WTS $\lambda\overline{\eta}.\lambda\overline{B}.([\![\Gamma;\emptyset|\emptyset \vdash MAP_{H_1\times H_2}]\!]\rho * \overline{\eta})_{\overline{B}}$

$= \lambda\overline{\eta}.\lambda\overline{B}.(([\![\Gamma;\emptyset|\emptyset\vdash MAP_{H_1}]\!]\rho*\overline{\eta})_{\overline{B}} \times ([\![\Gamma;\emptyset|\emptyset\vdash MAP_{H_2}]\!]\rho*\overline{\eta})_{\overline{B}})$

Proof

$\lambda\overline{\eta}.\lambda\overline{B}.([\![\Gamma;\emptyset|\emptyset\vdash MAP_{H_1\times H_2}]\!]\rho*\overline{\eta})_{\overline{B}}$

$=\lambda\overline{\eta}.\lambda\overline{B}.(Map_{[\![\Gamma;\Phi,\overline{\gamma}\vdash H_1\times H_2]\!]\rho[\overline{\gamma:=B}][\overline{\Phi:=-}]}(\overline{\lambda\overline{A}.\eta_{\overline{A}\overline{B}}}))$

$=\lambda\overline{\eta}.\lambda\overline{B}.(map_{[\![\Gamma;\Phi,\overline{\gamma}\vdash H_1]\!]\rho[\overline{\gamma:=B}][\overline{\Phi:=-}]\times[\![\Gamma;\Phi,\overline{\gamma}\vdash H_2]\!]\rho[\overline{\gamma:=B}][\overline{\Phi:=-}]}(\overline{\lambda\overline{A}.\eta_{\overline{A}\overline{B}}}))$

$=\lambda\overline{\eta}.\lambda\overline{B}.(map_{[\![\Gamma;\Phi,\overline{\gamma}\vdash H_1]\!]\rho[\overline{\gamma:=B}][\overline{\Phi:=-}]}(\overline{\lambda\overline{A}.\eta_{\overline{A}\overline{B}}}) \times Map_{[\![\Gamma;\Phi,\overline{\gamma}\vdash H_2]\!]\rho[\overline{\gamma:=B}][\overline{\Phi:=-}]}(\overline{\lambda\overline{A}.\eta_{\overline{A}\overline{B}}}))$

$=\lambda\overline{\eta}.\lambda\overline{B}.(([\![\Gamma;\emptyset|\emptyset\vdash MAP_{H_1}]\!]\rho*\overline{\eta})_{\overline{B}} \times ([\![\Gamma;\emptyset|\emptyset\vdash MAP_{H_2}]\!]\rho*\overline{\eta})_{\overline{B}})$

WTS $\lambda\overline{\eta}.\lambda\overline{B}.([\![\Gamma;\emptyset|\emptyset\vdash MAP_{\mathbb{0}} : Nat^{\emptyset}(\overline{Nat^{\overline{\alpha},\overline{\gamma}} F G})(Nat^{\overline{\gamma}}\,\mathbb{0}\,\mathbb{0})]\!]\rho*\overline{\eta})_{\overline{B}}$

$=\lambda\overline{\eta}.\lambda\overline{B}.([\![\Gamma;\emptyset|\emptyset\vdash L_{\overline{\gamma}}\,x.x : Nat^{\overline{\gamma}}\,\mathbb{0}\,\mathbb{0}]\!]\rho *)_{\overline{B}}$

Proof

$\lambda\overline{\eta}.\lambda\overline{B}.([\![\Gamma;\emptyset|\emptyset\vdash MAP_{\mathbb{0}} : Nat^{\emptyset}(\overline{Nat^{\overline{\alpha},\overline{\gamma}} F G})(Nat^{\overline{\gamma}}\,\mathbb{0}\,\mathbb{0})]\!]\rho*\overline{\eta})_{\overline{B}}$

$=\lambda\overline{\eta}.\lambda\overline{B}.\,map_{[\![\Gamma;\Phi,\overline{\gamma}\vdash\mathbb{0}]\!]\rho[\overline{\gamma:=B}][\overline{\Phi:=-}]}(\overline{\lambda\overline{A}.\eta_{\overline{A}\overline{B}}})$

