Practical Parametricity for GADTs

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Abstract goes here

Maybe develop our theory for *any* $\lambda \geq \omega_1$, and then specialize to ω_1 when discussing GADTs? Can we do that? It seems we really use properties of Set to get that interpretations of Nat-types are well-defined.

Locally presentable categories are the closest approximation to small complete categories. It's remarkable that local presentability is sufficient not just to give semantics to GADTs, but is also all that is needed to ensure that the model induced by that semantics is actually parametric.

We should ask what, intuitively, it means to have parametricity for a calculus with GADTs. In particular it should mean that any polymorphic function that takes a GADT as input should operate uniformly over all type instances of the GADT. That means that the GADT must itself be "parametric", i.e., uniform in its type indices. But what does this mean? For ADTs and nested types it means that the datatype elements are constructed in exactly the same ways at each type instance. For GADTs, it means that as well, but here we also have to take into account which instances are actually inhabited, since not all type instances of a GADT are inhabited. [Of course we did not have to do this for ADTs and nested types because *all* instances of ADTs and nested types are inhabited.] So, intuitively, we consider a GADT to be "parametric" if 1) the corresponding instances of the GADT are inhabited for for every instantiation of its type parameters $\overline{\alpha}$, and 2) the objects for each instantiation of the αs are constructed in exactly the same way. That is, which instances are inhabited, as well as how the data in each of those instances are constructed, is type independent. We capture this in our calculus by requiring that, for $\Gamma; \Phi \vdash (\mathsf{Lan}^{\overline{\alpha^0}}_{\overline{K}} F)\,\overline{A}$, we have that $\emptyset; \overline{\beta} \vdash K$, and that K does not involve any constants. (The latter ensures that a GADT with a constructor of type $\alpha \to Int$ is not parametric, which makes sense since there will be more ways to construct Int instances of the GADT than ways to construct its other type instances.) Further, to ensure that the categorical Lans interpreting Lan-constructs are ω-cocontinuous we also require that K is a polynomial functor in the αs (that necessarily will not involve any constant functors). This means that our constructors have types of the form $F\overline{\beta} \to F(K\overline{\beta})$ with no constants in K, so that exactly the corresponding instances of the GADT are inhabited for every instantiation of the αs, and the objects for each instantiation of the αs are constructed in exactly the same way. That is, the inhabited instances and the construction of the data in those instances are type independent.

1 THE CALCULUS

1.1 Types

For each $k \geq 0$, we assume countable sets $\mathbb{T}^k$ of *type constructor variables of arity* k and $\mathbb{F}^k$ of *functorial variables of arity* k, all mutually disjoint. The sets of all type constructor variables and functorial variables are $\mathbb{T} = \bigcup_{k\geq 0} \mathbb{T}^k$ and $\mathbb{F} = \bigcup_{k\geq 0} \mathbb{F}^k$, respectively, and a *type variable* is any element of $\mathbb{T} \cup \mathbb{F}$. We use lower case Greek letters for type variables, writing ϕ^k to indicate that $\phi \in \mathbb{T}^k \cup \mathbb{F}^k$, and omitting the arity indicator k when convenient, unimportant, or clear from context. We reserve letters from the beginning of the alphabet to denote type variables of arity 0, i.e., elements of $\mathbb{T}^0 \cup \mathbb{F}^0$. We write $\overline{\zeta}$ for either a set $\{\zeta_1, \ldots, \zeta_n\}$ of type constructor variables or a set of functorial variables when the cardinality n of the set is unimportant or clear from context. If

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P is a set of type variables we write $P, \overline{\phi}$ for $P \cup \overline{\phi}$ when $P \cap \overline{\phi} = \emptyset$. We omit the vector notation for a singleton set, thus writing ϕ, instead of $\overline{\phi}$, for $\{\phi\}$.

Definition 1. *Let V be a finite subset of $\mathbb{T}$, P be a finite subset of $\mathbb{F}$, $\overline{\alpha}$ be a finite subset of $\mathbb{F}^0$ disjoint from P, and $\phi^k \in \mathbb{F}^k \setminus P$. The set $\mathcal{F}^P(V)$ of* functorial expressions *over P and V are given by*

$$\begin{aligned}\mathcal{F}^P(V) \;::=\; & \mathbb{0} \mid \mathbb{1} \mid \mathsf{Nat}^P \mathcal{F}^P(V)\, \mathcal{F}^P(V) \mid P\, \overline{\mathcal{F}^P(V)} \mid V\, \overline{\mathcal{F}^P(V)} \mid \mathcal{F}^P(V) + \mathcal{F}^P(V) \\ & \mid \mathcal{F}^P(V) \times \mathcal{F}^P(V) \mid \left(\mu\phi^k.\lambda\alpha_1 \ldots \alpha_k.\mathcal{F}^{P,\alpha_1,\ldots,\alpha_k,\phi}(V)\right) \overline{\mathcal{F}^P(V)} \\ & \mid (\mathsf{Lan}^{\overline{\alpha}}_{\overline{\mathcal{F}^{\overline{\alpha}}}} \mathcal{F}^{P,\overline{\alpha}}) \overline{\mathcal{F}^P}\end{aligned}$$

A *type* over P and V is any element of $\mathcal{F}^P(V)$. The difference with [Johann et al. 2020] here lies solely in the incorporation of functorial expressions constructed from Lan.

The notation for types entails that an application $FF_1 \ldots F_k$ is allowed only when F is a type variable of arity k, or F is a subexpression of the form $\mu\phi^k.\lambda\alpha_1 \ldots \alpha_k.F'$ or $\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F'$. Moreover, if F has arity k then F must be applied to exactly k arguments. Accordingly, an overbar indicates a sequence of subexpressions whose length matches the arity of the type applied to it. The fact that types are always in *η-long normal form* avoids having to consider β-conversion at the level of types. In a subexpression $\mathsf{Nat}^\Phi F\, G$, the Nat operator binds all occurrences of the variables in Φ in F and G. Note that, by contrast with [Johann et al. 2020], variables of arity greater than 0 are allowed in Φ; this is necessary to construct well-typed terms of Lan types. In a subexpression $\mu\phi^k.\lambda\overline{\alpha}.F$, the μ operator binds all occurrences of the variable ϕ, and the λ operator binds all occurrences of the variables in $\overline{\alpha}$, in the body F. And in a subexpression $(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}$, the Lan operator binds all occurrences of the variables in $\overline{\alpha}$ in every element of $\overline{K}$, as well as in F.

A *type constructor context* is a finite set Γ of type constructor variables, and a *functorial context* is a finite set Φ of functorial variables. In Definition 2, a judgment of the form $\Gamma; \Phi \vdash F$ indicates that the type F is intended to be functorial in the variables in Φ but not necessarily in those in Γ.

Definition 2. *The formation rules for the set $\mathcal{F} \subseteq \bigcup_{V \subseteq \mathbb{T}, P \subseteq \mathbb{F}} \mathcal{F}^P(V)$ of* well-formed types *are*

$$\frac{}{\Gamma; \Phi \vdash \mathbb{0}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{1}}$$

$$\frac{\Gamma; \Phi \vdash F \qquad \Gamma; \Phi \vdash G}{\Gamma; \Phi \vdash F + G} \qquad \frac{\Gamma; \Phi \vdash F \qquad \Gamma; \Phi \vdash G}{\Gamma; \Phi \vdash F \times G}$$

$$\frac{\Gamma; \Phi \vdash F \qquad \Gamma; \Phi \vdash G}{\Gamma; \emptyset \vdash \mathsf{Nat}^\Phi F\, G}$$

$$\frac{\phi^k \in \Gamma \cup \Phi \qquad \overline{\Gamma; \Phi \vdash F}}{\Gamma; \Phi \vdash \phi^k \overline{F}}$$

$$\frac{\Gamma; \overline{\gamma^0}, \overline{\alpha^0}, \phi^k \vdash F \qquad \overline{\Gamma; \Phi, \overline{\gamma^0} \vdash G}}{\Gamma; \Phi, \overline{\gamma^0} \vdash (\mu\phi^k.\lambda\overline{\alpha^0}.\, F)\, \overline{G}}$$

$$\frac{\Gamma; \Phi, \overline{\alpha}^0 \vdash F \qquad \overline{\Gamma; \overline{\alpha}^0 \vdash K} \qquad \overline{\Gamma; \Phi \vdash A}}{\Gamma; \Phi \vdash (\mathsf{Lan}^{\overline{\alpha}^0}_{\overline{K}} F)\, \overline{A}}$$

In addition to textual replacement, we also have a proper substitution operation on types. If F is a type over P and V, if $P$and V contain only type variables of arity 0, and if $k = 0$ for every occurrence of ϕ^k bound by μ in F, then we say that F is *first-order*; otherwise we say that F is *second-order*. Substitution for first-order types is the usual capture-avoiding textual substitution. We write $F[\alpha := \sigma]$ for the result of substituting σ for α in F, and $F[\alpha_1 := F_1, ..., \alpha_k := F_k]$, or $F[\overline{\alpha := F}]$ when convenient, for $F[\alpha_1 := F_1][\alpha_2 := F_2, ..., \alpha_k := F_k]$. Substitution for second-order types is defined below, where we adopt a similar notational convention for vectors of types. Note that it is not correct to substitute along non-functorial variables.

Definition 3. *If $\Gamma; \Phi, \phi^k \vdash H$ and if $\Gamma; \Phi, \overline{\alpha} \vdash F$ with $|\overline{\alpha}| = k$, then $\Gamma; \Phi \vdash H[\phi :=_{\overline{\alpha}} F]$. Similarly, if $\Gamma, \phi^k; \Phi \vdash H$, and if $\Gamma; \overline{\psi}, \overline{\alpha} \vdash F$ with $|\overline{\alpha}| = k$ and $\Phi \cap \overline{\psi} = \emptyset$, then $\Gamma, \overline{\psi}'; \Phi \vdash H[\phi :=_{\overline{\alpha}} F[\overline{\psi :== \psi'}]]$. Here, the operation $(\cdot)[\phi :=_{\overline{\alpha}} F]$ of* second-order type substitution *along $\overline{\alpha}$ is defined by:*

$$
\begin{array}{lcl}
\mathbb{0}[\phi :=_{\overline{\alpha}} F] & = & \mathbb{0} \\
\mathbb{1}[\phi :=_{\overline{\alpha}} F] & = & \mathbb{1} \\
(\mathsf{Nat}^{\overline{\beta}} G\, K)[\phi :=_{\overline{\alpha}} F] & = & \mathsf{Nat}^{\overline{\beta}}\,(G[\phi :=_{\overline{\alpha}} F])\,(K[\phi :=_{\overline{\alpha}} F]) \\
(\psi \overline{G})[\phi :=_{\overline{\alpha}} F] & = & \begin{cases} \psi\, \overline{G[\phi :=_{\overline{\alpha}} F]} & \text{if } \psi \neq \phi \\ F[\overline{\alpha := G[\phi :=_{\overline{\alpha}} F]}] & \text{if } \psi = \phi \end{cases} \\
(G + K)[\phi :=_{\overline{\alpha}} F] & = & G[\phi :=_{\overline{\alpha}} F] + K[\phi :=_{\overline{\alpha}} F] \\
(G \times K)[\phi :=_{\overline{\alpha}} F] & = & G[\phi :=_{\overline{\alpha}} F] \times K[\phi :=_{\overline{\alpha}} F] \\
((\mu\psi.\lambda\overline{\beta}.\, G)\overline{K})[\phi :=_{\overline{\alpha}} F] & = & (\mu\psi.\lambda\overline{\beta}.\, G[\phi :=_{\overline{\alpha}} F])\, \overline{K[\phi :=_{\overline{\alpha}} F]} \\
((\mathsf{Lan}^{\overline{\beta}}_{\overline{H}} G)\overline{K})[\phi :=_{\overline{\alpha}} F] & = & (\mathsf{Lan}^{\overline{\beta}}_{\overline{H}} G[\phi :=_{\overline{\alpha}} F])\, \overline{K[\phi :=_{\overline{\alpha}} F]}
\end{array}
$$

We note that $(\cdot)[\phi^0 :=_{\emptyset} F]$ coincides with first-order substitution. We also omit $\overline{\alpha}$ when convenient.

