Kinematics, Dynamics, & Linearization

Dynamics & Modeling

Major sources: Kaare Brandt Petersen
Michael Syskind Pedersen

Major references: The Matrix Cookbook - Petersen, Pedersen

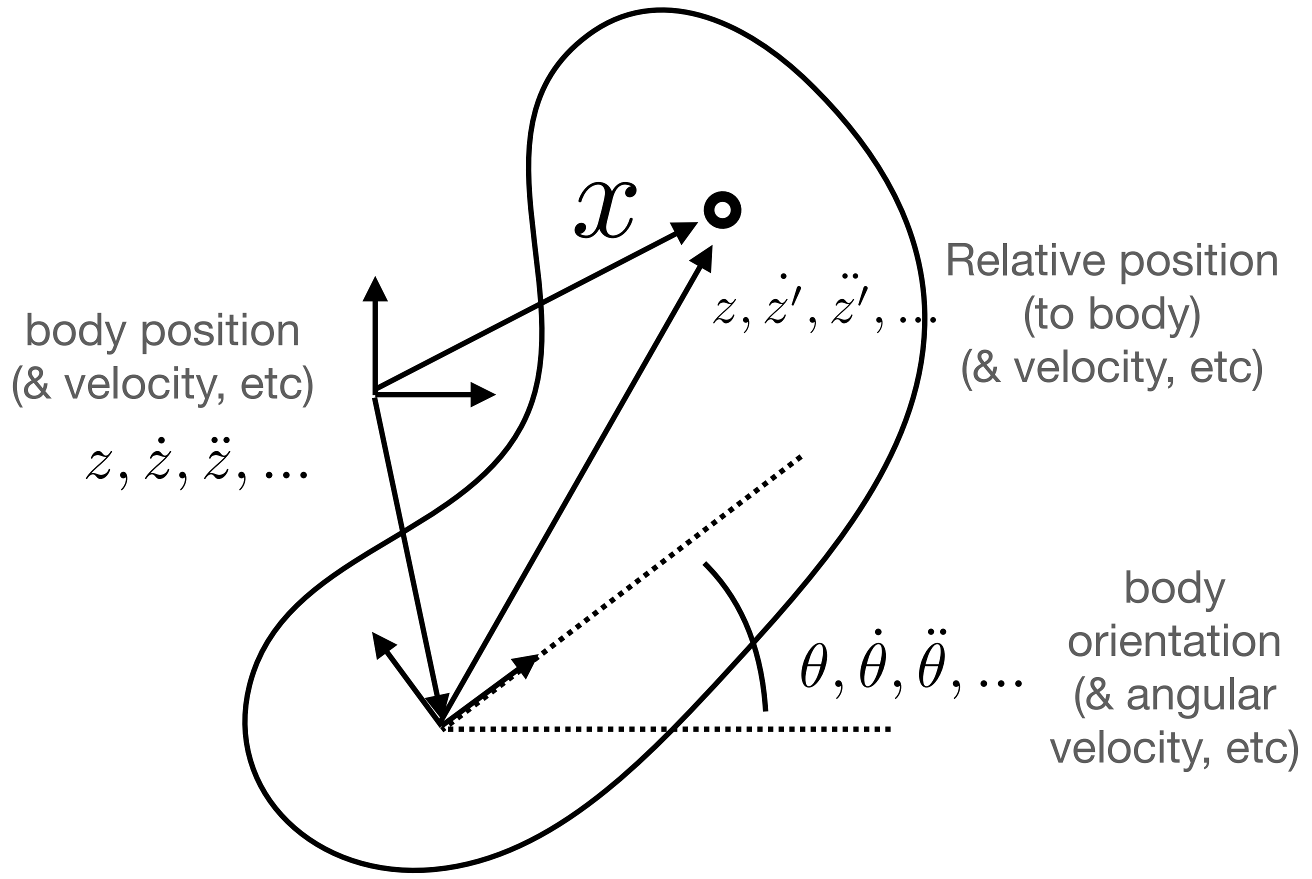
Winter 2022 - Dan Calderone

Kinematics 2D

useful for most homework problems

… for mechanical systems,
higher order motion can be quite complicated

2D:



Rotation Matrix Derivation:

Note:
rotation "axis"
- out of the page

$$\hat{\omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$R(\theta) = e^{\hat{\omega}\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\dot{R}(\theta) = \dot{\theta}\hat{\omega}e^{\hat{\omega}\theta} = \dot{\theta}\hat{\omega}R(\theta)$$

$$\ddot{R}(\theta) = \ddot{\theta}\hat{\omega}e^{\hat{\omega}\theta} + \dot{\theta}^2\hat{\omega}^2 e^{\hat{\omega}\theta}$$

$$= \ddot{\theta}\hat{\omega}R(\theta) + \dot{\theta}^2\hat{\omega}^2 R(\theta)$$

Velocity Terms:

$$\dot{x} = \underbrace{\dot{z}}_{\text{Body velocity}} + \underbrace{\dot{\theta}\hat{\omega}R(\theta)z'}_{\text{Angular velocity}} + \underbrace{R(\theta)\dot{z}'}_{\text{Relative velocity}}$$

Position: $x = z + R(\theta)z'$

Velocity:

$$\dot{x} = \dot{z} + \dot{R}(\theta)z' + R(\theta)\dot{z}'$$

$$= \dot{z} + \dot{\theta}\hat{\omega}R(\theta)z' + R(\theta)\dot{z}'$$

Acceleration Terms:

$$\ddot{x} = \underbrace{\ddot{z}}_{\text{Body acceleration}} + \underbrace{\ddot{\theta}\hat{\omega}R(\theta)z'}_{\text{Angular acceleration}} + \underbrace{\dot{\theta}^2\hat{\omega}^2R(\theta)z'}_{\text{Centripetal acceleration}} + \underbrace{2\dot{\theta}\hat{\omega}R(\theta)\dot{z}'}_{\text{Coriolis acceleration}} + \underbrace{R(\theta)\ddot{z}'}_{\text{Relative acceleration}}$$

Acceleration:

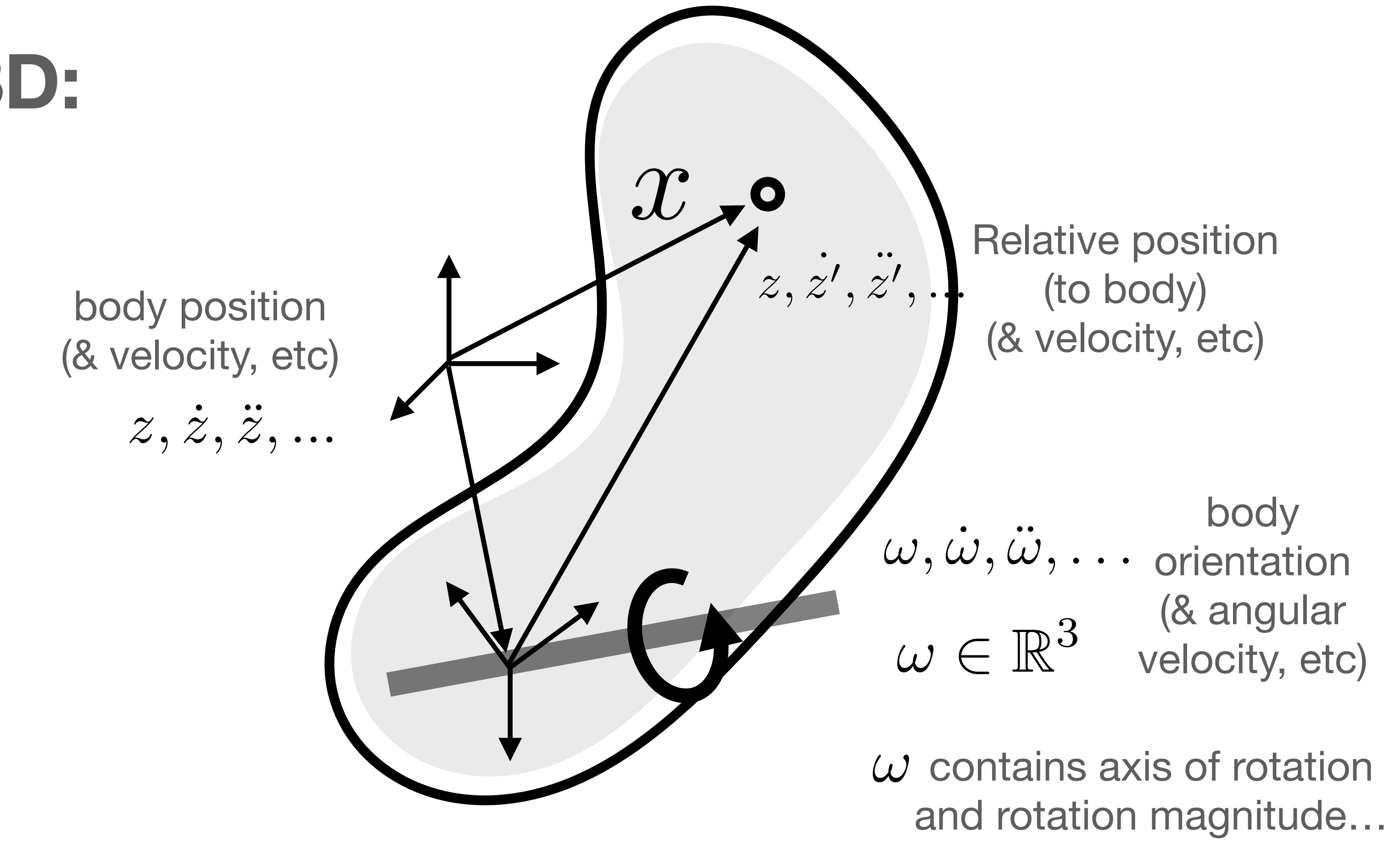
$$\ddot{x} = \ddot{z} + \ddot{R}(\theta)z' + 2\dot{R}(\theta)\dot{z}' + R(\theta)\ddot{z}'$$

$$= \ddot{z} + \ddot{\theta}\hat{\omega}R(\theta)z' + \dot{\theta}^2\hat{\omega}^2R(\theta)z' + 2\dot{\theta}\hat{\omega}R(\theta)\dot{z}' + R(\theta)\ddot{z}'$$