$=\lambda\overline{\eta}.\lambda\overline{B}.\,map_{-\mapsto 0}(\overline{\lambda\overline{A}.\eta_{\overline{A}\overline{B}}})$

$=\lambda\overline{\eta}.\lambda\overline{B}.(id_0)_{\overline{B}}$

$=\lambda\overline{\eta}.\lambda\overline{B}.(\lambda z.\pi_1\langle z\rangle)_{\overline{B}}$

$=\lambda\overline{\eta}.\lambda\overline{B}.(curry\ \pi_1\ *)_{\overline{B}}$

$=\lambda\overline{\eta}.\lambda\overline{B}.(curry([\![\Gamma;\overline{\gamma}|x:\mathbb{0}\vdash x:\mathbb{0}]\!]\rho)*)_{\overline{B}}$

$=\lambda\overline{\eta}.\lambda\overline{B}.([\![\Gamma;\emptyset|\emptyset\vdash L_{\overline{\gamma}}\,x.x:Nat^{\overline{\gamma}}\,\mathbb{0}\,\mathbb{0}]\!]\rho*)_{\overline{B}}$

$\mathbb{1}$ and $v:\Pi$ cases analogous. Replace $\mathbb{0}/0$ with $\mathbb{1}/1$ or $v/\rho v$ where appropriate.

$$\Gamma;\emptyset \vdash \mathrm{Nat}^{\overline{\beta}}\, H\, K$$

$$\Gamma;\emptyset|\emptyset \vdash \mathrm{MAP}_{\mathrm{Nat}^{\overline{\beta}} H K} : \mathrm{Nat}^{\emptyset}\,(\mathrm{Nat}^{\emptyset}\, F\, G)(\mathrm{Nat}^{\emptyset}(\mathrm{Nat}^{\overline{\beta}}\, H K)(\mathrm{Nat}^{\overline{\beta}}\, H K))$$

Same as ①, ② case but no gammas and ψ_s (0-length vector)

WTS $[\![\Gamma;\emptyset|\emptyset \vdash \mathrm{MAP}_{\mathrm{Nat}^{\overline{\beta}} H K} : \sim\!\sim]\!]\rho * ()$

$= [\![\Gamma;\emptyset|\emptyset \vdash L_{\emptyset}\, x.x : \mathrm{Nat}^{\emptyset}\,(\mathrm{Nat}^{\overline{\beta}}\, H K)\,(\mathrm{Nat}^{\overline{\beta}}\, H\, K)]\!]\rho *$

<u>Proof</u>

$[\![\Gamma;\emptyset|\emptyset \vdash \mathrm{MAP}_{\mathrm{Nat}^{\overline{\beta}} H K} : \sim\!\sim]\!]\rho * ()$

$= \mathrm{map}_{[\![\Gamma;\emptyset \vdash \mathrm{Nat}^{\overline{\beta}}\, H\, K]\!]\rho}\ ()$

$= \mathrm{id}_{[\![\Gamma;\emptyset \vdash \mathrm{Nat}^{\overline{\beta}}\, H\, K]\!]\rho}$

$= \lambda\eta.\ \pi_1 \langle \eta \rangle$

$= \mathrm{curry}\ \pi_1\ *$

$= \mathrm{curry}\,([\![\Gamma;\emptyset|\eta:\mathrm{Nat}^{\overline{\beta}}\, H K \vdash \eta : \mathrm{Nat}^{\overline{\beta}}\, H K]\!]\rho)\ *$

$= [\![\Gamma;\emptyset|\emptyset \vdash L_{\emptyset}\, \eta.\eta\ : \mathrm{Nat}^{\emptyset}\,(\mathrm{Nat}^{\overline{\beta}}\, H K)\,(\mathrm{Nat}^{\overline{\beta}}\, H K)]\!]\rho *$

