Practical Parametricity for GADTs

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Abstract goes here

We should ask what, intuitively, it means to have parametricity for a calculus with GADTs. In particular it should mean that any polymorphic function that takes a GADT as input should operate uniformly over all type instances of the GADT. That means that the GADT must itself be "parametric", i.e., uniform in its type indices. But what does this mean? For ADTs and nested types it means that the datatype elements are constructed in exactly the same ways at each type instance. For GADTs, it means that as well. That is, we require related elements of a GADT to be constructed with the same sequence of constructors acting on related data. This entails that which instances are inhabited, as well as how the data in each of those instances are constructed, is type independent.

Locally presentable categories are the closest approximation to small complete categories. It's remarkable that local presentability is sufficient not just to give semantics to GADTs, but is also all that is needed to ensure that the model induced by that semantics is actually parametric.

Maybe develop our theory for *any* $\lambda \geq \omega$, and then specialize to ω_1 (for ωCPO) when discussing GADTs in future work? Can we do that? It seems we really use properties of Set to get that interpretations of Nat-types are well-defined. It's not clear how to put ωCPO structures on the interpretations of Lan-types.

1 THE CALCULUS

1.1 Types

For each $k \geq 0$, we assume countable sets $\mathbb{T}^k$ of *type constructor variables of arity* k and $\mathbb{F}^k$ of *functorial variables of arity* k, all mutually disjoint. The sets of all type constructor variables and functorial variables are $\mathbb{T} = \bigcup_{k \geq 0} \mathbb{T}^k$ and $\mathbb{F} = \bigcup_{k \geq 0} \mathbb{F}^k$, respectively, and a *type variable* is any element of $\mathbb{T} \cup \mathbb{F}$. We use lower case Greek letters for type variables, writing ϕ^k to indicate that $\phi \in \mathbb{T}^k \cup \mathbb{F}^k$, and omitting the arity indicator k when convenient, unimportant, or clear from context. We reserve letters from the beginning of the alphabet to denote type variables of arity 0, i.e., elements of $\mathbb{T}^0 \cup \mathbb{F}^0$. We write $\overline{\zeta}$ for either a set $\{\zeta_1, ..., \zeta_n\}$ of type constructor variables or a set of functorial variables when the cardinality n of the set is unimportant or clear from context. If P is a set of type variables we write $P, \overline{\phi}$ for $P \cup \overline{\phi}$ when $P \cap \overline{\phi} = \emptyset$. We omit the vector notation for a singleton set, thus writing ϕ, instead of $\overline{\phi}$, for $\{\phi\}$.

Definition 1. *Let V be a finite subset of $\mathbb{T}$, P be a finite subset of $\mathbb{F}$, $\overline{\alpha}$ be a finite subset of $\mathbb{F}^0$ disjoint from P, and $\phi^k \in \mathbb{F}^k \setminus P$. The set $\mathcal{F}^P(V)$ of* functorial expressions *over P and V are given by*

$$\begin{aligned}\mathcal{F}^P(V) \;::=\; & \mathbb{0} \mid \mathbb{1} \mid \mathsf{Nat}^P \mathcal{F}^P(V)\, \mathcal{F}^P(V) \mid P\,\overline{\mathcal{F}^P(V)} \mid V\,\overline{\mathcal{F}^P(V)} \mid \mathcal{F}^P(V) + \mathcal{F}^P(V) \\ & \mid \mathcal{F}^P(V) \times \mathcal{F}^P(V) \mid \left(\mu\phi^k.\lambda\alpha_1 \ldots \alpha_k.\mathcal{F}^{P,\alpha_1,\ldots,\alpha_k,\phi}(V)\right)\overline{\mathcal{F}^P(V)} \\ & \mid (\mathsf{Lan}^{\overline{\alpha}}_{\overline{\mathcal{F}^{\overline{\alpha}}}} \mathcal{F}^{P,\overline{\alpha}})\overline{\mathcal{F}^P}\end{aligned}$$

A *type* over P and V is any element of $\mathcal{F}^P(V)$. The difference with [Johann et al. 2020] here lies solely in the incorporation of functorial expressions constructed using Lan.

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The notation for types entails that an application $FF_1...F_k$ is allowed only when F is a type variable of arity k, or F is a subexpression of the form $\mu\phi^k.\lambda\alpha_1...\alpha_k.F'$ or $\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F'$. Moreover, if F has arity k then F must be applied to exactly k arguments. Accordingly, an overbar indicates a sequence of subexpressions whose length matches the arity of the type applied to it. The fact that types are always in *η-long normal form* avoids having to consider β-conversion at the level of types. In a subexpression $\mathsf{Nat}^{\Phi}F\,G$, the Nat operator binds all occurrences of the variables in Φ in F and G. Note that, by contrast with [Johann et al. 2020], variables of arity greater than 0 are allowed in Φ; this is necessary to construct well-typed terms of Lan types. In a subexpression $\mu\phi^k.\lambda\overline{\alpha}.F$, the μ operator binds all occurrences of the variable ϕ, and the λ operator binds all occurrences of the variables in $\overline{\alpha}$, in the body F. And in a subexpression $(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}$, the Lan operator binds all occurrences of the variables in $\overline{\alpha}$ in every element of $\overline{K}$, as well as in F.

A *type constructor context* is a finite set Γ of type constructor variables, and a *functorial context* is a finite set Φ of functorial variables. In Definition 2, a judgment of the form $\Gamma;\Phi \vdash F$ indicates that the type F is intended to be functorial in the variables in Φ but not necessarily in those in Γ.

Definition 2. *The formation rules for the set $\mathcal{F} \subseteq \bigcup_{V\subseteq\mathbb{T},P\subseteq\mathbb{F}}\mathcal{F}^P(V)$ of* well-formed types *are*

$$\frac{}{\Gamma;\Phi \vdash \mathbb{0}} \qquad \frac{}{\Gamma;\Phi \vdash \mathbb{1}}$$

$$\frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G}{\Gamma;\Phi \vdash F+G} \qquad \frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G}{\Gamma;\Phi \vdash F\times G}$$

$$\frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G}{\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}F\,G}$$

$$\frac{\phi^k \in \Gamma\cup\Phi \qquad \overline{\Gamma;\Phi \vdash F}}{\Gamma;\Phi \vdash \phi^k\overline{F}}$$

$$\frac{\Gamma;\overline{\gamma^0},\overline{\alpha^0},\phi^k \vdash F \qquad \overline{\Gamma;\Phi,\overline{\gamma^0} \vdash G}}{\Gamma;\Phi,\overline{\gamma^0} \vdash (\mu\phi^k.\lambda\overline{\alpha^0}.\,F)\,\overline{G}}$$

$$\frac{\Gamma;\Phi,\overline{\alpha}^0 \vdash F \qquad \overline{\Gamma;\overline{\alpha}^0 \vdash K} \qquad \overline{\Gamma;\Phi \vdash A}}{\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}^0}_{\overline{K}}F)\,\overline{A}}$$

In addition to textual replacement, we also have a proper substitution operation on types. If F is a type over P and V, if P and V contain only type variables of arity 0, and if $k = 0$ for every occurrence of ϕ^k bound by μ in F, then we say that F is *first-order*; otherwise we say that F is *second-order*. Substitution for first-order types is the usual capture-avoiding textual substitution. We write $F[\alpha := \sigma]$ for the result of substituting σ for α in F, and $F[\alpha_1 := F_1, ..., \alpha_k := F_k]$, or $F[\overline{\alpha := F}]$ when convenient, for $F[\alpha_1 := F_1][\alpha_2 := F_2, ..., \alpha_k := F_k]$. Substitution for second-order types is defined below, where we adopt a similar notational convention for vectors of types. Note that it is not correct to substitute along non-functorial variables.

Definition 3. *If* $\Gamma;\Phi,\phi^k \vdash H$ *and if* $\Gamma;\Phi,\overline{\alpha} \vdash F$ *with* $|\overline{\alpha}| = k$*, then* $\Gamma;\Phi \vdash H[\phi :=_{\overline{\alpha}} F]$*. Similarly, if* $\Gamma,\phi^k;\Phi \vdash H$*, and if* $\Gamma;\overline{\psi},\overline{\alpha} \vdash F$ *with* $|\overline{\alpha}| = k$ *and* $\Phi \cap \overline{\psi} = \emptyset$*, then* $\Gamma,\overline{\psi}';\Phi \vdash H[\phi :=_{\overline{\alpha}} F[\overline{\psi :== \psi'}]]$.

Here, the operation $(\cdot)[\phi :=_{\overline{\alpha}} F]$ *of* second-order type substitution along $\overline{\alpha}$ *is defined by:*

$$
\begin{array}{lcl}
\mathbb{0}[\phi :=_{\overline{\alpha}} F] & = & \mathbb{0} \\
\mathbb{1}[\phi :=_{\overline{\alpha}} F] & = & \mathbb{1} \\
(\mathsf{Nat}^{\overline{\beta}} G\, K)[\phi :=_{\overline{\alpha}} F] & = & \mathsf{Nat}^{\overline{\beta}}\, (G[\phi :=_{\overline{\alpha}} F])\, (K[\phi :=_{\overline{\alpha}} F]) \\
(\psi \overline{G})[\phi :=_{\overline{\alpha}} F] & = & \begin{cases} \psi\, \overline{G[\phi :=_{\overline{\alpha}} F]} & \text{if } \psi \neq \phi \\ F[\overline{\alpha := G[\phi :=_{\overline{\alpha}} F]}] & \text{if } \psi = \phi \end{cases} \\
(G + K)[\phi :=_{\overline{\alpha}} F] & = & G[\phi :=_{\overline{\alpha}} F] + K[\phi :=_{\overline{\alpha}} F] \\
(G \times K)[\phi :=_{\overline{\alpha}} F] & = & G[\phi :=_{\overline{\alpha}} F] \times K[\phi :=_{\overline{\alpha}} F] \\
((\mu\psi.\lambda\overline{\beta}.\, G)\overline{K})[\phi :=_{\overline{\alpha}} F] & = & (\mu\psi.\lambda\overline{\beta}.\, G[\phi :=_{\overline{\alpha}} F])\, \overline{K[\phi :=_{\overline{\alpha}} F]} \\
((\mathsf{Lan}^{\overline{\beta}}_{\overline{H}}\, G)\overline{K})[\phi :=_{\overline{\alpha}} F] & = & (\mathsf{Lan}^{\overline{\beta}}_{\overline{H}}\, G[\phi :=_{\overline{\alpha}} F])\, \overline{K[\phi :=_{\overline{\alpha}} F]}
\end{array}
$$

We note that $(\cdot)[\phi^0 :=_\emptyset F]$ coincides with first-order substitution. We also omit $\overline{\alpha}$ when convenient.

