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Free Theorems for Nested Types

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1 SHORT CUT FUSION FOR ARBITRARY ADTS

THEOREM 1. *Let* $\overline{\vdash \tau : \mathcal{F}}$, *let* $\vdash \tau' : \mathcal{F}$, *let* $\alpha, \beta; \emptyset \vdash F : \mathcal{F}$, *and let* $\beta; \emptyset \,|\, \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta)\,\beta$. *If we regard*

$$\begin{array}{rcl} H & = & [\![\beta; \emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \\ G & = & [\![\beta; \emptyset \,|\, \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}\, F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Set}} \end{array}$$

as functors in β, *then for every* $B \in H[\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}}$ *we have*

$$fold_H\, B\, (G\, \mu H\; in_H) = G\, [\![\vdash \tau']\!]^{\mathsf{Set}}\, B$$

PROOF. Theorem ?? gives that, for any relation environment ρ and any $(a, b) \in [\![\beta; \emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$(G\,(\pi_1\rho), G\,(\pi_2\rho)) \in [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Rel}}\rho$$

Since

$$\begin{array}{rl} & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Rel}}\rho \\ = & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho \to \rho\beta \end{array}$$

we have that if $(A, B) \in [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho$ then

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \in \rho\beta$$

Now note that

$$[\![\vdash \mathsf{fold}^{\tau'}_{F[\overline{\alpha := \tau}]} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\, F[\overline{\alpha := \tau}][\beta := \tau']\,\tau')\,(\mathsf{Nat}^{\emptyset}\,(\mu\beta.F[\overline{\alpha := \tau}]\,\tau')]\!]^{\mathsf{Set}} = fold_H$$

and consider the instantiation

$$\begin{array}{rcl} A & = & in_H : H(\mu H) \to \mu H \\ B & : & H[\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & = & \langle fold_H\, B \rangle \end{array}$$

This gives

$$\begin{array}{rcl} \pi_1\rho\beta & = & [\![\vdash \mu\beta.F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \;=\; \mu H \\ \pi_2\rho\beta & = & [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & : & \mathsf{Rel}(\pi_1\rho\beta, \pi_2\rho\beta) \\ A & : & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_1\rho) \\ B & : & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_2\rho) \\ (A, B) & \in & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho \end{array}$$

so that

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \in \langle fold_H\, B \rangle$$

i.e.,

$$fold_H\, B\, (G\,(\pi_1\rho)\; in_H) = G\,(\pi_2\rho)\,B$$

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Since β is the only free variable in G, this simplifies to

$$fold_H\, B\, (G\, \mu H\, in_H) = G\, [\![\vdash \tau']\!]^{\mathrm{Set}}\, B$$

$\square$