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Free Theorems for Nested Types

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1 INTRODUCTION

- Bob has forall types. But we have data types. So we each add somethign different to the simply typed lambda calculus. We'll treat simply typed lambda calculus with data types first, and may add poly types later. This will require additional hypotheses on the semantic categories.
- We're not (obviously) using the exponential between functor categories anywhere.
- Couldn't do this before LICS paper? Or could Bob have done it? What's new?

1.1 Preliminaries

We write Set for the category of sets and functions.

Definition 1. *The category* Rel *is defined as follows.*

- *An object of* Rel *is a triple* (A, B, R) *where* R *is a relation between the objects* A *and* B *in* Set*, i.e., a subset of* $A \times B$*. We write* $R : \mathsf{Rel}(A, B)$ *when convenient.*
- *A morphism between objects* $R : \mathsf{Rel}(A, B)$ *and* $R' : \mathsf{Rel}(A', B')$ *of* Rel *is a pair* $(f : A \to A', g : B \to B')$ *of morphisms in* Set *such that* $(f\,a, g\,b) \in R'$ *whenever* $(a, b) \in R$*.*

If $R : \mathsf{Rel}(A, B)$ we write $\pi_1 R$ and $\pi_2 R$ for the *domain* A of R and the *codomain* B of R, respectively. If $A : \mathsf{Set}$, then we write $\mathsf{Eq}_A = (A, A, \{(x, x) \mid x \in A\})$ for the *equality relation* on A.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, we write $[\mathcal{C}, \mathcal{D}]$ for the set of ω-cocontinuous functors from $\mathcal{C}$ to $\mathcal{D}$.

2 THE CALCULUS

2.1 Types

For each $k \geq 0$, we assume a countable set $\mathbb{T}^k$ of *type constructor variables of arity* k, disjoint for distinct k. We use lower case Greek letters for type constructor variables, and write ϕ^k to indicate that $\phi \in \mathbb{T}^k$. When convenient we may write α, β, etc., rather than α^0, β^0, etc., for elements of $\mathbb{T}^0$. The set of all type constructor variables is $\mathbb{T} = \bigcup_{k \geq 0} \mathbb{T}^k$. We further assume an infinite set $\mathbb{V}$ of *type variables* disjoint from $\mathbb{T}$. We write $\overline{\zeta}$ for either a set $\{\zeta_1, ..., \zeta_n\}$ of type variables or a set of type constructor variables when the cardinality n of the set is unimportant. If $\mathcal{P}$ is a set of type constructor variables then we write $\mathcal{P}, \overline{\phi}$ for $\mathcal{P} \cup \overline{\phi}$ when $\mathcal{P} \cap \overline{\phi} = \emptyset$. We omit the boldface for a singleton set, thus writing ϕ, rather than $\overline{\phi}$, for $\{\phi\}$.

Definition 2. *Let* V *be a finite subset of* $\mathbb{V}$*, and let* $\mathcal{P}$ *and* $\overline{\alpha}$ *be finite subsets of* $\mathbb{T}$*. The sets* $\mathcal{T}(V)$ *of* type expressions *over* V *and* $\mathcal{F}^{\mathcal{P}}(V)$ *of* type constructor expressions *over* V *are given by:*

$$\mathcal{T}(V) \;::=\; V \mid \mathcal{T}(V) \to \mathcal{T}(V) \mid \forall v.\, \mathcal{T}(V, v) \mid \mathsf{Nat}^{\overline{\alpha}}(\mathcal{F}^{\overline{\alpha}}(V), \mathcal{F}^{\overline{\alpha}}(V))$$

and

$$\begin{aligned}\mathcal{F}^{\mathcal{P}}(V) \;::=\; & \mathcal{T}(V) \mid \mathbb{0} \mid \mathbb{1} \mid \mathcal{P}\overline{\mathcal{F}^{\mathcal{P}}(V)} \mid \mathcal{F}^{\mathcal{P}}(V) + \mathcal{F}^{\mathcal{P}}(V) \mid \mathcal{F}^{\mathcal{P}}(V) \times \mathcal{F}^{\mathcal{P}}(V) \\ & \mid \left(\mu\phi^k.\lambda\alpha_1 ... \alpha_k.\mathcal{F}^{\mathcal{P},\alpha_1,...,\alpha_k,\phi}(V)\right)\overline{\mathcal{F}^{\mathcal{P}}(V)}\end{aligned}$$

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The above notation entails that an application $\tau\tau_1...\tau_k$ is allowed only when τ is a type constructor variable of arity k, or τ is a subexpression of the form $\mu\phi^k.\lambda\alpha_1...\alpha_k.\tau$. Moreover, if τ has arity k then τ must be applied to exactly k arguments. Accordingly, an overbar indicates a sequence of subexpressions whose length matches the arity of the functorial expression applied to it. The fact that functorial expressions are always in *η-long normal form* avoids having to consider β-conversion at the level of type constructors, and the fact that the standard type formers are all defined pointwise avoids having to relate functorial expressions at different kinds.

If $\tau \in \mathcal{F}^{\mathcal{P}}(V)$, if $\mathcal{P}$ contains only type constructor variables of arity 0, and if $k = 0$ for every occurrence of ϕ^k bound by μ in τ, then we say that τ is *first-order*. Otherwise we say that τ is *second-order*. The intuition here is that variables in V can be substituted by any types, but those in $\mathcal{P}$ can only be substituted by type constructors, even if of arity 0. In this case, they'd be substituted by type constructors of arity 0 — i.e., type constants — such as Nat or Bool.

Definition 3. *Let* Γ *be a* type context, *i.e., a finite set of type variables, and let* Φ *be a* type constructor context, *i.e., a finite set of type constructor variables. The formation rules for the set* $\mathcal{T} \subseteq \bigcup_{V\subseteq\mathbb{V}} \mathcal{T}(V)$ *of* well-formed type expressions *are*

$$\frac{}{\Gamma, \upsilon; \emptyset \vdash \upsilon : \mathcal{T}} \qquad \frac{\Gamma; \emptyset \vdash \sigma : \mathcal{T} \qquad \Gamma; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \vdash \sigma \to \tau : \mathcal{T}}$$

$$\frac{\Gamma, \upsilon; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \vdash \forall \upsilon.\, \tau : \mathcal{T}} \qquad \frac{\Gamma; \overline{\alpha} \vdash \sigma : \mathcal{F} \qquad \Gamma; \overline{\alpha} \vdash \tau : \mathcal{F}}{\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, \sigma\, \tau : \mathcal{T}}$$

The formation rules for the set $\mathcal{F} \subseteq \bigcup_{V\subseteq\mathbb{V}, \mathcal{P}\subseteq\mathbb{T}} \mathcal{F}^{\mathcal{P}}(V)$ *of* well-formed type constructor expressions *are*

$$\frac{\Gamma; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \vdash \tau : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi, \upsilon \vdash \upsilon : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{0} : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{1} : \mathcal{F}}$$

$$\frac{\Gamma; \Phi \vdash \phi^k : \mathcal{F} \qquad \overline{\Gamma; \Phi \vdash \tau : \mathcal{F}}}{\Gamma; \Phi \vdash \phi^k \overline{\tau}}$$

$$\frac{\Gamma; \Phi, \overline{\alpha}, \phi^k \vdash \tau : \mathcal{F} \qquad \overline{\Gamma; \Phi \vdash \tau : \mathcal{F}}}{\Gamma; \Phi \vdash (\mu\phi^k.\lambda\overline{\alpha}.\, \tau)\overline{\tau}}$$

$$\frac{\Gamma; \Phi \vdash \sigma : \mathcal{F} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \vdash \sigma + \tau : \mathcal{F}} \qquad \frac{\Gamma; \Phi \vdash \sigma : \mathcal{F} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \vdash \sigma \times \tau : \mathcal{F}}$$

Our formation rules allow type constructor expressions like $\mathsf{List}\,\gamma = (\mu\beta.\lambda\alpha.\mathbb{1} + \alpha \times \beta)\gamma$ either to be natural in γ or not, according to whether it is well-formed in the context $\emptyset; \gamma$ or $\gamma; \emptyset$. If the former, then we can derive $\vdash \mathsf{Nat}^{\gamma}\,\mathbb{1}(\mathsf{List}\,\gamma) : \mathcal{T}$. If the latter, then we cannot. Our formation rules also allow the derivation of, e.g., $\delta; \emptyset \vdash \mathsf{Nat}^{\gamma}(\mathsf{List}\,\gamma)\,(\mathsf{Tree}\,\gamma\,\delta)$, which represents a natural transformation between lists and trees that is natural in γ but not in δ.

Substitution for first-order type constructor expressions is the usual capture-avoiding textual substitution. We write $\tau[\alpha := \sigma]$ for the result of substituting σ for α in τ, and $\tau[\alpha_1 := \tau_1, ..., \alpha_k := \tau_k]$, or $\tau[\overline{\alpha := \tau}]$ when convenient, for $\tau[\alpha_1 := \tau_1][\alpha_2 := \tau_2, ..., \alpha_k := \tau_k]$. Substitution for second-order type constructor expressions is given in the next definition.

Definition 4. *If* $\Gamma; \Phi, \phi^k \vdash h[\phi] : \mathcal{F}$ *and* $\Gamma; \Phi, \overline{\alpha} \vdash F : \mathcal{F}$ *with* $\overline{\alpha} = \{\alpha_1, ..., \alpha_k\}$ *and* $k \geq 1$, *then* $\Gamma; \Phi \vdash h[\phi := F] : \mathcal{F}$, *where the operation* $(\cdot)[\phi := F]$ *of* second-order type constructor substitution *is defined by:*

$$\begin{array}{lcl}
\tau[\phi := F] & = & \tau \text{ if } \tau \in \mathcal{T} \\
\mathbb{1}[\phi := F] & = & \mathbb{1} \\
\mathbb{0}[\phi := F] & = & \mathbb{0} \\
(\psi\overline{\tau})[\phi := F] & = & \begin{cases} \psi\,\overline{\tau[\phi := F]} & \text{if } \psi \neq \phi \\ F[\overline{\alpha := \tau[\phi := F]}] & \text{if } \psi = \phi \end{cases} \\
(\sigma + \tau)[\phi := F] & = & \sigma[\phi := F] + \tau[\phi := F] \\
(\sigma \times \tau)[\phi := F] & = & \sigma[\phi := F] \times \tau[\phi := F] \\
((\mu\psi.\lambda\overline{\beta}.\,G)\overline{\tau}[\phi := F] & = & (\mu\psi.\lambda\overline{\beta}.\,G[\phi := F])\,\overline{\tau[\phi := F]}
\end{array}$$

Note that, since an arity 0 type constructor is first-order, substitution into it is just the usual textual replacement, i.e., the usual notion of substitution, as expected.

2.2 Terms

We assume an infinite set $\mathcal{V}$ of term variables disjoint from $\mathbb{T}$ and $\mathbb{V}$.

Definition 5. *Let* Γ *be a type context and* Φ *be a type constructor context. A* term context for Γ and Φ *is a finite set of bindings of the form* $x : \tau$*, where* $x \in \mathcal{V}$ *and* $\Gamma;\Phi \vdash \tau : \mathcal{F}$*.*

We adopt the same conventions for denoting disjoint unions in term contexts as in type contexts and type constructor contexts.

Definition 6. *Let* Δ *be a term context for* Γ *and* Φ*. The formation rules for the set of* well-formed terms over Δ *are*

$$\frac{\Gamma;\emptyset \vdash \tau : \mathcal{T}}{\Gamma;\emptyset \mid \Delta, x : \tau \vdash x : \tau} \qquad \frac{\Gamma;\Phi \vdash \tau : \mathcal{F}}{\Gamma;\Phi \mid \Delta, x : \tau \vdash x : \tau}$$

$$\frac{\Gamma;\emptyset \mid \Delta, x : \sigma \vdash t : \tau}{\Gamma;\emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau} \qquad \frac{\Gamma;\emptyset \mid \Delta \vdash s : \sigma \to \tau \qquad \Gamma;\emptyset \mid \Delta \vdash t : \sigma}{\Gamma;\emptyset \mid \Delta \vdash st : \tau}$$

$$\frac{\Gamma,\alpha;\Phi \vdash \tau : \mathcal{T} \qquad \Gamma,\alpha;\Phi \mid \Delta \vdash t : \tau}{\Gamma;\Phi \mid \Delta \vdash \Lambda\alpha.t : \forall\alpha.\tau} \qquad \frac{\Gamma,\alpha;\Phi \vdash \sigma : \mathcal{T} \qquad \Gamma;\Phi \vdash \tau : \mathcal{T} \qquad \Gamma;\Phi \mid \Delta \vdash t : \forall\alpha.\sigma}{\Gamma;\Phi \mid \Delta \vdash t\tau : \sigma[\alpha := \tau]}$$

No intro $\mathbb{0}$

$$\frac{\Gamma;\Phi \mid \Delta \vdash t : \mathbb{0} \qquad \Gamma;\Phi \vdash \tau : \mathcal{F}}{\Gamma;\Phi \mid \Delta \vdash \bot_\tau t : \tau}$$

$$\frac{}{\Gamma;\Phi \mid \Delta \vdash \top : \mathbb{1}}$$

No elim $\mathbb{1}$

$$\frac{\Gamma;\Phi \mid \Delta \vdash s : \sigma}{\Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau} \qquad \frac{\Gamma;\Phi \mid \Delta \vdash t : \tau}{\Gamma;\Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau}$$

$$\frac{\Gamma;\Phi \vdash \tau,\sigma : \mathcal{F} \qquad \Gamma;\Phi \mid \Delta \vdash t : \sigma + \tau \qquad \Gamma;\Phi \mid \Delta, x : \sigma \vdash l : \gamma \qquad \Gamma;\Phi \mid \Delta, y : \tau \vdash r : \gamma}{\Gamma;\Phi \mid \Delta \vdash \mathsf{case}\ t\ \mathsf{of}\ \{x \mapsto l;\ y \mapsto r\} : \gamma}$$

$$\frac{\Gamma;\Phi \mid \Delta \vdash s : \sigma \qquad \Gamma;\Phi \mid \Delta \vdash t : \tau}{\Gamma;\Phi \mid \Delta \vdash (s,t) : \sigma \times \tau} \qquad \frac{\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau}{\Gamma;\Phi \mid \Delta \vdash \pi_1 t : \sigma} \qquad \frac{\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau}{\Gamma;\Phi \mid \Delta \vdash \pi_2 t : \tau}$$

$$\frac{\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G : \mathcal{T} \qquad \Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G}{\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G}$$

$$\frac{\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \qquad \overline{\Gamma;\Phi \vdash \tau : \mathcal{F}} \qquad \Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]}{\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]}$$

$$\frac{\Gamma;\Phi \mid \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \qquad \overline{\Gamma;\Phi \vdash \tau}}{\Gamma;\Phi \mid \Delta \vdash \mathsf{in}_H\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}}$$

$$\frac{\Gamma;\overline{\alpha} \vdash F : \mathcal{F} \qquad \Gamma;\phi,\overline{\beta} \vdash H : \mathcal{F} \qquad \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, H[\phi := F][\overline{\beta := \alpha}]\, F}{\Gamma;\emptyset \mid \Delta \vdash \mathsf{fold}_H\, t : \mathsf{Nat}^{\overline{\alpha}}\, ((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\, F}$$

3 INTERPRETING TYPES

Parametricity requires that set interpretations of types are defined concurrently with their relational interpretations. In this section we give the set interpretations for types; in the next section we give their relational interpretations. While the set interpretations are relatively straightforward, their relation interpretations are less so, mainly because of the cocontinuity conditions we must impose to ensure that they are well-behaved. We take some effort to develop these in Section 3.2, which separates Definitions 8 and 19 in space but otherwise has no impact on the fact that they are given by mutual induction.

3.1 Interpreting Types as Sets

Definition 7. *A* set environment *maps each type variable to a set, and each type constructor variable of arity k to an element of* $[\mathsf{Set}^k, \mathsf{Set}]$. *A morphism $f : \rho \to \rho'$ from a set environment ρ to a set environment ρ' with $\rho|_{\mathbb{V}} = \rho'|_{\mathbb{V}}$ maps each type variable v to $id_{\rho v}$, and each type constructor variable ϕ of arity k to a natural transformation from the k-ary functor $\rho\phi$ on* Set *to the k-ary functor $\rho'\phi$ on* Set. *Composition of morphisms on set environments is given componentwise, with the identity morphism mapping each set environment to itself. This gives a category of set environments and morphisms between them, which we denote* SetEnv.

When convenient we identify a functor $F : [\mathsf{Set}^0, \mathsf{Set}]$ with the set that is its codomain. With this convention, a set environment maps a type constructor variable of arity 0 to an ω-cocontinuous functor from Set^0 to Set — i.e., to a set — just as it does a type variable. If $\overline{\alpha} = \{\alpha_1, ..., \alpha_k\}$ and $\overline{A} = \{A_1, ..., A_k\}$, then we write $\rho[\overline{\alpha := A}]$ for the set environment ρ' such that $\rho'\alpha_i = A_i$ for $i = 1, ..., k$ and $\rho'\alpha = \rho\alpha$ if $\alpha \notin \{\alpha_1, ..., \alpha_k\}$.

If ρ is a set environment we write Eq_ρ for the relation environment such that $\mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v}$ for every type variable or type constructor variable v; see Definition 17 below for the complete definition of a relation environment. The relational interpretations referred to in the condition on the natural transformations in the clause of Definition 8 for types of the form $\mathsf{Nat}^{\overline{\alpha}}\, F\, G$ are given in full in Definition 19. Intuitively, this condition can be thought of as ensuring that set interpretations of such terms are sufficiently uniform.

Definition 8. *Let ρ be a set environment. The* set interpretation $[\![\cdot]\!]^{\mathsf{Set}} : \mathcal{F} \to [\mathsf{SetEnv}, \mathsf{Set}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma; \emptyset \vdash v]\!]^{\mathsf{Set}}\rho &= \rho v \text{ if } v \in \mathbb{V} \\
[\![\Gamma; \emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho \to [\![\Gamma; \emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho
\end{aligned}
$$

need to interpret forall types if we include them

$$
\begin{aligned}
[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho &= \{\eta : [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}] \Rightarrow [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}] \\
&\qquad \mid \forall \overline{A}, \overline{B} : \mathsf{Set}. \forall \overline{R : \mathsf{Rel}(A,B)}. \\
&\qquad\quad (\eta_{\overline{A}}, \eta_{\overline{B}}) : [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \to [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]\} \\
[\![\Gamma; \Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}}\rho &= 0 \\
[\![\Gamma; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho &= 1 \\
[\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho &= (\rho\phi)\, \overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \\
[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho + [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma; \Phi \vdash \sigma \times \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho \times [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho &= (\mu T^{\mathsf{Set}}_\rho)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \\
\text{where } T^{\mathsf{Set}}_\rho F &= \lambda\overline{A}.[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\phi := F][\overline{\alpha := A}] \\
\text{and } T^{\mathsf{Set}}_\rho \eta &= \lambda\overline{A}.[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_\rho[\phi := \eta][\overline{\alpha := id_A}]
\end{aligned}
$$

If ρ is a set environment and $\vdash \tau : \mathcal{F}$ then we may write $[\![\vdash \tau]\!]^{\mathsf{Set}}$ instead of $[\![\emptyset; \emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho$ since the environment is immaterial. Definition 8 ensures that

$$
[\![\Gamma; \Phi \vdash F\overline{\tau}]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \Phi, \overline{\alpha} \vdash F\overline{\alpha}]\!]^{\mathsf{Set}}(\rho[\overline{\alpha := [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}}])
$$

Moreover, the third fourth clause does indeed define a set. Indeed, local finite presentability of Set and ω-cocontinuity of $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho$ ensure that $\{\eta : [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho \Rightarrow [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho\}$ (which contains $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$) is a subset of

$$
\left\{ ([\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}])^{([\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}])} \;\middle|\; \overline{S} = (S_1, \ldots, S_{|\overline{\alpha}|}),\ \text{and } S_i \text{ is a finite set for } i = 1, \ldots, |\overline{\alpha}| \right\}
$$

There are countably many choices for tuples $\overline{S}$, and each of these gives rise to a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}]$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}]$. But there are only Set-many choices of morphisms between these (or any) two objects because Set is locally small.

In order to make sense of the last clause in Definition 8, we need to know that T^{Set}_ρ is an ω-cocontinuous endofunctor on $[\mathsf{Set}^k, \mathsf{Set}]$, so that it admits a fixed point. Since T^{Set}_ρ is defined in terms of $[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, this means that set interpretations of types must be functors. This in turn means that the actions of set interpretations of types on objects and on morphisms in SetEnv are intertwined. In fact, we know from [Johann and Polonsky 2019] that, for every $\Gamma; \overline{\alpha} \vdash \tau : \mathcal{F}$, $[\![\Gamma; \overline{\alpha} \vdash \tau]\!]^{\mathsf{Set}}$ is actually functorial in $\overline{\alpha}$ and ω-cocontinuous. What remains is to define the actions of each of these functors on morphisms between environments.

