Parametricity and Free Theorems for Nested Types

ANONYMOUS AUTHOR(S)

Abstract goes here

1 INTRODUCTION

Suppose we wanted to prove some property of programs over an algebraic data type (ADT) such as that of lists, coded in Agda as

```
data List (A : Set) : Set where
    nil : List A
    Cons : A → List A → List A
```

A natural approach to the problem uses structural induction on the input data structure in question. This requires knowing not just the definition of the ADT of which the input data structure is an instance, but also the program text for the functions involved in the properties to be proved. For example, to prove by induction that mapping a polymorphic function over a list and then reversing the resulting list is the same as reversing the original list and then mapping the function over the result, we unwind the (recursive) definitions of the `reverse` and `map` functions over lists to according to the inductive structure of the input list. Such data-driven induction proofs over ADTs are so routine that they are often included in, say, undergraduate functional programming courses.

An alternative technique for proving results like the above `map-reverse` property for lists is to use parametricity, a formalization of extensional type-uniformity in polymorphic languages. Parametricity captures the intuition that a polymorphic program must act uniformly on all of its possible type instantiations; it is formalized as the requirement that every polymorphic program preserves all relations between any pair of types that it is instantiated with. Parametricity was originally put forth by Reynolds [Reynolds 1983] for System F [Girard et al. 1989], the formal calculus at the core of all polymorphic functional languages. It was later popularized as Wadler's "theorems for free" [Walder 1989] because it allows the deduction of many properties of programs in such languages solely from their types, i.e., with no knowledge whatsoever of the text of the programs involved. To get interesting free theorems, Wadler's calculus included, implicitly, built-in list types; indeed, most of the free theorems in [Walder 1989] are consequences of naturality for polymorphic list-processing functions. However, parametricity can also be used to prove naturality properties for non-list ADTs, as well as results, like correctness of program optimizations like *short cut fusion* [Gill et al. 1993; Johann 2002, 2003], that go beyond simple naturality.

This paper is about parametricity and free theorems for a polymorphic calculus with explicit syntax not just for ADTs, but for nested types as well. An ADT defines a *family of inductive data types*, one for each input type. For example, the `List` data type definition above defines a collection of data types `List A`, `List B`, `List (A × B)`, `List (List A)`, etc., each independent of all the others. By contrast, a nested type [Bird and Meertens 1998] is an *inductive family of data types* that is defined over, or is defined mutually recursively with, (other) such data types. Since the structures of the data type at one type can depend on those at other types, the entire family of types must be defined at once. Examples of nested types include, trivially, ordinary ADTs, such as list and tree types; simple nested types, such as the data type

```
data PTree (A : Set) : Set where
    pleaf : A → PTree A
    pnode : PTree (A × A) → PTree A
```

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```
reversePTree : ∀{A : Set} → PTree A → PTree A
reversePTree {A} = foldPTree {A} {PTree}
                      pleaf
                      (λp → pnode (mapPTree swap p))

foldPTree : ∀{A : Set} → {F : Set → Set} →
              ({B : Set} → B → FB) →
              ({B : Set} → F(B × B) → FB) →
              PTree A → F A
foldPTree n c (pleaf x) = n x
foldPTree n c (pnode p) = c (foldPTree n c p)

mapPTree : ∀{A B : Set} → (A → B) → PLeaves A → PLeaves B
mapPTree f (pleaf x) = pleaf (f x)
mapPTree f (pnode p) = pnode (mapPTree (λp → (f(π₁ p), f(π₂ p))) p)

swap : ∀{A : Set} → (A × A) → (A × A)
swap (x, y) = (y, x)
```

Fig. 1. reversePTree and auxiliary functions in Agda

```
reverseBush : ∀{A : Set} → Bush A → Bush A
reverseBush {A} = foldBush {A} {Bush} bnil balg

foldBush : ∀{A : Set} → {F : Set → Set} →
        ({B : Set} → FB) →
        ({B : Set} → B → F (F B) → F B) →
        Bush A → F A
foldBush bn bc bnil = bn
foldBush bn bc (bcons x bb) =
        bc x (foldBush bn bc (mapBush (foldBush bn bc) bb))

mapBush : ∀{A B : Set} → (A → B) → (Bush A) → (Bush B)
mapBush _ bnil = bnil
mapBush f (bcons x bb) = bcons (f x) (mapBush (mapBush f) bb)

balg : ∀{B : Set} → B → Bush (Bush B) → Bush B
balg x bnil = bcons x bnil
balg x (bcons bnil bbbx) = bcons x (bcons bnil bbbx)
balg x (bcons (bcons y bx) bbbx) =
        bcons y (bcons (bcons x bx) bbbx)
```

Fig. 2. reverseBush and auxiliary functions in Agda

of perfect trees, whose recursive occurrences never appear below other type constructors; "deep" nested types [Johann and Polonsky 2020], such as the data type

```
data Forest (A : Set) : Set where
        fempty : Forest A
        fnode  : A → PTree (Forest A) → Forest A
```

of perfect forests, whose recursive occurrences appear below type constructors for other nested types; and truly nested types[1], such as the data type

