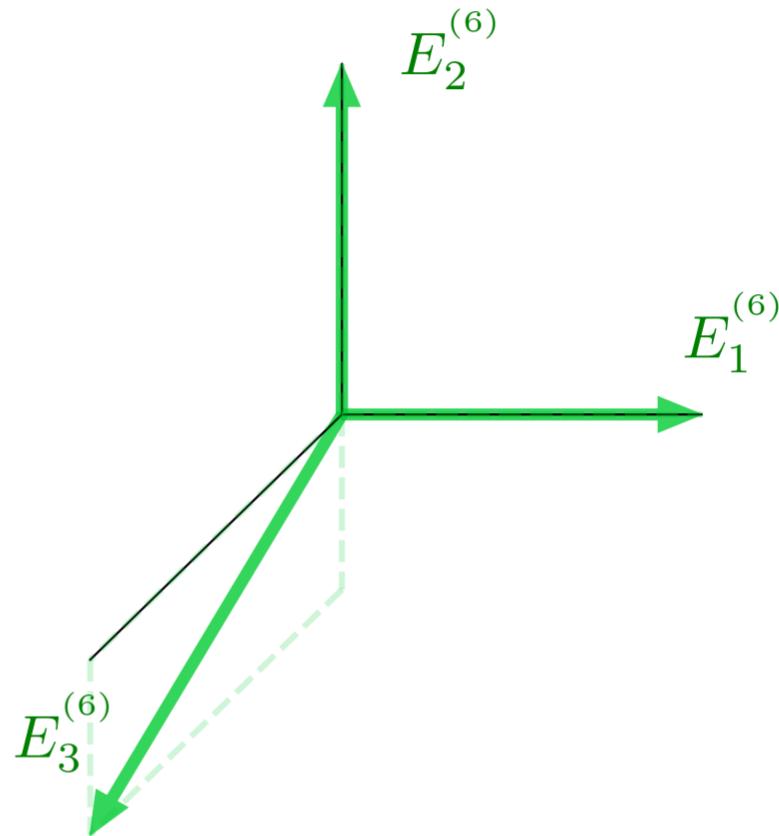
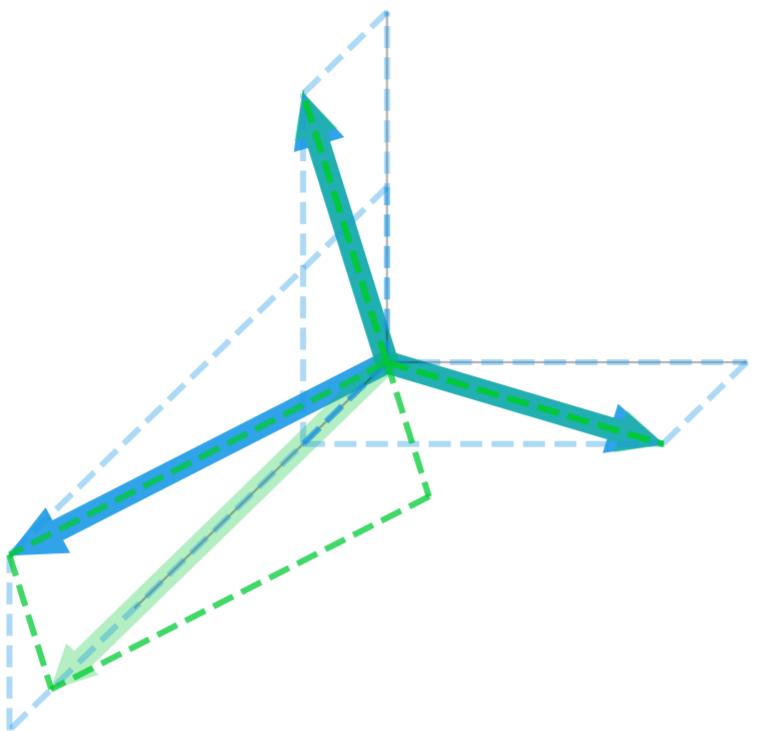
$$IE \underbrace{E^{(4)} E^{(5)}}_{I'} E^{(1)} E^{(2)} E^{(3)}$$

\times

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$E^{(6)}$

Column-reduction: operation 6...



$$AE^{(1)}E^{(2)}E^{(3)} \\ \underbrace{E^{(4)}E^{(5)}}_{A'}$$

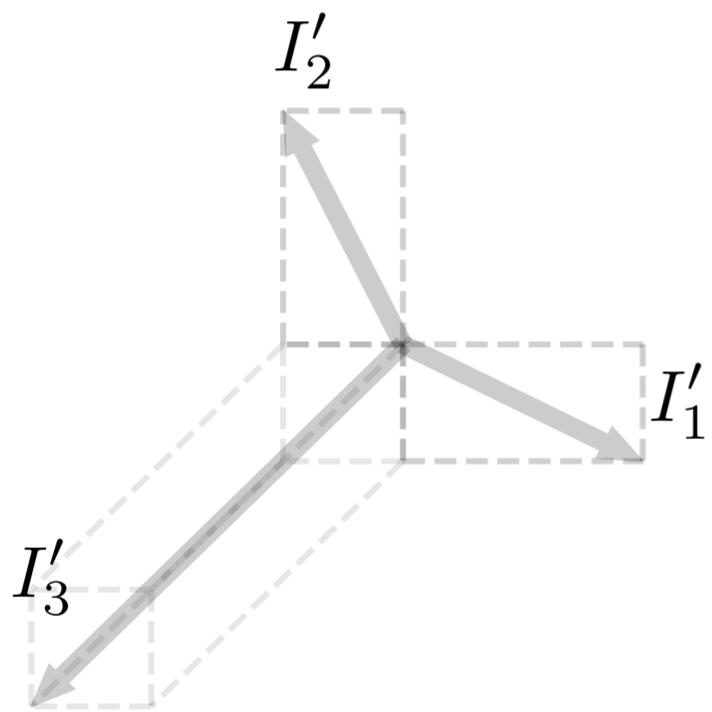
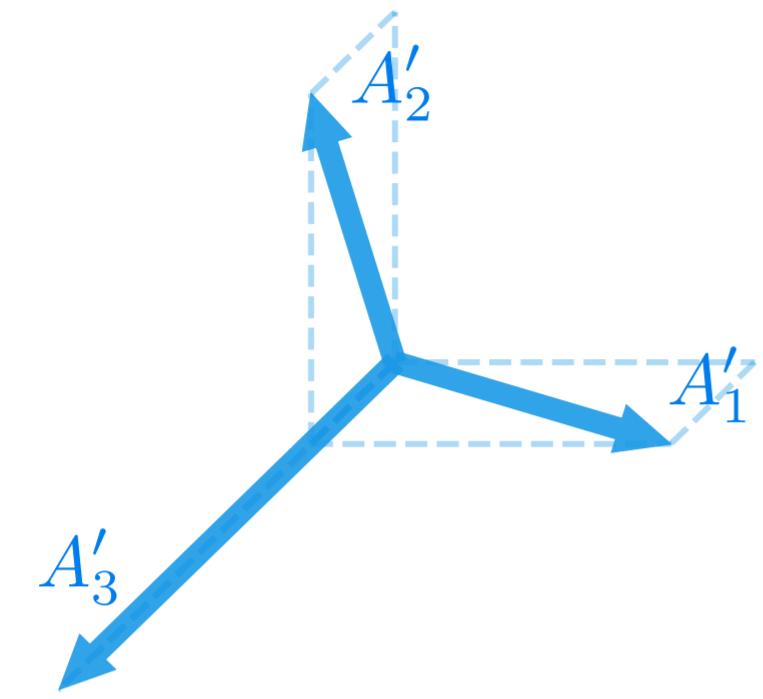
$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 1/3 & 1/3 & 3/2 \end{array} \right]$$

