

Row & Column Geometry

Linear Algebra

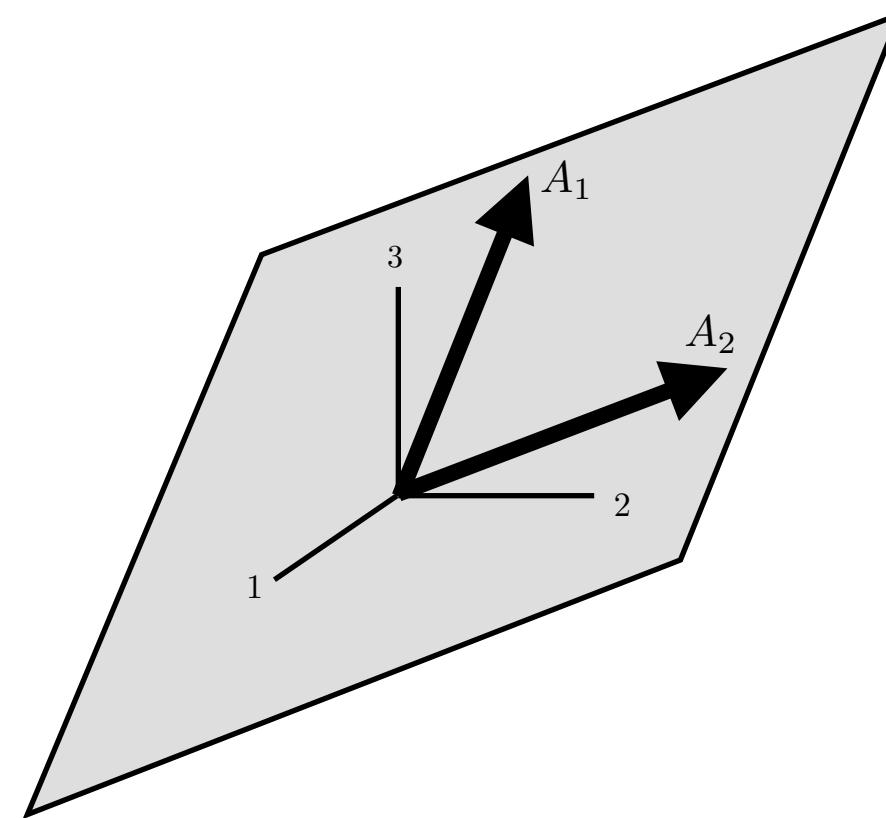
Winter 2022 - Dan Calderone

Block Matrix Multiplication

Range Space
 $\mathcal{R}(A)$

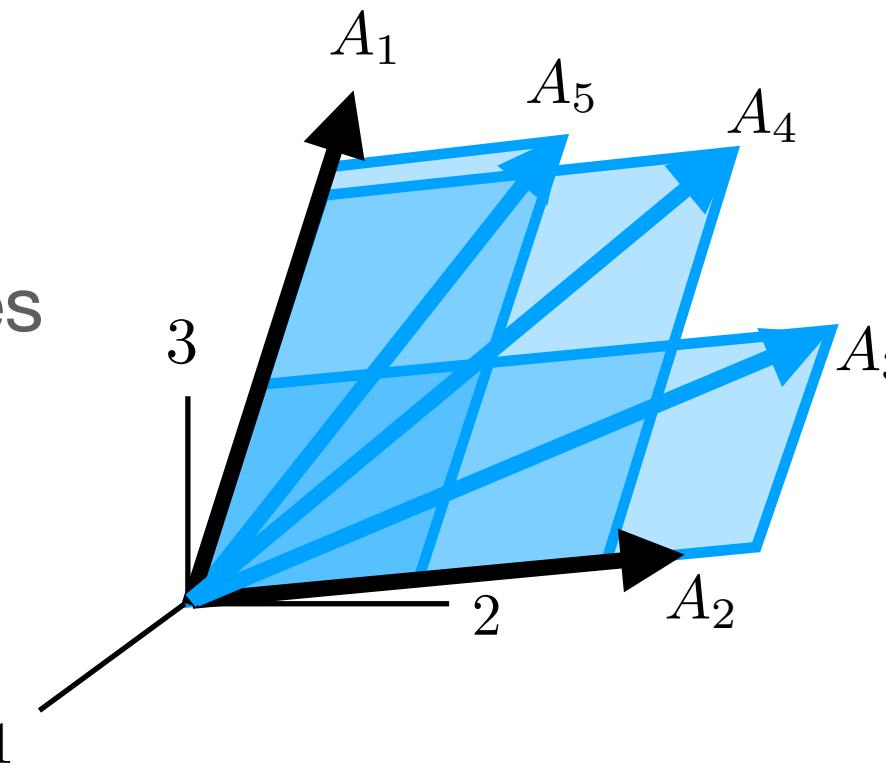
Column Geometry

$\mathcal{R}(A)$
“span of columns”



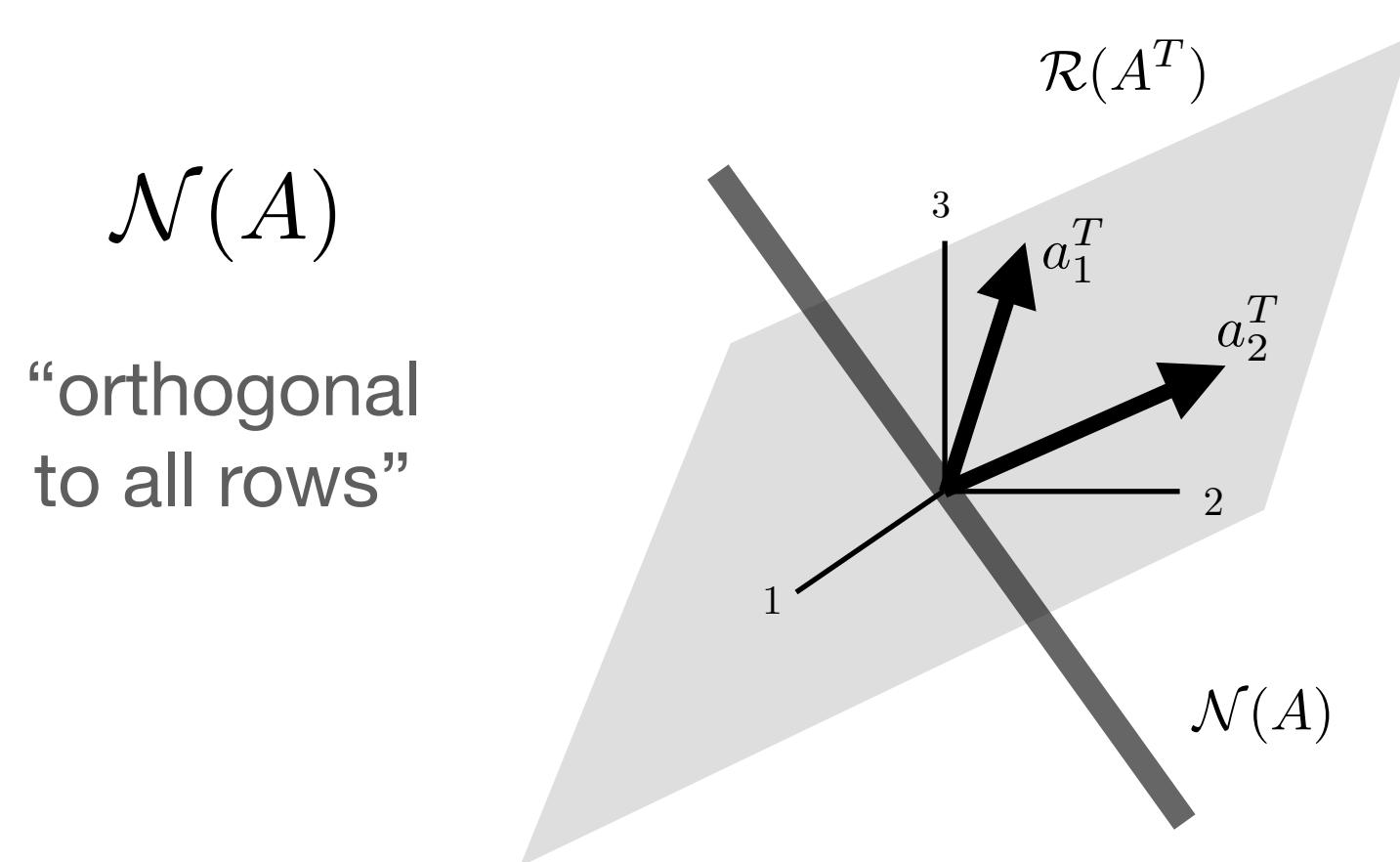
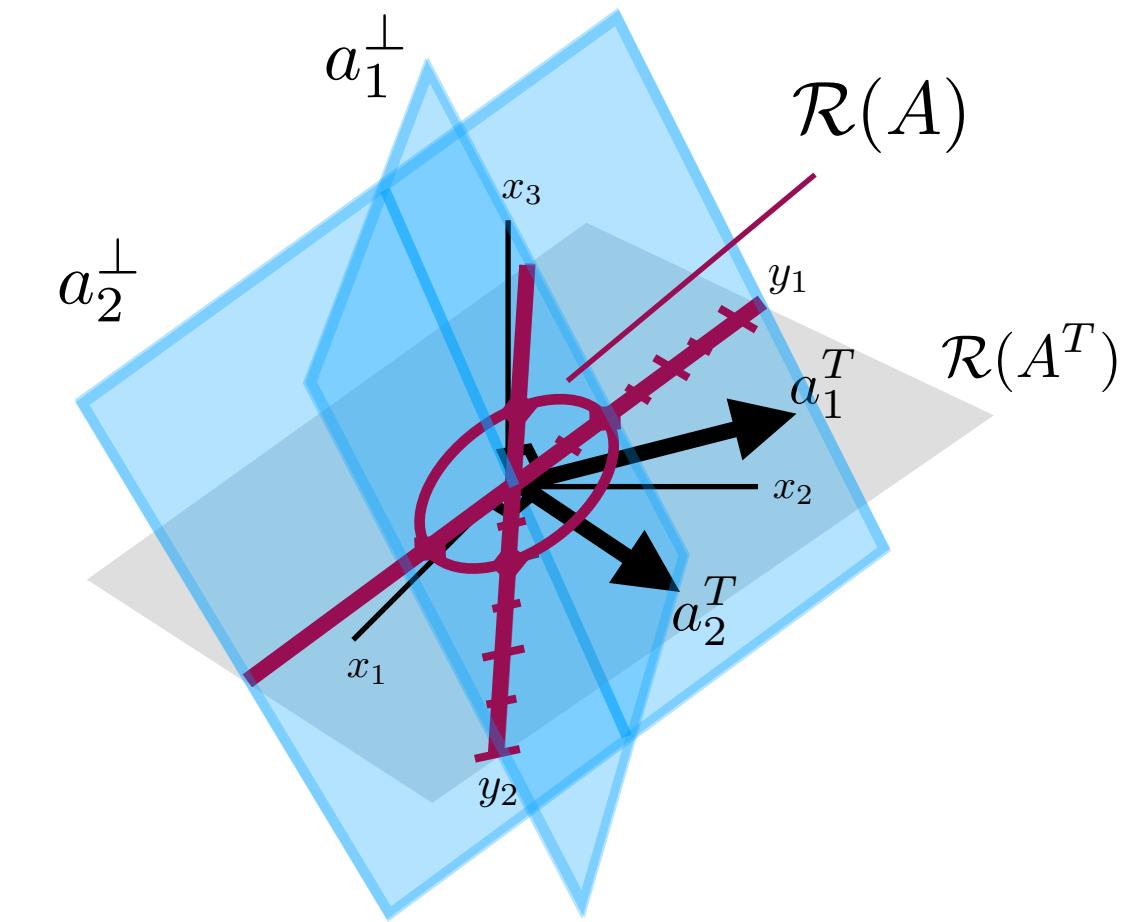
Nullspace
 $\mathcal{N}(A)$