Kinematics 3D
generalization

… for mechanical systems,
higher order motion can be quite complicated

3D:



…for fixed axis rotations with axis ξ: $\omega = \theta\xi,\ \dot\omega = \dot\theta\xi,\ \ddot\omega = \ddot\theta\xi$

Rotation Matrix Derivation:

$$\omega = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \qquad \hat\omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$$

$$R(\omega) = e^{\hat\omega}$$

$$\dot R(\omega) = \dot{\hat\omega} e^{\hat\omega}$$

$$\ddot R(\omega) = \ddot{\hat\omega} e^{\hat\omega} + \dot{\hat\omega}^2 e^{\hat\omega}$$

Velocity Terms:

$$\dot x = \underbrace{\dot z}_{\text{Body velocity}} + \underbrace{\dot{\hat\omega}}_{\text{Angular velocity}} R(\omega) z' + R(\omega) \underbrace{\dot z'}_{\text{Relative velocity}}$$

Position: $x = z + R(\omega) z'$

Velocity:
$$\dot x = \dot z + \dot R(\omega) z' + R(\omega)\dot z'$$
$$= \dot z + \dot{\hat\omega} R(\omega) z' + R(\omega)\dot z'$$

Acceleration Terms:

$$\ddot x = \underbrace{\ddot z}_{\text{Body acceleration}} + \underbrace{\ddot{\hat\omega}}_{\text{Angular acceleration}} R(\omega) z' + \underbrace{\dot{\hat\omega}^2 R(\omega) z'}_{\text{Centripetal acceleration}} + \underbrace{2\dot{\hat\omega} R(\omega) \dot z'}_{\text{Coriolis acceleration}} + R(\omega) \underbrace{\ddot z'}_{\text{Relative acceleration}}$$

Acceleration:
$$\ddot x = \ddot z + \ddot R(\omega) z' + 2\dot R(\omega)\dot z' + R(\omega)\ddot z'$$
$$= \ddot z + \ddot{\hat\omega} R(\omega) z' + \dot{\hat\omega}^2 R(\omega) z' + 2\dot{\hat\omega} R(\omega)\dot z' + R(\omega)\ddot z'$$

Vector Derivatives

Function: $f : \mathcal{X} \to \mathcal{Y} \qquad y = f(x)$

Derivative: linear map that estimates Δf given Δx

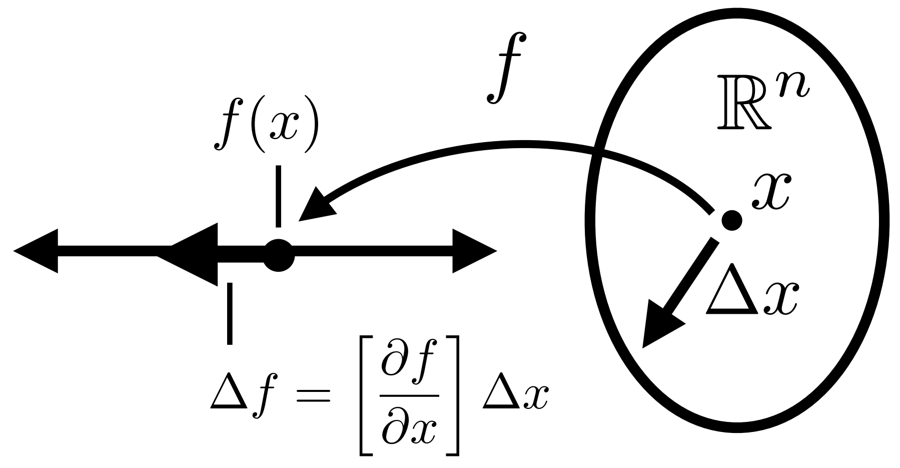
$$\Delta f = \left[\frac{\partial f}{\partial x}\right] \Delta x \qquad \boxed{\Delta f} = \frac{\boxed{\partial f}}{\cancel{\partial x}} \cancel{\Delta x}$$

Scalar Derivatives:

$$f : \mathbb{R} \to \mathbb{R} \qquad \Delta y = \Delta f = \frac{\partial f}{\partial x} \Delta x$$

Vector Derivatives: **scalar functions**

$$f : \mathbb{R}^n \to \mathbb{R}$$



Vector perturbation

$$\Delta f = \left[\frac{\partial f}{\partial x}\right] \Delta x = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}}_{\text{row vector}} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

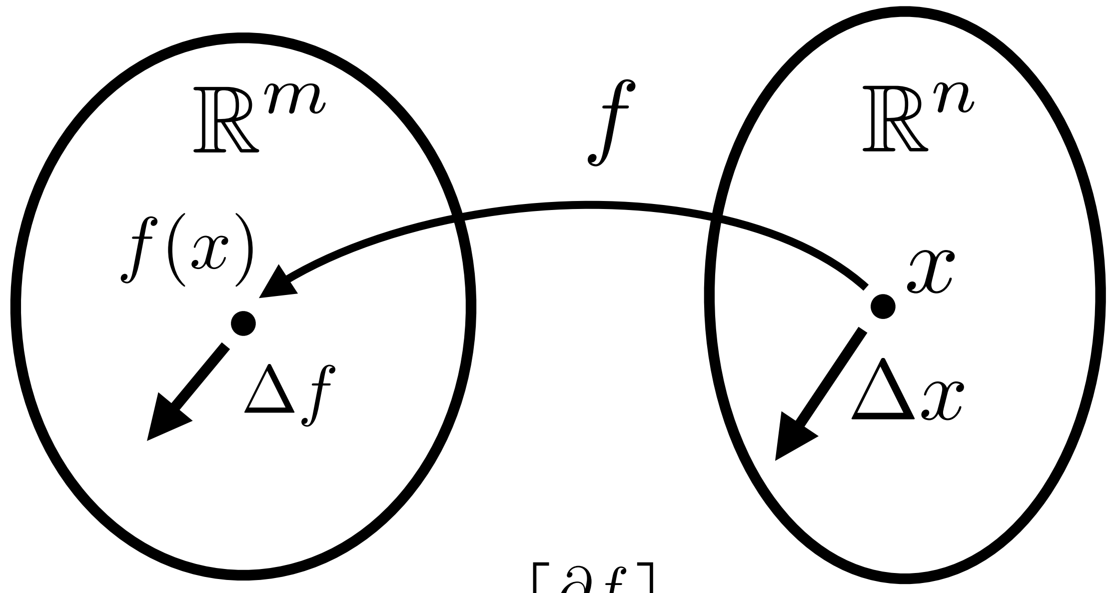
$$= \frac{\partial f}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial f}{\partial x_n} \Delta x_n$$

…partial derivative rule

Vector Derivatives:

vector functions

$$f : \mathbb{R}^n \to \mathbb{R}^m$$



$$\Delta f = \left[\frac{\partial f}{\partial x}\right]_x \Delta x$$

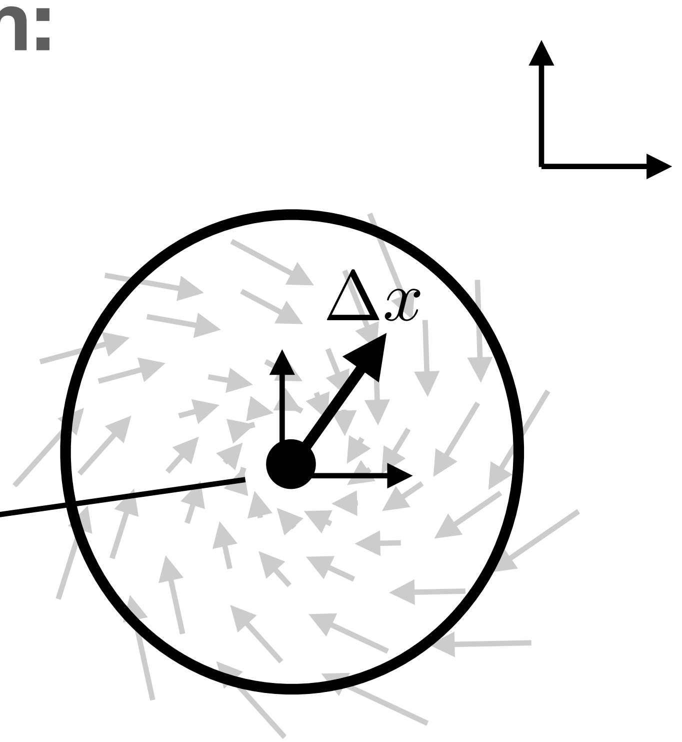
$$\Delta f = \left[\frac{\partial f}{\partial x}\right] \Delta x = \begin{array}{cc} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f}{\partial x_1} & \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \end{array} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

$$= \begin{bmatrix} | \\ \frac{\partial f}{\partial x_1} \\ | \end{bmatrix} \Delta x_1 + \cdots + \begin{bmatrix} | \\ \frac{\partial f}{\partial x_n} \\ | \end{bmatrix} \Delta x_n = \begin{bmatrix} \frac{\partial f_1}{\partial x} \Delta x \\ \vdots \\ \frac{\partial f_1}{\partial x} \Delta x \end{bmatrix}$$