1.2 Terms

We now define our term calculus. To do so we assume an infinite set $\mathcal{V}$ of term variables disjoint from $\mathbb{T}$ and $\mathbb{F}$. If Γ is a type constructor context and Φ is a functorial context, then a *term context for* Γ *and* Φ is a finite set of bindings of the form $x : F$, where $x \in \mathcal{V}$ and $\Gamma; \Phi \vdash F$. We adopt the same conventions for denoting disjoint unions and for vectors in term contexts as for type constructor contexts and functorial contexts.

Definition 4. *Let* Δ *be a term context for* Γ *and* Φ*. The formation rules for the set of* well-formed terms over Δ *are*

$$
\frac{\Gamma; \Phi \vdash F}{\Gamma; \Phi \mid \Delta, x : F \vdash x : F}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : \mathbb{0} \qquad \Gamma; \Phi \vdash F}{\Gamma; \Phi \mid \Delta \vdash \bot_F t : F}
\qquad
\frac{}{\Gamma; \Phi \mid \Delta \vdash \top : \mathbb{1}}
$$

$$
\frac{\Gamma; \Phi \mid \Delta \vdash s : F}{\Gamma; \Phi \mid \Delta \vdash \mathsf{inl}\, s : F + G}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : G}{\Gamma; \Phi \mid \Delta \vdash \mathsf{inr}\, t : F + G}
$$

$$
\frac{\Gamma; \Phi \vdash F, G \qquad \Gamma; \Phi \mid \Delta \vdash t : F + G \qquad \Gamma; \Phi \mid \Delta, x : F \vdash l : K \qquad \Gamma; \Phi \mid \Delta, y : G \vdash r : K}{\Gamma; \Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{x \mapsto l;\, y \mapsto r\} : K}
$$

$$
\frac{\Gamma; \Phi \mid \Delta \vdash s : F \qquad \Gamma; \Phi \mid \Delta \vdash t : G}{\Gamma; \Phi \mid \Delta \vdash (s, t) : F \times G}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : F \times G}{\Gamma; \Phi \mid \Delta \vdash \pi_1 t : F}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : F \times G}{\Gamma; \Phi \mid \Delta \vdash \pi_2 t : G}
$$

$$\frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G \qquad \Gamma;\Phi \,|\, \Delta, x : F \vdash t : G}{\Gamma;\emptyset \,|\, \Delta \vdash L_{\Phi} x.t : \mathsf{Nat}^{\Phi}\, F\, G}$$

$$\frac{\overline{\Gamma;\Phi,\overline{\beta} \vdash K} \qquad \Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\psi}}\, F\, G \qquad \Gamma;\Phi \,|\, \Delta \vdash s : F[\overline{\psi :=_{\overline{\beta}} K}]}{\Gamma;\Phi \,|\, \Delta \vdash t_{\overline{K}} s : G[\overline{\psi :=_{\overline{\beta}} K}]}$$

$$\frac{\Gamma;\Phi,\overline{\phi} \vdash H \qquad \overline{\Gamma;\Phi,\overline{\beta} \vdash F} \qquad \overline{\Gamma;\Phi,\overline{\beta} \vdash G}}{\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\Phi,\overline{\beta}}\, F\, G})\, (\mathsf{Nat}^{\Phi}\, H[\overline{\phi :=_{\overline{\beta}} F}]\; H[\overline{\phi :=_{\overline{\beta}} G}])}$$

$$\frac{\Gamma;\Phi,\phi,\overline{\alpha} \vdash H}{\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\Phi,\overline{\beta}} H[\phi :=_{\overline{\beta}} (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}$$

$$\frac{\Gamma;\phi,\Phi,\overline{\alpha} \vdash H \qquad \Gamma;\Phi,\overline{\beta} \vdash F}{\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{fold}_H^F : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\Phi,\overline{\beta}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\Phi,\overline{\beta}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)}$$

$$\frac{\Gamma;\Phi,\overline{\alpha} \vdash F \qquad \overline{\Gamma;\overline{\alpha} \vdash K}}{\Gamma;\emptyset \,|\, \emptyset \vdash \int_{K,F} : \mathsf{Nat}^{\Phi,\overline{\alpha}}\, F\, (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}\, F)\overline{K}}$$

$$\frac{\Gamma;\emptyset \,|\, \Delta \vdash \eta : \mathsf{Nat}^{\Phi,\overline{\alpha}}\, F\; G[\overline{\beta := K}]}{\Gamma;\emptyset \,|\, \Delta \vdash \partial_F^{G,\overline{K}} \eta : \mathsf{Nat}^{\Phi,\overline{\beta}}\, (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{\beta}\; G}$$

Sum and product intro and elim rules should be annotated with constituent types for consistency?

In the rule for $L_{\overline{\alpha}} x.t$, the L operator binds all occurrences of the type variables in $\overline{\alpha}$ in the type of the term variable x and in the body t, as well as all occurrences of x in t. In the rule for $t_{\overline{K}} s$ there is one functorial expression in $\overline{K}$ for every functorial variable in $\overline{\alpha}$. In the rule for $\mathsf{map}_H^{\overline{F},\overline{G}}$ there is one functorial expression F and one functorial expression G for each functorial variable in $\overline{\phi}$. Moreover, for each ϕ^k in $\overline{\phi}$ the number of functorial variables in $\overline{\beta}$ in the judgments for its corresponding functorial expresssions F and G is k. In the rules for in_H and fold_H^F, the functorial variables in $\overline{\beta}$ are fresh with respect to H, and there is one β for every α. (Recall from above that, in order for the types of in_H and fold_H^F to be well-formed, the length of α must equal the arity of ϕ.)

Substitution for terms is the obvious extension of the usual capture-avoiding textual substitution, and Definition 4 ensures that the expected weakening rules for well-formed terms hold.

We should have a computation rule along the lines of: If $\eta : \mathsf{Nat}^{\overline{\alpha}} F\, G[\overline{\beta := K}]$ then

$$\begin{aligned}
& (\partial_F^{G,\overline{K}} \eta)_{\overline{K[\overline{\alpha := A}]}} \circ \overline{(\int_{K,F})_{\overline{A}}} \;\to\; \eta_{\overline{A}} \\
: \quad & F[\overline{\alpha := A}] \to G[\overline{\beta := K[\overline{\alpha := A}]}] \\
= \quad & F[\overline{\alpha := A}] \to G[\overline{\beta := K}][\overline{\alpha := A}]
\end{aligned}$$

This will appear as a computational property of the term interpretations.

2 INTERPRETING TYPES

We denote the category of sets and functions by Set. The category Rel has as its objects triples (A, B, R) where R is a relation between the objects A and B in Set, i.e., a subset of $A \times B$, and has as its morphisms from (A, B, R) to (A', B', R') pairs $(f : A \to A', g : B \to B')$ of morphisms in Set such that $(f\,a, g\,b) \in R'$ whenever $(a, b) \in R$. We write $R : \mathsf{Rel}(A, B)$ in place of (A, B, R) when convenient. If $R : \mathsf{Rel}(A, B)$ we write $\pi_1 R$ and $\pi_2 R$ for the *domain* A of R and the *codomain* B of R, respectively. If A : Set, then we write $\mathsf{Eq}_A = (A, A, \{(x, x) \mid x \in A\})$ for the *equality relation* on A.

The fundamental idea underlying Reynolds' parametricity is to give each type $F(\alpha)$ with one free variable α both an *object interpretation* F_0 taking sets to sets and a *relational interpretation* F_1 taking relations $R : \mathsf{Rel}(A, B)$ to relations $F_1(R) : \mathsf{Rel}(F_0(A), F_0(B))$, and to interpret each term $t(\alpha, x) : F(\alpha)$ with one free term variable $x : G(\alpha)$ as a map t_0 associating to each set A a function $t_0(A) : G_0(A) \to F_0(A)$, and to each relation R a morphism $t_1(R) : G_1(R) \to F_1(R)$. These interpretations are to be given inductively on the structures of F and t in such a way that they imply two fundamental theorems. The first is an *Identity Extension Lemma*, which states that $F_1(\mathsf{Eq}_A) = \mathsf{Eq}_{F_0(A)}$, and is the essential property that makes a model relationally parametric rather than just induced by a logical relation. The second is an *Abstraction Theorem*, which states that, for any $R : \mathsf{Rel}(A, B)$, $(t_0(A), t_0(B))$ is a morphism in Rel from $(G_0(A), G_0(B), G_1(R))$ to $(F_0(A), F_0(B), F_1(R))$. The Identity Extension Lemma is similar to the Abstraction Theorem except that it holds for *all* elements of a type's interpretation, not just those that are interpretations of terms. Similar theorems are expected to hold for types and terms with any number of free variables.

In the remainder of this section we first inductively define, for each type, an object interpretation in Set and a relational interpretation in Rel. In Section 3 we show that these interpretations satisfy both an Identity Extension Lemma (Theorem 22) and an Abstraction Theorem (Theorem **??**) appropriate to the GADT setting. The key to proving our Identity Extension Lemma is a familiar "cutting down" of the interpretations of universally quantified types to include only the "parametric" elements; the relevant types of the calculus defined above are the Nat-types (which are now richer than those of [Johann et al. 2020]) and the Lan-types. The requisite cutting down requires that the object interpretations of our types in Set are defined simultaneously with their relational interpretations in Rel. We give the object interpretations for our types in Section 2.1 and give their relational interpretations in Section 2.2. While the former are relatively straightforward, the latter are less so because of the delicacy of interpreting the Lan-types. (The object and relational interpretations of Lan-types are not defined in the same "parallel" way, as are the two interpretations of the other types.) Another subtle point is guaranteeing the cocontinuity conditions, adapted to our richer setting from [Johann et al. 2020], that must hold if the type interpretations are to be well-defined. We develop these conditions in Section 2.2, which separates Definitions 6 and 16 in space, but otherwise has no impact on the fact that these definitions are given by mutual induction.

