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Lane Pricing via Decision–Theoretic Lane Changing Model of Driver Behavior

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Abstract— **We propose a new macroscopic model of traffic flow that incorporates multiple lanes, multiple populations of drivers, and local decision theoretic lane changing behavior. We show that the resulting nonlinear system of PDEs is weakly hyperbolic. We then use this model and an adjoint method to design tolling schemes to improve traffic flow on a two lane road example.**

I. INTRODUCTION

Traffic congestion is one of the major issues facing urban dwellers as cities and their surrounding suburbs continue to grow. According to study by Centre for Economics and Business Research (Cebr) created for INRIX[1], in 2013 traffic related expenditures were approximately \$124 billion and Cebr further estimates that Americans will waste \$2.8 trillion on transportation costs by 2030 if gridlock persists [1].

Reports suggest that congestion can be alleviated, or even completely eliminated, and travel time reduced by introducing appropriate incentives such as congestion pricing and similar mechanisms [2]–[5].

One typical mechanism for alleviating congestion and reducing travel time is the introduction of high–occupancy toll (HOT) lanes. The prevailing idea is that HOT lanes will give free passage to vehicles with multiple passengers—typically three or more—and will allow others to travel in these special lanes for a price during peak-hours in order to better utilize the lane's capacity [5]. As reported by the Urban Land Institute, as of 2012 there are 294 corridor miles of existing HOT and express lanes [6], and this number will likely increase as city planners seek to solve the congestion problem.

Largely the theory of designing incentives for roadways has been addressed from the economic policy development perspective, e.g. [7]–[9]. In addition, there have been many reports assessing the quality and effectiveness of lane pricing, e.g. [4], [10]–[12]. What is lacking is a framework in which incentives can be designed in a systematic way while also capturing lane changing behavior from a decision–theoretic point of view.

In this paper, we present a framework for optimizing lane pricing given a particular objective such as maximizing overall throughput. We first develop a decision–theoretic model for lane changing behavior on highways. We then leverage this decision–theoretic model in the design of incentives in the form of lane pricing. Our framework includes modeling of multiple populations of vehicles with different driving preferences on roadways with multiple lanes.

The paper is organized as follows. In Section II we formulate the decision–theoretic model for traffic flow on a multilane highway In Section III, we formulate the optimization problem for determining lane prices. We demonstrate the design of a toll for a simulated two lane road with an HOT lane in Section IV. Finally, we conclude with a discussion of the results and future directions in Section V.

II. DECISION–THEORETIC LANE CHANGING MODEL

A. Previous Macroscopic Traffic Models

Macroscopic modeling of traffic flow has been well studied in the past. In their seminal paper, Lighthill and Whitman describe traffic flow using kinematic waves [13]. Richards independently proposed a similar model in which the crucial underlying assumption is a relationship between flow and density [14]. The Lighthill, Whitman, and Richards (LWR) model, as it came to be known, can be described as follows. Consider the flow of cars on a highway. Let ρ denote the density—e.g. vehicles per mile—and let v denote the velocity. The density is restricted to live in the interval $[0, \rho_{\max}]$ where $\rho_{\max}$ is the *jam density*, i.e. the density at which cars are *bumper–to–bumper*. In this model vehicles are conserved; hence, the density and velocity are related by the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0 \tag{1}$$

where $q(\rho) = v(\rho)\rho$ is the traffic flow rate. Note we assume that the velocity is a function of ρ. This assumption is reasonable since on a highway vehicles would like to drive at some free flow speed, v_f, but when there is traffic congestion vehicles will slow down, with velocity decreasing as density increases.

From empirical studies, several different models have been proposed for the velocity–density relationship [15], [16] . The simplest of these models, which we will use in this paper, is the Greenshields model [17] which models velocity as a linear function of density:

$$v(\rho) = v_f \left(1 - \frac{\rho}{\rho_{\max}}\right) \tag{2}$$

where v_f is the *free–flow* velocity. Combining this relationship with the identity $q(\rho) = v(\rho)\rho$ we get the following:

$$q(\rho) = v_f \left(\rho - \frac{\rho^2}{\rho_{\max}}\right). \tag{3}$$

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The graph of the traffic flow as described in the above equation is referred to as the *fundamental diagram*.

Versions of the LWR models with multiple populations of drivers [16], [18], [19] and multiple lanes have been studied in the literature [20]–[25]. In these models, a separate density conservation equation is added for each population and/or each lane. In the case of lane changing, the exchange of density between lanes is modeled by source terms. For example, a two lane road with no on/off ramps would be modeled by the following system of equations

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial q_1(\rho_1)}{\partial x} &= s_1(x,t) \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial q_2(\rho_2)}{\partial x} &= s_2(x,t). \end{aligned} \tag{4}$$

In order to ensure conservation of mass between the two lanes, we enforce the following condition:

$$s_1(x,t) + s_2(x,t) = 0, \quad \forall\ x, t. \tag{5}$$

The source terms s_1 and s_2 describe the flow exchanged between the lanes. They are often modeled as probabilistic functions. So far, however, they have not been chosen from a decision–theoretic view point.