Definition 9. *Let $f : \rho \to \rho'$ for set environments ρ and ρ' such that $\rho|_{\mathbb{V}} = \rho'|_{\mathbb{V}}$. The action $[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} f$ of $[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}$ on the morphism f is given as follows:*

- *If $\Gamma, v; \emptyset \vdash v$ then $[\![\Gamma, v; \emptyset \vdash v]\!]^{\mathsf{Set}} f = id_{\rho v}$.*
- *If $\Gamma; \emptyset \vdash \sigma \to \tau$ then $[\![\Gamma; \emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Set}} f = id_{[\![\Gamma; \emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Set}}\rho}$.*
- *If $\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G$, then we define $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} f = id_{[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho}$.*

- *If* $\Gamma;\Phi \vdash \mathbb{0}$ *then* $[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}} f = id_0$.
- *If* $\Gamma;\Phi \vdash \mathbb{1}$ *then* $[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}} f = id_1$.
- *If* $\Gamma;\Phi \vdash \phi\overline{\tau}$, *then we have that* $[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} f : [\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho' = (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \to (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}$ *is defined by* $[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} f = (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f} = (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f} \circ (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}$. *This equality holds because* $\rho\phi$ *and* $\rho'\phi$ *are functors and* $f\phi : \rho\phi \to \rho'\phi$ *is a natural transformation, so that the following naturality square commutes:*

$$\begin{tikzcd}
(\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\rho\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}"'] & (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \arrow[d, "(\rho'\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}"'] \\
(\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}"] & (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd} \tag{1}$$

- *If* $\Gamma;\Phi \vdash \sigma+\tau$ *then* $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Set}} f$ *is defined by* $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Set}} f(\mathsf{inl}\,x) = \mathsf{inl}\,([\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}} f x)$ *and* $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Set}} f(\mathsf{inr}\,y) = \mathsf{inr}\,([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f y)$.
- *If* $\Gamma;\Phi \vdash \sigma\times\tau$ *then* $[\![\Gamma;\Phi \vdash \sigma\times\tau]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f$.
- *If* $\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$ *then letting* $\sigma_f^{\mathsf{Set}} : T_\rho^{\mathsf{Set}} \to T_{\rho'}^{\mathsf{Set}}$ *be the map*

$$F \mapsto \lambda\overline{A}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} f[\phi := id_F][\overline{\alpha := id_A}]$$

we define

$$\begin{array}{rl}
& [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}} f \\
: & [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho' \\
= & (\mu T_\rho^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \to (\mu T_{\rho'}^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}
\end{array}$$

by

$$\begin{array}{rl}
& (\mu\sigma_f^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'} \circ (\mu T_\rho^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f} \\
= & (\mu T_{\rho'}^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f} \circ (\mu\sigma_f^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}
\end{array}$$

Again, this equality holds because $\mu T_\rho^{\mathsf{Set}}$ *and* $\mu T_{\rho'}^{\mathsf{Set}}$ *are functors and* $\mu\sigma_f^{\mathsf{Set}} : \mu T_\rho^{\mathsf{Set}} \to \mu T_{\rho'}^{\mathsf{Set}}$ *is a natural transformation, so that the following naturality square commutes:*

$$\begin{tikzcd}
(\mu T_\rho^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \arrow[r, "(\mu\sigma_f^{\mathsf{Set}})_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\mu T_\rho^{\mathsf{Set}})\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f})"'] & (\mu T_{\rho'}^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \arrow[d, "(\mu T_{\rho'}^{\mathsf{Set}})\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}"'] \\
(\mu T_\rho^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'} \arrow[r, "(\mu\sigma_f^{\mathsf{Set}})_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}"] & (\mu T_{\rho'}^{\mathsf{Set}})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd} \tag{2}$$

3.2 Interpreting Types as Relations

Definition 10. *A* k-ary relation transformer F *is a triple* (F^0, F^1, F^*), *where* $F^0, F^1 : [\mathsf{Set}^k, \mathsf{Set}]$ *are functors,* $F^* : [\mathsf{Rel}^k, \mathsf{Rel}]$ *is a functor, if* $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, *then* $F^*\overline{R} : \mathsf{Rel}(F^0\overline{A}, F^1\overline{B})$, *and if* $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$ *then* $F^*\overline{(\alpha,\beta)} = (F^0\overline{\alpha}, F^1\overline{\beta})$. *We define* $F\overline{R}$ *to be* $F^*\overline{R}$ *and* $F\overline{(\alpha,\beta)}$ *to be* $F^*\overline{(\alpha,\beta)}$.

Expanding the last clause of Definition 10 is equivalent to: if $\overline{(a,b) \in R}$ implies $\overline{(\alpha\, a, \beta\, b) \in S}$ then $(c,d) \in F^*\overline{R}$ implies $(F^0\overline{\alpha}\, c, F^1\overline{\beta}\, d) \in F^*\overline{S}$.

When convenient we identify a 0-ary relation transformer (A, B, R) with $R : \mathsf{Rel}(A, B)$. We may also write F^0 and F^1 for $\pi_1 F$ and $\pi_2 F$. We extend these conventions to relation environments, introduced in Definition 17 below, as well.

Definition 11. *The category RT_k of k-ary relation transformers is given by the following data:*

- *An object of RT_k is a relation transformer.*
- *A morphism $\delta : (G^0, G^1, G^*) \to (H^0, H^1, H^*)$ in RT_k is a pair of natural transformations (δ^0, δ^1) where $\delta^0 : G^0 \to H^0$, $\delta^1 : G^1 \to H^1$ such that, for all $\overline{R : \mathsf{Rel}(A, B)}$, if $(x, y) \in G^*\overline{R}$ then $(\delta^0_{\overline{A}} x, \delta^1_{\overline{B}} y) \in H^*\overline{R}$. This is basically a fibred natural transformation, but for heterogeneous relations.*
- *Identity morphisms and composition are inherited from the category of functors on* Set.

Definition 12. *An endofunctor H on RT_k is a triple $H = (H^0, H^1, H^*)$, where*

- *H^0 and H^1 are functors from $[\mathsf{Set}^k, \mathsf{Set}]$ to $[\mathsf{Set}^k, \mathsf{Set}]$*
- *H^* is a functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]$*
- *for all $\overline{R : \mathsf{Rel}(A, B)}$, $\pi_1((H^*(\delta^0, \delta^1))_{\overline{R}}) = (H^0\delta^0)_{\overline{A}}$ and $\pi_2((H^*(\delta^0, \delta^1))_{\overline{R}}) = (H^1\delta^1)_{\overline{B}}$*
- *The action of H on objects is given by $H\,(F^0, F^1, F^*) = (H^0F^0,\ H^1F^1,\ H^*(F^0, F^1, F^*))$*
- *The action of H on morphisms is given by $H\,(\delta^0, \delta^1) = (H^0\delta^0, H^1\delta^1)$ for $(\delta^0, \delta^1) : (F^0, F^1, F^*) \to (G^0, G^1, G^*)$*

Since the results of applying H to k-ary relation transformers and morphisms between them must again be k-ary relation transformers and morphisms between them, respectively, Definition 12 implicitly requires that the following three conditions hold:

(1) if $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then
$$H^*(F^0, F^1, F^*)\overline{R} : \mathsf{Rel}(H^0F^0\overline{A}, H^1F^1\overline{B})$$
In other words, $\pi_1(H^*(F^0, F^1, F^*)\overline{R}) = H^0F^0\overline{A}$ and $\pi_2(H^*(F^0, F^1, F^*)\overline{R}) = H^1F^1\overline{B}$.

(2) if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then
$$H^*(F^0, F^1, F^*)\overline{(\alpha, \beta)} = (H^0F^0\overline{\alpha}, H^1F^1\overline{\beta})$$
In other words, $\pi_1(H^*(F^0, F^1, F^*)\overline{(\alpha, \beta)}) = H^0F^0\overline{\alpha}$ and $\pi_2(H^*(F^0, F^1, F^*)\overline{(\alpha, \beta)}) = H^1F^1\overline{\beta}$.

(3) if $(\delta^0, \delta^1) : (F^0, F^1, F^*) \to (G^0, G^1, G^*)$ and $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then
$$\text{if } (x, y) \in H^*(F^0, F^1, F^*)\overline{R} \text{ then } ((H^0\delta^0)_{\overline{A}} x, (H^1\delta^1)_{\overline{B}} y) \in H^*(G^0, G^1, G^*)\overline{R}$$
Note, however, that this condition is automatically satisfied because it is implied by the third bullet point of Definition 12.

Definition 13. *If H and K are endofunctors on RT_k, then a* natural transformation *$\sigma : H \to K$ is a pair $\sigma = (\sigma^0, \sigma^1)$, where $\sigma^0 : H^0 \to K^0$ and $\sigma^1 : H^1 \to K^1$ are natural transformations between endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]$ and the component of σ at the k-ary relation transformer F is given by $\sigma_F = (\sigma^0_{F^0}, \sigma^1_{F^1})$.*

Definition 13 entails that $\sigma^i_{F^i}$ must be natural in $F^i : [\mathsf{Set}^k, \mathsf{Set}]$, and, for every F, both $(\sigma^0_{F^0})_{\overline{A}}$ and $(\sigma^1_{F^1})_{\overline{A}}$ must be natural in $\overline{A}$. Moreover, since the results of applying σ to k-ary relation transformers must be morphisms of k-ary relation transformers, Definition 13 implicitly requires that $(\sigma_F)_{\overline{R}} = ((\sigma^0_{F^0})_{\overline{A}}, (\sigma^1_{F^1})_{\overline{B}})$ is a morphism in Rel for any k-tuple of relations $\overline{R : \mathsf{Rel}(A, B)}$, i.e., if $(x, y) \in H^*F\overline{R}$, then $((\sigma^0_{F^0})_{\overline{A}} x, (\sigma^1_{F^1})_{\overline{B}} y) \in K^*F\overline{R}$.

Next, we see that we can compute colimits in RT_k.

Lemma 14. $\varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d) = (\varinjlim_{d\in\mathcal{D}} F^0_d, \varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$

Proof. We first observe that $(\varinjlim_{d\in\mathcal{D}} F^0_d, \varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$ is in RT_k. If $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then $\varinjlim_{d\in\mathcal{D}} F^*_d\overline{R} : \mathsf{Rel}(\varinjlim_{d\in\mathcal{D}} F^0_d\overline{A}, \varinjlim_{d\in\mathcal{D}} F^1_d\overline{B})$ because of how colimits are computed in Rel. Moreover, if $(\alpha_1, \beta_1) \in \mathrm{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathrm{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then

$$\begin{array}{cl} & (\varinjlim_{d\in\mathcal{D}} F^*_d)\overline{(\alpha,\beta)} \\ = & \varinjlim_{d\in\mathcal{D}} F^*_d\overline{(\alpha,\beta)} \\ = & \varinjlim_{d\in\mathcal{D}} (F^0_d\overline{\alpha},\ F^1_d\overline{\beta}) \\ = & (\varinjlim_{d\in\mathcal{D}} F^0_d\overline{\alpha},\ \varinjlim_{d\in\mathcal{D}} F^1_d\overline{\beta}) \end{array}$$

so $(\varinjlim_{d\in\mathcal{D}} F^0_d, \varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$ actually is in RT_k.

Now to see that $\varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d) = (\varinjlim_{d\in\mathcal{D}} F^0_d, \varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$, let $\gamma^0_d : F^0_d \to \varinjlim_{d\in\mathcal{D}} F^0_d$ and $\gamma^1_d : F^1_d \to \varinjlim_{d\in\mathcal{D}} F^1_d$ be the injections for the colimits $\varinjlim_{d\in\mathcal{D}} F^0_d$ and $\varinjlim_{d\in\mathcal{D}} F^1_d$, respectively. Then $(\gamma^0_d, \gamma^1_d) : (F^0_d, F^1_d, F^*_d) \to \varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d)$ is a morphism in RT_k because, for all $\overline{R : \mathsf{Rel}(A,B)}$, $((\gamma^0_d)_{\overline{A}}, (\gamma^1_d)_{\overline{B}}) : F^*_d\overline{R} \to \varinjlim_{d\in\mathcal{D}} F^*_d\overline{R}$ is a morphism in Rel. So $\{(\gamma^0_d, \gamma^1_d)\}_{d\in\mathcal{D}}$ are the mediating morphisms of a cocone in RT_k with vertex $\varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d)$. To see that this cocone is a colimiting cocone, let $C = (C^0, C^1, C^*)$ be the vertex of a cocone for $\{(F^0_d, F^1_d, F^*_d)\}_{d\in\mathcal{D}}$ with injections $(\delta^0_d, \delta^1_d) : (F^0_d, F^1_d, F^*_d) \to C$. If $\eta^0 : \varinjlim_{d\in\mathcal{D}} F^0_d \to C^0$ and $\eta^1 : \varinjlim_{d\in\mathcal{D}} F^1_d \to C^1$ are the mediating morphisms in $[\mathsf{Set}^k, \mathsf{Set}]$, then η^0 and η^1 are unique such that $\delta^0_d = \eta^0 \circ \gamma^0_d$ and $\delta^1_d = \eta^1 \circ \gamma^1_d$. We therefore have that $(\eta^0, \eta^1) : \varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d) \to C$ is the mediating morphism in RT_k. Indeed, for all $\overline{R : \mathsf{Rel}(A,B)}$ and $(x, y) \in \varinjlim_{d\in\mathcal{D}} F^*_d\overline{R}$, there exist d and $(x', y') \in F^*_d\overline{R}$ such that $(\gamma^0_d)_{\overline{A}}x' = x$ and $(\gamma^1_d)_{\overline{B}}y' = y$. But then $(\eta^0_{\overline{A}}x, \eta^1_{\overline{B}}y) = (\eta^0_{\overline{A}}((\gamma^0_d)_{\overline{A}}x'), \eta^1_{\overline{B}}((\gamma^1_d)_{\overline{B}}y')) = ((\delta^0_d)_{\overline{A}}x', (\delta^1_d)_{\overline{B}}y')$, and this pair is in $C^*\overline{R}$ because (δ^0_d, δ^1_d) is a morphism from (F^0_d, F^1_d, F^*_d) to C in RT_k. □

Definition 15. *An endofunctor $T = (T^0, T^1, T^*)$ on RT_k is ω-*cocontinuous *if T^0 and T^1 are ω-cocontinuous endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]$ and T^* is an ω-cocontinuous functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]$, i.e., is in $[RT_k, [\mathsf{Rel}^k, \mathsf{Rel}]]$.*

For any k and $R : \mathsf{Rel}(A, B)$, let K^{Rel}_R be the constantly R-valued functor from Rel^k to Rel, and for any k and set A, let K^{Set}_A be the constantly A-valued functor from Set^k to Set. Moreover, let 0 denote either the initial object of Set or the initial object of Rel, depending on the context. Observing that, for every k, K^{Set}_0 is initial in $[\mathsf{Set}^k, \mathsf{Set}]$, and similarly for K^{Rel}_0, we have that, for each k, $K_0 = (K^{\mathsf{Set}}_0, K^{\mathsf{Set}}_0, K^{\mathsf{Rel}}_0)$ is initial in RT_k. Thus, if $T = (T^0, T^1, T^*) : RT_k \to RT_k$ is an endofunctor on RT_k then we can define μT to be the relation transformer

$$\mu T = \varinjlim_{n\in\mathbb{N}} T^n K_0$$

Then Lemma 14 shows μT is indeed a relation transformer, and that it is given explicitly by

$$\varinjlim_{n\in\mathbb{N}} T^n K_0 = (\mu T^0, \mu T^1, \varinjlim_{n\in\mathbb{N}} (T^n K_0)^*) \tag{3}$$

Lemma 16. *For any $T : [RT_k, RT_k]$, $\mu T \cong T(\mu T)$.*

Proof. We have $T(\mu T) = T(\varinjlim_{n\in\mathbb{N}}(T^n K_0)) \cong \varinjlim_{n\in\mathbb{N}} T(T^n K_0) = \mu T$. □

In fact, the isomorphism in Lemma 16 is given by the morphisms $(in_0, in_1) : T(\mu T) \to \mu T$ and $(in_0^{-1}, in_1^{-1}) : \mu T \to T(\mu T)$ in RT_k. It is worth noting that the latter is always a morphism in RT_k, but the former isn't necessarily a morphism in RT_k unless T is ω-cocontinuous.

Say realizing that not being able to define third components directly, but rather only through the other two components, is an important conceptual contribution. Not all functors on Rel are third components of relation transformers. It's overly restrictive to require that the third component of a functor on RT_k be a functor on all of $[\mathsf{Rel}^k, \mathsf{Rel}]$. For example, we can define $T_\rho F$ when F is a relation transformer, but it is not clear how we could define $T_\rho F$ when $F : [\mathsf{Rel}^k, \mathsf{Rel}]$.

Definition 17. *A* relation environment *maps each type variable to a relation, and each type constructor variable of arity k to an ω-cocontinuous k-ary relation transformer. A morphism $f : \rho \to \rho'$ from a relation environment ρ to a relation environment ρ' such that $\rho|_{\mathbb{V}} = \rho'|_{\mathbb{V}}$ maps each type variable v to $id_{\rho v}$ and each type constructor variable ϕ of arity k to a natural transformation from the k-ary relation transformer $\rho\phi$ to the k-ary relation transformer $\rho'\phi$. Composition of morphisms on relation environments is given componentwise, with the identity morphism mapping each relation environment to itself. This gives a category of relation environments and morphisms between them, which we denote* RelEnv.

When convenient we identify a 0-ary relation transformer with the relation (transformer) that is its codomain. With this convention, a relation environment maps a type constructor variable of arity 0 to a 0-ary relation transformer — i.e., to a relation — just as it does a type variable. We write $\rho[\overline{\alpha := R}]$ for the relation environment ρ' such that $\rho'\alpha_i = R_i$ for $i = 1, \ldots, k$ and $\rho'\alpha = \rho\alpha$ if $\alpha \notin \{\alpha_1, \ldots, \alpha_k\}$. If ρ is a relation environment, we write $\pi_1\rho$ for the set environment mapping each type variable β to $\pi_1(\rho\beta)$ and each type constructor variable ϕ to the functor $(\rho\phi)^0$. The set environment $\pi_2\rho$ is defined analogously.

We define, for each k, the notion of an ω-cocontinuous functor from RelEnv to RT_k:

Definition 18. *A functor $H : [\mathsf{RelEnv}, RT_k]$ is a triple $H = (H^0, H^1, H^*)$, where*

- *H^0 and H^1 are objects in* $[\mathsf{SetEnv}, [\mathsf{Set}^k, \mathsf{Set}]]$
- *H^* is a an object in* $[\mathsf{RelEnv}, [\mathsf{Rel}^k, \mathsf{Rel}]]$
- *for all $\overline{R : \mathsf{Rel}(A, B)}$ and morphisms f in* RelEnv, *$\pi_1(H^* f\, \overline{R}) = H^0(\pi_1 f)\, \overline{A}$ and $\pi_2(H^* f\, \overline{R}) = H^1(\pi_2 f)\, \overline{B}$*
- *The action of H on ρ in* RelEnv *is given by $H\rho = (H^0(\pi_1\rho),\ H^1(\pi_2\rho),\ H^*\rho)$*
- *The action of H on morphisms $f : \rho \to \rho'$ in* RelEnv *is given by $Hf = (H^0(\pi_1 f), H^1(\pi_2 f))$*

Spelling out the last two bullet points above gives the following analogues of Conditions (1), (2), and (3) immediately following Definition 12:

(1) if $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then

$$H^*\rho\, \overline{R} : \mathsf{Rel}(H^0(\pi_1\rho)\, \overline{A}, H^1(\pi_2\rho)\, \overline{B})$$

In other words, $\pi_1(H^*\rho\overline{R}) = H^0(\pi_1\rho)\, \overline{A}$ and $\pi_2(H^*\rho\overline{R}) = H^1(\pi_2\rho)\, \overline{B}$.

(2) if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then

$$H^*\rho\, \overline{(\alpha, \beta)} = (H^0(\pi_1\rho)\, \overline{\alpha}, H^1(\pi_2\rho)\, \overline{\beta})$$

In other words, $\pi_1(H^*\rho\, \overline{(\alpha, \beta)}) = H^0(\pi_1\rho)\, \overline{\alpha}$ and $\pi_2(H^*\rho\, \overline{(\alpha, \beta)}) = H^1(\pi_2\rho)\, \overline{\beta}$.

(3) if $f : \rho \to \rho'$ and $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then

$$\text{if } (x, y) \in H^*\rho\, \overline{R} \text{ then } (H^0(\pi_1 f)\, \overline{A}\, x, H^1(\pi_2 f)\, \overline{B}\, y) \in H^*\rho'\, \overline{R}$$

Note, however, that this condition is automatically satisfied because it is implied by the third bullet point of Definition 18.

Considering RelEnv as a product $\Pi_{\phi^k \in \mathbb{V} \cup \mathbb{T}} RT_k$, we extend Lemma 14 to compute colimits in RelEnv componentwise, and similarly extend Definition 15 to give a componentwise notion of ω-cocontinuity of functors from RelEnv to RT_k.

We recall from the start of this section that Definition 19 is given mutually inductively with Definition 8. We can, at last, define:

Definition 19. *Let ρ be a relation environment. The* relation interpretation $[\![\cdot]\!]^{\mathsf{Rel}} : \mathcal{F} \to [\mathsf{RelEnv}, \mathsf{Rel}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma; \emptyset \vdash v]\!]^{\mathsf{Rel}}\rho &= \rho v \text{ if } v \in \mathbb{V} \\
[\![\Gamma; \emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma; \emptyset \vdash \tau]\!]^{\mathsf{Rel}}\rho
\end{aligned}
$$

need to interpret forall types if we include them

$$
\begin{aligned}
[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho &= \{\eta : [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := -}] \Rightarrow [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := -}]\} \\
&= \{(t, t') \in [\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid \\
&\qquad \forall R_1 : \mathsf{Rel}(A_1, B_1) \ldots R_k : \mathsf{Rel}(A_k, B_k). \\
&\qquad\quad (t_{\overline{A}}, t'_{\overline{B}}) \in ([\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}])^{[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]}\} \\
&= \{(t, t') \in [\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid \\
&\qquad \forall R_1 : \mathsf{Rel}(A_1, B_1) \ldots R_k : \mathsf{Rel}(A_k, B_k). \\
&\qquad\quad \forall (a, b) \in [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]. \\
&\qquad\qquad (t_{\overline{A}}a, t'_{\overline{B}}b) \in [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]\} \\
[\![\Gamma; \Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\rho &= 0 \\
[\![\Gamma; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\rho &= 1 \\
[\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho &= (\rho\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho} \\
[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho + [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho \\
[\![\Gamma; \Phi \vdash \sigma \times \tau]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho \times [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho \\
[\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\rho &= (\mu T_\rho)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho} \\
\text{where } T_\rho &= (T^{\mathsf{Set}}_{\pi_1\rho}, T^{\mathsf{Set}}_{\pi_2\rho}, T^{\mathsf{Rel}}_\rho) \\
\text{and } T^{\mathsf{Rel}}_\rho F &= \lambda\overline{R}.[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F][\overline{\alpha := R}] \\
\text{and } T^{\mathsf{Rel}}_\rho \delta &= \lambda\overline{R}.[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := \delta][\overline{\alpha := id_R}]
\end{aligned}
$$

If ρ is a relational environment and $\vdash \tau : \mathcal{F}$, then we write $[\![\vdash \tau]\!]^{\mathsf{Rel}}$ instead of $[\![\emptyset; \emptyset \vdash \tau]\!]^{\mathsf{Rel}}\rho$ as for set interpretations.