$\mathcal{N}(A)$
“coordinates of origin”



Row Geometry

$\mathcal{R}(A)$
“projection orthogonal to other rows”

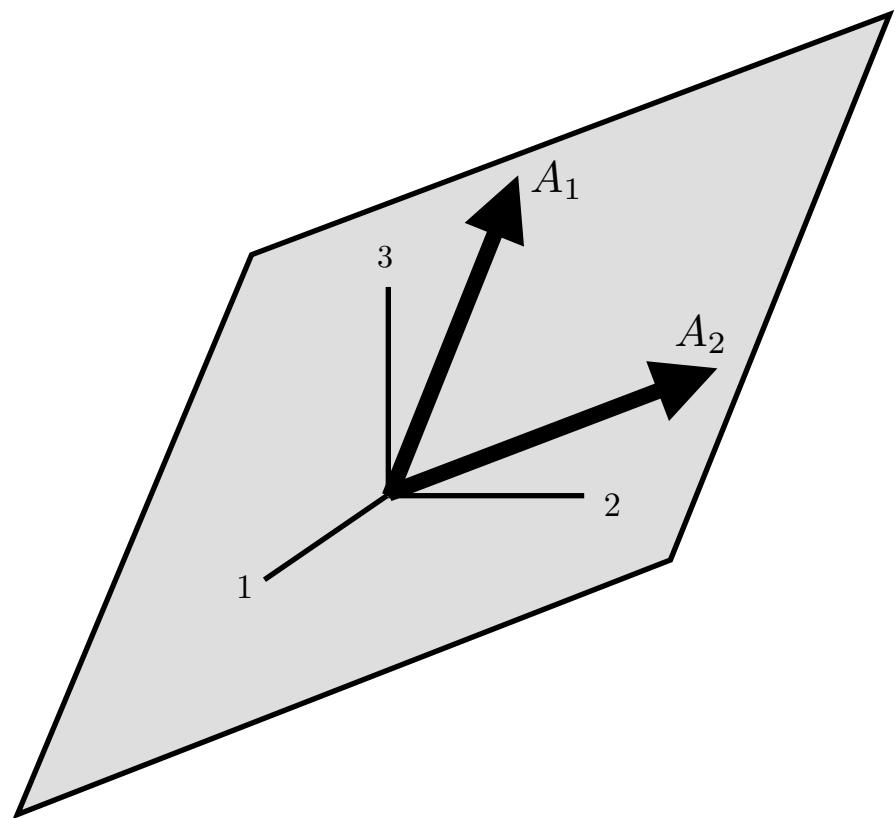


Column & Row Geometry

Range Space
 $\mathcal{R}(A)$

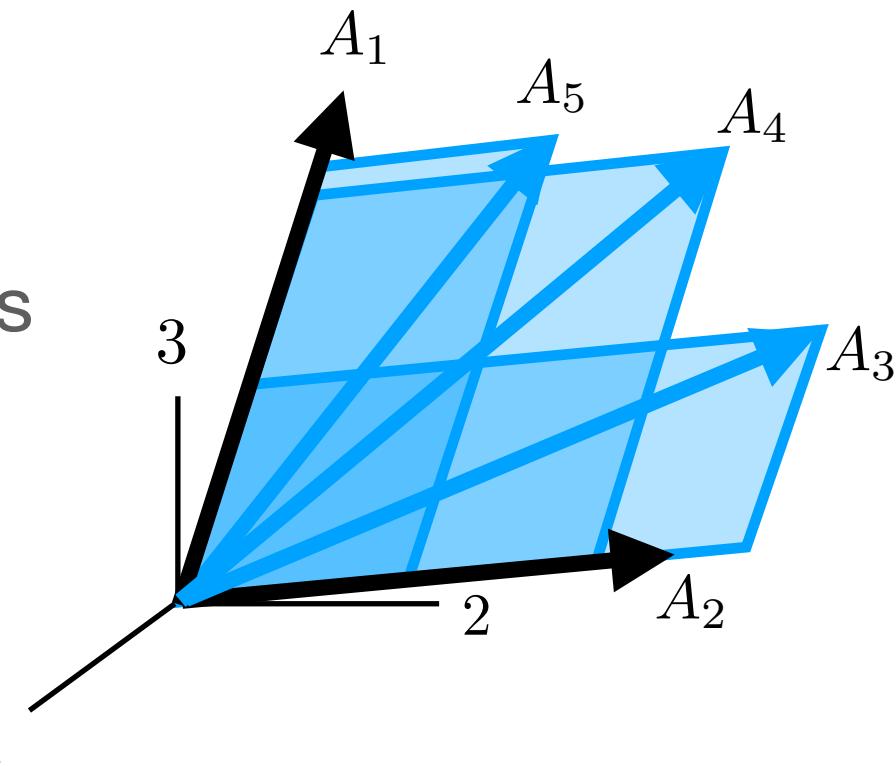
Column Geometry

$\mathcal{R}(A)$
“span of columns”



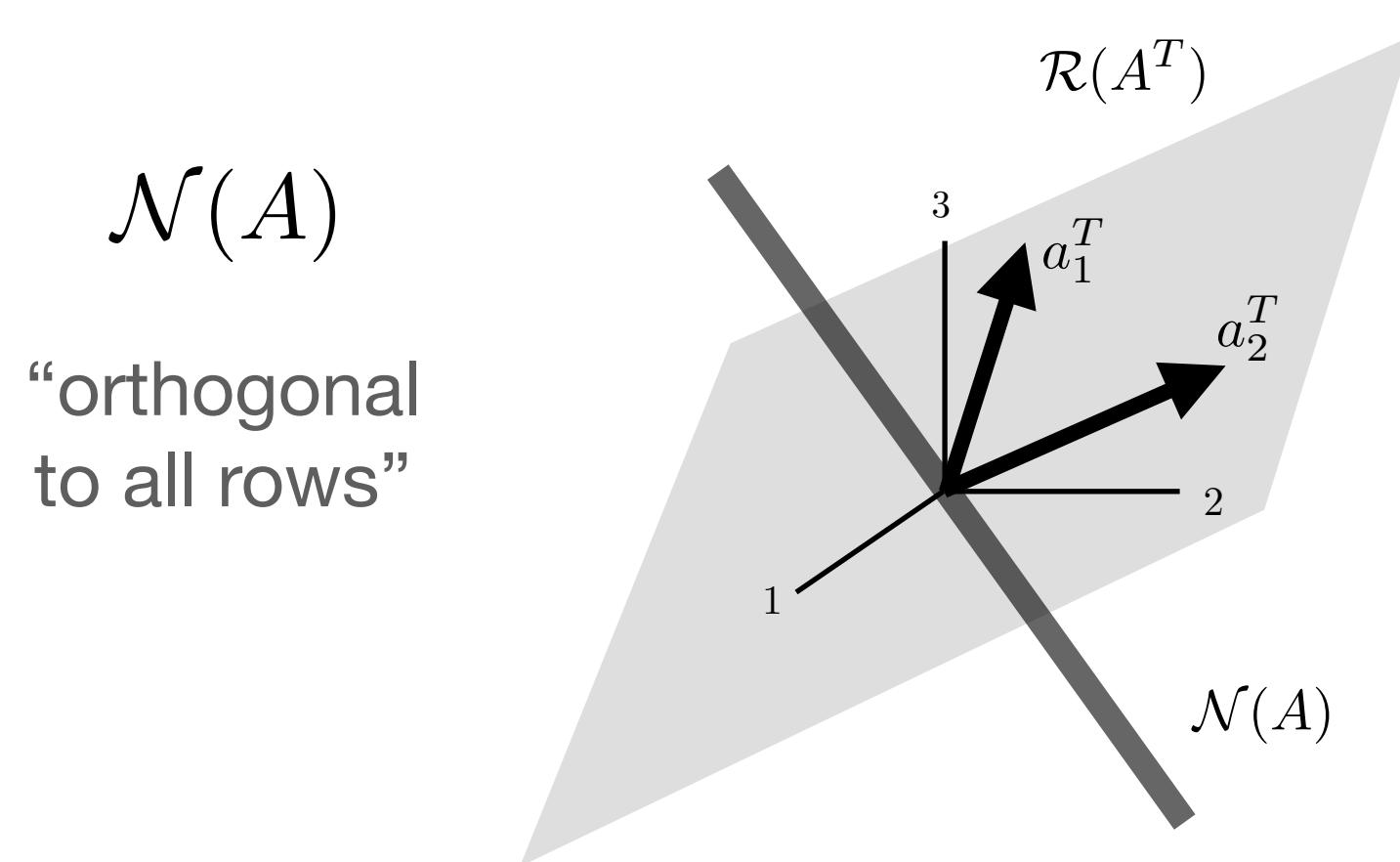
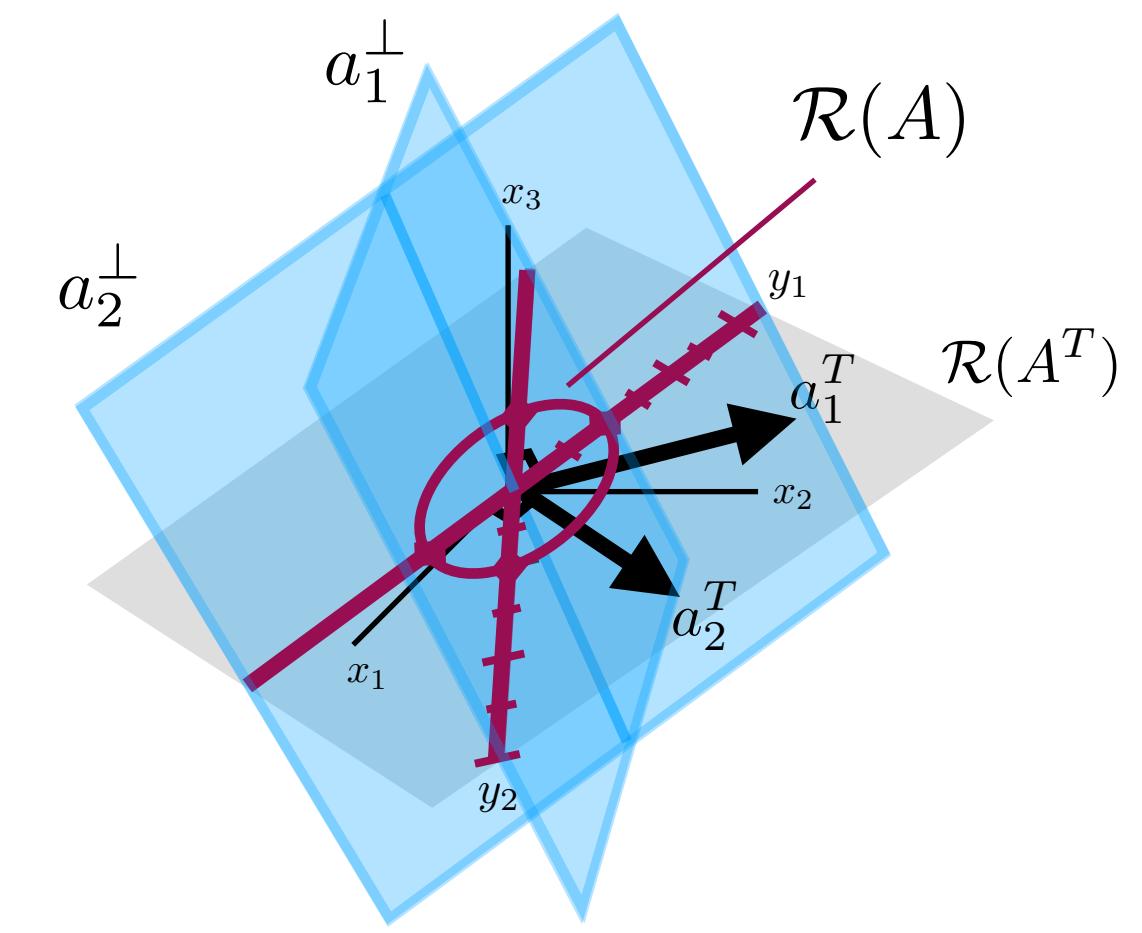
Nullspace
 $\mathcal{N}(A)$

