Supplementary Material

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Theorem (Identity Extension Lemma). If ρ is a set environment, and $\Gamma;\Phi \vdash F$, then $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}$.

Proof. By induction on F.

- $[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = 0_{\mathsf{Rel}} = \mathsf{Eq}_{0_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\mathbb{0}]\!]^{\mathsf{Set}}\rho}$
- $[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = 1_{\mathsf{Rel}} = \mathsf{Eq}_{1_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\mathbb{1}]\!]^{\mathsf{Set}}\rho}$
- By definition, $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho$ is the relation on $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$ relating t and t' if, for all $R_1 : \mathsf{Rel}(A_1,B_1), \ldots, R_k : \mathsf{Rel}(A_k,B_k)$, $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel. To prove that this is equal to $\mathsf{Eq}_{[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho}$ we need to show that $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1), \ldots, R_k : \mathsf{Rel}(A_k,B_k)$ if and only if $t = t'$ and $(t_{\overline{A}}, t_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1), \ldots, R_k : \mathsf{Rel}(A_k,B_k)$. The only interesting part of this equivalence is to show that if $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1)$, $\ldots, R_k : \mathsf{Rel}(A_k,B_k)$, then $t = t'$. By hypothesis, $(t_{\overline{A}}, t'_{\overline{A}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := \mathsf{Eq}_A]}$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := \mathsf{Eq}_A]}$ in Rel for all $A_1 \ldots A_k : \mathsf{Set}$. By the induction hypothesis, it is therefore a morphism from $\mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$ to $\mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$ in Rel. This means that, for every $x : \mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$, $t_{\overline{A}}x = t'_{\overline{A}}x$. Then, by extensionality, $t = t'$.
- The application case is proved by the following sequence of equalities, where the second equality is by the induction hypothesis and the definition of the relation environment Eq_ρ, the third is by the definition of application of relation transformers from Definition 9, and the fourth is by Lemma 21:

$$\begin{aligned}
[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mathsf{Eq}_\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \mathsf{Eq}_{\rho\phi}\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= (\mathsf{Eq}_{\rho\phi})^*\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{(\rho\phi)\,\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\phi\overline{F}]\!]^{\mathsf{Set}}\rho}
\end{aligned}$$

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- The fixpoint case is proven by the sequence of equalities

$$
\begin{aligned}
[\![\Gamma;\Phi,\overline{\gamma}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{F}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mu T_{H,\mathsf{Eq}_\rho})\,\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{H,\mathsf{Eq}_\rho}K_0\,\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{H,\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0})^*\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0\,\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{\varinjlim_{n\in\mathbb{N}}(T^{\mathsf{Set}}_{H,\rho})^n K_0\,\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi,\overline{\gamma}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{F}]\!]^{\mathsf{Set}}\rho}
\end{aligned}
$$

Here, the third equality is by induction hypothesis, the fifth is by Lemma 21 and the fourth equality is because, for every $n \in \mathbb{N}$, the following two statements can be proved by simultaneous induction: and for any H, ρ, A, and subformula J of H,

$$
T^n_{H,\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_A} = (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0})^*\,\overline{\mathsf{Eq}_A} \tag{1}
$$

and

$$
\begin{aligned}
&[\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash J]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}\\
&\qquad = [\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash J]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}
\end{aligned} \tag{2}
$$

(Notice that we don't know what's in the context Φ'.) We prove (1) by induction on n. The case $n = 0$ is trivial, because $T^0_{H,\mathsf{Eq}_\rho}K_0 = K_0$ and $(T^{\mathsf{Set}}_{H,\rho})^0 K_0 = K_0$; the inductive step is proved by the following sequence of equalities:

$$
\begin{aligned}
T^{n+1}_{H,\mathsf{Eq}_\rho}K_0\,\overline{\mathsf{Eq}_A} &= T^{\mathsf{Rel}}_{H,\mathsf{Eq}_\rho}(T^n_{H,\mathsf{Eq}_\rho}K_0)\overline{\mathsf{Eq}_A}\\
&= [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}\\
&= [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}\\
&= [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}\\
&= \mathsf{Eq}_{[\![\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}\rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}\\
&= \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^{n+1} K_0\overline{A}}\\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^{n+1} K_0})^*\,\overline{\mathsf{Eq}_A}
\end{aligned}
$$

Here, the third equality is by (2) for $J = H$, the fifth by the induction hypothesis of the IEL on H, and the last is by Lemma 21.