Recently, there has been some exciting work bridging the gap between microscopic and macroscopic models of traffic from a decision–theoretic perspective in the context of one lane roads and intersections [26], [27]. Lane changing specifically has been studied from a decision–theoretic perspective at a microscopic level [28]–[30]; however, these models have not been incorporated into macroscopic models. To this end, we propose a multi-population, multi-lane macroscopic model for freeway traffic flow that takes a decision–theoretic view of lane changing with the goal of incentivizing drivers to drive in specific lanes.

B. Decision–Theoretic, Macroscopic Traffic Model

We will consider a section of highway with K lanes and N populations of vehicles. Each population consists of vehicles with similar characteristics, be they driving capabilities, travel time preferences, etc. We use the notation $\rho_k^i(x,t)$ to represent the traffic density of the i–th population in lane k. We will also use the notation $\boldsymbol{\rho}_k := [\rho_k^1 \ \cdots \ \rho_k^N]^T$ to represent the full density vector in lane k, $\bar{\rho}_k := \sum_i \rho_k^i$ to represent the total density in lane k, and $\boldsymbol{\rho} := [\boldsymbol{\rho}_1^T \ \cdots \ \boldsymbol{\rho}_K^T]^T$.

We model traffic flow on a multi-lane highway as a system of $K \times N$ conservation laws:

$$\frac{\partial \rho_k^i}{\partial t} + \frac{\partial q_k^i(\boldsymbol{\rho}_k)}{\partial x} = s_k^i(\boldsymbol{\rho}, x, t) \tag{6}$$

for each $i \in \{1, \dots, N\}$ and $k \in \{1, \dots, K\}$. In the above system, $q_k^i(\boldsymbol{\rho}_k)$ is the flow rate of population i in lane k defined by the relationship

$$q_k^i(\boldsymbol{\rho}_k) = v_k(\bar{\rho}_k)\rho_k^i \tag{7}$$

where v_k is the velocity–density relation for lane k and has the form given in Equation 2.

Remark 1: We note that the traffic flow rate as described in (7) presumes that velocity–density relationship is isotropic with respect to the dependence on the various population densities [18]. Modeling the velocity as having a different dependence on each individual population would be a useful extension which we leave for future work. ■

Our goal is to define source functions s_k^i that capture decision–theoretic lane changing behavior but first we analyze some properties of this system of partial differential equations (PDEs). In particular, we show that the system is hyperbolic for all physically valid values of $\boldsymbol{\rho}_k$.

1) Hyperbolicity of the Proposed Conservation Law: The qualitative behavior of the solution of the system of equations defined in (6) depends on the systems type—i.e. hyperbolic, parabolic, or elliptic. In particular, if the system of equations is hyperbolic, we can expect the formation of weak solutions in the form of shock and rarefaction waves. As an extension of the LWR model, which is a hyperbolic conservation law, we would expect the system to be hyperbolic. In this section, we will show that this is indeed the case.

Let $Q_k(\boldsymbol{\rho}_k) := [q_k^1(\boldsymbol{\rho}_k) \ \cdots \ q_k^N(\boldsymbol{\rho}_k)]^T$ and $Q(\boldsymbol{\rho}) := [Q_1(\boldsymbol{\rho}_1)^T \ \cdots \ Q_K(\boldsymbol{\rho}_K)^T]^T$. We note that we can write the system in (6) as

$$\frac{\partial \boldsymbol{\rho}}{\partial t} + \frac{\partial Q(\boldsymbol{\rho})}{\partial x} = \boldsymbol{s}(x,t) \tag{8}$$

where $\boldsymbol{s}(x,t) = [s_1^1, \dots, s_1^N, s_2^1, \dots, s_K^1, \dots, s_K^N]^T$. The type of the system in (6) is determined by the eigenstructure of the matrix

$$\frac{\partial Q}{\partial \boldsymbol{\rho}} = \begin{bmatrix} \frac{\partial Q_1}{\partial \boldsymbol{\rho}_1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \frac{\partial Q_K}{\partial \boldsymbol{\rho}_K} \end{bmatrix} \tag{9}$$

In particular, it is hyperbolic if $\frac{\partial Q}{\partial \boldsymbol{\rho}}$ has $K \times N$ linearly independent eigenvectors. Further, the system is *weakly* hyperbolic if it has repeated eigenvalues.

