EE578B - Convex Optimization - Winter 2021

Homework 5 - Solution

Due Date: Wednesday, Feb 17^{th}, 2021 at 11:59 pm

1. **Linear Program Duality**

   Consider the linear program

$$p^* = \min_x \quad c^T x \qquad \text{s.t.} \quad Ax = b,\ Cx \geq d$$

   for $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{p\times n}$, $d \in \mathbb{R}^p$.

   (a) **(PTS: 0-2)** Write the linear program in it's game form

$$p^* = \min_{x\in\mathbb{R}^n} \max_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \quad \mathcal{L}(x, v, w)$$

   for dual variables $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^p_+$

   Solution:

$$p^* = \min_{x\in\mathbb{R}^n} \max_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \quad c^T x - v^T(Ax - b) - w^T(Cx - d)$$

   Note that the sign on the v-term doesn't matter since it's an equality constraint. The sign on the w-term must be negative in order to force $Cx - d \geq 0$. If $Cx - d < 0$ then w can push $-w^T(Cx - d)$ to positive ∞ while still remaining positive. This forces the primal variable x to keep $Cx - d \geq 0$.

   (b) **(PTS: 0-2)** The game form of the dual problem is given by swapping the minimum and maximum

$$d^* = \max_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \min_{x\in\mathbb{R}^n} \mathcal{L}(x, v, w)$$

   How does p^* relate to d^*?

   Solution: Swapping the min and max gives that

$$p^* = \min_{x\in\mathbb{R}^n} \max_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \mathcal{L} \geq \max_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \min_{x\in\mathbb{R}^n} \mathcal{L}$$

(c) **(PTS: 0-2)** Now suppose x is chosen to solve $\min_x \ \mathcal{L}(x, v, w)$. What constraints does this imply on v and w? (Hint: use the condition $\frac{\partial \mathcal{L}}{\partial x} = 0$ to compute the constraints.).

Solution: Solving the inner optimization

$$\min_{x \in \mathbb{R}^n} \ \mathcal{L}$$

by setting $\frac{\partial \mathcal{L}}{\partial x} = 0$ gives the constraint on the dual variables

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 0 \\ c^T - v^T A - w^T C &= 0 \end{aligned}$$

(d) **(PTS: 0-2)** Replace the $\min_x \ \mathcal{L}(x, v, w)$ with the constraints computed in the previous part and the appropriate objective function of v and w, $\ell(v, w)$. ie. write the dual problem in the form

$$\begin{aligned} \max_{v \in \mathbb{R}^m, \ w \in \mathbb{R}^p_+} \quad & \ell(v, w) \\ \text{s.t.} \quad & g(v, w) = 0, \\ & h(v, w) \geq 0 \end{aligned}$$

Solution:

The remaining terms in the Lagrangian (that only involve dual variables v, w and constants) are $v^T b$ and $w^T d$. By replacing the inner problem with the constraints previously computed (and explicitly expressing the positivity of w) we get the dual problem as

$$\begin{aligned} \max_{v \in \mathbb{R}^m, \ w \in \mathbb{R}^p} \quad & v^T b + w^T d \\ \text{s.t.} \quad & c^T - v^T A - w^T C = 0, \ w \geq 0 \end{aligned}$$

(e) **(PTS: 0-2)** For the following matrices, solve both the primal and dual versions of the linear program using `cvx` (in Matlab) or `cvxpy` in Python for $x \in \mathbb{R}^5$.

$$c^T = \begin{bmatrix} 1 & 2 & 4 & 5 & 6 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} I_{5\times 5} \\ C' \end{bmatrix}, \quad C' = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ -0.5 \end{bmatrix}$$

(f) **(PTS: 0-2)** Show numerically that the optimal dual variables for the constraints of the primal problem are the same as the optimizers of the dual problem. Show also that the optimizers of the primal problem are the optimal dual variables for the constraints of the dual problem.