$\boxed{map_H\ id_F = id_{HF}}$ $: Nat^{\emptyset}\ (\overline{Nat^{\overline{\alpha}\,\overline{\gamma}}\ F\, G})\ (Nat^{\overline{\gamma}}\ H[\overline{\varphi = F}]\ \ H[\overline{\varphi = G}])$

$$\text{WTS } ([\![\Gamma;\emptyset|\emptyset \vdash MAP_H]\!]\rho\, *)\ id_{[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash F]\!]\rho[\overline{\gamma := -}][\overline{\alpha := -}]}$$

$$= [\![\Gamma;\emptyset|\emptyset \vdash L_{\overline{\gamma}}\, x.x : Nat^{\overline{\gamma}}\ H[\overline{\varphi = F}]\ \ H[\overline{\varphi = F}]]\!]\rho\, *$$

Proof

$$\lambda\overline{B}.\left([\![\Gamma;\emptyset|\emptyset \vdash MAP_H]\!]\rho\, *\ id_{[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash F]\!]\rho[\overline{\gamma := -}][\overline{\alpha := -}]}\right)_{\overline{B}}$$

$$= \lambda\overline{B}.\left([\![\Gamma;\emptyset|\emptyset \vdash MAP_H]\!]\rho\, *\ id_{[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash F]\!]\rho[\overline{\gamma = B}][\overline{\alpha := -}]}\right)$$

$$= \lambda\overline{B}.\ map_{[\![\Gamma;\overline{\varphi},\overline{\gamma} \vdash H]\!]\rho[\overline{\gamma = B}][\overline{\varphi := -}]}\ \ id_{[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash F]\!]\rho[\overline{\gamma = B}][\overline{\alpha := -}]}$$

$$= \lambda\overline{B}.\ id_{[\![\Gamma;\overline{\varphi},\overline{\gamma} \vdash H]\!]\rho[\overline{\gamma = B}][\overline{\varphi = [\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash F]\!]\rho[\overline{\gamma = B}][\overline{\alpha := -}]}]}$$

$$\cancel{= \lambda\overline{B}.\ id_{[\![\Gamma;\overline{\gamma} \vdash H]\!]\rho[}}$$

$$= \lambda\overline{B}.\ id_{[\![\Gamma;\overline{\gamma} \vdash H[\overline{\varphi = F}]]\!]\rho[\overline{\gamma = B}]}$$

$$= \lambda\overline{B}.\left(curry\ \pi_1\ *\right)_{\overline{B}} \quad \text{where } \pi_1 : \langle [\![\Gamma;\overline{\gamma} \vdash H[\overline{\varphi = F}]]\!]\rho \rangle \to [\![\Gamma;\overline{\gamma} \vdash H[\overline{\varphi = F}]]\!]\rho$$

$$= \lambda\overline{B}.\left(curry\ ([\![\Gamma;\overline{\gamma}|x : H[\overline{\varphi = F}] \vdash x : H[\overline{\varphi = F}]]\!]\rho)\ *\right)_{\overline{B}}$$

$$= \lambda\overline{B}.\left([\![\Gamma;\emptyset|\emptyset \vdash L_{\overline{\gamma}}\, x.x : Nat^{\overline{\gamma}}\ H[\overline{\varphi = F}]\ \ H[\overline{\varphi = F}]]\!]\rho\, *\right)_{\overline{B}}$$

WTS $\llbracket \Gamma;\emptyset|\emptyset \vdash L_{\bar\gamma} x.((MAP_H)_\emptyset f)_{\bar\gamma}(((MAP_H)_\emptyset g)_{\bar\gamma} x) : Nat^{\bar\gamma}\ H[\bar\psi = \bar F]\ H[\bar\psi=\bar K]\rrbracket_\rho *$

$= \llbracket \Gamma;\emptyset|\emptyset \vdash (MAP_H)_\emptyset (L_{\bar\alpha\bar\gamma} x.\ f_{\bar\alpha\bar\gamma}(g_{\bar\alpha\bar\gamma} x)) : Nat^{\bar\gamma}\ H[\bar\psi=\bar F]\ H[\bar\psi = \bar K]\rrbracket_\rho *$

