

# Free Theorems for Nested Types

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Let  $\text{List } \alpha = (\mu\phi.\lambda\beta.\mathbb{1} + \beta \times \phi\beta)\alpha$ , and let  $\text{map} = \text{map}_{\lambda A. \llbracket \emptyset; \alpha \vdash \text{List } \alpha \rrbracket^{\text{Set}} \rho[\alpha := A]}$ .

LEMMA 1. *If  $g : A \rightarrow B$ ,  $\rho : \text{RelEnv}$ , and  $\rho\alpha = (A, B, \langle g \rangle)$ , then  $\llbracket \alpha; \emptyset \vdash \text{List } \alpha \rrbracket^{\text{Rel}} \rho = \langle \text{map } g \rangle$*

PROOF.

$$\begin{aligned}
 & \llbracket \alpha; \emptyset \vdash \text{List } \alpha \rrbracket^{\text{Rel}} \rho \\
 &= \mu T_\rho (\llbracket \alpha; \emptyset \vdash \alpha \rrbracket^{\text{Rel}} \rho) \\
 &= \mu T_\rho (A, B, \langle g \rangle) \\
 &= (\mu T_{\pi_1, \rho} A, \mu T_{\pi_2, \rho} B, \lim_{n \in \mathbb{N}} (T_\rho^n K_0)^* (A, B, \langle g \rangle)) \\
 &= (\text{List } A, \text{List } B, \lim_{n \in \mathbb{N}} \sum_{i=0}^n (A, B, \langle g \rangle)^i) \\
 &= (\text{List } A, \text{List } B, \text{List } (A, B, \langle g \rangle)) \\
 &= (\text{List } A, \text{List } B, \langle \text{map } g \rangle)
 \end{aligned}$$

The first equality is by Definition ??, the third equality is by Equation ??, and the fourth and sixth equalities are by Equations 1 and 2 below.

The following sequence of equalities shows

$$(T_\rho^n K_0)^* R = \sum_{i=0}^n R^i \quad (1)$$

by induction on  $n$ :

$$\begin{aligned}
 & (T_\rho^n K_0)^* R \\
 &= T_\rho^{\text{Rel}} (T_\rho^{n-1} K_0)^* R \\
 &= \llbracket \alpha; \phi, \beta \vdash \mathbb{1} + \beta \times \phi\beta \rrbracket^{\text{Set}} \rho[\phi := (T_\rho^{n-1} K_0)^*][\beta := R] \\
 &= \mathbb{1} + R \times (T_\rho^{n-1} K_0)^* R \\
 &= \mathbb{1} + R \times (\sum_{i=0}^{n-1} R^i) \\
 &= \sum_{i=0}^n R^i
 \end{aligned}$$

The following reasoning shows

$$\text{List } (A, B, \langle g \rangle) = \langle \text{map } g \rangle \quad (2)$$

By showing that  $(xs, xs') \in \text{List } (A, B, \langle g \rangle)$  if and only if  $(xs, xs') \in \langle \text{map } g \rangle$ :

$$\begin{aligned}
 & (xs, xs') \in \text{List } (A, B, \langle g \rangle) \\
 & \iff \forall i. (xs_i, xs'_i) \in \langle g \rangle \\
 & \iff \forall i. xs'_i = g(xs_i) \\
 & \iff xs' = \text{map } g \, xs \\
 & \iff (xs, xs') \in \langle \text{map } g \rangle
 \end{aligned}$$

□

THEOREM 2. If  $\Gamma; \Phi \mid \Delta \vdash t : \tau$  and  $\rho \in \text{RelEnv}$ , and if  $(a, b) \in \llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\text{Rel}} \rho$ , then  
 $(\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\text{Set}}(\pi_1 \rho) a, \llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\text{Set}}(\pi_2 \rho) b) \in \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\text{Rel}} \rho$