$$IE^{(1)}E^{(2)}E^{(3)} \\ \underbrace{E^{(4)}E^{(5)}}_{I'}$$

$$\left[\begin{array}{ccc} 2/3 & -1/3 & -1/2 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$E^{(6)}$$

Column-reduction: operation 6...

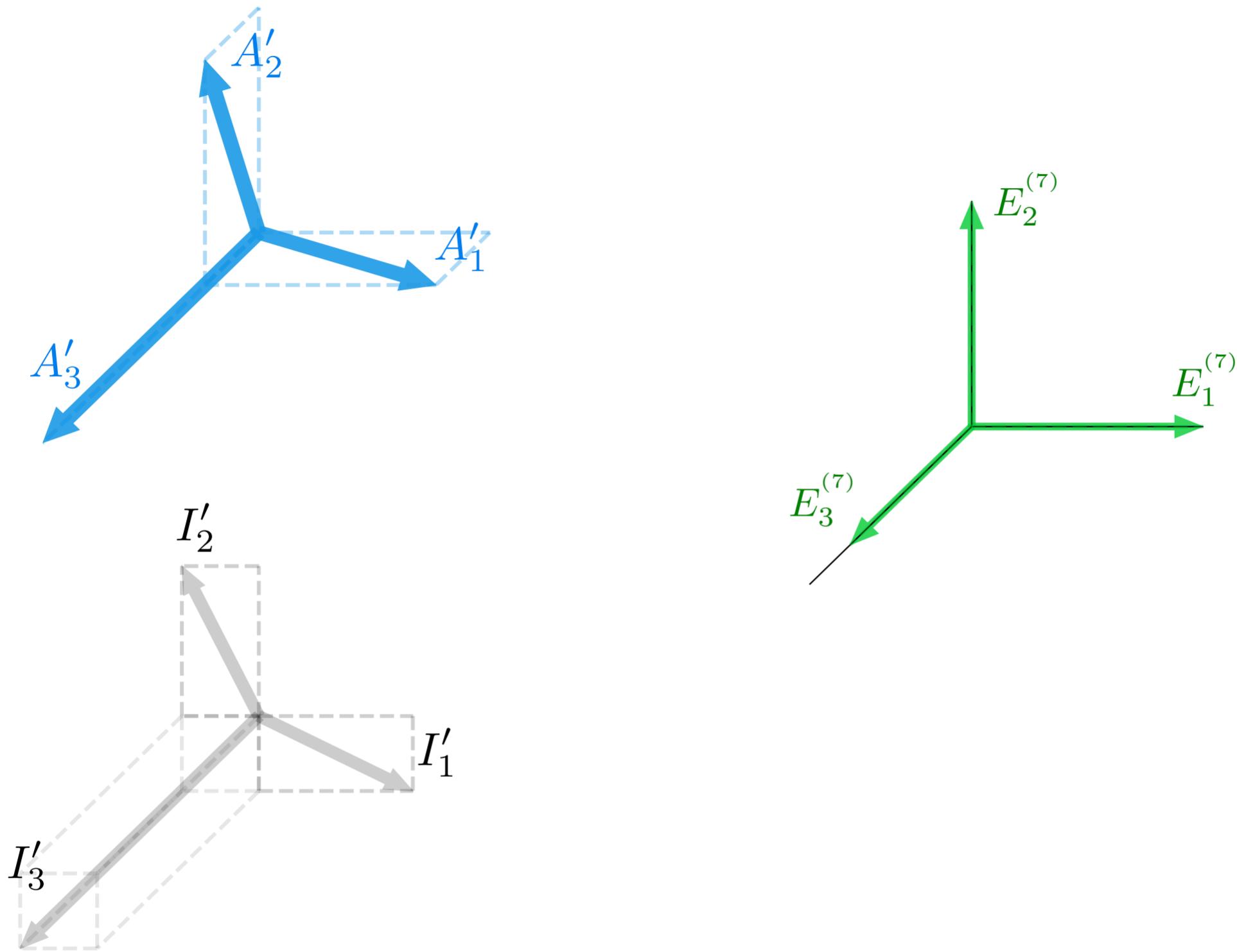


$$AE^{(1)}E^{(2)}E^{(3)} \\ \underbrace{E^{(4)}E^{(5)}E^{(6)}}_{A'}$$

$$IE^{(1)}E^{(2)}E^{(3)} \\ \underbrace{E^{(4)}E^{(5)}E^{(6)}}_{I'}$$

1	0	0
0	1	0
1/3	1/3	4/3
2/3	-1/3	-1/3
-1/3	2/3	-1/3
0	0	1

Column-reduction: operation 7...



