Parametricity and Free Theorems for Nested Types

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Abstract goes here

1 INTRODUCTION

Suppose we wanted to prove some property of programs over an algebraic data type (ADT) such as that of lists, coded in Agda as

```
data List (A : Set) : Set where
        nil  : List A
        Cons : A → List A → List A
```

A natural approach to the problem uses structural induction on the input data structure in question. This requires knowing not just the definition of the ADT of which the input data structure is an instance, but also the program text for the functions involved in the properties to be proved. For example, to prove by induction that mapping a polymorphic function over a list and then reversing the resulting list is the same as reversing the original list and then mapping the function over the result, we unwind the (recursive) definitions of the `reverse` and `map` functions over lists to according to the inductive structure of the input list. Such data-driven induction proofs over ADTs are so routine that they are often included in, say, undergraduate functional programming courses.

An alternative technique for proving results like the above `map-reverse` property for lists is to use parametricity, a formalization of extensional type-uniformity in polymorphic languages. Parametricity captures the intuition that a polymorphic program must act uniformly on all of its possible type instantiations; it is formalized as the requirement that every polymorphic program preserves all relations between any pair of types that it is instantiated with. Parametricity was originally put forth by Reynolds [Reynolds 1983] for System F [Girard et al. 1989], the formal calculus at the core of all polymorphic functional languages. It was later popularized for System F with a primitive list types as Wadler's so-called "theorems for free" [Walder 1989] because it allows the deduction of many properties of programs in such languages solely from their types, i.e., with no knowledge whatsoever of the text of the programs involved. While most of the free theorems derived by Wadler's are essentially naturality properties of polymorphic list-processing functions, parametricity can also be used to prove naturality properties for non-list ADTs, as well as properties, like correctness of the program optimization known as *short cut fusion* [Gill et al. 1993; Johann 2002, 2003], that go beyond simple naturality.

This paper is about parametricity for a variant of System F supporting not just ADTs, but nested types as well. An ADT defines a *family of inductive data types*, one for each input type. For example, the `List` data type definition above defines a collection of data types `List A`, `List B`, `List (A × B)`, `List (List A)`, etc., each independent of all the others. By contrast, a nested type [Bird and Meertens 1998] is an *inductive family of data types* that is defined over, or is defined mutually recursively with, (other) such data types. Since the structures of the data type at one type can depend on those at other types, the entire family of types must be defined at once. Examples of nested types include, trivially, ordinary ADTs, such as list and tree types; simple nested types, such as the data type

```
data PTree (A : Set) : Set where
        pleaf : A → PTree A
        pnode : PTree (A × A) → PTree A
```

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```
reversePTree : ∀{A : Set} → PTree A → PTree A
reversePTree {A} = foldPTree {A} {PTree}
                        pleaf
                        (λp → pnode (mapPTree swap p))

foldPTree : ∀{A : Set} → {F : Set → Set} →
               ({B : Set} → B → FB) →
               ({B : Set} → F(B × B) → FB)
               → PTree A → F A
foldPTree n c (pleaf x) = n x
foldPTree n c (pnode p) = c (foldPTree n c p)

mapPTree : ∀{A B : Set} → (A → B) → PLeaves A → PLeaves B
mapPTree f (pleaf x) = pleaf (f x)
mapPTree f (pnode p) = pnode (mapPTree (λp → (f(π₁ p), f(π₂ p))) p)

swap : ∀{A : Set} → (A × A) → (A × A)
swap (x, y) = (y, x)
```

Fig. 1. reversePTree and auxiliary functions in Agda

```
reverseBush : ∀{A : Set} → Bush A → Bush A
reverseBush {A} = bfold {A} {Bush} bnil balg

bfold : ∀{A : Set} → {F : Set → Set} →
          ({B : Set} → FB) →
          ({B : Set} → B → F (F B) → F B) →
          Bush A → F A
bfold bn bc bnil = bn
bfold bn bc (bcons x bb) =
          bc x (bfold bn bc (bmap (bfold bn bc) bb))

balg : ∀{B : Set} → B → Bush (Bush B) → Bush B
balg x bnil = bcons x bnil
balg x (bcons bnil bbbx) = bcons x (bcons bnil bbbx)
balg x (bcons (bcons y bx) bbbx) =
          bcons y (bcons (bcons x bx) bbbx)
```

Fig. 2. reverseBush and auxiliary functions in Agda

of perfect trees, whose recursive occurrences never appear below other type constructors; "deep" nested types [Johann and Polonsky 2020], such as the data type

```
data Forest (A : Set) : Set where
    fempty : Forest A
    fnode  : A → PTree (Forest A) → Forest A
```

of perfect forests, whose recursive occurrences appear below type constructors for other nested types; and truly nested types[1], such as the data type

