Free Theorems for Nested Types

ANONYMOUS AUTHOR(S)

1 SHORT CUT FUSION FOR ARBITRARY NESTED TYPES

Can take $\emptyset; \alpha \vdash c$ with $[\![\emptyset; \alpha \vdash c]\!]^{\mathsf{Set}}\rho = C$ for all ρ, i.e., can take c to denote a constant C. We then get a free theorem whose conclusion is $\mathit{fold}_H\, B \;\circ\; G\;\mu H\; \mathit{in}_H = G\,[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}\, B$.

Theorem 1. *Let* $\emptyset; \phi, \alpha \vdash F : \mathcal{F}$, *let* $\emptyset; \alpha \vdash K : \mathcal{F}$, *and let* $\phi; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha))$. *If we let* $H : [\mathsf{Set}, \mathsf{Set}] \to [\mathsf{Set}, \mathsf{Set}]$ *be defined by*

$$H\,f\,x \quad = \quad [\![\emptyset; \phi, \alpha \vdash F]\!]^{\mathsf{Set}}[\phi := f][\alpha := x]$$

and let

$$G = [\![\phi; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha))]\!]^{\mathsf{Set}}$$

then we have that, for every $B \in H[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \to [\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}$,

$$\mathit{fold}_H\, B\,(G\;\mu H\; \mathit{in}_H) = G\,[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}}\, B$$

Proof. We first note that the type of g is well-formed since $\emptyset; \phi, \alpha \vdash F : \mathcal{F}$ so our promotion theorem gives that $\phi; \alpha \vdash F : \mathcal{F}$, and $\phi; \alpha \vdash \phi\alpha : \mathcal{F}$, so that $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha) : \mathcal{T}$ and $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\mathbb{1}\,(\phi\alpha) : \mathcal{T}$. Then $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha) : \mathcal{F}$ and $\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\mathbb{1}\,(\phi\alpha) : \mathcal{F}$ also hold, and, finally, $\phi; \emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha}\, F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha)) : \mathcal{T}$

Theorem **??** gives that, for any relation environment ρ and any $(a, b) \in [\![\phi, \alpha; \emptyset \vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$\begin{array}{rcl}
(G\,(\pi_1\rho), G\,(\pi_2\rho)) & \in & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\alpha} F\,(\phi\alpha))\,(\mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha))]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha}\,\mathbb{1}\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to (\,\lambda A.1 \Rightarrow \lambda A.\,(\rho\phi)A\,) \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to (1 \Rightarrow \rho\phi) \\
& = & [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho \to \rho\phi
\end{array}$$

So if $(A, B) \in [\![\phi; \emptyset \vdash \mathsf{Nat}^{\alpha} F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho$ then

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B) \quad \in \quad \rho\phi$$

Now note that

$$[\![\vdash \mathsf{fold}_F^K : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\alpha} F[\phi := K]\,K)\,(\mathsf{Nat}^{\alpha}((\mu\phi.\lambda\alpha.F)\alpha)\,K)]\!]^{\mathsf{Set}} = \mathit{fold}_H$$

and consider the instantiation

$$\begin{array}{rcll}
A & = & \mathit{in}_H : H(\mu H) \Rightarrow \mu H & \\
B & : & H[\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} \Rightarrow [\![\emptyset; \alpha \vdash K]\!]^{\mathsf{Set}} & \\
\rho\phi & = & \langle \mathit{fold}_H\, B\rangle & \text{a graph of a natural transformation, defined in Enrico's notes}
\end{array}$$