For the last clause in Definition 19 to be well-defined, we need to know that T_ρ is an ω-cocontinuous endofunctor on RT so that, by Definition 16, it admits a fixed point. Since T_ρ is defined in terms of $[\![\Gamma; \Phi, \phi^k, \overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}$, this means that relational interpretations of types must be ω-cocontinuous functors from RelEnv to RT_0. This in turn means that the actions of relational interpretations of types on objects and on morphisms in RelEnv are intertwined. In fact, we already know from [Johann and Polonsky 2019] that, for every $\Gamma; \overline{\alpha} \vdash \tau : \mathcal{F}$, $[\![\Gamma; \overline{\alpha} \vdash \tau]\!]^{\mathsf{Rel}}$ is actually functorial in $\overline{\alpha}$ and ω-cocontinuous. We first define the actions of each of these functors on morphisms between environments, and then argue that the functors given by Definitions 19 and 20 are well-defined and have the required properties.

Definition 20. *Let $f : \rho \to \rho'$ for relation environments ρ and ρ' such that $\rho|_{\mathbb{V}} = \rho'|_{\mathbb{V}}$. The action $\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f$ of $\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}$ on the morphism f is given as follows:*

- *If $\Gamma, v; \emptyset \vdash v$ then $\llbracket\Gamma, v;\emptyset\vdash v\rrbracket^{\mathsf{Rel}}f = id_{\rho v}$.*
- *If $\Gamma;\emptyset\vdash\sigma\to\tau$ then $\llbracket\Gamma;\emptyset\vdash\sigma\to\tau\rrbracket^{\mathsf{Rel}}f = id_{\llbracket\Gamma;\emptyset\vdash\sigma\to\tau\rrbracket^{\mathsf{Rel}}\rho}$.*
- *If $\Gamma;\emptyset\vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G$, then we define $\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Rel}}f = id_{\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Rel}}\rho}$.*
- *If $\Gamma;\Phi\vdash\mathbb{0}$ then $\llbracket\Gamma;\Phi\vdash\mathbb{0}\rrbracket^{\mathsf{Rel}}f = id_0$.*
- *If $\Gamma;\Phi\vdash\mathbb{1}$ then $\llbracket\Gamma;\Phi\vdash\mathbb{1}\rrbracket^{\mathsf{Rel}}f = id_1$.*
- *If $\Gamma;\Phi\vdash\phi\overline{\tau}$, then we have that $\llbracket\Gamma;\Phi\vdash\phi\overline{\tau}\rrbracket^{\mathsf{Rel}}f : \llbracket\Gamma;\Phi\vdash\phi\overline{\tau}\rrbracket^{\mathsf{Rel}}\rho \to \llbracket\Gamma;\Phi\vdash\phi\overline{\tau}\rrbracket^{\mathsf{Rel}}\rho' = (\rho\phi)\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho} \to (\rho'\phi)\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho'}$ is defined by $\llbracket\Gamma;\Phi\vdash\phi\tau A\rrbracket^{\mathsf{Rel}}f = (f\phi)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho'}} \circ (\rho\phi)\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f} = (\rho'\phi)\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f} \circ (f\phi)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho}}$.*
- *If $\Gamma;\Phi\vdash\sigma{+}\tau$ then $\llbracket\Gamma;\Phi\vdash\sigma+\tau\rrbracket^{\mathsf{Rel}}f$ is defined by $\llbracket\Gamma;\Phi\vdash\sigma+\tau\rrbracket^{\mathsf{Rel}}f(\mathsf{inl}\,x) = \mathsf{inl}\,(\llbracket\Gamma;\Phi\vdash\sigma\rrbracket^{\mathsf{Rel}}f x)$ and $\llbracket\Gamma;\Phi\vdash\sigma+\tau\rrbracket^{\mathsf{Rel}}f(\mathsf{inr}\,y) = \mathsf{inr}\,(\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f y)$.*
- *If $\Gamma;\Phi\vdash\sigma\times\tau$ then $\llbracket\Gamma;\Phi\vdash\sigma\times\tau\rrbracket^{\mathsf{Rel}}f = \llbracket\Gamma;\Phi\vdash\sigma\rrbracket^{\mathsf{Rel}}f \times \llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f$.*
- *If $\Gamma;\Phi\vdash(\mu\phi^k.\lambda\overline{\alpha}.H)\overline{\tau}$ then letting $\sigma_f : T_\rho \to T_{\rho'}$ be the map*

$$F \mapsto \lambda\overline{R}.\llbracket\Gamma;\Phi,\phi,\overline{\alpha}\vdash H\rrbracket^{\mathsf{Rel}}f[\phi := id_F][\overline{\alpha := id_R}]$$

 we define

$$\begin{array}{rl} & \llbracket\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}\rrbracket^{\mathsf{Rel}}f \\ = & (\mu\sigma_f)\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho'} \circ (\mu T_\rho)\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f}) \\ = & (\mu T_{\rho'})\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f} \circ (\mu\sigma_f)\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho} \end{array}$$

To see that the functors given by Definitions 19 and 20 are well-defined we must show that $T_\rho F$ is a relation transformer for any relation transformer F, and that $\sigma_f F : T_\rho F \to T_{\rho'} F$ is a morphism of relation transformers for every relation transformer F and every morphism $f : \rho \to \rho'$ in RelEnv.

Lemma 21. *The interpretations in Definitions 19 and 20 are well-defined and, for every $\Gamma;\Phi\vdash\tau$,*

$$\llbracket\Gamma;\Phi\vdash\tau\rrbracket = (\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}, \llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}, \llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}})$$

is an ω-cocontinuous functor from RelEnv to RT_0, i.e., is an element of $[\mathsf{RelEnv}, RT_0]$.

Proof. By induction on the structure of τ. The only interesting cases are when $\tau = \phi\overline{\tau}$ and when $\tau = (\mu\phi^k.\lambda\overline{\alpha}.H)\overline{\tau}$. We consider each in turn.

- When $\tau = \Gamma;\Phi\vdash\phi\overline{\tau}$, we have

$$\begin{array}{rl} & \pi_i(\llbracket\Gamma;\Phi\vdash\phi\overline{\tau}\rrbracket^{\mathsf{Rel}}\rho) \\ = & \pi_i((\rho\phi)\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho}) \\ = & (\pi_i(\rho\phi))(\overline{\pi_i(\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho)}) \\ = & ((\pi_i\rho)\phi)(\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}(\pi_i\rho)}) \\ = & \llbracket\Gamma;\Phi\vdash\phi\overline{\tau}\rrbracket^{\mathsf{Set}}(\pi_i\rho) \end{array}$$

 and, for $f : \rho \to \rho'$ in RelEnv,

$$\begin{array}{rl} & \pi_i(\llbracket\Gamma;\Phi\vdash\phi\overline{\tau}\rrbracket^{\mathsf{Rel}}f) \\ = & \pi_i((f\phi)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho'}}) \circ \pi_i((\rho\phi)(\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f})) \\ = & (\pi_i(f\phi))_{\overline{\pi_i(\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}\rho')}} \circ (\pi_i(\rho\phi))(\overline{\pi_i(\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Rel}}f)}) \\ = & ((\pi_i f)\phi)_{\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}(\pi_i\rho')}} \circ ((\pi_i\rho)\phi)(\overline{\llbracket\Gamma;\Phi\vdash\tau\rrbracket^{\mathsf{Set}}(\pi_i f)}) \\ = & \llbracket\Gamma;\Phi\vdash\phi\overline{\tau}\rrbracket^{\mathsf{Set}}(\pi_i f) \end{array}$$

The third equalities of each of the above derivations are by the induction hypothesis. That $[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]$ is ω-cocontinuous is an immediate consequence of the facts that Set and Rel are locally finitely presentable, together with Corollary 12 of [Johann and Polonsky 2019].

- When $\tau = (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$ we first show that $[\![(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]$ is well-defined.
  - $\underline{T_\rho : [RT_k, RT_k]}$: We must show that, for any relation transformer $F = (F^0, F^1, F^*)$, the triple $\underline{T_\rho F = (T^{\mathsf{Set}}_{\pi_1\rho}F^0, T^{\mathsf{Set}}_{\pi_2\rho}F^1, T^{\mathsf{Rel}}_\rho F)}$ is also a relation transformer. Let $\overline{R : \mathsf{Rel}(A,B)}$. Then for $i = 1, 2$, we have

$$\begin{aligned}\pi_i(T^{\mathsf{Rel}}_\rho\, F\, \overline{R}) &= \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F]\overline{[\alpha := R]})\\ &= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i(\rho[\phi := F]\overline{[\alpha := R]}))\\ &= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i\rho)[\phi := \pi_i F]\overline{[\alpha := \pi_i R]})\\ &= T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i F)(\overline{\pi_i R})\end{aligned}$$

  and

$$\begin{aligned}\pi_i(T^{\mathsf{Rel}}_\rho\, F\, \overline{\gamma}) &= \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := id_F]\overline{[\alpha := \gamma]})\\ &= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i(id_\rho[\phi := id_F]\overline{[\alpha := \gamma]}))\\ &= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_{\pi_i\rho}[\phi := id_{\pi_i F}]\overline{[\alpha := \pi_i\gamma]}\\ &= T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i F)(\overline{\pi_i\gamma})\end{aligned}$$

  Here, the second equality in each of the above chains of equalities is by the induction hypothesis.
  We also have that, for every morphism $\delta = (\delta^0, \delta^1) : F \to G$ in RT_k and all $\overline{R : \mathsf{Rel}(A,B)}$,

$$\begin{array}{ll} & \pi_i((T^{\mathsf{Rel}}_\rho\delta)_{\overline{R}})\\ = & \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := \delta]\overline{[\alpha := id_R]})\\ = & [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_{\pi_i\rho}[\phi := \pi_i\delta]\overline{[\alpha := id_{\pi_i R}]}\\ = & (T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i\delta))_{\overline{\pi_i R}}\end{array}$$

  Here, the second equality is by the induction hypothesis. That T_ρ is ω-cocontinuous follows immediately from the induction hypothesis on $[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]$ and the fact that colimts are computed componentwise in RT.
  - $\underline{\sigma_f = (\sigma^{\mathsf{Set}}_{\pi_1 f}, \sigma^{\mathsf{Set}}_{\pi_2 f})\text{ is a natural transformation from }T_\rho\text{ to }T_{\rho'}}$: We must show that $(\sigma_f)_F = ((\sigma^{\mathsf{Set}}_{\pi_1 f})_{F^0}, (\sigma^{\mathsf{Set}}_{\pi_2 f})_{F^1})$ is a morphism in RT_k for all relation transformers $F = (F^0, F^1, F^*)$, i.e., that $((\sigma_f)_F)_{\overline{R}} = (((\sigma^{\mathsf{Set}}_{\pi_1 f})_{F^0})_{\overline{A}}, ((\sigma^{\mathsf{Set}}_{\pi_2 f})_{F_1})_{\overline{B}})$ is a morphism in Rel for all relations $\overline{R : \mathsf{Rel}(A,B)}$. Indeed, we have that

$$((\sigma_f)_F)_{\overline{R}} = [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} f[\phi := id_F]\overline{[\alpha := id_R]}$$

  is a morphism in RT_0 (and thus in Rel) by the induction hypothesis.

The relation transformer μT_ρ is therefore a fixed point of T_ρ by Lemma 16, and $\mu\sigma_f$ is a morphism in RT_k from μT_ρ to $\mu T_{\rho'}$. (μ is shown to be a functor in [Johann and Polonsky 2019].) So $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}$, and thus $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]$, is well-defined.

To see that $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!] : [\mathsf{RelEnv}, RT_0]$, we must verify three conditions:

– Condition (1) after Definition 18 is satisfied since

$$\begin{aligned}
\pi_i([\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\rho) &= \pi_i((\mu T_\rho)(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}))\\
&= \pi_i(\mu T_\rho)(\overline{\pi_i([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho)})\\
&= \mu T^{\mathsf{Set}}_{\pi_i\rho}(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}(\pi_i\rho)})\\
&= [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}(\pi_i\rho)
\end{aligned}$$

The third equality is by Equation 3 and the induction hypothesis.

– Condition (2) after Definition 18 is satisfied since it is subsumed by the previous condition because $k = 0$.

– The third bullet point of Definition 18 is satisfied because

$$\begin{aligned}
&\pi_i([\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}} f)\\
&= \pi_i((\mu T_{\rho'})(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f}) \circ (\mu\sigma_f)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}})\\
&= \pi_i((\mu T_{\rho'})(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f})) \circ \pi_i((\mu\sigma_f)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}})\\
&= \pi_i(\mu T_{\rho'})(\overline{\pi_i([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f)}) \circ \pi_i(\mu\sigma_f)_{\overline{\pi_i([\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho)}}\\
&= (\mu T^{\mathsf{Set}}_{\pi_i\rho'})(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}(\pi_i f)}) \circ (\mu\sigma^{\mathsf{Set}}_{\pi_i f})_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}(\pi_i\rho)}}\\
&= [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}(\pi_i f).
\end{aligned}$$

The fourth equality is by 3 and the induction hypothesis.

As before, that $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]$ is ω-concontinuous follows from the facts that Set and Rel are locally finitely presentable, and that colimits in RelEnv are computed componentwise, together with Corollary 12 of [Johann and Polonsky 2019].

□

The following lemma ensures that substitution interacts well with type interpretations. It is a consequence of Definitions 4, 27, and 28. Double check that no results from the next lemma are used in the preceding proof.

Lemma 22. *Let ρ be a set environment ρ and $f : \rho \to \rho'$ be a morphism of set environments.*

- *If $\Gamma;\Phi,\overline{\alpha} \vdash F$ and $\Gamma;\Phi \vdash \tau$, then*

$$[\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}] \tag{4}$$

and

$$[\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f}] \tag{5}$$

- *If $\Gamma;\Phi,\phi^k \vdash F$ and $\Gamma;\Phi,\alpha_1...\alpha_k \vdash H$, then*

$$[\![\Gamma;\Phi \vdash F[\phi := H]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}\rho[\phi := [\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}]] \tag{6}$$

and

$$[\![\Gamma;\Phi \vdash F[\phi := H]]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}} f[\phi := [\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} f[\overline{\alpha := -}]] \tag{7}$$

Analogous identities hold for relation environments and morphisms between them.

ISSUE: the substitution $[\phi := H]$ should specify that the αs in H correspond to the arguments of ϕ. For example, we could write $[\phi :=_\alpha H]$. Also add $\overline{\alpha}$ notation to Definition 4.

Proof. The proofs for the set and relational interpretations are completely analogous, so we just prove the former. Likewise, we only prove Equations 4 and 6, since the proofs for Equations 5 and 7 are again analogous. Finally, we prove Equation 4 for substitution for just a single type or type constructor variable since the proof for multiple simultaneous substitutions proceeds similarly.

Although Equation 4 is a special case of Equation 6, it is convenient to prove Equation 4 first, and then use it to prove Equation 6. We prove Equation 4 by induction on the structure of F as follows:

- If $\Gamma, \emptyset \vdash F : \mathcal{T}$, or if F is $\mathbb{1}$ or $\mathbb{0}$, then F does not contain any type constructor variables to replace, so there is nothing to prove.
- If F is $F_1 \times F_2$ or $F_1 + F_2$, then the substitution distributes over the product or coproduct as appropriate, so the result follows immediately from the induction hypothesis.
- If $F = \beta$ with $\beta \neq \alpha$, then there is nothing to prove.
- If $F = \alpha$, then

$$[\![\Gamma;\Phi \vdash \alpha[\alpha := \tau]]\!]^{\mathsf{Set}}\rho \;=\; [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho \;=\; [\![\Gamma;\Phi,\alpha \vdash \alpha]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho]$$

- If $F = \phi\overline{\sigma}$ with $\phi \neq \alpha$, then

$$\begin{array}{cl} & [\![\Gamma;\Phi \vdash (\phi\overline{\sigma})[\alpha := \tau]]\!]^{\mathsf{Set}}\rho \\ = & [\![\Gamma;\Phi \vdash \phi(\overline{\sigma[\alpha := \tau]})]\!]^{\mathsf{Set}}\rho \\ = & (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \sigma[\alpha := \tau]]\!]^{\mathsf{Set}}\rho} \\ = & (\rho\phi)\overline{[\![\Gamma;\Phi,\alpha \vdash \sigma]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}]} \\ = & [\![\Gamma;\Phi,\alpha \vdash \phi\overline{\sigma}]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}] \end{array}$$

  Here, the third equality is by the induction hypothesis.
- If $F = (\mu\phi.\lambda\overline{\beta}.G)\overline{\sigma}$, then

$$\begin{array}{cl} & [\![\Gamma;\Phi \vdash ((\mu\phi.\lambda\overline{\beta}.G)\overline{\sigma})[\alpha := \tau]]\!]^{\mathsf{Set}}\rho \\ = & [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\beta}.G[\alpha := \tau])(\overline{\sigma[\alpha := \tau]})]\!]^{\mathsf{Set}}\rho \\ = & \mu([\![\Gamma;\Phi,\phi,\overline{\beta} \vdash G[\alpha := \tau]]\!]^{\mathsf{Set}}\rho[\phi := -][\overline{\beta := -}])(\overline{[\![\Gamma;\Phi \vdash \sigma[\alpha := \tau]]\!]^{\mathsf{Set}}\rho}) \\ = & \mu([\![\Gamma;\Phi,\phi,\overline{\beta},\alpha \vdash G]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho][\phi := -][\overline{\beta := -}]) \\ & \qquad (\overline{[\![\Gamma;\Phi,\alpha \vdash \sigma]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho]}) \\ = & [\![\Gamma;\Phi,\alpha \vdash (\mu\phi.\lambda\overline{\beta}.G)\overline{\sigma}]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho] \end{array}$$

  Here, the third equality is by the induction hypothesis and weakening.

We now prove Equation 6, again by induction on the structure of F.

- If $\Gamma, \emptyset \vdash F : \mathcal{T}$, or if F is $\mathbb{1}$ or $\mathbb{0}$, then F does not contain any type constructor variables to replace, so there is nothing to prove.
- If F is $F_1 \times F_2$ or $F_1 + F_2$, then the substitution distributes over the product or coproduct as appropriate, so the result follows immediately from the induction hypothesis.
- If $F = \phi\overline{\tau}$, then

$$\begin{array}{cl} & [\![\Gamma;\Phi \vdash (\phi\overline{\tau})[\phi := H]]\!]^{\mathsf{Set}}\rho \\ = & [\![\Gamma;\Phi \vdash H[\overline{\alpha := \tau[\phi := H]}]]\!]^{\mathsf{Set}}\rho \\ = & [\![\Gamma;\Phi \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau[\phi := H]]\!]^{\mathsf{Set}}\rho}] \\ = & [\![\Gamma;\Phi \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}}\rho[\phi := [\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}]]}] \\ = & [\![\Gamma;\Phi,\phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho[\phi := [\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}]] \end{array}$$

  Here, the first equality is by Definition 4, the second is by Equation 4, the third is by the induction hypothesis, and the fourth is by Definition 8.
- If $F = \psi\overline{\tau}$ with $\psi \neq \phi$, then the proof is similar to that for the previous case, but simpler, because ϕ only needs to be substituted in the arguments $\overline{\tau}$ of ψ.

- If $F = (\mu\psi.\lambda\overline{\beta}.G)\overline{\tau}$, then

$$\begin{array}{ll} & [\![\Gamma;\Phi \vdash ((\mu\psi.\lambda\overline{\beta}.G)\overline{\tau})[\phi := H]]\!]^{\mathsf{Set}}\rho \\ = & [\![\Gamma;\Phi \vdash (\mu\psi.\lambda\overline{\beta}.G[\phi := H])(\overline{\tau[\phi := H]})]\!]^{\mathsf{Set}}\rho \\ = & \mu([\![\Gamma;\Phi,\psi,\overline{\beta} \vdash G[\phi := H]]\!]^{\mathsf{Set}}\rho[\psi := -][\overline{\beta := -}])(\overline{[\![\Gamma;\Phi\vdash \tau[\phi:=H]]\!]^{\mathsf{Set}}\rho}) \\ = & \mu([\![\Gamma;\Phi,\psi,\overline{\beta},\phi \vdash G]\!]^{\mathsf{Set}}\rho[\phi := [\![\Gamma;\Phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}]][\psi:=-][\overline{\beta := -}]) \\ & \qquad (\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho[\phi := [\![\Gamma;\Phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}]]}) \\ = & [\![\Gamma;\Phi,\phi \vdash (\mu\psi.\lambda\overline{\beta}.G)\overline{\tau}]\!]^{\mathsf{Set}}\rho[\phi := [\![\Gamma;\Phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}]] \end{array}$$

  Here, the first equality is by Definition 4, the second and fourth are by Definition 8, and the third is by the induction hypothesis and weakening.