Solution:

```python
# Import packages.
import cvxpy as cp
import numpy as np

n = 5;
c = np.array([1,2,4,5,6]);
A = np.array([[1,1,1,1,1],[1,1,-1,0,0],[0,0,0,1,-1]]);
b = np.array([1,0,0]);
C = np.block([[np.eye(5)],
              [np.array([[-1,-1,-1,0,0],
                         [0,0,0,-1,-1]])]]);
d = np.array([0,0,0,0,0,-0.5,-0.5])

# Define and solve the CVXPY problem.
x = cp.Variable(n)
obj = c.T @ x;
constraints = [A @ x == b, C @ x >= d];
primal = cp.Problem(cp.Minimize(obj),constraints)
primal.solve()

print('Primal Problem: ')
print("The optimal value for the primal problem is ", np.round(primal.value,4))
print(' ');
print("Optimal x: ",np.round(x.value,4))
print(' ');
print('Equality constraint dual variable: ',np.round(primal.constraints[0].dual_value,4))
print('Inequality constraint dual variable: ',np.round(primal.constraints[1].dual_value,4))

v = cp.Variable(3);
w = cp.Variable(7);
dual_obj = v.T@b + w.T@d
dual_constraints = [c -v@A - w@C == 0, w >= 0]
dual = cp.Problem(cp.Maximize(dual_obj),dual_constraints);
dual.solve();


print(' ');print(' ');
print('Dual Problem: ')
print("The optimal value for the dual problem is ", np.round(dual.value,4))
print(' ');
print("Optimal v: ",np.round(v.value,4))
print("Optimal w: ",np.round(w.value,4))
print(' ');
print('Equality dual variable: ',np.round(dual.constraints[0].dual_value,4))
print('Inequality dual variable: ',np.round(dual.constraints[1].dual_value,4))
```

```
Primal Problem:
The optimal value for the primal problem is  4.0

Optimal x:  [0.25 0.   0.25 0.25 0.25]

Equality constraint dual variable:  [-5.8041  1.5     0.5   ]
Inequality constraint dual variable:  [0.     1.     0.     0.     0.     3.3041 0.3041]


Dual Problem:
The optimal value for the dual problem is  4.0

Optimal v:  [ 5.8041 -1.5    -0.5   ]
Optimal w:  [ 0.      1.      0.     -0.     -0.      3.3041  0.3041]

Equality dual variable:  [-0.25 -0.   -0.25 -0.25 -0.25]
Inequality dual variable:  [0.25 0.   0.25 0.25 0.25 0.   0.  ]
```

2. **Quadratic Program Duality**

Consider the quadratic program

$$p^* = \max_x \quad \tfrac{1}{2}x^T Q x + r^T x \qquad \text{s.t.} \quad Ax = b,\ Cx \geq d$$

for $Q = Q^T \prec 0$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{p\times n}$, $d \in \mathbb{R}^p$.

(a) **(PTS: 0-2)** Write the quadratic program in it's game form

$$p^* = \max_{x\in\mathbb{R}^n} \min_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \mathcal{L}(x, v, w)$$

Solution:

$$p^* = \max_{x\in\mathbb{R}^n} \min_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \tfrac{1}{2}x^T Q x + r^T x - v^T(Ax - b) + w^T(Cx - d)$$

Note that the sign on the v-term doesn't matter since it's an equality constraint. The sign on the w-term must be positive in order to force $Cx - d \geq 0$. If $Cx - d < 0$ then w can push $w^T(Cx - d)$ to negative ∞ while still remaining positive. This forces the primal variable x to keep $Cx - d \geq 0$.

(b) **(PTS: 0-2)** The game form of the dual problem is given by swapping the minimum and maximum

$$d^* = \min_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \max_{x\in\mathbb{R}^n} \mathcal{L}(x, v, w)$$

How does p^* relate to d^*?

Solution: Swapping the max and min gives that

$$p^* = \max_{x\in\mathbb{R}^n} \min_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \mathcal{L} \leq \min_{\substack{v \in \mathbb{R}^m \\ w \in \mathbb{R}^p_+}} \max_{x\in\mathbb{R}^n} \mathcal{L} = d^*$$

(c) **(PTS: 0-2)** Now suppose x is chosen to solve $\max_x \mathcal{L}(x, v, w)$. What constraints does this imply on v and w? (Hint: use the condition $\frac{\partial \mathcal{L}}{\partial x} = 0$ to compute the constraints.)

Solution: Solving the inner optimization $\max_x \mathcal{L}$ by setting $\frac{\partial \mathcal{L}}{\partial x} = 0$ gives the constraint on the dual variables

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 0 \\ x^T Q + r^T - v^T A + w^T C &= 0 \end{aligned}$$

(d) **(PTS: 0-2)** Replace the $\max_x \mathcal{L}(x, v, w)$ with the constraints computed in the previous part and the appropriate objective function of the dual variables to write the dual problem.

