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Multi-Dimensional Continuous Type Population Potential Games

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***Abstract*— We consider an extension of continuous population potential games where the population mass is represented by a distribution over a multi-dimensional continuous type space. Each dimension of the type space corresponds to a subset of strategies in the potential game. The difference between two elements in each population member's type vector encodes their relative preference for strategies in each subset. We define an extension of a Wardrop equilibrium for this type based model and show how to compute the equilibrium by means of optimizing a specific potential function.**

I. Introduction

Continuous population potential games [1] have been a well studied fixture of the game theory community particularly in the traffic assignment problem [2]. In these games, individual agents are modeled as infinitesimal masses in a continuous population choosing from a finite set of strategies. The Nash-like equilibrium concept for these games is known as a *Wardrop equilibrium* [3] and in many cases can be characterized as the optimizer of a potential function within some constrained set [1], [4].

A restriction of standard population potential games is that they treat each member of the population as being identical and anonymous. Several lines of research have sought to alleviate this restriction. One of simplest and most common models is what is called the *variable demand* version of potential games where the total mass playing the game fluctuates with the equilibrium reward value population members receive [5], [6]. One can conceptualize this as some members of the population opting out of playing if the reward is too low. Another more complex class of models considers users as making decisions based on more than one reward criteria. These *multi-criteria equilibrium models* [7]–[9] often involve users considering the trade-off between travel time and monetary cost (tolls). Some models consider population users to belong to discrete type sets with different time-money trade-off parameters [10]–[12]. Another interesting class of models considers population members as a distribution over a continuous parameter space [13]–[17]. The equilibrium concepts in these continuous type models make use of variational inequality formulations to describe both how the population divides itself among routes and how the type space is partitioned into distinct regions in which population members choose specific strategies.

In this work, we propose a multi-dimensional continuous type model where the population is represented by a distribution supported over a multi-dimensional type space. Each population member has a type-vector that encodes their arbitrary preferences for specific strategy options in the potential game. We define an equilibrium concept that extends the traditional Wardrop equilibrium to this continuous type model and define a potential function and the corresponding optimization problem for computing the equilibrium. Our model is related to the bicriterion continuous type models listed above and reduces to the variable demand model in the one-dimensional case. The analysis in this paper bears some resemblance to the the design of multidimensional screening mechanisms or contracts [18], [19].

The rest of the paper is organized as follows. In Section II, we summarize potential game models. In Section III, we present variable demand potential games as a precursor to our model. In Section IV, we present our multi-dimensional model, define an equilibrium concept, and detail a potential function optimization program for equilibrium computation. In Section V, we give a brief numerical example and in Section VI we discuss future work.

II. Continuous Player Potential Games

Continuous population potential games are games where individual agents are modeled as infinitesimal masses in a continuous population. The population chooses from a finite number of actions $\mathcal{A}$. The vector $z \in \mathbb{R}_+^{|\mathcal{A}|}$ represents the portion of the population that chooses each action. The population vector z lives on a simplex scaled by the total population mass m—that is, the set of feasible population vectors is given by

$$\{z \in \mathbb{R}^{|\mathcal{A}|} \mid \mathbf{1}^T z = m, \quad z \geq 0\}. \tag{1}$$

We will sometimes use the notation $z \in m\Delta_{|\mathcal{A}|}$ where $\Delta_{|\mathcal{A}|}$ has the usual meaning—i.e., a simplex on $\mathcal{A}$. Competition is introduced in the game by a vector of rewards functions $r(z) \in \mathbb{R}^{|\mathcal{A}|}$ where $r_a(z)$, the reward for choosing action a, depends on the population distribution z.

A. Equilibrium Definition and Computation

The *Wardrop Equilibrium* is the most commonly adopted equilibrium notion in continuous population games as it is analogous to the Nash equilibrium concept. In particular, Wardrop equilibria are a continuous population analogs to Nash equilibria in that *no infinitesimal player can improve their reward by switching strategies*.

Definition 1 (Wardrop Equilibrium [3])**.** *A mass distribution $z \in m\Delta_{|\mathcal{A}|}$ is a Wardrop equilibrium if*

$$r_a(z) \geq r_{a'}(z) \quad \textit{when } z_a > 0 \quad \forall a' \in \mathcal{A} \tag{2}$$

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We say the game is a *congestion game* if the reward functions r_a are decreasing.

Definition 2 (Congestion Game)**.** *A continuous population potential game defined by* $(r, \mathcal{A}, m\Delta_{|\mathcal{A}|})$ *is a congestion game if*

$$\partial_{z_{a'}} r_a(z) \leq 0, \qquad \forall a, a' \in \mathcal{A},\ \forall z \in m\Delta_{|\mathcal{A}|}. \tag{3}$$

We say the game is a *potential game* if the reward vector $r(z)$ defines a conservative vector field. Equivalently, there exists a transformation of coordinates under which the vector field induced by $r(z)$ is transformed to a gradient vector field corresponding to some function; this function is the *potential function* for the game.

Definition 3 (Potential Game [1])**.** *We say the game* $(r, \mathcal{A}, m\Delta_{|\mathcal{A}|})$ *is a potential game if* $\exists\ f(z) \in \mathcal{C}^1$ *such that* $\partial_{z_a} f(z) = r_a(z)$.

Remark 1. *It is well-known that for differentiable rewards,* $f(z)$ *exists iff* $\partial_{z_{a'}} r_a(z) = \partial_{z_a} r_{a'}(z)$ *[20].*

If we assume that the reward vector has the structure

$$r(z)^T = \ell(\mathbf{R}z)^T \mathbf{R} \tag{4}$$

where $\mathbf{R} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{A}|}$ for some index set $\mathcal{E}$ and for each $e \in \mathcal{E}$, $\ell_e(x_e)$ is a function of single argument, then a potential function can be written as $f(z) = h(\mathbf{R}z)$ where

$$h(x) = \sum_e \int_0^{x_e} \ell_e(s)\ ds. \tag{5}$$

Indeed, $\partial_x h = \ell(x)^T$, and $\partial_z f = r(z)^T$ follows from the chain rule so that f is clearly a potential function.

This type of potential arises in one of the most common types of potential games, *non-atomic routing games*. In these games, population members choose between a set of routes $\mathcal{A}$ through a network represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ from an origin node $n_o \in \mathcal{N}$ to a destination node $n_d \in \mathcal{N}$ in a cost minimization framework where costs are captured by accumulated travel time on the links. That is, travel time on an individual link e is represented by a latency function $\ell_e : x_e \mapsto \mathbb{R}$ that gives the travel time on link e as a function of traffic mass x_e on that link. The flow z_a on route a gets added to each link in that route and we compute the vector of link flows $x \in \mathbb{R}^{|\mathcal{E}|}$ as $x = \mathbf{R}z$ where $\mathbf{R} \in \{0,1\}^{|\mathcal{E}| \times |\mathcal{A}|}$ is a route indicator matrix such that

$$[\mathbf{R}]_{ea} = \begin{cases} 1 & ; \text{ if edge } e \text{ is in route } a \\ 0 & ; \text{ otherwise} \end{cases} \tag{6}$$

The latency vector $\ell(x) \in \mathbb{R}^{|\mathcal{E}|}$ is composed of edge latencies. The reward associated with each route is then given by $r_a(z) = -[\ell(\mathbf{R}z)^T \mathbf{R}]_a$ as in (4).

A potential function allows us to compute equilibria by solving

$$\max_z\ \{f(z) \mid \mathbf{1}^T z = m,\ z \geq 0\} \tag{P-7}$$

as shown in the following theorem.

Theorem 1 (Potential Optimization [1])**.** *If z satisfies the Karush-Khun-Tucker (KKT) conditions for* (P-7)*, then z is a Wardrop equilibrium.*

Proof. The KKT conditions and the definition of the potential function imply that

$$r(z)^T = \lambda \mathbf{1}^T - \mu^T, \quad z, \mu \geq 0, \quad \mu^T z = 0 \tag{8}$$

for multipliers $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}_+^{|\mathcal{A}|}$. For two actions a and a', (8) gives

$$r_a(z) + \mu_a = \lambda = r_{a'}(z) + \mu_{a'}. \tag{9}$$

Moreover, $\mu_a z_a = 0$ and $\mu \geq 0$ implies

$$r_a(z) = \lambda \geq r_{a'}(z) \tag{10}$$

which is exactly the condition in (2) in Definition 1. □

Note that a strictly concave potential f guarantees uniqueness of the equilibrium [20].

Remark 2. *The multipliers λ and μ_a can be interpreted as the maximum reward and inefficiency of action a, respectively. Complementary slackness indicates that no member of the population chooses an inefficient action and thus the entire population achieves equal maximal reward.*

B. Dual Formulations

When the potential function has the form $f(z) = h(\mathbf{R}z)$ for $h(\cdot)$ defined as in (5), the dual of (P-7) is given by

$$\min_{c,\lambda}\ \{\lambda m - \tilde{h}(c) \mid \lambda \mathbf{1}^T \geq c^T \mathbf{R}\} \tag{P-11}$$

with $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}^{|\mathcal{E}|}$ and where $\tilde{h}(\cdot)$ is defined by

$$\tilde{h}(c) = \sum_e \int_{\ell_e(0)}^{c_e} \ell_e^{-1}(s)\ ds \tag{12}$$

Using dual variable $z \in \mathbb{R}_+^{|\mathcal{A}|}$ for the constraint, the optimality conditions for (P-11) are

$$m = \mathbf{1}^T z, \quad \ell^{-1}(c) = \mathbf{R}z, \quad z \geq 0, \tag{13a}$$
$$\lambda \mathbf{1}^T \geq c^T \mathbf{R}, \quad (\lambda \mathbf{1}^T - c^T \mathbf{R}) z = 0 \tag{13b}$$

where $\ell^{-1}(c) = [\ell_1^{-1}(c_1), \ldots, \ell_{|\mathcal{E}|}^{-1}(c_e)]^T$. In solving the dual problem, we solve for the maximum reward λ and the reward variables c_e which, when aggregated, give the reward for each action $[c^T \mathbf{R}]_a$. Conditions (13a) ensure that the dual variable z represents a feasible mass distribution. Conditions (13b) then guarantee that that mass distribution is a Wardrop equilibrium.

III. Variable Demand

In some cases, we consider the population mass m to be a variable that can change with the maximum reward of an action. Intuitively, different members of the population have a threshhold reward they must receive in order for them to participate in the game. We can define a demand function $\phi : \lambda \mapsto m$ that gives us the portion of the total population mass that participates in the game given a maximal reward λ. Since we are modeling agents as reward maximizers, we assume that ϕ is an increasing function of λ. We can solve for the equilibrium with variable demand by incorporating the demand function into (P-7). Indeed, consider the modified optimization problem given by

$$\max_{z,m} \left\{ f(z) - g(m) \mid \mathbf{1}^T z = m,\ z \geq 0 \right\} \tag{P-14}$$

where

$$g(m) = \int_0^m \phi^{-1}(s)\ ds. \tag{15}$$

This modification adds the optimality condition $\lambda = \phi^{-1}(m)$ which guarantees the maximum reward is consistent with the demand.

The corresponding dual problem is then

$$\min_{c,\lambda} \left\{ \tilde{g}(\lambda) - \tilde{h}(c) \mid \lambda \mathbf{1}^T \geq c^T \mathbf{R} \right\} \tag{P-16}$$

with

$$\tilde{g}(\lambda) = \int_{\phi^{-1}(0)}^{\lambda} \phi(s)\ ds. \tag{17}$$

Note that $\partial_\lambda \tilde{g}(\lambda) = \phi(\lambda) = m$ and the optimality condition corresponding to λ becomes $\phi(\lambda) = m = \mathbf{1}^T z$.

We note that we could derive the demand function by conceptualizing the population not as a single mass but as a measure $M(\theta_1)$ over a continuous type space $\Theta = \mathbb{R}$. Each bit of mass in the population is supported on a type parameter $\theta_1 \in \Theta$ which defines the minimum reward required for that population member to participate in the game. We can then get the demand function $\phi(\lambda_1)$ by integrating the mass up to the equilibrium reward λ_1, i.e., over a subset $\Theta_1(\lambda_1) = \{\theta_1 \in \Theta \mid \theta_1 \leq \lambda_1\}$. Indeed,

$$\phi(\lambda_1) = \int_{-\infty}^{\lambda_1} M(\theta_1)\ d\theta_1. \tag{18}$$

Note that we could alternatively write (18) as

$$\phi(\lambda_1) = -\int_0^{\infty} M(\lambda_1 - s)\ ds \tag{19}$$

using a change of variables. The one dimensional type space and region of integration are illustrated in Fig. 1.

IV. Multi-Parameter Type-Based Demand

We now consider how to extend this demand function to a multi-dimensional parameter space where the paramters represent the population's *preferences* or *type*.

A. Multi-dimensional Type Space

We divide the set of actions in the potential game $\mathcal{A}$ into different subsets, $\mathcal{A}_i$, for $i \in \mathcal{I}$ for some index set $\mathcal{I}$. Each subset of actions may overlap or not.

The population is now represented by measure $M(\theta)$ over a multi-dimensional continuous type space $\Theta = R^{|\mathcal{I}|}$. A type vector $\theta \in R^{|\mathcal{I}|}$ represents a population member's arbitrary preference for actions in each subset. In a routing game, for example, these subsets might refer to different types of transportation options such as biking, riding the bus, or driving. Different individuals might have arbitrary preferences for one type of transportation over another. When a population member chooses $a \in \mathcal{A}_i$, they receive a reward of $r_a(z) - \theta_i$. Each population member's goal is to pick the subset and action therein in order to maximize their reward for their particular type vector θ. In addition, θ_i represents the minimum reward required in order to choose an action in subset $\mathcal{A}_i$. If $r_a(z) \leq \theta_i$ for all $i \in \mathcal{I}, a \in \mathcal{A}_i$, then the population member *opts out* and does not participate in the game.

Given a particular reward vector $\lambda \in \Theta$, we can divide up the type space into different regions $\{\Theta_i(\lambda)\}_{i \in \mathcal{I}}$ where

$$\Theta_i(\lambda) = \left\{ \theta \in \Theta \;\middle|\; \lambda_i \geq \theta_i,\ \lambda_i - \theta_i \geq \lambda_j - \theta_j\ \forall j \in \mathcal{I} \right\}. \tag{20}$$

which we also find useful to express as

$$\Theta_i(\lambda) = \left\{ \theta \in \Theta \middle| \ \theta_i = \lambda_i - s, \theta_j \geq \lambda_j - s, s \geq 0\ \forall j \in \mathcal{I} \right\}. \tag{21}$$

Note that $\Theta_i(\lambda)$ is defined to be the subset of the type space that will chooose actions in subset $\mathcal{A}_i$ if

$$\lambda_i = \max_{a \in \mathcal{A}_i} \left\{ r_a(z) \right\} \tag{22}$$

We can also define the portion of the type space that would not choose any action in any subset, ie. $\theta > \lambda$:

$$\Theta_+ = \left\{ \theta \in \Theta \;\middle|\; \theta_i > \lambda_i\ \forall i \in \mathcal{I} \right\}. \tag{23}$$

These type space regions are illustrated in Fig. 1. Note that $\Theta = \cup_i \Theta_i \cup \Theta_+$. We now define a map $m : \lambda \mapsto \mathbb{R}_+^{|\mathcal{I}|}$ given by

$$m_i(\lambda) = \int_{\theta \in \Theta_i(\lambda)} M(\theta) d\theta. \tag{24}$$

Given $\Theta_i(\lambda)$, characterized in (21), we can explicitly compute

$$m_i(\lambda) = -\int_0^{\infty} ds \int_{\lambda_{\mathcal{I}-\{i\}} - s}^{\infty} d\theta_{\mathcal{I}-\{i\}} M\Big(\theta_i = \lambda_i - s\Big) \tag{25}$$

where we have used the short hand

$$M(\theta_i = s) = M(\theta_1, \ldots, \theta_{i-1}, s, \theta_{i+1}, \ldots, \theta_{|\mathcal{I}|}). \tag{26}$$

and

$$\int_{\lambda_{\{ijk\}} - s}^{\infty} d\theta_{\{ijk\}} = \int_{\lambda_i - s}^{\infty} \int_{\lambda_j - s}^{\infty} \int_{\lambda_k - s}^{\infty} d\theta_i d\theta_j d\theta_k \tag{27}$$

and similarly for subsets of $\mathcal{I}$ other than $\{ijk\}$ such as $\mathcal{I} - \{i\}$. Note that the inner integrals in (25) are over θ_j for $j \in \mathcal{I} - \{i\}$ and the outer integral is over s.

We can think of $m(\lambda)$ as a multi-dimensional demand function given the reward vector λ. The type space and regions of integration for one, two, and three dimensions are illustrated in Fig. 1.

The mass $m_i(\lambda)$ that chooses subset $\mathcal{A}_i$ then divides itself up over the actions in $\mathcal{A}_i$. The elements of the mass vector $z^i \in \mathbb{R}_+^{|\mathcal{A}_i|}$ give how much mass chooses each action. In order to ensure that the mass vectors are consistent with the underlying measure, we give the following definition of feasibility.

Definition 4. *We say that a set of mass vectors $\{z^i\}_{i \in \mathcal{I}}$ is* feasible *for reward vector λ if $\mathbf{1}^T z^i = m_i(\lambda)$*

If an action a is in multiple subsets than the total mass that chooses that action is given by $z_a = \sum_i z_a^i$. Given rewards of the form (4), we will sometimes use the notation $\mathbf{R}^i = \mathbf{R}[:, \mathcal{A}_i]$ to refer to the columns of $\mathbf{R}$ associated with actions in $\mathcal{A}_i$. Note that given this notation, the reward for an action $a \in \mathcal{A}_i$ can be computed as

$$r_a(z)^T = \left[\ell\Big(\sum_i \mathbf{R}^i z^i\Big)^T \mathbf{R}^i\right]_a. \tag{28}$$

Before we continue, we briefly consider the specific case where $m_i(\lambda) = 0$, ie. the region of integration $\Theta_i(\lambda)$ has no mass in it. We prove the following lemma.

Lemma 1. *Consider two reward vectors λ and λ' where $m_i(\lambda) = 0$ and $\lambda_i' \leq \lambda_i$ and $\lambda_j' = \lambda_j$ for $j \neq i$. It follows that $m_j(\lambda) = m_j(\lambda')$ for all $j \in \mathcal{I}$.*

Intuitively, if the integration region $\Theta_j(\lambda)$ is empty, reducing λ_i does not change any of the masses.

Proof. We consider how each region of integration changes as λ changes to λ'. Since $\lambda_i \geq \lambda_i'$, it follows that $\Theta_i(\lambda') \subseteq \Theta_i(\lambda)$ and thus $m_i(\lambda) = m_i(\lambda') = 0$. We now define the set

$$\begin{aligned}\Delta\Theta_j(\lambda, \lambda') = \Big\{\theta \in \Theta \;\Big|\; & \lambda_j - \theta_j \geq 0, \\ & \lambda_i - \theta_i > \lambda_j - \theta_j \geq \lambda_i' - \theta_i, \\ & \lambda_j - \theta_j \geq \lambda_k - \theta_k \quad \forall k \neq i\Big\}\end{aligned} \tag{29}$$

Note that $\Theta_j(\lambda') = \Theta_j(\lambda) \cup \Delta\Theta_j(\lambda, \lambda')$ and $\Theta_j(\lambda) \cap \Delta\Theta_j(\lambda, \lambda') = \emptyset$. Note also that $\Delta\Theta_j(\lambda, \lambda') \subseteq \Theta_i(\lambda)$. It follows then that $\int_{\Delta\Theta_j(\lambda,\lambda')} M(\theta) d\theta = 0$. and that

$$\begin{aligned} m_j(\lambda') &= \int_{\Theta_j(\lambda')} M(\theta) d\theta = \int_{\Theta_j(\lambda) + \Delta\Theta_j} M(\theta) d\theta \\ &= \int_{\Theta_j(\lambda)} M(\theta) d\theta + \int_{\Delta\Theta_j} M(\theta) d\theta = m_j(\lambda) \end{aligned} \tag{30}$$

□

B. Type-Based Equilibrium Concept

We now extend the notion of a Wardrop equilibrium to this type-based model. For any set of actions $\mathcal{A}_i$, we expect the classic notion of a Wardrop equilibrium to hold. We

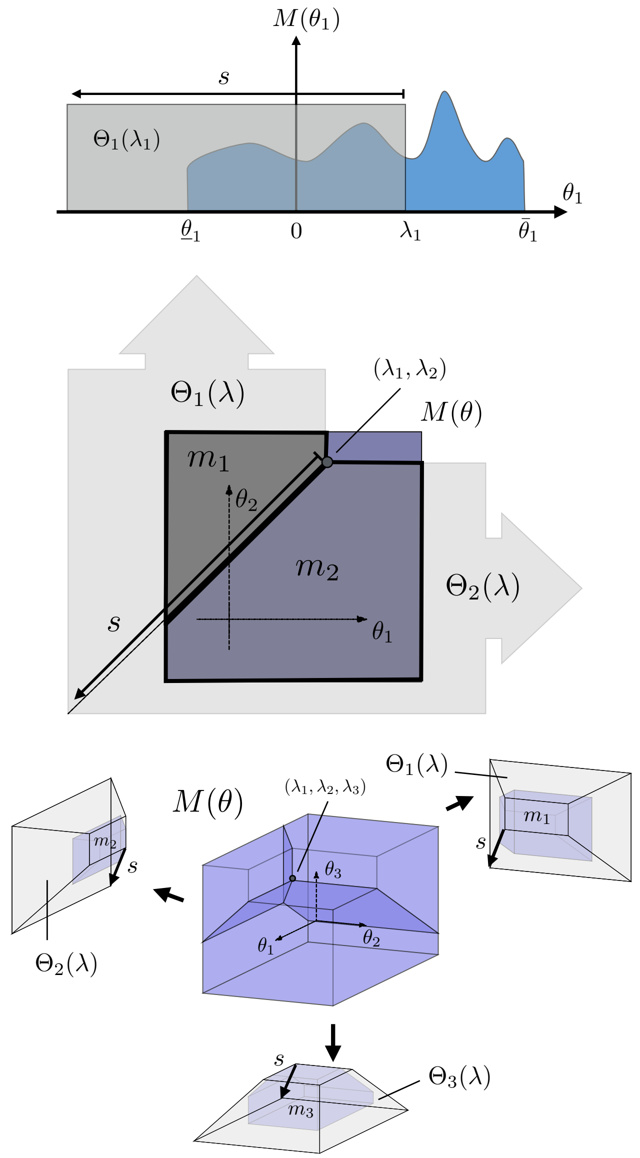


Fig. 1. Type spaces and regions of integration for one, two, and three dimensions.

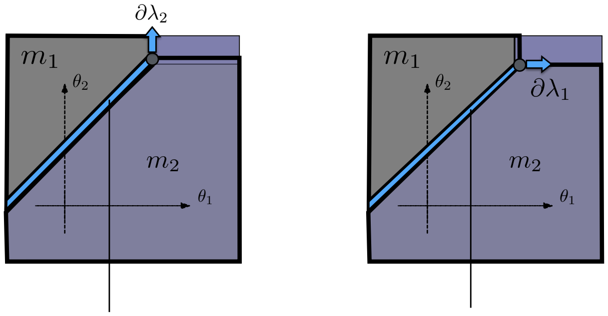
also expect the mass that chooses any set of actions to be consistent with the portion of the population that values that set over others. To this end, we define a continuous type Wardrop equilibrium.

Definition 5. *We say that a set of mass vectors $\{z^i\}_{i \in \mathcal{I}}$ is a* continuous type Wardrop equilibrium *if it is feasible for $\lambda_i = \max_{a \in \mathcal{A}_i} \{r_a(z)\}$ and if, whenever $z_a^i > 0$,*

$$r_a(z) \geq \theta_i, \quad r_a(z) - \theta_i \geq r_{a'}(z) - \theta_{i'} \tag{31a}$$

for all $\theta \in \Theta_i(\lambda)$, $i' \in \mathcal{I}$, and $a' \in \mathcal{A}_{i'}$

The above is the standard definition of the Wardrop equilibrium with the type parameter subtracted from the reward and the added condition that each mass vector z^i is consistent with the mass in the region of integration $\Theta_i(\lambda)$.



$$\frac{\partial m_1}{\partial \lambda_2} = \int_0^\infty ds\ M(\theta_1 = \lambda_1 - s, \theta_2 = \lambda_2 - s) = \frac{\partial m_2}{\partial \lambda_1}$$

Fig. 2. Illustration of conservative vector field condition in two dimensions.

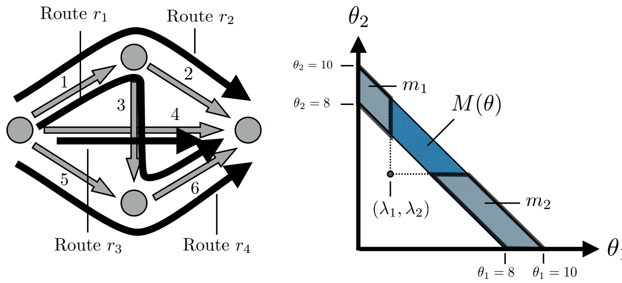


Fig. 3. Illustration of routing network and type distribution.

C. Potential Function

In order to compute this equilibrium we first analyze the map $m(\lambda)$.

Proposition 1. $\partial_{\lambda_j} m_i(\lambda) = \partial_{\lambda_i} m_j(\lambda)$.

Indeed, we can compute

$$\frac{\partial m_i}{\partial \lambda_j} = \int_0^\infty ds \int_{\lambda_{\mathcal{I}-\{ij\}}-s}^\infty d\theta_{\mathcal{I}-\{ij\}} \\ M\Big(\theta_i = \lambda_i - s, \theta_j = \lambda_j - s,\Big) \tag{32}$$

In two dimensions, (32) becomes

$$\frac{\partial m_1}{\partial \lambda_2} = \int_0^\infty ds\ M\Big(\theta_1 = \lambda_1 - s, \theta_2 = \lambda_2 - s,\Big) \tag{33}$$

illustrated in Figure 2. From this we get that $m(\lambda)$ is a conservative vector field and we have hope of finding a potential function $\tilde{g}(\lambda)$ such that $\partial_\lambda \tilde{g}(\lambda) = m(\lambda)$.

The potential function $\tilde{g}(\lambda)$ is given by

$$\tilde{g}(\lambda) = \int_0^\infty ds \int_{\lambda_{\mathcal{I}}-s}^\infty d\theta_{\mathcal{I}}\ M(\theta) \tag{34}$$

Proposition 2. $\partial_\lambda \tilde{g}(\lambda) = m(\lambda)^T$

Indeed, a straightforward calculation gives

$$\frac{\partial \tilde{g}}{\partial \lambda_i} = \int_0^\infty ds \int_{\lambda_{\mathcal{I}-\{i\}}-s}^\infty d\theta_{\mathcal{I}-\{i\}}\ M(\theta_i = \lambda_i - s) \tag{35}$$

D. Equilibrium Computation

We now frame the equilibrium computation as an optimization problem related to the dual problem (P-16). We have the following theorem.

Theorem 2. *Let $c \in \mathbb{R}^{|\mathcal{E}|}$ and $\lambda \in \mathbb{R}^{|\mathcal{I}|}$ optimize*

$$\min_{c,\lambda} \quad \{\tilde{g}(\lambda) - \tilde{h}(c) \ \ s.t.\ \ \lambda_i \mathbf{1}^T \geq c^T \mathbf{R}^i,\ \forall\ i \in \mathcal{I}\} \tag{36}$$

with

$$\tilde{g}(\lambda) = \int_0^\infty ds \int_{\lambda_{\mathcal{I}}-s}^\infty d\theta_{\mathcal{I}}\ M(\theta)$$

Then the optimal dual variables associated with the constraints, $\{z^i \in \mathbb{R}_+^{|\mathcal{A}_i|}\}_{i\in\mathcal{I}}$ are a set of feasible mass vectors for the reward vector λ', where $\lambda'_i = \max_{a\in\mathcal{A}_i} r_a(z)$, and form a continuous type Wardrop equilibrium.

Proof. We consider the KKT conditions. Stationarity with respect to λ_i along with the fact that $z^i \geq 0$ gives that

$$\partial_{\lambda_i} \tilde{g}(\lambda) = m_i(\lambda) = \mathbf{1}^T z^i, \qquad z^i \geq 0, \quad \forall\ i \in \mathcal{I} \tag{37}$$

Stationarity with respect to c gives

$$\partial_c \tilde{h}(c)^T = \ell^{-1}(c) = \sum_i \mathbf{R}^i z^i \ \Rightarrow\ c = \ell\Big(\sum_i \mathbf{R}^i z^i\Big) \tag{38}$$

From the constraints in (36) we get that

$$\lambda_i \mathbf{1}^T \geq \ell\Big(\sum_i \mathbf{R}^i z^i\Big)^T \mathbf{R}^i \tag{39}$$

or more succinctly $\lambda_i \geq r_a(z)$ for all $a \in \mathcal{A}_i$ Note that $\lambda_i \geq \lambda'_i$ and by complementary slackness $\lambda_i = \lambda'_i$ whenever some $z^i_a > 0$. Thus, we have that $\lambda_i > \lambda'_i$ implies that $m_i(\lambda) = 0$ and by Lemma 1, we have that $m(\lambda) = m(\lambda')$. It follows that $\{z^i\}_{i\in\mathcal{I}}$ are feasible mass vectors for λ'.

Now suppose that $z^i_a > 0$. It follows that

$$r_a(z) - \lambda'_i = 0 \geq r_{a'}(z) - \lambda'_{i'}, \quad \forall\ a' \in \mathcal{A}_{i'}, i' \in \mathcal{I} \tag{40}$$

For $\theta \in \Theta_i(\lambda')$, we have that $\lambda'_i - \theta_i \geq 0$ and $\lambda'_i - \theta_i \geq \lambda'_{i'} - \theta_{i'}$. Adding these conditions with (40) give

$$r_a(z) - \theta_i \geq 0 \tag{41a}$$

$$r_a(z) - \theta_i \geq r_{a'}(z) - \theta_{i'} \tag{41b}$$

for all $a' \in \mathcal{A}_{i'}$ and $i' \in \mathcal{I}$. □

V. Numerical Example

We consider a simple two parameter example where $M(\theta)$ is a uniform distribution over the region illustrated in Figure 3. We assume that the action sets are routes in a simple routing game with network, origin and destination, and routes illustrated in Figure 3.

