

# **Graph Structures & Matrices**

## **Algebraic Graph Theory**

**Acknowledgements:** Mehran Mesbahi  
Mathias Colbert Russelson,  
Sarah Li  
Shahriar Talebi

**Spring 2022 - Dan Calderone**

# Graphs

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

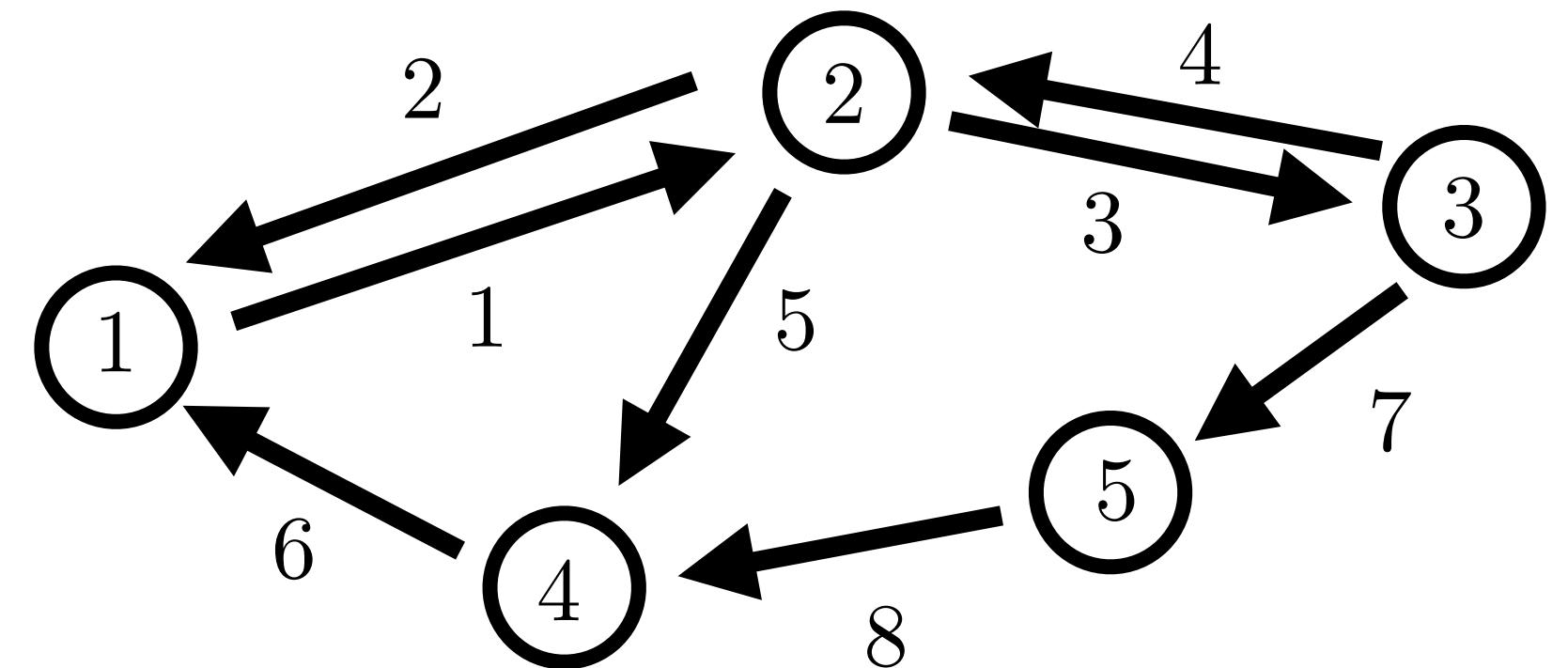
**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

... directed or undirected



# Incidence Matrix

**Graph:**

Vertices  $v \in \mathcal{V}$   
 Edges  $e \in \mathcal{E}$        $e = (v, v')$   
 ... directed or undirected

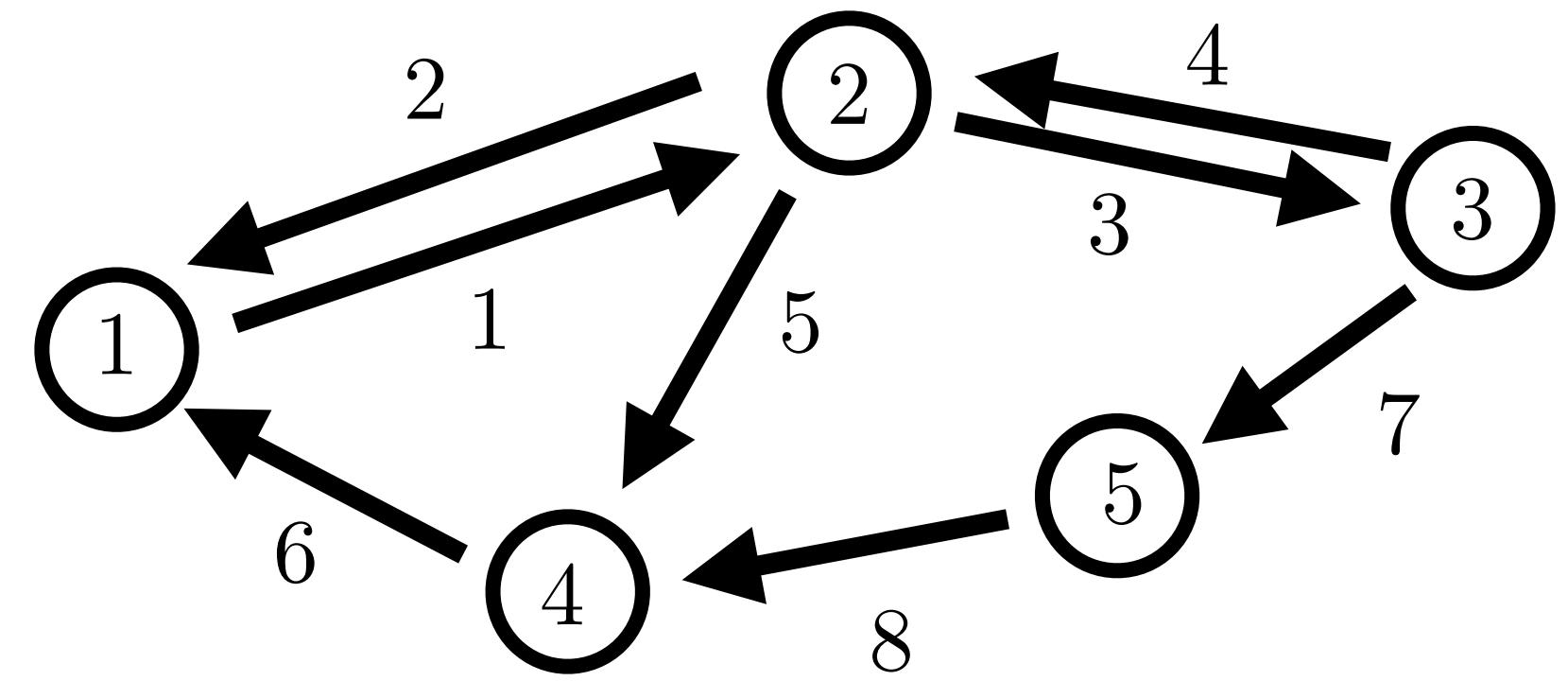
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$$E = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

← edges →

↑ vertices ↓



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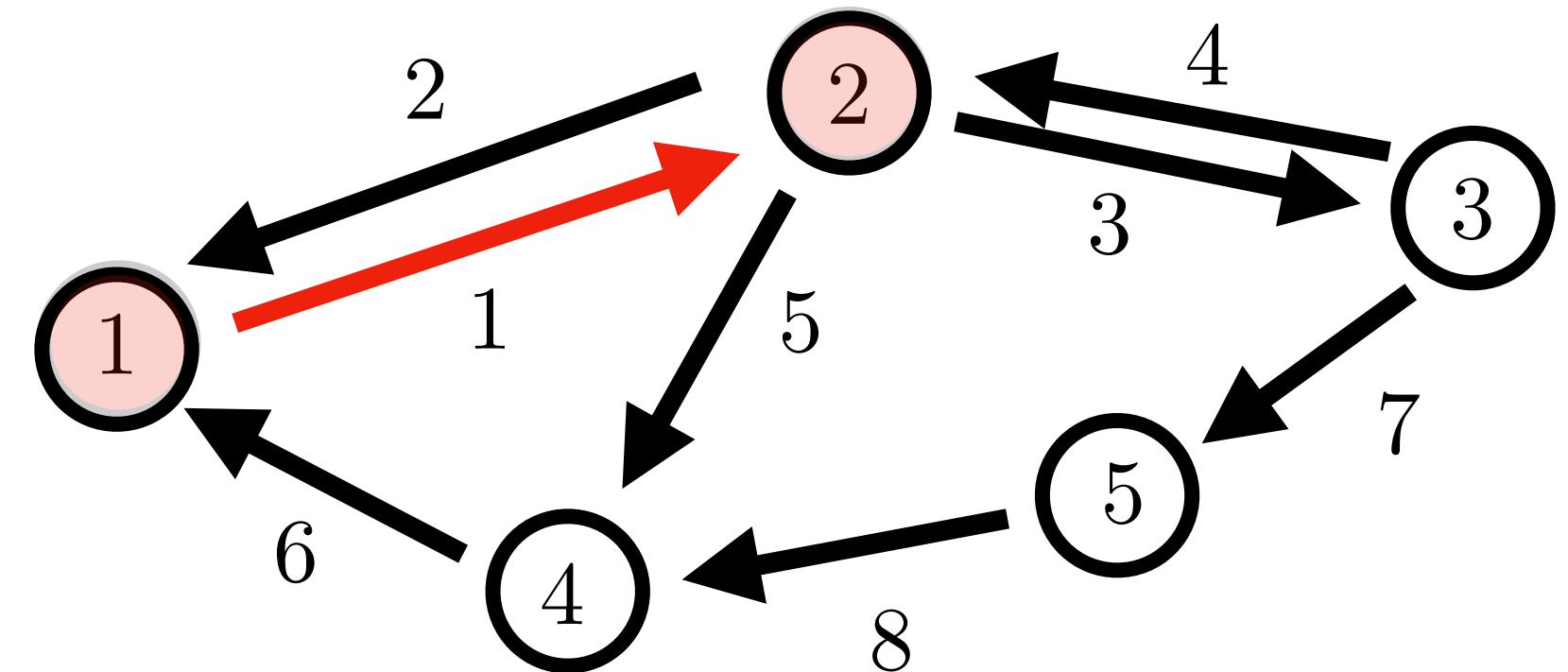
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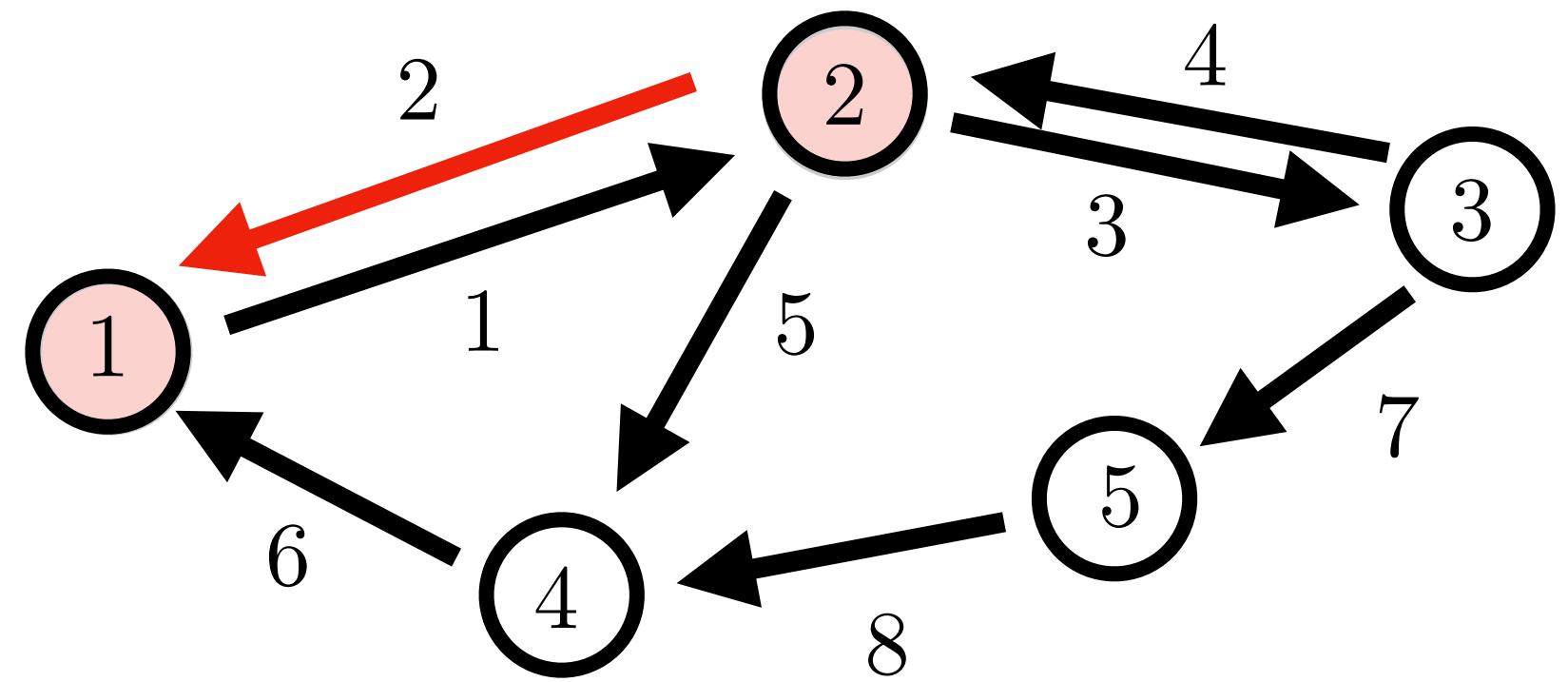
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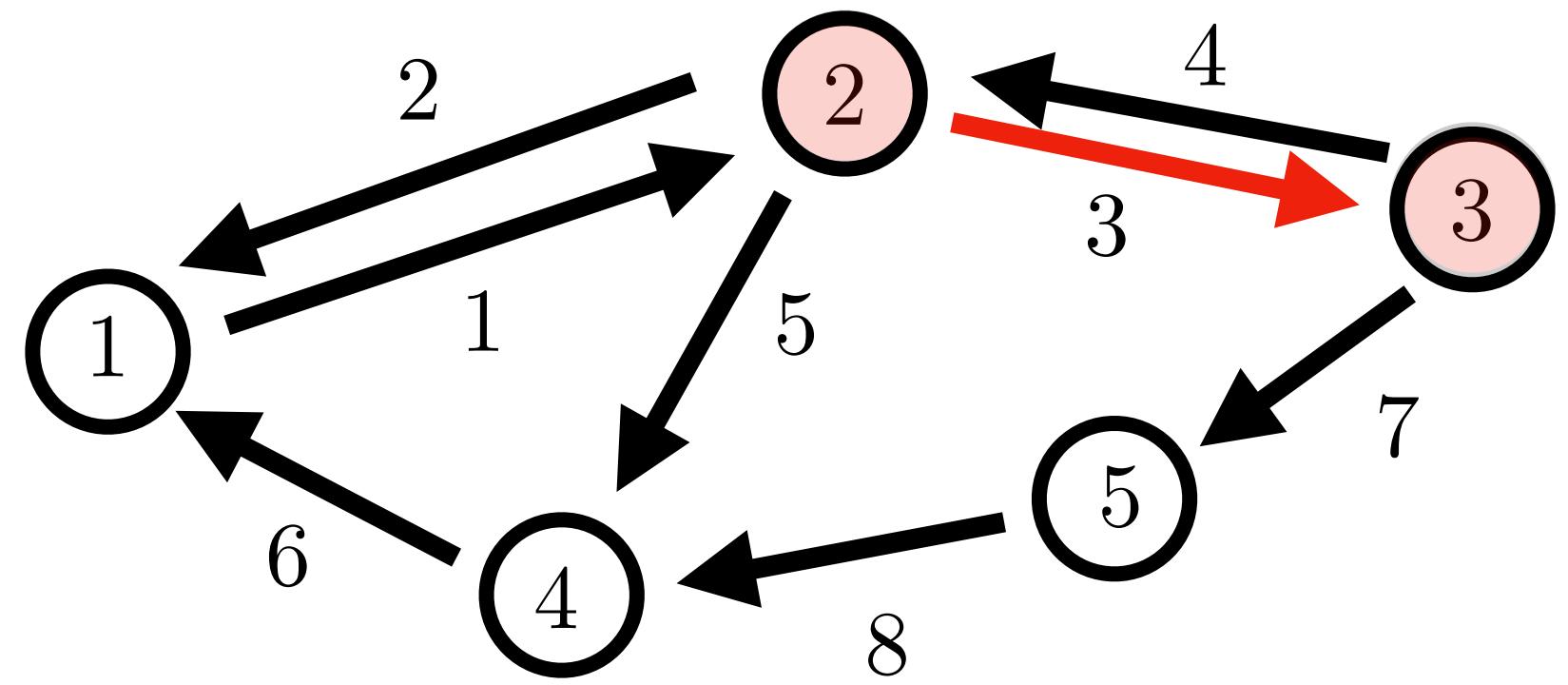
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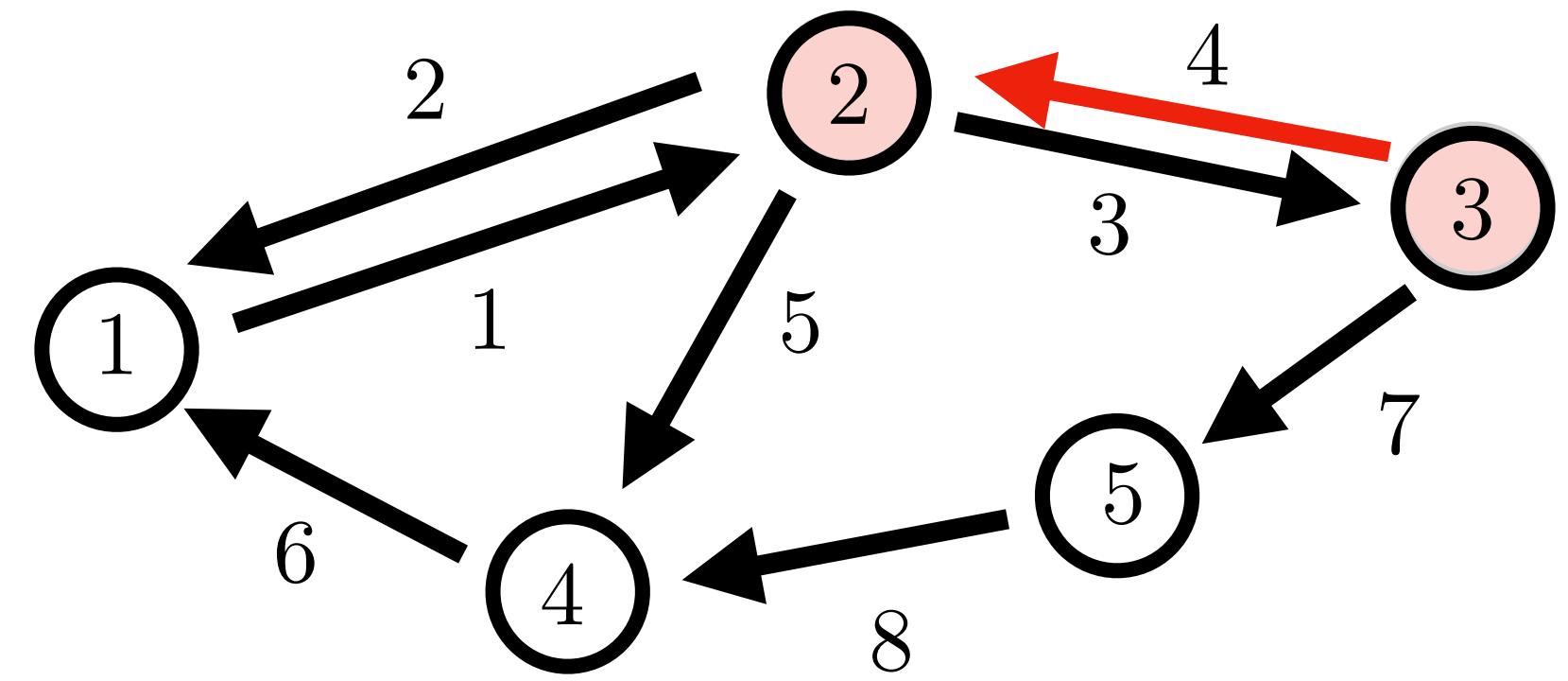
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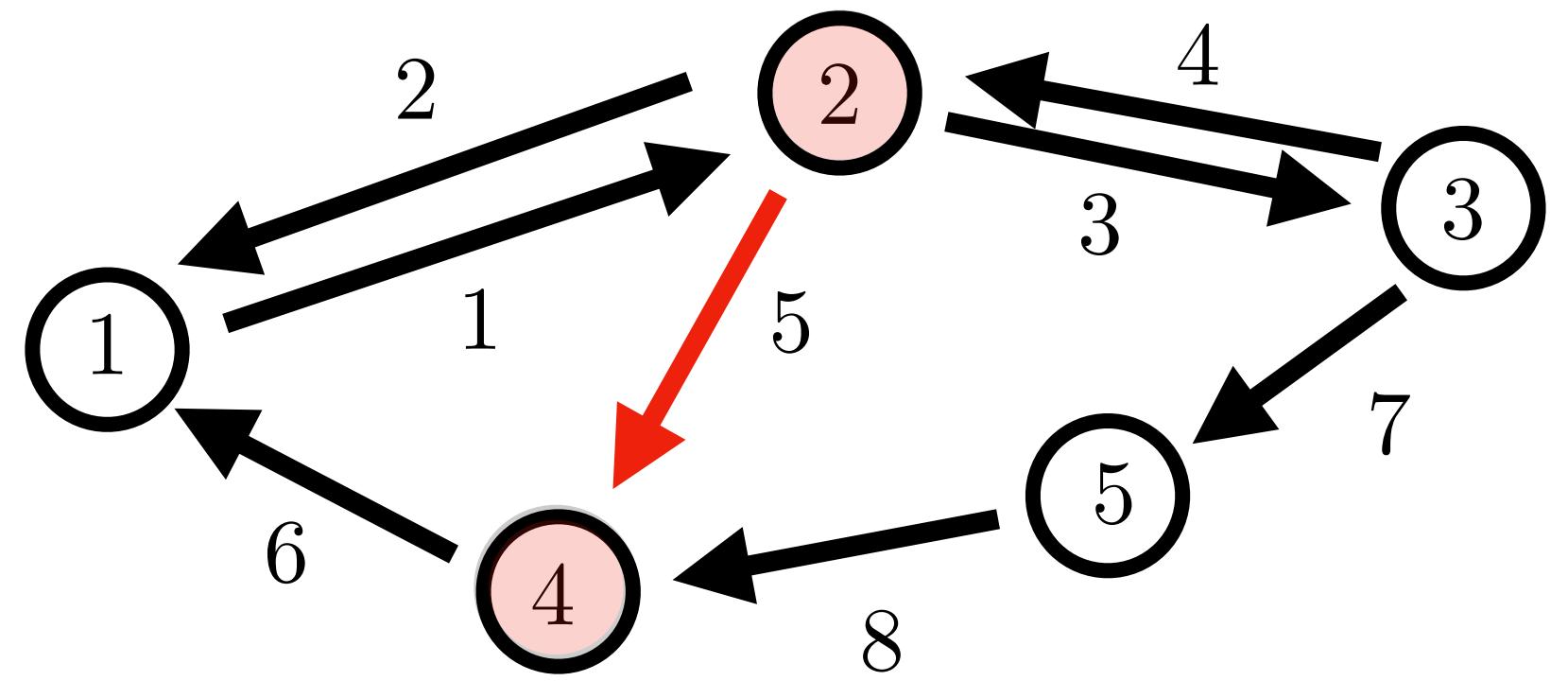
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edges ← → vertices ↑ ↓