```
data Bush (A : Set) : Set where
        bnil  : Bush A
        bcons : A → Bush (Bush A) → Bush A
```

of bushes (also called *bootstrapped heaps* in [Okasaki 1999]), whose recursive occurrences appear below their own type constructors.

Suppose we now want to prove properties of functions over nested types. We might, for example, want to prove a `map-reverse` property for the functions on perfect trees in Figure 1, or for those on bushes[2] in Figure 2. A few well-chosen examples quickly convince us that such a property should indeed hold for perfect trees, and, drawing inspiration from the situation for ADTs, we easily construct a proof by induction on the input perfect tree. To formally establish this result, we could even prove it in Coq or Agda: each of these provers actually generates an induction rule for perfect trees and the generated rule gives the expected result because proving properties of perfect trees requires only that we induct over the top-level perfect tree in the recursive position, leaving any data internal to the input tree untouched.

Unfortunately, it is nowhere near as clear that analogous intuitive or formal inductive arguments can be made for the `map-reverse` property for bushes. Indeed, a proof by induction on the input bush must recursively induct over the bushes that are internal to the top-level bush in the recursive position. This is sufficiently delicate that no induction rule for bushes or other truly nested types was known until very recently, when *deep induction* [Johann and Polonsky 2020] was developed as a way to induct over *all* of the structured data present in an input. Deep induction thus not only gave the first principled and practically useful structural induction rules for bushes and other truly

[1]Nested types that are defined over themselves are known as *truly nested types*.

[2]To define the `foldBush` and `mapBush` functions in Figure 2 it is necessary to turn off Agda's termination checker.

nested types, and has also opened the way for incorporating automatic generation of such rules for (truly) nested data types — and, eventually, even GADTs — into modern proof assistants.

Of course it is great to know that we *can*, at last, prove properties of programs over (truly) nested types by induction. But recalling that inductive proofs over ADTs can sometimes be circumvented in the presence of parametricity, we might naturally ask:

Can we derive properties of functions over (truly) nested types from parametricity?

This paper answers the above question in the affirmative by constructing a parametric model for a polymorphic calculus that provides primitives for constructing nested types.

We introduce our calculus in Section 2. At the type level, it is the level-2-truncation of the higher-kinded calculus from [Johann and Polonsky 2019], augmented with a primitive type of natural transformations. To represent nested types, it constructs type expressions not just from standard type variables, but also from type constructor variables of various arities. In addition, it includes an explicit μ-construct to represent type-level recursion with respect to these variables. The class of nested types thus represented is very robust and includes all (truly) nested types known from the literature. In Section 3 we construct set and relational interpretations for the types From Section 2. As is usual when modeling parametricity, types are interpreted as functors from environments interpreting their type variable contexts to set or relations, as appropriate. But in order to ensure that these functors satisfy the cocontinuity properties needed to know that the fixpoints interpreting μ-types exist, set environments must map each k-ary type constructor variable to an appropriately cocontinuous k-ary functor on sets and relation environments must map each k-ary type constructor variable to an appropriately cocontinuous k-ary relation transformer, and these cocontinuity conditions must be threaded throughout our type interpretations in such a way that the resulting model satisfies an appropriate Identity Extension Lemma (Theorem 23). Properly progagating cocontinuity turns out to be both subtle and challenging, and Section 4, where it is done, is where the bulk of the work in our model construction lies. At the term level, our calculus includes constructs representing the actions on morphisms of functors interpreting types, initial algebras for fixpoints of these functors, and structured recursion over elements of these initial algebras (i.e., `map`, `in`, and fold constructs, respectively). While our calculus does not support general recursion at the term level, it is strongly normalizing, so does perhaps edge us toward the kind of provably total practical programming language proposed at the end of [Walder 1989]. In Section 5, we construct set and relational interpretations for the terms of our calculus. As usual in parametric models, terms are interpreted as natural transformations from interpretations of the term contexts in which they are formed to the interpretations of their types, and these must cohere in what is essentially a fibrational way [Ghani et al. 2015]. Immediately from the definition of our interpretation we prove in Section 5.4 a scheme deriving free theorems that are consequences of naturality of functions that are polymorphic over nested types. This scheme is very general, is parameterized over both the data type and the polymorphic function at hand, and it has all of the above `map-reverse` results as instances. The relationship between naturality and parametricity has long been of interest, and our inclusion of a primitive type of natural transformations makes it possible for us clearly delineate those free theorems that are consequences of naturality, and thus would hold even in non-parametric models of our calculus, from those that use the full power of parametricity to go beyond naturality. In Section 5.5 we prove that our model satisfies an Abstraction Theorem (Theorem 28), from which we derive several of this latter kind of free theorem in Section 6. Specifically, we state and prove (non-)inhabitation results in Sections 6.1 and 6.2, a free theorem for the type of a filter function on generalized rose trees in Section 6.4, and the correctness of short cut fusion for nested types in Section 6.7.

We are not the first to consider parametricity at higher kinds. Atkey [Atkey 2012] constructs a parametric model for full System Fω, but within the impredicative Calculus of Inductive Constructions (iCIC) rather than in a semantic category. Although Atkey's construction is similar to ours, he does nor provide primitives for constructing data types, so he is not concerned that his type constructors represent functors or that fixpoints for these functors exist. As a result, his relation transformers include no cocontinuity conditions, and no such conditions are propagated throughout his model construction. Atkey does, however, verify the existence of initial algebras in iCIC for (what are intended to be) syntactic reflections of functors, provided each is *given*, together with an identity- and composition-preserving *fmap* function. (Unfortunately, the *fmap* function given with F is not otherwise required to behave like the functorial action of F.) Atkey's "functors" are therefore postulated rather than constructed. He also does not in any way indicate which type constructors can be endowed with the required *fmap* functions, and we suspect that doing so would essentially result in a higher-kinded extension of the results presented here for a calculus giving an explicit syntax for the higher-kinded types it guarantees are properly functorial by construction.

Like us, Pitts [Pitts 1998, 2000] extends parametricity from pure System F to accommodate direct representations of data types. Only list types are added in [Pitts 2000], although other ADTs are easily accommodated as in [Pitts 1998]. But even there, Pitts considers only polynomial data types, all of which can all be modeled as fixpoints of *first-order* functors. Part of Pitts' motivation is to show that ADTs and their Church encodings have the same operational behavior in his system, which he accomplishes via an operational analogue of short cut fusion for ADTs. (We cannot even ask this question about our system, since it does not support Church encodings of nested types, or even of ADTs such as list types, or even of pair or sum types.) Particularly interesting would be the operational analogues of functoriality and cocontinuity required to extend Pitts' parametricity results to a calculus supplying primitives for constructing nested types and other data types modeled as fixpoints of *higher-order* functors.

Finally, there is a long line of work on categorical models of parametricity for System F; see, for example, [Bainbridge et al. 1990; Birkedal and Møgelberg 2005; Dunphy and Reddy 2004; Ghani et al. 2015; Hasegawa 1994; Jacobs 1999; Ma and Reynolds 1992; Robinson and Rosolini 1994]. Much of this work does ultimately extend parametricity to data types that are modeled as fixpoints of first-order functors, but to our knowledge this has never been done by building the data types directly into the calculus whose parametricity is to be modeled categorically. Indeed, all extensions we know of use the categorical equivalent of Atkey's approach. That is, they verify in the just-constructed parametric model the existence of initial algebras for type constructors obtained by reflecting back into syntax the functors induced by the data types' Church encodings and supporting self-related universal strengths. The present paper draws on the rich tradition of categorical models of parametricity for System F, but it extends parametricity to primitive nested types *as the model is being constructed* by taking care to ensure that System F's full impredicative polymorphism does not interfere with the functoriality needed to support primitive nested types.

2 THE CALCULUS

2.1 Types

For each $k \geq 0$, we assume countable sets $\mathbb{T}^k$ of *type constructor variables of arity* k and $\mathbb{F}^k$ of *functorial variables of arity* k, all mutually disjoint. The sets of all type constructor variables and functorial variables are $\mathbb{T} = \bigcup_{k\geq 0} \mathbb{T}^k$ and $\mathbb{F} = \bigcup_{k\geq 0} \mathbb{F}^k$, respectively, and a *type variable* is any element of $\mathbb{T} \cup \mathbb{F}$. We use lower case Greek letters for type variables, writing ϕ^k to indicate that $\phi \in \mathbb{T}^k \cup \mathbb{F}^k$, and omitting the arity indicator k when convenient, unimportant, or clear from context. We reserve letters from the beginning of the alphabet to denote type variables of arity 0, i.e., elements of $\mathbb{T}^0 \cup \mathbb{F}^0$. We write $\overline{\zeta}$ for either a set $\{\zeta_1, ..., \zeta_n\}$ of type constructor variables or a set of functorial variables when the cardinality n of the set is unimportant or clear from context. If P is a set of type variables we write $P, \overline{\phi}$ for $P \cup \overline{\phi}$ when $P \cap \overline{\phi} = \emptyset$. We omit the vector notation for a singleton set, thus writing ϕ, instead of $\overline{\phi}$, for $\{\phi\}$.

Definition 1. *Let V be a finite subset of $\mathbb{T}$, let P be a finite subset of $\mathbb{F}$, let $\overline{\alpha}$ be a finite subset of $\mathbb{F}^0$ disjoint from P, and let $\phi^k \in \mathbb{F}^k \setminus P$. The sets $\mathcal{T}(V)$ of* type constructor expressions *over V and $\mathcal{F}^P(V)$ of* functorial expressions *over P and V are given by*

$$\mathcal{T}(V) \;::=\; V \mid \mathsf{Nat}^{\overline{\alpha}}\, \mathcal{F}^{\overline{\alpha}}(V)\; \mathcal{F}^{\overline{\alpha}}(V) \mid V\overline{\mathcal{T}(V)}$$

and

$$\begin{aligned}\mathcal{F}^P(V) \;::=\; & \mathcal{T}(V) \mid \mathbb{0} \mid \mathbb{1} \mid P\overline{\mathcal{F}^P(V)} \mid V\,\overline{\mathcal{F}^P(V)} \mid \mathcal{F}^P(V) + \mathcal{F}^P(V) \mid \mathcal{F}^P(V) \times \mathcal{F}^P(V) \\ & \mid \left(\mu\phi^{\,k}.\lambda\alpha_1...\alpha_k.\mathcal{F}^{P,\alpha_1,...,\alpha_k,\phi}(V)\right)\overline{\mathcal{F}^P(V)}\end{aligned}$$

A *type* over P and V is any element of $\mathcal{T}(V) \cup \mathcal{F}^P(V)$.

The notation for types entails that an application $\tau\tau_1...\tau_k$ is allowed only when τ is a type variable of arity k, or τ is a subexpression of the form $\mu\phi^k.\lambda\alpha_1...\alpha_k.\tau'$. Moreover, if τ has arity k then τ must be applied to exactly k arguments. Accordingly, an overbar indicates a sequence of subexpressions whose length matches the arity of the type applied to it. The fact that types are always in *η-long normal form* avoids having to consider β-conversion at the level of types. In a subexpression $\mathsf{Nat}^{\overline{\alpha}}\sigma\,\tau$, the Nat operator binds all occurrences of the variables in $\overline{\alpha}$ in σ and τ. Similarly, in a subexpression $\mu\phi^k.\lambda\overline{\alpha}.\tau$, the μ operator binds all occurrences of the variable ϕ, and the λ operator binds all occurrences of the variables in $\overline{\alpha}$, in the body τ.

A *type constructor context* is a finite set Γ of type constructor variables, and a *functorial context* is a finite set Φ of functorial variables. In Definition 2, a judgment of the form $\Gamma; \Phi \vdash \tau : \mathcal{T}$ or $\Gamma; \Phi \vdash \tau : \mathcal{F}$ indicates that the type τ is intended to be functorial in the variables in Φ but not necessarily in the variables in Γ.

Definition 2. *The formation rules for the set $\mathcal{T} \subseteq \bigcup_{V \subseteq \mathbb{T}} \mathcal{T}(V)$ of* well-formed type constructor expressions *are*

$$\frac{}{\Gamma, \alpha^0; \emptyset \vdash \alpha^0 : \mathcal{T}}$$

$$\frac{\Gamma; \overline{\alpha^0} \vdash \sigma : \mathcal{F} \qquad \Gamma; \overline{\alpha^0} \vdash \tau : \mathcal{F}}{\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha^0}}\sigma\,\tau : \mathcal{T}}$$

The formation rules for the set $\mathcal{F} \subseteq \bigcup_{V \subseteq \mathbb{T}, P \subseteq \mathbb{F}} \mathcal{F}^P(V)$ of well-formed functorial expressions *are*

$$\frac{\Gamma; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \vdash \tau : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi, \alpha^0 \vdash \alpha^0 : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{0} : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{1} : \mathcal{F}}$$

$$\frac{\phi^k \in \Gamma \cup \Phi \qquad \overline{\Gamma; \Phi \vdash \tau : \mathcal{F}}}{\Gamma; \Phi \vdash \phi^k\overline{\tau} : \mathcal{F}}$$

$$\frac{\Gamma; \Phi, \overline{\alpha}, \phi^k \vdash \tau : \mathcal{F} \qquad \overline{\Gamma; \Phi \vdash \tau : \mathcal{F}}}{\Gamma; \Phi \vdash (\mu\phi^k.\lambda\overline{\alpha}.\,\tau)\overline{\tau} : \mathcal{F}}$$

$$\frac{\Gamma; \Phi \vdash \sigma : \mathcal{F} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \vdash \sigma + \tau : \mathcal{F}} \qquad \frac{\Gamma; \Phi \vdash \sigma : \mathcal{F} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \vdash \sigma \times \tau : \mathcal{F}}$$

A type τ is well-formed *if it is either a well-formed type constructor expression or a well-formed functorial expression.*

If τ is a closed type we may write $\vdash \tau$, rather than $\emptyset;\emptyset \vdash \tau$, for the judgment that it is well-formed. Definition 2 ensures that the expected weakening rules for well-formed types hold — although weakening does not change the contexts in which Nat-types can be formed. If $\Gamma;\emptyset \vdash \sigma : \mathcal{T}$ and $\Gamma;\emptyset \vdash \tau : \mathcal{T}$, then our rules allow formation of the type $\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\sigma\,\tau$. Since a type $\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\sigma\,\tau$ represents a natural transformation in $\overline{\alpha}$ from σ to τ, the type $\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\sigma\,\tau$ represents the standard arrow type $\Gamma \vdash \sigma \to \tau$ in our calculus. We similarly represent a standard $\forall$-type $\Gamma;\emptyset \vdash \forall\overline{\alpha}.\tau$ as $\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,\mathbb{1}\,\tau : \mathcal{F}$ in our calculus. However, if $\overline{\alpha}$ is non-empty then τ cannot be of the form $\mathsf{Nat}^{\overline{\beta}} H\,K$ since $\Gamma;\overline{\alpha} \vdash \mathsf{Nat}^{\overline{\beta}} H\,K$ is not a valid type judgment in our calculus (except by weakening).

Definition 2 allows the formation of all of the (closed) nested types from the introduction:

$$\begin{array}{lcl} \mathit{List}\,\alpha & = & \mu\beta.\,\mathbb{1} + \alpha\times\beta \;=\; (\mu\phi.\lambda\beta.\,\mathbb{1} + \beta\times\phi\beta)\,\alpha \\ \mathit{PTree}\,\alpha & = & (\mu\phi.\lambda\beta.\,\beta + \phi\,(\beta\times\beta))\,\alpha \\ \mathit{Forest}\,\alpha & = & (\mu\phi.\lambda\beta.\,\mathbb{1} + \beta\times \mathit{PTree}\,(\phi\beta))\,\alpha \\ \mathit{Bush}\,\alpha & = & (\mu\phi.\lambda\beta.\,\mathbb{1} + \beta\times\phi\,(\phi\beta))\,\alpha \end{array}$$

Each of these types can be considered either functorial in α or not, according to whether $\alpha \in \Gamma$ or $\alpha \in \Phi$. For example, if $\emptyset;\alpha \vdash \mathit{List}\,\alpha$, then the type $\vdash \mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\mathit{List}\,\alpha) : \mathcal{T}$ is well-formed; if $\alpha;\emptyset \vdash \mathsf{List}\ \alpha$, then it is not. If $\mathit{Tree}\,\alpha\,\gamma = \mu\beta.\,\alpha + \beta\times\gamma\times\beta$, then Definition 2 also allows the derivation of, e.g., the type $\gamma;\emptyset \vdash \mathsf{Nat}^{\alpha}(\mathit{List}\,\alpha)\,(\mathit{Tree}\,\alpha\,\gamma)$ representing a natural transformation from lists to trees that is natural in α but not necessarily in γ. We emphasize that types can be functorial in variables of arity greater than 0. For example, the type $\mathit{GRose}\,\phi\,\alpha = \mu\beta.\alpha\times\phi\beta$ can be functorial in ϕ if $\phi \in \Phi$. As usual, whether $\phi \in \Gamma$ or $\phi \in \Phi$ determines whether types such as $\mathsf{Nat}^{\alpha}(\mathit{GRose}\,\phi\,\alpha)\,(\mathit{List}\,\alpha)$ are well-formed. But even if *GRose* is functorial in ϕ, it still cannot be the (co)domain of a Nat type representing a natural transformation in ϕ. This is because our calculus does not allow naturality in variables of arity greater than 0.

Definition 2 explicitly considers types in $\mathcal{T}$ to be types in $\mathcal{F}$ that are functorial in no variables. This allows the formation of types such as $\mathit{List}\,(\sigma\to\tau)$ and $\mathit{PTree}\,(\forall\alpha.\tau)$. Functorial variables in a well-formed type τ can also be demoted to non-functorial status. The proof is by induction on τ.

Lemma 3. *If $\Gamma;\Phi,\phi^k \vdash \tau : \mathcal{F}$, then $\Gamma,\psi^k;\Phi \vdash \tau[\phi^k :== \psi^k]$ is also derivable. Here, $\tau[\phi :== \psi]$ is the textual replacement of ϕ in τ, meaning that all occurences of $\phi\overline{\sigma}$ in τ become $\psi\overline{\sigma}$.*

In addition to textual replacement, we also have a proper substitution operation on types. If τ is a type over P and V, if P and V contain only type variables of arity 0, and if $k = 0$ for every occurrence of ϕ^k bound by μ in τ, then we say that τ is *first-order*; otherwise we say that τ is *second-order*. Substitution for first-order types is the usual capture-avoiding textual substitution. We write $\tau[\alpha := \sigma]$ for the result of substituting σ for α in τ, and $\tau[\alpha_1 := \tau_1, ..., \alpha_k := \tau_k]$, or $\tau[\overline{\alpha := \tau}]$ when convenient, for $\tau[\alpha_1 := \tau_1][\alpha_2 := \tau_2, ..., \alpha_k := \tau_k]$. Substitution for second-order types is defined below, where we adopt a similar notational convention for vectors of types.

Definition 4. *If $\phi^k \in \Gamma\cup\Phi$ with $k \geq 1$, if $\Gamma;\Phi \vdash F : \mathcal{F}$, and if $\Gamma,\overline{\beta};\Phi,\overline{\alpha} \vdash H : \mathcal{F}$ with $|\overline{\alpha}| + |\overline{\beta}| = k$, then $\Gamma\setminus\phi^k;\Phi\setminus\phi^k \vdash F[\phi :=_{\overline{\beta},\overline{\alpha}} H] : \mathcal{F}$, where the operation $(\cdot)[\phi := H]$ of* second-order type substitution *is defined by:*

$$
\begin{array}{lcl}
(\mathsf{Nat}^{\overline{\gamma}} G\, K)[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \mathsf{Nat}^{\overline{\gamma}}\,(G[\phi :=_{\overline{\beta},\overline{\alpha}} H])\,(K[\phi :=_{\overline{\beta},\overline{\alpha}} H]) \\
\mathbb{1}[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \mathbb{1} \\
\mathbb{0}[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \mathbb{0} \\
(\psi \overline{\sigma \tau})[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \begin{cases} \psi\, \overline{\tau[\phi :=_{\overline{\beta},\overline{\alpha}} H]} & \text{if } \psi \neq \phi \\ H[\overline{\alpha := \tau[\phi :=_{\overline{\beta},\overline{\alpha}} H]}][\overline{\beta := \sigma[\phi :=_{\overline{\beta},\overline{\alpha}} H]}] & \text{if } \psi = \phi \end{cases} \\
(\sigma + \tau)[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \sigma[\phi :=_{\overline{\beta},\overline{\alpha}} H] + \tau[\phi :=_{\overline{\beta},\overline{\alpha}} H] \\
(\sigma \times \tau)[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & \sigma[\phi :=_{\overline{\beta},\overline{\alpha}} H] \times \tau[\phi :=_{\overline{\beta},\overline{\alpha}} H] \\
((\mu\psi.\lambda\overline{\gamma}.\, G)\overline{\tau})[\phi :=_{\overline{\beta},\overline{\alpha}} H] & = & (\mu\psi.\lambda\overline{\gamma}.\, G[\phi :=_{\overline{\beta},\overline{\alpha}} H])\,\overline{\tau[\phi :=_{\overline{\beta},\overline{\alpha}} H]}
\end{array}
$$

We omit the variable subscripts in second-order type constructor substitution when convenient.

2.2 Terms

We assume an infinite set $\mathcal{V}$ of term variables disjoint from $\mathbb{T}$ and $\mathbb{F}$. If Γ be a type constructor context and Φ is a functorial context, then a *term context for* Γ *and* Φ is a finite set of bindings of the form $x : \tau$, where $x \in \mathcal{V}$ and $\Gamma; \Phi \vdash \tau : \mathcal{F}$. We adopt the same conventions for denoting disjoint unions and for vectors in term contexts as for type constructor contexts and functorial contexts.

Definition 5. *Let Δ be a term context for Γ and Φ. The formation rules for the set of* well-formed terms over Δ *are*

$$
\frac{\Gamma; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \,|\, \Delta, x : \tau \vdash x : \tau}
\qquad
\frac{\Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \,|\, \Delta, x : \tau \vdash x : \tau}
$$

$$
\frac{}{\Gamma; \Phi \,|\, \Delta \vdash \top : \mathbb{1}}
\qquad
\frac{\Gamma; \Phi \,|\, \Delta \vdash t : \mathbb{0} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \,|\, \Delta \vdash \bot_\tau t : \tau}
$$

$$
\frac{\Gamma; \Phi \,|\, \Delta \vdash s : \sigma}{\Gamma; \Phi \,|\, \Delta \vdash \mathsf{inl}\, s : \sigma + \tau}
\qquad
\frac{\Gamma; \Phi \,|\, \Delta \vdash t : \tau}{\Gamma; \Phi \,|\, \Delta \vdash \mathsf{inr}\, t : \sigma + \tau}
$$

$$
\frac{\Gamma; \Phi \vdash \tau, \sigma : \mathcal{F} \qquad \Gamma; \Phi \,|\, \Delta \vdash t : \sigma + \tau \qquad \Gamma; \Phi \,|\, \Delta, x : \sigma \vdash l : \gamma \qquad \Gamma; \Phi \,|\, \Delta, y : \tau \vdash r : \gamma}{\Gamma; \Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{\mathsf{inl}\, x \mapsto l;\ \mathsf{inr}\, y \mapsto r\} : \gamma}
$$

$$
\frac{\Gamma; \Phi \,|\, \Delta \vdash s : \sigma \qquad \Gamma; \Phi \,|\, \Delta \vdash t : \tau}{\Gamma; \Phi \,|\, \Delta \vdash (s, t) : \sigma \times \tau}
\qquad
\frac{\Gamma; \Phi \,|\, \Delta \vdash t : \sigma \times \tau}{\Gamma; \Phi \,|\, \Delta \vdash \pi_1 t : \sigma}
\qquad
\frac{\Gamma; \Phi \,|\, \Delta \vdash t : \sigma \times \tau}{\Gamma; \Phi \,|\, \Delta \vdash \pi_2 t : \tau}
$$

$$\frac{\Gamma;\overline{\alpha} \vdash F : \mathcal{F} \qquad \Gamma;\overline{\alpha} \vdash G : \mathcal{F} \qquad \Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G}{\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G}$$

$$\frac{\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \qquad \overline{\Gamma;\Phi \vdash \tau : \mathcal{F}} \qquad \Gamma;\Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}]}{\Gamma;\Phi \,|\, \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]}$$

$$\frac{\Gamma;\overline{\phi},\overline{\gamma} \vdash H : \mathcal{F} \qquad \overline{\Gamma;\overline{\beta},\overline{\gamma} \vdash F : \mathcal{F}} \qquad \overline{\Gamma;\overline{\beta},\overline{\gamma} \vdash G : \mathcal{F}}}{\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])}$$

$$\frac{\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H : \mathcal{F}}{\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi :=_{\overline{\beta}} (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}$$

$$\frac{\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H : \mathcal{F} \qquad \Gamma;\overline{\beta},\overline{\gamma} \vdash F : \mathcal{F}}{\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{fold}_H^F : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta})\, F)}$$

In the rule for $L_{\overline{\alpha}} x.t$, the L operator binds all occurrences of the type variables in $\overline{\alpha}$ in the type of the term variable x and in the body t, as well as all occurrences of x in t. In the rule for $t_{\overline{\tau}} s$ there is one functorial expression τ for every functorial variable α. In the rule for $\mathsf{map}_H^{\overline{F},\overline{G}}$ there is one functorial expression F and one functorial expression G for each functorial variable in $\overline{\phi}$. Moreover, for each $\phi^k \in \overline{\phi}$ the number of functorial variables β in the judgments for its corresponding functorial expresssions F and G is k. In the rules for in_H and fold_H^F, the functorial variables in $\overline{\beta}$ are fresh with respect to H, and there is one β for every α. (Recall from above that, in order for the types of in_H and fold_H^F to be well-formed, the length of α must equal the arity of ϕ.) Substitution for terms is the obvious extension of the usual capture-avoiding textual substitution, and Definition 5 ensures that the expected weakening rules for well-formed terms hold.

Using Definition 5 we can represent the `reversePTree` function from Figure 1 in our calculus as

$$\vdash \mathsf{fold}_{\beta+\phi(\beta\times\beta)}^{PTree\,\alpha}\, (\mathsf{in}_{\beta+\phi(\beta\times\beta)} \circ s) : \mathsf{Nat}^{\alpha}(PTree\,\alpha)\,(PTree\,\alpha)$$

where

$$\begin{array}{lcl}
\vdash \mathsf{fold}_{\beta+\phi(\beta\times\beta)}^{PTree\,\alpha} & : & \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\alpha}(\alpha + PTree\,(\alpha\times\alpha))\,(PTree\,\alpha))\,(\mathsf{Nat}^{\alpha}(PTree\,\alpha)\,(PTree\,\alpha)) \\
\vdash \mathsf{in}_{\beta+\phi(\beta\times\beta)} & : & \mathsf{Nat}^{\alpha}(\alpha + PTree\,(\alpha\times\alpha))\,(PTree\,\alpha) \\
\vdash \mathsf{map}_{PTree\,\alpha}^{\alpha\times\alpha,\alpha\times\alpha} & : & \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\alpha}(\alpha\times\alpha)\,(\alpha\times\alpha))\,(\mathsf{Nat}^{\alpha}(PTree\,(\alpha\times\alpha))\,(PTree\,(\alpha\times\alpha)))
\end{array}$$

and *swap* and s are the terms

$$\vdash L_{\alpha} p.\,(\pi_2 p, \pi_1 p) : \mathsf{Nat}^{\alpha}(\alpha\times\alpha)\,(\alpha\times\alpha)$$

and

$$\vdash L_{\alpha} t.\,\mathsf{case}\; t\; \mathsf{of}\; \{b \mapsto \mathsf{inl}\, b;\; t' \mapsto \mathsf{inr}\,(\mathsf{map}_{PTree\,\alpha}^{\alpha\times\alpha,\alpha\times\alpha}\; swap\; t')\} : \mathsf{Nat}^{\alpha}(\alpha + PTree\,(\alpha\times\alpha))\,(\alpha + PTree\,(\alpha\times\alpha))$$

respectively. We can similarly represent the `reverseBush` function from Figure 2 as

