Matrix Shape, Polar Decomposition, Singular Value Decomposition

Linear Algebra

Winter 2022 - Dan Calderone

Matrix Shape

$$A \in \mathbb{R}^{m\times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Inner products between

... columns $\quad A^T A \in \mathbb{R}^{n\times n}$

$$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix} = \begin{bmatrix} |A_1||A_1| & \cdots & |A_1||A_n|\cos\theta_{1n} \\ \vdots & & \vdots \\ |A_n||A_1|\cos\theta_{1n} & \cdots & |A_n||A_n| \end{bmatrix}$$

... rows $\quad AA^T \in \mathbb{R}^{m\times m}$

$$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix} = \begin{bmatrix} |a_1||a_1| & \cdots & |a_1||a_n|\cos\theta_{1n} \\ \vdots & & \vdots \\ |a_n||a_1|\cos\theta_{1n} & \cdots & |a_n||a_n| \end{bmatrix}$$

Properties

Positive semi-definite $\quad A^T A \succeq 0 \qquad AA^T \succeq 0$ $\qquad$ **Proof:** $\quad x^T A^T A x = ||Ax||_2^2 \geq 0$

Rank $\quad \mathrm{rk}(A) = \mathrm{rk}(A^T) = \mathrm{rk}(A^T A) = \mathrm{rk}(AA^T) = k$

Proof: $\quad Ax = 0 \quad \Rightarrow \quad A^T A x = 0$

$$A^T A x = 0 \quad \Rightarrow \quad 0 = x A^T A x = ||Ax||_2^2 \quad \Rightarrow \quad Ax = 0$$

$\mathcal{N}(A) = \mathcal{N}(A^T A)$ $\qquad$ **rank-nullity theorem**

Matrix Shape

$$A \in \mathbb{R}^{m\times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Inner products between

… columns $\quad A^TA \in \mathbb{R}^{n\times n}$

$$A^TA = \begin{bmatrix} A_1^TA_1 & \cdots & A_1^TA_n \\ \vdots & & \vdots \\ A_n^TA_1 & \cdots & A_n^TA_n \end{bmatrix} = \begin{bmatrix} |A_1||A_1| & \cdots & |A_1||A_n|\cos\theta_{1n} \\ \vdots & & \vdots \\ |A_n||A_1|\cos\theta_{1n} & \cdots & |A_n||A_n| \end{bmatrix}$$

… rows $\quad AA^T \in \mathbb{R}^{m\times m}$

$$AA^T = \begin{bmatrix} a_1^Ta_1 & \cdots & a_1^Ta_m \\ \vdots & & \vdots \\ a_m^Ta_1 & \cdots & a_m^Ta_m \end{bmatrix} = \begin{bmatrix} |a_1||a_1| & \cdots & |a_1||a_n|\cos\theta_{1n} \\ \vdots & & \vdots \\ |a_n||a_1|\cos\theta_{1n} & \cdots & |a_n||a_n| \end{bmatrix}$$

Diagonalization: $\quad \mathrm{rk}(A) = k$

$$A^TA = V\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}V^T = \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{k} & \overbrace{\quad}^{n-k} \end{array} \\ \begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} \end{array} \begin{array}{c} k\{ \\ n-k\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{k} & \overbrace{\quad}^{n-k} \end{array} \\ \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} \begin{array}{c} \}\ k \\ \}\ n-k \end{array}$$

$$AA^T = U\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}U^T = \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{k} & \overbrace{\quad}^{m-k} \end{array} \\ \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} \end{array} \begin{array}{c} k\{ \\ m-k\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{k} & \overbrace{\quad}^{m-k} \end{array} \\ \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} - & U'^T & - \\ - & U''^T & - \end{bmatrix} \begin{array}{c} \}\ k \\ \}\ m-k \end{array}$$

$$\Sigma \in \mathbb{R}^{k\times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

Note: need to show same Σ works for A^TA & AA^T… (later)