$f : Nat^{\bar\alpha\bar\gamma}\ G\ K$

$g : Nat^{\bar\alpha\bar\gamma}\ F\ G$

$\llbracket \Gamma;\emptyset|\emptyset \vdash L_{\bar\gamma} x.((MAP_H)_\emptyset f)_{\bar\gamma}(((MAP_H)_\emptyset g)_{\bar\gamma} x) : Nat^{\bar\gamma}\ H[\bar\psi=\bar F]\ \ H[\bar\psi=\bar K]\rrbracket_\rho *$

$curry(\llbracket \Gamma;\bar\gamma|x : H[\bar\psi=\bar F] \vdash ((MAP_H)_\emptyset f)_{\bar\gamma}(((MAP_H)_\emptyset g)_{\bar\gamma} x) : H[\bar\psi=\bar K]\rrbracket_\rho) *$

$map\llbracket f\rrbracket \circ map\llbracket g\rrbracket$

$\lambda\llbracket x\rrbracket.\ \llbracket \Gamma;\bar\gamma/x : H[\bar\psi=\bar F]\ \vdash (MAP_H)_\emptyset f)_{\bar\gamma}(((MAP_H)_\emptyset g)_{\bar\gamma} x)\rrbracket_\rho \llbracket x\rrbracket$

$\lambda\llbracket x\rrbracket.(\llbracket \Gamma;\emptyset|x : H[\bar\psi=\bar F]\vdash (MAP_H)_\emptyset f\rrbracket_\rho \llbracket x\rrbracket)_{\overline{\llbracket \Gamma;\bar\gamma\vdash\gamma\rrbracket_\rho}}(\llbracket\Gamma;\bar\gamma|x : H[\bar\psi=\bar F]\ \vdash ((MAP_H)_\emptyset g)_{\bar\gamma}\ x : H[\bar\psi=\bar G]\rrbracket_\rho \llbracket x\rrbracket)$

$\lambda\llbracket x\rrbracket.(\lambda\bar B.\ map_{\llbracket\Gamma;\bar\psi,\bar\gamma\vdash H\rrbracket_\rho[\bar\gamma=\bar B][\bar\psi=-]}(\lambda\bar A.(\llbracket\Gamma;\emptyset|x : H[\bar\psi=\bar F]\vdash f\rrbracket_\rho \llbracket x\rrbracket)_{\bar A\bar B}))_{\overline{\llbracket\Gamma;\bar\gamma\vdash\gamma\rrbracket_\rho}}(\llbracket\Gamma;\bar\gamma|x : H[\bar\psi=\bar F]\vdash ((MAP_H)_\emptyset g)_{\bar\gamma}\ x\rrbracket_\rho \llbracket x\rrbracket)$

$\lambda\llbracket x\rrbracket.(\lambda\bar B.\ map_{\llbracket\Gamma;\bar\psi,\bar\gamma\vdash H\rrbracket_\rho[\bar\gamma=\bar B][\bar\psi=-]}(\lambda\bar A.(\llbracket\Gamma;\emptyset|x : H[\bar\psi=\bar F]\vdash f\rrbracket_\rho\llbracket x\rrbracket)_{\bar A\bar B}))_{\overline{\llbracket\Gamma;\bar\gamma\vdash\gamma\rrbracket_\rho}}((\lambda\bar B.\ map_{\llbracket\Gamma;\bar\psi,\bar\gamma\vdash H\rrbracket_\rho[\bar\gamma=\bar B][\bar\psi=-]}(\lambda\bar A.(\llbracket\Gamma;\emptyset|x : H[\bar\psi=\bar F]\vdash g\rrbracket_\rho\llbracket x\rrbracket)_{\bar A\bar B}))_{\overline{\llbracket\Gamma;\bar\gamma\vdash\gamma\rrbracket_\rho}}\llbracket x\rrbracket)$

$(\ map_H\llbracket f\rrbracket \circ map_H\llbracket g\rrbracket = map_H\ \llbracket f\rrbracket\circ\llbracket g\rrbracket\ )$