$$\vdash \mathsf{fold}_{\mathbb{1}+\beta\times\phi(\phi\beta)}^{Bush\,\alpha}\, (\mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)} \circ (\mathbb{1} + t \circ i \circ i')) : \mathsf{Nat}^{\alpha}(Bush\,\alpha)\,(Bush\,\alpha)$$

where

$$\vdash \mathsf{fold}_{\mathbb{1}+\beta\times\phi(\phi\beta)}^{Bush\,\alpha} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\,(\mathbb{1} + \alpha \times Bush\,(Bush\,\alpha)))\,(Bush\,\alpha))\,(\mathsf{Nat}^{\alpha}\,(Bush\,\alpha)\,(Bush\,\alpha))$$
$$\vdash \mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)} : \mathsf{Nat}^{\alpha}\,(\mathbb{1} + \alpha \times Bush\,(Bush\,\alpha))\,(Bush\,\alpha)$$

and *bnil*, *bcons*, $\mathsf{in}^{-1}_{\mathbb{1}+\beta\times\phi(\phi\beta)}$, t, i, and i' are the terms

$$
\begin{array}{l}
\vdash \mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)} \circ (L_\alpha\, x.\, \mathsf{inl}\, x) \ : \ \mathsf{Nat}^\alpha\ \mathbb{1}\ (\mathit{Bush}\,\alpha)\\
\vdash \mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)} \circ (L_\alpha\, x.\, \mathsf{inr}\, x) \ : \ \mathsf{Nat}^\alpha\ (\alpha\times \mathit{Bush}\,(\mathit{Bush}\,\alpha))\ (\mathit{Bush}\,\alpha)\\
\vdash \mathsf{fold}^{(\mathbb{1}+\beta\times\phi(\phi\beta))[\phi:=\mathit{Bush}\,\alpha]}_{\mathbb{1}+\beta\times\phi(\phi\beta)}\ (\mathsf{map}^{(\mathbb{1}+\beta\times\phi(\phi\beta))[\phi:=\mathit{Bush}\,\alpha][\beta:=\alpha],\,\mathit{Bush}\,\alpha}_{\mathbb{1}+\beta\times\phi(\phi\beta)}\ \mathsf{in}_{\mathbb{1}+\beta\times\phi(\phi\beta)})\\
\qquad : \mathsf{Nat}^\alpha\ (\mathit{Bush}\,\alpha)\ (\mathbb{1}+\alpha\times \mathit{Bush}\,(\mathit{Bush}\,\alpha))\\
\vdash L_\alpha\,(b,s).\ \mathsf{case}\ s\ \{\quad * \mapsto \mathit{bcons}_\alpha\, b\,(\mathit{bnil}_\alpha\, *);\\
\qquad\qquad\qquad\qquad (s',u)\mapsto \mathsf{case}\ s'\{\quad * \mapsto \mathit{bcons}_\alpha\, b\,(\mathit{bcons}_{\mathit{Bush}\,\alpha}\,(\mathit{bnil}_\alpha\, *)\, u);\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad (b',u')\mapsto \mathit{bcons}_\alpha\, b'(\mathit{bcons}_{\mathit{Bush}\,\alpha}\,(\mathit{bcons}_\alpha\, b\, u)\, u')\}\}\\
\qquad : \mathsf{Nat}^\alpha(\alpha\times(\mathbb{1}+(\mathbb{1}+\alpha\times \mathit{Bush}\,(\mathit{Bush}\,\alpha)))\times \mathit{Bush}\,(\mathit{Bush}\,(\mathit{Bush}\,\alpha)))\ (\alpha\times \mathit{Bush}\,(\mathit{Bush}\,\alpha))\\
\vdash \alpha\times(\mathbb{1}+\mathsf{in}^{-1}_{\mathbb{1}+\beta\times\phi(\phi\beta)}\times \mathit{Bush}\,(\mathit{Bush}\,(\mathit{Bush}\,\alpha)))\\
\qquad : \mathsf{Nat}^\alpha(\alpha\times(\mathbb{1}+\mathit{Bush}\,\alpha\times \mathit{Bush}\,(\mathit{Bush}\,(\mathit{Bush}\,\alpha))))\\
\qquad\qquad (\alpha\times(\mathbb{1}+(\mathbb{1}+\alpha\times \mathit{Bush}\,(\mathit{Bush}\,\alpha))\times \mathit{Bush}\,(\mathit{Bush}\,(\mathit{Bush}\,\alpha))))\\
\vdash \alpha\times(L_\alpha\, x.\, (\mathsf{in}^{-1}_{\mathbb{1}+\beta\times\phi(\phi\beta)})_{\mathit{Bush}\,\alpha}\, x)\\
\qquad : \mathsf{Nat}^\alpha(\alpha\times \mathit{Bush}\,(\mathit{Bush}\,\alpha))\ (\alpha\times(\mathbb{1}+\mathit{Bush}\,(\alpha\times \mathit{Bush}\,(\mathit{Bush}\,(\mathit{Bush}\,\alpha)))))
\end{array}
$$

respectively. Here, $\Gamma;\emptyset\,|\,\Delta\vdash\sigma+\eta : \mathsf{Nat}^{\overline{\alpha}}(\sigma+F)\ (\sigma+G)$ and $\Gamma;\emptyset\,|\,\Delta\vdash\sigma\times\eta : \mathsf{Nat}^{\overline{\alpha}}(\sigma\times F)\ (\sigma\times G)$ for $\sigma+\eta := L_{\overline{\alpha}}\, x.\ \mathsf{case}\, x\ \mathsf{of}\ \{s\mapsto \mathsf{inl}\, s;\ t\mapsto \mathsf{inr}\,(\eta_{\overline{\alpha}} t)\}$ and $\sigma\times\eta := L_{\overline{\alpha}}\, x.\, (\pi_1 x, \eta_{\overline{\alpha}}(\pi_2 x))$ for $\Gamma;\emptyset\,|\,\Delta\vdash \eta : \mathsf{Nat}^{\overline{\alpha}} F\ G$ and $\Gamma;\overline{\alpha}\vdash\sigma:\mathcal{F}$.

Because of its functoriality restrictions, our system cannot express nontrivial recursive functions, such as $\mathit{concat} : \mathit{PTree}\,\alpha\to\mathit{PTree}\,\alpha\to\mathit{PTree}\,\alpha$, that take any nontrivial arguments other than just a single non-algebraic nested type as input. Indeed, since recursion is only expressible via folds, the type of the function would have to be expressed in curried form. But since a fold applied to an algebra must be a natural transformation from a μ-type to another functor, the codomain of the function's curried type would have to be a functor, which clearly is not the case if the function has more than one input. Moreover, even some recursive functions of a single non-algebraic nested type — e.g., a `reverseBush` function that is a true involution — cannot be expressed as folds. This is because Nat-types must be formed in empty functorial contexts, and this conflicts with the fact that the algebra arguments to folds must be polymorphic. Unfortunately, generalized folds don't mitigate these difficulties. MAKE THIS ABOUT THREADING FUNCTORIALITY, NOT A PECULIARITY OF OUR SYSTEM.

The presence of the "extra" functorial variables in $\overline{\gamma}$ in the rules for $\mathsf{map}^{\overline{F},\overline{G}}_H$, in_H, and fold^F_H merit special mention. They allows us to map or fold polymorphic functions over nested types. Consider, for example, the function $\mathit{flatten} : \mathsf{Nat}^\beta(\mathit{PTree}\,\beta)\,(\mathit{List}\,\beta)$ that maps perfect trees to lists. Even in the absence of extra variables the instance of map required to map each non-functorial monomorphic instantiation of *flatten* over a list of perfect trees is well-typed:

$$
\frac{\Gamma;\alpha\vdash \mathit{List}\,\alpha \qquad \Gamma;\emptyset\vdash\sigma \qquad \Gamma;\emptyset\vdash\tau \qquad \Gamma;\emptyset\vdash \mathit{PTree}\,\sigma \qquad \Gamma;\emptyset\vdash \mathit{List}\,\tau}{\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}^{\mathit{PTree}\,\sigma,\ \mathit{List}\,\tau}_{\mathit{List}\,\alpha} : \mathsf{Nat}^\emptyset\ (\mathsf{Nat}^\emptyset\ (\mathit{PTree}\,\sigma)\,(\mathit{List}\,\tau))\ (\mathsf{Nat}^\emptyset\ (\mathit{List}\,(\mathit{PTree}\,\sigma))\,(\mathit{List}\,(\mathit{List}\,\tau)))}
$$

But in the absence of $\overline{\gamma}$, the instance

$$
\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}^{\mathit{PTree}\,\beta,\mathit{List}\,\beta}_{\mathit{List}\,\alpha} : \mathsf{Nat}^\emptyset\ (\mathsf{Nat}^\beta(\mathit{PTree}\,\beta)\,(\mathit{List}\,\beta))\ (\mathsf{Nat}^\beta\ (\mathit{List}\,(\mathit{PTree}\,\beta))\,(\mathit{List}\,(\mathit{List}\,\beta)))
$$

of `map` required to map the *polymorphic flatten* function over a list of perfect trees is not: in that setting the functorial contexts for F and G in the rule for $\mathsf{map}^{F,G}_H$ would have to be empty, but the fact that the polymorphic *flatten* function is functorial in some variable, say δ, means that it cannot possibly have a type of the form $\mathsf{Nat}^\emptyset F\, G$ that would be required for it to be the function input to map. Since untypeability of this instance of map is unsatisfactory in a polymorphic calculus, where we naturally expect to be able to manipulate entire polymorphic functions rather than just their monomorphic instances, we use the "extra" variables in $\overline{\gamma}$ to remedy the situation. Specifrically, the rules from Definition 5 ensure that the instance of map needed to map the polymorphic *flatten* function is typeable as follows:

$$\frac{\Gamma; \alpha, \beta \vdash \mathit{List}\, \alpha \qquad \Gamma; \beta \vdash \mathit{PTree}\, \beta \qquad \Gamma; \beta \vdash \mathit{List}\, \beta}{\Gamma; \emptyset \mid \emptyset \vdash \mathsf{map}^{F,G}_{\mathit{List}} : \mathsf{Nat}^{\emptyset}\; (\mathsf{Nat}^{\beta}(\mathit{PTree}\, \beta)\, (\mathit{List}\, \beta))\; (\mathsf{Nat}^{\beta}\; (\mathit{List}\, (\mathit{PTree}\, \beta))\, (\mathit{List}\, (\mathit{List}\, \beta)))}$$

Similar remarks explain the appearance of $\overline{\gamma}$ in the typing rules for in and fold.

3 INTERPRETING TYPES

We denote the category of sets and functions by Set. The category Rel has as its objects triples (A, B, R) where R is a relation between the objects A and B in Set, i.e., a subset of $A \times B$, and has as its morphisms from (A, B, R) to (A', B', R') pairs $(f : A \to A', g : B \to B')$ of morphisms in Set such that $(f\, a, g\, b) \in R'$ whenever $(a, b) \in R$. We write $R : \mathsf{Rel}(A, B)$ in place of (A, B, R) when convenient. If $R : \mathsf{Rel}(A, B)$ we write $\pi_1 R$ and $\pi_2 R$ for the *domain* A of R and the *codomain* B of R, respectively. If A : Set, then we write $\mathsf{Eq}_A = (A, A, \{(x, x) \mid x \in A\})$ for the *equality relation* on A.

The key idea underlying Reynolds' parametricity is to give each type $\tau(\alpha)$ with one free variable α both an *object interpretation* τ_0 taking sets to sets and a *relational interpretation* τ_1 taking relations $R : \mathsf{Rel}(A, B)$ to relations $\tau_1(R) : \mathsf{Rel}(\tau_0(A), \tau_0(B))$, and to interpret each term $t(\alpha, x) : \tau(\alpha)$ with one free term variable $x : \sigma(\alpha)$ as a map t_0 associating to each set A a function $t_0(A) : \sigma_0(A) \to \tau_0(A)$. These interpretations are to be given inductively on the structures of τ and t in such a way that they imply two fundamental theorems. The first is an *Identity Extension Lemma*, which states that $\tau_1(\mathsf{Eq}_A) = \mathsf{Eq}_{\tau_0(A)}$, and is the essential property that makes a model relationally parametric rather than just induced by a logical relation. The second is an *Abstraction Theorem*, which states that, for any $R : \mathsf{Rel}(A, B)$, $(t_0(A), t_0(B))$ is a morphism in Rel from $(\sigma_0(A), \sigma_0(B), \sigma_1(R))$ to $(\tau_0(A), \tau_0(B), \tau_1(R))$. The Identity Extension Lemma is similar to the Abstraction Theorem except that it holds for *all* elements of a type's interpretation, not just those that are interpretations of terms. Similar results are expected to hold for types and terms with any number of free variables.

The key to proving the Identity Extension Lemma (Theorem 23) in our setting is a familiar "cutting down" of the interpretations of universally quantified types, such as our Nat-types, to include only the "parametric" elements. This requires that set interpretations of types are defined simultaneously with their relational interpretations. We give set interpretations for our types in Section 3.1 and give their relational interpretations in Section 3.2. While the set interpretations are relatively straightforward, their relation interpretations are less so, mainly because of the cocontinuity conditions we must impose to ensure that they are well-defined. We take some effort to develop conditions in Section 3.2, which separates Definitions 7 and 16 in space, but otherwise has no impact on the fact that they are given by mutual induction.

3.1 Interpreting Types as Sets

We will interpret the types in our calculus as ω-cocontinuous functors on locally finitely presentable categories [Adámek and Rosický 1994]. Since functor categories of locally finitely presentable categories are again locally finitely presentable, this will ensure, in particular, that the fixed points interpreting μ-types in Set and Rel exist, and thus that both the set and relational interpretations of all of the types in Definition 2 are well-defined [Johann and Polonsky 2019]. To bootstrap this process, we interpret type variables themselves as ω-cocontinuous functors in Definitions 6 and 14. If C and $\mathcal{D}$ are locally finitely presentable categories, we write $[C, \mathcal{D}]$ for the set of ω-cocontinuous functors from C to $\mathcal{D}$.

Definition 6. *A* set environment *maps each type variable in* $\mathbb{T}^k \cup \mathbb{F}^k$ *to an element of* $[\mathsf{Set}^k, \mathsf{Set}]$*. A morphism* $f : \rho \to \rho'$ *for set environments* ρ *and* ρ' *with* $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$ *maps each type constructor variable* $\psi^k \in \mathbb{T}$ *to the identity natural transformation on* $\rho\psi^k = \rho'\psi^k$ *and each functorial variable* $\phi^k \in \mathbb{F}$ *to a natural transformation from the* k*-ary functor* $\rho\phi^k$ *on* Set *to the* k*-ary functor* $\rho'\phi^k$ *on* Set*. Composition of morphisms on set environments is given componentwise, with the identity morphism mapping each set environment to itself. This gives a category of set environments and morphisms between them, which we denote* SetEnv*.*

When convenient we identify a functor $F : [\mathsf{Set}^0, \mathsf{Set}]$ with the set that is its codomain and consider a set environment to map a type variable of arity 0 to a set. If $\overline{\alpha} = \{\alpha_1, ..., \alpha_k\}$ and $\overline{A} = \{A_1, ..., A_k\}$, then we write $\rho[\overline{\alpha := A}]$ for the set environment ρ' such that $\rho'\alpha_i = A_i$ for $i = 1, ..., k$ and $\rho'\alpha = \rho\alpha$ if $\alpha \notin \{\alpha_1, ..., \alpha_k\}$. If ρ is a set environment we write Eq_ρ for the relation environment (see Definition 14) such that $\mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v}$ for every type variable v. The relational interpretations appearing in the second clause of Definition 7 are given in full in Definition 16.

Definition 7. *The* set interpretation $[\![\cdot]\!]^{\mathsf{Set}} : \mathcal{F} \to [\mathsf{SetEnv}, \mathsf{Set}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma; \emptyset \vdash v]\!]^{\mathsf{Set}}\rho &= \rho v \text{ if } v \in \mathbb{T}^0\\
[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho &= \{\eta : \lambda\overline{A}.\, [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}] \Rightarrow \lambda\overline{A}.\, [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]\\
&\qquad \mid \forall \overline{A}, \overline{B} : \mathsf{Set}. \forall \overline{R : \mathsf{Rel}(A, B)}.\\
&\qquad\qquad (\eta_{\overline{A}}, \eta_{\overline{B}}) : [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \to [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]\}\\
[\![\Gamma; \Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}}\rho &= 0\\
[\![\Gamma; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho &= 1\\
[\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho &= (\rho\phi)\, \overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}\\
[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho + [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma; \Phi \vdash \sigma \times \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho \times [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho &= (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}\\
\text{where } T^{\mathsf{Set}}_{H,\rho}\, F &= \lambda\overline{A}.[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\phi := F][\overline{\alpha := A}]\\
\text{and } T^{\mathsf{Set}}_{H,\rho}\, \eta &= \lambda\overline{A}.[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_\rho[\phi := \eta][\overline{\alpha := id_A}]
\end{aligned}
$$

The interpretations in Definition 7 respect weakening, i.e., a type and its weakenings all have the same set interpretations. The same holds for the actions of these interpretations on morphisms in Definition 8 below. Moreover, the interpretation of Nat types ensures that $[\![\Gamma \vdash \sigma \to \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma \vdash \sigma]\!]^{\mathsf{Set}}\rho \to [\![\Gamma \vdash \tau]\!]^{\mathsf{Set}}\rho$, as expected. If ρ is a set environment and $\vdash \tau : \mathcal{F}$ then we may write $[\![\vdash \tau]\!]^{\mathsf{Set}}$ instead of $[\![\vdash \tau]\!]^{\mathsf{Set}}\rho$ since the environment is immaterial. We note that the second clause of Definition 7 does indeed define a set: local finite presentability of Set and ω-cocontinuity of $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho$ ensure that $\{\eta : [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho \Rightarrow [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho\}$ (which contains $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$) is a subset of $\big\{([\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}])^{([\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=S}])} \,\big|\, \overline{S} = (S_1, ..., S_{|\overline{\alpha}|})$, and S_i is a finite set for $i = 1, ..., |\overline{\alpha}|\big\}$. There are countably many choices for tuples $\overline{S}$, and each of these gives rise to a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}]$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}]$. But there are only Set-many choices of morphisms between these (or any) two objects because Set is locally small.

In order to make sense of the last clause in Definition 7, we need to know that, for each $\rho \in \mathsf{SetEnv}$, $T^{\mathsf{Set}}_{H,\rho}$ is an ω-cocontinuous endofunctor on $[\mathsf{Set}^k, \mathsf{Set}]$, and thus admits a fixed point. Since $T^{\mathsf{Set}}_{H,\rho}$ is defined in terms of $[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, this means that interpretations of types must be such functors, which in turn means that the actions of set interpretations of types on objects and on morphisms in SetEnv are intertwined. Fortunately, we know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash \tau : \mathcal{F}$, $\llbracket \Gamma;\overline{\alpha} \vdash \tau \rrbracket^{\mathsf{Set}}$ is actually in $[\mathsf{Set}^k, \mathsf{Set}]$ where $k = |\overline{\alpha}|$. This means that for each $\llbracket \Gamma;\Phi,\phi^k,\overline{\alpha} \vdash H \rrbracket^{\mathsf{Set}}$, the corresponding operator T_H^{Set} can be extended to a *functor* from SetEnv to $[[\mathsf{Set}^k, \mathsf{Set}], [\mathsf{Set}^k, \mathsf{Set}]]$. The action of T_H^{Set} on an object $\rho \in \mathsf{SetEnv}$ is given by the higher-order functor $T_{H,\rho}^{\mathsf{Set}}$, whose actions on objects (functors in $[\mathsf{Set}^k, \mathsf{Set}]$) and morphisms (natural transformations between such functors) are given in Definition 7. Its action on a morphism $f : \rho \to \rho'$ is the higher-order natural transformation $T_{H,f}^{\mathsf{Set}} : T_{H,\rho}^{\mathsf{Set}} \to T_{H,\rho'}^{\mathsf{Set}}$ whose action on $F : [\mathsf{Set}^k, \mathsf{Set}]$ is the natural transformation $T_{H,f}^{\mathsf{Set}}\,F : T_{H,\rho}^{\mathsf{Set}}\,F \to T_{H,\rho'}^{\mathsf{Set}}\,F$ whose component at $\overline{A}$ is $(T_{H,f}^{\mathsf{Set}}\,F)_{\overline{A}} = \llbracket \Gamma;\Phi,\phi,\overline{\alpha} \vdash H \rrbracket^{\mathsf{Set}} f[\phi := id_F]\overline{[\alpha := id_A]}$. The next definition uses the functor T_H^{Set} to define the actions of functors interpreting types on morphisms between set environments.

Definition 8. *Let $f : \rho \to \rho'$ for set environments ρ and ρ' (so that $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$). The action $\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f$ of $\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}$ on the morphism f is given as follows:*

- *If $\Gamma, v;\emptyset \vdash v$ then $\llbracket \Gamma, v;\emptyset \vdash v \rrbracket^{\mathsf{Set}} f = id_{\rho v}$*
- *If $\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G$ then $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G \rrbracket^{\mathsf{Set}} f = id_{\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G \rrbracket^{\mathsf{Set}}\rho}$*
- *If $\Gamma;\Phi \vdash \mathbb{0}$ then $\llbracket \Gamma;\Phi \vdash \mathbb{0} \rrbracket^{\mathsf{Set}} f = id_0$*
- *If $\Gamma;\Phi \vdash \mathbb{1}$ then $\llbracket \Gamma;\Phi \vdash \mathbb{1} \rrbracket^{\mathsf{Set}} f = id_1$*
- *If $\Gamma;\Phi \vdash \phi\overline{\tau}$ then $\llbracket \Gamma;\Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Set}} f : \llbracket \Gamma;\Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Set}}\rho \to \llbracket \Gamma;\Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Set}}\rho' = (\rho\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho} \to (\rho'\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'}$ is defined by $\llbracket \Gamma;\Phi \vdash \phi\overline{\tau} \rrbracket^{\mathsf{Set}} f = (f\phi)_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'}} \circ (\rho\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f} = (\rho'\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f} \circ (f\phi)_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho}}$. The latter equality holds because $\rho\phi$ and $\rho'\phi$ are functors and $f\phi : \rho\phi \to \rho'\phi$ is a natural transformation, so the following naturality square commutes:*

$$\begin{tikzcd}
(\rho\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho} \arrow[r, "(f\phi)_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho}}"] \arrow[d, "(\rho\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f}"'] & (\rho'\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho} \arrow[d, "(\rho'\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f}"'] \\
(\rho\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'} \arrow[r, "(f\phi)_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'}}"] & (\rho'\phi)\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'}
\end{tikzcd} \tag{1}$$

- *If $\Gamma;\Phi \vdash \sigma + \tau$ then $\llbracket \Gamma;\Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Set}} f$ is defined by $\llbracket \Gamma;\Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Set}} f(\mathsf{inl}\,x) = \mathsf{inl}\,(\llbracket \Gamma;\Phi \vdash \sigma \rrbracket^{\mathsf{Set}} f x)$ and $\llbracket \Gamma;\Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Set}} f(\mathsf{inr}\,y) = \mathsf{inr}\,(\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f y)$.*
- *If $\Gamma;\Phi \vdash \sigma \times \tau$ then $\llbracket \Gamma;\Phi \vdash \sigma \times \tau \rrbracket^{\mathsf{Set}} f = \llbracket \Gamma;\Phi \vdash \sigma \rrbracket^{\mathsf{Set}} f \times \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f$.*
- *If $\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$ then $\llbracket \Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Set}} f : \llbracket \Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Set}}\rho \to \llbracket \Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Set}}\rho'$ $= (\mu T_{H,\rho}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho} \to (\mu T_{H,\rho'}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'}$ is defined by $(\mu T_{H,f}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'} \circ (\mu T_{H,\rho}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f} = (\mu T_{H,\rho'}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f} \circ (\mu T_{H,f}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho}$. The latter equality holds because $\mu T_{H,\rho}^{\mathsf{Set}}$ and $\mu T_{H,\rho'}^{\mathsf{Set}}$ are functors and $\mu T_{H,f}^{\mathsf{Set}} : \mu T_{H,\rho}^{\mathsf{Set}} \to \mu T_{H,\rho'}^{\mathsf{Set}}$ is a natural transformation, so the following naturality square commutes:*

$$\begin{tikzcd}
(\mu T_{H,\rho}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho} \arrow[r, "(\mu T_{H,f}^{\mathsf{Set}})_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho}}"] \arrow[d, "(\mu T_{H,\rho}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f}"'] & (\mu T_{H,\rho'}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho} \arrow[d, "(\mu T_{H,\rho'}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}} f}"'] \\
(\mu T_{H,\rho}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'} \arrow[r, "(\mu T_{H,f}^{\mathsf{Set}})_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'}}"] & (\mu T_{H,\rho'}^{\mathsf{Set}})\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho'}
\end{tikzcd} \tag{2}$$

3.2 Interpreting Types as Relations

Definition 9. *A k-ary relation transformer F is a triple (F^1, F^2, F^*), where $F^1, F^2 : [\mathsf{Set}^k, \mathsf{Set}]$ are functors, $F^* : [\mathsf{Rel}^k, \mathsf{Rel}]$ is a functor, if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $F^*\overline{R} : \mathsf{Rel}(F^1\overline{A}, F^2\overline{B})$, and if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$ then $F^*\overline{(\alpha, \beta)} = (F^1\overline{\alpha}, F^2\overline{\beta})$. We define $F\overline{R}$ to be $F^*\overline{R}$ and $F\overline{(\alpha, \beta)}$ to be $F^*\overline{(\alpha, \beta)}$.*

The last clause of Definition 9 expands to: if $\overline{(a, b) \in R}$ implies $\overline{(\alpha\, a, \beta\, b) \in S}$ then $(c, d) \in F^*\overline{R}$ implies $(F^1\overline{\alpha}\, c, F^2\overline{\beta}\, d) \in F^*\overline{S}$. When convenient we identify a 0-ary relation transformer (A, B, R) with $R : \mathsf{Rel}(A, B)$. We may also write $\pi_1 F$ for F^1 and $\pi_2 F$ for F^2. We extend these conventions to relation environments, introduced in Definition 14 below, in the obvious way.

Definition 10. *The category RT_k of k-ary relation transformers is given by the following data:*

- *An object of RT_k is a relation transformer.*
- *A morphism $\delta : (G^1, G^2, G^*) \to (H^1, H^2, H^*)$ in RT_k is a pair of natural transformations (δ^1, δ^2) where $\delta^1 : G^1 \to H^1$, $\delta^2 : G^2 \to H^2$ such that, for all $\overline{R : \mathsf{Rel}(A, B)}$, if $(x, y) \in G^*\overline{R}$ then $(\delta^1_{\overline{A}} x, \delta^2_{\overline{B}} y) \in H^*\overline{R}$.*
- *Identity morphisms and composition are inherited from the category of functors on Set.*

Definition 11. *An endofunctor H on RT_k is a triple $H = (H^1, H^2, H^*)$, where*

- *H^1 and H^2 are functors from $[\mathsf{Set}^k, \mathsf{Set}]$ to $[\mathsf{Set}^k, \mathsf{Set}]$*
- *H^* is a functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]$*
- *for all $\overline{R : \mathsf{Rel}(A, B)}$, $\pi_1((H^*(\delta^1, \delta^2))_{\overline{R}}) = (H^1\delta^1)_{\overline{A}}$ and $\pi_2((H^*(\delta^1, \delta^2))_{\overline{R}}) = (H^2\delta^2)_{\overline{B}}$*
- *The action of H on objects is given by $H\,(F^1, F^2, F^*) = (H^1F^1,\ H^2F^2,\ H^*(F^1, F^2, F^*))$*
- *The action of H on morphisms is given by $H\,(\delta^1, \delta^2) = (H^1\delta^1, H^2\delta^2)$ for $(\delta^1, \delta^2) : (F^1, F^2, F^*) \to (G^1, G^2, G^*)$*

Since the results of applying an endofunctor H to k-ary relation transformers and morphisms between them must again be k-ary relation transformers and morphisms between them, respectively, Definition 11 implicitly requires that the following three conditions hold: *i)* if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $H^*(F^1, F^2, F^*)\overline{R} : \mathsf{Rel}(H^1F^1\overline{A}, H^2F^2\overline{B})$; *ii)* if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then $H^*(F^1, F^2, F^*)\,\overline{(\alpha, \beta)} = (H^1F^1\overline{\alpha}, H^2F^2\overline{\beta})$; and $(\delta^1, \delta^2) : (F^1, F^2, F^*) \to (G^1, G^2, G^*)$ and $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $((H^1\delta^1)_{\overline{A}} x, (H^2\delta^2)_{\overline{B}} y) \in H^*(G^1, G^2, G^*)\overline{R}$ whenever $(x, y) \in H^*(F^1, F^2, F^*)\overline{R}$. Note, however, that this last condition is automatically satisfied because it is implied by the third bullet point of Definition 11.

Definition 12. *If H and K are endofunctors on RT_k, then a* natural transformation $\sigma : H \to K$ *is a pair $\sigma = (\sigma^1, \sigma^2)$, where $\sigma^1 : H^1 \to K^1$ and $\sigma^2 : H^2 \to K^2$ are natural transformations between endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]$ and the component of σ at $F \in RT_k$ is given by $\sigma_F = (\sigma^1_{F^1}, \sigma^2_{F^2})$.*

Definition 12 entails that $\sigma^i_{F^i}$ must be natural in $F^i : [\mathsf{Set}^k, \mathsf{Set}]$, and, for every F, both $(\sigma^1_{F^1})_{\overline{A}}$ and $(\sigma^2_{F^2})_{\overline{A}}$ must be natural in $\overline{A}$. Moreover, since the results of applying σ to k-ary relation transformers must be morphisms of k-ary relation transformers, Definition 12 implicitly requires that $(\sigma_F)_{\overline{R}} = ((\sigma^1_{F^1})_{\overline{A}}, (\sigma^2_{F^2})_{\overline{B}})$ is a morphism in Rel for any k-tuple of relations $\overline{R : \mathsf{Rel}(A, B)}$, i.e., that if $(x, y) \in H^*F\overline{R}$, then $((\sigma^1_{F^1})_{\overline{A}} x, (\sigma^2_{F^2})_{\overline{B}} y) \in K^*F\overline{R}$.