$V \in \mathbb{R}^{n\times n}$

$$V^TV = \begin{bmatrix} V'^T \\ V''^T \end{bmatrix}\begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} = \begin{bmatrix} V'^TV' & V'^TV'' \\ V''^TV' & V''^TV'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$U \in \mathbb{R}^{m\times m}$

$$U^TU = \begin{bmatrix} U'^T \\ U''^T \end{bmatrix}\begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} = \begin{bmatrix} U'^TU' & U'^TU'' \\ U''^TU' & U''^TU'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Matrix Shape - positive semi-definite shape

$$A \in \mathbb{R}^{m\times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Matrix shapes	relative shape of...	positive semi-definite
... columns	$\boxed{(A^TA)^{\frac{1}{2}}} \in \mathbb{R}^{n\times n}$	$(A^TA)^{\frac{1}{2}} \succeq 0$
... rows	$\boxed{(AA^T)^{\frac{1}{2}}} \in \mathbb{R}^{m\times m}$	$(AA^T)^{\frac{1}{2}} \succeq 0$

Complex Number Analogy:

$$z = \boxed{|z|} e^{i\phi} = a + bi$$

$$|z| = \boxed{(z^*z)^{\frac{1}{2}}} = \sqrt{a^2+b^2} \geq 0$$

complex number
"shape", ie. Magnitude

Diagonalization: $\mathrm{rk}(A) = k$

$$(A^TA)^{\frac{1}{2}} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = \begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} \begin{array}{c} k\{ \\ n-k\{ \end{array} \overset{\begin{array}{cc} k & n-k \end{array}}{\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} \begin{array}{l} \}\ k \\ \}\ n-k \end{array}$$

$$(AA^T)^{\frac{1}{2}} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} \begin{array}{c} k\{ \\ m-k\{ \end{array} \overset{\begin{array}{cc} k & m-k \end{array}}{\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} - & U'^T & - \\ - & U''^T & - \end{bmatrix} \begin{array}{l} \}\ k \\ \}\ m-k \end{array}$$

$$\Sigma \in \mathbb{R}^{k\times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

Note: need to show same Σ works for A^TA & AA^T... (later)

$V \in \mathbb{R}^{n\times n}$

$$V^TV = \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} \begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} = \begin{bmatrix} V'^TV' & V'^TV'' \\ V''^TV' & V''^TV'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$U \in \mathbb{R}^{m\times m}$

$$U^TU = \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} = \begin{bmatrix} U'^TU' & U'^TU'' \\ U''^TU' & U''^TU'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Matrix Shape - orientation

$$A \in \mathbb{R}^{m\times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Matrix shapes | **orientation of...** | **orthonormal**

... columns $\boxed{A(A^TA)^{-\frac{1}{2}}} \in \mathbb{R}^{m\times n}$ $\qquad \left((A^TA)^{-\frac{1}{2}}A^T\right)\left(A(A^TA)^{-\frac{1}{2}}\right) = I$ **...if full col rank**

... rows $\boxed{(AA^T)^{-\frac{1}{2}}A} \in \mathbb{R}^{m\times n}$ $\qquad \left((AA^T)^{-\frac{1}{2}}A\right)\left(A^T(AA^T)^{-\frac{1}{2}}\right) = I$ **...if full row rank**

Complex Number Analogy:

$$z = |z|\boxed{e^{i\phi}} = a + bi$$

$$\boxed{e^{i\phi},} \qquad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

complex number orientation, ie. phase

Shape matrix:

... if full column rank

$$(A^TA)^{\frac{1}{2}} = V\Sigma V^T = V'\Sigma V'^T$$

... if full row rank

$$(AA^T)^{\frac{1}{2}} = U\Sigma U^T = U'\Sigma U'^T$$

Orientation matrix:

Note: proof using SVD...

$$A(A^TA)^{-\frac{1}{2}} = UV^T$$

$$(AA^T)^{-\frac{1}{2}}A = UV^T$$

$$\Sigma \in \mathbb{R}^{k\times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