$\mathcal{N}(A)$
“coordinates of origin”



Row Geometry

$\mathcal{R}(A)$
“projection orthogonal to other rows”



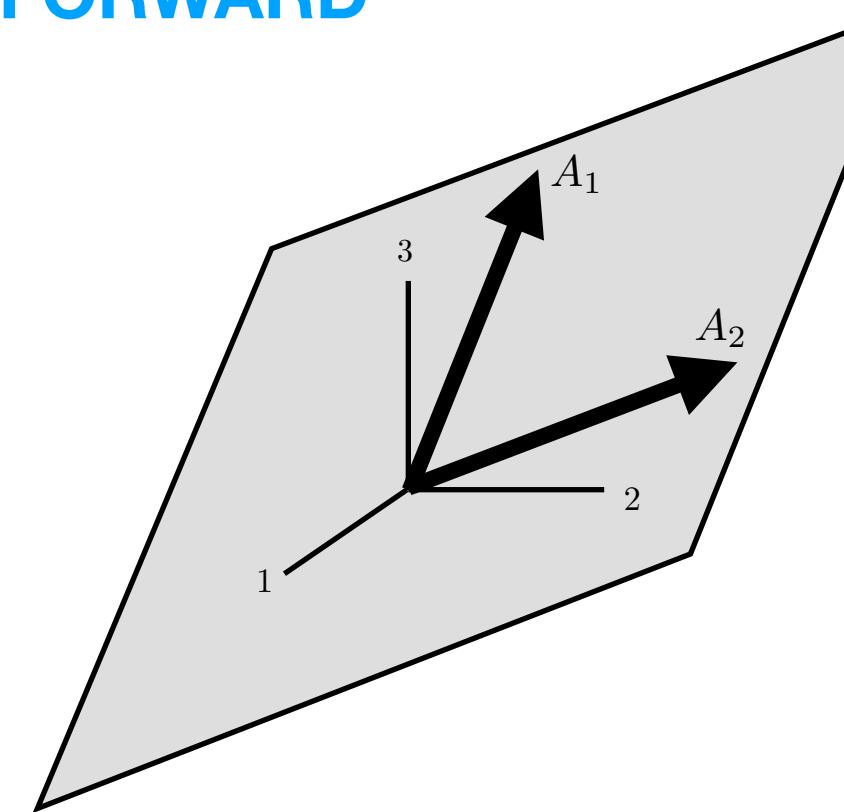
Column & Row Geometry

Range Space
 $\mathcal{R}(A)$

Column Geometry

STRAIGHTFORWARD

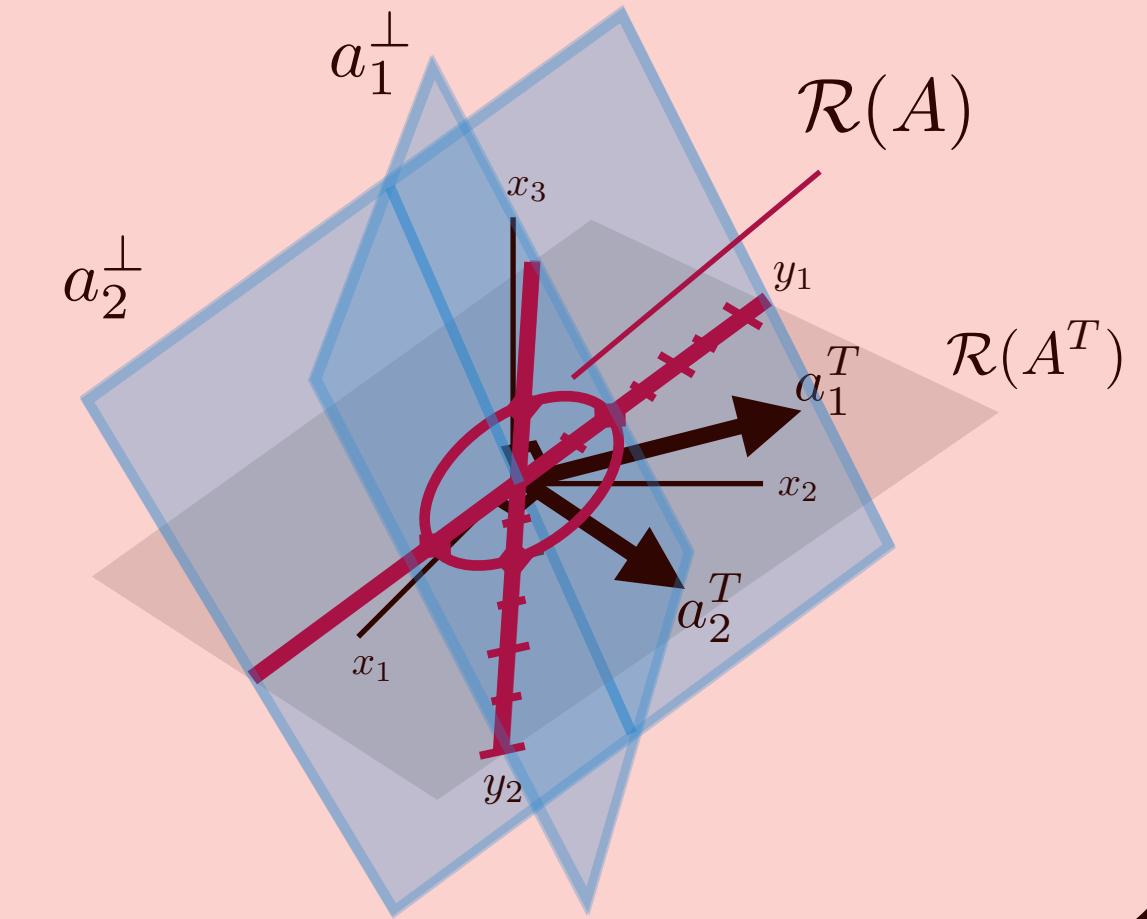
$\mathcal{R}(A)$
“span of columns”



Row Geometry

HARDEST

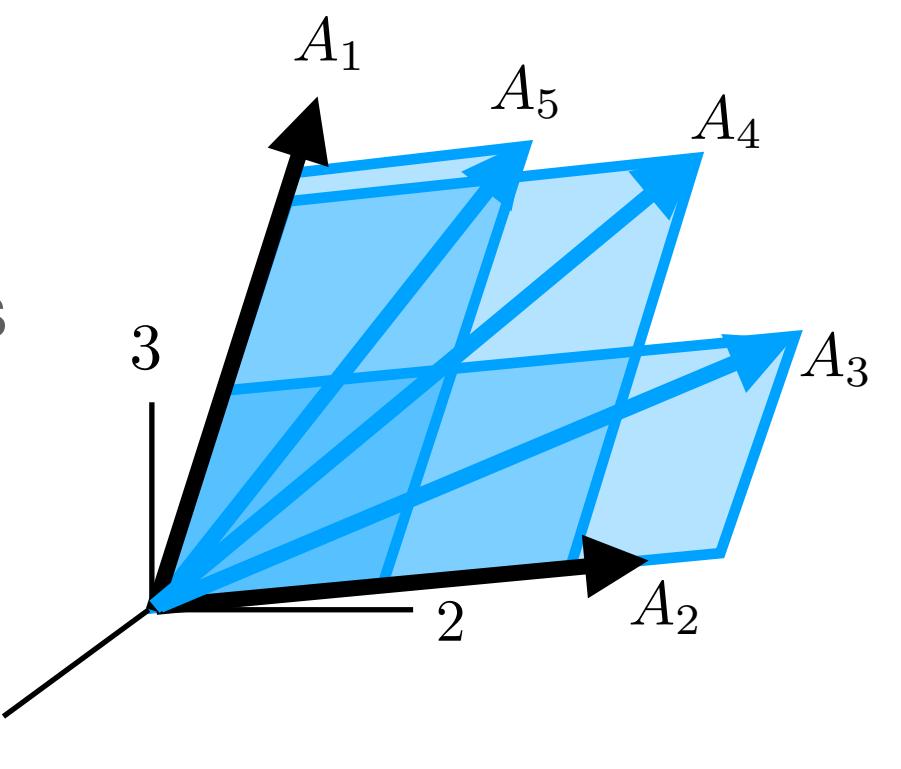
$\mathcal{R}(A)$
“projection orthogonal to other rows”



Nullspace
 $\mathcal{N}(A)$

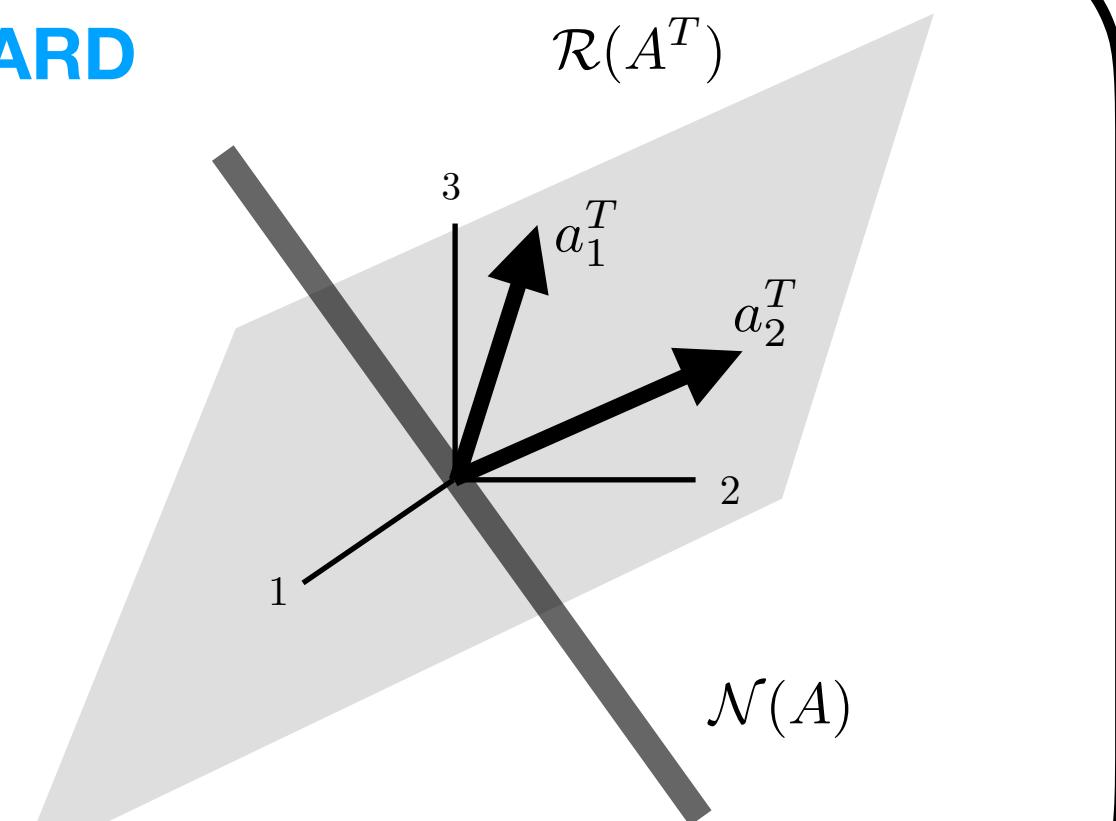
HARDER

$\mathcal{N}(A)$
“coordinates of origin”