We prove (2) by structural induction on J. The only interesting cases, though, are when $J = \phi\overline{G}$ and when $J = (\mu\psi.\lambda\overline{\beta}.G)\overline{K}$.

– The case $J = \phi\overline{G}$ is proved by the sequence of equalities:

$$\begin{aligned}
&[\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash \phi\overline{G}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}\\
&= T^n_{H,\mathsf{Eq}_\rho}K_0\;\overline{[\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}}\\
&= T^n_{H,\mathsf{Eq}_\rho}K_0\;\overline{[\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}}\\
&= T^n_{H,\mathsf{Eq}_\rho}K_0\;\overline{[\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}}\\
&= T^n_{H,\mathsf{Eq}_\rho}K_0\;\overline{\mathsf{Eq}_{[\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}}\\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0})^*\;\overline{\mathsf{Eq}_{[\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}}\\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0})^*\overline{[\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}}\\
&= [\![\Gamma;\Phi',\phi,\overline{\alpha}\vdash \phi\overline{G}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}
\end{aligned}$$

Here, the second equality is by the induction hypothesis for (2) on the Gs, the fourth is by the induction hypothesis for the IEL on the Gs, and the fifth is by the induction hypothesis on n for (1).

– The case $J = (\mu\psi.\lambda\overline{\beta}.G)\overline{K}$ is proved by the sequence of equalities (where $\Phi' = \Phi'',\overline{\gamma}$):

$$\begin{aligned}
&[\![\Gamma;\Phi'',\overline{\gamma},\phi,\overline{\alpha}\vdash (\mu\psi.\lambda\overline{\beta}.G)\overline{K}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}\\
&= (\mu T_{G,\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}})\;\overline{[\![\Gamma;\Phi'',\overline{\gamma},\phi,\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}}\\
&= \varinjlim_{m\in\mathbb{N}} T^m_{G,\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}}K_0\;(\overline{[\![\Gamma;\Phi'',\overline{\gamma},\phi,\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}})\\
&= \varinjlim_{m\in\mathbb{N}} T^m_{G,\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}}K_0\;(\overline{[\![\Gamma;\Phi'',\overline{\gamma},\phi,\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}})\\
&= \varinjlim_{m\in\mathbb{N}} T^m_{G,\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}}K_0\;(\overline{[\![\Gamma;\Phi'',\overline{\gamma},\phi,\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}})\\
&= (\mu T_{G,\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}})\;\overline{[\![\Gamma;\Phi'',\overline{\gamma},\phi,\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}}\\
&= [\![\Gamma;\Phi'',\overline{\gamma},\phi,\overline{\alpha}\vdash (\mu\psi.\lambda\overline{\beta}.G)\overline{K}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}
\end{aligned}$$

Here, the third equality is by the induction hypothesis for (2) on the Ks, and the fourth equality holds because we can prove that, for all $m \in \mathbb{N}$,

$$T^m_{G,\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}}K_0 \;=\; T^m_{G,\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}}K_0 \tag{3}$$

Indeed, the base case of (3) is trivial because

$$T^0_{G,\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0]\overline{[\alpha := \mathsf{Eq}_A]}}K_0 \;=\; K_0 = T^0_{G,\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}}K_0$$

and the inductive case is proved by:

$$
\begin{aligned}
& T^{m+1}_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}\, K_0 \\
&= T_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}(T^m_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}\, K_0) \\
&= T_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}(T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}][\overline{\alpha:=\mathsf{Eq}_A}]}\, K_0) \\
&= \lambda\overline{R}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha},\psi,\overline{\beta}\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha := \mathsf{Eq}_A}][\psi := T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}][\overline{\alpha:=\mathsf{Eq}_A}]}\, K_0][\overline{\beta := R}] \\
&= \lambda\overline{R}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha},\psi,\overline{\beta}\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}][\overline{\alpha := \mathsf{Eq}_A}][\psi := T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}][\overline{\alpha:=\mathsf{Eq}_A}]}\, K_0][\overline{\beta := R}] \\
&= T_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}][\overline{\alpha:=\mathsf{Eq}_A}]}(T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}][\overline{\alpha:=\mathsf{Eq}_A}]}\, K_0) \\
&= T^{m+1}_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}][\overline{\alpha:=\mathsf{Eq}_A}]}\, K_0
\end{aligned}
$$

Here, the second equality holds by the induction hypothesis for (3) on m. The fourth equality holds because, due to typing rule restrictions for the μ types, ϕ either does not appear in G, or must have arity 0, in which case $\overline{\alpha}$ must be empty, if ϕ appears in G, and uses (2) for G when ϕ has arity 0.