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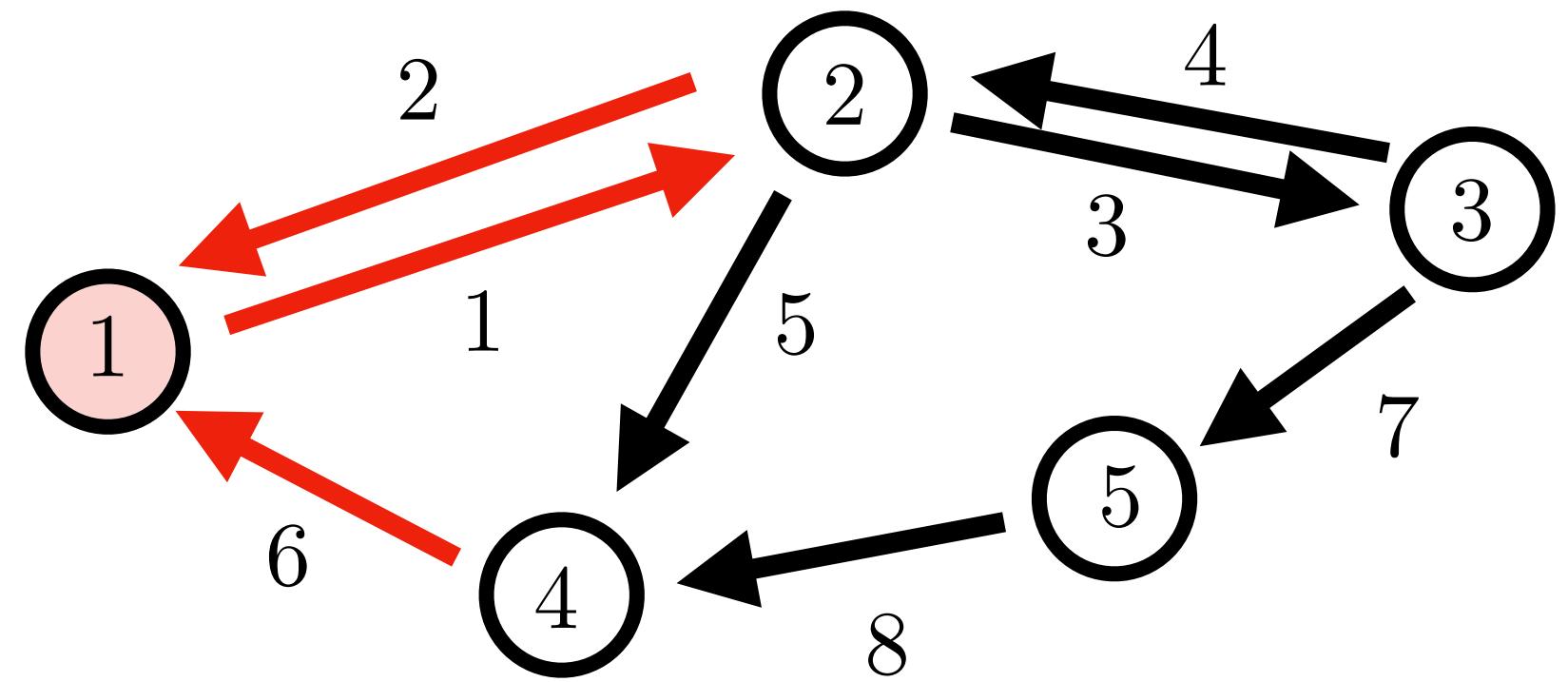
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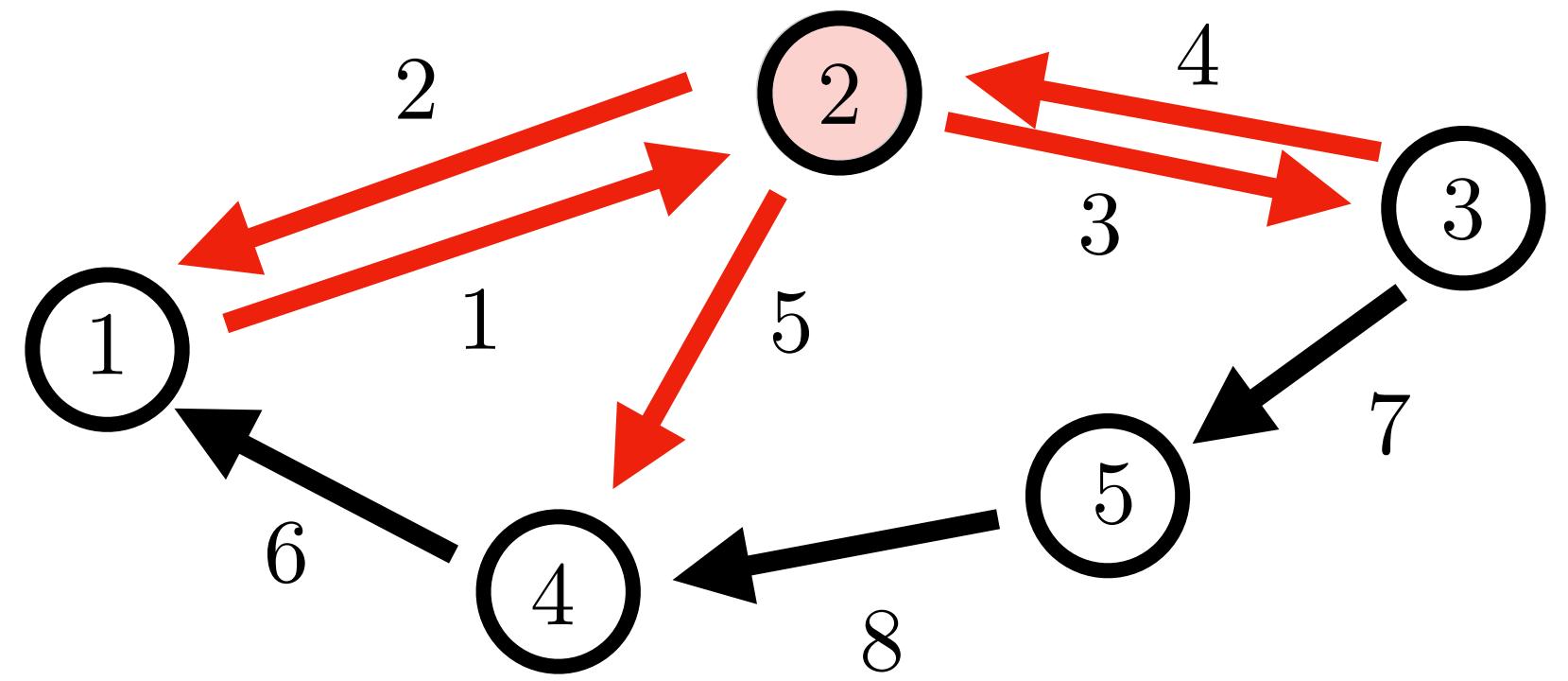
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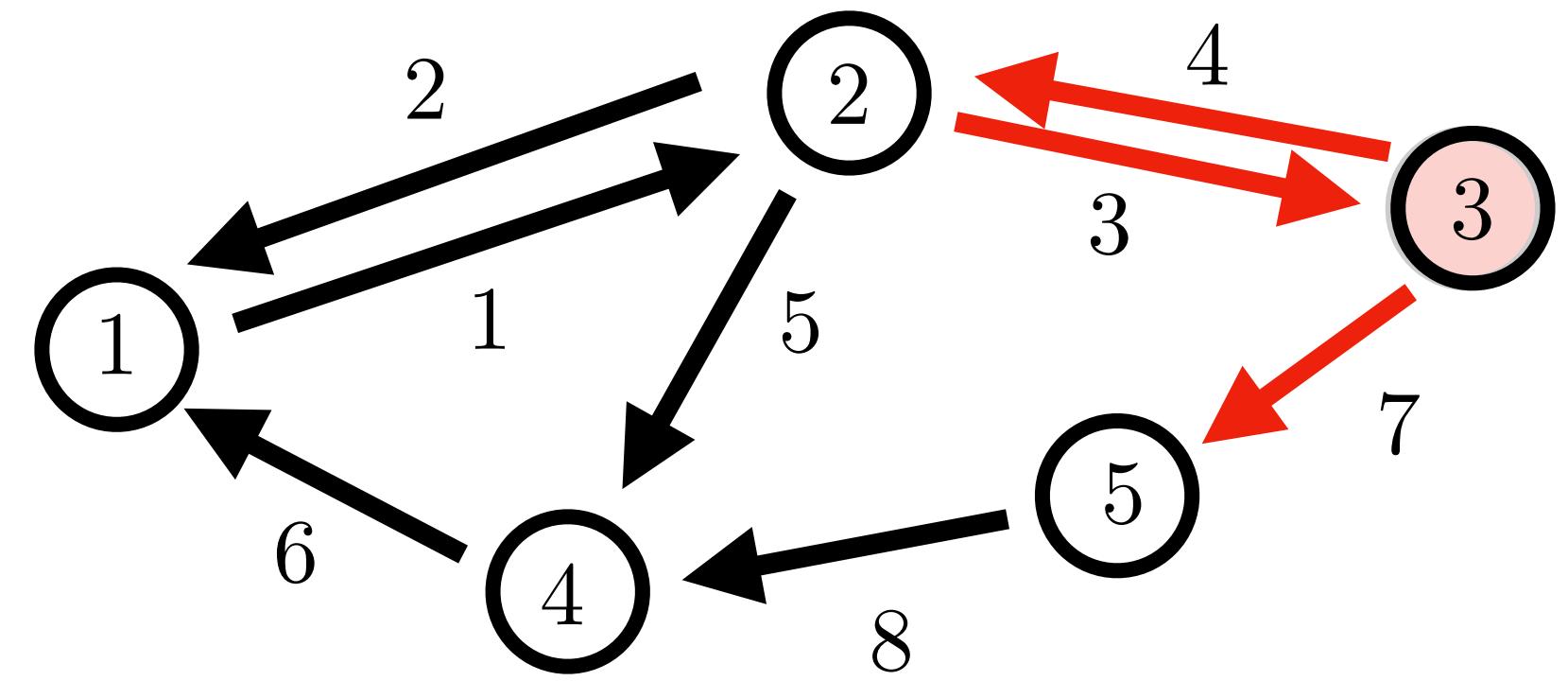
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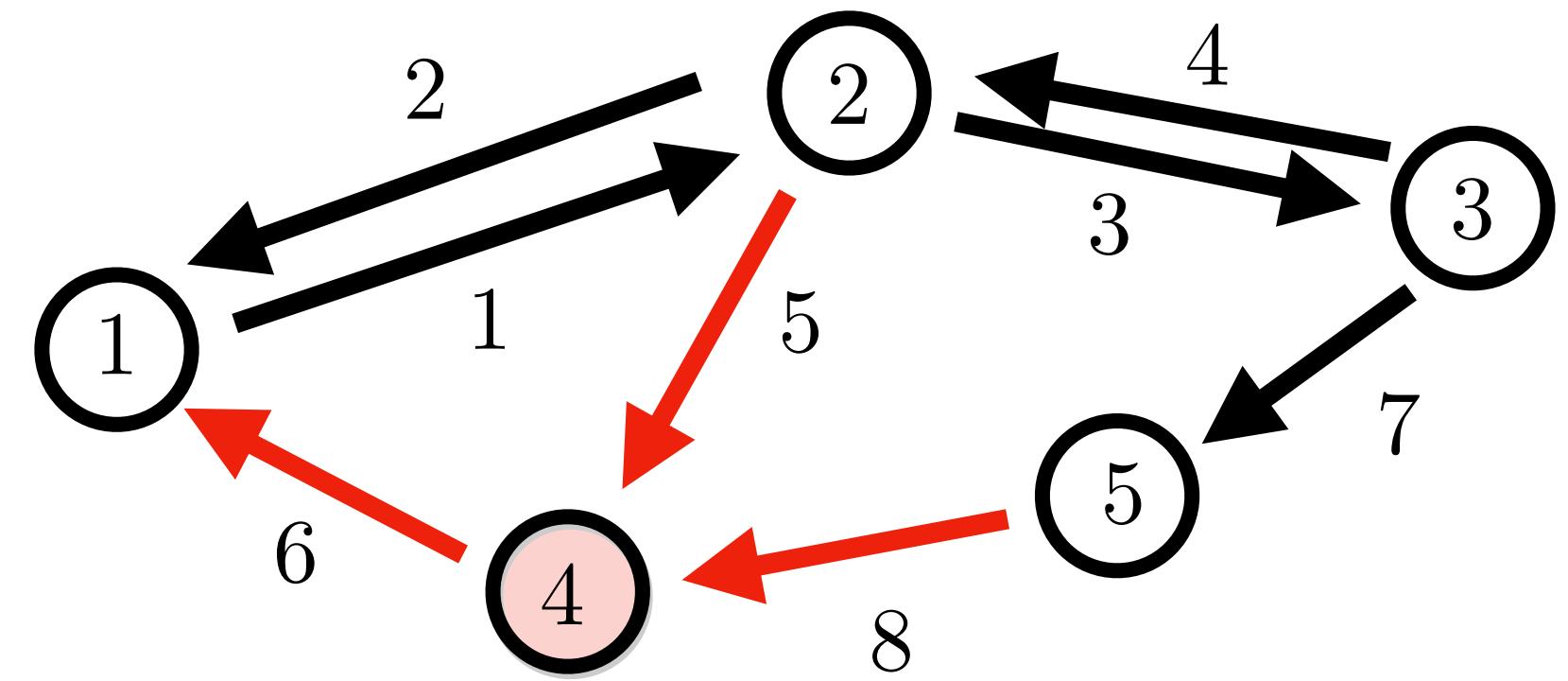
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# Incidence Matrix - Nullspaces

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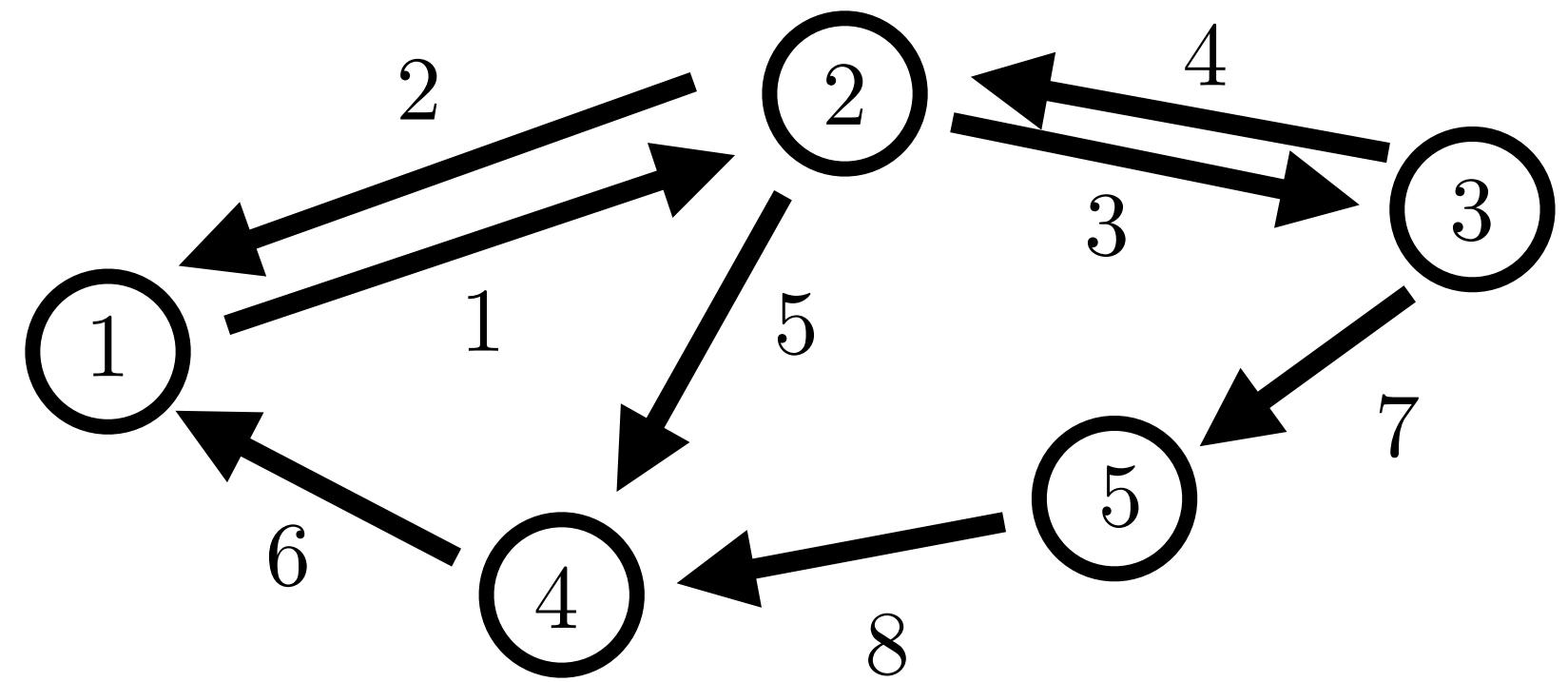
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Left  
Nullspace

$$\mathbf{1}^T E = 0$$



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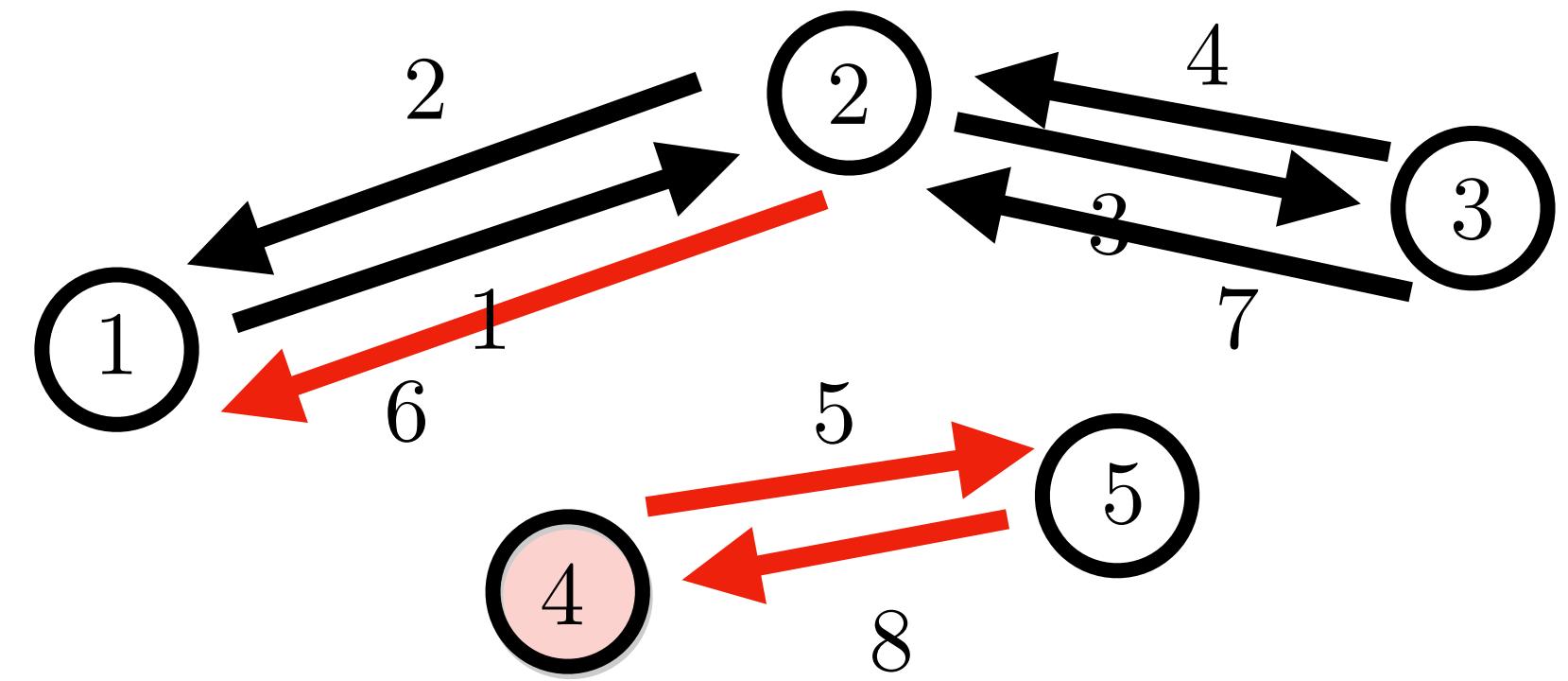
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Left  
Nullspace  
(General)

$$\begin{bmatrix} \mathbf{1}^T & 0 & \cdots & 0 \\ 0 & \mathbf{1}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} E \end{bmatrix} = \mathbf{0}$$

num = dim



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edge mass flows  $\rightarrow$

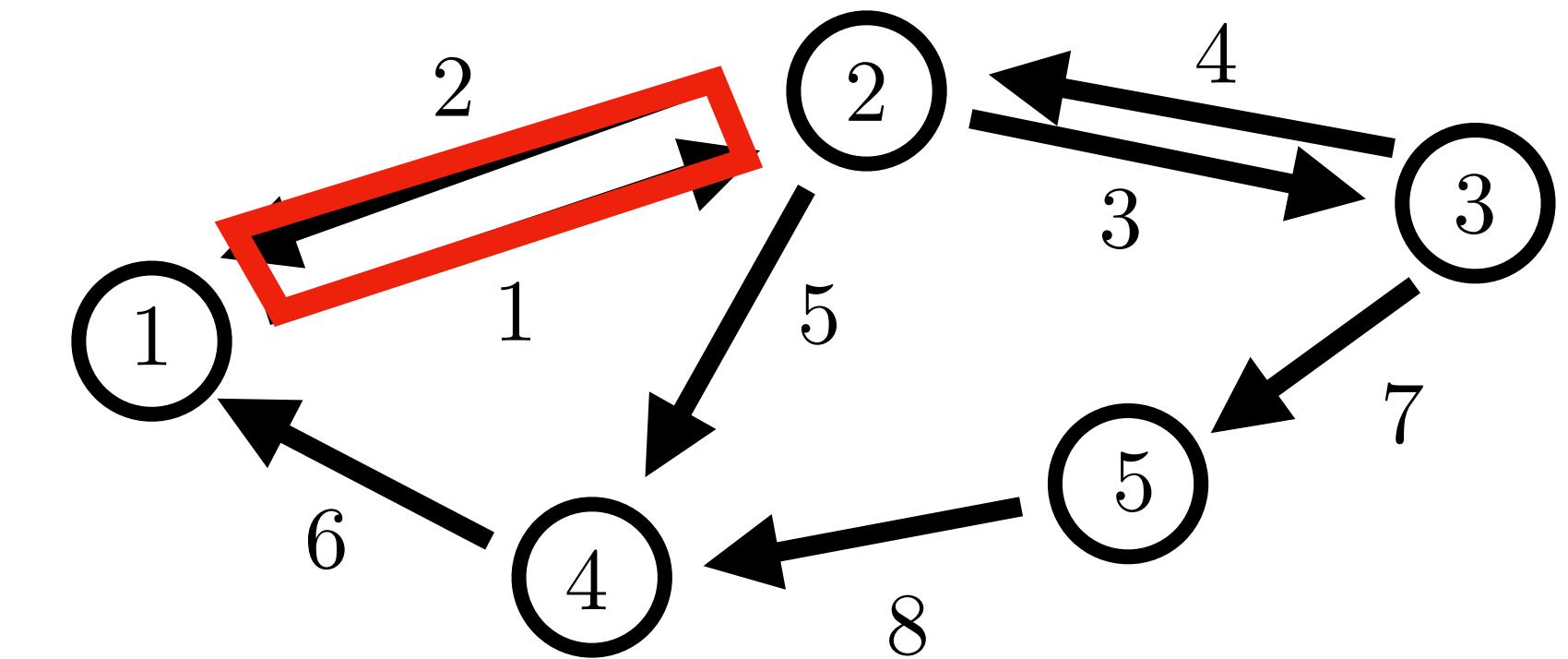
Right  
Nullspace

$$EC = 0$$

Cycle space  $\mathcal{C}$

$$\dim \mathcal{C} = |\mathcal{E}| - |\mathcal{V}| + 1$$

$$\text{Ex: } 8 - 5 + 1 = 4$$



Cycle indicator (basis) matrix:

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Sign indicates  
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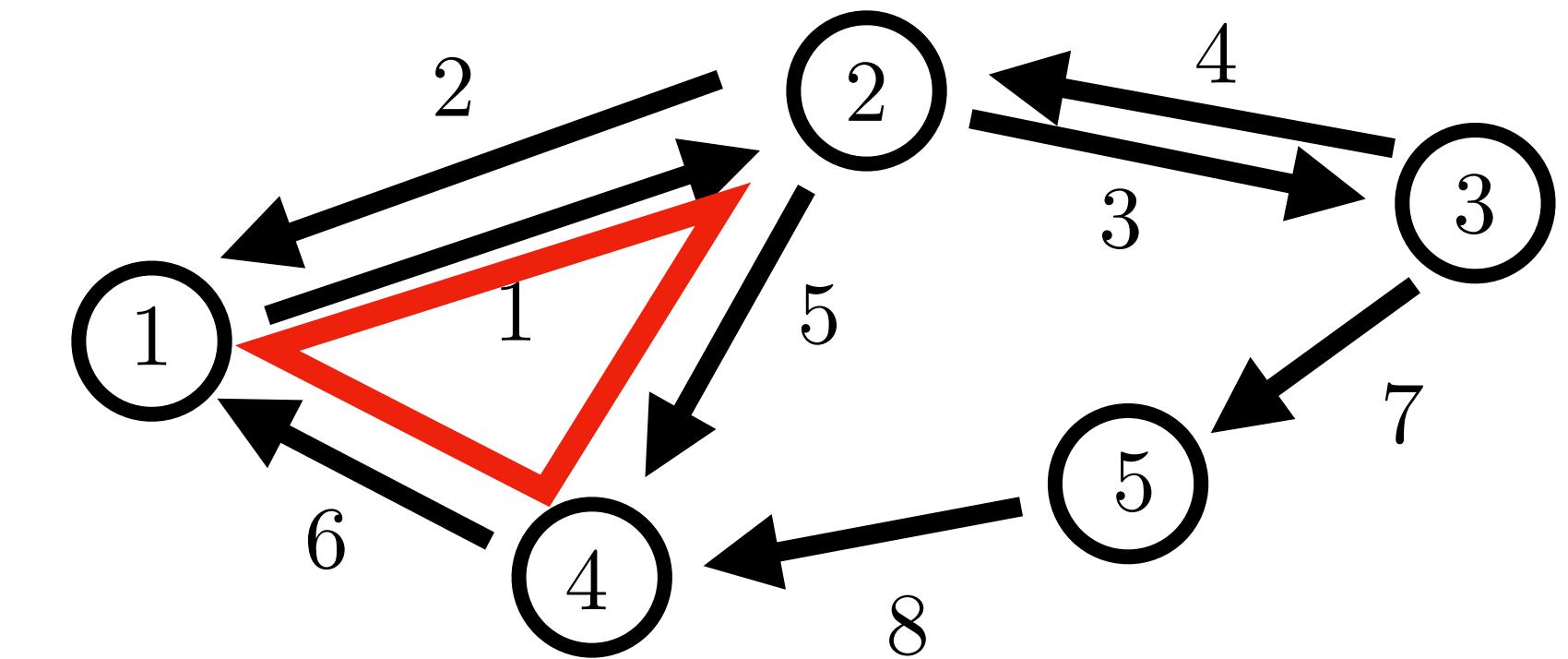
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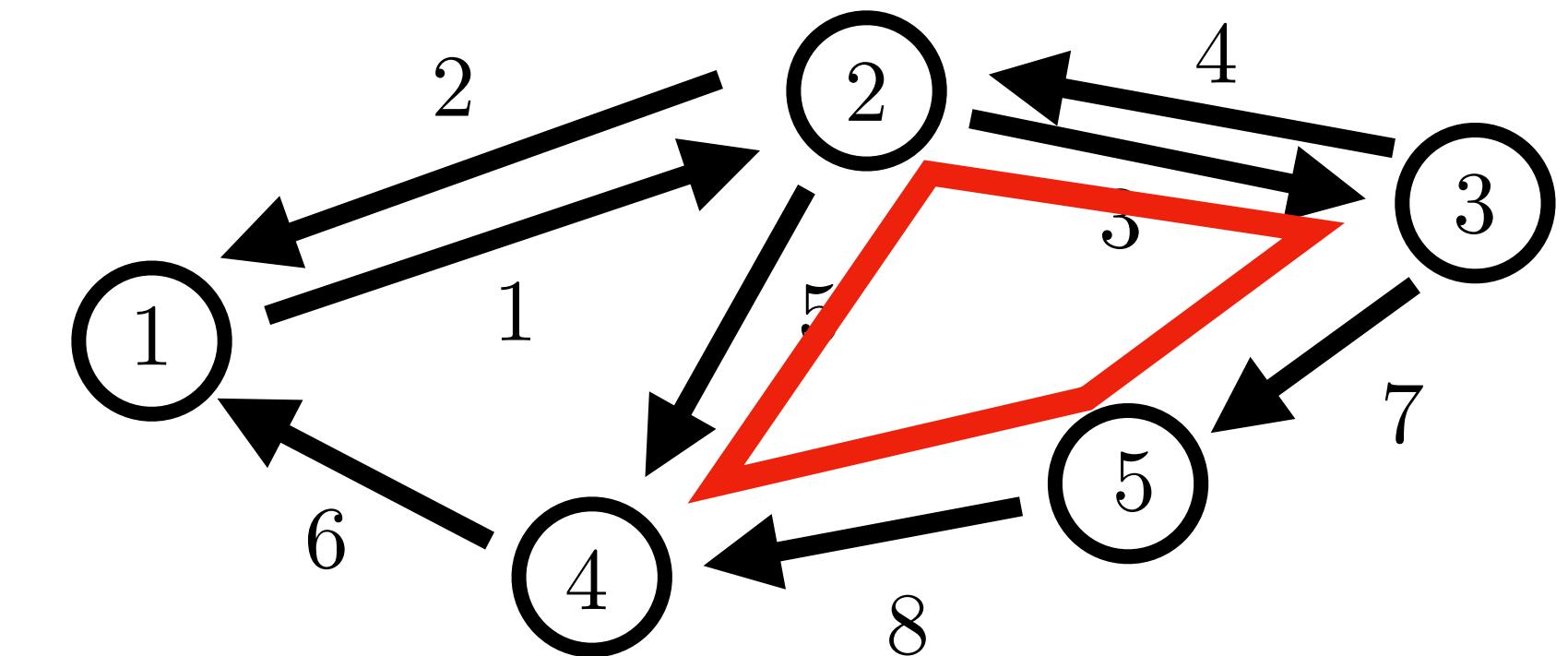
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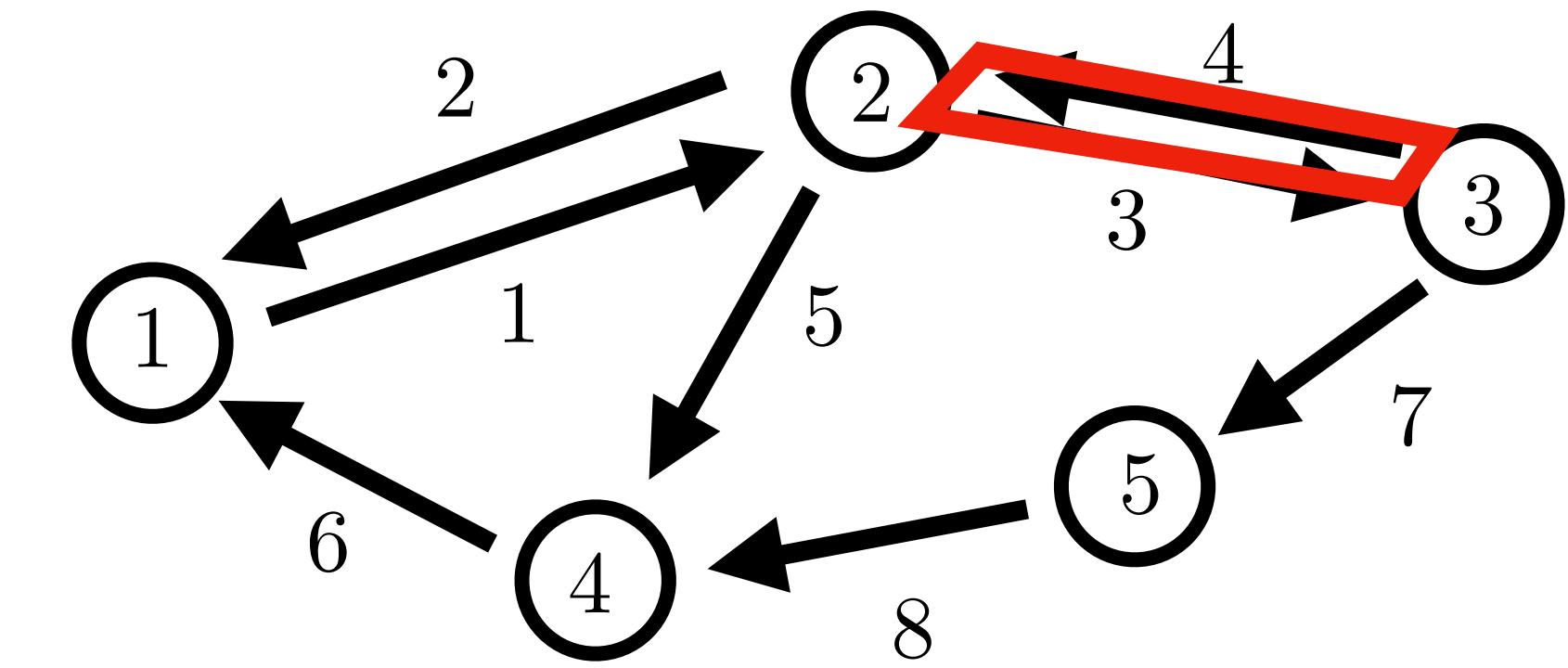
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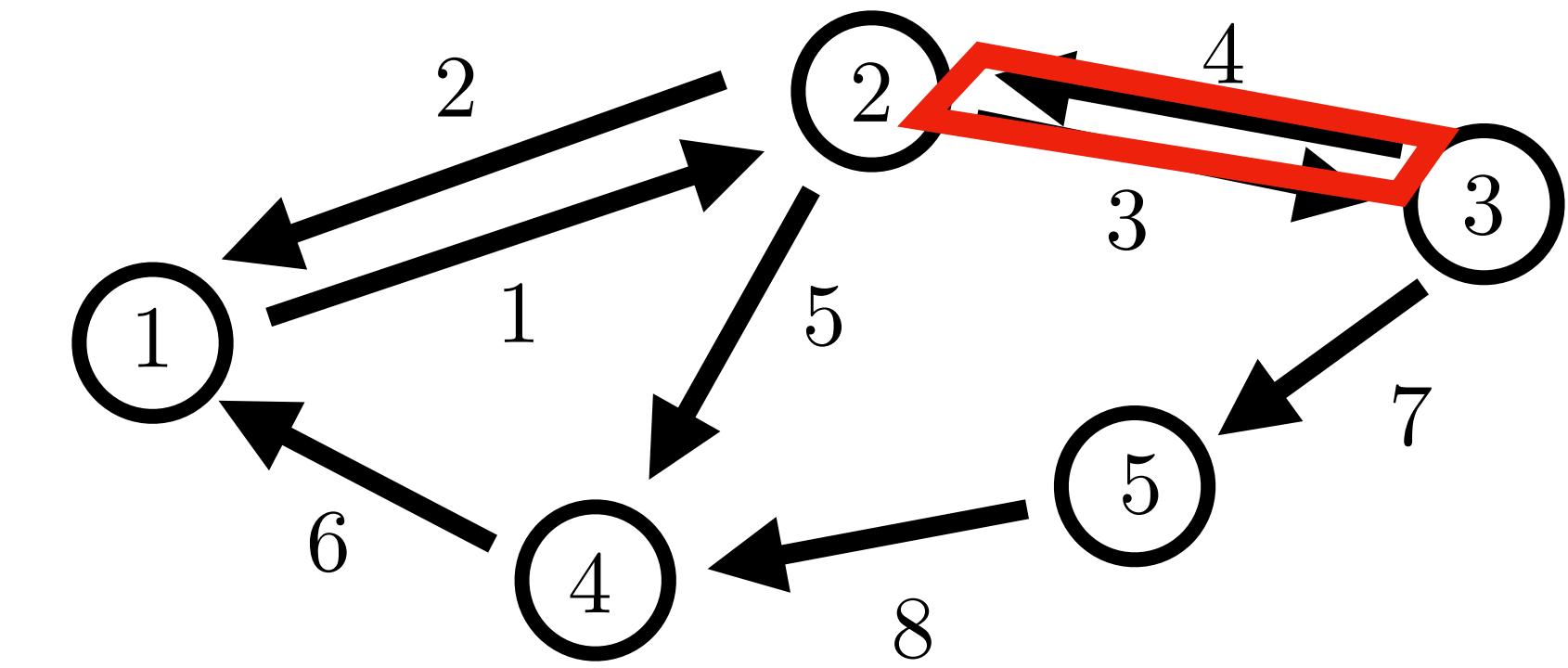
Subspace  
Constraint

$$Ex = 0 \quad \Rightarrow \quad x = Cz$$

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$$e \in \mathcal{E} \quad e = (v, v')$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows →

Affine  
Constraint

$$Ex = S \Rightarrow x = \bar{x} + Cz$$

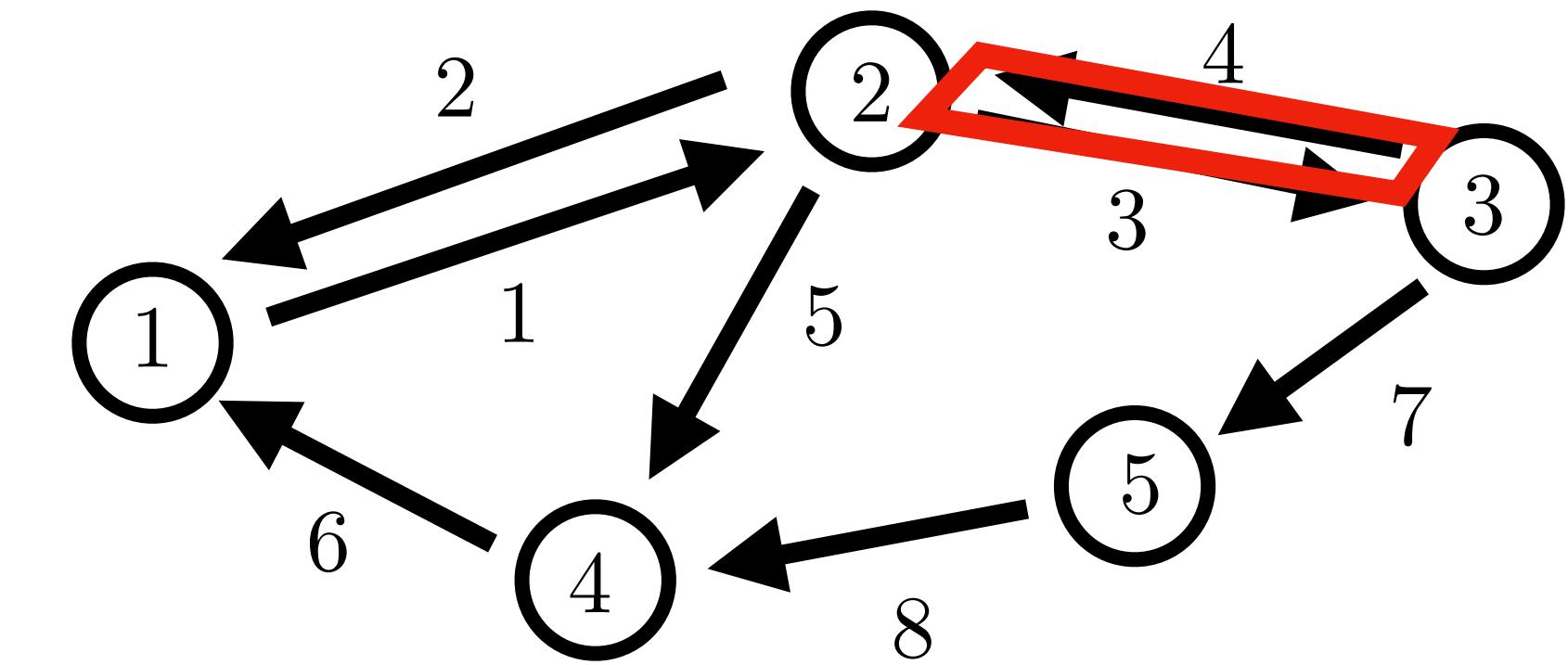
Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$

Specific  
Solution

Cyclic  
Flow

Columns = basis for  $\mathcal{C}$



Cycle indicator (basis) matrix:

$$[C]_{ec} = \begin{cases} 1 & ; \text{ if } e \text{ flows with cycle } c \\ -1 & ; \text{ if } e \text{ flows against cycle } c \\ 0 & ; e \text{ not in cycle } c \end{cases}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sign indicates  
if cycle goes  
with or against  
edge direction

# Incidence Matrix - Geometry

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

$$e = (v, v')$$

... directed or undirected

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edge mass flows →

Affine  
Constraint

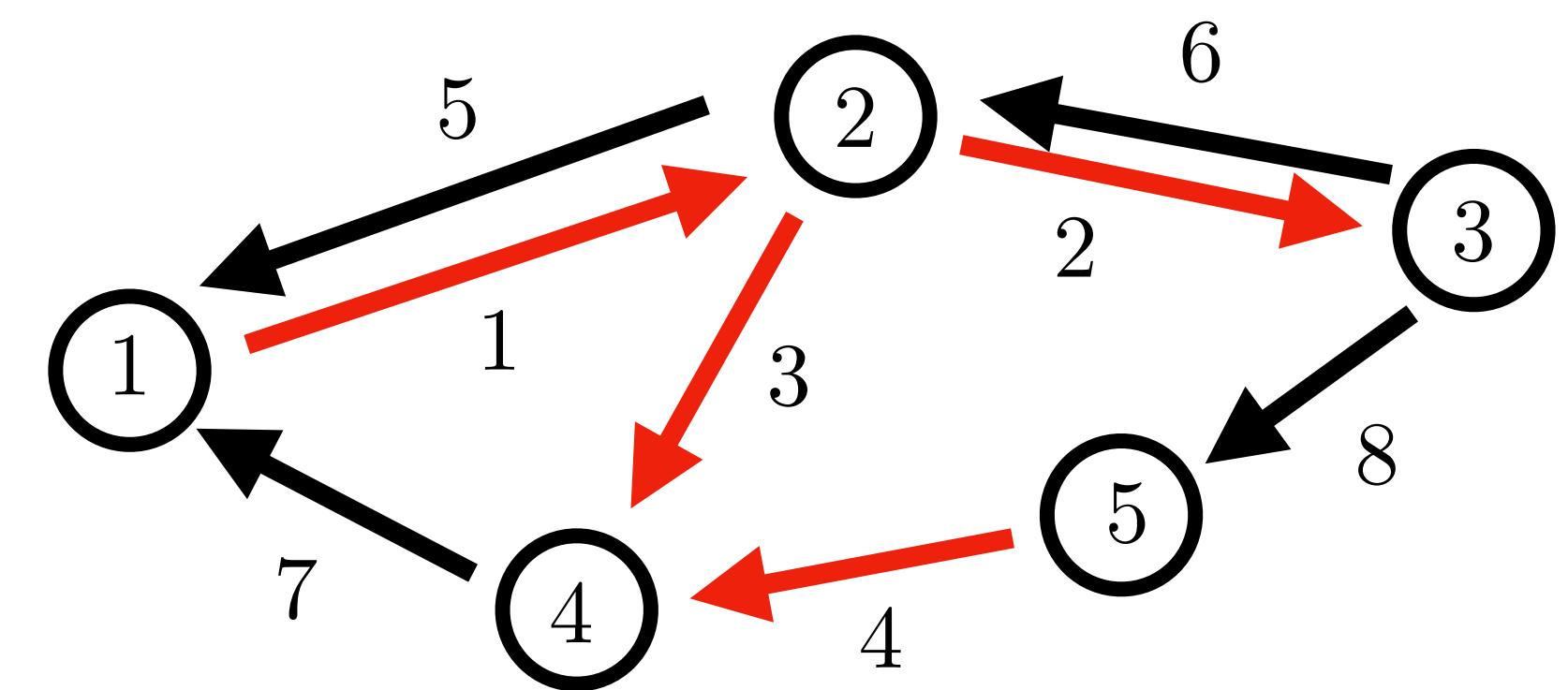
$$Ex = S \Rightarrow x = \bar{x} + Cz$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$

Specific  
Solution

Cyclic  
Flow



**Spanning Tree Construction:**

$$E = [E' \ E''] = [E' \ E'M] = E'[I \ M]$$

$$|\mathcal{V}| \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}_{|\mathcal{V}| - 1} \quad |E'|$$

# Incidence Matrix - Geometry

**Graph:**

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**Vertices**

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edge mass flows →

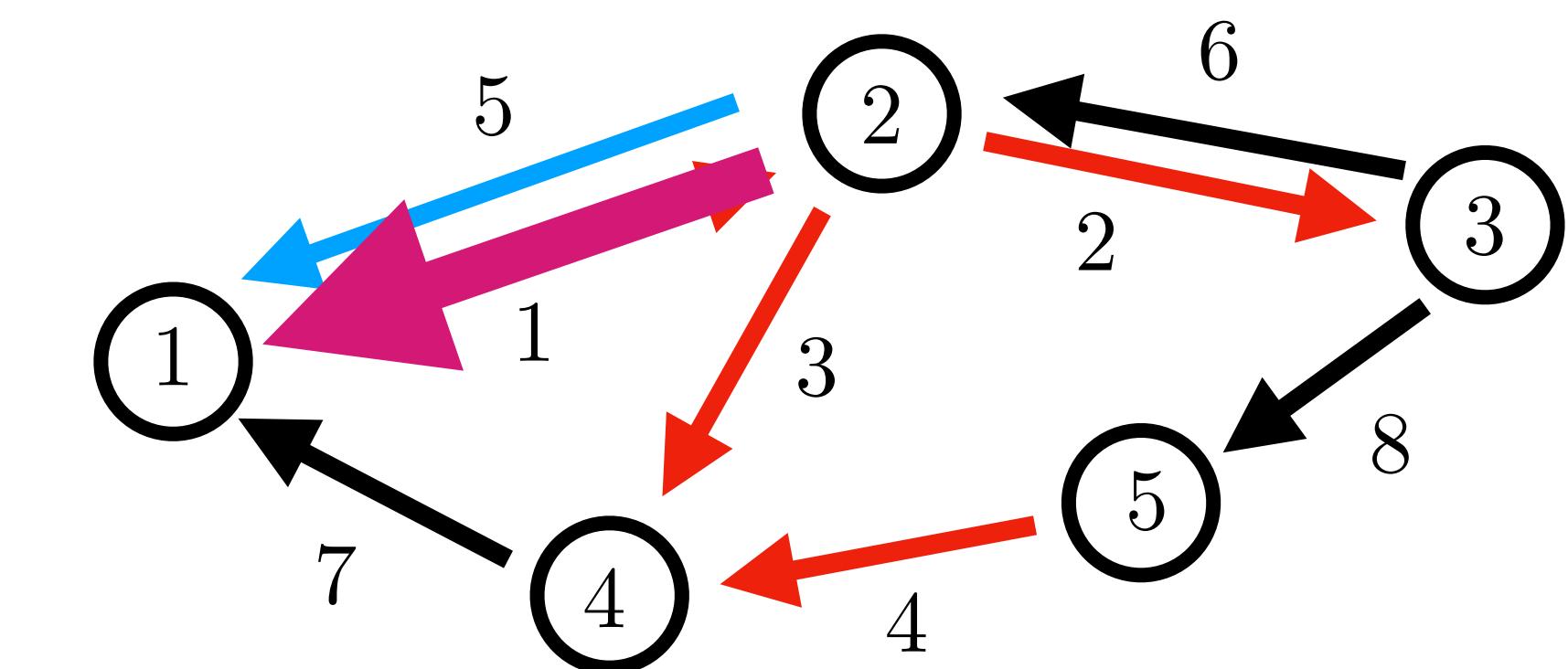
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Specific  
Solution



**Spanning Tree Construction:**

$$E = [E' \ E''] = [E' \ E'M] = E'[I \ M]$$

$$|V| \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} |V| - 1 \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} M$$

Cyclic  
Flow

# Incidence Matrix - Geometry

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

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**Edges**

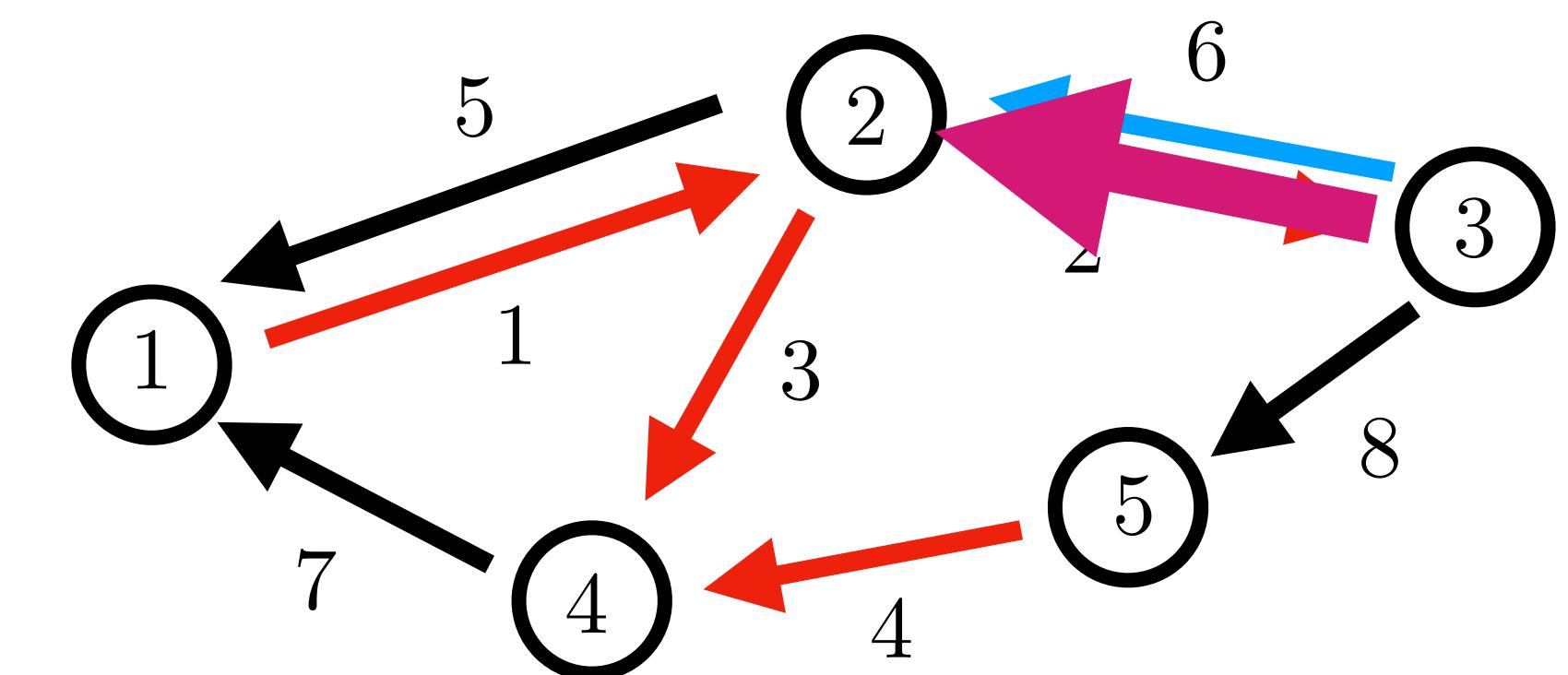
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$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} M$$

# Incidence Matrix - Geometry

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edge mass flows →

Affine  
Constraint

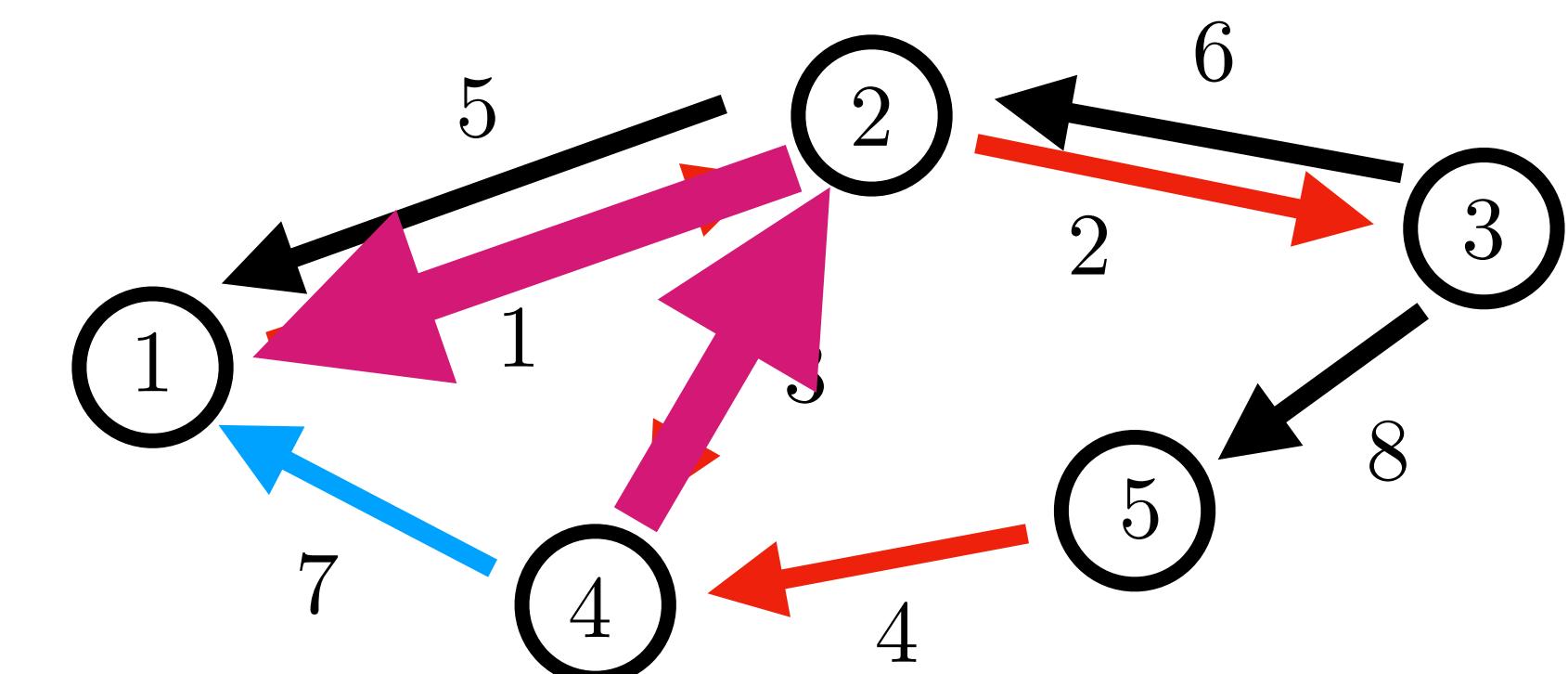
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Specific  
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$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} M$$

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edge mass flows →

Affine  
Constraint

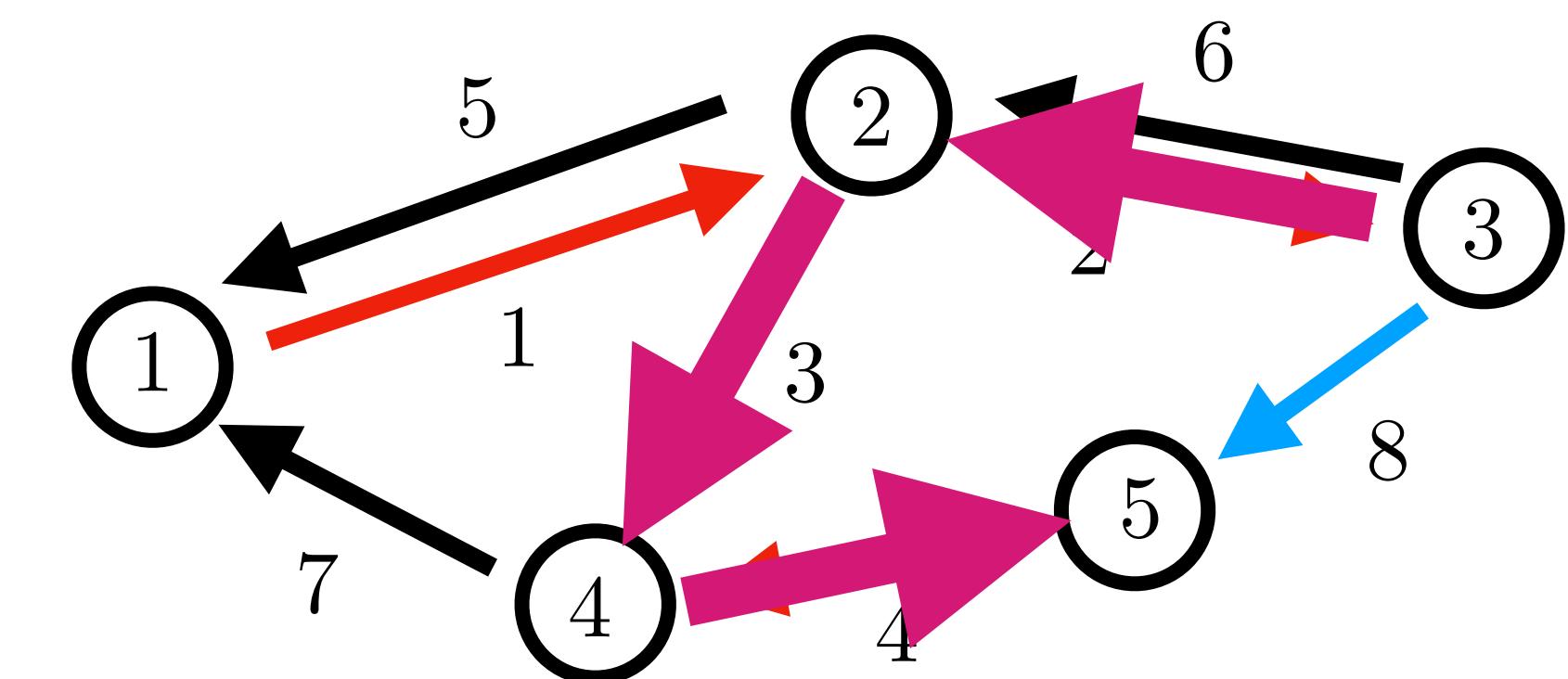
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$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} M$$

$M$

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edge mass flows →

Affine  
Constraint

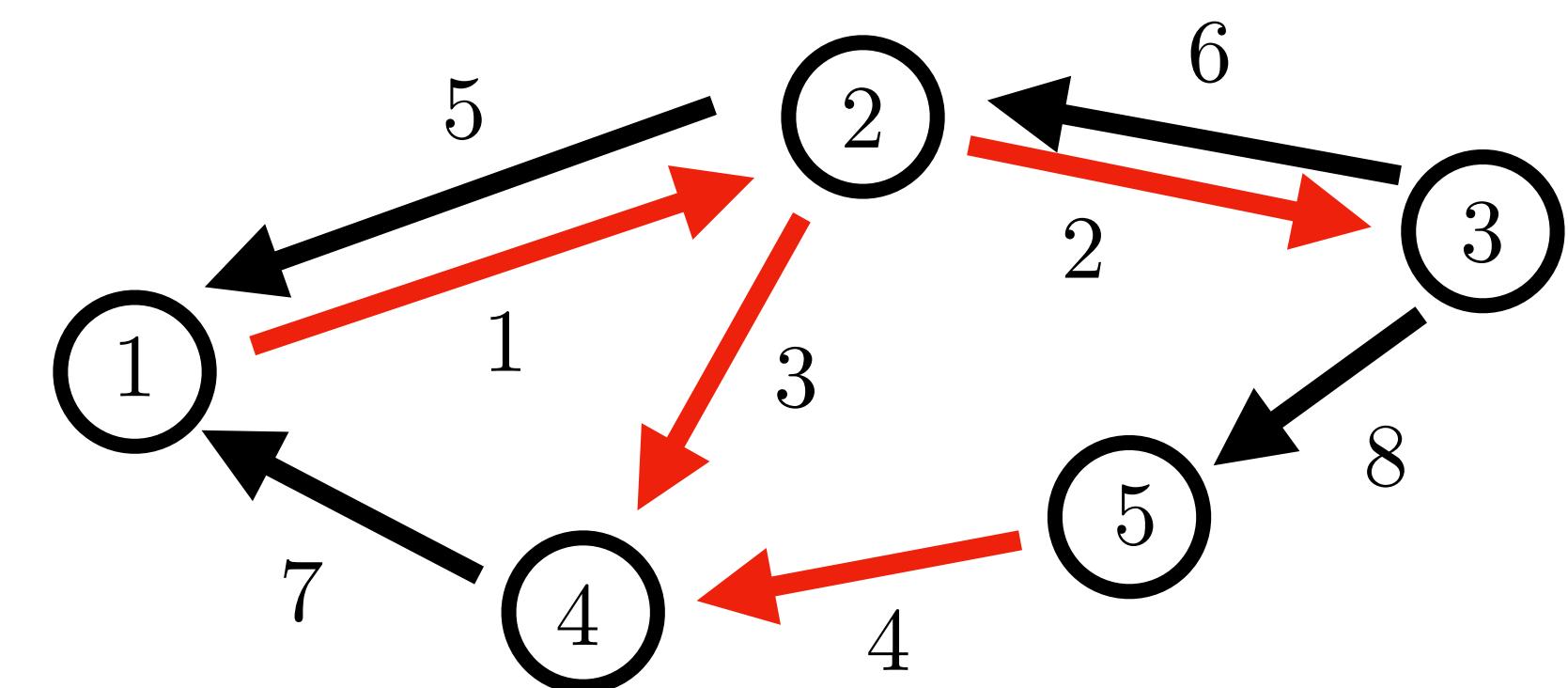
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Specific  
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$$E = [E' \ E''] = [E' \ E'M] = E'[I \ M]$$

$$E = \left[ \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{cccc} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array} \right]$$

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edge mass flows →

Affine  
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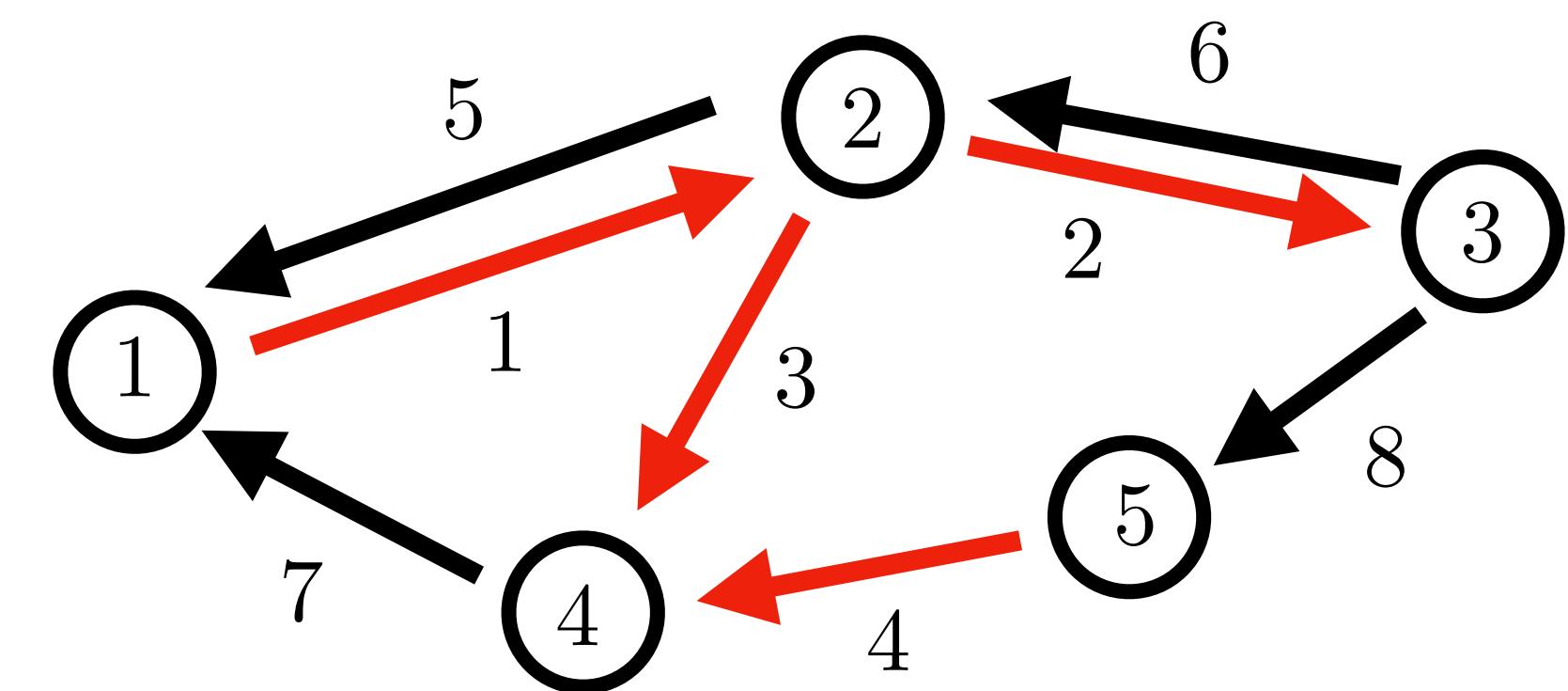
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# Incidence Matrix - Geometry