Critically, we can compute ω-directed colimits in RT_k: indeed, if $\mathcal{D}$ is an ω-directed set, then $\varinjlim_{d \in \mathcal{D}} (F^1_d, F^2_d, F^*_d) = (\varinjlim_{d \in \mathcal{D}} F^1_d, \varinjlim_{d \in \mathcal{D}} F^2_d, \varinjlim_{d \in \mathcal{D}} F^*_d)$. We then define an endofunctor $T = (T^1, T^2, T^*)$ on RT_k to be *ω-cocontinuous* if T^1 and T^2 are ω-cocontinuous endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]$ and T^* is an ω-cocontinuous functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]$, i.e., is in $[RT_k, [\mathsf{Rel}^k, \mathsf{Rel}]]$. Now, for any k and $R : \mathsf{Rel}(A, B)$, let K_R^{Rel} be the constantly R-valued functor from Rel^k to Rel, and for any k and set A, let K_A^{Set} be the constantly A-valued functor from Set^k to Set, and let 0 denote either the initial object of Set or the initial object of Rel, as appropriate. Observing that, for every k, K_0^{Set} is initial in $[\mathsf{Set}^k, \mathsf{Set}]$, and K_0^{Rel} is initial in $[\mathsf{Rel}^k, \mathsf{Rel}]$, we have that, for each k, $K_0 = (K_0^{\mathsf{Set}}, K_0^{\mathsf{Set}}, K_0^{\mathsf{Rel}})$ is initial in RT_k. Thus, if $T = (T^1, T^2, T^*) : RT_k \to RT_k$ is an endofunctor on RT_k then we can define the relation transformer μT to be $\varinjlim_{n\in\mathbb{N}} T^n K_0$. It is not hard to see that μT is given explicitly as

$$\mu T = (\mu T^1, \mu T^2, \varinjlim_{n\in\mathbb{N}} (T^n K_0)^*) \tag{3}$$

and that, as our notation suggests, it really is a fixpoint for T if T is ω-cocontinuous:

Lemma 13. *For any $T : [RT_k, RT_k]$, $\mu T \cong T(\mu T)$.*

The isomorphism is given by the morphisms $(in_1, in_2) : T(\mu T) \to \mu T$ and $(in_1^{-1}, in_2^{-1}) : \mu T \to T(\mu T)$ in RT_k. The latter is always a morphism in RT_k, but the former need not be if T is not ω-cocontinuous.

It is worth noting that the third component in Equation (3) is the colimit in $[\mathsf{Rel}^k, \mathsf{Rel}]$ of third components of relation transformers, rather than a fixpoint of an endofunctor on $[\mathsf{Rel}^k, \mathsf{Rel}]$. That there is an asymmetry between the first two components of μT and its third is an important conceptual observation, and reflects the fact that the third component of an endofunctor on RT_k need not be a functor on all of $[\mathsf{Rel}^k, \mathsf{Rel}]$. In particular, although we can define $T_{H,\rho}\, F$ for a relation transformer F in Definition 16 below, it is not clear how we could define it for $F : [\mathsf{Rel}^k, \mathsf{Rel}]$.

Definition 14. *A* relation environment *maps each each type variable in $\mathbb{T}^k \cup \mathbb{F}^k$ to a k-ary relation transformer. A morphism $f : \rho \to \rho'$ for relation environments ρ and ρ' with $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$ maps each type constructor variable $\psi^k \in \mathbb{T}$ to the identity morphism on $\rho\psi^k = \rho'\psi^k$ and each functorial variable $\phi^k \in \mathbb{F}$ to a morphism from the k-ary relation transformer $\rho\phi$ to the k-ary relation transformer $\rho'\phi$. Composition of morphisms on relation environments is given componentwise, with the identity morphism mapping each relation environment to itself. This gives a category of relation environments and morphisms between them, which we denote* RelEnv.

When convenient we identify a 0-ary relation transformer with the relation (transformer) that is its codomain and consider a relation environment to map a type variable of arity 0 to a relation. We write $\rho[\overline{\alpha := R}]$ for the relation environment ρ' such that $\rho'\alpha_i = R_i$ for $i = 1, ..., k$ and $\rho'\alpha = \rho\alpha$ if $\alpha \notin \{\alpha_1, ..., \alpha_k\}$. If ρ is a relation environment, we write $\pi_1\rho$ and $\pi_2\rho$ for the set environments mapping each type variable ϕ to the functors $(\rho\phi)^1$ and $(\rho\phi)^2$, respectively.

We define, for each k, the notion of an ω-cocontinuous functor from RelEnv to RT_k:

Definition 15. *A functor $H : [\mathsf{RelEnv}, RT_k]$ is a triple $H = (H^1, H^2, H^*)$, where*

- *H^1 and H^2 are objects in* $[\mathsf{SetEnv}, [\mathsf{Set}^k, \mathsf{Set}]]$
- *H^* is a an object in* $[\mathsf{RelEnv}, [\mathsf{Rel}^k, \mathsf{Rel}]]$
- *for all $\overline{R : \mathsf{Rel}(A, B)}$ and morphisms f in* RelEnv, *$\pi_1(H^* f\, \overline{R}) = H^1(\pi_1 f)\, \overline{A}$ and $\pi_2(H^* f\, \overline{R}) = H^2(\pi_2 f)\, \overline{B}$*
- *The action of H on ρ in* RelEnv *is given by $H\rho = (H^1(\pi_1\rho), H^2(\pi_2\rho), H^*\rho)$*
- *The action of H on morphisms $f : \rho \to \rho'$ in* RelEnv *is given by $Hf = (H^1(\pi_1 f), H^2(\pi_2 f))$*

Spelling out the last two bullet points above gives the following analogues of the three conditions immediately following Definition 11: *i*) if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $H^*\rho\, \overline{R} : \mathsf{Rel}(H^1(\pi_1\rho)\, \overline{A}, H^2(\pi_2\rho)\, \overline{B})$; *ii*) if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then $H^*\rho\, \overline{(\alpha, \beta)} = (H^1(\pi_1\rho)\, \overline{\alpha}, H^2(\pi_2\rho)\, \overline{\beta})$; and *iii*) if $f : \rho \to \rho'$ and $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k :$

$\mathsf{Rel}(A_k, B_k)$, then $(H^1(\pi_1 f)\,\overline{A}\,x, H^2(\pi_2 f)\,\overline{B}\,y) \in H^*\rho'\,\overline{R}$ whenever $(x,y) \in H^*\rho\,\overline{R}$. As before, the last condition is automatically satisfied because it is implied by the third bullet point of Definition 15.

Considering RelEnv as a product $\Pi_{\phi^k \in \mathbb{T}\cup\mathbb{F}} RT_k$, we extend the computation of ω-directed colimits in RT_k to compute colimits in RelEnv componentwise. We similarly extend the notion of an ω-cocontinuous endofunctor on RT_k componentwise to give a notion of ω-cocontinuity for functors from RelEnv to RT_k. Recalling from the start of this subsection that Definition 16 is given mutually inductively with Definition 7 we can, at last, define:

Definition 16. *The* relational interpretation $[\![\cdot]\!]^{\mathsf{Rel}} : \mathcal{F} \to [\mathsf{RelEnv}, \mathsf{Rel}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma;\emptyset \vdash v]\!]^{\mathsf{Rel}}\rho &= \rho v \;\; \textit{if}\; v \in \mathbb{T}^0\\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Rel}}\rho &= \{\eta : \lambda\overline{R}.\,[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] \Rightarrow \lambda\overline{R}.\,[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]\}\\
&= \{(t,t') \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid\\
&\qquad \forall R_1 : \mathsf{Rel}(A_1,B_1)\ldots R_k : \mathsf{Rel}(A_k,B_k).\\
&\qquad\qquad (t_{\overline{A}}, t'_{\overline{B}}) \in ([\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}])^{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]}\}\\
[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\rho &= 0\\
[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\rho &= 1\\
[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho &= (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}\\
[\![\Gamma;\Phi \vdash \sigma + \tau]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho + [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho\\
[\![\Gamma;\Phi \vdash \sigma \times \tau]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho \times [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho\\
[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\rho &= (\mu T_{H,\rho})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}\\
\textit{where}\; T_{H,\rho} &= (T^{\mathsf{Set}}_{H,\pi_1\rho}, T^{\mathsf{Set}}_{H,\pi_2\rho}, T^{\mathsf{Rel}}_{H,\rho})\\
\textit{and}\; T^{\mathsf{Rel}}_{H,\rho}\,F &= \lambda\overline{R}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F][\overline{\alpha := R}]\\
\textit{and}\; T^{\mathsf{Rel}}_{H,\rho}\,\delta &= \lambda\overline{R}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := \delta][\overline{\alpha := id_{\overline{R}}}]
\end{aligned}
$$

The interpretations in Definition 16, as well as in Definition 17 below, respect weakening. Definition 16 also ensures that $[\![\Gamma \vdash \sigma \to \tau]\!]^{\mathsf{Rel}}\rho = [\![\Gamma \vdash \sigma]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma \vdash \tau]\!]^{\mathsf{Rel}}\rho$. If ρ is a relational environment and $\vdash \tau : \mathcal{F}$, then we write $[\![\vdash \tau]\!]^{\mathsf{Rel}}$ instead of $[\![\vdash \tau]\!]^{\mathsf{Rel}}\rho$ as for set interpretations. For the last clause in Definition 16 to be well-defined, we need to know that T_ρ is an ω-cocontinuous endofunctor on RT so that, by Lemma 13, it admits a fixed point. Since T_ρ is defined in terms of $[\![\Gamma;\Phi,\phi^k,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}$, this means that relational interpretations of types must be ω-cocontinuous functors from RelEnv to RT_0, which in turn entails that the actions of relational interpretations of types on objects and on morphisms in RelEnv are intertwined. As for set interpretations, we know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash \tau : \mathcal{F}$, $[\![\Gamma;\overline{\alpha} \vdash \tau]\!]^{\mathsf{Set}}$ is actually in $[\mathsf{Rel}^k, \mathsf{Rel}]$ where $k = |\overline{\alpha}|$. We first define the actions of each of these functors on morphisms between environments in Definition 17, and then argue that the functors given by Definitions 16 and 17 are well-defined and have the required properties. To do this, we extend T_H to a *functor* from RelEnv to $[[\mathsf{Rel}^k, \mathsf{Rel}], [\mathsf{Rel}^k, \mathsf{Rel}]]$. Its action on an object $\rho \in \mathsf{RelEnv}$ is given by the higher-order functor $T^{\mathsf{Rel}}_{H,\rho}$ whose actions on objects and morphisms are given in Definition 17. Its action on a morphism $f : \rho \to \rho'$ is the higher-order natural transformation $T_{H,f} : T_{H,\rho} \to T_{H,\rho'}$ whose action on any $F : [\mathsf{Rel}^k, \mathsf{Rel}]$ is the natural transformation $T_{H,f}\,F : T_{H,\rho}\,F \to T_{H,\rho'}\,F$ whose component at $\overline{R}$ is $(T_{H,f}\,F)_{\overline{R}} = [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} f[\phi := id_F][\overline{\alpha := id_R}]$. The next definition uses the functor T_H to define the actions of functors interpreting types on morphisms between relation environments.

Definition 17. *Let $f : \rho \to \rho'$ for relation environments ρ and ρ' (so that $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$). The action $[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f$ of $[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}$ on the morphism f is given as follows:*

- *If $\Gamma, v; \emptyset \vdash v$ then $[\![\Gamma, v; \emptyset \vdash v]\!]^{\mathsf{Rel}} f = id_{\rho v}$*
- *If $\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G$, then $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}} f = id_{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho}$*
- *If $\Gamma;\Phi \vdash \mathbb{0}$ then $[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}} f = id_0$*
- *If $\Gamma;\Phi \vdash \mathbb{1}$ then $[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}} f = id_1$*
- *If $\Gamma;\Phi \vdash \phi\overline{\tau}$, then $[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}} f : [\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho' = (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho} \to (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho'}$ is defined by $[\![\Gamma;\Phi \vdash \phi\tau A]\!]^{\mathsf{Rel}} f = (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f} = (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f} \circ (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}}$*
- *If $\Gamma;\Phi \vdash \sigma{+}\tau$ then $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Rel}} f$ is defined by $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Rel}} f(\mathsf{inl}\, x) = \mathsf{inl}\,([\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}} f x)$ and $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Rel}} f(\mathsf{inr}\, y) = \mathsf{inr}\,([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f y)$*
- *If $\Gamma;\Phi \vdash \sigma\times\tau$ then $[\![\Gamma;\Phi \vdash \sigma\times\tau]\!]^{\mathsf{Rel}} f = [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}} f \times [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f$*
- *If $\Gamma;\Phi \vdash (\mu\phi^k.\lambda\overline{\alpha}.H)\overline{\tau}$ then $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}} f = (\mu T_{H,f})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho'} \circ (\mu T_{H,\rho})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f} = (\mu T_{H,\rho'})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f} \circ (\mu T_{H,f})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}$*

To see that the functors given by Definitions 16 and 17 are well-defined we must show that, for every H, $T_{H,\rho}\, F$ is a relation transformer for any relation transformer F, and that $T_{H,f}\, F : T_{H,\rho}\, F \to T_{H,\rho'}\, F$ is a morphism of relation transformers for every relation transformer F and every morphism $f : \rho \to \rho'$ in RelEnv. This is an immediate consequence of

Lemma 18. *For every $\Gamma;\Phi \vdash \tau$, $[\![\Gamma;\Phi \vdash \tau]\!] = ([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}) \in [\mathsf{RelEnv}, RT_0]$.*

The proof is a straightforward induction on the structure of τ, using an appropriate result from [Johann and Polonsky 2019] to deduce ω-cocontinuity of $[\![\Gamma;\Phi \vdash \tau]\!]$ in each case.

We can also prove by induction that our interpretations of types interact well with demotion of functorial variables and substitution. Indeed, we have

Lemma 19. *Let $\rho, \rho' : \mathsf{SetEnv}$ be such that $\rho\phi = \rho\psi = \rho'\phi = \rho'\psi$, and let $f : \rho \to \rho'$ be a morphism of set environments such that $f\phi = f\psi = id_{\rho\phi}$. If $\Gamma;\Phi,\phi^k \vdash F : \mathcal{F}$, $\Gamma;\Phi,\overline{\alpha} \vdash G$ $\Gamma;\Phi,\alpha_1...\alpha_k \vdash H$, and $\Gamma;\Phi \vdash \tau$, then*

$$[\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}\rho = [\![\Gamma,\psi;\Phi \vdash F[\phi :== \psi]]\!]^{\mathsf{Set}}\rho \tag{4}$$

$$[\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}} f = [\![\Gamma,\psi;\Phi \vdash F[\phi :== \psi]]\!]^{\mathsf{Set}} f \tag{5}$$

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}] \tag{6}$$

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi,\overline{\alpha} \vdash G]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f}] \tag{7}$$

$$[\![\Gamma;\Phi \vdash F[\phi := H]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}\rho[\phi := \lambda\overline{A}.\, [\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]] \tag{8}$$

$$[\![\Gamma;\Phi \vdash F[\phi := H]]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}} f[\phi := \lambda\overline{A}.\, [\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} f[\overline{\alpha := id_{A}}]] \tag{9}$$

Analogous identities hold for relational interpretations.

4 THE IDENTITY EXTENSION LEMMA

Rather than taking equality relations as given, e.g., by reflexive graphs, and using them to define graph relations, as is traditionally done, we give a categorical definition of graph relations that works for morphisms between functors (i.e., natural transformations) and derive equality relations as particular graph relations. FUNCTORIALITY IS ESSENTIAL TO THIS CONSTRUCTION. HOW MUCH OF THIS COULD BE DONE IN A FIBRATIONAL SETTING?

The standard definition of the *graph* for a morphism $f : A \to B$ in Set is the relation $\langle f \rangle : \mathsf{Rel}(A, B)$ defined by $(x, y) \in \langle f \rangle$ iff $fx = y$. This definition naturally generalizes to associate to each natural transformation between k-ary functors on Set a k-ary relation transformer as follows:

Definition 20. *If $F, G : \mathsf{Set}^k \to \mathsf{Set}$ and $\alpha : F \to G$ is a natural transformation, then the functor $\langle \alpha \rangle^* : \mathsf{Rel}^k \to \mathsf{Rel}$ is defined as follows. Given $R_1 : \mathsf{Rel}(A_1, B_1), \dots, R_k : \mathsf{Rel}(A_k, B_k)$, let $\iota_{R_i} : R_i \hookrightarrow A_i \times B_i$, for $i = 1, \dots, k$, be the inclusion of R_i as a subset of $A_i \times B_i$, let $h_{\overline{A\times B}}$ be the unique morphism making the diagram*

$$\begin{tikzcd}
F\overline{A} & F(\overline{A\times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, dashed, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
 & F\overline{A}\times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

commute, and let $h_{\overline{R}} : F\overline{R} \to F\overline{A} \times G\overline{B}$ be $h_{\overline{A\times B}} \circ F\overline{\iota_R}$. Further, let $\alpha^\wedge\overline{R}$ be the subobject through which $h_{\overline{R}}$ is factorized by the mono-epi factorization system in Set, *as shown in the following diagram:*

$$\begin{tikzcd}
F\overline{R} \arrow[rr, "h_{\overline{R}}"] \arrow[dr, two heads, "q_{\alpha^\wedge\overline{R}}"'] & & F\overline{A}\times G\overline{B} \\
 & \alpha^\wedge\overline{R} \arrow[ur, hook, "\iota_{\alpha^\wedge\overline{R}}"'] &
\end{tikzcd}$$

Then $\alpha^\wedge\overline{R} : \mathsf{Rel}(F\overline{A}, G\overline{B})$ by construction, so the action of $\langle\alpha\rangle^*$ on objects can be given by $\langle\alpha\rangle^*\overline{(A, B, R)} = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge\overline{R}}\alpha^\wedge\overline{R})$. Its action on morphisms is given by $\langle\alpha\rangle^*\overline{(\beta,\beta')} = (F\overline{\beta}, G\overline{\beta}')$.

The data in Definition 20 yield a relation transformer $\langle\alpha\rangle = (F, G, \langle\alpha\rangle^*)$, called the *graph relation transformer for* α.

Lemma 21. *If $F, G : [\mathsf{Set}^k, \mathsf{Set}]$, and if $\alpha : F \to G$ is a natural transformation, then $\langle\alpha\rangle$ is in RT_k.*

Proof. Clearly, $\langle\alpha\rangle^*$ is ω-cocontinuous, so $\langle\alpha\rangle^* : [\mathsf{Rel}^k, \mathsf{Rel}]$. Now, suppose $\overline{R : \mathsf{Rel}(A, B)}$, $\overline{S : \mathsf{Rel}(C, D)}$, and $\overline{(\beta, \beta') : R \to S}$. We want to show that there exists a morphism $\epsilon : \alpha^\wedge\overline{R} \to \alpha^\wedge\overline{S}$ such that

$$\begin{tikzcd}
\alpha^\wedge\overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{R}}"] \arrow[d, "\epsilon"'] & F\overline{A}\times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
\alpha^\wedge\overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{S}}"'] & F\overline{C}\times G\overline{D}
\end{tikzcd}$$

commutes. Since $\overline{(\beta, \beta') : R \to S}$, there exist $\overline{\gamma : R \to S}$ such that each diagram

$$\begin{tikzcd}
R_i \arrow[r, hook, "\iota_{R_i}"] \arrow[d, "\gamma_i"'] & A_i \times B_i \arrow[d, "\beta_i\times\beta_i'"] \\
S_i \arrow[r, hook, "\iota_{S_i}"] & C_i \times D_i
\end{tikzcd}$$

commutes. Moreover, since both $h_{\overline{C\times D}} \circ F(\overline{\beta\times\beta'})$ and $(F\overline{\beta}\times G\overline{\beta'}) \circ h_{\overline{A\times B}}$ make

$$\begin{tikzcd}
F\overline{C} & F\overline{C}\times F\overline{D} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F\overline{D} \arrow[r, "\alpha_{\overline{D}}"] & G\overline{D} \\
 & F(\overline{A\times B}) \arrow[ul, "F\overline{\pi_1}\circ F(\overline{\beta\times\beta'})"] \arrow[u, dashed, "\exists!"] \arrow[urr, "\alpha_{\overline{D}}\circ F\overline{\pi_2}\circ F(\overline{\beta\times\beta'})"'] & &
\end{tikzcd}$$

commute, they must be equal. We therefore get that the right-hand square below commutes, and thus that the entire following diagram does as well:

$$\begin{tikzcd}
F\overline{R} \arrow[r, hook, "F\iota_{\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] \arrow[rr, bend left, "h_{\overline{R}}"] & F(\overline{A\times B}) \arrow[r, "h_{\overline{A\times B}}"] \arrow[d, "F(\overline{\beta\times\beta'})"] & F\overline{A}\times G\overline{B} \arrow[d, "F\overline{\beta}\times F\overline{\beta'}"] \\
F\overline{S} \arrow[r, hook, "F\iota_{\overline{S}}"'] \arrow[rr, bend right, "h_{\overline{S}}"'] & F(\overline{C\times D}) \arrow[r, "h_{\overline{C\times D}}"'] & F\overline{C}\times G\overline{D}
\end{tikzcd}$$

Finally, by the left-lifting property of $q_{F^\wedge\overline{R}}$ with respect to $\iota_{F^\wedge\overline{S}}$ given by the epi-mono factorization system, there exists an ϵ such that the following diagram commutes:

$$\begin{tikzcd}
F\overline{R} \arrow[r, two heads, "q_{\alpha^\wedge\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] & \alpha^\wedge\overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{R}}"] \arrow[d, dashed, "\epsilon"] & F\overline{A}\times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
F\overline{S} \arrow[r, two heads, "q_{\alpha^\wedge\overline{S}}"'] & \alpha^\wedge\overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{S}}"'] & F\overline{C}\times G\overline{D}
\end{tikzcd}$$

□

It is not hard to see that if $f : A \to B$ is a morphism in Set then the standard definition of its graph $\langle f\rangle$ in Rel coincides with the definition the graph relation transformer for f as a natural transformation between the 0-ary functors A and B. Graph relation transformers are thus a reasonable generalization of graph relations in Rel.

To prove the IEL, we will need to know that the equality relation transformer preserves equality relations; see Equation 10 below. This will follow from the next lemma, which shows how to compute the action of a graph relation transformer on any graph relation.

Lemma 22. *If $\alpha : F \to G$ is a morphism in $[\mathsf{Set}^k, \mathsf{Set}]$ and $f_1 : A_1 \to B_1, \ldots, f_k : A_k \to B_k$, then $\langle\alpha\rangle^*\langle\overline{f}\rangle = \langle G\overline{f}\circ\alpha_{\overline{A}}\rangle = \langle\alpha_{\overline{B}}\circ F\overline{f}\rangle$.*

Proof. Since $h_{\overline{A\times B}}$ is the unique morphism making the bottom triangle of this diagram commute

$$\begin{tikzcd}
& F\overline{A} \arrow[dl, equals] \arrow[d, "F\overline{\langle id_A, f\rangle}"] \arrow[dr, "F\overline{f}"] & & \\
F\overline{A} & F(\overline{A\times B}) \arrow[l, "F\pi_1"'] \arrow[r, "F\pi_2"] \arrow[d, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
& F\overline{A}\times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

and since $h_{\langle\overline{f}\rangle} = h_{\overline{A\times B}} \circ F\,\overline{\iota_{\langle f\rangle}} = h_{\overline{A\times B}} \circ F\overline{\langle id_A, f\rangle}$, the universal property of the product

$$\begin{tikzcd}
F\overline{A} & F\overline{A}\times G\overline{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & G\overline{B} \\
& F\overline{A} \arrow[ul, equals] \arrow[u, dashed, "\exists!"] \arrow[r, "F\overline{f}"'] & F\overline{B} \arrow[u, "\alpha_{\overline{B}}"']
\end{tikzcd}$$

gives $h_{\langle\overline{f}\rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle : F\overline{A} \to F\overline{A} \times G\overline{B}$. Moreover, $\langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle$ is a monomorphism in Set because $id_{F\overline{A}}$ is, so its epi-mono factorization gives $\iota_{\alpha^\wedge\langle\overline{f}\rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle$, and thus $\alpha^\wedge\langle\overline{f}\rangle = F\overline{A}$. Then $\iota_{\alpha^\wedge\langle\overline{f}\rangle}\alpha^\wedge\langle\overline{f}\rangle = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle(F\overline{A}) = \langle \alpha_{\overline{B}} \circ F\overline{f}\rangle^*$, so that $\langle\alpha\rangle^*\langle\overline{f}\rangle = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge\langle\overline{f}\rangle}\,\alpha^\wedge\langle\overline{f}\rangle) = (F\overline{A}, G\overline{B}, \langle \alpha_{\overline{B}} \circ F\overline{f}\rangle^*) = \langle \alpha_{\overline{B}} \circ F\overline{f}\rangle$. Finally, $\alpha_{\overline{B}} \circ F\overline{f} = G\overline{f} \circ \alpha_{\overline{A}}$ by naturality of α. □

The *equality relation transformer* on $F : [\mathsf{Set}^k, \mathsf{Set}]$ is defined to be $\mathsf{Eq}_F = \langle id_F\rangle$. Specifically, $Eq_F = (F, F, \mathsf{Eq}^*_F)$ with $\mathsf{Eq}^*_F = \langle id_F\rangle^*$, and Lemma 22 indeed ensures that

$$\mathsf{Eq}^*_F\overline{\mathsf{Eq}_A} = \langle id_F\rangle^*\langle id_{\overline{A}}\rangle = \langle F id_{\overline{A}} \circ (id_F)_{\overline{A}}\rangle = \langle id_{F\overline{A}} \circ id_{F\overline{A}}\rangle = \langle id_{F\overline{A}}\rangle = \mathsf{Eq}_{F\overline{A}} \tag{10}$$

for all $\overline{A : \mathsf{Set}}$. Graph relation transformers in general, and equality relation transformers in particular, extend to relation environments in the obvious ways. Indeed, if $\rho, \rho' : \mathsf{SetEnv}$ and $f : \rho \to \rho'$, then the *graph relation environment* $\langle f\rangle$ is defined pointwise by $\langle f\rangle\phi = \langle f\phi\rangle$ for every ϕ. This entails that $\pi_1\langle f\rangle = \rho$ and $\pi_2\langle f\rangle = \rho'$. In particular, the *equality relation environment* Eq_ρ is defined to be $\langle id_\rho\rangle$. This entails that $\mathsf{Eq}_\rho\phi = \mathsf{Eq}_{\rho\phi}$ for every ϕ. With these definitions in hand, we can state and prove both an Identity Extension Lemma and a Graph Lemma for our calculus.

Theorem 23 (IEL). *If $\rho : \mathsf{SetEnv}$ and $\Gamma; \Phi \vdash \tau : \mathcal{F}$ then $[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}$.*

The proof is by induction on the structure of τ. Only the application and fixpoint cases are non-routine. Both use Lemma 22. The latter also uses the observation that, for every $n \in \mathbb{N}$, the following intermediate results can be proved by simultaneous induction with Theorem 23: $T^n_{\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} = (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0})^*\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}$ and $[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]} = [\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}$.

Lemma 24 (Graph Lemma). *If $\rho, \rho' : \mathsf{SetEnv}$, $f : \rho \to \rho'$, and $\Gamma; \Phi \vdash F : \mathcal{F}$, then $\langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\rangle = [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$.*