□

3.3 The Identity Extension Lemma

Definition 23. *If F is a functor from Set^k to Set, we define the functor $\mathsf{Eq}^*_F : \mathsf{Rel}^k \to \mathsf{Rel}$ as follows. Given $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, let $\iota_{R_i} : R_i \hookrightarrow A_i \times B_i$, for $i = 1, \ldots, k$, be the inclusion of R_i as a subset of $A_i \times B_i$. By the universal property of the product, there exists a unique $h_{\overline{A\times B}}$ making the diagram*

$$\begin{tikzcd} F\overline{A} & F(\overline{A\times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, dashed, "h_{\overline{A\times B}}"] & F\overline{B} \\ & F\overline{A}\times F\overline{B} \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

commute. Let $h_{\overline{R}} : F\overline{R} \to F\overline{A} \times F\overline{B}$ be $h_{\overline{A\times B}} \circ F\overline{\iota_R}$. Define $F^\wedge\overline{R}$ to be the subobject through which $h_{\overline{R}}$ is factorized by the mono-epi factorization system in Set, as shown in the following diagram:

$$\begin{tikzcd} F\overline{R} \arrow[rr, "h_{\overline{R}}"] \arrow[dr, two heads, "q_{F^\wedge\overline{R}}"'] & & F\overline{A}\times F\overline{B} \\ & F^\wedge\overline{R} \arrow[ur, hook, "\iota_{F^\wedge\overline{R}}"'] & \end{tikzcd}$$

Note that $F^\wedge\overline{R} : \mathsf{Rel}(F\overline{A}, F\overline{B})$ by construction, so we can define $\mathsf{Eq}^*_F\overline{(A,B,R)} = (F\overline{A}, F\overline{B}, \iota_{F^\wedge\overline{R}}F^\wedge\overline{R})$. Moreover, if $\overline{(\alpha,\beta) : (A,B,R) \to (C,D,S)}$ are morphisms in Rel, then we define $\mathsf{Eq}^*_F\overline{(\alpha,\beta)}$ to be $(F\overline{\alpha}, F\overline{\beta})$.

If F is a functor from Set^k to Set, let $\mathsf{Eq}_F = (F, F, \mathsf{Eq}^*_F)$. Note that if $A : \mathsf{Set}$ then Eq_A is precisely as defined in Section 1.1. This is consistent with the fact that a set can be seen as a 0-ary functor on sets and a relation can be seen as a 0-ary functor on relations.

Lemma 24. *If $F : [\mathsf{Set}^k, \mathsf{Set}]$ then Eq_F is in RT_k.*

Proof. Clearly, Eq^*_F is ω-cocontinuous, so $\mathsf{Eq}^*_F : [\mathsf{Rel}^k, \mathsf{Rel}]$.

Now, consider $\overline{(\alpha,\beta) : R \to S}$, where $\overline{R : \mathsf{Rel}(A,B)}$ and $\overline{S : \mathsf{Rel}(C,D)}$. We want to show that there exists a morphism $\epsilon : F^\wedge\overline{R} \to F^\wedge\overline{S}$ such that

$$\begin{tikzcd} F^\wedge\overline{R} \arrow[r, hook, "\iota_{F^\wedge\overline{R}}"] \arrow[d, "\epsilon"'] & F\overline{A}\times F\overline{B} \arrow[d, "F\overline{\alpha}\times F\overline{\beta}"] \\ F^\wedge\overline{S} \arrow[r, hook, "\iota_{F^\wedge\overline{S}}"'] & F\overline{C}\times F\overline{D} \end{tikzcd}$$

commutes. By hypothesis, there exist $\overline{\gamma : R \longrightarrow S}$ such that each diagram

$$\begin{tikzcd}
R_i \arrow[r, hook, "\iota_{R_i}"] \arrow[d, "\gamma_i"'] & A_i \times B_i \arrow[d, "\alpha_i \times \beta_i"] \\
S_i \arrow[r, hook, "\iota_{S_i}"] & C_i \times D_i
\end{tikzcd}$$

commutes. Now note that both $h_{\overline{C \times D}} \circ F(\overline{\alpha \times \beta})$ and $(F\overline{\alpha} \times F\overline{\beta}) \circ h_{\overline{A \times B}}$ make

$$\begin{tikzcd}
F\overline{C} & F\overline{C} \times F\overline{D} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F\overline{D} \\
& F(\overline{A \times B}) \arrow[ul, "F\pi_1 \circ F(\overline{\alpha \times \beta})"] \arrow[ur, "F\pi_2 \circ F(\overline{\alpha \times \beta})"'] \arrow[u, dashed, "\exists!"] &
\end{tikzcd}$$

commute, so they must be equal. We therefore get that the right-hand square below commutes, and thus that the entire following diagram does as well:

$$\begin{tikzcd}
F\overline{R} \arrow[r, hook, "F\overline{\iota_R}"] \arrow[d, "F\overline{\gamma}"'] \arrow[rr, bend left, "h_{\overline{R}}"] & F(\overline{A \times B}) \arrow[r, "h_{\overline{A \times B}}"] \arrow[d, "F(\overline{\alpha \times \beta})"] & F\overline{A} \times F\overline{B} \arrow[d, "F\overline{\alpha} \times F\overline{\beta}"] \\
F\overline{S} \arrow[r, hook, "F\overline{\iota_S}"'] \arrow[rr, bend right, "h_{\overline{S}}"'] & F(\overline{C \times D}) \arrow[r, "h_{\overline{C \times D}}"'] & F\overline{C} \times F\overline{D}
\end{tikzcd}$$

Finally, by the left-lifting property of $q_{F^\wedge\overline{R}}$ with respect to $\iota_{F^\wedge\overline{S}}$ given by the epi-mono factorization system, there exists an ϵ such that the diagram

$$\begin{tikzcd}
F\overline{R} \arrow[r, two heads, "q_{F^\wedge\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] & F^\wedge\overline{R} \arrow[r, hook, "\iota_{F^\wedge\overline{R}}"] \arrow[d, dashed, "\epsilon"] & F\overline{A} \times F\overline{B} \arrow[d, "F\overline{\alpha} \times F\overline{\beta}"] \\
F\overline{S} \arrow[r, two heads, "q_{F^\wedge\overline{S}}"'] & F^\wedge\overline{S} \arrow[r, hook, "\iota_{F^\wedge\overline{S}}"'] & F\overline{C} \times F\overline{D}
\end{tikzcd}$$

commutes. ◻

Lemma 25. *If* $F : [\mathsf{Set}^k, \mathsf{Set}]$ *and* $A_1, \ldots, A_k : \mathsf{Set}$, *then* $\mathsf{Eq}^*_F \overline{\mathsf{Eq}_A} = \mathsf{Eq}_{F\overline{A}}$.

Proof. Each $\mathsf{Eq}_A : \mathsf{Rel}$ has as its third component $\Delta_A A$, where $\Delta_A : A \longrightarrow A \times A$ is given by $\Delta_A A = \{(x, x) \mid x \in A\}$. Since $h_{\overline{A \times A}}$ is the unique morphism making the bottom triangle of the following diagram commute

$$\begin{tikzcd}
& F\overline{A} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, "F\overline{\Delta_A}"] & \\
F\overline{A} & F(\overline{A \times A}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, "h_{\overline{A \times A}}"] & F\overline{A} \\
& F\overline{A} \times F\overline{A} \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

and since $h_{\overline{\mathsf{Eq}^*_A}} = h_{\overline{A \times A}} \circ F\overline{\Delta_A}$, the universal property of the product

$$
\begin{tikzcd}
F\overline{A} & F\overline{A} \times F\overline{A} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F\overline{A} \\
 & F\overline{A} \arrow[ul, equal] \arrow[ur, equal] \arrow[u, dashed, "\exists!"] &
\end{tikzcd}
$$

gives that $h_{\overline{\mathsf{Eq}^*_A}} = \Delta_{F\overline{A}}$. Moreover, since $\Delta_{F\overline{A}}$ is a monomorphism, its epi-mono factorization gives that $\Delta_{F\overline{A}} = \iota_{F^\wedge \overline{\Delta_A}}$, and thus that $F^\wedge \overline{\Delta_A} = F\overline{A}$. Therefore, $\mathsf{Eq}^*_F \overline{\mathsf{Eq}_A} = (F\overline{A}, F\overline{A}, \iota_{F^\wedge \overline{\Delta_A}} F^\wedge \overline{\Delta_A}) = (F\overline{A}, F\overline{A}, \Delta_{F\overline{A}} F\overline{A}) = \mathsf{Eq}_{F\overline{A}}$. □

We now show that the Identity Extension Lemma holds for the interpretations given in Definitions 8 and 19. If ρ is a set environment, define Eq_ρ to be the relation environment such that $\mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v}$ for all $v \in \mathbb{V} \cup \mathbb{T}$. This is exactly the same definition that was given informally in Section 3.1. The Identity Extension Lemma can then be stated and proved as follows:

Theorem 26. *If ρ is a set environment, and $\Gamma; \Phi \vdash \tau : \mathcal{F}$, then $[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}} \rho}$.*

Proof. By induction on the structure of τ.

- $[\![\Gamma; \emptyset \vdash v]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v} = \mathsf{Eq}_{[\![\Gamma; \emptyset \vdash v]\!]^{\mathsf{Set}} \rho}$ where $v \in \Gamma$.
- $[\![\Gamma; \emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = [\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \to [\![\Gamma; \emptyset \vdash \tau]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Set}} \rho} \to \mathsf{Eq}_{[\![\Gamma; \emptyset \vdash \tau]\!]^{\mathsf{Set}} \rho} = \mathsf{Eq}_{[\![\Gamma; \emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Set}} \rho}$, where the second equality is by the induction hypothesis.
- $\tau = \forall v. \tau_1$
- By definition, $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho$ is the relation on $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho$ relating t and t' if, for all $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ in Rel. To prove that this is equal to $\mathsf{Eq}_{[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho}$ we need to show that $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$ if and only if $t = t'$ and $(t_{\overline{A}}, t_{\overline{B}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$. The only interesting part of this equivalence is to show that if $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then $t = t'$. By hypothesis, $(t_{\overline{A}}, t'_{\overline{A}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := \mathsf{Eq}_A]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := \mathsf{Eq}_A]}$ in Rel for all $A_1 \ldots A_k$: Set. By the induction hypothesis, it is therefore a morphism from $\mathsf{Eq}_{[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \rho \overline{[\alpha := A]}}$ to $\mathsf{Eq}_{[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}} \rho \overline{[\alpha := A]}}$ in Rel. This means that, for every $x : \mathsf{Eq}_{[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \rho \overline{[\alpha := A]}}$, $t_{\overline{A}} x = t'_{\overline{A}} x$. Then, by extensionality, $t = t'$.
- $[\![\Gamma; \Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = 0_{\mathsf{Rel}} = \mathsf{Eq}_{0_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma; \Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}} \rho}$
- $[\![\Gamma; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = 1_{\mathsf{Rel}} = \mathsf{Eq}_{1_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}} \rho}$
- The application case is proved by the following sequence of equalities, where the second equality is by the induction hypothesis and the definition of the relation environment Eq_ρ, the third is by the definition of application of relation transformers from Definition 10, and

the fourth is by Lemma 25:

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mathsf{Eq}_\rho\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \mathsf{Eq}_{\rho\phi}\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= (\mathsf{Eq}_{\rho\phi})^*\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{(\rho\phi)\,\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\phi\overline{\tau}]\!]^{\mathsf{Set}}\rho}
\end{aligned}
$$

- The fixed point case is proven by the sequence of equalities

$$
\begin{aligned}
[\![\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mu T_{\mathsf{Eq}_\rho})\,\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{\mathsf{Eq}_\rho}K_0\,\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0})^*\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0\,\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{\varinjlim_{n\in\mathbb{N}}(T^{\mathsf{Set}}_\rho)^nK_0\,\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho}
\end{aligned}
$$

Here, the third equality is by induction hypothesis, the fifth is by Lemma 25 and the fourth equality is because, for every $n \in \mathbb{N}$, the following two statements can be proved by simultaneous induction:

$$
T^n_{\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} = (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0})^*\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} \tag{8}
$$

and

$$
\begin{aligned}
&[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&\qquad = [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0}]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}
\end{aligned} \tag{9}
$$

We prove (8). The case $n = 0$ is trivial, because $T^0_{\mathsf{Eq}_\rho}K_0 = K_0$ and $(T^{\mathsf{Set}}_\rho)^0K_0 = K_0$; the inductive step is proved by the following sequence of equalities:

$$
\begin{aligned}
T^{n+1}_{\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} &= T^{\mathsf{Rel}}_{\mathsf{Eq}_\rho}(T^n_{\mathsf{Eq}_\rho}K_0)\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0}]\overline{[\alpha := \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho[\phi:=(T^{\mathsf{Set}}_\rho)^nK_0]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\phi:=(T^{\mathsf{Set}}_\rho)^nK_0]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}\\
&= \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^{n+1}K_0\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^{n+1}K_0})^*\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}
\end{aligned}
$$

Here, the third equality is by (9), the fifth by the induction hypothesis on H, and the last by Lemma 25. We prove the induction step of (9) by structural induction on H: the only interesting case, though, is when ϕ is applied, i.e., for $H = \phi\overline{\sigma}$, which is proved by the sequence of equalities:

$$
\begin{aligned}
&\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \phi\overline{\sigma} \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho [\phi := T^n_{\mathsf{Eq}_\rho} K_0] \overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho}]} \\
&= T^n_{\mathsf{Eq}_\rho} K_0 \, \overline{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho [\phi := T^n_{\mathsf{Eq}_\rho} K_0] \overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho}]}} \\
&= T^n_{\mathsf{Eq}_\rho} K_0 \, \overline{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho [\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}] \overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho}]}} \\
&= T^n_{\mathsf{Eq}_\rho} K_0 \, \overline{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_{\rho[\phi := (T^{\mathsf{Set}}_\rho)^n K_0] \overline{[\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho]}}} \\
&= T^n_{\mathsf{Eq}_\rho} K_0 \, \overline{\mathsf{Eq}_{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Set}} \rho[\phi := (T^{\mathsf{Set}}_\rho)^n K_0] \overline{[\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho]}}} \\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0})^* \, \overline{\mathsf{Eq}_{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Set}} \rho[\phi := (T^{\mathsf{Set}}_\rho)^n K_0] \overline{[\alpha := \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho]}}} \\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0})^* \overline{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho [\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}] \overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho}]}} \\
&= \llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \phi\overline{\sigma} \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho [\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0}] \overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho}]}
\end{aligned}
$$

Here, the second equality is by the induction hypothesis for (9) on the σs, the fourth is by the induction hypothesis for Theorem 26 on the σs, and the fifth is by the induction hypothesis on n for (8).

- $\llbracket \Gamma; \Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho = \llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho + \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Set}} \rho} + \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho} = \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Set}} \rho + \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho} = \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Set}} \rho}$
- $\llbracket \Gamma; \Phi \vdash \sigma \times \tau \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho = \llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho \times \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Set}} \rho} \times \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho} = \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Set}} \rho \times \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}} \rho} = \mathsf{Eq}_{\llbracket \Gamma; \Phi \vdash \sigma \times \tau \rrbracket^{\mathsf{Set}} \rho}$

□

4 INTERPRETING TERMS

If $\Delta = x_1 : \tau_1, ..., x_n : \tau_n$ is a term context for Γ and Φ, then the interpretations $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}}$ and $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}$ are defined by

$$
\begin{aligned}
\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}} &= \llbracket \Gamma; \Phi \vdash \tau_1 \rrbracket^{\mathsf{Set}} \times ... \times \llbracket \Gamma; \Phi \vdash \tau_n \rrbracket^{\mathsf{Set}} \\
\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}} &= \llbracket \Gamma; \Phi \vdash \tau_1 \rrbracket^{\mathsf{Rel}} \times ... \times \llbracket \Gamma; \Phi \vdash \tau_n \rrbracket^{\mathsf{Rel}}
\end{aligned}
$$

Every well-formed term $\Gamma; \Phi \mid \Delta \vdash t : \tau$ then has a set interpretation $\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}$ as a natural transformation from $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}}$ to $\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}}$, and a relational interpretation $\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}$ as a natural transformation from $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}$ to $\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}}$. These are given in the next two definitions.

Definition 27. *If ρ is a set environment and $\Gamma;\Phi \mid \Delta \vdash t : \tau$ then $[\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho$ is defined as follows:*

$$
\begin{array}{lcl}
[\![\Gamma;\emptyset \mid \Delta, x:\tau \vdash x : \tau]\!]^{\mathsf{Set}}\rho & = & \pi_{|\Delta|+1} \\
[\![\Gamma;\emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau]\!]^{\mathsf{Set}}\rho & = & \mathsf{curry}([\![\Gamma;\emptyset \mid \Delta, x:\sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho) \\
[\![\Gamma;\emptyset \mid \Delta \vdash st : \tau]\!]^{\mathsf{Set}}\rho & = & \mathsf{eval} \circ \langle [\![\Gamma;\emptyset \mid \Delta \vdash s : \sigma \to \tau]\!]^{\mathsf{Set}}\rho, [\![\Gamma;\emptyset \mid \Delta \vdash t : \sigma]\!]^{\mathsf{Set}}\rho \rangle \\
[\![\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho & = & \mathsf{curry}([\![\Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]) \\
[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho & = & \mathsf{eval} \circ \langle ([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}}, \\
 & & \qquad [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho \rangle
\end{array}
$$

Add rules for $\forall$ if we include it

$$
\begin{array}{lcl}
[\![\Gamma;\Phi \mid \Delta, x:\tau \vdash x : \tau]\!]^{\mathsf{Set}}\rho & = & \pi_{|\Delta|+1} \\
[\![\Gamma;\Phi \mid \Delta \vdash \bot_\tau t : \tau]\!]^{\mathsf{Set}}\rho & = & !^0_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \mathbb{0}]\!]^{\mathsf{Set}}\rho,\ \textit{where} \\
 & & \quad !^0_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}\ \textit{is the unique morphism from } 0 \\
 & & \quad \textit{to } [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \top : \mathbb{1}]\!]^{\mathsf{Set}}\rho & = & !^{[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}}\rho}_1,\ \textit{where } !^{[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}}\rho}_1 \\
 & & \quad \textit{is the unique morphism from } [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho \textit{ to } 1 \\
[\![\Gamma;\Phi \mid \Delta \vdash (s,t) : \sigma \times \tau]\!]^{\mathsf{Set}}\rho & = & [\![\Gamma;\Phi \mid \Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \pi_1 t : \sigma]\!]^{\mathsf{Set}}\rho & = & \pi_1 \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \pi_2 t : \sigma]\!]^{\mathsf{Set}}\rho & = & \pi_2 \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{case}\ t\ \mathsf{of}\ \{x \mapsto l;\ y \mapsto r\} : \gamma]\!]^{\mathsf{Set}}\rho & = & \mathsf{eval} \circ \langle \mathsf{curry}\,[[\![\Gamma;\Phi \mid \Delta, x:\sigma \vdash l : \gamma]\!]^{\mathsf{Set}}\rho, \\
 & & \qquad\qquad [\![\Gamma;\Phi \mid \Delta, y:\tau \vdash r : \gamma]\!]^{\mathsf{Set}}\rho], \\
 & & \qquad [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma + \tau]\!]^{\mathsf{Set}}\rho \rangle \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\,s : \sigma + \tau]\!]^{\mathsf{Set}}\rho & = & \mathsf{inl} \circ [\![\Gamma;\Phi \mid \Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{inr}\,t : \sigma + \tau]\!]^{\mathsf{Set}}\rho & = & \mathsf{inr} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho \\
[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{in}\,t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho & = & \mathit{in} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho, \\
 & & \quad \textit{where the variables in } \overline{\beta} \textit{ are fresh} \\
[\![\Gamma;\emptyset \mid \Delta \vdash \mathsf{fold}_{H,F}\,t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}\rho & = & \mathit{fold} \circ [\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho
\end{array}
$$

Definition 28. *If ρ is a relation environment and $\Gamma;\Phi \mid \Delta \vdash t : \tau$ then $\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho$ is defined as follows:*

$$\begin{array}{lcl}
\llbracket \Gamma;\emptyset \mid \Delta, x:\tau \vdash x : \tau \rrbracket^{\mathsf{Rel}}\rho & = & \pi_{|\Delta|+1} \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{curry}(\llbracket \Gamma;\emptyset \mid \Delta, x:\sigma \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho) \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash st : \tau \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{eval} \circ \langle \llbracket \Gamma;\emptyset \mid \Delta \vdash s : \sigma \to \tau \rrbracket^{\mathsf{Rel}}\rho, \llbracket \Gamma;\emptyset \mid \Delta \vdash t : \sigma \rrbracket^{\mathsf{Rel}}\rho \rangle \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{curry}(\llbracket \Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G \rrbracket^{\mathsf{Rel}}\rho[\overline{\alpha := _}]) \\
\llbracket \Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{eval} \circ \langle (\llbracket \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket^{\mathsf{Rel}}\rho\ _)_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}\rho}}, \\
& & \qquad \llbracket \Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}}\rho \rangle
\end{array}$$

Add rules for ∀ if we include it

$$\begin{array}{lcl}
\llbracket \Gamma;\Phi \mid \Delta, x:\tau \vdash x : \tau \rrbracket^{\mathsf{Rel}}\rho & = & \pi_{|\Delta|+1} \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \bot_\tau t : \tau \rrbracket^{\mathsf{Rel}}\rho & = & !^{0}_{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}\rho} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \mathbb{0} \rrbracket^{\mathsf{Rel}}\rho,\ \textit{where} \\
& & \quad !^{0}_{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}\rho}\ \textit{is the unique morphism from } 0 \\
& & \quad \textit{to } \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \top : \mathbb{1} \rrbracket^{\mathsf{Rel}}\rho & = & !^{\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}\rho}_{1},\ \textit{where } !^{\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}\rho}_{1} \\
& & \quad \textit{is the unique morphism from } \llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}\rho \textit{ to } 1 \\
\llbracket \Gamma;\Phi \mid \Delta \vdash (s,t) : \sigma \times \tau \rrbracket^{\mathsf{Rel}}\rho & = & \llbracket \Gamma;\Phi \mid \Delta \vdash s : \sigma \rrbracket^{\mathsf{Rel}}\rho \times \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \pi_1 t : \sigma \rrbracket^{\mathsf{Rel}}\rho & = & \pi_1 \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \pi_2 t : \sigma \rrbracket^{\mathsf{Rel}}\rho & = & \pi_2 \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{x \mapsto l;\, y \mapsto r\} : \gamma \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{eval} \circ \langle \mathsf{curry}\, [\llbracket \Gamma;\Phi \mid \Delta, x:\sigma \vdash l : \gamma \rrbracket^{\mathsf{Rel}}\rho, \\
& & \qquad\qquad \llbracket \Gamma;\Phi \mid \Delta, y:\tau \vdash r : \gamma \rrbracket^{\mathsf{Rel}}\rho], \\
& & \qquad \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma + \tau \rrbracket^{\mathsf{Rel}}\rho \rangle \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{inl} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash s : \sigma \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{inr} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Rel}}\rho & = & \mathit{in} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}}\rho \\
& & \quad \textit{where the variables in } \overline{\beta} \textit{ are fresh} \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash \mathsf{fold}_{H,F}\, t : \mathsf{Nat}^{\overline{\alpha}}\, ((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\, F \rrbracket^{\mathsf{Rel}}\rho & = & \mathit{fold} \circ \llbracket \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, (H[\phi := F][\overline{\beta := \alpha}])\, F \rrbracket^{\mathsf{Rel}}\rho
\end{array}$$

If t is closed, i.e., if $\emptyset;\emptyset \mid \emptyset \vdash t : \tau$, then we write $\llbracket \vdash t : \tau \rrbracket^{\mathsf{Set}}$ instead of $\llbracket \emptyset;\emptyset \mid \emptyset \vdash t : \tau \rrbracket^{\mathsf{Set}}$, and similarly for $\llbracket \emptyset;\emptyset \mid \emptyset \vdash t : \tau \rrbracket^{\mathsf{Rel}}$.

