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Free Theorems for Nested Types

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1 FREE THEOREM FOR *filter* ON GRose

THEOREM 1. *Let* $g : A \to B$ *be a function,* $\eta : F \to G$ *a natural transformation of* Set *functors,* ρ : RelEnv, $\rho\alpha = (A, B, \langle g\rangle)$, $\rho\psi = (F, G, \langle\eta\rangle)$, $(a, b) \in [\![\alpha, \psi; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, *and* $(s \circ g, s) \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset} \alpha\, Bool]\!]^{\mathsf{Rel}}\rho$. *Then, for any well-formed term filter, if we call*

$$t = [\![\alpha, \psi; \emptyset \mid \Delta \vdash \mathit{filter} : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\emptyset}\, \alpha\, Bool)(\mathsf{Nat}^{\emptyset}\, (\mathsf{GRose}\ \psi\ \alpha)\, (\mathsf{GRose}\ \psi\ (\alpha + \mathbb{1})))]\!]^{\mathsf{Set}}$$

we have that

$$\mathsf{map}\, \eta\, (g + 1) \circ t(\pi_1\rho)\, a\, (s \circ g) = t(\pi_2\rho)\, b\, s \circ \mathsf{map}\, \eta\, g$$

PROOF. By Theorem ??,

$$(t(\pi_1\rho)a, t(\pi_2\rho)b) \in [\![\alpha, \psi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\emptyset}\, \alpha\, Bool)(\mathsf{Nat}^{\emptyset}\, (\mathsf{GRose}\ \psi\ \alpha)\, (\mathsf{GRose}\ \psi\ (\alpha + \mathbb{1})))]\!]^{\mathsf{Rel}}\rho$$

Thus if $(s, s') \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset} \alpha\, Bool]\!]^{\mathsf{Rel}}\rho = \rho\alpha \to \mathsf{Eq}_{Bool}$, then

$$\begin{aligned}(t(\pi_1\rho)\, a\, s, t(\pi_2\rho)\, b\, s') &\in [\![\alpha, \psi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\, (\mathsf{GRose}\ \psi\ \alpha)\, (\mathsf{GRose}\ \psi\ (\alpha + \mathbb{1}))]\!]^{\mathsf{Rel}}\rho \\ &= [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\ \alpha]\!]^{\mathsf{Rel}}\rho \to [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\ (\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho\end{aligned}$$

So if $(xs, xs') \in [\![\alpha; \emptyset \vdash \mathsf{GRose}\ \psi\ \alpha]\!]^{\mathsf{Rel}}\rho$ then,

$$(t(\pi_1\rho)\, a\, s\, xs, t(\pi_2\rho)\, b\, s'\, xs') \in [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\ (\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho \tag{1}$$

Since $\rho\alpha = (A, B, \langle g\rangle)$ and $\rho\psi = (F, G, \langle\psi\rangle)$, then $[\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\ \alpha]\!]^{\mathsf{Rel}}\rho = \langle \mathsf{map}\ \eta g\rangle$ and $[\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\ (\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho = \langle \mathsf{map}\ \eta(g + 1)\rangle$, by Lemma ??. Moreover, $(xs, xs') \in \langle \mathsf{map}\ \eta\, g\rangle$ implies $xs' = \mathsf{map}\ \eta\, g\, xs$. We also have that $(s, s') \in \langle g\rangle \to \mathsf{Eq}_{Bool}$ implies $\forall (x, gx) \in \langle g\rangle.\ sx = s'(gx)$ and thus $s = s' \circ g$ due to the definition of morphisms between relations. With these instantiations, Equation 1 becomes

$$\begin{aligned}&(t(\pi_1\rho)\, a\, (s' \circ g)\, xs, t(\pi_2\rho)\, b\, s'\, (\mathsf{map}\, \eta\, g\, xs)) \in \langle \mathsf{map}\, \eta\, (g + 1)\rangle, \\ &\textit{i.e.}, \\ &\mathsf{map}\, \eta\, (g + 1)\, (t(\pi_1\rho)\, a\, (s' \circ g)\, xs) = t(\pi_2\rho)\, b\, s'\, (\mathsf{map}\, \eta\, g\, xs), \\ &\textit{i.e.}, \\ &\mathsf{map}\, \eta\, (g + 1) \circ t(\pi_1\rho)\, a\, (s' \circ g) = t(\pi_2\rho)\, b\, s' \circ \mathsf{map}\, \eta\, g\end{aligned}$$

as desired. □

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