Since $\frac{\partial Q}{\partial \boldsymbol{\rho}}$ is block diagonal it suffices to show that each block has N linearly independent eigenvectors. We can show that this is indeed the case by explicitly calculating the eigenvectors and showing that they are linearly independent for all $\boldsymbol{\rho}$.

Proposition 1: The multilane, multipopulation model for traffic flow described in (8) is *weakly* hyperbolic.

Proof: We first decompose $\frac{\partial Q_k}{\partial \boldsymbol{\rho}_k}$ as

$$\frac{\partial Q_k}{\partial \boldsymbol{\rho}_k} = v_k(\bar{\rho}_k) I_{N \times N} + \boldsymbol{\rho}_k v_k'(\bar{\rho}_k) \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \tag{10}$$

Note that $\boldsymbol{\rho}_k$ is a column vector and $v_k'(\bar{\rho}_k)$ is a scalar–valued function evaluated at the total density in lane k, $\bar{\rho}_k$. Let $\{w_1, \dots, w_{N-1}\}$ be any set of linearly independent vectors orthogonal to $[1 \ \cdots \ 1]^T$. From the above decomposition, one can check that the eigenvectors are given by

$$\Big\{\boldsymbol{\rho}_k, w_1, \dots, w_{N-1}\Big\} \tag{11}$$

and they have corresponding eigenvalues given by

$$\Big\{v_k(\bar{\rho}_k) + \bar{\rho}_k v_k'(\bar{\rho}_k),\ v_k(\bar{\rho}_k),\ \dots,\ v_k(\bar{\rho}_k)\Big\} \tag{12}$$

We then need to show that these eigenvectors are linearly independent. The eigenvectors $\{w_1, \ldots, w_{N-1}\}$ are linearly independent by construction. If $\boldsymbol{\rho}_k$ were linearly dependent on $\{w_1, \ldots, w_{N-1}\}$, then $\boldsymbol{\rho}_k$ would be orthogonal to $[1 \ \cdots \ 1]^T$; this cannot be the case for any physically meaningful $\boldsymbol{\rho}_k$ since all the entries of $\boldsymbol{\rho}_k$ must be positive. Therefore, for any physically meaningful $\boldsymbol{\rho}$ (i.e. whose values are positive), the matrix $\frac{\partial Q}{\partial \boldsymbol{\rho}}$ has $K \times N$ linearly independent eigenvectors. Further, there are repeated eigenvalues. Thus, the system in (6) is weakly hyperbolic. ■

2) Decision–Theoretic Source Functions: We now return to designing source functions that model the decision–theoretic behavior of drivers changing lanes. In this paper, we focus on a simplified scenario where there are no on/off ramps and drivers only switch lanes in order to improve their travel time. In future work, we plan to consider lane changing for exiting/entering the highway as well as to avoid obstacles.

When deciding to switch between lanes in our model, a driver from a given population assigns a utility to each lane. We assume the following form for the utility of population i in lane k considering switching into lane k':

$$u^i_{k,k'} = v_{k'} - w^i \tau_{k'} - d_{k,k'} c^i \tag{13}$$

where

- $v_{k'}$ is the velocity in lane k',
- $\tau_{k'}$ is a toll for driving in lane k' (maybe 0),
- w^i is the weight population i puts on toll money (vs. velocity),
- c^i is a weight that encodes population i's aversion to switching lanes, and
- $d_{k,k'}$ is the number of lane switches required to switch from lane k to k'.

We note that this utility can be thought of as a myopic approach to minimizing travel time with penalty terms for paying a toll and the inconvenience of switching lanes.

Remark 2: This utility is somewhat simplistic in that drivers only consider the velocity of traffic at their current location without looking ahead. A similar utility could be formulated where, rather than just trying to maximize their current velocity, drivers seek to minimize travel time over some finite time horizon. It could also be argued that the penalty for changing into a lane, c^i, should be a function of the density in that lane. We leave both of these extensions to future work. ■

At any given time, drivers seek to switch into the lane that maximizes their population–specific utility. We would expect, however, that even if a driver wants to switch lanes, they might have difficulty doing so if the density in the lane next to them is high. For this reason, we want the source functions to incorporate both whether drivers from a given population want to switch lanes as well as how many of them are able to switch at any given moment. This ability to switch will be captured by the rate functions, $r_k(\bar{\rho}_k)$. The exact form of $r_k(\bar{\rho}_k)$ is a topic for future research though it should be a decreasing function of the total density in lane k that reaches 0 at jam density. For now, we will take $r_k(\bar{\rho}_k) = v_k(\bar{\rho}_k)$.