Move factoids about interpretations here.

4.1 The Abstraction Theorem

To prove the Abstraction Theorem we actually prove a more general result in Theorem 29 about possibly open terms. We then recover the Abstraction Theorem as the special case of Theorem 29 for closed terms of closed type.

Theorem 29. *Every well-formed term $\Gamma;\Phi \mid \Delta \vdash t : \tau$ induces a natural transformation from $\llbracket \Gamma;\Phi \vdash \Delta \rrbracket$ to $\llbracket \Gamma;\Phi \vdash \tau \rrbracket$, i.e., a triple of natural transformations*

$$(\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}, \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}, \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}})$$

where

$$\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}} \quad : \quad \llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}} \to \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}$$

has as its component at $\rho : \mathsf{SetEnv}$ a morphism

$$\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}\rho \quad : \quad \llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}\rho \to \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho$$

in Set, *and*

$$[\![\Gamma;\Phi\mid\Delta\vdash t:\tau]\!]^{\mathsf{Rel}}\quad:\quad[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}}\to[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}$$

has as its component at $\rho:\mathsf{RelEnv}$ *a morphism*

$$[\![\Gamma;\Phi\mid\Delta\vdash t:\tau]\!]^{\mathsf{Rel}}\rho\quad:\quad[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}}\rho\to[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho$$

in Rel, *and for all* $\rho:\mathsf{RelEnv}$,

$$[\![\Gamma;\Phi\mid\Delta\vdash t:\tau]\!]^{\mathsf{Rel}}\rho=([\![\Gamma;\Phi\mid\Delta\vdash t:\tau]\!]^{\mathsf{Set}}(\pi_1\rho),[\![\Gamma;\Phi\mid\Delta\vdash t:\tau]\!]^{\mathsf{Set}}(\pi_2\rho))$$

Proof. We proceed by structural induction, showing only the interesting cases.

- We first consider $\Gamma;\emptyset\mid\Delta\vdash L_{\overline{\alpha}}x.t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G$.
  - To see that $[\![\Gamma;\emptyset\mid\Delta\vdash L_{\overline{\alpha}}x.t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\emptyset\vdash\Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}$, since the functorial part Φ of the type context is empty, we need only show that, for every $\rho:\mathsf{SetEnv}$, $[\![\Gamma;\emptyset\mid\Delta\vdash L_{\overline{\alpha}}x.t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$ is a morphism in Set from $[\![\Gamma;\emptyset\vdash\Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$. For this, recall that

$$[\![\Gamma;\emptyset\mid\Delta\vdash L_{\overline{\alpha}}x.t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho=\mathsf{curry}\,([\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=\_}])$$

By the induction hypothesis, $[\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]\rho[\overline{\alpha:=\_}]$ induces a natural transformation

$$\begin{array}{rl}
& [\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]\\
: & [\![\Gamma;\overline{\alpha}\vdash\Delta,x:F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]\to[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]\\
= & [\![\Gamma;\overline{\alpha}\vdash\Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]\times[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]\to[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]
\end{array}$$

and thus a family of morphisms

$$\begin{array}{rl}
& \mathsf{curry}\,([\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]\rho[\overline{\alpha:=_}])\\
: & [\![\Gamma;\overline{\alpha}\vdash\Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]\to([\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]\to[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}])
\end{array}$$

That is, for each $\overline{A:\mathsf{Set}}$ and each $d:[\![\Gamma;\emptyset\vdash\Delta]\!]^{\mathsf{Set}}\rho=[\![\Gamma;\overline{\alpha}\vdash\Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}]$ by weakening, we have

$$\begin{array}{rl}
& ([\![\Gamma;\emptyset\mid\Delta\vdash L_{\overline{\alpha}}x.t:\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho\,d)_{\overline{A}}\\
= & \mathsf{curry}\,([\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}])\,d\\
: & [\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}]\to[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}]
\end{array}$$

Moreover, these maps actually form a natural transformation $\eta:[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=\_}]\to[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=\_}]$ because each

$$\eta_{\overline{A}}\;=\;\mathsf{curry}\,([\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}])\,d$$

is the partial specialization to d of the natural transformation $[\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=\_}]$. To see that the components of η also satisfy the additional condition necessary for η to be in $[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$, let $\overline{R:\mathsf{Rel}(A,B)}$ and

$$(u,v)\in[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha:=R}]=([\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}],[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=B}])$$

Then the induction hypothesis on the term t ensures that

$$\begin{array}{rl}
& [\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha:=R}]\\
: & [\![\Gamma;\overline{\alpha}\vdash\Delta,x:F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha:=R}]\to[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha:=R}]
\end{array}$$

and

$$\begin{array}{rl}
& [\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha:=R}] \qquad (*)\\
= & ([\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=A}],[\![\Gamma;\overline{\alpha}\mid\Delta,x:F\vdash t:G]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=B}])
\end{array}$$

Since $(d, d) \in [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]$ we therefore have that

$$
\begin{array}{ll}
& (\eta_{\overline{A}} u, \eta_{\overline{B}} v) \\
= & (\mathsf{curry}\,([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}])\, d\, u, \mathsf{curry}\,([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := B}])\, d\, v) \\
= & \mathsf{curry}\,([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}])\,(d,d)\,(u,v) \\
: & [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]
\end{array}
$$

as desired. Here, the second equality is by $(*)$.

– To see that $[\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}$ is a natural transformation from $[\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}$, since the functorial part Φ of the type context is empty, we need only show that, for every $\rho : \mathsf{RelEnv}$, $[\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ is a morphism in Rel from $[\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$. The proof is similar to that in the previous bullet point for the set semantics. The only difference is in the additional condition on the components of the natural transformation

$$\eta \;=\; \mathsf{curry}\,([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}])\, d$$

for $d : [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$ that must be verified to know that η is in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho$. For that, let $\overline{R : \mathsf{Rel}(A, B)}$ and

$$(u, v) \in [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] = ([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho)[\overline{\alpha := A}], [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_2\rho)[\overline{\alpha := B}])$$

Then the induction hypothesis on the term t ensures that

$$[\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] \;:\; [\![\Gamma;\overline{\alpha} \vdash \Delta, x : F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] \longrightarrow [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]$$

and

$$\eta_{\overline{R}} = ((\pi_1\eta)_{\overline{A}}, (\pi_1\eta)_{\overline{B}})$$

so that

$$
\begin{array}{ll}
& ((\pi_1\eta)_{\overline{A}} u, (\pi_2\eta)_{\overline{B}} v) \\
= & \eta_{\overline{R}}(u, v) \\
= & \mathsf{curry}\,[\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]\, d\,(u, v) \\
: & [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]
\end{array}
$$

– Finally, to see that $\pi_i([\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho) = [\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_i\rho)$ we compute

$$
\begin{array}{l}
\pi_i([\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho) \\
= \pi_i(\mathsf{curry}([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}])) \\
= \mathsf{curry}(\pi_i([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \_}])) \\
= \mathsf{curry}([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}(\pi_i(\rho[\overline{\alpha := \_}]))) \\
= \mathsf{curry}([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}(\pi_i\rho)[\overline{\alpha := \_}]) \\
= [\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_i\rho)
\end{array}
$$

- We now consider $\Gamma;\Phi \,|\, \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]$.
  – To see that $[\![\Gamma;\Phi \,|\, \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ we need to show that, for every $\rho : \mathsf{SetEnv}$, $[\![\Gamma;\Phi \,|\, \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$ is a morphism from $[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$, and that these this family of

morphisms is natural in ρ. Let ρ : SetEnv and $d : [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$. Then

$$
\begin{aligned}
& [\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\, \rho\, d \\
= \;& (\mathsf{eval} \circ \langle ([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho\; \_)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}},\, [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho \rangle)\, d \\
= \;& \mathsf{eval}(([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho\; \_)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}}\, d,\, [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho\, d) \\
= \;& \mathsf{eval}(([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho\; d)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}},\, [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho\, d)
\end{aligned}
$$

By the induction hypothesis, $([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho\; d)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}}$ has type

$$[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}]$$

and $[\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho\, d$ has type

$$
\begin{aligned}
& [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho \\
= \;& [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}] \\
= \;& [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}]
\end{aligned}
$$

by Equation 4, and by weakening in the last step, since the type $\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}} F\, G$ is only well-formed if $\Gamma;\overline{\alpha} \vdash F : \mathcal{F}$ and $\Gamma;\overline{\alpha} \vdash G : \mathcal{F}$. Thus, $[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\, \rho\, d$ has type $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}] = [\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$, as desired.

To see that the family of maps comprising $[\![\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ form a natural transformation, i.e., are natural in their relation environment argument, we need to show that the following diagram commutes:

$$
\begin{tikzcd}
[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho \arrow[rr, "{[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}} f}"] \arrow[d, "{\langle [\![\Gamma;\emptyset \mid \Delta \vdash t:\mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho,\, [\![\Gamma;\Phi \mid \Delta \vdash s:F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}}\rho \rangle}"'] & & [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho' \arrow[d, "{\langle [\![\Gamma;\emptyset \mid \Delta \vdash t:\mathsf{Nat}^{\overline{\alpha}}}"] \\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho \arrow[rr, "{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi \vdash F[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}} f}"] \arrow[d, "{\mathsf{eval} \circ ((-)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]\rho}} \times id)}"'] & & [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho' \times [\![\Gamma;\Phi \vdash F[\overline{\alpha :} \arrow[d, "{\mathsf{eval} \circ ((-)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]\rho'}}}"] \\
[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho \arrow[rr, "{[\![\Gamma;\Phi \vdash G[\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}} f}"'] & & [\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho'
\end{tikzcd}
$$

The top diagram commutes because the induction hypothesis ensures $[\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ and $[\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ are natural in ρ. To see that the bottom diagram commutes, we first note that since $\rho|_\Gamma = \rho'|_\Gamma$, $\Gamma;\overline{\alpha} \vdash F : \mathcal{F}$, and $\Gamma;\overline{\alpha} \vdash G : \mathcal{F}$ we can replace both instances of f in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f$ with id. Using the fact that $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ and $[\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ are both functorial, we have that if $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ and $x \in [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$, then $([\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f)\,(\eta, x) = (\eta, x)$. We must therefore prove that, for every $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ and $x \in [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$, we have

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f(\eta_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]\rho}}\, x) = \eta_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]\rho'}}([\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f\, x)$$

i.e., for every $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f \circ \eta_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]\rho}} = \eta_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]\rho'}} \circ [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f$$

But this follows from the naturality of η. Indeed, $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ implies that η is a natural transformation from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$. For each

τ, consider the morphism $[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f : [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} \rho \to [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} \rho'$. The following diagram commutes by naturality of η:

$$\begin{tikzcd}
{[\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho} \arrow[d, equal] & {[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho} \arrow[d, equal] \\
{[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}]} \arrow[r, "\eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}]}"] & {[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}]} \arrow[d, "{[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}]}"] \\
{[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}]} \arrow[r, "\eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}"] & {[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}]}
\end{tikzcd}$$

That is, $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f}] \circ \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} = \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash}$ But since the only variables in the functorial contexts for F and G are $\overline{\alpha}$, we have that $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f}] = [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f}] = [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f$, and similarly for G. Commutativity of this last diagram thus gives that $[\![\Gamma;\Phi \vdash G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f \circ \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} = \eta_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ [\![\Gamma;\Phi \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} f$, as desired.

– The proof that the relational interpretation of the same term is a natural transformation of the right type is analogous.

– Finally, to see that $\pi_i([\![\Gamma;\Phi \mid \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho) = [\![\Gamma;\Phi \mid \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_i\rho)$ we compute

$$\begin{array}{ll}
& \pi_i([\![\Gamma;\Phi \mid \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho) \\
= & \pi_i(\mathsf{eval} \circ \langle ([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho\ _)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}}, [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho\rangle) \\
= & \mathsf{eval} \circ \langle \pi_i(([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho\ _)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}}), \pi_i([\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho)\rangle \\
= & \mathsf{eval} \circ \langle \pi_i([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho\ _)_{\overline{\pi_i([\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho)}}, \pi_i([\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho)\rangle \\
= & \mathsf{eval} \circ \langle ([\![\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}(\pi_i\rho)\ _)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}(\pi_i\rho)}}, [\![\Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_i\rho)\rangle \\
= & [\![\Gamma;\Phi \mid \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_i\rho)
\end{array}$$

- We now consider $\Gamma;\Phi \mid \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$.
  – To see that $[\![\Gamma;\Phi \mid \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}$, we first note that it has the right domain and codomain. Indeed, by the induction hypothesis we have that $[\![\Gamma;\Phi \mid \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\Phi \vdash H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}$. We

therefore have that, for each $\rho \in \mathsf{SetEnv}$,

$$
\begin{tikzcd}
\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}\rho \arrow[d, "\llbracket \Gamma;\Phi \,|\, \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\rho"] \\
\llbracket \Gamma;\Phi \vdash H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\rho \arrow[d, equal] \\
\llbracket \Gamma;\Phi,\phi,\overline{\alpha} \vdash H \rrbracket^{\mathsf{Set}}\rho[\phi := \mu T^{\mathsf{Set}}_\rho][\overline{\alpha := \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho}] \arrow[d, equal] \\
T^{\mathsf{Set}}_\rho(\mu T^{\mathsf{Set}}_\rho)(\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho}) \arrow[d, "\mathsf{in}"] \\
\mu T^{\mathsf{Set}}_\rho(\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho}) \arrow[d, equal] \\
\llbracket \Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Set}}\rho
\end{tikzcd}
$$

Here, the first equality is by Equations 4 and 6 in Lemma 22, the second equality uses the definition of T^{Set}_ρ, and the final equality is by Definition 8. The entire morphism above from $\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}\rho$ to $\llbracket \Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Set}}\rho$ is, by Definition 27, $\llbracket \Gamma;\Phi \,|\, \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Set}}$.

To see that $\llbracket \Gamma;\Phi \,|\, \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau} \rrbracket^{\mathsf{Set}}$ is natural, we first note that since, by the induction hypothesis, $\llbracket \Gamma;\Phi \,|\, \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}$ is natural, the following diagram commutes:

$$
\begin{tikzcd}
\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}\rho \arrow[r, "\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}} f"] \arrow[d, "\llbracket \Gamma;\Phi \,|\, \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\rho"'] & \llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}\rho' \arrow[d, "\llbracket \Gamma;\Phi \,|\, \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\rho'"] \\
\llbracket \Gamma;\Phi \vdash H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\rho \arrow[r, "\llbracket \Gamma;\Phi \vdash H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}} f"] & \llbracket \Gamma;\Phi \vdash H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\rho'
\end{tikzcd}
$$

Naturality of $[\![\Gamma; \Phi \,|\, \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}$ then follows from the naturality of *in* with respect to the set arguments to the functor in the third commuting square below. Indeed, for any morphism of set environments $f : \rho \to \rho'$, the following diagram commutes:

$$
\begin{tikzcd}
{[\![\Gamma; \Phi \vdash H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho} \arrow[r, bend left, "{[\![\Gamma;\Phi\vdash H[\phi:=\mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha:=\tau}]]\!]^{\mathsf{Set}} f}"] \arrow[d, equals] & {[\![\Gamma; \Phi \vdash H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho'} \arrow[d, equals] \\
{[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\phi := \mu T^{\mathsf{Set}}_{\rho}][\overline{\alpha := [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}]} \arrow[r, bend left, "{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}} f[\phi:=\mu\sigma^{\mathsf{Set}}_f][\overline{\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}]}"] \arrow[d, equals] & {[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho'[\phi := \mu T^{\mathsf{Set}}_{\rho'}][\overline{\alpha := [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{S}}}} \arrow[d, equals] \\
{T^{\mathsf{Set}}_{\rho}(\mu T^{\mathsf{Set}}_{\rho})(\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho})} \arrow[r, "{\sigma^{\mathsf{Set}}_f(\mu\sigma^{\mathsf{Set}}_f)(\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f})}"] \arrow[d, "{in_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f}}}"] & {T^{\mathsf{Set}}_{\rho'}(\mu T^{\mathsf{Set}}_{\rho'})(\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'})} \arrow[d, "{in_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}}"] \\
{\mu T^{\mathsf{Set}}_{\rho}(\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho})} \arrow[r, "{\mu\sigma^{\mathsf{Set}}_f(\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}} f})}"] \arrow[d, equals] & {\mu T^{\mathsf{Set}}_{\rho'}(\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'})} \arrow[d, equals] \\
{[\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho} \arrow[r, "{[\![\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}} f}"] & {[\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.)\overline{\tau}]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd}
$$

- The proof that the relational interpretation of the same term is a natural transformation of the right type is analogous.
- Finally, to see that $\pi_i([\![\Gamma; \Phi \,|\, \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\rho) = [\![\Gamma; \Phi \,|\, \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}(\pi_i\rho)$ we compute

$$
\begin{aligned}
&\pi_i([\![\Gamma; \Phi \,|\, \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\rho) \\
&= \pi_i(in \circ [\![\Gamma; \Phi \,|\, \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho) \\
&= in \circ \pi_i([\![\Gamma; \Phi \,|\, \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho) \\
&= in \circ [\![\Gamma; \Phi \,|\, \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_i\rho) \\
&= [\![\Gamma; \Phi \,|\, \Delta \vdash \mathsf{in}\, t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}(\pi_i\rho)
\end{aligned}
$$

- We now consider $\Gamma; \emptyset \,|\, \Delta \vdash \mathsf{fold}_{H,F}\, t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F$.
  - To see that $[\![\Gamma; \emptyset \,|\, \Delta \vdash \mathsf{fold}_{H,F}\, t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma; \Phi \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}$, since Φ is empty, we need only show that, for every $\rho : \mathsf{SetEnv}$, $[\![\Gamma; \emptyset \,|\, \Delta \vdash \mathsf{fold}_{H,F}\, t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}$ is a morphism in Set from $[\![\Gamma; \Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}\rho$. We first note that it has the right domain and codomain. By the induction hypothesis, $[\![\Gamma; \emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma; \Phi \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}$. Thus,

for each $\rho \in \mathsf{SetEnv}$, we have

$$\begin{array}{c}
[\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho \\
\Big\downarrow{\scriptstyle [\![\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho} \\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho \\
\Big\downarrow{\scriptstyle \mathit{fold}_{H,F}} \\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}\rho
\end{array}$$

The result follows from observing that $[\![\Gamma;\emptyset \,|\, \Delta \vdash \mathsf{fold}_{H,F}\,t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}$ is exactly $\mathit{fold} \circ [\![\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho$ by Definition 27, and that the second arrow above is justified because, for every $d \in [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho$, we have that

$$\begin{array}{rl}
& [\![\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho\, d \\
: & [\![\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho \\
= & \{\eta : [\![\Gamma;\overline{\alpha} \vdash H[\phi := F][\overline{\beta := \alpha}]]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}] \Rightarrow [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}] \mid \ldots\} \\
= & \{\eta : T_{\rho[\overline{\alpha := _}]}([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}]) \Rightarrow [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}] \mid \ldots\}
\end{array}$$

Here, the final equality holds because using weakening in the third fifth, and sixth equalities below, and using Equations 4 and 6 in the first and fourth, respectively, ensures that

$$\begin{array}{rl}
& [\![\Gamma;\overline{\alpha} \vdash H[\phi := F][\overline{\beta := \alpha}]]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}] \\
= & [\![\Gamma;\overline{\alpha};\overline{\beta} \vdash H[\phi := F]]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}][\overline{\beta := [\![\Gamma;\overline{\alpha} \vdash \alpha]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}]}] \\
= & [\![\Gamma;\overline{\alpha},\overline{\beta} \vdash H[\phi := F]]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}][\overline{\beta := _}] \\
= & [\![\Gamma;\overline{\beta} \vdash H[\phi := F]]\!]^{\mathsf{Set}}\rho[\overline{\beta := _}] \\
= & [\![\Gamma;\overline{\beta},\phi \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\beta := _}]\rho[\phi := [\![\Gamma;\overline{\beta},\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := _}][\overline{\alpha := _}] \\
= & [\![\Gamma;\overline{\beta},\phi \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\beta := _}][\phi := [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}]] \\
= & [\![\Gamma;\overline{\alpha},\overline{\beta},\phi \vdash H]\!]^{\mathsf{Set}}\rho[\phi := [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}]][\overline{\beta := _}] \\
= & T_{\rho[\overline{\alpha := _}]}([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := _}])
\end{array}$$

That is, for each $d : [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ by weakening, we have that $[\![\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho\, d$ comprises a family of natural transformations, each of which is a $T^{\mathsf{Set}}_{\rho[\overline{\alpha := \_}]}$-algebra with carrier $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$. Thus, if $\eta : [\![\Gamma;\emptyset \,|\, \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho\, d$ then $\mathit{fold}_{H,F}$ can actually be applied to η to get a new natural transformation $\mathit{fold}\,\eta$ from $\mu T_{\rho[\overline{\alpha := \_}]}$ to $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$, i.e., from $[\![\Gamma;\overline{\alpha} \vdash (\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha}]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ to $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$.