Proof. Applying Lemma 18 to the morphisms $(f, id_{\rho'}) : \langle f\rangle \to \mathsf{Eq}_{\rho'}$ and $(id_\rho, f) : \mathsf{Eq}_\rho \to \langle f\rangle$ of relation environments gives that $([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}) = [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}(f, id_{\rho'}) : [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle \to [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}$ and $([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_\rho, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f) = [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}(id_\rho, f) : [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho \to [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$. Expanding the first equation gives that if $(x, y) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$ then $([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\,x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}\,y) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}$. Then $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}\,y = id_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}\,y = y$ and $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}$, so if $(x, y) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$ then $([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\,x, y) \in \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}$, i.e., $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\,x = y$, i.e., $(x, y) \in \langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$. So, we have that $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle \subseteq \langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$. Expanding the second equation gives that if $x \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}}\rho$ then $([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_\rho\,x, [\![\Gamma; \Phi \vdash$ $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$. Then $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_\rho\,x = id_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}x = x$, so for any $x \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}}\rho$ we have that $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\,x) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$. Moreover, $x \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}}\rho$ if and only if $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\,x) \in \langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$ and, if $x \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}}\rho$ then $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\,x) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$, so if $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\,x) \in \langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\rangle$ then $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\,x) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$, i.e., $\langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\rangle \subseteq [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f\rangle$.

□

5 INTERPRETING TERMS

Here, we are using angle bracket notation for both the graph relation of a function and for the pairing of functions with the same domain. This is justified by the relationship between the two notions observed immediately after Lemma 21.

If $\Delta = x_1 : \tau_1, ..., x_n : \tau_n$ is a term context for Γ and Φ, then the interpretations $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}}$ and $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}$ are defined by

$$
\begin{array}{lcl}
\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}} & = & \llbracket \Gamma; \Phi \vdash \tau_1 \rrbracket^{\mathsf{Set}} \times ... \times \llbracket \Gamma; \Phi \vdash \tau_n \rrbracket^{\mathsf{Set}} \\
\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}} & = & \llbracket \Gamma; \Phi \vdash \tau_1 \rrbracket^{\mathsf{Rel}} \times ... \times \llbracket \Gamma; \Phi \vdash \tau_n \rrbracket^{\mathsf{Rel}}
\end{array}
$$

Every well-formed term $\Gamma; \Phi \mid \Delta \vdash t : \tau$ then has, for every $\rho \in \mathsf{SetEnv}$, set interpretations $\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}} \rho$ as natural transformations from $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}} \rho$ to $\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho$, and, for every $\rho \in \mathsf{RelEnv}$, relational interpretations $\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}} \rho$ as natural transformations from $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}} \rho$ to $\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \rho$. These are given in the next (two) definitions.

Definition 25. *If ρ is a set (resp., relation) environment and $\Gamma; \Phi \mid \Delta \vdash t : \tau$ then $\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}} \rho$ (resp., $\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}} \rho$) is defined as follows, where* D *is either* Set *or* Rel *as appropriate:*

$$
\begin{array}{lcl}
\llbracket \Gamma; \emptyset \mid \Delta, x : \tau \vdash x : \tau \rrbracket^{\mathsf{D}} \rho & = & \pi_{|\Delta|+1} \\
\llbracket \Gamma; \emptyset \mid \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{D}} \rho & = & \mathsf{curry}(\llbracket \Gamma; \overline{\alpha} \mid \Delta, x : F \vdash t : G \rrbracket^{\mathsf{D}} \rho[\overline{\alpha := \_}]) \\
\llbracket \Gamma; \Phi \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{D}} \rho & = & \mathsf{eval} \circ \langle \lambda d. (\llbracket \Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{D}} \rho\; d)_{\overline{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{D}}}}, \\
& & \qquad \llbracket \Gamma; \Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{D}} \rho \rangle \\
\\
\llbracket \Gamma; \Phi \mid \Delta, x : \tau \vdash x : \tau \rrbracket^{\mathsf{D}} \rho & = & \pi_{|\Delta|+1} \\
\llbracket \Gamma; \Phi \mid \Delta \vdash \bot_{\tau} t : \tau \rrbracket^{\mathsf{D}} \rho & = & !^{0}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{D}} \rho} \circ \llbracket \Gamma; \Phi \mid \Delta \vdash t : \mathbb{0} \rrbracket^{\mathsf{D}} \rho, \; \textit{where} \\
& & \quad !^{0}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{D}} \rho} \; \textit{is the unique morphism from } 0 \\
& & \quad \textit{to } \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{D}} \rho \\
\llbracket \Gamma; \Phi \mid \Delta \vdash \top : \mathbb{1} \rrbracket^{\mathsf{D}} \rho & = & !^{\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{D}} \rho}_{1}, \; \textit{where} \; !^{\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{D}} \rho}_{1} \\
& & \quad \textit{is the unique morphism from } \llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{D}} \rho \textit{ to} \\
\llbracket \Gamma; \Phi \mid \Delta \vdash (s, t) : \sigma \times \tau \rrbracket^{\mathsf{D}} \rho & = & \llbracket \Gamma; \Phi \mid \Delta \vdash s : \sigma \rrbracket^{\mathsf{D}} \rho \times \llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{D}} \rho \\
\llbracket \Gamma; \Phi \mid \Delta \vdash \pi_1 t : \sigma \rrbracket^{\mathsf{D}} \rho & = & \pi_1 \circ \llbracket \Gamma; \Phi \mid \Delta \vdash t : \sigma \times \tau \rrbracket^{\mathsf{D}} \rho \\
\llbracket \Gamma; \Phi \mid \Delta \vdash \pi_2 t : \sigma \rrbracket^{\mathsf{D}} \rho & = & \pi_2 \circ \llbracket \Gamma; \Phi \mid \Delta \vdash t : \sigma \times \tau \rrbracket^{\mathsf{D}} \rho \\
\llbracket \Gamma; \Phi \mid \Delta \vdash \mathsf{case}\; t\; \mathsf{of}\; \{x \mapsto l;\, y \mapsto r\} : \gamma \rrbracket^{\mathsf{D}} \rho & = & \mathsf{eval} \circ \langle \mathsf{curry}\, [\llbracket \Gamma; \Phi \mid \Delta, x : \sigma \vdash l : \gamma \rrbracket^{\mathsf{D}} \rho, \\
& & \qquad \llbracket \Gamma; \Phi \mid \Delta, y : \tau \vdash r : \gamma \rrbracket^{\mathsf{D}} \rho], \\
& & \qquad \llbracket \Gamma; \Phi \mid \Delta \vdash t : \sigma + \tau \rrbracket^{\mathsf{D}} \rho \rangle \\
\llbracket \Gamma; \Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau \rrbracket^{\mathsf{D}} \rho & = & \mathsf{inl} \circ \llbracket \Gamma; \Phi \mid \Delta \vdash s : \sigma \rrbracket^{\mathsf{D}} \rho \\
\llbracket \Gamma; \Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau \rrbracket^{\mathsf{D}} \rho & = & \mathsf{inr} \circ \llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{D}} \rho \\
\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{F}, \overline{G}}_{H} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta}, \overline{\gamma}}\, F\, G}) & = & \lambda d\, \overline{\eta}\, \overline{B}.\, \llbracket \Gamma; \overline{\phi}, \overline{\gamma} \vdash H \rrbracket^{\mathsf{D}} id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda \overline{A}. \eta_{\overline{A}\, \overline{B}}}] \\
\qquad (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]) \rrbracket^{\mathsf{Set}} \rho & & \\
\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta}, \overline{\gamma}}\, H[\phi := (\mu \phi. \lambda \overline{\alpha}. H) \overline{\beta}][\overline{\alpha := \beta}] & = & \lambda d\, \overline{B}\, \overline{C}.\, (in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}})_{\overline{B}} \\
\qquad (\mu \phi. \lambda \overline{\alpha}. H) \overline{\beta} \rrbracket^{\mathsf{Set}} \rho & & \\
\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{fold}^{F}_{H} : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta}, \overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F) & = & \lambda d\, \eta\, \overline{B}\, \overline{C}.\, (fold_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}\, (\lambda \overline{A}.\, \eta_{\overline{A}\, \overline{C}}))_{\overline{B}} \\
\qquad (\mathsf{Nat}^{\overline{\beta}, \overline{\gamma}}\, (\mu \phi. \lambda \overline{\alpha}. H) \overline{\beta}\; F) \rrbracket^{\mathsf{Set}} \rho & &
\end{array}
$$

The return type for the semantic fold in the last clause is $\llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{D}} \rho[\overline{\gamma := C}]$. This interpretation gives that $\llbracket \Gamma; \emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau \rrbracket^{\mathsf{D}} \rho = \mathsf{curry}(\llbracket \Gamma; \emptyset \mid \Delta, x : \sigma \vdash t : \tau \rrbracket^{\mathsf{D}} \rho)$ and $\llbracket \Gamma; \emptyset \mid \Delta \vdash st : \tau \rrbracket^{\mathsf{D}} \rho = \mathsf{eval} \circ \langle \llbracket \Gamma; \emptyset \mid \Delta \vdash s : \sigma \to \tau \rrbracket^{\mathsf{D}} \rho, \llbracket \Gamma; \emptyset \mid \Delta \vdash t : \sigma \rrbracket^{\mathsf{D}} \rho \rangle$, so it specializes to the standard interpretations for System F terms. If t is closed, i.e., if $\emptyset; \emptyset \mid \emptyset \vdash t : \tau$, then we write $\llbracket \vdash t : \tau \rrbracket^{\mathsf{D}}$ instead of $\llbracket \emptyset; \emptyset \mid \emptyset \vdash t : \tau \rrbracket^{\mathsf{D}}$.

5.1 Basic Properties of Term Interpretations

The interpretations in Definition 25 respect weakening, i.e., a term and its weakenings all have the same set and relational interpretations. In particular, for any $\rho \in \mathsf{SetEnv}$,

$$[\![\Gamma;\Phi \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho = ([\![\Gamma;\Phi \,|\, \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho) \circ \pi_\Delta$$

where π_Δ is the projection $[\![\Gamma;\Phi \vdash \Delta, x : \sigma]\!]^{\mathsf{Set}} \to [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}$, and similarly for relational interpretations. Moreover, if $\Gamma,\alpha;\Phi \,|\, \Delta \vdash t : \tau$ and $\Gamma;\Phi,\alpha \,|\, \Delta \vdash t' : \tau$ and $\Gamma;\Phi \vdash \sigma : \mathcal{F}$ then

- $[\![\Gamma;\Phi \,|\, \Delta[\alpha := \sigma] \vdash t[\alpha := \sigma] : \tau[\alpha := \sigma]]\!]^{\mathsf{Set}}\rho = [\![\Gamma,\alpha;\Phi \,|\, \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho]$
- $[\![\Gamma;\Phi \,|\, \Delta[\alpha := \sigma] \vdash t'[\alpha := \sigma] : \tau[\alpha := \sigma]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\alpha \,|\, \Delta \vdash t' : \tau]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho]$

and if $\Gamma;\Phi \,|\, \Delta, x : \sigma \vdash t : \tau$ and $\Gamma;\Phi \,|\, \Delta \vdash s : \sigma$ then

- $\lambda A.\, [\![\Gamma;\Phi \,|\, \Delta \vdash t[x := s] : \tau]\!]^{\mathsf{Set}}\rho\, A = \lambda A.\, [\![\Gamma;\Phi \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho\, (A, [\![\Gamma;\Phi \,|\, \Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho\, A)$

Direct calculation reveals that the set interpretations of terms also satisfy

- $[\![\Gamma;\Phi \mid \Delta \vdash (L_{\overline{\alpha}}x.t)_{\overline{\tau}}s]\!]^{\mathsf{Set}} = [\![\Gamma;\Phi \mid \Delta \vdash t[\overline{\alpha := \tau}][x := s]]\!]^{\mathsf{Set}}$

Term extensionality with respect to types and terms — i.e., $[\![\Gamma;\Phi \vdash (L_\alpha x.t)_\alpha \top]\!]^{\mathsf{Set}} = [\![\Gamma;\Phi \vdash t]\!]^{\mathsf{Set}}$ and $[\![\Gamma;\Phi \vdash (L_\alpha x.t)_\alpha x]\!]^{\mathsf{Set}} = [\![\Gamma;\Phi \vdash t]\!]^{\mathsf{Set}}$ — follow. Similar properties hold for relational interpretations.

5.2 Properties of Terms of Nat-Type

Define, for $\Gamma;\overline{\alpha} \vdash F$, the term id_F to be $\Gamma;\emptyset \,|\, \emptyset \vdash L_{\overline{\alpha}}x.x : \mathsf{Nat}^{\overline{\alpha}} F\, F$ and, for terms $\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\, G$ and $\Gamma;\emptyset \,|\, \Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\, H$, the *composition* $s \circ t$ of t and s to be $\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.s_{\overline{\alpha}}(t_{\overline{\alpha}}x) : \mathsf{Nat}^{\overline{\alpha}} F\, H$. Then

$$\begin{array}{l} [\![\Gamma;\emptyset \,|\, \emptyset \vdash id_F : \mathsf{Nat}^{\overline{\alpha}} F\, F]\!]^{\mathsf{Set}}\rho\, * = id_{\lambda\overline{A}.[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]} \quad \text{for any set environment } \rho \\ [\![\Gamma;\emptyset \,|\, \Delta \vdash s \circ t : \mathsf{Nat}^{\overline{\alpha}} F\, H]\!]^{\mathsf{Set}} = [\![\Gamma;\emptyset \,|\, \Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\, H]\!]^{\mathsf{Set}} \circ [\![\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\, G]\!]^{\mathsf{Set}} \end{array}$$

Moreover, terms of Nat type behave as natural transformations with respect to their source and target functorial types, i.e.,

$$\begin{array}{l} [\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) \circ (L_{\overline{\gamma}}z.x_{\overline{\sigma},\overline{\gamma}}z) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \\ = [\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash (L_{\overline{\gamma}}z.x_{\overline{\tau},\overline{\gamma}}z) \circ ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \end{array}$$

In particular, when $x = in_H$ the above equality specializes to

Theorem 26. *If* $\xi = \mathsf{Nat}^{\overline{\gamma}} H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \sigma}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$ *then*

$$\begin{array}{l} [\![\Gamma;\emptyset \,|\, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash ((\mathsf{map}^{\overline{\sigma},\overline{\tau}}_{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \overline{y}) \circ (L_{\overline{\gamma}}z.(\mathsf{in}_H)_{\overline{\sigma},\overline{\gamma}}z) : \xi]\!]^{\mathsf{Set}} \\ = [\![\Gamma;\emptyset \,|\, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash (L_{\overline{\gamma}}z.(\mathsf{in}_H)_{\overline{\tau},\overline{\gamma}}z) \circ ((\mathsf{map}^{\overline{\sigma},\overline{\tau}}_{H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset \overline{y}) : \xi]\!]^{\mathsf{Set}} \end{array}$$

Analogous results hold for relational interpretations of terms and relational environments.

As we observe in Section 5.4, Theorem 26 gives a family of free theorems that are consequences of naturality, and thus do not require the full power of parametricity. Most, but not all, of the free theorems derived in [Walder 1989] are in this family.

5.3 Properties of Initial Algebraic Constructs

We first observe that map-terms are interpreted as semantic *map*s: Contrast with Bob's $fmaps_F$s, which are not required to be the functorial action of F, only to preserve identities and compositions.

Let $\Gamma; \overline{\phi}, \overline{\gamma} \vdash H : \mathcal{F}$, $\overline{\Gamma; \overline{\beta}, \overline{\gamma} \vdash F : \mathcal{F}}$ and $\overline{\Gamma; \overline{\beta}, \overline{\gamma} \vdash G : \mathcal{F}}$. By definition of the semantic interpretation of map terms, we have

$$\begin{aligned}
&[\![\Gamma; \emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\rho \\
&\qquad\qquad = \lambda d\, \overline{\eta}\, \overline{B}.\, [\![\Gamma; \overline{\phi}, \overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda \overline{A}.\eta_{\overline{A}\,\overline{B}}}] \qquad (11)
\end{aligned}$$

Then let $\Gamma; \overline{\alpha} \vdash F : \mathcal{F}$, $\overline{\Gamma; \emptyset \vdash \sigma : \mathcal{F}}$, $\overline{\Gamma; \emptyset \vdash \tau : \mathcal{F}}$ and $*$ be the unique element of $[\![\Gamma; \emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$. As a special case of the above definition, we have

$$\begin{aligned}
& [\![\Gamma; \emptyset \,|\, \emptyset \vdash \mathsf{map}_F^{\overline{\sigma},\overline{\tau}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\emptyset}\, \sigma\, \tau})\, (\mathsf{Nat}^{\emptyset}\, F[\overline{\alpha := \sigma}]\, F[\overline{\alpha := \tau}])]\!]^{\mathsf{Set}}\rho\, * \\
= \;& \lambda \overline{f : [\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho \to [\![\Gamma; \emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho}.\, [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}} id_{\rho}[\overline{\alpha := f}] \\
= \;& \lambda \overline{f : [\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho \to [\![\Gamma; \emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho}.\, map_{\lambda \overline{A}.\, [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]}\, \overline{f} \\
= \;& map_{\lambda \overline{A}.\, [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]}
\end{aligned}$$

where the first equality is by Equation 11, the second equality is obtained by noting that $\lambda \overline{A}.\, [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]$ is a functor in α, and map_G denotes the action of the functor G on morphisms.

We also have the expected relationships between interpetations of terms involving map, in, and fold:

- If $\Gamma; \overline{\psi}, \overline{\gamma} \vdash H$, $\overline{\Gamma; \overline{\alpha}, \overline{\gamma}, \overline{\phi} \vdash K}$, $\overline{\Gamma; \overline{\beta}, \overline{\gamma} \vdash F}$, and $\overline{\Gamma; \overline{\beta}, \overline{\gamma} \vdash G}$, then
$$[\![\Gamma; \emptyset \,|\, \emptyset \vdash \mathsf{map}_{H[\overline{\psi := K}]}^{\overline{F},\overline{G}} : \xi]\!]^{\mathsf{Set}} = [\![\Gamma; \emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{K[\overline{\phi := F}]}, \overline{K[\overline{\phi := G}]}} \circ \overline{\mathsf{map}_K^{\overline{F},\overline{G}}} : \xi]\!]^{\mathsf{Set}}$$
at type $\xi = \mathsf{Nat}^{\emptyset}(\overline{\mathsf{Nat}^{\overline{\alpha},\overline{\beta},\overline{\gamma}} F G})(\mathsf{Nat}^{\overline{\gamma}} H[\overline{\psi := K}][\overline{\phi := F}] H[\overline{\psi := K}][\overline{\phi := G}])$
- If $\Gamma; \overline{\beta}, \overline{\gamma} \vdash H$, $\Gamma; \overline{\beta}, \overline{\gamma} \vdash K$, $\overline{\Gamma; \overline{\alpha}, \overline{\gamma} \vdash F}$, $\overline{\Gamma; \overline{\alpha}, \overline{\gamma} \vdash G}$, $\overline{\Gamma; \phi, \overline{\psi}, \overline{\gamma} \vdash \tau}$, $\overline{I}$ is the sequence $\overline{F}, H$ and $\overline{J}$ is the sequence $\overline{G}, K$ then

$$[\![\Gamma; \emptyset \,|\, \emptyset \vdash L_{\emptyset}(x, \overline{y}).L_{\overline{\gamma}} z.x_{\overline{\tau[\overline{\psi := G}][\phi := K]}, \overline{\gamma}}\Big(((\mathsf{map}_H^{\overline{\tau[\overline{\psi := F}][\phi := H]}, \overline{\tau[\overline{\psi := G}][\phi := K]}})_{\emptyset}(\overline{(\mathsf{map}_{\tau}^{\overline{I},\overline{J}})_{\emptyset}(x, \overline{y})}))_{\overline{\gamma}} z\Big) : \xi]\!]^{\mathsf{Set}}$$

$$= [\![\Gamma; \emptyset \,|\, \emptyset \vdash L_{\emptyset}(x, \overline{y}).L_{\overline{\gamma}} z.\big((\mathsf{map}_K^{\overline{\tau[\overline{\psi := F}][\phi := H]}, \overline{\tau[\overline{\psi := G}][\phi := K]}})_{\emptyset}(\overline{(\mathsf{map}_{\tau}^{\overline{I},\overline{J}})_{\emptyset}(x, \overline{y})})\big)_{\overline{\gamma}}\Big(x_{\overline{\tau[\overline{\psi := F}][\phi := H]}, \overline{\gamma}} z\Big) : \xi]\!]^{\mathsf{Set}}$$

  at type $\xi = \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H\, K \times \overline{\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G})\, (\mathsf{Nat}^{\overline{\gamma}} H[\overline{\beta := \tau}][\overline{\psi := F}][\phi := H]\, K[\overline{\beta := \tau}][\overline{\psi := G}][\phi := K])$.
- $[\![\Gamma; \emptyset \,|\, x : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := F][\overline{\alpha := \beta}]\, F \vdash ((\mathsf{fold}_{H,F})_{\emptyset} x) \circ \mathsf{in}_H : \xi]\!]^{\mathsf{Set}}$
  $= [\![\Gamma; \emptyset \,|\, x : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := F][\overline{\alpha := \beta}]\, F \vdash x \circ \big((\mathsf{map}_{H[\overline{\alpha := \beta}]}^{(\mu\phi.\lambda\overline{\alpha}H)\overline{\beta}, F})_{\emptyset}((\mathsf{fold}_{H,F})_{\emptyset} x)\big) : \xi]\!]^{\mathsf{Set}}$
  at type $\xi = \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, F$
- $[\![\Gamma; \emptyset \,|\, \emptyset \vdash \mathsf{in}_H \circ (\mathsf{fold}_{H, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_{\emptyset}((\mathsf{map}_H^{H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}], (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_{\emptyset} \mathsf{in}_H) : \xi]\!]^{\mathsf{Set}}$
  $= [\![\Gamma; \emptyset \,|\, \emptyset \vdash Id_{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}} : \xi]\!]^{\mathsf{Set}}$
  at type $\xi = \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}$
- $[\![\Gamma; \emptyset \,|\, \emptyset \vdash (\mathsf{fold}_{H, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_{\emptyset}((\mathsf{map}_H^{H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}], (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_{\emptyset} \mathsf{in}_H) \circ \mathsf{in}_H : \xi]\!]^{\mathsf{Set}}$
  $= [\![\Gamma; \emptyset \,|\, \emptyset \vdash Id_{H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]} : \xi]\!]^{\mathsf{Set}}$
  at type $\xi = \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]$.

Analogous results hold for relational interpretations of terms and relational environments. The set and relational interpretations of terms therefore respect the congruence closed equational theory obtained by adding these judgments to those generating the usual congruence closed equational theory induced by the other term formers.

5.4 Free Theorems Derived from Naturality

Foralls in Nat-types are at the object level, whereas the foralls in contexts are at the meta-level. So par results in subst theorem internalize parametricity in the calculus, whereas those parametricity results that do not follow from the interpretation of Nat-types are externalized at the meta-level.

Make this not about *subst* Note that the free theorem for a type is always independent of the particular term of that type, so the proof below is independent of the choice of function *subst*. In addition, it is independent of the particular data type — in this case, Lam — over which *subst* acts. Also independent of the functor arguments — in this case $+\mathbb{1}$ and *id* — to the data type. Indeed, the following result is just a consequence of naturality.

We already know from Theorem 26 that

$$\begin{aligned}&[\![\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G,\overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y})\circ(L_{\overline{\gamma}}z.x_{\overline{\sigma},\overline{\gamma}}z):\mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}\\&=[\![\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G,\overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash(L_{\overline{\gamma}}z.x_{\overline{\tau},\overline{\gamma}}z)\circ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y}):\mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}\end{aligned}\tag{12}$$

In particular, if we instantiate x with any term *subst* of type $\vdash \mathsf{Nat}^{\alpha}(\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha)\,\mathsf{Lam}\,\alpha$ (and thus there is a single α and no γ's) we have

$$\begin{aligned}&[\![\Gamma;\emptyset\,|\,y:\mathsf{Nat}^{\emptyset}\sigma\,\tau\vdash((\mathsf{map}_{\mathsf{Lam}\,\alpha}^{\sigma,\tau})_{\emptyset}y)\circ(L_{\emptyset}z.subst_{\sigma}z):\mathsf{Nat}^{\emptyset}(\mathsf{Lam}\,(\sigma+\mathbb{1})\times\mathsf{Lam}\,\sigma)\,\mathsf{Lam}\,\tau]\!]^{\mathsf{Set}}\\&=[\![\Gamma;\emptyset\,|\,y:\mathsf{Nat}^{\emptyset}\sigma\,\tau\vdash(L_{\emptyset}z.subst_{\tau}z)\circ((\mathsf{map}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha}^{\sigma,\tau})_{\emptyset}y):\mathsf{Nat}^{\emptyset}(\mathsf{Lam}\,(\sigma+\mathbb{1})\times\mathsf{Lam}\,\sigma)\,\mathsf{Lam}\,\tau]\!]^{\mathsf{Set}}\end{aligned}\tag{13}$$

Thus, for any set environment ρ and any function $f:[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\emptyset}\sigma\,\tau]\!]^{\mathsf{Set}}\rho$, we have that

$$\begin{aligned}&[\![\Gamma;\emptyset\,|\,y:\mathsf{Nat}^{\emptyset}\sigma\,\tau\vdash((\mathsf{map}_{\mathsf{Lam}\,\alpha}^{\sigma,\tau})_{\emptyset}y)\circ(L_{\emptyset}z.subst_{\sigma}z):\mathsf{Nat}^{\emptyset}(\mathsf{Lam}\,(\sigma+\mathbb{1})\times\mathsf{Lam}\,\sigma)\,\mathsf{Lam}\,\tau]\!]^{\mathsf{Set}}\rho f\\&=[\![\Gamma;\emptyset\,|\,y:\mathsf{Nat}^{\emptyset}\sigma\,\tau\vdash((\mathsf{map}_{\mathsf{Lam}\,\alpha}^{\sigma,\tau})_{\emptyset}y)]\!]^{\mathsf{Set}}\rho f\circ[\![\Gamma;\emptyset\,|\,y:\mathsf{Nat}^{\emptyset}\sigma\,\tau\vdash L_{\emptyset}z.subst_{\sigma}z]\!]^{\mathsf{Set}}\rho f\\&=[\![\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{map}_{\mathsf{Lam}\,\alpha}^{\overline{\sigma},\overline{\tau}}]\!]^{\mathsf{Set}}\rho f\circ[\![\Gamma;\emptyset\,|\,\emptyset\vdash L_{\emptyset}z.subst_{\sigma}z]\!]^{\mathsf{Set}}\rho\\&=\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]}f\circ([\![\vdash subst]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset\vdash\sigma]\!]^{\mathsf{Set}}\rho}\end{aligned}\tag{14}$$

and

$$\begin{aligned}&[\![\Gamma;\emptyset\,|\,y:\mathsf{Nat}^{\emptyset}\sigma\,\tau\vdash(L_{\emptyset}z.subst_{\tau}z)\circ((\mathsf{map}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha}^{\sigma,\tau})_{\emptyset}y):\mathsf{Nat}^{\emptyset}(\mathsf{Lam}\,(\sigma+\mathbb{1})\times\mathsf{Lam}\,\sigma)\,\mathsf{Lam}\,\tau]\!]^{\mathsf{Set}}\rho f\\&=[\![\Gamma;\emptyset\,|\,y:\mathsf{Nat}^{\emptyset}\sigma\,\tau\vdash L_{\emptyset}z.subst_{\tau}z]\!]^{\mathsf{Set}}\rho f\circ[\![\Gamma;\emptyset\,|\,y:\mathsf{Nat}^{\emptyset}\sigma\,\tau\vdash(\mathsf{map}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha}^{\sigma,\tau})_{\emptyset}y]\!]^{\mathsf{Set}}\rho f\\&=[\![\Gamma;\emptyset\,|\,\emptyset\vdash L_{\emptyset}z.subst_{\tau}z]\!]^{\mathsf{Set}}\rho\circ[\![\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{map}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha}^{\sigma,\tau}]\!]^{\mathsf{Set}}\rho f\\&=([\![\vdash subst]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset\vdash\tau]\!]^{\mathsf{Set}}\rho}\circ\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,(\alpha+\mathbb{1})\times\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]}f\\&=([\![\vdash subst]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset\vdash\tau]\!]^{\mathsf{Set}}\rho}\circ(\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]}(f+\mathbb{1})\times\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]}f)\end{aligned}\tag{15}$$

So, we can conclude that

$$\begin{aligned}&\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]}f\circ([\![\vdash subst]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset\vdash\sigma]\!]^{\mathsf{Set}}\rho}\\&=([\![\vdash subst]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset\vdash\tau]\!]^{\mathsf{Set}}\rho}\circ(\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]}(f+\mathbb{1})\times\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]}f)\end{aligned}\tag{16}$$

Moreover, for any A, B : Set, we can choose $\sigma = v$ and $\tau = w$ to be variables such that $\rho v = A$ and $\rho w = B$. Then for any function $f : A \to B$ we have that

$$\begin{aligned}&\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]} f \circ ([\![\vdash \mathit{subst}]\!]^{\mathsf{Set}})_A \\ &\qquad = ([\![\vdash \mathit{subst}]\!]^{\mathsf{Set}})_B \circ (\mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]}(f + \mathbb{1}) \times \mathsf{map}_{[\![\emptyset;\alpha\vdash\mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha:=\_]} f) \qquad (17)\end{aligned}$$