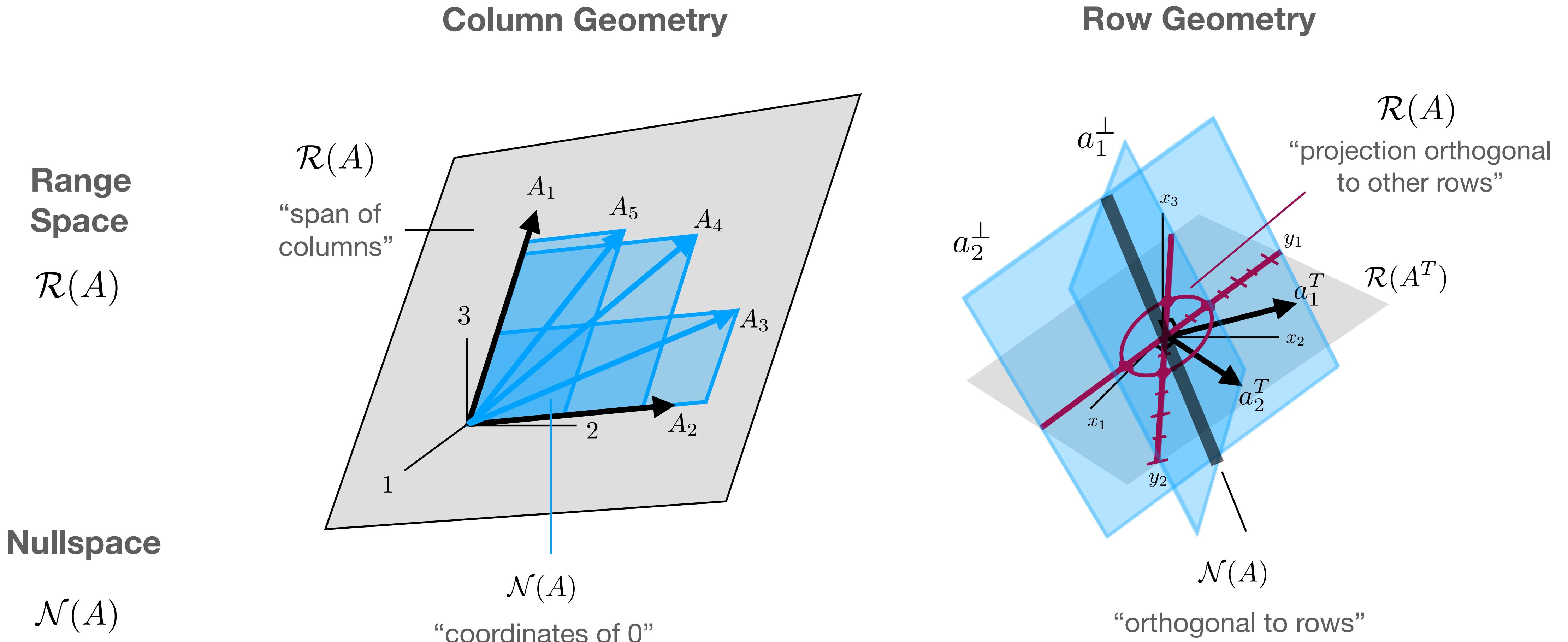


STRAIGHTFORWARD

$\mathcal{N}(A)$
“orthogonal to all rows”



Column & Row Geometry



$$A = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix} \quad \text{rk}(A) = 2$$

$$A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \end{bmatrix} \quad \text{rk}(A) = 2$$

Range - Column Geometry

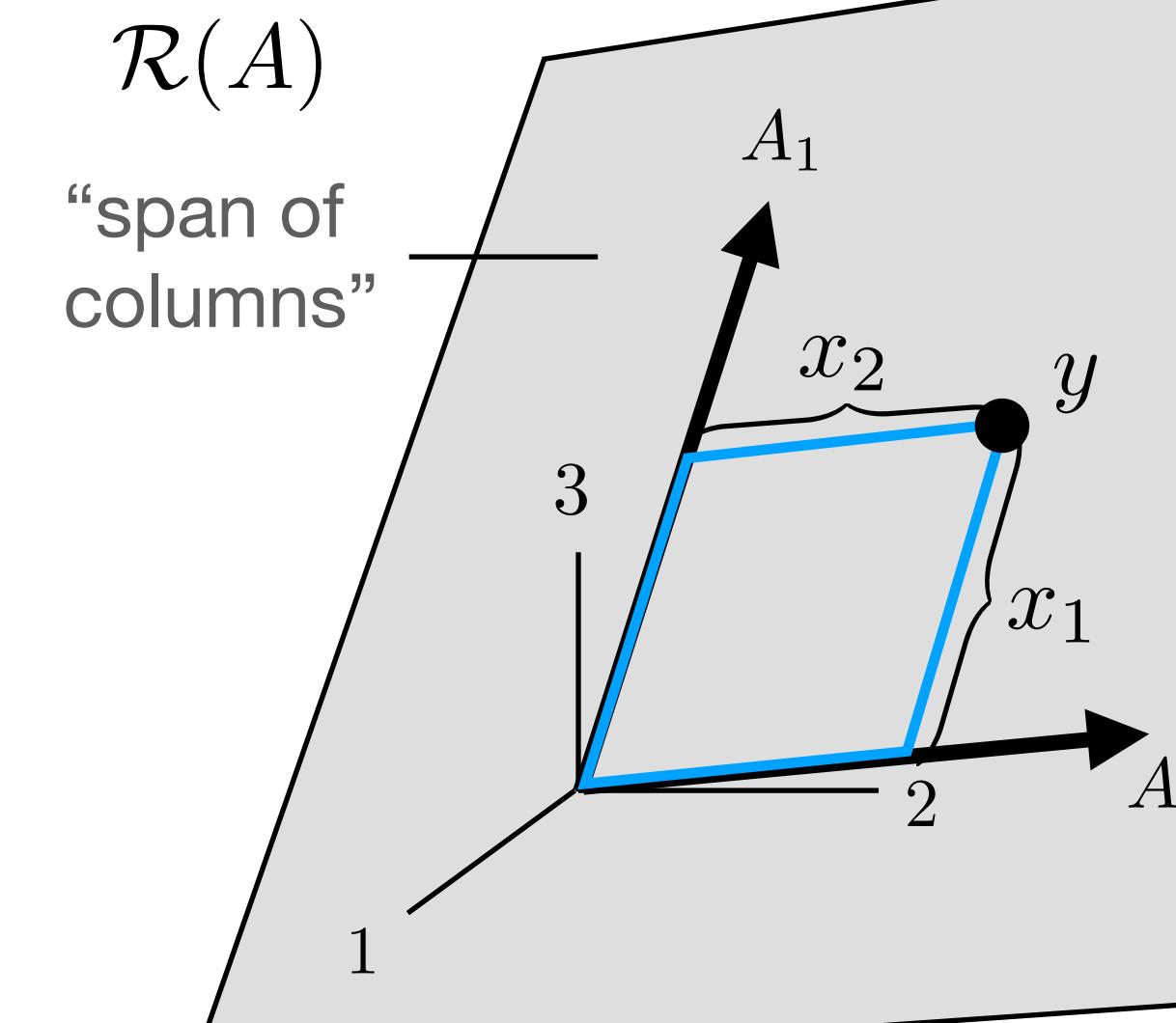
$$y \in \mathcal{R}(A) \quad y = Ax$$

$$y = \sum_i A_i x_i$$

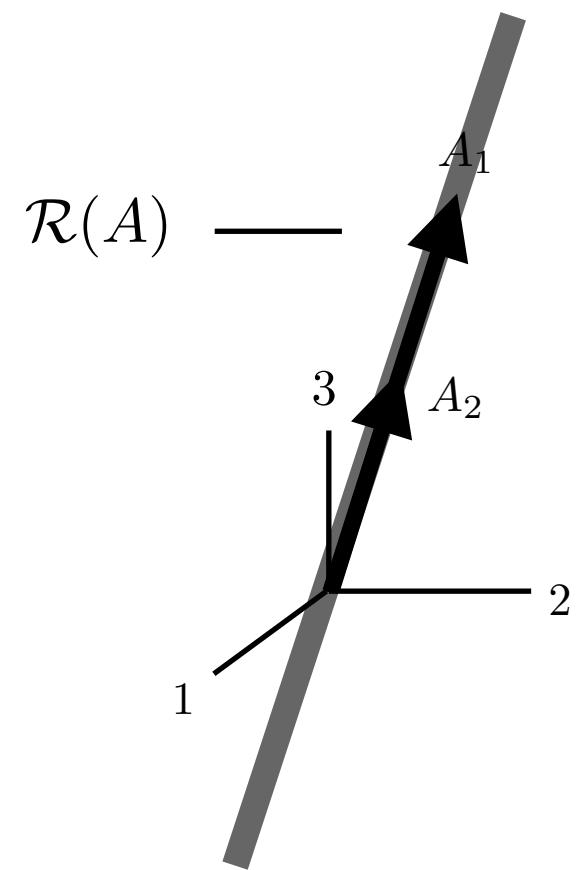
$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1 + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} x_n$$