```
data Bush (A : Set) : Set where
    bnil  : Bush A
    bcons : A → Bush (Bush A) → Bush A
```

of bushes (also called *bootstrapped heaps* in [Okasaki 1999]), whose recursive occurrences appear below their own type constructors.

Mention {−#TERMINATING#−} in figure. Suppose we now want to prove properties of functions over nested types. We might, for example, want to prove a map-reverse property for the functions on perfect trees in Figure 1, or for those on bushes in Figure 2. A few well-chosen examples quickly convince us that such a property should indeed hold for perfect trees, and, drawing inspiration from the situation for ADTs, we easily construct a proof by induction on the input perfect tree. To formally establish this result, we could even prove it in Coq or Agda: each of these provers actually generates an induction rule for perfect trees and the generated rule gives the expected result because proving properties of perfect trees requires only that we induct over the top-level perfect tree in the recursive position, leaving any data internal to the input tree untouched.

Unfortunately, it is nowhere near as clear that analogous intuitive or formal inductive arguments can be made for the map-reverse property for bushes. Indeed, a proof by induction on the input bush must recursively induct over the bushes that are internal to the top-level bush in the recursive position. This is sufficiently delicate that no induction rule for bushes or other truly nested types was known until very recently, when *deep induction* [Johann and Polonsky 2020] was developed as a way to induct over *all* of the structured data present in an input. Deep induction thus not only gave the first principled and practically useful structural induction rules for bushes and other truly nested types, and has also opened the way for incorporating automatic generation of such rules for (truly) nested data types — and, eventually, even GADTs — into modern proof assistants.

[1]Nested types that are defined over themselves are known as *truly nested types*.

Of course it is great to know that we *can*, at last, prove properties of programs over (truly) nested types by induction. But recalling that inductive proofs over ADTs can sometimes be circumvented in the presence of parametricity, we might naturally ask:

Can we prove properties of functions over (truly) nested types via paramertricity?

This paper answers the above question in the affirmative. To achieve this, we first introduce in Section 2 a polymorphic calculus supporting nested types as generated by the grammar of [Johann and Polonsky 2019]. At the type level, the calculus is the level-2-truncated version of the calculus from [Johann and Polonsky 2019]; the class of data types it includes as primitives is very robust and includes all (truly) nested types known from the literature. At the term level, our calculus features primitive constructs for functors, their initial algebras, and structured recursion (map, in, and fold, respectively). While our calculus does not support general recursion at the term level, it does provide strong termination guarantees since it is strongly normalizing, and it moves us toward the practical programming language based on ... sought by Wadler. Next, in Sections 3 and 4, we construct set and relational interpretations of our types and terms, respectively. Along the way, we prove that the model so constructed is parametric, i.e., that it satisfies an Identity Extension Lemma (Theorem 28) and an Abstraction Theorem (Theorem 34). Free theorems are just instantiations of the Abstraction Theorem. Our interpretations of types rely on the result from [Johann and Polonsky 2019] ensuring that all of the data types that can be represented in our calculus have well-defined semantics appropriately structured categories. Finally, in Section sec:ftnt, we formulate and prove in our calculus a variety of free theorems for nested types. We prove not only actually one generic theorem naturality-style free theorems in Section **??**, including those described above, but also give a formal proof of the short cut fusion for nested types first proposed in [**?**] in Section **??**.

In fact, we do much more than just build a parametric model for this calculus. The relationship between parametricity and naturality has long been of interest. By incorporating explicit Nat-types at the object level and interpreting them as natural transformations, our calculus allows us to clearly delineate those standard consequences of parametricity — like most of Wadler's — that are really consequences of naturality and those — like free theorems for the type of filter for lists, ADTs, and even nested types, or the standard short cut fusion for ADTs and its extension, new here, to nested types, or non-existence of terms of bottom type, or uniqueness of terms of the type of the polymorphic identity function — that actually go beyond naturality. This is made explicit in Enrico's subst theorem, which is an equality that is simply a consequence of our semantics and would hold even in a non-parametric model (i.e., one that didn't include the extra condition in the Set interpretation of Nat-types), and does not in any way reply on the abstraction theorem. (We will undoubtedly have to mention Bernardy et al. if we mention internalizing parametricity...)

Enrico's subst theorem is actually very general. Generic over data types, functors over which data types are built in Nat-types, and functions of Nat-type.

Foralls in Nat-types are at the object level, whereas the foralls in contexts are at the meta-level. So par results in subst theorem internalize parametricity in the calculus, whereas those parametricity results that do not follow from the interpretation of Nat-types are externalized at the meta-level

Couldn't do this before [Johann and Polonsky 2019] because we didn't know before that nested types (and then some) always have well-defined interpretations in locally finitely presentable categories like Set and Rel. In fact, could extend results here to "locally presentable fibrations", where these are yet to be defined, but would at least have locally presentable base and total categories with the locally presentable structure preserved by the fibration and appropriate reflection of the total category in the base (as in Alex's effects paper?).

Compare with Bob's paper. What's new? No ADTs, etc., at type level; no intro/elim rules for them at term level. Treatment of in Secs 4.4 and 4.5 therefore hand-wavey. Similar ideas, but hand-wavey. (Only gets one half of graph lemma, too.

mention Pitts categorical/denotational vs. operational semantics. Pitts has ADTs as primitives.

Mention that we don't have full polymorphism (like Bob) but just one level of foralls, but most free theorems only use that anyway (even short cut). We do have fixpoints at the term level, though — even h-o ones.

2 THE CALCULUS

2.1 Types

For each $k \geq 0$, we assume countable sets $\mathbb{T}^k$ of *type constructor variables of arity* k and $\mathbb{F}^k$ of *functorial variables of arity* k, all mutually disjoint. The sets of all type constructor variables and functorial variables are $\mathbb{T} = \bigcup_{k\geq 0} \mathbb{T}^k$ and $\mathbb{F} = \bigcup_{k\geq 0} \mathbb{F}^k$, respectively, and a *type variable* is any element of $\mathbb{T} \cup \mathbb{F}$. We use lower case Greek letters for type variables, writing ϕ^k to indicate that $\phi \in \mathbb{T}^k \cup \mathbb{F}^k$, and omitting the arity indicator k when convenient, unimportant, or clear from context. We reserve letters from the beginning of the alphabet to denote type variables of arity 0, i.e., elements of $\mathbb{T}^0 \cup \mathbb{F}^0$. We write $\overline{\zeta}$ for either a set $\{\zeta_1, ..., \zeta_n\}$ of type constructor variables or a set of functorial variables when the cardinality n of the set is unimportant or clear from context. If P is a set of type variables we write $P, \overline{\phi}$ for $P \cup \overline{\phi}$ when $P \cap \overline{\phi} = \emptyset$. We omit the vector notation for a singleton set, thus writing ϕ, instead of $\overline{\phi}$, for $\{\phi\}$.

Definition 1. *Let V be a finite subset of $\mathbb{T}$, let P be a finite subset of $\mathbb{F}$, let $\overline{\alpha}$ be a finite subset of $\mathbb{F}^0$ disjoint from P, and let $\phi^k \in \mathbb{F}^k \setminus P$. The sets $\mathcal{T}(V)$ of* type constructor expressions *over V and $\mathcal{F}^P(V)$ of* functorial expressions *over P and V are given by*

$$\mathcal{T}(V) \ ::= \ V \mid \mathsf{Nat}^{\overline{\alpha}}\, \mathcal{F}^{\overline{\alpha}}(V)\, \mathcal{F}^{\overline{\alpha}}(V) \mid V\overline{\mathcal{T}(V)}$$

and

$$\begin{aligned}\mathcal{F}^P(V) \ ::= \ & \mathcal{T}(V) \mid \mathbb{0} \mid \mathbb{1} \mid P\overline{\mathcal{F}^P(V)} \mid V\,\overline{\mathcal{F}^P(V)} \mid \mathcal{F}^P(V) + \mathcal{F}^P(V) \mid \mathcal{F}^P(V) \times \mathcal{F}^P(V) \\ & \mid \left(\mu\phi^{\,k}.\lambda\alpha_1...\alpha_k.\mathcal{F}^{P,\alpha_1,\ldots,\alpha_k,\phi}(V)\right)\overline{\mathcal{F}^P(V)}\end{aligned}$$

A *type* over P and V is any element of $\mathcal{T}(V) \cup \mathcal{F}^P(V)$.

The notation for types entails that an application $\tau\tau_1...\tau_k$ is allowed only when τ is a type variable of arity k, or τ is a subexpression of the form $\mu\phi^k.\lambda\alpha_1...\alpha_k.\tau'$. Moreover, if τ has arity k then τ must be applied to exactly k arguments. Accordingly, an overbar indicates a sequence of subexpressions whose length matches the arity of the type applied to it. The fact that types are always in *η-long normal form* avoids having to consider β-conversion at the level of types. In a subexpression $\mathsf{Nat}^{\overline{\alpha}}\sigma\,\tau$, the Nat operator binds all occurrences of the variables in $\overline{\alpha}$ in σ and τ. Similarly, in a subexpression $\mu\phi^k.\lambda\overline{\alpha}.\tau$, the μ operator binds all occurrences of the variable ϕ, and the λ operator binds all occurrences of the variables in $\overline{\alpha}$, in the body τ.

A *type constructor context* is a finite set Γ of type constructor variables, and a *functorial context* is a finite set Φ of functorial variables. In Definition 2, a judgment of the form $\Gamma; \Phi \vdash \tau : \mathcal{T}$ or $\Gamma; \Phi \vdash \tau : \mathcal{F}$ indicates that the type τ is intended to be functorial in the variables in Φ but not necessarily in the variables in Γ.

Definition 2. *The formation rules for the set* $\mathcal{T} \subseteq \bigcup_{V \subseteq \mathbb{T}} \mathcal{T}(V)$ *of* well-formed type constructor expressions *are*

$$\frac{}{\Gamma, v^0; \emptyset \vdash v^0 : \mathcal{T}}$$

$$\frac{\Gamma; \overline{\alpha} \vdash \sigma : \mathcal{F} \qquad \Gamma; \overline{\alpha} \vdash \tau : \mathcal{F}}{\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}} \sigma\, \tau : \mathcal{T}}$$

The formation rules for the set $\mathcal{F} \subseteq \bigcup_{V \subseteq \mathbb{T}, P \subseteq \mathbb{F}} \mathcal{F}^P(V)$ *of* well-formed functorial expressions *are*

$$\frac{\Gamma; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \vdash \tau : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi, v^0 \vdash v^0 : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{0} : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{1} : \mathcal{F}}$$

$$\frac{\phi^k \in \Gamma \cup \Phi \qquad \overline{\Gamma; \Phi \vdash \tau : \mathcal{F}}}{\Gamma; \Phi \vdash \phi^k \overline{\tau} : \mathcal{F}}$$

$$\frac{\Gamma; \Phi, \overline{\alpha}, \phi^k \vdash \tau : \mathcal{F} \qquad \overline{\Gamma; \Phi \vdash \tau : \mathcal{F}}}{\Gamma; \Phi \vdash (\mu \phi^k . \lambda \overline{\alpha}.\, \tau) \overline{\tau} : \mathcal{F}}$$

$$\frac{\Gamma; \Phi \vdash \sigma : \mathcal{F} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \vdash \sigma + \tau : \mathcal{F}} \qquad \frac{\Gamma; \Phi \vdash \sigma : \mathcal{F} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \vdash \sigma \times \tau : \mathcal{F}}$$

A type τ *is* well-formed *if it is either a well-formed type constructor expression or a well-formed functorial expression.*

If τ is a closed type we may write $\vdash \tau$, rather than $\emptyset; \emptyset \vdash \tau$, for the judgment that it is well-formed. Definition 2 ensures that the expected weakening rules for well-formed types hold — although weakening does not change the contexts in which Nat-types can be formed. If $\Gamma; \emptyset \vdash \sigma : \mathcal{T}$ and $\Gamma; \emptyset \vdash \tau : \mathcal{T}$, then our rules allow formation of the type $\Gamma; \emptyset \vdash \mathsf{Nat}^{\emptyset} \sigma\, \tau$. Since a type $\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}} \sigma\, \tau$ represents a natural transformation in $\overline{\alpha}$ from σ to τ, the type $\Gamma; \emptyset \vdash \mathsf{Nat}^{\emptyset} \sigma\, \tau$ represents the standard arrow type $\Gamma \vdash \sigma \to \tau$ in our calculus. We similarly represent a standard $\forall$-type $\Gamma; \emptyset \vdash \forall \overline{\alpha}.\tau$ as $\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, \mathbb{1}\, \tau : \mathcal{F}$ in our calculus. However, if $\overline{\alpha}$ is non-empty then τ cannot be of the form $\mathsf{Nat}^{\overline{\beta}} H\, K$ since $\Gamma; \overline{\alpha} \vdash \mathsf{Nat}^{\overline{\beta}} H\, K$ is not a valid type judgment in our calculus (except by weakening).

Definition 2 allows the formation of all of the (closed) nested types from the introduction:

$$\begin{array}{lcl} \mathit{List}\, \alpha & = & \mu\beta.\, \mathbb{1} + \alpha \times \beta \;=\; (\mu\phi.\lambda\beta.\, \mathbb{1} + \beta \times \phi\beta)\, \alpha \\ \mathit{PTree}\, \alpha & = & (\mu\phi.\lambda\beta.\, \beta + \phi\, (\beta \times \beta))\, \alpha \\ \mathit{Forest}\, \alpha & = & (\mu\phi.\lambda\beta.\, \mathbb{1} + \beta \times \mathit{PTree}(\phi\, \beta))\, \alpha \\ \mathit{Bush}\, \alpha & = & (\mu\phi.\lambda\beta.\, \mathbb{1} + \beta \times \phi\, (\phi\, \beta))\, \alpha \end{array}$$

Each of these types can either be natural in α or not, according to whether $\alpha \in \Gamma$ or $\alpha \in \Phi$. For example, if $\emptyset; \alpha \vdash \mathit{List}\, \alpha$, then the type $\vdash \mathsf{Nat}^{\alpha} \mathbb{1}\, (\mathit{List}\, \alpha) : \mathcal{T}$ is well-formed; If $\alpha; \emptyset \vdash \mathsf{List}\ \alpha$, then it is not. Definition 2 also allows the derivation of, e.g., the type $\alpha; \emptyset \vdash \mathsf{Nat}^{\alpha} (\mathit{List}\, \alpha)\, (\mathit{Tree}\, \alpha\, \gamma)$ representing a natural transformation from lists to trees that is natural in α but not necessarily in γ. We emphasize that types can be functorial in variables of arity greater than 0. For example, the type $\mathit{GRose}\, \phi\, \alpha = \mu\beta.\alpha \times \phi\beta$ can be functorial in ϕ if $\phi \in \Phi$. As usual, whether $\phi \in \Gamma$ or $\phi \in \Phi$ determines whether types such as $\mathsf{Nat}^{\alpha} (\mathit{GRose}\, \phi\, \alpha)\, (\mathit{List} \alpha)$ are well-formed. But even if *GRose* is functorial in ϕ, it still cannot be the (co)domain of a Nat type representing a natural transformation in ϕ. This is because our calculus does not allow naturality in variables of arity greater than 0.

Definition 2 explicitly considers types in $\mathcal{T}$ to be types in $\mathcal{F}$ that are functorial in no variables. It is not hard to see that this definition also supports the demotion of functorial variables in a well-formed type τ to non-functorial status. The proof is by induction on the structure of τ.

Lemma 3. *If $\Gamma;\Phi,\phi^k \vdash \tau : \mathcal{F}$, then $\Gamma,\psi^k;\Phi \vdash \tau[\phi^k :== \psi^k]$ is also derivable. Here, $\tau[\phi :== \psi]$ is the textual replacement of ϕ in τ, meaning that all occurences of $\phi\overline{\sigma}$ in τ become $\psi\overline{\sigma}$.*

In addition to textual replacement, we also have a proper substitution operation on types. If τ is a type over P and V, if P and V contain only type variables of arity 0, and if $k = 0$ for every occurrence of ϕ^k bound by μ in τ, then we say that τ is *first-order*; otherwise we say that τ is *second-order*. Substitution for first-order types is the usual capture-avoiding textual substitution. We write $\tau[\alpha := \sigma]$ for the result of substituting σ for α in τ, and $\tau[\alpha_1 := \tau_1, ..., \alpha_k := \tau_k]$, or $\tau[\overline{\alpha := \tau}]$ when convenient, for $\tau[\alpha_1 := \tau_1][\alpha_2 := \tau_2, ..., \alpha_k := \tau_k]$. Substitution for second-order types is defined below, where we adopt a similar notational convention for vectors of types.

Definition 4. *If $\phi^k \in \Gamma \cup \Phi$ with $k \geq 1$, if $\Gamma;\Phi \vdash F : \mathcal{F}$, and if $\Gamma,\overline{\beta};\Phi,\overline{\alpha} \vdash H : \mathcal{F}$ with $|\overline{\alpha}| + |\overline{\beta}| = k$, then $\Gamma \setminus \phi^k;\Phi \setminus \phi^k \vdash F[\phi :=_{\overline{\beta},\overline{\alpha}} H] : \mathcal{F}$, where the operation $(\cdot)[\phi := H]$ of* second-order type substitution *is defined by:*

$$
\begin{array}{lcl}
(\mathsf{Nat}^{\overline{\gamma}} G\, K)[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \mathsf{Nat}^{\overline{\gamma}}\, (G[\phi :=_{\overline{\beta},\overline{\alpha}} H])\, (K[\phi :=_{\overline{\beta},\overline{\alpha}} H]) \\
\mathbb{1}[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \mathbb{1} \\
\mathbb{0}[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \mathbb{0} \\
(\psi\overline{\sigma\tau})[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \begin{cases} \psi\, \overline{\tau[\phi :=_{\overline{\beta},\overline{\alpha}} H]} & \text{if } \psi \neq \phi \\ H[\overline{\alpha := \tau[\phi :=_{\overline{\beta},\overline{\alpha}} H]}][\overline{\beta := \sigma[\phi :=_{\overline{\beta},\overline{\alpha}} H]}] & \text{if } \psi = \phi \end{cases} \\
(\sigma + \tau)[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \sigma[\phi :=_{\overline{\beta},\overline{\alpha}} H] + \tau[\phi :=_{\overline{\beta},\overline{\alpha}} H] \\
(\sigma \times \tau)[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \sigma[\phi :=_{\overline{\beta},\overline{\alpha}} H] \times \tau[\phi :=_{\overline{\beta},\overline{\alpha}} H] \\
((\mu\psi.\lambda\overline{\gamma}.\, G)\overline{\tau})[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & (\mu\psi.\lambda\overline{\gamma}.\, G[\phi :=_{\overline{\beta},\overline{\alpha}} H])\, \overline{\tau[\phi :=_{\overline{\beta},\overline{\alpha}} H]}
\end{array}
$$

We omit the variable subscripts in second-order type constructor substitution when convenient.

2.2 Terms

We assume an infinite set $\mathcal{V}$ of term variables disjoint from $\mathbb{T}$ and $\mathbb{F}$. If Γ be a type constructor context and Φ is a functorial context, then a *term context for* Γ *and* Φ is a finite set of bindings of the form $x : \tau$, where $x \in \mathcal{V}$ and $\Gamma;\Phi \vdash \tau : \mathcal{F}$. We adopt the same conventions for denoting disjoint unions and for vectors in term contexts as for type constructor contexts and functorial contexts.

Definition 5. *Let Δ be a term context for Γ and Φ. The formation rules for the set of* well-formed terms over Δ *are*

$$
\frac{\Gamma;\emptyset \vdash \tau : \mathcal{T}}{\Gamma;\emptyset \mid \Delta, x : \tau \vdash x : \tau}
\qquad
\frac{\Gamma;\Phi \vdash \tau : \mathcal{F}}{\Gamma;\Phi \mid \Delta, x : \tau \vdash x : \tau}
$$

$$
\frac{}{\Gamma;\Phi \mid \Delta \vdash \top : \mathbb{1}}
\qquad
\frac{\Gamma;\Phi \mid \Delta \vdash t : \mathbb{0} \qquad \Gamma;\Phi \vdash \tau : \mathcal{F}}{\Gamma;\Phi \mid \Delta \vdash \bot_\tau t : \tau}
$$

$$
\frac{\Gamma;\Phi \mid \Delta \vdash s : \sigma}{\Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau}
\qquad
\frac{\Gamma;\Phi \mid \Delta \vdash t : \tau}{\Gamma;\Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau}
$$

$$
\frac{\Gamma;\Phi \vdash \tau,\sigma : \mathcal{F} \qquad \Gamma;\Phi \mid \Delta \vdash t : \sigma + \tau \qquad \Gamma;\Phi \mid \Delta, x : \sigma \vdash l : \gamma \qquad \Gamma;\Phi \mid \Delta, y : \tau \vdash r : \gamma}{\Gamma;\Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{\mathsf{inl}\, x \mapsto l;\ \mathsf{inr}\, y \mapsto r\} : \gamma}
$$

$$
\frac{\Gamma;\Phi \mid \Delta \vdash s : \sigma \qquad \Gamma;\Phi \mid \Delta \vdash t : \tau}{\Gamma;\Phi \mid \Delta \vdash (s, t) : \sigma \times \tau}
\qquad
\frac{\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau}{\Gamma;\Phi \mid \Delta \vdash \pi_1 t : \sigma}
\qquad
\frac{\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau}{\Gamma;\Phi \mid \Delta \vdash \pi_2 t : \tau}
$$

$$\frac{\Gamma;\overline{\alpha}\vdash F:\mathcal{F} \qquad \Gamma;\overline{\alpha}\vdash G:\mathcal{F} \qquad \Gamma;\overline{\alpha}\,|\,\Delta,x:F\vdash t:G}{\Gamma;\emptyset\,|\,\Delta\vdash L_{\overline{\alpha}}x.t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G}$$

$$\frac{\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G \qquad \overline{\Gamma;\Phi\vdash\tau:\mathcal{F}} \qquad \Gamma;\Phi\,|\,\Delta\vdash s:F[\overline{\alpha:=\tau}]}{\Gamma;\Phi\,|\,\Delta\vdash t_{\overline{\tau}}s:G[\overline{\alpha:=\tau}]}$$

$$\frac{\Gamma;\overline{\phi},\overline{\gamma}\vdash H:\mathcal{F} \qquad \overline{\Gamma;\overline{\beta},\overline{\gamma}\vdash F:\mathcal{F}} \qquad \overline{\Gamma;\overline{\beta},\overline{\gamma}\vdash G:\mathcal{F}}}{\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{map}_H^{\overline{F},\overline{G}}:\mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi:=_{\overline{\beta}}F}]\,H[\overline{\phi:=_{\overline{\beta}}G}])}$$

$$\frac{\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H:\mathcal{F}}{\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{in}_H:\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}H[\phi:=_{\overline{\beta}}(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}$$

$$\frac{\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H:\mathcal{F} \qquad \Gamma;\overline{\beta},\overline{\gamma}\vdash F:\mathcal{F}}{\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{fold}_H^F:\mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi:=_{\overline{\beta}}F][\overline{\alpha:=\beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta})\,F)}$$

In the rule for $L_{\overline{\alpha}}x.t$, the L operator binds all occurrences of the type variables in $\overline{\alpha}$ in the type of the term variable x and in the body t, as well as all occurrences of x in t. In the rule for $t_{\overline{\tau}}s$ there is one functorial expression τ for every functorial variable α. In the rule for $\mathsf{map}_H^{\overline{F},\overline{G}}$ there is one functorial expression F and one functorial expression G for each functorial variable in $\overline{\phi}$. Moreover, for each $\phi^k \in \overline{\phi}$ the number of functorial variables β in the judgments for its corresponding functorial expresssions F and G is k. In the rules for in_H and fold_H^F, the functorial variables in $\overline{\beta}$ are fresh with respect to H, and there is one β for every α. (Recall from above that, in order for the types of in_H and fold_H^F to be well-formed, the length of α must equal the arity of ϕ.) Substitution for terms is the obvious extension of the usual capture-avoiding textual substitution, and Definition 5 ensures that the expected weakening rules for well-formed terms hold.

Using Definition 5 we can represent the `reversePTree` function from Figure 1 in our calculus as

$$\vdash \mathsf{fold}_{\beta+\phi(\beta\times\beta)}^{PTree\,\alpha}\,(\mathsf{in}_{\beta+\phi(\beta\times\beta)}\circ s):\mathsf{Nat}^{\alpha}(PTree\,\alpha)\,(PTree\,\alpha)$$

where

$$\begin{array}{lcl}
\vdash \mathsf{fold}_{\beta+\phi(\beta\times\beta)}^{PTree\,\alpha} & : & \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\alpha}(\alpha+PTree\,(\alpha\times\alpha))\,(PTree\,\alpha))\,(\mathsf{Nat}^{\alpha}(PTree\,\alpha)\,(PTree\,\alpha))\\
\vdash \mathsf{in}_{\beta+\phi(\beta\times\beta)} & : & \mathsf{Nat}^{\alpha}(\alpha+PTree\,(\alpha\times\alpha))\,(PTree\,\alpha)\\
\vdash \mathsf{map}_{PTree\,\alpha}^{\alpha\times\alpha,\alpha\times\alpha} & : & \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\alpha}(\alpha\times\alpha)\,(\alpha\times\alpha))\,(\mathsf{Nat}^{\alpha}(PTree\,(\alpha\times\alpha))\,(PTree\,(\alpha\times\alpha)))
\end{array}$$

and *swap* and s are the terms

$$\vdash L_{\alpha}p.\,(\pi_2 p,\pi_1 p):\mathsf{Nat}^{\alpha}(\alpha\times\alpha)\,(\alpha\times\alpha)$$

and

$$\vdash L_{\alpha}t.\,\mathsf{case}\;t\;\mathsf{of}\;\{b\mapsto\mathsf{inl}\,b;\;t'\mapsto\mathsf{inr}\,(\mathsf{map}_{PTree\,\alpha}^{\alpha\times\alpha,\alpha\times\alpha}\;swap\,t')\}:\mathsf{Nat}^{\alpha}(\alpha+PTree\,(\alpha\times\alpha))\,(\alpha+PTree\,(\alpha\times\alpha))$$

respectively. We can similarly represent the `reverseBush` function from Figure 2 as

$$\vdash \mathsf{fold}_{\mathbb{1}+\beta\times\phi(\phi\beta)}^{Bush\,\alpha}\,(\mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)}\circ(\mathbb{1}+t\circ i\circ i')):\mathsf{Nat}^{\alpha}(Bush\,\alpha)\,(Bush\,\alpha)$$

where

$$\vdash \mathsf{fold}_{\mathbb{1}+\beta\times\phi(\phi\beta)}^{Bush\,\alpha}:\mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\,(\mathbb{1}+\alpha\times Bush\,(Bush\,\alpha)))\,(Bush\,\alpha))\,(\mathsf{Nat}^{\alpha}\,(Bush\,\alpha)\,(Bush\,\alpha))$$

$$\vdash \mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)}:\mathsf{Nat}^{\alpha}\,(\mathbb{1}+\alpha\times Bush\,(Bush\,\alpha))\,(Bush\,\alpha)$$

and *bnil*, *bcons*, $\mathsf{in}^{-1}_{\mathbb{1}+\beta\times\phi(\phi\beta)}$, t, i, and i' are the terms

$$
\begin{array}{l}
\vdash \mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)} \circ (L_\alpha\, x.\,\mathsf{inl}\, x) \;:\; \mathsf{Nat}^\alpha\, \mathbb{1}\, (Bush\,\alpha)\\
\vdash \mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)} \circ (L_\alpha\, x.\,\mathsf{inr}\, x) \;:\; \mathsf{Nat}^\alpha\, (\alpha\times Bush\,(Bush\,\alpha))\, (Bush\,\alpha)\\
\vdash \mathsf{fold}^{(\mathbb{1}+\beta\times\phi(\phi\beta))[\phi:=Bush\,\alpha]}_{\mathbb{1}+\beta\times\phi(\phi\beta)}\, (\mathsf{map}^{(\mathbb{1}+\beta\times\phi(\phi\beta))[\phi:=Bush\,\alpha][\beta:=\alpha],\,Bush\,\alpha}_{\mathbb{1}+\beta\times\phi(\phi\beta)}\; \mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)})\\
\quad : \mathsf{Nat}^\alpha\, (Bush\,\alpha)\, (\mathbb{1} + \alpha\times Bush\,(Bush\,\alpha))\\
\vdash L_\alpha\,(b,s).\,\mathsf{case}\; s\,\{\quad * \mapsto bcons_\alpha\, b\,(bnil_\alpha\, *);\\
\qquad\qquad\qquad (s',u) \mapsto \mathsf{case}\; s'\{\quad * \mapsto bcons_\alpha\, b\,(bcons_{Bush\,\alpha}\,(bnil_\alpha\,*)\,u);\\
\qquad\qquad\qquad\qquad\qquad\qquad (b',u') \mapsto bcons_\alpha\, b'(bcons_{Bush\,\alpha}\,(bcons_\alpha\, b\, u)\, u')\}\}\\
\quad : \mathsf{Nat}^\alpha(\alpha\times(\mathbb{1} + (\mathbb{1}+\alpha\times Bush\,(Bush\,\alpha)))\times Bush\,(Bush\,(Bush\,\alpha)))\; (\alpha\times Bush\,(Bush\,\alpha))\\
\vdash \alpha\times(\mathbb{1} + \mathsf{in}^{-1}_{\mathbb{1}+\beta\times\phi(\phi\beta)} \times Bush\,(Bush\,(Bush\,\alpha)))\\
\quad : \mathsf{Nat}^\alpha(\alpha\times(\mathbb{1} + Bush\,\alpha\times Bush\,(Bush\,(Bush\,\alpha))))\\
\qquad\quad (\alpha\times(\mathbb{1} + (\mathbb{1}+\alpha\times Bush\,(Bush\,\alpha))\times Bush\,(Bush\,(Bush\,\alpha))))\\
\vdash \alpha\times(L_\alpha\, x.\,(\mathsf{in}^{-1}_{\mathbb{1}+\beta\times\phi(\phi\beta)})_{Bush\,\alpha}\, x)\\
\quad : \mathsf{Nat}^\alpha(\alpha\times Bush\,(Bush\,\alpha))\; (\alpha\times(\mathbb{1} + Bush\,(\alpha\times Bush\,(Bush\,(Bush\,\alpha)))))
\end{array}
$$

respectively. Here, $\Gamma;\emptyset\,|\,\Delta \vdash \sigma+\eta : \mathsf{Nat}^{\overline{\alpha}}(\sigma+F)\,(\sigma+G)$ and $\Gamma;\emptyset\,|\,\Delta\vdash\sigma\times\eta : \mathsf{Nat}^{\overline{\alpha}}(\sigma\times F)\,(\sigma\times G)$ for $\sigma+\eta := L_{\overline{\alpha}}\,x.\,\mathsf{case}\; x\; \mathsf{of}\;\{s\mapsto \mathsf{inl}\, s;\; t\mapsto \mathsf{inr}\,(\eta_{\overline{\alpha}} t)\}$ and $\sigma\times\eta := L_{\overline{\alpha}}\,x.\,(\pi_1 x, \eta_{\overline{\alpha}}(\pi_2 x))$ for $\Gamma;\emptyset\,|\,\Delta\vdash \eta : \mathsf{Nat}^{\overline{\alpha}} F\; G$ and $\Gamma;\overline{\alpha}\vdash\sigma:\mathcal{F}$.

Unfortunately, we cannot write functions, such as $concat : PTree\,\alpha \to PTree\,\alpha \to PTree\,\alpha$, that take as input more than one non-algebraic nested type. This is because Nat-types must be formed in empty functorial contexts, and this conflicts with the need to feed folds algebras. (Can't fold over pairs to get the right functions; can't get them by using continuation style because of the aforementioned typing conflict.) Add Daniel's commentary about "real" involution reverse for bushes, too. Massage paragraph. Not a good restriction.

The presence of the "extra" functorial variables in $\overline{\gamma}$ in the rules for $\mathsf{map}^{\overline{F},\overline{G}}_H$, in_H, and fold^F_H merit special mention. They allows us to map or fold polymorphic functions over nested types. Consider, for example, the function $flatten : \mathsf{Nat}^\beta(PTree\,\beta)\,(List\,\beta)$ that maps perfect trees to lists. Even in the absence of extra variables the instance of map required to map each non-functorial monomorphic instantiation of *flatten* over a list of perfect trees is well-typed:

$$
\frac{\Gamma;\alpha\vdash List\,\alpha \qquad \Gamma;\emptyset\vdash\sigma \qquad \Gamma;\emptyset\vdash\tau \qquad \Gamma;\emptyset\vdash PTree\,\sigma \qquad \Gamma;\emptyset\vdash List\,\tau}{\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}^{PTree\,\sigma,\;List\,\tau}_{List\,\alpha} : \mathsf{Nat}^\emptyset\;(\mathsf{Nat}^\emptyset\,(PTree\,\sigma)\,(List\,\tau))\;(\mathsf{Nat}^\emptyset\,(List\,(PTree\,\sigma))\,(List\,(List\,\tau)))}
$$

But in the absence of $\overline{\gamma}$, the instance

$$
\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}^{PTree\,\beta,List\,\beta}_{List\,\alpha} : \mathsf{Nat}^\emptyset\;(\mathsf{Nat}^\beta(PTree\,\beta)\,(List\,\beta))\;(\mathsf{Nat}^\beta\,(List\,(PTree\,\beta))\,(List\,(List\,\beta)))
$$

of `map` required to map the *polymorphic flatten* function over a list of perfect trees is not: in that setting the functorial contexts for F and G in the rule for $\mathsf{map}^{F,G}_H$ would have to be empty, but the fact that the polymorphic *flatten* function is functorial in some variable, say δ, means that it cannot possibly have a type of the form $\mathsf{Nat}^\emptyset F\, G$ that would be required for it to be the function input to map. Since untypeability of this instance of map is unsatisfactory in a polymorphic calculus, where we naturally expect to be able to manipulate entire polymorphic functions rather than just their monomorphic instances, we use the "extra" variables in $\overline{\gamma}$ to remedy the situation. Specifrically, the rules from Definition 5 ensure that the instance of map needed to map the polymorphic *flatten* function is typeable as follows:

$$
\frac{\Gamma;\alpha,\beta\vdash List\,\alpha \qquad \Gamma;\beta\vdash PTree\,\beta \qquad \Gamma;\beta\vdash List\,\beta}{\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}^{F,G}_{List} : \mathsf{Nat}^\emptyset\;(\mathsf{Nat}^\beta(PTree\,\beta)\,(List\,\beta))\;(\mathsf{Nat}^\beta\,(List\,(PTree\,\beta))\,(List\,(List\,\beta)))}
$$

Similar remarks explain the appearance of $\overline{\gamma}$ in the typing rules for in and fold.

3 INTERPRETING TYPES

We denote the category of sets and functions by Set. The category Rel has as its objects triples (A, B, R) where R is a relation between the objects A and B in Set, i.e., a subset of $A \times B$, and has as its morphisms from (A, B, R) to (A', B', R') pairs $(f : A \to A', g : B \to B')$ of morphisms in Set such that $(f\,a, g\,b) \in R'$ whenever $(a, b) \in R$. We write $R : \mathsf{Rel}(A, B)$ in place of (A, B, R) when convenient. If $R : \mathsf{Rel}(A, B)$ we write $\pi_1 R$ and $\pi_2 R$ for the *domain* A of R and the *codomain* B of R, respectively. If A : Set, then we write $\mathsf{Eq}_A = (A, A, \{(x, x) \mid x \in A\})$ for the *equality relation* on A.