We expect the source functions to have the following form:

$$s^i_k = -(s^i_k)_{OL} - (s^i_k)_{OR} + (s^i_k)_{IL} + (s^i_k)_{IR} \tag{14}$$

where

- $(s^i_k)_{OL}$: rate pop. i switches out of lane k to the left.
- $(s^i_k)_{OR}$: rate pop. i switches out of lane k to the right.
- $(s^i_k)_{IL}$: rate pop. i switches into lane k from the left.
- $(s^i_k)_{IR}$: rate pop. i switches into lane k from the right.

Below we will define, $(s^i_k)_{OL}$, noting that the other terms are defined similarly.

For a given lane k, let k_l be the lane directly to the left and k_r be the lane directly to the right. Let L_k represent the set of all lanes to the left and R_k represent the set of all lanes to the right. We then define $(s^i_k)_{OL}$ as follows:

$$\begin{aligned}(s^i_k)_{OL} = r_{k_l}(\bar{\rho}_{k_l}) \rho^i_k H\big(\max_{k' \in L_k} u^i_{k,k'} - \max_{k' \in R_k} u^i_{k,k'}\big) \\ \cdot H\big(\max_{k' \in L_k} u^i_{k,k'} - u^i_{k,k}\big) \end{aligned} \tag{15}$$

where $H(x)$ is a heaviside function:

$$H(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases} \tag{16}$$

Intuitively, we can break down (15) from right to left as follows.

- $H\big(\max_{k' \in L_k} u^i_{k,k'} - u^i_{k,k}\big)$ compares the utility in the drivers current lane with the maximum utility in any lane to the left (taking into account how many switches are required to get to that lane) and indicates whether the driver prefers switching left over staying put.
- $H\big(\max_{k' \in L_k} u^i_{k,k'} - \max_{k' \in R_k} u^i_{k,k'}\big)$ then indicates whether the driver prefers switching to the left over switching to the right.
- Combining these functions with the density term ρ^i_k gives the amount of density trying to switch out of lane k to the left.
- Finally, multiplying by the rate function of the lane directly to the left, $r_{k_l}(\bar{\rho}_{k_l})$, gives the rate at which the density in lane k is shifting left into lane k_l.

We define $(s^i_k)_{OR}$ similarly to $(s^i_k)_{OL}$ and note that conservation of mass dictates that $(s^i_k)_{IL} = (s^i_{k_l})_{OR}$ and $(s^i_k)_{IR} = (s^i_{k_r})_{OL}$.

In practice, we use smoothed versions of $H(\cdot)$ and $\max(\cdot)$. In particular, we replace the heavyside function with

$$H(x) = 0.5 + 0.5 \tanh(\alpha x) \tag{17}$$

and the $\max$ function with its approximation

$$\max_i \left(\{x_i\}_{i=1}^n\right) \approx \frac{\ln \sum_i e^{\alpha x_i}}{\alpha} \tag{18}$$

where in both cases, α is a smoothing parameter. This is necessary to apply the adjoint method described in Section III.

Remark 3: In this model, we are assuming that cars always try to get in the lane that maximizes their utility. Different levels of driver aggressiveness can be captured by adding more populations with different values for c^i. ■

Remark 4: The max function approximation has provided scaling challenges in solving the adjoint equation (Section III). In this paper, we only present a two lane example that does not require them (see Section IV) though they have the potential to provide a rich modeling framework for decision–theoretic lane changing in the future. ■

III. Optimizing Lane Prices

Given the above PDE model, we can use an adjoint method to find toll prices (τ) or solve other design problems. Suppose we want to solve the optimization problem given by

$$\max_{\tau(x,t)} \qquad J(\boldsymbol{\rho}, x, t) = \int_0^T \int_0^X l(\boldsymbol{\rho}(x,T), x, T)\ dx\ dt \tag{19}$$

$$\text{s.t.} \qquad \text{Equation} \quad (8)$$

where τ is our decision variable, T is the final time and X is the maximum state.

We can calculate the gradient of J by solving the adjoint system of equations, (20), backward in time from the final condition, (21), where

$$-\frac{\partial \boldsymbol{p}}{\partial t} - \frac{\partial Q(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}^T \frac{\partial \boldsymbol{p}}{\partial x} = \frac{\partial \mathbf{s}(\boldsymbol{\rho}, \tau)}{\partial \boldsymbol{\rho}}^T \boldsymbol{p} + \frac{\partial l}{\partial \boldsymbol{\rho}}^T \tag{20}$$

$$\boldsymbol{p}(x, T) = \frac{\partial l(\boldsymbol{\rho}(x,T), x, T)}{\partial \boldsymbol{\rho}} \tag{21}$$

where $\boldsymbol{\rho}(x,t)$ is the solution to Equation (6) (written in vector form in Equation (8)). The gradient of J with respect to τ can then be calculated and is given by

$$\frac{\partial J}{\partial \tau}(x,t) = \frac{\partial \mathbf{s}(\boldsymbol{\rho}, \tau)}{\partial \tau}^T \boldsymbol{p}(x,t). \tag{22}$$

Once we have determined the gradient of J, we can perform gradient descent to optimize for $\tau(x,t)$.

