Free Theorems for Nested Types

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Let $List\,\alpha = (\mu\phi.\lambda\beta.\mathbb{1} + \beta \times \phi\beta)\alpha$, and let $map = \mathrm{map}_{\lambda A.[\![\emptyset;\alpha\vdash List\,\alpha]\!]^{\mathsf{Set}}\rho[\alpha:=A]}$.

Theorem 1. *If* $\Gamma;\Phi\,|\,\Delta \vdash t : \tau$ *and* $\rho \in \mathsf{RelEnv}$, *and if* $(a,b) \in [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, *then* $([\![\Gamma;\Phi\,|\,\Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_1\rho)\,a\,, [\![\Gamma;\Phi\,|\,\Delta \vdash t : \tau]\!]^{\mathsf{Set}}(\pi_2\rho)\,b) \in [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho$

Proof. Immediate from Theorem **??** (at-gen). □

Theorem 2. *If* $g : A \to B$, $\rho : \mathsf{RelEnv}$, $\rho\alpha = (A, B, \langle g\rangle)$, $(a,b) \in [\![\alpha;\emptyset \vdash \Delta]\!]^{\mathsf{Rel}}\rho$, $(s\circ g, s) \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\; Bool]\!]^{\mathsf{Rel}}\rho$, *and, for some well-formed term filter,*

$$t = [\![\alpha;\emptyset\,|\,\Delta \vdash \mathit{filter} : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\; Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Set}}, \text{ then}$$

$$map\,g \circ t(\pi_1\rho)\,a\,(s\circ g) = t(\pi_2\rho)\,b\,s \circ map\,g$$

Proof. By Theorem 1, $(t(\pi_1\rho)a, t(\pi_2\rho)b) \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\emptyset}\,\alpha\; Bool)(\mathsf{Nat}^{\emptyset}\,(List\,\alpha)\,(List\,\alpha))]\!]^{\mathsf{Rel}}\rho$. Thus if $(s,s') \in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}\alpha\; Bool]\!]^{\mathsf{Rel}}\rho = \rho\alpha \to \mathsf{Eq}_{Bool}$, then

$$\begin{aligned}(t(\pi_1\rho)\,a\,s, t(\pi_2\rho)\,b\,s') &\in [\![\alpha;\emptyset \vdash \mathsf{Nat}^{\emptyset}(List\,\alpha)\,(List\,\alpha)]\!]^{\mathsf{Rel}}\rho\\ &= [\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho \to [\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho\end{aligned}$$

So if $(xs, xs') \in [\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho$ then,

$$(t(\pi_1\rho)\,a\,s\,xs, t(\pi_2\rho)\,b\,s'\,xs') \in [\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho \tag{1}$$

Consider the case in which $\rho\alpha = (A, B, \langle g\rangle)$. Then $[\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho = \langle map\,g\rangle$. Indeed, $[\![\alpha;\emptyset \vdash List\,\alpha]\!]^{\mathsf{Rel}}\rho$ is equal to $[\![\emptyset;\alpha \vdash List\,\alpha]\!]^{\mathsf{Rel}}[\alpha := \langle g\rangle]$, which is equal to $\langle [\![\emptyset;\alpha \vdash List\,\alpha]\!]^{\mathsf{Set}}[\alpha := g]\rangle$ by the Graph Lemma (Lemma **??**), i.e., $\langle map\,g\rangle$. We also have that $(xs, xs') \in \langle map\,g\rangle$ implies $xs' = map\,g\,xs$, and that $(s,s') \in \langle g\rangle \to \mathsf{Eq}_{Bool}$ implies $\forall (x, gx) \in \langle g\rangle.\; sx = s'(gx)$ and thus $s = s'\circ g$ due to the definition of morphisms between relations. With these instantiations, Equation 1 becomes

$$\begin{aligned}&(t(\pi_1\rho)\,a\,(s'\circ g)\,xs, t(\pi_2\rho)\,b\,s'\,(map\,g\,xs)) \in \langle map\,g\rangle,\\ &i.e.,\\ &map\,g\,(t(\pi_1\rho)\,a\,(s'\circ g)\,xs) = t(\pi_2\rho)\,b\,s'\,(map\,g\,xs),\\ &i.e.,\\ &map\,g \circ t(\pi_1\rho)\,a\,(s'\circ g) = t(\pi_2\rho)\,b\,s' \circ map\,g\end{aligned}$$

as desired.

□

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