5.5 The Abstraction Theorem

To go beyond naturality and get *all* consequences of parametricity, we prove an Abstraction Theorem for our calculus. In fact, we actually prove a more general result in Theorem 27 about possibly open terms. We then recover the Abstraction Theorem as the special case of Theorem 27 for closed terms of closed type.

THEOREM 27. *Every well-formed term $\Gamma;\Phi \mid \Delta \vdash t : \tau$ induces a natural transformation from $[\![\Gamma;\Phi \vdash \Delta]\!]$ to $[\![\Gamma;\Phi \vdash \tau]\!]$, i.e., a triple of natural transformations*

$$([\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}})$$

where

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}} \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}} \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}$$

has as its component at ρ : SetEnv a morphism

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho$$

in Set, *and*

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}} \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}} \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}$$

has as its component at ρ : RelEnv a morphism

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho$$

in Rel, *and for all ρ : RelEnv,*

$$[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho = ([\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_1\rho), [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_2\rho))$$

PROOF. We proceed by structural induction, showing only the interesting cases.

- We first consider $\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G$.
  - To see that $[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}$, since the functorial part Φ of the context is empty, we need only show that, for every ρ : SetEnv, $[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$ is a morphism in Set from $[\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$. For this, recall that

    $$[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho = \mathsf{curry}\,([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}])$$

    By the induction hypothesis, $[\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]\rho[\overline{\alpha := \_}]$ induces a natural transformation

    $$\begin{aligned}&[\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \\ :\;& [\![\Gamma;\overline{\alpha} \vdash \Delta, x : F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \\ =\;& [\![\Gamma;\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \times [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]\end{aligned}$$

    and thus a family of morphisms

    $$\begin{aligned}&\mathsf{curry}\,([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]\rho[\overline{\alpha := \_}]) \\ :\;& [\![\Gamma;\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to ([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}])\end{aligned}$$

That is, for each $\overline{A : \mathsf{Set}}$ and each $d : [\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]$ by weakening, we have

$$\begin{array}{ll} & ([\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho\, d)_{\overline{A}} \\ = & \mathsf{curry}\, ([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}])\, d \\ : & [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}] \to [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}] \end{array}$$

Moreover, these maps actually form a natural transformation $\eta : [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ because each

$$\eta_{\overline{A}} \;=\; \mathsf{curry}\, ([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}])\, d$$

is the component at $\overline{A}$ of the partial specialization to d of the natural transformation $[\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$.
To see that the components of η also satisfy the additional condition necessary for η to be in $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$, let $\overline{R : \mathsf{Rel}(A, B)}$ and

$$(u, v) \in [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho}[\overline{\alpha := R}] = ([\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}], [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := B}])$$

Then the induction hypothesis on the term t ensures that

$$\begin{array}{ll} & [\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho}[\overline{\alpha := R}] \\ : & [\![\Gamma; \overline{\alpha} \vdash \Delta, x : F]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho}[\overline{\alpha := R}] \to [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho}[\overline{\alpha := R}] \end{array}$$

and

$$\begin{array}{ll} & [\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho}[\overline{\alpha := R}] \qquad (*) \\ = & ([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}], [\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := B}]) \end{array}$$

Since $(d, d) \in [\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho} = [\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho}[\overline{\alpha := R}]$ we therefore have that

$$\begin{array}{ll} & (\eta_{\overline{A}} u, \eta_{\overline{B}} v) \\ = & (\mathsf{curry}\, ([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}])\, d\, u, \mathsf{curry}\, ([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := B}])\, d\, v) \\ = & \mathsf{curry}\, ([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho}[\overline{\alpha := R}])\, (d, d)\, (u, v) \\ : & [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho}[\overline{\alpha := R}] \end{array}$$

Here, the second equality is by $(*)$.

– The proofs that $[\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}$ is a natural transformation from $[\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}}$ to $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}$ and that, for all $\rho : \mathsf{RelEnv}$ and $d : [\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}}$,

$$[\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\, \rho\, d$$

is a natural transformation from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}]$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}]$, are analogous.

– Finally, to see that $\pi_i([\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho) = [\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_i\rho)$ we observe that π_1 and π_2 are surjective and compute

$$\begin{array}{l} \pi_i([\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho) \\ = \pi_i(\mathsf{curry}([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}])) \\ = \mathsf{curry}(\pi_i([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}])) \\ = \mathsf{curry}([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}(\pi_i(\rho[\overline{\alpha := \_}]))) \\ = \mathsf{curry}([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}(\pi_i\rho)[\overline{\alpha := \_}]) \\ = [\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_i\rho) \end{array}$$

- We now consider $\Gamma; \Phi \,|\, \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]$.

– To see that $[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ we need to show that, for every $\rho : \mathsf{SetEnv}$, $[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$ is a morphism from $[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$, and that this family of morphisms is natural in ρ. Let $d : [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$. Then

$$\begin{array}{cl} & [\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\,\rho\, d \\ = & (\mathsf{eval} \circ \langle([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}},\ [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\rangle)\, d \\ = & \mathsf{eval}(([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\, d,\ [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\, d) \\ = & \mathsf{eval}(([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ d)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}},\ [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\, d) \end{array}$$

By the induction hypothesis, $([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ d)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}$ has type

$$[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho]} \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho\overline{[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho]}$$

and $[\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\, d$ has type

$$\begin{array}{cl} & [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho \\ = & [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho]} \\ = & [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho]} \end{array}$$

by Equation 6, and by weakening in the last step, since the type $\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}} F\, G$ is only well-formed if $\Gamma;\overline{\alpha} \vdash F : \mathcal{F}$ and $\Gamma;\overline{\alpha} \vdash G : \mathcal{F}$. Thus, $[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\,\rho\, d$ has type $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho\overline{[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho]} = [\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$, as desired.

To see that the family of maps comprising $[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ form a natural transformation, i.e., are natural in their set environment argument, we need to show that the following diagram commutes:

$$\begin{tikzcd}
{[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho} \arrow[r, "{[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}}f}"] \arrow[d, "{\langle[\![\Gamma;\emptyset \mid \Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho,[\![\Gamma;\Phi \mid \Delta\vdash s:F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}\rho\rangle}"'] & {[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho'} \arrow[d, "{\langle[\![\Gamma;\emptyset \mid \Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}}"] \\
{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho} \arrow[r, "{[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}f\times[\![\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}f}"] \arrow[d, "{\mathsf{eval}\circ((-)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]\rho}}\times id)}"'] & {[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho' \times [\![\Gamma;\Phi \vdash F[\overline{\alpha :}}} \arrow[d, "{\mathsf{eval}\circ((-)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]\rho'}}}"] \\
{[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho} \arrow[r, "{[\![\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}f}"'] & {[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd}$$

The top diagram commutes because the induction hypothesis ensures $[\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ and $[\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ are natural in ρ. To see that the bottom diagram commutes, we first note that since $\rho|_{\Gamma} = \rho'|_{\Gamma}$, $\Gamma;\overline{\alpha} \vdash F : \mathcal{F}$, and $\Gamma;\overline{\alpha} \vdash G : \mathcal{F}$ we can replace the instance of f in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} f$ with id. Then, using the fact that $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ is a functor, we have that $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} id = id$. To see that the bottom diagram commutes we must therefore prove that, for every $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ and $x \in [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$, we have

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f(\eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]\rho}}x) = \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]\rho'}}([\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f x)$$

i.e., that for every $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f \circ \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]\rho}} = \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]\rho'}} \circ [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f$$

But this follows from the naturality of η. Indeed, $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ implies that η is a natural transformation from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$. For each

τ, consider the morphism $[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f : [\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'$. The following diagram commutes by naturality of η:

$$\begin{tikzcd}
{[\![\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}\rho} \arrow[d, equal] & {[\![\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}\rho} \arrow[d, equal] \\
{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]} \arrow[r, "\eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}]}"] & {[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]} \arrow[d, "{[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}]}"] \\
{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}]} \arrow[r, "\eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}"] & {[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}]}
\end{tikzcd}$$

That is,

$$\begin{array}{rl}
& [\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}] \circ \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} \\
= & \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ [\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}]
\end{array}$$

But since the only variables in the functorial contexts for F and G are $\overline{\alpha}$, we have that

$$\begin{array}{rl}
& [\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}] \\
= & [\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}] \\
= & [\![\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}} f
\end{array}$$

and similarly for G. Commutativity of this last diagram thus gives that $[\![\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}} f \circ \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} = \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ [\![\Gamma;\Phi\vdash F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}} f$, as desired.

- The proof that $[\![\Gamma;\Phi\,|\,\Delta\vdash t_\tau s : G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Rel}}$ is a natural transformation from $[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}}$ to $[\![\Gamma;\Phi\vdash G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Rel}}$ is analogous.
- Finally, to see that $\pi_i([\![\Gamma;\Phi\,|\,\Delta\vdash t_\tau s : G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Rel}}\rho) = [\![\Gamma;\Phi\,|\,\Delta\vdash t_\tau s : G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}(\pi_i\rho)$ we compute

$$\begin{array}{rl}
& \pi_i([\![\Gamma;\Phi\,|\,\Delta\vdash t_\tau s : G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Rel}}\rho) \\
= & \pi_i(\mathsf{eval}\circ\langle([\![\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Rel}}\rho\ _)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}}, [\![\Gamma;\Phi\,|\,\Delta\vdash s : F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Rel}}\rho\rangle) \\
= & \mathsf{eval}\circ\langle\pi_i(([\![\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Rel}}\rho\ _)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}}), \pi_i([\![\Gamma;\Phi\,|\,\Delta\vdash s : F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Rel}}\rho)\rangle \\
= & \mathsf{eval}\circ\langle\pi_i([\![\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Rel}}\rho\ _)_{\overline{\pi_i([\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho)}}, \pi_i([\![\Gamma;\Phi\,|\,\Delta\vdash s : F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Rel}}\rho)\rangle \\
= & \mathsf{eval}\circ\langle([\![\Gamma;\emptyset\,|\,\Delta\vdash t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}(\pi_i\rho)\ _)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}(\pi_i\rho)}}, [\![\Gamma;\Phi\,|\,\Delta\vdash s : F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}(\pi_i\rho)\rangle \\
= & [\![\Gamma;\Phi\,|\,\Delta\vdash t_\tau s : G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}(\pi_i\rho)
\end{array}$$

- We now consider $\Gamma;\emptyset\mid\emptyset\vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi:=_{\overline{\beta}}F}]\,H[\overline{\phi:=_{\overline{\beta}}G}])$.
  - To see that $[\![\Gamma;\emptyset\mid\emptyset\vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi:=_{\overline{\beta}}F}]\,H[\overline{\phi:=_{\overline{\beta}}G}])]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\emptyset\vdash\emptyset]\!]^{\mathsf{Set}}$ to $[\![\mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi:=_{\overline{\beta}}F}]\,H[\overline{\phi:=_{\overline{\beta}}G}])]\!]^{\mathsf{Set}}$, since the functorial part Φ of the context is empty, we need only show that, for every $\rho : \mathsf{SetEnv}$,

$$[\![\Gamma;\emptyset\mid\emptyset\vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi:=_{\overline{\beta}}F}]\,H[\overline{\phi:=_{\overline{\beta}}G}])]\!]^{\mathsf{Set}}\,\rho$$

is a morphism in Set from $[\![\Gamma;\emptyset\vdash\emptyset]\!]^{\mathsf{Set}}\rho$ to

$$[\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi:=_{\overline{\beta}}F}]\,H[\overline{\phi:=_{\overline{\beta}}G}])]\!]^{\mathsf{Set}}\,\rho$$

i.e., that, for the unique $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\, \rho\, d$$

is a morphism from $\overline{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}}\rho}$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\rho$. For this we show that for all $\overline{\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}}\rho}$ we have

$$\begin{array}{rl} & [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\, \rho\, d\, \overline{\eta} \\ : & [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\rho \end{array}$$

To this end, we note that, for any $\overline{B}$,

$$\begin{array}{rl} & ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\, \rho\, d\, \overline{\eta})_{\overline{B}} \\ = & [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}] \end{array}$$

is indeed a morphism from

$$\begin{array}{rl} & [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}] \\ = & [\![\Gamma;\overline{\gamma},\overline{\phi} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}] \end{array}$$

to

$$\begin{array}{rl} & [\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}] \\ = & [\![\Gamma;\overline{\gamma},\overline{\phi} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}] \end{array}$$

since $[\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}}$ is a functor from SetEnv to Set and $id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}]$ is a morphism in SetEnv from

$$\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}]$$

to

$$\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}]$$

To see that this family of morphisms is natural in $\overline{B}$ we first observe that if $\overline{f : B \to B'}$ then, writing t for $[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\, \rho\, d\, \overline{\eta}$, we have

$$\begin{tikzcd} {[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}]} \arrow[r, "t_{\overline{B}}"] \arrow[d, "{[\![\Gamma;\overline{\gamma}\vdash H[\overline{\phi:=F}]]\!]^{\mathsf{Set}} id_{\rho}[\overline{\gamma:=f}]}"'] & {[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}]} \arrow[d, "{[\![\Gamma;\overline{\gamma}\vdash H[\overline{\phi:=G}]]\!]^{\mathsf{Set}} id_{\rho}[\overline{\gamma:=f}]}"] \\ {[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B'}]} \arrow[r, "t_{\overline{B'}}"] & {[\![\Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B'}]} \end{tikzcd}$$

This diagram commutes because $[\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}}$ is a functor from SetEnv to Set and because, letting

$$E_{F,B} = \rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.\, [\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}]$$

and

$$e_{F,f} = id_{\rho}[\overline{\gamma := f}][\overline{\phi := \lambda\overline{A}.\, [\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := f}][\overline{\beta := id_A}]}]$$

for all F and B and $\overline{f : B \to B'}$, the following diagram commutes by the fact that composition of environments is componentwise together with the naturality of η:

$$\begin{tikzcd}
E_{F,B} \arrow[rr, "id_\rho[\gamma:=id_B][\overline{\phi:=\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{B}}}]"] \arrow[d, "e_{F,f}"'] & & E_{G,B} \arrow[d, "e_{G,f}"] \\
E_{F,B'} \arrow[rr, "id_\rho[\gamma:=id_{B'}][\overline{\phi:=\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{B'}}}]"'] & & E_{G,B'}
\end{tikzcd}$$

We therefore have that

$$\lambda\overline{B}.\,[\![\Gamma;\emptyset\mid\emptyset\vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,\rho\,d\,\overline{\eta})_{\overline{B}}$$

is natural in $\overline{B}$ as desired.

– To see that, for every $\rho : \mathsf{SetEnv}$ and $d : [\![\Gamma;\emptyset\vdash\emptyset]\!]^{\mathsf{Set}}\rho$, and all $\overline{\eta : [\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G]\!]^{\mathsf{Set}}\rho}$,

$$[\![\Gamma;\emptyset\mid\emptyset\vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,\rho\,d\,\overline{\eta}$$

satisfies the additional condition necessary for it to be in $[\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\rho$, let $\overline{R : \mathsf{Rel}(B,B')}$ and $\overline{S : \mathsf{Rel}(C,C')}$. Since each map in $\overline{\eta}$ satisfies the extra condition necessary for it to be in its corresponding $[\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G]\!]^{\mathsf{Set}}\rho$ — i.e., since

$$(\eta_{\overline{B}\,\overline{C}},\eta_{\overline{B'}\,\overline{C'}}) \in [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}] \to [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}]$$

— we have that

$$\begin{array}{rl}
& (\,(\lambda e\,v\,\overline{Z}.\,[\![\Gamma;\overline{\phi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}\,id_{\rho[\overline{\gamma:=Z}]}[\overline{\phi := \lambda\overline{A}.v_{\overline{A}\,\overline{Z}}}])\,d\,\overline{\eta}\,\overline{B},\\
& \qquad(\lambda e\,v\,\overline{Z}.\,[\![\Gamma;\overline{\phi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}\,id_{\rho[\overline{\gamma:=Z}]}[\overline{\phi := \lambda\overline{A}.v_{\overline{A}\,\overline{Z}}}])\,d\,\overline{\eta}\,\overline{B'}\,)\\
= & (\,[\![\Gamma;\overline{\phi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}\,id_{\rho[\overline{\gamma:=B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}]),\ [\![\Gamma;\overline{\phi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}\,id_{\rho[\overline{\gamma:=B'}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B'}}}])\,)
\end{array}$$

has type

$$\begin{array}{rl}
& ([\![\Gamma;\overline{\gamma}\vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}] \to [\![\Gamma;\overline{\gamma}\vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B}],\\
& \qquad [\![\Gamma;\overline{\gamma}\vdash H[\overline{\phi := F}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B'}] \to [\![\Gamma;\overline{\gamma}\vdash H[\overline{\phi := G}]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := B'}])\\
= & [\![\Gamma;\overline{\gamma}\vdash H[\overline{\phi := F}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := R}] \to [\![\Gamma;\overline{\gamma}\vdash H[\overline{\phi := G}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := R}]
\end{array}$$

as desired.

– The proofs that

$$[\![\Gamma;\emptyset\mid\emptyset\vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}$$

is a natural transformation from $[\![\Gamma;\emptyset\vdash\emptyset]\!]^{\mathsf{Rel}}$ to

$$[\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}$$

and that, for every $\rho : \mathsf{RelEnv}$ and the unique $d : [\![\Gamma;\emptyset\vdash\emptyset]\!]^{\mathsf{Rel}}\rho$,

$$[\![\Gamma;\emptyset\mid\emptyset\vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}\,\rho\,d$$

is a morphism from $\overline{[\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G]\!]^{\mathsf{Rel}}}\,\rho$ to $[\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Rel}}\,\rho$, are analogous.

– Finally, to see that

$$
\begin{array}{ll}
& \pi_i([\![\Gamma;\emptyset \mid \emptyset\vdash \mathsf{map}_H^{\overline{F}\,\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}\,\rho)\\
= & [\![\Gamma;\emptyset \mid \emptyset\vdash \mathsf{map}_H^{\overline{F}\,\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,(\pi_i\rho)
\end{array}
$$

we compute

$$
\begin{array}{ll}
& \pi_i([\![\Gamma;\emptyset \mid \emptyset\vdash \mathsf{map}_H^{\overline{F}\,\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Rel}}\,\rho)\\
= & \pi_i\,(\lambda e\,\overline{v}\,\overline{R}.\,[\![\Gamma;\overline{\phi},\overline{\gamma}\vdash H]\!]^{\mathsf{Rel}}\,id_{\rho[\overline{\gamma := R}]}[\overline{\phi := \lambda\overline{S}.v_{\overline{S}\,\overline{R}}}])\\
= & \lambda e\,\overline{v}\,\overline{R}.\,[\![\Gamma;\overline{\phi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}\,id_{(\pi_i\rho)[\overline{\gamma := \pi_i R}]}[\overline{\phi := \lambda\overline{S}.(\pi_i v)_{\overline{\pi_i S}\,\overline{\pi_i R}}}]\\
= & \lambda d\,\overline{\eta}\,\overline{B}.\,[\![\Gamma;\overline{\phi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}\,id_{(\pi_i\rho)[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.\eta_{\overline{A}\,\overline{B}}}]\\
= & [\![\Gamma;\emptyset \mid \emptyset\vdash \mathsf{map}_H^{\overline{F}\,\overline{G}} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,(\pi_i\rho)
\end{array}
$$

- We now consider $\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\,H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}$.
  – To see that if $d : [\![\Gamma;\emptyset\vdash\emptyset]\!]^{\mathsf{Set}}\rho$ then

$$
[\![\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\,H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\,d
$$

is in $[\![\Gamma;\emptyset\vdash Nat^{\overline{\beta},\overline{\gamma}}\,H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho$, we first note that, for all $\overline{B}$ and $\overline{C}$,

$$
\begin{array}{ll}
& ([\![\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\,H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\,d)_{\overline{B}\,\overline{C}}\\
= & (in_{T_{\rho[\overline{\gamma := C}]}})_{\overline{B}}
\end{array}
$$

does indeed map

$$
\begin{array}{ll}
& [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]\\
= & [\![\Gamma;\overline{\beta},\overline{\gamma},\overline{\alpha}\vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}]\\
= & [\![\Gamma;\phi,\overline{\beta},\overline{\gamma},\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}]\\
& \qquad [\phi := \lambda\overline{D}.\,[\![\Gamma;\overline{\beta},\overline{\gamma},\overline{\alpha}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}][\overline{\beta := D}]]\\
= & [\![\Gamma;\phi,\overline{\gamma},\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\alpha := B}]\\
& \qquad [\phi := \lambda\overline{D}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := D}][\overline{\gamma := C}]]\\
= & T_{\rho[\overline{\gamma := C}]}\,(\lambda\overline{D}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := D}][\overline{\gamma := C}])\,\overline{B}\\
= & T_{\rho[\overline{\gamma := C}]}\,(\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B}
\end{array}
$$

to

$$
\begin{array}{ll}
& [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]\\
= & (\lambda\overline{D}.\,[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := D}][\overline{\gamma := C}])\overline{B}\\
= & (\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B}
\end{array}
$$

To see that

$$
\begin{array}{ll}
& [\![\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\,H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\,d\\
= & \lambda\overline{B}\,\overline{C}.\,(in_{T_{\rho[\overline{\gamma := C}]}})_{\overline{B}}