The running cost $l(\boldsymbol{\rho}, x, t)$ depends on the design problem. In Section IV, we show an HOT lane example where a social planner tries to utilize more of an HOV lane's capacity while still maintaining a reasonable travel speed in the lane. To this end the social planner might design a toll to maximize the flow rate in the toll lane (lane t) given by

$$l(\boldsymbol{\rho}, x, t) = \sum_i v_t(\bar{\rho}_t)\rho_t^i \tag{23}$$

The adjoint method can also be used to fit parameters to data. For example, rather than optimizing for toll prices, τ, we might optimize for the parameters, c^i, in order to learn different populations' aversion to changing lanes. In this case, we would define the running cost to be

$$l(\boldsymbol{\rho}, x, t) = \big|\big|\boldsymbol{\rho}(x,t) - \boldsymbol{\rho}_{\mathrm{D}}(x,t)\big|\big|^2 \tag{24}$$

where $\boldsymbol{\rho}_{\mathrm{D}}(x,t)$ is the observed traffic density and $\boldsymbol{\rho}(x,t)$ is the density predicted by the model.

Solutions to the adjoint equation for hyperbolic conservation laws are a topic of current research both theoretically and empirically. The theoretical basis for these methods has been established for scalar equations but not for systems of equations [31]–[37]. Empirically, different techniques have been tried with good results in some cases [38]–[42]. The main difficulties come from the formation of shocks which form discontinuities in the solution $\boldsymbol{\rho}(x,t)$ and which leads to non-differentiability in the terms of the adjoint equation. In this paper, we apply a technique similar to the technique used in [38]. We don't give a proof of convergence, but we find that we get good results in practice.

We solve both Equation (8) and the adjoint system (20) using code from the CLAWPACK library [43]. In both cases, we implement a conservative Godunov method using a Roe linear approximation of $\frac{\partial Q}{\partial \boldsymbol{\rho}}$, a van Leer limiter for higher order accuracy, and a Harten-Hyman entropy fix for rarefaction waves. We integrate the source terms using forward Euler and use first order Godunov splitting to combine the flux and source terms. See [44] for details.

IV. Numerical Examples

We illustrate the design of toll prices on a numerical example with 10 populations of drivers on a two lane road that forms a two mile loop. Our goal is to design a toll for Lane 1 that maximizes the flow rate in that lane. We note that any realistic tolling example would have non-periodic boundary conditions and have entry and exit points on the road. We will add these in future work.

The velocity-density curve for each lane is defined by

$$v_k(\bar{\rho}_k) = 60\text{mph}\left(1 - \frac{\bar{\rho}_k}{200\text{cars/mi}}\right) \tag{25}$$

with free flow velocity of 60 mph and jam density of 200 cars/mile. As discussed above, we use $r_k(\bar{\rho}_k) = v_k(\bar{\rho}_k)$.

In order to maximize flow rate in Lane 1, we use the running cost defined in (23).

Each population of drivers places a different utility on driving in each lane. The utilities for population i are given by

$$\text{Staying in Lane 1:} \qquad u_1^i = v_1(\bar{\rho}_1) - w^i\tau \tag{26}$$

$$\text{Switching to Lane 1:} \qquad u_{2,1}^i = v_1(\bar{\rho}_1) - w^i\tau - c^i \tag{27}$$

$$\text{Staying in Lane 2:} \qquad u_2^i = v_2(\bar{\rho}_2) \tag{28}$$

$$\text{Switching to Lane 2:} \qquad u_{1,2}^i = v_2(\bar{\rho}_2) - c^i \tag{29}$$

where τ is the toll for driving in Lane 1, w^i encodes population i's value of toll money, and c^i encodes population i's aversion to changing lanes. Table I gives the values of these parameters for each population. Note that Population 1 models high occupancy vehicles that are not required to pay a toll and thus have a weight of $w^1 = 0.0$.

For a two lane road, the s-function for population i in Lane 1 is

$$s_1^i = -r_2(\bar{\rho}_2)\rho_1^i H(u_{1,2}^i - u_{1,1}^i) + r_1(\bar{\rho}_1)\rho_2^i H(u_{2,1}^i - u_{2,2}^i) \tag{30}$$

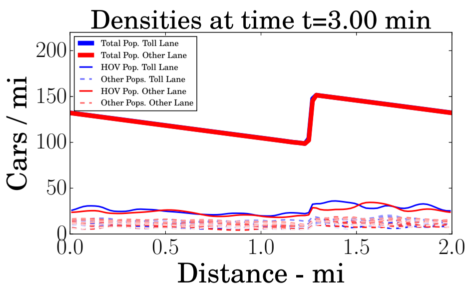
and similarly for s_2^i.