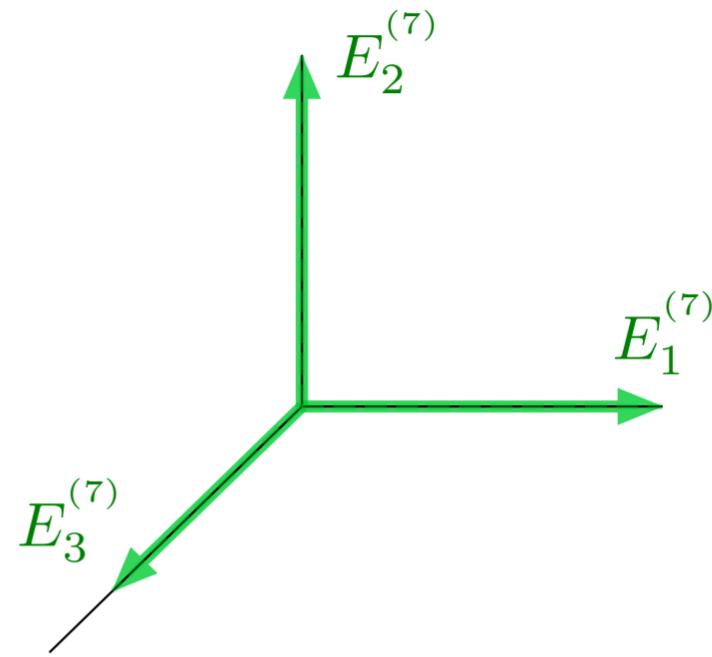
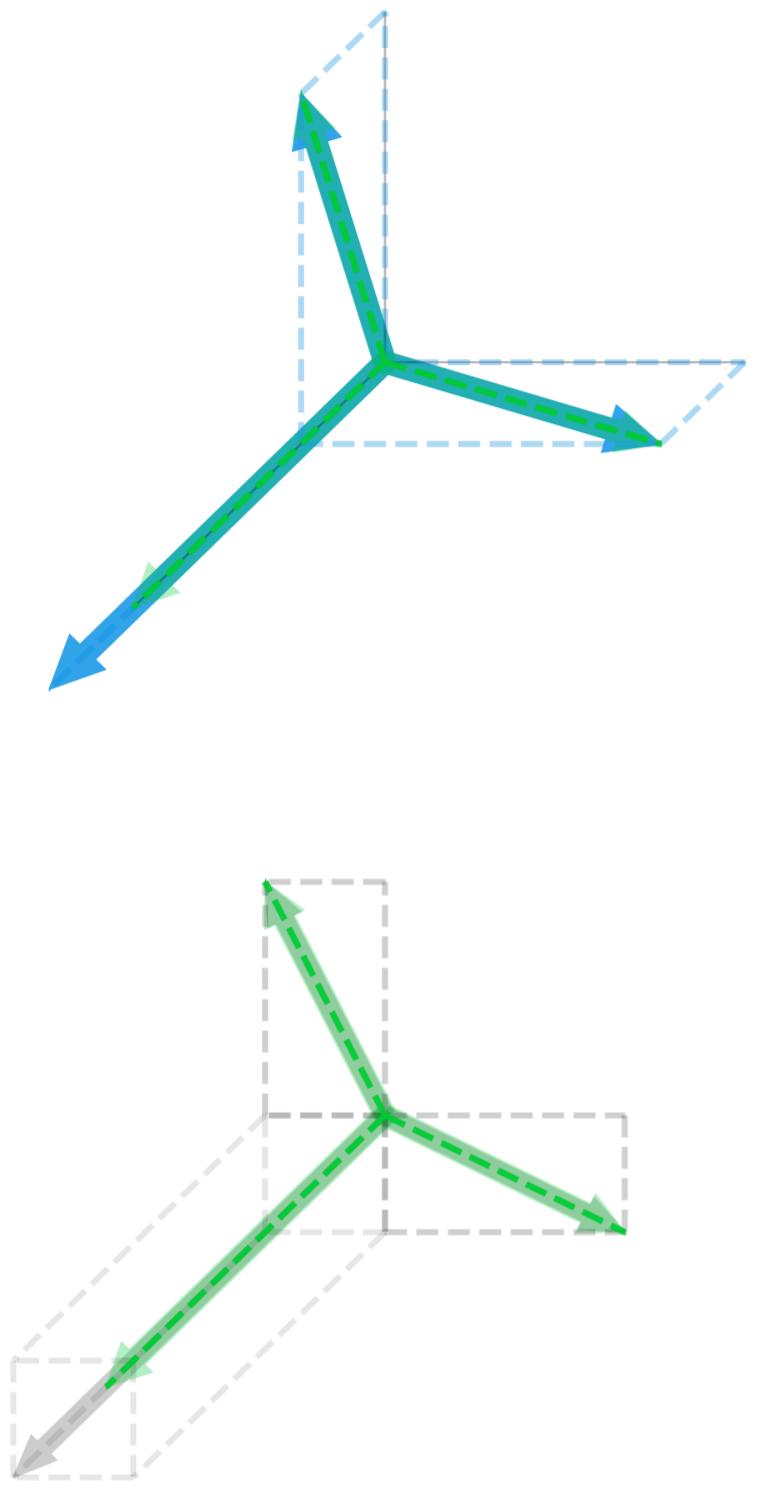
$$AE^{(1)}E^{(2)}E^{(3)} \\ \underbrace{E^{(4)}E^{(5)}E^{(6)}}_{A'}$$

$$IE^{(1)}E^{(2)}E^{(3)} \\ \underbrace{E^{(4)}E^{(5)}E^{(6)}}_{I'}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 4/3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{bmatrix}$$

$E^{(7)}$

Column-reduction: operation 7...



$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}}_{A'} \underbrace{E^{(4)}E^{(5)}E^{(6)}}_{A'}$$