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}) \quad \begin{array}{ll} \text{Vertices} & v \in \mathcal{V} \\ \text{Edges} & e \in \mathcal{E} \quad e = (v, v') \\ \dots & \text{directed or undirected} \end{array}$$

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edge mass flows  $\rightarrow$

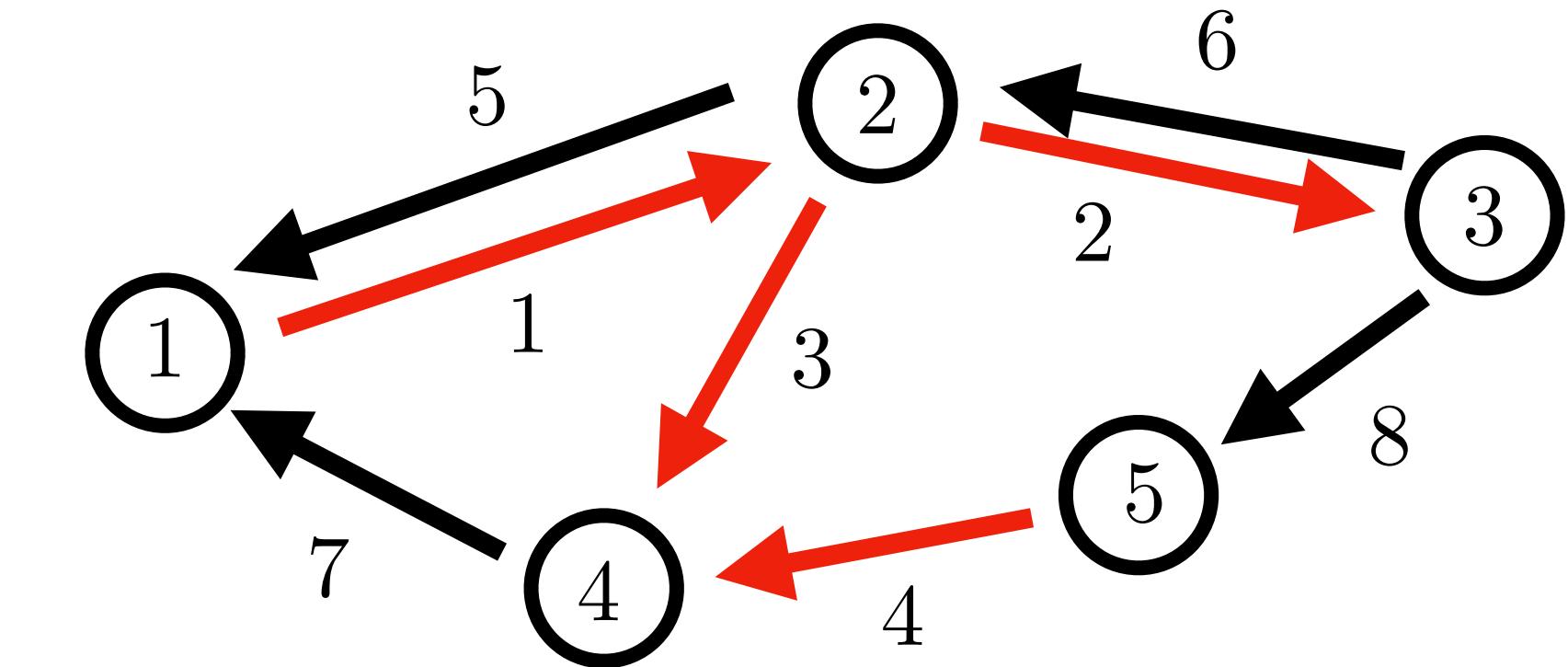
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**Spanning Tree Construction:**

$$E = [E' \quad E''] = [E' \quad E'M] = E'[I \quad M]$$

$$E = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} M \\ -I \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$EC = [E' \quad E''] \begin{bmatrix} M \\ -I \end{bmatrix} = E'M - E'' = 0$$

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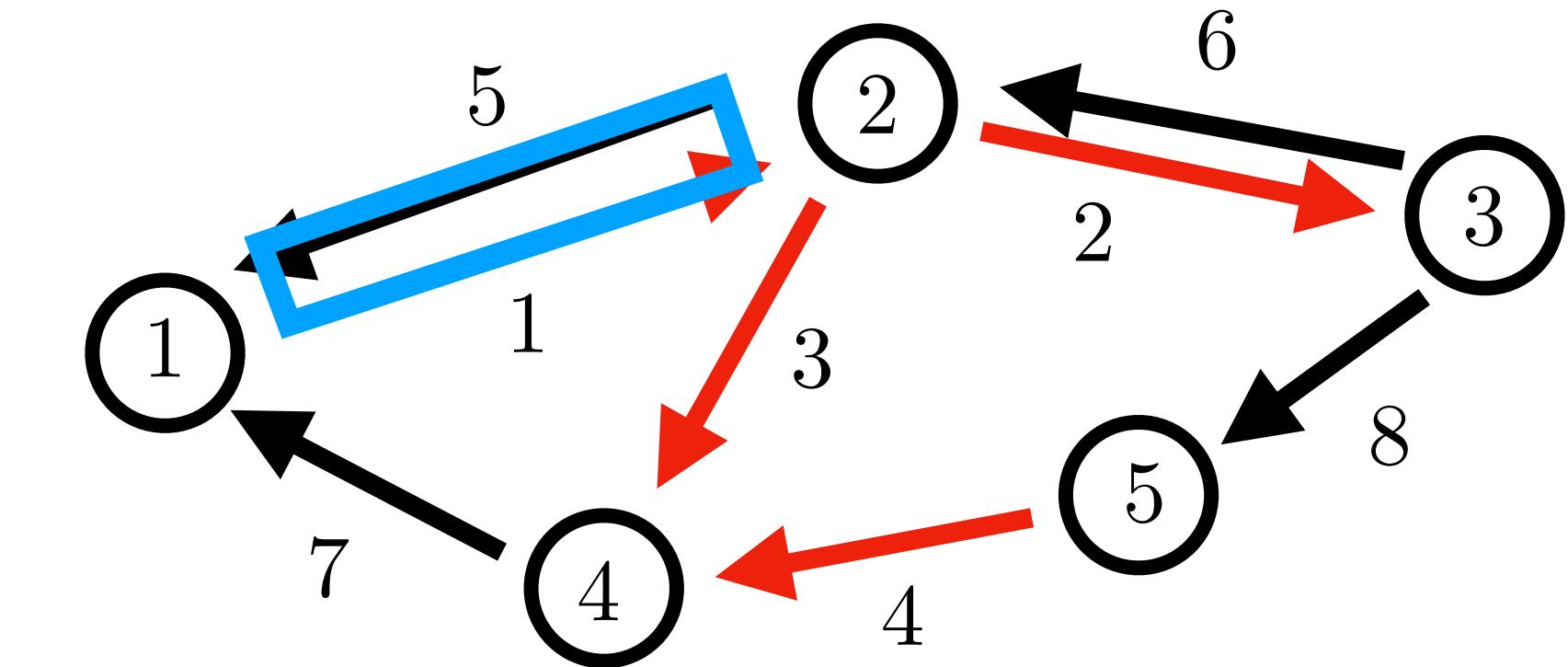
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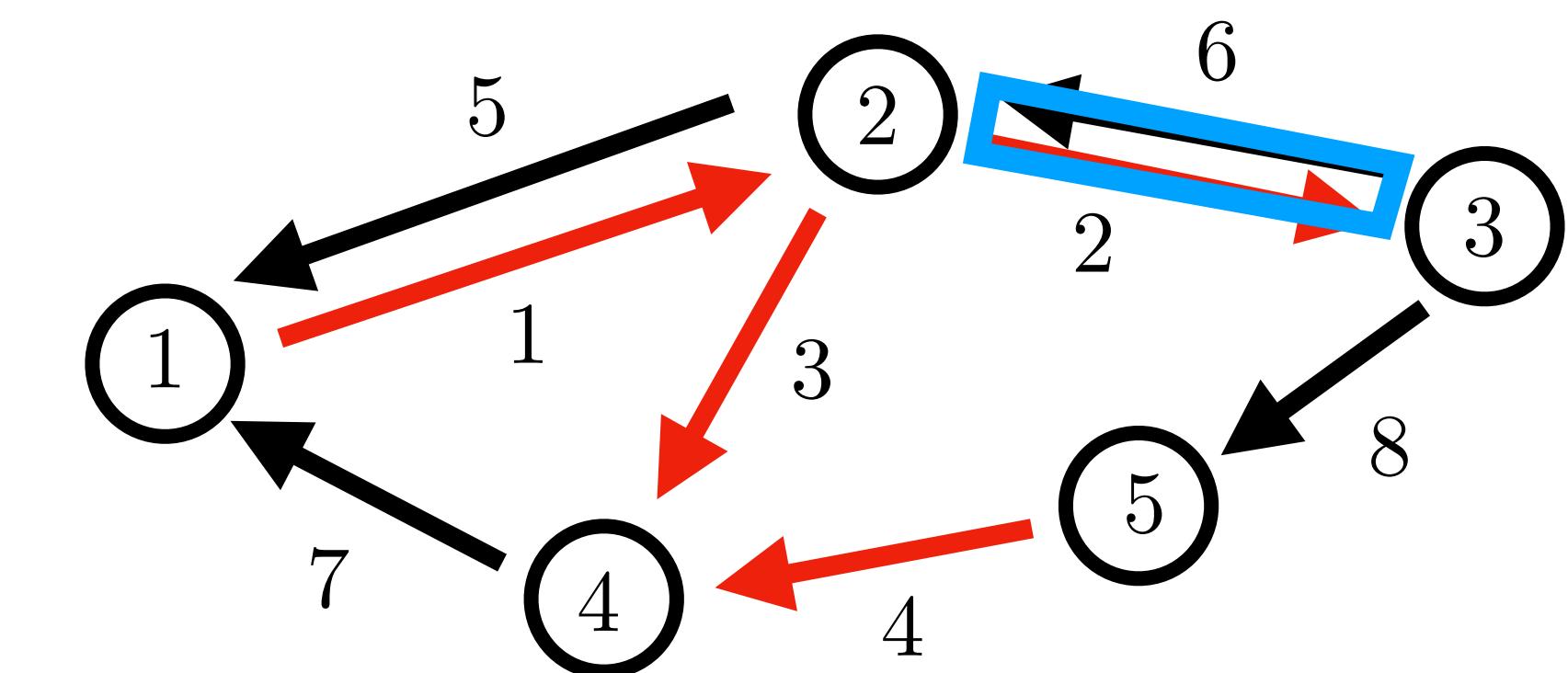
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$$EC = [E' \ E''] \begin{bmatrix} M \\ -I \end{bmatrix} = E'M - E'' = 0$$

Cyclic  
Flow

# Incidence Matrix - Geometry

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E} \quad e = (v, v')$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

edge mass flows  $\rightarrow$

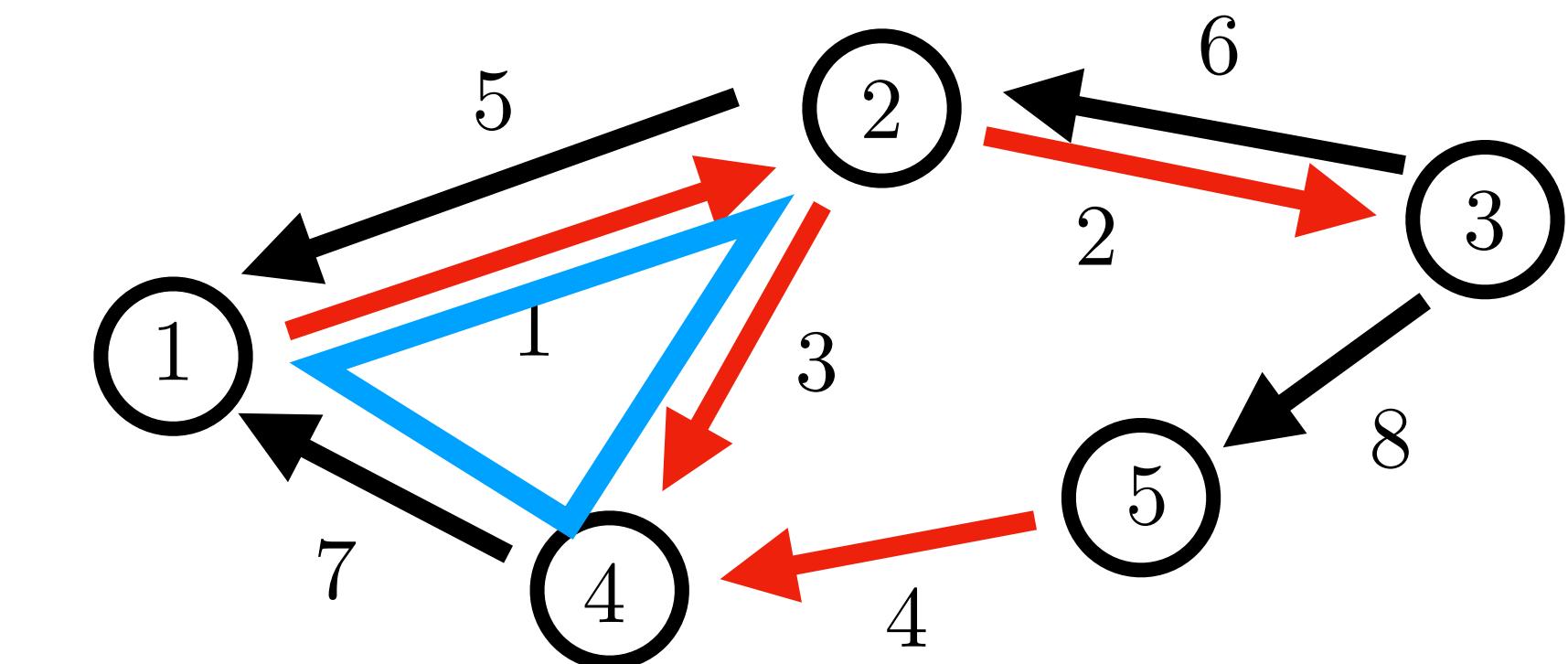
Affine  
Constraint

$$Ex = S \Rightarrow x = \bar{x} + Cz$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$

Specific  
Solution



**Spanning Tree Construction:**

$$E = [E' \ E''] = [E' \ E'M] = E'[I \ M]$$

$$E = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

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# Incidence Matrix - Geometry

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}) \quad \begin{array}{ll} \text{Vertices} & v \in \mathcal{V} \\ \text{Edges} & e \in \mathcal{E} \quad e = (v, v') \\ \dots & \text{directed or undirected} \end{array}$$

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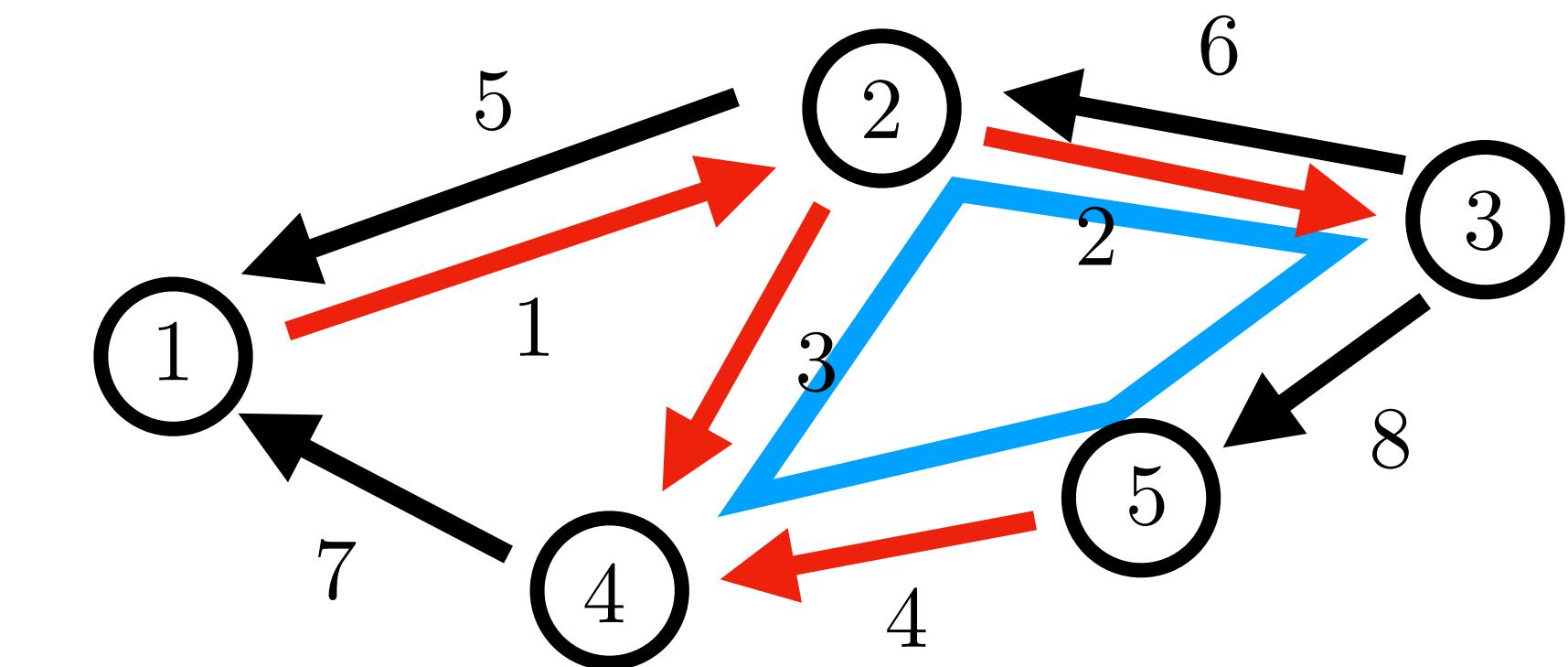
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Constraint

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Min Norm Solution

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Specific  
Solution



**Spanning Tree Construction:**

$$E = [E' \quad E''] = [E' \quad E'M] = E'[I \quad M]$$

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# Incidence Matrix - Geometry

**Graph:**

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**Vertices**

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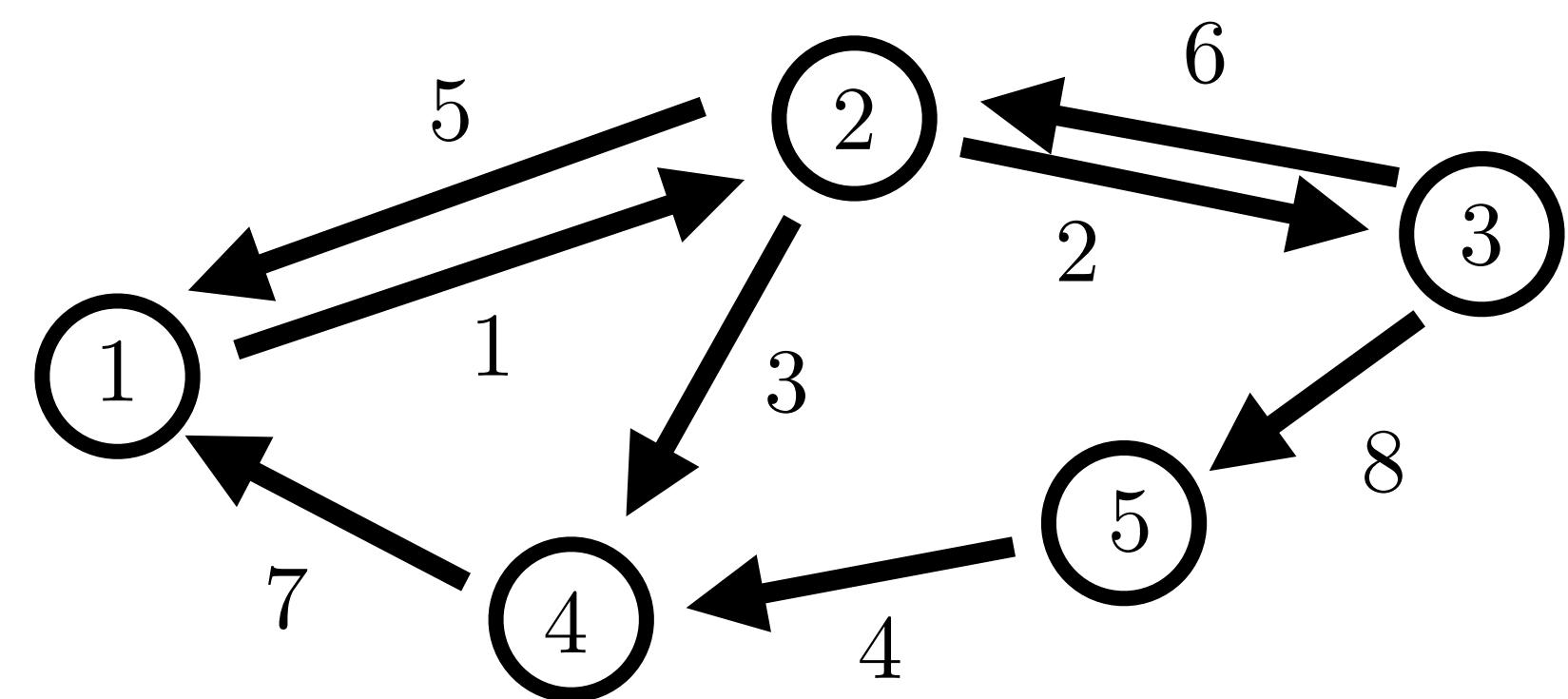
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Domain & Codomain Interpretations**

**Domain:**  $x \in \mathbb{R}^{|\mathcal{E}|}$  **Mass flow on edges**

**Co-domain:**  $v \in \mathbb{R}^{|\mathcal{V}|}$  **...source-sink on nodes**  
**...value function on nodes**

**value:** **Cost-to-go**  
**Potential value**

**“Height”**  
**gravitational potential**

# Incidence Matrix - Geometry

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E} \quad e = (v, v')$$

... directed or undirected

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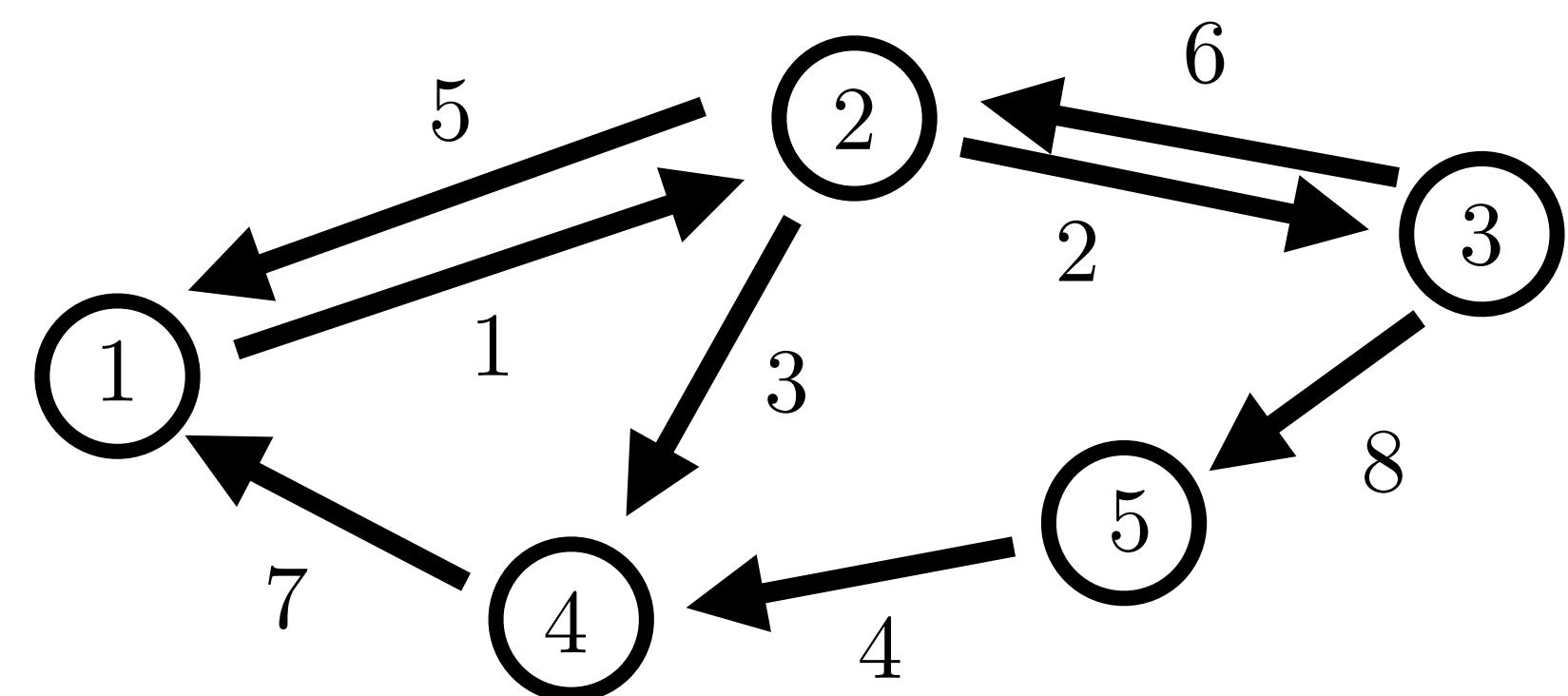
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



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**Co-domain:**  $v \in \mathbb{R}^{|\mathcal{V}|}$  **...source-sink on nodes**  
**...value function on nodes**

**Column Geometry**

$$S = Ex$$

**Non-conserved flow**

**Edge flow vector**

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

# Incidence Matrix - Geometry

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E} \quad e = (v, v')$$

... directed or undirected

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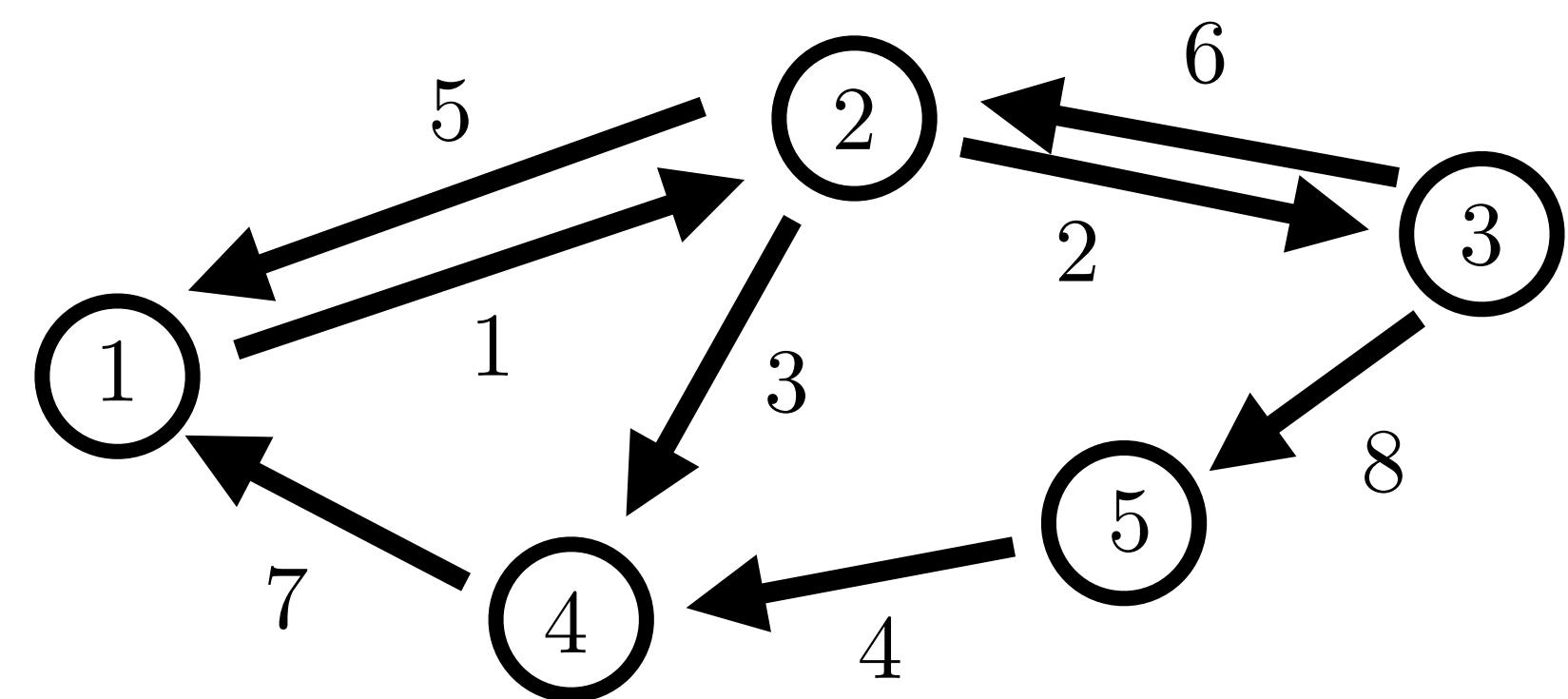
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

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**Domain & Codomain Interpretations**

**Domain:**  $x \in \mathbb{R}^{|\mathcal{E}|}$  **Mass flow on edges**

**Co-domain:**  $v \in \mathbb{R}^{|\mathcal{V}|}$  ...**source-sink on nodes**  
...**value function on nodes**

**Row Geometry**

$$v^T E = \tau^T$$

“Height” of nodes

Tension in edges

$$[v^T E]_e = v_i - v_{i'} = \tau_e$$

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

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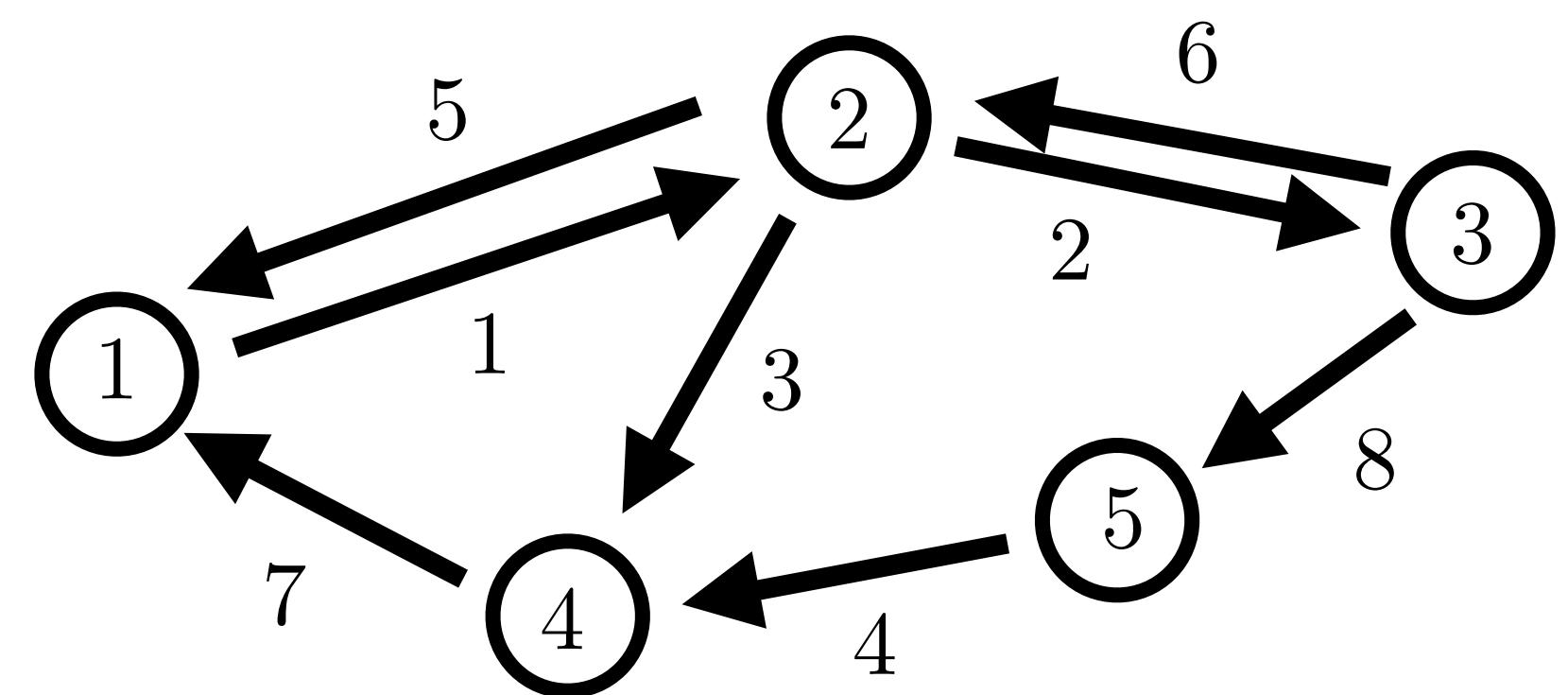
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

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**Shape Matrices**

...recall  $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ \vdots & \ddots & \vdots \\ - & a_m^T & - \end{bmatrix}$$

cols...

rows...

