1 ZIP FUNCTION FOR BUSH DATATYPE

We can define a zip function for bushes with the following type

$$\vdash \mathsf{Nat}^{\beta}\,(B\,\beta)\,(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta)))$$

as a fold in our calculus. The unapplied fold will be formed as

$$\begin{aligned}\vdash \mathsf{fold}^{\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta))}_{\mathbb{1}+\alpha\times\phi(\phi\alpha)} : \mathsf{Nat}^{\emptyset}(&\mathsf{Nat}^{\beta}(\mathbb{1}+\beta\times(\mathsf{Nat}^{\delta}B\,\delta\,(B\,((\mathsf{Nat}^{\delta'}B\,\delta'\,B\,(\beta\times\delta'))\times\delta))))\,(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta))))\\ &(\mathsf{Nat}^{\beta}B\,\beta\;(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta))))\end{aligned}$$

and the algebra components to which we will apply the fold have the following types:

$$\vdash \mathit{nilHelp} : \mathsf{Nat}^{\beta}\mathbb{1}\,(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta)))$$

$$\vdash \mathit{consHelp} : \mathsf{Nat}^{\beta}(\beta\times(\mathsf{Nat}^{\delta}B\,\delta\,(B\,((\mathsf{Nat}^{\delta'}B\,\delta'\,B\,(\beta\times\delta'))\times\delta))))\,(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta)))$$

We first define *nilHelp* as a constant function which always returns an empty bush:

$$\vdash L_{\beta}\,*\,.L_{\delta}\,b.\mathit{bnil}_{\beta\times\delta}* : \mathsf{Nat}^{\beta}\mathbb{1}\,(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta)))$$

consHelp has a more involved definition. It takes two arguments (an element of β, b, and a natural transformation over bushes, r) and returns a natural transformation from $B\,\delta$ to $B\,(\beta\times\delta)$. Since the return type is a natural transformation, we can see *consHelp* as taking three arguments and returning an element of $B\,(\beta\times\delta)$.

We define *consHelp* by pattern matching on the third argument. If this bush is empty, we return an empty bush. Otherwise we pair the head element of the third argument with the first argument and combine this pair with the recursively processed tail:

$$\begin{aligned}\vdash L_{\beta}\,(b,r).L_{\delta}\,bd.\,&\mathit{case}\,((\mathsf{in}^{-1}_{\mathbb{1}+\alpha\times\phi(\phi(\alpha))})_{\delta}\,bd)\,\mathsf{of}\\ &\{*\mapsto \mathit{bnil}_{\beta\times\delta}\,*;\\ &(d,bbd)\mapsto \mathit{bcons}_{\beta\times\delta}((b,d),(\mathsf{map}^{(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta)))\times B\,\delta,B\,(\beta\times\delta)}_{B\,\alpha}\;\mathit{app})_{\beta,\delta}(r_{(B\,\delta)}\,bbd))\}\end{aligned}$$

where $\mathit{app} : \mathsf{Nat}^{\beta,\delta}((\mathsf{Nat}^{\delta'}B\,\delta'\,B\,(\beta\times\delta'))\times B\,\delta)\,(B\,(\beta\times\delta))$ is the following application function:

$$\vdash L_{\beta,\delta}\,(\eta,bd).\eta_{\delta}\,bd$$

We can now define the zip function for bushes as

$$\vdash (\mathsf{fold}^{\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta))}_{\mathbb{1}+\alpha\times\phi(\phi\alpha)})_{\emptyset}\,\mathit{alg} : (\mathsf{Nat}^{\beta}(B\,\beta)\;(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta))))$$

where $\mathit{alg} : \mathsf{Nat}^{\beta}(\mathbb{1}+\beta\times(\mathsf{Nat}^{\delta}B\,\delta\,(B\,((\mathsf{Nat}^{\delta'}B\,\delta'\,B\,(\beta\times\delta'))\times\delta))))\,(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta)))$ is the sum of the algebra components:

$$\vdash L_{\beta}\,s.\mathsf{case}\,(s)\,\mathsf{of}\,\{*\mapsto \mathit{nilHelp}_{\beta}\,*;\;(b,r)\mapsto \mathit{consHelp}_{\beta}\,(b,r)\}$$

