Free Theorems for Nested Types

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1 FREE THEOREMS FOR NESTED TYPES

Let us recall the standard definition of graph relation of a function.

Definition 1. *If $f : A \to B$ then the relation $\langle f \rangle : \mathsf{Rel}(A,B)$ is defined by $(x,y) \in \langle f \rangle$ iff $fx = y$.*

Also, note that we are using an angle-bracket notation for both the graph relation of a function and for the pairing of functions with the same domain. Such notation is in part justified by the relation between the two notions, exposed in Remark 4.

Any set has an associated equality relation over itself, and any morphism in Set has an associated graph relation in Rel. Likewise, for a natural transformation between k-ary set functors, we shall define an associated k-ary relation transformer.

Definition 2. *If $F, G : \mathsf{Set}^k \to \mathsf{Set}$ are k-ary set functors and $\alpha : F \to G$ is a natural transformation, we define the functor $\langle \alpha \rangle^* : \mathsf{Rel}^k \to \mathsf{Rel}$ as follows. Given $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, let $\iota_{R_i} : R_i \hookrightarrow A_i \times B_i$, for $i = 1, \ldots, k$, be the inclusion of R_i as a subset of $A_i \times B_i$. By the universal property of the product, there exists a unique $h_{\overline{A \times B}}$ making the diagram*

$$\begin{tikzcd}
F\overline{A} & F(\overline{A \times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, dashed, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
& F\overline{A} \times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

commute. Let $h_{\overline{R}} : F\overline{R} \to F\overline{A} \times G\overline{B}$ be $h_{\overline{A \times B}} \circ F\overline{\iota_R}$. Define $\alpha^\wedge \overline{R}$ to be the subobject through which $h_{\overline{R}}$ is factorized by the mono-epi factorization system in Set*, as shown in the following diagram:*

$$\begin{tikzcd}
F\overline{R} \arrow[rr, "h_{\overline{R}}"] \arrow[dr, two heads, "q_{\alpha^\wedge \overline{R}}"'] & & F\overline{A} \times G\overline{B} \\
& \alpha^\wedge \overline{R} \arrow[ur, hook, "\iota_{\alpha^\wedge \overline{R}}"'] &
\end{tikzcd}$$

Note that $\alpha^\wedge \overline{R} : \mathsf{Rel}(F\overline{A}, G\overline{B})$ by construction, so we can define $\langle \alpha \rangle^* \overline{(A,B,R)} = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge \overline{R}} \alpha^\wedge \overline{R})$. Moreover, if $\overline{(\beta, \beta') : (A,B,R) \to (C,D,S)}$ are morphisms in Rel*, then we define $\langle \alpha \rangle^* \overline{(\beta, \beta')}$ to be $(F\overline{\beta}, G\overline{\beta}')$.*

We now show that the above data yield a relation transformer.

Lemma 3. *If $\alpha : F \to G$ is a morphism in $[\mathsf{Set}^k, \mathsf{Set}]$, i.e., a natural transformation between ω-cocontinuous functors, then $\langle \alpha \rangle = (F, G, \langle \alpha \rangle^*)$ is in RT_k.*

Proof. Clearly, $\langle \alpha \rangle^*$ is ω-cocontinuous, so $\langle \alpha \rangle^* : [\mathsf{Rel}^k, \mathsf{Rel}]$.

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Now, consider $\overline{(\beta, \beta') : R \longrightarrow S}$, where $\overline{R : \mathsf{Rel}(A, B)}$ and $\overline{S : \mathsf{Rel}(C, D)}$. We want to show that there exists a morphism $\epsilon : \alpha^\wedge\overline{R} \longrightarrow \alpha^\wedge\overline{S}$ such that

$$\begin{tikzcd}
\alpha^\wedge\overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{R}}"] \arrow[d, "\epsilon"'] & F\overline{A} \times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
\alpha^\wedge\overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{S}}"'] & F\overline{C} \times G\overline{D}
\end{tikzcd}$$