The total number of drivers in each lane is initialized as

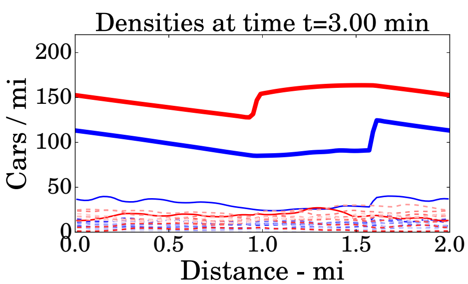
$$\rho_k(x,0) = \left(125 + 50\sin\left(\frac{2\pi}{2\text{mi}}x\right)\right)\frac{\text{cars}}{\text{mi}}, \qquad k = 1,2. \tag{31}$$

TABLE I: Population Parameters

Population i	1	2	3	4	5	6	7	8	9	10
Fraction of Population	0.2	0.06	0.08	0.08	0.1	0.12	0.12	0.1	0.08	0.06
Money-Speed Tradeoff: w^i	0.0	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0
Reluctance to Change Lanes: c^i	3.0	0.0	0.0	0.0	3.0	3.0	3.0	6.0	6.0	6.0



(a) Without toll



(b) With toll: $\tau = 0.6$.

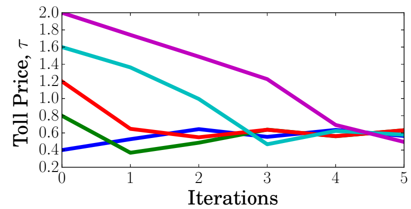
Fig. 1: Snapshots of traffic evolution

The proportion of drivers from different populations in each lane is initialized according to a Dirichlet distribution with the concentration parameters also shown in Table I. The forward dynamics and adjoint systems were simulated in CLAWPACK [43] using 100 spatial cells over 400 time steps using the methods outlined at the end of Section III.

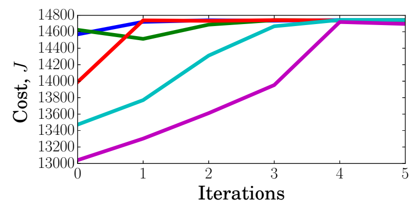
Simulations with and without a toll can be found at

- Without toll: `http://www.eecs.berkeley.edu/~danjc/cdc15/wotoll/sim.html`
- With toll: `http://www.eecs.berkeley.edu/~danjc/cdc15/wtoll/sim.html`

Figures (1a) and (1b) are snap shots of the simulations at $t = 3$ min. Note that without a toll, the lane populations evolve together. (Total Pop. Toll Lane is hidden under Total Pop. Other Lane in Figure (1a).) When the optimal toll is applied, however, much of the non-HOV population shifts out of the toll lane. Note also that the total density in the toll lane (the blue lane) approaches the critical density, 100 cars/mi, at which maximum flow rate occurs. Figures (2a) and (2b) show the convergence of the toll itself and the cost (flow rate in the toll lane) over several iterations of gradient descent.



(a) Tolls



(b) Costs

Fig. 2: Convergence of optimization

V. Discussion

In this paper, we have a presented a macroscopic model for traffic flow with decision theoretic lane changing incorporating multiple lanes and multiple populations. As indicated in the paper, there are numerous directions for further research outlined here. The first is to enrich various components of the model. Some examples are as follows.

- The velocities functions, v_k, could be designed to depend differently on different populations of vehicles. A simple way to do this would be to have v_k depend on a weighted sum of the densities in lane k. This could be used to model different types of vehicles such as cars and trucks. A similar thing could be done for the rate functions, r_k.
- The exact form of the rate functions should be determined from first principles or from data. In addition, r_k could be made a function of the relative velocities between lanes.
- The c^i penalty terms could be made a function of density as well.

Along with enriching the current model, more features should be added to the model in order to capture the many facets of lane changing. In particular, lane changing for entering and exiting the freeway should be incorporated.

Another aspect of lane changing, that we have not included, is its effect on the velocity in the lane cars are switching into. In [45], this is done by having the incoming density from a neighboring lane modify the velocity-density function in the destination lane. One difficulty here is that this model breaks the diagonal structure of the $\frac{\partial Q}{\partial \rho}$ matrix which makes analysis of the system of PDEs substantially more difficult.

Along with enriching the model, the parameters of the various functions should be determined from data. This is a difficult task, in general, because of the wide variety of driver populations on the road at any given time and the lack of available data.

Along with designing tolling strategies, this model and similar optimization techniques could be used to design ramping metering strategies.

