

$$\text{MAP}_H^{FK} f \circ g = \text{MAP}_H^{GK} f \circ \text{MAP}_H^{FG} g \quad \left| \begin{array}{l} f: \text{Nat}^{\sigma\tau} GK \\ g: \text{Nat}^{\sigma\tau} FG \end{array} \right.$$

$$\begin{aligned} & \llbracket \Gamma; \emptyset \vdash L_{\bar{y}} x ((\text{MAP}_H^{GK})_{\emptyset} f)_{\bar{r}} ((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{r}} x : \text{Nat}^{\sigma\tau} H[\bar{e}:=F] H[\bar{e}:=K] \rrbracket \rho * \\ &= \lambda \bar{b}. \llbracket \Gamma; \emptyset \vdash L_{\bar{y}} x. ((\text{MAP}_H^{GK})_{\emptyset} f)_{\bar{r}} ((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{r}} x : \text{Nat}^{\sigma\tau} H[\bar{e}:=F] H[\bar{e}:=K] \rrbracket \rho * \bar{b} \\ &= \lambda \bar{b}. (\text{convy} (\llbracket \Gamma; \bar{y} \vdash x : H[\bar{e}:=F] \rrbracket \vdash ((\text{MAP}_H^{GK})_{\emptyset} f)_{\bar{r}} ((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{r}} x : H[\bar{e}:=K] \rrbracket \rho [\bar{y}:=\bar{b}]) *) \\ &= \lambda \bar{b} \downarrow. \llbracket \Gamma; \bar{y} \vdash x : H[\bar{e}:=F] \rrbracket \vdash ((\text{MAP}_H^{GK})_{\emptyset} f)_{\bar{r}} ((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{r}} x : H[\bar{e}:=K] \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow \\ &= \lambda \bar{b} \downarrow. (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash (\text{MAP}_H^{GK})_{\emptyset} f : \text{Nat}^{\sigma\tau} H[\bar{e}:=G] H[\bar{e}:=K] \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} \\ & \quad (\llbracket \Gamma; \bar{y} \vdash x : H[\bar{e}:=F] \rrbracket \vdash ((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{r}} x : H[\bar{e}:=G] \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow) \\ &= \lambda \bar{b} \downarrow. (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash \text{MAP}_H^{GK} : \text{Nat}^{\sigma\tau} (\text{Nat}^{\sigma\tau} GK \times \text{Nat}^{\sigma\tau} H[\bar{e}:=G] H[\bar{e}:=K]) \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash f : \text{Nat}^{\sigma\tau} GK \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} \\ & \quad (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash ((\text{MAP}_H^{FG})_{\emptyset} g) : \text{Nat}^{\sigma\tau} H[\bar{e}:=F] H[\bar{e}:=G] \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} (\llbracket \Gamma; \bar{y} \vdash x : H[\bar{e}:=F] \rrbracket \vdash x : H[\bar{e}:=G] \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} \\ &= \lambda \bar{b} \downarrow. (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash \text{MAP}_H^{GK} : \text{Nat}^{\sigma\tau} (\text{Nat}^{\sigma\tau} GK \times \text{Nat}^{\sigma\tau} H[\bar{e}:=G] H[\bar{e}:=K]) \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash f : \text{Nat}^{\sigma\tau} GK \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} \\ & \quad (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash \text{MAP}_H^{FG} : \text{Nat}^{\sigma\tau} (\text{Nat}^{\sigma\tau} FG \times \text{Nat}^{\sigma\tau} H[\bar{e}:=F] H[\bar{e}:=G]) \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash g : \text{Nat}^{\sigma\tau} FG \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{b}} (\downarrow)) \\ &= \lambda \bar{b} \downarrow. (\lambda \bar{c}. \llbracket \Gamma; \bar{e}, \bar{y} \vdash H \rrbracket \text{id}_{\rho[\bar{e}:=\bar{b}]\bar{y}:=\bar{c}}} [\bar{e}:=\lambda \bar{A}. (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash f : \text{Nat}^{\sigma\tau} GK \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{A}\bar{c}}])_{\bar{b}} \\ & \quad (\lambda \bar{c}. \llbracket \Gamma; \bar{e}, \bar{y} \vdash H \rrbracket \text{id}_{\rho[\bar{e}:=\bar{b}]\bar{y}:=\bar{c}}} [\bar{e}:=\lambda \bar{A}. (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash g : \text{Nat}^{\sigma\tau} FG \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{A}\bar{c}}])_{\bar{b}} \downarrow) \\ &= \lambda \bar{b} \downarrow. (\llbracket \Gamma; \bar{e}, \bar{y} \vdash H \rrbracket \text{id}_{\rho[\bar{e}:=\bar{b}]\bar{y}:=\bar{c}}} [\bar{e}:=\lambda \bar{A}. (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash f : \text{Nat}^{\sigma\tau} GK \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{A}\bar{b}}])_{\bar{b}} \\ & \quad (\llbracket \Gamma; \bar{e}, \bar{y} \vdash H \rrbracket \text{id}_{\rho[\bar{e}:=\bar{b}]\bar{y}:=\bar{c}}} [\bar{e}:=\lambda \bar{A}. (\llbracket \Gamma; \emptyset \vdash x : H[\bar{e}:=F] \rrbracket \vdash g : \text{Nat}^{\sigma\tau} FG \rrbracket \rho [\bar{y}:=\bar{b}] \downarrow)_{\bar{A}\bar{b}}])_{\bar{b}} \downarrow) \\ &= \lambda \bar{b}. \llbracket \Gamma; \bar{e}, \bar{y} \vdash H \rrbracket \text{id}_{\rho[\bar{e}:=\bar{b}]\bar{y}:=\bar{c}}} [\bar{e}:=\lambda \bar{A}. (\lambda \bar{c} \downarrow. (\llbracket \Gamma; \emptyset \vdash x : F \vdash \text{Nat}^{\sigma\tau} GK \rrbracket \rho[\bar{e}:=\bar{c}]\bar{y}:=\bar{b}] \downarrow)_{\bar{c}\bar{b}} \\ & \quad (\llbracket \Gamma; \emptyset \vdash x : F \vdash g : \text{Nat}^{\sigma\tau} FG \rrbracket \rho[\bar{e}:=\bar{c}]\bar{y}:=\bar{b}] \downarrow)_{\bar{c}\bar{b}} \downarrow)_{\bar{A}\bar{b}}] \end{aligned}$$