Linearization

Dynamics $\dot{x} = f(x) \quad x \in \mathbb{R}^n$

...around Equilibrium:



Equilibrium:

$x: \ f(x) = 0$

Perturbed state

$$x + \Delta x(t)$$

$$\dot{x} + \dot{\Delta x} = f(x + \Delta x)$$

$$\underbrace{\cancel{\dot{x}}}_{0} + \dot{\Delta x} = \underbrace{\cancel{f(x)}}_{0} + \left[\frac{\partial f}{\partial x}\right]_x \Delta x$$

$$\dot{\Delta x} = \underbrace{\left[\frac{\partial f}{\partial x}\right]_x}_{A} \Delta x$$

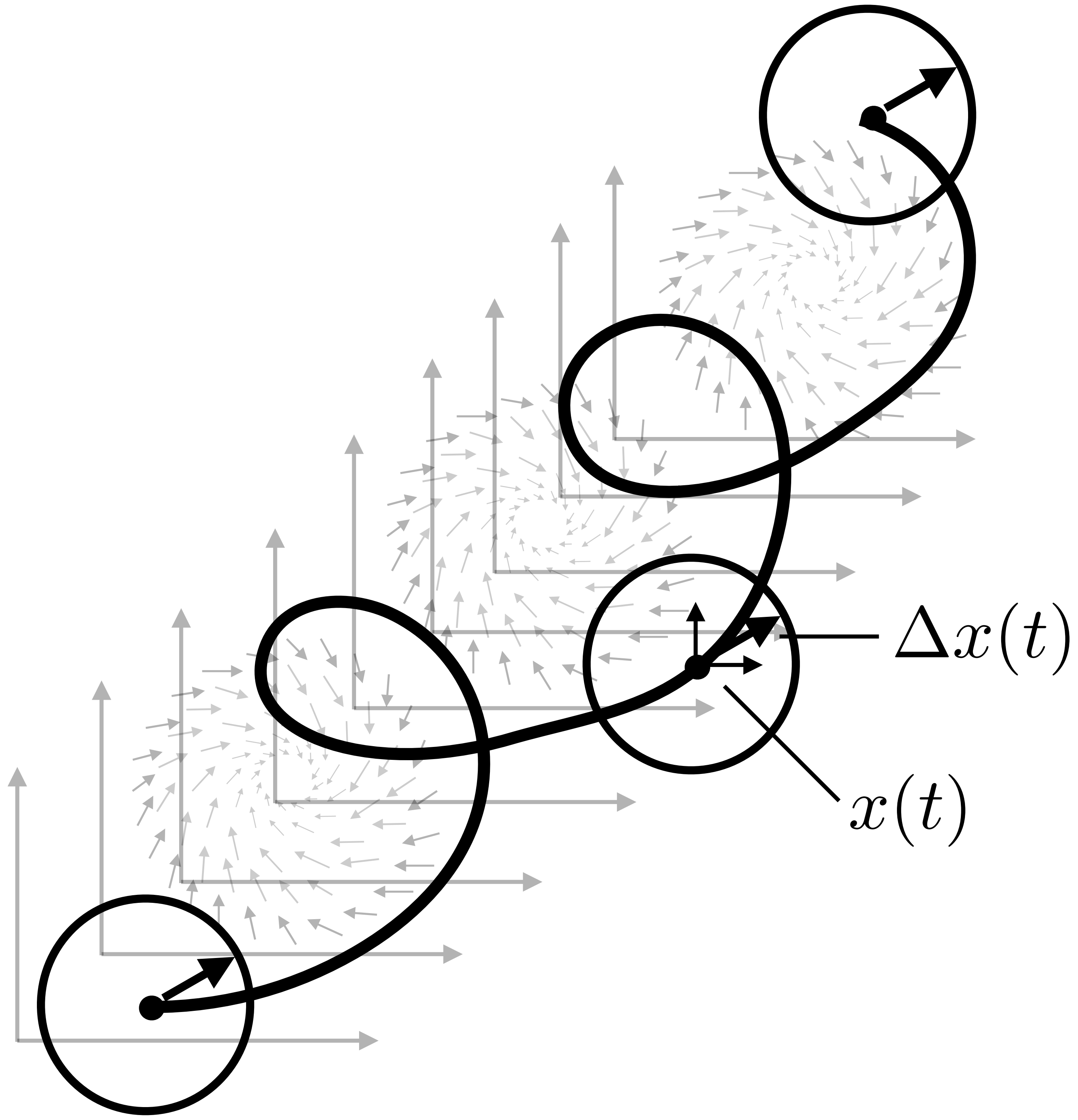
...around Trajectory

Nominal Trajectory:

$x(t): \quad \dot{x}(t) = f(x(t))$

Perturbed Trajectory:

$$x(t) + \Delta x(t)$$
$$u(t) + \Delta u(t)$$



$$\dot{x}(t) + \dot{\Delta x}(t) = f\big(x + \Delta x\big)$$

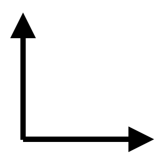
$$\underbrace{\cancel{\dot{x}(t)}}_{0} + \dot{\Delta x}(t) = \underbrace{\cancel{f(x)}}_{0} + \left[\frac{\partial f}{\partial x}\right]_{x(t)} \Delta x(t)$$

$$\dot{\Delta x}(t) = \underbrace{\left[\frac{\partial f}{\partial x}\right]_{x(t)}}_{A} \Delta x(t)$$

Linearization - with control

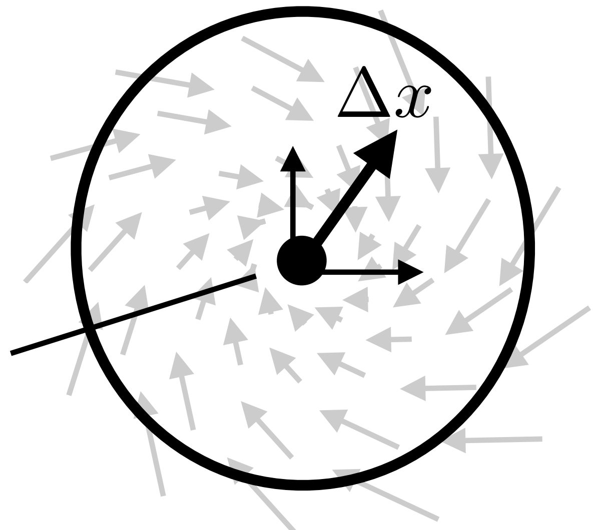
Dynamics $\dot{x} = f(x, u)$ $\quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$

...around Equilibrium:



Equilibrium:

$x, u : \quad f(x, u) = 0$



Perturbed state & control

$$x + \Delta x(t) \qquad u + \Delta u(t)$$

$$\dot{x} + \dot{\Delta x} = f(x + \Delta x, u + \Delta u)$$

$$\overset{0}{\cancel{\dot{x}(t)}} + \dot{\Delta x}(t) = \underset{0}{\cancel{f(x, u)}} + \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$

$$\dot{\Delta x} = \underbrace{\left[\frac{\partial f}{\partial x}\right]_x}_{A} \Delta x + \underbrace{\left[\frac{\partial f}{\partial u}\right]_u}_{B} \Delta u$$