Note: need to show same Σ works for A^TA & AA^T... (later)

$$V \in \mathbb{R}^{n\times n}$$

$$V^TV = \begin{bmatrix} V'^T \\ V''^T \end{bmatrix}\begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} = \begin{bmatrix} V'^TV' & V'^TV'' \\ V''^TV' & V''^TV'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$$U \in \mathbb{R}^{m\times m}$$

$$U^TU = \begin{bmatrix} U'^T \\ U''^T \end{bmatrix}\begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} = \begin{bmatrix} U'^TU' & U'^TU'' \\ U''^TU' & U''^TU'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Polar Decomposition

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Polar Decomposition

... columns

$$A = \boxed{\underbrace{A(A^TA)^{-\frac{1}{2}}}_{\text{orthonormal}}}\boxed{\underbrace{(A^TA)^{\frac{1}{2}}}_{\text{PSD}}}$$

...if full col rank

... rows

$$A = \boxed{\underbrace{(AA^T)^{\frac{1}{2}}}_{\text{PSD}}}\boxed{\underbrace{(AA^T)^{-\frac{1}{2}}A}_{\text{orthonormal}}}$$

...if full row rank

Shape matrix:

... if full column rank

$$(A^TA)^{\frac{1}{2}} = V\Sigma V^T = V'\Sigma V'^T$$

... if full row rank

$$(AA^T)^{\frac{1}{2}} = U\Sigma U^T = U'\Sigma U'^T$$

Orientation matrix:

Note: proof using SVD...

$$A(A^TA)^{-\frac{1}{2}} = UV^T$$

$$(AA^T)^{-\frac{1}{2}}A = UV^T$$

Complex Number Analogy:

$$z = \boxed{|z|}\boxed{e^{i\phi}} = a + bi$$

complex number "shape", ie. Magnitude

$$|z| = \boxed{(z^*z)^{\frac{1}{2}}} = \sqrt{a^2+b^2}$$

complex number orientation, ie. phase

$$\boxed{e^{i\phi},} \qquad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\Sigma \in \mathbb{R}^{k \times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

Note: need to show same Σ works for A^TA & AA^T... (later)

$$V \in \mathbb{R}^{n \times n}$$

$$V^TV = \begin{bmatrix} V'^T \\ V''^T \end{bmatrix}\begin{bmatrix} | & | \\ V' & V'' \\ | & | \end{bmatrix} = \begin{bmatrix} V'^TV' & V'^TV'' \\ V''^TV' & V''^TV'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$$U \in \mathbb{R}^{m \times m}$$

$$U^TU = \begin{bmatrix} U'^T \\ U''^T \end{bmatrix}\begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} = \begin{bmatrix} U'^TU' & U'^TU'' \\ U''^TU' & U''^TU'' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Singular Value Decomposition (SVD)

$$A \in \mathbb{R}^{m\times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Singular Value Decomposition

… take $U' = AV'\Sigma^{-1}$ …valid eigenvectors with eigenvalues Σ^2 $\quad AA^TU' = AA^TAV'\Sigma^{-1} = AV'\Sigma^{-1}\Sigma^2$

…orthonormal $\quad \Sigma^{-1}{V'}^T A^T A V' \Sigma^{-1} = \Sigma^{-1}{V'}^T V' \Sigma^2 \Sigma^{-1} = I$

$$\underbrace{\begin{bmatrix} U' & U'' \end{bmatrix}}_{U} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U'\Sigma & 0 \end{bmatrix} = \begin{bmatrix} AV' & 0 \end{bmatrix} = \begin{bmatrix} AV' & AV'' \end{bmatrix} = A\underbrace{\begin{bmatrix} V' & V'' \end{bmatrix}}_{V}$$