If C and $\mathcal{D}$ are locally finitely presentable categories [Adámek and Rosický 1994], write $[C, \mathcal{D}]$ for the set of ω-cocontinuous functors from C to $\mathcal{D}$. Both Set and Rel are locally finitely presentable.

The key idea underlying Reynolds' parametricity is to give each type $\tau(\alpha)$ with one free variable α both an *object interpretation* τ_0 taking sets to sets and a *relational interpretation* τ_1 taking relations $R : \mathsf{Rel}(A, B)$ to relations $\tau_1(R) : \mathsf{Rel}(\tau_0(A), \tau_0(B))$, and to interpret each term $t(\alpha, x) : \tau(\alpha)$ with one free term variable $x : \sigma(\alpha)$ as a map t_0 associating to each set A a function $t_0(A) : \sigma_0(A) \to \tau_0(A)$. These interpretations are to be given inductively on the structures of τ and t in such a way that they imply two fundamental theorems. The first is an *Identity Extension Lemma*, which states that if R is the equality relation on A then $\tau_1(R)$ is the equality relation on $\tau_0(A)$, and is the essential property that makes a model relationally parametric, rather than just induced by a logical relation. The second is an *Abstraction Theorem*, which states that, for any $R : \mathsf{Rel}(A, B)$, $(t_0(A), t_0(B))$ is a morphism in Rel from $(\sigma_0(A), \sigma_0(B), \sigma_1(R))$ to $(\tau_0(A), \tau_0(B), \tau_1(R))$. The Identity Extension Lemma is similar to the Abstraction Theorem except that it applies to, and thus can be used to reason about, *all* elements of a type's interpretation, not just those that are interpretations of terms. Similar results are expected to hold for types and terms with any number of free variables.

As usual, parametricity in our setting requires that set interpretations of types are defined simultaneously with their relational interpretations. This allows us to cut down the interpretations of Nat types to include only the "parametric" elements, as discussed in, e.g., [Reynolds 1983; ?; ?; ?], which is crucial to obtaining an Identity Extension Lemma (Theorem 28) in the present setting. We give set interpretations for our types in Section 3.1 and give their relational interpretations in Section 3.2. While the set interpretations are relatively straightforward, their relation interpretations are less so, mainly because of the cocontinuity conditions we must impose to ensure that they are well-behaved. We take some effort to develop conditions in Section 3.2, which separates Definitions 7 and 17 in space, but otherwise has no impact on the fact that they are given by mutual induction.

3.1 Interpreting Types as Sets

To ensure that we stay in the setting of [Johann and Polonsky 2019] — and thus to ensure that all of the types in our calculus have well-defined intepretations in the Set and Rel — we interpret all type variables as ω-cocontinuous functors in Definitions 6 and 15.

Definition 6. *A* set environment *maps each type variable in $\mathbb{T}^k \cup \mathbb{F}^k$ to an element of* $[\mathsf{Set}^k, \mathsf{Set}]$. *A morphism $f : \rho \to \rho'$ from a set environment ρ to a set environment ρ' with $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$ maps each type constructor variable $\psi^k \in \mathbb{T}$ to the identity natural transformation on $\rho\psi^k = \rho'\psi^k$ and maps each functorial variable $\phi^k \in \mathbb{F}$ to a natural transformation from the k-ary functor $\rho\phi^k$ on* Set *to the k-ary functor $\rho'\phi^k$ on* Set. *Composition of morphisms on set environments is given componentwise, with the identity morphism mapping each set environment to itself. This gives a category of set environments and morphisms between them, which we denote* SetEnv.

When convenient we identify a functor $F : [\mathsf{Set}^0, \mathsf{Set}]$ with the set that is its codomain. With this convention, a set environment maps a type variable of arity 0 to an ω-cocontinuous functor from Set^0 to Set, i.e., to a set. If $\overline{\alpha} = \{\alpha_1, ..., \alpha_k\}$ and $\overline{A} = \{A_1, ..., A_k\}$, then we write $\rho[\overline{\alpha := A}]$ for the set environment ρ' such that $\rho'\alpha_i = A_i$ for $i = 1, ..., k$ and $\rho'\alpha = \rho\alpha$ if $\alpha \notin \{\alpha_1, ..., \alpha_k\}$.

If ρ is a set environment we write Eq_ρ for the relation environment such that $\mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v}$ for every type variable v; see Definition 15 below for the complete definition of a relation environment. The relational interpretations in the second clause of Definition 7 are given in Definition 17.

Definition 7. *Let ρ be a set environment. The* set interpretation $[\![\cdot]\!]^{\mathsf{Set}} : \mathcal{F} \to [\mathsf{SetEnv}, \mathsf{Set}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma;\emptyset \vdash v]\!]^{\mathsf{Set}}\rho &= \rho v \text{ if } v \in \mathbb{T}^0\\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho &= \{\eta : \lambda\overline{A}.\, [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}] \Rightarrow \lambda\overline{A}.\, [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]\\
&\qquad\qquad \mid \forall \overline{A}, \overline{B} : \mathsf{Set}. \forall \overline{R : \mathsf{Rel}(A, B)}.\\
&\qquad\qquad\quad (\eta_{\overline{A}}, \eta_{\overline{B}}) : [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]\}\\
[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}}\rho &= 0\\
[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho &= 1\\
[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho &= (\rho\phi)\, \overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}\\
[\![\Gamma;\Phi \vdash \sigma + \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho + [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma;\Phi \vdash \sigma \times \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho &= (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}\\
\text{where } T^{\mathsf{Set}}_{H,\rho}\, F &= \lambda\overline{A}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\phi := F][\overline{\alpha := A}]\\
\text{and } T^{\mathsf{Set}}_{H,\rho}\, \eta &= \lambda\overline{A}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_\rho[\phi := \eta][\overline{\alpha := id_A}]
\end{aligned}
$$

The interpretations in Definition 7 respect weakening, i.e., a type and its weakenings all have the same set interpretations. The same holds for the actions of these interpretations on morphisms in Definition 8 below. Moreover, the interpretation of Nat types ensures that $[\![\Gamma \vdash \sigma \to \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma \vdash \sigma]\!]^{\mathsf{Set}}\rho \to [\![\Gamma \vdash \tau]\!]^{\mathsf{Set}}\rho$, as expected. If ρ is a set environment and $\vdash \tau : \mathcal{F}$ then we may write $[\![\vdash \tau]\!]^{\mathsf{Set}}$ instead of $[\![\vdash \tau]\!]^{\mathsf{Set}}\rho$ since the environment is immaterial. We note that the second clause of Definition 7 does indeed define a set: local finite presentability of Set and ω-cocontinuity of $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho$ ensure that $\{\eta : [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho \Rightarrow [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho\}$ (which contains $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$) is a subset of $\big\{([\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}])^{([\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=S}])} \,\big|\, \overline{S} = (S_1, ..., S_{|\overline{\alpha}|})$, and S_i is a finite set for $i = 1, ..., |\overline{\alpha}|\big\}$. There are countably many choices for tuples $\overline{S}$, and each of these gives rise to a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}]$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}]$. But there are only Set-many choices of morphisms between these (or any) two objects because Set is locally small.

In order to make sense of the last clause in Definition 7, we need to know that, for each $\rho \in \mathsf{SetEnv}$, $T^{\mathsf{Set}}_{H,\rho}$ is an ω-cocontinuous endofunctor on $[\mathsf{Set}^k, \mathsf{Set}]$, and thus admits a fixed point. Since $T^{\mathsf{Set}}_{H,\rho}$ is defined in terms of $[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, this means that interpretations of types must be such functors, which in turn means that the actions of set interpretations of types on objects and on morphisms in SetEnv are intertwined. Fortunately, we know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash \tau : \mathcal{F}$, $[\![\Gamma;\overline{\alpha} \vdash \tau]\!]^{\mathsf{Set}}$ is actually in $[\mathsf{Set}^k, \mathsf{Set}]$ where $k = |\overline{\alpha}|$. This means that for each $[\![\Gamma;\Phi,\phi^k,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, the corresponding operator T^{Set}_H can be extended to a *functor* from SetEnv to $[[\mathsf{Set}^k, \mathsf{Set}], [\mathsf{Set}^k, \mathsf{Set}]]$. The action of T^{Set}_H on an object $\rho \in \mathsf{SetEnv}$ is given by the higher-order functor $T^{\mathsf{Set}}_{H,\rho}$, whose actions on objects (functors in $[\mathsf{Set}^k, \mathsf{Set}]$) and morphisms (natural transformations between such functors) are as defined in Definition 7. The action of T^{Set}_H

on a morphism $f : \rho \to \rho'$ is the higher-order natural transformation $T^{\mathsf{Set}}_{H,f} : T^{\mathsf{Set}}_{H,\rho} \to T^{\mathsf{Set}}_{H,\rho'}$ whose action on $F : [\mathsf{Set}^k, \mathsf{Set}]$ is the natural transformation $T^{\mathsf{Set}}_{H,f}\, F : T^{\mathsf{Set}}_{H,\rho}\, F \to T^{\mathsf{Set}}_{H,\rho'}\, F$ whose component at $\overline{A}$ is $(T^{\mathsf{Set}}_{H,f}\, F)_{\overline{A}} = [\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}} f[\phi := id_F][\overline{\alpha := id_A}]$. The next definition uses the functor T^{Set}_H to define the actions of functors interpreting types on morphisms between set environments.

Definition 8. *Let $f : \rho \to \rho'$ for set environments ρ and ρ' (so that $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$). The action $[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} f$ of $[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}$ on the morphism f is given as follows:*

- *If $\Gamma, v; \emptyset \vdash v$ then $[\![\Gamma, v; \emptyset \vdash v]\!]^{\mathsf{Set}} f = id_{\rho v}$.*
- *If $\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G$, then we define $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} f = id_{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho}$.*
- *If $\Gamma; \Phi \vdash \mathbb{0}$ then $[\![\Gamma; \Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}} f = id_0$.*
- *If $\Gamma; \Phi \vdash \mathbb{1}$ then $[\![\Gamma; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}} f = id_1$.*
- *If $\Gamma; \Phi \vdash \phi\overline{\tau}$, then we have that $[\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} f : [\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} \rho \to [\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} \rho' = (\rho\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho} \to (\rho'\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho'}$ is defined by $[\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} f = (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} f} = (\rho'\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} f} \circ (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}$. This equality holds because $\rho\phi$ and $\rho'\phi$ are functors and $f\phi : \rho\phi \to \rho'\phi$ is a natural transformation, so that the following naturality square commutes:*

$$
\begin{tikzcd}
(\rho\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\rho\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}"'] & (\rho'\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho} \arrow[d, "(\rho'\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}"'] \\
(\rho\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho'} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}"] & (\rho'\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho'}
\end{tikzcd}
\tag{1}
$$

- *If $\Gamma; \Phi \vdash \sigma + \tau$ then $[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Set}} f$ is defined by $[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Set}} f(\mathsf{inl}\, x) = \mathsf{inl}\,([\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}} f x)$ and $[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Set}} f(\mathsf{inr}\, y) = \mathsf{inr}\,([\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} f y)$.*
- *If $\Gamma; \Phi \vdash \sigma \times \tau$ then $[\![\Gamma; \Phi \vdash \sigma \times \tau]\!]^{\mathsf{Set}} f = [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}} f \times [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} f$.*
- *If $\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$ we define*

$$
\begin{array}{rl}
& [\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}} f \\
: & [\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}} \rho \to [\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}} \rho' \\
= & (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho} \to (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho'}
\end{array}
$$

by

$$
\begin{array}{rl}
& (\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho'}} \circ (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} f} \\
= & (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} f} \circ (\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho}}
\end{array}
$$

This equality holds because $\mu T^{\mathsf{Set}}_{H,\rho}$ and $\mu T^{\mathsf{Set}}_{H,\rho'}$ are functors and $\mu T^{\mathsf{Set}}_{H,f} : \mu T^{\mathsf{Set}}_{H,\rho} \to \mu T^{\mathsf{Set}}_{H,\rho'}$ is a natural transformation, so that the following naturality square commutes:

$$
\begin{tikzcd}
(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho} \arrow[r, "(\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}"'] & (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho} \arrow[d, "(\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}"'] \\
(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho'} \arrow[r, "(\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}"] & (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho'}
\end{tikzcd}
\tag{2}
$$

3.2 Interpreting Types as Relations

Definition 9. *A k-ary relation transformer F is a triple (F^1, F^2, F^*), where $F^1, F^2 : [\mathsf{Set}^k, \mathsf{Set}]$ are functors, $F^* : [\mathsf{Rel}^k, \mathsf{Rel}]$ is a functor, if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $F^*\overline{R} : \mathsf{Rel}(F^1\overline{A}, F^2\overline{B})$, and if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$ then $F^*\overline{(\alpha, \beta)} = (F^1\overline{\alpha}, F^2\overline{\beta})$. We define $F\overline{R}$ to be $F^*\overline{R}$ and $F\overline{(\alpha, \beta)}$ to be $F^*\overline{(\alpha, \beta)}$.*

The last clause of Definition 9 expands to: if $\overline{(a, b) \in R}$ implies $\overline{(\alpha\, a, \beta\, b) \in S}$ then $(c, d) \in F^*\overline{R}$ implies $(F^1\overline{\alpha}\, c, F^2\overline{\beta}\, d) \in F^*\overline{S}$. When convenient we identify a 0-ary relation transformer (A, B, R) with $R : \mathsf{Rel}(A, B)$. We may also write $\pi_1 F$ for F^1 and $\pi_2 F$ for F^2. We further extend these conventions to relation environments, introduced in Definition 15 below.

Definition 10. *The category RT_k of k-ary relation transformers is given by the following data:*

- *An object of RT_k is a relation transformer.*
- *A morphism $\delta : (G^1, G^2, G^*) \to (H^1, H^2, H^*)$ in RT_k is a pair of natural transformations (δ^1, δ^2) where $\delta^1 : G^1 \to H^1$, $\delta^2 : G^2 \to H^2$ such that, for all $\overline{R : \mathsf{Rel}(A, B)}$, if $(x, y) \in G^*\overline{R}$ then $(\delta^1_{\overline{A}} x, \delta^2_{\overline{B}} y) \in H^*\overline{R}$.*
- *Identity morphisms and composition are inherited from the category of functors on* Set.

Definition 11. *An endofunctor H on RT_k is a triple $H = (H^1, H^2, H^*)$, where*

- *H^1 and H^2 are functors from $[\mathsf{Set}^k, \mathsf{Set}]$ to $[\mathsf{Set}^k, \mathsf{Set}]$*
- *H^* is a functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]$*
- *for all $\overline{R : \mathsf{Rel}(A, B)}$, $\pi_1((H^*(\delta^1, \delta^2))_{\overline{R}}) = (H^1\delta^1)_{\overline{A}}$ and $\pi_2((H^*(\delta^1, \delta^2))_{\overline{R}}) = (H^2\delta^2)_{\overline{B}}$*
- *The action of H on objects is given by $H\,(F^1, F^2, F^*) = (H^1F^1,\ H^2F^2,\ H^*(F^1, F^2, F^*))$*
- *The action of H on morphisms is given by $H\,(\delta^1, \delta^2) = (H^1\delta^1, H^2\delta^2)$ for $(\delta^1, \delta^2) : (F^1, F^2, F^*) \to (G^1, G^2, G^*)$*

Since the results of applying an endofunctor H to k-ary relation transformers and morphisms between them must again be k-ary relation transformers and morphisms between them, respectively, Definition 11 implicitly requires that the following three conditions hold:

(1) if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $H^*(F^1, F^2, F^*)\overline{R} : \mathsf{Rel}(H^1F^1\overline{A}, H^2F^2\overline{B})$
(2) if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then

$$H^*(F^1, F^2, F^*)\,\overline{(\alpha, \beta)} = (H^1F^1\overline{\alpha}, H^2F^2\overline{\beta})$$

(3) if $(\delta^1, \delta^2) : (F^1, F^2, F^*) \to (G^1, G^2, G^*)$ and $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then

$$\text{if } (x, y) \in H^*(F^1, F^2, F^*)\overline{R} \text{ then } ((H^1\delta^1)_{\overline{A}} x, (H^2\delta^2)_{\overline{B}} y) \in H^*(G^1, G^2, G^*)\overline{R}$$

Of course this condition is automatically satisfied because it is implied by the third bullet point of Definition 11.

Definition 12. *If H and K are endofunctors on RT_k, then a* natural transformation $\sigma : H \to K$ *is a pair $\sigma = (\sigma^1, \sigma^2)$, where $\sigma^1 : H^1 \to K^1$ and $\sigma^2 : H^2 \to K^2$ are natural transformations between endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]$ and the component of σ at $F \in RT_k$ is given by $\sigma_F = (\sigma^1_{F^1}, \sigma^2_{F^2})$.*

Definition 12 entails that $\sigma^i_{F^i}$ must be natural in $F^i : [\mathsf{Set}^k, \mathsf{Set}]$, and, for every F, both $(\sigma^1_{F^1})_{\overline{A}}$ and $(\sigma^2_{F^2})_{\overline{A}}$ must be natural in $\overline{A}$. Moreover, since the results of applying σ to k-ary relation transformers must be morphisms of k-ary relation transformers, Definition 12 implicitly requires that $(\sigma_F)_{\overline{R}} = ((\sigma^1_{F^1})_{\overline{A}}, (\sigma^2_{F^2})_{\overline{B}})$ is a morphism in Rel for any k-tuple of relations $\overline{R : \mathsf{Rel}(A, B)}$, i.e., that if $(x, y) \in H^* F\overline{R}$, then $((\sigma^1_{F^1})_{\overline{A}} x, (\sigma^2_{F^2})_{\overline{B}} y) \in K^* F\overline{R}$.

Next, we observe that we can compute ω-directed colimits in RT_k. It is straightforward to check that if $\mathcal{D}$ is an ω-directed set, then $\varinjlim_{d\in\mathcal{D}}(F^1_d, F^2_d, F^*_d) = (\varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^2_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$.

Definition 13. *An endofunctor $T = (T^1, T^2, T^*)$ on RT_k is* ω-cocontinuous *if T^1 and T^2 are ω-cocontinuous endofunctors on* $[\mathsf{Set}^k, \mathsf{Set}]$ *and T^* is an ω-cocontinuous functor from RT_k to* $[\mathsf{Rel}^k, \mathsf{Rel}]$*, i.e., is in* $[RT_k, [\mathsf{Rel}^k, \mathsf{Rel}]]$.

For any k and $R : \mathsf{Rel}(A, B)$, let K^{Rel}_R be the constantly R-valued functor from Rel^k to Rel, and for any k and set A, let K^{Set}_A be the constantly A-valued functor from Set^k to Set. Moreover, let 0 denote either the initial object of Set or the initial object of Rel, depending on the context. Observing that, for every k, K^{Set}_0 is initial in $[\mathsf{Set}^k, \mathsf{Set}]$, and similarly for K^{Rel}_0, we have that, for each k, $K_0 = (K^{\mathsf{Set}}_0, K^{\mathsf{Set}}_0, K^{\mathsf{Rel}}_0)$ is initial in RT_k. Thus, if $T = (T^1, T^2, T^*) : RT_k \to RT_k$ is an endofunctor on RT_k then we can define μT to be the relation transformer

$$\mu T = \varinjlim_{n\in\mathbb{N}} T^n K_0$$

Then Lemma **??** shows μT is indeed a relation transformer, and that it is given explicitly by

$$\varinjlim_{n\in\mathbb{N}} T^n K_0 = (\mu T^1, \mu T^2, \varinjlim_{n\in\mathbb{N}} (T^n K_0)^*) \tag{3}$$

Lemma 14. *For any $T : [RT_k, RT_k]$, $\mu T \cong T(\mu T)$.*

Proof. We have $T(\mu T) = T(\varinjlim_{n\in\mathbb{N}}(T^n K_0)) \cong \varinjlim_{n\in\mathbb{N}} T(T^n K_0) = \mu T$. □

In fact, the isomorphism in Lemma 14 is given by the morphisms $(in_1, in_2) : T(\mu T) \to \mu T$ and $(in_1^{-1}, in_2^{-1}) : \mu T \to T(\mu T)$ in RT_k. It is worth noting that the latter is always a morphism in RT_k, but the former isn't necessarily a morphism in RT_k unless T is ω-cocontinuous.

Say realizing that not being able to define third components directly, but rather only through the other two components, is an important conceptual contribution. Not all functors on Rel are third components of relation transformers. It's overly restrictive to require that the third component of a functor on RT_k be a functor on all of $[\mathsf{Rel}^k, \mathsf{Rel}]$. For example, we can define $T_\rho F$ when F is a relation transformer, but it is not clear how we could define $T_\rho F$ when $F : [\mathsf{Rel}^k, \mathsf{Rel}]$.

Definition 15. *A* relation environment *maps each each type variable in $\mathbb{T}^k \cup \mathbb{F}^k$ to an ω-cocontinuous k-ary relation transformer. A morphism $f : \rho \to \rho'$ from a relation environment ρ to a relation environment ρ' with $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$ maps each type constructor variable $\psi^k \in \mathbb{T}$ to the identity morphism on $\rho\psi^k = \rho'\psi^k$ and maps each functorial variable $\phi^k \in \mathbb{F}$ to a morphism from the k-ary relation transformer $\rho\phi$ to the k-ary relation transformer $\rho'\phi$. Composition of morphisms on relation environments is given componentwise, with the identity morphism mapping each relation environment to itself. This gives a category of relation environments and morphisms between them, which we denote* RelEnv.

When convenient we identify a 0-ary relation transformer with the relation (transformer) that is its codomain. With this convention, a relation environment maps a type variable of arity 0 to a 0-ary relation transformer, i.e., to a relation. We write $\rho[\overline{\alpha := R}]$ for the relation environment ρ' such that $\rho'\alpha_i = R_i$ for $i = 1, ..., k$ and $\rho'\alpha = \rho\alpha$ if $\alpha \notin \{\alpha_1, ..., \alpha_k\}$. If ρ is a relation environment, we write $\pi_1\rho$ for the set environment mapping each type variable ϕ to the functor $(\rho\phi)^1$. The set environment $\pi_2\rho$ similarly maps each type variable ϕ to the functor $(\rho\phi)^2$.

We define, for each k, the notion of an ω-cocontinuous functor from RelEnv to RT_k:

Definition 16. *A functor $H : [\mathsf{RelEnv}, RT_k]$ is a triple $H = (H^1, H^2, H^*)$, where*

- *H^1 and H^2 are objects in* $[\mathsf{SetEnv}, [\mathsf{Set}^k, \mathsf{Set}]]$
- *H^* is a an object in* $[\mathsf{RelEnv}, [\mathsf{Rel}^k, \mathsf{Rel}]]$
- *for all $\overline{R : \mathsf{Rel}(A, B)}$ and morphisms f in* RelEnv*, $\pi_1(H^* f\, \overline{R}) = H^1(\pi_1 f)\, \overline{A}$ and $\pi_2(H^* f\, \overline{R}) = H^2(\pi_2 f)\, \overline{B}$*
- *The action of H on ρ in* RelEnv *is given by $H\rho = (H^1(\pi_1\rho),\ H^2(\pi_2\rho),\ H^*\rho)$*
- *The action of H on morphisms $f : \rho \to \rho'$ in* RelEnv *is given by $Hf = (H^1(\pi_1 f), H^2(\pi_2 f))$*

Spelling out the last two bullet points above gives the following analogues of Conditions (1), (2), and (3) immediately following Definition 11:

(1) if $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then

$$H^* \rho\, \overline{R} : \mathsf{Rel}(H^1(\pi_1\rho)\, \overline{A}, H^2(\pi_2\rho)\, \overline{B})$$

In other words, $\pi_1(H^*\rho\overline{R}) = H^1(\pi_1\rho)\, \overline{A}$ and $\pi_2(H^*\rho\overline{R}) = H^2(\pi_2\rho)\, \overline{B}$.

(2) if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then

$$H^* \rho\, \overline{(\alpha, \beta)} = (H^1(\pi_1\rho)\, \overline{\alpha}, H^2(\pi_2\rho)\, \overline{\beta})$$

In other words, $\pi_1(H^*\rho\, \overline{(\alpha, \beta)}) = H^1(\pi_1\rho)\, \overline{\alpha}$ and $\pi_2(H^*\rho\, \overline{(\alpha, \beta)}) = H^2(\pi_2\rho)\, \overline{\beta}$.

(3) if $f : \rho \to \rho'$ and $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then

$$\text{if } (x, y) \in H^* \rho\, \overline{R} \text{ then } (H^1(\pi_1 f)\, \overline{A}\, x, H^2(\pi_2 f)\, \overline{B}\, y) \in H^* \rho'\, \overline{R}$$

Note, however, that this condition is automatically satisfied because it is implied by the third bullet point of Definition 16.

Considering RelEnv as a product $\Pi_{\phi^k \in \mathbb{T} \cup \mathbb{F}} RT_k$, we extend Lemma ?? to compute colimits in RelEnv componentwise, and similarly extend Definition 13 to give a componentwise notion of ω-cocontinuity of functors from RelEnv to RT_k.

We recall from the start of this section that Definition 17 is given mutually inductively with Definition 7. We can, at last, define:

Definition 17. *Let ρ be a relation environment. The* relation interpretation $[\![\cdot]\!]^{\mathsf{Rel}} : \mathcal{F} \to [\mathsf{RelEnv}, \mathsf{Rel}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma;\emptyset \vdash v]\!]^{\mathsf{Rel}}\rho &= \rho v \textit{ if } v \in \mathbb{T}^0\\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho &= \{\eta : \lambda\overline{R}.\, [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] \Rightarrow \lambda\overline{R}.\, [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]\}\\
&= \{(t,t') \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid\\
&\qquad \forall R_1 : \mathsf{Rel}(A_1,B_1) \ldots R_k : \mathsf{Rel}(A_k,B_k).\\
&\qquad\quad (t_{\overline{A}}, t'_{\overline{B}}) \in ([\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}])^{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=R}]}\}\\
&= \{(t,t') \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid\\
&\qquad \forall R_1 : \mathsf{Rel}(A_1,B_1) \ldots R_k : \mathsf{Rel}(A_k,B_k).\\
&\qquad\quad \forall (a,b) \in [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}].\\
&\qquad\qquad (t_{\overline{A}}a, t'_{\overline{B}}b) \in [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]\}\\
[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\rho &= 0\\
[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\rho &= 1\\
[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho &= (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}\\
[\![\Gamma;\Phi \vdash \sigma + \tau]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho + [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho\\
[\![\Gamma;\Phi \vdash \sigma \times \tau]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho \times [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho\\
[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\rho &= (\mu T_\rho)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}\\
\textit{where } T_\rho &= (T^{\mathsf{Set}}_{\pi_1\rho}, T^{\mathsf{Set}}_{\pi_2\rho}, T^{\mathsf{Rel}}_{\rho})\\
\textit{and } T^{\mathsf{Rel}}_{\rho} F &= \lambda\overline{R}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F][\overline{\alpha := R}]\\
\textit{and } T^{\mathsf{Rel}}_{\rho} \delta &= \lambda\overline{R}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := \delta][\overline{\alpha := id_R}]
\end{aligned}
$$

The interpretations in Definition 17 respect weakening, i.e., a type and its weakenings all have the same relational interpretations. The same holds for the actions of these interpretations on morphisms in Definition 18 below. Moreover, the interpretation of Nat types ensures that $[\![\Gamma \vdash \sigma \to \tau]\!]^{\mathsf{Rel}}\rho = [\![\Gamma \vdash \sigma]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma \vdash \tau]\!]^{\mathsf{Rel}}\rho$, as expected. If ρ is a relational environment and $\vdash \tau : \mathcal{F}$, then we write $[\![\vdash \tau]\!]^{\mathsf{Rel}}$ instead of $[\![\vdash \tau]\!]^{\mathsf{Rel}}\rho$ as for set interpretations.

For the last clause in Definition 17 to be well-defined, we need to know that T_ρ is an ω-cocontinuous endofunctor on RT so that, by Lemma 14, it admits a fixed point. Since T_ρ is defined in terms of $[\![\Gamma;\Phi,\phi^k,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}$, this means that relational interpretations of types must be ω-cocontinuous functors from RelEnv to RT_0. This in turn means that the actions of relational interpretations of types on objects and on morphisms in RelEnv are intertwined. In fact, we already know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash \tau : \mathcal{F}$, $[\![\Gamma;\overline{\alpha} \vdash \tau]\!]^{\mathsf{Rel}}$ is actually functorial in $\overline{\alpha}$ and ω-cocontinuous. We first define the actions of each of these functors on morphisms between environments, and then argue that the functors given by Definitions 17 and 18 are well-defined and have the required properties. As in the case of the set interpretations, each operator T Should be T_H, really, and similarly throughout. can be extended to a *functor* from RelEnv to $[[\mathsf{Rel}^k, \mathsf{Rel}], [\mathsf{Rel}^k, \mathsf{Rel}]]$. Its action on an object $\rho \in \mathsf{RelEnv}$ is given by the higher-order functor T^{Rel}_ρ $T^{\mathsf{Rel}}_{H,\rho}$ whose actions on objects and morphisms are as defined in Definition 18. The action of T on a morphism $f : \rho \to \rho'$ is the higher-order natural transformation May want to use $T_{H,f}$ rather than σ_f in the final version. $\sigma_f : T_\rho \to T_{\rho'}$ whose action on any $F : [\mathsf{Rel}^k, \mathsf{Rel}]$ is the natural transformation $\sigma_f F : T_\rho F \to T_{\rho'} F$ whose component at $\overline{R}$ is

$$(\sigma_f F)_{\overline{R}} = \llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash H \rrbracket^{\mathsf{Rel}} f[\phi := id_F][\overline{\alpha := id_R}]$$

The next definition uses this observation to define the action of each functor interpreting a type on morphisms between relation environments.

Definition 18. *Let $f : \rho \to \rho'$ for relation environments ρ and ρ' (so that $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$). The action $\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f$ of $\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}}$ on the morphism f is given as follows:*

- *If $\Gamma, v; \emptyset \vdash v$ then $\llbracket \Gamma, v; \emptyset \vdash v \rrbracket^{\mathsf{Rel}} f = id_{\rho v}$.*
- *If $\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G$, then we define $\llbracket \Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{Rel}} f = id_{\llbracket \Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{Rel}} \rho}$.*
- *If $\Gamma; \Phi \vdash \mathbb{0}$ then $\llbracket \Gamma; \Phi \vdash \mathbb{0} \rrbracket^{\mathsf{Rel}} f = id_0$.*
- *If $\Gamma; \Phi \vdash \mathbb{1}$ then $\llbracket \Gamma; \Phi \vdash \mathbb{1} \rrbracket^{\mathsf{Rel}} f = id_1$.*
- *If $\Gamma; \Phi \vdash \phi\overline{\tau}$, then we have that $\llbracket \Gamma; \Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Rel}} f : \llbracket \Gamma; \Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Rel}} \rho \to \llbracket \Gamma; \Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Rel}} \rho' = (\rho\phi)\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho} \to (\rho'\phi)\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho'}$ is defined by $\llbracket \Gamma; \Phi \vdash \phi\tau A \rrbracket^{\mathsf{Rel}} f = (f\phi)_{\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho'}} \circ (\rho\phi)\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f} = (\rho'\phi)\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f} \circ (f\phi)_{\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho}}$.*