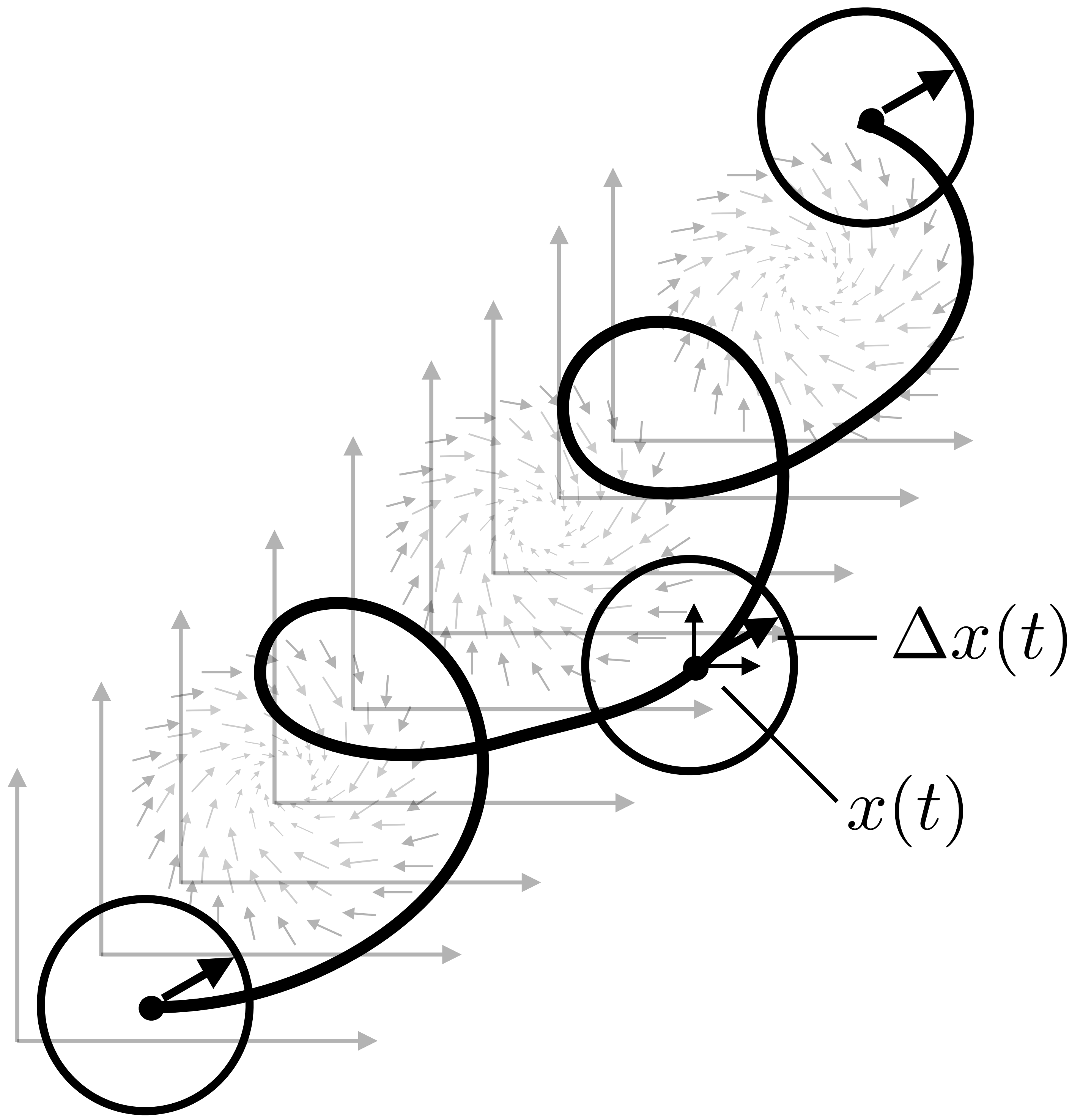
...around Trajectory

Nominal Trajectory:

$x(t) : \; \dot{x}(t) = f(x(t), u(t))$

Perturbed Trajectory:

$$x(t) + \Delta x(t)$$
$$u(t) + \Delta u(t)$$



$$\dot{x}(t) + \dot{\Delta x}(t) = f\Big(x + \Delta x, u + \Delta u\Big)$$

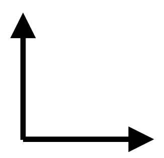
$$\cancel{\dot{x}(t)} + \dot{\Delta x}(t) = \cancel{f\Big(x(t), u(t)\Big)} + \left[\frac{\partial f}{\partial x}\right]_{x(t)} \Delta x(t) + \left[\frac{\partial f}{\partial u}\right]_{u(t)} \Delta u(t)$$

$$\dot{\Delta x}(t) = \underbrace{\left[\frac{\partial f}{\partial x}\right]_{x(t)}}_{A(t)} \Delta x(t) + \underbrace{\left[\frac{\partial f}{\partial u}\right]_{u(t)}}_{B(t)} \Delta u(t)$$

Linearization - Example

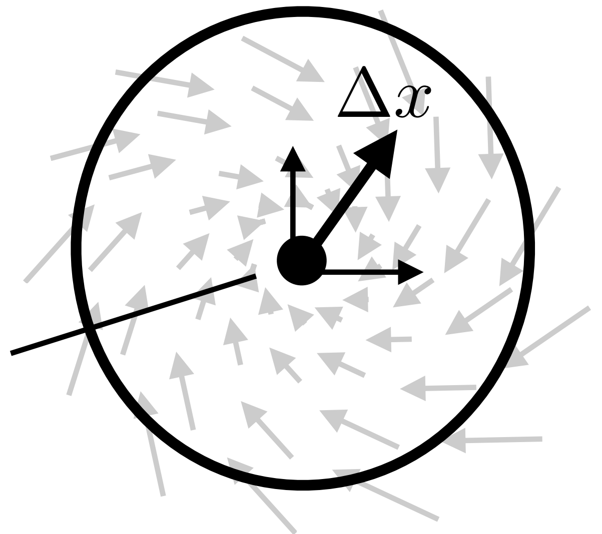
Dynamics $\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$

…around Equilibrium:



Equilibrium:

$x, u: \quad f(x, u) = 0$



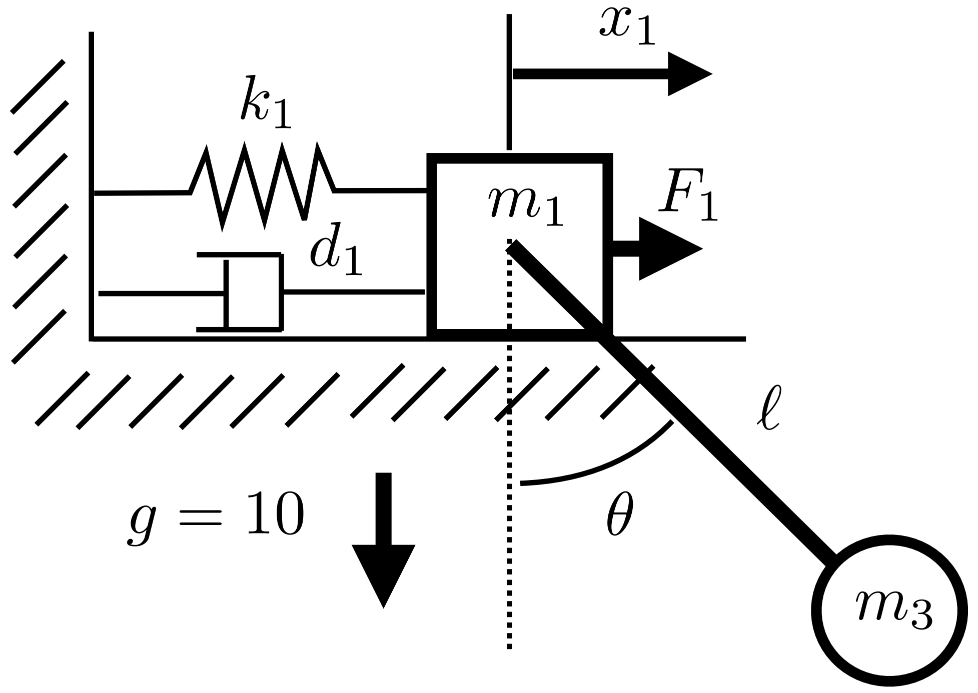
Perturbed state & control

$$x + \Delta x(t) \qquad u + \Delta u(t)$$

$$\dot{x} + \dot{\Delta x} = f(x + \Delta x, u + \Delta u)$$

$$\overset{0}{\cancel{\dot{x}(t)}} + \dot{\Delta x}(t) = \underset{0}{\cancel{f(x, u)}} + \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$

$$\dot{\Delta x} = \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$



Block dynamics

① $\quad M\ddot{x} = F_1 + F_p \sin\theta - kx - d\dot{x}$

Pendulum kinematics:

$$x_p = x + \ell \sin\theta \qquad y_p = -\ell \cos\theta$$

$$\dot{x}_p = \dot{x} + \dot{\theta}\ell \cos\theta \qquad \dot{y}_p = \dot{\theta}\ell \sin\theta$$

$$\ddot{x}_p = \ddot{x} + \ddot{\theta}\ell \cos\theta - \dot{\theta}^2\ell \sin\theta \qquad \ddot{y}_p = \ddot{\theta}\ell \sin\theta + \dot{\theta}^2\ell \cos\theta$$

Pendulum dynamics

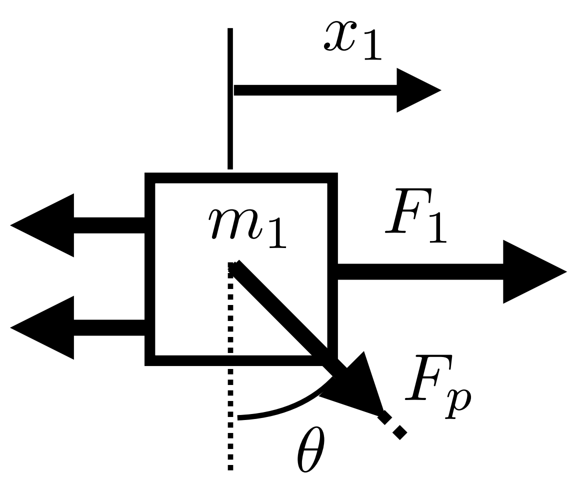
② $\quad m\ddot{x} + m\ddot{\theta}\ell \cos\theta - m\dot{\theta}^2\ell \sin\theta = -F_p \sin\theta$