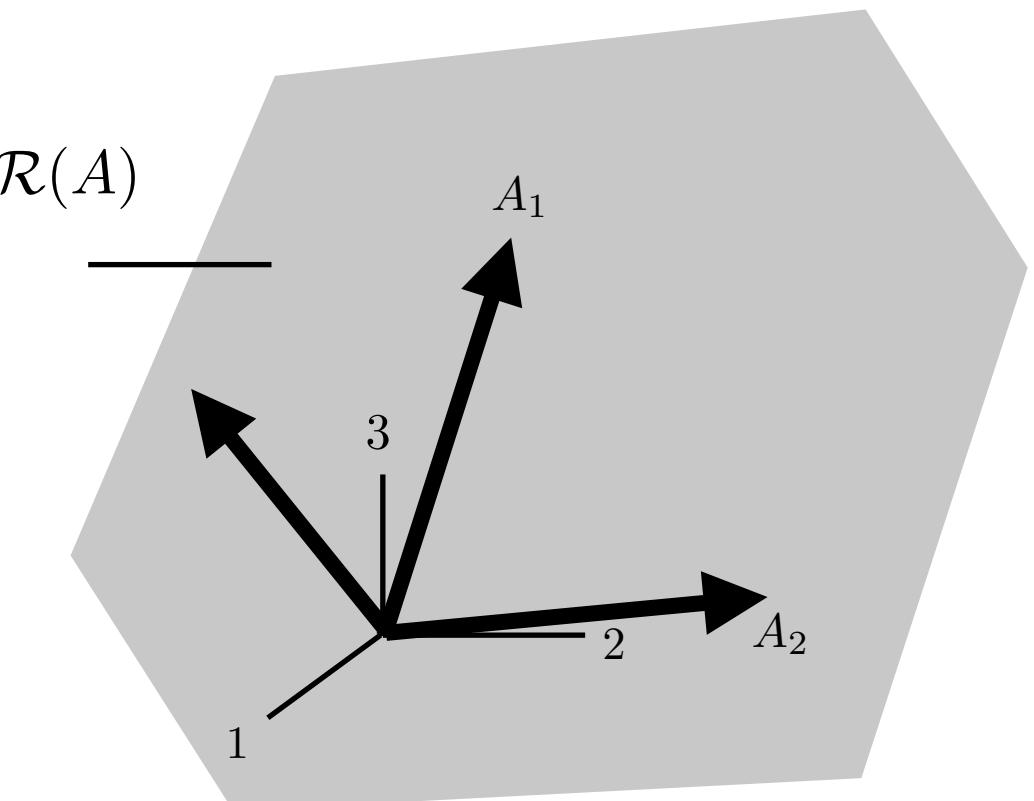
y are the coordinates of x w.r.t the columns of A



...1D span



...3D span



Range - Column Geometry

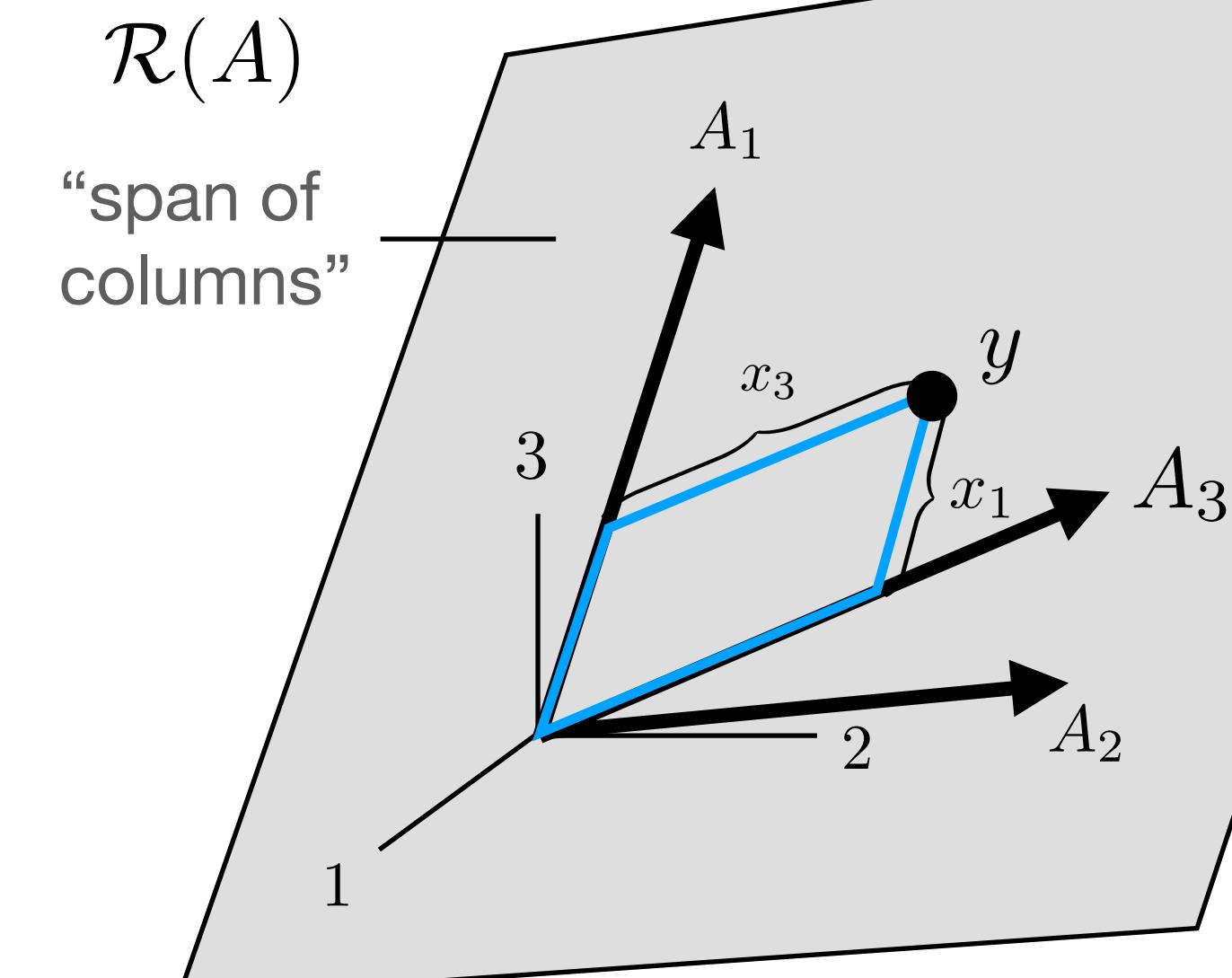
$$y \in \mathcal{R}(A) \quad y = Ax$$

$$y = \sum_i A_i x_i$$

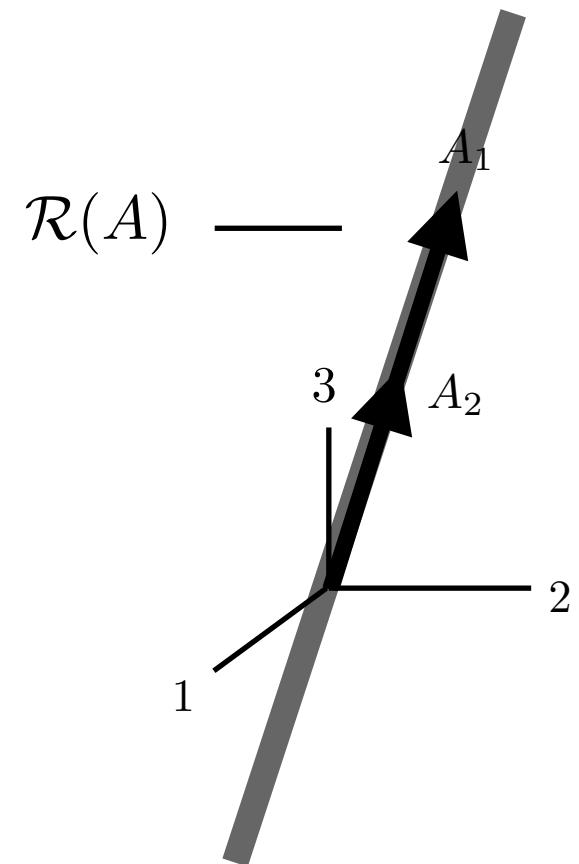
$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ A_1 & \dots & A_n \\ | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1 + \dots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} x_n$$

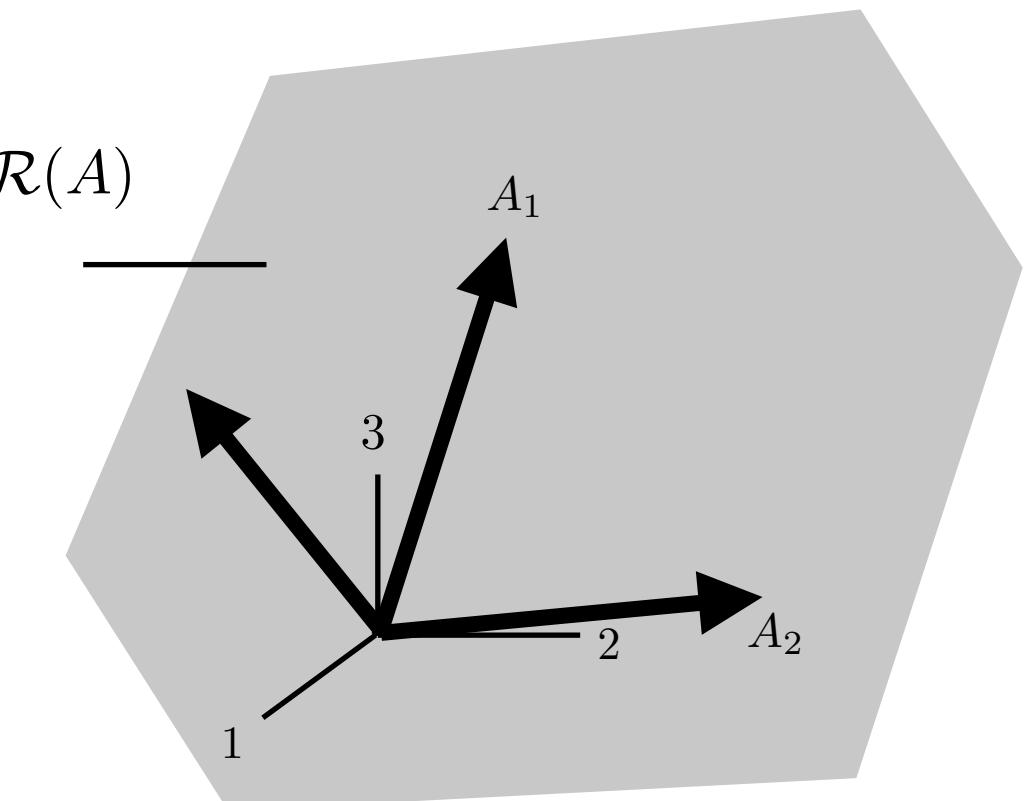
y are the coordinates of x w.r.t the columns of A