- *If $\Gamma; \Phi \vdash \sigma + \tau$ then $\llbracket \Gamma; \Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Rel}} f$ is defined by $\llbracket \Gamma; \Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Rel}} f (\mathsf{inl}\, x) = \mathsf{inl}\, (\llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Rel}} f x)$ and $\llbracket \Gamma; \Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Rel}} f (\mathsf{inr}\, y) = \mathsf{inr}\, (\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f y)$.*
- *If $\Gamma; \Phi \vdash \sigma \times \tau$ then $\llbracket \Gamma; \Phi \vdash \sigma \times \tau \rrbracket^{\mathsf{Rel}} f = \llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Rel}} f \times \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f$.*
- *If $\Gamma; \Phi \vdash (\mu\phi^k.\lambda\overline{\alpha}.H)\overline{\tau}$ we define*

$$\begin{array}{rl} & \llbracket \Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Rel}} f \\ = & (\mu\sigma_f)\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho'} \circ (\mu T_\rho)\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f} \\ = & (\mu T_{\rho'})\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f} \circ (\mu\sigma_f)\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho} \end{array}$$

To see that the functors given by Definitions 17 and 18 are well-defined we must show that $T_\rho F$ is a relation transformer for any relation transformer F, and that $\sigma_f F : T_\rho F \to T_{\rho'} F$ is a morphism of relation transformers for every relation transformer F and every morphism $f : \rho \to \rho'$ in RelEnv.

Lemma 19. *The interpretations in Definitions 17 and 18 are well-defined and, for every $\Gamma; \Phi \vdash \tau$,*

$$\llbracket \Gamma; \Phi \vdash \tau \rrbracket = (\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}}, \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}}, \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}})$$

is an ω-cocontinuous functor from RelEnv to RT_0, i.e., is an element of $[\mathsf{RelEnv}, RT_0]$.

Proof. By induction on the structure of τ. The only interesting cases are when $\tau = \phi\overline{\tau}$ and when $\tau = (\mu\phi^k.\lambda\overline{\alpha}.H)\overline{\tau}$. We consider each in turn.

- When $\tau = \Gamma; \Phi \vdash \phi\overline{\tau}$, we have

$$\begin{array}{rl} & \pi_i(\llbracket \Gamma; \Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Rel}} \rho) \\ = & \pi_i((\rho\phi)\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho}) \\ = & (\pi_i(\rho\phi))(\overline{\pi_i(\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho)}) \\ = & ((\pi_i\rho)\phi)(\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} (\pi_i\rho)}) \\ = & \llbracket \Gamma; \Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Set}} (\pi_i\rho) \end{array}$$

  and, for $f : \rho \to \rho'$ in RelEnv,

$$\begin{array}{rl} & \pi_i(\llbracket \Gamma; \Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Rel}} f) \\ = & \pi_i((f\phi)_{\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho'}}) \circ \pi_i((\rho\phi)(\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f})) \\ = & (\pi_i(f\phi))_{\overline{\pi_i(\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho')}} \circ (\pi_i(\rho\phi))(\overline{\pi_i(\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} f)}) \\ = & ((\pi_i f)\phi)_{\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} (\pi_i\rho')}} \circ ((\pi_i\rho)\phi)(\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} (\pi_i f)}) \\ = & \llbracket \Gamma; \Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Set}} (\pi_i f) \end{array}$$

The third equalities of each of the above derivations are by the induction hypothesis. That $[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]$ is ω-cocontinuous is an immediate consequence of the facts that Set and Rel are locally finitely presentable, together with Corollary 12 of [Johann and Polonsky 2019].

- When $\tau = (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$ we first show that $[\![(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]$ is well-defined.
  - $\underline{T_\rho : [RT_k, RT_k]}$: We must show that, for any relation transformer $F = (F^1, F^2, F^*)$, the triple $\underline{T_\rho F = (T^{\mathsf{Set}}_{\pi_1\rho}F^1, T^{\mathsf{Set}}_{\pi_2\rho}F^2, T^{\mathsf{Rel}}_{\rho}F)}$ is also a relation transformer. Let $\overline{R : \mathsf{Rel}(A,B)}$. Then for $i = 1, 2$, we have

$$\begin{aligned}\pi_i(T^{\mathsf{Rel}}_\rho\, F\, \overline{R}) &= \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F]\overline{[\alpha := R]})\\ &= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i(\rho[\phi := F]\overline{[\alpha := R]}))\\ &= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i\rho)[\phi := \pi_i F]\overline{[\alpha := \pi_i R]})\\ &= T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i F)(\overline{\pi_i R})\end{aligned}$$

  and

$$\begin{aligned}\pi_i(T^{\mathsf{Rel}}_\rho\, F\, \overline{\gamma}) &= \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := id_F]\overline{[\alpha := \gamma]})\\ &= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i(id_\rho[\phi := id_F]\overline{[\alpha := \gamma]}))\\ &= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_{\pi_i\rho}[\phi := id_{\pi_i F}]\overline{[\alpha := \pi_i\gamma]}\\ &= T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i F)(\overline{\pi_i\gamma})\end{aligned}$$

  Here, the second equality in each of the above chains of equalities is by the induction hypothesis.
  We also have that, for every morphism $\delta = (\delta^1, \delta^2) : F \to G$ in RT_k and all $\overline{R : \mathsf{Rel}(A,B)}$,

$$\begin{array}{ll} & \pi_i((T^{\mathsf{Rel}}_\rho\delta)_{\overline{R}})\\ = & \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := \delta]\overline{[\alpha := id_R]})\\ = & [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_{\pi_i\rho}[\phi := \pi_i\delta]\overline{[\alpha := id_{\pi_i R}]}\\ = & (T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i\delta))_{\overline{\pi_i R}}\end{array}$$

  Here, the second equality is by the induction hypothesis. That T_ρ is ω-cocontinuous follows immediately from the induction hypothesis on $[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]$ and the fact that colimts are computed componentwise in RT.
  - $\underline{\sigma_f = (\sigma^{\mathsf{Set}}_{\pi_1 f}, \sigma^{\mathsf{Set}}_{\pi_2 f}) \text{ is a natural transformation from } T_\rho \text{ to } T_{\rho'}}$: We must show that $\underline{(\sigma_f)_F = ((\sigma^{\mathsf{Set}}_{\pi_1 f})_{F^1}, (\sigma^{\mathsf{Set}}_{\pi_2 f})_{F^2})}$ is a morphism in RT_k for all relation transformers $F = (F^1, F^2, F^*)$, i.e., that $((\sigma_f)_F)_{\overline{R}} = (((\sigma^{\mathsf{Set}}_{\pi_1 f})_{F^1})_{\overline{A}}, ((\sigma^{\mathsf{Set}}_{\pi_2 f})_{F_2})_{\overline{B}})$ is a morphism in Rel for all relations $\overline{R : \mathsf{Rel}(A,B)}$. Indeed, we have that

$$((\sigma_f)_F)_{\overline{R}} = [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} f[\phi := id_F]\overline{[\alpha := id_R]}$$

  is a morphism in RT_0 (and thus in Rel) by the induction hypothesis.

The relation transformer μT_ρ is therefore a fixed point of T_ρ by Lemma 14, and $\mu\sigma_f$ is a morphism in RT_k from μT_ρ to $\mu T_{\rho'}$. (μ is shown to be a functor in [Johann and Polonsky 2019].) So $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}$, and thus $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]$, is well-defined.

To see that $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!] : [\mathsf{RelEnv}, RT_0]$, we must verify three conditions:

- Condition (1) after Definition 16 is satisfied since

$$\begin{aligned}\pi_i(\llbracket\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}\rrbracket^{\mathsf{Rel}}\rho) &= \pi_i((\mu T_\rho)(\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho}))\\ &= \pi_i(\mu T_\rho)(\overline{\pi_i(\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho)})\\ &= \mu T^{\mathsf{Set}}_{\pi_i\rho}(\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}(\pi_i\rho)})\\ &= \llbracket\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}\rrbracket^{\mathsf{Set}}(\pi_i\rho)\end{aligned}$$

 The third equality is by Equation 3 and the induction hypothesis.
- Condition (2) after Definition 16 is satisfied since it is subsumed by the previous condition because $k = 0$.
- The third bullet point of Definition 16 is satisfied because

$$\begin{aligned}&\pi_i(\llbracket\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}\rrbracket^{\mathsf{Rel}}f)\\ &= \pi_i((\mu T_{\rho'})(\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f})\circ(\mu\sigma_f)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho}})\\ &= \pi_i((\mu T_{\rho'})(\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f}))\circ\pi_i((\mu\sigma_f)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho}})\\ &= \pi_i(\mu T_{\rho'})(\overline{\pi_i(\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f)})\circ\pi_i(\mu\sigma_f)_{\overline{\pi_i(\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho)}}\\ &= (\mu T^{\mathsf{Set}}_{\pi_i\rho'})(\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}(\pi_i f)})\circ(\mu\sigma^{\mathsf{Set}}_{\pi_i f})_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}(\pi_i\rho)}}\\ &= \llbracket\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}\rrbracket^{\mathsf{Set}}(\pi_i f).\end{aligned}$$

 The fourth equality is by 3 and the induction hypothesis.

As before, that $\llbracket\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}\rrbracket$ is ω-concontinuous follows from the facts that Set and Rel are locally finitely presentable, and that colimits in RelEnv are computed componentwise, together with Corollary 12 of [Johann and Polonsky 2019].

□

— The next two theorems are proven by simultaneous induction. We are actually only interested in using Theorem 20, but in order to prove the μ case for this theorem, we need Theorem ?? to show that two functors have equal actions on morphisms.

Theorem 20. *Let $\Gamma;\Phi,\phi\vdash\tau:\mathcal{F}$. If ρ,ρ' : SetEnv are such that $\rho\phi=\rho\psi=\rho'\phi=\rho'\psi$, and if $f:\rho\to\rho'$ is a morphism of set environments such that $f\phi=f\psi=id_{\rho\phi}$, then*

$$\llbracket\Gamma;\Phi,\phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho = \llbracket\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]\rrbracket^{\mathsf{Set}}\rho$$

and

$$\llbracket\Gamma;\Phi,\phi\vdash\tau\rrbracket^{\mathsf{Set}}f = \llbracket\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]\rrbracket^{\mathsf{Set}}f$$

Analogously, if ρ,ρ' : RelEnv are such that $\rho\phi=\rho\psi=\rho'\phi=\rho'\psi$, and if $f:\rho\to\rho'$ is a morphism of relation environments such that $f\phi=f\psi=id_{\rho\phi}$, then

$$\llbracket\Gamma;\Phi,\phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho = \llbracket\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]\rrbracket^{\mathsf{Rel}}\rho$$

and

$$\llbracket\Gamma;\Phi,\phi\vdash\tau\rrbracket^{\mathsf{Rel}}f = \llbracket\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]\rrbracket^{\mathsf{Rel}}f$$

Proof. We prove the result for set interpretations by induction on the structure of τ. The case for relational interpretations proceeds analogously. Since the only interesting cases are the application case and the μ-case, we elide the others.

- If $\Gamma;\Phi,\phi \vdash \phi\overline{\tau} : \mathcal{F}$, then the induction hypothesis gives that

$$[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho$$

and

$$[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}f = [\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}f$$

for each τ. Then

$$\begin{aligned}
&[\![\Gamma;\Phi,\phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho\\
&= (\rho\phi)\overline{[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}\rho}\\
&= (\rho\phi)\overline{[\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho}\\
&= (\rho\psi)\overline{[\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho}\\
&= [\![\Gamma,\psi;\Phi \vdash \psi\overline{\tau[\phi :== \psi]}]\!]^{\mathsf{Set}}\rho\\
&= [\![\Gamma,\psi;\Phi \vdash (\phi\overline{\tau})[\phi :== \psi]]\!]^{\mathsf{Set}}\rho
\end{aligned}$$

Here, the first and fifth equalities are by Definition 7, and the fourth equality is by equality of the functors $\rho\phi$ and $\rho\psi$. We also have that

$$\begin{aligned}
&[\![\Gamma;\Phi,\phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}f\\
&= (f\phi)_{\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}f}\\
&= (id_{\rho\phi})_{\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}f}\\
&= (\rho\phi)\overline{[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}f}\\
&= (\rho\psi)\overline{[\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}f}\\
&= (id_{\rho\psi})_{\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\psi)\overline{[\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}f}\\
&= (f\psi)_{\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\psi)\overline{[\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}f}\\
&= [\![\Gamma,\psi;\Phi \vdash \psi\overline{\tau[\phi :== \psi]}]\!]^{\mathsf{Set}}f\\
&= [\![\Gamma,\psi;\Phi \vdash (\phi\overline{\tau})[\phi :== \psi]]\!]^{\mathsf{Set}}f
\end{aligned}$$

- If $\Gamma;\Phi,\phi \vdash (\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau} : \mathcal{F}$, then the induction hypothesis gives that

$$[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}}\rho = [\![\Gamma,\psi;\Phi,\phi',\overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}}\rho$$

and

$$[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho$$

as well as that

$$[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}}f = [\![\Gamma,\psi;\Phi,\phi',\overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}}f$$

and

$$[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}f = [\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}f$$

for each τ. Then

$$
\begin{aligned}
&[\![\Gamma;\Phi,\phi\vdash(\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]))\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho}\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]))\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho}\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}])\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho}\\
&=[\![\Gamma,\psi;\Phi\vdash(\mu\phi'.\lambda\overline{\alpha}.H[\phi:==\psi])\overline{\tau[\phi:==\psi]}]\!]^{\mathsf{Set}}\rho\\
&=[\![\Gamma,\psi;\Phi\vdash((\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau})[\phi:==\psi]]\!]^{\mathsf{Set}}\rho
\end{aligned}
$$

The first and fifth equalities are by Definition 7. The second equality follows from the following equality:

$$
\begin{aligned}
&\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]\\
&\quad=\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]
\end{aligned}
$$

These two maps have the same actions on objects and morphisms by the induction hypothesis on H, and the fact that the extended environment $\rho[\phi':=F][\overline{\alpha:=A}]$ satisfies the required hypothesis. They are thus equal as functors and so have the same fixed point. We also have that

$$
\begin{aligned}
&[\![\Gamma;\Phi,\phi\vdash(\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}f\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}f[\phi':=id_F][\overline{\alpha:=id_A}]))_{\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}\\
&\quad\circ(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]))\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}f}\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}f[\phi':=id_F][\overline{\alpha:=id_A}]))_{\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho'}}\\
&\quad\circ(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]))\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}f}\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}f[\phi':=id_F][\overline{\alpha:=id_A}]))_{\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho'}}\\
&\quad\circ(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]))\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}f}\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}f[\phi':=id_F][\overline{\alpha:=id_A}]))_{\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho'}}\\
&\quad\circ(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]))\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}f}\\
&=[\![\Gamma,\psi;\Phi\vdash(\mu\phi'.\lambda\overline{\alpha}.H[\phi:==\psi])\overline{\tau[\phi:==\psi]}]\!]^{\mathsf{Set}}f\\
&=[\![\Gamma,\psi;\Phi\vdash((\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau})[\phi:==\psi]]\!]^{\mathsf{Set}}f
\end{aligned}
$$

The first and fifth equalities are by Definition 7. The third equality is by the equality of the arguments to the first μ operator:

$$
\begin{aligned}
&\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}f[\phi':=id_F][\overline{\alpha:=id_A}]\\
&\quad=\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}f[\phi':=id_F][\overline{\alpha:=id_A}]
\end{aligned}
$$

By the induction hypothesis on H and the fact that the morphism $f[\phi':=id_F][\overline{\alpha:=id_A}]:\rho[\phi':=F][\overline{\alpha:=A}]\to\rho'[\phi':=F][\overline{\alpha:=A}]$ still satisfies the required hypotheses. The fourth equality is by the equality of the arguments to the second μ operator:

$$\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\phi' := F][\overline{\alpha := A}]$$
$$= \lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi :== \psi]]\!]^{\mathsf{Set}}\rho[\phi' := F][\overline{\alpha := A}]$$

By the same reasoning as above, these two maps are equal as functors, and thus have the same fixed point.

□

—

The following lemma ensures that substitution interacts well with type interpretations. It is a consequence of Definitions 4, 30, and 31.

Lemma 21. *Let ρ be a set environment ρ and $f:\rho\to\rho'$ be a morphism of set environments.*

- *If $\Gamma;\Phi,\overline{\alpha}\vdash F$ and $\Gamma;\Phi\vdash\tau$, then*

$$[\![\Gamma;\Phi\vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}] \tag{4}$$

  and

$$[\![\Gamma;\Phi\vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}f = [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}f[\overline{\alpha := [\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}] \tag{5}$$

- *If $\Gamma;\Phi,\phi^k\vdash F$ and $\Gamma;\Phi,\alpha_1...\alpha_k\vdash H$, then*

$$[\![\Gamma;\Phi\vdash F[\phi := H]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\phi\vdash F]\!]^{\mathsf{Set}}\rho[\phi := \lambda\overline{A}.\,[\![\Gamma;\Phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]] \tag{6}$$

  and

$$[\![\Gamma;\Phi\vdash F[\phi := H]]\!]^{\mathsf{Set}}f = [\![\Gamma;\Phi,\phi\vdash F]\!]^{\mathsf{Set}}f[\phi := \lambda\overline{A}.\,[\![\Gamma;\Phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}f[\overline{\alpha := id_{\overline{A}}}]] \tag{7}$$

Analogous identities hold for relation environments and morphisms between them.

Proof. The proofs for the set and relational interpretations are completely analogous, so we just prove the former. Likewise, we only prove Equations 4 and 6, since the proofs for Equations 5 and 7 are again analogous. Finally, we prove Equation 4 for substitution for just a single type variable since the proof for multiple simultaneous substitutions proceeds similarly.

Although Equation 4 is a special case of Equation 6, it is convenient to prove Equation 4 first, and then use it to prove Equation 6. We prove Equation 4 by induction on the structure of F as follows:

- If $\Gamma;\emptyset\vdash F:\mathcal{T}$, or if F is $\mathbb{1}$ or $\mathbb{0}$, then F does not contain any functorial variables to replace, so there is nothing to prove.
- If F is $F_1\times F_2$ or F_1+F_2, then the substitution distributes over the product or coproduct as appropriate, so the result follows immediately from the induction hypothesis.
- If $F=\beta$ with $\beta\neq\alpha$, then there is nothing to prove.
- If $F=\alpha$, then

$$[\![\Gamma;\Phi\vdash\alpha[\alpha:=\tau]]\!]^{\mathsf{Set}}\rho \;=\; [\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho \;=\; [\![\Gamma;\Phi,\alpha\vdash\alpha]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]$$

- If $F=\phi\overline{\sigma}$ with $\phi\neq\alpha$, then

$$\begin{array}{rl}
& [\![\Gamma;\Phi\vdash(\phi\overline{\sigma})[\alpha:=\tau]]\!]^{\mathsf{Set}}\rho\\
= & [\![\Gamma;\Phi\vdash\phi(\overline{\sigma[\alpha:=\tau]})]\!]^{\mathsf{Set}}\rho\\
= & (\rho\phi)\overline{[\![\Gamma;\Phi\vdash\sigma[\alpha:=\tau]]\!]^{\mathsf{Set}}\rho}\\
= & (\rho\phi)\overline{[\![\Gamma;\Phi,\alpha\vdash\sigma]\!]^{\mathsf{Set}}\rho[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}]}\\
= & [\![\Gamma;\Phi,\alpha\vdash\phi\overline{\sigma}]\!]^{\mathsf{Set}}\rho[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}]
\end{array}$$

  Here, the third equality is by the induction hypothesis.

- If $F = (\mu\phi.\lambda\overline{\beta}.G)\overline{\sigma}$, then

$$
\begin{aligned}
& \llbracket \Gamma;\Phi \vdash ((\mu\phi.\lambda\overline{\beta}.G)\overline{\sigma})[\alpha := \tau] \rrbracket^{\mathsf{Set}}\rho \\
= \; & \llbracket \Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\beta}.G[\alpha := \tau])(\overline{\sigma[\alpha := \tau]}) \rrbracket^{\mathsf{Set}}\rho \\
= \; & \mu(\llbracket \Gamma;\Phi,\phi,\overline{\beta} \vdash G[\alpha := \tau] \rrbracket^{\mathsf{Set}}\rho[\phi := -][\overline{\beta := -}])(\overline{\llbracket \Gamma;\Phi \vdash \sigma[\alpha := \tau] \rrbracket^{\mathsf{Set}}\rho}) \\
= \; & \mu(\llbracket \Gamma;\Phi,\phi,\overline{\beta},\alpha \vdash G \rrbracket^{\mathsf{Set}}\rho[\alpha := \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho][\phi := -][\overline{\beta := -}]) \\
& \qquad (\overline{\llbracket \Gamma;\Phi,\alpha \vdash \sigma \rrbracket^{\mathsf{Set}}\rho[\alpha := \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho]}) \\
= \; & \llbracket \Gamma;\Phi,\alpha \vdash (\mu\phi.\lambda\overline{\beta}.G)\overline{\sigma} \rrbracket^{\mathsf{Set}}\rho[\alpha := \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho]
\end{aligned}
$$

  Here, the third equality is by the induction hypothesis and weakening.

We now prove Equation 6, again by induction on the structure of F.

- If $\Gamma;\emptyset \vdash F : \mathcal{T}$, or if F is $\mathbb{1}$ or $\mathbb{0}$, then F does not contain any functorial variables to replace, so there is nothing to prove.
- If F is $F_1 \times F_2$ or $F_1 + F_2$, then the substitution distributes over the product or coproduct as appropriate, so the result follows immediately from the induction hypothesis.
- If $F = \phi\overline{\tau}$, then

$$
\begin{aligned}
& \llbracket \Gamma;\Phi \vdash (\phi\overline{\tau})[\phi := H] \rrbracket^{\mathsf{Set}}\rho \\
= \; & \llbracket \Gamma;\Phi \vdash H[\overline{\alpha := \tau[\phi := H]}] \rrbracket^{\mathsf{Set}}\rho \\
= \; & \llbracket \Gamma;\Phi \vdash H \rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket \Gamma;\Phi \vdash \tau[\phi := H] \rrbracket^{\mathsf{Set}}\rho}] \\
= \; & \llbracket \Gamma;\Phi \vdash H \rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket \Gamma;\Phi,\phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho[\phi := \llbracket \Gamma;\Phi,\overline{\alpha} \vdash H \rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := -}]]}] \\
= \; & \llbracket \Gamma;\Phi,\phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Set}}\rho[\phi := \llbracket \Gamma;\Phi,\overline{\alpha} \vdash H \rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := -}]]
\end{aligned}
$$

  Here, the first equality is by Definition 4, the second is by Equation 4, the third is by the induction hypothesis, and the fourth is by Definition 7.
- If $F = \psi\overline{\tau}$ with $\psi \neq \phi$, then the proof is similar to that for the previous case, but simpler, because ϕ only needs to be substituted in the arguments $\overline{\tau}$ of ψ.
- If $F = (\mu\psi.\lambda\overline{\beta}.G)\overline{\tau}$, then

$$
\begin{aligned}
& \llbracket \Gamma;\Phi \vdash ((\mu\psi.\lambda\overline{\beta}.G)\overline{\tau})[\phi := H] \rrbracket^{\mathsf{Set}}\rho \\
= \; & \llbracket \Gamma;\Phi \vdash (\mu\psi.\lambda\overline{\beta}.G[\phi := H])(\overline{\tau[\phi := H]}) \rrbracket^{\mathsf{Set}}\rho \\
= \; & \mu(\llbracket \Gamma;\Phi,\psi,\overline{\beta} \vdash G[\phi := H] \rrbracket^{\mathsf{Set}}\rho[\psi := -][\overline{\beta := -}])(\overline{\llbracket \Gamma;\Phi \vdash \tau[\phi := H] \rrbracket^{\mathsf{Set}}\rho}) \\
= \; & \mu(\llbracket \Gamma;\Phi,\psi,\overline{\beta},\phi \vdash G \rrbracket^{\mathsf{Set}}\rho[\phi := \llbracket \Gamma;\Phi,\overline{\alpha} \vdash H \rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := -}]][\psi := -][\overline{\beta := -}]) \\
& \qquad (\overline{\llbracket \Gamma;\Phi,\phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho[\phi := \llbracket \Gamma;\Phi,\overline{\alpha} \vdash H \rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := -}]]}) \\
= \; & \llbracket \Gamma;\Phi,\phi \vdash (\mu\psi.\lambda\overline{\beta}.G)\overline{\tau} \rrbracket^{\mathsf{Set}}\rho[\phi := \llbracket \Gamma;\Phi,\overline{\alpha} \vdash H \rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := -}]]
\end{aligned}
$$

  Here, the first equality is by Definition 4, the second and fourth are by Definition 7, and the third is by the induction hypothesis and weakening.

□

3.3 The Identity Extension Lemma

The standard definition of graph relation of a morphism in Set is

Definition 22. *If $f : A \to B$ then the relation $\langle f \rangle : \mathsf{Rel}(A, B)$ is defined by $(x, y) \in \langle f \rangle$ iff $fx = y$.*

Here, we are using angle bracket notation for both the graph relation of a function and for the pairing of functions with the same domain. This is justified by the relationship between the two notions observed immediately after Lemma ??.

We extend the notion of a graph to natural transformations between k-ary set functors by defining an associated k-ary relation transformer.

Definition 23. *If $F, G : \mathsf{Set}^k \to \mathsf{Set}$ are k-ary set functors and $\alpha : F \to G$ is a natural transformation, we define the functor $\langle\alpha\rangle^* : \mathsf{Rel}^k \to \mathsf{Rel}$ as follows. Given $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, let $\iota_{R_i} : R_i \hookrightarrow A_i \times B_i$, for $i = 1, \ldots, k$, be the inclusion of R_i as a subset of $A_i \times B_i$. By the universal property of the product, there exists a unique $h_{\overline{A\times B}}$ making the diagram*

$$\begin{tikzcd}
F\overline{A} & F(\overline{A\times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, dashed, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
& F\overline{A}\times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

commute. Let $h_{\overline{R}} : F\overline{R} \to F\overline{A}\times G\overline{B}$ be $h_{\overline{A\times B}} \circ F\overline{\iota_R}$. Define $\alpha^\wedge\overline{R}$ to be the subobject through which $h_{\overline{R}}$ is factorized by the mono-epi factorization system in Set, *as shown in the following diagram:*

$$\begin{tikzcd}
F\overline{R} \arrow[rr, "h_{\overline{R}}"] \arrow[dr, two heads, "q_{\alpha^\wedge\overline{R}}"'] & & F\overline{A}\times G\overline{B} \\
& \alpha^\wedge\overline{R} \arrow[ur, hook, "\iota_{\alpha^\wedge\overline{R}}"'] &
\end{tikzcd}$$

Since $\alpha^\wedge\overline{R} : \mathsf{Rel}(F\overline{A}, G\overline{B})$ by construction, we define $\langle\alpha\rangle^*\overline{(A,B,R)} = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge\overline{R}}\alpha^\wedge\overline{R})$. Moreover, if $\overline{(\beta,\beta') : (A,B,R) \to (C,D,S)}$ are morphisms in Rel, *then we define $\langle\alpha\rangle^*\overline{(\beta,\beta')}$ to be $(F\overline{\beta}, G\overline{\beta}')$.*

The data in Definition 23 yield a relation transformer.

Lemma 24. *If $\alpha : F \to G$ is a morphism in $[\mathsf{Set}^k, \mathsf{Set}]$, i.e., a natural transformation between ω-cocontinuous functors, then $\langle\alpha\rangle = (F, G, \langle\alpha\rangle^*)$ is in RT_k.*

Proof. Clearly, $\langle\alpha\rangle^*$ is ω-cocontinuous, so $\langle\alpha\rangle^* : [\mathsf{Rel}^k, \mathsf{Rel}]$.

Now, consider $\overline{(\beta,\beta') : R \to S}$, where $\overline{R : \mathsf{Rel}(A,B)}$ and $\overline{S : \mathsf{Rel}(C,D)}$. We want to show that there exists a morphism $\epsilon : \alpha^\wedge\overline{R} \to \alpha^\wedge\overline{S}$ such that

$$\begin{tikzcd}
\alpha^\wedge\overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{R}}"] \arrow[d, "\epsilon"'] & F\overline{A}\times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
\alpha^\wedge\overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{S}}"'] & F\overline{C}\times G\overline{D}
\end{tikzcd}$$

commutes. Since $\overline{(\beta,\beta') : R \to S}$, there exist $\overline{\gamma : R \to S}$ such that each diagram

$$\begin{tikzcd}
R_i \arrow[r, hook, "\iota_{R_i}"] \arrow[d, "\gamma_i"'] & A_i\times B_i \arrow[d, "\beta_i\times\beta_i'"] \\
S_i \arrow[r, hook, "\iota_{S_i}"] & C_i\times D_i
\end{tikzcd}$$

commutes. Now note that both $h_{\overline{C\times D}} \circ F(\overline{\beta\times\beta'})$ and $(F\overline{\beta}\times G\overline{\beta'}) \circ h_{\overline{A\times B}}$ make

$$\begin{tikzcd}
F\overline{C} & F\overline{C}\times F\overline{D} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F\overline{D} \arrow[r, "\alpha_{\overline{D}}"] & G\overline{D} \\
& F(\overline{A\times B}) \arrow[ul, "F\overline{\pi_1}\circ F(\overline{\beta\times\beta'})"] \arrow[u, dashed, "\exists!"] \arrow[urr, "\alpha_{\overline{D}}\circ F\overline{\pi_2}\circ F(\overline{\beta\times\beta'})"'] & &
\end{tikzcd}$$

commute, so they must be equal. We therefore get that the right-hand square below commutes, and thus that the entire following diagram does as well:

$$\begin{tikzcd}
F\overline{R} \arrow[r, hook, "F\iota_{\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] \arrow[rr, bend left, "h_{\overline{R}}"] & F(\overline{A \times B}) \arrow[r, "h_{\overline{A\times B}}"] \arrow[d, "F(\overline{\beta \times \beta'})"] & F\overline{A} \times G\overline{B} \arrow[d, "F\overline{\beta}\times F\overline{\beta'}"] \\
F\overline{S} \arrow[r, hook, "F\iota_{\overline{S}}"'] \arrow[rr, bend right, "h_{\overline{S}}"'] & F(\overline{C \times D}) \arrow[r, "h_{\overline{C\times D}}"'] & F\overline{C} \times G\overline{D}
\end{tikzcd}$$

Finally, by the left-lifting property of $q_{F^\wedge \overline{R}}$ with respect to $\iota_{F^\wedge \overline{S}}$ given by the epi-mono factorization system, there exists an ϵ such that the diagram

$$\begin{tikzcd}
F\overline{R} \arrow[r, two heads, "q_{\alpha^\wedge \overline{R}}"] \arrow[d, "F\overline{\gamma}"'] & \alpha^\wedge \overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge \overline{R}}"] \arrow[d, dashed, "\epsilon"] & F\overline{A} \times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
F\overline{S} \arrow[r, two heads, "q_{\alpha^\wedge \overline{S}}"'] & \alpha^\wedge \overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge \overline{S}}"'] & F\overline{C} \times G\overline{D}
\end{tikzcd}$$

commutes. □

If $f : A \to B$ be a function with graph relation $\langle f \rangle = (A, B, \langle f \rangle^*)$, then $\langle id_A, f \rangle : A \to A \times B$ and $\langle id_A, f \rangle A = \langle f \rangle^*$. Moreover, if $\iota_{\langle f \rangle} : \langle f \rangle^* \hookrightarrow A \times B$ is the inclusion of $\langle f \rangle^*$ into $A \times B$ then there is an isomorphism of subobjects