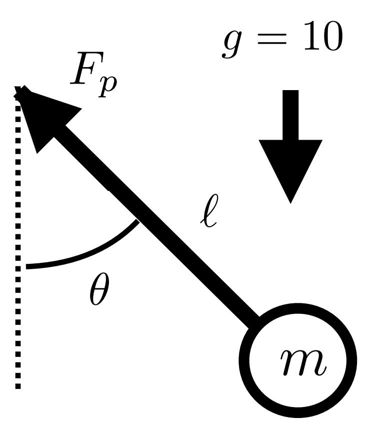
③ $\quad m\ddot{\theta}\ell \sin\theta + m\dot{\theta}^2\ell \cos\theta = F_p \cos\theta - mg$

Free Body Diagrams

$F_k = kx$

$F_d = d\dot{x}$





solving for F_p

and getting expressions for $\ddot{x}, \ddot{\theta}$

①+② $\quad M\ddot{x} + m\ddot{x} + m\ddot{\theta}\ell \cos\theta - m\dot{\theta}^2\ell \sin\theta = F_1 - kx - d\dot{x}$

$\sin\theta$ ③ $\quad m\ddot{\theta}\ell \sin^2\theta + m\dot{\theta}^2\ell \cos\theta \sin\theta = F_p \cos\theta \sin\theta - mg \sin\theta$

$+\cos\theta$ ② $\quad m\ddot{x}\cos\theta + m\ddot{\theta}\ell \cos^2\theta - m\dot{\theta}^2\ell \cos\theta \sin\theta = -F_p \cos\theta \sin\theta$

$$m\ell\ddot{\theta} + m\ddot{x}\cos\theta = -mg \sin\theta$$

$$\underbrace{\begin{bmatrix} M + m & m\ell \cos\theta \\ m\ell \cos\theta & m\ell^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m\dot{\theta}^2\ell \sin\theta + F_1 - kx - d\dot{x} \\ -mg \sin\theta \end{bmatrix}}_{\mathbf{F}}$$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

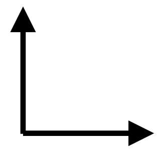
…explicitly

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{Mm\ell^2 + (m\ell \sin\theta)^2} \begin{bmatrix} m\ell^2 & -m\ell \cos\theta \\ -m\ell \cos\theta & M + m \end{bmatrix} \begin{bmatrix} m\dot{\theta}^2\ell \sin\theta + F_1 - kx - d\dot{x} \\ -mg \sin\theta \end{bmatrix}$$

Linearization - Example

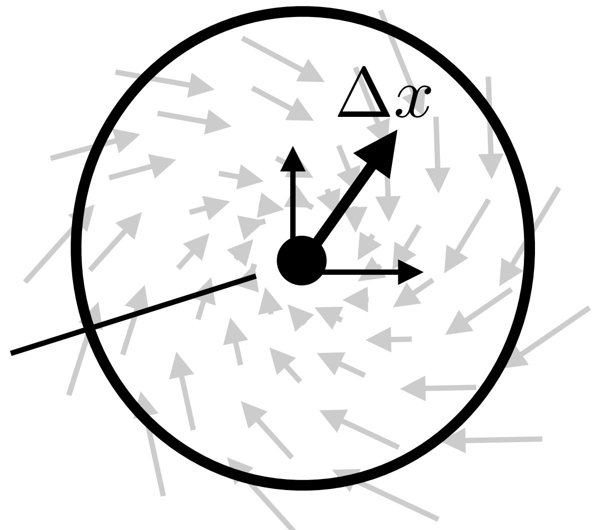
Dynamics $\dot{x} = f(x,u)$ $\quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$

...around Equilibrium:



Equilibrium:

$x, u : \quad f(x,u) = 0$



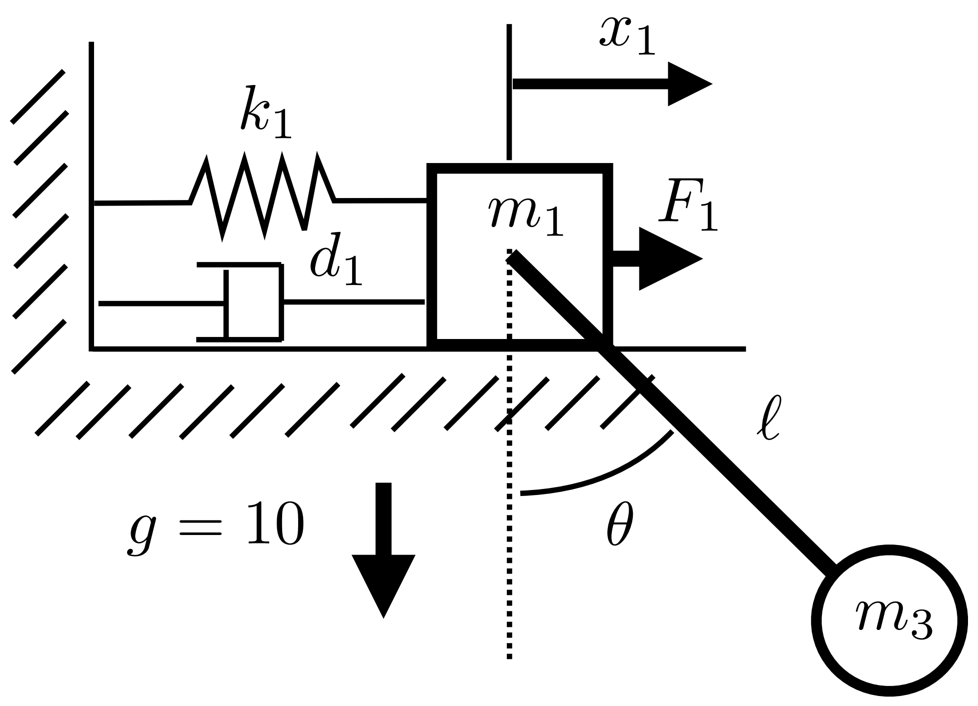
Perturbed state & control

$$x + \Delta x(t) \qquad u + \Delta u(t)$$

$$\dot{x} + \dot{\Delta x} = f(x + \Delta x, u + \Delta u)$$

$$\overset{0}{\cancel{\dot{x}(t)}} + \dot{\Delta x}(t) = \underset{0}{\cancel{f(x,u)}} + \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$

$$\dot{\Delta x} = \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$



Block dynamics

1. $M\ddot{x} = F_1 + F_p \sin\theta - kx - d\dot{x}$

Pendulum kinematics:

$x_p = x + \ell\sin\theta \qquad y_p = -\ell\cos\theta$

$\dot{x}_p = \dot{x} + \dot{\theta}\ell\cos\theta \qquad \dot{y}_p = \dot{\theta}\ell\sin\theta$

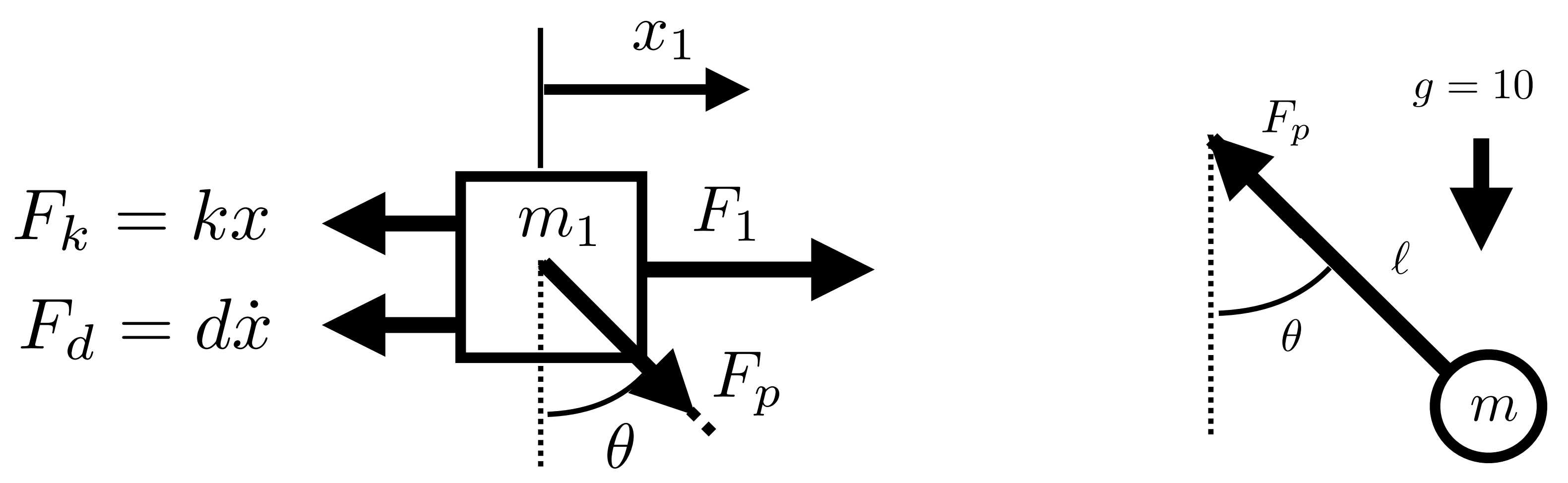
$\ddot{x}_p = \ddot{x} + \ddot{\theta}\ell\cos\theta - \dot{\theta}^2\ell\sin\theta \qquad \ddot{y}_p = \ddot{\theta}\ell\sin\theta + \dot{\theta}^2\ell\cos\theta$