$= \lambda\bar B.\ map_{\llbracket\Gamma;\bar\psi,\bar\gamma\vdash H\rrbracket_\rho[\bar\gamma=\bar B][\bar\psi=-]}\ (\lambda\bar A.(\lambda\llbracket y\rrbracket.(\llbracket\Gamma;\emptyset|y : H[\bar\psi=\bar F]\vdash f\rrbracket_\rho\llbracket y\rrbracket)_{\overline{\llbracket\Gamma;\alpha\vdash\alpha\rrbracket_\rho}\ \overline{\llbracket\Gamma;\gamma\vdash\gamma\rrbracket_\rho}}((\llbracket\Gamma;\emptyset|y : H[\bar\psi=\bar F]\vdash g\rrbracket_\rho\llbracket y\rrbracket)_{\overline{\llbracket\Gamma;\alpha\vdash\alpha\rrbracket_\rho}\ \overline{\llbracket\Gamma;\gamma\vdash\gamma\rrbracket_\rho}}\llbracket y\rrbracket))_{\bar A\bar B})$

$$= \lambda \overline{B}.\, map_{[\![\Gamma;\overline{\varphi},\overline{\gamma} \vdash H]\!]_\rho[\overline{\gamma}=\overline{B}][\overline{\varphi}=-]} \left(\lambda \overline{A}.\, (\lambda [\![y]\!].\, [\![\Gamma;\overline{\gamma},\overline{\alpha} \mid y : H[\overline{\varphi}=\overline{F}] \vdash f_{\overline{\alpha}\overline{\gamma}}(g_{\overline{\alpha}\overline{\gamma}}\, y) : H[\overline{\varphi}=\overline{K}]]\!]_\rho [\![y]\!])_{\overline{A}\overline{B}}\right)$$

$$= \lambda \overline{B}.\, map_{[\![\Gamma;\overline{\varphi},\overline{\gamma} \vdash H]\!]_\rho[\overline{\gamma}=\overline{B}][\overline{\varphi}=-]} \left(\lambda \overline{A}.\, (curry([\![\Gamma;\overline{\alpha},\overline{\gamma} \mid y : H[\overline{\varphi}=\overline{F}] \vdash f_{\overline{\alpha}\overline{\gamma}}(g_{\overline{\alpha}\overline{\gamma}}\, y)]\!]_\rho)*)_{\overline{A}\overline{B}}\right)$$

$$= \lambda \overline{B}.\, map_{[\![\Gamma;\overline{\varphi},\overline{\gamma} \vdash H]\!]_\rho[\overline{\gamma}=\overline{B}][\overline{\varphi}=-]} \left(\lambda \overline{A}.\, ([\![\Gamma;\emptyset \mid \emptyset \vdash L_{\overline{\alpha}\overline{\gamma}}\, y.\, f_{\overline{\alpha}\overline{\gamma}}(g_{\overline{\alpha}\overline{\gamma}}\, y) : Nat^{\overline{\alpha}\overline{\gamma}}\, \cancel{H[\overline{\varphi}=\overline{F}]}\; F \;\; K]\!]_\rho *)_{\overline{A}\overline{B}}\right)$$

$$= \left(\lambda \overline{\eta}.\, \lambda \overline{B}.\, map_{[\![\Gamma;\overline{\varphi},\overline{\gamma} \vdash H]\!]_\rho[\overline{\gamma}=\overline{B}][\overline{\varphi}=-]} (\lambda \overline{A}.\, \eta_{\overline{A}\overline{B}})\right)_\emptyset \left([\![\Gamma;\emptyset \mid \emptyset \vdash L_{\overline{\alpha}\overline{\gamma}}\, y.\, f_{\overline{\alpha}\overline{\gamma}}(g_{\overline{\alpha}\overline{\gamma}}\, y)]\!]_\rho *\right)$$

$$= [\![\Gamma;\emptyset \mid \emptyset \vdash (MAP_H)_\emptyset (L_{\overline{\alpha}\overline{\gamma}}\, y.\, f_{\overline{\alpha}\overline{\gamma}}(g_{\overline{\alpha}\overline{\gamma}}\, y)) : Nat^{\overline{\gamma}}\; H[\overline{\varphi}=\overline{F}] \;\; H[\overline{\varphi}=\overline{K}]]\!]_\rho *$$