# Free Theorems for Nested Types

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## 1 FREE THEOREM FOR subst

We already know, as a general fact, that

$$\begin{aligned}&\llbracket \Gamma;\emptyset \mid x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\,G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\,\tau} \vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y}) \circ (L_{\overline{\gamma}} z. x_{\overline{\sigma},\overline{\gamma}} z) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]\rrbracket^{\mathsf{Set}} \\ &= \llbracket \Gamma;\emptyset \mid x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\,G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\,\tau} \vdash (L_{\overline{\gamma}} z. x_{\overline{\tau},\overline{\gamma}} z) \circ ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y}) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]\rrbracket^{\mathsf{Set}}\end{aligned} \tag{1}$$

In particular, if we instantiate $x$ with any term subst of type $\vdash \mathsf{Nat}^{\alpha}(\mathrm{Lan}(\alpha + \mathbb{1}) \times \mathrm{Lan}\,\alpha)\ \mathrm{Lan}\,\alpha$ (and thus there is a single $\alpha$ and no $\gamma$'s) we have

$$\begin{aligned}&\llbracket \Gamma;\emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathrm{Lan}\,\alpha})_{\emptyset} y) \circ (L_{\emptyset} z.\,\mathrm{subst}_{\sigma}\, z) : \mathsf{Nat}^{\emptyset}(\mathrm{Lan}(\sigma+\mathbb{1}) \times \mathrm{Lan}\,\sigma)\ \mathrm{Lan}\,\tau\rrbracket^{\mathsf{Set}} \\ &= \llbracket \Gamma;\emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash (L_{\emptyset} z.\,\mathrm{subst}_{\tau}\, z) \circ ((\mathsf{map}^{\sigma,\tau}_{\mathrm{Lan}(\alpha+\mathbb{1})\times\mathrm{Lan}\,\alpha})_{\emptyset} y) : \mathsf{Nat}^{\emptyset}(\mathrm{Lan}(\sigma+\mathbb{1}) \times \mathrm{Lan}\,\sigma)\ \mathrm{Lan}\,\tau\rrbracket^{\mathsf{Set}}\end{aligned} \tag{2}$$

Thus, for any set environment $\rho$ and any function $f : \llbracket \Gamma;\emptyset \vdash \sigma\rrbracket^{\mathsf{Set}}\rho \to \llbracket \Gamma;\emptyset \vdash \tau\rrbracket^{\mathsf{Set}}\rho$, we have that

$$\begin{aligned}&\llbracket \Gamma;\emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathrm{Lan}\,\alpha})_{\emptyset} y) \circ (L_{\emptyset} z.\,\mathrm{subst}_{\sigma}\, z) : \mathsf{Nat}^{\emptyset}(\mathrm{Lan}(\sigma+\mathbb{1}) \times \mathrm{Lan}\,\sigma)\ \mathrm{Lan}\,\tau\rrbracket^{\mathsf{Set}}\rho f \\ &\quad \llbracket \Gamma;\emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash ((\mathsf{map}^{\sigma,\tau}_{\mathrm{Lan}\,\alpha})_{\emptyset} y)\rrbracket^{\mathsf{Set}}\rho f \circ \llbracket \Gamma;\emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash L_{\emptyset} z.\,\mathrm{subst}_{\sigma}\, z\rrbracket^{\mathsf{Set}}\rho f \\ &\quad = \llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{\sigma},\overline{\tau}}_{\mathrm{Lan}\,\alpha}\rrbracket^{\mathsf{Set}}\rho f \circ \llbracket \Gamma;\emptyset \mid \emptyset \vdash L_{\emptyset} z.\,\mathrm{subst}_{\sigma}\, z\rrbracket^{\mathsf{Set}}\rho \\ &\quad = \mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]} f \circ (\llbracket \vdash \mathrm{subst}\rrbracket^{\mathsf{Set}})_{\llbracket \Gamma;\emptyset\vdash\sigma\rrbracket^{\mathsf{Set}}\rho}\end{aligned} \tag{3}$$