We now show that if η satisfies the additional condition on its components necessary for it to be in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}\rho$ — i.e., if for all $\overline{R : \mathsf{Rel}(A,B)}$ we have $(\eta_{\overline{A}}, \eta_{\overline{B}}) \in [\![\Gamma;\overline{\alpha} \vdash H[\phi := F][\overline{\beta := \alpha}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \to [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]$ — then $\mathit{fold}\,\eta$ satisfies the additional condition necessary for it to be in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}\rho$. To show this, note that since $(\eta_{\overline{A}}, \eta_{\overline{B}})$ has type

$$\begin{array}{rl}
& [\![\Gamma;\overline{\alpha} \vdash H[\phi := F][\overline{\beta := \alpha}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \to [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \\
= & T_{\mathsf{Eq}_\rho[\overline{\alpha := R}]}([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]) \to [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]
\end{array}$$

we have that $((fold\,\eta)_{\overline{A}}, (fold\,\eta)_{\overline{B}}) = (fold\,\eta_{\overline{A}}, fold\,\eta_{\overline{B}})$ has type

$$\begin{array}{ll} & \mu T_{\mathsf{Eq}_\rho[\overline{\alpha := R}]} \to [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] \\ = & ?? \\ = & (\mu T_{\mathsf{Eq}_\rho[\overline{\alpha := R}]})\overline{R} \to [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] \\ = & (\mu T_{\mathsf{Eq}_\rho[\overline{\alpha := R}]})\overline{[\![\Gamma; \overline{\alpha} \vdash \alpha]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]} \to [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] \\ = & [\![\Gamma; \overline{\alpha} \vdash (\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \to [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}] \end{array}$$

as desired.

- The proofs that the relational interpretation of the same term has the right type, and that if η satisfies the additional condition on its components necessary for it to be in $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Rel}}\rho$ then $fold\,\eta$ satisfies the additional condition necessary for it to be in $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Rel}}\rho$, are analogous.
- Finally, to see that

$$\begin{array}{ll} & \pi_i([\![\Gamma; \emptyset \mid \Delta \vdash \mathsf{fold}_{H,F}\,t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Rel}}\rho) \\ = & [\![\Gamma; \emptyset \mid \Delta \vdash \mathsf{fold}_{H,F}\,t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}(\pi_i\rho) \end{array}$$

we compute

$$\begin{array}{ll} & \pi_i([\![\Gamma; \emptyset \mid \Delta \vdash \mathsf{fold}_{H,F}\,t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Rel}}\rho) \\ = & \pi_i(fold \circ [\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Rel}}\rho) \\ = & fold \circ \pi_i([\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Rel}}\rho) \\ = & fold \circ [\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,(H[\phi := F][\overline{\beta := \alpha}])\,F]\!]^{\mathsf{Set}}(\pi_i\rho) \\ = & [\![\Gamma; \emptyset \mid \Delta \vdash \mathsf{fold}_{H,F}\,t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F]\!]^{\mathsf{Set}}(\pi_i\rho) \end{array}$$

□

The Abstraction Theorem is now the special case of Theorem 29 for closed terms of close type:

Theorem 30. *If* $\vdash \tau : \mathcal{F}$ *and* $\vdash t : \tau$, *then* $([\![\vdash t : \tau]\!]^{\mathsf{Set}}, [\![\vdash t : \tau]\!]^{\mathsf{Set}}) \in [\![\vdash \tau]\!]^{\mathsf{Rel}}$.

Definition 31. *Let $\Gamma; \Phi, \alpha \mid \Delta \vdash t : \tau$ be a term and $\Gamma; \Phi \vdash \sigma : \mathcal{F}$ be a type. We define a term $\Gamma; \Phi \mid \Delta[\alpha := \sigma] \vdash t[\alpha := \sigma] : \tau[\alpha := \sigma]$ by structural induction on t as*

$$
\begin{aligned}
x[\alpha := \sigma] &= x \\
(\lambda x.t)[\alpha := \sigma] &= \lambda x.(t[\alpha := \sigma]) \\
(st)[\alpha := \sigma] &= (s[\alpha := \sigma])(t[\alpha := \sigma]) \\
(L_{\overline{\alpha}} x.t)[\beta := \sigma] &= L_{\overline{\alpha}} x.(t[\beta := \sigma]) \\
(t_{\overline{\tau}} s)[\alpha := \sigma] &= t_{\overline{\tau[\alpha:=\sigma]}}(s[\alpha := \sigma]) \\
(\bot_\tau t)[\alpha := \sigma] &= \bot_{\tau[\alpha:=\sigma]}(t[\alpha := \sigma]) \\
\top[\alpha := \sigma] &= \top \\
(s, t)[\alpha := \sigma] &= (s[\alpha := \sigma], t[\alpha := \sigma]) \\
(\pi_1 t)[\alpha := \sigma] &= \pi_1(t[\alpha := \sigma]) \\
(\pi_2 t)[\alpha := \sigma] &= \pi_2(t[\alpha := \sigma]) \\
(\mathsf{case}\, t \,\mathsf{of}\, \{x \mapsto l;\, y \mapsto r\})[\alpha := \sigma] &= \mathsf{case}\, t[\alpha := \sigma] \,\mathsf{of}\, \{x \mapsto l[\alpha := \sigma];\, y \mapsto r[\alpha := \sigma]\} \\
(\mathsf{inl}\, s)[\alpha := \sigma] &= \mathsf{inl}\,(s[\alpha := \sigma]) \\
(\mathsf{inr}\, s)[\alpha := \sigma] &= \mathsf{inr}\,(s[\alpha := \sigma]) \\
(\mathsf{in}\, t)[\alpha := \sigma] &= \mathsf{in}(t[\alpha := \sigma]) \\
(\mathsf{fold}_{H,F}\, t)[\alpha := \sigma] &= \mathsf{fold}_{H[\alpha:=\sigma],F}\,(t[\alpha := \sigma])
\end{aligned}
$$

Lemma 32. *Let $\Gamma; \Phi, \alpha \mid \Delta \vdash t : \tau$ be a term and $\Gamma; \Phi \vdash \sigma : \mathcal{F}$ be a type. Then, for any set environment ρ we have that*

$$[\![\Gamma; \Phi \mid \Delta[\alpha := \sigma] \vdash t[\alpha := \sigma] : \tau[\alpha := \sigma]]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \Phi, \alpha \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho]$$

and, analogously, for any relational environment ρ we have that

$$[\![\Gamma; \Phi \mid \Delta[\alpha := \sigma] \vdash t[\alpha := \sigma] : \tau[\alpha := \sigma]]\!]^{\mathsf{Rel}}\rho = [\![\Gamma; \Phi, \alpha \mid \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho[\alpha := [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho]$$

Proof. By induction on $\Gamma; \Phi, \alpha \mid \Delta \vdash t : \tau$

- $\Gamma; \Phi, \alpha \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\beta := \tau}]$

$$
\begin{aligned}
& [\![\Gamma; \Phi \mid \Delta[\alpha := \sigma] \vdash (t_{\overline{\tau}} s)[\alpha := \sigma] : G[\overline{\beta := \tau}][\alpha := \sigma]]\!]^{\mathsf{Set}}\rho \\
= \;& [\![\Gamma; \Phi \mid \Delta[\alpha := \sigma] \vdash t_{\overline{\tau[\alpha:=\sigma]}}(s[\alpha := \sigma]) : G[\overline{\beta := \tau[\alpha := \sigma]}]]\!]^{\mathsf{Set}}\rho \\
= \;& \mathsf{eval} \circ \langle ([\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta}}\, F\, G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi\vdash\tau[\alpha:=\sigma]]\!]^{\mathsf{Set}}\rho}}, \\
& \quad [\![\Gamma; \Phi \mid \Delta[\alpha := \sigma] \vdash s[\alpha := \sigma] : F[\overline{\beta := \tau[\alpha := \sigma]}]]\!]^{\mathsf{Set}}\rho\rangle \\
= \;& \mathsf{eval} \circ \langle ([\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta}}\, F\, G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi,\alpha\vdash\tau]\!]^{\mathsf{Set}}\rho[\alpha:=[\![\Gamma;\Phi\vdash\sigma]\!]^{\mathsf{Set}}\rho]}}, \\
& \quad [\![\Gamma; \Phi, \alpha \mid \Delta \vdash s : F[\overline{\beta := \tau}]]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho]\rangle \\
= \;& [\![\Gamma; \Phi, \alpha \mid \Delta \vdash t_{\overline{\tau}} s : G[\overline{\beta := \tau}]]\!]^{\mathsf{Set}}\rho[\alpha := [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho]
\end{aligned}
$$

□

Definition 33. *Let* $\Gamma;\Phi\,|\,\Delta, x : \sigma \vdash t : \tau$ *and* $\Gamma;\Phi\,|\,\Delta \vdash s : \sigma$ *be terms. We define a term* $\Gamma;\Phi\,|\,\Delta \vdash t[x := s] : \tau$ *by structural induction on* t *as*

$$
\begin{aligned}
x[x := s] &= s \\
y[x := s] &= y \qquad \text{if } x \neq y \\
(\lambda x.t)[y := s] &= \lambda x.(t[y := s]) \\
(st)[x := u] &= (s[x := u])(t[x := u]) \\
(L_{\overline{\alpha}} x.t)[y := s] &= L_{\overline{\alpha}} x.(t[y := s]) \\
(t_{\overline{\tau}} s)[x := u] &= (t[x := u])_{\overline{\tau}}(s[x := u]) \\
(\bot_\tau t)[x := s] &= \bot_\tau(t[x := s]) \\
\top[x := s] &= \top \\
(s, t)[x := u] &= (s[x := u], t[x := u]) \\
(\pi_1 t)[x := s] &= \pi_1(t[x := s]) \\
(\pi_2 t)[x := s] &= \pi_2(t[x := s]) \\
(\mathsf{case}\; t \;\mathsf{of}\; \{x \mapsto l;\; y \mapsto r\})[z := s] &= \mathsf{case}\; t[z := s] \;\mathsf{of}\; \{x \mapsto l[z := s];\; y \mapsto r[z := s]\} \\
(\mathsf{inl}\, s)[x := u] &= \mathsf{inl}\,(s[x := u]) \\
(\mathsf{inr}\, s)[x := u] &= \mathsf{inr}\,(s[x := u]) \\
(\mathsf{in}\, t)[x := s] &= \mathsf{in}(t[x := s]) \\
(\mathsf{fold}_{H,F}\; t)[x := s] &= \mathsf{fold}_{H,F}\,(t[x := s])
\end{aligned}
$$

Lemma 34. *Let* $\Gamma;\Phi\,|\,\Delta, x : \sigma \vdash t : \tau$ *and* $\Gamma;\Phi\,|\,\Delta \vdash s : \sigma$ *be terms. Then, for any set environment* ρ *we have that*

$$[\![\Gamma;\Phi\,|\,\Delta \vdash t[x := s] : \tau]\!]^{\mathsf{Set}}\rho(\_) = [\![\Gamma;\Phi\,|\,\Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho(\_, [\![\Gamma;\Phi\,|\,\Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho(\_))$$

and, analogously, for any relational environment ρ *we have that*

$$[\![\Gamma;\Phi\,|\,\Delta \vdash t[x := s] : \tau]\!]^{\mathsf{Rel}}\rho(\_) = [\![\Gamma;\Phi\,|\,\Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Rel}}\rho(\_, [\![\Gamma;\Phi\,|\,\Delta \vdash s : \sigma]\!]^{\mathsf{Rel}}\rho(\_))$$

Proof. By induction on $\Gamma;\Phi,\alpha\,|\,\Delta, x : \sigma \vdash t : \tau$

- $\Gamma;\Phi\,|\,x : \sigma \vdash x : \sigma$

$$
\begin{aligned}
[\![\Gamma;\Phi\,|\,\Delta \vdash x[x := s] : \sigma]\!]^{\mathsf{Set}}\rho(\_) &= [\![\Gamma;\Phi\,|\,\Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho(\_) \\
&= [\![\Gamma;\Phi\,|\,\Delta, x : \sigma \vdash x : \sigma]\!]^{\mathsf{Set}}\rho(\_, [\![\Gamma;\Phi\,|\,\Delta \vdash s : \sigma]\!]^{\mathsf{Set}}\rho(\_))
\end{aligned}
$$

- $\Gamma;\Phi\,|\,\Delta \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]$

$$
\begin{aligned}
& [\![\Gamma;\Phi\,|\,\Delta \vdash (t_{\overline{\tau}} s)[x := u] : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho \\
= \;& [\![\Gamma;\Phi\,|\,\Delta \vdash (t[x := u])_{\overline{\tau}}(s[x := u]) : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho \\
= \;& \mathsf{eval} \circ \langle ([\![\Gamma;\emptyset\,|\,\Delta \vdash t[x := u] : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\;\_)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}, \\
& \qquad [\![\Gamma;\Phi\,|\,\Delta \vdash s[x := u] : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\rangle \\
= \;& \mathsf{eval} \circ \langle ([\![\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho(\_, [\![\Gamma;\Phi\,|\,\Delta \vdash u : \sigma]\!]^{\mathsf{Set}}\rho\;\_\;))_{\overline{[\![\Gamma;\Phi,\vdash\tau]\!]^{\mathsf{Set}}\rho}}, \\
& \qquad [\![\Gamma;\Phi,\,|\,\Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho(\_, [\![\Gamma;\Phi\,|\,\Delta \vdash u : \sigma]\!]^{\mathsf{Set}}\rho\;\_\rangle \\
= \;& [\![\Gamma;\Phi\,|\,\Delta, x : \sigma \vdash t_{\overline{\tau}} s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho(\_, [\![\Gamma;\Phi\,|\,\Delta \vdash u : \sigma]\!]^{\mathsf{Set}}\rho\;\_)
\end{aligned}
$$

□

Lemma 35 (Weakening). *Weakening is supported by the following facts:*

- *Let $\Gamma; \emptyset \vdash \sigma : \mathcal{T}$ be a type. Then, $\Gamma, v; \emptyset \vdash \sigma : \mathcal{T}$ is well-typed, and for any set environment ρ we have that*
$$[\![\Gamma, v; \emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho$$
and for any relation environment ρ we have that
$$[\![\Gamma, v; \emptyset \vdash \sigma]\!]^{\mathsf{Rel}}\rho = [\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Rel}}\rho$$
- *Let $\Gamma; \Phi \vdash \sigma : \mathcal{F}$ be a type. Then, $\Gamma; \Phi, \alpha \vdash \sigma : \mathcal{F}$ is well-typed, and for any set environment ρ we have that*
$$[\![\Gamma; \Phi, \alpha \vdash \sigma]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho$$
and for any relation environment ρ we have that
$$[\![\Gamma; \Phi, \alpha \vdash \sigma]\!]^{\mathsf{Rel}}\rho = [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho$$
- *Let $\Gamma; \Phi \,|\, \Delta \vdash t : \tau$ be a term and $\Gamma; \Phi \vdash \sigma$ be a type. Then, $\Gamma; \Phi \,|\, \Delta, x : \sigma \vdash t : \tau$ is well-formed, and for any set environment ρ we have that*
$$[\![\Gamma; \Phi \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \Phi \,|\, \Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho\pi_\Delta$$
where π_Δ is the projection $[\![\Gamma; \Phi \vdash \Delta, x : \sigma]\!]^{\mathsf{Set}} \to [\![\Gamma; \Phi \vdash \Delta]\!]^{\mathsf{Set}}$, and for any relation environment ρ we have that
$$[\![\Gamma; \Phi \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Rel}}\rho = [\![\Gamma; \Phi \,|\, \Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho\pi_\Delta$$
where π_Δ is the projection $[\![\Gamma; \Phi \vdash \Delta, x : \sigma]\!]^{\mathsf{Rel}} \to [\![\Gamma; \Phi \vdash \Delta]\!]^{\mathsf{Rel}}$.

Proof. Do we need a proof for this? □

All of the following lemmas are stated and proved for set semantics, but they hold for relational semantics just as well.

Lemma 36. *Let $\Gamma; \Phi \,|\, \Delta, x : \sigma \vdash t : \tau$ be a term. Then, for any set environment ρ we have that $[\![\Gamma; \emptyset \,|\, \Delta, y : \sigma \vdash t[x := y] : \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho$*

Proof. We have that
$$\begin{aligned}
& [\![\Gamma; \emptyset \,|\, \Delta, y : \sigma \vdash t[x := y] : \tau]\!]^{\mathsf{Set}}\rho(_{\Delta}, _{\sigma}) \\
= \;& [\![\Gamma; \emptyset \,|\, \Delta, y : \sigma, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho(_{\Delta}, _{\sigma}, [\![\Gamma; \emptyset \,|\, \Delta, y : \sigma \vdash y : \sigma]\!]^{\mathsf{Set}}\rho(_{\Delta}, _{\sigma})) \\
= \;& [\![\Gamma; \emptyset \,|\, \Delta, y : \sigma, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho(_{\Delta}, _{\sigma}, _{\sigma}) \\
= \;& [\![\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho\pi_{\Delta\times\sigma}(_{\Delta}, _{\sigma}, _{\sigma}) \\
= \;& [\![\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho(_{\Delta}, _{\sigma})
\end{aligned}$$
where the first equality is by Lemma 34 and the third equality is given by Lemma 35. □

Lemma 37. *Let $\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t : \tau$ be a term. Then, for any set environment ρ we have that $[\![\Gamma; \emptyset \,|\, \Delta \vdash \lambda y.t[x := y] : \sigma \to \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \emptyset \,|\, \Delta \vdash \lambda x.t : \sigma \to \tau]\!]^{\mathsf{Set}}\rho$*

Proof. We have that
$$\begin{aligned}
& [\![\Gamma; \emptyset \,|\, \Delta \vdash \lambda y.t[x := y] : \sigma \to \tau]\!]^{\mathsf{Set}}\rho \\
= \;& \mathsf{curry} \circ [\![\Gamma; \emptyset \,|\, \Delta, y : \sigma \vdash t[x := y] : \tau]\!]^{\mathsf{Set}}\rho \\
= \;& \mathsf{curry} \circ [\![\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}}\rho \\
= \;& [\![\Gamma; \emptyset \,|\, \Delta \vdash \lambda x.t : \sigma \to \tau]\!]^{\mathsf{Set}}\rho
\end{aligned}$$
where the second equality is by Lemma 36. □

Lemma 38. *Let $\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t : G$ be a term. Then, for any set environment ρ we have that $[\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\beta}}y.(t[\overline{\alpha := \beta}][x := y]) : \mathsf{Nat}^{\beta} F[\overline{\alpha := \beta}]G[\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\alpha} FG]\!]^{\mathsf{Set}}\rho$*

Proof. We have that

$$
\begin{array}{rl}
& [\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\beta}} y.(t[\overline{\alpha := \beta}][x := y]) : \mathsf{Nat}^{\beta} F[\overline{\alpha := \beta}] G[\overline{\alpha := \beta}]]\!]^{\mathsf{Set}} \rho \\
= & \mathsf{curry}([\![\Gamma;\overline{\beta} \,|\, \Delta, y : F[\overline{\alpha := \beta}] \vdash t[\overline{\alpha := \beta}][x := y] : G[\overline{\alpha := \beta}]]\!]^{\mathsf{Set}} \rho[\overline{\beta := \_}]) \\
= & \mathsf{curry}([\![\Gamma;\overline{\beta} \,|\, \Delta, x : F[\overline{\alpha := \beta}] \vdash t[\overline{\alpha := \beta}] : G[\overline{\alpha := \beta}]]\!]^{\mathsf{Set}} \rho[\overline{\beta := \_}]) \\
= & \mathsf{curry}([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\beta := \_}][\overline{\alpha := [\![\Gamma;\overline{\beta} \vdash \beta]\!]^{\mathsf{Set}} \rho[\overline{\beta := \_}]}]) \\
= & \mathsf{curry}([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := \_}]) \\
= & [\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.(t) : \mathsf{Nat}^{\alpha} F G]\!]^{\mathsf{Set}} \rho
\end{array}
$$

where the second equality is by Lemma 36 and the third equality is by Lemma 32. □

Lemma 39. *Let* $\Gamma;\emptyset \,|\, \Delta, x : \sigma \vdash t : \tau$ *be a term. Then, for any set environment* ρ *we have that* $[\![\Gamma;\emptyset \,|\, \Delta \vdash (\lambda x.t)s : \tau]\!]^{\mathsf{Set}} \rho = [\![\Gamma;\emptyset \,|\, \Delta \vdash t[x := s] : \tau]\!]^{\mathsf{Set}} \rho$