commutes. Since $\overline{(\beta, \beta') : R \longrightarrow S}$, there exist $\overline{\gamma : R \longrightarrow S}$ such that each diagram

$$\begin{tikzcd}
R_i \arrow[r, hook, "\iota_{R_i}"] \arrow[d, "\gamma_i"'] & A_i \times B_i \arrow[d, "\beta_i\times\beta'_i"] \\
S_i \arrow[r, hook, "\iota_{S_i}"] & C_i \times D_i
\end{tikzcd}$$

commutes. Now note that both $h_{\overline{C\times D}} \circ F(\overline{\beta \times \beta'})$ and $(F\overline{\beta} \times G\overline{\beta'}) \circ h_{\overline{A\times B}}$ make

$$\begin{tikzcd}
F\overline{C} & F\overline{C} \times F\overline{D} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F\overline{D} \arrow[r, "\alpha_{\overline{D}}"] & G\overline{D} \\
 & F(\overline{A\times B}) \arrow[ul, "F\overline{\pi_1}\circ F(\overline{\beta\times\beta'})"] \arrow[u, dashed, "\exists!"] \arrow[urr, "\alpha_{\overline{D}}\circ F\overline{\pi_2}\circ F(\overline{\beta\times\beta'})"'] & &
\end{tikzcd}$$

commute, so they must be equal. We therefore get that the right-hand square below commutes, and thus that the entire following diagram does as well:

$$\begin{tikzcd}
F\overline{R} \arrow[r, hook, "F\iota_{\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] \arrow[rr, bend left, "h_{\overline{R}}"] & F(\overline{A\times B}) \arrow[r, "h_{\overline{A\times B}}"] \arrow[d, "F(\overline{\beta\times\beta'})"] & F\overline{A}\times G\overline{B} \arrow[d, "F\overline{\beta}\times F\overline{\beta'}"] \\
F\overline{S} \arrow[r, hook, "F\iota_{\overline{S}}"'] \arrow[rr, bend right, "h_{\overline{S}}"'] & F(\overline{C\times D}) \arrow[r, "h_{\overline{C\times D}}"'] & F\overline{C}\times G\overline{D}
\end{tikzcd}$$

Finally, by the left-lifting property of $q_{F^\wedge\overline{R}}$ with respect to $\iota_{F^\wedge\overline{S}}$ given by the epi-mono factorization system, there exists an ϵ such that the diagram

$$\begin{tikzcd}
F\overline{R} \arrow[r, two heads, "q_{\alpha^\wedge\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] & \alpha^\wedge\overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{R}}"] \arrow[d, dashed, "\epsilon"] & F\overline{A}\times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
F\overline{S} \arrow[r, two heads, "q_{\alpha^\wedge\overline{S}}"'] & \alpha^\wedge\overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{S}}"'] & F\overline{C}\times G\overline{D}
\end{tikzcd}$$

commutes. □

Remark 4. *Let $f : A \longrightarrow B$ be a function with graph relation $\langle f\rangle = (A, B, \langle f\rangle^*)$. Consider the function $\langle id_A, f\rangle : A \longrightarrow A \times B$. Then $\langle id_A, f\rangle A = \langle f\rangle^*$. Moreover, if $\iota_{\langle f\rangle} : \langle f\rangle^* \hookrightarrow A \times B$ is the* *inclusion of $\langle f\rangle^*$ into $A \times B$, there is an isomorphism of subobjects*

$$\begin{tikzcd}
A \arrow[rr, "\cong"] \arrow[dr, "\langle id_A, f\rangle"'] & & \langle f\rangle^* \arrow[dl, "\iota_{\langle f\rangle}"] \\
& A \times B &
\end{tikzcd}$$