Diagonalization: $\mathrm{rk}(A) = k$

$$\Sigma \in \mathbb{R}^{k\times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

$$A^TA = V\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T = \begin{bmatrix} \overbrace{|}^{k} & \overbrace{|}^{n-k} \\ V' & V'' \\ | & | \end{bmatrix} \begin{array}{r} k\{ \\ n-k\{ \end{array} \begin{bmatrix} \overbrace{\Sigma^2}^{k} & \overbrace{0}^{n-k} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & {V'}^T & - \\ - & {V''}^T & - \end{bmatrix} \begin{array}{l} \}\, k \\ \}\, n-k \end{array}$$

$$AA^T = U\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} \overbrace{|}^{k} & \overbrace{|}^{m-k} \\ U' & U'' \\ | & | \end{bmatrix} \begin{array}{r} k\{ \\ m-k\{ \end{array} \begin{bmatrix} \overbrace{\Sigma^2}^{k} & \overbrace{0}^{m-k} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & {U'}^T & - \\ - & {U''}^T & - \end{bmatrix} \begin{array}{l} \}\, k \\ \}\, m-k \end{array}$$

…eigenvalues of $(A^TA)^{\frac{1}{2}}$ $\quad (AA^T)^{\frac{1}{2}}$

$V \in \mathbb{R}^{n\times n}$ …eigenvectors of $(A^TA)^{\frac{1}{2}}$

$U \in \mathbb{R}^{m\times m}$ …eigenvectors of $(AA^T)^{\frac{1}{2}}$

Singular Value Decomposition (SVD)

$A \in \mathbb{R}^{m\times n}$ $\qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$

Singular Value Decomposition

… take $\quad U' = AV'\Sigma^{-1}$ $\quad$ …valid eigenvectors with eigenvalues Σ^2 $\quad AA^TU' = AA^TAV'\Sigma^{-1} = AV'\Sigma^{-1}\Sigma^2$

…orthonormal $\quad \Sigma^{-1}V'^TA^TAV'\Sigma^{-1} = \Sigma^{-1}V'^TV'\Sigma^2\Sigma^{-1} = I$

$$\underbrace{\begin{bmatrix} U' & U'' \end{bmatrix}}_{U} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U'\Sigma & 0 \end{bmatrix} = \begin{bmatrix} AV' & 0 \end{bmatrix} = \begin{bmatrix} AV' & AV'' \end{bmatrix} = A\underbrace{\begin{bmatrix} V' & V'' \end{bmatrix}}_{V}$$

$$A = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^T = \begin{array}{cc} \overbrace{\quad}^{k} & \overbrace{\quad}^{m-k} \\ \multicolumn{2}{c}{\begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix}} \end{array} \begin{array}{r} k\{ \\ m-k\{ \end{array} \begin{array}{cc} \overbrace{\quad}^{k} & \overbrace{\quad}^{n-k} \\ \multicolumn{2}{c}{\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}} \end{array} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} \begin{array}{l} \}\, k \\ \}\, n-k \end{array}$$

$\Sigma \in \mathbb{R}^{k\times k}$ $\quad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix}$ $\quad \sigma_i > 0$

…eigenvalues of $(A^TA)^{\frac{1}{2}}$ $\quad (AA^T)^{\frac{1}{2}}$

U' orthonormal basis for $\mathcal{R}(A)$

U'' orthonormal basis for $\mathcal{N}(A^T)$

V' orthonormal basis for $\mathcal{R}(A^T)$

V'' orthonormal basis for $\mathcal{N}(A)$

$V \in \mathbb{R}^{n\times n}$ …eigenvectors of $(A^TA)^{\frac{1}{2}}$

$U \in \mathbb{R}^{m\times m}$ …eigenvectors of $(AA^T)^{\frac{1}{2}}$

Singular Value Decomposition (SVD)

$$A \in \mathbb{R}^{m\times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Singular Value Decomposition & Polar Decomposition