...1D span



...3D span



Range - Column Geometry

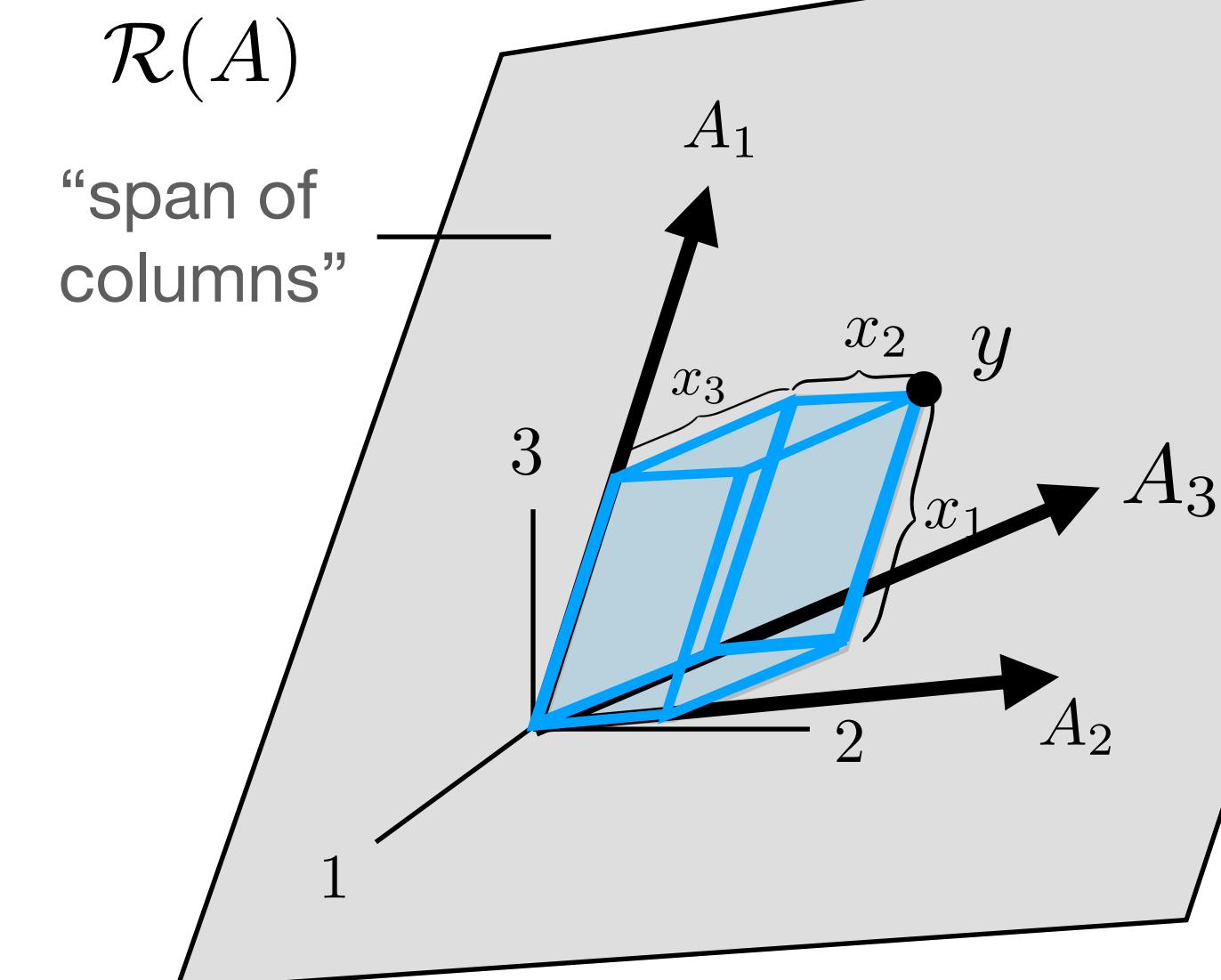
$$y \in \mathcal{R}(A) \quad y = Ax$$

$$y = \sum_i A_i x_i$$

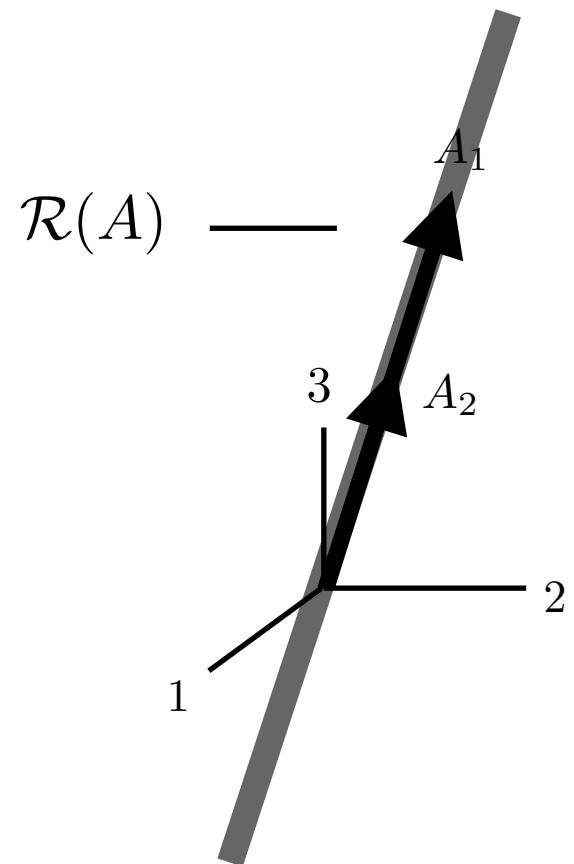
$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ A_1 & \dots & A_n \\ | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1 + \dots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} x_n$$

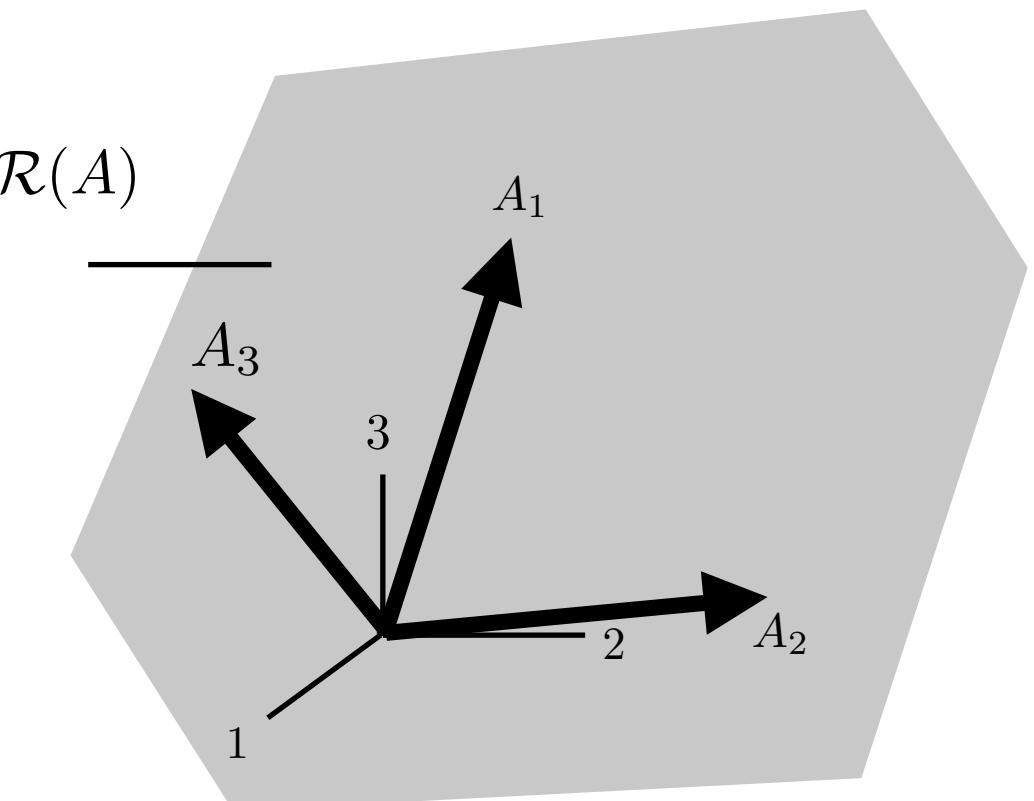
y are the coordinates of x w.r.t the columns of A



...1D span



...3D span



Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{array}{|c|} \hline A_1 \\ \hline A_2 \\ \hline \end{array}}_{A' \text{ Linear independent columns}} & \underbrace{\begin{array}{|c|} \hline A_3 \\ \hline A_4 \\ \hline A_5 \\ \hline \end{array}}_{A'' \text{ Linear dependent columns}} \end{bmatrix}$$

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \end{bmatrix} \quad A'' = A'B$$

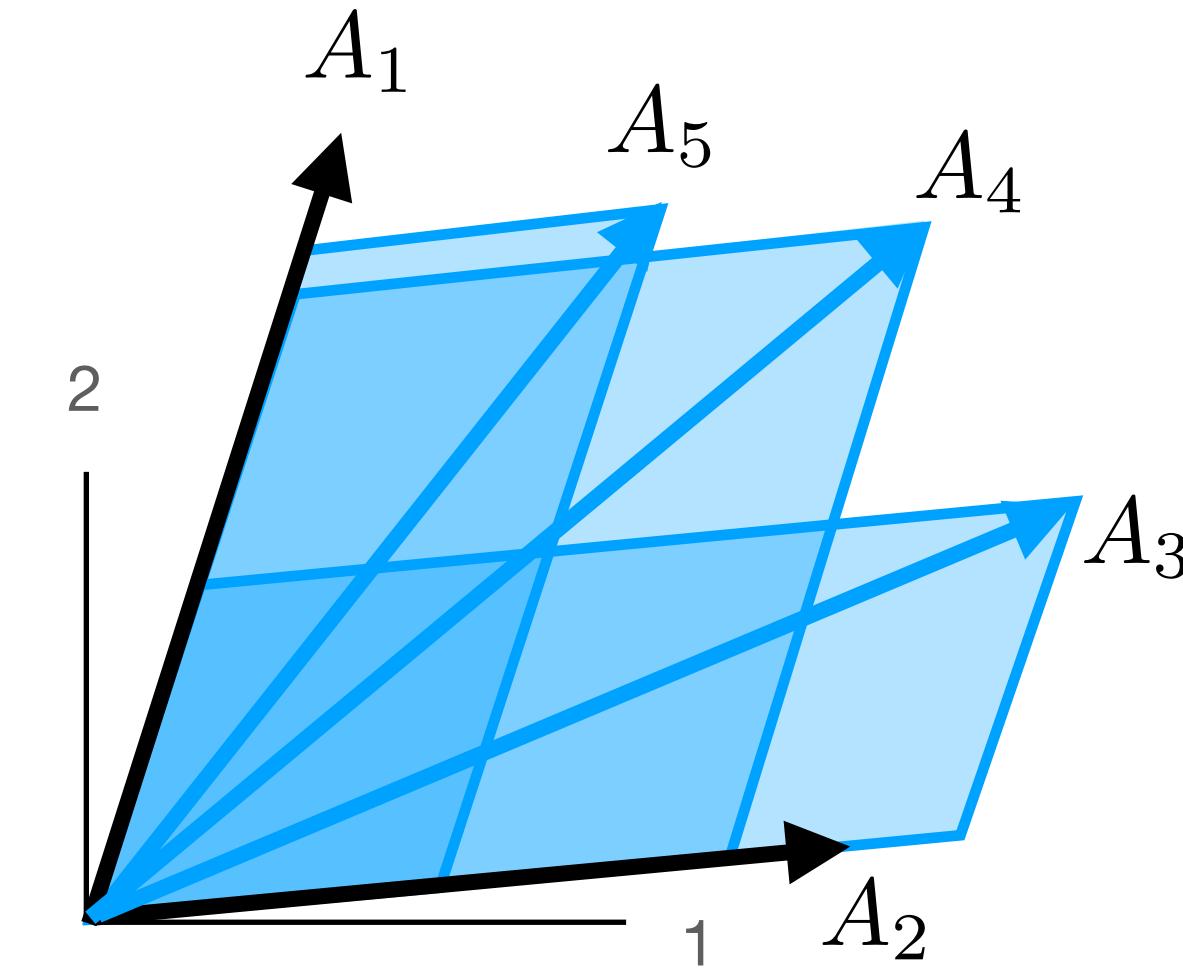
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} = A' \begin{bmatrix} I & B \end{bmatrix}$$

$$\begin{bmatrix} | & | & | \\ A_3 & A_4 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} A_1 B_{13} + A_2 B_{23} & A_1 B_{14} + A_2 B_{24} & A_1 B_{15} + A_2 B_{25} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$AN = 0$$

$$N = \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \Rightarrow z = 0$

Span: $x \in \mathcal{N}(A)$ A' lin. ind.