1.2 Terms

We now define our term calculus. To do so we assume an infinite set $\mathcal{V}$ of term variables disjoint from $\mathbb{T}$ and $\mathbb{F}$. If Γ is a type constructor context and Φ is a functorial context, then a *term context for* Γ *and* Φ is a finite set of bindings of the form $x : F$, where $x \in \mathcal{V}$ and $\Gamma; \Phi \vdash F$. We adopt the same conventions for denoting disjoint unions and for vectors in term contexts as for type constructor contexts and functorial contexts.

Definition 4. *Let Δ be a term context for Γ and Φ. The formation rules for the set of* well-formed terms over Δ *are*

$$
\frac{\Gamma; \Phi \vdash F}{\Gamma; \Phi \mid \Delta, x : F \vdash x : F}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : \mathbb{0} \qquad \Gamma; \Phi \vdash F}{\Gamma; \Phi \mid \Delta \vdash \bot_F t : F}
\qquad
\frac{}{\Gamma; \Phi \mid \Delta \vdash \top : \mathbb{1}}
$$

$$
\frac{\Gamma; \Phi \mid \Delta \vdash s : F}{\Gamma; \Phi \mid \Delta \vdash \mathsf{inl}\, s : F + G}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : G}{\Gamma; \Phi \mid \Delta \vdash \mathsf{inr}\, t : F + G}
$$

$$
\frac{\Gamma; \Phi \vdash F, G \qquad \Gamma; \Phi \mid \Delta \vdash t : F + G \qquad \Gamma; \Phi \mid \Delta, x : F \vdash l : K \qquad \Gamma; \Phi \mid \Delta, y : G \vdash r : K}{\Gamma; \Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{x \mapsto l;\, y \mapsto r\} : K}
$$

$$
\frac{\Gamma; \Phi \mid \Delta \vdash s : F \qquad \Gamma; \Phi \mid \Delta \vdash t : G}{\Gamma; \Phi \mid \Delta \vdash (s, t) : F \times G}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : F \times G}{\Gamma; \Phi \mid \Delta \vdash \pi_1 t : F}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : F \times G}{\Gamma; \Phi \mid \Delta \vdash \pi_2 t : G}
$$

$$\frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G \qquad \Gamma;\Phi \mid \Delta, x : F \vdash t : G}{\Gamma;\emptyset \mid \Delta \vdash L_{\Phi} x.t : \mathsf{Nat}^{\Phi}\, F\, G}$$

$$\frac{\overline{\Gamma;\Phi,\overline{\beta} \vdash K} \qquad \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\psi}}\, F\, G \qquad \Gamma;\Phi \mid \Delta \vdash s : F[\overline{\psi :=_{\overline{\beta}} K}]}{\Gamma;\Phi \mid \Delta \vdash t_{\overline{K}} s : G[\overline{\psi :=_{\overline{\beta}} K}]}$$

$$\frac{\Gamma;\Phi,\overline{\phi} \vdash H \qquad \overline{\Gamma;\Phi,\overline{\beta} \vdash F} \qquad \overline{\Gamma;\Phi,\overline{\beta} \vdash G}}{\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\Phi,\overline{\beta}}\, F\, G})\, (\mathsf{Nat}^{\Phi}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}])}$$

$$\frac{\Gamma;\Phi,\phi,\overline{\alpha} \vdash H}{\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\Phi,\overline{\beta}} H[\phi :=_{\overline{\beta}} (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}$$

$$\frac{\Gamma;\phi,\Phi,\overline{\alpha} \vdash H \qquad \Gamma;\Phi,\overline{\beta} \vdash F}{\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}_H^F : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\Phi,\overline{\beta}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\Phi,\overline{\beta}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)}$$

$$\frac{\Gamma;\Phi,\overline{\alpha} \vdash F \qquad \overline{\Gamma;\overline{\alpha} \vdash K} \qquad \overline{\Gamma;\Phi \vdash A} \qquad \Gamma;\Phi \mid \Delta \vdash t : F[\overline{\alpha := A}]}{\Gamma;\Phi \mid \Delta \vdash \int_{\overline{K},F} t : (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\, \overline{K[\overline{\alpha := A}]}}$$

$$\frac{\Gamma;\emptyset \mid \Delta \vdash \eta : \mathsf{Nat}^{\Phi,\overline{\alpha}}\, F\, G[\overline{\beta := K}] \qquad \overline{\Gamma;\Phi \vdash B} \qquad \Gamma;\Phi \mid \Delta \vdash t : (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\, \overline{B}}{\Gamma;\Phi \mid \Delta \vdash \partial_F^{G,\overline{K}} \eta\, t : G[\overline{\beta := B}]}$$

In the rule for $L_{\overline{\alpha}} x.t$, the L operator binds all occurrences of the type variables in $\overline{\alpha}$ in the type of the term variable x and in the body t, as well as all occurrences of x in t. In the rule for $t_{\overline{K}} s$ there is one functorial expression in $\overline{K}$ for every functorial variable in $\overline{\alpha}$. In the rule for $\mathsf{map}_H^{\overline{F},\overline{G}}$ there is one functorial expression F and one functorial expression G for each functorial variable in $\overline{\phi}$. Moreover, for each ϕ^k in $\overline{\phi}$ the number of functorial variables in $\overline{\beta}$ in the judgments for its corresponding functorial expresssions F and G is k. In the rules for in_H and fold_H^F, the functorial variables in $\overline{\beta}$ are fresh with respect to H, and there is one β for every α. (Recall from above that, in order for the types of in_H and fold_H^F to be well-formed, the length of α must equal the arity of ϕ.) In the rule for $\int_{\overline{K},H} t$, there is one functorial expression A for every functorial variable in $\overline{\alpha}$, and in the rule for $\partial_F^{G,\overline{K}} \eta\, t$, there is one functorial expression A for every functorial expression in $\overline{K}$ (and hence for every functorial variable in $\overline{\beta}$).

Substitution for terms is the obvious extension of the usual capture-avoiding textual substitution, and Definition 4 ensures that the expected weakening rules for well-formed terms hold.

Sum and product intro and elim rules should be annotated with constituent types for consistency?

We should have a computation rule along the lines of: If $\eta : \mathsf{Nat}^{\overline{\alpha}} F\, G[\overline{\beta := K}]$ then

$$\begin{array}{ll} & (\partial_F^{G,\overline{K}} \eta)_{\overline{K[\overline{\alpha := A}]}} \circ (\int_{K,F})_{\overline{A}} \;\to\; \eta_{\overline{A}} \\ : & F[\overline{\alpha := A}] \to G[\overline{\beta := K[\overline{\alpha := A}]}] \\ = & F[\overline{\alpha := A}] \to G[\overline{\beta := K}][\overline{\alpha := A}] \end{array}$$

This will appear as a computational property of the term interpretations.

2 INTERPRETING TYPES

The fundamental idea underlying Reynolds' parametricity is to give each type $F(\alpha)$ with one free variable α both an *object interpretation* F_0 taking sets to sets and a *relational interpretation* F_1 taking relations $R : \mathsf{Rel}(A, B)$ to relations $F_1(R) : \mathsf{Rel}(F_0(A), F_0(B))$, and to interpret each term $t(\alpha, x) : F(\alpha)$ with one free term variable $x : G(\alpha)$ as a map t_0 associating to each set A a function $t_0(A) : G_0(A) \to F_0(A)$, and to each relation R a morphism $t_1(R) : G_1(R) \to F_1(R)$. These interpretations are to be given inductively on the structures of F and t in such a way that they imply two fundamental theorems. The first is an *Identity Extension Lemma*, which states that $F_1(\mathsf{Eq}_A) = \mathsf{Eq}_{F_0(A)}$, and is the essential property that makes a model relationally parametric rather than just induced by a logical relation. The second is an *Abstraction Theorem*, which states that, for any $R : \mathsf{Rel}(A, B)$, $(t_0(A), t_0(B))$ is a morphism in Rel from $(G_0(A), G_0(B), G_1(R))$ to $(F_0(A), F_0(B), F_1(R))$. The Identity Extension Lemma is similar to the Abstraction Theorem except that it holds for *all* elements of a type's interpretation, not just those that are interpretations of terms. Similar theorems are expected to hold for types and terms with any number of free variables.

NOT NEEDED ANYMORE?? To accommodate GADTs, we will need to transition Reynolds' approach from a Set-based semantics to a semantics based on ω-complete partial orders. We denote the category of ω-complete partial orders (Sets) and their monotone? sup-preserving morphisms by ωCPO. The underlying set of an Set A is denoted $|A|$. The category Rel of Set relations has as its objects triples (A, B, R), where A, B : Set; $R : \mathsf{Rel}(|A|, |B|)$; $(a, b) \in R$, $a \leq a'$, and $b \leq b'$ imply $(a', b') \in R$; so next part is redundant? and $(\bigvee_{i<\omega} a_i, \bigvee_{i<\omega} b_i) \in R$ whenever $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ are chains in A and B, respectively, such that $(a_i, b_i) \in R$ for all i. The category Rel has as its morphisms from (A, B, R) to (A', B', R') pairs $(f : A \to A', g : B \to B')$ of morphisms in Set such that $(f\,a, g\,b) \in R'$ whenever $(a, b) \in R$. We note that if $(f, g) : (A, B, R) \to (A', B', R')$ and $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ are chains in A and B, respectively, then $(f(\bigvee_{i<\omega} a_i), g(\bigvee_{i<\omega} b_i)) = (\bigvee_{i<\omega}(f\,a_i), \bigvee_{i<\omega}(g b_i)) \in R'$ necessarily holds. We write $R : \mathsf{Rel}(A, B)$ in place of (A, B, R) : Rel when convenient. If $R : \mathsf{Rel}(A, B)$ then we write $\pi_1 R$ and $\pi_2 R$ for the *domain* A of R and the *codomain* B of R, respectively. If A : Set, then we write $\mathsf{Eq}_A = (A, A, \mathsf{Eq}_{|A|})$ for the *equality relation* on A.

To adapt Reynolds' approach, we first inductively define, for each type, an object interpretation in Set and a relational interpretation in Rel. Next, we show that these interpretations satisfy both an Identity Extension Lemma (Theorem 24) and an Abstraction Theorem (Theorem **??**) appropriate to the Set setting. The key to proving our Identity Extension Lemma is a familiar "cutting down" of the interpretations of universally quantified types to include only the "parametric" elements; as in [Johann et al. 2020], the relevant types of the calculus defined above are the (now richer) Nat-types. The requisite cutting down requires that the object interpretations of our types in ωCPO are defined simultaneously with their relational interpretations in Rel. We give the object interpretations for our types in Section 2.1 and give their relational interpretations in Section 2.2. While the former are relatively straightforward, the latter are less so, mainly because of the cocontinuity conditions, adapted from the Set-based setting of [Johann et al. 2020], that must hold if they are to be well-defined. We develop these conditions in Section 2.2, which separates Definitions 8 and 18 in space, but otherwise has no impact on the fact that they are given by mutual induction.