$$\begin{tikzcd}
A \arrow[rr, "\cong"] \arrow[dr, "\langle id_A, f \rangle"'] & & \langle f \rangle^* \arrow[dl, "\iota_{\langle f \rangle}"] \\
& A \times B &
\end{tikzcd}$$

We also note that if $f : A \to B$ is a function seen as a natural transformation between 0-ary functors, then $\langle f \rangle$ is (the 0-ary relation transformer associated with) the graph relation of f. Indeed, we need to apply Definition 23 with $k = 0$, i.e., to the degenerate relation $* : \mathsf{Rel}(*, *)$. As degenerate 0-ary functors, A and B are constant functors, i.e., $A* = A$ and $B* = B$. By the universal property of the product, there exists a unique h making the diagram

$$\begin{tikzcd}
A \arrow[r, equal] & A \arrow[r, equal] \arrow[d, dashed, "h"] & A \arrow[r, "f"] & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

commute. Since $\iota_* : * \to *$ is the identity on $*$, and $A\, id_* = id_A$, we have $h_* = h$. Moreover, $h_{\overline{A\times B}} = \langle id_A, f \rangle$ is a monomorphism in Set because id_A is. Then, $\iota_{f^\wedge *} = \langle id_A, f \rangle$ and $f^\wedge * = A$, from which we deduce that $\iota_{f^\wedge *} f^\wedge * = \langle id_A, f \rangle A = \langle f \rangle^*$. This ensures that the graph of f as a 0-ary natural transformation coincides with the graph of f as a morphism in Set, and so that Definition 23 is a reasonable generalization of Definition 22.

Just as the equality relation Eq_B on a set B coincides with $\langle id_B \rangle$, the graph of the identity on the set, so we can define the equality relation transformer to be the graph of the identity natural transformation. This gives

Definition 25. *Let $F : [\mathsf{Set}^k, \mathsf{Set}]$. The equality relation transformer on F is defined to be $\mathsf{Eq}_F = \langle id_F \rangle$. This entails that $Eq_F = (F, F, \mathsf{Eq}^*_F)$ with $\mathsf{Eq}^*_F = \langle id_F \rangle^*$.*

The graph relation transformer on a graph relation is easily computed:

Lemma 26. *If $\alpha : F \to G$ is a morphism in $[\mathsf{Set}^k, \mathsf{Set}]$ and $f_1 : A_1 \to B_1, \ldots, f_k : A_k \to B_k$, then $\langle \alpha \rangle^* \langle \overline{f} \rangle = \langle G\overline{f} \circ \alpha_{\overline{A}} \rangle = \langle \alpha_{\overline{B}} \circ F\overline{f} \rangle$.*

Proof. Since $h_{\overline{A \times B}}$ is the unique morphism making the bottom triangle of the following diagram commute

$$\begin{tikzcd}
& F\overline{A} \arrow[dl, equal] \arrow[d, dashed, "F\overline{\langle id_A, f\rangle}"] \arrow[dr, "F\overline{f}"] & & \\
F\overline{A} & F(\overline{A \times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
& F\overline{A} \times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

and since $h_{\langle \overline{f} \rangle} = h_{\overline{A\times B}} \circ F\,\overline{\iota_{\langle f \rangle}} = h_{\overline{A\times B}} \circ F\overline{\langle id_A, f \rangle}$, the universal property of the product

$$\begin{tikzcd}
F\overline{A} & F\overline{A}\times G\overline{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & G\overline{B} \\
& F\overline{A} \arrow[ul, equal] \arrow[u, dashed, "\exists!"] \arrow[r, "F\overline{f}"'] & F\overline{B} \arrow[u, "\alpha_{\overline{B}}"']
\end{tikzcd}$$

gives that $h_{\langle \overline{f} \rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f} \rangle : F\overline{A} \to F\overline{A} \times G\overline{B}$. Moreover, $\langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f} \rangle$ is a monomorphism in Set because $id_{F\overline{A}}$ is, so its epi-mono factorization gives that $\iota_{\alpha^\wedge \langle \overline{f} \rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f} \rangle$, and thus that $\alpha^\wedge \langle \overline{f} \rangle$, the domain of $\iota_{\alpha^\wedge \langle \overline{f} \rangle}$ is equal to $F\overline{A}$. Then, $\iota_{\alpha^\wedge \langle \overline{f} \rangle} \alpha^\wedge \langle \overline{f} \rangle = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f} \rangle (F\overline{A}) = \langle \alpha_{\overline{B}} \circ F\overline{f} \rangle^*$. We therefore conclude that $\langle \alpha \rangle^* \langle \overline{f} \rangle = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge \langle \overline{f} \rangle}\, \alpha^\wedge \langle \overline{f} \rangle) = (F\overline{A}, G\overline{B}, \langle \alpha_{\overline{B}} \circ F\overline{f} \rangle^*) = \langle \alpha_{\overline{B}} \circ F\overline{f} \rangle$.

Finally, note that $\alpha_{\overline{B}} \circ F\overline{f} = G\overline{f} \circ \alpha_{\overline{A}}$ by naturality of α. □

As an immediate corollary we have

Corollary 27. *If $F : [\mathsf{Set}^k, \mathsf{Set}]$ and $\overline{A : \mathsf{Set}}$, then $\mathsf{Eq}^*_F \overline{\mathsf{Eq}_A} = \mathsf{Eq}_{F\overline{A}}$.*

Proof. We have that

$$\mathsf{Eq}^*_F \overline{\mathsf{Eq}_A} = \langle id_F \rangle^* \langle id_{\overline{A}} \rangle = \langle F id_{\overline{A}} \circ (id_F)_{\overline{A}} \rangle = \langle id_{F\overline{A}} \circ id_{F\overline{A}} \rangle = \langle id_{F\overline{A}} \rangle = \mathsf{Eq}_{F\overline{A}}$$

The second identity here is by Lemma 26. □

We can extend the notions of the graph of a natural transformation and the equality relation transformer to environments as follows:

Definition 3.1. Let $f : \rho \to \rho'$ is a morphism of set environments. The graph relation environment $\langle f \rangle$ is defined pointwise, i.e., for any variable ϕ, we define $\langle f \rangle \phi = \langle f \phi \rangle$. This ensures that $\pi_1 \langle f \rangle = \rho$ and $\pi_2 \langle f \rangle = \rho'$. Then the equality relation environment Eq_ρ for any set environment ρ is then defined to be $\langle id_\rho \rangle$.

With this definition in hand, we can prove an Identity Extension Lemma for our type interpretations.

Theorem 28. *If ρ is a set environment, and $\Gamma;\Phi \vdash \tau : \mathcal{F}$, then* $[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}$.

Proof. By induction on the structure of τ.

- $[\![\Gamma;\emptyset \vdash v]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v} = \mathsf{Eq}_{[\![\Gamma;\emptyset\vdash v]\!]^{\mathsf{Set}}\rho}$ where $v \in \Gamma$.
- By definition, $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho$ is the relation on $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$ relating t and t' if, for all $R_1 : \mathsf{Rel}(A_1,B_1),\ldots,R_k : \mathsf{Rel}(A_k,B_k)$, $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel. To prove that this is equal to $\mathsf{Eq}_{[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho}$ we need to show that $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1),\ldots,R_k : \mathsf{Rel}(A_k,B_k)$ if and only if $t = t'$ and $(t_{\overline{A}}, t_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1),\ldots,$ $R_k : \mathsf{Rel}(A_k,B_k)$. The only interesting part of this equivalence is to show that if $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1),$ $\ldots, R_k : \mathsf{Rel}(A_k,B_k)$, then $t = t'$. By hypothesis, $(t_{\overline{A}}, t'_{\overline{A}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := \mathsf{Eq}_A]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := \mathsf{Eq}_A]}$ in Rel for all $A_1 \ldots A_k : \mathsf{Set}$. By the induction hypothesis, it is therefore a morphism from $\mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$ to $\mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$ in Rel. This means that, for every $x : \mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$, $t_{\overline{A}}x = t'_{\overline{A}}x$. Then, by extensionality, $t = t'$.
- $[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = 0_{\mathsf{Rel}} = \mathsf{Eq}_{0_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\mathbb{0}]\!]^{\mathsf{Set}}\rho}$
- $[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = 1_{\mathsf{Rel}} = \mathsf{Eq}_{1_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\mathbb{1}]\!]^{\mathsf{Set}}\rho}$
- The application case is proved by the following sequence of equalities, where the second equality is by the induction hypothesis and the definition of the relation environment Eq_ρ, the third is by the definition of application of relation transformers, and the fourth is by Lemma 26:

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mathsf{Eq}_\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \mathsf{Eq}_{\rho\phi}\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= (\mathsf{Eq}_{\rho\phi})^*\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{(\rho\phi)\,\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\phi\overline{\tau}]\!]^{\mathsf{Set}}\rho}
\end{aligned}
$$

- The fixed point case is proven by the sequence of equalities

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mu T_{\mathsf{Eq}_\rho})\,\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{\mathsf{Eq}_\rho}K_0\,\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0})^*\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0\,\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{\varinjlim_{n\in\mathbb{N}}(T^{\mathsf{Set}}_\rho)^n K_0\,\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho}
\end{aligned}
$$

Here, the third equality is by induction hypothesis, the fifth is by Lemma 26 and the fourth equality is because, for every $n \in \mathbb{N}$, the following two statements can be proved by simultaneous induction:

$$T^n_{\mathsf{Eq}_\rho} K_0\, \overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} = (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0})^* \overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} \tag{8}$$

and

$$\begin{aligned}&[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\ &\qquad = [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\end{aligned} \tag{9}$$

We prove (8). The case $n = 0$ is trivial, because $T^0_{\mathsf{Eq}_\rho} K_0 = K_0$ and $(T^{\mathsf{Set}}_\rho)^0 K_0 = K_0$; the inductive step is proved by the following sequence of equalities:

$$\begin{aligned}T^{n+1}_{\mathsf{Eq}_\rho} K_0\, \overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} &= T^{\mathsf{Rel}}_{\mathsf{Eq}_\rho}(T^n_{\mathsf{Eq}_\rho} K_0)\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho[\phi:=(T^{\mathsf{Set}}_\rho)^n K_0]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\phi:=(T^{\mathsf{Set}}_\rho)^n K_0]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}\\
&= \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^{n+1} K_0\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^{n+1} K_0})^*\, \overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\end{aligned}$$

Here, the third equality is by (9), the fifth by the induction hypothesis on H, and the last by Lemma 26. We prove the induction step of (9) by structural induction on H: the only interesting case, though, is when ϕ is applied, i.e., for $H = \phi\overline{\sigma}$, which is proved by the sequence of equalities:

$$\begin{aligned}&[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash \phi\overline{\sigma}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&= T^n_{\mathsf{Eq}_\rho} K_0\, \overline{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash \sigma]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}}\\
&= T^n_{\mathsf{Eq}_\rho} K_0\, \overline{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash \sigma]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}}\\
&= T^n_{\mathsf{Eq}_\rho} K_0\, \overline{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash \sigma]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho[\phi:=(T^{\mathsf{Set}}_\rho)^n K_0]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}}\\
&= T^n_{\mathsf{Eq}_\rho} K_0\, \overline{\mathsf{Eq}_{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash\sigma]\!]^{\mathsf{Set}}\rho[\phi:=(T^{\mathsf{Set}}_\rho)^n K_0]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}}\\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0})^*\, \overline{\mathsf{Eq}_{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash\sigma]\!]^{\mathsf{Set}}\rho[\phi:=(T^{\mathsf{Set}}_\rho)^n K_0]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}}\\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0})^*\overline{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash \sigma]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash \phi\overline{\sigma}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\end{aligned}$$

Here, the second equality is by the induction hypothesis for (9) on the σs, the fourth is by the induction hypothesis for Theorem 28 on the σs, and the fifth is by the induction hypothesis on n for (8).

- $[\![\Gamma;\Phi\vdash\sigma+\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\Phi\vdash\sigma]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho + [\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\sigma]\!]^{\mathsf{Set}}\rho} + \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\sigma]\!]^{\mathsf{Set}}\rho+[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\sigma+\tau]\!]^{\mathsf{Set}}\rho}$

- $[\![\Gamma;\Phi\vdash\sigma\times\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\Phi\vdash\sigma]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho \times [\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\sigma]\!]^{\mathsf{Set}}\rho} \times \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\sigma]\!]^{\mathsf{Set}}\rho\times[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\sigma\times\tau]\!]^{\mathsf{Set}}\rho}$

□

It follows from Theorem 28 that $[\![\Gamma\vdash\sigma\to\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma\vdash\sigma\to\tau]\!]^{\mathsf{Set}}\rho}$, as expected. Moreover, the Identity Extension Lemma also allows us to prove a Graph Lemma.

Lemma 29 (Graph Lemma). *If $f:\rho\to\rho'$ is a morphism of set environments and $\Gamma;\Phi\vdash F:\mathcal{F}$, then $\langle[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\rangle = [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$*

Proof. First observe that $(f, id_{\rho'}) : \langle f\rangle \to \mathsf{Eq}_{\rho'}$ and $(id_\rho, f) : \mathsf{Eq}_\rho \to \langle f\rangle$ are morphisms of relation environments. Applying Lemma 19 to each of these observations gives that

$$([\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}id_{\rho'}) = [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}(f, id_{\rho'}) : [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle \to [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'} \tag{10}$$

and

$$([\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}id_\rho, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f) = [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}(id_\rho, f) : [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho \to [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle \tag{11}$$

Expanding Equation 10 gives that if $(x,y)\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$ then

$$([\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}id_{\rho'}\,y) \in [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}$$

Observe that $[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}id_{\rho'}\,y = id_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}\,y = y$ and $[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}$ So, if $(x,y)\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$ then $([\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x, y) \in \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}$, i.e., $[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x = y$, i.e., $(x,y)\in\langle[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\rangle$. So, we have that $[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle \subseteq \langle[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\rangle$

Expanding Equation 11 gives that, for any $x\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho$, then

$$([\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}id_\rho\,x, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x) \in [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$$

Observe that $[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}id_\rho\,x = id_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}x = x$ so, for any $x\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho$, we have that $(x, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x)\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$. Moreover, $x\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho$ if and only if $(x, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x)\in\langle[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\rangle$ and, if $x\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho$ then $(x, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x)\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$, so if $(x, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x)\in\langle[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\rangle$ then $(x, [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\,x)\in[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$, i.e., $\langle[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\rangle\subseteq[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$. We conclude that $[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\langle f\rangle = \langle[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}f\rangle$ as desired. □

4 INTERPRETING TERMS

If $\Delta = x_1:\tau_1,\ldots,x_n:\tau_n$ is a term context for Γ and Φ, then the interpretations $[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}}$ and $[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}}$ are defined by

$$\begin{array}{rcl}
[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}} & = & [\![\Gamma;\Phi\vdash\tau_1]\!]^{\mathsf{Set}}\times\ldots\times[\![\Gamma;\Phi\vdash\tau_n]\!]^{\mathsf{Set}}\\
[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}} & = & [\![\Gamma;\Phi\vdash\tau_1]\!]^{\mathsf{Rel}}\times\ldots\times[\![\Gamma;\Phi\vdash\tau_n]\!]^{\mathsf{Rel}}
\end{array}$$

Every well-formed term $\Gamma;\Phi\mid\Delta\vdash t:\tau$ then has, for every $\rho\in\mathsf{SetEnv}$, set interpretations $[\![\Gamma;\Phi\mid\Delta\vdash t:\tau]\!]^{\mathsf{Set}}\rho$ as natural transformations from $[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho$, and, for every $\rho\in\mathsf{RelEnv}$, relational interpretations $[\![\Gamma;\Phi\mid\Delta\vdash t:\tau]\!]^{\mathsf{Rel}}\rho$ as natural transformations from $[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}}\rho$ to $[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho$. These are given in the next two definitions.

Definition 30. *If ρ is a set environment and $\Gamma;\Phi \mid \Delta \vdash t : \tau$ then $[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho$ is defined as follows:*

$$
\begin{array}{lcl}
[\![\Gamma;\emptyset \mid \Delta, x : \tau \vdash x : \tau]\!]^{\mathsf{Set}}\rho & = & \pi_{|\Delta|+1} \\
[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho & = & \mathsf{curry}([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]) \\
[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho & = & \mathsf{eval} \circ \langle \lambda d.\, ([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ d)_{\overline{[\![\Gamma;\Phi\vdash\tau]}} \\
 & & \qquad [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\rangle \\
 & & \\
[\![\Gamma;\Phi \mid \Delta, x : \tau \vdash x : \tau]\!]^{\mathsf{Set}}\rho & = & \pi_{|\Delta|+1} \\
[\![\Gamma;\Phi \mid \Delta \vdash \bot_\tau t : \tau]\!]^{\mathsf{Set}}\rho & = & !^{0}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \mathbb{0}]\!]^{\mathsf{Set}}\rho,\ \textit{where} \\
 & & \quad !^{0}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}\ \textit{is the unique morphism from } 0 \\
 & & \quad \textit{to } [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \top : \mathbb{1}]\!]^{\mathsf{Set}}\rho & = & !_{1}^{[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}}\rho},\ \textit{where}\ !_{1}^{[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}}\rho} \\
 & & \quad \textit{is the unique morphism from } [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho\ \textit{to} \\
[\![\Gamma;\Phi \mid \Delta \vdash (s,t) : \sigma \times \tau]\!]^{\mathsf{Set}}\rho & = & [\![\Gamma;\Phi \mid \Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \pi_1 t : \sigma]\!]^{\mathsf{Set}}\rho & = & \pi_1 \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \pi_2 t : \sigma]\!]^{\mathsf{Set}}\rho & = & \pi_2 \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{case}\ t\ \mathsf{of}\ \{x \mapsto l;\ y \mapsto r\} : \gamma]\!]^{\mathsf{Set}}\rho & = & \mathsf{eval} \circ \langle \mathsf{curry}\,[[\![\Gamma;\Phi \mid \Delta, x : \sigma \vdash l : \gamma]\!]^{\mathsf{Set}}\rho, \\
 & & \qquad\qquad [\![\Gamma;\Phi \mid \Delta, y : \tau \vdash r : \gamma]\!]^{\mathsf{Set}}\rho], \\
 & & \qquad [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma + \tau]\!]^{\mathsf{Set}}\rho\rangle \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau]\!]^{\mathsf{Set}}\rho & = & \mathsf{inl} \circ [\![\Gamma;\Phi \mid \Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau]\!]^{\mathsf{Set}}\rho & = & \mathsf{inr} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\ \overline{(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G)} & = & \lambda d\, \overline{\eta}\, \overline{B}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}] \\
\qquad (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\rho & & \\
[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}] & = & \lambda d\, \overline{B}\, \overline{C}.\, (in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}})_{\overline{B}} \\
\qquad (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho & & \\
[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}_H^{F} : \mathsf{Nat}^{\emptyset}\ (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F) & = & \lambda d\, \eta\, \overline{B}\, \overline{C}.\, (fold_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}\ (\lambda\overline{A}.\, \eta_{\overline{A}\,\overline{C}}))_{\overline{B}} \\
\qquad (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\ F)]\!]^{\mathsf{Set}}\rho & &
\end{array}
$$

Add return type for fold in last clause? Should be $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho]\overline{\gamma := C}]$.

This interpretation gives that $[\![\Gamma;\emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau]\!]^{\mathsf{Set}}\rho = \mathsf{curry}([\![\Gamma;\emptyset \mid \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho)$ and $[\![\Gamma;\emptyset \mid \Delta \vdash st : \tau]\!]^{\mathsf{Set}}\rho = \mathsf{eval} \circ \langle [\![\Gamma;\emptyset \mid \Delta \vdash s : \sigma \to \tau]\!]^{\mathsf{Set}}\rho, [\![\Gamma;\emptyset \mid \Delta \vdash t : \sigma]\!]^{\mathsf{Set}}\rho\rangle$, as expected.

Definition 31. *If ρ is a relation environment and $\Gamma;\Phi \mid \Delta \vdash t : \tau$ then $[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho$ is defined as follows:*

$$
\begin{array}{lcl}
[\![\Gamma;\emptyset \mid \Delta, x : \tau \vdash x : \tau]\!]^{\mathsf{Rel}}\rho & = & \pi_{|\Delta|+1} \\
[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho & = & \mathsf{curry}([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}]) \\
[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho & = & \mathsf{eval} \circ \langle \lambda e.\, ([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho\; e)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]}} \\
 & & \qquad [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho\rangle \\
 & & \\
[\![\Gamma;\Phi \mid \Delta, x : \tau \vdash x : \tau]\!]^{\mathsf{Rel}}\rho & = & \pi_{|\Delta|+1} \\
[\![\Gamma;\Phi \mid \Delta \vdash \bot_\tau t : \tau]\!]^{\mathsf{Rel}}\rho & = & !^{0}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \mathbb{0}]\!]^{\mathsf{Rel}}\rho,\ \textit{where} \\
 & & \quad !^{0}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}\ \textit{is the unique morphism from } 0 \\
 & & \quad \textit{to } [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \top : \mathbb{1}]\!]^{\mathsf{Rel}}\rho & = & !^{[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}}\rho}_{1},\ \textit{where } !^{[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}}\rho}_{1} \\
 & & \quad \textit{is the unique morphism from } [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho \textit{ to} \\
[\![\Gamma;\Phi \mid \Delta \vdash (s,t) : \sigma \times \tau]\!]^{\mathsf{Rel}}\rho & = & [\![\Gamma;\Phi \mid \Delta \vdash s : \sigma]\!]^{\mathsf{Rel}}\rho \times [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \pi_1 t : \sigma]\!]^{\mathsf{Rel}}\rho & = & \pi_1 \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau]\!]^{\mathsf{Rel}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \pi_2 t : \sigma]\!]^{\mathsf{Rel}}\rho & = & \pi_2 \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau]\!]^{\mathsf{Rel}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{case}\; t \;\mathsf{of}\; \{x \mapsto l;\, y \mapsto r\} : \gamma]\!]^{\mathsf{Rel}}\rho & = & \mathsf{eval} \circ \langle \mathsf{curry}\, [[\![\Gamma;\Phi \mid \Delta, x : \sigma \vdash l : \gamma]\!]^{\mathsf{Rel}}\rho, \\
 & & \qquad\qquad [\![\Gamma;\Phi \mid \Delta, y : \tau \vdash r : \gamma]\!]^{\mathsf{Rel}}\rho], \\
 & & \qquad [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma + \tau]\!]^{\mathsf{Rel}}\rho\rangle \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau]\!]^{\mathsf{Rel}}\rho & = & \mathsf{inl} \circ [\![\Gamma;\Phi \mid \Delta \vdash s : \sigma]\!]^{\mathsf{Rel}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau]\!]^{\mathsf{Rel}}\rho & = & \mathsf{inr} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho \\
[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{F},\overline{G}}_{H} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G}) & = & \lambda d\, \overline{\eta}\, \overline{R}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Rel}} id_{\rho[\overline{\gamma := R}]}[\overline{\phi := \lambda \overline{S}.\eta_{\overline{S}\,\overline{R}}}] \\
\qquad (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}\rho & & \\
[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}] & = & \lambda d\, \overline{R}\, \overline{S}.\, (in_{T_{\rho[\overline{\gamma := S}]}})_{\overline{R}} \\
\qquad (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\rho & & \\
[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^{F}_{H} : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F) & = & \lambda d\, \eta\, \overline{R}\, \overline{S}.\, (fold_{T_{\rho[\overline{\gamma := S}]}}\, (\lambda \overline{Z}.\, \eta_{\overline{Z}\,\overline{S}}))_{\overline{R}} \\
\qquad (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F)]\!]^{\mathsf{Rel}}\rho & &
\end{array}
$$

Add return type for fold in last clause? Should be $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Rel}}\rho]\overline{\gamma := C}]$.

If t is closed, i.e., if $\emptyset;\emptyset \mid \emptyset \vdash t : \tau$, then we write $[\![\vdash t : \tau]\!]^{\mathsf{Set}}$ instead of $[\![\emptyset;\emptyset \mid \emptyset \vdash t : \tau]\!]^{\mathsf{Set}}$, and similarly for $[\![\emptyset;\emptyset \mid \emptyset \vdash t : \tau]\!]^{\mathsf{Rel}}$.

4.1 Basic Properties of Term Interpretations

This interpretation gives that $[\![\Gamma;\emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau]\!]^{\mathsf{Rel}}\rho = \mathsf{curry}([\![\Gamma;\emptyset \mid \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Rel}}\rho)$ and $[\![\Gamma;\emptyset \mid \Delta \vdash st : \tau]\!]^{\mathsf{Rel}}\rho = \mathsf{eval} \circ \langle [\![\Gamma;\emptyset \mid \Delta \vdash s : \sigma \to \tau]\!]^{\mathsf{Rel}}\rho, [\![\Gamma;\emptyset \mid \Delta \vdash t : \sigma]\!]^{\mathsf{Rel}}\rho\rangle$, as expected.

The interpretations in Definitions 30 and 31 respect weakening, i.e., a term and its weakenings all have the same set and relational interpretations. In particular, for any $\rho \in \mathsf{SetEnv}$,

$$[\![\Gamma;\Phi \mid \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho = ([\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho) \circ \pi_\Delta$$

where π_Δ is the projection $[\![\Gamma;\Phi \vdash \Delta, x : \sigma]\!]^{\mathsf{Set}} \to [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}$, and for any $\rho \in \mathsf{RelEnv}$,

$$[\![\Gamma;\Phi \mid \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Rel}}\rho = ([\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho) \circ \pi_\Delta$$

where π_Δ is the projection $[\![\Gamma;\Phi \vdash \Delta, x : \sigma]\!]^{\mathsf{Rel}} \to [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}$. Moreover, if $\Gamma, \alpha;\Phi \mid \Delta \vdash t : \tau$ and $\Gamma;\Phi,\alpha \mid \Delta \vdash t' : \tau$ and $\Gamma;\Phi \vdash \sigma : \mathcal{F}$ then

- $[\![\Gamma;\Phi \mid \Delta[\alpha := \sigma] \vdash t[\alpha := \sigma] : \tau[\alpha := \sigma]]\!]^{\mathsf{Set}}\rho = [\![\Gamma,\alpha;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho]$

- $[\![\Gamma;\Phi \,|\, \Delta[\alpha := \sigma] \vdash t[\alpha := \sigma] : \tau[\alpha := \sigma]]\!]^{\mathsf{Rel}}\rho = [\![\Gamma,\alpha;\Phi \,|\, \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho[\alpha := [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho]$
- $[\![\Gamma;\Phi \,|\, \Delta[\alpha := \sigma] \vdash t'[\alpha := \sigma] : \tau[\alpha := \sigma]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\alpha \,|\, \Delta \vdash t' : \tau]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho]$
- $[\![\Gamma;\Phi \,|\, \Delta[\alpha := \sigma] \vdash t'[\alpha := \sigma] : \tau[\alpha := \sigma]]\!]^{\mathsf{Rel}}\rho = [\![\Gamma;\Phi,\alpha \,|\, \Delta \vdash t' : \tau]\!]^{\mathsf{Rel}}\rho[\alpha := [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho]$

and if $\Gamma;\Phi \,|\, \Delta, x : \sigma \vdash t : \tau$ and $\Gamma;\Phi \,|\, \Delta \vdash s : \sigma$ then

- $\lambda A.\, [\![\Gamma;\Phi \,|\, \Delta \vdash t[x := s] : \tau]\!]^{\mathsf{Set}}\rho\, A = \lambda A.\, [\![\Gamma;\Phi \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho(A, [\![\Gamma;\Phi \,|\, \Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho\, A)$
- $\lambda R.\, [\![\Gamma;\Phi \,|\, \Delta \vdash t[x := s] : \tau]\!]^{\mathsf{Rel}}\rho\, R = \lambda R.\, [\![\Gamma;\Phi \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Rel}}\rho\, (R, [\![\Gamma;\Phi \,|\, \Delta \vdash s : \sigma]\!]^{\mathsf{Rel}}\rho\, R)$

Direct calculation reveals that the set interpretations of terms also satisfy

- $[\![\Gamma;\Phi \mid \Delta \vdash (L_{\overline{\alpha}}x.t)_{\overline{\tau}}s]\!]^{\mathsf{Set}} = [\![\Gamma;\Phi \mid \Delta \vdash t[\overline{\alpha := \tau}][x := s]]\!]^{\mathsf{Set}}$

Standard type extensionality $[\![\Gamma;\Phi \vdash (L_\alpha x.t)_\alpha t]\!]^{\mathsf{Set}} = [\![\Gamma;\Phi \vdash t]\!]^{\mathsf{Set}}$ and $[\![\Gamma;\Phi \vdash (L_\alpha x.t)_\alpha t]\!]^{\mathsf{Rel}} = [\![\Gamma;\Phi \vdash t]\!]^{\mathsf{Rel}}$, as well as term extensionality $[\![\Gamma;\Phi \vdash (L_\alpha x.t)_\alpha \top]\!]^{\mathsf{Set}} = [\![\Gamma;\Phi \vdash t]\!]^{\mathsf{Set}}$ and $[\![\Gamma;\Phi \vdash (L_\alpha x.t)_\alpha \top]\!]^{\mathsf{Rel}} = [\![\Gamma;\Phi \vdash t]\!]^{\mathsf{Rel}}$, for terms are immediate consequences.

4.2 Properties of Terms of Nat-Type

If we define, for $\Gamma;\overline{\alpha} \vdash F$, the term id_F to be $\Gamma;\emptyset \,|\, \emptyset \vdash L_{\overline{\alpha}}x.x : \mathsf{Nat}^{\overline{\alpha}} F\, F$ and, for terms $\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\, G$ and $\Gamma;\emptyset \,|\, \Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\, H$, the *composition* $s \circ t$ of t and s to be $\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.s_{\overline{\alpha}}(t_{\overline{\alpha}}x) : \mathsf{Nat}^{\overline{\alpha}} F\, H$, then

