Free Theorems for Nested Types

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1 SHORT CUT FUSION FOR ARBITRARY NESTED TYPES

Can take $\emptyset;\alpha \vdash c$ with $[\![\emptyset;\alpha \vdash c]\!]^{\mathsf{Set}}\rho = C$ for all ρ, i.e., can take c to denote a constant C. We then get a free theorem whose conclusion is $\mathit{fold}_H\, B \circ G\, \mu H\; \mathit{in}_H = G\,[\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}}\, B$.

Theorem 1. *Let* $\emptyset;\phi,\alpha \vdash F : \mathcal{F}$, *let* $\emptyset;\alpha \vdash K : \mathcal{F}$, *and let* $\phi;\emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\,F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha))$. *If we let* $H : [\mathsf{Set},\mathsf{Set}] \to [\mathsf{Set},\mathsf{Set}]$ *be defined by*

$$H\,f\,x \quad = \quad [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}[\phi := f][\alpha := x]$$

and let

$$G = [\![\phi;\emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\,F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha))]\!]^{\mathsf{Set}}$$

then we have that, for every $B \in H[\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}} \to [\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}}$,

$$\mathit{fold}_H\, B\,(G\; \mu H\; \mathit{in}_H) = G\,[\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}}\, B$$

Proof. We first note that the type of g is well-formed since $\emptyset;\phi,\alpha \vdash F : \mathcal{F}$ so our promotion theorem gives that $\phi,\alpha \vdash F : \mathcal{F}$, and $\phi;\alpha \vdash \phi\alpha : \mathcal{F}$, so that $\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha) : \mathcal{T}$ and $\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}\mathbb{1}\,(\phi\alpha) : \mathcal{T}$. Then $\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha) : \mathcal{F}$ and $\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}\mathbb{1}\,(\phi\alpha) : \mathcal{F}$ also hold, and, finally, $\phi;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\,F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha)) : \mathcal{T}$

Theorem **??** gives that, for any relation environment ρ and any $(a,b) \in [\![\phi,\alpha;\emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$\begin{array}{rcl}
(G\,(\pi_1\rho), G\,(\pi_2\rho)) & \in & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\alpha}F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha))]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to (\,\lambda A.1 \Rightarrow \lambda A.\,(\rho\phi)A\,) \\
& = & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to (1 \Rightarrow \rho\phi) \\
& = & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to \rho\phi
\end{array}$$

where $H^{\mathsf{Rel}} : [\mathsf{Rel},\mathsf{Rel}] \to [\mathsf{Rel},\mathsf{Rel}]$ is defined by $H^{\mathsf{Rel}}\,f\,x = [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Rel}}[\phi := f][\alpha := x]$. So if $(A,B) \in [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho$ then

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \quad \in \quad \rho\phi$$

Now note that

$$[\![\vdash \mathsf{fold}_F^K : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}F[\phi := K]\,K)\,(\mathsf{Nat}^{\alpha}((\mu\phi.\lambda\alpha.F)\alpha)\,K)]\!]^{\mathsf{Set}} = \mathit{fold}_H$$

and consider the instantiation

$$\begin{array}{rcl}
A & = & \mathit{in}_H : H(\mu H) \Rightarrow \mu H \\
B & : & H[\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}} \Rightarrow [\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}} \\
\rho\beta & = & \langle \mathit{fold}_H\, B\rangle \qquad \text{a graph of a natural transformation, defined in Enrico's notes}
\end{array}$$

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(Note that all the types here are well-formed.) This gives

$$\begin{array}{lcl} \pi_1\rho\beta & = & \mu H \\ \pi_2\rho\beta & = & [\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}} \\ \rho\beta & : & \mathsf{Rel}(\pi_1\rho\beta,\pi_2\rho\beta) \\ A & : & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_1\rho) \\ B & : & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_2\rho) \end{array}$$

since

$$\begin{array}{lll} A = in_H & : & H(\mu H) \Rightarrow \mu H \\ & = & [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}[\phi := \mu[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}] \Rightarrow \mu[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}} \\ & = & [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho) \Rightarrow [\![\emptyset;\phi,\alpha \vdash \phi\alpha]\!]^{\mathsf{Set}}(\pi_1\rho) \\ & = & [\![\phi;\alpha \vdash F]\!]^{\mathsf{Set}}(\pi_1\rho) \Rightarrow [\![\phi;\alpha \vdash \phi\alpha]\!]^{\mathsf{Set}}(\pi_1\rho) \qquad \text{Daniel's trick; now a theorem} \\ & = & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_1\rho) \end{array}$$

We also have

$$\begin{array}{lll} (A,B) = (in_H, B) & \in & [\![\phi;\emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \\ & = & \lambda A.[\![\phi;\alpha \vdash F]\!]^{\mathsf{Rel}}\rho[\alpha := A] \Rightarrow \lambda A.(\rho\phi)A \\ & = & \lambda A.[\![\phi;\alpha \vdash F]\!]^{\mathsf{Rel}}[\phi := \langle fold_H\, B\rangle][\alpha := A] \Rightarrow \langle fold_H\, B\rangle \\ & = & \lambda A.[\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Rel}}[\phi := \langle fold_H\, B\rangle][\alpha := A] \Rightarrow \langle fold_H\, B\rangle \qquad \text{Daniel's trick; now a the} \\ & = & [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Rel}}\langle fold_H\, B\rangle \Rightarrow \langle fold_H\, B\rangle \\ & = & \langle [\![\emptyset;\phi,\alpha \vdash F]\!]^{\mathsf{Set}}\,(fold_H\, B)\rangle \Rightarrow \langle fold_H\, B\rangle \qquad \text{Graph Lemma} \\ & = & \langle map_H\,(fold_H\, B)\rangle \Rightarrow \langle fold_H\, B\rangle \end{array}$$

since if $(x,y) \in \langle map_H\,(fold_H\, B)\rangle$, i.e., if $map_H\,(fold_H\, B)\, x = y$, then $fold_H\, B\,(in_H\, x) = B\, y = B\,(map_H\,(fold_H\, B)\, x)$ by the definition of $fold_H$ as a (indeed, the unique) morphism from in_H to B. Thus,

$$(G\,(\pi_1\rho)\, A, G\,(\pi_2\rho)\, B) \in \langle fold_H\, B\rangle$$

i.e.,

$$fold_H\, B\,(G\,(\pi_1\rho)\; in_H) = G\,(\pi_2\rho)\, B$$

Since β is the only free variable in G, this simplifies to

$$fold_H\, B\,(G\,\mu H\; in_H) = G\,[\![\emptyset;\alpha \vdash K]\!]^{\mathsf{Set}}\, B$$

□