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \Rightarrow x' + Bx'' = 0$$

$$\Rightarrow x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{33} & B_{34} & B_{35} \end{bmatrix}$$

$$A'' = A'B$$

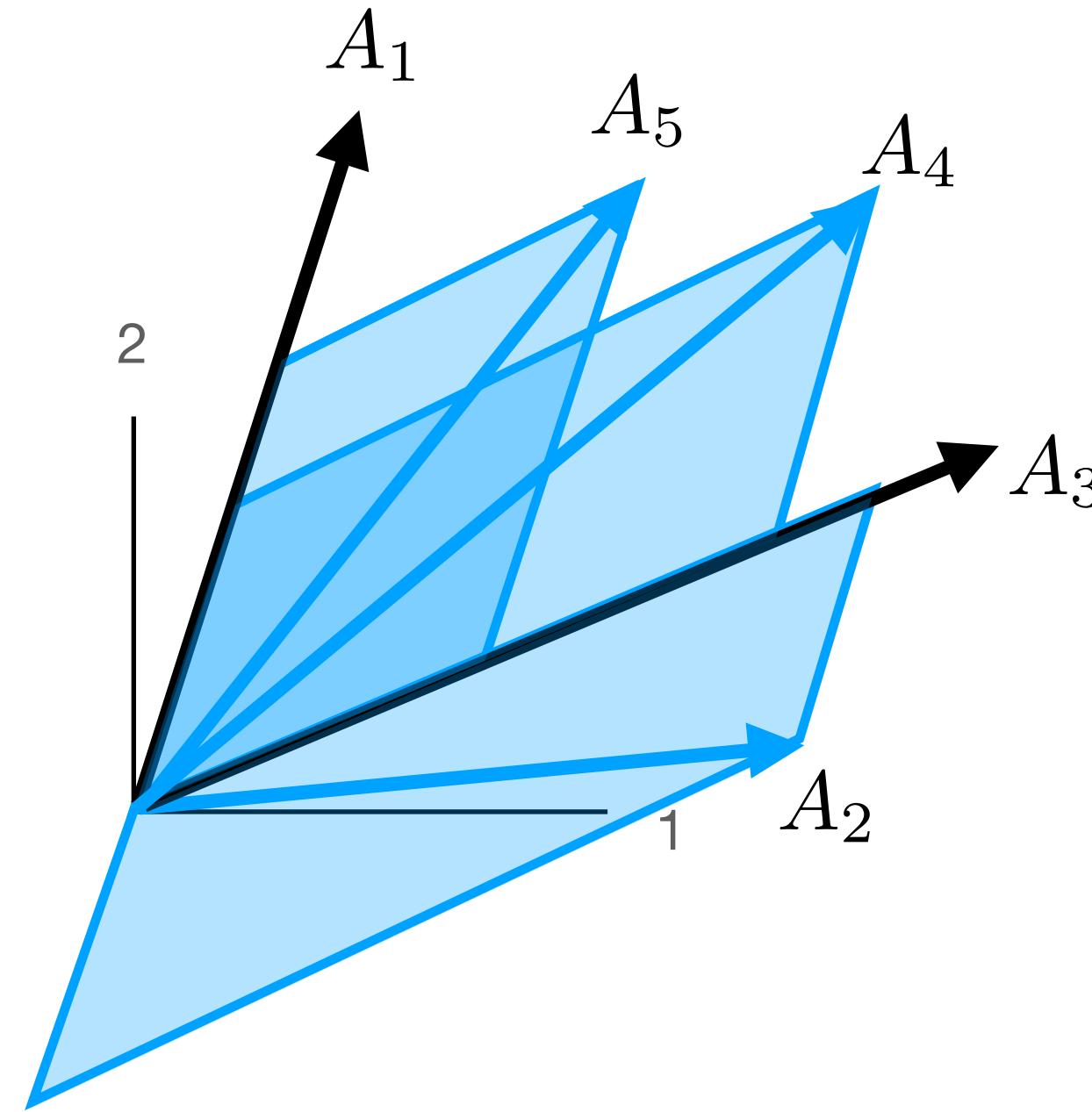
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_4 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_3 B_{32} & A_1 B_{14} + A_3 B_{34} & A_1 B_{15} + A_3 B_{35} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ B_{33} & B_{34} & B_{35} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$



PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \Rightarrow z = 0$

Span: $x \in \mathcal{N}(A)$ A' lin. ind.

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \Rightarrow x' + Bx'' = 0$$

$$\Rightarrow x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{12} & B_{13} & B_{15} \\ B_{42} & B_{43} & B_{45} \end{bmatrix}$$

$$A'' = A'B$$

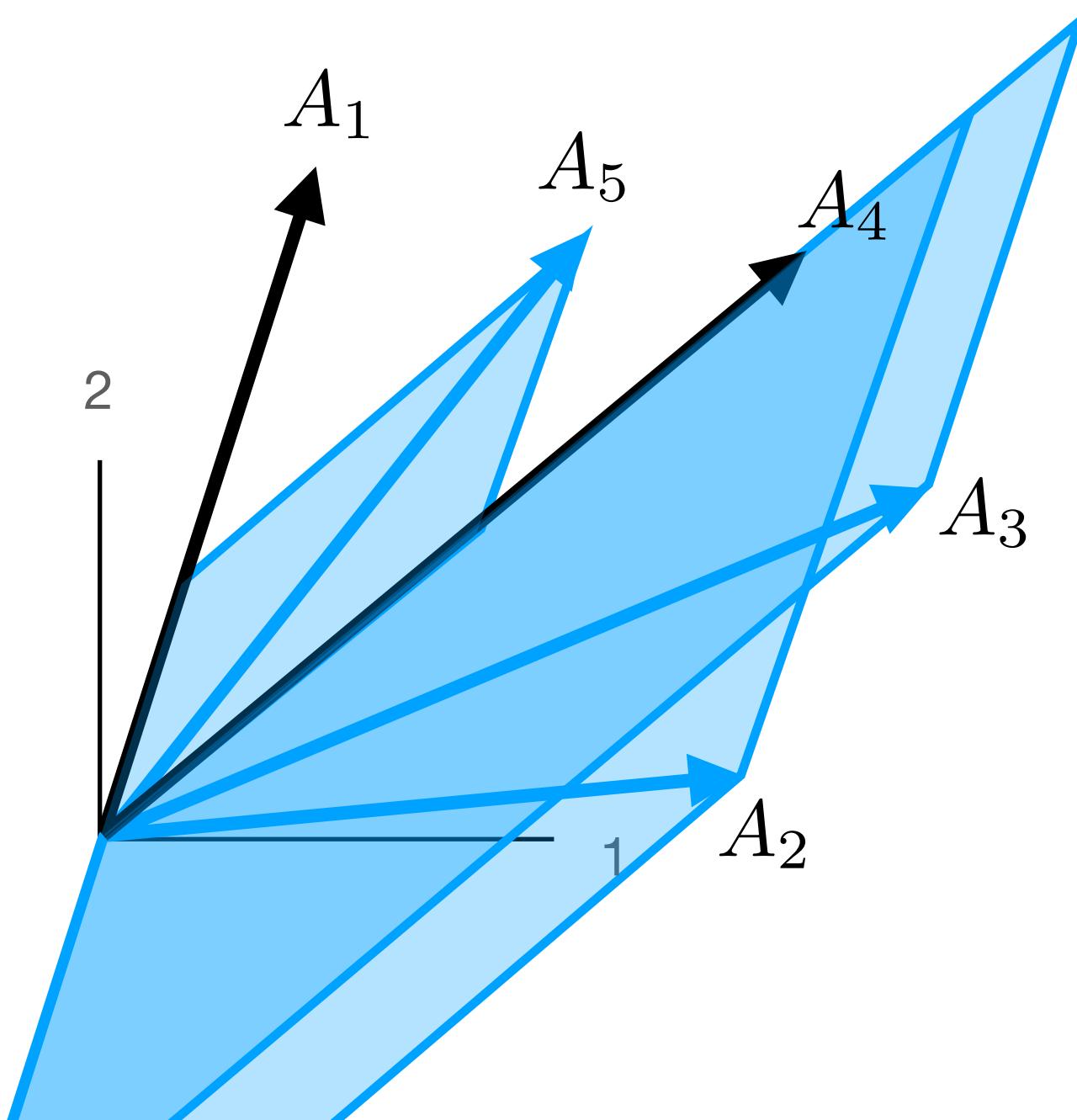
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_3 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_4 B_{42} & A_1 B_{13} + A_4 B_{43} & A_1 B_{15} + A_4 B_{45} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ B_{43} & B_{44} & B_{45} \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$



PROOF:

$$\text{Lin ind: } \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \Rightarrow z = 0$$

Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \Rightarrow x' + Bx'' = 0$$

$$\Rightarrow x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{12} & B_{13} & B_{14} \\ B_{52} & B_{53} & B_{54} \end{bmatrix}$$

$$A'' = A'B$$

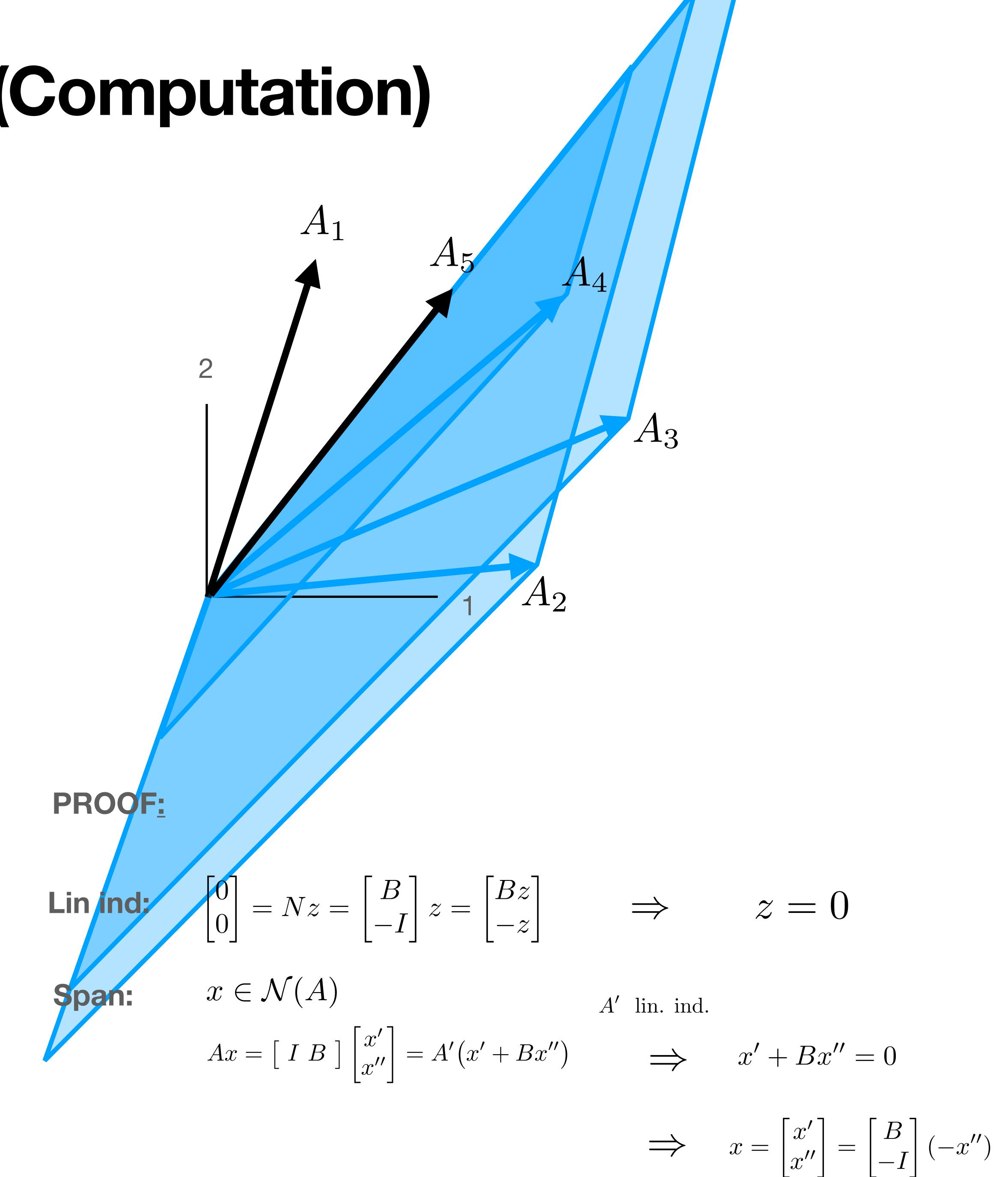
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} A_2 & A_3 & A_4 \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_5 B_{52} & A_1 B_{13} + A_5 B_{53} & A_1 B_{14} + A_5 B_{54} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ B_{53} & B_{54} & B_{55} \end{bmatrix}$$