\end{array}
$$

is natural in $\overline{B}$ and $\overline{C}$, we observe that the following diagram commutes for all $\overline{f : B \to B'}$ and $\overline{g : C \to C'}$:

$$\begin{tikzcd}
T_{\rho[\overline{\gamma := C}]}\,(\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B} \arrow[rr, "(in_{T_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}"] \arrow[d, "\sigma_{id_{\rho[\overline{\gamma:=g}]}}\,(\mu\sigma_{id_{\rho[\overline{\gamma:=g}]}})\,\overline{B}"] & & (\mu T_{\rho[\overline{\gamma := C}]})\,\overline{B} \arrow[d, "(\mu\sigma_{id_{\rho[\overline{\gamma:=g}]}})\,\overline{B}"] \\
T_{\rho[\overline{\gamma := C'}]}\,(\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B} \arrow[rr, "(in_{T_{\rho[\overline{\gamma:=C'}]}})_{\overline{B}}"] \arrow[d, "T_{\rho[\overline{\gamma:=C'}]}\,(\mu T_{\rho[\overline{\gamma:=C'}]})\,\overline{f}"] & & (\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B} \arrow[d, "(in_{T_{\rho[\overline{\gamma:=C'}]}})_{\overline{B}}"] \\
T_{\rho[\overline{\gamma := C'}]}\,(\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B'} \arrow[rr, "(in_{T_{\rho[\overline{\gamma:=C'}]}})_{\overline{B'}}"] & & (\mu T_{\rho[\overline{\gamma := C'}]})\,\overline{B'}
\end{tikzcd}$$

Indeed, naturality of in with respect to its functor argument ensures that the top diagram commutes, and naturality of $in_{T_{\rho[\overline{\gamma:=C'}]}}$ ensures that the bottom one commutes.

– To see that $[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d$ satisfies the additional property necessary for it to be in

$$[\![\Gamma;\emptyset \vdash Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho$$

let $\overline{R : \mathsf{Rel}(B,B')}$ and $\overline{S : \mathsf{Rel}(C,C')}$. Then

$$\begin{array}{ll}
& (\,([\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d)_{\overline{B},\overline{C}},\\
& \qquad ([\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d)_{\overline{B'},\overline{C'}}\,)\\
= & (\,(in_{T_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}, (in_{T_{\rho[\overline{\gamma:=C'}]}})_{\overline{B'}}\,)
\end{array}$$

has type

$$\begin{array}{ll}
& (\,[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \to\\
& \qquad [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}],\\
& \;[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B'}][\overline{\gamma := C'}] \to\\
& \qquad [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B'}][\overline{\gamma := C'}]\,)\\
= & (\,T_{\rho[\overline{\gamma:=C}]}\,(\mu T_{\rho[\overline{\gamma:=C}]})\,\overline{B} \to (\mu T_{\rho[\overline{\gamma:=C}]})\,\overline{B},\; T_{\rho[\overline{\gamma:=C'}]}\,(\mu T_{\rho[\overline{\gamma:=C'}]})\,\overline{B'} \to (\mu T_{\rho[\overline{\gamma:=C'}]})\,\overline{B'}\,)\\
= & [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}] \to\\
& \qquad [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}]
\end{array}$$

– The proofs that $[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}$ is a natural transformation from $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}$ to $[\![\Gamma;\emptyset \vdash Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{R}}$ and that, for all $\rho : \mathsf{RelEnv}$ and $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}$,

$$[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\,\rho\, d$$

is a natural transformation from $\lambda\overline{R}\,\overline{S}.\,[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Rel}}\rho[\overline{\beta := R}][\overline{\gamma := S}]$ to $\lambda\overline{R}\,\overline{S}.\,[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\rho[\overline{\beta := R}][\overline{\gamma := S}]$, are analogous.

– Finally, to see that $\pi_i([\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\,\rho\, d) = [\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,(\pi_i\rho)\,(\pi_i d)$ we first note that $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho$ and $\pi_i d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}(\pi_i\rho)$ are uniquely determined. Further, the definition of natural transformations in Rel ensures that, for any $\overline{R}$ and $\overline{S}$,

$$
\begin{aligned}
&(in_{T_{\rho[\overline{\gamma:=S}]}})_{\overline{R}}\\
&= ((in_{\pi_1(T_{\rho[\overline{\gamma:=S}]})})_{\overline{\pi_1 R}},\ (in_{\pi_2(T_{\rho[\overline{\gamma:=S}]})})_{\overline{\pi_2 R}})\\
&= ((in_{T^{\mathsf{Set}}_{(\pi_1\rho)[\overline{\gamma:=\pi_1 S}]}})_{\overline{\pi_1 R}},\ (in_{T^{\mathsf{Set}}_{(\pi_2\rho)[\overline{\gamma:=\pi_2 S}]}})_{\overline{\pi_2 R}})
\end{aligned}
$$

Observing that π_1 and π_2 are surjective, we therefore have that

$$
\begin{aligned}
&\pi_i([\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\,\rho\, d)\\
&= \pi_i(\lambda\overline{R}\,\overline{S}.\,(in_{T_{\rho[\overline{\gamma:=S}]}})_{\overline{R}})\\
&= \lambda\overline{B}\,\overline{C}.\,(in_{T^{\mathsf{Set}}_{(\pi_i\rho)[\overline{\gamma:=C}]}})_{\overline{B}}\\
&= [\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{in}_H : Nat^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,(\pi_i\rho)\,(\pi_i d)
\end{aligned}
$$

• We now consider $\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F)$.

– To see that $[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}$ to

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}$$

since the functorial part Φ of the context is empty, we need only show that, for every $\rho : \mathsf{SetEnv}$,

$$[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}\,\rho$$

is a morphism in Set from $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$ to

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}\,\rho$$

i.e., that, for the unique $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}\,\rho\, d$$

is a morphism from $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}\rho$. For this we show that for every $\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F]\!]^{\mathsf{Set}}\rho$ we have

$$
\begin{aligned}
&[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}\,\rho\, d\,\eta\\
:\;\; &[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}\,\rho
\end{aligned}
$$

To this end we show that, for any $\overline{B}$ and $\overline{C}$,

$$([\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\;(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}\,\rho\, d\,\eta)_{\overline{B}\,\overline{C}}$$

is a morphism from

$$[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \;=\; (\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]})\overline{B}$$

to

$$[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

To see this, we use Equations 6 and 8 for the first and second equalities below, together with weakening, to see that η is itself a natural transformation from

$$\begin{array}{rl}
& \lambda \overline{B}\,\overline{C}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}] \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}] \\
= & \lambda \overline{B}\,\overline{C}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma}, \overline{\alpha} \vdash H[\phi := F] \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}] \\
= & \lambda \overline{B}\,\overline{C}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma}, \overline{\alpha}, \phi \vdash H \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}] \\
& \qquad [\phi := \lambda \overline{A}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma}, \overline{\alpha} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}][\overline{\alpha := B}][\overline{\beta := A}]] \\
= & \lambda \overline{B}\,\overline{C}.\, \llbracket \Gamma; \overline{\gamma}, \overline{\alpha}, \phi \vdash H \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C}][\overline{\alpha := B}][\phi := \lambda \overline{A}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C}][\overline{\beta := A}]] \\
= & \lambda \overline{B}\,\overline{C}.\, T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}\, (\lambda \overline{A}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := A}][\overline{\gamma := C}])\, \overline{B}
\end{array}$$

to

$$\lambda \overline{B}\,\overline{C}.\, (\lambda \overline{A}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := A}][\overline{\gamma := C}])\overline{B} \quad = \quad \lambda \overline{B}\,\overline{C}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}]$$

Thus, if $x : \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta} \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}] \;=\; (\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]})\overline{B}$, then

$$\begin{array}{rl}
& (\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta}, \overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta}, \overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F \rrbracket^{\mathsf{Set}}\, \rho\, d\, \eta)_{\overline{B}\,\overline{C}}\, x \\
= & (\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}\, (\lambda \overline{A}.\, \eta_{\overline{A}\,\overline{C}}))_{\overline{B}}\, x \\
: & (\lambda \overline{A}.\, \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := A}][\overline{\gamma := C}])\overline{B}
\end{array}$$

i.e., for each $\overline{B}$ and $\overline{C}$

$$(\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta}, \overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta}, \overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F \rrbracket^{\mathsf{Set}}\, \rho\, d\, \eta)_{\overline{B}\,\overline{C}}$$

is a morphism from $(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]})\overline{B}$ to $\llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}]$.

To see that this family of morphisms is natural in $\overline{B}$ and $\overline{C}$, we observe that the following diagram commutes for all $\overline{f : B \to B'}$ and $\overline{g : C \to C'}$:

$$\begin{tikzcd}
(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]})\, \overline{B} \arrow[rr, "(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}\, (\lambda \overline{A}.\, \eta_{\overline{A}\,\overline{C}}))_{\overline{B}}"] \arrow[d, "(\mu\sigma_{id_{\rho[\overline{\gamma := g}]}})\, \overline{B}"'] & & \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C}][\overline{\beta := B}] \arrow[d, "\llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} id_{\rho}[\overline{\gamma := g}][\overline{\beta := id_B}]"] \\
(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]})\, \overline{B} \arrow[rr, "(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}\, (\lambda \overline{A}.\, \eta_{\overline{A}\,\overline{C'}}))_{\overline{B}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]})\, \overline{f}"'] & & \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C'}][\overline{\beta := B}] \arrow[d, "\llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} id_{\rho}[\overline{\gamma := id_{C'}}][\overline{\beta := f}]"] \\
(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]})\, \overline{B'} \arrow[rr, "(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}\, (\lambda \overline{A}.\, \eta_{\overline{A}\,\overline{C'}}))_{\overline{B'}}"] & & \llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash F \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C'}][\overline{\beta := B'}]
\end{tikzcd}$$

Indeed, naturality of $\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}\, (\lambda \overline{A}.\, \eta_{\overline{A}\,\overline{C'}})$ ensures that the bottom diagram commutes. To see that the top one commutes is considerably more delicate.

To see that the top diagram commutes we first observe that, given a natural transformation $\Theta : H \to K : [\mathsf{Set}^k, \mathsf{Set}] \to [\mathsf{Set}^k, \mathsf{Set}]$, the fixpoint natural transformation $\mu\Theta : \mu H \to$

$\mu K : \mathsf{Set}^k \to \mathsf{Set}$ is defined to be $fold_H(\Theta\,(\mu K) \circ in_K)$, i.e., the unique morphism making the following square commute:

$$\begin{tikzcd}
H(\mu H) \arrow[r, "H(\mu\Theta)"] \arrow[dd, "in_H"'] & H(\mu K) \arrow[d, "\Theta(\mu K)"] \\
 & K(\mu K) \arrow[d, "in_K"] \\
\mu H \arrow[r, "\mu\Theta"'] & \mu K
\end{tikzcd}$$

Taking $\Theta = \sigma^{\mathsf{Set}}_f : T^{\mathsf{Set}}_\rho \to T^{\mathsf{Set}}_{\rho'}$ gives that the following diagram commutes for any morphism of set environments $f : \rho \to \rho'$:

$$\begin{tikzcd}
T^{\mathsf{Set}}_\rho(\mu T^{\mathsf{Set}}_\rho) \arrow[r, "T^{\mathsf{Set}}_\rho(\mu\sigma^{\mathsf{Set}}_f)"] \arrow[dd, "in_{T^{\mathsf{Set}}_\rho}"'] & T^{\mathsf{Set}}_\rho(\mu T^{\mathsf{Set}}_{\rho'}) \arrow[d, "\sigma^{\mathsf{Set}}_f(\mu T^{\mathsf{Set}}_{\rho'})"] \\
 & T^{\mathsf{Set}}_{\rho'}(\mu T^{\mathsf{Set}}_{\rho'}) \arrow[d, "in_{T^{\mathsf{Set}}_{\rho'}}"] \\
\mu T^{\mathsf{Set}}_\rho \arrow[r, "\mu\sigma^{\mathsf{Set}}_f"'] & \mu T^{\mathsf{Set}}_{\rho'}
\end{tikzcd} \tag{18}$$

We next observe that the action of the functor

$$\lambda\overline{B}.\lambda\overline{C}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash H[\phi := F][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

on the morphisms $\overline{f : B \to B'}, \overline{g : C \to C'}$ is given by

$$\begin{aligned}
& \llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash H[\phi := F][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Set}} id_\rho[\overline{\beta := f}][\overline{\gamma := g}] \\
= \;& \llbracket\Gamma;\overline{\alpha},\overline{\gamma}\vdash H[\phi := F]\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := f}][\overline{\gamma := g}] \\
= \;& \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := f}][\overline{\gamma := g}][\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta:=A}]}[\overline{\gamma := g}]] \\
= \;& \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H\rrbracket^{\mathsf{Set}}(id_{\rho[\overline{\gamma:=C'}][\phi:=\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C'}]]}[\overline{\alpha := f}] \\
& \qquad \circ id_{\rho[\overline{\alpha:=B}][\phi:=\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C'}]]}[\overline{\gamma := g}] \\
& \qquad \circ id_{\rho[\overline{\alpha:=B}][\overline{\gamma:=C}]}[\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta:=A}]}[\overline{\gamma := g}]]) \\
= \;& \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\gamma:=C'}][\phi:=\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C'}]]}[\overline{\alpha := f}] \\
& \qquad \circ \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\alpha:=B}][\phi:=\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C'}]]}[\overline{\gamma := g}] \\
& \qquad \circ \llbracket\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\alpha:=B}][\overline{\gamma:=C}]}[\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta:=A}]}[\overline{\gamma := g}]] \\
= \;& T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}(\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\overline{f} \\
& \qquad \circ \big(\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\big)_{\overline{B}} \\
& \qquad \circ \big(T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\lambda\overline{A}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta:=A}]}[\overline{\gamma := g}])\big)_{\overline{B}}
\end{aligned}$$

So, if η is a natural transformation from

$$\lambda\overline{B}\,\overline{C}.\llbracket\Gamma;\overline{\alpha},\overline{\gamma}\vdash H[\phi := F][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

to

$$\lambda\overline{B}\,\overline{C}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

then, by naturality,

$$
\begin{aligned}
& [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\beta := f}][\overline{\gamma := g}] \circ \eta_{\overline{B},\overline{C}} \\
= \;& \eta_{\overline{B'},\overline{C'}} \circ [\![\Gamma;\overline{\alpha},\overline{\gamma}\vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}} id_\rho[\overline{\beta := f}][\overline{\gamma := g}] \\
= \;& \eta_{\overline{B'},\overline{C'}} \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\overline{f} \\
& \circ\big(\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma := g}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\big)_{\overline{B}} \\
& \circ\big(T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])\big)_{\overline{B}}
\end{aligned}
$$

As a special case when $\overline{f = id_B}$ we have

$$
\begin{aligned}
& [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \eta_{\overline{B},\overline{C}} \\
= \;& \eta_{\overline{B},\overline{C'}} \circ \big(\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma := g}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\big)_{\overline{B}} \\
& \quad \circ\big(T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])\big)_{\overline{B}}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
& \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \lambda\overline{B}.\eta_{\overline{B},\overline{C}} \\
= \;& \lambda\overline{B}.\eta_{\overline{B},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma := g}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]) \\
& \qquad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])
\end{aligned}
\tag{19}
$$

Now, to see that the top diagram in the diagram on page 34 commutes we first note that the diagram

$$
\begin{tikzcd}
T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}) \arrow[rr, "T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})\circ\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]})"] \arrow[dd, "\mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}"'] & & T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}])"] \\
& & T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}"] \\
\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]} \arrow[r, "\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}"'] & \mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]} \arrow[r, "\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})"'] & \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]
\end{tikzcd}
$$

commutes because

$$
\begin{aligned}
& \lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \\
& \qquad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}) \\
= \;& \lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \\
& \qquad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})) \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}) \\
= \;& \lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})) \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}) \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}) \\
= \;& \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}} \circ \sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}) \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}(\mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]}) \\
= \;& \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu\sigma^{\mathsf{Set}}_{id_\rho[\overline{\gamma:=g}]} \circ \mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}
\end{aligned}
$$

Here, the first equality is by functoriality of $T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}$, the second equality is by naturality of $\sigma^{\mathsf{Set}}_{id_{\rho}[\overline{\gamma := g}]}$, the third equality by the universal property of $\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})$ and the last equality by Equation 18. That is, we have

$$\begin{aligned}
&\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu\sigma^{\mathsf{Set}}_{id_{\rho}[\overline{\gamma := g}]} \\
= \quad &\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_{\rho}[\overline{\gamma := g}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]))
\end{aligned} \tag{20}$$

Next, we note that the diagram

$$\begin{tikzcd}
T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}) \arrow[rr, "T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}))"] \arrow[dd, "\mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}"'] & & T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\cdots \arrow[d, "\sigma^{\mathsf{Set}}_{id_{\rho}[\overline{\gamma := g}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\cdots"] \\
& & T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\cdots \arrow[d, "\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}"] \\
\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]} \arrow[r, bend right, "\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})"'] & \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \arrow[r, bend right, "\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}]"'] & \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\cdots
\end{tikzcd}$$

commutes because

$$\begin{aligned}
&\lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_{\rho}[\overline{\gamma := g}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]) \\
&\qquad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})) \\
&= \lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_{\rho}[\overline{\gamma := g}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]) \\
&\qquad \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}]) \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})) \\
&= \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \lambda\overline{A}.\eta_{\overline{A},\overline{C}} \circ T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}(\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})) \\
&= \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}) \circ \mathsf{in}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}
\end{aligned}$$

Here, the first equality is by functoriality of $T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}$, the second equality is by Equation 19, and the last equality is by the universal property of $\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})$. That is, we have

$$\begin{aligned}
&\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}) \\
= \quad &\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}} \circ \sigma^{\mathsf{Set}}_{id_{\rho}[\overline{\gamma := g}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C'}]))
\end{aligned} \tag{21}$$

Combining Equations 20 and 21 we get that

$$\mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu\sigma^{\mathsf{Set}}_{id_{\rho}[\overline{\gamma := g}]} = \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma} \vdash F\rrbracket^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \mathsf{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})$$

i.e., that the top diagram in the diagram on page 34 commutes. We therefore have that

$$(\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F\rrbracket^{\mathsf{Set}}\,\rho\,d\,\eta)_{\overline{B}\,\overline{C}}$$

is natural in $\overline{B}$ and $\overline{C}$ as desired.

– To see that, for every $\rho : \mathsf{SetEnv}$, $d \in \llbracket \Gamma;\emptyset \vdash \emptyset\rrbracket^{\mathsf{Set}}\rho$, and $\eta : \llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F\rrbracket^{\mathsf{Set}}\rho$,

$$\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F\rrbracket^{\mathsf{Set}}\,\rho\,d\,\eta$$

satisfies the additional condition necessary for it to be in $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F\rrbracket^{\mathsf{Set}}\,\rho$, let $\overline{R : \mathsf{Rel}(B,B')}$ and $\overline{S : \mathsf{Rel}(C,C')}$. Since η satisfies the additional condition necessary for it to be in $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(H[\phi := F][\overline{\alpha := \beta}])\,F\rrbracket^{\mathsf{Set}}\rho$ — i.e., since

$$\begin{aligned}
(\eta_{\overline{B}\,\overline{C}}, \eta_{\overline{B'}\,\overline{C'}}) \;&\in\; \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash H[\phi := F][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \to \\
&\qquad\qquad \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \\
&=\; T_{\mathsf{Eq}_\rho[\overline{\gamma := S}]}\,(\llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}]) \to \\
&\qquad\qquad \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}]
\end{aligned}$$

— we have that

$$(\,(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C}]}}\,(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C}}))_{\overline{B}},\;(\mathit{fold}_{T^{\mathsf{Set}}_{\rho[\overline{\gamma := C'}]}}\,(\lambda\overline{A}.\eta_{\overline{A}\,\overline{C'}}))_{\overline{B'}}\,)$$

has type

$$\begin{aligned}
&(\mu T_{\mathsf{Eq}_\rho[\overline{\gamma := S}]})\,\overline{R} \to \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \\
=\;& (\mu T_{\mathsf{Eq}_\rho[\overline{\gamma := S}]})\,\overline{\llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash \beta\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}]} \to \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \\
=\;& \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}] \to \llbracket \Gamma;\overline{\gamma},\overline{\beta} \vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma := S}][\overline{\beta := R}]
\end{aligned}$$

as desired.

– The proofs that

$$\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)\rrbracket^{\mathsf{Rel}}$$

is a natural transformation from $\llbracket \Gamma;\emptyset \vdash \emptyset\rrbracket^{\mathsf{Rel}}$ to

$$\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^\emptyset\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)\rrbracket^{\mathsf{Rel}}$$

and that, for all $\rho : \mathsf{RelEnv}$ and the unique $d : \llbracket \Gamma;\emptyset \vdash \emptyset\rrbracket^{\mathsf{Rel}}\rho$,

$$\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)\rrbracket^{\mathsf{Rel}}\,\rho\,d$$

is a morphism from $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\rrbracket^{\mathsf{Rel}}\rho$ to $\llbracket \Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F\rrbracket^{\mathsf{Rel}}\rho$, are analogous.

– Finally, to see that

$$\begin{aligned}
&\pi_i(\llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)\rrbracket^{\mathsf{Rel}}\rho) \\
=\;& \llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^\emptyset\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)\rrbracket^{\mathsf{Set}}(\pi_i\rho)
\end{aligned}$$

we compute

$$
\begin{array}{rl}
& \pi_i(\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)\rrbracket^{\mathsf{Rel}}) \\
= & \pi_i(\lambda e\, \eta\, \overline{R}\, \overline{S}.\, (fold_{T_{\rho[\overline{\gamma := S}]}}\, (\lambda \overline{Z}.\, \eta_{\overline{Z}\, \overline{S}}))_{\overline{R}}) \\
= & \lambda e\, \eta\, \overline{R}\, \overline{S}.\, (fold_{T_{(\pi_i\rho)[\overline{\gamma := \pi_i S}]}}\, (\lambda \overline{Z}.\, (\pi_i \eta)_{\overline{\pi_i Z}\, \overline{\pi_i S}}))_{\overline{\pi_i R}} \\
= & \lambda d\, \eta\, \overline{B}\, \overline{C}.\, (fold_{T_{(\pi_i\rho)[\overline{\gamma := C}]}}\, (\lambda \overline{A}. \eta_{\overline{A}\, \overline{C}}))_{\overline{B}} \\
= & \llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)\rrbracket^{\mathsf{Rel}}(\pi_i \rho)
\end{array}
$$

Here, we are again using the fact that π_1 and π_2 are surjective.

□

The Abstraction Theorem is now the special case of Theorem 27 for closed terms of close type: State more generally as: If $(a, b) \in \llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}} \rho$ then $(\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}}(\pi_1\rho)a, \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}}(\pi_2\rho)b) \in \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho$. Get the next theorem as a corollary for closed terms of closed type.

Theorem 28. *If $\vdash \tau : \mathcal{F}$ and $\vdash t : \tau$, then* $(\llbracket \vdash t : \tau \rrbracket^{\mathsf{Set}}, \llbracket \vdash t : \tau \rrbracket^{\mathsf{Set}}) \in \llbracket \vdash \tau \rrbracket^{\mathsf{Rel}}$.

6 FREE THEOREMS FOR NESTED TYPES

6.1 Free Theorem for Type of Polymorphic Bottom

Suppose $\vdash g : \mathsf{Nat}^{\alpha}\, \mathbb{1}\, \alpha$, let $G^{\mathsf{Set}} = \llbracket \vdash g : \mathsf{Nat}^{\alpha}\, \mathbb{1}\, \alpha \rrbracket^{\mathsf{Set}}$, and let $G^{\mathsf{Rel}} = \llbracket \vdash g : \mathsf{Nat}^{\alpha}\, \mathbb{1}\, \alpha \rrbracket^{\mathsf{Rel}}$. By Theorem 28, $(G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) = G^{\mathsf{Rel}}\rho$. Thus, for all $\rho \in \mathsf{RelEnv}$ and any $(a, b) \in \llbracket \vdash \emptyset \rrbracket^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$