- $[\![\Gamma;\emptyset \,|\, \emptyset \vdash id_F : \mathsf{Nat}^{\overline{\alpha}} F\, F]\!]^{\mathsf{Set}}\rho \,* = id_{\lambda \overline{A}.[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]}$ for any set environment ρ
- $[\![\Gamma;\emptyset \,|\, \Delta \vdash s \circ t : \mathsf{Nat}^{\overline{\alpha}} F\, H]\!]^{\mathsf{Set}} = [\![\Gamma;\emptyset \,|\, \Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\, H]\!]^{\mathsf{Set}} \circ [\![\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\, G]\!]^{\mathsf{Set}}$

Moreover, terms of Nat type behave as natural transformations with respect to their source and target functorial types.

- $[\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) \circ (L_{\overline{\gamma}}z.x_{\overline{\sigma},\overline{\gamma}}z) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$
  $= [\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash (L_{\overline{\gamma}}z.x_{\overline{\tau},\overline{\gamma}}z) \circ ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$

As the special case of the previous equality when $x = in_H$ we have

Theorem 32.

- $[\![\Gamma;\emptyset \,|\, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash ((\mathsf{map}^{\overline{\sigma},\overline{\tau}}_{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \overline{y}) \circ (L_{\overline{\gamma}}z.(\mathsf{in}_H)_{\overline{\sigma},\overline{\gamma}}z) : \xi]\!]^{\mathsf{Set}}$
  $= [\![\Gamma;\emptyset \,|\, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash (L_{\overline{\gamma}}z.(\mathsf{in}_H)_{\overline{\tau},\overline{\gamma}}z) \circ ((\mathsf{map}^{\overline{\sigma},\overline{\tau}}_{H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset \overline{y}) : \xi]\!]^{\mathsf{Set}}$
  at type $\xi = \mathsf{Nat}^{\overline{\gamma}} H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \sigma}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$

Analogous results hold for relational interpretations of terms and relational environments.

As we observe in Section 5.1, Theorem 32 gives a family of results that we normally derive as free theorems but actually are consequences of naturality. Most of Wadler's fall into this family, for example, but not the free theorem for filter (even for lists) or short cut fusion.

4.3 Properties of Initial Algebraic Constructs

We first observe that map-terms are interpreted as semantic *maps*:

Let $\Gamma;\overline{\phi},\overline{\gamma} \vdash H : \mathcal{F}$, $\Gamma;\overline{\beta},\overline{\gamma} \vdash F : \mathcal{F}$ and $\Gamma;\overline{\beta},\overline{\gamma} \vdash G : \mathcal{F}$. By definition of the semantic interpretation of map terms, we have

$$[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^\emptyset\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\rho$$
$$= \lambda d\, \overline{\eta}\, \overline{B}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}] \qquad (12)$$

Then let $\Gamma;\overline{\alpha} \vdash F : \mathcal{F}$, $\overline{\Gamma;\emptyset \vdash \sigma : \mathcal{F}}$, $\overline{\Gamma;\emptyset \vdash \tau : \mathcal{F}}$ and $*$ be the unique element of $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$. As a special case of the above definition, we have

$$\begin{array}{rl} & [\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{map}_F^{\overline{\sigma},\overline{\tau}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\emptyset}\,\sigma\,\tau})\,(\mathsf{Nat}^{\emptyset}\,F[\overline{\alpha := \sigma}]\,F[\overline{\alpha := \tau}])]\!]^{\mathsf{Set}}\rho\, * \\ = & \lambda \overline{f : [\![\Gamma;\emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho}.\ [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}id_{\rho}[\overline{\alpha := f}] \\ = & \lambda \overline{f : [\![\Gamma;\emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho}.\ map_{\lambda\overline{A}.\,[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}]}\overline{f} \\ = & map_{\lambda\overline{A}.\,[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}]} \end{array}$$

where the first equality is by Equation 12, the second equality is obtained by noting that $\lambda\overline{A}.\ [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]$ is a functor in α, and map_G denotes the action of the functor G on morphisms.

We also have the expected relationships between interpetations of terms involving map, in, and fold:

- If $\Gamma;\overline{\psi},\overline{\gamma} \vdash H$, $\overline{\Gamma;\overline{\alpha},\overline{\gamma},\overline{\phi} \vdash K}$, $\overline{\Gamma;\overline{\beta},\overline{\gamma} \vdash F}$, and $\overline{\Gamma;\overline{\beta},\overline{\gamma} \vdash G}$, then

$$[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{map}^{\overline{F},\overline{G}}_{\overline{H[\psi:=K]}} : \xi]\!]^{\mathsf{Set}} = [\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{K[\overline{\phi:=F}]},\overline{K[\overline{\phi:=G}]}} \circ \overline{\mathsf{map}_K^{\overline{F},\overline{G}}} : \xi]\!]^{\mathsf{Set}}$$

  at type $\xi = \mathsf{Nat}^{\emptyset}(\overline{\mathsf{Nat}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}FG})(\mathsf{Nat}^{\overline{\gamma}}H[\overline{\psi := K}][\overline{\phi := F}]H[\overline{\psi := K}][\overline{\phi := G}])$
- If $\Gamma;\overline{\beta},\overline{\gamma} \vdash H$, $\Gamma;\overline{\beta},\overline{\gamma} \vdash K$, $\overline{\Gamma;\overline{\alpha},\overline{\gamma} \vdash F}$, $\overline{\Gamma;\overline{\alpha},\overline{\gamma} \vdash G}$, $\overline{\Gamma;\phi,\overline{\psi},\overline{\gamma} \vdash \tau}$, $\overline{I}$ is the sequence $\overline{F},H$ and $\overline{J}$ is the sequence $\overline{G},K$ then

$$[\![\Gamma;\emptyset \,|\, \emptyset \vdash L_{\emptyset}(x,\overline{y}).L_{\overline{\gamma}}z.x_{\overline{\tau[\overline{\psi:=G}][\phi:=K]},\overline{\gamma}}\Big(((\mathsf{map}_H^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}})_{\emptyset}(\overline{(\mathsf{map}_{\tau}^{\overline{I},\overline{J}})_{\emptyset}(x,\overline{y})}))_{\overline{\gamma}}z\Big) : \xi]\!]^{\mathsf{Set}}$$

$$= [\![\Gamma;\emptyset \,|\, \emptyset \vdash L_{\emptyset}(x,\overline{y}).L_{\overline{\gamma}}z.((\mathsf{map}_K^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}})_{\emptyset}(\overline{(\mathsf{map}_{\tau}^{\overline{I},\overline{J}})_{\emptyset}(x,\overline{y})}))_{\overline{\gamma}}\Big(x_{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\gamma}}z\Big) : \xi]\!]^{\mathsf{Set}}$$

  at type $\xi = \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}H\,K \times \overline{\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}H[\overline{\beta := \tau}][\overline{\psi := F}][\phi := H]\,K[\overline{\beta := \tau}][\overline{\psi := G}][\phi := K])$.
- $[\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}H[\phi := F][\overline{\alpha := \beta}]\,F \vdash ((\mathsf{fold}_{H,F})_{\emptyset}x) \circ \mathsf{in}_H : \xi]\!]^{\mathsf{Set}}$
  $= [\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}H[\phi := F][\overline{\alpha := \beta}]\,F \vdash x \circ \big((\mathsf{map}^{(\mu\phi.\lambda\overline{\alpha}H)\overline{\beta},F}_{H[\overline{\alpha:=\beta}]})_{\emptyset}((\mathsf{fold}_{H,F})_{\emptyset}x)\big) : \xi]\!]^{\mathsf{Set}}$
  at type $\xi = \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\,F$
- $[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H \circ (\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_{\emptyset}((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_{\emptyset}\mathsf{in}_H) : \xi]\!]^{\mathsf{Set}}$
  $= [\![\Gamma;\emptyset \,|\, \emptyset \vdash Id_{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}} : \xi]\!]^{\mathsf{Set}}$
  at type $\xi = \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}$
- $[\![\Gamma;\emptyset \,|\, \emptyset \vdash (\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_{\emptyset}((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_{\emptyset}\mathsf{in}_H) \circ \mathsf{in}_H : \xi]\!]^{\mathsf{Set}}$
  $= [\![\Gamma;\emptyset \,|\, \emptyset \vdash Id_{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]} : \xi]\!]^{\mathsf{Set}}$
  at type $\xi = \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\,H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]$.

Analogous results hold for relational interpretations of terms and relational environments. The set and relational interpretations of terms therefore respect the congruence closed equational theory obtained by adding these judgments to those generating the usual congruence closed equational theory induced by the other term formers.

5 FREE THEOREMS FOR NESTED TYPES

5.1 Free Theorems Derived from Naturality

Make this not about *subst* Note that the free theorem for a type is always independent of the particular term of that type, so the proof below is independent of the choice of function *subst*. In addition, it is independent of the particular data type — in this case, Lam — over which *subst* acts. Also independent of the functor arguments — in this case $+\mathbb{1}$ and *id* — to the data type. Indeed, the following result is just a consequence of naturality.

We already know from Theorem 32 that

$$\begin{aligned}&\llbracket \Gamma; \emptyset \mid x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\,\tau} \vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) \circ (L_{\overline{\gamma}} z. x_{\overline{\sigma},\overline{\gamma}} z) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} \\ &= \llbracket \Gamma; \emptyset \mid x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\,\tau} \vdash (L_{\overline{\gamma}} z. x_{\overline{\tau},\overline{\gamma}} z) \circ ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\end{aligned} \tag{13}$$

In particular, if we instantiate x with any term *subst* of type $\vdash \mathsf{Nat}^{\alpha}(\mathsf{Lam}\,(\alpha + \mathbb{1}) \times \mathsf{Lam}\,\alpha)\,\mathsf{Lam}\,\alpha$ (and thus there is a single α and no γ's) we have

$$\begin{aligned}&\llbracket \Gamma; \emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,\alpha})_\emptyset y) \circ (L_\emptyset z. subst_\sigma z) : \mathsf{Nat}^{\emptyset}(\mathsf{Lam}\,(\sigma+\mathbb{1}) \times \mathsf{Lam}\,\sigma)\,\mathsf{Lam}\,\tau \rrbracket^{\mathsf{Set}} \\ &= \llbracket \Gamma; \emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash (L_\emptyset z. subst_\tau z) \circ ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha})_\emptyset y) : \mathsf{Nat}^{\emptyset}(\mathsf{Lam}\,(\sigma+\mathbb{1}) \times \mathsf{Lam}\,\sigma)\,\mathsf{Lam}\,\tau \rrbracket^{\mathsf{Set}}\end{aligned} \tag{14}$$

Thus, for any set environment ρ and any function $f : \llbracket \Gamma; \emptyset \vdash \mathsf{Nat}^{\emptyset} \sigma\,\tau \rrbracket^{\mathsf{Set}} \rho$, we have that

$$\begin{aligned}&\llbracket \Gamma; \emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,\alpha})_\emptyset y) \circ (L_\emptyset z. subst_\sigma z) : \mathsf{Nat}^{\emptyset}(\mathsf{Lam}\,(\sigma+\mathbb{1}) \times \mathsf{Lam}\,\sigma)\,\mathsf{Lam}\,\tau \rrbracket^{\mathsf{Set}} \rho f \\ &= \llbracket \Gamma; \emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,\alpha})_\emptyset y)\rrbracket^{\mathsf{Set}} \rho f \circ \llbracket \Gamma; \emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash L_\emptyset z. subst_\sigma z \rrbracket^{\mathsf{Set}} \rho f \\ &= \llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{\sigma},\overline{\tau}}_{\mathsf{Lam}\,\alpha} \rrbracket^{\mathsf{Set}} \rho f \circ \llbracket \Gamma; \emptyset \mid \emptyset \vdash L_\emptyset z. subst_\sigma z \rrbracket^{\mathsf{Set}} \rho \\ &= \mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} f \circ (\llbracket \vdash subst \rrbracket^{\mathsf{Set}})_{\llbracket \Gamma; \emptyset \vdash \sigma \rrbracket^{\mathsf{Set}} \rho}\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\llbracket \Gamma; \emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash (L_\emptyset z. subst_\tau z) \circ ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha})_\emptyset y) : \mathsf{Nat}^{\emptyset}(\mathsf{Lam}\,(\sigma+\mathbb{1}) \times \mathsf{Lam}\,\sigma)\,\mathsf{Lam}\,\tau \rrbracket^{\mathsf{Set}} \rho f \\ &= \llbracket \Gamma; \emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash L_\emptyset z. subst_\tau z \rrbracket^{\mathsf{Set}} \rho f \circ \llbracket \Gamma; \emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash (\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha})_\emptyset y \rrbracket^{\mathsf{Set}} \rho f \\ &= \llbracket \Gamma; \emptyset \mid \emptyset \vdash L_\emptyset z. subst_\tau z \rrbracket^{\mathsf{Set}} \rho \circ \llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha} \rrbracket^{\mathsf{Set}} \rho f \\ &= (\llbracket \vdash subst \rrbracket^{\mathsf{Set}})_{\llbracket \Gamma; \emptyset \vdash \tau \rrbracket^{\mathsf{Set}} \rho} \circ \mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} f \\ &= (\llbracket \vdash subst \rrbracket^{\mathsf{Set}})_{\llbracket \Gamma; \emptyset \vdash \tau \rrbracket^{\mathsf{Set}} \rho} \circ (\mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} (f + \mathbb{1}) \times \mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} f)\end{aligned} \tag{16}$$

So, we can conclude that

$$\begin{aligned}&\mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} f \circ (\llbracket \vdash subst \rrbracket^{\mathsf{Set}})_{\llbracket \Gamma; \emptyset \vdash \sigma \rrbracket^{\mathsf{Set}} \rho} \\ &= (\llbracket \vdash subst \rrbracket^{\mathsf{Set}})_{\llbracket \Gamma; \emptyset \vdash \tau \rrbracket^{\mathsf{Set}} \rho} \circ (\mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} (f + \mathbb{1}) \times \mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} f)\end{aligned} \tag{17}$$

Moreover, for any $A, B : \mathsf{Set}$, we can choose $\sigma = v$ and $\tau = w$ to be variables such that $\rho v = A$ and $\rho w = B$. Then for any function $f : A \to B$ we have that

$$\begin{aligned}&\mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} f \circ (\llbracket \vdash subst \rrbracket^{\mathsf{Set}})_A \\ &= (\llbracket \vdash subst \rrbracket^{\mathsf{Set}})_B \circ (\mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} (f + \mathbb{1}) \times \mathsf{map}_{\llbracket \emptyset; \alpha \vdash \mathsf{Lam}\,\alpha \rrbracket^{\mathsf{Set}}[\alpha := \_]} f)\end{aligned} \tag{18}$$

5.2 The Abstraction Theorem

To go beyond naturality and get *all* consequences of parametricity, we prove an Abstraction Theorem for our calculus. In fact, we actually prove a more general result in Theorem 33 about possibly open terms. We then recover the Abstraction Theorem as the special case of Theorem 33 for closed terms of closed type.

THEOREM 33. *Every well-formed term $\Gamma;\Phi \mid \Delta \vdash t : \tau$ induces a natural transformation from $[\![\Gamma;\Phi \vdash \Delta]\!]$ to $[\![\Gamma;\Phi \vdash \tau]\!]$, i.e., a triple of natural transformations*

$$([\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}})$$

where

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}} \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}} \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}$$

has as its component at $\rho : \mathsf{SetEnv}$ a morphism

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho$$

in Set, *and*

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}} \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}} \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}$$

has as its component at $\rho : \mathsf{RelEnv}$ a morphism

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho$$

in Rel, *and for all $\rho : \mathsf{RelEnv}$,*

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho = ([\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_1\rho), [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_2\rho))$$

PROOF. We proceed by structural induction, showing only the interesting cases.

- We first consider $\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G$.
  - To see that $[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}$, since the functorial part Φ of the context is empty, we need only show that, for every $\rho : \mathsf{SetEnv}$, $[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$ is a morphism in Set from $[\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$. For this, recall that

    $$[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho = \mathsf{curry}\,([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}])$$

    By the induction hypothesis, $[\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]\rho[\overline{\alpha := \_}]$ induces a natural transformation

    $$\begin{array}{ll} & [\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \\ : & [\![\Gamma;\overline{\alpha} \vdash \Delta, x : F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \\ = & [\![\Gamma;\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \times [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \end{array}$$

    and thus a family of morphisms

    $$\begin{array}{ll} & \mathsf{curry}\,([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]\rho[\overline{\alpha := \_}]) \\ : & [\![\Gamma;\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to ([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]) \end{array}$$

    That is, for each $\overline{A : \mathsf{Set}}$ and each $d : [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]$ by weakening, we have

    $$\begin{array}{ll} & ([\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho\,d)_{\overline{A}} \\ = & \mathsf{curry}\,([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}])\,d \\ : & [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}] \end{array}$$

    Moreover, these maps actually form a natural transformation $\eta : [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ because each

    $$\eta_{\overline{A}} = \mathsf{curry}\,([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}])\,d$$

is the component at $\overline{A}$ of the partial specialization to d of the natural transformation $[\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$.

To see that the components of η also satisfy the additional condition necessary for η to be in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$, let $\overline{R : \mathsf{Rel}(A, B)}$ and

$$(u, v) \in [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] = ([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}], [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := B}])$$

Then the induction hypothesis on the term t ensures that

$$\begin{array}{ll} & [\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \\ : & [\![\Gamma;\overline{\alpha} \vdash \Delta, x : F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \end{array}$$

and

$$\begin{array}{ll} & [\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \qquad (*) \\ = & ([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}], [\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := B}]) \end{array}$$

Since $(d, d) \in [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]$ we therefore have that

$$\begin{array}{ll} & (\eta_{\overline{A}} u, \eta_{\overline{B}} v) \\ = & (\mathsf{curry}\,([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}])\, d\, u, \mathsf{curry}\,([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := B}])\, d\, v) \\ = & \mathsf{curry}\,([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}])\,(d, d)\,(u, v) \\ : & [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \end{array}$$

Here, the second equality is by $(*)$.

– The proofs that $[\![\Gamma;\emptyset\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}$ is a natural transformation from $[\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}$ and that, for all $\rho : \mathsf{RelEnv}$ and $d : [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}$,

$$[\![\Gamma;\emptyset\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\,\rho\, d$$

is a natural transformation from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}]$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}]$, are analogous.

– Finally, to see that $\pi_i([\![\Gamma;\emptyset\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho) = [\![\Gamma;\emptyset\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_i\rho)$ we observe that π_1 and π_2 are surjective and compute

$$\begin{aligned} & \pi_i([\![\Gamma;\emptyset\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho) \\ &= \pi_i(\mathsf{curry}([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}])) \\ &= \mathsf{curry}(\pi_i([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}])) \\ &= \mathsf{curry}([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}(\pi_i(\rho[\overline{\alpha := \_}]))) \\ &= \mathsf{curry}([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}(\pi_i\rho)[\overline{\alpha := \_}]) \\ &= [\![\Gamma;\emptyset\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_i\rho) \end{aligned}$$

- We now consider $\Gamma;\Phi\,|\,\Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]$.
  – To see that $[\![\Gamma;\Phi\,|\,\Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ we need to show that, for every $\rho : \mathsf{SetEnv}$, $[\![\Gamma;\Phi\,|\,\Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$ is a morphism from $[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$, and that this family of morphisms is natural in ρ. Let $d : [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$. Then

$$\begin{array}{ll} & [\![\Gamma;\Phi\,|\,\Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\,\rho\, d \\ = & (\mathsf{eval} \circ \langle ([\![\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}}, [\![\Gamma;\Phi\,|\,\Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\rangle)\, d \\ = & \mathsf{eval}(([\![\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}}\, d, [\![\Gamma;\Phi\,|\,\Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\, d) \\ = & \mathsf{eval}(([\![\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\, d)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}}, [\![\Gamma;\Phi\,|\,\Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\, d) \end{array}$$

By the induction hypothesis, $(\llbracket\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\;d)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho}}$ has type

$$\llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho}]\to\llbracket\Gamma;\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho}]$$

and $\llbracket\Gamma;\Phi\,|\,\Delta\vdash s:F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho\,d$ has type

$$\begin{array}{rl} & \llbracket\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho \\ = & \llbracket\Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho}] \\ = & \llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho}] \end{array}$$

by Equation 4, and by weakening in the last step, since the type $\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}F\,G$ is only well-formed if $\Gamma;\overline{\alpha}\vdash F:\mathcal{F}$ and $\Gamma;\overline{\alpha}\vdash G:\mathcal{F}$. Thus, $\llbracket\Gamma;\Phi\,|\,\Delta\vdash t_{\overline{\tau}}s:G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho\,d$ has type $\llbracket\Gamma;\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho}]=\llbracket\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho$, as desired.

To see that the family of maps comprising $\llbracket\Gamma;\Phi\,|\,\Delta\vdash t_{\overline{\tau}}s:G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}$ form a natural transformation, i.e., are natural in their set environment argument, we need to show that the following diagram commutes:

$$\begin{tikzcd}
\llbracket\Gamma;\Phi\vdash\Delta\rrbracket^{\mathsf{Set}}\rho \arrow[r, "\llbracket\Gamma;\Phi\vdash\Delta\rrbracket^{\mathsf{Set}}f"] \arrow[d, "\langle\llbracket\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho,\,\llbracket\Gamma;\Phi\,|\,\Delta\vdash s:F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho\rangle"'] & \llbracket\Gamma;\Phi\vdash\Delta\rrbracket^{\mathsf{Set}}\rho' \arrow[d, "\langle\llbracket\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}"] \\
\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\times\llbracket\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho \arrow[r, "\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}f\times\llbracket\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}f"] \arrow[d, "\mathsf{eval}\circ((-)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket\rho}}\times id)"'] & \llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho'\times\llbracket\Gamma;\Phi\vdash F[\alpha: \arrow[d, "\mathsf{eval}\circ((-)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket\rho'}}"] \\
\llbracket\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho \arrow[r, "\llbracket\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}f"'] & \llbracket\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho'
\end{tikzcd}$$

The top diagram commutes because the induction hypothesis ensures $\llbracket\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}$ and $\llbracket\Gamma;\Phi\,|\,\Delta\vdash s:F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}$ are natural in ρ. To see that the bottom diagram commutes, we first note that since $\rho|_\Gamma=\rho'|_\Gamma$, $\Gamma;\overline{\alpha}\vdash F:\mathcal{F}$, and $\Gamma;\overline{\alpha}\vdash G:\mathcal{F}$ we can replace the instance of f in $\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}f$ with id. Then, using the fact that $\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}$ is a functor, we have that $\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}id=id$. To see that the bottom diagram commutes we must therefore prove that, for every $\eta\in\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho$ and $x\in\llbracket\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho$, we have

$$\llbracket\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}f(\eta_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket\rho}}x)=\eta_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket\rho'}}(\llbracket\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}f\,x)$$

i.e., that for every $\eta\in\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho$,

$$\llbracket\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}f\circ\eta_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket\rho}}=\eta_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket\rho'}}\circ\llbracket\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}f$$

But this follows from the naturality of η. Indeed, $\eta\in\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho$ implies that η is a natural transformation from $\llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\_}]$ to $\llbracket\Gamma;\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\_}]$. For each τ, consider the morphism $\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}f:\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho\to\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}\rho'$. The following diagram commutes by naturality of η:

$$\begin{tikzcd}
\llbracket \Gamma; \Phi \vdash F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} \rho \arrow[d, equal] & \llbracket \Gamma; \Phi \vdash G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} \rho \arrow[d, equal] \\
\llbracket \Gamma; \overline{\alpha} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho}] \arrow[r, "\eta_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} \rho}}"] \arrow[d, "\llbracket \Gamma;\overline{\alpha}\vdash F \rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := \llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} f}]"] & \llbracket \Gamma; \overline{\alpha} \vdash G \rrbracket^{\mathsf{Set}} \rho[\overline{\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho}] \arrow[d, "\llbracket \Gamma;\overline{\alpha}\vdash G \rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := \llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} f}]"] \\
\llbracket \Gamma; \overline{\alpha} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho'}] \arrow[r, "\eta_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} \rho'}}"] & \llbracket \Gamma; \overline{\alpha} \vdash G \rrbracket^{\mathsf{Set}} \rho[\overline{\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho'}]
\end{tikzcd}$$

That is,

$$\begin{array}{ll}
& \llbracket \Gamma; \overline{\alpha} \vdash G \rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} f}] \circ \eta_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} \rho}} \\
= & \eta_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} \rho'}} \circ \llbracket \Gamma; \overline{\alpha} \vdash F \rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} f}]
\end{array}$$

But since the only variables in the functorial contexts for F and G are $\overline{\alpha}$, we have that

$$\begin{array}{ll}
& \llbracket \Gamma; \overline{\alpha} \vdash F \rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} f}] \\
= & \llbracket \Gamma; \overline{\alpha} \vdash F \rrbracket^{\mathsf{Set}} f[\overline{\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} f}] \\
= & \llbracket \Gamma; \Phi \vdash F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} f
\end{array}$$

and similarly for G. Commutativity of this last diagram thus gives that $\llbracket \Gamma; \Phi \vdash G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} f \circ \eta_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} \rho}} = \eta_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} \rho'}} \circ \llbracket \Gamma; \Phi \vdash F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} f$, as desired.

– The proof that $\llbracket \Gamma; \Phi \,|\, \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}}$ is a natural transformation from $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}$ to $\llbracket \Gamma; \Phi \vdash G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}}$ is analogous.

– Finally, to see that $\pi_i(\llbracket \Gamma; \Phi \,|\, \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}} \rho) = \llbracket \Gamma; \Phi \,|\, \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} (\pi_i \rho)$ we compute

$$\begin{array}{ll}
& \pi_i(\llbracket \Gamma; \Phi \,|\, \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}} \rho) \\
= & \pi_i(\mathsf{eval} \circ \langle (\llbracket \Gamma; \emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{Rel}} \rho\ _)_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Rel}} \rho}}, \llbracket \Gamma; \Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}} \rho \rangle) \\
= & \mathsf{eval} \circ \langle \pi_i((\llbracket \Gamma; \emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{Rel}} \rho\ _)_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Rel}} \rho}}), \pi_i(\llbracket \Gamma; \Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}} \rho) \rangle \\
= & \mathsf{eval} \circ \langle \pi_i(\llbracket \Gamma; \emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{Rel}} \rho\ _)_{\overline{\pi_i(\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Rel}} \rho)}}, \pi_i(\llbracket \Gamma; \Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}} \rho) \rangle \\
= & \mathsf{eval} \circ \langle (\llbracket \Gamma; \emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{Set}} (\pi_i \rho)\ _)_{\overline{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}} (\pi_i \rho)}}, \llbracket \Gamma; \Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} (\pi_i \rho) \rangle \\
= & \llbracket \Gamma; \Phi \,|\, \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} (\pi_i \rho)
\end{array}$$

- We now consider $\Gamma; \emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])$.
  – To see that $\llbracket \Gamma; \emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]) \rrbracket^{\mathsf{Set}}$ is a natural transformation from $\llbracket \Gamma; \emptyset \vdash \emptyset \rrbracket^{\mathsf{Set}}$ to $\llbracket \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]) \rrbracket^{\mathsf{Set}}$, since the functorial part Φ of the context is empty, we need only show that, for every $\rho : \mathsf{SetEnv}$,

$$\llbracket \Gamma; \emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]) \rrbracket^{\mathsf{Set}} \rho$$

is a morphism in Set from $\llbracket \Gamma; \emptyset \vdash \emptyset \rrbracket^{\mathsf{Set}} \rho$ to

$$\llbracket \Gamma; \emptyset \vdash \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]) \rrbracket^{\mathsf{Set}} \rho$$

i.e., that, for the unique $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{F},\overline{G}}_H : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,\rho\,d$$

is a morphism from $[\![\Gamma;\emptyset \vdash \overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G}]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\rho$. For this we show that for all $\overline{\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G]\!]^{\mathsf{Set}}\rho}$ we have

$$\begin{array}{ll} & [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{F},\overline{G}}_H : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,\rho\,d\,\overline{\eta} \\ : & [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\rho \end{array}$$

To this end, we note that, for any $\overline{B}$,

$$\begin{array}{ll} & ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{F},\overline{G}}_H : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,\rho\,d\,\overline{\eta})_{\overline{B}} \\ = & [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}}\,id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}] \end{array}$$

is indeed a morphism from

$$\begin{array}{ll} & [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}] \\ = & [\![\Gamma;\overline{\gamma},\overline{\phi} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}] \end{array}$$

to

$$\begin{array}{ll} & [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}] \\ = & [\![\Gamma;\overline{\gamma},\overline{\phi} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}] \end{array}$$

since $[\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}}$ is a functor from SetEnv to Set and $id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}]$ is a morphism in SetEnv from

$$\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}]$$

to

$$\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}]$$

To see that this family of morphisms is natural in $\overline{B}$ we first observe that if $\overline{f : B \to B'}$ then, writing t for $[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{F},\overline{G}}_H : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,\rho\,d\,\overline{\eta}$, we have

$$\begin{tikzcd}
{[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}]} \arrow[r, "t_{\overline{B}}"] \arrow[d, "{[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\,id_\rho[\overline{\gamma := f}]}"'] & {[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}]} \arrow[d, "{[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\,id_\rho[\overline{\gamma := f}]}"] \\
{[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B'}]} \arrow[r, "t_{\overline{B'}}"] & {[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B'}]}
\end{tikzcd}$$

This diagram commutes because $[\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}}$ is a functor from SetEnv to Set and because, letting

$$E_{F,B} = \rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.\,[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}]$$

and

$$e_{F,f} = id_\rho[\overline{\gamma := f}][\overline{\phi := \lambda\overline{A}.\,[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := f}][\overline{\beta := id_A}]}]$$

for all F and B and $\overline{f : B \to B'}$, the following diagram commutes by the fact that composition of environments is componentwise together with the naturality of η:

$$\begin{tikzcd}
E_{F,B} \arrow[rr, "id_\rho[\gamma := id_B][\overline{\phi := \lambda\overline{A}.\,\eta_{\overline{A}\,\overline{B}}}]"] \arrow[d, "e_{F,f}"'] & & E_{G,B} \arrow[d, "e_{G,f}"] \\
E_{F,B'} \arrow[rr, "id_\rho[\gamma := id_{B'}][\overline{\phi := \lambda\overline{A}.\,\eta_{\overline{A}\,\overline{B'}}}]"'] & & E_{G,B'}
\end{tikzcd}$$

We therefore have that

$$\lambda\overline{B}.\, [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^\emptyset\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\, \rho\, d\, \overline{\eta})_{\overline{B}}$$

is natural in $\overline{B}$ as desired.

– To see that, for every $\rho : \mathsf{SetEnv}$ and $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$, and all $\overline{\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}}\rho}$,

$$[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^\emptyset\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\, \rho\, d\, \overline{\eta}$$

satisfies the additional condition necessary for it to be in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\rho$, let $\overline{R : \mathsf{Rel}(B, B')}$ and $\overline{S : \mathsf{Rel}(C, C')}$. Since each map in $\overline{\eta}$ satisfies the extra condition necessary for it to be in its corresponding $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}}\rho$ — i.e., since

$$(\eta_{\overline{B}\,\overline{C}}, \eta_{\overline{B'}\,\overline{C'}}) \in [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}] \to [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}]$$

— we have that

$$\begin{array}{rl}
& (\,(\lambda e\, v\, \overline{Z}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := Z}]}[\overline{\phi := \lambda\overline{A}.v_{\overline{A}\,\overline{Z}}}])\, d\, \overline{\eta}\, \overline{B}, \\
& \qquad (\lambda e\, v\, \overline{Z}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := Z}]}[\overline{\phi := \lambda\overline{A}.v_{\overline{A}\,\overline{Z}}}])\, d\, \overline{\eta}\, \overline{B'}\,) \\
= & (\,[\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}]),\ [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := B'}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B'}}}])\,)
\end{array}$$

has type

$$\begin{array}{rl}
& ([\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}] \to [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}], \\
& \qquad [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B'}] \to [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B'}]) \\
= & [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := R}] \to [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := R}]
\end{array}$$

as desired.

– The proofs that

$$[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^\emptyset\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}$$

is a natural transformation from $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}$ to

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^\emptyset\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}$$

and that, for every $\rho : \mathsf{RelEnv}$ and the unique $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho$,

$$[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^\emptyset\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}\, \rho\, d$$

is a morphism from $[\![\Gamma;\emptyset \vdash \overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G}]\!]^{\mathsf{Rel}}\,\rho$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Rel}}\,\rho$, are analogous.

– Finally, to see that

$$
\begin{array}{rl}
& \pi_i([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F}\,\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}\,\rho) \\
= & [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F}\,\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,(\pi_i\rho)
\end{array}
$$

we compute

$$
\begin{array}{rl}
& \pi_i([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F}\,\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}\,\rho) \\
= & \pi_i\,(\lambda e\,\overline{v}\,\overline{R}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Rel}} id_{\rho[\overline{\gamma := R}]}[\overline{\phi := \lambda\overline{S}.v_{\overline{S}\,\overline{R}}}]) \\
= & \lambda e\,\overline{v}\,\overline{R}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{(\pi_i\rho)[\overline{\gamma := \pi_i R}]}[\overline{\phi := \lambda\overline{S}.(\pi_i v)_{\overline{\pi_i S}\,\overline{\pi_i R}}}] \\
= & \lambda d\,\overline{\eta}\,\overline{B}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{(\pi_i\rho)[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}] \\
= & [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F}\,\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,(\pi_i\rho)
\end{array}
$$

- We now consider $\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}$.
  – To see that if $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$ then

$$
[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d
$$

is in $[\![\Gamma;\emptyset \vdash Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho$, we first note that, for all $\overline{B}$ and $\overline{C}$,

$$
\begin{array}{rl}
& ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d)_{\overline{B}\,\overline{C}} \\
= & (in_{T_{\rho[\overline{\gamma := C}]}})_{\overline{B}}
\end{array}
$$

does indeed map

$$
\begin{array}{rl}
& [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \\
= & [\![\Gamma;\overline{\beta},\overline{\gamma},\overline{\alpha} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}] \\
= & [\![\Gamma;\phi,\overline{\beta},\overline{\gamma},\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}] \\
& \qquad [\phi := \lambda\overline{D}.\, [\![\Gamma;\overline{\beta},\overline{\gamma},\overline{\alpha} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}][\overline{\beta := D}]] \\
= & [\![\Gamma;\phi,\overline{\gamma},\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\alpha := B}] \\
& \qquad [\phi := \lambda\overline{D}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := D}][\overline{\gamma := C}]] \\
= & T_{\rho[\overline{\gamma := C}]}\,(\lambda\overline{D}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := D}][\overline{\gamma := C}])\,\overline{B} \\
= & T_{\rho[\overline{\gamma := C}]}\,(\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B}
\end{array}
$$

to

$$
\begin{array}{rl}
& [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \\
= & (\lambda\overline{D}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := D}][\overline{\gamma := C}])\overline{B} \\
= & (\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B}
\end{array}
$$

To see that

$$
\begin{array}{rl}
& [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d \\
= & \lambda\overline{B}\,\overline{C}.\,(in_{T_{\rho[\overline{\gamma := C}]}})_{\overline{B}}
\end{array}
$$

is natural in $\overline{B}$ and $\overline{C}$, we observe that the following diagram commutes for all $\overline{f : B \to B'}$ and $\overline{g : C \to C'}$:

$$\begin{tikzcd}
T_{\rho[\overline{\gamma := C}]}\,(\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B} \arrow[rr, "(in_{T_{\rho[\overline{\gamma := C}]}})_{\overline{B}}"] \arrow[d, "\sigma_{id_{\rho[\overline{\gamma := g}]}}\,(\mu \sigma_{id_{\rho[\overline{\gamma := g}]}})\,\overline{B}"] & & (\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B} \arrow[d, "(\mu \sigma_{id_{\rho[\overline{\gamma := g}]}})\,\overline{B}"] \\
T_{\rho[\overline{\gamma := C'}]}\,(\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B} \arrow[rr, "(in_{T_{\rho[\overline{\gamma := C'}]}})_{\overline{B}}"] \arrow[d, "T_{\rho[\overline{\gamma := C'}]}\,(\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{f}"] & & (\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B} \arrow[d, "(in_{T_{\rho[\overline{\gamma := C'}]}})_{\overline{B}}"] \\
T_{\rho[\overline{\gamma := C'}]}\,(\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B'} \arrow[rr, "(in_{T_{\rho[\overline{\gamma := C'}]}})_{\overline{B'}}"] & & (\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B'}
\end{tikzcd}$$

Indeed, naturality of in with respect to its functor argument ensures that the top diagram commutes, and naturality of $in_{T_{\rho[\overline{\gamma := C'}]}}$ ensures that the bottom one commutes.

- To see that $[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\, \rho\, d$ satisfies the additional property necessary for it to be in

$$[\![\Gamma;\emptyset \vdash Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\, \rho$$

let $\overline{R : \mathsf{Rel}(B, B')}$ and $\overline{S : \mathsf{Rel}(C, C')}$. Then

$$\begin{array}{rl}
& (\,([\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\, \rho\, d)_{\overline{B},\overline{C}}, \\
& \qquad ([\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\, \rho\, d)_{\overline{B'},\overline{C'}}\,) \\
= & (\,(in_{T_{\rho[\overline{\gamma := C}]}})_{\overline{B}}, (in_{T_{\rho[\overline{\gamma := C'}]}})_{\overline{B'}}\,)
\end{array}$$

has type

$$\begin{array}{rl}
& (\,[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \to \\
& \qquad [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}], \\
& \quad [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B'}][\overline{\gamma := C'}] \to \\
& \qquad [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B'}][\overline{\gamma := C'}]\,) \\
= & (\,T_{\rho[\overline{\gamma := C}]}\,(\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B} \to (\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B},\; T_{\rho[\overline{\gamma := C'}]}\,(\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B'} \to (\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B'}\,) \\
= & [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}] \to \\
& \qquad [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}]
\end{array}$$

- The proofs that $[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}$ is a natural transformation from $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}$ to $[\![\Gamma;\emptyset \vdash Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{R}}$ and that, for all $\rho : \mathsf{RelEnv}$ and $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}$,

$$[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\, \rho\, d$$

is a natural transformation from $\lambda\overline{R}\,\overline{S}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Rel}}\rho[\overline{\beta := R}][\overline{\gamma := S}]$ to $\lambda\overline{R}\,\overline{S}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\rho[\overline{\beta := R}][\overline{\gamma := S}]$, are analogous.

– Finally, to see that $\pi_i([\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\,\rho\, d) = [\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,(\pi_i\rho)\,(\pi_i d)$ we first note that $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho$ and $\pi_i d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}(\pi_i\rho)$ are uniquely determined. Further, the definition of natural transformations in Rel ensures that, for any $\overline{R}$ and $\overline{S}$,

$$\begin{aligned}
&(in_{T_{\rho[\overline{\gamma:=S}]}})_{\overline{R}}\\
&= ((in_{\pi_1(T_{\rho[\overline{\gamma:=S}]})})_{\overline{\pi_1 R}},\, (in_{\pi_2(T_{\rho[\overline{\gamma:=S}]})})_{\overline{\pi_2 R}})\\
&= ((in_{T^{\mathsf{Set}}_{(\pi_1\rho)[\overline{\gamma:=\pi_1 S}]}})_{\overline{\pi_1 R}},\, (in_{T^{\mathsf{Set}}_{(\pi_2\rho)[\overline{\gamma:=\pi_2 S}]}})_{\overline{\pi_2 R}})
\end{aligned}$$

Observing that π_1 and π_2 are surjective, we therefore have that

$$\begin{aligned}
&\pi_i([\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\,\rho\, d)\\
&= \pi_i(\lambda\overline{R}\,\overline{S}.\,(in_{T_{\rho[\overline{\gamma:=S}]}})_{\overline{R}})\\
&= \lambda\overline{B}\,\overline{C}.\,(in_{T^{\mathsf{Set}}_{(\pi_i\rho)[\overline{\gamma:=C}]}})_{\overline{B}}\\
&= [\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,(\pi_i\rho)\,(\pi_i d)
\end{aligned}$$

- We now consider $\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F)$.
  – To see that $[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}$ to

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}$$

since the functorial part Φ of the context is empty, we need only show that, for every $\rho : \mathsf{SetEnv}$,

$$[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}\,\rho$$

is a morphism in Set from $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$ to

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}\,\rho$$

i.e., that, for the unique $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}\,\rho\, d$$

is a morphism from $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}\rho$. For this we show that for every $\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F]\!]^{\mathsf{Set}}\rho$ we have

$$\begin{aligned}
&[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}\,\rho\, d\,\eta\\
:\;&[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}\,\rho
\end{aligned}$$

To this end we show that, for any $\overline{B}$ and $\overline{C}$,

$$([\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F]\!]^{\mathsf{Set}}\,\rho\, d\,\eta)_{\overline{B}\,\overline{C}}$$

is a morphism from

$$[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \;=\; (\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]})\overline{B}$$

to

$$[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

To see this, we use Equations 4 and 6 for the first and second equalities below, together with weakening, to see that η is itself a natural transformation from

$$\begin{array}{cl} & \lambda\overline{B}\,\overline{C}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash H[\phi:=F][\overline{\alpha:=\beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}] \\ = & \lambda\overline{B}\,\overline{C}.\,[\![\Gamma;\overline{\beta},\overline{\gamma},\overline{\alpha}\vdash H[\phi:=F]]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}][\overline{\alpha:=B}] \\ = & \lambda\overline{B}\,\overline{C}.\,[\![\Gamma;\overline{\beta},\overline{\gamma},\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}][\overline{\alpha:=B}] \\ & \qquad [\phi:=\lambda\overline{A}.\,[\![\Gamma;\overline{\beta},\overline{\gamma},\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}][\overline{\alpha:=B}][\overline{\beta:=A}]] \\ = & \lambda\overline{B}\,\overline{C}.\,[\![\Gamma;\overline{\gamma},\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\alpha:=B}][\phi:=\lambda\overline{A}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\beta:=A}]] \\ = & \lambda\overline{B}\,\overline{C}.\,T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}\,(\lambda\overline{A}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C}])\,\overline{B} \end{array}$$

to

$$\lambda\overline{B}\,\overline{C}.\,(\lambda\overline{A}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C}])\overline{B} \;=\; \lambda\overline{B}\,\overline{C}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}]$$

Thus, if $x : [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}] \;=\; (\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]})\overline{B}$, then

$$\begin{array}{cl} & ([\![\Gamma;\emptyset\mid\emptyset\vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi:=_{\overline{\beta}}F][\overline{\alpha:=\beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F]\!]^{\mathsf{Set}}\,\rho\,d\,\eta)_{\overline{B}\,\overline{C}}\,x \\ = & (\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}\,(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C}}))_{\overline{B}}\,x \\ : & (\lambda\overline{A}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C}])\overline{B} \end{array}$$

i.e., for each $\overline{B}$ and $\overline{C}$

$$([\![\Gamma;\emptyset\mid\emptyset\vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi:=_{\overline{\beta}}F][\overline{\alpha:=\beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F]\!]^{\mathsf{Set}}\,\rho\,d\,\eta)_{\overline{B}\,\overline{C}}$$

is a morphism from $(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]})\overline{B}$ to $[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}]$.

To see that this family of morphisms is natural in $\overline{B}$ and $\overline{C}$, we observe that the following diagram commutes for all $\overline{f:B\to B'}$ and $\overline{g:C\to C'}$:

$$\begin{tikzcd}
(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]})\,\overline{B} \arrow[r, "(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}\,(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C}}))_{\overline{B}}"] \arrow[d, "(\mu\sigma_{id_{\rho}[\overline{\gamma:=g}]})\,\overline{B}"'] & {[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\beta:=B}]} \arrow[d, "{[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}id_{\rho}[\overline{\gamma:=g}][\overline{\beta:=id_B}]}"] \\
(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]})\,\overline{B} \arrow[r, "(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}\,(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}}))_{\overline{B}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]})\,\overline{f}"'] & {[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C'}][\overline{\beta:=B}]} \arrow[d, "{[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}id_{\rho}[\overline{\gamma:=id_{C'}}][\overline{\beta:=f}]}"] \\
(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]})\,\overline{B'} \arrow[r, "(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}\,(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}}))_{\overline{B'}}"] & {[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C'}][\overline{\beta:=B'}]}
\end{tikzcd}$$

Indeed, naturality of $\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}\,(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}})$ ensures that the bottom diagram commutes. To see that the top one commutes is considerably more delicate.

To see that the top diagram commutes we first observe that, given a natural transformation $\Theta : H\to K : [\mathsf{Set}^k,\mathsf{Set}]\to[\mathsf{Set}^k,\mathsf{Set}]$, the fixpoint natural transformation $\mu\Theta : \mu H\to$

$\mu K : \mathsf{Set}^k \to \mathsf{Set}$ is defined to be $fold_H(\Theta\,(\mu K) \circ in_K)$, i.e., the unique morphism making the following square commute:

$$\begin{tikzcd}
H(\mu H) \arrow[r, "H(\mu\Theta)"] \arrow[dd, "in_H"'] & H(\mu K) \arrow[d, "\Theta(\mu K)"] \\
& K(\mu K) \arrow[d, "in_K"] \\
\mu H \arrow[r, "\mu\Theta"'] & \mu K
\end{tikzcd}$$

Taking $\Theta = \sigma_f^{\mathsf{Set}} : T_\rho^{\mathsf{Set}} \to T_{\rho'}^{\mathsf{Set}}$ gives that the following diagram commutes for any morphism of set environments $f : \rho \to \rho'$:

$$\begin{tikzcd}
T_\rho^{\mathsf{Set}}(\mu T_\rho^{\mathsf{Set}}) \arrow[r, "T_\rho^{\mathsf{Set}}(\mu\sigma_f^{\mathsf{Set}})"] \arrow[dd, "in_{T_\rho^{\mathsf{Set}}}"'] & T_\rho^{\mathsf{Set}}(\mu T_{\rho'}^{\mathsf{Set}}) \arrow[d, "\sigma_f^{\mathsf{Set}}(\mu T_{\rho'}^{\mathsf{Set}})"] \\
& T_{\rho'}^{\mathsf{Set}}(\mu T_{\rho'}^{\mathsf{Set}}) \arrow[d, "in_{T_{\rho'}^{\mathsf{Set}}}"] \\
\mu T_\rho^{\mathsf{Set}} \arrow[r, "\mu\sigma_f^{\mathsf{Set}}"'] & \mu T_{\rho'}^{\mathsf{Set}}
\end{tikzcd} \tag{19}$$

We next observe that the action of the functor

$$\lambda\overline{B}.\lambda\overline{C}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

on the morphisms $\overline{f : B \to B'}, \overline{g : C \to C'}$ is given by

$$\begin{aligned}
& \llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Set}} id_\rho[\overline{\beta := f}][\overline{\gamma := g}] \\
= \; & \llbracket\Gamma;\overline{\alpha},\overline{\gamma} \vdash H[\phi := F]\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := f}][\overline{\gamma := g}] \\
= \; & \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := f}][\overline{\gamma := g}][\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}]] \\
= \; & \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H\rrbracket^{\mathsf{Set}}(id_{\rho[\overline{\gamma := C'}][\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]]}[\overline{\alpha := f}] \\
& \qquad \circ id_{\rho[\overline{\alpha := B}][\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]]}[\overline{\gamma := g}] \\
& \qquad \circ id_{\rho[\overline{\alpha := B}][\overline{\gamma := C}]}[\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}]]) \\
= \; & \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\gamma := C'}][\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]]}[\overline{\alpha := f}] \\
& \qquad \circ \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\alpha := B}][\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]]}[\overline{\gamma := g}] \\
& \qquad \circ \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\alpha := B}][\overline{\gamma := C}]}[\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}]] \\
= \; & T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}(\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\overline{f} \\
& \qquad \circ (\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma := g}]}(\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]))_{\overline{B}} \\
& \qquad \circ (T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}]))_{\overline{B}}
\end{aligned}$$

So, if η is a natural transformation from

$$\lambda\overline{B}\,\overline{C}.\llbracket\Gamma;\overline{\alpha},\overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

to

$$\lambda\overline{B}\,\overline{C}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

then, by naturality,

$$\begin{aligned}
& [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\, id_\rho[\overline{\beta := f}][\overline{\gamma := g}] \circ \eta_{\overline{B},\overline{C}} \\
= \;& \eta_{\overline{B'},\overline{C'}} \circ [\![\Gamma;\overline{\alpha},\overline{\gamma}\vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\, id_\rho[\overline{\beta := f}][\overline{\gamma := g}] \\
= \;& \eta_{\overline{B'},\overline{C'}} \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\overline{f} \\
& \circ\big(\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\big)_{\overline{B}} \\
& \circ\big(T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\, id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])\big)_{\overline{B}}
\end{aligned}$$

As a special case when $\overline{f = id_B}$ we have

$$\begin{aligned}
& [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\, id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \eta_{\overline{B},\overline{C}} \\
= \;& \eta_{\overline{B},\overline{C'}} \circ \big(\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\big)_{\overline{B}} \\
& \circ\big(T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\, id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])\big)_{\overline{B}}
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\, id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \lambda\overline{B}.\eta_{\overline{B},\overline{C}} \\
= \;& \lambda\overline{B}.\eta_{\overline{B},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]) \\
& \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\, id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])
\end{aligned} \tag{20}$$

Now, to see that the top diagram in the diagram on page 44 commutes we first note that the diagram

$$\begin{tikzcd}
T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}) \arrow[rr, "T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})\circ\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]})"] \arrow[ddd, "\mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}"'] & & T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]) \arrow[d, "\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}])"] \\
& & T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]) \arrow[dd, "\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}"] \\
& & \\
\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]} \arrow[r, "\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}"'] & \mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]} \arrow[r, "\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})"'] & \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]
\end{tikzcd}$$

commutes because

$$\begin{aligned}
& \lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]) \\
& \quad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}) \\
= \;& \lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]) \\
& \quad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})) \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}) \\
= \;& \lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})) \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}) \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}) \\
= \;& \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}) \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}) \\
= \;& \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]} \circ \mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}
\end{aligned}$$

Here, the first equality is by functoriality of $T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}$, the second equality is by naturality of $\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}$, the third equality by the universal property of $\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})$ and the last equality by Equation 19. That is, we have

$$\begin{aligned}
&\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]} \\
= \;& \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]))
\end{aligned} \tag{21}$$

Next, we note that the diagram

$$\begin{tikzcd}
T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}) \arrow[rr, "T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}]\circ\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}))"] \arrow[ddd, "\mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}"'] & & T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}])"] \\
& & T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[dd, "\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}"] \\
& & \\
\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]} \arrow[r, bend right, "\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})"'] & \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}] \arrow[r, bend right, "\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}]"'] & \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]
\end{tikzcd}$$

commutes because

$$\begin{aligned}
&\lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \\
&\qquad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})) \\
&= \lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \\
&\qquad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}]) \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})) \\
&= \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}] \circ \lambda\overline{A}.\eta_{\overline{A},\overline{C}} \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})) \\
&= \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}) \circ \mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}
\end{aligned}$$

Here, the first equality is by functoriality of $T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}$, the second equality is by Equation 20, and the last equality is by the universal property of $\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})$. That is, we have

$$\begin{aligned}
&\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}) \\
= \;& \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]))
\end{aligned} \tag{22}$$

Combining Equations 21 and 22 we get that

$$\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]} = \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})$$

i.e., that the top diagram in the diagram on page 44 commutes. We therefore have that

$$(\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F\rrbracket^{\mathsf{Set}}\, \rho\, d\, \eta)_{\overline{B}\,\overline{C}}$$

is natural in $\overline{B}$ and $\overline{C}$ as desired.

– To see that, for every $\rho : \mathsf{SetEnv}$, $d \in \llbracket \Gamma;\emptyset \vdash \emptyset\rrbracket^{\mathsf{Set}}\rho$, and $\eta : \llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F\rrbracket^{\mathsf{Set}}\rho$,

$$\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F\rrbracket^{\mathsf{Set}}\, \rho\, d\, \eta$$

satisfies the additional condition necessary for it to be in $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F\rrbracket^{\mathsf{Set}}\rho$, let $\overline{R : \mathsf{Rel}(B, B')}$ and $\overline{S : \mathsf{Rel}(C, C')}$. Since η satisfies the additional condition necessary for it to be in $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (H[\phi := F][\overline{\alpha := \beta}])\, F\rrbracket^{\mathsf{Set}}\rho$ — i.e., since

$$\begin{array}{rcl}(\eta_{\overline{B}\,\overline{C}}, \eta_{\overline{B'}\,\overline{C'}}) & \in & \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash H[\phi := F][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \to \\ & & \qquad\qquad \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \\ & = & T_{\mathsf{Eq}_\rho[\overline{\gamma := S}]}\, (\llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}]) \to \\ & & \qquad\qquad \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}]\end{array}$$

— we have that

$$(\,(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}\, (\lambda\overline{A}.\, \eta_{\overline{A}\,\overline{C}}))_{\overline{B}},\, (\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}\, (\lambda\overline{A}.\eta_{\overline{A}\,\overline{C'}}))_{\overline{B'}}\,)$$

has type

$$\begin{array}{rl} & (\mu T_{\mathsf{Eq}_\rho[\overline{\gamma := S}]})\, \overline{R} \to \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \\ = & (\mu T_{\mathsf{Eq}_\rho[\overline{\gamma := S}]})\, \overline{\llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash \beta\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}]} \to \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \\ = & \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \to \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}]\end{array}$$

as desired.

– The proofs that

$$\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)\rrbracket^{\mathsf{Rel}}$$

is a natural transformation from $\llbracket \Gamma;\emptyset \vdash \emptyset\rrbracket^{\mathsf{Rel}}$ to

$$\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^\emptyset\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)\rrbracket^{\mathsf{Rel}}$$

and that, for all $\rho : \mathsf{RelEnv}$ and the unique $d : \llbracket \Gamma;\emptyset \vdash \emptyset\rrbracket^{\mathsf{Rel}}\rho$,

$$\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)\rrbracket^{\mathsf{Rel}}\, \rho\, d$$

is a morphism from $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\rrbracket^{\mathsf{Rel}}\rho$ to $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F\rrbracket^{\mathsf{Rel}}\rho$, are analogous.

– Finally, to see that

$$\begin{array}{rl} & \pi_i(\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)\rrbracket^{\mathsf{Rel}}\rho) \\ = & \llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)\rrbracket^{\mathsf{Set}}(\pi_i\rho)\end{array}$$

we compute

$$
\begin{array}{rl}
& \pi_i([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)]\!]^{\mathsf{Rel}}) \\
= & \pi_i(\lambda e\,\eta\,\overline{R}\,\overline{S}.\,(fold_{T_{\rho[\overline{\gamma := S}]}}\,(\lambda\overline{Z}.\,\eta_{\overline{Z}\,\overline{S}}))_{\overline{R}}) \\
= & \lambda e\,\eta\,\overline{R}\,\overline{S}.\,(fold_{T_{(\pi_i\rho)[\overline{\gamma := \pi_i S}]}}\,(\lambda\overline{Z}.\,(\pi_i\eta)_{\overline{\pi_i Z}\,\overline{\pi_i S}}))_{\overline{\pi_i R}} \\
= & \lambda d\,\eta\,\overline{B}\,\overline{C}.\,(fold_{T_{(\pi_i\rho)[\overline{\gamma := C}]}}\,(\lambda\overline{A}.\eta_{\overline{A}\,\overline{C}}))_{\overline{B}} \\
= & [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)]\!]^{\mathsf{Rel}}(\pi_i\rho)
\end{array}
$$

Here, we are again using the fact that π_1 and π_2 are surjective.

□

The Abstraction Theorem is now the special case of Theorem 33 for closed terms of close type: State more generally as: If $(a, b) \in [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho$ then $([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}(\pi_1\rho)a, [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}(\pi_2\rho)b) \in [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho$. Get the next theorem as a corollary for closed terms of closed type.

Theorem 34. *If $\vdash \tau : \mathcal{F}$ and $\vdash t : \tau$, then $([\![\vdash t : \tau]\!]^{\mathsf{Set}}, [\![\vdash t : \tau]\!]^{\mathsf{Set}}) \in [\![\vdash \tau]\!]^{\mathsf{Rel}}$.*

Our calculus does not support Church encodings of data types like pair or sum or list types because all of the "forall"s in our calculus must be at the top level. Nevertheless, our calculus does admit actual sum and product and list types because they are coded by μ-terms in our calculus. We just don't have an equivalence of these types and their Church encodings in our calculus, that's all.

6 FREE THEOREMS FOR NESTED TYPES

6.1 Free Theorem for Type of Polymorphic Bottom

Suppose $\vdash g : \mathsf{Nat}^{\alpha}\,\mathbb{1}\,\alpha$, let $G^{\mathsf{Set}} = [\![\vdash g : \mathsf{Nat}^{\alpha}\,\mathbb{1}\,\alpha]\!]^{\mathsf{Set}}$, and let $G^{\mathsf{Rel}} = [\![\vdash g : \mathsf{Nat}^{\alpha}\,\mathbb{1}\,\alpha]\!]^{\mathsf{Rel}}$. By Theorem 34, $(G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) = G^{\mathsf{Rel}}\rho$. Thus, for all $\rho \in \mathsf{RelEnv}$ and any $(a, b) \in [\![\vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$
\begin{array}{rcl}
(G^{\mathsf{Set}}, G^{\mathsf{Set}}) = (G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) & \in & [\![\vdash \mathsf{Nat}^{\alpha}\,\mathbb{1}\,\alpha]\!]^{\mathsf{Rel}}\rho \\
& = & \{\eta : K_1 \Rightarrow id\} \\
& = & \{(\eta_1 : K_1 \Rightarrow id, \eta_2 : K_1 \Rightarrow id)\}
\end{array}
$$

That is, G^{Set} is a natural transformation from the constantly 1-valued functor to the identity functor in Set. In particular, for every $S : \mathsf{Set}$, $G^{\mathsf{Set}}_S : 1 \to S$. Note, however, that if $S = \emptyset$, then there can be no such morphism, so no such natural transformation can exist in Set, and thus no term $\vdash g : \mathsf{Nat}^{\alpha}\mathbb{1}\,\alpha$ can exist in our calculus. That is, our calculus does not admit any terms with the closed type $\mathsf{Nat}^{\alpha}\mathbb{1}\,\alpha$ of the polymorphic bottom.

6.2 Free Theorem for Type of Polymorphic Identity

Suppose $\vdash g : \mathsf{Nat}^{\alpha}\,\alpha\,\alpha$, let $G^{\mathsf{Set}} = [\![\vdash g : \mathsf{Nat}^{\alpha}\,\alpha\,\alpha]\!]^{\mathsf{Set}}$, and let $G^{\mathsf{Rel}} = [\![\vdash g : \mathsf{Nat}^{\alpha}\,\alpha\,\alpha]\!]^{\mathsf{Rel}}$. By Theorem 34, $(G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) = G^{\mathsf{Rel}}\rho$. Thus, for all $\rho \in \mathsf{RelEnv}$ and any $(a, b) \in [\![\vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$
\begin{array}{rcl}
(G^{\mathsf{Set}}, G^{\mathsf{Set}}) = (G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) & \in & [\![\vdash \mathsf{Nat}^{\alpha}\,\alpha\,\alpha]\!]^{\mathsf{Rel}}\rho \\
& = & \{\eta : id \Rightarrow id\} \\
& = & \{(\eta_1 : id \Rightarrow id, \eta_2 : id \Rightarrow id)\}
\end{array}
$$

That is, G^{Set} is a natural transformation from the identity functor on Set to itself.

Now let S be any set. If $S = \emptyset$, then there is exactly one morphism $id_S : S \to S$, so $G^{\mathsf{Set}}_S : S \to S$ must be id_S. If $S \neq \emptyset$, then if a is any element of S and $K_a : S \to S$ is the constantly a-valued morphism on S, then instantiating the naturality square implied by the above equality gives that

$G_S^{\mathsf{Set}} \circ K_a = K_a \circ G_S^{\mathsf{Set}}$, i.e., $G_S^{\mathsf{Set}}\, a = a$, i.e., $G_S^{\mathsf{Set}} = id_S$. Putting these two cases together we have that for every $S : \mathsf{Set}$, $G_S^{\mathsf{Set}} = id_S$, i.e., G^{Set} is the identity natural transformation for the identity functor on Set. So every closed term g of closed type $\mathsf{Nat}^{\alpha}\alpha\,\alpha$ always denotes the identity natural transformation for the identity functor on Set, i.e., every closed term g of type $\mathsf{Nat}^{\alpha}\alpha\,\alpha$ denotes the polymorphic identity function.

6.3 Free Theorem for Type of *filter* for Lists

Let $List\,\alpha = (\mu\phi.\lambda\beta.\mathbb{1} + \beta \times \phi\beta)\alpha$, and let $map = \mathsf{map}_{\lambda A.[\![\emptyset;\alpha\vdash List\,\alpha]\!]^{\mathsf{Set}}\rho[\alpha:=A]}$.

Lemma 35. *If $g : A \to B$, $\rho : \mathsf{RelEnv}$, and $\rho\alpha = (A, B, \langle g\rangle)$, then $[\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho = \langle map\,g\rangle$*

Proof.

$$
\begin{aligned}
&[\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho\\
&= \mu T_\rho([\![\alpha;\emptyset \vdash \alpha]\!]^{\mathsf{Rel}}\rho)\\
&= \mu T_\rho(A, B, \langle g\rangle)\\
&= (\mu T_{\pi_1\rho}A, \mu T_{\pi_2\rho}B, \varinjlim_{n\in\mathbb{N}}(T_\rho^n K_0)^*(A, B, \langle g\rangle))\\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \varinjlim_{n\in\mathbb{N}}\Sigma_{i=0}^n(A, B, \langle g\rangle)^i)\\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \mathsf{List}\,(A, B, \langle g\rangle))\\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \langle map\,g\rangle)
\end{aligned}
$$

The first equality is by Definition 17, the third equality is by Equation 3, and the fourth and sixth equalities are by Equations 23 and 24 below.
The following sequence of equalities shows

$$(T_\rho^n K_0)^*\,R = \Sigma_{i=0}^n R^i \tag{23}$$

by induction on n:

$$
\begin{aligned}
&(T_\rho^n K_0)^*\,R\\
&= T_\rho^{\mathsf{Rel}}(T_\rho^{n-1}K_0)^*\,R\\
&= [\![\alpha;\phi,\beta \vdash \mathbb{1} + \beta \times \phi\beta]\!]^{\mathsf{Set}}\rho[\phi := (T_\rho^{n-1}K_0)^*][\beta := R]\\
&= \mathbb{1} + R \times (T_\rho^{n-1}K_0)^*\,R\\
&= \mathbb{1} + R \times (\Sigma_{i=0}^{n-1}R^i)\\
&= \Sigma_{i=0}^n R^i
\end{aligned}
$$

The following reasoning shows

$$\mathsf{List}\,(A, B, \langle g\rangle) = \langle map\,g\rangle \tag{24}$$

By showing that $(xs, xs') \in \mathsf{List}\,(A, B, \langle g\rangle)$ if and only if $(xs, xs') \in \langle map\,g\rangle$:

$$
\begin{aligned}
&(xs, xs') \in \mathsf{List}\,(A, B, \langle g\rangle)\\
\iff\ & \forall i.(xs_i, xs'_i) \in \langle g\rangle\\
\iff\ & \forall i.xs'_i = g(xs_i)\\
\iff\ & xs' = map\,g\,xs\\
\iff\ & (xs, xs') \in \langle map\,g\rangle
\end{aligned}
$$

□

Theorem 36. *If* $\Gamma;\Phi \,|\, \Delta \vdash t : \tau$ *and* $\rho \in \mathsf{RelEnv}$, *and if* $(a,b) \in [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, *then* $([\![\Gamma;\Phi \,|\, \Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_1\rho)\, a\,, [\![\Gamma;\Phi \,|\, \Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_2\rho)\, b) \in [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho$

Proof. Immediate from Theorem 33 (at-gen). □

Theorem 37. *If* $g : A \to B$, $\rho : \mathsf{RelEnv}$, $\rho\alpha = (A, B, \langle g\rangle)$, $(a,b) \in [\![\alpha;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, $(s \circ g, s) \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho$, *and, for some well-formed term filter,*

$$t = [\![\alpha;\emptyset \,|\, \Delta \vdash \mathit{filter} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Set}}, \text{ then}$$

$$map\, g \circ t(\pi_1\rho)\, a\, (s \circ g) = t(\pi_2\rho)\, b\, s \circ map\, g$$

Proof. By Theorem 36, $(t(\pi_1\rho)a, t(\pi_2\rho)b) \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Rel}}\rho$. Thus if $(s,s') \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho = \rho\alpha \to \mathsf{Eq}_{Bool}$, then

$$\begin{aligned}(t(\pi_1\rho)\, a\, s, t(\pi_2\rho)\, b\, s') &\in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}(List\,\alpha)\,(List\,\alpha)]\!]^{\mathsf{Rel}}\rho\\ &= [\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho \to [\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho\end{aligned}$$

So if $(xs, xs') \in [\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho$ then,

$$(t(\pi_1\rho)\, a\, s\, xs, t(\pi_2\rho)\, b\, s'\, xs') \in [\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho \tag{25}$$

Consider the case in which $\rho\alpha = (A, B, \langle g\rangle)$. Then $[\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho = \langle map\, g\rangle$, by Lemma 35, and $(xs, xs') \in \langle map\, g\rangle$ implies $xs' = map\, g\, xs$. We also have that $(s,s') \in \langle g\rangle \to \mathsf{Eq}_{Bool}$ implies $\forall (x, gx) \in \langle g\rangle.\ sx = s'(gx)$ and thus $s = s' \circ g$ due to the definition of morphisms between relations. With these instantiations, Equation 26 becomes

$$\begin{aligned}&(t(\pi_1\rho)\, a\, (s' \circ g)\, xs, t(\pi_2\rho)\, b\, s'\, (map\, g\, xs)) \in \langle map\, g\rangle,\\ &i.e.,\\ &map\, g\, (t(\pi_1\rho)\, a\, (s' \circ g)\, xs) = t(\pi_2\rho)\, b\, s'\, (map\, g\, xs),\\ &i.e.,\\ &map\, g \circ t(\pi_1\rho)\, a\, (s' \circ g) = t(\pi_2\rho)\, b\, s' \circ map\, g\end{aligned}$$

as desired.

□

6.4 Free Theorem for Type of *filter* for GRose

Theorem 38. *Let* $g : A \to B$ *be a function,* $\eta : F \to G$ *a natural transformation of* Set *functors,* $\rho : \mathsf{RelEnv}$, $\rho\alpha = (A, B, \langle g\rangle)$, $\rho\psi = (F, G, \langle\eta\rangle)$, $(a,b) \in [\![\alpha,\psi;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, *and* $(s \circ g, s) \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho$. *Then, for any well-formed term filter, if we call*

$$t = [\![\alpha,\psi;\emptyset \,|\, \Delta \vdash \mathit{filter} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Set}}$$

we have that

$$map\,\eta\,(g+1) \circ t(\pi_1\rho)\, a\, (s \circ g) = t(\pi_2\rho)\, b\, s \circ map\,\eta\, g$$

Proof. By Theorem 36,

$$(t(\pi_1\rho)a, t(\pi_2\rho)b) \in [\![\alpha,\psi;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Rel}}\rho$$

Thus if $(s, s') \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, \mathit{Bool}]\!]^{\mathsf{Rel}}\rho = \rho\alpha \to \mathsf{Eq}_{\mathit{Bool}}$, then

$$\begin{aligned}(t(\pi_1\rho)\, a\, s, t(\pi_2\rho)\, b\, s') &\in [\![\alpha,\psi;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{GRose}\;\psi\;\alpha)\,(\mathsf{GRose}\;\psi\,(\alpha+\mathbb{1}))]\!]^{\mathsf{Rel}}\rho\\ &= [\![\alpha,\psi;\emptyset \vdash \mathsf{GRose}\;\psi\;\alpha]\!]^{\mathsf{Rel}}\rho \to [\![\alpha,\psi;\emptyset \vdash \mathsf{GRose}\;\psi\,(\alpha+\mathbb{1})]\!]^{\mathsf{Rel}}\rho\end{aligned}$$

So if $(xs, xs') \in [\![\alpha;\emptyset \vdash \mathsf{GRose}\;\psi\;\alpha]\!]^{\mathsf{Rel}}\rho$ then,

$$(t(\pi_1\rho)\, a\, s\, xs, t(\pi_2\rho)\, b\, s'\, xs') \in [\![\alpha,\psi;\emptyset \vdash \mathsf{GRose}\;\psi\,(\alpha+\mathbb{1})]\!]^{\mathsf{Rel}}\rho \tag{26}$$

Since $\rho\alpha = (A, B, \langle g\rangle)$ and $\rho\psi = (F, G, \langle\psi\rangle)$, then $[\![\alpha,\psi;\emptyset \vdash \mathsf{GRose}\;\psi\;\alpha]\!]^{\mathsf{Rel}}\rho = \langle \mathit{map}\,\eta g\rangle$ and $[\![\alpha,\psi;\emptyset \vdash \mathsf{GRose}\;\psi\,(\alpha+\mathbb{1})]\!]^{\mathsf{Rel}}\rho = \langle \mathit{map}\,\eta(g+1)\rangle$, by Lemma 35. Moreover, $(xs, xs') \in \langle \mathit{map}\,\eta\, g\rangle$ implies $xs' = \mathit{map}\,\eta\, g\, xs$. We also have that $(s, s') \in \langle g\rangle \to \mathsf{Eq}_{\mathit{Bool}}$ implies $\forall (x, gx) \in \langle g\rangle.\; sx = s'(gx)$ and thus $s = s' \circ g$ due to the definition of morphisms between relations. With these instantiations, Equation 26 becomes

$$\begin{aligned}&(t(\pi_1\rho)\, a\,(s'\circ g)\, xs, t(\pi_2\rho)\, b\, s'\,(\mathit{map}\,\eta\, g\, xs)) \in \langle \mathit{map}\,\eta\,(g+1)\rangle,\\ &i.e.,\\ &\mathit{map}\,\eta\,(g+1)\,(t(\pi_1\rho)\, a\,(s'\circ g)\, xs) = t(\pi_2\rho)\, b\, s'\,(\mathit{map}\,\eta\, g\, xs),\\ &i.e.,\\ &\mathit{map}\,\eta\,(g+1)\circ t(\pi_1\rho)\, a\,(s'\circ g) = t(\pi_2\rho)\, b\, s'\circ \mathit{map}\,\eta\, g\end{aligned}$$

as desired. □

6.5 Short Cut Fusion for Lists

THEOREM 39. *Let* $\vdash \tau : \mathcal{F}$, $\vdash \tau' : \mathcal{F}$, *and* $\beta;\emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\beta)\,\beta)\,\beta$. *If*

$$G \;=\; [\![\beta;\emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\beta)\,\beta)\,\beta]\!]^{\mathsf{Set}}$$

then

$$\mathit{fold}_{1+\tau\times\_}\; n\, c\;(G\;(\mathsf{List}\,[\![\vdash\tau]\!]^{\mathsf{Set}})\;\mathit{nil}\;\mathit{cons}) = G\,[\![\vdash\tau']\!]^{\mathsf{Set}}\, n\, c$$

PROOF. Let $\vdash \tau : \mathcal{F}$ and $\vdash \tau' : \mathcal{F}$, let

$$\beta;\emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\beta)\,\beta)\,\beta$$

and let

$$G = [\![\beta;\emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\beta)\,\beta)\,\beta]\!]^{\mathsf{Set}}$$

Then Theorem 34 gives that, for any relation environment ρ and any $(a, b) \in [\![\beta;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, then (eliding the only possible instantiations of a and b) we have

$$(G\,(\pi_1\rho), G\,(\pi_2\rho)) \in [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\beta)\,\beta)\,\beta]\!]^{\mathsf{Rel}}\rho$$

Since

$$\begin{aligned}&[\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\beta)\,\beta)\,\beta]\!]^{\mathsf{Rel}}\rho\\ =\;& [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\beta)\,\beta]\!]^{\mathsf{Rel}}\rho \to \rho\beta\\ =\;& ([\![\beta;\emptyset \vdash \mathbb{1}+\tau\times\beta]\!]^{\mathsf{Rel}}\rho \to \rho\beta) \to \rho\beta\\ =\;& ((\mathbb{1}+[\![\vdash\tau]\!]^{\mathsf{Rel}}\rho\times\rho\beta) \to \rho\beta) \to \rho\beta\\ \cong\;& ((([\![\vdash\tau]\!]^{\mathsf{Rel}}\rho\times\rho\beta) \to \rho\beta)\times\rho\beta) \to \rho\beta\end{aligned}$$

we have that if $(c', c) \in [\![\vdash\tau]\!]^{\mathsf{Rel}}\rho\times\rho\beta \to \rho\beta$ and $(n', n) \in \rho\beta$, then

$$(G\,(\pi_1\rho)\, n'\, c', G\,(\pi_2\rho)\, n\, c) \in \rho\beta$$

Now note that

$$[\![\vdash \mathsf{fold}^{\tau'}_{\mathbb{1}+\tau\times\beta} : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\tau')\,\tau')\,(\mathsf{Nat}^{\emptyset}(\mu\alpha.\mathbb{1}+\tau\times\alpha)\,\tau')]\!]^{\mathsf{Set}} = \mathit{fold}_{1+\tau\times\_}$$

and observe that if $c \in [\![\vdash \tau]\!]^{\mathsf{Set}} \times [\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}}$ and $n \in [\![\vdash \tau']\!]^{\mathsf{Set}}$, then

$$(n, c) \in [\![\vdash \mathsf{Nat}^{\emptyset}(\mathbb{1}+\tau\times\tau')\,\tau']\!]^{\mathsf{Set}}$$

Consider the instantiation:

$$\begin{array}{rcl} \pi_1\rho\beta &=& [\![\vdash \mu\alpha.\mathbb{1}+\tau\times\alpha]\!]^{\mathsf{Set}} \;=\; \mathsf{List}\;[\![\vdash \tau]\!]^{\mathsf{Set}} \\ \pi_2\rho\beta &=& [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta &=& \langle \mathit{fold}_{1+\tau\times\_}\;n\,c\rangle : \mathsf{Rel}(\pi_1\rho\beta, \pi_2\rho\beta) \\ c' &=& \mathit{cons} \\ n' &=& \mathit{nil} \end{array}$$

Clearly, $(\mathit{nil}, n) \in \rho\beta = \langle \mathit{fold}_{1+\tau\times\_}\;n\,c\rangle$ because $\mathit{fold}_{1+\tau\times\_}\;n\,c\;\mathit{nil} = n$. Moreover, $(\mathit{cons}, c) \in [\![\vdash \tau]\!]^{\mathsf{Rel}} \times \rho\beta \to \rho\beta$ since if $(x, x') \in [\![\vdash \tau]\!]^{\mathsf{Rel}}$, i.e., $x = x'$, and if $(y, y') \in \rho\beta = \langle \mathit{fold}_{1+\tau\times\_}\;n\,c\rangle$, i.e., $y' = \mathit{fold}_{1+\tau\times\_}\;n\,c\,y$, then

$$(\mathit{cons}\,x\,y, c\,x\,(\mathit{fold}_{1+\tau\times\_}\;n\,c\,y)) \in \langle \mathit{fold}_{1+\tau\times\_}\;n\,c\rangle$$

i.e.,

$$c\,x\,(\mathit{fold}_{1+\tau\times\_}\;n\,c\,y) = \mathit{fold}_{1+\tau\times\_}\;n\,c\,(\mathit{cons}\,x\,y)$$

holds by definition of $\mathit{fold}_{1+\tau\times\_}$. We therefore conclude that

$$(G\,(\mathsf{List}\;[\![\vdash \tau]\!]^{\mathsf{Set}})\;\mathit{nil}\;\mathit{cons}, G\,[\![\vdash \tau']\!]^{\mathsf{Set}}\;n\,c) \in \langle \mathit{fold}_{1+\tau\times\_}\;n\,c\rangle$$

i.e., that

$$\mathit{fold}_{1+\tau\times\_}\;n\,c\;(G\,(\mathsf{List}\;[\![\vdash \tau]\!]^{\mathsf{Set}})\;\mathit{nil}\;\mathit{cons}) = G\,[\![\vdash \tau']\!]^{\mathsf{Set}}\;n\,c$$

□

6.6 Short Cut Fusion for Arbitrary ADTs

THEOREM 40. *Let* $\overline{\vdash \tau : \mathcal{F}}$, *let* $\vdash \tau' : \mathcal{F}$, *let* $\overline{\alpha}; \beta \vdash F : \mathcal{F}$, *and let* $\beta; \emptyset \,|\, \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta$. *If we regard*

$$\begin{array}{rcl} H &=& [\![\emptyset; \beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \\ G &=& [\![\beta; \emptyset \,|\, \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}\,F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Set}} \end{array}$$

as functors in β, *then for every* $B \in H[\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}}$ *we have*

$$\mathit{fold}_H\,B\,(G\,\mu H\,\mathit{in}_H) = G\,[\![\vdash \tau']\!]^{\mathsf{Set}}\,B$$

PROOF. We first note that the type of g is well-formed, since $\emptyset; \beta \vdash F[\overline{\alpha := \tau}] : \mathcal{F}$ so our promotion theorem gives that $\beta; \emptyset \vdash F[\overline{\alpha := \tau}] : \mathcal{F}$, and $\emptyset; \beta \vdash \beta : \mathcal{F}$ so that our promotion theorem gives $\beta; \emptyset \vdash \beta : \mathcal{F}$. From these facts we deduce that $\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,F[\overline{\alpha := \tau}]\,\beta : \mathcal{T}$, and thus that $\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta : \mathcal{T}$.

Theorem 34 gives that, for any relation environment ρ and any $(a, b) \in [\![\beta; \emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$(G\,(\pi_1\rho), G\,(\pi_2\rho)) \in [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Rel}}\rho$$

Since

$$\begin{array}{rl} & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Rel}}\rho \\ = & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho \to \rho\beta \end{array}$$

we have that if $(A,B) \in [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho$ then

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \in \rho\beta$$

Now note that

$$[\![\vdash \mathsf{fold}^{\tau'}_{F[\overline{\alpha:=\tau}]} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,F[\overline{\alpha := \tau}][\beta := \tau']\,\tau')\,(\mathsf{Nat}^{\emptyset}\,(\mu\beta.F[\overline{\alpha := \tau}]\,\tau')]\!]^{\mathsf{Set}} = \mathit{fold}_H$$

and consider the instantiation

$$\begin{array}{lcl} A & = & \mathit{in}_H : H(\mu H) \to \mu H \\ B & : & H[\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & = & \langle \mathit{fold}_H\,B\rangle \end{array}$$

(Note that all the types here are well-formed.) This gives

$$\begin{array}{lcl} \pi_1\rho\beta & = & [\![\vdash \mu\beta.F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \;=\; \mu H \\ \pi_2\rho\beta & = & [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & : & \mathsf{Rel}(\pi_1\rho\beta, \pi_2\rho\beta) \\ A & : & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_1\rho) \\ B & : & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_2\rho) \end{array}$$

since

$$\begin{array}{lcl} A \;=\; \mathit{in}_H & : & H(\mu H) \to \mu H \\ & = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\mu[\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}) \to \mu[\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \\ & = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_1\rho) \to [\![\emptyset;\beta \vdash \beta]\!]^{\mathsf{Set}}(\pi_1\rho) \\ & = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_1\rho) \to [\![\beta;\emptyset \vdash \beta]\!]^{\mathsf{Set}}(\pi_1\rho) \qquad \text{Daniel's trick; now a theorem} \\ & = & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_1\rho) \end{array}$$

where "Daniel's trick" is the observation that a functor can be seen as non-functorial when we only care about its action on objects. This is now a theorem. We also have

$$\begin{array}{lcl} (A,B) \;=\; (\mathit{in}_H, B) & \in & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho \\ & = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho[\beta := \langle \mathit{fold}_H\,B\rangle] \to \langle \mathit{fold}_H\,B\rangle \\ & = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\langle \mathit{fold}_H\,B\rangle \to \langle \mathit{fold}_H\,B\rangle \\ & = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\langle \mathit{fold}_H\,B\rangle \to \langle \mathit{fold}_H\,B\rangle \qquad \text{Daniel's trick; now a theorem} \\ & = & \langle [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\,(\mathit{fold}_H\,B)\rangle \to \langle \mathit{fold}_H\,B\rangle \qquad \text{by the graph lemma} \\ & = & \langle \mathit{map}_H\,(\mathit{fold}_H\,B)\rangle \to \langle \mathit{fold}_H\,B\rangle \end{array}$$

since if $(x,y) \in \langle \mathit{map}_H\,(\mathit{fold}_H\,B)\rangle$, i.e., if $\mathit{map}_H\,(\mathit{fold}_H\,B)\,x = y$, then $\mathit{fold}_H\,B\,(\mathit{in}_H\,x) = B\,y = B\,(\mathit{map}_H\,(\mathit{fold}_H\,B)\,x)$ by the definition of fold_H as a (indeed, the unique) morphism from in_H to B. Thus,

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \in \langle \mathit{fold}_H\,B\rangle$$

i.e.,

$$\mathit{fold}_H\,B\,(G\,(\pi_1\rho)\,\mathit{in}_H) = G\,(\pi_2\rho)\,B$$

Since β is the only free variable in G, this simplifies to

$$\mathit{fold}_H\,B\,(G\,\mu H\,\mathit{in}_H) = G\,[\![\vdash \tau']\!]^{\mathsf{Set}}\,B$$

$\square$

6.7 Short Cut Fusion for Arbitrary Nested Types

Can take $\emptyset; \alpha \vdash c$ with $[\![\emptyset; \alpha \vdash c]\!]^{\mathsf{Set}}\rho = C$ for all ρ, i.e., can take c to denote a constant C. We then get a free theorem whose conclusion is $fold_H\, B\, \circ\, G\, \mu H\; in_H = G\,[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}\, B$.

Can do Hinze's bit-reversal protocol in our system with
$\mathsf{cat} :: \alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathsf{List}\,\alpha)(\mathsf{List}\,\alpha))(\mathsf{List}\,\alpha)$
$\mathsf{zip} :: \alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathsf{List}\,\alpha)(\mathsf{List}\,\beta))(\mathsf{List}\,(\alpha \times \beta))$
?

Theorem 41. *Let* $\emptyset; \phi, \alpha \vdash F : \mathcal{F}$, *let* $\emptyset; \alpha \vdash K : \mathcal{F}$, *and let* $\phi; \emptyset \,|\, \emptyset \vdash g : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha))$. *If we let* $H : [\mathsf{Set}, \mathsf{Set}] \to [\mathsf{Set}, \mathsf{Set}]$ *be defined by*

$$H\, f\, x \quad = \quad [\![\emptyset; \phi, \alpha \vdash F]\!]^{\mathsf{Set}}[\phi := f][\alpha := x]$$

and let

$$G = [\![\phi; \emptyset \,|\, \emptyset \vdash g : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha))]\!]^{\mathsf{Set}}$$

then we have that, for every $B \in H[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \to [\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}$,

$$fold_H\, B\,(G\, \mu H\; in_H) = G\,[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}\, B$$

Proof. We first note that the type of g is well-formed since $\emptyset; \phi, \alpha \vdash F : \mathcal{F}$ so our promotion theorem gives that $\phi; \alpha \vdash F : \mathcal{F}$, and $\phi; \alpha \vdash \phi\alpha : \mathcal{F}$, so that $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha) : \mathcal{T}$ and $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\mathbb{1}\,(\phi\alpha) : \mathcal{T}$. Then $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha) : \mathcal{F}$ and $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\mathbb{1}\,(\phi\alpha) : \mathcal{F}$ also hold, and, finally, $\phi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha)) : \mathcal{T}$

Theorem 34 gives that, for any relation environment ρ and any $(a, b) \in [\![\phi, \alpha; \emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$\begin{array}{rcl}
(G\,(\pi_1\rho), G\,(\pi_2\rho)) & \in & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\alpha} F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha))]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to (\,\lambda A.1 \Rightarrow \lambda A.\,(\rho\phi)A\,) \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to (1 \Rightarrow \rho\phi) \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to \rho\phi
\end{array}$$

So if $(A, B) \in [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho$ then

$$(G\,(\pi_1\rho)\, A, G\,(\pi_2\rho)\, B) \quad \in \quad \rho\phi$$

Now note that

$$[\![\vdash \mathsf{fold}_F^K : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha} F[\phi := K]\, K)\,(\mathsf{Nat}^{\alpha}((\mu\phi.\lambda\alpha.F)\alpha)\, K)]\!]^{\mathsf{Set}} = fold_H$$

and consider the instantiation

$$\begin{array}{rcl}
A & = & in_H : H(\mu H) \Rightarrow \mu H \\
B & : & H[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \Rightarrow [\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \\
\rho\phi & = & \langle fold_H\, B\rangle \qquad \text{a graph of a natural transformation, defined in Enrico's notes}
\end{array}$$

(Note that all the types here are well-formed.) This gives

$$\begin{array}{rcl}
\pi_1\rho\phi & = & \mu H \\
\pi_2\rho\phi & = & [\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \\
\rho\phi & : & \mathsf{Rel}(\pi_1\rho\phi, \pi_2\rho\phi) \\
A & : & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_1\rho) \\
B & : & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_2\rho)
\end{array}$$

since

$$\begin{aligned}
A \;=\; in_H \quad &: \quad H(\mu H) \Rightarrow \mu H \\
&= \quad [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}[\phi := \mu[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}] \Rightarrow \mu[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}} \\
&= \quad [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho) \Rightarrow [\![\emptyset;\phi,\alpha \vdash \phi\alpha]\!]^{\mathsf{Set}}(\pi_1\rho) \\
&= \quad [\![\phi;\alpha \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho) \Rightarrow [\![\phi;\alpha \vdash \phi\alpha]\!]^{\mathsf{Set}}(\pi_1\rho) \qquad \text{Daniel's trick; now a theorem} \\
&= \quad [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_1\rho)
\end{aligned}$$

We also have

$$\begin{aligned}
(A,B) \;=\; (in_H, B) \quad &\in \quad [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \\
&= \quad \lambda A.[\![\phi;\alpha \vdash F]\!]^{\mathsf{Rel}}\rho[\alpha := A] \Rightarrow \lambda A.(\rho\phi)A \\
&= \quad \lambda A.[\![\phi;\alpha \vdash F]\!]^{\mathsf{Rel}}[\phi := \langle fold_H\, B\rangle][\alpha := A] \Rightarrow \langle fold_H\, B\rangle \\
&= \quad \lambda A.[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Rel}}[\phi := \langle fold_H\, B\rangle][\alpha := A] \Rightarrow \langle fold_H\, B\rangle \qquad \text{Daniel's trick; now a the} \\
&= \quad [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Rel}}\langle fold_H\, B\rangle \Rightarrow \langle fold_H\, B\rangle \\
&= \quad \langle [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}\,(fold_H\, B)\rangle \Rightarrow \langle fold_H\, B\rangle \qquad \text{Graph Lemma} \\
&= \quad \langle map_H\,(fold_H\, B)\rangle \Rightarrow \langle fold_H\, B\rangle
\end{aligned}$$

since if $(x,y) \in \langle map_H\,(fold_H\, B)\rangle$, i.e., if $map_H\,(fold_H\, B)\,x = y$, then $fold_H\, B\,(in_H\, x) = B\,y = B\,(map_H\,(fold_H\, B)\,x)$ by the definition of $fold_H$ as a (indeed, the unique) morphism from in_H to B. Thus,

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \in \langle fold_H\, B\rangle$$

i.e.,

$$fold_H\, B\,(G\,(\pi_1\rho)\; in_H) = G\,(\pi_2\rho)\,B$$

Since ϕ is the only free variable in G, this simplifies to

$$fold_H\, B\,(G\,\mu H\; in_H) = G\,[\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}}\,B$$

□

7 CONCLUSION

Can do everything in abstract locally presentable cartesian closed category.

Give definitions for arb lpccc, but compute free theorems in Set/Rel.

Future Work (in progress): extend calculus to GADTs

Add more polymorphisms (all foralls), even though most free theorems only use one level (or maybe two, like short cut).

fixed points at term level ala Pitts

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