Free Theorems for Nested Types

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1 FREE THEOREM FOR subst

We already know, as a general fact, that

$$\begin{aligned}&[\![\Gamma;\emptyset \mid x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) \circ (L_{\overline{\gamma}} z. x_{\overline{\sigma},\overline{\gamma}} z) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\\ &= [\![\Gamma;\emptyset \mid x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash (L_{\overline{\gamma}} z. x_{\overline{\tau},\overline{\gamma}} z) \circ ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}\end{aligned} \tag{1}$$

In particular, if we instantiate x with any term subst of type $\vdash \mathsf{Nat}^\alpha (\mathsf{Lam}\,(\alpha + \mathbb{1}) \times \mathsf{Lam}\,\alpha)\, \mathsf{Lam}\,\alpha$ (and thus there is a single α and no γ's) we have

$$\begin{aligned}&[\![\Gamma;\emptyset \mid y : \mathsf{Nat}^\emptyset \sigma\, \tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,\alpha})_\emptyset y) \circ (L_\emptyset z.\, \mathsf{subst}_\sigma\, z) : \mathsf{Nat}^\emptyset (\mathsf{Lam}\,(\sigma + \mathbb{1}) \times \mathsf{Lam}\,\sigma)\, \mathsf{Lam}\,\tau]\!]^{\mathsf{Set}}\\ &= [\![\Gamma;\emptyset \mid y : \mathsf{Nat}^\emptyset \sigma\, \tau \vdash (L_\emptyset z.\, \mathsf{subst}_\tau\, z) \circ ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times \mathsf{Lam}\,\alpha})_\emptyset y) : \mathsf{Nat}^\emptyset (\mathsf{Lam}\,(\sigma + \mathbb{1}) \times \mathsf{Lam}\,\sigma)\, \mathsf{Lam}\,\tau]\!]^{\mathsf{Set}}\end{aligned} \tag{2}$$

Thus, for any set environment ρ and any function $f : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^\emptyset \sigma\, \tau]\!]^{\mathsf{Set}}\rho$, we have that

$$\begin{aligned}&[\![\Gamma;\emptyset \mid y : \mathsf{Nat}^\emptyset \sigma\, \tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,\alpha})_\emptyset y) \circ (L_\emptyset z.\, \mathsf{subst}_\sigma\, z) : \mathsf{Nat}^\emptyset (\mathsf{Lam}\,(\sigma + \mathbb{1}) \times \mathsf{Lam}\,\sigma)\, \mathsf{Lam}\,\tau]\!]^{\mathsf{Set}}\rho f\\ &= [\![\Gamma;\emptyset \mid y : \mathsf{Nat}^\emptyset \sigma\, \tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,\alpha})_\emptyset y)]\!]^{\mathsf{Set}}\rho f \circ [\![\Gamma;\emptyset \mid y : \mathsf{Nat}^\emptyset \sigma\, \tau \vdash L_\emptyset z.\, \mathsf{subst}_\sigma\, z]\!]^{\mathsf{Set}}\rho f\\ &= [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{\sigma},\overline{\tau}}_{\mathsf{Lam}\,\alpha}]\!]^{\mathsf{Set}}\rho f \circ [\![\Gamma;\emptyset \mid \emptyset \vdash L_\emptyset z.\, \mathsf{subst}_\sigma\, z]\!]^{\mathsf{Set}}\rho\\ &= \mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} f \circ ([\![\vdash \mathsf{subst}]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho}\end{aligned} \tag{3}$$

and

$$\begin{aligned}&[\![\Gamma;\emptyset \mid y : \mathsf{Nat}^\emptyset \sigma\, \tau \vdash (L_\emptyset z.\, \mathsf{subst}_\tau\, z) \circ ((\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times \mathsf{Lam}\,\alpha})_\emptyset y) : \mathsf{Nat}^\emptyset (\mathsf{Lam}\,(\sigma + \mathbb{1}) \times \mathsf{Lam}\,\sigma)\, \mathsf{Lam}\,\tau]\!]^{\mathsf{Set}}\rho f\\ &= [\![\Gamma;\emptyset \mid y : \mathsf{Nat}^\emptyset \sigma\, \tau \vdash L_\emptyset z.\, \mathsf{subst}_\tau\, z]\!]^{\mathsf{Set}}\rho f \circ [\![\Gamma;\emptyset \mid y : \mathsf{Nat}^\emptyset \sigma\, \tau \vdash (\mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times \mathsf{Lam}\,\alpha})_\emptyset y]\!]^{\mathsf{Set}}\rho f\\ &= [\![\Gamma;\emptyset \mid \emptyset \vdash L_\emptyset z.\, \mathsf{subst}_\tau\, z]\!]^{\mathsf{Set}}\rho \circ [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\sigma,\tau}_{\mathsf{Lam}\,(\alpha+\mathbb{1})\times \mathsf{Lam}\,\alpha}]\!]^{\mathsf{Set}}\rho f\\ &= ([\![\vdash \mathsf{subst}]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho} \circ \mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,(\alpha+\mathbb{1})\times \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} f\\ &= ([\![\vdash \mathsf{subst}]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho} \circ (\mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} (f + \mathbb{1}) \times \mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} f)\end{aligned} \tag{4}$$

So, we can conclude that

$$\begin{aligned}&\mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} f \circ ([\![\vdash \mathsf{subst}]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho}\\ &= ([\![\vdash \mathsf{subst}]\!]^{\mathsf{Set}})_{[\![\Gamma;\emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho} \circ (\mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} (f + \mathbb{1}) \times \mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} f)\end{aligned} \tag{5}$$

Moreover, for any $A, B : \mathsf{Set}$, we can choose $\sigma = v$ and $\tau = w$ to be variables such that $\rho v = A$ and $\rho w = B$. Then for any function $f : A \longrightarrow B$ we have that

$$\begin{aligned}&\mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} f \circ ([\![\vdash \mathsf{subst}]\!]^{\mathsf{Set}})_A\\ &= ([\![\vdash \mathsf{subst}]\!]^{\mathsf{Set}})_B \circ (\mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} (f + \mathbb{1}) \times \mathsf{map}_{[\![\emptyset;\alpha \vdash \mathsf{Lam}\,\alpha]\!]^{\mathsf{Set}}[\alpha := \_]} f)\end{aligned} \tag{6}$$

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REMARK 1. *The free theorem for a type is always independent of the particular term of that type, so the above proof is independent of the choice of function* subst. *In addition, the above proof is independent of the particular data type — in this case,* Lam *— over which* subst *acts. Indeed, the result holds for any data type, and thus it is just a consequence of naturality.*