Pendulum dynamics

2. $m\ddot{x} + m\ddot{\theta}\ell\cos\theta - m\dot{\theta}^2\ell\sin\theta = -F_p\sin\theta$

3. $m\ddot{\theta}\ell\sin\theta + m\dot{\theta}^2\ell\cos\theta = F_p\cos\theta - mg$

Free Body Diagrams



Full State Space Model - 2nd Order $\quad \dot{z} = f(z,u)$

$$\dot{z} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I \end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} \\ \frac{1}{Mm\ell^2 + (m\ell\sin\theta)^2}\begin{bmatrix} m\ell^2 & -m\ell\cos\theta\\ -m\ell\cos\theta & M+m\end{bmatrix}\begin{bmatrix} m\dot{\theta}^2\ell\sin\theta + F_1 - kx - d\dot{x}\\ -mg\sin\theta\end{bmatrix}\end{bmatrix}$$

Linearization

Option 1: from explicit formula…(expand out, take partial derivatives)

Option 2: using matrix forms…(more general)

> **Option 1:** from explicit formula…(expand out, take partial derivatives)
>
> *…conceptually simpler…**more chances for arithmetic errors***
>
> **Option 2:** using matrix forms…(more general)
>
> *…conceptually harder…**less chances for arithmetic errors***

$$\frac{\partial \mathbf{a}}{\partial z} = \begin{bmatrix} \mathbf{M}(z)^{-1}\frac{\partial \mathbf{M}}{\partial z_1}\mathbf{M}(z)^{-1}\mathbf{F} & \cdots & \mathbf{M}(z)^{-1}\frac{\partial \mathbf{M}}{\partial z_n}\mathbf{M}(z)^{-1}\mathbf{F}\end{bmatrix} + \mathbf{M}(z)^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

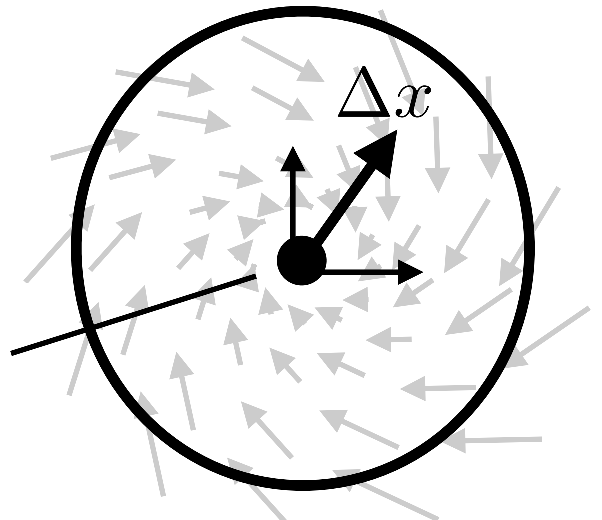
Linearization - Example

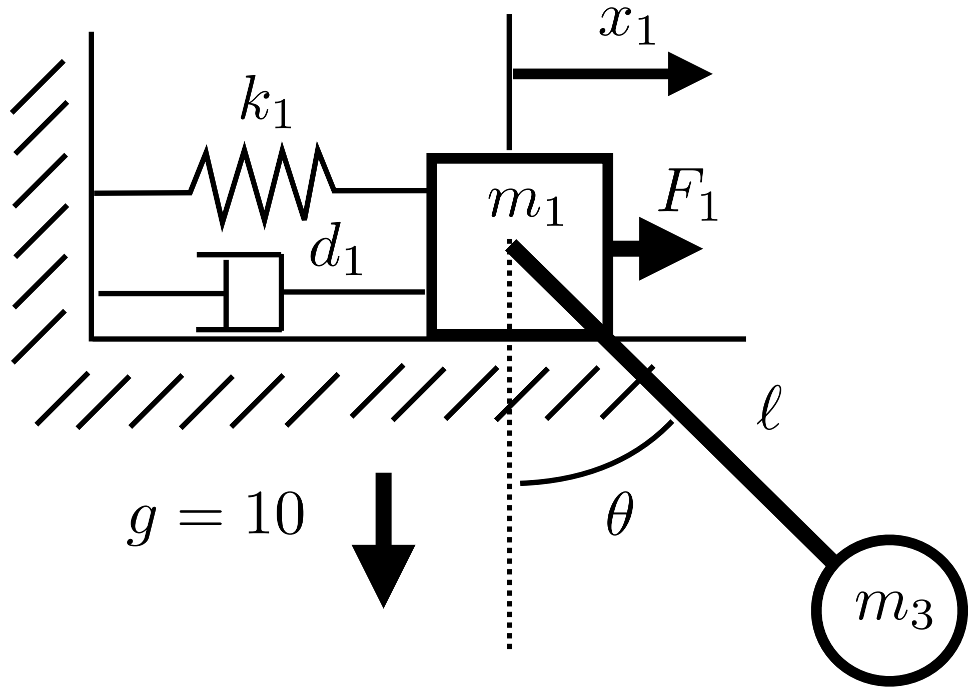
Dynamics $\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$

…around Equilibrium:

Equilibrium:

$$x, u : \quad f(x,u) = 0$$





Perturbed state & control

$$x + \Delta x(t) \qquad u + \Delta u(t)$$

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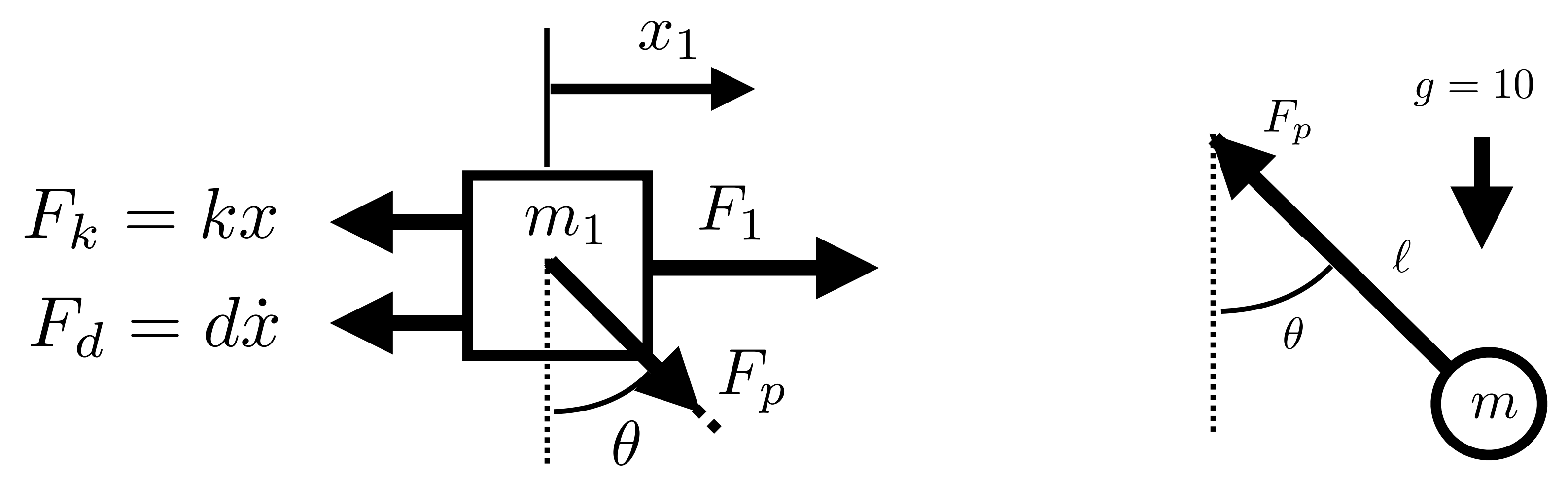
$$\overset{0}{\cancel{\dot{x}(t)}} + \dot{\Delta x}(t) = \underset{0}{\cancel{f(x,u)}} + \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$

$$\dot{\Delta x} = \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$

Free Body Diagrams



Block dynamics

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$$\ddot{x}_p = \ddot{x} + \ddot{\theta}\ell\cos\theta - \dot{\theta}^2\ell\sin\theta \qquad \ddot{y}_p = \ddot{\theta}\ell\sin\theta + \dot{\theta}^2\ell\cos\theta$$

Pendulum dynamics

$$② \quad m\ddot{x} + m\ddot{\theta}\ell\cos\theta - m\dot{\theta}^2\ell\sin\theta = -F_p\sin\theta$$

$$③ \quad m\ddot{\theta}\ell\sin\theta + m\dot{\theta}^2\ell\cos\theta = F_p\cos\theta - mg$$

Full State Space Model - 2nd Order $\dot{z} = f(z,u)$

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} \\ \frac{1}{M m \ell^2 + (m\ell\sin\theta)^2} \begin{bmatrix} m\ell^2 & -m\ell\cos\theta \\ -m\ell\cos\theta & M+m \end{bmatrix} \begin{bmatrix} m\dot{\theta}^2\ell\sin\theta + F_1 - kx - d\dot{x} \\ -mg\sin\theta \end{bmatrix} \end{bmatrix}$$

Linearization

Option 1: from explicit formula…(expand out, take partial derivatives)