2.1 Object Interpretations of Types

The object interpretations of the types in our calculus will be ω-cocontinuous functors between categories of ω-cocontinuous functors between categories constructed from the locally finitely presentable category Set [Adámek and Rosický 1994]; the relational interpretations will be similar for the locally finitely presentable category Rel. The fact that functor categories of locally finitely presentable categories are again locally finitely presentable ensures that the fixpoints interpreting μ-types in Set and Rel exist, and thus that both the set and relational interpretations of all of the types in Definition 2 are well-defined [Johann and Polonsky 2019].

To define the interpretations of the types in Definition 2 we must first interpret their variables. Writing $[\mathcal{C}, \mathcal{D}]$ for the category of ω-cocontinuous functors from $\mathcal{C}$ to $\mathcal{D}$ for the locally finitely presentable categories $\mathcal{C}$ and $\mathcal{D}$, for our Set interpretations we have:

Definition 5. *A* Set environment *maps each type variable in* $\mathbb{T}^k \cup \mathbb{F}^k$ *to an element of* $[\mathsf{Set}^k, \mathsf{Set}]$*. A morphism* $f : \rho \to \rho'$ *for set environments* ρ *and* ρ' *with* $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$ *maps each type constructor variable* $\psi^k \in \mathbb{T}$ *to the identity natural transformation on* $\rho\psi^k = \rho'\psi^k$ *and each functorial variable* $\phi^k \in \mathbb{F}$ *to a natural transformation from the* k*-ary functor* $\rho\phi^k$ *on* Set *to the* k*-ary functor* $\rho'\phi^k$ *on* Set*. Composition of morphisms on set environments is given componentwise, with the identity morphism mapping each set environment to itself. This gives a category of set environments and morphisms between them, which we denote* SetEnv*.*

When convenient we identify a functor in $[\mathsf{Set}^0, \mathsf{Set}]$ with its value on $*$ and consider a Set environment to map a type variable of arity 0 to an ω-cocontinuous functor from Set^0 to Set, i.e., to an Set. If $\Phi = \{\phi_1^{k_1}, ..., \phi_n^{k_n}\}$ and $\overline{K} = \{K_1, ..., K_n\}$ where $K_i : [\mathsf{Set}^{k_i}, \mathsf{Set}]_\omega$ for $i = 1, ..., n$, then we write $\rho[\Phi := \overline{K}]$ for the Set environment ρ' such that $\rho'\phi_i = K_i$ for $i = 1, ..., n$ and $\rho'\phi = \rho\phi$ if $\phi \notin \Phi$. If ρ is an Set environment, we write Eq_ρ for the Rel environment (see Definition 14) such that $\mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v}$ for every type variable v. The categories RT_k and relational interpretations appearing in the third clause of Definition 6 are given in full in Section 2.2.

Definition 6. *The* object interpretation $[\![\cdot]\!]^{\mathsf{Set}} : \mathcal{F} \to [\mathsf{SetEnv}, \mathsf{Set}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}}\rho &= 0\\
[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho &= 1\\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}\, F\, G]\!]^{\mathsf{Set}}\rho &= \{\eta : \lambda\overline{K}.\, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho[\Phi := \overline{K}] \Rightarrow \lambda\overline{K}.\, [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho[\Phi := \overline{K}]\\
&\qquad \mid \forall \overline{K = (K^1, K^2, K^*) : \mathsf{RT}_k}.\\
&\qquad\quad (\eta_{\overline{K^1}}, \eta_{\overline{K^2}}) : [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\Phi := \overline{K}] \to [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\Phi := \overline{K}]\}\\
[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}}\rho &= (\rho\phi)\,\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho}\\
[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho + [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma;\Phi \vdash F \times G]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Set}}\rho &= (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho}\\
\textit{where } T^{\mathsf{Set}}_{H,\rho}\, F &= \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\phi := F][\overline{\alpha := A}]\\
\textit{and } T^{\mathsf{Set}}_{H,\rho}\, \eta &= \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_\rho[\phi := \eta][\overline{\alpha := id_A}]\\
[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho &= \{t : (Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}}[\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}])\overline{[\![\Gamma;\Phi \vdash A]\!]^{\mathsf{Set}}\rho}\\
&\qquad \mid (t,t) \in [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\}
\end{aligned}
$$

WHERE DOES THIS ALL GO? Must explain Lans, say why they are functors, say they can be computed using colimits in Set and Rel, define injections, etc. If $f : B \to B'$ in Set, then the functorial action of $Lan_{\overline{K}}\, F$ on f is the unique morphism $(Lan_{\overline{K}}\, F)f$ such that

$$(Lan_{\overline{K}}\, F)f \circ \iota_{X,g} = \iota'_{X,f\circ g} \textit{ MISSING SOME BARS} \tag{1}$$

Here, $\iota_{X,g}$ is the injection of FX into $(Lan_{\overline{K}}\, F)\, B = \varinjlim_{X:\mathrm{Set}_0,\ g:KX\to B} FX$ and $\iota'_{X,g'}$ is the injection of FX into $(Lan_{\overline{K}}\, F)\, B' = \varinjlim_{X:\mathrm{Set}_0,\ g':KX\to B'} FX$.

Say somewhere that recognizing the non-parallel nature of the Lan interps, and yet still being able to define the set and rel interps so that everything is intertwined in exactly the right way to make the proofs (e.g., of the IEL) work, is what's hard here. This is where the insight is required.

If ρ : SetEnv and $\vdash F$ then we write $[\![\vdash F]\!]^{\mathsf{Set}}$ instead of $[\![\vdash F]\!]^{\mathsf{Set}}\rho$ since the environment is immaterial. To know that Definition 6 is well-defined we have to check that each object interpretation is in Set. First, we have

Lemma 7. *The collection of all natural transformations*

$$\eta : \lambda\overline{K}.[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho[\Phi := \overline{K}] \Rightarrow \lambda\overline{K}.[\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho[\Phi := \overline{K}]$$

defines a set.

Proof. We first note that $\lambda\overline{K}.[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho[\Phi := \overline{K}]$ and $\lambda\overline{K}.[\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho[\Phi := \overline{K}]$ are both in $[\overline{[\mathsf{Set}^k,\mathsf{Set}]},\mathsf{Set}]$. Since $[\overline{[\mathsf{Set}^k,\mathsf{Set}]},\mathsf{Set}]$ is locally finitely presentable it is locally small. There are therefore only Set-many morphisms (i.e., natural transformations) between any two functors in $[\overline{[\mathsf{Set}^k,\mathsf{Set}]},\mathsf{Set}]$. In particular, there are only Set-many natural transformations from $\lambda\overline{K}.[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho[\Phi := \overline{K}]$ to $\lambda\overline{K}.[\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho[\Phi := \overline{K}]$. □

Moreover, for any ρ : SetEnv we have that $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\, G]\!]^{\mathsf{Set}}\rho$ is an ω-cocontinuous functor because it is a constant functor. Interpretations of Nat-types ensure that $[\![\Gamma \vdash F \to G]\!]^{\mathsf{Set}}\rho$ and $[\![\Gamma \vdash \forall\overline{\alpha}.F]\!]^{\mathsf{Set}}\rho$ are as expected in any parametric model.

To make sense of the penultimate clause of Definition 6, we need to know that, for each ρ : SetEnv, $T^{\mathsf{Set}}_{H,\rho}$ is an ω-cocontinuous endofunctor on $[\mathsf{Set}^k,\mathsf{Set}]$, and thus admits a fixpoint. Since $T^{\mathsf{Set}}_{H,\rho}$ is defined in terms of $[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, this means that interpretations of types must be such functors, which in turn means that the actions of set interpretations of types on the objects and morphisms in SetEnv are intertwined. Fortunately, we know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash F$, $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}$ is actually in $[\mathsf{Set}^k,\mathsf{Set}]$, where $k = |\overline{\alpha}|$. Therefore, for each $[\![\Gamma;\overline{\gamma},\phi^k,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, the corresponding operator T^{Set}_H can be extended to a *functor* from SetEnv to $[[\mathsf{Set}^k,\mathsf{Set}],[\mathsf{Set}^k,\mathsf{Set}]]$. The action of T^{Set}_H on an object ρ : SetEnv is given by the higher-order functor $T^{\mathsf{Set}}_{H,\rho}$, whose actions on objects (i.e., functors in $[\mathsf{Set}^k,\mathsf{Set}]$) and on morphisms (natural transformations between such functors) are given in the penultimate clause of Definition 6. The action of T^{Set}_H on a morphism $f : \rho \to \rho'$ is the higher-order natural transformation $T^{\mathsf{Set}}_{H,f} : T^{\mathsf{Set}}_{H,\rho} \to T^{\mathsf{Set}}_{H,\rho'}$ whose action on $F : [\mathsf{Set}^k,\mathsf{Set}]$ is the natural transformation $T^{\mathsf{Set}}_{H,f}\, F : T^{\mathsf{Set}}_{H,\rho}\, F \to T^{\mathsf{Set}}_{H,\rho'}\, F$ whose component at $\overline{A}$ is given by $(T^{\mathsf{Set}}_{H,f}\, F)_{\overline{A}} = [\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} f[\phi := id_F][\overline{\alpha := id_A}]$.

Finally, to make sense of the final clause Definition 6, we first observe that, for any ρ : SetEnv, $Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}}[\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ is a functor from Set^m to Set for some m. This ensures that $(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}}[\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}])\overline{[\![\Gamma;\Phi \vdash A]\!]^{\mathsf{Set}}\rho}$ is a set. Moreover, for any ρ : SetEnv, the fact that $[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho$ is an ω-cocontinuous functor follows from Corollary 12 of [Johann and Polonsky 2019] provided each functor in $\overline{[\![\Gamma;\overline{\alpha} \vdash K]\!]^{\mathsf{Set}}}$ is polynomial in the variables in $\overline{\alpha}$. We also note that, for each vector of functors $\overline{K}$, $Lan_{\overline{K}}$ is itself a (higher-order) functor. Specifically, given functors $F, F' : \mathcal{C} \to \mathcal{D}$, a sequence of functors $\overline{K} = K_1,\ldots,K_n$ with $K_i : \mathcal{C} \to \mathcal{C}_i$ for $i = 1,\ldots,n$, and a natural transformation $\alpha : F \to F'$, the functorial action $Lan_{\overline{K}}\alpha : Lan_{\overline{K}}F \to Lan_{\overline{K}}F'$ of $Lan_{\overline{K}}$ on α is defined to be the unique natural transformation such that $((Lan_{\overline{K}}\alpha) \circ \langle K_1, ..., K_n\rangle) \circ \eta_F = \eta_{F'} \circ \alpha$. Here, $\eta_F : F \to (Lan_{\overline{K}}F) \circ \langle K_1, ..., K_n\rangle$ and $\eta_{F'} : F' \to (Lan_{\overline{K}}F') \circ \langle K_1, ..., K_n\rangle$ are the natural transformations associated with the functors $Lan_{\overline{K}}F$ and $Lan_{\overline{K}}F'$ from $\Pi_{i\in\{1,...,n\}}\mathcal{C}_i$ to $\mathcal{D}$, respectively. In Set, we have the slightly more general property

$$(Lan_{\overline{K}}\,\alpha)\overline{B} \circ \iota_{X,g} = \iota'_{X,g} \circ \alpha_X \tag{2}$$

Here, $\iota_{X,g}$ is the injection of FX into $\varinjlim_{X:\mathsf{Set}_0,\ g:KX\to B} FX$ and $\iota'_{X,g}$ is the injection of $F'X$ into $\varinjlim_{X:\mathsf{Set}_0,\ g:KX\to B} F'X$.