Solution:

The constraints computed in the previous part still involve x and we want to write the dual problem purely in terms of the dual variables. In order to do this, we need to solve for x directly

$$x^T = -Q^{-1}(r^T - v^T A + w^T C)$$

and plug it back in to the Lagrangian. Rather than doing this explicitly, it's a little simpler notationally to just define the piece of the constraint that depends on x as a new dual variable $z^T = x^T Q$. The constraint is then written as

$$z^T + r^T - v^T A + w^T C = 0$$

purely in terms of dual variables. (Another way to think about this is that we are just defining the quantity $-r^T + v^T A - w^T C$ to be z^T to make it easier to plug in to the Lagrangian.) After plugging in we get that

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2} z^T Q^{-1} z + (r^T - v^T A + w^T C) Q^{-1} z + v^T b - w^T d \\
&= \tfrac{1}{2} z^T Q^{-1} z - z^T Q^{-1} z + v^T b - w^T d \\
\mathcal{L} &= -\tfrac{1}{2} z^T Q^{-1} z + v^T b - w^T d
\end{aligned}$$

The dual form of the optimization problem is then given by

$$\begin{aligned}
\min_{v,w,z} \quad & -\tfrac{1}{2} z^T Q^{-1} z + v^T b - w^T d \\
\text{s.t.} \quad & z^T + r^T - v^T A + w^T C = 0, \;\; w \geq 0
\end{aligned}$$

It would also be fine to plug in the equality constraint into the objective function.

(e) **(PTS: 0-2)** For the following matrices, solve both the primal and dual versions of the linear program using `cvx` (in Matlab) or `cvxpy` in Python for $x \in \mathbb{R}^5$.

$$Q = -\text{diag}\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}\right), \quad r^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} I_{5\times 5} \\ C' \end{bmatrix}, \quad C' = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.5 \\ -0.5 \end{bmatrix}$$

(f) **(PTS: 0-2)** Show numerically that the optimal dual variables for the constraints of the primal problem are the same as the optimizers of the dual problem. Show also that the optimizers of the primal problem are the optimal dual variables for the constraints of the dual problem.

Solution:

```python
# Import packages.
import cvxpy as cp
import numpy as np
import numpy.linalg as mat

# Generate a random non-trivial quadratic program.
n = 5;
r = np.array([1,2,4,5,6]);
A = np.array([[1,1,1,1,1],[1,1,-1,0,0],[0,0,0,1,-1]]);
b = np.array([1,0,0]);
C = np.block([[np.eye(5)],
              [np.array([[-1,-1,-1,0,0],
                         [0,0,0,-1,-1 ]])]])
d = np.array([0,0,0,0,0,-0.5,-0.5])

Q = -np.diag([1,2,3,4,5]);
r = np.array([1,2,3,4,5]);

# Define and solve the CVXPY problem.
x = cp.Variable(n)
obj = 0.5*cp.quad_form(x, Q) + r.T @ x;
constraints = [A @ x == b, C @ x >= d];
primal = cp.Problem(cp.Maximize(obj),constraints)
primal.solve()

print('Primal Problem: ')
print("The optimal value for the primal problem is ", np.round(primal.value,4)) print(' ');
print("Optimal x: ",np.round(x.value,4)) print(' ');
print('Equality dual: ',np.round(primal.constraints[0].dual_value,4))
print('Inequality dual: ',np.round(primal.constraints[1].dual_value,4))

z = cp.Variable(5);
v = cp.Variable(3);
w = cp.Variable(7);
dual_obj = -0.5*cp.quad_form(z,mat.inv(Q)) + v.T@b - w.T@d
dual_constraints = [z + r  - v@A + w@C == 0, w >= 0]
dual = cp.Problem(cp.Minimize(dual_obj),dual_constraints);
dual.solve();

print(' ');
print(' ');
print('Dual Problem: ')
print("The optimal value for the dual problem is ", np.round(dual.value,4))
print(' ');
print("Optimal z: ",np.round(z.value,4))
print("Optimal v: ",np.round(v.value,4))
print("Optimal w: ",np.round(w.value,4))
print(' ');
print('Equality dual: ',np.round(dual.constraints[0].dual_value,4))
print('Inequality dual: ',np.round(dual.constraints[1].dual_value,4))


print('Blip: ', v.value@A + w.value@C)
```

```
Primal Problem:
The optimal value for the primal problem is  3.0625

Optimal x:  [0.   0.25 0.25 0.25 0.25]

Equality dual:  [ 0.7857 -0.375  -0.375 ]
Inequality dual:  [0.5    0.     0.     0.     0.     1.0893 2.5893]


Dual Problem:
The optimal value for the dual problem is  3.0625

Optimal z:  [ 0.   -0.5  -0.75 -1.   -1.25]
Optimal v:  [ 1.6017 -0.375  -0.375 ]
Optimal w:  [0.5    0.     0.     0.     0.     0.2733 1.7733]

Equality dual:  [0.   0.25 0.25 0.25 0.25]
Inequality dual:  [0.   0.25 0.25 0.25 0.25 0.   0.  ]
```