$$\text{MAP}_H^{\text{FK}} \circ f \circ g = \text{MAP}_H^{\text{GK}} \circ f \circ \text{MAP}_H^{\text{FG}} \circ g \quad \text{continued}$$

$$= (\lambda \eta \ \overline{B} \cdot \overline{[ \Gamma, \overline{e}, \overline{\gamma} \vdash H ]}_{\text{id}}}_{\overline{e} \vdash \overline{B}} \left[ e \vdash \lambda \overline{A} \cdot \eta \right]_{\overline{A} \overline{B}} \left( \lambda \overline{C} \overline{D} d \cdot (\overline{[ \Gamma, \overline{d} \vdash F \vdash f : N \overline{A}^{\overline{A} \overline{B}} \overline{G} \vdash K ]}_{\overline{d} \vdash \overline{C} \overline{D} [ \overline{\gamma} \vdash \overline{D} ]}} d)_{\overline{C} \overline{D}} \right. \\ \left. (\overline{[ \Gamma, \overline{d} \vdash F \vdash g : N \overline{A}^{\overline{A} \overline{B}} \overline{F} \overline{G} ]}_{\overline{d} \vdash \overline{C} \overline{D} [ \overline{\gamma} \vdash \overline{D} ]}} d)_{\overline{C} \overline{D}} d) \right)$$

$$= (\langle \langle \Gamma, \delta \vdash M \rangle_{\Gamma}^{\text{FK}} : \text{N}^{\text{FK}}(\text{N}^{\text{FK}} \vdash \text{FK}) (\text{N}^{\text{FK}} \vdash \text{H} \vdash \text{FK}) \text{H} \vdash \text{FK} \rangle \rangle_{\text{FK}}) (\langle \langle \Gamma \vdash \langle \langle \Gamma, \delta \vdash F \vdash F : \text{N}^{\text{FK}} \vdash G \rangle \rangle_{\text{FK}} \rangle \rangle_{\text{FK}} [\text{A} := \text{C}] [\text{G} := \text{D}] \rangle \rangle_{\text{FK}}) (\langle \langle \Gamma, \delta \vdash x : F \vdash g_{\delta \delta} \ x : G \rangle \rangle_{\text{FK}} [\text{A} := \text{C}] [\text{G} := \text{D}] \rangle \rangle_{\text{FK}}) )^{\text{C} \text{ D}}$$