# Graph Laplacian

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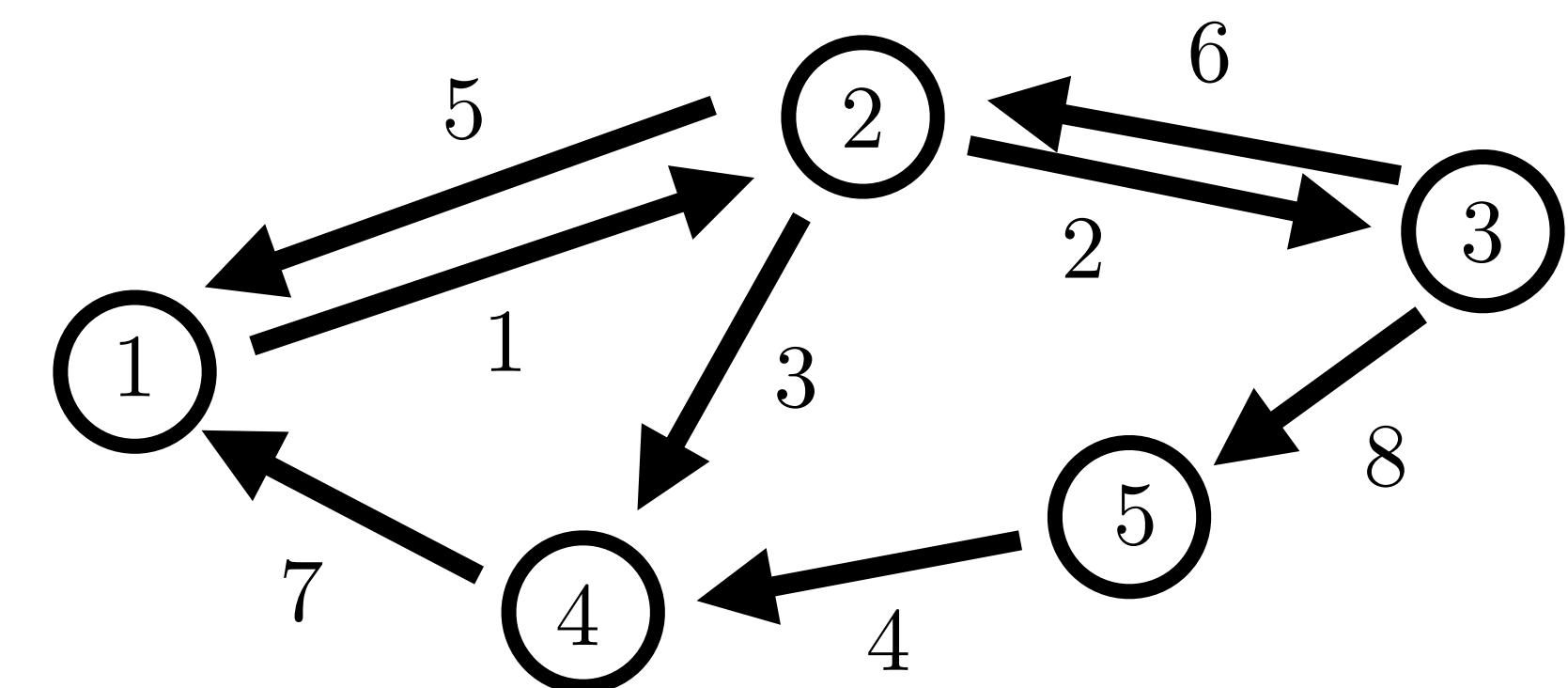
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cols...

rows...

$$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix}$$

Inner products  
of columns

“Relative geometry  
of columns”

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

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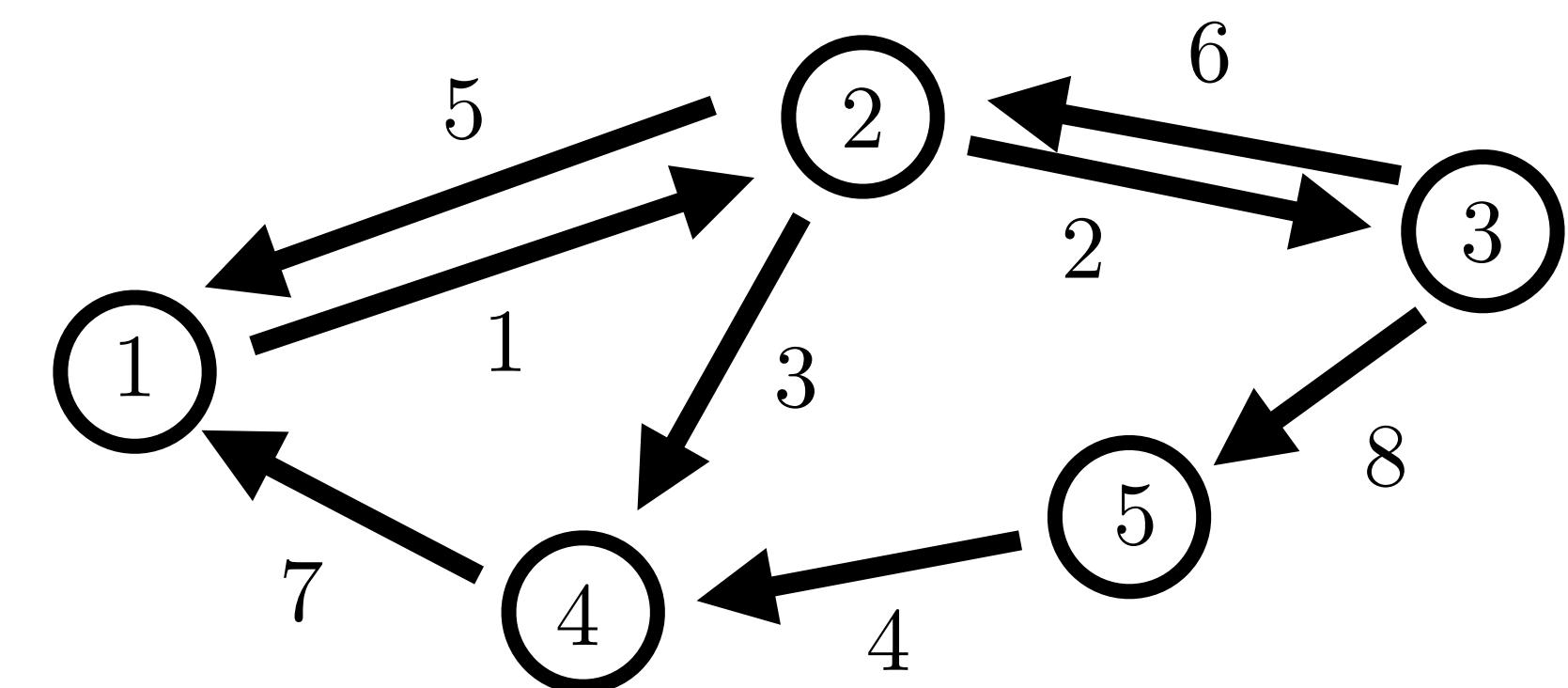
edge mass flows  $\rightarrow$

Affine  
Constraint

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cols...

rows...

$$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix}$$

Inner products  
of rows

“Relative geometry  
of rows”

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

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**Edges**

$$e \in \mathcal{E}$$

... directed or undirected

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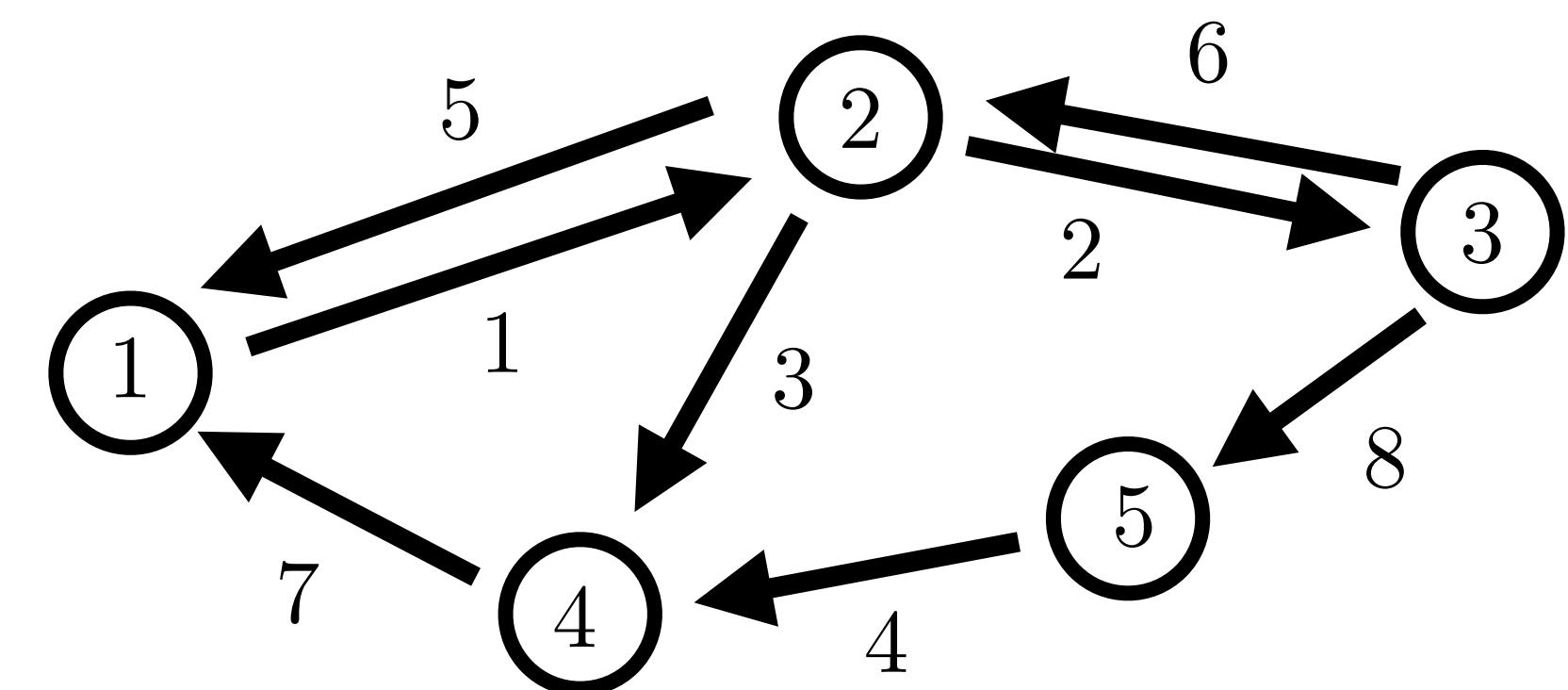
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

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**cols...**

**rows...**

$$RA$$

rotate columns of A...

relative geometry stays the same.

$$(AR)^T (RA) = A^T R^T RA = A^T A$$

# Graph Laplacian

**Graph:**

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**Edges**

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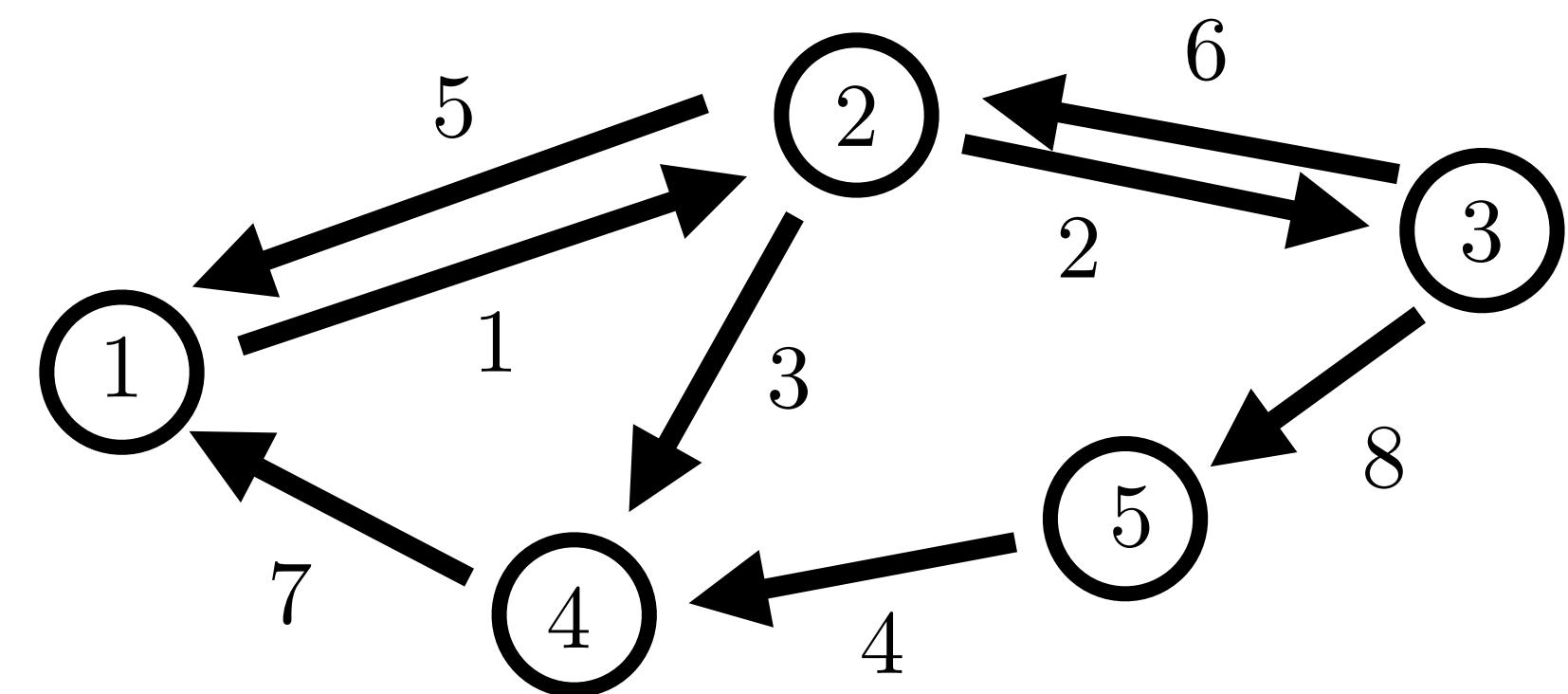
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Affine  
Constraint

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cols...

rows...

$$A^T A \quad \text{“Shape” of the columns of A}$$

# Graph Laplacian

**Graph:**

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**Vertices**

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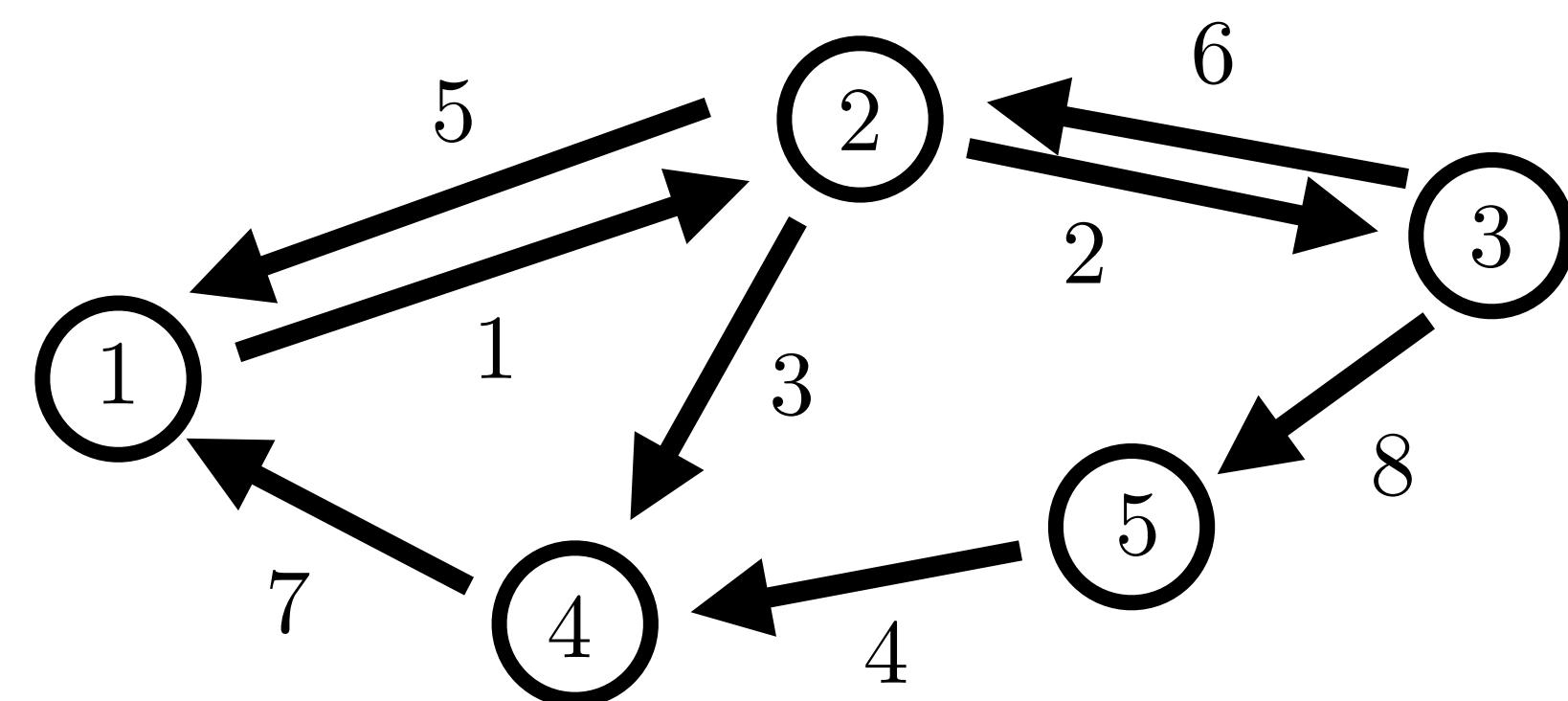
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cols...

rows...

$A^T A$     ~~“Shape” of the columns of A~~

$(A^T A)^{1/2}$     “Shape” of the columns of A

More  
accurate

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}) \quad \begin{array}{ll} \text{Vertices} & v \in \mathcal{V} \\ \text{Edges} & e \in \mathcal{E} \quad e = (v, v') \\ \dots \text{ directed or undirected} & \end{array}$$

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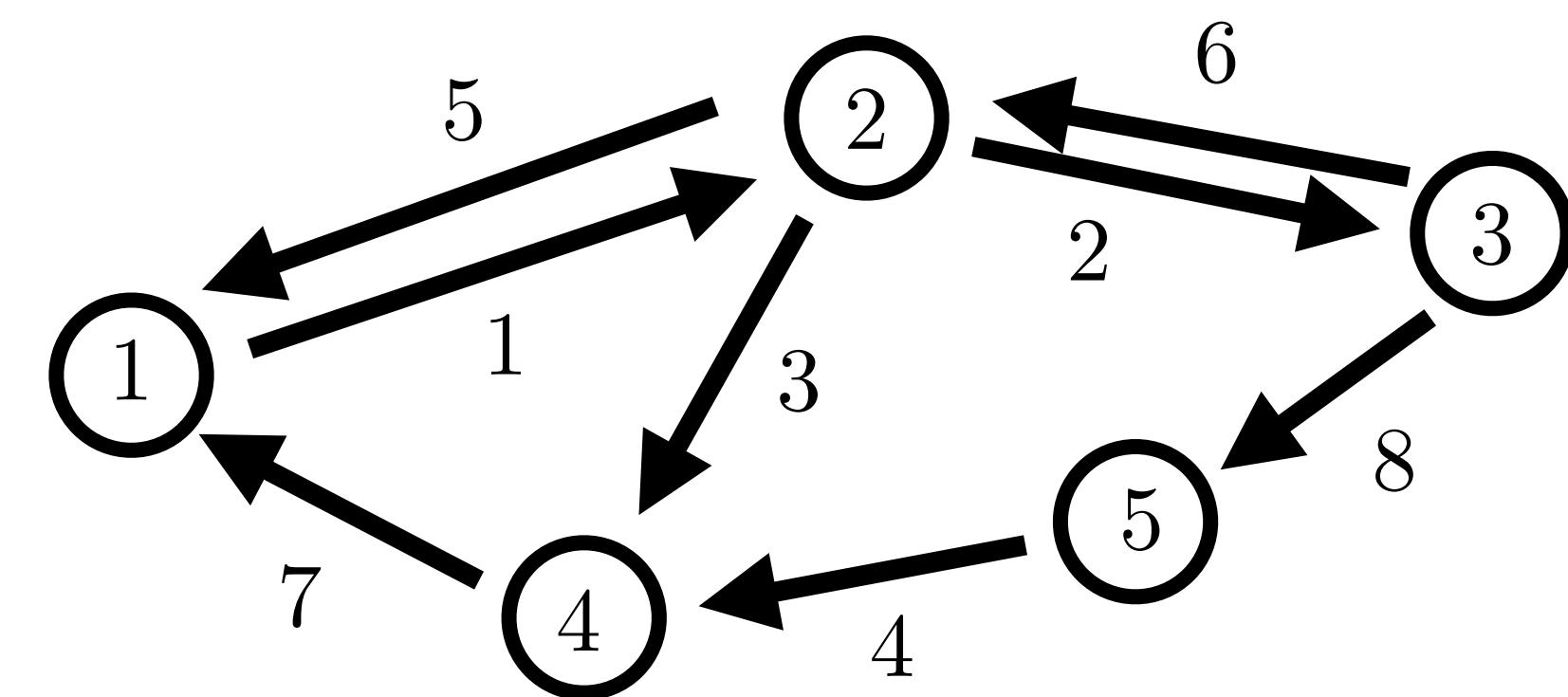
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

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**Shape Matrices**

...recall

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cols...

rows...

$$(A^T A)^{1/2}$$

“Shape” of the columns of A

$$(AA^T)^{1/2}$$

“Shape” of rows of A

Analogy:

$$z \in \mathbb{C} \quad |z| = \sqrt{z^* z} \quad z = |z|e^{i\phi}$$

Polar  
Decomposition  
“column version”

$$A = A(A^T A)^{-1/2} \cdot (A^T A)^{1/2}$$

Rotation

PSD “shape”

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}) \quad \begin{array}{ll} \text{Vertices} & v \in \mathcal{V} \\ \text{Edges} & e \in \mathcal{E} \quad e = (v, v') \\ \dots \text{ directed or undirected} & \end{array}$$

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

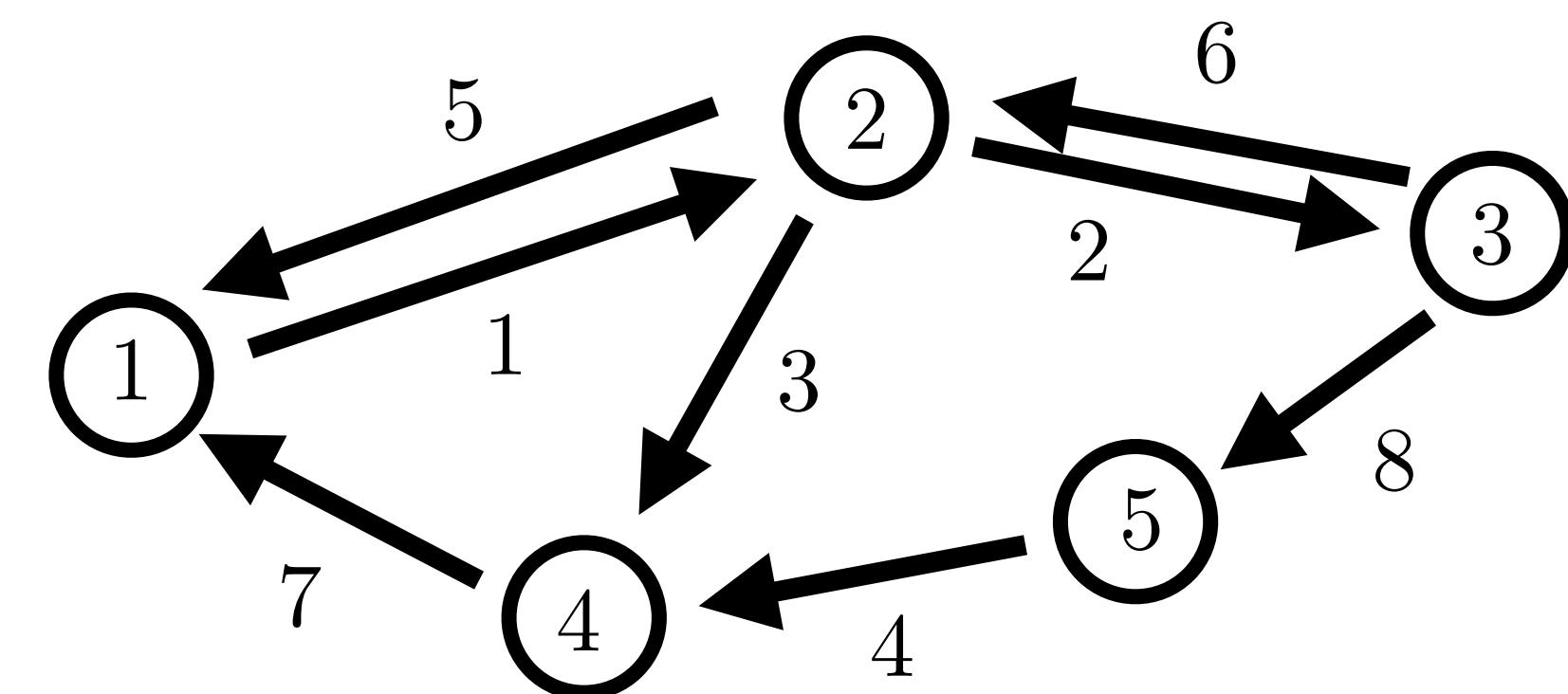
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Shape Matrices**

...recall

$$A \in \mathbb{R}^{m \times n}$$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ \vdots & \ddots & \vdots \\ - & a_m^T & - \end{bmatrix}$$

cols...

rows...

$$(A^T A)^{1/2}$$

“Shape” of the columns of A

$$(AA^T)^{1/2}$$

“Shape” of rows of A

Analogy:

$$z \in \mathbb{C} \quad |z| = \sqrt{z^* z} \quad z = |z|e^{i\phi}$$

Polar  
Decomposition  
“row version”