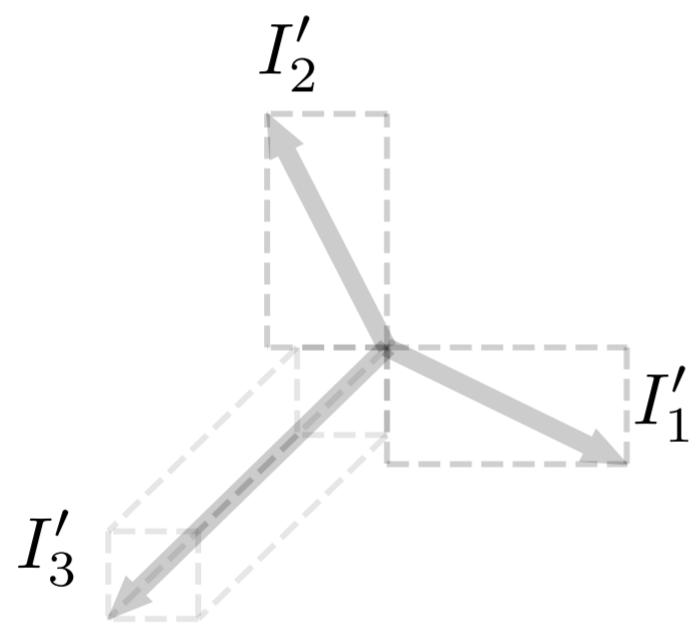
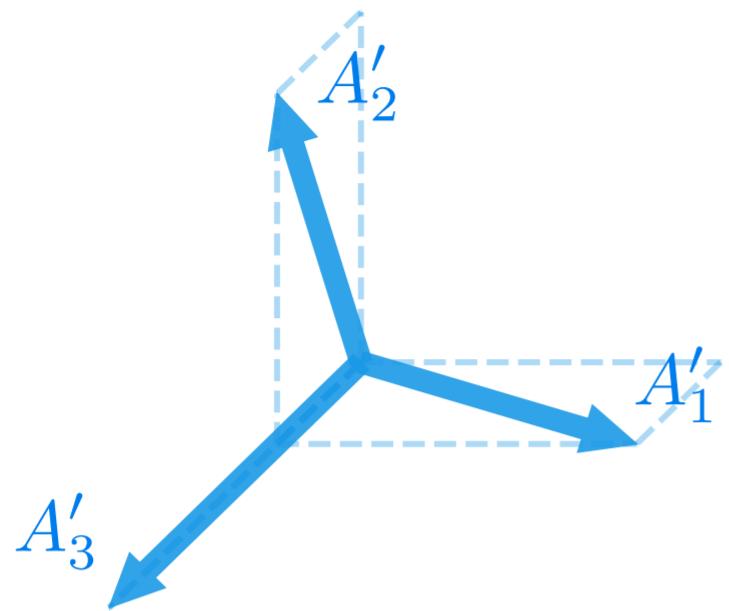
$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}}_{I'} \underbrace{E^{(4)}E^{(5)}E^{(6)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 4/3 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array} \right]$$

$$E^{(7)}$$

$$\left[\begin{array}{ccc} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ 0 & 0 & 1 \end{array} \right]$$

Column-reduction: operation 7...

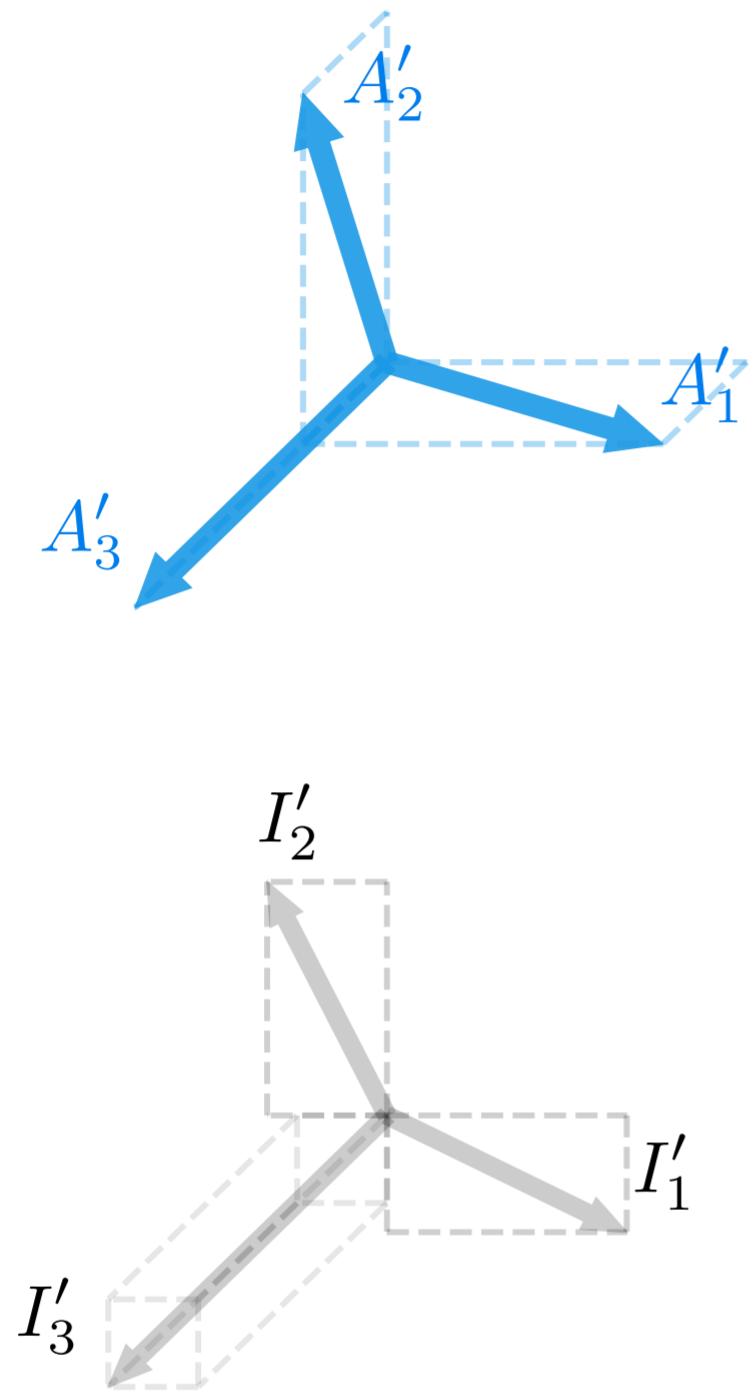


$$\underbrace{AE^{(1)}E^{(2)}E^{(3)} \\ E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{A'}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)} \\ E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{I'}$$

1	0	0
0	1	0
1/3	1/3	1
2/3	-1/3	-1/4
-1/3	2/3	-1/4
0	0	3/4

Column-reduction: operation 8...



$$\underbrace{AE^{(1)}E^{(2)}E^{(3)}}_{A'} \quad \underbrace{E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{}$$

$$\underbrace{IE^{(1)}E^{(2)}E^{(3)}}_{I'} \quad \underbrace{E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{}$$