The next definition uses the functors T_H^{Set} and $Lan_{\overline{K}}$ to define the actions of functors interpreting types on morphisms between set environments.

Definition 8. *Let $f : \rho \to \rho'$ be a morphism between* Set *environments ρ and ρ' (so that $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$). The action $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f$ of $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}$ on the morphism f is given as follows:*

- *If $\Gamma;\Phi \vdash \mathbb{0}$ then $[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}} f = id_0$*
- *If $\Gamma;\Phi \vdash \mathbb{1}$ then $[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}} f = id_1$*
- *If $\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\,G$ then $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\,G]\!]^{\mathsf{Set}} f = id_{[\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\Phi} F\,G]\!]^{\mathsf{Set}}\rho}$*
- *If $\Gamma;\Phi \vdash \phi\overline{F}$ then*

$$\begin{aligned}[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}} f : [\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}}\rho &\to [\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}}\rho' \\ &= (\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho} \to (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho'}\end{aligned}$$

is defined by

$$\begin{aligned}[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}} f &= (f\phi)_{\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f} \\ &= (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f} \circ (f\phi)_{\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}\end{aligned}$$

The latter equality holds because $\rho\phi$ and $\rho'\phi$ are functors and $f\phi : \rho\phi \to \rho'\phi$ is a natural transformation, so the following naturality square commutes:

$$\begin{tikzcd}
(\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\rho\phi)\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}} f}"'] & (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho} \arrow[d, "(\rho'\phi)\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}} f}"'] \\
(\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho'} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}}"] & (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd} \tag{3}$$

- *If $\Gamma;\Phi \vdash F + G$ then $[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Set}} f$ is defined by*

$$[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Set}} f(\mathsf{inl}\,x) = \mathsf{inl}\,([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f x)$$
$$[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Set}} f(\mathsf{inr}\,y) = \mathsf{inr}\,([\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}} f y)$$

- *If $\Gamma;\Phi \vdash F \times G$ then $[\![\Gamma;\Phi \vdash F \times G]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}} f$*
- *If $\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}$ then*

$$\begin{aligned}[\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Set}} f \quad &: \quad [\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Set}}\rho' \\ &= \quad (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho} \to (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho'}\end{aligned}$$

is defined by

$$\begin{aligned}&(\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho'}} \circ (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} f} \\ = \quad &(\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} f} \circ (\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho}}\end{aligned}$$

The latter equality holds because $\mu T^{\mathsf{Set}}_{H,\rho}$ *and* $\mu T^{\mathsf{Set}}_{H,\rho'}$ *are functors and* $\mu T^{\mathsf{Set}}_{H,f} : \mu T^{\mathsf{Set}}_{H,\rho} \to \mu T^{\mathsf{Set}}_{H,\rho'}$ *is a natural transformation, so the following naturality square commutes:*

$$
\begin{tikzcd}
(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho} \arrow[r, "(\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}f}"'] & (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho} \arrow[d, "(\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}f}"'] \\
(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho'} \arrow[r, "(\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho'}}"] & (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd}
\tag{4}
$$

- *If* $\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}$ *then*

$$[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} \rho \to [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} \rho'$$

is defined by

$$
\begin{aligned}
& \big(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}} [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := id\_}]\big)\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}\rho'} \\
& \circ \big(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}} [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := \_}]\big)\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}f} \\
= \; & \big(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}} [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} \rho'[\overline{\alpha := \_}]\big)\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}f} \\
& \circ \big(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}} [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := id\_}]\big)\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}\rho}
\end{aligned}
$$

where the above equality holds by naturality of $Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}} [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := id\_}]$.

To see that the functorial action on Lan types is well-defined, we need to check that it preserves the extra condition in the interpretation of Lan types. That is, we need to show that if $t : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho$, then $[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f t : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho'$. By definition, if $t : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho$, then $(t,t) \in [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho$, i.e., there exist $\overline{Z : \mathsf{Set}_0}$, $t_1 : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := Z}]$, $t_2 : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := Z}]$, and $g : \overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := \mathsf{Eq}_Z}] \to [\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\mathsf{Eq}}$ such that $\iota_{\overline{Z},\overline{\pi_1 g}} t_1 = t$, $\kappa_{\overline{Z},\overline{\pi_2 g}} t_2 = t$, and $(t_1,t_2) : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := \mathsf{Eq}_Z}]$, where ι and κ are as in the final clause of Definition 16 below. Now, to prove that $[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f t : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho'$, let $\overline{Z' = Z}$, let $t'_i = [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := id_Z}] t_i$ for $i = 1,2$, and let $\overline{g' = [\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\mathsf{Eq}_f \circ g}$. We have to show that $\iota'_{\overline{Z'},\overline{\pi_1 g'}} t'_1 = [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f t$, $\kappa'_{\overline{Z'},\overline{\pi_2 g'}} t'_2 = [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f t$, and $(t'_1,t'_2) : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}[\overline{\alpha := \mathsf{Eq}_{Z'}}]$, where ι' and κ' are the injections for ρ' corresponding to ι and κ for ρ. The latter is immediate from the facts that $(t'_1,t'_2) = [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_f[\overline{\alpha := id_{\mathsf{Eq}_Z}}](t_1,t_2)$ and $(t_1,t_2) : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := \mathsf{Eq}_Z}]$. That $\iota'_{\overline{Z'},\overline{\pi_1 g'}} t'_1 = [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f t$ is because

$$
\begin{aligned}
\iota'_{\overline{Z'},\overline{\pi_1 g'}} t'_1 &= \iota'_{\overline{Z'},\overline{\pi_1 g'}}([\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := id_Z}] t_1) \\
&= [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f(\iota_{\overline{Z},\overline{\pi_1 g}} t_1) \\
&= [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f t
\end{aligned}
$$

where the second equality here is justified by the commutativity of the following diagram, and ι'' is the obvious injection indicated:

$$\begin{tikzcd}
\llbracket \Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := Z}] \arrow[r, "\iota_{\overline{Z},\overline{\pi_1 g}}"] \arrow[d, "\llbracket\Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}} f[\overline{\alpha:=id_Z}]"'] & \varinjlim_{\overline{\overline{S}, h:\llbracket\Gamma;\overline{\alpha}\vdash K\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=S}]\to\llbracket\Gamma;\Phi\vdash A\rrbracket^{\mathsf{Set}}\rho}} \llbracket \Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := S}] \arrow[d, "(Lan_{\overline{\llbracket\Gamma;\overline{\alpha}\vdash K\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=_}]}}\llbracket\Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}} f[\overline{\alpha:=id_}])\overline{\llbracket\Gamma;\Phi\vdash A\rrbracket^{\mathsf{Set}}\rho}"] \\
\llbracket \Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha := Z}] \arrow[r, "\iota''_{\overline{Z},\overline{\pi_1 g}}"] \arrow[d, equal] & \varinjlim_{\overline{\overline{S}, h:\llbracket\Gamma;\overline{\alpha}\vdash K\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha:=S}]\to\llbracket\Gamma;\Phi\vdash A\rrbracket^{\mathsf{Set}}\rho}} \llbracket \Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha := S}] \arrow[d, "(Lan_{\overline{\llbracket\Gamma;\overline{\alpha}\vdash K\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=_}]}}\llbracket\Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha:=_}])\overline{\llbracket\Gamma;\Phi\vdash A\rrbracket^{\mathsf{Set}} f}"] \\
\llbracket \Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha := Z}] \arrow[r, "\iota'_{\overline{Z'},\overline{\pi_1(\llbracket\Gamma;\Phi\vdash A\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_f\circ g)}}"'] & \varinjlim_{\overline{\overline{S}, h:\llbracket\Gamma;\overline{\alpha}\vdash K\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha:=S}]\to\llbracket\Gamma;\Phi\vdash A\rrbracket^{\mathsf{Set}}\rho'}} \llbracket \Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha := S}]
\end{tikzcd} \tag{5}$$

Here, the top diagram commutes by the definition of the "higher" functorial action of $Lan_{\overline{\llbracket\Gamma;\overline{\alpha}\vdash K\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\_}]}}$ on $\llbracket\Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id\_}]$ as given in (2), and the bottom one commutes by the definition of the functorial action of $Lan_{\overline{\llbracket\Gamma;\overline{\alpha}\vdash K\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha:=\_}]}}\llbracket\Gamma;\Phi,\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha := \_}]$ on $\overline{\llbracket\Gamma;\Phi\vdash A\rrbracket^{\mathsf{Set}} f}$ as given in (1). The proof that $\kappa_{Z,\pi_2 g'} t_2' = \llbracket\Gamma;\Phi\vdash(\mathsf{Lan}_{\overline{K}}^{\overline{\alpha}} F)\overline{A}\rrbracket^{\mathsf{Set}} f t$ is analogous.

Definitions 6 and 8 respect weakening. That is, they ensure that a type and its weakenings all have the same set interpretations.