2.1 Object Interpretations of Types

The object interpretations of the types in our calculus will be ω-cocontinuous functors between categories of ω-cocontinuous functors on categories constructed from the locally ω-presentable category Set. We therefore begin by recording some important facts about locally ω-presentable categories and functors on them, and verifying verify the properties needed to interpret our syntax.

2.1.1 Preliminaries. Perhaps have as preliminaries to entire paper.

A category is *small* if its collection of morphisms is a set. It is *locally small* if, for any two objects A and B, the collection of morphisms from A to B is a set. A *small (co)limit* in a category C is a (co)limit of a diagram $F : \mathcal{A} \to C$, where $\mathcal{A}$ is a small category. A category C is *(co)complete* if it has all small (co)limits.

A poset $\mathcal{D} = (D, \leq)$ is ω*-directed* if every countable subset of D has a supremum. When $\mathcal{D}$ is considered as a category, we write $d \in \mathcal{D}$ to indicate that d is an object of $\mathcal{D}$ (i.e., $d \in D$). An ω*-directed colimit* in a category C is a colimit of a diagram $F : \mathcal{D} \to C$, where $\mathcal{D}$ is an ω-directed poset. A category C is ω*-cocomplete* if it has all ω-directed colimits; a *cocomplete* category is one that has all colimits.

If $\mathcal{A}$ and C are ω-cocomplete, then a functor $F : \mathcal{A} \to C$ is ω*-cocontinuous* if it preserves ω-directed colimits. We write $[\mathcal{A}, C]_\omega$ for the category of ω-cocontinuous functors from $\mathcal{A}$ to C, and $C^{\mathcal{A}}$ for the category of *all* functors from $\mathcal{A}$ to C. Since (co)limits of functors are computed pointwise, $C^{\mathcal{A}}$ has all (co)limits that C has, and (co)limits of (co)continuous functors are again (co)continuous. It follows that $[\mathcal{A}, C]_{(\omega)}$ is (ω-)(co)complete whenever C is. In particular, if $F : [C, C]_{(\omega)}$ for a(n) (ω-)cocomplete category C, then we can define $F^0 = Id$, $F^{\alpha+1} = F \circ F^\alpha$ at successor ordinals, and $F^{\bigcup_{\beta<\alpha} \beta} = \varinjlim_{\beta<\alpha} F^\beta$ at limit ordinals.

If $\mathcal{A}$ is locally small, then an object A of $\mathcal{A}$ is ω*-presentable* if the functor $\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathrm{Set}$ preserves ω-directed colimits, i.e., if for every ω-directed poset $\mathcal{D}$ and every functor $F : \mathcal{D} \to \mathcal{A}$, there is a canonical isomorphism $\varinjlim_{d\in\mathcal{D}} \mathrm{Hom}_{\mathcal{A}}(A, Fd) \simeq \mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{d\in\mathcal{D}} Fd)$. A locally small category $\mathcal{A}$ is ω*-accessible* if it is ω-cocomplete and has a set $\mathcal{A}_0$ of ω-presentable objects such that every object is an ω-directed colimit of objects in $\mathcal{A}_0$; a locally small category is *locally* ω*-presentable* if it is ω-accessible and cocomplete.

The category Set is locally presentable; its presentable objects are precisely the finite sets. In the next subsection we will interpret type variables in $\mathbb{T}^k \cup \mathbb{F}^k$ as elements of $[\mathrm{Set}^k, \mathrm{Set}]_\omega$; the following special cases of standard results (see, e.g., [Adámek and Rosický 1994]) will therefore be critical to deducing important properties of our object interpretations of types:

Proposition 5. (1) *If $C_1, \ldots, C_n$ are locally ω-presentable categories then so is $C_1 \times \ldots \times C_n$. Moreover, the presentable objects of $C_1 \times \ldots \times C_n$ are exactly the tuples of the form $(P_1, \ldots, P_n)$, where, for each $i = 1, \ldots, n$, the object P_i is presentable in C_i.*
(2) *If $\mathcal{A}$ is ω-accessible and C is λ-cocomplete, then the category $[\mathcal{A}, C]_\omega$ is naturally equivalent to the category $C^{\mathcal{A}_0}$.*
(3) *If C is locally ω-presentable and $\mathcal{A}_0$ is essentially small, then $C^{\mathcal{A}_0}$ is locally ω-presentable.*

Together, the statements in Proposition 5 give that if $\mathcal{A}$ and C are locally ω-presentable, then $[\mathcal{A}, C]_\omega$ is naturally equivalent to $C^{\mathcal{A}_0}$, and thus is ω-presentable. Thus, for all $k_1, \ldots, k_n \in \mathbb{N}^n$, $[\mathcal{A}^{k_1}, C]_\omega \times \ldots \times [\mathcal{A}^{k_n}, C]_\omega$ is locally ω-presentable, and therefore $[[\mathcal{A}^{k_1}, C]_\omega \times \ldots \times [\mathcal{A}^{k_n}, C]_\omega, C]_\omega$

is as well. Taking both $\mathscr{A}$ and C to be Set — as we will to ensure that the fixpoints interpreting μ-types in Set exist — we have

Proposition 6. *For all $k_1, \ldots, k_n \in \mathbb{N}^n$,*

$$[[\mathsf{Set}^{k_1}, \mathsf{Set}]_\omega \times \ldots \times [\mathsf{Set}^{k_n}, \mathsf{Set}]_\omega, \mathsf{Set}]_\omega$$

is locally ω-presentable.

2.1.2 Object Interpretations. To define the object interpretations of the types in Definition 2 we must first interpret their variables. We have:

Definition 7. *A* Set environment *maps each type variable in $\mathbb{T}^k \cup \mathbb{F}^k$ to an element of* $[\mathsf{Set}^k, \mathsf{Set}]_\omega$. *A morphism $f : \rho \to \rho'$ for set environments ρ and ρ' with $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$ maps each type constructor variable $\psi^k \in \mathbb{T}$ to the identity natural transformation on $\rho\psi^k = \rho'\psi^k$ and each functorial variable $\phi^k \in \mathbb{F}$ to a natural transformation from the k-ary functor $\rho\phi^k$ on* Set *to the k-ary functor $\rho'\phi^k$ on* Set. *Composition of morphisms on set environments is given componentwise, with the identity morphism mapping each set environment to itself. This gives a category of set environments and morphisms between them, which we denote* SetEnv.

When convenient we identify a functor in $[\mathsf{Set}^0, \mathsf{Set}]_\omega$ with its value on $*$ and consider a Set environment to map a type variable of arity 0 to an ω-cocontinuous functor from Set^0 to Set, i.e., to an Set. If $\Phi = \{\phi_1^{k_1}, \ldots, \phi_n^{k_n}\}$ and $\overline{K} = \{K_1, \ldots, K_n\}$ where $K_i : [\mathsf{Set}^{k_i}, \mathsf{Set}]_\omega$ for $i = 1, \ldots, n$, then we write either $\rho[\overline{\Phi := K}]$ or $\rho[\overline{\phi := K}]$ for the Set environment ρ' such that $\rho'\phi_i = K_i$ for $i = 1, \ldots, n$ and $\rho'\phi = \rho\phi$ if $\phi \notin \Phi$. If ρ is an Set environment, we write Eq_ρ for the Rel environment (see Definition 16) such that $\mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v}$ for every type variable v. The categories RT_k and relational interpretations appearing in the third clause of Definition 8 are given in full in Section 2.2. We note that ω-directed colimits in SetEnv are taken pointwise.

Definition 8. *The* object interpretation $[\![\cdot]\!]^{\mathsf{Set}} : \mathcal{F} \to [\mathsf{SetEnv}, \mathsf{Set}]_\omega$ *is defined by*

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}}\rho &= 0\\
[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho &= 1\\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}\, F\, G]\!]^{\mathsf{Set}}\rho &= \{\eta : \lambda\overline{K}.\, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\Phi := K}] \Rightarrow \lambda\overline{K}.\, [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\Phi := K}]\\
&\qquad \mid \forall \overline{K = (K^1, K^2, K^*) : \mathsf{RT}_k}.\\
&\qquad\quad (\eta_{\overline{K^1}}, \eta_{\overline{K^2}}) : [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\Phi := K}] \to [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\Phi := K}]\}\\
[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}}\rho &= (\rho\phi)\,\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho}\\
[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho + [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma;\Phi \vdash F \times G]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Set}}\rho &= (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho}\\
\textit{where } T^{\mathsf{Set}}_{H,\rho}\, F &= \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\phi := F][\overline{\alpha := A}]\\
\textit{and } T^{\mathsf{Set}}_{H,\rho}\, \eta &= \lambda\overline{A}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_\rho[\phi := \eta][\overline{\alpha := id_A}]\\
[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho &= \{t : (Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]}}[\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}])\overline{[\![\Gamma;\Phi \vdash A]\!]^{\mathsf{Set}}\rho}\\
&\qquad \mid (t,t) \in [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\}
\end{aligned}
$$

If $\rho \in \mathsf{SetEnv}$ and $\vdash F$ then we write $[\![\vdash F]\!]^{\mathsf{Set}}$ instead of $[\![\vdash F]\!]^{\mathsf{Set}}\rho$ since the environment is immaterial.

For Definition 8 to be well-defined, we have to check that each object interpretation is in Set and, in particular, that each contains sups of all ω-chains. This will be proved by induction on types, and in most cases existence of sups of ω-chains will follow from the induction hypotheses. However, well-definedness needs to be proved directly for object interpretations of Nat-types. First, we have

Lemma 9. *The collection of all natural transformations*

$$\eta : \lambda\overline{K}.[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\Phi := K}] \Rightarrow \lambda\overline{K}.[\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\Phi := K}]$$

defines a set.

Proof. We first note that $\lambda\overline{K}.[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\Phi := K}]$ and $\lambda\overline{K}.[\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\Phi := K}]$ are both in $[[\mathsf{Set}^k, \mathsf{Set}]_\omega, \mathsf{Set}]_\omega$. By Proposition 6, $[[\mathsf{Set}^k, \mathsf{Set}]_\omega, \mathsf{Set}]_\omega$ is locally ω-presentable. It is therefore locally small, so there are only Set-many morphisms (i.e., natural transformations) between any two functors in $[[\mathsf{Set}^k, \mathsf{Set}]_\omega, \mathsf{Set}]_\omega$. In particular, there are only Set-many natural transformations from $\lambda\overline{K}.[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\Phi := K}]$ to $\lambda\overline{K}.[\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\Phi := K}]$. □

That each of the above interpretations is ω-cocontinuous follows from Corollary 12 of [Johann and Polonsky 2019] if we APPROPRIATELY RESTRICT THE SUBSCRIPTS OF Lans; we will require that K is a polynomial functor in the αs that does not involve any constant functors. For Nat-types, we note that $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\, G]\!]^{\mathsf{Set}}$ is an ω-cocontinuous functor because, in accordance with Definition 10, it is constant on ω-directed sets. Interpretations of Nat-types ensure that $[\![\Gamma \vdash F \to G]\!]^{\mathsf{Set}}$ and $[\![\Gamma \vdash \forall\overline{\alpha}.F]\!]^{\mathsf{Set}}$ are as expected in any parametric model.