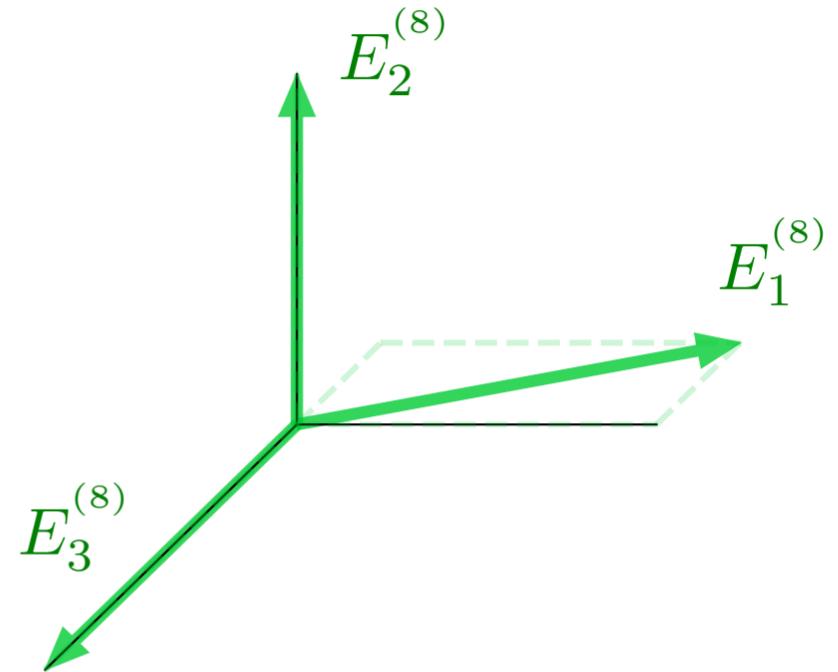
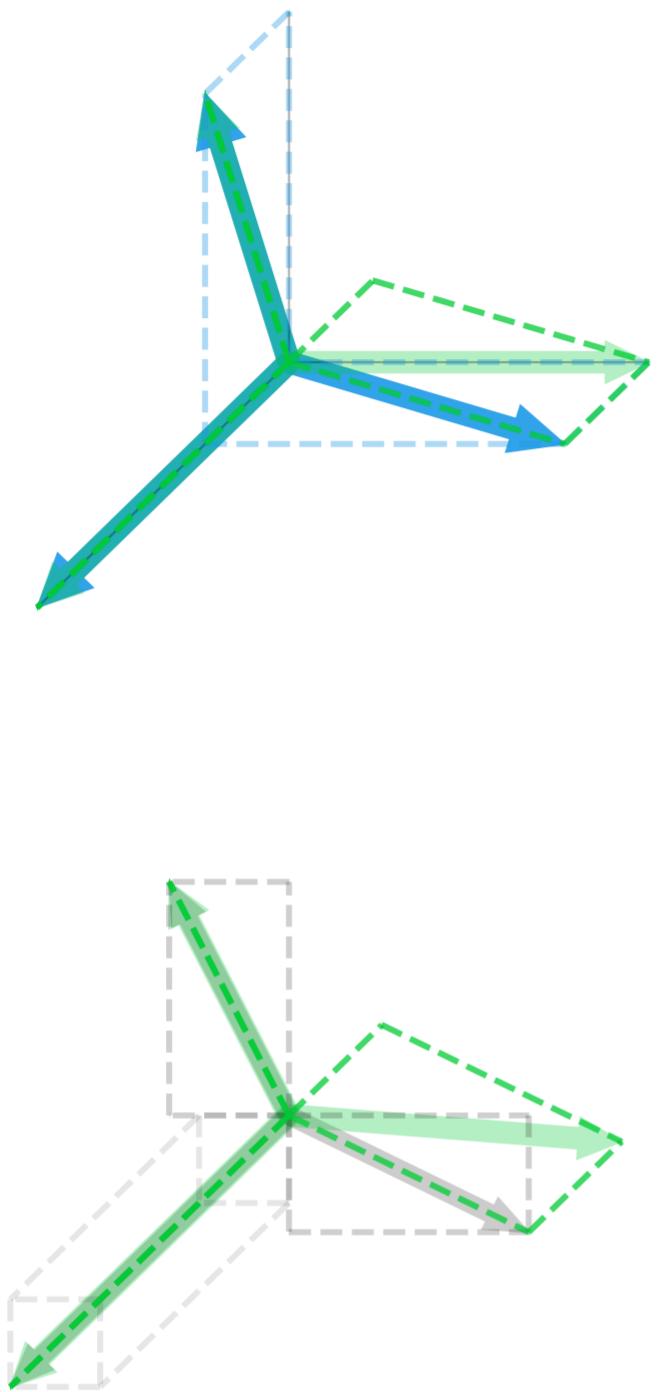
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

\times

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

$E^{(8)}$

Column-reduction: operation 8...



$$\underbrace{AE^{(1)}E^{(2)}E^{(3)} \\ E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{A'}$$