- $[\![\Gamma;\Phi\vdash F+G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho + [\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho} + \mathsf{Eq}_{[\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho+[\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F+G]\!]^{\mathsf{Set}}\rho}$
- $[\![\Gamma;\Phi\vdash F\times G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho \times [\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho} \times \mathsf{Eq}_{[\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho\times[\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F\times G]\!]^{\mathsf{Set}}\rho}$

□

Theorem (Abstraction Theorem). Every well-formed term $\Gamma;\Phi \mid \Delta \vdash t : F$ induces a natural transformation from $[\![\Gamma;\Phi\vdash\Delta]\!]$ to $[\![\Gamma;\Phi\vdash F]\!]$, i.e., a triple of natural transformations

$$([\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}},\ [\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}},\ [\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Rel}})$$

where

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}} \quad : \quad [\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}} \to [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}$$

has as its component at $\rho : \mathsf{SetEnv}$ a morphism

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}}\rho \quad : \quad [\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho$$

in Set,

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Rel}} \quad : \quad [\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}} \to [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}$$

has as its component at $\rho : \mathsf{RelEnv}$ a morphism

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Rel}}\rho \quad : \quad [\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\rho$$

in Rel, and, for all $\rho : \mathsf{RelEnv}$,

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Rel}}\rho = ([\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}}(\pi_1\rho),\ [\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}}(\pi_2\rho)) \tag{4}$$

Proof. By induction on t. The only interesting cases are the cases for abstraction, application, map, in, and fold so we omit the others.

- $\underline{\Gamma; \emptyset \mid \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G}$ To see that $[\![\Gamma; \emptyset \mid \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ we need show that, for every $\rho : \mathsf{SetEnv}$, $[\![\Gamma; \emptyset \mid \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ is a morphism in Set from $[\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$, and that such family of morphisms is natural. First, we need to show that, for all $\overline{A : \mathsf{Set}}$ and all $d : [\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho = [\![\Gamma; \overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]$, we have $([\![\Gamma; \emptyset \mid \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\, d)_{\overline{A}}$ $: [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}] \to [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]$, but this follows easily from the induction hypothesis. That these maps comprise a natural transformation $\eta : [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ is clear since $\eta_{\overline{A}} = \mathsf{curry}\,([\![\Gamma; \overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}])\, d$ is the component at $\overline{A}$ of the partial specialization to d of the natural transformation

$$[\![\Gamma; \overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$$

To see that the components of η also satisfy the additional condition needed for η to be in $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$, let $\overline{R : \mathsf{Rel}(A, B)}$ and suppose

$$\begin{array}{lcl}(u, v) & \in & [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \\ & = & ([\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}], [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := B}], ([\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}])^*)\end{array}$$

Then the induction hypothesis and $(d, d) \in [\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]$ ensure that

$$\begin{array}{ll} & (\eta_{\overline{A}} u, \eta_{\overline{B}} v) \\ = & (\mathsf{curry}\,([\![\Gamma; \overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}])\, d\, u, \mathsf{curry}\,([\![\Gamma; \overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := B}])\, d\, v) \\ = & \mathsf{curry}\,([\![\Gamma; \overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}])\,(d, d)\,(u, v) \\ : & [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]\end{array}$$

Moreover, $[\![\Gamma; \emptyset \mid \Delta \vdash L_{\overline{\alpha}} x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ is trivially natural in ρ, as the functorial action of $[\![\Gamma; \emptyset \vdash \Delta]\!]^{\mathsf{Set}}$ and $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ on morphisms is the identity.
- $\underline{\Gamma; \Phi \mid \Delta \vdash t_{\overline{K}} s : G[\overline{\alpha := K}]}$ To see that $[\![\Gamma; \Phi \mid \Delta \vdash t_{\overline{K}} s : G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma; \Phi \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma; \Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}$ we must show that, for every $\rho : \mathsf{SetEnv}$, $[\![\Gamma; \Phi \mid \Delta \vdash t_{\overline{K}} s : G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho$ is a morphism from $[\![\Gamma; \Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma; \Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho$, and that this family of morphisms is natural in ρ. Let $d : [\![\Gamma; \Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho$. Then