Option 2: using matrix forms…(more general)

$$\mathbf{a} = \mathbf{M}(z)^{-1}\mathbf{F}(z,u)$$

Note: $\mathbf{M}\mathbf{M}^{-1} = I \quad \Rightarrow \quad \frac{\partial \mathbf{M}}{\partial z_k}\mathbf{M}^{-1} + \mathbf{M}\frac{\partial \mathbf{M}^{-1}}{\partial z_k} = 0$

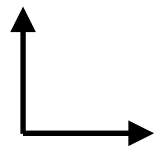
$$\Rightarrow \quad \frac{\partial \mathbf{M}^{-1}}{\partial z_k} = -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial z_k}\mathbf{M}^{-1}$$

$$\frac{\partial \mathbf{a}}{\partial z} = \begin{bmatrix} \mathbf{M}(z)^{-1}\frac{\partial \mathbf{M}}{\partial z_1}\mathbf{M}(z)^{-1}\mathbf{F} & \cdots & \mathbf{M}(z)^{-1}\frac{\partial \mathbf{M}}{\partial z_n}\mathbf{M}(z)^{-1}\mathbf{F} \end{bmatrix} + \mathbf{M}(z)^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

Linearization - Example

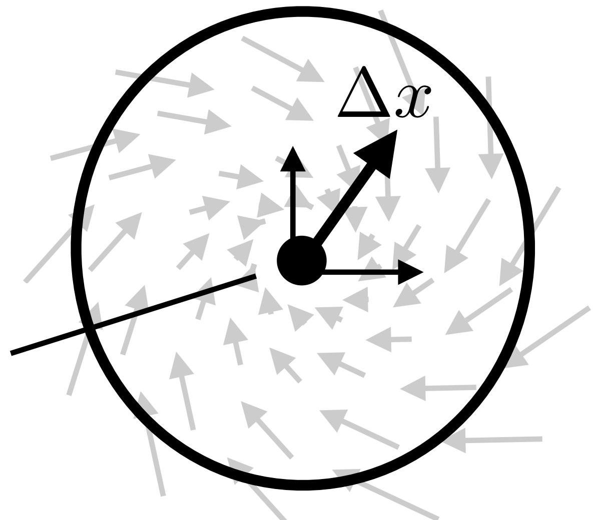
Dynamics $\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$

…around Equilibrium:



Equilibrium:

$$x, u: \quad f(x,u) = 0$$



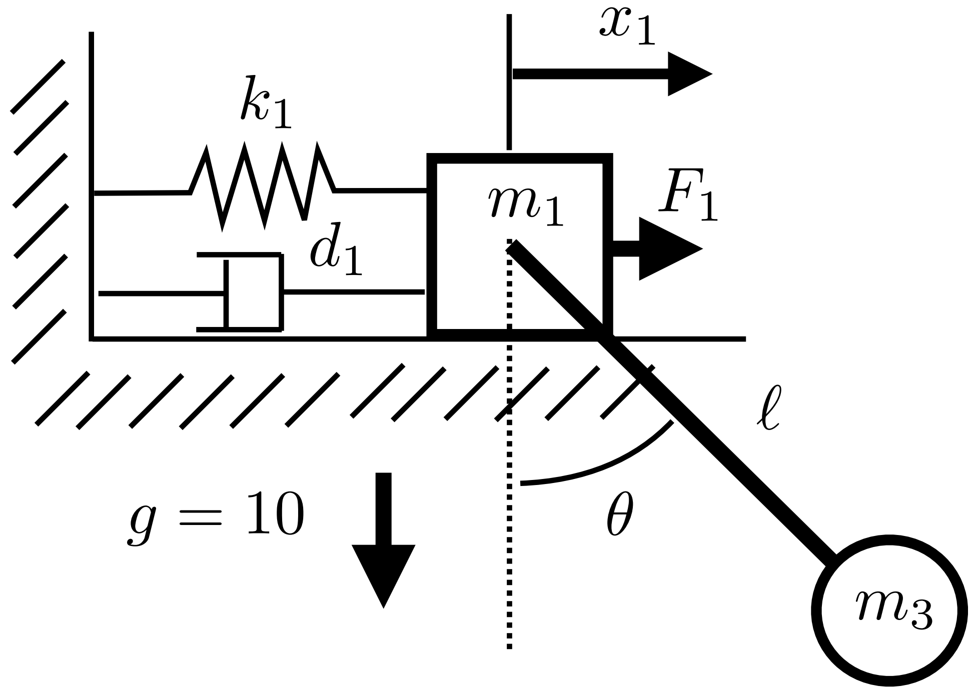
Perturbed state & control

$$x + \Delta x(t) \qquad u + \Delta u(t)$$

$$\dot{x} + \dot{\Delta x} = f(x + \Delta x, u + \Delta u)$$

$$\overset{0}{\cancel{\dot{x}(t)}} + \dot{\Delta x}(t) = \underset{0}{\cancel{f(x,u)}} + \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$

$$\dot{\Delta x} = \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$



Block dynamics

(1) $M\ddot{x} = F_1 + F_p \sin\theta - kx - d\dot{x}$

Pendulum kinematics:

$x_p = x + \ell \sin\theta \qquad y_p = -\ell\cos\theta$

$\dot{x}_p = \dot{x} + \dot{\theta}\ell\cos\theta \qquad \dot{y}_p = \dot{\theta}\ell\sin\theta$

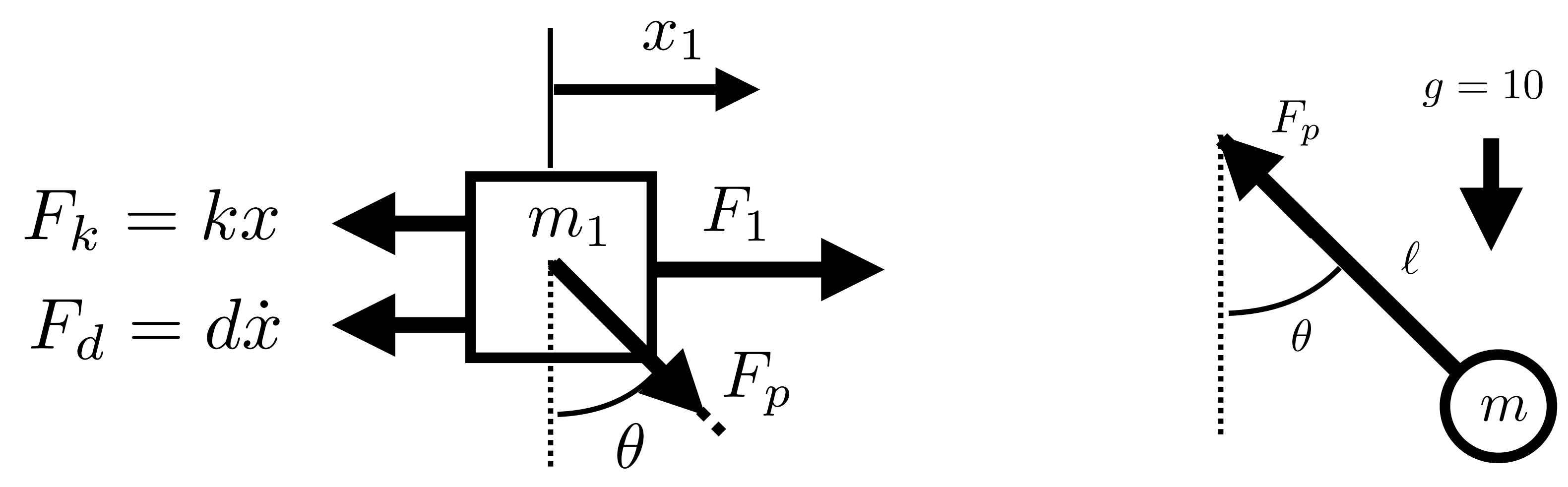
$\ddot{x}_p = \ddot{x} + \ddot{\theta}\ell\cos\theta - \dot{\theta}^2\ell\sin\theta \qquad \ddot{y}_p = \ddot{\theta}\ell\sin\theta + \dot{\theta}^2\ell\cos\theta$

Pendulum dynamics

(2) $m\ddot{x} + m\ddot{\theta}\ell\cos\theta - m\dot{\theta}^2\ell\sin\theta = -F_p\sin\theta$

(3) $m\ddot{\theta}\ell\sin\theta + m\dot{\theta}^2\ell\cos\theta = F_p\cos\theta - mg$

Free Body Diagrams



Full State Space Model - 2nd Order $\dot{z} = f(z,u)$

$$\dot{z} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I \end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} \\ \frac{1}{Mm\ell^2 + (m\ell\sin\theta)^2}\begin{bmatrix} m\ell^2 & -m\ell\cos\theta \\ -m\ell\cos\theta & M+m\end{bmatrix}\begin{bmatrix} m\dot{\theta}^2\ell\sin\theta + F_1 - kx - d\dot{x} \\ -mg\sin\theta\end{bmatrix}\end{bmatrix}$$

Linearization

Option 1: from explicit formula…(expand out, take partial derivatives)

Option 2: using matrix forms…(more general)

$$\mathbf{a} = \mathbf{M}(z)^{-1}\mathbf{F}(z,u)$$

Note: $\mathbf{M}\mathbf{M}^{-1} = I \quad \Rightarrow \quad \frac{\partial \mathbf{M}^{-1}}{\partial z_k} = -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial z_k}\mathbf{M}^{-1}$

$$\frac{\partial \mathbf{a}}{\partial u} = \mathbf{M}(z)^{-1}\frac{\partial \mathbf{F}}{\partial u}$$