2.2 Relational Interpretations of Types

Definition 9. *A k-ary relation transformer F is a triple (F^1, F^2, F^*), where $F^1, F^2 : [\mathsf{Set}^k, \mathsf{Set}]_\omega$ and $F^* : [\mathsf{Rel}^k, \mathsf{Rel}]_\omega$ are such that if $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$ then $F^*\overline{R} : \mathsf{Rel}(F^1\overline{A}, F^2\overline{B})$, and if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$ then $F^*\overline{(\alpha,\beta)} = (F^1\overline{\alpha}, F^2\overline{\beta})$. We define $F\overline{R}$ to be $F^*\overline{R}$ and $F\overline{(\alpha,\beta)}$ to be $F^*\overline{(\alpha,\beta)}$.*

The first condition of the first sentence of Definition 9 entails that $F^*\overline{R}$ relates sups of chains of pairwise related elements in $F^1\overline{A}$ and $F^2\overline{B}$. The last condition of the first sentence of Definition 9 expands to: if $\overline{(a,b) \in R}$ implies $\overline{(\alpha\, a, \beta\, b) \in S}$ then $(c,d) \in F^*\overline{R}$ implies $(F^1\overline{\alpha}\, c, F^2\overline{\beta}\, d) \in F^*\overline{S}$. When convenient we identify a 0-ary relation transformer (A, B, R) with $R : \mathsf{Rel}(A, B)$. We may also write $\pi_1 F$ for F^1 and $\pi_2 F$ for F^2. Without loss of generality, we assume that π_1 and π_2 are surjective on relations. We extend these conventions to Set relation environments, introduced in Definition 14 below, in the obvious way.

Definition 10. *The category RT_k of k-ary relation transformers is given by the following data:*

- *An object of RT_k is a k-ary relation transformer.*
- *A morphism $\delta : (G^1, G^2, G^*) \to (H^1, H^2, H^*)$ in RT_k is a pair of natural transformations (δ^1, δ^2), where $\delta^1 : G^1 \to H^1$ and $\delta^2 : G^2 \to H^2$ are such that, for all $\overline{R : \mathsf{Rel}(A, B)}$, if $(x, y) \in G^*\overline{R}$ then $(\delta^1_{\overline{A}} x, \delta^2_{\overline{B}} y) \in H^*\overline{R}$.*
- *Identity morphisms and composition are inherited from the category of functors on* Set.

Definition 11. *An endofunctor H on RT_k is a triple $H = (H^1, H^2, H^*)$, where*

- *H^1 and H^2 are functors from $[\mathsf{Set}^k, \mathsf{Set}]$ to $[\mathsf{Set}^k, \mathsf{Set}]$*
- *H^* is a functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]$*

- *for all* $\overline{R : \mathsf{Rel}(A,B)}$, $\pi_1((H^*(\delta^1,\delta^2))_{\overline{R}}) = (H^1\delta^1)_{\overline{A}}$ *and* $\pi_2((H^*(\delta^1,\delta^2))_{\overline{R}}) = (H^2\delta^2)_{\overline{B}}$
- *The action of* H *on objects is given by* $H\,(F^1, F^2, F^*) = (H^1F^1,\ H^2F^2,\ H^*(F^1,F^2,F^*))$
- *The action of* H *on morphisms is given by* $H\,(\delta^1,\delta^2) = (H^1\delta^1, H^2\delta^2)$ *for* $(\delta^1,\delta^2) : (F^1,F^2,F^*) \to (G^1,G^2,G^*)$

Since the results of applying an endofunctor H to k-ary relation transformers and morphisms between them must again be k-ary relation transformers and morphisms between them, respectively, Definition 11 implicitly requires that the following three conditions hold: *i)* if $R_1 : \mathsf{Rel}(A_1,B_1), \ldots, R_k : \mathsf{Rel}(A_k,B_k)$, then $H^*(F^1,F^2,F^*)\overline{R} : \mathsf{Rel}(H^1F^1\overline{A}, H^2F^2\overline{B})$; *ii)* if $(\alpha_1,\beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1,S_1), \ldots, (\alpha_k,\beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k,S_k)$, then $H^*(F^1,F^2,F^*)\,\overline{(\alpha,\beta)} = (H^1F^1\overline{\alpha}, H^2F^2\overline{\beta})$; and *iii)* if $(\delta^1,\delta^2) : (F^1,F^2,F^*) \to (G^1,G^2,G^*)$ and $R_1 : \mathsf{Rel}(A_1,B_1), \ldots, R_k : \mathsf{Rel}(A_k,B_k)$, then $((H^1\delta^1)_{\overline{A}}x, (H^2\delta^2)_{\overline{B}}y) \in H^*(G^1,G^2,G^*)\overline{R}$ whenever $(x,y) \in H^*(F^1,F^2,F^*)\overline{R}$. Note, however, that this last condition is automatically satisfied because it is implied by the third bullet point of Definition 11.

Definition 12. *If* H *and* K *are endofunctors on* RT_k*, then a* natural transformation $\sigma : H \to K$ *is a pair* $\sigma = (\sigma^1, \sigma^2)$*, where* $\sigma^1 : H^1 \to K^1$ *and* $\sigma^2 : H^2 \to K^2$ *are natural transformations between endofunctors on* $[\mathsf{Set}^k, \mathsf{Set}]$ *and the component of* σ *at* $F = (F^1,F^2,F^*) \in \mathsf{RT}_k$ *is* $\sigma_F = (\sigma^1_{F^1}, \sigma^2_{F^2})$.

Definition 12 entails that $\sigma^i_{F^i}$ must be natural in $F^i : [\mathsf{Set}^k, \mathsf{Set}]$, and, for every F, $(\sigma^1_{F^1})_{\overline{A}}$ and $(\sigma^2_{F^2})_{\overline{B}}$ must be natural in $\overline{A}$ and $\overline{B}$, respectively. Moreover, since the results of applying σ to k-ary relation transformers must be morphisms of k-ary relation transformers, Definition 12 implicitly requires that $(\sigma_F)_{\overline{R}} = ((\sigma^1_{F^1})_{\overline{A}}, (\sigma^2_{F^2})_{\overline{B}})$ is a morphism in Rel for any k-tuple of relations $\overline{R : \mathsf{Rel}(A,B)}$, i.e., that if $(x,y) \in H^*F\overline{R}$, then $((\sigma^1_{F^1})_{\overline{A}}x, (\sigma^2_{F^2})_{\overline{B}}y) \in K^*F\overline{R}$.

Critically, we can compute ω-directed colimits in RT_k: it is not hard to see that if $\mathcal{D}$ is an ω-directed set then $\varinjlim_{d\in\mathcal{D}}(F^1_d, F^2_d, F^*_d) = (\varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^2_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$. We define an endofunctor $T = (T^1, T^2, T^*)$ on RT_k to be ω*-cocontinuous* if T^1 and T^2 are ω-cocontinuous endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]$ and T^* is an ω-cocontinuous functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]$, i.e., is in $[\mathsf{RT}_k, [\mathsf{Rel}^k, \mathsf{Rel}]]$.

Now, for any k, any $A : \mathsf{Set}$, and any $R : \mathsf{Rel}(A,B)$, let K^{Set}_A be the constantly A-valued functor from Set^k to Set and K^{Rel}_R be the constantly R-valued functor from Rel^k to Rel. Also let 0 denote either the initial object of Set or the initial object of Rel, as appropriate. Observing that, for every k, K^{Set}_0 is initial in $[\mathsf{Set}^k, \mathsf{Set}]$, and K^{Rel}_0 is initial in $[\mathsf{Rel}^k, \mathsf{Rel}]$, we have that, for each k, $K_0 = (K^{\mathsf{Set}}_0, K^{\mathsf{Set}}_0, K^{\mathsf{Rel}}_0)$ is initial in RT_k. Thus, if $T = (T^1, T^2, T^*) : \mathsf{RT}_k \to \mathsf{RT}_k$ is an endofunctor on RT_k then we can define the relation transformer μT to be $\varinjlim_{i<\omega} T^i K_0$. It is not hard to see that μT is given explicitly as

$$\mu T = (\mu T^1, \mu T^2, \varinjlim_{i<\omega} (T^i K_0)^*) \tag{6}$$

and that, as our notation suggests, it really is a fixpoint for T if T is ω-cocontinuous:

Lemma 13. *For any* $T : [\mathsf{RT}_k, \mathsf{RT}_k]$*,* $\mu T \cong T(\mu T)$*.*

The isomorphism is given by the morphisms $(in_1, in_2) : T(\mu T) \to \mu T$ and $(in_1^{-1}, in_2^{-1}) : \mu T \to T(\mu T)$ in RT_k. The latter is always a morphism in RT_k, but the former need not be if T is not ω-cocontinuous.

It is worth noting that the third component in Equation (6) is the colimit in $[\mathsf{Rel}^k, \mathsf{Rel}]$ of third components of relation transformers, rather than a fixpoint of an endofunctor on $[\mathsf{Set}^k, \mathsf{Set}]$. That there is an asymmetry between the first two components of μT and its third reflects the important conceptual observation that the third component of an endofunctor on RT_k need not be a functor on all of $[\mathsf{Rel}^k, \mathsf{Rel}]$. In particular, although we can define $T_{H,\rho}\,F$ for a relation transformer F in Definition 16 below, it is not clear how we could define it for an arbitrary $F : [\mathsf{Rel}^k, \mathsf{Rel}]$.

Definition 14. *An* Set relation environment *maps each type variable in* $\mathbb{T}^k \cup \mathbb{F}^k$ *to a k-ary relation transformer. A morphism* $f : \rho \to \rho'$ *between* Set *relation environments* ρ *and* ρ' *with* $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$ *maps each type constructor variable* $\psi^k \in \mathbb{T}$ *to the identity morphism on* $\rho\psi^k = \rho'\psi^k$ *and each functorial variable* $\phi^k \in \mathbb{F}$ *to a morphism from the k-ary relation transformer* $\rho\phi$ *to the k-ary relation transformer* $\rho'\phi$*. Composition of morphisms on* Set *relation environments is given componentwise, with the identity morphism mapping each* Set *relation environment to itself. This gives a category of* Set *relation environments and morphisms between them, which we denote* RelEnv.

When convenient we identify a 0-ary relation transformer with the Set relation (transformer) that is its codomain and consider an Set relation environment to map a type variable of arity 0 to an Set relation. If $\Phi = \{\phi_1^{k_1}, ..., \phi_n^{k_n}\}$ and $\overline{K} = \{K_1, ..., K_n\}$ where $K_i : [\mathsf{Rel}^{k_i}, \mathsf{Rel}]$ for $i = 1, ..., n$, then we write either $\rho[\overline{\Phi := K}]$ or $\rho[\overline{\phi := K}]$ for the Set relation environment ρ' such that $\rho'\phi_i = K_i$ for $i = 1, ..., n$ and $\rho'\phi = \rho\phi$ if $\phi \notin \Phi$. If ρ is an Set relation environment, we write $\pi_1\rho$ and $\pi_2\rho$ for the Set relation environments mapping each type variable ϕ to the functors $(\rho\phi)^1$ and $(\rho\phi)^2$, respectively.