Remark 5. *If $f : A \to B$ is a function seen as a natural transformation between 0-ary functors, then $\langle f\rangle$ is (the 0-ary relation transformer associated with) the graph relation of f. Indeed, we need to apply Definition 2 with $k = 0$, i.e., to the degenerate relation $* : \mathsf{Rel}(*, *)$. As degenerate 0-ary functors, A and B are constant functors, i.e., $A* = A$ and $B* = B$. By the universal property of the product, there exists a unique h making the diagram*

$$\begin{tikzcd}
A \arrow[r, equal] & A \arrow[r, equal] & A \arrow[r, "f"] & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[u, dashed, "h"'] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

commute. Notice that $\iota_* : * \to *$ is the identity on $*$, and $A\, id_* = id_A$, so $h_* = h$. Notice that $h_{\overline{A \times B}} = \langle id_A, f\rangle$ is a monomorphism in Set *because id_A is. Then, $\iota_{f^\wedge *} = \langle id_A, f\rangle$ and $f^\wedge * = A$, from which we deduce that $\iota_{f^\wedge *} f^\wedge * = \langle id_A, f\rangle\, A = \langle f\rangle^*$, meaning that the graph of f as a 0-ary natural transformation coincides with the graph of f as a function.*

Just as the equality relation Eq_B on a set B coincides with $\langle id_B\rangle$, the graph of the identity on the set, we can define the equality relation transformer to be the graph of the identity natural transformation.

Definition 6. *Let $F : [\mathsf{Set}^k, \mathsf{Set}]$. Define the equality relation transformer on F as $\mathsf{Eq}_F = \langle id_F\rangle$. Then, $Eq_F = (F, F, \mathsf{Eq}^*_F)$ with $\mathsf{Eq}^*_F = \langle id_F\rangle^*$.*

Next, we show a useful formula to compute the graph relation transformer on a graph relation.

Lemma 7. *If $\alpha : F \to G$ is a morphism in $[\mathsf{Set}^k, \mathsf{Set}]$ and $f_1 : A_1 \to B_1, \ldots, f_k : A_k \to B_k$, then $\langle \alpha\rangle^* \langle \overline{f}\rangle = \langle G\overline{f} \circ \alpha_{\overline{A}}\rangle = \langle \alpha_{\overline{B}} \circ F\overline{f}\rangle$.*

Proof. Since $h_{\overline{A \times B}}$ is the unique morphism making the bottom triangle of the diagram

$$\begin{tikzcd}
& F\overline{A} \arrow[dl, equal] \arrow[d, "F\overline{\langle id_A, f\rangle}"] \arrow[dr, "F\overline{f}"] & & \\
F\overline{A} & F(\overline{A \times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, "h_{\overline{A \times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
& F\overline{A} \times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

commute, and since $h_{\langle \overline{f}\rangle} = h_{\overline{A\times B}} \circ F\,\overline{\iota_{\langle f\rangle}} = h_{\overline{A\times B}} \circ F\overline{\langle id_A, f\rangle}$ (the last equality being by Remark 4), the universal property of the product

$$\begin{tikzcd}
F\overline{A} & F\overline{A}\times G\overline{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & G\overline{B} \\
& F\overline{A} \arrow[ul, equal] \arrow[u, dashed, "\exists!"] \arrow[r, "F\overline{f}"'] & F\overline{B} \arrow[u, "\alpha_{\overline{B}}"']
\end{tikzcd}$$

gives that $h_{\langle \overline{f}\rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle : F\overline{A} \to F\overline{A}\times G\overline{B}$. Moreover, $\langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle$ is a monomorphism in Set because $id_{F\overline{A}}$ is, so its epi-mono factorization gives that $\iota_{\alpha^\wedge\langle \overline{f}\rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle$, and thus that $\alpha^\wedge\langle \overline{f}\rangle$, the domain of $\iota_{\alpha^\wedge\langle \overline{f}\rangle}$ is equal to $F\overline{A}$. Then, $\iota_{\alpha^\wedge\langle \overline{f}\rangle}\alpha^\wedge\langle \overline{f}\rangle = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle(F\overline{A}) = \langle \alpha_{\overline{B}} \circ F\overline{f}\rangle^*$ (where the last equality is by Remark 4). Therefore, we deduce that $\langle\alpha\rangle^*\langle\overline{f}\rangle = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge\langle \overline{f}\rangle}\,\alpha^\wedge\langle \overline{f}\rangle) = (F\overline{A}, G\overline{B}, \langle \alpha_{\overline{B}} \circ F\overline{f}\rangle^*) = \langle \alpha_{\overline{B}} \circ F\overline{f}\rangle$.