Proof. We have that

$$
\begin{array}{rl}
& [\![\Gamma;\emptyset \,|\, \Delta \vdash (\lambda x.t)s : \tau]\!]^{\mathsf{Set}} \rho \\
= & \mathsf{eval} \circ \langle [\![\Gamma;\emptyset \,|\, \Delta \vdash (\lambda x.t) : \sigma \to \tau]\!]^{\mathsf{Set}} \rho, [\![\Gamma;\emptyset \,|\, \Delta \vdash s : \sigma]\!]^{\mathsf{Set}} \rho \rangle \\
= & \mathsf{eval} \circ \langle \mathsf{curry} [\![\Gamma;\emptyset \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}} \rho, [\![\Gamma;\emptyset \,|\, \Delta \vdash s : \sigma]\!]^{\mathsf{Set}} \rho \rangle \\
= & [\![\Gamma;\emptyset \,|\, \Delta, x : \sigma \vdash t : \tau]\!]^{\mathsf{Set}} \rho(\_, [\![\Gamma;\emptyset \,|\, \Delta \vdash s : \sigma]\!]^{\mathsf{Set}} \rho\_) \\
= & [\![\Gamma;\emptyset \,|\, \Delta \vdash t[x := s] : \tau]\!]^{\mathsf{Set}} \rho
\end{array}
$$

where the last equality is by Lemma 34. □

Lemma 40. *Let* $\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G$, $\Gamma;\Phi,\overline{\alpha} \,|\, \Delta \vdash s : F$, *and* $\Gamma;\Phi \vdash \tau$ *be terms. Then, for any set environment* ρ *we have that* $[\![\Gamma;\Phi \,|\, \Delta \vdash (L_{\overline{\alpha}} x.t)_{\overline{\tau}} s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho = [\![\Gamma;\Phi \,|\, \Delta \vdash t[\overline{\alpha := \tau}][x := s] : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho$,

Proof. We have that

$$
\begin{array}{rl}
& [\![\Gamma;\Phi \,|\, \Delta \vdash (L_{\overline{\alpha}} x.t)_{\overline{\tau}} s : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho \\
= & \mathsf{eval} \circ \langle ([\![\Gamma;\emptyset \,|\, \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}} \rho\_)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} \rho}}, [\![\Gamma;\Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho \rangle \\
= & \mathsf{eval} \circ \langle \mathsf{curry}([\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} \rho}])\_, [\![\Gamma;\Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho \rangle \\
= & [\![\Gamma;\overline{\alpha} \,|\, \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} \rho}](\_, [\![\Gamma;\Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho\_) \\
= & [\![\Gamma;\Phi \,|\, \Delta, x : F[\overline{\alpha := \tau}] \vdash t[\overline{\alpha := \tau}] : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho(\_, [\![\Gamma;\Phi \,|\, \Delta \vdash s : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho\_) \\
= & [\![\Gamma;\Phi \,|\, \Delta \vdash t[\overline{\alpha := \tau}][x := s] : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho
\end{array}
$$

where the second-to-last equality is by Lemma 32 and the last equality is by Lemma 34. □

Lemma 41. *Let* $\Gamma;\Phi \,|\, \Delta \vdash t_1 : \sigma_1$ *and* $\Gamma;\Phi \,|\, \Delta \vdash t_2 : \sigma_2$ *be terms. Then, for any set environment* ρ *we have that* $[\![\Gamma;\Phi \,|\, \Delta \vdash \pi_i(t_1, t_2) : \sigma_i]\!]^{\mathsf{Set}} \rho = [\![\Gamma;\Phi \,|\, \Delta \vdash t_i : \sigma_i]\!]^{\mathsf{Set}} \rho$ *for* $i = 1, 2$.

Proof. We have that

$$
\begin{array}{rl}
& [\![\Gamma;\Phi \,|\, \Delta \vdash \pi_i(t_1, t_2) : \sigma_i]\!]^{\mathsf{Set}} \rho \\
= & \pi_i \circ [\![\Gamma;\Phi \,|\, \Delta \vdash (t_1, t_2) : \sigma_1 \times \sigma_2]\!]^{\mathsf{Set}} \rho \\
= & \pi_i \circ ([\![\Gamma;\Phi \,|\, \Delta \vdash t_1 : \sigma_1]\!]^{\mathsf{Set}} \rho \times [\![\Gamma;\Phi \,|\, \Delta \vdash t_2 : \sigma_2]\!]^{\mathsf{Set}} \rho) \\
= & [\![\Gamma;\Phi \,|\, \Delta \vdash t_i : \sigma_i]\!]^{\mathsf{Set}} \rho
\end{array}
$$

□

Lemma 42. *Let* $\Gamma;\Phi \,|\, \Delta, x : \sigma \vdash t_1 : \gamma$ *and* $\Gamma;\Phi \,|\, \Delta, x : \tau \vdash t_2 : \gamma$ *be terms. Then, for any set environment* ρ *we have that* $[\![\Gamma;\Phi \,|\, \Delta \vdash \mathsf{case}\ \mathsf{inl}\ t\ \mathsf{of}\ \{x_1 \mapsto t_1;\ x_2 \mapsto t_2\} : \gamma]\!]^{\mathsf{Set}} \rho = [\![\Gamma;\Phi \,|\, \Delta \vdash t_1[x := t] : \gamma]\!]^{\mathsf{Set}} \rho$ *and* $[\![\Gamma;\Phi \,|\, \Delta \vdash \mathsf{case}\ \mathsf{inr}\ t\ \mathsf{of}\ \{x_1 \mapsto t_1;\ x_2 \mapsto t_2\} : \gamma]\!]^{\mathsf{Set}} \rho = [\![\Gamma;\Phi \,|\, \Delta \vdash t_2[x := t] : \gamma]\!]^{\mathsf{Set}} \rho$

Proof. We prove the inl case, as the inr case is completely analogous. We have that

$$
\begin{array}{ll}
& [\![\Gamma;\Phi \mid \Delta \vdash \mathsf{case}\ \mathsf{inl}\, t\ \mathsf{of}\ \{x_1 \mapsto t_1;\ x_2 \mapsto t_2\} : \gamma]\!]^{\mathsf{Set}}\rho \\
= & \mathsf{eval} \circ \langle \mathsf{curry}[[\![\Gamma;\Phi \mid \Delta, x : \sigma \vdash t_1 : \gamma]\!]^{\mathsf{Set}}\rho, [\![\Gamma;\Phi \mid \Delta, x : \tau \vdash t_2 : \gamma]\!]^{\mathsf{Set}}\rho], [\![\Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\, t : \sigma + \tau]\!]^{\mathsf{Set}}\rho\rangle \\
= & \mathsf{eval} \circ \langle \mathsf{curry}[[\![\Gamma;\Phi \mid \Delta, x : \sigma \vdash t_1 : \gamma]\!]^{\mathsf{Set}}\rho, [\![\Gamma;\Phi \mid \Delta, x : \tau \vdash t_2 : \gamma]\!]^{\mathsf{Set}}\rho], \mathsf{inl} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma]\!]^{\mathsf{Set}}\rho\rangle \\
= & [[\![\Gamma;\Phi \mid \Delta, x : \sigma \vdash t_1 : \gamma]\!]^{\mathsf{Set}}\rho, [\![\Gamma;\Phi \mid \Delta, x : \tau \vdash t_2 : \gamma]\!]^{\mathsf{Set}}\rho](\_, \mathsf{inl} \circ [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma]\!]^{\mathsf{Set}}\rho\_) \\
= & [\![\Gamma;\Phi \mid \Delta, x : \sigma \vdash t_1 : \gamma]\!]^{\mathsf{Set}}\rho(\_, [\![\Gamma;\Phi \mid \Delta \vdash t : \sigma]\!]^{\mathsf{Set}}\rho\_) \\
= & [\![\Gamma;\Phi \mid \Delta \vdash t_1[x := t] : \gamma]\!]^{\mathsf{Set}}\rho
\end{array}
$$

where the last equality is by Lemma 34. □

Lemma 43. *Let*

$$\Gamma;\Phi \mid \Delta \vdash t_1 : \sigma \to \gamma$$

$$\Gamma;\Phi \mid \Delta \vdash t_2 : \sigma \to \gamma$$

$$\Gamma;\Phi \mid \Delta \vdash s_1 : \sigma$$

$$\Gamma;\Phi \mid \Delta \vdash s_2 : \sigma$$

be terms such that, for any set environment ρ,

$$[\![\Gamma;\Phi \mid \Delta \vdash t_1 : \sigma \to \gamma]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi \mid \Delta \vdash t_2 : \sigma \to \gamma]\!]^{\mathsf{Set}}\rho$$

and

$$[\![\Gamma;\Phi \mid \Delta \vdash s_1 : \sigma]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi \mid \Delta \vdash s_2 : \sigma]\!]^{\mathsf{Set}}\rho$$

Then, for any set environment ρ we have that $[\![\Gamma;\Phi \mid \Delta \vdash t_1 s_1 : \gamma]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi \mid \Delta \vdash t_2 s_2 : \gamma]\!]^{\mathsf{Set}}\rho$

Proof. We have that

$$
\begin{array}{rcl}
[\![\Gamma;\Phi \mid \Delta \vdash t_1 s_1 : \gamma]\!]^{\mathsf{Set}}\rho & = & \mathsf{eval}\langle [\![\Gamma;\Phi \mid \Delta \vdash t_1 : \sigma \to \gamma]\!]^{\mathsf{Set}}\rho, [\![\Gamma;\Phi \mid \Delta \vdash s_1 : \sigma]\!]^{\mathsf{Set}}\rho\rangle \\
& = & \mathsf{eval}\langle [\![\Gamma;\Phi \mid \Delta \vdash t_2 : \sigma \to \gamma]\!]^{\mathsf{Set}}\rho, [\![\Gamma;\Phi \mid \Delta \vdash s_2 : \sigma]\!]^{\mathsf{Set}}\rho\rangle \\
& = & [\![\Gamma;\Phi \mid \Delta \vdash t_2 s_2 : \gamma]\!]^{\mathsf{Set}}\rho
\end{array}
$$

□

Lemma 44. *Let $\Gamma;\Phi \vdash \tau$ be a type and*

$$\Gamma;\emptyset \mid \Delta \vdash t_1 : \mathsf{Nat}^{\overline{\alpha}} F G$$

$$\Gamma;\emptyset \mid \Delta \vdash t_2 : \mathsf{Nat}^{\overline{\alpha}} F G$$

$$\Gamma;\Phi \mid \Delta \vdash s_1 : F[\overline{\alpha := \tau}]$$

$$\Gamma;\Phi \mid \Delta \vdash s_2 : F[\overline{\alpha := \tau}]$$

be terms such that, for any set environment ρ,

$$[\![\Gamma;\emptyset \mid \Delta \vdash t_1 : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\emptyset \mid \Delta \vdash t_2 : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}}\rho$$

and

$$[\![\Gamma;\Phi \mid \Delta \vdash s_1 : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi \mid \Delta \vdash s_2 : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$$

Then, for any set environment ρ we have that

$$[\![\Gamma;\Phi \mid \Delta \vdash (t_1)_\tau s_1 : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi \mid \Delta \vdash (t_2)_\tau s_2 : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho$$

Proof. We have that

$$
\begin{array}{ll}
& [\![\Gamma;\Phi \mid \Delta \vdash (t_1)_\tau s_1 : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho \\
= & \mathsf{eval}\langle ([\![\Gamma;\emptyset \mid \Delta \vdash t_1 : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}}\rho\_)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}}, [\![\Gamma;\Phi \mid \Delta \vdash s_1 : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\rangle \\
= & \mathsf{eval}\langle ([\![\Gamma;\emptyset \mid \Delta \vdash t_2 : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}}\rho\_)_{\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}}, [\![\Gamma;\Phi \mid \Delta \vdash s_2 : F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho\rangle \\
= & [\![\Gamma;\Phi \mid \Delta \vdash (t_2)_\tau s_2 : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\rho
\end{array}
$$

□

Lemma 45. *Let*

$$\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t_1 : \tau$$
$$\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t_2 : \tau$$

be terms such that, for any set environment ρ,

$$[\![\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t_1 : \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t_2 : \tau]\!]^{\mathsf{Set}}\rho$$

Then, for any set environment ρ *we have that*

$$[\![\Gamma; \emptyset \,|\, \Delta \vdash \lambda x.t_1 : \sigma \to \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \emptyset \,|\, \Delta \vdash \lambda x.t_1 : \sigma \to \tau]\!]^{\mathsf{Set}}\rho$$

Proof. We have that

$$\begin{array}{rcl} [\![\Gamma; \emptyset \,|\, \Delta \vdash \lambda x.t_1 : \sigma \to \tau]\!]^{\mathsf{Set}}\rho & = & \mathsf{curry}([\![\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t_1 : \tau]\!]^{\mathsf{Set}}\rho) \\ & = & \mathsf{curry}([\![\Gamma; \emptyset \,|\, \Delta, x : \sigma \vdash t_2 : \tau]\!]^{\mathsf{Set}}\rho) \\ & = & [\![\Gamma; \emptyset \,|\, \Delta \vdash \lambda x.t_2 : \sigma \to \tau]\!]^{\mathsf{Set}}\rho \end{array}$$

□

Lemma 46. *Let*

$$\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t_1 : G$$
$$\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t_2 : G$$

be terms such that, for any set environment ρ,

$$[\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t_1 : G]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t_2 : G]\!]^{\mathsf{Set}}\rho$$

Then, for any set environment ρ *we have that*

$$[\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t_1 : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t_2 : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}}\rho$$

Proof. We have that

$$\begin{array}{rcl} [\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t_1 : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}}\rho & = & \mathsf{curry}([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t_1 : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]) \\ & = & \mathsf{curry}([\![\Gamma; \overline{\alpha} \,|\, \Delta, x : F \vdash t_2 : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]) \\ & = & [\![\Gamma; \emptyset \,|\, \Delta \vdash L_{\overline{\alpha}}x.t_2 : \mathsf{Nat}^{\overline{\alpha}} F G]\!]^{\mathsf{Set}}\rho \end{array}$$

□

There might be more congruence lemmas (one for each term introduction/elimination rule)

We will need to go back and add typing rules for well-formed terms involving $\mathsf{map}^{\mathcal{F}}$ and $\mathsf{map}^{\mathcal{T}}$ in Def 5, set and relational interpretations of these maps (just the actual functorial actions), and cases for map to all of our proofs thus far having to do with terms.

Next we will want to sanity-check our model by showing that term interps respect conversion rules. These are

- $\lambda x.t = \lambda y.t[x := y]$
- $L_{\boldsymbol{\alpha}}x.t = L_{\boldsymbol{\beta}}y.(t[\boldsymbol{\alpha} := \boldsymbol{\beta}][x := y])$
- $(\lambda x.t)s = t[x := s]$
- $(L_{\boldsymbol{\alpha}}x.t)_{\boldsymbol{\tau}}s = t[\boldsymbol{\alpha} := \boldsymbol{\tau}][x := s]$
- $\pi_i(t_1, t_2) = t_i$
- $\mathsf{case}\ \mathsf{inl}\ t\ \mathsf{of}\ \{x_1 \mapsto t_1;\ x_2 \mapsto t_2\} = t_i[x_i := t]$
- and other conversion rules as on page 18 of MFPS paper
- perhaps add weakening rules explicitly here?

- All of the above are shorthands for saying that the interps of the LHSs are the same as the interps of the RHSs. For this conversion rule: $\mathsf{fold}\,k\,(\mathsf{in}\,t) = k\,(\mathsf{map}\,(\mathsf{fold}\,k)\,t)$, we can't express it in syntax. So what we really want to say here is that some semantic equivalent of this syntactic rule holds. And similarly for the next rules.
- Maybe we want to show that $(\llbracket \mu\alpha.F[\alpha]\rrbracket, \llbracket \mathsf{in}\rrbracket)$ is an initial $\llbracket F\rrbracket$-algebra in the model? See Birkedal and Mogelberg Section 5.4. As part of this we would have the next bullet point, plus some other intermediate results as in 5.17, 5.18, and 5.19 there. We would also need representations of map functions. Perhaps we can define them syntactically as in Plotkin and Abadi section 2.1? (But isn't this precisely what we tried?)
- $\mathsf{fold}_H\,\mathsf{in}_H x = x$ (Intuitively, this is the syntactic counterpart to initiality of in.)
- $\mathsf{map}^{\mathcal{F}}_H\,\overline{(L_{\boldsymbol{\alpha}} x.x)} = L_{\bigcup \boldsymbol{\alpha}} x.x$ for all H
- $\mathsf{map}^{\mathcal{F}}_H\,(\overline{L_{\boldsymbol{\alpha}} x.\eta_{\boldsymbol{\alpha}}(\mu_{\boldsymbol{\alpha}} x)}) = L_{\bigcup \boldsymbol{\alpha}} x.(\mathsf{map}^{\mathcal{F}}_H\,\overline{\eta})_{\bigcup \boldsymbol{\alpha}}((\mathsf{map}^{\mathcal{F}}_H\,\overline{\mu})_{\bigcup \boldsymbol{\alpha}} x)$
- $\lambda x.\mathsf{map}^{\mathcal{F}}_G\,\overline{f}\,(\eta_{\overline{\sigma}} x) = \lambda x.\eta_{\overline{\tau}}\,(\mathsf{map}^{\mathcal{F}}_F\,\overline{f}\,x)$ (note that $.. \vdash f : \mathsf{Nat}^{\emptyset}\,F\,G$)
- $\mathsf{map}^{\mathcal{F}}_H(\overline{\mathsf{map}^{\mathcal{F}}_{K_i}\,\overline{t_i}}) = \mathsf{map}^{\mathcal{F}}_{H[\overline{\psi := K}]}\,\overline{t}$
- $\mathsf{map}^{\mathcal{F}}_{\phi}\,\eta = \eta$

Note that there are no computation rules for types because types are always fully applied in our syntax.

Show $\llbracket \Gamma; \emptyset \vdash \sigma \to \tau\rrbracket = \llbracket \Gamma; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,\sigma\,\tau\rrbracket$. Oh, this doesn't appear to hold. Unfolding the definitions, the latter appears to impose a commutativity condition $(\llbracket \Gamma \vdash \tau\rrbracket^{\mathsf{Rel}}(\mathsf{Eq}\,\rho) \circ \eta = \eta \circ \llbracket \Gamma \vdash \sigma\rrbracket^{\mathsf{Rel}}(\mathsf{Eq}\,\rho)$ that the former does not require.

Other sanity checks?

Note that our calculus does not support Church encodings of data types like pair or sum or list types because all of the "forall"s in our calculus must be at the top level. Nevertheless, our calculus does admit actual sum and product and list types because they are coded by μ-terms in our calculus. We just don't have an equivalence of these types and their Church encodings in our calculus, that's all.

5 FREE THEOREMS FOR NESTED TYPES

We can use the results of Section 4.1 to prove interesting results about nested types. To this end, let α_i have arity n_i for $i = 1, \ldots, k$, and suppose further that $\emptyset; \boldsymbol{\alpha} \vdash E : \mathcal{F}$, that $F = \lambda \boldsymbol{A}.\llbracket \emptyset; \boldsymbol{\alpha} \vdash E\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$, and that $F^* = \lambda \boldsymbol{R}.\llbracket \emptyset; \boldsymbol{\alpha} \vdash E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$.

The next proposition is the only place where we use the syntactic structure of E.

Propagate contexts?

Proposition 47. *If $(\beta_i, \gamma_i) \in \mathsf{Hom}_{\mathsf{Rel}^{n_i}}(R_i, R_i')$ for $i = 1, \ldots, k$, then $(F\boldsymbol{\beta}, F\boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}}(F^*\boldsymbol{R}, F^*\boldsymbol{R'})$.*

Proof. By induction on the structure of E.

- If $\emptyset; \boldsymbol{\alpha} \vdash E : \mathcal{T}$, then the functor F is constant in $\boldsymbol{\alpha}$. Since F therefore maps every morphism in Set to id, we need only show that $(id, id) \in \mathsf{Hom}_{\mathsf{Rel}}(F^*\boldsymbol{R}, F^*\boldsymbol{R'})$ for all $\boldsymbol{R}$ and $\boldsymbol{R'}$. But since the functor F^* is also constant in $\boldsymbol{\alpha}$, this holds trivially.
- $E = \mathbb{0}$. Similar to previous case.
- $E = \mathbb{1}$. Similar to previous case.
- $E = E_1 * E_2$. If $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A}, \boldsymbol{B})$, $\boldsymbol{R'} : \mathsf{Rel}(\boldsymbol{A'}, \boldsymbol{B'})$, $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}^{\boldsymbol{n}}}(\boldsymbol{R}, \boldsymbol{R'})$, and $(x, y) \in \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, then $x \in \llbracket \vdash E\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$ and $y \in \llbracket E\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$, so $x = (x_1, x_2)$ where $x_i \in \llbracket \emptyset; \boldsymbol{\alpha} \vdash E_i\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$ and $y = (y_1, y_2)$ where $y_i \in \llbracket E_i\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$. Therefore $(x_1, y_1) \in \llbracket \emptyset; \boldsymbol{\alpha} \vdash E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$ and $(x_2, y_2) \in \llbracket E_2\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$. Using the induction hypothesis twice we get that $(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\beta} x_1, \llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma} y_1) \in \llbracket E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$ and $(\llbracket E_2\rrbracket^{\mathsf{Set}}\boldsymbol{\beta} x_2, \llbracket E_2\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma} y_2) \in$

$[\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $(([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_1, [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_2), ([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_1, [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_2)) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \times [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $(([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta} \times [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\beta})(x_1, x_2), ([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma} \times [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\gamma})(y_1, y_2)) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \times [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $([\![E]\!]^{\mathsf{Set}}\boldsymbol{\beta}x, [\![E]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$.