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD “shape”      Rotation

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

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... directed or undirected

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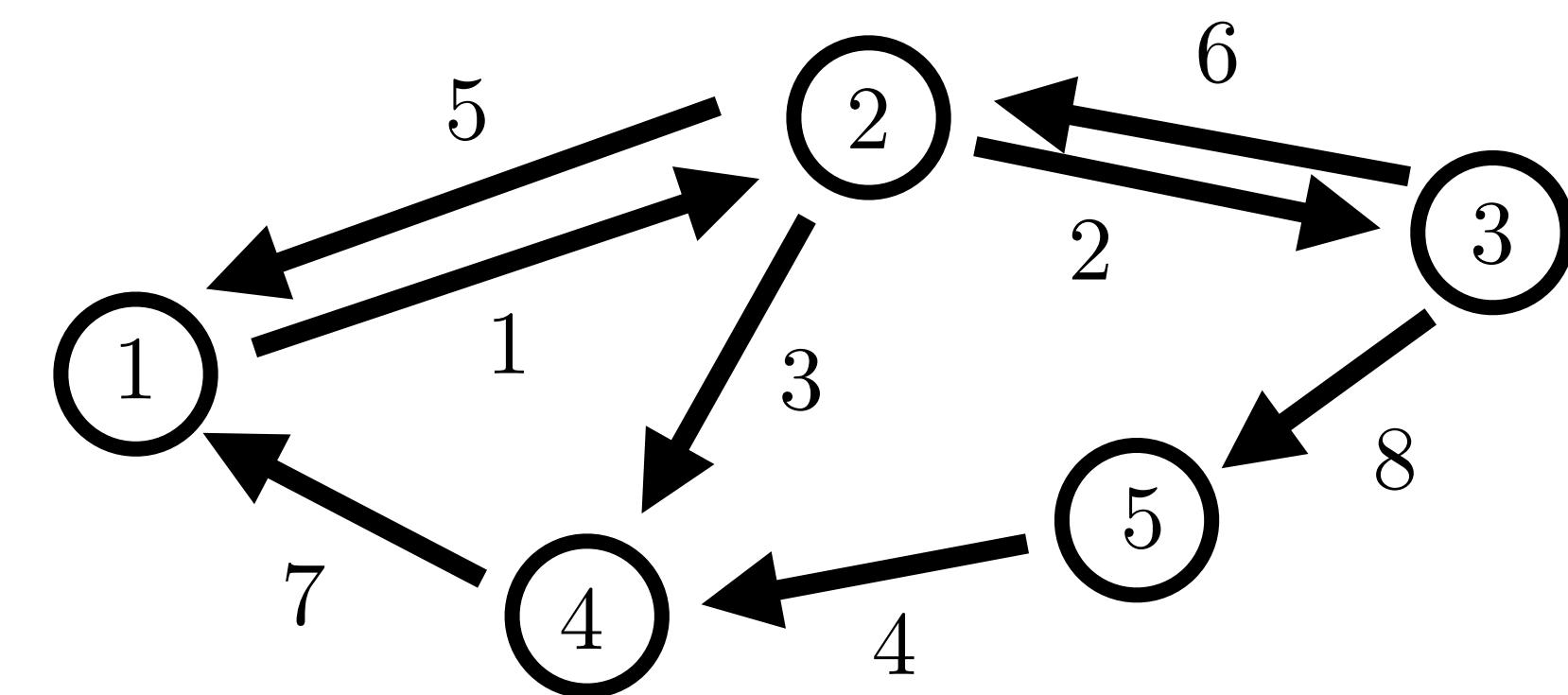
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Constraint

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$$(AA^T)^{1/2}$$

“Shape” of rows of A

$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \quad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

**EVD of shapes**

**Polar  
Decomposition**  
“row version”

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**PSD “shape”**      **Rotation**

# Graph Laplacian

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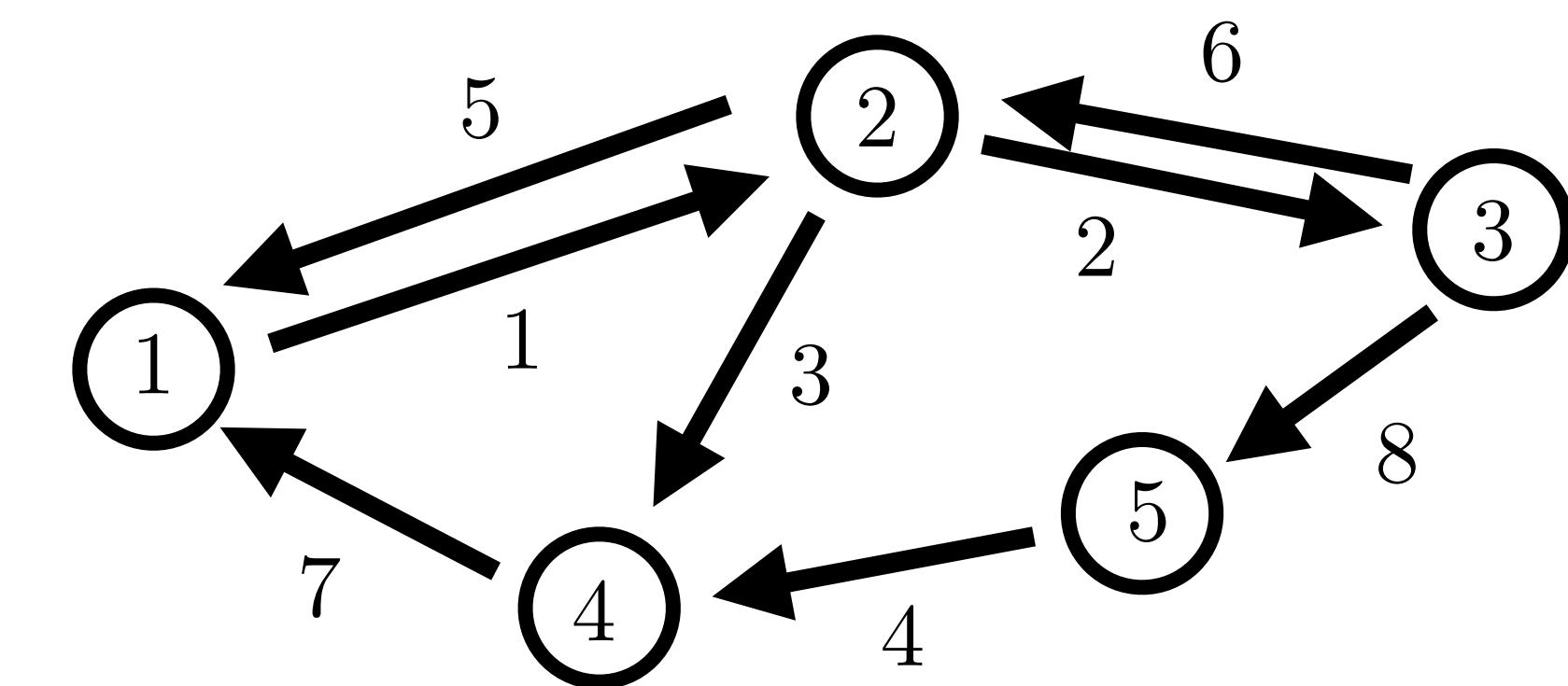
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Constraint

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Min Norm Solution

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**Polar  
Decomposition  
“row version”**

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T \cdot U V^T$$

**PSD “shape”      Rotation**

# Graph Laplacian

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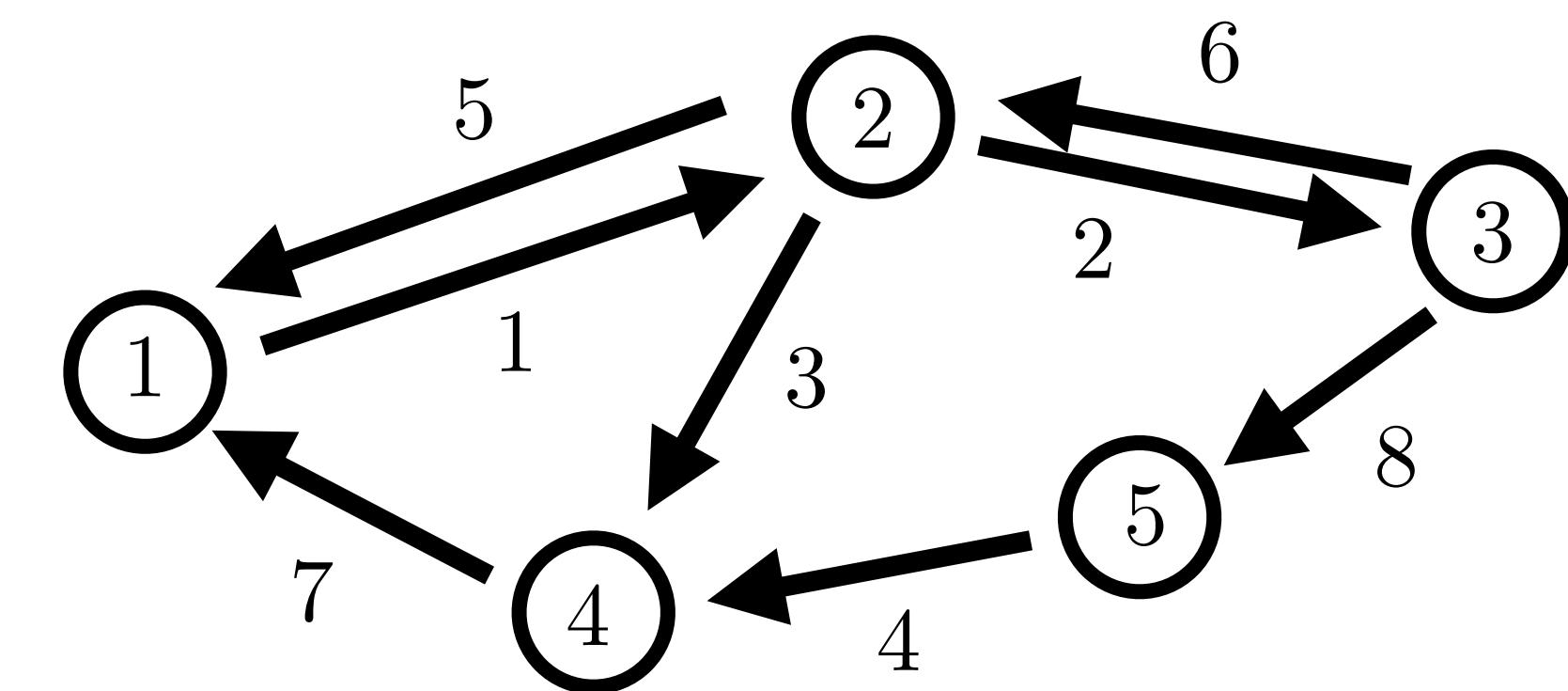
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**Polar  
Decomposition  
“column version”**

$$A = UV^T \cdot V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

**Rotation PSD “shape”**

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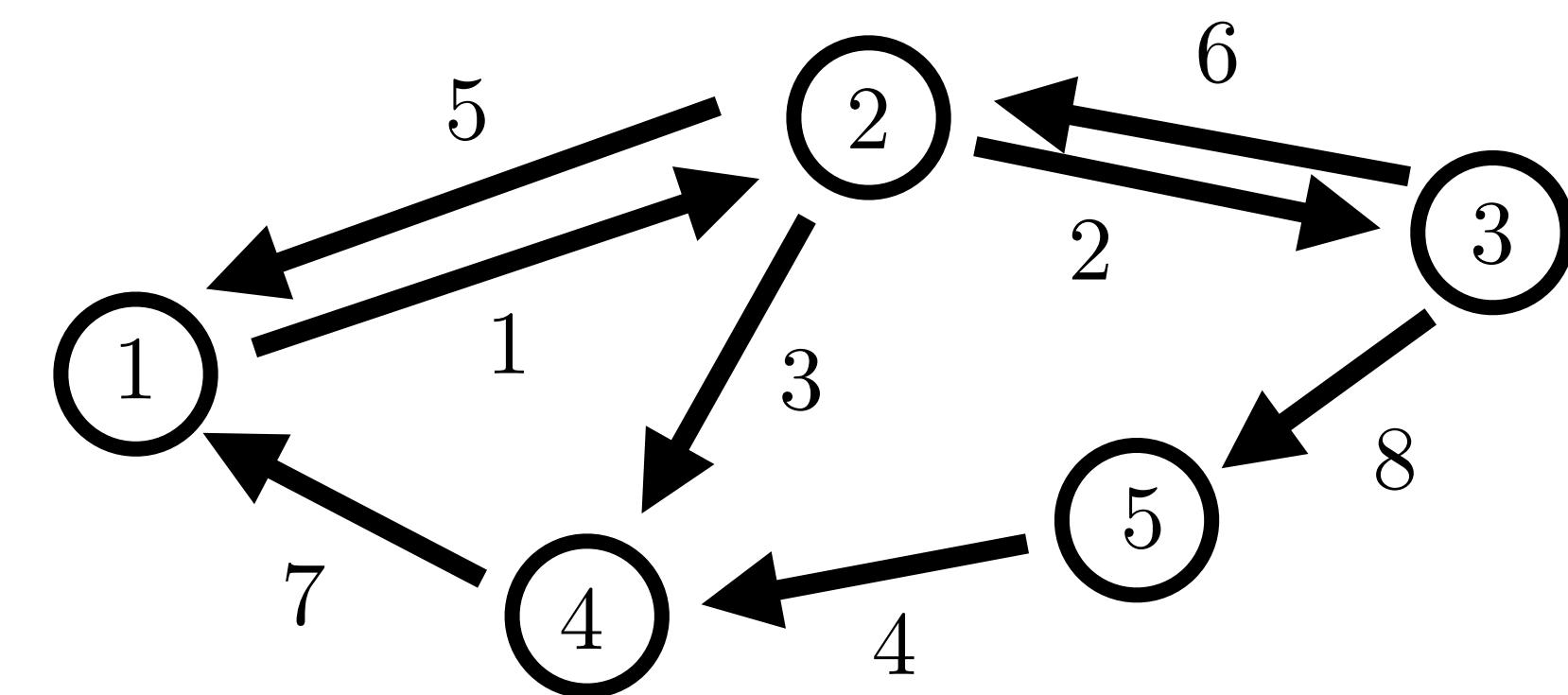
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**Singular Value  
Decomposition**

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

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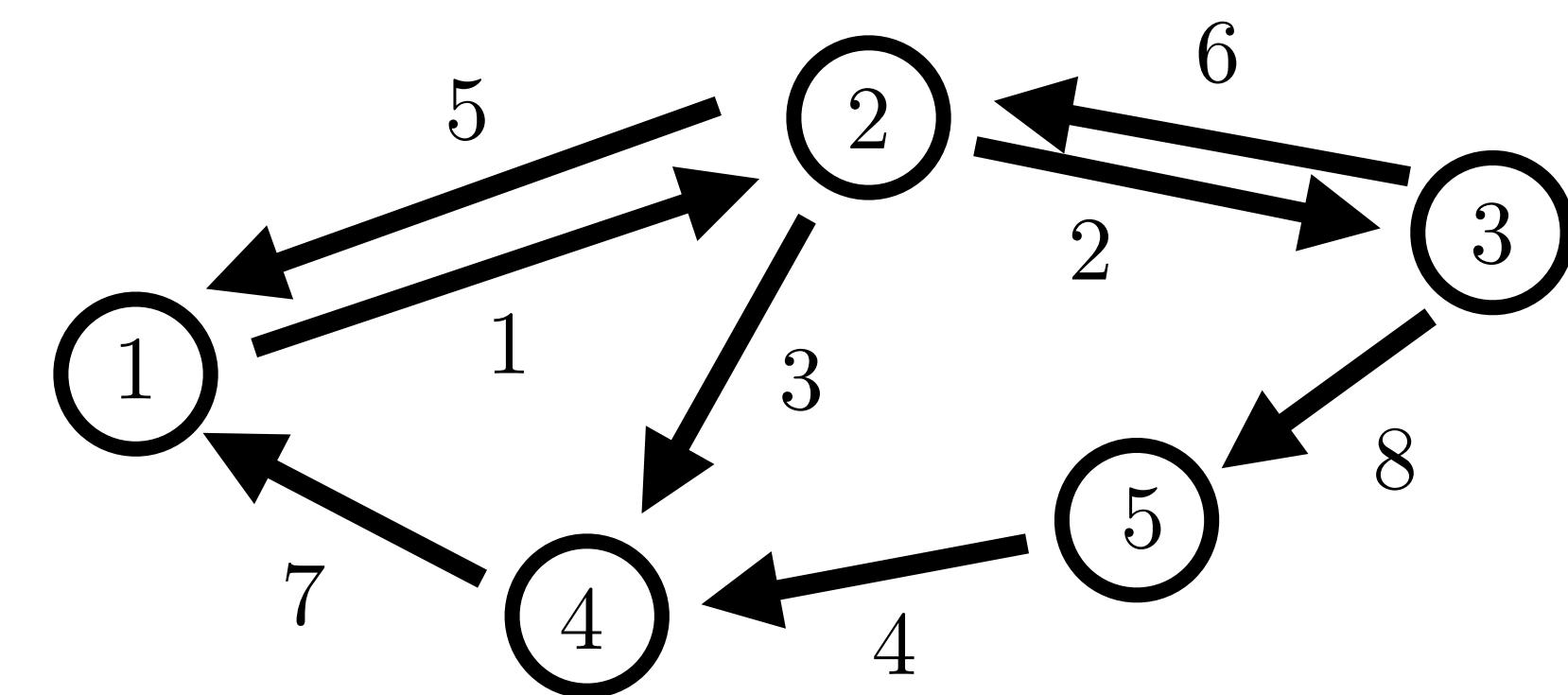
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$$U' = AV'\Sigma^{-1}$$

**Singular Value  
Decomposition**

$$V'^T = \Sigma^{-1} U'^T A$$

for singular vectors  
w/ non-zero values

# Graph Laplacian

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**Vertices**

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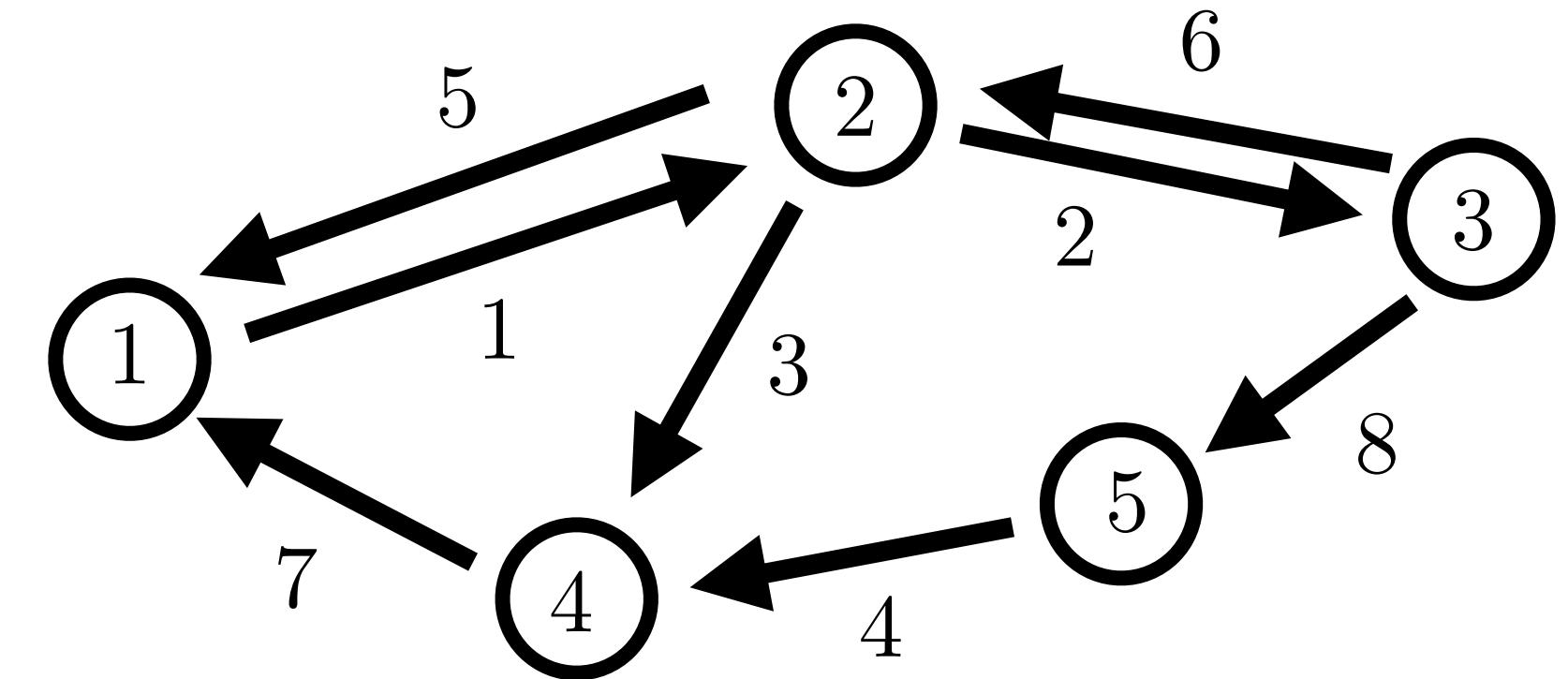
edge mass flows →

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian**

“shape” matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

SVD of  
incidence  
matrix

$$E = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

# Graph Laplacian

**Graph:**

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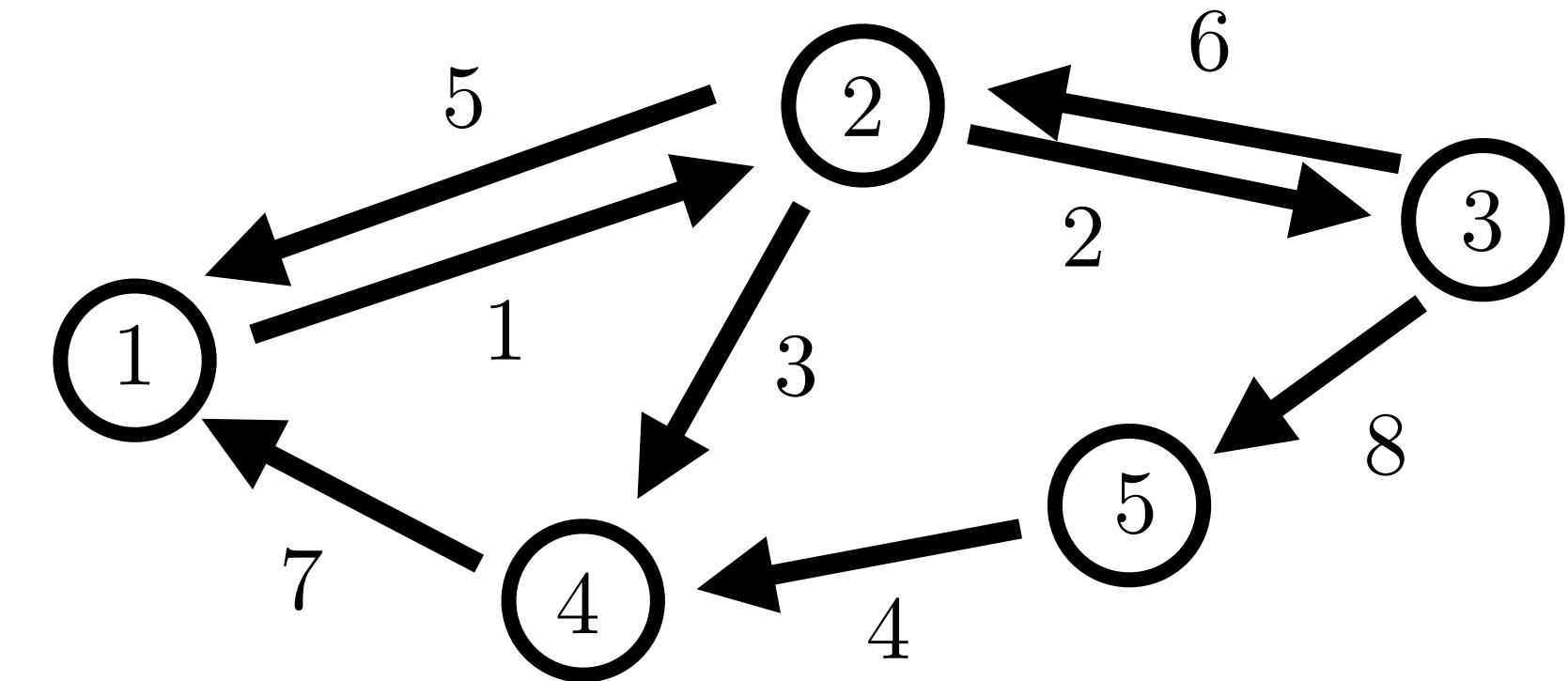
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**Graph Laplacian** “shape” matrix of rows (squared)

$$\begin{aligned} L = EE^T &= U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T \\ &= \begin{bmatrix} | & | \\ U' & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & U'^T & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix} \quad \bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix} \\ &\text{Laplacian eigenvectors} \quad \text{(Incidence matrix left singular vectors)} \end{aligned}$$

**SVD of incidence matrix**

$$E = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

# Graph Laplacian

**Graph:**

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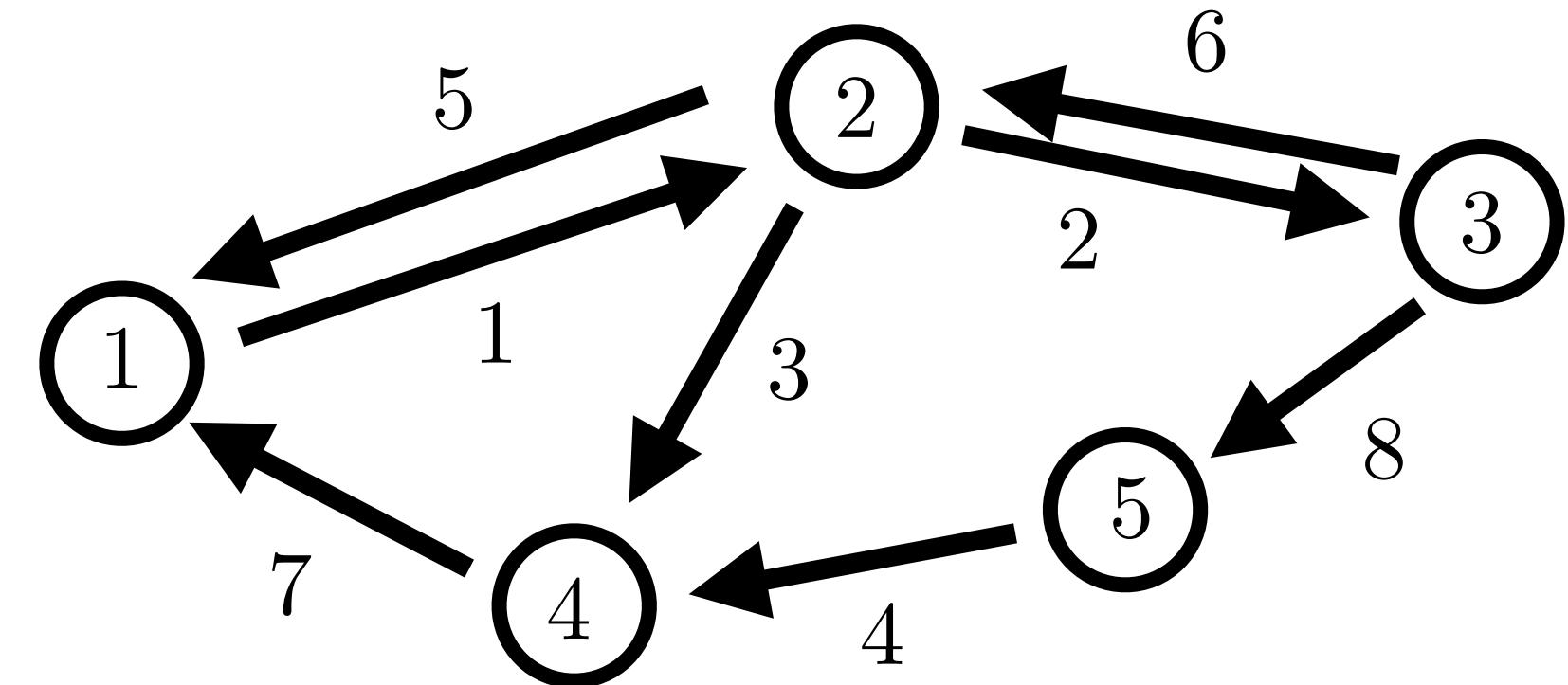
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**Graph Laplacian** “shape” matrix of rows (squared)

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Action of  
Laplacian:

$$Lu = [E][E^T] \begin{bmatrix} | \\ u \\ | \end{bmatrix}$$

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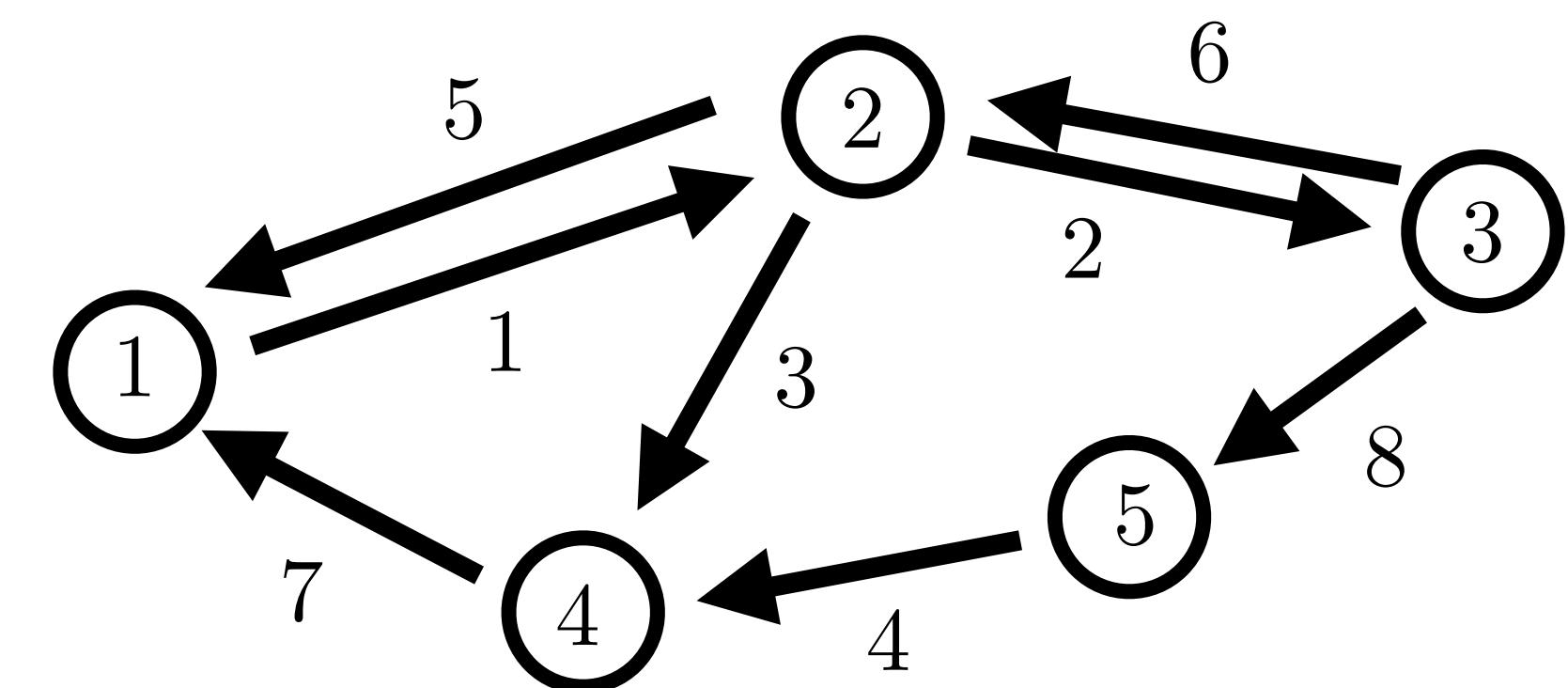
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Action of  
Laplacian:

$$Lu = \underbrace{[E]}_{\text{...tension created in edges}} \underbrace{[E^T]}_{\text{... summed resulting tension on nodes}} \underbrace{\begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{“heights” of nodes}}$$