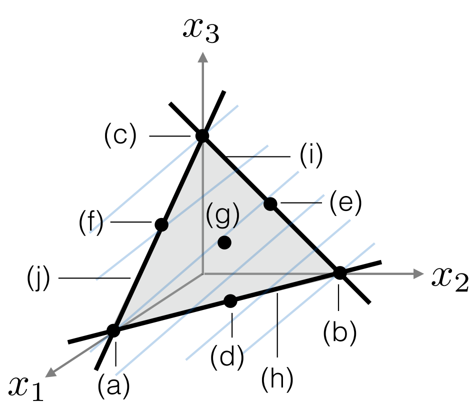
3. **Simplex Optimization**

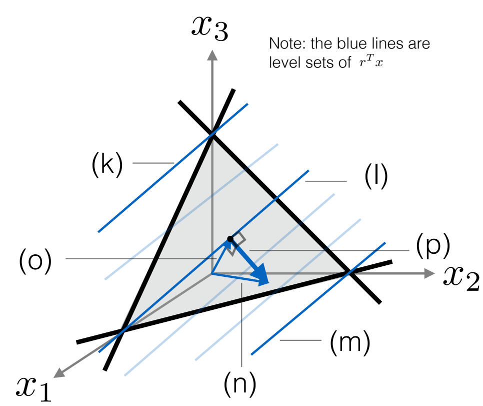
- Consider the primal form of optimization on a simplex in $\mathbb{R}^3$.

$$\begin{aligned} \max_{x\in\mathbb{R}^3} \quad & r^T x \\ \text{s.t.} \quad & \mathbf{1}^T x = 1,\ x \geq 0 \end{aligned}$$

for $\mathbf{1}^T = [1\ 1\ 1]$. Consider the following illustrations of the primal problems. Label each indicated part of the diagrams.

(PTS: 0-2) Label: (a)-(j) **(PTS: 0-2)** Label: (k)-(p)





Solution:

$$(a) = \begin{bmatrix}1\\0\\0\end{bmatrix},\ (b) = \begin{bmatrix}0\\1\\0\end{bmatrix},\ (c) = \begin{bmatrix}0\\0\\1\end{bmatrix},\ (d) = \begin{bmatrix}0.5\\0.5\\0\end{bmatrix},\ (e) = \begin{bmatrix}0\\0.5\\0.5\end{bmatrix},\ (f) = \begin{bmatrix}0.5\\0\\0.5\end{bmatrix},\ (g) = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$$

$$\begin{aligned} (h) &= \{x \mid \mathbf{1}^T x = 1,\ x_3 = 0\} \\ (i) &= \{x \mid \mathbf{1}^T x = 1,\ x_1 = 0\} \\ (j) &= \{x \mid \mathbf{1}^T x = 1,\ x_2 = 0\} \end{aligned}$$

$$\begin{aligned} (k) &= \{x \mid \mathbf{1}^T x = 1,\ r^T x = r_3\} \\ (l) &= \{x \mid \mathbf{1}^T x = 1,\ r^T x = r_1\} \\ (m) &= \{x \mid \mathbf{1}^T x = 1,\ r^T x = r_2\} \end{aligned}$$

Let P be the projection matrix onto the range of $\mathbf{1}$, ie. $P = \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T$.

