

# Vector Visualizations

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**Abstract**—This tutorial paper gives basic vector visualization techniques. We first present two approaches for vector visualization we term the *spatial (orthogonal) axis* approach and the *parallel axis* approach and give a discussion of the strength and limitations of both. A brief dive into visualization of geometric shapes is included. We then present two visualization techniques for inner products: one based on the spatial axis representation and one based on a hybrid spatial-parallel axis representation.

## I. INTRODUCTION

This paper is an introduction to basic vector visualization techniques. The results are meant to give a foundation and reference for several later monographs on matrix visualization. We start with two vector visualization techniques, one we refer to as the *spatial or orthogonal* axis representation and what we call the *parallel axis* representation. The spatial (orthogonal) axis representation is the traditional approach where vectors are represented as arrows in a space defined relative a coordinate axis (usually presumed to be orthogonal). We discuss this representation in two and three dimensions and then extend it to higher dimensional spaces. The technique is simple and general to apply; however, the pictures produced are (of course) 2D projections of the higher dimensional space. We discuss the problem of “depth” (information lost in projection) for higher dimensional spatial visualization along with brief thoughts on how it can be fruitful despite its limitations. We then present the parallel axis representation where vectors are simply seen as displacements along parallel axes which is much simpler.

The next section contains visualizations of basic shapes in the spatial axis representation. While this is somewhat interesting in its own right, its primary purpose is to serve as a basic reference for results in later monographs.

The last section of this tutorial discusses two methods for visualizing inner products. The first method extends the traditional notion of projecting one vector onto another to include the case where neither of the two vectors is a unit vector. We discuss the general technique, show how it reduces to the projection picture when one of the vectors is a unit vector, and then briefly tease several interesting cases such as when two vectors are

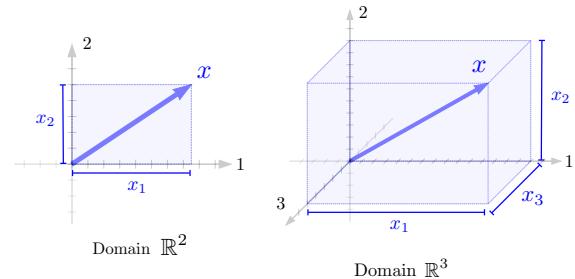
orthogonal. We also include a brief discussion of how it reduces in the scalar case to a rather natural visualization of scalar multiplication. The authors think this technique is quite fruitful in that combines the scalar (magnitude) and vector (direction) nature of an inner product as a mathematical operator. The second method is a hybrid spatial-parallel axis presentation of an inner product. It is much less traditional than the projection approach and teases the visualization techniques presented in later manuscripts for more complicated matrix operations. The general technique as well several examples are given with a brief discussion of strengths and limitations. Again, this section may seem esoteric on its own; however, the authors will make extensive use of this technique in later monographs.<sup>1</sup>

## II. VECTOR VISUALIZATIONS

Finite dimensional vectors are represented as a string of digits, each of which gives a displacement relative to an axis. Visual representations of vectors show these displacements in various ways. We will focus on two methods which we will refer to as the *orthogonal (or spatial) axis* representation and the *parallel axis* representation.

### A. Spatial (Orthogonal) Axes

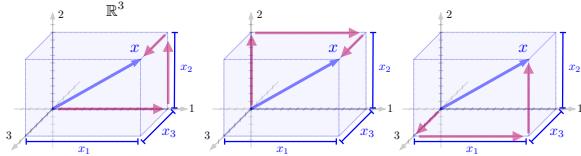
The natural (spatial) way to represent the various displacements is along axes that are orthogonal to each other visualized in 2D and 3D here.



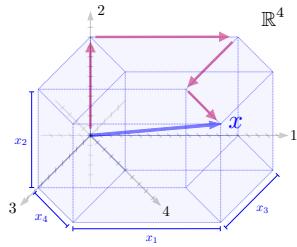
Cubes (or rectangles) can be used to visualize each individual coordinate of the vector. The length of each cube edge shows the size of each coordinate. Note also

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that we can visualize building up a vector one coordinate at a time as walking along edges of the cube from the origin to the tip. Note there are multiple different paths that all result in reaching the tip corresponding to the fact that coordinates can be added in any order.