2020. 2475-1421/2020/1-ART1 \$15.00
https://doi.org/

(Note that all the types here are well-formed.) This gives

$$\begin{array}{lcl} \pi_1\rho\phi & = & \mu H \\ \pi_2\rho\phi & = & [\![\emptyset;\alpha\vdash K]\!]^{\mathsf{Set}} \\ \rho\phi & : & \mathsf{Rel}(\pi_1\rho\phi,\pi_2\rho\phi) \\ A & : & [\![\phi;\emptyset\vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_1\rho) \\ B & : & [\![\phi;\emptyset\vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_2\rho) \end{array}$$

since

$$\begin{array}{lcll} A = in_H & : & H(\mu H)\Rightarrow \mu H & \\ & = & [\![\emptyset;\phi,\alpha\vdash F]\!]^{\mathsf{Set}}[\phi := \mu[\![\emptyset;\phi,\alpha\vdash F]\!]^{\mathsf{Set}}]\Rightarrow \mu[\![\emptyset;\phi,\alpha\vdash F]\!]^{\mathsf{Set}} & \\ & = & [\![\emptyset;\phi,\alpha\vdash F]\!]^{\mathsf{Set}}(\pi_1\rho)\Rightarrow [\![\emptyset;\phi,\alpha\vdash \phi\alpha]\!]^{\mathsf{Set}}(\pi_1\rho) & \\ & = & [\![\phi;\alpha\vdash F]\!]^{\mathsf{Set}}(\pi_1\rho)\Rightarrow [\![\phi;\alpha\vdash \phi\alpha]\!]^{\mathsf{Set}}(\pi_1\rho) & \text{Daniel's trick; now a theorem} \\ & = & [\![\phi;\emptyset\vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Set}}(\pi_1\rho) & \end{array}$$

We also have

$$\begin{array}{lcll} (A,B) = (in_H,B) & \in & [\![\phi;\emptyset\vdash \mathsf{Nat}^{\alpha}F\,(\phi\alpha)]\!]^{\mathsf{Rel}}\rho & \\ & = & \lambda A.[\![\phi;\alpha\vdash F]\!]^{\mathsf{Rel}}\rho[\alpha := A]\Rightarrow \lambda A.(\rho\phi)A & \\ & = & \lambda A.[\![\phi;\alpha\vdash F]\!]^{\mathsf{Rel}}[\phi := \langle fold_H\,B\rangle][\alpha := A]\Rightarrow \langle fold_H\,B\rangle & \\ & = & \lambda A.[\![\emptyset;\phi,\alpha\vdash F]\!]^{\mathsf{Rel}}[\phi := \langle fold_H\,B\rangle][\alpha := A]\Rightarrow \langle fold_H\,B\rangle & \text{Daniel's trick; now a the} \\ & = & [\![\emptyset;\phi,\alpha\vdash F]\!]^{\mathsf{Rel}}\langle fold_H\,B\rangle\Rightarrow \langle fold_H\,B\rangle & \\ & = & \langle [\![\emptyset;\phi,\alpha\vdash F]\!]^{\mathsf{Set}}\,(fold_H\,B)\rangle\Rightarrow \langle fold_H\,B\rangle & \text{Graph Lemma} \\ & = & \langle map_H\,(fold_H\,B)\rangle\Rightarrow \langle fold_H\,B\rangle & \end{array}$$

since if $(x,y)\in\langle map_H\,(fold_H\,B)\rangle$, i.e., if $map_H\,(fold_H\,B)\,x = y$, then $fold_H\,B\,(in_H\,x) = B\,y = B\,(map_H\,(fold_H\,B)\,x)$ by the definition of $fold_H$ as a (indeed, the unique) morphism from in_H to B. Thus,

$$(G\,(\pi_1\rho)\,A, G\,(\pi_2\rho)\,B)\in\langle fold_H\,B\rangle$$

i.e.,

$$fold_H\,B\,(G\,(\pi_1\rho)\,in_H) = G\,(\pi_2\rho)\,B$$

Since ϕ is the only free variable in G, this simplifies to

$$fold_H\,B\,(G\,\mu H\,in_H) = G\,[\![\emptyset;\alpha\vdash K]\!]^{\mathsf{Set}}\,B$$

□