To make sense of the next-to-last clause in Definition 8, we need to know that, for each $\rho \in \mathsf{SetEnv}$, $T^{\mathsf{Set}}_{H,\rho}$ is an ω-cocontinuous endofunctor on $[\mathsf{Set}^k, \mathsf{Set}]_\omega$, and thus admits a fixpoint. Since $T^{\mathsf{Set}}_{H,\rho}$ is defined in terms of $[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, this means that interpretations of types must be such functors, which in turn means that the actions of set interpretations of types on objects and on morphisms in SetEnv are intertwined. Fortunately, we know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash F$, $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}$ is actually in $[\mathsf{Set}^k, \mathsf{Set}]_\omega$, where $k = |\overline{\alpha}|$. Therefore, for each $[\![\Gamma;\overline{\gamma},\phi^k,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, the corresponding operator T^{Set}_H can be extended to a *functor* from SetEnv to $[[\mathsf{Set}^k, \mathsf{Set}]_\omega, [\mathsf{Set}^k, \mathsf{Set}]_\omega]_\omega$. The action of T^{Set}_H on an object $\rho \in \mathsf{SetEnv}$ is given by the higher-order functor $T^{\mathsf{Set}}_{H,\rho}$, whose actions on objects (functors in $[\mathsf{Set}^k, \mathsf{Set}]_\omega$) and on morphisms (natural transformations between such functors) are given in Definition 8. The action of T^{Set}_H on a morphism $f : \rho \to \rho'$ is the higher-order natural transformation $T^{\mathsf{Set}}_{H,f} : T^{\mathsf{Set}}_{H,\rho} \to T^{\mathsf{Set}}_{H,\rho'}$ whose action on $F : [\mathsf{Set}^k, \mathsf{Set}]_\omega$ is the natural transformation $T^{\mathsf{Set}}_{H,f}\, F : T^{\mathsf{Set}}_{H,\rho}\, F \to T^{\mathsf{Set}}_{H,\rho'}\, F$ whose component at $\overline{A}$ is $(T^{\mathsf{Set}}_{H,f}\, F)_{\overline{A}} = [\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} f[\phi := id_F][\overline{\alpha := id_A}]$.

In addition, for each $\overline{K}$, we have that $Lan_{\overline{K}}$ is itself a (higher-order) functor. Specifically, given functors $F, F' : \mathcal{C} \to \mathcal{D}$, a sequence of functors $\overline{K} = K_1, ..., K_n$ with $K_i : \mathcal{C} \to \mathcal{C}_i$ for $i = 1, ..., n$, and a natural transformation $\alpha : F \to F'$, the functorial action $Lan_{\overline{K}}\alpha : Lan_{\overline{K}}F \to Lan_{\overline{K}}F'$ of $Lan_{\overline{K}}$ on α is defined to be the unique natural transformation such that $((Lan_{\overline{K}}\alpha) \circ \langle K_1, ..., K_n\rangle) \circ \eta_F = \eta_{F'} \circ \alpha$. Here, $\eta_F : F \to (Lan_{\overline{K}}F) \circ \langle K_1, ..., K_n\rangle$ and $\eta_{F'} : F' \to (Lan_{\overline{K}}F') \circ \langle K_1, ..., K_n\rangle$ are the natural transformations associated with the functors $Lan_{\overline{K}}F$ and $Lan_{\overline{K}}F'$ from $\Pi_{i\in\{1,...,n\}}\mathcal{C}_i$ to $\mathcal{D}$, respectively. It is not hard to see that $Lan_{\overline{K}}$ is a (higher-order) functor under this definition.

The next definition uses the functors T^{Set}_H and Lan_K to define the actions of functors interpreting types on morphisms between set environments.

Definition 10. *Let $f : \rho \to \rho'$ be a morphism between* Set *environments ρ and ρ' (so that $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$). The action $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f$ of $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}$ on the morphism f is given as follows:*

- *If $\Gamma;\Phi \vdash \mathbb{0}$ then $[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}} f = id_0$*
- *If $\Gamma;\Phi \vdash \mathbb{1}$ then $[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}} f = id_1$*
- *If $\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\, G$ then $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\, G]\!]^{\mathsf{Set}} f = id_{[\![\Gamma;\emptyset\vdash \mathsf{Nat}^{\Phi} F\, G]\!]^{\mathsf{Set}}\rho}$*
- *If $\Gamma;\Phi \vdash \phi\overline{F}$ then*

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}} f : [\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}}\rho &\to [\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}}\rho' \\
&= (\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho} \to (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho'}
\end{aligned}
$$

is defined by

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Set}} f &= (f\phi)_{\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f} \\
&= (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f} \circ (f\phi)_{\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}
\end{aligned}
$$

The latter equality holds because $\rho\phi$ and $\rho'\phi$ are functors and $f\phi : \rho\phi \to \rho'\phi$ is a natural transformation, so the following naturality square commutes:

$$
\begin{tikzcd}
(\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\rho\phi)\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}} f}"'] & (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho} \arrow[d, "(\rho'\phi)\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}} f}"'] \\
(\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho'} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}}"] & (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd}
\tag{1}
$$

- *If $\Gamma;\Phi \vdash F + G$ then $[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Set}} f$ is defined by*

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Set}} f(\mathsf{inl}\, x) &= \mathsf{inl}\,([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f x) \\
[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Set}} f(\mathsf{inr}\, y) &= \mathsf{inr}\,([\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}} f y)
\end{aligned}
$$

- *If $\Gamma;\Phi \vdash F \times G$ then $[\![\Gamma;\Phi \vdash F \times G]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Set}} f$*
- *If $\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}$ then*

$$
\begin{aligned}
[\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Set}} f \quad &: \quad [\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Set}}\rho' \\
&= \quad (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho} \to (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho'}
\end{aligned}
$$

is defined by

$$
\begin{aligned}
& (\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho'}} \circ (\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} f} \\
= \quad & (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} f} \circ (\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho}}
\end{aligned}
$$

The latter equality holds because $\mu T^{\mathsf{Set}}_{H,\rho}$ and $\mu T^{\mathsf{Set}}_{H,\rho'}$ are functors and $\mu T^{\mathsf{Set}}_{H,f} : \mu T^{\mathsf{Set}}_{H,\rho} \to \mu T^{\mathsf{Set}}_{H,\rho'}$ is a natural transformation, so the following naturality square commutes:

$$
\begin{tikzcd}
(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho} \arrow[r, "(\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}} f}"'] & (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho} \arrow[d, "(\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}} f}"'] \\
(\mu T^{\mathsf{Set}}_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho'} \arrow[r, "(\mu T^{\mathsf{Set}}_{H,f})_{\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho'}}"] & (\mu T^{\mathsf{Set}}_{H,\rho'})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd}
\tag{2}
$$

- *If $\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}$ then*

$$
[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} f : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}}\rho'
$$

is defined by

$$\begin{aligned}
&\big(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]}}\,[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := id_}]\big)\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}\rho'}\\
&\circ\big(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]}}\,[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := _}]\big)\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}f}\\
=\;&\big(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]}}\,[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} \rho'[\overline{\alpha := _}]\big)\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}f}\\
&\circ\big(Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=_}]}}\,[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := id_}]\big)\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}\rho}
\end{aligned}$$

where the above equality holds by naturality of $Lan_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=\_}]}}\,[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := id\_}]$.

We need to check that the functorial action on Lan types is well-defined. That amounts to showing that the extra condition in the interpretation of Lan types is preserved by the functorial action, so that, if $t : [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}\rho$, then $[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}} f\,t : [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}\rho'$. By definition, if $t : [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}\rho$, then $(t,t)\in[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho$, i.e., there exist $\overline{Z:\mathsf{Set}_0}$, $t_1 : \overline{[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=Z}]}$, $t_2 : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha:=Z}]$, and $g : [\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha:=\mathsf{Eq}_Z}]\to[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\mathsf{Eq}$ such that $\iota_{\overline{Z},\overline{\pi_1 g}}t_1 = t$, $\kappa_{\overline{Z},\overline{\pi_2 g}}t_2 = t$, and $(t_1,t_2) : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha:=\mathsf{Eq}_Z}]$. To prove that $[\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}} f\,t : [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}\rho'$, let $\overline{Z'=Z}$, $t_i' = [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha:=id_Z}]t_i$ for $i=1,2$, and $\overline{g'=[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}} f\circ g}$. We thus have to show that $\iota'_{\overline{Z'},\overline{\pi_1 g'}}t_1' = [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}} f\,t$, $\kappa'_{\overline{Z'},\overline{\pi_2 g'}}t_2' = [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}} f\,t$, and $(t_1',t_2') : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}[\overline{\alpha:=\mathsf{Eq}_{Z'}}]$. That $(t_1',t_2') : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}[\overline{\alpha:=\mathsf{Eq}_{Z'}}]$ follows from the fact that $(t_1',t_2') = [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}} Eq_f[\overline{\alpha:=id_{\mathsf{Eq}_Z}}](t_1,t_2)$ and $(t_1,t_2) : [\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha:=\mathsf{Eq}_Z}]$. That $\iota'_{\overline{Z'},\overline{\pi_1 g'}}t_1' = [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}} f\,t$ is because

$$\begin{aligned}
\iota'_{\overline{Z'},\overline{\pi_1 g'}}t_1' &= \iota'_{\overline{Z'},\overline{\pi_1 g'}}([\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha:=id_Z}]t_1)\\
&= [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}} f(\iota_{\overline{Z},\overline{\pi_1 g}}t_1)\\
&= [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}} f\,t
\end{aligned}$$

where the second equality is justified by the commutativity of the diagram

$$\begin{tikzcd}
{[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=Z}]} \arrow[r, "\iota_{\overline{Z},\overline{\pi_1 g}}"] \arrow[d, "{[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}} f[\overline{\alpha:=id_Z}]}"] & \lim\limits_{\overrightarrow{\overline{S},\,h:[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=S}]\to[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\rho}}[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=S}] \arrow[d, "{(\mathsf{Lan}_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=_}]}}[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}} f[\overline{\alpha:=id_}])\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\rho}}"] \\
{[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha:=Z}]} \arrow[r, "\iota''_{\overline{Z},\overline{\pi_1 g}}"] \arrow[d, equal] & \lim\limits_{\overrightarrow{\overline{S},\,h:[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha:=S}]\to[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\rho'}}[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha:=S}] \arrow[d, "{(\mathsf{Lan}_{\overline{[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\rho[\overline{\alpha:=_}]}}[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha:=_}])\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}} f}}"] \\
{[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha:=Z}]} \arrow[r, "\iota'_{\overline{Z'},\overline{\pi_1([\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}} f\circ g)}}"'] & \lim\limits_{\overrightarrow{\overline{S},\,h:[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha:=S}]\to[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Rel}}\rho'}}[\![\Gamma;\Phi,\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha:=S}]
\end{tikzcd} \tag{3}$$

Analogously, we show that $\kappa_{Z,\pi_2 g'}t_2' = [\![\Gamma;\Phi\vdash(\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}} f\,t$.

Definitions 8 and 10 respect weakening, i.e., ensure that a type and its weakenings have the same set interpretations.