As mentioned previously, improving numerical methods for optimization of these types of systems is an open area of research. For our particular optimization problem, we are interested in understanding and mitigating the effects of scaling.

REFERENCES

[1] "The future economic and environmental costs of gridlock in 2030," Centre for Economics and Business Research, 2014.

[2] A. A. Kurzhanskiy and P. Varaiya, "Traffic management: An outlook," University of California, Berkeley, Tech. Rep., Dec. 2014.

[3] "Traffic congestion: Road pricing can help reduce congestion but equity concerns may grow," United States Government Accountability Office, January 2012.

[4] R. W. Poole Jr. and C. K. Orski, "Building a case for hot lanes: A new approach to reducing urban highway congestion," *Policy Study*, no. 257, 1999.

[5] G. J. Fielding and D. B. Klein, "High occupancy/toll lanes: Phasing in conjestion pricing a lane at a time," *Reason Foundation, Policy Study*, no. 170, Nov. 1993.

[6] "When the road price is right: Land use, tolls, and congestion pricing," Urban Land Institute: Infrastructure Initiative, 2013.

[7] J. Rouwendal and E. T. Verhoef, "Basic economic principles of road pricing: From theory to applications," *Transport Policy*, vol. 13, no. 2, pp. 106 – 114, 2006.

[8] K. A. Small, C. Winston, J. Yan, N. Baum-Snow, and J. A. Gómez-Ibáñez, "Differentiated road pricing, express lanes, and carpools: Exploiting heterogeneous preferences in policy design," *Brookings-Wharton Papers on Urban Affairs*, pp. 53–96, 2006.

[9] M. Smirti, A. Evans, M. Gougherty, and E. Morris, "Politics, public opinion, and project design in california road pricin," *Transportation Research Board*, vol. 1996, pp. 41–48, Nov. 2007.

[10] M. W. Burris and B. R. Stockton, "Hot lanes in houston—six years of experience," *Journal of Public Transportation*, vol. 7, Nov. 2004.

[11] R. W. Poole Jr and C. K. Orski, "Hot lanes: a better way to attack urban highway congestion," *Regulation*, vol. 23, p. 15, 2000.

[12] M. Burris and E. Sullivan, "Benefit-cost analysis of variable pricing projects: Quickride hot lanes," *Journal of Transportation Engineering*, vol. 132, no. 3, pp. 183–190, 2006.

[13] M. J. Lighthill and G. B. Whitham, "On kinematic waves. i:flow movement in long rivers. ii:a theory of traffic on long crowded roods," in *Proc. Royal Soc.*, no. A229, 1955, pp. 281–345.

[14] P. I. Richards, "Shock waves on the highway," *Operations Research*, vol. 4, no. 1, pp. pp. 42–51, Feb. 1956.

[15] L. Fermo and A. Tosin, "Fundamental diagrams for kinetic equations of traffic flow," *Discrete and Continuous Dynamical Systems - Series S*, vol. 7, no. 3, pp. 449–462, 2014. [Online]. Available: http://aimsciences.org/journals/displayArticlesnew.jsp?paperID=9590

[16] G. Puppo, M. Semplice, A. Tosin, and G. Visconti, "Fundamental diagrams in traffic flow: the case of heterogeneous kinetic models," *arXiv preprint arXiv:1411.4988*, 2014.

[17] B. D. Greenshields, "A study in highway capacity," *Highway Research Board*, vol. 14, p. 458, 1935.

[18] G. Wong and S. Wong, "A multi-class traffic flow model–an extension of LWR model with heterogeneous drivers," *Transportation Research Part A: Policy and Practice*, vol. 36, no. 9, pp. 827–841, 2002.

[19] S. P. Hoogendoorn and P. H. Bovy, "Continuum modeling of multiclass traffic flow," *Transportation Research Part B: Methodological*, vol. 34, no. 2, pp. 123–146, 2000.

[20] T. Tang and H. Huang, "Continuum models for freeways with two lanes and numerical tests," *Chinese Science Bulletin*, vol. 49, no. 19, pp. 2097–2104, 2004.

[21] T. Tang, H. Huang, S. Wong, and R. Jiang, "Lane changing analysis for two-lane traffic flow," *Acta Mechanica Sinica*, vol. 23, no. 1, pp. 49–54, 2007.

[22] C. F. Daganzo, "A continuum theory of traffic dynamics for freeways with special lanes," *Transportation Research Part B: Methodological*, vol. 31, no. 2, pp. 83–102, 1997.

[23] E. N. Holland and A. W. Woods, "A continuum model for the dispersion of traffic on two-lane roads," *Transportation Research Part B: Methodological*, vol. 31, no. 6, pp. 473–485, 1997.