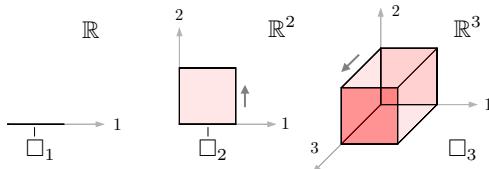


Our brains are highly adapted for visualizing vectors orthogonally in two and three dimensions. If we wish to use the orthogonal representation for vectors in dimensions higher than three, the best we can do is to draw a 2D projection of the higher dimensional vectors. One projection method can be achieved by simply drawing a direction (in 2D) for each axis and then simply showing displacements along these axes. This process is illustrated here for a vector in  $\mathbb{R}^4$ .

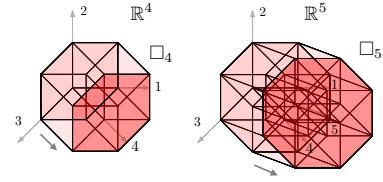


Since there is only room for two orthogonal axes in 2D, we must imagine that the axes we draw are actually orthogonal.

**Remark 1.** *The following exercise is useful for visualizing hypercubes in higher dimension. A 1D cube is a basic line segment along the interval  $[0, 1]$ . We can obtain a 2D cube, a square, by sweeping that interval along a 2nd axis. Sweeping the square along a 3rd axis produces a 3D cube.*



Higher dimensional cubes can be produced by continuing this process. A 3D cube swept along a 4th axis gives a 4D-hypercube; a 4D-hypercube swept along a 5th axis produces a 5D-hypercube; etc.

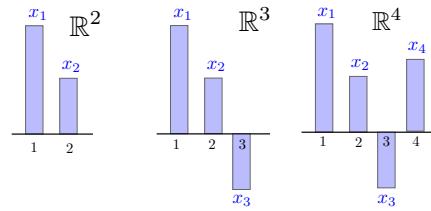


**Remark 2.** *When projecting higher dimensional shapes onto 2D images, a certain amount of information, the "depth" direction(s), in the image get lost. If we're drawing a 3D vector, depth is one dimensional (out of the page). If we're drawing a 4D vector, depth is 2-dimensional; for a 5D vector, depth is 3-dimensional, etc. Any intuition derived from projections of higher dimensional sets should be verified with rigorous proof.*

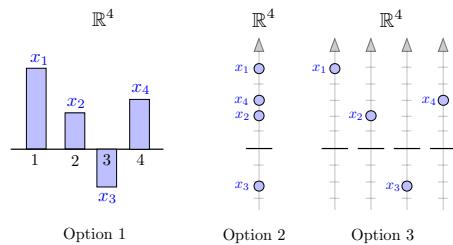
Referring to these visualization techniques as an "orthogonal" representation is somewhat of a misnomer because the axes are only truly orthogonal in 2D. We use this term to reference the fact that we are conceptualizing the axes as orthogonal in some higher dimensional space even though we can only view 2D projections. The term *spatial representation* is perhaps more accurate.

### B. Parallel Axes

Another less traditional way to represent vectors is to place the axes parallel to each other and show each coordinate displacement in the same direction. We will often use rectangles of the appropriate heights to visualize this type of representation as demonstrated here for vectors in 2D, 3D, and 4D.



Note that for negative coordinates the rectangles extend down from the zero level. Other options are possible particularly visualizing coordinates as points on parallel axes or on the same axis.

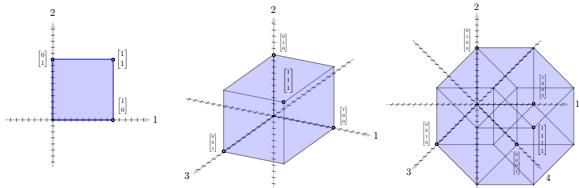


This parallel axis representation does not suffer from the pathology of depth; however this is simply because it does not seek to leverage our 3D spatial intuition. For example, even in 2D or 3D it is quite difficult to see immediately that two vectors are orthogonal in their parallel axes representation. (Readers are encouraged to try this.) This parallel axis representation will arise in several ways in our treatment of these subjects. First, when we represent columns of a matrix using a spatial representation, the rows naturally appear in a parallel representation along each axis (and vice versa if the rows are represented spatially, the columns appear in a parallel representation.) While not used significantly in this paper, this is an interesting fact that the authors hope to explore in future work. Second, a hybrid parallel-spatial geometric technique can be quite useful in visualizing inner products (discussed below).

### III. BASIC GEOMETRIC SETS

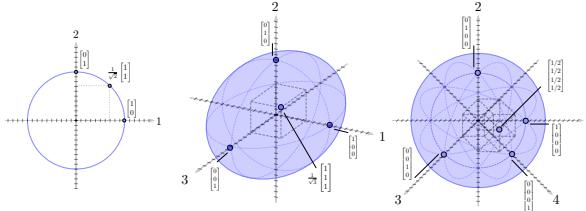
We briefly discuss several geometric sets that we will refer to later. For each set, we label several examples of interesting points. We have already discussed unit cubes briefly. Similarly, let  $\square_n$  represent the  $n$ -dimensional unit cube.

$$\square_n = \left\{ x \in \mathbb{R}^n \mid \mathbf{0} \leq x \leq \mathbf{1} \right\}$$



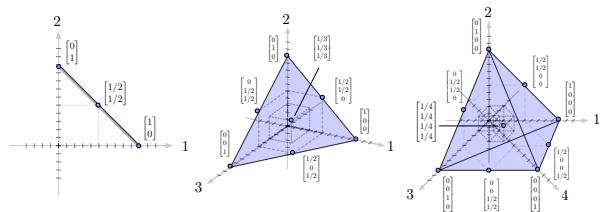
Let  $\bigcirc_n$  represent the  $n$ -dimensional unit sphere.

$$\bigcirc_n = \left\{ x \in \mathbb{R}^n \mid \|x\|_2 = 1 \right\}$$



Note that depending on the axes are drawn a unit circle may not appear perfectly circular. Let  $\Delta_n$  represent the  $n$ -dimensional simplex.

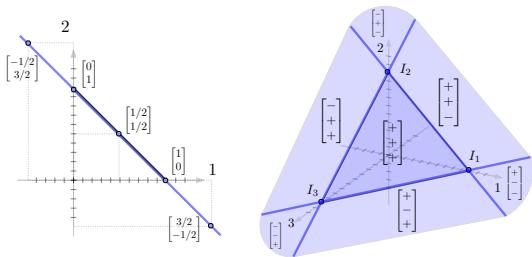
$$\Delta_n = \left\{ x \in \mathbb{R}^n \mid \mathbf{1}^T x = 1, x \geq 0 \right\}$$



Simplexes are often used to represent probability distributions on finite sets. We note that if we remove the constraints that each element  $x_i \geq 0$ , we get a particular  $n - 1$ -dimensional affine space

$$\ell_n = \left\{ x \in \mathbb{R}^n \mid \mathbf{1}^T x = 1 \right\}$$

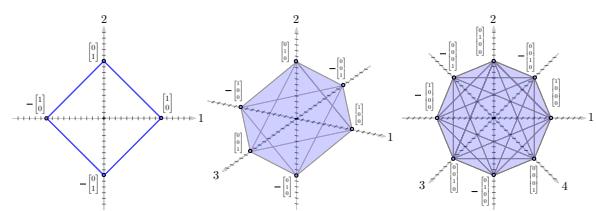
For  $n = 2$  this is a line through the points  $I_1, I_2$ ; for  $n = 3$  this is a plane through the points  $I_1, I_2, I_3$ . Later on, this set will be useful in defining many affine spaces via transformation through a matrix.



It will also be useful to represent unit balls of other norms. The 2-norm unit ball is simply the Euclidean unit sphere above. Let  $\diamond_n$  represent the  $n$ -dimensional unit 1-norm ball.

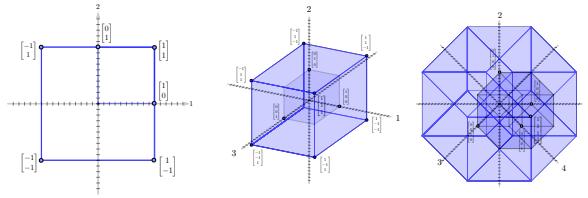
$$\diamond_n = \left\{ x \in \mathbb{R}^n \mid \|x\|_1 = 1 \right\}$$

Note that the “faces” of the 1-norm ball are simplices with different sign patterns.