2 TYPE CHECK

To see that *consHelp* is well-typed, first observe that $(r_{(B\,\delta)}\; bbd)$ has type $\emptyset;\beta,\delta \vdash B((\mathsf{Nat}^{\delta'} B\,\delta'\, B(\beta\times\delta'))\times B\,\delta))$. Using the map rule, we can derive

$$\frac{\emptyset;\alpha,\beta,\delta \vdash B\,\alpha \qquad \emptyset;\beta,\delta \vdash (\mathsf{Nat}^{\delta'}(B\,\delta')\,(B\,(\beta\times\delta')))\times B\,\delta \qquad \emptyset;\beta,\delta \vdash B(\beta\times\delta) \qquad \vdash app : \mathsf{Nat}^{\beta,\delta}((\mathsf{Nat}}{\vdash \mathsf{map}_{B\,\alpha}^{\cdot\,\cdot\,\cdot}\; app : \mathsf{Nat}^{\beta,\delta}(B\,(\mathsf{Nat}^{\delta'}(B\,\delta')\,(B\,(\beta\times\delta'))\times B\,\delta))(B\,(B\,(\beta\times\delta)))}$$

and we can apply this result to $(r_{(B\,\delta)} bbd)$ to get (omitting the type of r for space)

$$\emptyset;\beta,\delta\,|\,b:\beta, r:-, bd: B\,\delta, d:\delta, bbd: B\,(B\,\delta) \vdash (\mathsf{map}_{B\,\alpha}^{\cdot\,\cdot\,\cdot}\; app)_{\beta,\delta}\,(r_{(B\,\delta)}\; bbd) : B\,(B\,(\beta\times\delta))$$

Applying *bcons* to this last result gives

$$\emptyset;\beta,\delta\,|\,b:\beta, r:-, bd: B\,\delta, d:\delta, bbd: B\,(B\,\delta) \vdash bcons_{\beta\times\delta}\,((b,d),(\mathsf{map}_{B\,\alpha}^{\cdot\,\cdot\,\cdot}\; app)_{\beta,\delta}\,(r_{(B\,\delta)}\; bbd)) : B\,(\beta\times\delta)$$

We can now show the case term has the correct type. Let R stand for the last term above, let $\Delta = b:\beta, r:-, bd: B\,\delta$ and let $bd' = (\mathsf{in}^{-1}_{\mathbb{1}+\alpha\times\phi(\phi(\alpha))})_\delta\; bd$

$$\frac{\emptyset;\beta,\delta\,|\,\Delta, *:\mathbb{1} \vdash bnil_{\beta\times\delta}\, * : B\,(\beta\times\delta) \qquad \emptyset;\beta,\delta\,|\,\Delta, d:\delta, bbd: B\,(B\,\delta) \vdash R : B\,(\beta\times\delta) \qquad \emptyset;\beta,\delta\,|\,\Delta \vdash bd' :}{\emptyset;\beta,\delta\,|\,\Delta \vdash \mathsf{case}\; bd'\; \mathsf{of}\;\{* \mapsto bnil_{\beta\times\delta}\, *;\ (d, bbd)\mapsto R\} : B\,(\beta\times\delta)}$$

Let C stand for the entire *case* term above. Now we can derive

$$\frac{\emptyset;\delta \vdash B\,\delta \qquad \emptyset;\beta,\delta \vdash B\,(\beta\times\delta) \qquad \emptyset;\beta,\delta\,|\,b:\beta, r:-,\ bd: B\,\delta \vdash C : B\,(\beta\times\delta)}{\emptyset;\beta\,|\,b:\beta, r:- \vdash L_\delta\; bd.C : \mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta))}$$

and finally, if $F = \beta\times(\mathsf{Nat}^{\delta} B\,\delta\,(B\,((\mathsf{Nat}^{\delta'} B\,\delta'\, B\,(\beta\times\delta'))\times\delta)))$

$$\frac{\emptyset;\beta \vdash F \qquad \emptyset;\beta \vdash \mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta)) \qquad \emptyset;\beta\,|\,b:\beta, r:- \vdash L_\delta\; bd.C : \mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta))}{\vdash L_\beta\,(b,r).\,L_\delta\; bd.C : \mathsf{Nat}^{\beta} F\,(\mathsf{Nat}^{\delta}(B\,\delta)\,(B\,(\beta\times\delta)))}$$