... summed resulting tension on nodes

# Graph Laplacian

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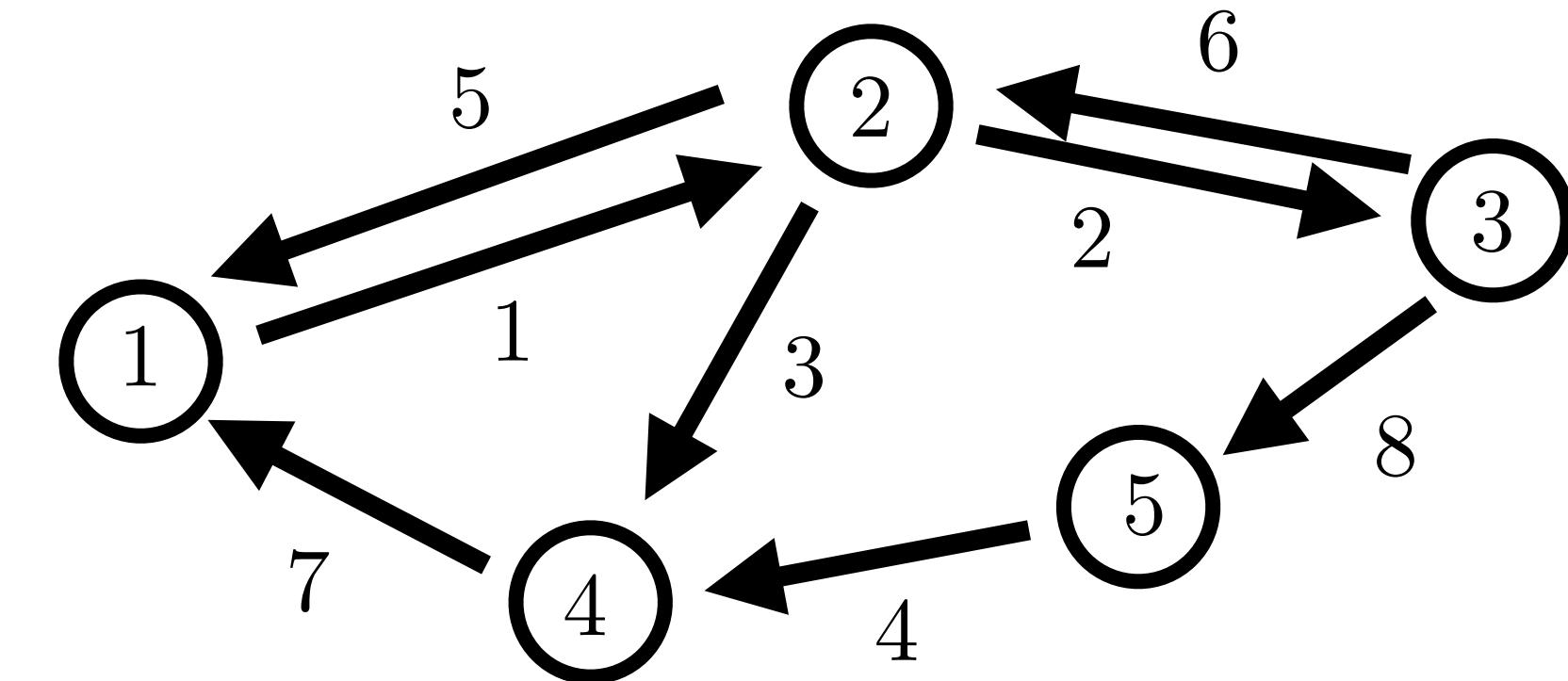
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Action of  
Laplacian:

$$Lu = \underbrace{[E][E^T]}_{\dots \text{ summed tension from node height differences}} \begin{bmatrix} || \\ u \\ || \end{bmatrix} \quad \text{“heights” of nodes}$$

Linear ODE

$$\dot{u} = -Lu$$

Eigenvectors are oscillation modes

# Graph Laplacian

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... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

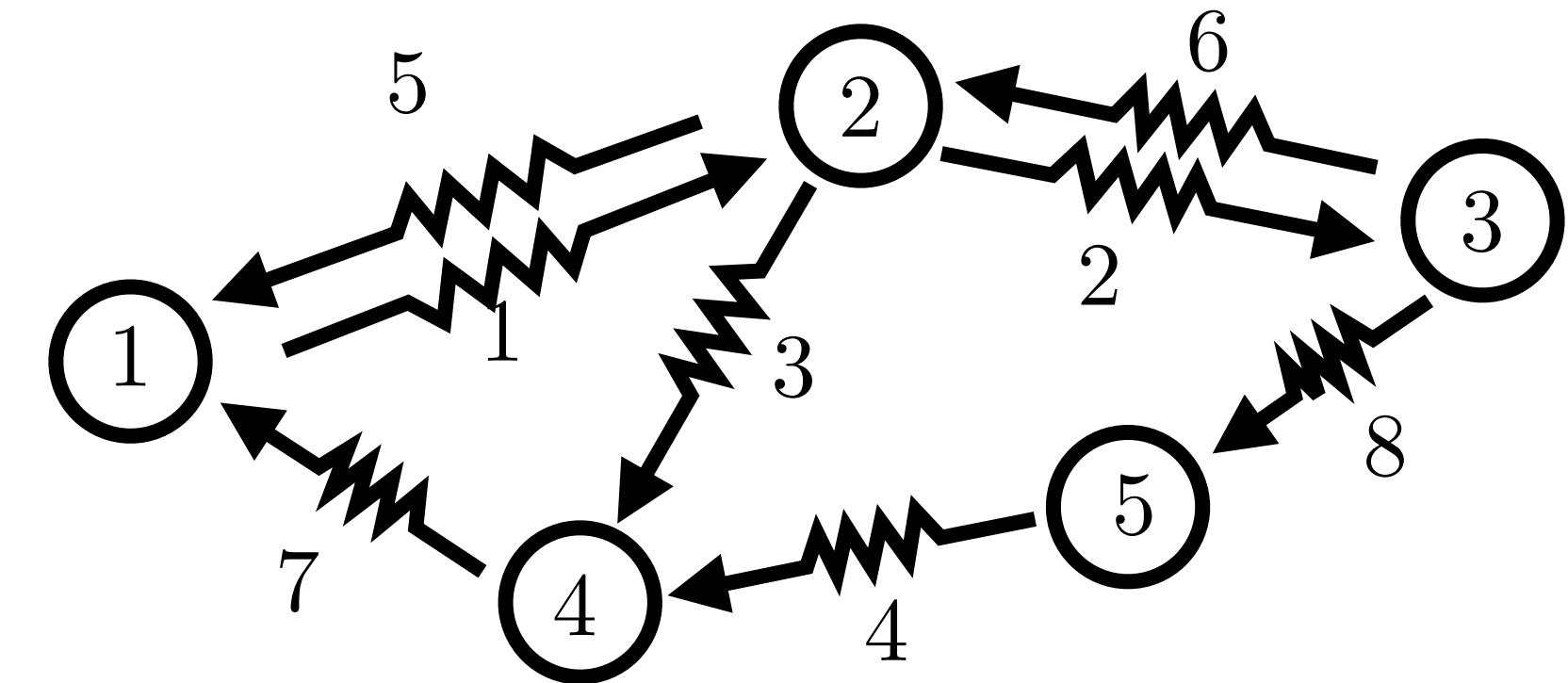
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Weighted Graph Laplacian**

“shape” matrix of rows (squared)

$$L = EWE^T = U_W \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} U_W^T$$

$$= \begin{bmatrix} | & | \\ U'_W & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & U'^T_W & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix}^T \quad \bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix}$$

Action of  
Laplacian:

$$Lu = \underbrace{[E][W][E^T]}_{\dots \text{tension created in edges scaled by weights}} \underbrace{\begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\dots \text{summed resulting tension on nodes}} \text{ “heights” of nodes}$$

... summed resulting tension on nodes

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

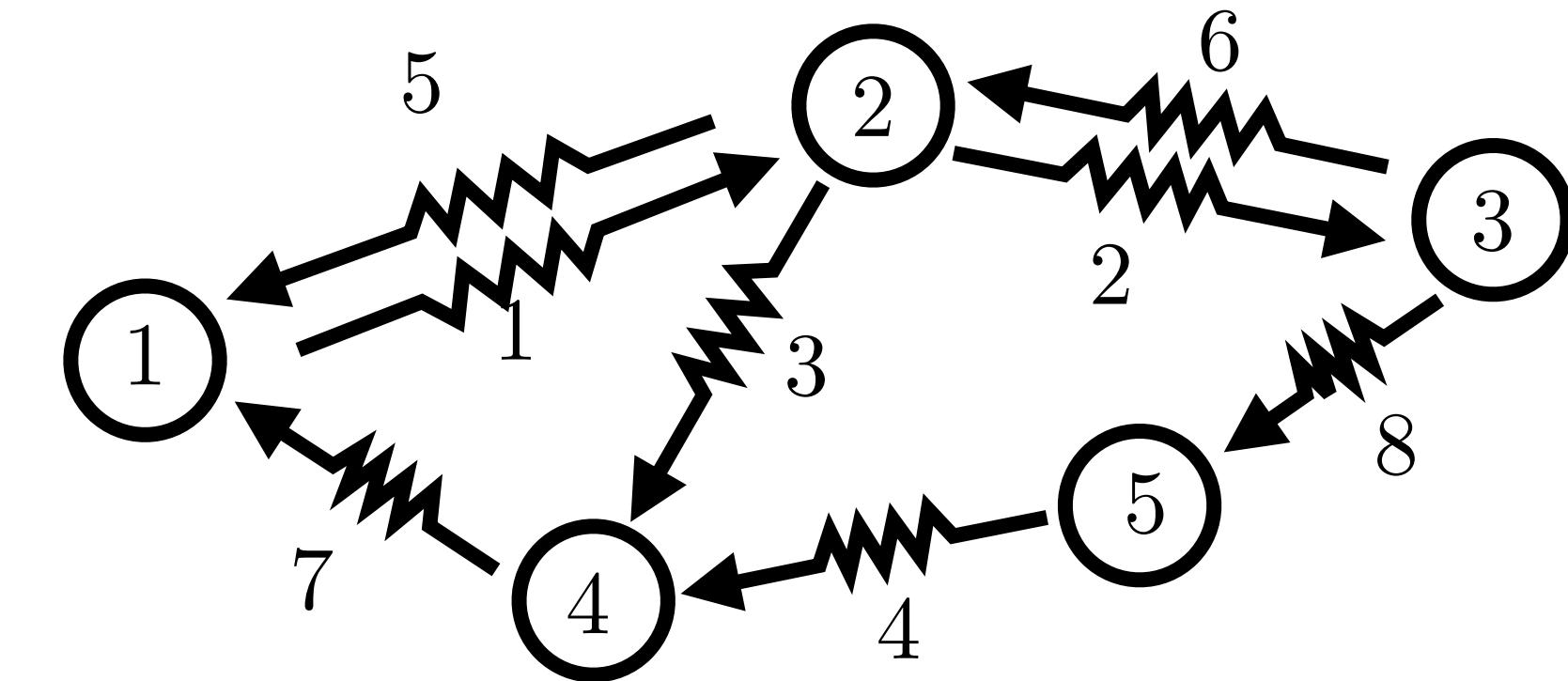
edge mass flows  $\rightarrow$

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Weighted Graph Laplacian** “shape” matrix of rows (squared)

$$\begin{aligned} L = EWE^T &= U_W \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} U_W^T \\ &= \begin{bmatrix} | & | \\ U'_W & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & U'^T_W & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix}^T \quad \bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix} \end{aligned}$$

Action of  
Laplacian:

$$Lu = \underbrace{[E][W][E^T]}_{\dots \text{summed scaled tension from node height differences}} \begin{bmatrix} \| \\ u \\ \| \end{bmatrix} \quad \text{“heights” of nodes}$$

Linear ODE

$$\dot{u} = -Lu$$

Eigenvectors are oscillation modes

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

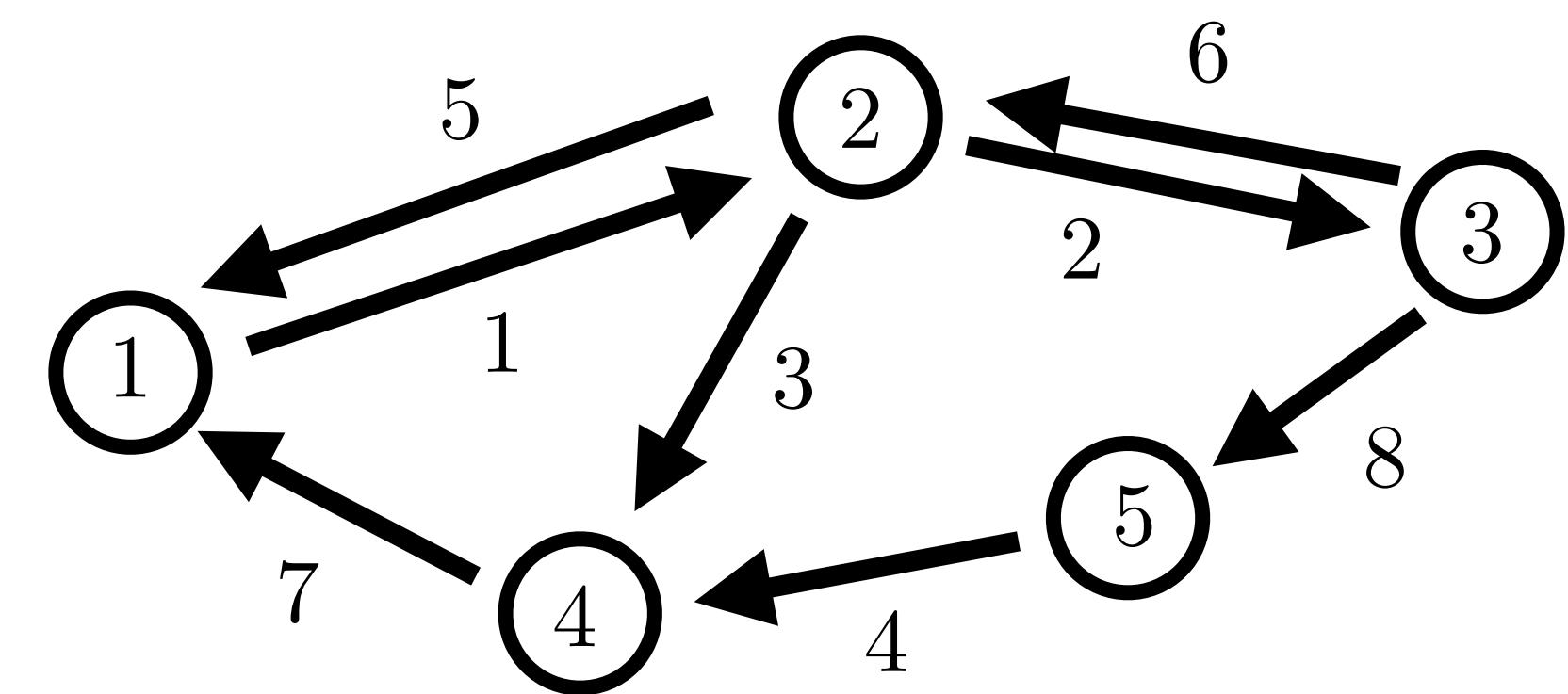
edge mass flows →

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian** “shape” matrix of rows (squared)

$$\begin{aligned} L = EE^T &= U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T \\ &= \begin{bmatrix} 3 & -2 & 0 & -1 & 0 \\ -2 & 5 & -2 & -1 & 0 \\ 0 & -2 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \end{aligned}$$

# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

$$e = (v, v')$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

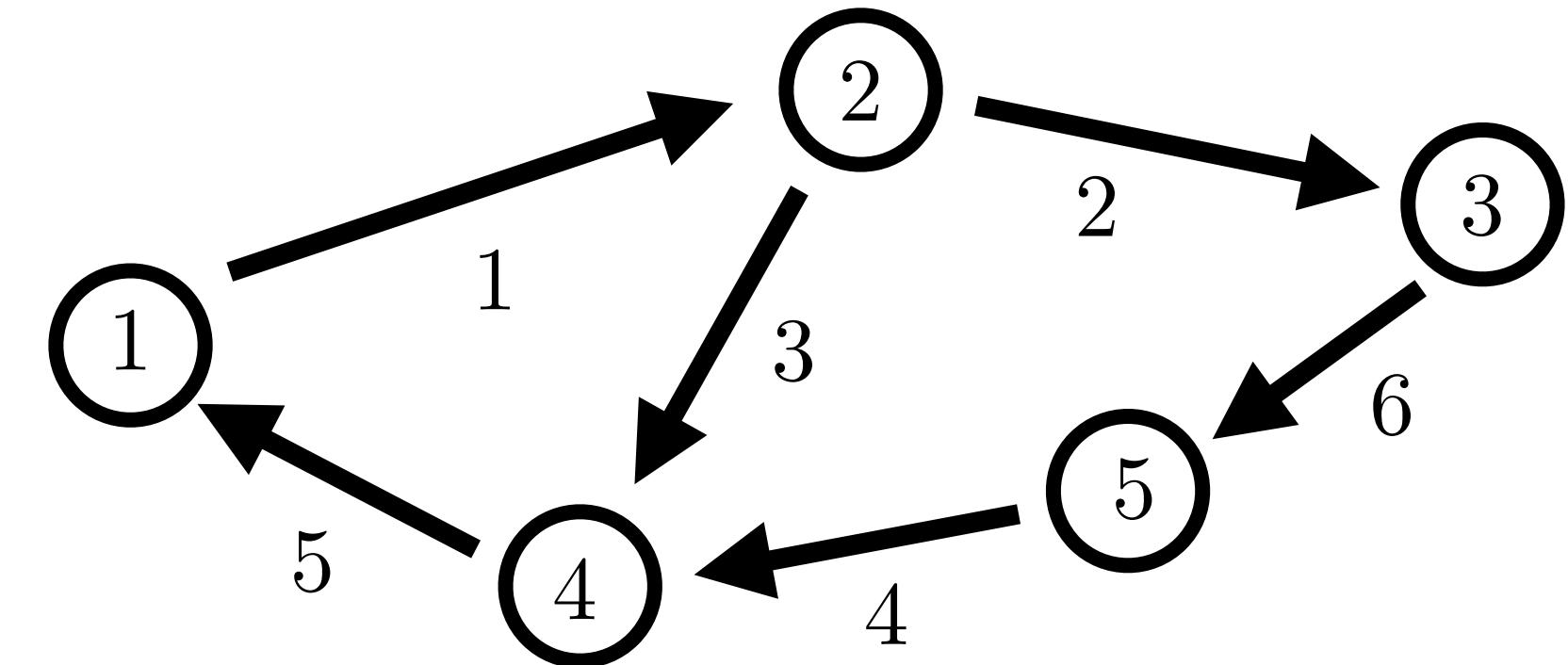
edge mass flows

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian** “shape” matrix of rows (squared)

$$\begin{aligned} L = EE^T &= U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T \\ &= \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \end{aligned}$$

Independent  
of edge direction

# Degree & Adjacency Matrices

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

$$e = (v, v')$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

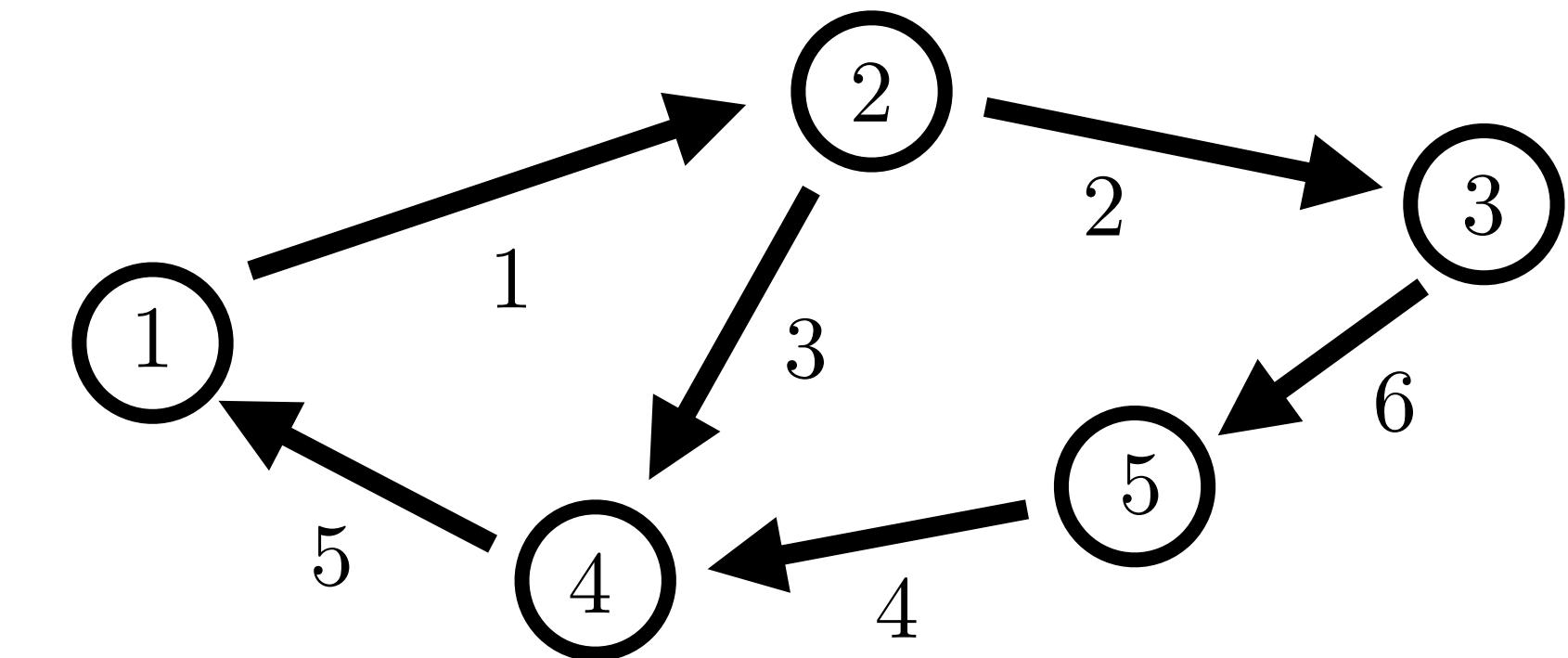
edge mass flows

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian**

“shape” matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Independent  
of edge direction

$$L = D - A$$

**Degree  
Matrix**

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

**Adjacency  
Matrix**

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

# Degree & Adjacency Matrices

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

$$e = (v, v')$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

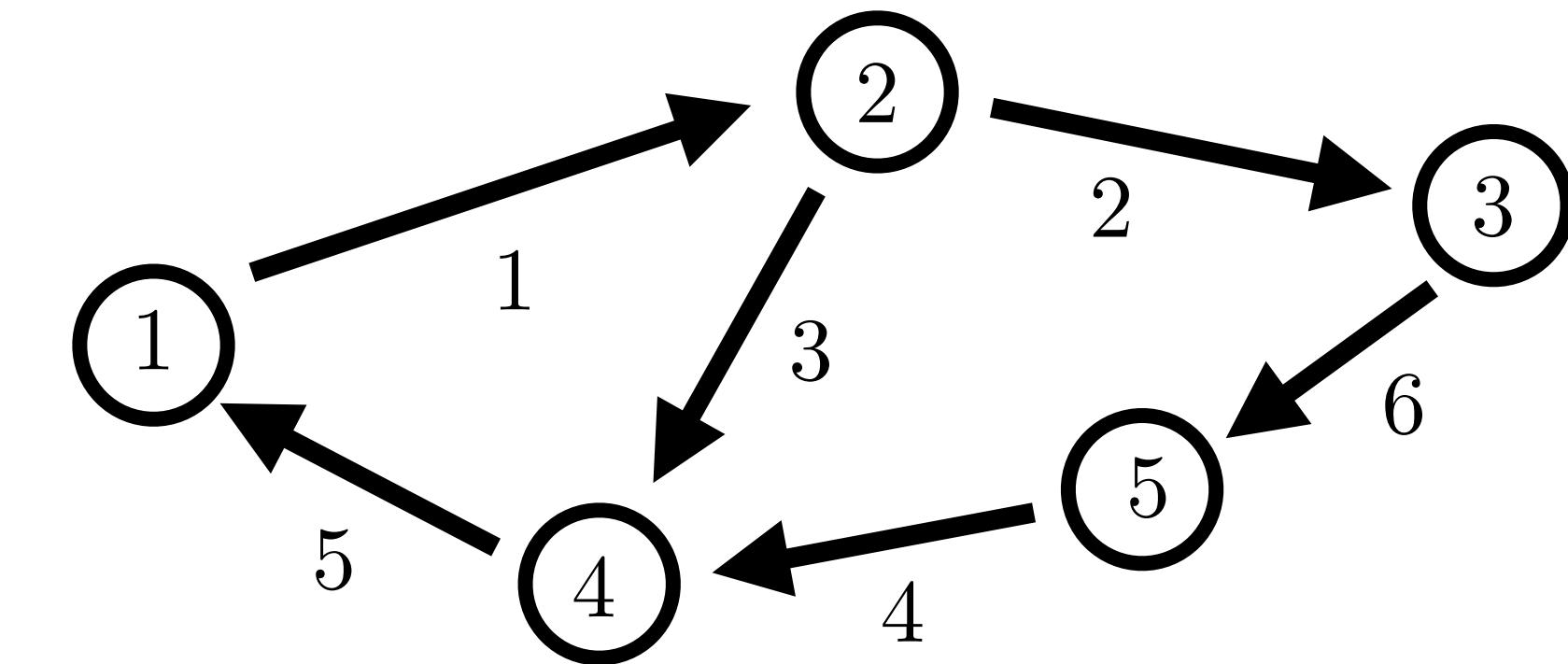
edge mass flows

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian**

“shape” matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Independent  
of edge direction

$$L = D - A$$

**Degree Matrix**

$$[D]_{vv} = \begin{array}{l} \text{diagonal} \\ |N_v| \\ \# \text{ edges} \\ (\text{in \& out}) \end{array}$$

**Adjacency Matrix**

$$[A]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$$

# Adjacency Matrix

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

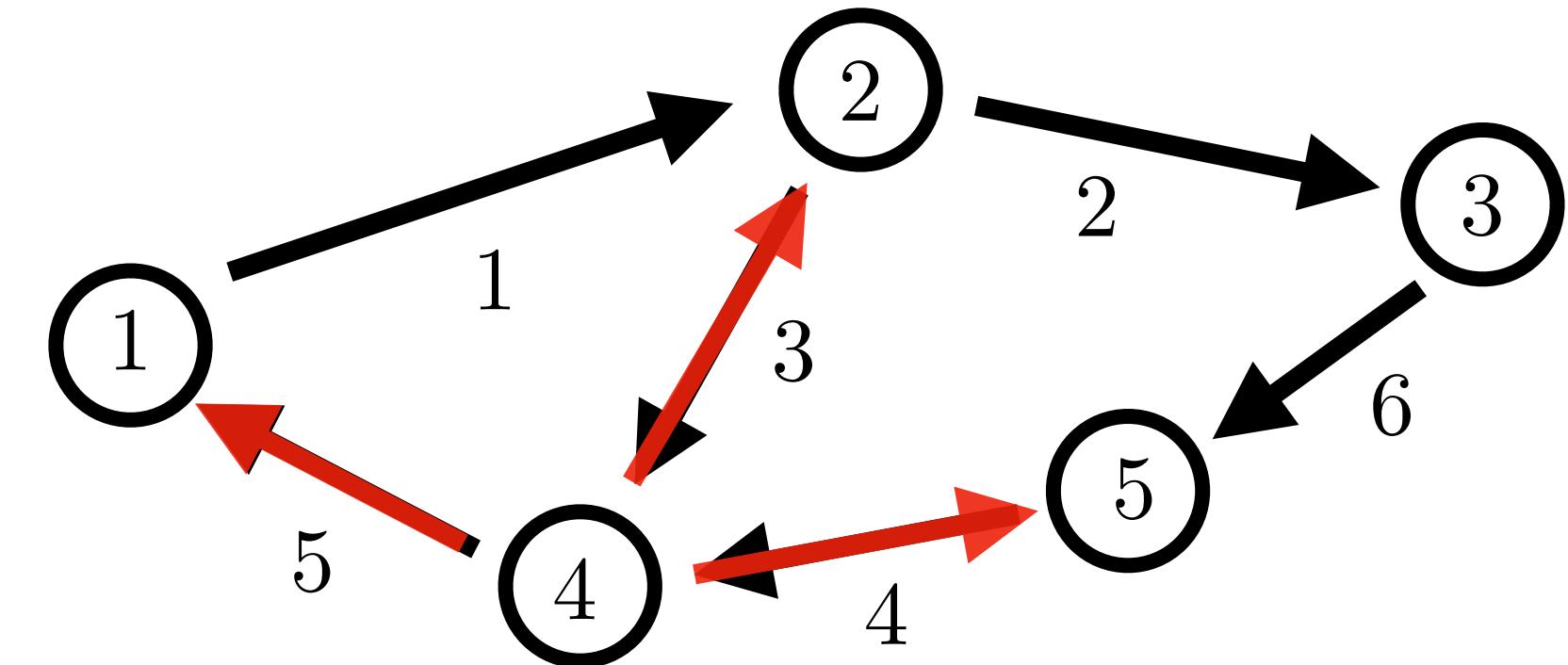
edge mass flows

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian**

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

**Degree  
Matrix**  
diagonal

$$[D]_{vv} = |\mathcal{N}_v|$$

**Adjacency  
Matrix**

$$[A]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$$

**Adjacency  
Matrix**

Edges to  
Nodes  
1,2, & 5

From Node 4

$$\begin{array}{c} \downarrow \\ \leftarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

# Adjacency Matrix

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

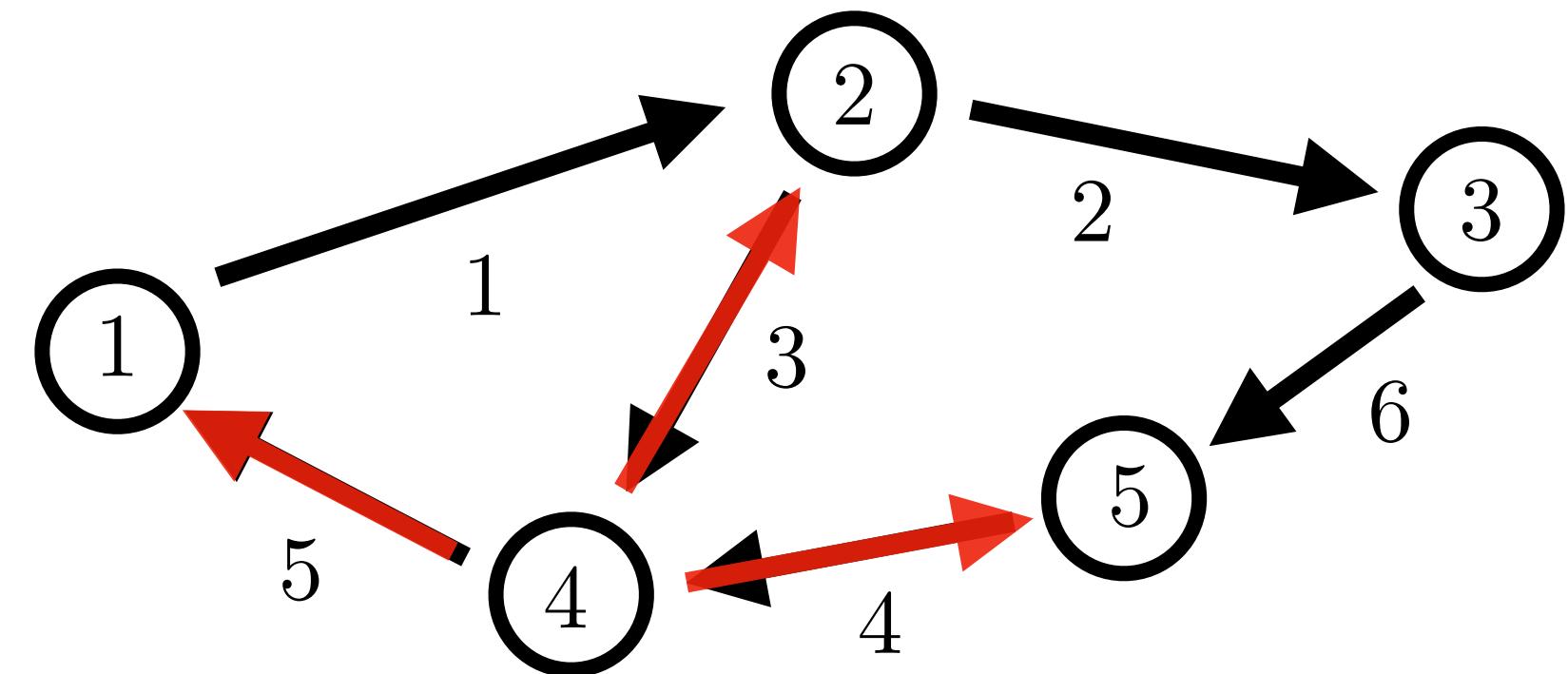
edge mass flows

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian**

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

**Degree  
Matrix**  
diagonal

$$[D]_{vv} = |\mathcal{N}_v|$$

**Adjacency  
Matrix**

$$[A]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$$

Start @  
node 4

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

←  
Edges to  
Nodes  
1,2, & 5

**Powers of  
Adjacency**

# Adjacency Matrix

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

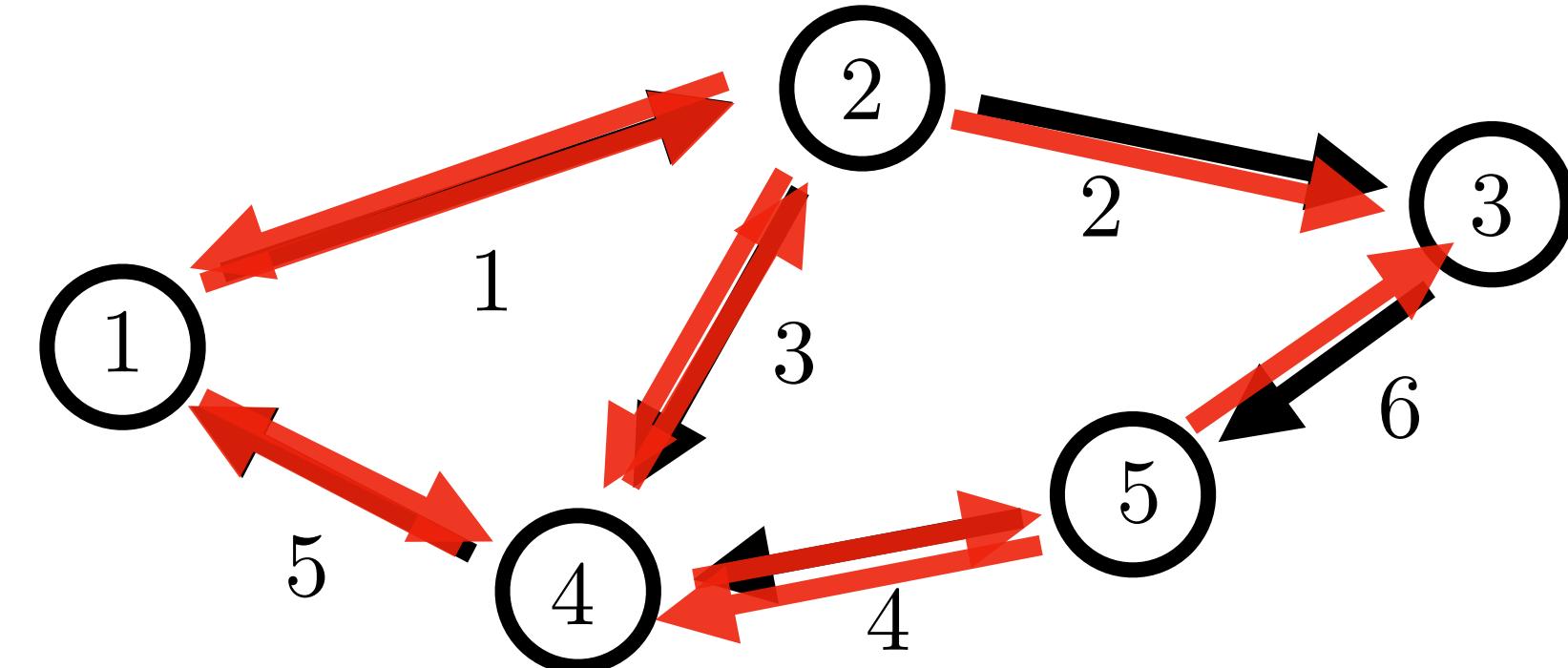
edge mass flows

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian**

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

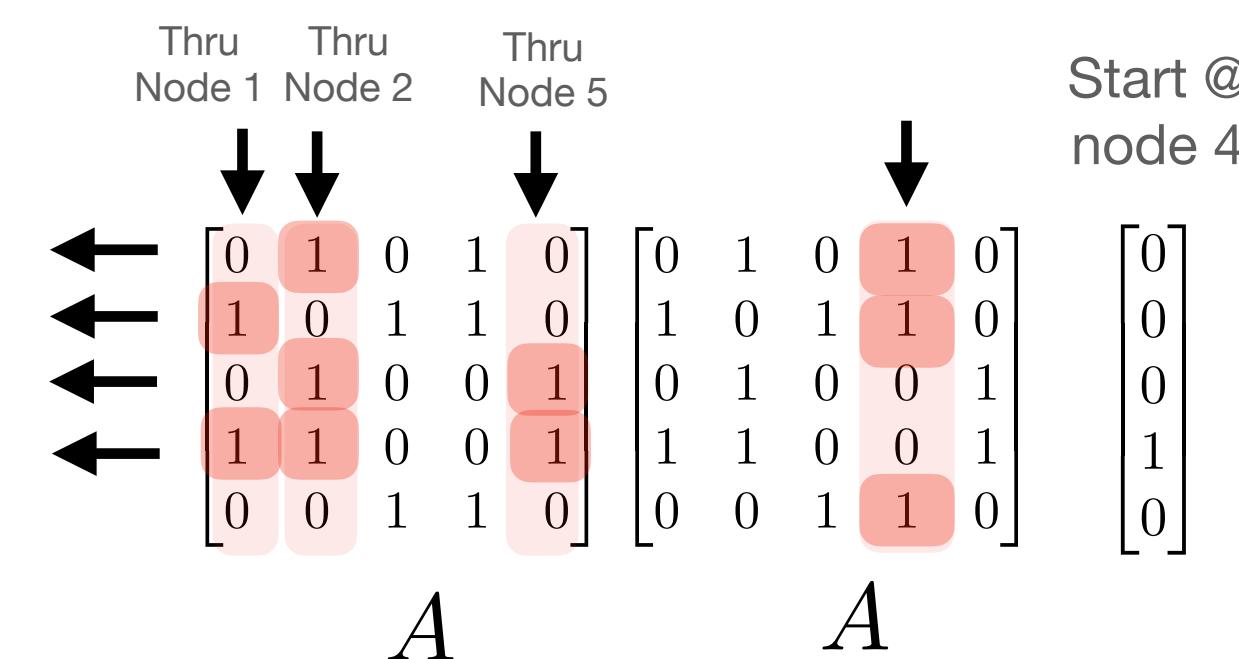
**Degree  
Matrix**  
diagonal

$$[D]_{vv} = |\mathcal{N}_v|$$

**Adjacency  
Matrix**

$$[A]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$$

**Powers of  
Adjacency**



# Graph Laplacian

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}) \quad \begin{array}{ll} \text{Vertices} & v \in \mathcal{V} \\ \text{Edges} & e \in \mathcal{E} \quad e = (v, v') \\ \dots \text{ directed or undirected} & \end{array}$$

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

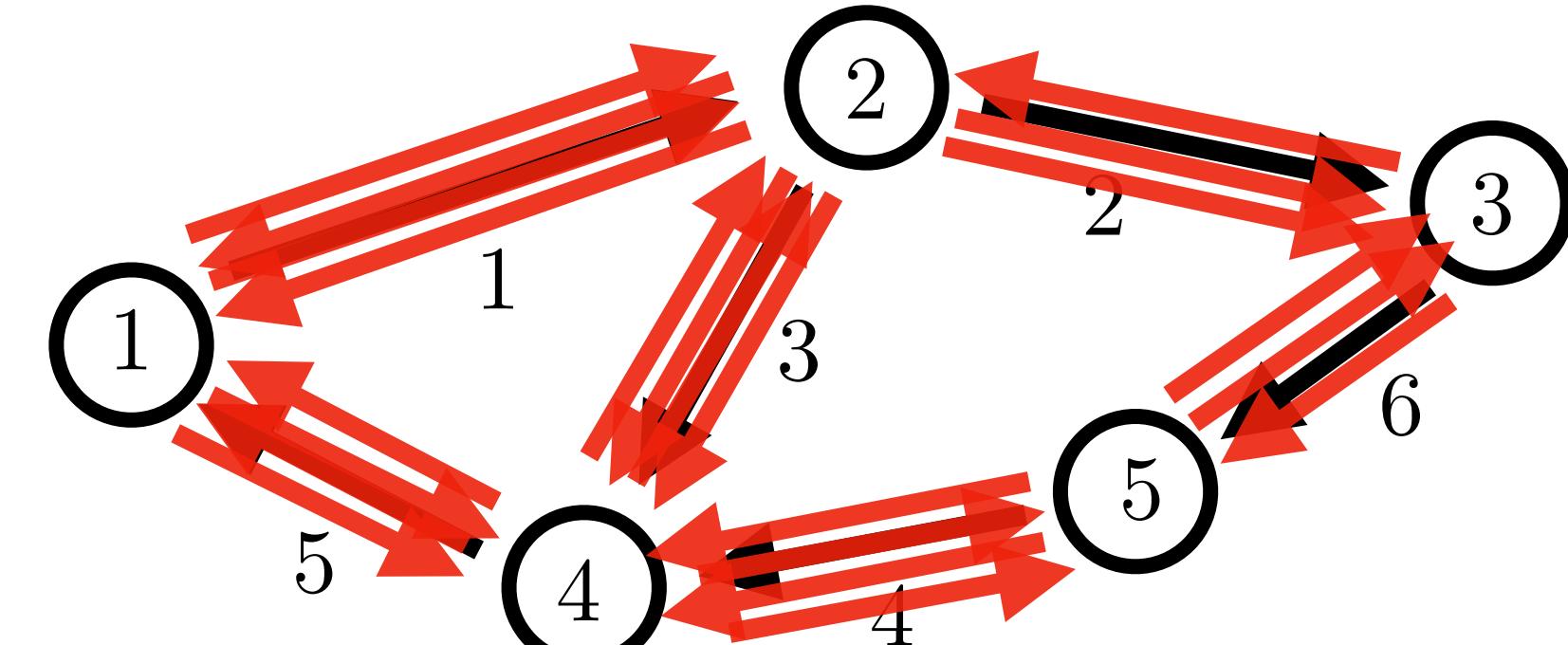
edge mass flows

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian**

“shape” matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

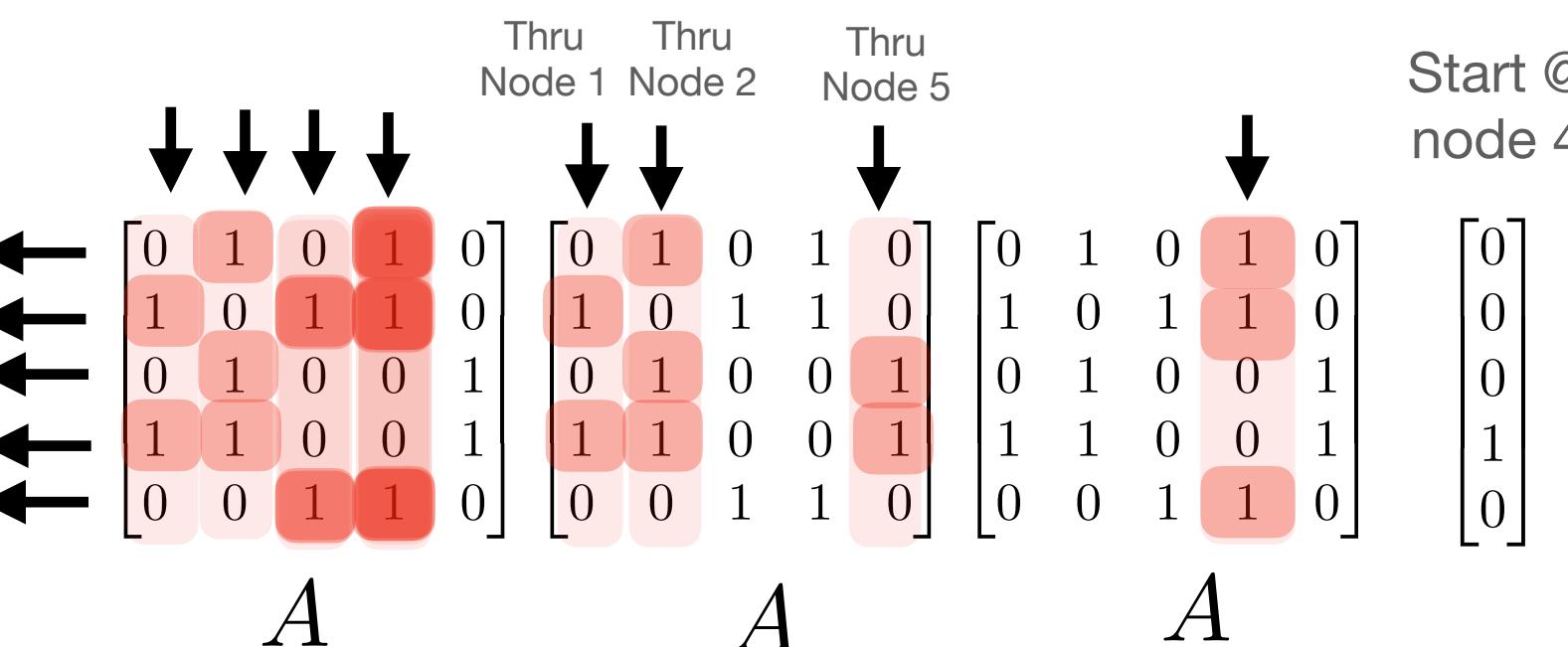
**Degree Matrix**      diagonal  
 $[D]_{vv} = |\mathcal{N}_v|$

**Adjacency Matrix**

$$[A]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$$

**Powers of  
Adjacency**

- # 3-step paths from node 4 to node 1
- # 3-step paths from node 4 to node 2
- ⋮
- # 3-step paths from node 4 to node 5



# Adjacency Matrix

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

**Vertices**

$$v \in \mathcal{V}$$

**Edges**

$$e \in \mathcal{E}$$

... directed or undirected

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}$$

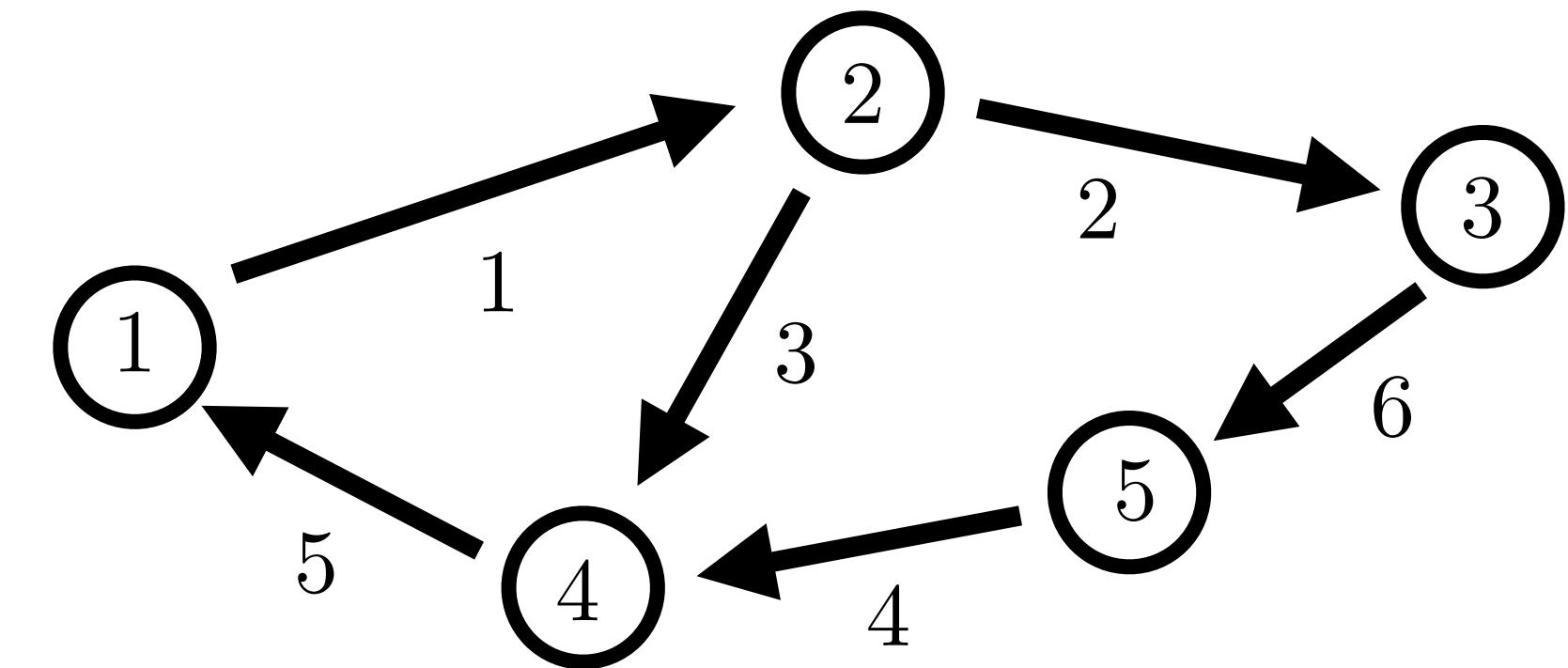
edge mass flows

Affine  
Constraint

$$Ex = S$$

Min Norm Solution

$$\bar{x} = E^T (EE^T)^\dagger S$$



**Graph Laplacian**

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

**Degree  
Matrix**  
diagonal

$$[D]_{vv} = |\mathcal{N}_v|$$

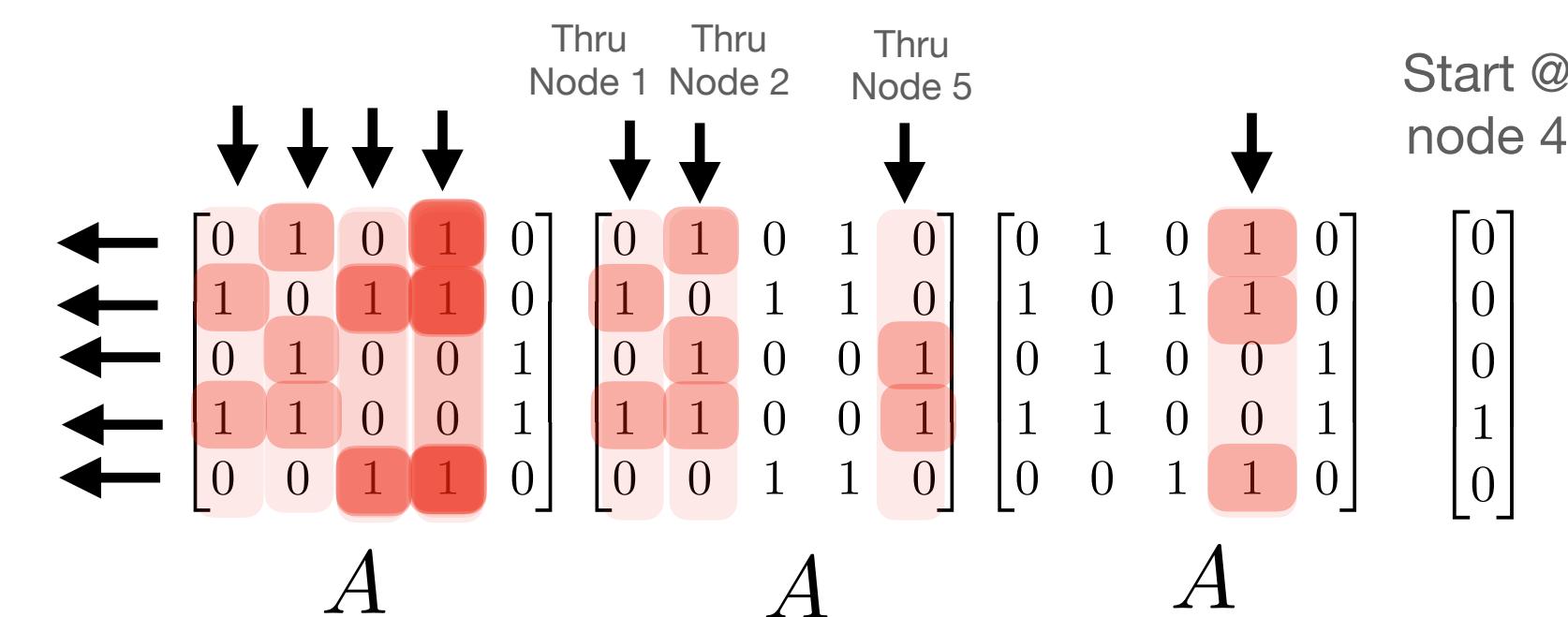
**Adjacency  
Matrix**

$$[A]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$$

**Powers of  
Adjacency**

$$[A^k]_{vv'}$$

# k-step paths  
from node v to node v'



# In/out Incidence Matrices

**Graph:**

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}) \quad \begin{array}{ll} \text{Vertices} & v \in \mathcal{V} \\ \text{Edges} & e \in \mathcal{E} \quad e = (v, v') \\ & \dots \text{ directed or undirected} \end{array}$$

**Incidence Matrix:**  $E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

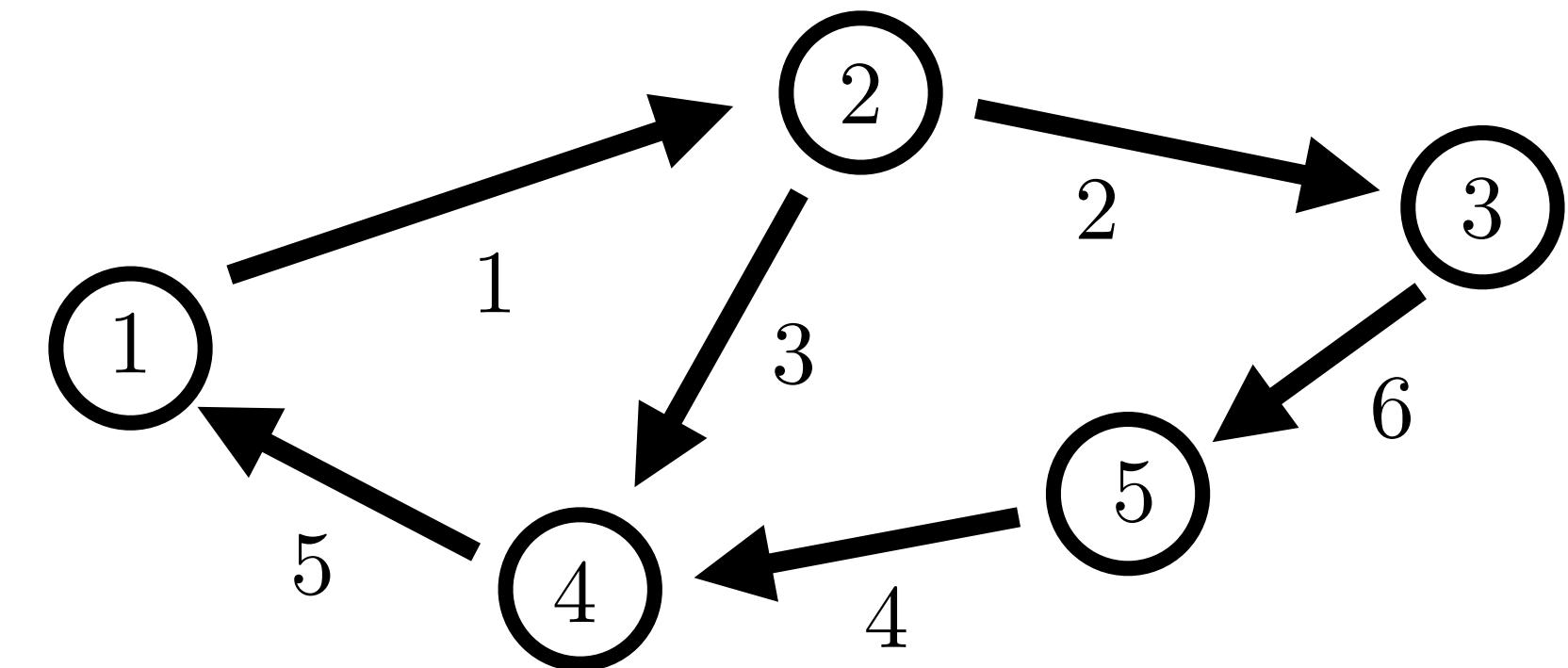
$$[E]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

**In/out Incidence Matrices:**

$$[E_{\text{out}}]_{ve} = \begin{cases} 1; & \text{if } e \text{ out of } v \\ 0; & \text{otherwise} \end{cases}$$

$$[E_{\text{in}}]_{ve} = \begin{cases} 1; & \text{if } e \text{ into } v \\ 0; & \text{otherwise} \end{cases}$$

$$E = E_{\text{in}} - E_{\text{out}}$$



**Graph Laplacian** “shape” matrix of rows (squared)

$$L = EE^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$L = D - A$$

**Degree Matrix** diagonal  
 $[D]_{vv} = |\mathcal{N}_v|$

**Powers of Adjacency**

$$[A^k]_{vv'} \quad \begin{array}{l} \# k\text{-step paths} \\ \text{from node } v \text{ to node } v' \end{array}$$

Start @ node 4


Thru Node 1 Thru Node 2 Thru Node 5

0	1	0	1	0	0
1	0	1	1	0	0
0	1	0	0	1	0
1	1	0	0	0	1
0	0	1	1	0	0

A

0	1	0	1	0	0
1	0	1	1	0	0
0	1	0	0	1	0
1	1	0	0	0	1
0	0	1	1	0	0

A

0	1	0	1	0	0
1	0	1	1	0	0
0	1	0	0	1	0
1	1	0	0	0	1
0	0	1	1	0	0

A