

Block Matrix Multiplication

Linear Algebra

Winter 2022 - Dan Calderone

Block Matrix Multiplication

**Matrix
Multiplication**

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix}$$

**Block Matrix
Multiplication**

$$AB = \begin{matrix} & \overset{n_1}{\text{---}} & & \overset{n_N}{\text{---}} \\ \begin{matrix} m_1 \\ \vdots \\ m_M \end{matrix} \text{I} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} & \begin{matrix} n_1 \\ \vdots \\ n_N \end{matrix} \text{I} & \begin{matrix} \overset{p_1}{\text{---}} & & \overset{p_P}{\text{---}} \\ \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{matrix}$$

*General
Case*

$$= \begin{matrix} & \overset{p_1}{\text{---}} & & \overset{p_P}{\text{---}} \\ \begin{matrix} m_1 \\ \vdots \\ m_M \end{matrix} \text{I} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{matrix}$$

Block Matrix Multiplication

Block Matrix
Multiplication

$$AB = \begin{matrix} & \overbrace{\hspace{1cm}}^{n_1} & & \overbrace{\hspace{1cm}}^{n_N} \\ \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} & \begin{matrix} \overbrace{\hspace{1cm}}^{p_1} & & \overbrace{\hspace{1cm}}^{p_P} \\ \mathbf{I} & & \mathbf{I} \end{matrix} & \begin{bmatrix} B_{11} & \cdots & B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{matrix} = \begin{matrix} & \overbrace{\hspace{2cm}}^{p_1} & & \overbrace{\hspace{2cm}}^{p_P} \\ \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{matrix}$$

Case 1

“Linear
Combination
of Columns”

$$Ax = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} x_1 + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} x_n$$

Case 2

“Inner product
with rows”

$$Ax = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

Block Matrix Multiplication

Block Matrix Multiplication

$$AB = \begin{bmatrix} \overset{m_1}{\mathbf{I}} \begin{bmatrix} \overset{n_1}{A_{11}} \\ \vdots \\ A_{M1} \end{bmatrix} & \cdots & \begin{bmatrix} \overset{n_N}{A_{1N}} \\ \vdots \\ A_{MN} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overset{n_1}{\mathbf{I}} \begin{bmatrix} \overset{p_1}{B_{11}} & \cdots & B_{1P} \end{bmatrix} \\ \vdots \\ \overset{n_N}{\mathbf{I}} \begin{bmatrix} B_{N1} & \cdots & \overset{p_P}{B_{NP}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \overset{m_1}{\mathbf{I}} \begin{bmatrix} \overset{p_1}{A_{11}B_{11} + \cdots + A_{1N}B_{N1}} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \end{bmatrix} \\ \vdots \\ \overset{m_M}{\mathbf{I}} \begin{bmatrix} A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{bmatrix}$$

Case 3

“Inner product with columns”

$$y^T A = \begin{bmatrix} - & y^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} y^T A_1 & \cdots & y^T A_n \end{bmatrix}$$

Case 4

“Linear Combination of Rows”

$$y^T A = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} - & a_1^T & - \\ \vdots & \vdots & \\ - & a_m^T & - \end{bmatrix} = y_1 \begin{bmatrix} - & a_1^T & - \end{bmatrix} + \cdots + y_m \begin{bmatrix} - & a_m^T & - \end{bmatrix}$$

Block Matrix Multiplication

Block Matrix
Multiplication

$$AB = \begin{matrix} & \overbrace{\hspace{1cm}}^{n_1} & & \overbrace{\hspace{1cm}}^{n_N} \\ \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} & \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} & \begin{matrix} \overbrace{\hspace{1cm}}^{n_1} \mathbf{I} \\ \vdots \\ \overbrace{\hspace{1cm}}^{n_N} \mathbf{I} \end{matrix} & \begin{bmatrix} \overbrace{\hspace{1cm}}^{p_1} B_{11} & \cdots & \overbrace{\hspace{1cm}}^{p_P} B_{1P} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NP} \end{bmatrix} \end{matrix} = \begin{matrix} & \overbrace{\hspace{2cm}}^{p_1} & & \overbrace{\hspace{2cm}}^{p_P} \\ \begin{matrix} m_1 \mathbf{I} \\ \vdots \\ m_M \mathbf{I} \end{matrix} & \begin{bmatrix} A_{11}B_{11} + \cdots + A_{1N}B_{N1} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{matrix}$$

Case 5

“A times each
column of B”

$$AB = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | & & | \\ B_1 & \cdots & B_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ AB_1 & \cdots & AB_p \\ | & & | \end{bmatrix}$$

Case 6

“B times each
row of A”

$$AB = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} - & a_1^T B & - \\ & \vdots & \\ - & a_m^T B & - \end{bmatrix}$$