… full column rank

$$A = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T = \underbrace{A\left(A^TA\right)^{-\frac{1}{2}}}_{\text{orthonormal}} \underbrace{\left(A^TA\right)^{\frac{1}{2}}}_{\text{PD}} = \underbrace{(U'V^T)}_{\text{orthonormal}} \underbrace{(V\Sigma V^T)}_{\text{PD}}$$

… full row rank

$$A = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \underbrace{\left(AA^T\right)^{\frac{1}{2}}}_{\text{PD}} \underbrace{\left(AA^T\right)^{-\frac{1}{2}} A}_{\text{orthonormal}} = \underbrace{(U\Sigma U^T)}_{\text{PD}} \underbrace{(UV'^T)}_{\text{orthonormal}}$$

…general

$$A = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \underbrace{A\left(A^TA\right)^{\frac{\dagger}{2}}}_{\text{orthonormal}} \underbrace{\left(A^TA\right)^{\frac{1}{2}}}_{\text{PD}} = \underbrace{(U'V'^T)}_{\text{orthonormal}} \underbrace{(V'\Sigma V'^T)}_{\text{PD}}$$

$$= \underbrace{\left(AA^T\right)^{\frac{1}{2}}}_{\text{PD}} \underbrace{\left(AA^T\right)^{\frac{\dagger}{2}} A}_{\text{orthonormal}} = \underbrace{(U'\Sigma U'^T)}_{\text{PD}} \underbrace{(U'V'^T)}_{\text{orthonormal}}$$

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = \begin{bmatrix} | & | \\ \overbrace{U'}^{k} & \overbrace{U''}^{m-k} \\ | & | \end{bmatrix} \begin{array}{c} k\{ \\ m-k\{ \end{array} \begin{bmatrix} \overbrace{\Sigma}^{k} & \overbrace{0}^{n-k} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix} \begin{array}{l} \}\, k \\ \}\, n-k \end{array}$$

$$\Sigma \in \mathbb{R}^{k\times k} \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_k \end{bmatrix} \qquad \sigma_i > 0$$

…eigenvalues of $(A^TA)^{\frac{1}{2}}$ $(AA^T)^{\frac{1}{2}}$

$V \in \mathbb{R}^{n\times n}$ …eigenvectors of $(A^TA)^{\frac{1}{2}}$

$U \in \mathbb{R}^{m\times m}$ …eigenvectors of $(AA^T)^{\frac{1}{2}}$

SVD & Polar Decomposition

$$A \in \mathbb{R}^{m\times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Related Matrices *…use -1 to represent the Moore-Penrose pseudo inverse*

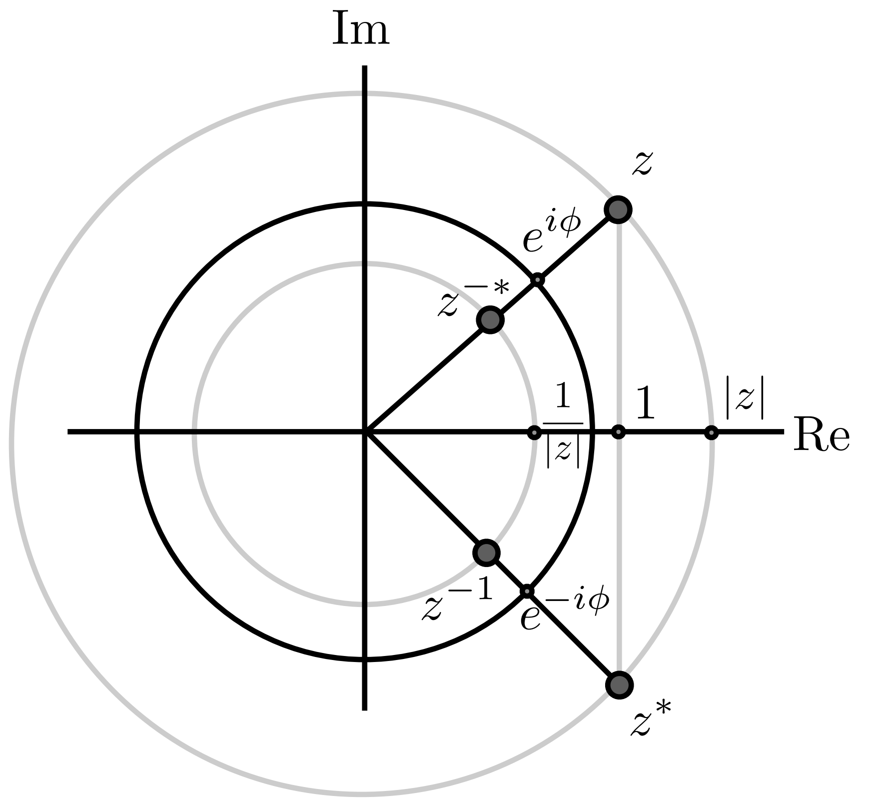
$$A = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \quad A\left(A^TA\right)^{-\frac{1}{2}}\left(A^TA\right)^{\frac{1}{2}} = \left(AA^T\right)^{\frac{1}{2}}\left(AA^T\right)^{-\frac{1}{2}}A \quad = \left(U'V'^T\right)\left(V'\Sigma V'^T\right) = \left(U'\Sigma U'^T\right)\left(U'V'^T\right)$$