Finally, let  $\square_n^\infty$  represent the  $n$ -dimensional unit  $\infty$ -norm ball.

$$\square_n^\infty = \left\{ x \in \mathbb{R}^n \mid -\mathbf{1} \leq x \leq \mathbf{1} \right\}$$



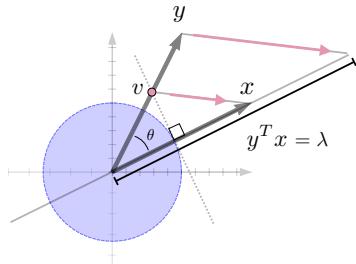
Note that the unit  $\infty$ -ball is substantially larger than the unit cube though it has the same shape. The unit cube always has unit volume regardless of the dimension. The volume of the unit  $\infty$ -ball grows rapidly as  $2^n$ .

#### IV. INNER PRODUCTS

Visualizing inner products,  $y^T x$  for  $x, y \in \mathbb{R}^n$  is critical to any geometric visualization. We detail two methods here: one based on the spatial axis representations of vectors and one based on a hybrid parallel-spatial axis representation. Note that while we will present these techniques with the two vectors  $y$  and  $x$  playing different roles, inner products are symmetric and the roles of these vectors can always be reversed.

##### A. Spatial Visualization

The first technique is a somewhat traditional approach to inner product visualization where an inner product is thought of as a projection from one vector onto another. Consider two vectors  $x, y \in \mathbb{R}^n$ . First define the unit vector in the  $x$  direction and an  $n-1$  dimensional affine space tangent to the unit sphere at this point. Next, define a vector  $v$  in the  $y$ -direction on this plane. If we drag this point  $v$  to the tip of  $x$  (and move  $y$  along with it), then  $y$  moves to a vector with length  $y^T x$ . This process is illustrated (for  $y, x \in \mathbb{R}^2$ ) here.



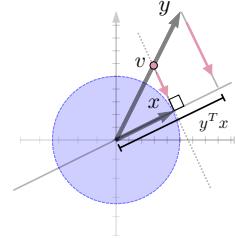
A brief algebraic justification for this is warranted.

*Proof:* From properties of similar triangles and the definition of  $v$  we can obtain that

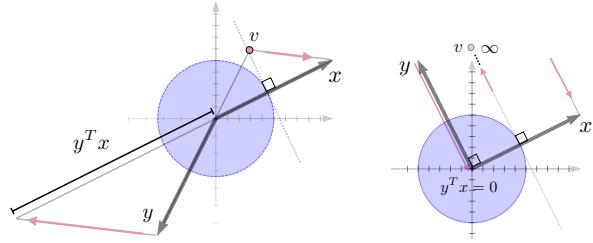
$$\frac{\lambda}{\|x\|_2} = \frac{\|y\|_2}{\|v\|_2}, \quad \|v\|_2 \cos \theta = 1$$

It follows that  $\lambda = \|y\|_2 \|x\|_2 \cos \theta = y^T x$ . If  $x$  is a unit vector, ie.  $\|x\|_2 = 1$ , this visualization becomes

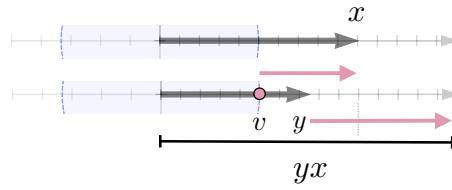
the traditional visualization of an inner product as a projection. One can think of this technique as extending the projection idea to the case when  $x$  and  $y$  are both non-unit vectors.



The reader is encouraged to experiment with other values of  $x$  and  $y$ . Here we show the case where the inner product is negative and the important limiting case where  $y$  and  $x$  are orthogonal ( $y^T x = 0$ ) and the point  $v$  goes to  $\infty$ , ie.  $y$  does not intersect the tangent plane.



We also note that this technique reduces to perhaps the most natural way to visualize scalar multiplication. For two values  $x, y \in \mathbb{R}$  on a number line, one can think of taking the product  $yx$  as stretching a unit value for  $y$  to be the number  $x$ , ie. treating  $x$  as the “units” for  $y$ . This “rescaling” of the  $y$  number line (by  $x$ ) moves  $y$  to  $yx$ .