PROOF. Immediate from Theorem ?? (at-gen).  $\square$

THEOREM 3. If  $g : A \rightarrow B$ ,  $\rho : \text{RelEnv}$ ,  $\rho\alpha = (A, B, \langle g \rangle)$ ,  $(a, b) \in \llbracket \alpha; \emptyset \vdash \Delta \rrbracket^{\text{Rel}} \rho$ ,  $(s \circ g, s) \in \llbracket \alpha; \emptyset \vdash \text{Nat}^0 \alpha \text{Bool} \rrbracket^{\text{Rel}} \rho$ , and, for some well-formed term filter,

$$t = \llbracket \alpha; \emptyset \mid \Delta \vdash \text{filter} : \text{Nat}^0 (\text{Nat}^0 \alpha \text{Bool}) (\text{Nat}^0 (\text{List } \alpha) (\text{List } \alpha)) \rrbracket^{\text{Set}}, \text{ then}$$

$$\text{map } g \circ t(\pi_1 \rho) a (s \circ g) = t(\pi_2 \rho) b s \circ \text{map } g$$

PROOF. By Theorem 2,  $(t(\pi_1 \rho) a, t(\pi_2 \rho) b) \in \llbracket \alpha; \emptyset \vdash \text{Nat}^0 (\text{Nat}^0 \alpha \text{Bool}) (\text{Nat}^0 (\text{List } \alpha) (\text{List } \alpha)) \rrbracket^{\text{Rel}} \rho$ .  
 Thus if  $(s, s') \in \llbracket \alpha; \emptyset \vdash \text{Nat}^0 \alpha \text{Bool} \rrbracket^{\text{Rel}} \rho = \rho\alpha \rightarrow \text{Eq}_{\text{Bool}}$ , then

$$\begin{aligned} (t(\pi_1 \rho) a s, t(\pi_2 \rho) b s') &\in \llbracket \alpha; \emptyset \vdash \text{Nat}^0 (\text{List } \alpha) (\text{List } \alpha) \rrbracket^{\text{Rel}} \rho \\ &= \llbracket \alpha; \emptyset \vdash \text{List } \alpha \rrbracket^{\text{Rel}} \rho \rightarrow \llbracket \alpha; \emptyset \vdash \text{List } \alpha \rrbracket^{\text{Rel}} \rho \end{aligned}$$

So if  $(xs, xs') \in \llbracket \alpha; \emptyset \vdash \text{List } \alpha \rrbracket^{\text{Rel}} \rho$  then,

$$(t(\pi_1 \rho) a s xs, t(\pi_2 \rho) b s' xs') \in \llbracket \alpha; \emptyset \vdash \text{List } \alpha \rrbracket^{\text{Rel}} \rho \quad (3)$$

Consider the case in which  $\rho\alpha = (A, B, \langle g \rangle)$ . Then  $\llbracket \alpha; \emptyset \vdash \text{List } \alpha \rrbracket^{\text{Rel}} \rho = \langle \text{map } g \rangle$ , by Lemma 1, and  $(xs, xs') \in \langle \text{map } g \rangle$  implies  $xs' = \text{map } g xs$ . We also have that  $(s, s') \in \langle g \rangle \rightarrow \text{Eq}_{\text{Bool}}$  implies  $\forall (x, gx) \in \langle g \rangle. sx = s'(gx)$  and thus  $s = s' \circ g$  due to the definition of morphisms between relations. With these instantiations, Equation 3 becomes

$$(t(\pi_1 \rho) a (s' \circ g) xs, t(\pi_2 \rho) b s' (\text{map } g xs)) \in \langle \text{map } g \rangle,$$

i.e.,

$$\text{map } g (t(\pi_1 \rho) a (s' \circ g) xs) = t(\pi_2 \rho) b s' (\text{map } g xs),$$

i.e.,

$$\text{map } g \circ t(\pi_1 \rho) a (s' \circ g) = t(\pi_2 \rho) b s' \circ \text{map } g$$

as desired.  $\square$