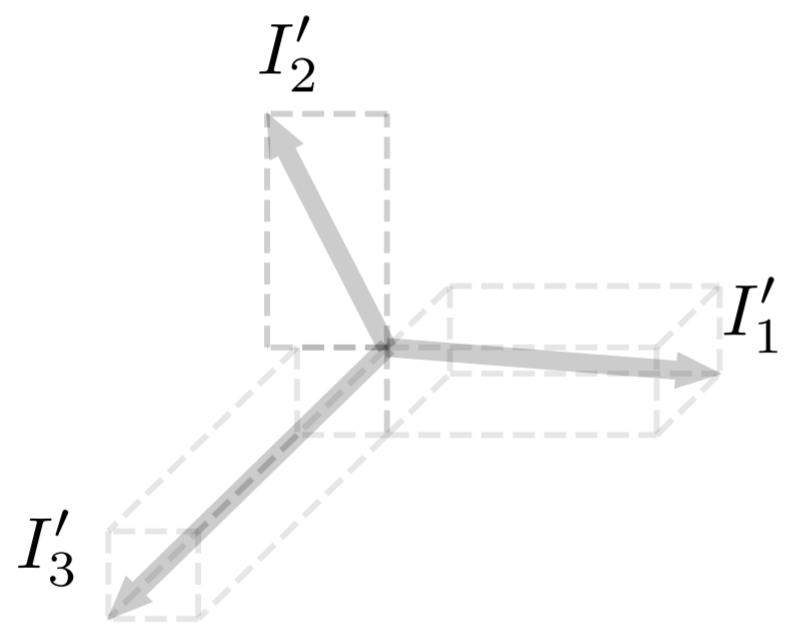
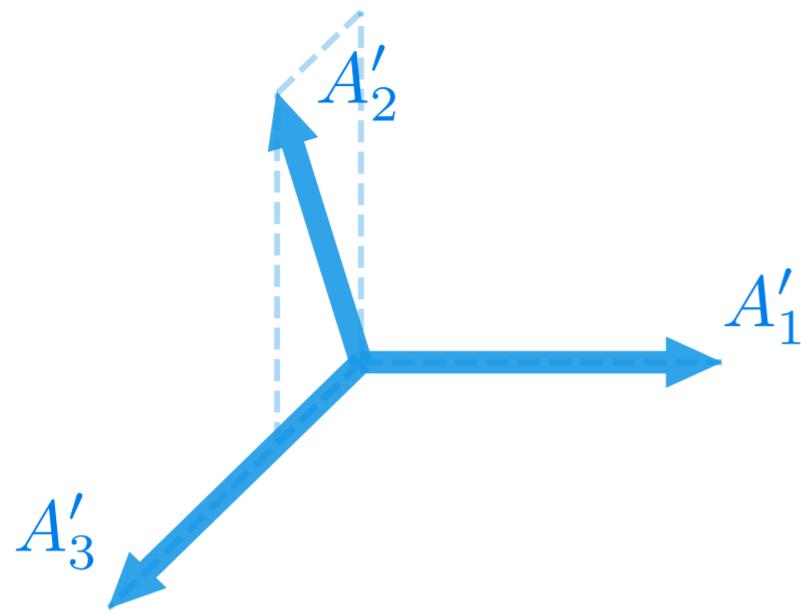
$$\underbrace{IE^{(1)}E^{(2)}E^{(3)} \\ E^{(4)}E^{(5)}E^{(6)}E^{(7)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{array} \right]$$

$$E^{(8)}$$

$$\left[\begin{array}{ccc} 2/3 & -1/3 & -1/4 \\ -1/3 & 2/3 & -1/4 \\ 0 & 0 & 3/4 \end{array} \right]$$

Column-reduction: operation 8...

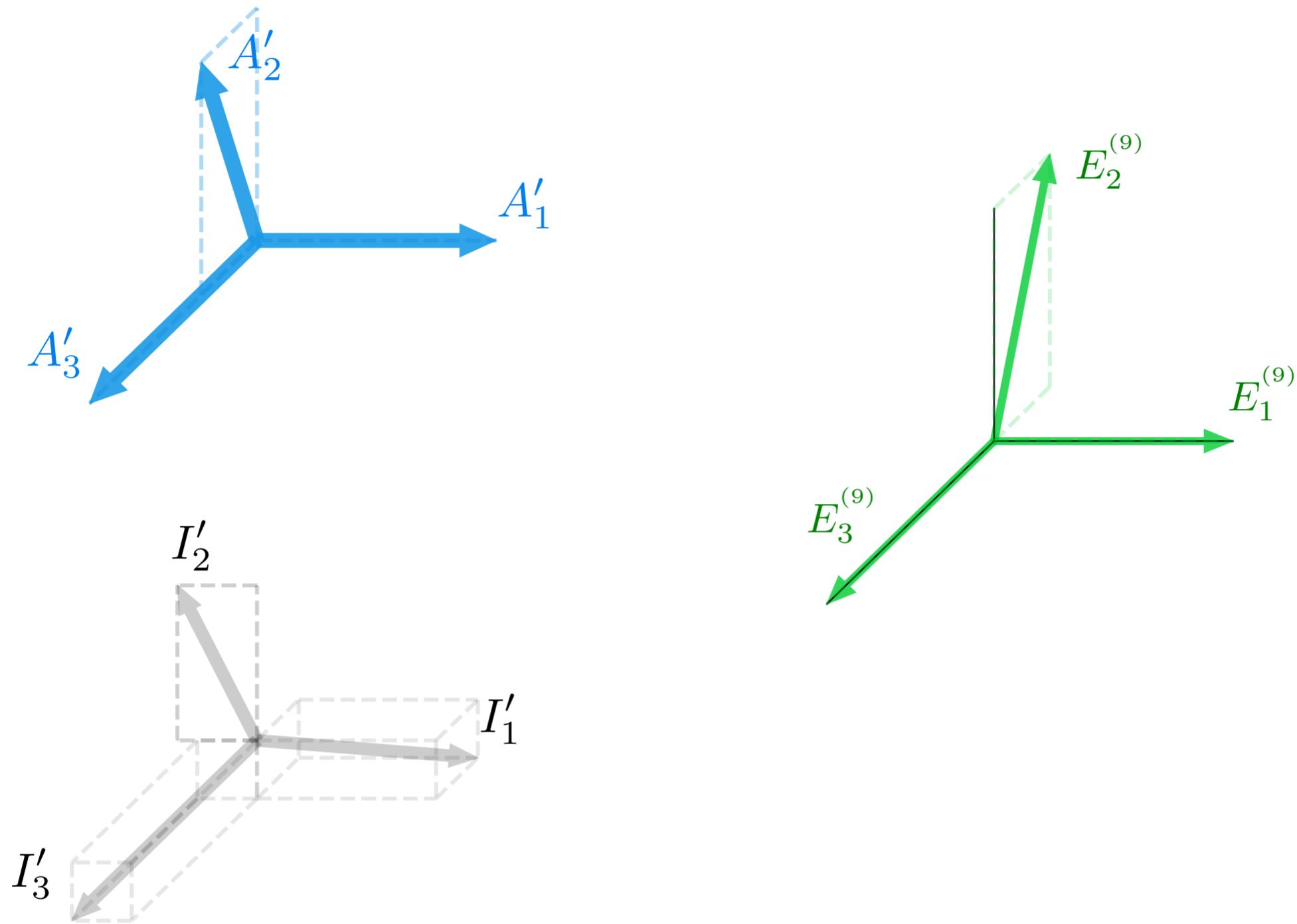


$$AE^{(1)}E^{(2)}E^{(3)} \\ E^{(4)}E^{(5)}E^{(6)} \\ \underbrace{E^{(7)}E^{(8)}}_{A'}$$

$$IE^{(1)}E^{(2)}E^{(3)} \\ E^{(4)}E^{(5)}E^{(6)} \\ \underbrace{E^{(7)}E^{(8)}}_{I'}$$

1	0	0
0	1	0
0	1/3	1
3/4	-1/3	-1/4
-1/4	2/3	-1/4
-1/4	0	3/4

Column-reduction: operation 9...