[24] P. Munjal and L. A. Pipes, "Propagation of on-ramp density perturbations on unidirectional two-and three-lane freeways," *Transportation Research*, vol. 5, no. 4, pp. 241–255, 1971.

[25] P. Munjal, Y.-S. Hsu, and R. Lawrence, "Analysis and validation of lane-drop effects on multi-lane freeways," *Transportation Research*, vol. 5, no. 4, pp. 257–266, 1971.

[26] A. Tosin, "From generalized kinetic theory to discrete velocity modeling of vehicular traffic. a stochastic game approach," *Applied Mathematics Letters*, vol. 22, no. 7, pp. 1122–1125, 2009.

[27] E. Cristiani, B. Piccoli, and A. Tosin, "How can macroscopic models reveal self-organization in traffic flow?" in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*. IEEE, 2012, pp. 6989–6994.

[28] N. A. Webster, T. Suzuki, and M. Kuwahara, "Tactical lane change model with sequential maneuver planning," *Transportmetrica*, vol. 4, no. 1, pp. 63–78, 2008.

[29] H. X. Liu, W. Xin, Z. Adam, and J. Ban, "A game theoretical approach for modelling merging and yielding behaviour at freeway on-ramp sections," *Elsevier, London*, pp. 197–211, 2007.

[30] F. Marczak, W. Daamen, and C. Buisson, "Key variables of merging behaviour: empirical comparison between two sites and assessment of gap acceptance theory," *Procedia-Social and Behavioral Sciences*, vol. 80, pp. 678–697, 2013.

[31] S. Ulbrich, "A sensitivity and adjoint calculus for discontinuous solutions of hyperbolic conservation laws with source terms," *SIAM journal on control and optimization*, vol. 41, no. 3, pp. 740–797, 2002.

[32] M. Giles and S. Ulbrich, "Convergence of linearized and adjoint approximations for discontinuous solutions of conservation laws. part 1: Linearized approximations and linearized output functionals," *SIAM Journal on Numerical Analysis*, vol. 48, no. 3, pp. 882–904, 2010.

[33] ——, "Convergence of linearized and adjoint approximations for discontinuous solutions of conservation laws. part 2: Adjoint approximations and extensions," *SIAM Journal on Numerical Analysis*, vol. 48, no. 3, pp. 905–921, 2010.

[34] S. Ulbrich, "Optimal control of nonlinear hyperbolic conservation laws with source terms," *Technische Universitaet Muenchen*, 2001.

[35] C. Castro, F. Palacios, and E. Zuazua, "An alternating descent method for the optimal control of the inviscid burgers equation in the presence of shocks," *Mathematical Models and Methods in Applied Sciences*, vol. 18, no. 03, pp. 369–416, 2008.

[36] A. Bressan and A. Marson, "A variational calculus for discontinuous solutions of systems of conservation laws," *Communications in partial differential equations*, vol. 20, no. 9, pp. 1491–1552, 1995.

[37] A. Bressan and W. Shen, "Optimality conditions for solutions to hyperbolic balance laws," *Contemporary Mathematics*, vol. 426, pp. 129–152, 2007.

[38] M. Herty, A. Kurganov, and D. Kurochkin, "Numerical method for optimal control problems governed by nonlinear hyperbolic systems of pdes," *Commun. Math. Sci*, vol. 13, pp. 15–48, 2015.

[39] L. C. Wilcox, G. Stadler, T. Bui-Thanh, and O. Ghattas, "Discretely exact derivatives for hyperbolic pde-constrained optimization problems discretized by the discontinuous galerkin method," *Journal of Scientific Computing*, pp. 1–25, 2013.

[40] R. Herzog and K. Kunisch, "Algorithms for pde-constrained optimization," *GAMM-Mitteilungen*, vol. 33, no. 2, pp. 163–176, 2010.

[41] J. Reilly, "On cybersecurity of freeway control systems: Analysis of coordinated ramp metering attacks 2," Ph.D. dissertation, University of California, Berkeley, 1755.

[42] A. Chertock, M. Herty, and A. Kurganov, "An eulerian–lagrangian method for optimization problems governed by multidimensional nonlinear hyperbolic pdes," *Computational Optimization and Applications*, vol. 59, no. 3, pp. 689–724, 2014.

[43] Clawpack Development Team, "Clawpack software," 2014, version 5.2.2. [Online]. Available: http://www.clawpack.org

[44] R. J. LeVeque, "Finite volume methods for hyperbolic problems," *Meccanica*, vol. 39, pp. 88–89, 2004.

[45] T.-Q. Tang, S. Wong, H.-J. Huang, and P. Zhang, "Macroscopic modeling of lane-changing for two-lane traffic flow," *Journal of Advanced Transportation*, vol. 43, no. 3, pp. 245–273, 2009.