$$\begin{array}{ll} & [\![\Gamma; \Phi \mid \Delta \vdash t_{\overline{K}} s : G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\,\rho\, d \\ = & (\mathsf{eval} \circ \langle ([\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}}, [\![\Gamma; \Phi \mid \Delta \vdash s : F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho\rangle)\, d \\ = & \mathsf{eval}(([\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ \_)_{\overline{[\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}}\, d,\ [\![\Gamma; \Phi \mid \Delta \vdash s : F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho\, d) \\ = & \mathsf{eval}(([\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\ d)_{\overline{[\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}},\ [\![\Gamma; \Phi \mid \Delta \vdash s : F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho\, d)\end{array}$$

The induction hypothesis ensures that $([\![\Gamma; \emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\, d)_{\overline{[\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}}$ has type $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma; \Phi \vdash K]\!]^{\mathsf{Set}}\rho}] \to [\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma; \Phi \vdash K]\!]^{\mathsf{Set}}\rho}]$. Since, in addition,

$$\begin{array}{ll} & [\![\Gamma; \Phi \mid \Delta \vdash s : F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho\, d : [\![\Gamma; \Phi \vdash F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho \\ = & [\![\Gamma; \Phi, \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma; \Phi \vdash K]\!]^{\mathsf{Set}}\rho}] \\ = & [\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma; \Phi \vdash K]\!]^{\mathsf{Set}}\rho}]\end{array}$$

by Equation (6) from the paper, we have that

$$\begin{array}{ll} & [\![\Gamma; \Phi \mid \Delta \vdash t_{\overline{K}} s : G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\,\rho\, d : [\![\Gamma; \Phi, \overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma; \Phi \vdash K]\!]^{\mathsf{Set}}\rho}] \\ = & [\![\Gamma; \Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho\end{array}$$

as desired.

To see that the family of maps comprising $[\![\Gamma;\Phi\,|\,\Delta \vdash t_{\overline{K}} s : G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}$ is natural in ρ we need to show that, if $f : \rho \to \rho'$ in SetEnv, then the following diagram commutes, where $g = [\![\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}$ and $h = [\![\Gamma;\Phi\,|\,\Delta \vdash s : F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}$:

$$\begin{tikzcd}
{[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho} \arrow[r, "{[\![\Gamma;\Phi\vdash\Delta]\!]^{\mathsf{Set}} f}"] \arrow[d, "{\langle g\rho, h\rho\rangle}"'] & {[\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho'} \arrow[d, "{\langle g\rho', h\rho'\rangle}"] \\
{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \vdash F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho} \arrow[r, "{[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi\vdash F[\overline{\alpha:=K}]]\!]^{\mathsf{Set}} f}"] \arrow[d, "{\mathsf{eval}\circ((-)_{\overline{[\![\Gamma;\Phi\vdash K]\!]\rho}} \times id)}"'] & {[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho' \times [\![\Gamma;\Phi \vdash F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho'} \arrow[d, "{\mathsf{eval}\circ((-)_{\overline{[\![\Gamma;\Phi\vdash K]\!]\rho'}} \times id)}"] \\
{[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho} \arrow[r, "{[\![\Gamma;\Phi\vdash G[\overline{\alpha:=K}]]\!]^{\mathsf{Set}} f}"'] & {[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd}$$

The top diagram commutes because g and h are natural in ρ by the induction hypothesis. To see that the bottom diagram commutes, we need to show that $[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}} f (\eta_{\overline{[\![\Gamma;\Phi\vdash K]\!]\rho}} x) = ([\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}} f\,\eta)_{\overline{[\![\Gamma;\Phi\vdash K]\!]\rho'}}([\![\Gamma;\Phi \vdash F[\overline{\alpha := K}]]\!]^{\mathsf{Set}} f x)$ holds for all $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$ and $x \in [\![\Gamma;\Phi \vdash F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho$, i.e., by remembering the following facts,

$$\begin{aligned}
[\![\Gamma;\Phi \vdash F[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}] \\
[\![\Gamma;\Phi \vdash F[\overline{\alpha := K}]]\!]^{\mathsf{Set}} f &= [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}} f}] \\
[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}] \\
[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}} f &= [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}} f}]
\end{aligned}$$