Finally, notice that $\alpha_{\overline{B}} \circ F\overline{f} = G\overline{f} \circ \alpha_{\overline{A}}$ by naturality of α. □

We have an immediate corollary of the previous result.

Corollary 8. *If $F : [\mathsf{Set}^k, \mathsf{Set}]$ and $\overline{A : \mathsf{Set}}$, then $\mathsf{Eq}^*_F\overline{\mathsf{Eq}_A} = \mathsf{Eq}_{F\overline{A}}$.*

Proof. We have that

$$\mathsf{Eq}^*_F\overline{\mathsf{Eq}_A} = \langle id_F\rangle^*\langle id_{\overline{A}}\rangle = \langle F id_{\overline{A}} \circ (id_F)_{\overline{A}}\rangle = \langle id_{F\overline{A}} \circ id_{F\overline{A}}\rangle = \langle id_{F\overline{A}}\rangle = \mathsf{Eq}_{F\overline{A}}$$

where the second identity is by Lemma 7. □

We can extend the previous notions of graph of a natural transformation and equality relation transformer to environments.

Definition 1.1. Let $f : \rho \to \rho'$ is a morphism of set environments. Then, we define a graph relation environment $\langle f\rangle$ pointwise, i.e., for any variable ϕ, we define $\langle f\rangle\phi = \langle f\phi\rangle$. Notice that $\pi_1\langle f\rangle = \rho$ and $\pi_2\langle f\rangle = \rho'$.

Likewise, if ρ is a set environment, we define the equality relation environment Eq_ρ as $\langle id_\rho\rangle$.

With the previous definitions, we can prove an Identity Extension Lemma for our interpretations.
Insert Identity Extension Lemma here
Moreover, by making use of the Identity Extension Lemma we can also prove a Graph Lemma.

Lemma 9 (Graph Lemma). *If $f : \rho \to \rho'$ is a morphism of set environments and $\Gamma; \Phi \vdash F : \mathcal{F}$, then $\langle [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\rangle = [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$*

Proof. First observe that $(f, id_{\rho'}) : \langle f\rangle \to \mathsf{Eq}_{\rho'}$ and $(id_\rho, f) : \mathsf{Eq}_\rho \to \langle f\rangle$ are morphisms of relation environments. Applying Lemma ?? to each of these observations gives that

$$([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}) = [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}(f, id_{\rho'}) : [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle \to [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'} \tag{1}$$

and

$$([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} id_\rho, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f) = [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}(id_\rho, f) : [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho \to [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle \tag{2}$$

Expanding Equation 1 gives that if $(x, y) \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$ then

$$([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}\, y) \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}$$

Observe that $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}\, y = id_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}\, y = y$ and $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho'} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}$ So, if $(x, y) \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$ then $([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x, y) \in \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}$, i.e., $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x = y$, i.e., $(x, y) \in \langle [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$. So, we have that $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle \subseteq \langle [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$

Expanding Equation 2 gives that, for any $x \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho$, then

$$([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho}\, x, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$$

Observe that $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho}\, x = id_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho} x = x$ so, for any $x \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho$, we have that $(x, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$. Moreover, $x \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho$ if and only if $(x, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in \langle [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$ and, if $x \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho$ then $(x, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$, so if $(x, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in \langle [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$ then $(x, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$, i.e., $\langle [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\rangle \subseteq [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$.

So, we conclude that $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle = \langle [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$. $\square$