and

$$\begin{aligned}&\llbracket \Gamma;\emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash (L_{\emptyset} z.\,\mathrm{subst}_{\tau}\, z) \circ ((\mathsf{map}^{\sigma,\tau}_{\mathrm{Lan}(\alpha+\mathbb{1})\times\mathrm{Lan}\,\alpha})_{\emptyset} y) : \mathsf{Nat}^{\emptyset}(\mathrm{Lan}(\sigma+\mathbb{1}) \times \mathrm{Lan}\,\sigma)\ \mathrm{Lan}\,\tau\rrbracket^{\mathsf{Set}}\rho f \\ &= \llbracket \Gamma;\emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash L_{\emptyset} z.\,\mathrm{subst}_{\tau}\, z\rrbracket^{\mathsf{Set}}\rho f \circ \llbracket \Gamma;\emptyset \mid y : \mathsf{Nat}^{\emptyset} \sigma\,\tau \vdash (\mathsf{map}^{\sigma,\tau}_{\mathrm{Lan}(\alpha+\mathbb{1})\times\mathrm{Lan}\,\alpha})_{\emptyset} y\rrbracket^{\mathsf{Set}}\rho f \\ &= \llbracket \Gamma;\emptyset \mid \emptyset \vdash L_{\emptyset} z.\,\mathrm{subst}_{\tau}\, z\rrbracket^{\mathsf{Set}}\rho \circ \llbracket \Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\sigma,\tau}_{\mathrm{Lan}(\alpha+\mathbb{1})\times\mathrm{Lan}\,\alpha}\rrbracket^{\mathsf{Set}}\rho f \\ &= (\llbracket \vdash \mathrm{subst}\rrbracket^{\mathsf{Set}})_{\llbracket \Gamma;\emptyset\vdash\tau\rrbracket^{\mathsf{Set}}\rho} \circ \mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}(\alpha+\mathbb{1})\times\mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]} f \\ &= (\llbracket \vdash \mathrm{subst}\rrbracket^{\mathsf{Set}})_{\llbracket \Gamma;\emptyset\vdash\tau\rrbracket^{\mathsf{Set}}\rho} \circ (\mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]}(f+\mathbb{1}) \times \mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]} f)\end{aligned} \tag{4}$$

So, we can conclude that

$$\begin{aligned}&\mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]} f \circ (\llbracket \vdash \mathrm{subst}\rrbracket^{\mathsf{Set}})_{\llbracket \Gamma;\emptyset\vdash\sigma\rrbracket^{\mathsf{Set}}\rho} \\ &\quad = (\llbracket \vdash \mathrm{subst}\rrbracket^{\mathsf{Set}})_{\llbracket \Gamma;\emptyset\vdash\tau\rrbracket^{\mathsf{Set}}\rho} \circ (\mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]}(f+\mathbb{1}) \times \mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]} f)\end{aligned} \tag{5}$$

Moreover, for any $A, B : \mathsf{Set}$, we can choose $\sigma = v$ and $\tau = w$ to be variables such that $\rho v = A$ and $\rho w = B$. Then for any function $f : A \to B$ we have that

$$\begin{aligned}&\mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]} f \circ (\llbracket \vdash \mathrm{subst}\rrbracket^{\mathsf{Set}})_{A} \\ &\qquad = (\llbracket \vdash \mathrm{subst}\rrbracket^{\mathsf{Set}})_{B} \circ (\mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]}(f+\mathbb{1}) \times \mathsf{map}_{\llbracket \emptyset;\alpha\vdash \mathrm{Lan}\,\alpha\rrbracket^{\mathsf{Set}}[\alpha:=\_]} f)\end{aligned} \tag{6}$$

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Remark 1. *The above discussion does not depend on the definition of type* Lan. *Indeed the result holds for any type, and thus it is just a consequence of naturality.*