Free Theorems for Nested Types

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1 FREE THEOREM FOR *filter* ON GRose

Theorem 1. *Let $g : A \to B$ be a function, $\eta : F \to G$ a natural transformation of* Set *functors, ρ : RelEnv, $\rho\alpha = (A, B, \langle g\rangle)$, $\rho\psi = (F, G, \langle\eta\rangle)$, $(a, b) \in [\![\alpha, \psi; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, and $(s \circ g, s) \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho$. Then, for any well-formed term filter, if we call*

$$t = [\![\alpha, \psi; \emptyset \mid \Delta \vdash \mathit{filter} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(\mathsf{GRose}\;\psi\;\alpha)\;(\mathsf{GRose}\;\psi\;(\alpha + \mathbb{1})))]\!]^{\mathsf{Set}}$$

we have that

$$\mathsf{map}\,\eta\;(g+1) \circ t(\pi_1\rho)\;a\;(s \circ g) = t(\pi_2\rho)\;b\;s \circ \mathsf{map}\,\eta\,g$$

Proof. By Theorem ??,

$$(t(\pi_1\rho)a, t(\pi_2\rho)b) \in [\![\alpha, \psi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(\mathsf{GRose}\;\psi\;\alpha)\;(\mathsf{GRose}\;\psi\;(\alpha + \mathbb{1})))]\!]^{\mathsf{Rel}}\rho$$

Thus if $(s, s') \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho = \rho\alpha \to \mathsf{Eq}_{Bool}$, then

$$\begin{aligned}(t(\pi_1\rho)\;a\,s, t(\pi_2\rho)\;b\,s') &\in [\![\alpha, \psi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{GRose}\;\psi\;\alpha)\;(\mathsf{GRose}\;\psi\;(\alpha + \mathbb{1}))]\!]^{\mathsf{Rel}}\rho \\ &= [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\;\psi\;\alpha]\!]^{\mathsf{Rel}}\rho \to [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\;\psi\;(\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho\end{aligned}$$

So if $(xs, xs') \in [\![\alpha; \emptyset \vdash \mathsf{GRose}\;\psi\;\alpha]\!]^{\mathsf{Rel}}\rho$ then,

$$(t(\pi_1\rho)\;a\,s\,xs, t(\pi_2\rho)\;b\,s'\,xs') \in [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\;\psi\;(\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho \tag{1}$$

We know that $[\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\;\psi\;\alpha]\!]^{\mathsf{Rel}}\rho$ is equal to $[\![\emptyset; \alpha, \psi \vdash \mathsf{GRose}\;\psi\;\alpha]\!]^{\mathsf{Rel}}\rho$, which, since $\rho\alpha = (A, B, \langle g\rangle)$ and $\rho\psi = (F, G, \langle\eta\rangle)$, is equal to $\langle[\![\emptyset; \alpha, \psi \vdash \mathsf{GRose}\;\psi\;\alpha]\!]^{\mathsf{Set}}\,[\alpha := g]\;[\psi := \eta]\rangle$ by the Graph Lemma (Lemma ??), i.e., $\langle\mathsf{map}\,\eta\,g\rangle$. Analogously, $[\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\;\psi\;(\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho = \langle\mathsf{map}\;\eta(g + 1)\rangle$. Moreover, $(xs, xs') \in \langle\mathsf{map}\;\eta\,g\rangle$ implies $xs' = \mathsf{map}\;\eta\,g\,xs$. We also have that $(s, s') \in \langle g\rangle \to \mathsf{Eq}_{Bool}$ implies $\forall (x, gx) \in \langle g\rangle.\ sx = s'(gx)$ and thus $s = s' \circ g$ due to the definition of morphisms between relations. With these instantiations, Equation 1 becomes

$$\begin{aligned}&(t(\pi_1\rho)\;a\;(s' \circ g)\;xs, t(\pi_2\rho)\;b\,s'\;(\mathsf{map}\,\eta\,g\,xs)) \in \langle\mathsf{map}\,\eta\;(g+1)\rangle, \\ &\textit{i.e.}, \\ &\mathsf{map}\,\eta\;(g+1)\;(t(\pi_1\rho)\;a\;(s' \circ g)\;xs) = t(\pi_2\rho)\;b\,s'\;(\mathsf{map}\,\eta\,g\,xs), \\ &\textit{i.e.}, \\ &\mathsf{map}\,\eta\;(g+1) \circ t(\pi_1\rho)\;a\;(s' \circ g) = t(\pi_2\rho)\;b\,s' \circ \mathsf{map}\,\eta\,g\end{aligned}$$

as desired.

□

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