$$AE^{(1)}E^{(2)}E^{(3)} \\ E^{(4)}E^{(5)}E^{(6)} \\ \underbrace{E^{(7)}E^{(8)}}_{A'}$$

$$IE^{(1)}E^{(2)}E^{(3)} \\ E^{(4)}E^{(5)}E^{(6)} \\ \underbrace{E^{(7)}E^{(8)}}_{I'}$$

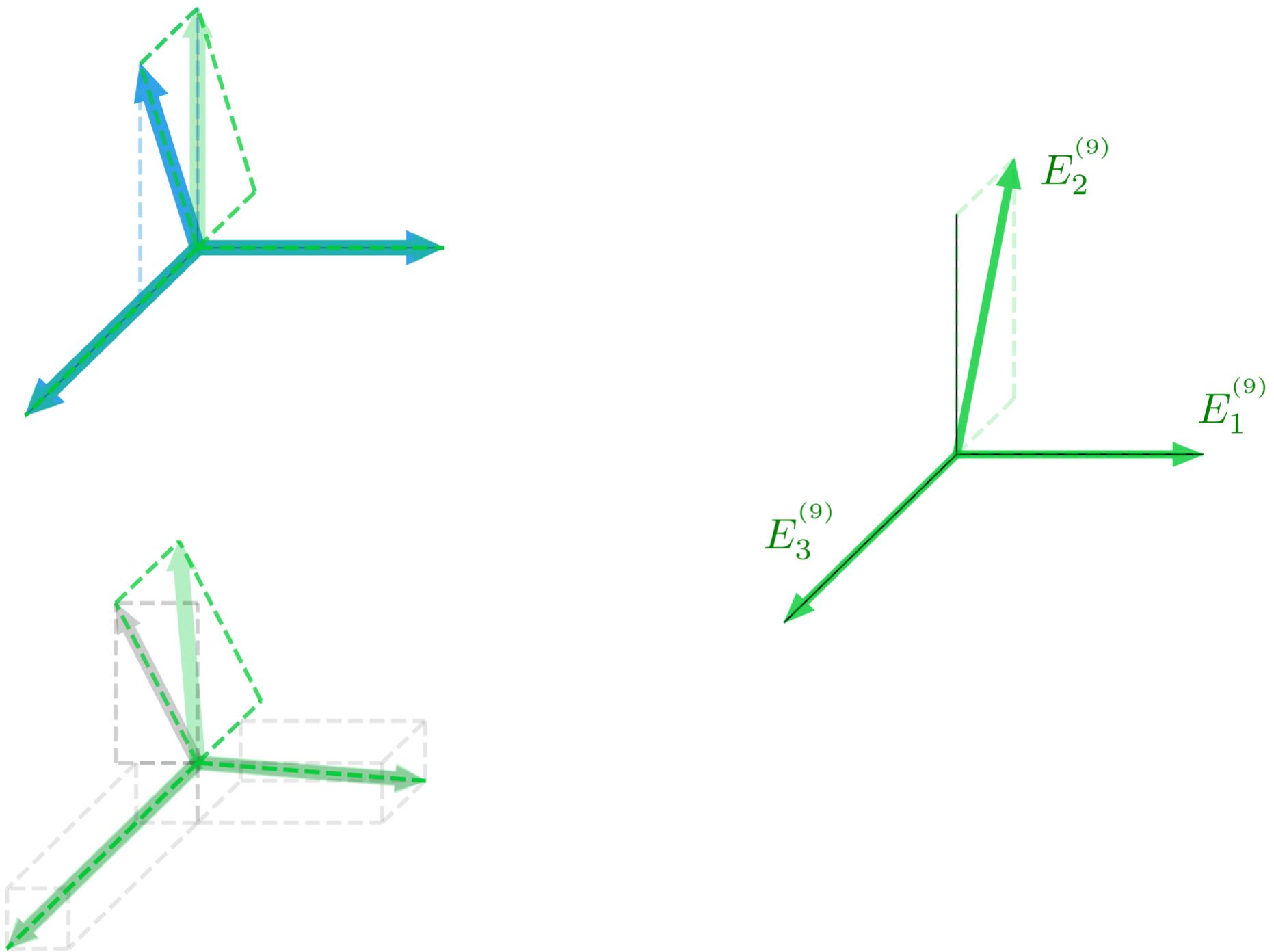
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

\times

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix}$$

$E^{(9)}$

Column-reduction: operation 9...



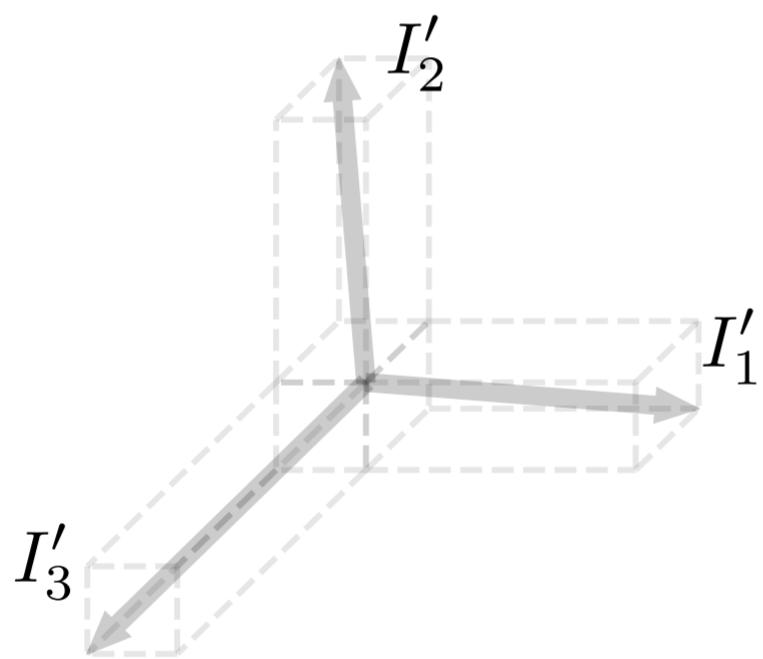
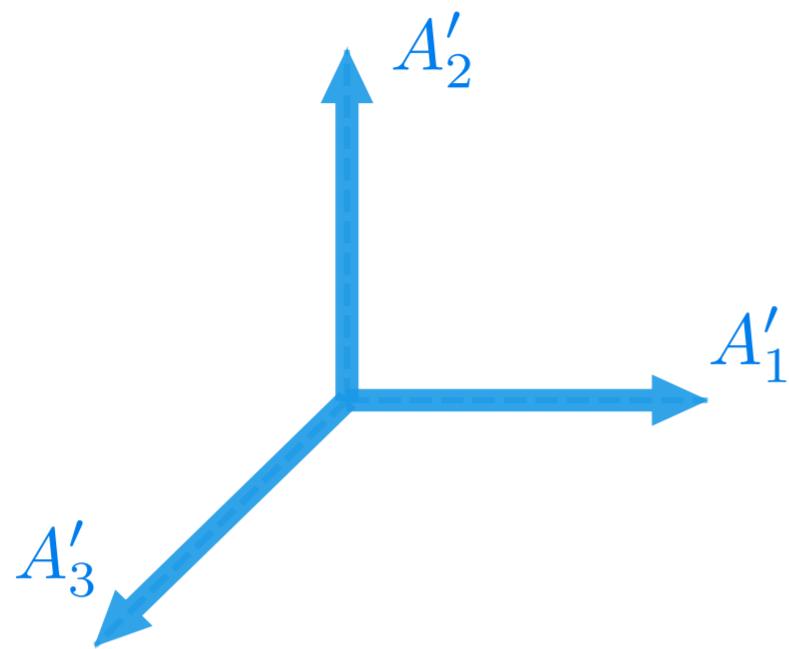
$$A \underbrace{E^{(1)} E^{(2)} E^{(3)} \\ E^{(4)} E^{(5)} E^{(6)} \\ E^{(7)} E^{(8)}}_{A'}$$