we need to show that

$$\begin{aligned}
&[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}} f}] \circ \eta_{\overline{[\![\Gamma;\Phi\vdash K]\!]^{\mathsf{Set}}\rho}} \\
&\qquad\qquad = \eta_{\overline{[\![\Gamma;\Phi\vdash K]\!]^{\mathsf{Set}}\rho'}} \circ [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}} f}]
\end{aligned}$$

for all $\eta \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$. But this follows from the naturality of η, which ensures the commutativity of

$$\begin{tikzcd}
{[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}]} \arrow[r, "{\eta_{\overline{[\![\Gamma;\Phi\vdash K]\!]^{\mathsf{Set}}\rho}}}"] \arrow[d, "{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi\vdash K]\!]^{\mathsf{Set}} f}]}"'] & {[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}]} \arrow[d, "{[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi\vdash K]\!]^{\mathsf{Set}} f}]}"] \\
{[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho'[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho'}]} \arrow[r, "{\eta_{\overline{[\![\Gamma;\Phi\vdash K]\!]^{\mathsf{Set}}\rho'}}}"] & {[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho'[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho'}]}
\end{tikzcd}$$

- $\underline{\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{map}^{\overline{F},\overline{G}}_{H} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])}$ To see that

$$[\![\Gamma;\emptyset\,|\,\emptyset \vdash \mathsf{map}^{\overline{F},\overline{G}}_{H} : \mathsf{Nat}^{\emptyset}\,(\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G})\,(\mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}])]\!]^{\mathsf{Set}}\,\rho\,d\,\overline{\eta}$$

is in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\rho$ for all $\rho : \mathsf{SetEnv}$, all $\eta : \overline{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G]\!]^{\mathsf{Set}}\rho}$, and d the unique element of $[\![\Gamma;\emptyset \vdash \mathbb{0}]\!]^{\mathsf{Set}}\rho$, we first note that $[\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}}$ is a functor from SetEnv to Set and, for any $\overline{C}$, $id_{\rho[\overline{\gamma := C}]}[\overline{\phi := \lambda\overline{B}.\eta_{\overline{B}\,\overline{C}}}]$ is a morphism in SetEnv from

$$\rho[\overline{\gamma := C}][\overline{\phi := \lambda\overline{B}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\beta := B}]}]$$

to

$$\rho[\overline{\gamma := C}][\overline{\phi := \lambda\overline{B}.[\![\Gamma;\overline{\gamma},\overline{\beta} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\beta := B}]}]$$

so that $(\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])\rrbracket^{\mathsf{Set}}\, \rho\, d\, \overline{\eta})_{\overline{C}} = \llbracket \Gamma; \overline{\phi}, \overline{\gamma} \vdash H \rrbracket^{\mathsf{Set}} id_{\rho[\overline{\gamma := C}]}[\overline{\phi := \lambda \overline{B}.\eta_{\overline{B}\,\overline{C}}}]$ is indeed a morphism from $\llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := F}]\rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C}]$ to $\llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := G}]\rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C}]$. This family of morphisms is natural in $\overline{C}$: if $\overline{f : C \to C'}$ then, writing ξ for

$$\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])\rrbracket^{\mathsf{Set}}\, \rho\, d\, \overline{\eta}$$

the naturality of η, together with the fact that composition of environments is computed componentwise, ensure that the following naturality diagram for ξ commutes:

$$\begin{tikzcd}
\llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := F}]\rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C}] \arrow[r, "\xi_{\overline{C}}"] \arrow[d, "\llbracket \Gamma;\overline{\gamma} \vdash H[\overline{\phi := F}]\rrbracket^{\mathsf{Set}} id_{\rho}[\overline{\gamma := f}]"'] & \llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := G}]\rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C}] \arrow[d, "\llbracket \Gamma;\overline{\gamma} \vdash H[\overline{\phi := G}]\rrbracket^{\mathsf{Set}} id_{\rho}[\overline{\gamma := f}]"] \\
\llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := F}]\rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C'}] \arrow[r, "\xi_{\overline{C'}}"] & \llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := G}]\rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := C'}]
\end{tikzcd}$$

That, for all $\rho : \mathsf{SetEnv}$ and $d : \llbracket \Gamma; \emptyset \vdash \emptyset \rrbracket^{\mathsf{Set}} \rho$, ξ satisfies the additional condition needed for it to be in $\llbracket \Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]\rrbracket^{\mathsf{Set}} \rho$ follows from the fact that η satisfies the extra condition needed for it to be in its corresponding $\llbracket \Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G\rrbracket^{\mathsf{Set}} \rho$.
Finally, since $\Phi = \emptyset$, the naturality of

$$\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G})\, (\mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])\rrbracket^{\mathsf{Set}} \rho$$

in ρ is trivial.