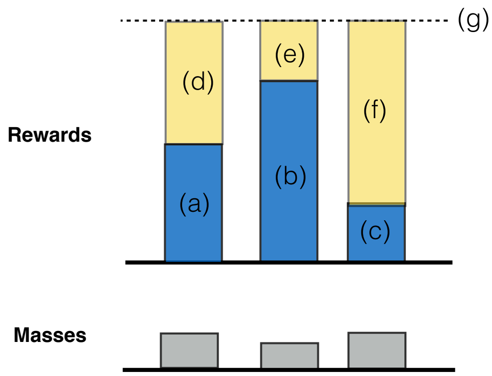
$$(n) = r^T, \quad (o) = r^T P, \quad (p) = r^T(I - P)$$

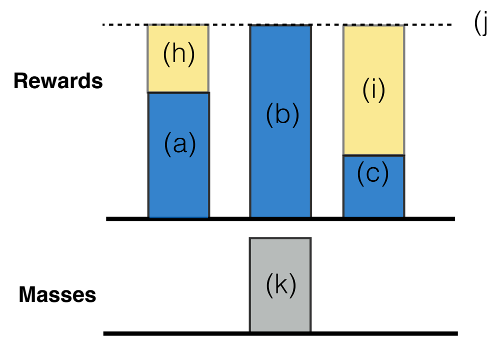
- Consider the dual form of the optimization problem

$$\begin{aligned} \min_{\lambda \in \mathbb{R},\ \mu \in \mathbb{R}^3_+} \quad & \lambda \\ \text{s.t.} \quad & \lambda \mathbf{1}^T = r^T + \mu^T,\ \ \mu \geq 0 \end{aligned}$$

Consider the following illustrations of the dual problem. Label each indicated part of the diagrams.

(PTS: 0-2) Label: (a)-(f) **(PTS: 0-2)** Label: (h),(i),(j),(k)





Solution:

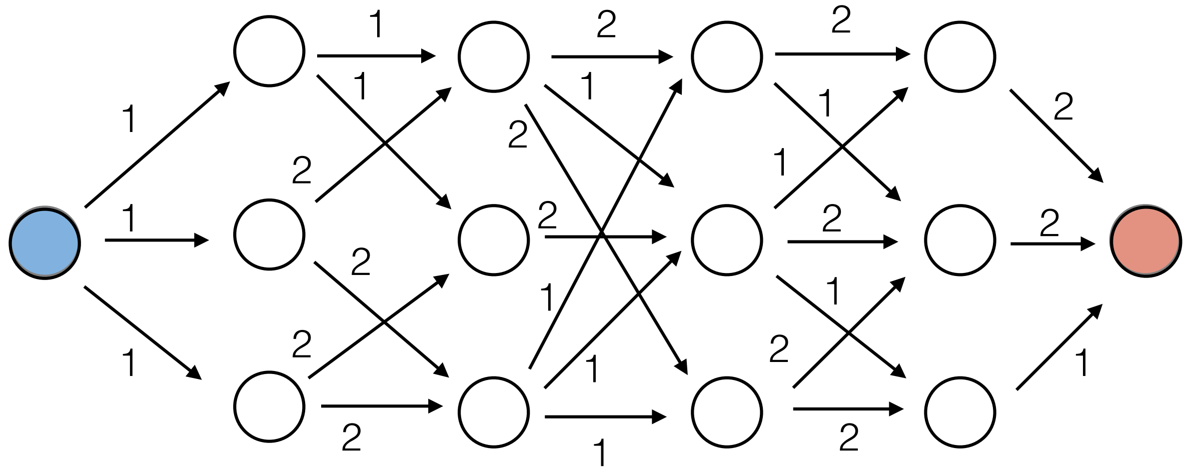
$$(a) = r_1, \quad (b) = r_2, \quad (c) = r_3, \quad (d) = \mu_1, \quad (e) = \mu_2, \quad (f) = \mu_3, \quad (g) = \lambda,$$

Let λ^*, μ^* be the optimal variables and x^* be the optimal dual (of the dual) variables associated with the constraint.

$$(h) = \mu_1^*, \quad (i) = \mu_3^*, \quad (j) = \lambda^*, \quad (k) = x_2^*$$

4. **Dynamic Programming**

   Use dynamic programming to compute the shortest path from the blue node to the red node (given the travel costs on each edge given in the diagram).



- **(PTS:0-2)** Compute the optimal "cost-to-go" from each node to the end.
- **(PTS:0-2)** What is the shortest path?

  Solution: