Dual Programs: LP

Primal Program

$$\max_{x} \quad r^\top x$$

$$\text{s.t.} \quad \boxed{Ax = b \quad \lambda} \qquad \boxed{Cx \geq d \quad \mu}$$

$$\mathcal{L}(x, \lambda, \mu) = r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\max_{x} \min_{\lambda, \mu \geq 0} \quad r^\top x + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

$$\min_{\lambda, \mu \geq 0} \max_{x} \quad \underbrace{(r^\top + \lambda^\top A + \mu^\top C)}_{\substack{\text{must be 0 for inner} \\ \text{problem to be bounded}}} x \quad -\lambda^\top b - \mu^\top d$$

Dual Program

$$\min_{\lambda, \mu} \quad -\lambda^\top b - \mu^\top d$$

$$\text{s.t.} \quad r^\top + \lambda^\top A + \mu^\top C = 0 \,, \quad \mu \geq 0$$

Dual Programs: QP

$$\max_{x} \quad \tfrac{1}{2}x^\top Q x \;+\; r^\top x$$

Primal Program

$$\text{s.t.} \quad \boxed{Ax = b \;\; \lambda} \quad \boxed{Cx \geq d \;\; \mu}$$

Note: $Q = Q^\top \prec 0$

$$\mathcal{L}(x,\lambda,\mu) = \tfrac{1}{2}x^\top Q x \;+\; r^\top x \;+\; \lambda^\top(Ax-b) + \mu^\top(Cx-d)$$

$$\max_{x} \; \min_{\lambda,\mu\geq 0} \quad \tfrac{1}{2}x^\top Q x \;+\; r^\top x \;+\; \lambda^\top(Ax-b) + \mu^\top(Cx-d)$$

$$\min_{\lambda,\mu\geq 0} \; \max_{x} \quad \underbrace{\tfrac{1}{2}x^\top Q x \;+\; (r^\top + \lambda^\top A + \mu^\top C)x}_{\text{maximize explicitly...}} \; -\lambda^\top b - \mu^\top d$$

Define $\xi^\top = r^\top + \lambda^\top A + \mu^\top C \qquad \frac{\partial}{\partial x}\left(\tfrac{1}{2}x^\top Q x + \xi^\top x\right) = 0 \quad \Rightarrow \quad x = -Q^{-1}\xi$

Plug in x...

$$\min_{\xi,\lambda,\mu} \quad -\tfrac{1}{2}\xi^\top Q^{-1}\xi \;\; -\lambda^\top b - \mu^\top d$$

Dual Program

$$\text{s.t.} \quad \xi^\top = r^\top + \lambda^\top A + \mu^\top C\,, \quad \mu \geq 0$$

Note: $-Q^{-1} = -Q^{-\top} \succ 0$