- $E = E_1 + E_2$. If $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A}, \boldsymbol{B})$, $\boldsymbol{R'} : \mathsf{Rel}(\boldsymbol{A'}, \boldsymbol{B'})$, $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\boldsymbol{R}, \boldsymbol{R'})$, and $(x, y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, then $x \in [\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}] = [\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}] + [\![E_2]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$ and $y \in [\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}] = [\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}] + [\![E_2]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$. Since $(x, y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, we must have either $x = \mathsf{inl}\, x_1$ for $x_1 \in [\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$, $y = \mathsf{inl}\, y_1$ for $y_1 \in [\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$, and $(x_1, y_1) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, or $x = \mathsf{inr}\, x_2$ for $x_2 \in [\![E_2]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$, $y = \mathsf{inr}\, y_2$ for $y_2 \in [\![E_2]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$, and $(x_2, y_2) \in [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$. We prove the result for the first case; the second is analogous. By the induction hypothesis, $([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_1, [\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_1) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, so $(\mathsf{inl}\,([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_1), \mathsf{inl}\,([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_1)) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] + [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $([\![E]\!]^{\mathsf{Set}}\boldsymbol{\beta}(\mathsf{inl}\, x_1), [\![E]\!]^{\mathsf{Set}}\boldsymbol{\gamma}(\mathsf{inl}\, y_1)) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $([\![E]\!]^{\mathsf{Set}}\boldsymbol{\beta}x, [\![E]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$.
- $E = \phi^m E_1 \ldots E_m$. Suppose $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A}, \boldsymbol{B})$, $\boldsymbol{R'} : \mathsf{Rel}(\boldsymbol{A'}, \boldsymbol{B'})$, $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\boldsymbol{R}, \boldsymbol{R'})$, $R_\phi = (R^0_\phi, R^1_\phi, R^*_\phi)$, and $R'_\phi = ({R'}^0_\phi, {R'}^1_\phi, {R'}^*_\phi)$. If

$$(x, y) \in [\![\phi^m E_1 \ldots E_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}] = R^*_\phi([\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]) \ldots ([\![E_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])$$

(since $\phi \in \boldsymbol{\alpha}$), then

$$x \in R^0_\phi([\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]) \ldots ([\![E_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])$$

and

$$y \in R^1_\phi([\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]) \ldots ([\![E_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])$$

Since $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathsf{Hom}(\boldsymbol{R}, \boldsymbol{R'})$, the induction hypothesis gives that, for each $i = 1, \ldots, m$, $(w, z) \in [\![E_i]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$ implies $([\![E_i]\!]^{\mathsf{Set}}\boldsymbol{\beta}w, [\![E_i]\!]^{\mathsf{Set}}\boldsymbol{\gamma}z) \in [\![E_i]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $([\![E_i]\!]^{\mathsf{Set}}\boldsymbol{\beta}, [\![E_i]\!]^{\mathsf{Set}}\boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}}([\![E_i]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}], [\![E_i]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}])$. The remark after Definition 10 thus gives that $(R^0_\phi([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}) \ldots ([\![E_m]\!]^{\mathsf{Set}}\boldsymbol{\beta}), R^1_\phi([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}) \ldots ([\![E_m]\!]^{\mathsf{Set}}\boldsymbol{\gamma})) \in \mathsf{Hom}_{\mathsf{Rel}}(R^*_\phi([\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]) \ldots ([\![E_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]), R^*_\phi([\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \ldots ([\![E_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]))$. Then since $(x, y) \in R^*_\phi([\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]) \ldots ([\![E_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])$, we have that

$$\begin{aligned} & (R^0_\phi([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}) \ldots ([\![E_m]\!]^{\mathsf{Set}}\boldsymbol{\beta})x, R^1_\phi([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}) \ldots ([\![E_m]\!]^{\mathsf{Set}}\boldsymbol{\gamma})y) \\ \in\ & R^*_\phi([\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \ldots ([\![E_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \end{aligned} \tag{10}$$

By hypothesis, $(\beta_\phi, \gamma_\phi) : R^*_\phi \to {R'}^*_\phi$. Since β_ϕ and γ_ϕ are natural transformations, this gives that for all $\boldsymbol{S} : \mathsf{Rel}(\boldsymbol{C}, \boldsymbol{D})$, $((\beta_\phi)_{\boldsymbol{C}}, (\gamma_\phi)_{\boldsymbol{D}}) \in \mathsf{Hom}_{\mathsf{Rel}}(R_\phi{}^*\boldsymbol{S}, {R'}^*_\phi\boldsymbol{S})$. Letting $\boldsymbol{S} = ([\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]), \ldots, ([\![E_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R'}])$, $\boldsymbol{C} = ([\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}]), \ldots, ([\![E_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])$, and $\boldsymbol{D} = ([\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}]), \ldots, ([\![E_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}])$, and noting that

$$(R^0_\phi([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}) \ldots ([\![E_m]\!]^{\mathsf{Set}}\boldsymbol{\beta})x, R^1_\phi([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}) \ldots ([\![E_m]\!]^{\mathsf{Set}}\boldsymbol{\gamma})y) \in R^*_\phi\boldsymbol{S}$$

by Equation 12, our hypothesis gives that

$$\begin{aligned} & ((\beta_\phi)_{\boldsymbol{C}}(R^0_\phi([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}) \ldots ([\![E_m]\!]^{\mathsf{Set}}\boldsymbol{\beta})x), (\gamma_\phi)_{\boldsymbol{D}}(R^1_\phi([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}) \ldots ([\![E_m]\!]^{\mathsf{Set}}\boldsymbol{\gamma})y)) \\ \in\ & {R'}^*_\phi\boldsymbol{S} = {R'}^*_\phi([\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \ldots ([\![E_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) = [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \end{aligned} \tag{11}$$

Using the definition of the action of $[\![E]\!]^{\mathsf{Set}}\boldsymbol{\beta}$ on morphisms (see Diagram 1) twice — once with instantiations $\rho = \boldsymbol{A}$, $\rho' = \boldsymbol{A'}$, $f = \boldsymbol{\beta}$ and $\phi\rho = R^0_\phi$, and once with instantiations $\rho = \boldsymbol{B}$, $\rho' = \boldsymbol{B'}$, $f = \boldsymbol{\gamma}$ and $\phi\rho = R^1_\phi$ — Equation 11 is exactly $([\![E]\!]^{\mathsf{Set}}\boldsymbol{\beta}x, [\![E]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$.

- $E = (\mu\phi^m.\lambda\delta_1...\delta_m.h)T_1...T_m$. Suppose $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A},\boldsymbol{B})$, $\boldsymbol{R'} : \mathsf{Rel}(\boldsymbol{A'},\boldsymbol{B'})$, $(\boldsymbol{\beta},\boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\boldsymbol{R},\boldsymbol{R'})$, and $(x,y) \in F^*\boldsymbol{R} = [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$. If $(x,y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, then $x \in [\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$ and $y \in [\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$. Consider the relation transformers (L^0, L^1, L^*) and (G^0, G^1, G^*), where

$$\begin{array}{lcl} L^0 & = & \mu(H \mapsto \lambda\boldsymbol{X}.[\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}]) \\ L^1 & = & \mu(H \mapsto \lambda\boldsymbol{X}.[\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}]) \\ L^* & = & \mu(W \mapsto \lambda\boldsymbol{S}.[\![h]\!]^{\mathsf{Rel}}[\phi := W][\boldsymbol{\delta} := \boldsymbol{S}][\boldsymbol{\alpha} := \boldsymbol{R}]) \\ G^0 & = & \mu(H \mapsto \lambda\boldsymbol{X}.[\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}]) \\ G^1 & = & \mu(H \mapsto \lambda\boldsymbol{X}.[\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B'}]) \\ G^* & = & \mu(W \mapsto \lambda\boldsymbol{S}.[\![h]\!]^{\mathsf{Rel}}[\phi := W][\boldsymbol{\delta} := \boldsymbol{S}][\boldsymbol{\alpha} := \boldsymbol{R'}]) \end{array}$$

Then $(x,y) \in L^*([\![T_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])...([\![T_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])$, i.e., $x \in L^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])...([\![T_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{A}])$ and $y \in L^1([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}])...([\![T_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{B}])$. Lemma ?? ensures that each $i = 1, ...m$, $([\![T_i]\!]^{\mathsf{Set}}, [\![T_i]\!]^{\mathsf{Set}}, [\![T_i]\!]^{\mathsf{Rel}})$ is a relation transformer, so the induction hypothesis gives that $(w,z) \in [\![T_i]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$ implies $([\![T_i]\!]^{\mathsf{Set}}\boldsymbol{\beta}w, [\![T_i]\!]^{\mathsf{Set}}\boldsymbol{\gamma}z) \in [\![T_i]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$ for all $i = 1, ..., m$, i.e., $([\![T_i]\!]^{\mathsf{Set}}\boldsymbol{\beta}, [\![T_i]\!]^{\mathsf{Set}}\boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}}([\![T_i]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}], [\![T_i]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}])$. The remark after Definition 10 thus gives that

$$\begin{array}{ll} & (L^0([\![T_1]\!]^{\mathsf{Set}}\boldsymbol{\beta})...([\![T_m]\!]^{\mathsf{Set}}\boldsymbol{\beta}), L^1([\![T_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma})...([\![T_m]\!]^{\mathsf{Set}}\boldsymbol{\gamma})) \\ \in & \mathsf{Hom}_{\mathsf{Rel}}(L^*([\![T_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])...([\![T_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]), \\ & \qquad L^*([\![T_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}])...([\![T_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}])) \end{array}$$

Then since $(x,y) \in L^*([\![T_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])...([\![T_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])$, we have that

$$\begin{array}{ll} & (L^0([\![T_1]\!]^{\mathsf{Set}}\boldsymbol{\beta})...([\![T_m]\!]^{\mathsf{Set}}\boldsymbol{\beta})x, L^1([\![T_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma})...([\![T_m]\!]^{\mathsf{Set}}\boldsymbol{\gamma})y) \\ \in & L^*([\![T_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}])...([\![T_m]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \end{array} \tag{12}$$

Now, note that for every functor H and sequence of sets $\boldsymbol{X}$,

$$\begin{array}{lcl} \eta^0_{H,\boldsymbol{X}} & = & [\![h]\!]^{\mathsf{Set}}[\phi := id][\boldsymbol{\delta} := \boldsymbol{id}][\boldsymbol{\alpha} := \boldsymbol{\beta}] \\ & : & [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}] \to [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}] \end{array}$$

is a morphism in Set^k, so

$$\begin{array}{lcl} \eta^0 & = & (H \mapsto \lambda\boldsymbol{X}.\, [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{\beta}]) \\ & & \quad : (H \mapsto \lambda\boldsymbol{X}.\, [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}]) \\ & & \qquad \to (H \mapsto \lambda\boldsymbol{X}.\, [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}]) \end{array}$$

is a morphism (i.e., a higher-order natural transformation) between higher-order functors between functors on $\mathsf{Set}^m \to \mathsf{Set}$: indeed, for every natural transformation $f : H \to H'$ we have that

$$\begin{tikzcd} [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}] \arrow[r, "\eta^0_{H,\boldsymbol{X}}"] \arrow[d, "{[\![h]\!]^{\mathsf{Set}}[\phi:=f][\boldsymbol{\delta}:=\boldsymbol{id}_{\boldsymbol{X}}][\boldsymbol{\alpha}:=\boldsymbol{id}_{\boldsymbol{A}}]}"'] & [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}] \arrow[d, "{[\![h]\!]^{\mathsf{Set}}[\phi:=f][\boldsymbol{\delta}:=\boldsymbol{id}_{\boldsymbol{X}}][\boldsymbol{\alpha}:=\boldsymbol{id}_{\boldsymbol{A'}}]}"'] \\ [\![h]\!]^{\mathsf{Set}}[\phi := H'][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}] \arrow[r, "\eta^0_{H',\boldsymbol{X}}"] & [\![h]\!]^{\mathsf{Set}}[\phi := H'][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}] \end{tikzcd} \tag{13}$$

commutes because the vertical arrows are the $\boldsymbol{A}$ and $\boldsymbol{A'}$ components of the natural transformation $[\![h]\!]^{\mathsf{Set}}[\phi := f][\boldsymbol{\delta} := \boldsymbol{id}_{\boldsymbol{X}}][\boldsymbol{\alpha} := \boldsymbol{id}\_]$ induced by f between the functors $[\![h]\!]^{\mathsf{Set}}[\phi :=$

$H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \_]$ and $[\![h]\!]^{\mathsf{Set}}[\phi := H'][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \_]$. Similarly, if

$$\begin{array}{rcl} \eta^1_{H,\boldsymbol{X}} & = & [\![h]\!]^{\mathsf{Set}}[\phi := id][\boldsymbol{\delta} := \boldsymbol{id}][\boldsymbol{\alpha} := \boldsymbol{\gamma}] \\ & : & [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}] \to [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}'] \end{array}$$

and

$$\begin{array}{rcl} \eta^1 & = & (H \mapsto \lambda \boldsymbol{X}.\, [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{\gamma}]) \\ & & \quad : (H \mapsto \lambda \boldsymbol{X}.\, [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}]) \\ & & \quad\quad \to (H \mapsto \lambda \boldsymbol{X}.\, [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}']) \end{array}$$

then η^1 is a morphism between higher-order functors between functors on $\mathsf{Set}^m \to \mathsf{Set}$. Since μ is functorial, it has an action on morphisms, so $\mu\eta^0 : L^0 \to G^0$ and $\mu\eta^1 : L^1 \to G^1$ are well-defined. Moreover, since $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}}(\boldsymbol{R}, \boldsymbol{R}')$, the following diagram commutes:

$$\begin{tikzcd}
L^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}']) \arrow[r, "(\mu\eta^0)([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])"] \arrow[d, leftrightarrow, "L^*([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R}'])"] & G^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}']) \arrow[d, leftrightarrow, "G^*([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R}'])"'] \\
L^1([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}']) \arrow[r, "(\mu\eta^1)([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])"] & G^1([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}'])
\end{tikzcd} \tag{14}$$

Together with Equation 12, Equation 14 gives

$$\begin{array}{l} (\,(\mu\eta^0)([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])(L^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])x), \\ \;(\mu\eta^1)([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])(L^1([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])y)\,) \\ \quad \in G^*([\![T_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R}']) \\ \quad = [\![(\mu\phi.\lambda\boldsymbol{\delta}.h)\boldsymbol{T}]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'] \\ \quad = [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'] \end{array} \tag{15}$$

We also have that if ψ is a fresh type constructor variable, then

$$[\![\psi T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}][\psi := L^0] \;=\; L^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])$$

and

$$[\![\psi T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'][\psi := G^0] \;=\; G^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])$$

so that

$$\begin{array}{rl} & [\![\psi T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}][\psi := \mu\eta^0] \\ = & (\mu\eta^0)([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}']) \circ L^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]) \\ : & L^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]) \to G^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}']) \end{array} \tag{16}$$

Similarly,

$$\begin{array}{rl} & [\![\psi T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}][\psi := \mu\eta^1] \\ = & (\mu\eta^1)([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}']) \circ L^1([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]) \\ : & L^1([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]) \to G^1([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}'])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}']) \end{array} \tag{17}$$

Rewriting Equation 15 using Equations 16 and 17 gives

$$([\![\psi T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}][\psi := \mu\eta^0]x, [\![\psi T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}][\psi := \mu\eta^1]y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'] \tag{18}$$

Now we have that

$$\begin{array}{ll} & [\![\psi T_1 ... T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}][\psi := \mu\eta^0] \\ = & \mu\eta^0([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]) \\ = & \mu(H \mapsto \lambda \boldsymbol{X}.\, [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{\beta}])([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]) \\ = & [\![(\mu\phi.\lambda\boldsymbol{\delta}.h)T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}] \end{array}$$

and

$$\begin{array}{ll} & [\![\psi T_1 ... T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}][\psi := \mu\eta^1] \\ = & \mu\eta^1([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]) \\ = & \mu(H \mapsto \lambda \boldsymbol{X}.\, [\![h]\!]^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{\gamma}])([\![T_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])...([\![T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]) \\ = & [\![(\mu\phi.\lambda\boldsymbol{\delta}.h)T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}] \end{array}$$

so (18) becomes

$$([\![(\mu\phi.\lambda\boldsymbol{\delta}.h)T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]x, [\![(\mu\phi.\lambda\boldsymbol{\delta}.h)T_1...T_m]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \tag{19}$$

i.e., $([\![E]\!]^{\mathsf{Set}}\boldsymbol{\beta}x, [\![E]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$.

□

With the following standard definition, we can prove that our interpretations give rise to a Graph Lemma.

Definition 48. *If $f : A \to B$ then the relation $\langle f \rangle : \mathsf{Rel}(A, B)$ is defined by $(x, y) \in \langle f \rangle$ iff $fx = y$.*

Note that $\langle id_B \rangle = \mathsf{Eq}_B$.

Theorem 49. *If $f_i : A_i \to B_i$ for $i = 1, ..., k$ then $F^*\langle f \rangle_1 ... \langle f \rangle_k = \langle F f_1 ... f_k \rangle$.*

Proof. First observe that

$$((f_1, ..., f_k), (id_{B_1}, ..., id_{B_k})) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\boldsymbol{\langle f \rangle}, \mathbf{Eq}_{\boldsymbol{B_i}})$$

and

$$((id_{A_1}, ..., id_{A_k}), (f_1, ..., f_k)) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\mathbf{Eq}_{\boldsymbol{A_i}}, \boldsymbol{\langle f \rangle})$$

Applying Proposition 47 to each of these observations gives that

$$(F\boldsymbol{f}, F\boldsymbol{id}_{B_i}) \in \mathsf{Hom}_{\mathsf{Rel}}(F^*\boldsymbol{\langle f \rangle}, F^*\mathbf{Eq}_{\boldsymbol{B_i}}) \tag{20}$$

and

$$(F\boldsymbol{id}_{A_i}, F\boldsymbol{f}) \in \mathsf{Hom}_{\mathsf{Rel}}(F^*\mathbf{Eq}_{\boldsymbol{A_i}}, F^*\boldsymbol{\langle f \rangle}) \tag{21}$$

Expanding Equation 20 gives that if $(x, y) \in F^*\boldsymbol{\langle f \rangle}$ then $(F\boldsymbol{f}x, F\boldsymbol{id}_{B_i}y) \in F^*\mathbf{Eq}_{\boldsymbol{B_i}} = [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \mathbf{Eq}_{\boldsymbol{B_i}}] = \mathsf{Eq}_{[\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B_i}]} = \mathsf{Eq}_{F\boldsymbol{B}}$, where the penultimate equality holds by Theorem 26. That is, if $(x, y) \in F^*\boldsymbol{\langle f \rangle}$ then $(F\boldsymbol{f}x, y) \in \mathsf{Eq}_{F\boldsymbol{B}}$, i.e., if $(x, y) \in F^*\boldsymbol{\langle f \rangle}$ then $F\boldsymbol{f}x = y$, i.e., if $(x, y) \in F^*\boldsymbol{\langle f \rangle}$ then $(x, y) \in \langle F\boldsymbol{f} \rangle$. Thus $F^*\boldsymbol{\langle f \rangle} \subseteq \langle F\boldsymbol{f} \rangle$.

Similar analysis of Equation 21 gives that $\langle F\boldsymbol{f} \rangle \subseteq F^*\boldsymbol{\langle f \rangle}$. □

Inlining the definitions of F and F^* in the statement of Theorem 49 gives

$$[\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{\langle f \rangle}] = \langle [\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] \rangle \tag{22}$$

We can use Equation 22 to prove that the set interpretation of a closed term of (closed) type $\mathsf{Nat}^{\boldsymbol{\alpha}}\, F\, G$ is a natural transformation.

Theorem 50. *If* $\vdash t : \mathsf{Nat}^{\boldsymbol{\alpha}}\, F\, G$ *and* $f : \boldsymbol{A} \to \boldsymbol{B}$, *then* $[\![t]\!]^{\mathsf{Set}}_{\boldsymbol{B}} \circ [\![F]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] = [\![G]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] \circ [\![t]\!]^{\mathsf{Set}}_{\boldsymbol{A}}$.

Proof. Theorem 30 ensures that $([\![t]\!]^{\mathsf{Set}}, [\![t]\!]^{\mathsf{Set}}) \in [\![\mathsf{Nat}^{\boldsymbol{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}$, i.e., that for all $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A}, \boldsymbol{B})$, $x : F\boldsymbol{A}$, and $x' : F\boldsymbol{B}$, if $(x, x') \in [\![F]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$ then $([\![t]\!]^{\mathsf{Set}}_{\boldsymbol{A}} x, [\![t]\!]^{\mathsf{Set}}_{\boldsymbol{B}} x') \in [\![G]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$. If $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$, then taking $\boldsymbol{R} = \langle \boldsymbol{f} \rangle$ and instantiating gives that if $(x, x') \in [\![F]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \langle \boldsymbol{f} \rangle]$ then $([\![t]\!]^{\mathsf{Set}}_{\mathrm{A}} x, [\![t]\!]^{\mathsf{Set}}_{\mathrm{B}} x') \in [\![G]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \langle \boldsymbol{f} \rangle]$. By Equation 22 this is the same as the requirement that if $(x, x') \in \langle [\![F]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] \rangle$ then $([\![t]\!]^{\mathsf{Set}}_{\boldsymbol{A}} x, [\![t]\!]^{\mathsf{Set}}_{\boldsymbol{B}} x') \in \langle [\![G]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] \rangle$ i.e., that if $x' = [\![F]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]x$ then $[\![t]\!]^{\mathsf{Set}}_{\boldsymbol{B}} x' = [\![G]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]([\![t]\!]^{\mathsf{Set}}_{\boldsymbol{A}})$, i.e., that $[\![t]\!]^{\mathsf{Set}}_{\boldsymbol{B}}([\![F]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]x) = [\![G]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]([\![t]\!]^{\mathsf{Set}}_{\boldsymbol{A}} x)$ for all $x : F\boldsymbol{A}$, i.e., that $[\![t]\!]^{\mathsf{Set}}_{\boldsymbol{B}} \circ [\![F]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] = [\![G]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] \circ [\![t]\!]^{\mathsf{Set}}_{\boldsymbol{A}}$. □

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