Free Theorems for Nested Types

ANONYMOUS AUTHOR(S)

Let $List\ \alpha = (\mu\phi.\lambda\beta.\mathbb{1} + \beta \times \phi\beta)\alpha$, and let $map = \mathsf{map}_{\lambda A.[\![\emptyset;\alpha \vdash List\ \alpha]\!]^{\mathsf{Set}}\rho[\alpha:=A]}$.

Lemma 1. *If $g : A \to B$, $\rho : \mathsf{RelEnv}$, and $\rho\alpha = (A, B, \langle g\rangle)$, then $[\![\alpha; \emptyset \vdash List\ \alpha]\!]^{\mathsf{Rel}}\rho = \langle map\, g\rangle$*

Proof.

$$
\begin{aligned}
&[\![\alpha; \emptyset \vdash List\ \alpha]\!]^{\mathsf{Rel}}\rho \\
&= \mu T_\rho([\![\alpha; \emptyset \vdash \alpha]\!]^{\mathsf{Rel}}\rho) \\
&= \mu T_\rho(A, B\langle g\rangle) \\
&= (\mu T_{\pi_1\rho}A, \mu T_{\pi_2\rho}B, \varinjlim_{n\in\mathbb{N}}(T^n_\rho K_0)^*(A, B, \langle g\rangle)) \\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \varinjlim_{n\in\mathbb{N}}\Sigma^n_{i=0}(A, B, \langle g\rangle)^i) \\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \mathsf{List}\,(A, B, \langle g\rangle)) \\
&= (\mathsf{List}\ A, \mathsf{List}\ B, \langle map\, g\rangle)
\end{aligned}
$$

The first equality is by Definition ??, the third equality is by Equation ??, and the fourth and sixth equalities are by Equations 1 and 2 below.
The following sequence of equalities shows

$$(T^n_\rho K_0)^*\, R = \Sigma^n_{i=0} R^i \tag{1}$$

by induction on n:

$$
\begin{aligned}
&(T^n_\rho K_0)^*\, R \\
&= T^{\mathsf{Rel}}_\rho (T^{n-1}_\rho K_0)^*\, R \\
&= [\![\alpha; \phi, \beta \vdash \mathbb{1} + \beta \times \phi\beta]\!]^{\mathsf{Set}}\rho[\phi := (T^{n-1}_\rho K_0)^*][\beta := R] \\
&= \mathbb{1} + R \times (T^{n-1}_\rho K_0)^*\, R \\
&= \mathbb{1} + R \times (\Sigma^{n-1}_{i=0} R^i) \\
&= \Sigma^n_{i=0} R^i
\end{aligned}
$$

The following reasoning shows

$$\mathsf{List}\,(A, B, \langle g\rangle) = \langle map\, g\rangle \tag{2}$$

By showing that $(xs, xs') \in \mathsf{List}\,(A, B, \langle g\rangle)$ if and only if $(xs, xs') \in \langle map\, g\rangle$:

$$
\begin{aligned}
&(xs, xs') \in \mathsf{List}\,(A, B, \langle g\rangle) \\
\iff\ & \forall i.(xs_i, xs'_i) \in \langle g\rangle \\
\iff\ & \forall i.xs'_i = g(xs_i) \\
\iff\ & xs' = map\, g\, xs \\
\iff\ & (xs, xs') \in \langle map\, g\rangle
\end{aligned}
$$

□

2020. 2475-1421/2020/1-ART1 $15.00
https://doi.org/

Theorem 2. *If* $\Gamma;\Phi \mid \Delta \vdash t : \tau$ *and* $\rho \in \mathsf{RelEnv}$, *and if* $(a,b) \in [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, *then* $([\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_1\rho)\, a\,, [\![\Gamma;\Phi \mid \Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_2\rho)\, b) \in [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho$

Proof. Immediate from Theorem **??** (at-gen). ◻

Theorem 3. *If* $g : A \to B$, $\rho : \mathsf{RelEnv}$, $\rho\alpha = (A, B, \langle g \rangle)$, $(a,b) \in [\![\alpha;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, $(s \circ g, s) \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho$, *and, for some well-formed term filter,*

$$t = [\![\alpha;\emptyset \mid \Delta \vdash filter : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\emptyset}\, \alpha\, Bool)(\mathsf{Nat}^{\emptyset}\, List\, \alpha\, List\, \alpha)]\!]^{\mathsf{Set}}, \text{ then}$$

$$map\, g \circ t(\pi_1\rho)\, a\, (s \circ g) = t(\pi_2\rho)\, b\, s \circ map\, g$$

Proof. By Theorem 2, $(t(\pi_1\rho)a, t(\pi_2\rho)b) \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\emptyset}\, \alpha\, Bool)(\mathsf{Nat}^{\emptyset}\, List\, \alpha\, List\, \alpha)]\!]^{\mathsf{Rel}}\rho$. Thus if $(s,s') \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\, Bool]\!]^{\mathsf{Rel}}\rho = \rho\alpha \to \mathsf{Eq}_{Bool}$, then

$$\begin{aligned}(t(\pi_1\rho)\, a\, p, t(\pi_2\rho)\, b\, p) &\in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset} List\, \alpha\, List\, \alpha]\!]^{\mathsf{Rel}}\rho \\ &= [\![\alpha;\emptyset \vdash List\, \alpha]\!]^{\mathsf{Rel}}\rho \to [\![\alpha;\emptyset \vdash List\, \alpha]\!]^{\mathsf{Rel}}\rho\end{aligned}$$

So if $(xs, xs') \in [\![\alpha;\emptyset \vdash List\, \alpha]\!]^{\mathsf{Rel}}\rho$ then,

$$(t(\pi_1\rho)\, a\, s\, xs, t(\pi_2\rho)\, b\, s'\, xs') \in [\![\alpha;\emptyset \vdash List\, \alpha]\!]^{\mathsf{Rel}}\rho \tag{3}$$

Consider the case in which $\rho\alpha = (A, B, \langle g \rangle)$. Then $[\![\alpha;\emptyset \vdash List\, \alpha]\!]^{\mathsf{Rel}}\rho = \langle map\, g \rangle$, by Lemma 1, and $(xs, xs') \in \langle map\, g \rangle$ implies $xs' = map\, g\, xs$. We also have that $(s,s') \in \langle g \rangle \to \mathsf{Eq}_{Bool}$ implies $\forall (x, gx) \in \langle g \rangle.\ sx = s'(gx)$ and thus $s = s' \circ g$ due to the definition of morphisms between relations. With these instantiations, Equation 3 becomes

$$\begin{aligned}&(t(\pi_1\rho)\, a\, (s' \circ g)\, xs, t(\pi_2\rho)\, b\, s'\, (map\, g\, xs)) \in \langle map\, g \rangle, \\ &i.e., \\ &map\, g\, (t(\pi_1\rho)\, a\, (s' \circ g)\, xs) = t(\pi_2\rho)\, b\, s'\, (map\, g\, xs), \\ &i.e., \\ &map\, g \circ t(\pi_1\rho)\, a\, (s' \circ g) = t(\pi_2\rho)\, b\, s' \circ map\, g\end{aligned}$$

as desired.

◻