[illegible]

$$= (\prod_{i=1}^n \phi_i \circ \pi_i) \circ \pi_i^* : N_{\pi}(\pi_i^* \pi_i) \rightarrow N_{\pi}(\pi_i^* \pi_i) \quad \text{with } \pi_i^* \pi_i = \pi_i^* \pi_i \quad \text{and } \pi_i^* \pi_i = \pi_i^* \pi_i$$

$$= (\Gamma, \emptyset \vdash H_{\bar{a}}^{FK} : \text{Nat}^{\bar{a}\bar{b}} FK \times \text{Nat}^{\bar{c}\bar{d}} [H_{\bar{c}\bar{d}}^{\bar{a}\bar{b}}] [H_{\bar{a}\bar{b}}^{\bar{c}\bar{d}}] \prod_{p \times}) \otimes (\Gamma, \emptyset \vdash H_{\bar{a}\bar{b}} x. f_{\bar{a}\bar{b}}(g_{\bar{a}\bar{b}} x) : \text{Nat}^{\bar{c}\bar{d}} FK \prod_{p \times})$$

$$= [\Gamma, \emptyset / \emptyset H(\text{MAP}_H^{\text{FK}})(L_{\bar{\alpha}\bar{\beta}} x, f_{\bar{\alpha}\bar{\beta}}(g_{\bar{\alpha}\bar{\beta}} x)) : \text{Nat}^{\bar{\sigma}} H[\bar{y} := \bar{p}] H[\bar{y} := \bar{k}]] \uparrow_p \neq$$



$$\text{MAP}_{H}^{\bar{F}, \bar{F}} \overline{\text{id}_F} = \text{id}_{H\bar{F}}$$

$$\begin{aligned}
 & \lambda \bar{b}. (\llbracket \Gamma; \emptyset \mid \emptyset \vdash (\text{MAP}_{H}^{\bar{F}, \bar{F}})_{\emptyset} (\overline{L_{\bar{\sigma}\bar{\gamma}} x, x} : \text{Nat}^{\bar{\sigma}} H[\bar{e} := \bar{F}] H[\bar{e} := \bar{F}]] \rho * \rrbracket \bar{b}) \\
 &= \lambda \bar{b}. ((\text{eval} \circ \langle \llbracket \Gamma; \emptyset \mid \emptyset \vdash \text{MAP}_{H}^{\bar{F}, \bar{F}} : \text{Nat}^{\bar{\sigma}} (\text{Nat}^{\bar{\sigma}} \bar{F} \bar{F}) (\text{Nat}^{\bar{\sigma}} H[\bar{e} := \bar{F}] H[\bar{e} := \bar{F}]] \rho \rrbracket \bar{\sigma}, \\
 & \quad \llbracket \Gamma; \emptyset \mid \emptyset \vdash L_{\bar{\sigma}\bar{\gamma}} x, x : \text{Nat}^{\bar{\sigma}} \bar{F} \bar{F} \rrbracket \rho \rrbracket * ) \bar{b}) \\
 &= \lambda \bar{b}. ((\llbracket \Gamma; \emptyset \mid \emptyset \vdash \text{MAP}_{H}^{\bar{F}, \bar{F}} : \text{Nat}^{\bar{\sigma}} (\text{Nat}^{\bar{\sigma}} \bar{F} \bar{F}) (\text{Nat}^{\bar{\sigma}} H[\bar{e} := \bar{F}] H[\bar{e} := \bar{F}]] \rho \rrbracket \bar{\sigma}, \\
 & \quad (\llbracket \Gamma; \emptyset \mid \emptyset \vdash L_{\bar{\sigma}\bar{\gamma}} x, x : \text{Nat}^{\bar{\sigma}} \bar{F} \bar{F} \rrbracket \rho \rrbracket * ) \bar{b}) \\
 &= \lambda \bar{b}. ((\lambda \bar{\eta}. \bar{b}', \llbracket \Gamma; \bar{e}, \bar{\gamma} \vdash H \rrbracket \text{id}_{\llbracket \bar{\gamma} := \bar{b}' \rrbracket} [\bar{e} := \lambda \bar{A}. \eta_{\bar{A} \bar{b}'}] \rrbracket \\
 & \quad (\text{curry} (\llbracket \Gamma; \bar{\sigma}, \bar{\gamma} \mid x : \bar{F} \vdash x : \bar{F} \rrbracket \rho [\bar{\alpha} := -] [\bar{\gamma} := -]) * ) \bar{b}) \\
 &= \lambda \bar{b}. ((\lambda \bar{\eta}. \bar{b}', \llbracket \Gamma; \bar{e}, \bar{\gamma} \vdash H \rrbracket \text{id}_{\rho [\bar{\gamma} := \bar{b}']} [\bar{e} := \lambda \bar{A}. \eta_{\bar{A} \bar{b}'}] \rrbracket \text{id}_{\llbracket \Gamma; \bar{\sigma}, \bar{\gamma} \vdash \bar{F} \rrbracket \rho [\bar{\alpha} := -] [\bar{\gamma} := -]}) \bar{b}) \\
 &= \lambda \bar{b}. (\lambda \bar{b}'. \llbracket \Gamma; \bar{e}, \bar{\gamma} \vdash H \rrbracket \text{id}_{\rho [\bar{\gamma} := \bar{b}']} [\bar{e} := \lambda \bar{A}. \text{id}_{\llbracket \Gamma; \bar{\sigma}, \bar{\gamma} \vdash \bar{F} \rrbracket \rho [\bar{\alpha} := \bar{A}] [\bar{\gamma} := \bar{b}']} \rrbracket) \bar{b}) \\
 &= \lambda \bar{b}. \llbracket \Gamma; \bar{e}, \bar{\gamma} \vdash H \rrbracket \text{id}_{\rho [\bar{\gamma} := \bar{b}']} [\bar{e} := \lambda \bar{A}. \text{id}_{\llbracket \Gamma; \bar{\sigma}, \bar{\gamma} \vdash \bar{F} \rrbracket \rho [\bar{\alpha} := \bar{A}] [\bar{\gamma} := \bar{b}']} \rrbracket \\
 &= \lambda \bar{b}. \text{id}_{\llbracket \Gamma; \bar{e}, \bar{\gamma} \vdash H \rrbracket \rho [\bar{\gamma} := \bar{b}']} [\bar{e} := \lambda \bar{A}. \llbracket \Gamma; \bar{\sigma}, \bar{\gamma} \vdash \bar{F} \rrbracket \rho [\bar{\alpha} := \bar{A}] [\bar{\gamma} := \bar{b}']} \\
 &= \lambda \bar{b}. \text{id}_{\llbracket \Gamma; \bar{\gamma} \vdash H[\bar{e} := \bar{F}] \rrbracket \rho [\bar{\gamma} := \bar{b}]} \\
 &= \lambda \bar{b}. (\text{curry } \pi, *) \bar{b}) \\
 &= \lambda \bar{b}. (\text{curry} (\llbracket \Gamma; \bar{\gamma} \mid x : H[\bar{e} := \bar{F}] \vdash x : H[\bar{e} := \bar{F}] \rrbracket \rho [\bar{\gamma} := -] * ) \bar{b}) \\
 &= \lambda \bar{b}. \llbracket \Gamma; \emptyset \mid \emptyset \vdash L_{\bar{\gamma}} x, x : \text{Nat}^{\bar{\sigma}} H[\bar{e} := \bar{F}] H[\bar{e} := \bar{F}] \rrbracket \rho * \bar{b})
 \end{aligned}$$