2.2 Relational Interpretations of Types

Definition 11. *A k-ary* Set *relation transformer F is a triple (F^1, F^2, F^*), where $F^1, F^2 : [\mathsf{Set}^k, \mathsf{Set}]_\omega$ and $F^* : [\mathsf{Rel}^k, \mathsf{Rel}]_\omega$ are such that if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$ then $F^*\overline{R} : \mathsf{Rel}(F^1\overline{A}, F^2\overline{B})$, and if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$ then $F^*\overline{(\alpha, \beta)} = (F^1\overline{\alpha}, F^2\overline{\beta})$. We define $F\overline{R}$ to be $F^*\overline{R}$ and $F\overline{(\alpha, \beta)}$ to be $F^*\overline{(\alpha, \beta)}$.*

The first condition of the first sentence of Definition 11 entails that $F^*\overline{R}$ relates sups of chains of pairwise related elements in $F^1\overline{A}$ and $F^2\overline{B}$. The last condition of the first sentence of Definition 11 expands to: if $\overline{(a, b) \in R}$ implies $\overline{(\alpha\, a, \beta\, b) \in S}$ then $(c, d) \in F^*\overline{R}$ implies $(F^1\overline{\alpha}\, c, F^2\overline{\beta}\, d) \in F^*\overline{S}$. When convenient we identify a 0-ary Set relation transformer (A, B, R) with $R : \mathsf{Rel}(A, B)$. We may also write $\pi_1 F$ for F^1 and $\pi_2 F$ for F^2. Without loss of generality, we may assume that π_1 and π_2 are surjective on relations. We extend these conventions to Set relation environments, introduced in Definition 16 below, in the obvious way.

Definition 12. *The category RT_k of k-ary* Set *relation transformers is given by the following data:*

- *An object of RT_k is a k-ary* Set *relation transformer.*
- *A morphism $\delta : (G^1, G^2, G^*) \to (H^1, H^2, H^*)$ in RT_k is a pair of natural transformations (δ^1, δ^2), where $\delta^1 : G^1 \to H^1$ and $\delta^2 : G^2 \to H^2$ are such that, for all $\overline{R : \mathsf{Rel}(A, B)}$, if $(x, y) \in G^*\overline{R}$ then $(\delta^1_{\overline{A}} x, \delta^2_{\overline{B}} y) \in H^*\overline{R}$.*
- *Identity morphisms and composition are inherited from the category of functors on* Set.

Definition 13. *An endofunctor H on RT_k is a triple $H = (H^1, H^2, H^*)$, where*

- *H^1 and H^2 are functors from $[\mathsf{Set}^k, \mathsf{Set}]_\omega$ to $[\mathsf{Set}^k, \mathsf{Set}]_\omega$*
- *H^* is a functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]_\omega$*
- *for all $\overline{R : \mathsf{Rel}(A, B)}$, $\pi_1((H^*(\delta^1, \delta^2))_{\overline{R}}) = (H^1\delta^1)_{\overline{A}}$ and $\pi_2((H^*(\delta^1, \delta^2))_{\overline{R}}) = (H^2\delta^2)_{\overline{B}}$*
- *The action of H on objects is given by $H\,(F^1, F^2, F^*) = (H^1F^1,\ H^2F^2,\ H^*(F^1, F^2, F^*))$*
- *The action of H on morphisms is given by $H\,(\delta^1, \delta^2) = (H^1\delta^1, H^2\delta^2)$ for $(\delta^1, \delta^2) : (F^1, F^2, F^*) \to (G^1, G^2, G^*)$*

Since the results of applying an endofunctor H to k-ary Set relation transformers and morphisms between them must again be k-ary Set relation transformers and morphisms between them, respectively, Definition 13 implicitly requires that the following three conditions hold: *i)* if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $H^*(F^1, F^2, F^*)\overline{R} : \mathsf{Rel}(H^1F^1\overline{A}, H^2F^2\overline{B})$; *ii)* if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then $H^*(F^1, F^2, F^*)\,\overline{(\alpha, \beta)} = (H^1F^1\overline{\alpha}, H^2F^2\overline{\beta})$; and *iii)* if $(\delta^1, \delta^2) : (F^1, F^2, F^*) \to (G^1, G^2, G^*)$ and $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then $((H^1\delta^1)_{\overline{A}} x, (H^2\delta^2)_{\overline{B}} y) \in H^*(G^1, G^2, G^*)\overline{R}$ whenever $(x, y) \in H^*(F^1, F^2, F^*)\overline{R}$. Note, however, that this last condition is automatically satisfied because it is implied by the third bullet point of Definition 13.

Definition 14. *If H and K are endofunctors on RT_k, then a* natural transformation *$\sigma : H \to K$ is a pair $\sigma = (\sigma^1, \sigma^2)$, where $\sigma^1 : H^1 \to K^1$ and $\sigma^2 : H^2 \to K^2$ are natural transformations between endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]_\omega$ and the component of σ at $F \in \mathsf{RT}_k$ is given by $\sigma_F = (\sigma^1_{F^1}, \sigma^2_{F^2})$.*

Definition 14 entails that $\sigma^i_{F^i}$ must be natural in $F^i : [\mathsf{Set}^k, \mathsf{Set}]_\omega$, and, for every F, $(\sigma^1_{F^1})_{\overline{A}}$ and $(\sigma^2_{F^2})_{\overline{B}}$ must be natural in $\overline{A}$ and $\overline{B}$, respectively. Moreover, since the results of applying σ to k-ary Set relation transformers must be morphisms of k-ary relation transformers, Definition 14 implicitly requires that $(\sigma_F)_{\overline{R}} = ((\sigma^1_{F^1})_{\overline{A}}, (\sigma^2_{F^2})_{\overline{B}})$ is a morphism in Rel for any k-tuple of relations $\overline{R : \mathsf{Rel}(A, B)}$, i.e., that if $(x, y) \in H^*F\overline{R}$, then $((\sigma^1_{F^1})_{\overline{A}} x, (\sigma^2_{F^2})_{\overline{B}} y) \in K^*F\overline{R}$.

Critically, we can compute ω-directed colimits in RT_k: it is not hard to see that if $\mathcal{D}$ is an ω-directed set then $\varinjlim_{d\in\mathcal{D}}(F^1_d, F^2_d, F^*_d) = (\varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^2_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$. We define an endofunctor $T = (T^1, T^2, T^*)$ on RT_k to be *ω-cocontinuous* if T^1 and T^2 are ω-cocontinuous endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]_\omega$ and T^* is an ω-cocontinuous functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]_\omega$, i.e., is in $[\mathsf{RT}_k, [\mathsf{Rel}^k, \mathsf{Rel}]_\omega]_\omega$.

Now, for any k, any $A : \mathsf{Set}$, and any $R : \mathsf{Rel}(A, B)$, let K^{Set}_A be the constantly A-valued functor from Set^k to Set and K^{Rel}_R be the constantly R-valued functor from Rel^k to Rel. Also let 0 denote either the initial object of Set or the initial object of Rel, as appropriate. Observing that, for every k, K^{Set}_0 is initial in $[\mathsf{Set}^k, \mathsf{Set}]_\omega$, and K^{Rel}_0 is initial in $[\mathsf{Rel}^k, \mathsf{Rel}]_\omega$, we have that, for each k, $K_0 = (K^{\mathsf{Set}}_0, K^{\mathsf{Set}}_0, K^{\mathsf{Rel}}_0)$ is initial in RT_k. Thus, if $T = (T^1, T^2, T^*) : \mathsf{RT}_k \to \mathsf{RT}_k$ is an endofunctor on RT_k then we can define the Set relation transformer μT to be $\varinjlim_{i<\omega} T^i K_0$. It is not hard to see that μT is given explicitly as

$$\mu T = (\mu T^1, \mu T^2, \varinjlim_{i<\omega} (T^i K_0)^*) \tag{4}$$

and that, as our notation suggests, it really is a fixpoint for T if T is ω-cocontinuous:

Lemma 15. *For any $T : [\mathsf{RT}_k, \mathsf{RT}_k]_\omega$, $\mu T \cong T(\mu T)$.*

The isomorphism is given by the morphisms $(in_1, in_2) : T(\mu T) \to \mu T$ and $(in_1^{-1}, in_2^{-1}) : \mu T \to T(\mu T)$ in RT_k. The latter is always a morphism in RT_k, but the former need not be if T is not ω-cocontinuous.

It is worth noting that the third component in Equation (4) is the colimit in $[\mathsf{Rel}^k, \mathsf{Rel}]_\omega$ of third components of Set relation transformers, rather than a fixpoint of an endofunctor on $[\mathsf{Set}^k, \mathsf{Set}]_\omega$. That there is an asymmetry between the first two components of μT and its third reflects the important conceptual observation that the third component of an endofunctor on RT_k need not be a functor on all of $[\mathsf{Rel}^k, \mathsf{Rel}]_\omega$. In particular, although we can define $T_{H,\rho}\, F$ for an Set relation transformer F in Definition 18 below, it is not clear how we could define it for an arbitrary $F : [\mathsf{Rel}^k, \mathsf{Rel}]_\omega$.

Definition 16. *An* Set relation environment *maps each type variable in $\mathbb{T}^k \cup \mathbb{F}^k$ to a k-ary* Set *relation transformer. A morphism $f : \rho \to \rho'$ between* Set *relation environments ρ and ρ' with $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$ maps each type constructor variable $\psi^k \in \mathbb{T}$ to the identity morphism on $\rho\psi^k = \rho'\psi^k$ and each functorial variable $\phi^k \in \mathbb{F}$ to a morphism from the k-ary* Set *relation transformer $\rho\phi$ to the k-ary* Set *relation transformer $\rho'\phi$. Composition of morphisms on* Set *relation environments is given componentwise, with the identity morphism mapping each* Set *relation environment to itself. This gives a category of* Set *relation environments and morphisms between them, which we denote* RelEnv.

When convenient we identify a 0-ary Set relation transformer with the Set relation (transformer) that is its codomain and consider an Set relation environment to map a type variable of arity 0 to an Set relation. If $\Phi = \{\phi_1^{k_1}, ..., \phi_n^{k_n}\}$ and $\overline{K} = \{K_1, ..., K_n\}$ where $K_i : [\mathsf{Rel}^{k_i}, \mathsf{Rel}]_\omega$ for $i = 1, ..., n$, then we write either $\rho[\overline{\Phi := K}]$ or $\rho[\overline{\phi := K}]$ for the Set relation environment ρ' such that $\rho'\phi_i = K_i$ for $i = 1, ..., n$ and $\rho'\phi = \rho\phi$ if $\phi \notin \Phi$. If ρ is an Set relation environment, we write $\pi_1\rho$ and $\pi_2\rho$ for the Set relation environments mapping each type variable ϕ to the functors $(\rho\phi)^1$ and $(\rho\phi)^2$, respectively.