- $\underline{\Gamma; \emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\beta,\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}$ To see that if $d : \llbracket \Gamma; \emptyset \vdash \emptyset \rrbracket^{\mathsf{Set}} \rho$ then $\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Set}}\, \rho\, d$ is in $\llbracket \Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Set}}\, \rho$, we first note that, for all $\overline{B}$ and $\overline{C}$, $(\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Set}}\, \rho\, d)_{\overline{B}\,\overline{C}} = (in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}})_{\overline{B}}$ maps $\llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}] = T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}\, (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\, \overline{B}$ to $\llbracket \Gamma; \overline{\beta}, \overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}] = (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\, \overline{B}$. Secondly, we observe that $\llbracket \Gamma; \emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Set}}\, \rho\, d = \lambda\overline{B}\,\overline{C}.\, (in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}})_{\overline{B}}$ is natural in $\overline{B}$ and $\overline{C}$, since naturality of *in* with respect to its functor argument and naturality of $in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}$ ensure that the following diagram commutes for all $\overline{f : B \to B'}$ and $\overline{g : C \to C'}$:

$$\begin{tikzcd}
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\,\overline{B} \arrow[r, "(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}})_{\overline{B}}"] \arrow[d, "T^{\mathsf{Set}}_{H,id_\rho[\overline{\gamma:=g}]}\,(\mu T^{\mathsf{Set}}_{H,id_\rho[\overline{\gamma:=g}]})\,\overline{B}"] & (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\,\overline{B} \arrow[d, "(\mu T^{\mathsf{Set}}_{H,id_\rho[\overline{\gamma:=g}]})\,\overline{B}"] \\
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{B} \arrow[r, "(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}})_{\overline{B}}"] \arrow[d, "T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{f}"] & (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{B} \arrow[d, "\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}\,\overline{f}"] \\
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{B'} \arrow[r, "(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}})_{\overline{B'}}"] & (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{B'}
\end{tikzcd}$$

That, for all $\rho : \mathsf{SetEnv}$ and $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d$$

satisfies the additional property needed for it to be in

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho$$

let $\overline{R : \mathsf{Rel}(B,B')}$ and $\overline{S : \mathsf{Rel}(C,C')}$ follows from the fact that

$$\begin{aligned}
&(\,([\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d)_{\overline{B},\overline{C}},\\
&\qquad([\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\, d)_{\overline{B'},\overline{C'}}\,)\\
=\;&(\,(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}})_{\overline{B}}, (in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}})_{\overline{B'}}\,)
\end{aligned}$$

has type

$$\begin{aligned}
&(\,T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\,\overline{B} \to (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\,\overline{B},\\
&\qquad T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{B'} \to (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{B'}\,)\\
=\;&[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}] \to\\
&\qquad[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}]
\end{aligned}$$

Finally, since $\Phi = \emptyset$, the naturality of

$$[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}$$

in ρ is trivial.

- $\underline{\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F)}$ Since Φ is empty, to see that $[\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\; F)]\!]^{\mathsf{Set}}$ is a natural transformation $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}$ to

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)]\!]^{\mathsf{Set}}$$

we need only show that, for all ρ : SetEnv, the unique $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$, and all $\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)]\!]^{\mathsf{Set}}\, \rho\, d\, \eta$$

has type $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F]\!]^{\mathsf{Set}}\, \rho$ i.e., for any $\overline{B}$ and $\overline{C}$,

$$([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)]\!]^{\mathsf{Set}}\, \rho\, d\, \eta)_{\overline{B}\,\overline{C}}$$

is a morphism from $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \;=\; (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\overline{B}$ to $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$. To see this, note that η is a natural transformation from

$$\begin{array}{rl} & \lambda\overline{B}\,\overline{C}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \\ = & \lambda\overline{B}\,\overline{C}.\, T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}\, (\lambda\overline{A}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C}])\,\overline{B} \end{array}$$