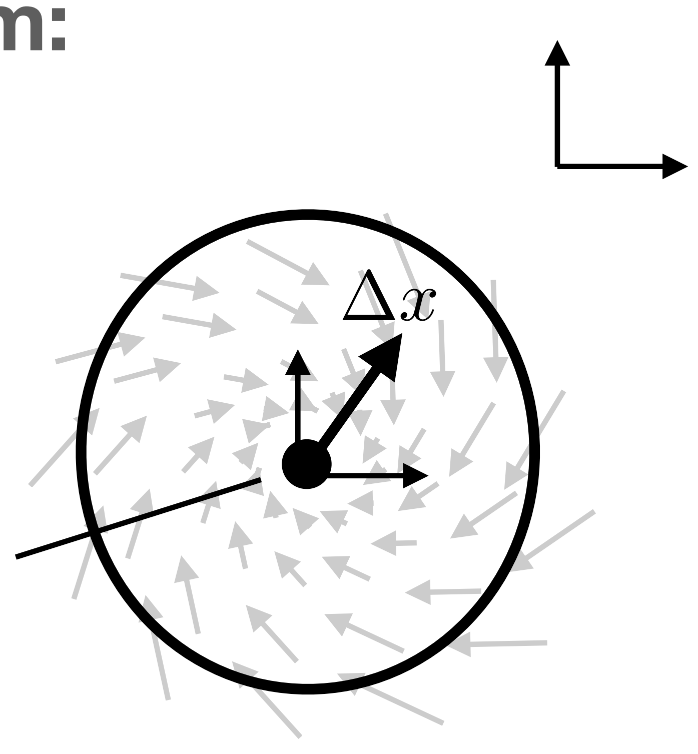
Linearization - Example

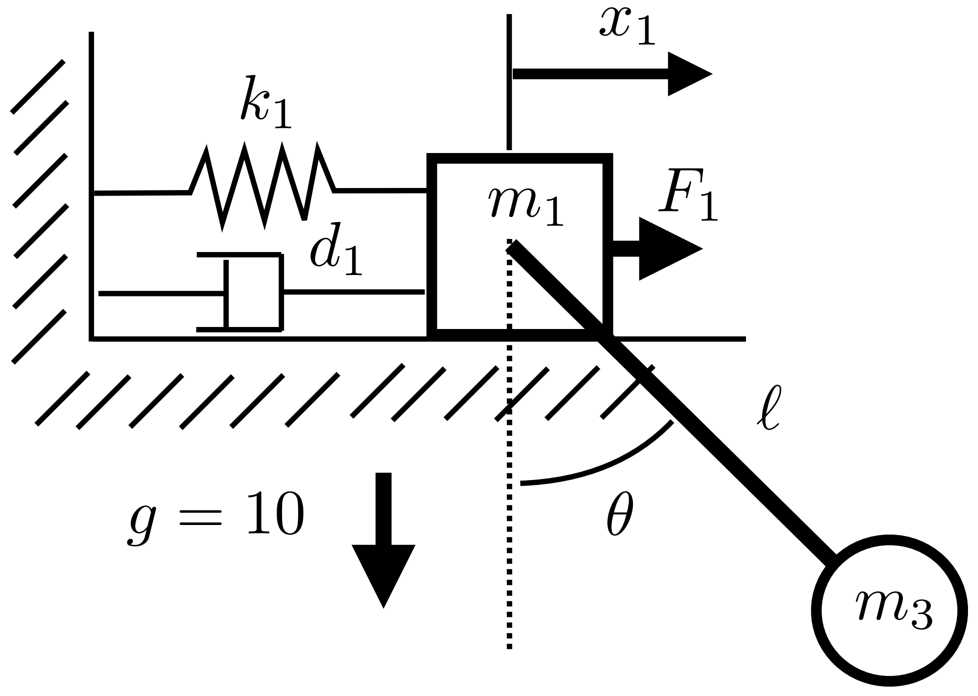
Dynamics $\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$

...around Equilibrium:

Equilibrium:

$$x,u: \quad f(x,u) = 0$$





Perturbed state & control

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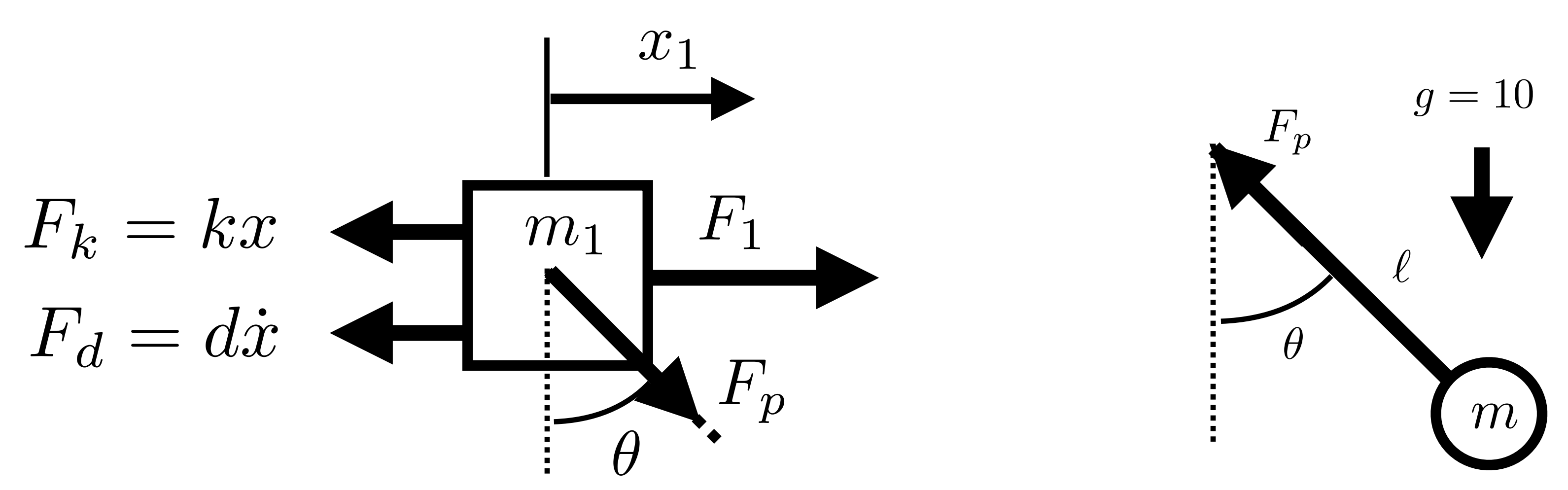
$$\dot{\Delta x} = \left[\frac{\partial f}{\partial x}\right]_x \Delta x + \left[\frac{\partial f}{\partial u}\right]_u \Delta u$$

Free Body Diagrams



Block dynamics

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Pendulum dynamics

$$② \quad m\ddot{x} + m\ddot{\theta}\ell\cos\theta - m\dot{\theta}^2\ell\sin\theta = -F_p\sin\theta$$

$$③ \quad m\ddot{\theta}\ell\sin\theta + m\dot{\theta}^2\ell\cos\theta = F_p\cos\theta - mg$$

Full State Space Model - 2nd Order $\dot{z} = f(z,u)$

$$\dot{z} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I \end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} \\ \frac{1}{Mm\ell^2 + (m\ell\sin\theta)^2}\begin{bmatrix} m\ell^2 & -m\ell\cos\theta \\ -m\ell\cos\theta & M+m\end{bmatrix}\begin{bmatrix} m\dot{\theta}^2\ell\sin\theta + F_1 - kx - d\dot{x} \\ -mg\sin\theta\end{bmatrix}\end{bmatrix}$$

Linearization $\mathbf{a} = \mathbf{M}(z)^{-1}\mathbf{F}(z,u)$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \begin{bmatrix} \mathbf{0} & -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \theta}\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{0}\end{bmatrix} + & \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}\end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix}\mathbf{0} \\ \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u}\end{bmatrix}$$

$$\mathbf{M}(z) = \begin{bmatrix} M+m & m\ell\cos\theta \\ m\ell\cos\theta & m\ell^2\end{bmatrix} \qquad \mathbf{F}(z,u) = \begin{bmatrix} m\dot{\theta}^2\ell\sin\theta + F_1 - kx - d\dot{x} \\ -mg\sin\theta\end{bmatrix}$$

$$\frac{\partial \mathbf{M}}{\partial \theta} = \begin{bmatrix} 0 & -m\ell\sin\theta \\ -m\ell\sin\theta & 0\end{bmatrix} \qquad \frac{\partial \mathbf{M}}{\partial x} = \frac{\partial \mathbf{M}}{\partial \dot{x}} = \frac{\partial \mathbf{M}}{\partial \dot{\theta}} = \mathbf{0}$$

$$\frac{\partial \mathbf{F}}{\partial z} = \begin{bmatrix} -k & m\dot{\theta}^2\ell\cos\theta & -d & 2m\dot{\theta}\ell\sin\theta \\ 0 & -mg\cos\theta & 0 & 0\end{bmatrix} \qquad \frac{\partial \mathbf{F}}{\partial u} = \begin{bmatrix}1\\0\end{bmatrix}$$