Block Matrix Multiplication

**Block Matrix
Multiplication**

$$AB = \begin{bmatrix} \overset{m_1}{\textcolor{blue}{I}} \begin{bmatrix} \overset{n_1}{\textcolor{red}{A}_{11}} & \cdots & \overset{n_N}{\textcolor{red}{A}_{1N}} \\ \vdots & & \vdots \\ \overset{m_M}{\textcolor{blue}{I}} \begin{bmatrix} \textcolor{red}{A}_{M1} & \cdots & \textcolor{red}{A}_{MN} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overset{n_1}{\textcolor{red}{I}} \begin{bmatrix} \overset{p_1}{\textcolor{red}{B}_{11}} & \cdots & \overset{p_P}{\textcolor{red}{B}_{1P}} \\ \vdots & & \vdots \\ \textcolor{red}{I}_{n_N} \begin{bmatrix} \textcolor{red}{B}_{N1} & \cdots & \textcolor{red}{B}_{NP} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \overset{m_1}{\textcolor{blue}{I}} \begin{bmatrix} \overset{p_1}{A_{11}B_{11} + \cdots + A_{1N}B_{N1}} & \cdots & A_{11}B_{1P} + \cdots + A_{1N}B_{NP} \\ \vdots & & \vdots \\ \overset{m_M}{\textcolor{blue}{I}} \begin{bmatrix} A_{M1}B_{11} + \cdots + A_{MN}B_{N1} & \cdots & A_{M1}B_{1P} + \cdots + A_{MN}B_{NP} \end{bmatrix} \end{bmatrix}$$

Case 7
“Pairwise
inner products
of rows of A
& columns of B”

$$AB = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ B_1 & \cdots & B_p \\ | & & | \end{bmatrix} = \begin{bmatrix} a_1^T B_1 & \cdots & a_1^T B_p \\ \vdots & & \vdots \\ a_m^T B_1 & \cdots & a_m^T B_p \end{bmatrix}$$

Case 8
“Sum of
outer products
of columns of A
and rows of B”

$$AB = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \end{bmatrix} + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} \begin{bmatrix} - & b_n^T & - \end{bmatrix}$$

Block Matrix Multiplication

Block Matrix Multiplication

$$AB = \begin{bmatrix} m_1 \mathbf{I} & & \\ & \ddots & \\ & & m_M \mathbf{I} \end{bmatrix} \begin{bmatrix} \overbrace{A_{11} \cdots A_{1N}}^{n_1} & \cdots & \overbrace{A_{1N} \cdots A_{1N}}^{n_N} \\ \vdots & & \vdots \\ \overbrace{A_{M1} \cdots A_{MN}}^{n_1} & \cdots & \overbrace{A_{MN} \cdots A_{MN}}^{n_N} \end{bmatrix} \begin{bmatrix} \overbrace{B_{11} \cdots B_{1P}}^{p_1} \\ \vdots \\ \overbrace{B_{N1} \cdots B_{NP}}^{p_P} \end{bmatrix} = \begin{bmatrix} m_1 \mathbf{I} & & \\ & \ddots & \\ & & m_M \mathbf{I} \end{bmatrix} \begin{bmatrix} \overbrace{A_{11}B_{11} + \cdots + A_{1N}B_{N1}}^{p_1} & \cdots & \overbrace{A_{11}B_{1P} + \cdots + A_{1N}B_{NP}}^{p_P} \\ \vdots & & \vdots \\ \overbrace{A_{M1}B_{11} + \cdots + A_{MN}B_{N1}}^{p_1} & \cdots & \overbrace{A_{M1}B_{1P} + \cdots + A_{MN}B_{NP}}^{p_P} \end{bmatrix}$$

Case 9
“Pairwise
inner products
of rows of A
& columns of B
around D ”

$$AB = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} & & \\ & D & \\ & & \end{bmatrix} \begin{bmatrix} | & & | \\ B_1 & \cdots & B_p \\ | & & | \end{bmatrix} = \begin{bmatrix} a_1^T DB_1 & \cdots & a_1^T DB_p \\ \vdots & & \vdots \\ a_m^T DB_1 & \cdots & a_m^T DB_p \end{bmatrix}$$

Case 10
“Sum of scaled
pairwise
outer products”

$$AB = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} d_{11} \begin{bmatrix} - & b_1^T & - \end{bmatrix} + \cdots + \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} d_{1n} \begin{bmatrix} - & b_n^T & - \end{bmatrix} + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} d_{n1} \begin{bmatrix} - & b_1^T & - \end{bmatrix} + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} d_{nn} \begin{bmatrix} - & b_n^T & - \end{bmatrix} = \sum_i \sum_j \begin{bmatrix} | \\ A_i \\ | \end{bmatrix} d_{ij} \begin{bmatrix} - & b_j^T & - \end{bmatrix}$$

Case 10b
“Sum of scaled
outer products
(diagonal)”

$$AB = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} d_{11} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} d_{11} \begin{bmatrix} - & b_1^T & - \end{bmatrix} + \cdots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} d_{nn} \begin{bmatrix} - & b_n^T & - \end{bmatrix} = \sum_i \begin{bmatrix} | \\ A_i \\ | \end{bmatrix} d_{ii} \begin{bmatrix} - & b_i^T & - \end{bmatrix}$$