We define, for each k, the notion of an ω-cocontinuous functor from RelEnv to RT_k:

Definition 17. *A functor $H : [\mathsf{RelEnv}, \mathsf{RT}_k]_\omega$ is a triple $H = (H^1, H^2, H^*)$, where*

- *H^1 and H^2 are objects in $[\mathsf{SetEnv}, [\mathsf{Set}^k, \mathsf{Set}]_\omega]_\omega$*
- *H^* is a an object in $[\mathsf{RelEnv}, [\mathsf{Rel}^k, \mathsf{Rel}]_\omega]_\omega$*
- *for all $\overline{R : \mathsf{Rel}(A, B)}$ and morphisms f in RelEnv, $\pi_1((H^* f)_{\overline{R}}) = (H^1(\pi_1 f))_{\overline{A}}$ and $\pi_2((H^* f)_{\overline{R}}) = (H^2(\pi_2 f))_{\overline{B}}$*

- *The action of H on ρ in* RelEnv *is given by* $H\rho = (H^1(\pi_1\rho),\ H^2(\pi_2\rho),\ H^*\rho)$
- *The action of H on morphisms $f : \rho \to \rho'$ in* RelEnv *is given by* $Hf = (H^1(\pi_1 f), H^2(\pi_2 f))$

Spelling out the last two bullet points above gives the following analogues of the three conditions immediately following Definition 13: *i)* if $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then $H^*\rho\,\overline{R} : \mathsf{Rel}(H^1(\pi_1\rho)\,\overline{A}, H^2(\pi_2\rho)\,\overline{B})$; *ii)* if $(\alpha_1, \beta_1) \in \mathrm{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathrm{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then $H^*\rho\,\overline{(\alpha,\beta)} = (H^1(\pi_1\rho)\,\overline{\alpha}, H^2(\pi_2\rho)\,\overline{\beta})$; and *iii)* if $f : \rho \to \rho'$ and $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then $((H^1(\pi_1 f))_{\overline{A}}\,x, (H^2(\pi_2 f))_{\overline{B}}\,y) \in H^*\rho'\,\overline{R}$ whenever $(x, y) \in H^*\rho\,\overline{R}$. As before, the last condition is automatically satisfied because it is implied by the third bullet point of Definition 17.

Considering RelEnv as a product $\Pi_{\phi^k \in \mathbb{T}\cup\mathbb{F}}\mathsf{RT}_k$, we extend the computation of ω-directed colimits in RT_k to compute colimits in RelEnv componentwise. We similarly extend the notion of an ω-cocontinuous endofunctor on RT_k componentwise to give a notion of ω-cocontinuity for functors from RelEnv to RT_k. Recalling from the start of this subsection that Definition 18 is given mutually inductively with Definition 8 we can, at last, define:

Definition 18. *The* relational interpretation $[\![\cdot]\!]^{\mathsf{Rel}} : \mathcal{F} \to [\mathsf{RelEnv}, \mathsf{Rel}]_\omega$ *is defined by*

$$
\begin{aligned}
[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\rho &= 0\\
[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\rho &= 1\\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}\,F\,G]\!]^{\mathsf{Rel}}\rho &= \{\eta : \lambda\overline{K}.\,[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\Phi := K}] \Rightarrow \lambda\overline{K}.\,[\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\phi := K}]\}\\
&= \{(\eta_1,\eta_2) \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}\,F\,G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi}\,F\,G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid\\
&\qquad \forall\overline{K = (K^1, K^2, K^*) : \mathsf{RT}_k}.\\
&\qquad\quad ((\eta_1)_{\overline{K^1}}, (\eta_2)_{\overline{K^2}}) \in ([\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\Phi := K}])^{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\Phi:=K}]}\}\\
[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Rel}}\rho &= (\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho}\\
[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho + [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\rho\\
[\![\Gamma;\Phi \vdash F \times G]\!]^{\mathsf{Rel}}\rho &= [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho \times [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\rho\\
[\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{G}]\!]^{\mathsf{Rel}}\rho &= (\mu T_{H,\rho})\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash G]\!]^{\mathsf{Rel}}\rho}\\
\textit{where } T_{H,\rho} &= (T^{\mathsf{Set}}_{H,\pi_1\rho}, T^{\mathsf{Set}}_{H,\pi_2\rho}, T^{\mathsf{Rel}}_{H,\rho})\\
\textit{and } T^{\mathsf{Rel}}_{H,\rho}\,F &= \lambda\overline{R}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F][\overline{\alpha := R}]\\
\textit{and } T^{\mathsf{Rel}}_{H,\rho}\,\delta &= \lambda\overline{R}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}id_\rho[\phi := \delta][\overline{\alpha := id_{\overline{R}}}]\\
[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Rel}}\rho &= \{(t_1,t_2) : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}\pi_1\rho \times [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}F)\overline{A}]\!]^{\mathsf{Set}}\pi_2\rho\\
&\qquad \mid \exists\,\overline{Z : \mathsf{Set}_0}, t_1' : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\pi_1\rho[\overline{\alpha := Z}],\ t_2' : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\pi_2\rho[\overline{\alpha := Z}],\\
&\qquad\ \overline{f : [\![\Gamma;\overline{\alpha} \vdash K]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \mathsf{Eq}_Z}] \to [\![\Gamma;\Phi \vdash A]\!]^{\mathsf{Rel}}\rho}\\
&\qquad\ \textit{such that } \iota_{\overline{Z},\overline{\pi_1 f}}t_1' = t_1,\ \kappa_{\overline{Z},\overline{\pi_2 f}}t_2' = t_2,\\
&\qquad\ \textit{and } (t_1', t_2') : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := \mathsf{Eq}_Z}]\,\}
\end{aligned}
$$

In the final clause of Definition 18, $\iota_{Z,h}$ is the injection into

$$\varinjlim_{Z\,:\,\mathsf{Set}_0,\ h\,:\,[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\pi_1\rho[\overline{\alpha:=Z}]\to[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}\pi_1\rho} [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\pi_1\rho[\overline{\alpha := Z}]$$

and $\kappa_{Z,h}$ is the injection into

$$\varinjlim_{Z\,:\,\mathsf{Set}_0,\ h\,:\,[\![\Gamma;\overline{\alpha}\vdash K]\!]^{\mathsf{Set}}\pi_2\rho[\overline{\alpha:=Z}]\to[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}\pi_2\rho} [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\pi_2\rho[\overline{\alpha := Z}]$$

For Definition 18 to be well-defined, we have to check that each relational interpretation is in Rel and, in particular, that each relates sups of pairwise related ω-chains. This will be proved by induction on types, and in most cases it will follow from the induction hypotheses. However, well-definedness needs to be proved directly for relational interpretations of Nat-types.

The proof that relational interpretations of Nat-types define sets is analogous to the proof of Lemma 9.

Moreover, ω-cocontinuity of each of the above interpretations follows from Corollary 12 of [Johann and Polonsky 2019] if we APPROPRIATELY RESTRICT THE SUBSCRIPTS of Lans. For Nat-types, $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\, G]\!]^{\mathsf{Rel}}$ is an ω-cocontinuous functor because it is constant on ω-directed sets. Interpretations of Nat-types ensure that $[\![\Gamma \vdash F \to G]\!]^{\mathsf{Rel}}$ and $[\![\Gamma \vdash \forall\overline{\alpha}.F]\!]^{\mathsf{Rel}}$ are as expected in any parametric model.

For the next-to-last clause in Definition 18 to be well-defined we need $T_{H,\rho}$ to be an ω-cocontinuous endofunctor on RT so that, by Lemma 15, it admits a fixpoint. Since $T_{H,\rho}$ is defined in terms of $[\![\Gamma;\overline{\gamma},\phi^k,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}$, this means that relational interpretations of types must be ω-cocontinuous functors from RelEnv to RT_0, which in turn entails that the actions of relational interpretations of types on objects and on morphisms in RelEnv are intertwined. As for Set interpretations, we know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash F$, $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}$ is actually in $[\mathsf{Rel}^k,\mathsf{Rel}]_\omega$, where $k = |\overline{\alpha}|$. We first define the actions of each of these functors on morphisms between Set relation environments in Definition 19, and then argue that the functors given by Definitions 18 and 19 are well-defined and have the required properties. To do this, we extend T_H to a *functor* from RelEnv to $[[\mathsf{Rel}^k,\mathsf{Rel}]_\omega,[\mathsf{Rel}^k,\mathsf{Rel}]_\omega]_\omega$. Its action on an object $\rho \in$ RelEnv is given by the higher-order functor $T_{H,\rho}$ whose actions on objects and morphisms are given in Definition 18. Its action on a morphism $f : \rho \to \rho'$ is the higher-order natural transformation $T_{H,f} : T_{H,\rho} \to T_{H,\rho'}$ whose action on any $F : [\mathsf{Rel}^k,\mathsf{Rel}]_\omega$ is the higher-order natural transformation $T_{H,f}\, F : T_{H,\rho}\, F \to T_{H,\rho'}\, F$ whose component at $\overline{R}$ is $(T_{H,f}\, F)_{\overline{R}} = [\![\Gamma;\overline{\gamma},\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} f[\phi := id_F][\overline{\alpha := id_R}]$. The next definition uses T_H to define the actions of functors interpreting types on morphisms between Set relation environments.

If $\rho \in$ RelEnv and $\vdash F$, then we write $[\![\vdash F]\!]^{\mathsf{Rel}}$ instead of $[\![\vdash F]\!]^{\mathsf{Rel}}\rho$. The interpretations in Definitions 18 and in Definition 19 below respect weakening.

Definition 19. *Let $f : \rho \to \rho'$ for* Set *relation environments ρ and ρ' (so that $\rho|_{\mathbb{T}} = \rho'|_{\mathbb{T}}$). The action $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}} f$ of $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}$ on the morphism f for all types but* Lan *is given exactly as in Definition 10, except that all interpretations are* Set *relational interpretations and all occurrences of $T^{\mathsf{Set}}_{H,f}$ are replaced by $T_{H,f}$. In the* Lan *case, instead, define*

$$[\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}} f = ([\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}} \pi_1 f, [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}} \pi_2 f)$$

We need to show that the functorial action for the Lan case is well-defined, i.e., that for every $(t_1,t_2) : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}}$ we have that

$$([\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} \pi_1 f\, t_1, [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} \pi_2 f\, t_2) : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}} \rho'$$

From $(t_1,t_2) : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}}$ we have, by definition, that there exist $\overline{Z : \mathsf{Set}_0}$, $t_1' : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \pi_1 \rho[\overline{\alpha :=}$ $t_2' : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \pi_2 \rho[\overline{\alpha := Z}]$, $\overline{g : [\![\Gamma;\overline{\alpha} \vdash K]\!]^{\mathsf{Rel}} \rho[\overline{\alpha := \mathsf{Eq}_Z}] \to [\![\Gamma;\Phi \vdash A]\!]^{\mathsf{Rel}} \rho}$ such that $\iota_{\overline{Z},\overline{\pi_1 g}} t_1' = t_1$, $\kappa_{\overline{Z},\overline{\pi_2 g}} t_2' = t_2$, and $(t_1',t_2') : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \rho[\overline{\alpha := \mathsf{Eq}_Z}]$. To show that

$$([\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} \pi_1 f\, t_1, [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Set}} \pi_2 f\, t_2) : [\![\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\overline{A}]\!]^{\mathsf{Rel}} \rho'$$

let $\overline{S = Z : \mathsf{Set}_0,\ s_1 = [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_1 f)[\overline{\alpha := Z}]t'_1,\ s_2 = [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_2 f)[\overline{\alpha := Z}]t'_2, \text{ and } h = [\![\Gamma;\Phi \vdash A]\!]^{\mathsf{Rel}} f \circ g}$. The first thing we need to show is that $(s_1, s_2) : [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho'[\overline{\alpha := \mathsf{Eq}_S}]$, and that is because $(s_1,s_2) = [\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} f[\overline{\alpha := \mathsf{Eq}_Z}](t'_1, t'_2)$. The second thing we need to show is that $\iota'_{\overline{S},\overline{\pi_1 h}} s'_1 = s_1$ and $\kappa'_{\overline{S},\overline{\pi_2 h}} s'_2 = s_2$. That is because

$$\begin{aligned}
\iota'_{\overline{S},\overline{\pi_1 h}} s'_1 &= \iota'_{\overline{S},\overline{[\![\Gamma;\Phi\vdash A]\!]^{\mathsf{Set}}\pi_1 f \circ \pi_1 g}}([\![\Gamma;\Phi,\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}(\pi_1 f)[\overline{\alpha := Z}]t'_1)\\
&= ([\![\Gamma;\Phi \vdash (\mathsf{Lan}_{\overline{K}}^{\overline{\alpha}} F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1 f) \circ \iota_{\overline{S},\overline{\pi_1 g}})t'_1\\
&= [\![\Gamma;\Phi \vdash (\mathsf{Lan}_{\overline{K}}^{\overline{\alpha}} F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1 f)(\iota_{\overline{S},\overline{\pi_1 g}} t'_1)\\
&= [\![\Gamma;\Phi \vdash (\mathsf{Lan}_{\overline{K}}^{\overline{\alpha}} F)\overline{A}]\!]^{\mathsf{Set}}(\pi_1 f)t_1\\
&= s_1
\end{aligned}$$

where the second equality is by the commutativity of diagram 3. Analogously, we show that $\kappa'_{\overline{S},\overline{\pi_2 h}} s'_2 = s_2$.