to

$$\begin{array}{rl} & \lambda\overline{B}\,\overline{C}.\, (\lambda\overline{A}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C}])\overline{B} \\ = & \lambda\overline{B}\,\overline{C}.\, [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \end{array}$$

and thus for each $\overline{B}$ and $\overline{C}$,

$$([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\; (\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)]\!]^{\mathsf{Set}}\, \rho\, d\, \eta)_{\overline{B}\,\overline{C}}$$

is a morphism from $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \;=\; (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\overline{B}$ to $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$.
To see that this family of morphisms is natural in $\overline{B}$ and $\overline{C}$, we observe that the following diagram commutes for all $\overline{f : B \to B'}$ and $\overline{g : C \to C'}$:

$$\begin{tikzcd}
(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\overline{B} \arrow[r, "(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C}}))_{\overline{B}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{H,id_\rho[\overline{\gamma := g}]})\overline{B}"'] & {[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\beta := B}]} \arrow[d, "{[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\gamma := g}][\overline{\beta := id_B}]}"] \\
(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]})\overline{B} \arrow[r, "(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}}))_{\overline{B}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]})\overline{f}"'] & {[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C'}][\overline{\beta := B}]} \arrow[d, "{[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\gamma := id_{C'}}][\overline{\beta := f}]}"] \\
(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]})\overline{B'} \arrow[r, "(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}}))_{\overline{B'}}"] & {[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C'}][\overline{\beta := B'}]}
\end{tikzcd}$$

Indeed, naturality of $fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}})$ ensures that the bottom diagram commutes. To see that the top one commutes we first observe that, given a natural transformation $\Theta : H \to K : [\mathsf{Set}^k, \mathsf{Set}] \to [\mathsf{Set}^k, \mathsf{Set}]$, the fixpoint natural transformation $\mu\Theta : \mu H \to \mu K : \mathsf{Set}^k \to \mathsf{Set}$ is defined to be $fold_H(\Theta\,(\mu K) \circ in_K)$, i.e., the unique morphism making the following diagram commute:

$$\begin{tikzcd} H(\mu H) \arrow[r, "H(\mu\Theta)"] \arrow[dd, "in_H"'] & H(\mu K) \arrow[d, "\Theta(\mu K)"] \\ & K(\mu K) \arrow[d, "in_K"] \\ \mu H \arrow[r, "\mu\Theta"'] & \mu K \end{tikzcd}$$

Taking $\Theta = T^{\mathsf{Set}}_{H,f} : T^{\mathsf{Set}}_{H,\rho} \to T^{\mathsf{Set}}_{H,\rho'}$ thus gives that, for any $f : \rho \to \rho'$ in SetEnv,

$$in_{T^{\mathsf{Set}}_{H,\rho'}} \circ T^{\mathsf{Set}}_{H,f}(\mu T^{\mathsf{Set}}_{H,\rho'}) \circ T^{\mathsf{Set}}_{H,\rho}(\mu T^{\mathsf{Set}}_{H,f}) \;=\; \mu T^{\mathsf{Set}}_{H,f} \circ in_{T^{\mathsf{Set}}_{H,\rho}} \tag{5}$$

Next, note that the action of the functor $\lambda\overline{B}.\lambda\overline{C}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$ on the morphisms $\overline{f : B \to B'}, \overline{g : C \to C'}$ is given by

$$\begin{aligned}
& [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}} id_\rho[\overline{\beta := f}][\overline{\gamma := g}] \\
= \; & [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_\rho[\overline{\alpha := f}][\overline{\gamma := g}][\phi := \lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}]] \\
= \; & [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := C'}][\phi := \lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]]}[\overline{\alpha := f}] \\
& \quad \circ [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\alpha := B}][\phi := \lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]]}[\overline{\gamma := g}] \\
& \quad \circ [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\alpha := B}][\overline{\gamma := C}]}[\phi := \lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}]] \\
= \; & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\overline{f} \\
& \quad \circ \big(T^{\mathsf{Set}}_{H, id_{\rho[\overline{\gamma := g}]}}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\big)_{\overline{B}} \\
& \quad \circ \big(T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])\big)_{\overline{B}}
\end{aligned}$$