$$I \underbrace{E^{(1)} E^{(2)} E^{(3)} \\ E^{(4)} E^{(5)} E^{(6)} \\ E^{(7)} E^{(8)}}_{I'}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{array} \right]$$

$$E^{(9)}$$

**Column-reduction:
operation 9...**

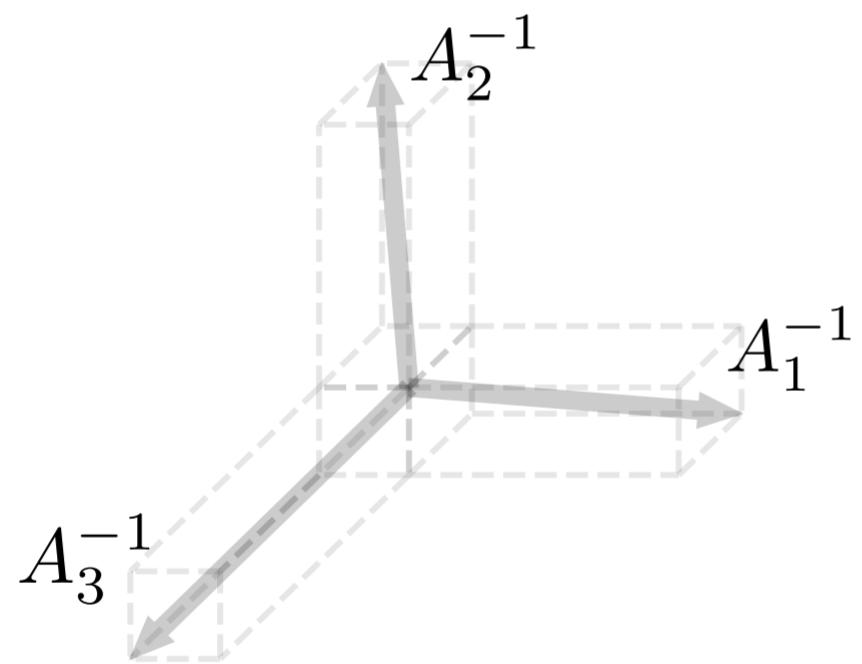
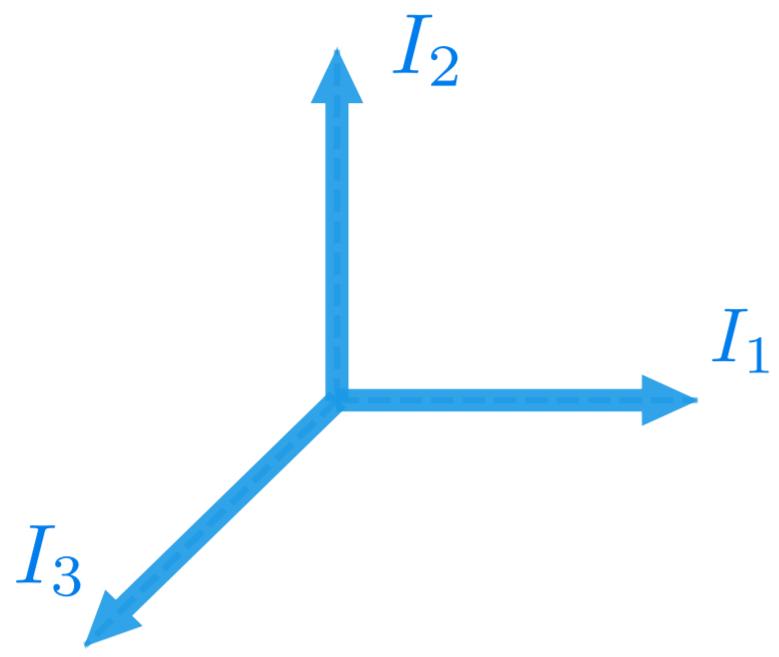


$$AE^{(1)}E^{(2)}E^{(3)}$$
$$E^{(4)}E^{(5)}E^{(6)}$$
$$\underbrace{E^{(7)}E^{(8)}E^{(9)}}_{A'}$$

$$IE^{(1)}E^{(2)}E^{(3)}$$
$$E^{(4)}E^{(5)}E^{(6)}$$
$$\underbrace{E^{(7)}E^{(8)}E^{(9)}}_{I'}$$

1	0	0
0	1	0
0	0	1
<hr/>		
3/4	-1/4	-1/4
-1/4	3/4	-1/4
-1/4	-1/4	3/4

**final
system...**



$$\begin{bmatrix} \mathbf{I} \\ \mathbf{A}^{-1} \end{bmatrix}$$

The matrix \mathbf{I} is a 3x3 identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix \mathbf{A}^{-1} is a 3x3 matrix:

$$\mathbf{A}^{-1} = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

**final
system...**