\begin{array}{rcl}
(G^{\mathsf{Set}}, G^{\mathsf{Set}}) \;=\; (G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) & \in & \llbracket \vdash \mathsf{Nat}^{\alpha}\, \mathbb{1}\, \alpha \rrbracket^{\mathsf{Rel}}\rho \\
& = & \{\eta : K_1 \Rightarrow id\} \\
& = & \{(\eta_1 : K_1 \Rightarrow id, \eta_2 : K_1 \Rightarrow id)\}
\end{array}
$$

That is, G^{Set} is a natural transformation from the constantly 1-valued functor to the identity functor in Set. In particular, for every $S : \mathsf{Set}$, $G^{\mathsf{Set}}_S : 1 \to S$. Note, however, that if $S = \emptyset$, then there can be no such morphism, so no such natural transformation can exist in Set, and thus no term $\vdash g : \mathsf{Nat}^{\alpha}\mathbb{1}\, \alpha$ can exist in our calculus. That is, our calculus does not admit any terms with the closed type $\mathsf{Nat}^{\alpha}\mathbb{1}\, \alpha$ of the polymorphic bottom.

6.2 Free Theorem for Type of Polymorphic Identity

Suppose $\vdash g : \mathsf{Nat}^{\alpha}\, \alpha\, \alpha$, let $G^{\mathsf{Set}} = \llbracket \vdash g : \mathsf{Nat}^{\alpha}\, \alpha\, \alpha \rrbracket^{\mathsf{Set}}$, and let $G^{\mathsf{Rel}} = \llbracket \vdash g : \mathsf{Nat}^{\alpha}\, \alpha\, \alpha \rrbracket^{\mathsf{Rel}}$. By Theorem 28, $(G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) = G^{\mathsf{Rel}}\rho$. Thus, for all $\rho \in \mathsf{RelEnv}$ and any $(a, b) \in \llbracket \vdash \emptyset \rrbracket^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$
\begin{array}{rcl}
(G^{\mathsf{Set}}, G^{\mathsf{Set}}) \;=\; (G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) & \in & \llbracket \vdash \mathsf{Nat}^{\alpha}\, \alpha\, \alpha \rrbracket^{\mathsf{Rel}}\rho \\
& = & \{\eta : id \Rightarrow id\} \\
& = & \{(\eta_1 : id \Rightarrow id, \eta_2 : id \Rightarrow id)\}
\end{array}
$$

That is, G^{Set} is a natural transformation from the identity functor on Set to itself.

Now let S be any set. If $S = \emptyset$, then there is exactly one morphism $id_S : S \to S$, so $G^{\mathsf{Set}}_S : S \to S$ must be id_S. If $S \neq \emptyset$, then if a is any element of S and $K_a : S \to S$ is the constantly a-valued morphism on S, then instantiating the naturality square implied by the above equality gives that $G^{\mathsf{Set}}_S \circ K_a = K_a \circ G^{\mathsf{Set}}_S$, i.e., $G^{\mathsf{Set}}_S\, a = a$, i.e., $G^{\mathsf{Set}}_S = id_S$. Putting these two cases together we have that for every $S : \mathsf{Set}$, $G^{\mathsf{Set}}_S = id_S$, i.e., G^{Set} is the identity natural transformation for the identity functor on Set. So every closed term g of closed type $\mathsf{Nat}^{\alpha}\alpha\, \alpha$ always denotes the identity natural transformation for the identity functor on Set, i.e., every closed term g of type $\mathsf{Nat}^{\alpha}\,\alpha\,\alpha$ denotes the polymorphic identity function.

6.3 Free Theorem for Type of *filter* for Lists

Let $List\,\alpha = (\mu\phi.\lambda\beta.\mathbb{1} + \beta \times \phi\beta)\alpha$, and let $map = \mathsf{map}_{\lambda A.[\![\emptyset;\alpha\vdash List\,\alpha]\!]^{\mathsf{Set}}\rho[\alpha:=A]}$.

Lemma 29. *If $g : A \to B$, $\rho : \mathsf{RelEnv}$, and $\rho\alpha = (A, B, \langle g\rangle)$, then $[\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho = \langle map\,g\rangle$*

Proof.

$$
\begin{aligned}
&[\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho\\
&= \mu T_\rho([\![\alpha;\emptyset \vdash \alpha]\!]^{\mathsf{Rel}}\rho)\\
&= \mu T_\rho(A, B, \langle g\rangle)\\
&= (\mu T_{\pi_1\rho}A, \mu T_{\pi_2\rho}B, \varinjlim_{n\in\mathbb{N}}(T^n_\rho K_0)^*(A, B, \langle g\rangle))\\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \varinjlim_{n\in\mathbb{N}}\Sigma^n_{i=0}(A, B, \langle g\rangle)^i)\\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \mathsf{List}\,(A, B, \langle g\rangle))\\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \langle map\,g\rangle)
\end{aligned}
$$

The first equality is by Definition 16, the third equality is by Equation 3, and the fourth and sixth equalities are by Equations 22 and 23 below.
The following sequence of equalities shows

$$(T^n_\rho K_0)^*\,R = \Sigma^n_{i=0}R^i \tag{22}$$

by induction on n:

$$
\begin{aligned}
&(T^n_\rho K_0)^*\,R\\
&= T^{\mathsf{Rel}}_\rho(T^{n-1}_\rho K_0)^*\,R\\
&= [\![\alpha;\phi,\beta \vdash \mathbb{1} + \beta\times\phi\beta]\!]^{\mathsf{Set}}\rho[\phi := (T^{n-1}_\rho K_0)^*][\beta := R]\\
&= \mathbb{1} + R\times(T^{n-1}_\rho K_0)^*\,R\\
&= \mathbb{1} + R\times(\Sigma^{n-1}_{i=0}R^i)\\
&= \Sigma^n_{i=0}R^i
\end{aligned}
$$

The following reasoning shows

$$\mathsf{List}\,(A, B, \langle g\rangle) = \langle map\,g\rangle \tag{23}$$

By showing that $(xs, xs') \in \mathsf{List}\,(A, B, \langle g\rangle)$ if and only if $(xs, xs') \in \langle map\,g\rangle$:

$$
\begin{aligned}
&(xs, xs') \in \mathsf{List}\,(A, B, \langle g\rangle)\\
\Longleftrightarrow\ &\forall i.(xs_i, xs'_i) \in \langle g\rangle\\
\Longleftrightarrow\ &\forall i.xs'_i = g(xs_i)\\
\Longleftrightarrow\ &xs' = map\,g\,xs\\
\Longleftrightarrow\ &(xs, xs') \in \langle map\,g\rangle
\end{aligned}
$$

□

Theorem 30. *If $\Gamma;\Phi\,|\,\Delta \vdash t : \tau$ and $\rho \in \mathsf{RelEnv}$, and if $(a, b) \in [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, then $([\![\Gamma;\Phi\,|\,\Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_1\rho)\,a\,, [\![\Gamma;\Phi\,|\,\Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_2\rho)\,b) \in [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho$*

Proof. Immediate from Theorem 27 (at-gen). ☐

Theorem 31. *If* $g : A \to B$, ρ : RelEnv, $\rho\alpha = (A, B, \langle g\rangle)$, $(a, b) \in [\![\alpha; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, $(s \circ g, s) \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho$, *and, for some well-formed term filter,*

$$t = [\![\alpha; \emptyset \,|\, \Delta \vdash \mathit{filter} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Set}}, \text{ then}$$

$$map\, g \circ t(\pi_1\rho)\, a\, (s \circ g) = t(\pi_2\rho)\, b\, s \circ map\, g$$

Proof. By Theorem 30, $(t(\pi_1\rho)a, t(\pi_2\rho)b) \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Rel}}\rho$. Thus if $(s, s') \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho = \rho\alpha \to \mathsf{Eq}_{Bool}$, then

$$\begin{aligned}(t(\pi_1\rho)\, a\, s, t(\pi_2\rho)\, b\, s') &\in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}(List\,\alpha)\,(List\,\alpha)]\!]^{\mathsf{Rel}}\rho \\ &= [\![\alpha; \emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho \to [\![\alpha; \emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho\end{aligned}$$

So if $(xs, xs') \in [\![\alpha; \emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho$ then,

$$(t(\pi_1\rho)\, a\, s\, xs, t(\pi_2\rho)\, b\, s'\, xs') \in [\![\alpha; \emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho \tag{24}$$

Consider the case in which $\rho\alpha = (A, B, \langle g\rangle)$. Then $[\![\alpha; \emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho = \langle map\, g\rangle$, by Lemma 29, and $(xs, xs') \in \langle map\, g\rangle$ implies $xs' = map\, g\, xs$. We also have that $(s, s') \in \langle g\rangle \to \mathsf{Eq}_{Bool}$ implies $\forall (x, gx) \in \langle g\rangle.\ sx = s'(gx)$ and thus $s = s' \circ g$ due to the definition of morphisms between relations. With these instantiations, Equation 25 becomes

$$\begin{aligned}&(t(\pi_1\rho)\, a\, (s' \circ g)\, xs, t(\pi_2\rho)\, b\, s'\, (map\, g\, xs)) \in \langle map\, g\rangle,\\ &i.e.,\\ &map\, g\, (t(\pi_1\rho)\, a\, (s' \circ g)\, xs) = t(\pi_2\rho)\, b\, s'\, (map\, g\, xs),\\ &i.e.,\\ &map\, g \circ t(\pi_1\rho)\, a\, (s' \circ g) = t(\pi_2\rho)\, b\, s' \circ map\, g\end{aligned}$$

as desired.

☐

6.4 Free Theorem for Type of *filter* for GRose

Theorem 32. *Let* $g : A \to B$ *be a function,* $\eta : F \to G$ *a natural transformation of* Set *functors,* ρ : RelEnv, $\rho\alpha = (A, B, \langle g\rangle)$, $\rho\psi = (F, G, \langle \eta\rangle)$, $(a, b) \in [\![\alpha, \psi; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, *and* $(s \circ g, s) \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho$. *Then, for any well-formed term filter, if we call*

$$t = [\![\alpha, \psi; \emptyset \,|\, \Delta \vdash \mathit{filter} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Set}}$$

we have that

$$map\,\eta\,(g + 1) \circ t(\pi_1\rho)\, a\, (s \circ g) = t(\pi_2\rho)\, b\, s \circ map\,\eta\, g$$

Proof. By Theorem 30,

$$(t(\pi_1\rho)a, t(\pi_2\rho)b) \in [\![\alpha, \psi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\, Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Rel}}\rho$$

Thus if $(s, s') \in [\![\alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho = \rho\alpha \to \mathsf{Eq}_{Bool}$, then

$$\begin{aligned}(t(\pi_1\rho)\, a\, s, t(\pi_2\rho)\, b\, s') &\in [\![\alpha, \psi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{GRose}\ \psi\,\alpha)\,(\mathsf{GRose}\ \psi\,(\alpha + \mathbb{1}))]\!]^{\mathsf{Rel}}\rho \\ &= [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\,\alpha]\!]^{\mathsf{Rel}}\rho \to [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\,(\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho\end{aligned}$$

So if $(xs, xs') \in [\![\alpha; \emptyset \vdash \mathsf{GRose}\ \psi\ \alpha]\!]^{\mathsf{Rel}}\rho$ then,

$$(t(\pi_1\rho)\, a\, s\, xs, t(\pi_2\rho)\, b\, s'\, xs') \in [\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\, (\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho \tag{25}$$

Since $\rho\alpha = (A, B, \langle g\rangle)$ and $\rho\psi = (F, G, \langle\psi\rangle)$, then $[\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\ \alpha]\!]^{\mathsf{Rel}}\rho = \langle map\, \eta g\rangle$ and $[\![\alpha, \psi; \emptyset \vdash \mathsf{GRose}\ \psi\, (\alpha + \mathbb{1})]\!]^{\mathsf{Rel}}\rho = \langle map\, \eta(g + 1)\rangle$, by Lemma 29. Moreover, $(xs, xs') \in \langle map\, \eta\, g\rangle$ implies $xs' = map\, \eta\, g\, xs$. We also have that $(s, s') \in \langle g\rangle \to \mathsf{Eq}_{Bool}$ implies $\forall (x, gx) \in \langle g\rangle.\ sx = s'(gx)$ and thus $s = s' \circ g$ due to the definition of morphisms between relations. With these instantiations, Equation 25 becomes

$$\begin{array}{l}
(t(\pi_1\rho)\, a\, (s' \circ g)\, xs, t(\pi_2\rho)\, b\, s'\, (map\, \eta\, g\, xs)) \in \langle map\, \eta\, (g + 1)\rangle,\\
i.e.,\\
map\, \eta\, (g + 1)\, (t(\pi_1\rho)\, a\, (s' \circ g)\, xs) = t(\pi_2\rho)\, b\, s'\, (map\, \eta\, g\, xs),\\
i.e.,\\
map\, \eta\, (g + 1) \circ t(\pi_1\rho)\, a\, (s' \circ g) = t(\pi_2\rho)\, b\, s' \circ map\, \eta\, g
\end{array}$$

as desired.

□

6.5 Short Cut Fusion for Lists

Theorem 33. *Let* $\vdash \tau : \mathcal{F}$, $\vdash \tau' : \mathcal{F}$, *and* $\beta; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \beta)\, \beta)\, \beta$. *If*

$$G = [\![\beta; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \beta)\, \beta)\, \beta]\!]^{\mathsf{Set}}$$

then

$$fold_{1+\tau\times\_}\, n\, c\, (G\, (\mathsf{List}\, [\![\vdash \tau]\!]^{\mathsf{Set}})\, nil\, cons) = G\, [\![\vdash \tau']\!]^{\mathsf{Set}}\, n\, c$$

Proof. Let $\vdash \tau : \mathcal{F}$ and $\vdash \tau' : \mathcal{F}$, let

$$\beta; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \beta)\, \beta)\, \beta$$

and let

$$G = [\![\beta; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \beta)\, \beta)\, \beta]\!]^{\mathsf{Set}}$$

Then Theorem 28 gives that, for any relation environment ρ and any $(a, b) \in [\![\beta; \emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, then (eliding the only possible instantiations of a and b) we have

$$(G\, (\pi_1\rho), G\, (\pi_2\rho)) \in [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \beta)\, \beta)\, \beta]\!]^{\mathsf{Rel}}\rho$$

Since

$$\begin{array}{cl}
& [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \beta)\, \beta)\, \beta]\!]^{\mathsf{Rel}}\rho\\
= & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \beta)\, \beta]\!]^{\mathsf{Rel}}\rho \to \rho\beta\\
= & ([\![\beta; \emptyset \vdash \mathbb{1} + \tau \times \beta]\!]^{\mathsf{Rel}}\rho \to \rho\beta) \to \rho\beta\\
= & ((\mathbb{1} + [\![\vdash \tau]\!]^{\mathsf{Rel}}\rho \times \rho\beta) \to \rho\beta) \to \rho\beta\\
\cong & ((([\![\vdash \tau]\!]^{\mathsf{Rel}}\rho \times \rho\beta) \to \rho\beta) \times \rho\beta) \to \rho\beta
\end{array}$$

we have that if $(c', c) \in [\![\vdash \tau]\!]^{\mathsf{Rel}}\rho \times \rho\beta \to \rho\beta$ and $(n', n) \in \rho\beta$, then

$$(G\, (\pi_1\rho)\, n'\, c', G\, (\pi_2\rho)\, n\, c) \in \rho\beta$$

Now note that

$$[\![\vdash \mathsf{fold}^{\tau'}_{\mathbb{1}+\tau\times\beta} : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \tau')\, \tau')\, (\mathsf{Nat}^{\emptyset}(\mu\alpha.\mathbb{1} + \tau \times \alpha)\, \tau')]\!]^{\mathsf{Set}} = fold_{1+\tau\times\_}$$

and observe that if $c \in [\![\vdash \tau]\!]^{\mathsf{Set}} \times [\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}}$ and $n \in [\![\vdash \tau']\!]^{\mathsf{Set}}$, then

$$(n, c) \in [\![\vdash \mathsf{Nat}^{\emptyset}(\mathbb{1} + \tau \times \tau')\,\tau']\!]^{\mathsf{Set}}$$

Consider the instantiation:

$$\begin{array}{rcl} \pi_1\rho\beta & = & [\![\vdash \mu\alpha.\mathbb{1} + \tau \times \alpha]\!]^{\mathsf{Set}} \;=\; \mathsf{List}\ [\![\vdash \tau]\!]^{\mathsf{Set}} \\ \pi_2\rho\beta & = & [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & = & \langle \mathit{fold}_{1+\tau\times\_}\ n\, c\rangle : \mathsf{Rel}(\pi_1\rho\beta, \pi_2\rho\beta) \\ c' & = & \mathit{cons} \\ n' & = & \mathit{nil} \end{array}$$

Clearly, $(\mathit{nil}, n) \in \rho\beta = \langle \mathit{fold}_{1+\tau\times\_}\ n\, c\rangle$ because $\mathit{fold}_{1+\tau\times\_}\ n\, c\ \mathit{nil} = n$. Moreover, $(\mathit{cons}, c) \in [\![\vdash \tau]\!]^{\mathsf{Rel}} \times \rho\beta \to \rho\beta$ since if $(x, x') \in [\![\vdash \tau]\!]^{\mathsf{Rel}}$, i.e., $x = x'$, and if $(y, y') \in \rho\beta = \langle \mathit{fold}_{1+\tau\times\_}\ n\, c\rangle$, i.e., $y' = \mathit{fold}_{1+\tau\times\_}\ n\, c\, y$, then

$$(\mathit{cons}\, x\, y, c\, x\, (\mathit{fold}_{1+\tau\times\_}\ n\, c\, y)) \in \langle \mathit{fold}_{1+\tau\times\_}\ n\, c\rangle$$

i.e.,

$$c\, x\, (\mathit{fold}_{1+\tau\times\_}\ n\, c\, y) = \mathit{fold}_{1+\tau\times\_}\ n\, c\, (\mathit{cons}\, x\, y)$$

holds by definition of $\mathit{fold}_{1+\tau\times\_}$. We therefore conclude that

$$(G\ (\mathsf{List}\ [\![\vdash \tau]\!]^{\mathsf{Set}})\ \mathit{nil}\ \mathit{cons}, G\, [\![\vdash \tau']\!]^{\mathsf{Set}}\ n\, c) \in \langle \mathit{fold}_{1+\tau\times\_}\ n\, c\rangle$$

i.e., that

$$\mathit{fold}_{1+\tau\times\_}\ n\, c\ (G\ (\mathsf{List}\ [\![\vdash \tau]\!]^{\mathsf{Set}})\ \mathit{nil}\ \mathit{cons}) = G\, [\![\vdash \tau']\!]^{\mathsf{Set}}\ n\, c$$

□

6.6 Short Cut Fusion for Arbitrary ADTs

Theorem 34. *Let* $\overline{\vdash \tau : \mathcal{F}}$*, let* $\vdash \tau' : \mathcal{F}$*, let* $\overline{\alpha}; \beta \vdash F : \mathcal{F}$*, and let* $\beta; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta$*. If we regard*

$$\begin{array}{rcl} H & = & [\![\emptyset; \beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \\ G & = & [\![\beta; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}\, F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Set}} \end{array}$$

as functors in β*, then for every* $B \in H[\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}}$ *we have*

$$\mathit{fold}_H\, B\ (G\ \mu H\ \mathit{in}_H) = G\, [\![\vdash \tau']\!]^{\mathsf{Set}}\ B$$

Proof. We first note that the type of g is well-formed, since $\emptyset; \beta \vdash F[\overline{\alpha := \tau}] : \mathcal{F}$ so our promotion theorem gives that $\beta; \emptyset \vdash F[\overline{\alpha := \tau}] : \mathcal{F}$, and $\emptyset; \beta \vdash \beta : \mathcal{F}$ so that our promotion theorem gives $\beta; \emptyset \vdash \beta : \mathcal{F}$. From these facts we deduce that $\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}\, F[\overline{\alpha := \tau}]\,\beta : \mathcal{T}$, and thus that $\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta : \mathcal{T}$.

Theorem 28 gives that, for any relation environment ρ and any $(a, b) \in [\![\beta; \emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$(G\,(\pi_1\rho), G\,(\pi_2\rho)) \in [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Rel}}\rho$$

Since

$$\begin{array}{rl} & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta)\,\beta]\!]^{\mathsf{Rel}}\rho \\ = & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho \to \rho\beta \end{array}$$

we have that if $(A, B) \in [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho$ then

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \in \rho\beta$$

Now note that

$$[\![\vdash \mathsf{fold}^{\tau'}_{F[\overline{\alpha := \tau}]} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,F[\overline{\alpha := \tau}][\beta := \tau']\,\tau')\,(\mathsf{Nat}^{\emptyset}\,(\mu\beta.F[\overline{\alpha := \tau}]\,\tau')]\!]^{\mathsf{Set}} = \mathit{fold}_H$$

and consider the instantiation

$$\begin{array}{rcl} A & = & \mathit{in}_H : H(\mu H) \to \mu H \\ B & : & H[\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & = & \langle \mathit{fold}_H\, B\rangle \end{array}$$

(Note that all the types here are well-formed.) This gives

$$\begin{array}{rcl} \pi_1\rho\beta & = & [\![\vdash \mu\beta.F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \;=\; \mu H \\ \pi_2\rho\beta & = & [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & : & \mathsf{Rel}(\pi_1\rho\beta, \pi_2\rho\beta) \\ A & : & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_1\rho) \\ B & : & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_2\rho) \end{array}$$

since

$$\begin{array}{rcll} A = \mathit{in}_H & : & H(\mu H) \to \mu H & \\ & = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\mu[\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}) \to \mu[\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} & \\ & = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_1\rho) \to [\![\emptyset;\beta \vdash \beta]\!]^{\mathsf{Set}}(\pi_1\rho) & \\ & = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_1\rho) \to [\![\beta;\emptyset \vdash \beta]\!]^{\mathsf{Set}}(\pi_1\rho) & \text{Daniel's trick; now a theorem} \\ & = & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_1\rho) & \end{array}$$

where "Daniel's trick" is the observation that a functor can be seen as non-functorial when we only care about its action on objects. This is now a theorem. We also have

$$\begin{array}{rcll} (A,B) = (\mathit{in}_H, B) & \in & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho & \\ & = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho[\beta := \langle \mathit{fold}_H\, B\rangle] \to \langle \mathit{fold}_H\, B\rangle & \\ & = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\langle \mathit{fold}_H\, B\rangle \to \langle \mathit{fold}_H\, B\rangle & \\ & = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\langle \mathit{fold}_H\, B\rangle \to \langle \mathit{fold}_H\, B\rangle & \text{Daniel's trick; now a theorem} \\ & = & \langle [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\,(\mathit{fold}_H\, B)\rangle \to \langle \mathit{fold}_H\, B\rangle & \text{by the graph lemma} \\ & = & \langle \mathit{map}_H\,(\mathit{fold}_H\, B)\rangle \to \langle \mathit{fold}_H\, B\rangle & \end{array}$$

since if $(x,y) \in \langle \mathit{map}_H\,(\mathit{fold}_H\, B)\rangle$, i.e., if $\mathit{map}_H\,(\mathit{fold}_H\, B)\,x = y$, then $\mathit{fold}_H\, B\,(\mathit{in}_H\, x) = B\,y = B\,(\mathit{map}_H\,(\mathit{fold}_H\, B)\,x)$ by the definition of fold_H as a (indeed, the unique) morphism from in_H to B. Thus,

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \in \langle \mathit{fold}_H\, B\rangle$$

i.e.,

$$\mathit{fold}_H\, B\,(G\,(\pi_1\rho)\,\mathit{in}_H) = G\,(\pi_2\rho)\,B$$

Since β is the only free variable in G, this simplifies to

$$\mathit{fold}_H\, B\,(G\,\mu H\,\mathit{in}_H) = G\,[\![\vdash \tau']\!]^{\mathsf{Set}}\,B$$

□

6.7 Short Cut Fusion for Arbitrary Nested Types

fold/build rule shows that Church encodings for nested types are iso (have the same interp) as the nested types themselves. Just like Ex 2.8 in parpoe, but categorical rather than operational semantics. But built-in data types are more versatile in implementations. They can be inducted on, e.g. And they are stored in the heap, not the run-time stack, and are therefore more efficient.

"Most higher order type languages meant for human programmers eschew fully impredicative polymorphism" says Andy on p322.

Can take $\emptyset; \alpha \vdash c$ with $[\![\emptyset; \alpha \vdash c]\!]^{\mathsf{Set}}\rho = C$ for all ρ, i.e., can take c to denote a constant C. We then get a free theorem whose conclusion is $\mathit{fold}_H\, B \circ G\, \mu H\, \mathit{in}_H = G\,[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}\, B$.

Can do Hinze's bit-reversal protocol in our system with
$\mathsf{cat} :: \alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathsf{List}\,\alpha)(\mathsf{List}\,\alpha))(\mathsf{List}\,\alpha)$
$\mathsf{zip} :: \alpha; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}(\mathsf{List}\,\alpha)(\mathsf{List}\,\beta))(\mathsf{List}\,(\alpha \times \beta))$
?

Theorem 35. *Let* $\emptyset; \phi, \alpha \vdash F : \mathcal{F}$, *let* $\emptyset; \alpha \vdash K : \mathcal{F}$, *and let* $\phi; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha))$. *If we let* $H : [\mathsf{Set}, \mathsf{Set}] \to [\mathsf{Set}, \mathsf{Set}]$ *be defined by*

$$H\, f\, x \quad = \quad [\![\emptyset; \phi, \alpha \vdash F]\!]^{\mathsf{Set}}[\phi := f][\alpha := x]$$

and let

$$G = [\![\phi; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha))]\!]^{\mathsf{Set}}$$

then we have that, for every $B \in H[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \to [\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}$,

$$\mathit{fold}_H\, B\,(G\, \mu H\, \mathit{in}_H) = G\,[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}\, B$$

Proof. We first note that the type of g is well-formed since $\emptyset; \phi, \alpha \vdash F : \mathcal{F}$ so our promotion theorem gives that $\phi; \alpha \vdash F : \mathcal{F}$, and $\phi; \alpha \vdash \phi\alpha : \mathcal{F}$, so that $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha) : \mathcal{T}$ and $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\mathbb{1}\,(\phi\alpha) : \mathcal{T}$. Then $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha) : \mathcal{F}$ and $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\mathbb{1}\,(\phi\alpha) : \mathcal{F}$ also hold, and, finally, $\phi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha)) : \mathcal{T}$

Theorem 28 gives that, for any relation environment ρ and any $(a, b) \in [\![\phi, \alpha; \emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$\begin{array}{rcl}
(G\,(\pi_1\rho), G\,(\pi_2\rho)) & \in & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\alpha} F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha))]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\, \mathbb{1}\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to (\,\lambda A.1 \Rightarrow \lambda A.\,(\rho\phi)A\,) \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to (1 \Rightarrow \rho\phi) \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to \rho\phi
\end{array}$$

So if $(A, B) \in [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho$ then

$$(G\,(\pi_1\rho)\, A, G\,(\pi_2\rho)\, B) \quad \in \quad \rho\phi$$

Now note that

$$[\![\vdash \mathsf{fold}^K_F : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha} F[\phi := K]\, K)\,(\mathsf{Nat}^{\alpha}((\mu\phi.\lambda\alpha.F)\alpha)\, K)]\!]^{\mathsf{Set}} = \mathit{fold}_H$$

and consider the instantiation

$$\begin{array}{rcl}
A & = & \mathit{in}_H : H(\mu H) \Rightarrow \mu H \\
B & : & H[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \Rightarrow [\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \\
\rho\phi & = & \langle \mathit{fold}_H\, B \rangle \qquad \text{a graph of a natural transformation, defined in Enrico's notes}
\end{array}$$

(Note that all the types here are well-formed.) This gives

$$\begin{array}{rcl}
\pi_1\rho\phi & = & \mu H \\
\pi_2\rho\phi & = & [\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \\
\rho\phi & : & \mathsf{Rel}(\pi_1\rho\phi, \pi_2\rho\phi) \\
A & : & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_1\rho) \\
B & : & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_2\rho)
\end{array}$$

since

$$\begin{array}{rcl} A = in_H & : & H(\mu H) \Rightarrow \mu H \\ & = & [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}[\phi := \mu[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}] \Rightarrow \mu[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}} \\ & = & [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho) \Rightarrow [\![\emptyset;\phi,\alpha \vdash \phi\alpha]\!]^{\mathsf{Set}}(\pi_1\rho) \\ & = & [\![\phi;\alpha \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho) \Rightarrow [\![\phi;\alpha \vdash \phi\alpha]\!]^{\mathsf{Set}}(\pi_1\rho) \qquad \text{Daniel's trick; now a theorem} \\ & = & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_1\rho) \end{array}$$

We also have

$$\begin{array}{rcl} (A,B) = (in_H, B) & \in & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \\ & = & \lambda A.[\![\phi;\alpha \vdash F]\!]^{\mathsf{Rel}}\rho[\alpha := A] \Rightarrow \lambda A.(\rho\phi)A \\ & = & \lambda A.[\![\phi;\alpha \vdash F]\!]^{\mathsf{Rel}}[\phi := \langle fold_H\, B\rangle][\alpha := A] \Rightarrow \langle fold_H\, B\rangle \\ & = & \lambda A.[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Rel}}[\phi := \langle fold_H\, B\rangle][\alpha := A] \Rightarrow \langle fold_H\, B\rangle \qquad \text{Daniel's trick; now a the} \\ & = & [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Rel}}\langle fold_H\, B\rangle \Rightarrow \langle fold_H\, B\rangle \\ & = & \langle [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}\,(fold_H\, B)\rangle \Rightarrow \langle fold_H\, B\rangle \qquad \text{Graph Lemma} \\ & = & \langle map_H\,(fold_H\, B)\rangle \Rightarrow \langle fold_H\, B\rangle \end{array}$$

since if $(x,y) \in \langle map_H\,(fold_H\,B)\rangle$, i.e., if $map_H\,(fold_H\,B)\,x = y$, then $fold_H\,B\,(in_H\,x) = B\,y = B\,(map_H\,(fold_H\,B)\,x)$ by the definition of $fold_H$ as a (indeed, the unique) morphism from in_H to B. Thus,

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \in \langle fold_H\,B\rangle$$

i.e.,

$$fold_H\,B\,(G\,(\pi_1\rho)\;in_H) = G\,(\pi_2\rho)\,B$$

Since ϕ is the only free variable in G, this simplifies to

$$fold_H\,B\,(G\,\mu H\;in_H) = G\,[\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}}\,B$$

□

7 CONCLUSION AND DIRECTIONS FOR FUTURE WORK

We have forall-types.

Can do everything in abstract locally presentable cartesian closed category.

Give definitions for arb lpccc, but compute free theorems in Set/Rel.

Future Work (in progress): extend calculus to GADTs

Add more polymorphisms (all foralls), even though most free theorems only use one level (or maybe two, like short cut).

Couldn't do this before [Johann and Polonsky 2019] because we didn't know before that nested types (and then some) always have well-defined interpretations in locally finitely presentable categories like Set and Rel. In fact, could extend results here to "locally presentable fibrations", where these are yet to be defined, but would at least have locally presentable base and total categories with the locally presentable structure preserved by the fibration and appropriate reflection of the total category in the base (as in Alex's effects paper?).

fixed points at term level ala Pitts

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