To see that the functors given by Definitions 18 and 19 are well-defined we must also show that, for every H, $T_{H,\rho}\, F$ is an Set relation transformer for any Set relation transformer F, and that $T_{H,f}\, F : T_{H,\rho}\, F \to T_{H,\rho'}\, F$ is a morphism of Set relation transformers for every Set relation transformer F and every morphism $f : \rho \to \rho'$ in RelEnv. This is an immediate consequence of the following Lemma.

Lemma 20. *For every* $\Gamma;\Phi \vdash F$, $[\![\Gamma;\Phi \vdash F]\!] = ([\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}, [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}) \in [\mathsf{RelEnv}, \mathsf{RT}_0]_\omega$.

The proof is a straightforward induction on the structure of F, using an appropriate result from [Johann and Polonsky 2019] to deduce ω-cocontinuity of $[\![\Gamma;\Phi \vdash F]\!]$ in each case, together with Lemma 15 and Equation 4 for μ-types. Lan types need restriction on their subscripts.

We can also prove by simultaneous induction that our interpretations of types interact well with demotion of functorial variables. Indeed, we have that, if $\rho, \rho' : \mathsf{SetEnv}$, $f : \rho \to \rho'$, $\rho\phi = \rho\psi = \rho'\phi = \rho'\psi$, $f\phi = f\psi = id_{\rho\phi}$, $\Gamma;\Phi,\phi^k \vdash F$, $\Gamma;\Phi,\overline{\alpha} \vdash G$, $\Gamma;\Phi,\alpha_1 ... \alpha_k \vdash H$, and $\overline{\Gamma;\Phi \vdash K}$, then

$$[\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}\rho = [\![\Gamma,\psi;\Phi \vdash F[\phi :== \psi]]\!]^{\mathsf{Set}}\rho \tag{5}$$

$$[\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}} f = [\![\Gamma,\psi;\Phi \vdash F[\phi :== \psi]]\!]^{\mathsf{Set}} f \tag{6}$$

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}}\rho}] \tag{7}$$

$$[\![\Gamma;\Phi \vdash G[\overline{\alpha := K}]]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi,\overline{\alpha} \vdash G]\!]^{\mathsf{Set}} f[\overline{\alpha := [\![\Gamma;\Phi \vdash K]\!]^{\mathsf{Set}} f}] \tag{8}$$

$$[\![\Gamma;\Phi \vdash F[\phi := H]]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}}\rho[\phi := \lambda\overline{A}.\, [\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]] \tag{9}$$

$$[\![\Gamma;\Phi \vdash F[\phi := H]]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi,\phi \vdash F]\!]^{\mathsf{Set}} f[\phi := \lambda\overline{A}.\, [\![\Gamma;\Phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} f[\overline{\alpha := id_{\overline{A}}}]] \tag{10}$$

Identities analogous to (5) through (10) hold for Set relational interpretations as well.

3 THE IDENTITY EXTENSION LEMMA

The standard definition of the graph for a morphism $f : A \to B$ in Set is the relation $\langle f \rangle : \mathsf{Rel}(A,B)$ defined by $(x,y) \in \langle f \rangle$ iff $fx = y$. This definition naturally generalizes to associate to each natural transformation between k-ary functors on Set a k-ary Set relation transformer as follows.

First, let $f : A \to B$ be a morphism in Set. Then the graph of the underlying function $|f| : |A| \to |B|$ in Set is an Set relation. Indeed, if $(a_i, b_i) : \langle |f| \rangle$ for all $i < \omega$ is a chain, then $|f|a_i = b_i$ for all i, and consequently $|f|(\bigvee_{i<\omega} a_i) = \bigvee_{i<\omega}(|f|a_i) = \bigvee_{i<\omega} b_i$, i.e., $(\bigvee_{i<\omega} a_i, \bigvee_{i<\omega} b_i) : \langle |f| \rangle$. We therefore define the Set *graph of* f in Set to be $\langle f \rangle = \langle |f| \rangle$. Note in particular that if A : Set then $\langle id_A \rangle$ is an Set relation. We denote this Set relation, called the *equality relation on* A, by Eq_A. It coincides exactly with the equality relation on the underlying set $|A|$ of A.

The notion of an Set graph relation naturally generalizes to associate to each natural transformation between k-ary functors on Set a k-ary Set relation transformer as follows. Recall that, since Set is a locally ω-presentable category, Proposition 1.6.1 of [Adámek and Rosický 1994] ensures that it has a (strong epi, mono) factorization system. We then have:

Definition 21. *If $F, G : \mathsf{Set}^k \to \mathsf{Set}$ and $\alpha : F \to G$ is a natural transformation, then the functor $\langle\alpha\rangle^* : \mathsf{Rel}^k \to \mathsf{Rel}$ is defined as follows. Given $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, let $\iota_{R_i} : R_i \hookrightarrow A_i \times B_i$, for $i = 1, \ldots, k$, be the inclusion of R_i as a sub*Set *of $A_i \times B_i$, let $h_{\overline{A\times B}}$ be the unique morphism making the diagram*

$$\begin{tikzcd}
F\overline{A} & F(\overline{A\times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, dashed, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
 & F\overline{A}\times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

commute, and let $h_{\overline{R}} : F\overline{R} \to F\overline{A}\times G\overline{B}$ be $h_{\overline{A\times B}} \circ F\overline{\iota_R}$. Further, let $\alpha^\wedge\overline{R}$ be the subobject through which $h_{\overline{R}}$ is factorized by the (strong epi, mono) factorization system of Set, *as shown in the following diagram:*

$$\begin{tikzcd}
F\overline{R} \arrow[rr, "h_{\overline{R}}"] \arrow[dr, two heads, "q_{\alpha^\wedge\overline{R}}"'] & & F\overline{A}\times G\overline{B} \\
 & \alpha^\wedge\overline{R} \arrow[ur, hook, "\iota_{\alpha^\wedge\overline{R}}"'] &
\end{tikzcd}$$

Then $\alpha^\wedge\overline{R} : \mathsf{Rel}(F\overline{A}, G\overline{B})$ by construction, so the action of $\langle\alpha\rangle^*$ on objects of Rel *can be given by $\langle\alpha\rangle^*\overline{(A, B, R)} = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge\overline{R}}\alpha^\wedge\overline{R})$. The action of $\langle\alpha\rangle^*$ on morphisms of* Rel *is given by $\langle\alpha\rangle^*\overline{(\beta, \beta')} = (F\overline{\beta}, G\overline{\beta}')$.*

The next lemma shows that the data in Definition 21 actually yield a relation transformer $\langle\alpha\rangle = (F, G, \langle\alpha\rangle^*)$. We call this the Set *graph relation transformer* for α.

Lemma 22. *If $F, G : [\mathsf{Set}^k, \mathsf{Set}]_\omega$, and if $\alpha : F \to G$ is a natural transformation, then $\langle\alpha\rangle$ is in RT_k.*

Proof. Clearly, $\langle\alpha\rangle^*$ is ω-cocontinuous, so $\langle\alpha\rangle^* : [\mathsf{Rel}^k, \mathsf{Rel}]_\omega$. Now, suppose $\overline{R : \mathsf{Rel}(A, B)}$, $\overline{S : \mathsf{Rel}(C, D)}$, and $\overline{(\beta, \beta') : R \to S}$. We want to show that there exists a morphism $\epsilon : \alpha^\wedge\overline{R} \to \alpha^\wedge\overline{S}$ such that the diagram on the left below commutes. Since $\overline{(\beta, \beta') : R \to S}$, there exist $\overline{\gamma : R \to S}$ such that each diagram in the middle commutes. Moreover, since both $h_{\overline{C\times D}} \circ F(\overline{\beta\times\beta'})$ and $(F\overline{\beta}\times G\overline{\beta'}) \circ h_{\overline{A\times B}}$ make the diagram on the right commute, they must be equal. We therefore get

$$\begin{tikzcd}
\alpha^\wedge\overline{R} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{R}}"] \arrow[d, "\epsilon"'] & F\overline{A}\times G\overline{B} \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
\alpha^\wedge\overline{S} \arrow[r, hook, "\iota_{\alpha^\wedge\overline{S}}"'] & F\overline{C}\times G\overline{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
R_i \arrow[r, hook, "\iota_{R_i}"] \arrow[d, "\gamma_i"'] & A_i\times B_i \arrow[d, "\beta_i\times\beta_i'"] \\
S_i \arrow[r, hook, "\iota_{S_i}"'] & C_i\times D_i
\end{tikzcd}
\qquad
\begin{tikzcd}
F\overline{C} & F\overline{C}\times F\overline{D} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F\overline{D} \arrow[r, "\alpha_{\overline{D}}"] & G\overline{D} \\
 & F(\overline{A\times B}) \arrow[ul, "F\overline{\pi_1}\circ F(\overline{\beta\times\beta'})"] \arrow[u, dashed, "\exists!"] \arrow[urr, "\alpha_{\overline{D}}\circ F\overline{\pi_2}\circ F(\overline{\beta\times\beta'})"'] & &
\end{tikzcd}$$

that the right-hand square in the diagram on the left below commutes, and thus that the entire diagram does as well. Finally, by the left-lifting property of $q_{\alpha^\wedge\overline{R}}$ with respect to $\iota_{\alpha^\wedge\overline{S}}$ given by the (strong epi, mono) factorization system there exists an ϵ such that the diagram on the right below commutes.

$$
\begin{tikzcd}
F\overline{R} \arrow[r, hook, "F\iota_{\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] \arrow[rr, bend left, "h_{\overline{R}}"] & F(\overline{A \times B}) \arrow[r, "h_{\overline{A\times B}}"] \arrow[d, "F(\overline{\beta\times\beta'})"] & F\overline{A} \times G\overline{B} \arrow[d, "F\overline{\beta}\times F\overline{\beta'}"] \\
F\overline{S} \arrow[r, hook, "F\iota_{\overline{S}}"'] \arrow[rr, bend right, "h_{\overline{S}}"'] & F(\overline{C \times D}) \arrow[r, "h_{\overline{C\times D}}"'] & F\overline{C} \times G\overline{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
F\overline{R} \arrow[r, two heads, "q_{\alpha^\wedge\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] & \alpha^\wedge\overline{R} \arrow[d, dashed, "\epsilon"] & F\overline{A} \times G\overline{B} \arrow[l, hook', "\iota_{\alpha^\wedge\overline{R}}"'] \arrow[d, "F\overline{\beta}\times G\overline{\beta'}"] \\
F\overline{S} \arrow[r, two heads, "q_{\alpha^\wedge\overline{S}}"'] & \alpha^\wedge\overline{S} & F\overline{C} \times G\overline{D} \arrow[l, hook', "\iota_{\alpha^\wedge\overline{S}}"]
\end{tikzcd}
$$

□

If $f : A \to B$ is a morphism in Set then the definition of the Set graph relation transformer $\langle f \rangle$ for f as a natural transformation between 0-ary functors A and B coincides with the definition of $\langle f \rangle$ for f as a morphism in Set given in the second paragraph of this section. As a result, Set graph relation transformers are a reasonable extension of Set graph relations to functors.