We consider two sets of routes $\mathcal{A}_1 = \{r_1, r_3\}$ and $\mathcal{A}_2 = \{r_2, r_4\}$. Routing matrices for each subset $\mathbf{R}^1$ and $\mathbf{R}^2$ are given by

$$\mathbf{R}^1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

$$\mathbf{R}^2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T$$

The routing portion of the potential function is given by $f()$ where $f(\cdot)$ is the standard routing potential

$$f\big(\mathbf{R}^1 z^1 + \mathbf{R}^2 z^2\big) = -\sum_e \int_0^{\left[\mathbf{R}^1 z^1 + \mathbf{R}^2 z^2\right]_e} \ell_e(u)\ du$$

The latency functions are given by $\ell(x) = Ax + b$ where $A = \mathrm{diag}\big([1, 1, 1, 1, 1, 1]\big)$ and $b = [1, 1, 1, 1, 1, 1]^T$ and

$$\ell^{-1}(c) = A^{-1}c - A^{-1}b \tag{42}$$

For simplicity, we construct the example such that $\lambda_1 + \lambda_2 \le 8$ and thus we can compute m_1 and m_2 as

$$m_1(\lambda) = 2\lambda_1, \qquad m_2(\lambda) = 2\lambda_2 \tag{43}$$

as shown in Figure 3. The demand portion of the potential function is then given by

$$\tilde{g}(\lambda) = \lambda_1^2 + \lambda_2^2 = \lambda^T \lambda; \tag{44}$$

We write out Problem (36) as

$$\min_{\lambda, c} \quad \lambda^T \lambda + \frac{1}{2}(c - b)^T A^{-1}(c - b) \tag{45}$$

$$\text{s.t.} \quad \lambda_1 \mathbf{1}^T \ge c^T \mathbf{R}^1,\ \lambda_2 \mathbf{1}^T \ge c^T \mathbf{R}^2 \tag{46}$$

The resulting equilibrium distribution mass is given by

$$z^1 = \begin{bmatrix} z_{r_1} & z_{r_3} \end{bmatrix}^T = \begin{bmatrix} 2.50 & 3.16 \end{bmatrix}^T$$
$$z^2 = \begin{bmatrix} z_{r_2} & z_{r_4} \end{bmatrix}^T = \begin{bmatrix} 0.83 & 2.83 \end{bmatrix}^T$$

with $\lambda_1 = 2.83$ and $\lambda_2 = 1.83$. Computing the rewards and total masses gives

$$\begin{aligned} \begin{bmatrix} r_{r_1}(z) & r_{r_3}(z) \end{bmatrix}^T &= \Big(A\big(\mathbf{R}^1 z^1 + \mathbf{R}^2 z^2\big) + b\Big)^T \mathbf{R}^1 \\ &= \begin{bmatrix} 2.83 & 2.83 \end{bmatrix} \\ \begin{bmatrix} r_{r_1}(z) & r_{r_3}(z) \end{bmatrix}^T &= \Big(A\big(\mathbf{R}^1 z^1 + \mathbf{R}^2 z^2\big) + b\Big)^T \mathbf{R}^2 \\ &= \begin{bmatrix} 1.83 & 1.83 \end{bmatrix} \end{aligned}$$

and

$$\mathbf{1}^T z^1 = z_{r_1} + z_{r_3} = 5.67 = 2\lambda_1 = m_1(\lambda) \tag{47}$$

$$\mathbf{1}^T z^2 = z_{r_2} + z_{r_4} = 3.67 = 2\lambda_2 = m_2(\lambda) \tag{48}$$

as expected.

VI. Discussion and Conclusion

In this paper, we develop a potential game model where the population is no longer anonymous but is represented by a distribution over a continuous multi-dimensional type space that encodes a population members arbitrary preference for different subsets of actions. We note that one of the advantages of this model is that it is able to encode a truly wide range of population variations since no ordering on the different options $i \in \mathcal{I}$ is required for all population members. We believe this model can have wide application in population potential games specifically in multi-modal transportation problems. Future work includes analyzing cases where the type parameter does not enter linearly into the total reward as well as potentially an inverse utility learning problem where the mass distribution of the potential game is observed under a variety of conditions and these observations are used to estimate the underlying population type distribution.

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