We define, for each k, the notion of an ω-cocontinuous functor from RelEnv to RT_k:

Definition 15. *A functor* $H : [\mathsf{RelEnv}, \mathsf{RT}_k]$ *is a triple* $H = (H^1, H^2, H^*)$*, where*

- H^1 *and* H^2 *are objects in* $[\mathsf{SetEnv}, [\mathsf{Set}^k, \mathsf{Set}]]$
- H^* *is a an object in* $[\mathsf{RelEnv}, [\mathsf{Rel}^k, \mathsf{Rel}]]$
- *for all* $\overline{R : \mathsf{Rel}(A, B)}$ *and morphisms* f *in* RelEnv, $\pi_1((H^* f)_{\overline{R}}) = (H^1(\pi_1 f))_{\overline{A}}$ *and* $\pi_2((H^* f)_{\overline{R}}) = (H^2(\pi_2 f))_{\overline{B}}$
- *The action of* H *on* ρ *in* RelEnv *is given by* $H\rho = (H^1(\pi_1\rho),\ H^2(\pi_2\rho),\ H^*\rho)$
- *The action of* H *on morphisms* $f : \rho \to \rho'$ *in* RelEnv *is given by* $Hf = (H^1(\pi_1 f), H^2(\pi_2 f))$

Spelling out the last two bullet points above gives the following analogues of the three conditions immediately following Definition 11: *i*) if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $H^*\rho\,\overline{R} : \mathsf{Rel}(H^1(\pi_1\rho)\,\overline{A}, H^2(\pi_2\rho)\,\overline{B})$; *ii*) if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then $H^*\rho\,\overline{(\alpha, \beta)} = (H^1(\pi_1\rho)\,\overline{\alpha}, H^2(\pi_2\rho)\,\overline{\beta})$; and *iii*) if $f : \rho \to \rho'$ and $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $((H^1(\pi_1 f))_{\overline{A}}\, x, (H^2(\pi_2 f))_{\overline{B}}\, y) \in H^*\rho'\,\overline{R}$ whenever $(x, y) \in H^*\rho\,\overline{R}$. As before, the last condition is automatically satisfied because it is implied by the third bullet point of Definition 15.

Considering RelEnv as a product $\Pi_{\phi^k \in \mathbb{T} \cup \mathbb{F}} \mathsf{RT}_k$, we extend the computation of ω-directed colimits in RT_k to compute colimits in RelEnv componentwise. We similarly extend the notion of an ω-cocontinuous endofunctor on RT_k componentwise to give a notion of ω-cocontinuity for functors from RelEnv to RT_k. Recalling from the start of this subsection that Definition 16 is given mutually inductively with Definition 6 we can, at last, define:

Definition 16. *The* relational interpretation $[\![\cdot]\!]^{\mathsf{Rel}} : \mathcal{F} \to [\mathsf{RelEnv}, \mathsf{Rel}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\rho &= 0\\
[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\rho &= 1\\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}\, F\, G]\!]^{\mathsf{Rel}}\rho &= \{\eta : \lambda\overline{K}.\,[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\Phi := K}] \Rightarrow \lambda\overline{K}.\,[\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\phi := K}]\}\\
&= \{(\eta_1,\eta_2) \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}\, F\, G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}\, F\, G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid\\
&\qquad \forall \overline{K = (K^1, K^2, K^*) : \mathsf{RT}_k}.\\
&\qquad\quad ((\eta_1)_{\overline{K^1}}, (\eta_2)_{\overline{K^2}}) \in ([\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\Phi := K}])^{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\Phi:=K}]}\}\\
[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Rel}}\rho &= (\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho}\\
[\![\Gamma;\Phi \vdash F+G]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho + [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\rho\\
[\![\Gamma;\Phi \vdash F\times G]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho \times [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\rho\\
[\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Rel}}\rho &= (\mu T_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Rel}}\rho}\\
\textit{where } T_{H,\rho} &= (T^{\mathsf{Set}}_{H,\pi_1\rho}, T^{\mathsf{Set}}_{H,\pi_2\rho}, T^{\mathsf{Rel}}_{H,\rho})\\
\textit{and } T^{\mathsf{Rel}}_{H,\rho}\, F &= \lambda\overline{R}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F][\overline{\alpha := R}]\\
\textit{and } T^{\mathsf{Rel}}_{H,\rho}\, \delta &= \lambda\overline{R}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_{\rho}[\phi := \delta][\overline{\alpha := id_{\overline{R}}}]\\
[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}}\rho &= \{(t_1,t_2) : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}(\pi_2\rho)\\
&\qquad \mid \exists\, \overline{Z : \mathsf{Set}_0},\, t_1' : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho)[\overline{\alpha := Z}],\ t_2' : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_2\rho)[\overline{\alpha := Z}],\\
&\qquad \overline{f : [\![\Gamma;\overline{\alpha} \vdash K]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \mathsf{Eq}_Z}] \to [\![\Gamma;\Phi \vdash A]\!]^{\mathsf{Rel}}\rho} \textit{ such that}\\
&\qquad \iota_{\overline{Z},\overline{\pi_1 f}} t_1' = t_1,\ \kappa_{\overline{Z},\overline{\pi_2 f}} t_2' = t_2,\ \textit{and } (t_1',t_2') : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \mathsf{Eq}_Z}]\,\}
\end{aligned}
$$

In the final clause of Definition 16, $\iota_{Z,h}$ is the injection into

$$\varinjlim_{Z : \mathsf{Set}_0,\ h : [\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}(\pi_1\rho)[\overline{\alpha := Z}] \to [\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}(\pi_1\rho)} [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho)[\overline{\alpha := Z}]$$

and $\kappa_{Z,h}$ is the injection into

$$\varinjlim_{Z : \mathsf{Set}_0,\ h : [\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}(\pi_2\rho)[\overline{\alpha := Z}] \to [\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}(\pi_2\rho)} [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_2\rho)[\overline{\alpha := Z}]$$

We note that the pair $(\iota_{Z,h}, \kappa_{Z,h})$ need not coincide with the injection j_{Z,Eq_h} into

$$\varinjlim_{Z : \mathsf{Set}_0,\ g : [\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \mathsf{Eq}_Z}] \to [\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\rho} [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \mathsf{Eq}_Z}]$$

In fact, it is precisely this that keeps the object and relational interpretations of Lan types in from being "parallel". Of course ι and κ depend on ρ, too. Perhaps the notation should reflect this.

If ρ : RelEnv and $\vdash F$, then we write $[\![\vdash F]\!]^{\mathsf{Rel}}$ instead of $[\![\vdash F]\!]^{\mathsf{Rel}}\rho$. The interpretations in Definitions 16 and in Definition 17 below respect weakening.

As for Definition 6, to know that Definition 16 is well-defined we have to check that, for every ρ : RelEnv, each relational interpretation is in Rel. The proof that relational interpretations of Nat-types define relations is analogous to the proof of Lemma 7. Moreover, for any ρ : RelEnv, we have that $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\, G]\!]^{\mathsf{Rel}}\rho$ is an ω-cocontinuous functor because it is a constant functor. Interpretations of Nat-types ensure that $[\![\Gamma \vdash F \to G]\!]^{\mathsf{Rel}}\rho$ and $[\![\Gamma \vdash \forall\overline{\alpha}.F]\!]^{\mathsf{Rel}}\rho$ are as expected in any parametric model.

To make sense of the penultimate clause of Definition 16, we need to know that, for each ρ : RelEnv, $T_{H,\rho}$ is an ω-cocontinuous endofunctor on RT, and thus admits a fixpoint. Since $T_{H,\rho}$ is defined in terms of $[\![\Gamma;\overline{\gamma},\phi^k,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}$, this means that relational interpretations of types must be ω-cocontinuous functors from RelEnv to RT_0, which in turn entails that the actions of relational interpretations of types on objects and on morphisms in RelEnv are intertwined. As for Set interpretations, we know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha}\vdash F$, $[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}$ is actually in $[\mathsf{Rel}^k,\mathsf{Rel}]_\omega$, where $k=|\overline{\alpha}|$. Therefore, for each $[\![\Gamma;\overline{\gamma},\phi^k,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}$, the corresponding operator T_H can be extended to a *functor* from RelEnv to $[[\mathsf{Rel}^k,\mathsf{Rel}],[\mathsf{Rel}^k,\mathsf{Rel}]]$. The action of T_H on an object ρ : RelEnv is given by the higher-order functor $T_{H,\rho}$ whose actions on objects and morphisms are given in the penultimate clause of Definition 16. The action of T_H on a morphism $f:\rho\to\rho'$ is the higher-order natural transformation $T_{H,f}:T_{H,\rho}\to T_{H,\rho'}$ whose action on any $F:[\mathsf{Rel}^k,\mathsf{Rel}]_\omega$ is the higher-order natural transformation $T_{H,f}\,F:T_{H,\rho}\,F\to T_{H,\rho'}\,F$ whose component at $\overline{R}$ is $(T_{H,f}\,F)_{\overline{R}}=[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}f[\phi:=id_F][\overline{\alpha:=id_R}]$. Finally, to see that the functor given by the penultimate clause of Definition 16 is well-defined we must also show that, for every H, $T_{H,\rho}\,F$ is a relation transformer for any relation transformer F. To know that the functor given by Definition 17 is well-defined we will similarly need to show and that $T_{H,f}\,F:T_{H,\rho}\,F\to T_{H,\rho'}\,F$ is a morphism of relation transformers for every relation transformer F and every morphism $f:\rho\to\rho'$ in RelEnv. These results will be immediate consequences of Lemma 18 below.

To make sense of the final clause of Definition 16, we first observe that, for any ρ : RelEnv, $[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}\rho$ is a set because $[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1\rho)$ and $[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_2\rho)$ are. Moreover, for any ρ : RelEnv, the fact that $[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}\rho$ is an ω-cocontinuous functor follows from Corollary 12 of [Johann and Polonsky 2019] provided each functor in $\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}}$ is polynomial in the variables in $\overline{\alpha}$.

The next definition gives the actions of functors interpreting types on morphisms between relation environments.