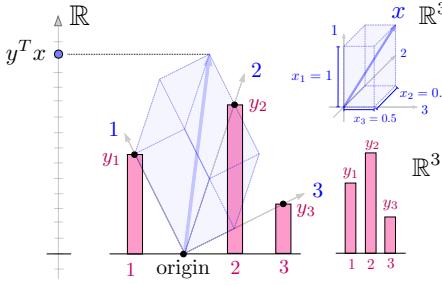


##### B. Parallel-Spatial

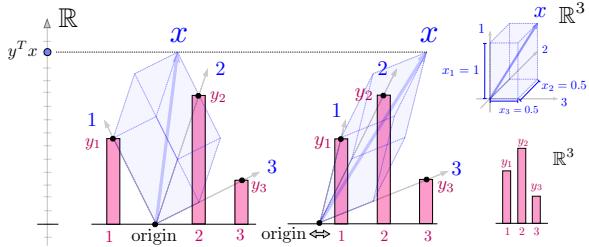
The second inner product visualization technique combines the parallel axis and spatial axis representations of vectors. This technique is subtle and actually quite powerful and is closely related to the idea of column geometry visualization which is at the heart of this paper.

For this technique, we will one vector,  $y$ , in its parallel representation, and the other vector,  $x$ , in its

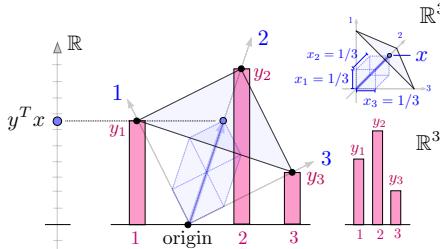
spatial representation. We illustrate this for  $y, x \in \mathbb{R}^3$ . First, draw  $y$  as a set of heights (or parallel displacements). Next, draw axes with the tips of unit vectors at the ends of each coordinate values of  $y$  and place the origin anywhere along the zero-value line. Finally, draw the spatial representation of  $x$  relative to this axis. The height of the corresponding point (or, more generally, the displacement along the parallel axis direction) is  $y^T x$ . This technique is illustrated in the figure below.



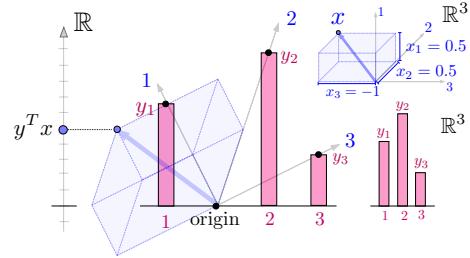
It should be noted that the location of the origin and actually all the horizontal values in this picture are irrelevant to the final result and can be shifted freely. We will often take advantage of this to improve the legibility of various illustrations. Again, the reader is encouraged to try this.



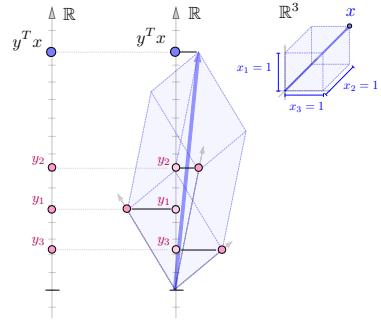
This technique is particularly useful when the vector  $x$  is a convex combination, ie.  $1^T x = 1$ ,  $x \geq 0$ . In this case, the image of  $x$  in the hybrid visualization will appear in the convex hull of the heights of the  $y'_i$ 's as illustrated here. This image and then the value  $y^T x$  are easy to see.



On the other hand, the visualization can be more difficult to use when the values of  $x$  are large or negative. It still works, however.



It is worth noting that the worst option for the horizontal position of the elements of  $y$  is to line them all up on the same axis. Providing variation in the horizontal configuration allows us to differentiate the vector elements and “see” the spatial representation of  $x$  better. In fact, many times even if the points naturally lie on the same axis, we will simply “bump” each of them off the axis an arbitrary (but different) amount. When we project the result back onto the original axis to visualize  $y^T x$  we are removing the horizontal information that we added initially.



Also it should be noted that there is nothing special about “horizontal” and “vertical” in these examples and their roles can be swapped with the same intuition. However, it is important that the “perturbation” direction is orthogonal to the primary direction.