$$\text{MAP}_{\text{Nat}^{\bar{B}} H K} = \text{id}_{\text{Nat}^{\bar{B}} H K}$$

$$[\Gamma; \emptyset \vdash \text{MAP}^{\emptyset}_{\text{Nat}^{\bar{B}} H K} : \text{Nat}^{\emptyset}(\overline{\text{Nat}^{\emptyset} FG})(\text{Nat}^{\emptyset}(\text{Nat}^{\bar{B}} H K)(\text{Nat}^{\bar{B}} H K))]_p ***$$

$$= [\Gamma; \emptyset \vdash \text{Nat}^{\bar{B}} H K]_p \text{id}_p$$

$$= \text{id}_{[\Gamma; \emptyset \vdash \text{Nat}^{\bar{B}} H K]_p}$$

$$= (\lambda \bar{A}. \text{id}_{[\Gamma; \emptyset \vdash \text{Nat}^{\bar{B}} H K]_p[\emptyset := \bar{A}]}) *$$

$$= (\lambda \bar{A}. \text{curry } \pi, *) *$$

$$= (\text{curry } ([\Gamma; \emptyset \vdash \eta : \text{Nat}^{\bar{B}} H K \vdash \eta : \text{Nat}^{\bar{B}} H K]_p[\emptyset := -]) *) *) *$$

$$= ([\Gamma; \emptyset \vdash L_{\emptyset} \eta. \eta : \text{Nat}^{\emptyset}(\text{Nat}^{\bar{B}} H K)(\text{Nat}^{\bar{B}} H K)]_p *) *) *$$