Definition 17. *Let $f:\rho\to\rho'$ for relation environments ρ and ρ' (so that $\rho|_{\mathbb{T}}=\rho'|_{\mathbb{T}}$). The actions $[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}f$ of $[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}$ on the morphism f for all types other than* Lan*-types are given exactly as in Definition 8, except that all interpretations involved are now relational interpretations and all occurrences of $T^{\mathsf{Set}}_{H,f}$ are replaced by $T_{H,f}$. For* Lan*-types, we define*

$$[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}f=\left([\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}(\pi_1 f),\ [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}(\pi_2 f)\right)$$

We now need to show that the functorial actions of Lan-types are well-defined, i.e., that for every $(t_1,t_2):[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}\rho$ we have that

$$\left([\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1 f)t_1,[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_2 f)t_2\right):[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}\rho' \tag{7}$$

Since $(t_1,t_2):[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}$ there exist $\overline{Z:\mathsf{Set}_0}$, $t_1':[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}(\pi_1\rho)[\overline{\alpha:=Z}]$, $t_2':[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}(\pi_2\rho)[\overline{\alpha:=Z}]$, $g:\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=\mathsf{Eq}_Z}]\to[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\rho}$ such that $\iota_{\overline{Z},\overline{\pi_1 g}}t_1'=t_1$, $\kappa_{\overline{Z},\overline{\pi_2 g}}t_2'=t_2$, and $(t_1',t_2'):[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=\mathsf{Eq}_Z}]$. To show that (7) holds, let $\overline{S=Z}$, $s_1'=[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}(\pi_1 f)[\overline{\alpha:=Z}]t_1'$, $s_2'=[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}(\pi_2 f)[\overline{\alpha:=Z}]t_2'$, and $\overline{h=[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}f\circ g}$. Then $(s_1',s_2'):[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha:=\mathsf{Eq}_Z}]$ because $(t_1',t_2'):[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=\mathsf{Eq}_Z}]$ and $(s_1',s_2')=[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}f[\overline{\alpha:=\mathsf{Eq}_Z}](t_1',t_2')$. In addition, $\iota'_{\overline{Z},\overline{\pi_1 h}}s_1'=[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1 f)t_1$ and $\kappa'_{\overline{Z},\overline{\pi_2 h}}s_2'=$

$[\![\Gamma;\Phi \vdash (\mathsf{Lan}_{\overline{K}}^{\overline{\alpha}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_2 f)t_2$ hold because

$$\begin{aligned}
\iota'_{\overline{Z},\overline{\pi_1 h}} s'_1 &= \iota'_{\overline{Z},\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}(\pi_1 f)\circ\pi_1 g}}([\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_1 f)[\overline{\alpha := id_Z}]t'_1)\\
&= [\![\Gamma;\Phi \vdash (\mathsf{Lan}_{\overline{K}}^{\overline{\alpha}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1 f)(\iota_{\overline{Z},\overline{\pi_1 g}} t'_1)\\
&= [\![\Gamma;\Phi \vdash (\mathsf{Lan}_{\overline{K}}^{\overline{\alpha}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1 f)t_1
\end{aligned}$$

where the second equality is by (5), and similarly for $\kappa'_{\overline{Z},\overline{\pi_2 h}} s'_2 = [\![\Gamma;\Phi \vdash (\mathsf{Lan}_{\overline{K}}^{\overline{\alpha}}F)\overline{A}]\!]^{\mathsf{Set}}(\pi_2 f)t_2$.

With the relational interpretations of types laid out we can now prove:

Lemma 18. *For every* $\Gamma;\Phi \vdash F$,

$$[\![\Gamma;\Phi \vdash F]\!] = ([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}) \in [\mathsf{RelEnv}, \mathsf{RT}_0]$$

The proof is a straightforward induction on the structure of F, using an appropriate result from [Johann and Polonsky 2019] to deduce ω-cocontinuity of $[\![\Gamma;\Phi \vdash F]\!]$ in each case, together with Lemma 13 and Equation 6 for μ-types. The result holds by construction for Lan-types.

We can also prove by simultaneous induction that our interpretations of types interact well with demotion of functorial variables. Indeed, we have that, if $\rho, \rho' : \mathsf{SetEnv}$, $f : \rho \to \rho'$, $\rho\phi = \rho\psi = \rho'\phi = \rho'\psi$, $f\phi = f\psi = id_{\rho\phi}$, $\Gamma;\Phi,\phi^k \vdash F$, $\Gamma;\Phi,\overline{\alpha} \vdash G$, $\Gamma;\Phi,\alpha_1...\alpha_k \vdash H$, and $\overline{\Gamma;\Phi \vdash K}$, then WE NEED TO CHECK LAN CASES HERE!

$$[\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}\rho = [\![\Gamma,\psi;\Phi \vdash F[\phi :== \psi]]\!]^{\mathsf{Set}}\rho \tag{8}$$

$$[\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}f = [\![\Gamma,\psi;\Phi \vdash F[\phi :== \psi]]\!]^{\mathsf{Set}}f \tag{9}$$

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}] \tag{10}$$

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}f = [\![\Gamma;\Phi,\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}f[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}f}] \tag{11}$$

$$[\![\Gamma;\Phi \vdash F[\phi := H]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}\rho[\phi := \lambda\overline{A}.\,[\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]] \tag{12}$$

$$[\![\Gamma;\Phi \vdash F[\phi := H]]\!]^{\mathsf{Set}}f = [\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}f[\phi := \lambda\overline{A}.\,[\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}f[\overline{\alpha := id_{\overline{A}}}]] \tag{13}$$

Identities analogous to (8) through (13) hold for relational interpretations as well.

3 THE IDENTITY EXTENSION LEMMA

The standard definition of the graph for a morphism $f : A \to B$ in Set is the relation $\langle f \rangle : \mathsf{Rel}(A, B)$ defined by $(x, y) \in \langle f \rangle$ iff $fx = y$. This definition naturally generalizes to associate to each natural transformation between k-ary functors on Set a k-ary relation transformer as follows.

The notion of a graph relation naturally generalizes to associate to each natural transformation between k-ary functors on Set a k-ary relation transformer as follows. Recall that, since Set is a locally finitely presentable category, Proposition 1.6.1 of [Adámek and Rosický 1994] ensures that it has a (strong epi, mono) factorization system. We then have:

Definition 19. *If* $F, G : \mathsf{Set}^k \to \mathsf{Set}$ *and* $\alpha : F \to G$ *is a natural transformation, then the functor* $\langle\alpha\rangle^* : \mathsf{Rel}^k \to \mathsf{Rel}$ *is defined as follows. Given* $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, *let* $\iota_{R_i} : R_i \hookrightarrow A_i \times B_i$, *for* $i = 1, ..., k$, *be the inclusion of* R_i *as a sub*Set *of* $A_i \times B_i$, *let* $h_{\overline{A\times B}}$ *be the unique morphism making the diagram*

$$\begin{tikzcd}
F\overline{A} & F(\overline{A\times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, dashed, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
 & F\overline{A}\times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

commute, and let $h_{\overline{R}} : F\overline{R} \to F\overline{A} \times G\overline{B}$ be $h_{\overline{A\times B}} \circ F\iota_{\overline{R}}$. Further, let $\alpha^\wedge\overline{R}$ be the subobject through which $h_{\overline{R}}$ is factorized by the (strong epi, mono) factorization system of Set*, as shown in the following diagram:*

$$\begin{tikzcd}
F\overline{R} \arrow[rr, "h_{\overline{R}}"] \arrow[dr, two heads, "q_{\alpha^\wedge\overline{R}}"'] & & F\overline{A} \times G\overline{B} \\
& \alpha^\wedge\overline{R} \arrow[ur, hook, "\iota_{\alpha^\wedge\overline{R}}"'] &
\end{tikzcd}$$

Then $\alpha^\wedge\overline{R}$: $\mathsf{Rel}(F\overline{A}, G\overline{B})$ *by construction, so the action of $\langle\alpha\rangle^*$ on objects of* Rel *can be given by $\langle\alpha\rangle^*\overline{(A, B, R)} = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge\overline{R}}\alpha^\wedge\overline{R})$. The action of $\langle\alpha\rangle^*$ on morphisms of* Rel *is given by $\langle\alpha\rangle^*\overline{(\beta, \beta')} = (F\overline{\beta}, G\overline{\beta}')$.*

The next lemma shows that the data in Definition 19 actually yield a relation transformer $\langle\alpha\rangle = (F, G, \langle\alpha\rangle^*)$. We call this the *graph relation transformer* for α.

Lemma 20. *If $F, G : [\mathsf{Set}^k, \mathsf{Set}]$, and if $\alpha : F \to G$ is a natural transformation, then $\langle\alpha\rangle$ is in RT_k.*

Proof. Clearly, $\langle\alpha\rangle^*$ is ω-cocontinuous, so $\langle\alpha\rangle^* : [\mathsf{Rel}^k, \mathsf{Rel}]$. Now, let $\overline{R : \mathsf{Rel}(A, B)}$, $\overline{S : \mathsf{Rel}(C, D)}$, and $\overline{(\beta, \beta') : R \to S}$. We want to show that there exists a morphism $\epsilon : \alpha^\wedge\overline{R} \to \alpha^\wedge\overline{S}$ such that the diagram on the left below commutes. Since $\overline{(\beta, \beta') : R \to S}$, there exist $\overline{\gamma : R \to S}$ such that each diagram in the middle commutes. Moreover, since both $h_{\overline{C\times D}} \circ F(\overline{\beta \times \beta'})$ and $(F\overline{\beta} \times G\overline{\beta'}) \circ h_{\overline{A\times B}}$ make the diagram on the right commute, they must be equal. We therefore get that the right-hand

$$\begin{tikzcd}
\alpha^\wedge\overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{R}}"] \arrow[d, "\epsilon"'] & F\overline{A} \times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
\alpha^\wedge\overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{S}}"'] & F\overline{C} \times G\overline{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
R_i \arrow[r, hook, "\iota_{R_i}"] \arrow[d, "\gamma_i"'] & A_i \times B_i \arrow[d, "\beta_i\times\beta'_i"] \\
S_i \arrow[r, hook, "\iota_{S_i}"] & C_i \times D_i
\end{tikzcd}
\qquad
\begin{tikzcd}
F\overline{C} & F\overline{C} \times F\overline{D} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F\overline{D} \arrow[r, "\alpha_{\overline{D}}"] & G\overline{D} \\
& F(\overline{A \times B}) \arrow[ul, "F\pi_1\circ F(\overline{\beta\times\beta'})"] \arrow[u, dashed, "\exists!"] \arrow[urr, "\alpha_{\overline{D}}\circ F\pi_2\circ F(\overline{\beta\times\beta'})"'] & &
\end{tikzcd}$$

square in the diagram on the left below commutes, and thus that the entire diagram does as well. Finally, by the left-lifting property of $q_{\alpha^\wedge\overline{R}}$ with respect to $\iota_{\alpha^\wedge\overline{S}}$ given by the (strong epi, mono) factorization system there exists an ϵ such that the diagram on the right below commutes.