$$A^T = \begin{bmatrix} V' & V'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} = \quad A^T\left(AA^T\right)^{-\frac{1}{2}}\left(AA^T\right)^{\frac{1}{2}} = \left(A^TA\right)^{\frac{1}{2}}\left(A^TA\right)^{-\frac{1}{2}}A^T \quad = \left(V'\Sigma V'^T\right)\left(V'U'^T\right) = \left(V'U'^T\right)\left(U'\Sigma U'^T\right)$$

$$A^{-1} = \begin{bmatrix} V' & V'' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} = \quad A^{-1}\left(AA^T\right)^{\frac{1}{2}}\left(AA^T\right)^{-\frac{1}{2}} = \left(A^TA\right)^{-\frac{1}{2}}\left(A^TA\right)^{\frac{1}{2}}A^{-1} \quad = \left(V'\Sigma^{-1}V'^T\right)\left(V'U'^T\right) = \left(V'U'^T\right)\left(U'\Sigma^{-1}U'^T\right)$$

$$A^{-T} = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \quad A^{-T}\left(A^TA\right)^{\frac{1}{2}}\left(A^TA\right)^{-\frac{1}{2}} = \left(AA^T\right)^{-\frac{1}{2}}\left(AA^T\right)^{\frac{1}{2}}A^{-T} \quad = \left(U'V'^T\right)\left(V'\Sigma^{-1}V'^T\right) = \left(U'\Sigma^{-1}U'^T\right)\left(U'V'^T\right)$$

SVD & Polar Decomposition



$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

Related Matrices *…use -1 to represent the Moore-Penrose pseudo inverse*

$$A = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \left(U'V'^T\right)\left(V'\Sigma V'^T\right) = \left(U'\Sigma U'^T\right)\left(U'V'^T\right)$$

$$A^T = \begin{bmatrix} V' & V'' \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} = \left(V'\Sigma V'^T\right)\left(V'U'^T\right) = \left(V'U'^T\right)\left(U'\Sigma U'^T\right)$$

$$A^{-1} = \begin{bmatrix} V' & V'' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U'^T \\ U''^T \end{bmatrix} = \left(V'\Sigma^{-1} V'^T\right)\left(V'U'^T\right) = \left(V'U'^T\right)\left(U'\Sigma^{-1} U'^T\right)$$

$$A^{-T} = \begin{bmatrix} U' & U'' \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'^T \\ V''^T \end{bmatrix} = \left(U'V'^T\right)\left(V'\Sigma^{-1} V'^T\right) = \left(U'\Sigma^{-1} U'^T\right)\left(U'V'^T\right)$$