$$AN = 0$$



Range - Row Geometry

$$y \in \mathcal{R}(A) \quad y = Ax$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & \vdots & - \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

Pre-image of

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

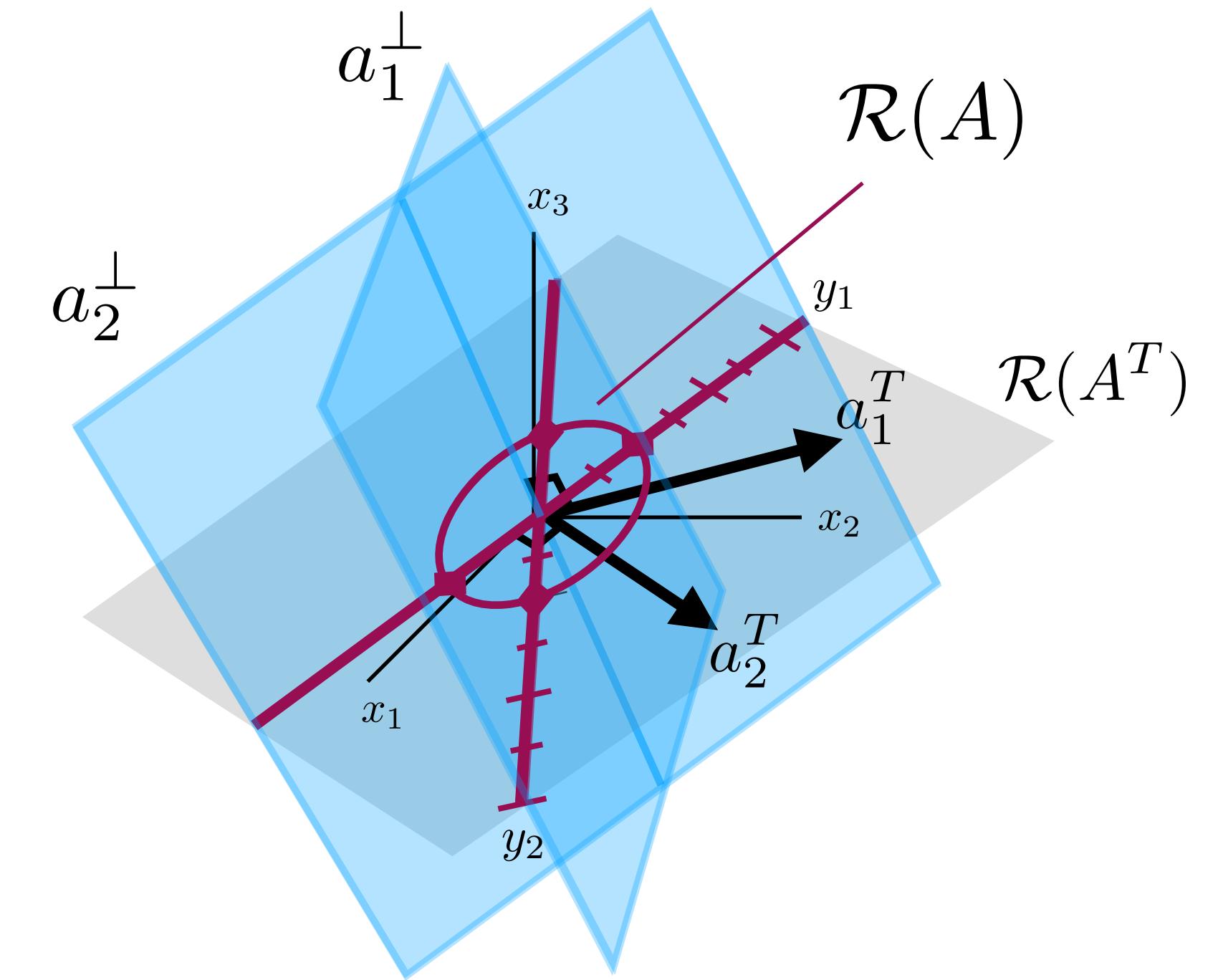
is...

$$\left\{ \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

...intersection of
m-1 subspaces
(each with dim n-1)

$$\mathcal{R}(A)$$

“projection
orthogonal
to other rows”



$$\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Pre-image of

$$\left\{ \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

etc...

Nullspace - Row Geometry

$$x \in \mathcal{N}(A) \quad Ax = 0$$

$$\begin{bmatrix} - & a_1^T & - \\ \vdots & \vdots & \vdots \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

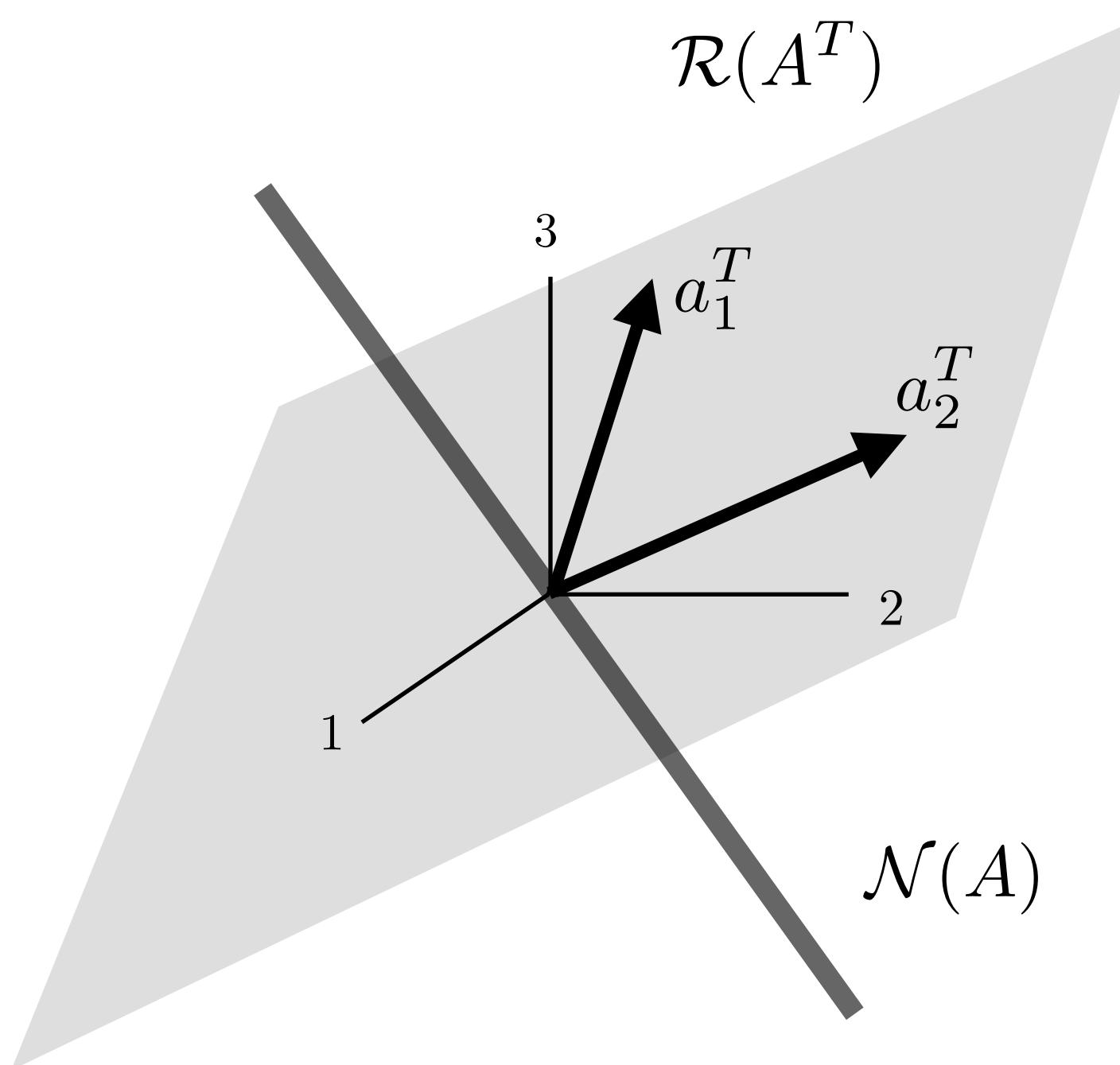
$$\begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

x orthogonal to all rows.

(pre-image of 0)

$$\mathcal{N}(A)$$

“orthogonal
to all rows”



Rank = Column Rank = Row Rank

Column Rank

num. of linear independent columns

$$\dim(\mathcal{R}(A))$$

Row Rank

num. of linear independent rows

$$\dim(\mathcal{R}(A^T))$$

Column Rank = Row Rank = Rank

$$A \in \mathbb{R}^{m \times n}$$

... if col rank = k

write... $A = CV$

$$A = m \begin{bmatrix} C_1 & \dots & C_k \end{bmatrix} \begin{bmatrix} V_1 & \dots & V_n \end{bmatrix}$$

lin. ind. (basis) for col space coords of cols w.r.t basis

... if row rank = k

write... $A = WR$

$$A = m \begin{bmatrix} w_1^T & \dots & w_m^T \end{bmatrix} \begin{bmatrix} r_1^T & \dots & r_k^T \end{bmatrix}$$

coords of rows w.r.t basis lin. ind. (basis) for row space

change perspective...



row rank $\leq k$

$$A = m \begin{bmatrix} c_1^T & \dots & c_m^T \end{bmatrix} \begin{bmatrix} v_1^T & \dots & v_k^T \end{bmatrix}$$

coords of rows... rows in span...

col rank $\leq k$

$$A = m \begin{bmatrix} W_1 & \dots & W_k \end{bmatrix} \begin{bmatrix} R_1 & \dots & R_n \end{bmatrix}$$

cols in span... coords of cols...