$$\begin{tikzcd}
F\overline{R} \arrow[r, hook, "F\iota_{\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] \arrow[rr, bend left, "h_{\overline{R}}"] & F(\overline{A \times B}) \arrow[r, "h_{\overline{A\times B}}"] \arrow[d, "F(\overline{\beta\times\beta'})"] & F\overline{A} \times G\overline{B} \arrow[d, "F\overline{\beta}\times F\overline{\beta'}"] \\
F\overline{S} \arrow[r, hook, "F\iota_{\overline{S}}"'] \arrow[rr, bend right, "h_{\overline{S}}"'] & F(\overline{C \times D}) \arrow[r, "h_{\overline{C\times D}}"'] & F\overline{C} \times G\overline{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
F\overline{R} \arrow[r, two heads, "q_{\alpha^\wedge\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] & \alpha^\wedge\overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{R}}"] \arrow[d, dashed, "\epsilon"] & F\overline{A} \times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
F\overline{S} \arrow[r, two heads, "q_{\alpha^\wedge\overline{S}}"'] & \alpha^\wedge\overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{S}}"'] & F\overline{C} \times G\overline{D}
\end{tikzcd}$$

□

If $f : A \to B$ is a morphism in Set then the definition of the graph relation transformer $\langle f\rangle$ for f as a natural transformation between 0-ary functors A and B coincides with the definition of $\langle f\rangle$ for f as a morphism in Set given in the second paragraph of this section. As a result, graph relation transformers are a reasonable extension of graph relations to functors.

To prove the IEL, we will need to know that the equality relation transformer preserves equality relations in Rel; see Equation 14 below. This will follow from the next lemma, which shows how to compute the action of a graph relation transformer on any graph relation.

Lemma 21. *If $\alpha : F \to G$ is a morphism in $[\mathsf{Set}^k, \mathsf{Set}]$ and $f_1 : A_1 \to B_1, \ldots, f_k : A_k \to B_k$, then* $\langle\alpha\rangle^*\langle\overline{f}\rangle = \langle G\overline{f} \circ \alpha_{\overline{A}}\rangle = \langle\alpha_{\overline{B}} \circ F\overline{f}\rangle$.

Proof. Since $h_{\overline{A\times B}}$ is the unique morphism making the bottom triangle of the diagram on the left below commute, and since $h_{\langle\overline{f}\rangle} = h_{\overline{A\times B}} \circ F\,\overline{\iota_{\langle f\rangle}} = h_{\overline{A\times B}} \circ F\overline{\langle id_A, f\rangle}$, the universal property of the product depicted in the diagram on the right gives $h_{\langle\overline{f}\rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle : F\overline{A} \to F\overline{A} \times G\overline{B}$.

$$\begin{tikzcd}
& F\overline{A} \arrow[dl, equal] \arrow[d, "F\overline{\langle id_A, f\rangle}"] \arrow[dr, "F\overline{f}"] & & \\
F\overline{A} & F(\overline{A\times B}) \arrow[l, "F\pi_1"'] \arrow[r, "F\pi_2"] \arrow[d, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
& F\overline{A}\times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F\overline{A} & F\overline{A}\times G\overline{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & G\overline{B} \\
& F\overline{A} \arrow[ul, equal] \arrow[u, dashed, "\exists!"] \arrow[r, "F\overline{f}"'] & F\overline{B} \arrow[u, "\alpha_{\overline{B}}"']
\end{tikzcd}$$

Moreover, $\langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle$ is a monomorphism in Set because $id_{F\overline{A}}$ is, so its (strong epi, mono) factorization gives $\iota_{\alpha^\wedge\langle\overline{f}\rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle$, and thus that $\alpha^\wedge\langle\overline{f}\rangle = F\overline{A}$. Then $\iota_{\alpha^\wedge\langle\overline{f}\rangle}\alpha^\wedge\langle\overline{f}\rangle = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f}\rangle(F\overline{A}) = \langle\alpha_{\overline{B}} \circ F\overline{f}\rangle^*$. Then $\langle\alpha\rangle^*\langle\overline{f}\rangle = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge\langle\overline{f}\rangle}\,\alpha^\wedge\langle\overline{f}\rangle) = (F\overline{A}, G\overline{B}, \langle\alpha_{\overline{B}} \circ F\overline{f}\rangle^*) = \langle\alpha_{\overline{B}} \circ F\overline{f}\rangle$, and, finally, $\alpha_{\overline{B}} \circ F\overline{f} = G\overline{f} \circ \alpha_{\overline{A}}$ by naturality of α. □

The *equality relation transformer* on $F : [\mathsf{Set}^k, \mathsf{Set}]$ is defined to be $\mathsf{Eq}_F = \langle id_F\rangle$. Specifically, $\mathsf{Eq}_F = (F, F, \mathsf{Eq}_F^*)$ with $\mathsf{Eq}_F^* = \langle id_F\rangle^*$, and Lemma 21 indeed ensures that

$$\mathsf{Eq}_F^*\overline{\mathsf{Eq}_A} = \langle id_F\rangle^*\langle id_{\overline{A}}\rangle = \langle F id_{\overline{A}} \circ (id_F)_{\overline{A}}\rangle = \langle id_{F\overline{A}} \circ id_{F\overline{A}}\rangle = \langle id_{F\overline{A}}\rangle = \mathsf{Eq}_{F\overline{A}} \tag{14}$$

for all $\overline{A : \mathsf{Set}}$. Graph relation transformers in general, and equality relation transformers in particular, extend to relation environments in the obvious ways. Indeed, if $\rho, \rho' : \mathsf{SetEnv}$ and $f : \rho \to \rho'$, then the *graph relation environment* $\langle f\rangle$ is defined pointwise by $\langle f\rangle\phi = \langle f\phi\rangle$ for every ϕ, which entails that $\pi_1\langle f\rangle = \rho$ and $\pi_2\langle f\rangle = \rho'$. In particular, the *equality relation environment* Eq_ρ is defined to be $\langle id_\rho\rangle$, which entails that $\mathsf{Eq}_\rho\phi = \mathsf{Eq}_{\rho\phi}$ for every ϕ. With these definitions in hand, we can state and prove both an Identity Extension Lemma and a Graph Lemma for our calculus.

Theorem 22 (IEL). *If $\rho : \mathsf{SetEnv}$ and $\Gamma; \Phi \vdash F$ then* $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}$.

Proof. By induction on the structure of F. Only the Nat, application, fixpoint, and Lan cases are non-routine. The application and fixpoint cases use Equation 14. The fixpoint case $\Gamma; \Phi, \overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{F}$ also uses the observation that, for every $i < \omega$, the following intermediate results can be proved by simultaneous induction with Theorem 22: for any H, ρ, $\overline{A}$, and any subformula J of H, $T^i_{H,\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_A} = (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^i K_0})^*\overline{\mathsf{Eq}_A}$ and

$$\begin{array}{rl}
& [\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash J]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^i_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]} \\
= & [\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash J]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^i K_0}]\overline{[\alpha := \mathsf{Eq}_A]}
\end{array}$$

The case of the latter equation for $J = (\mathsf{Lan}^{\overline{\beta}}_{\overline{K}}\, F)\overline{B}$ uses the facts that $\pi_1(T^i_{H,\mathsf{Eq}_\rho}K_0) = (T^{\mathsf{Set}}_{H,\rho})^i K_0$ for all $i < \omega$, $\pi_2(T^i_{H,\mathsf{Eq}_\rho}K_0) = (T^{\mathsf{Set}}_{H,\rho})^i K_0$ for all $i < \omega$, and that, without loss of generality, the variables in $\overline{\beta}$ do not appear free in H. With these results in hand, the proof of Theorem 22 follows. The simultaneous induction required to make the various parts of the proof hang together appropriately is somewhat delicate, so we give the proof in its entirety in the appendix. As noted there, if functorial variables of arity greater than 0 were allowed in the bodies of μ-types then the IEL would fail. □

Lemma 23 (Graph Lemma). *If* $\rho, \rho' : \mathsf{SetEnv}$, $f : \rho \to \rho'$, *and* $\Gamma; \Phi \vdash F$, *then* $\langle \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f \rangle = \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$.

Proof. Applying Lemma 18 to the morphisms $(f, id_{\rho'}) : \langle f \rangle \to \mathsf{Eq}_{\rho'}$ and $(id_\rho, f) : \mathsf{Eq}_\rho \to \langle f \rangle$ of relation environments gives that

$$(\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f, \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} id_{\rho'}) = \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} (f, id_{\rho'}) \\ : \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle \to \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_{\rho'}$$

and

$$(\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} id_\rho, \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f) = \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} (id_\rho, f) \\ : \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho \to \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$$

Expanding the first equation gives that if $(x, y) \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$ then

$$(\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x, \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} id_{\rho'}\, y) \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_{\rho'}$$

So $\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} id_{\rho'}\, y = id_{\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} \rho'}\, y = y$ and $\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_{\rho'} = \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} \rho'}$, and if $(x, y) \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$ then $(\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x, y) \in \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} \rho'}$, i.e., $\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x = y$, i.e., $(x, y) \in \langle \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f \rangle$. So, we have that $\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle \subseteq \langle \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f \rangle$. Expanding the second equation gives that if $x \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} \rho$ then $(\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} id_\rho\, x,$ $\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x) \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$. Then $\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} id_\rho\, x = id_{\llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} \rho} x = x$, so for any $x \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} \rho$ we have that $(x, \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x) \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$. Moreover, $x \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} \rho$ if and only if $(x, \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x) \in \langle \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f \rangle$ and, if $x \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} \rho$ then $(x, \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x) \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$, so if $(x, \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x) \in \langle \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f \rangle$ then $(x, \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f\, x) \in \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$, i.e., $\langle \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Set}} f \rangle \subseteq \llbracket \Gamma; \Phi \vdash F \rrbracket^{\mathsf{Rel}} \langle f \rangle$. ◻

4 INTERPRETING TERMS

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