To prove the IEL, we will need to know that the equality Set relation transformer preserves equality relations in Rel; see Equation 11 below. This will follow from the next lemma, which shows how to compute the action of an Set graph relation transformer on any Set graph relation.

Lemma 23. *If $\alpha : F \to G$ is a morphism in $[\mathrm{Set}^k, \mathrm{Set}]_\omega$ and $f_1 : A_1 \to B_1, \ldots, f_k : A_k \to B_k$, then $\langle \alpha \rangle^* \langle \overline{f} \rangle = \langle G\overline{f} \circ \alpha_{\overline{A}} \rangle = \langle \alpha_{\overline{B}} \circ F\overline{f} \rangle$.*

Proof. Since $h_{\overline{A\times B}}$ is the unique morphism making the bottom triangle of the diagram on the left below commute, and since $h_{\langle \overline{f} \rangle} = h_{\overline{A\times B}} \circ F\,\overline{\iota_{\langle f \rangle}} = h_{\overline{A\times B}} \circ F\overline{\langle id_A, f \rangle}$, the universal property of the product depicted in the diagram on the right gives $h_{\langle \overline{f} \rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f} \rangle : F\overline{A} \to F\overline{A} \times G\overline{B}$.

$$
\begin{tikzcd}
 & F\overline{A} \arrow[dl, equal] \arrow[d, dashed, "F\overline{\langle id_A, f \rangle}"] \arrow[dr, "F\overline{f}"] & & \\
F\overline{A} & F(\overline{A \times B}) \arrow[l, "F\pi_1"'] \arrow[r, "F\pi_2"] \arrow[d, "h_{\overline{A\times B}}"] & F\overline{B} \arrow[r, "\alpha_{\overline{B}}"] & G\overline{B} \\
 & F\overline{A} \times G\overline{B} \arrow[ul, "\pi_1"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F\overline{A} & F\overline{A} \times G\overline{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & G\overline{B} \\
 & F\overline{A} \arrow[ul, equal] \arrow[u, dashed, "\exists!"] \arrow[r, "F\overline{f}"'] & F\overline{B} \arrow[u, "\alpha_{\overline{B}}"']
\end{tikzcd}
$$

Moreover, $\langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f} \rangle$ is a monomorphism in Set because $id_{F\overline{A}}$ is, so its (strong epi, mono) factorization gives $\iota_{\alpha^\wedge\langle \overline{f} \rangle} = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f} \rangle$, and thus that $\alpha^\wedge \langle \overline{f} \rangle = F\overline{A}$. Then $\iota_{\alpha^\wedge\langle \overline{f} \rangle} \alpha^\wedge \langle \overline{f} \rangle = \langle id_{F\overline{A}}, \alpha_{\overline{B}} \circ F\overline{f} \rangle (F\overline{A}) = \langle \alpha_{\overline{B}} \circ F\overline{f} \rangle^*$. Then $\langle \alpha \rangle^* \langle \overline{f} \rangle = (F\overline{A}, G\overline{B}, \iota_{\alpha^\wedge\langle \overline{f} \rangle}\, \alpha^\wedge \langle \overline{f} \rangle) = (F\overline{A}, G\overline{B}, \langle \alpha_{\overline{B}} \circ F\overline{f} \rangle^*) = \langle \alpha_{\overline{B}} \circ F\overline{f} \rangle$, and, finally, $\alpha_{\overline{B}} \circ F\overline{f} = G\overline{f} \circ \alpha_{\overline{A}}$ by naturality of α. □

The *equality* Set *relation transformer* on $F : [\mathrm{Set}^k, \mathrm{Set}]_\omega$ is defined to be $\mathsf{Eq}_F = \langle id_F \rangle$. Specifically, $\mathsf{Eq}_F = (F, F, \mathsf{Eq}_F^*)$ with $\mathsf{Eq}_F^* = \langle id_F \rangle^*$, and Lemma 23 indeed ensures that

$$
\mathsf{Eq}_F^* \overline{\mathsf{Eq}_A} = \langle id_F \rangle^* \langle id_{\overline{A}} \rangle = \langle F id_{\overline{A}} \circ (id_F)_{\overline{A}} \rangle = \langle id_{F\overline{A}} \circ id_{F\overline{A}} \rangle = \langle id_{F\overline{A}} \rangle = \mathsf{Eq}_{F\overline{A}} \tag{11}
$$

for all $\overline{A : \mathrm{Set}}$. Graph Set relation transformers in general, and equality Set relation transformers in particular, extend to Set relation environments in the obvious ways. Indeed, if $\rho, \rho' :$ SetEnv and $f : \rho \to \rho'$, then the *graph* Set *relation environment* $\langle f \rangle$ is defined pointwise by $\langle f \rangle \phi = \langle f \phi \rangle$ for every ϕ, which entails that $\pi_1 \langle f \rangle = \rho$ and $\pi_2 \langle f \rangle = \rho'$. In particular, the *equality* Set *relation* *environment* Eq_ρ is defined to be $\langle id_\rho \rangle$, which entails that $\mathsf{Eq}_\rho\phi = \mathsf{Eq}_{\rho\phi}$ for every ϕ. With these definitions in hand, we can state and prove both an Identity Extension Lemma and a Graph Lemma for our calculus.

Theorem 24 (IEL). *If* $\rho : \mathsf{SetEnv}$ *and* $\Gamma; \Phi \vdash F$ *then* $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}$.

Proof. By induction on the structure of F. Only the Nat, application, fixpoint, and Lan cases are non-routine.

The latter two use Equation 11. The fixpoint case also uses the observation that, for every $i < \omega$, the following intermediate results can be proved by simultaneous induction with Theorem 24: CHECK THIS! for any H, ρ, A, and any subformula J of H, both $T^i_{H,\mathsf{Eq}_\rho} K_0\, \overline{\mathsf{Eq}_A} = (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^i K_0})^* \overline{\mathsf{Eq}_A}$ and

$$
\begin{aligned}
& [\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash J]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^i_{H,\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_A]} \\
= \quad & [\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash J]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^i K_0}]\overline{[\alpha := \mathsf{Eq}_A]}
\end{aligned}
$$

hold. With these results in hand, the proof follows easily for all but Lan-types. The entire proof is given in detail in the accompanying anonymous supplementary material. As noted there, if functorial variables of arity greater than 0 were allowed to appear in the bodies of μ-types, then the IEL would fail.

For Lan-types,....

□

Lemma 25 (Graph Lemma). *If* $\rho, \rho' : \mathsf{SetEnv}$, $f : \rho \to \rho'$, *and* $\Gamma; \Phi \vdash F$, *then* $\langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f \rangle = [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle$.

Proof. Applying Lemma 20 to the morphisms $(f, id_{\rho'}) : \langle f \rangle \to \mathsf{Eq}_{\rho'}$ and $(id_\rho, f) : \mathsf{Eq}_\rho \to \langle f \rangle$ of relation environments gives that

$$
\begin{aligned}
& ([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}) = [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}(f, id_{\rho'}) \\
& \qquad : [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle \to [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}
\end{aligned}
$$

and

$$
\begin{aligned}
& ([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_\rho, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f) = [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}(id_\rho, f) \\
& \qquad : [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho \to [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle
\end{aligned}
$$

Expanding the first equation gives that if $(x, y) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle$ then

$$
([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}\, y) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'}
$$

So $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_{\rho'}\, y = id_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}\, y = y$ and $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho'} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}$, and if $(x, y) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle$ then $([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x, y) \in \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho'}$, i.e., $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x = y$, i.e., $(x, y) \in \langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f \rangle$. So, we have that $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle \subseteq \langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f \rangle$. Expanding the second equation gives that if $x \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}}\rho$ then $([\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_\rho\, x,$ $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle$. Then $[\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} id_\rho\, x = id_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho} x = x$, so for any $x \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}}\rho$ we have that $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle$. Moreover, $x \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}}\rho$ if and only if $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in \langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f \rangle$ and, if $x \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}}\rho$ then $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle$, so if $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in \langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f \rangle$ then $(x, [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f\, x) \in [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle$, i.e., $\langle [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Set}} f \rangle \subseteq [\![\Gamma; \Phi \vdash F]\!]^{\mathsf{Rel}}\langle f \rangle$. □

Some questions/issues:

- Can we write zipBush and appendBush with ∂ and $\int$? We could already represent the uncurried type of appendBush (although not its curried type), but couldn't recurse over both input bushes because folds take natural transformations as inputs.
- More generally, how do we compute with ∂ and $\int$? Can we use the colimit formulation of Lans (see Lemma 6.3.7 of [Riehl 2016]) to get a handle on this?
- What is the connection between exponentials and natural transformations? (Should we assume only small objects are exponentiable?) Do we want the former or the latter for computational purposes? (I suspect the latter.)
  [From nlab: In a functor category D^C, a natural transformation $\alpha : F \to G$ is exponentiable if (though probably not "only if") it is cartesian and each component $\alpha_c : Fc \to Gc$ is exponentiable in D. Given $H \to F$ we define $(\Pi_\alpha H)c = \Pi_{\alpha_c}(Hc)$; then for $u : c \to c'$ to obtain a map $\Pi_{\alpha_c}(Hc) \to \Pi_{\alpha_{c'}}(Hc')$ we need a map $\alpha^*_{c'}(\Pi_{\alpha_c}(Hc)) \to Hc'$. But since α is cartesian, $\alpha^*_{c'}(\Pi_{\alpha_c}(Hc)) \cong \alpha^*_c(\Pi_{\alpha_c}(Hc))$, so we have the counit $\alpha^*_c(\Pi_{\alpha_c}(Hc)) \to Hc$ that we can compose with Hu.]
- After we understand what we can do with Lans and folds on GADTs we might want to try to extend calculus with term-level fixpoints. This would give a categorical analogue for GADTs of [Pitts 1998, 2000] for ADTs. Would it also more accurately reflect how GADTs are used in practice, or are functions over GADTs usually folds? Investigate applications in the literature and/or in implementations.
- ωCPO is a natural choice for modeling general recursion. We know $(Lan^{\gamma}_{C}\mathbb{1})D$ is $C \to D$ for any closed type C. (Also for select classes of open types?) So can model $\mathsf{Nat} \to \gamma$. But the functor $NX = \mathsf{Nat} \to X$ isn't ω-cocontinuous. It also doesn't preserve ω-presentable objects, i.e., countable ωCPOs since $\mathsf{Nat} \to \mathsf{Nat}$ is not countable. So we cannot have a functor like N as the subscript to Lan and expect the resulting Lan to be ω-cocontinuous.
- What functors can be subscripts to Lan and produce ω-cocontinuous functors? We can use functors that preserve presentable objects by theorem in [Johann and Polonsky 2019], and possibly others as well. These include polynomial functors, ADTs and nested types seen as functors, certain (which?) GADTs seen as functors? How big can GADTs get?

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