So if η is a natural transformation such that $\eta_{\overline{B},\overline{C}}$ has type

$$[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \to [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$$

then, by naturality,

$$\begin{aligned}
& [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_\rho[\overline{\beta := f}][\overline{\gamma := g}] \circ \eta_{\overline{B},\overline{C}} \\
= \; & \eta_{\overline{B'},\overline{C'}} \circ T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\overline{f} \\
& \circ \big(T^{\mathsf{Set}}_{H, id_{\rho[\overline{\gamma := g}]}}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\big)_{\overline{B}} \\
& \circ \big(T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])\big)_{\overline{B}}
\end{aligned}$$

As a special case when $\overline{f = id_B}$ we have

$$\begin{aligned}
& \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \lambda\overline{B}.\eta_{\overline{B},\overline{C}} \\
= \; & \lambda\overline{B}.\eta_{\overline{B},\overline{C'}} \circ T^{\mathsf{Set}}_{H, id_{\rho[\overline{\gamma := g}]}}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}]) \\
& \qquad \circ T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])
\end{aligned} \tag{6}$$

Finally, to see that the top diagram in the diagram on page 10 commutes we first note that functoriality of $T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}$, naturality of $T^{\mathsf{Set}}_{H, id_{\rho[\overline{\gamma := g}]}}$, the universal property of $fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})$

and Equation 5 ensure that the following diagram commutes:

$$
\begin{tikzcd}
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}) \arrow[rr, "T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})\circ\mu T^{\mathsf{Set}}_{H,id_{\rho}[\overline{\gamma:=g}]})"] \arrow[d, "in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}"'] & & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "T^{\mathsf{Set}}_{H,id_{\rho}[\overline{\gamma:=g}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}])"] \\
& & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}"] \\
\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]} \arrow[r, "\mu T^{\mathsf{Set}}_{H,id_{\rho}[\overline{\gamma:=g}]}"'] & \mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]} \arrow[r, bend right, "fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C'}})"'] & \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]
\end{tikzcd}
\tag{7}
$$

Next, we note that functoriality of $T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}$, Equation 6, and the universal property of $fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})$ ensure that the following diagram commutes:

$$
\begin{tikzcd}
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}) \arrow[rr, "T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}]\circ fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}))"] \arrow[d, "in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}"'] & & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "T^{\mathsf{Set}}_{H,id_{\rho}[\overline{\gamma:=g}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}])"] \\
& & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}"] \\
\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]} \arrow[r, bend right, "fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})"'] & \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}] \arrow[r, bend right, "\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}id_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}]"'] & \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]
\end{tikzcd}
\tag{8}
$$

Combining the equations entailed by 7 and 8, we get that the top diagram in the diagram on page 10 commutes, as desired. To see that, for all $\rho : \mathsf{SetEnv}$, $d \in \llbracket\Gamma;\emptyset\vdash\emptyset\rrbracket^{\mathsf{Set}}\rho$, and $\eta : \llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F\rrbracket^{\mathsf{Set}}\rho$,

$$\llbracket\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)\rrbracket^{\mathsf{Set}}\,\rho\, d\, \eta$$

satisfies the additional condition needed for it to be in $\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F\rrbracket^{\mathsf{Set}}\rho$, let $\overline{R : \mathsf{Rel}(B,B')}$ and $\overline{S : \mathsf{Rel}(C,C')}$. Since η satisfies the additional condition needed for it to be in $\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(H[\phi := F][\overline{\alpha := \beta}])\, F\rrbracket^{\mathsf{Set}}\rho$,

$$(\,(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C}}))_{\overline{B}},\,(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A}\,\overline{C'}}))_{\overline{B'}}\,)$$

has type

$$
\begin{aligned}
& (\mu T_{H,\mathsf{Eq}_{\rho}[\overline{\gamma:=S}]})\,\overline{R} \to \llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_{\rho}[\overline{\gamma:=S}][\overline{\beta:=R}] \\
= \; & (\mu T_{H,\mathsf{Eq}_{\rho}[\overline{\gamma:=S}]})\,\overline{\llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash\beta\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_{\rho}[\overline{\gamma:=S}][\overline{\beta:=R}]} \to \llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_{\rho}[\overline{\gamma:=S}][\overline{\beta:=R}] \\
= \; & \llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_{\rho}[\overline{\gamma:=S}][\overline{\beta:=R}] \to \llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_{\rho}[\overline{\gamma:=S}][\overline{\beta:=R}] \qquad \square
\end{aligned}
$$