Supplementary Material

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Theorem (Identity Extension Lemma). *If ρ is a set environment, and $\Gamma;\Phi \vdash F$, then $[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}$.*

Proof. By induction on F.

- $[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = 0_{\mathsf{Rel}} = \mathsf{Eq}_{0_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash \mathbb{0}]\!]^{\mathsf{Set}}\rho}$
- $[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = 1_{\mathsf{Rel}} = \mathsf{Eq}_{1_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho}$
- By definition, $[\![\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho$ is the relation on $[\![\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$ relating t and t' if, for all $R_1 : \mathsf{Rel}(A_1,B_1),\ldots,R_k : \mathsf{Rel}(A_k,B_k)$, $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel. To prove that this is equal to $\mathsf{Eq}_{[\![\Gamma;\overline{\gamma}\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho}$ we need to show that $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1),\ldots,R_k : \mathsf{Rel}(A_k,B_k)$ if and only if $t = t'$ and $(t_{\overline{A}}, t_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1),\ldots,R_k : \mathsf{Rel}(A_k,B_k)$. The only interesting part of this equivalence is to show that if $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ to $[\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1,B_1),\ldots,R_k : \mathsf{Rel}(A_k,B_k)$, then $t = t'$. By hypothesis, $(t_{\overline{A}}, t'_{\overline{A}})$ is a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := \mathsf{Eq}_A]}$ to $[\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho\overline{[\alpha := \mathsf{Eq}_A]}$ in Rel for all $A_1 \ldots A_k$: Set. By the induction hypothesis, it is therefore a morphism from $\mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$ to $\mathsf{Eq}_{[\![\Gamma;\overline{\gamma},\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$ in Rel. This means that, for every $x : \mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$, $t_{\overline{A}}x = t'_{\overline{A}}x$. Then, by extensionality, $t = t'$.
- The application case is proved by the following sequence of equalities, where the second equality is by the induction hypothesis and the definition of the relation environment Eq_ρ, the third is by the definition of application of relation transformers from Definition 9, and the fourth is by Lemma 21:

$$\begin{aligned}
[\![\Gamma;\Phi \vdash \phi\overline{F}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mathsf{Eq}_\rho\phi)\overline{[\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \mathsf{Eq}_{\rho\phi}\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= (\mathsf{Eq}_{\rho\phi})^*\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{(\rho\phi)\,\overline{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi\vdash \phi\overline{F}]\!]^{\mathsf{Set}}\rho}
\end{aligned}$$

2020. 2475-1421/2020/1-ART1 $15.00
https://doi.org/

- The fixpoint case is proven by the sequence of equalities

$$
\begin{aligned}
[\![\Gamma;\Phi,\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{F}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mu T_{H,\mathsf{Eq}_\rho})\,\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{H,\mathsf{Eq}_\rho} K_0\,\overline{[\![\Gamma;\Phi,\overline{\gamma} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{H,\mathsf{Eq}_\rho} K_0\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0})^*\,\overline{\mathsf{Eq}_{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0\,\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{\varinjlim_{n\in\mathbb{N}}(T^{\mathsf{Set}}_{H,\rho})^n K_0\,\overline{[\![\Gamma;\Phi,\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi,\overline{\gamma}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{F}]\!]^{\mathsf{Set}}\rho}
\end{aligned}
$$

Here, the third equality is by induction hypothesis, the fifth is by Lemma 21 and the fourth equality is because, for every $n \in \mathbb{N}$, the following two statements can be proved by simultaneous induction: and for any H, ρ, A, and subformula J of H,

$$
T^n_{H,\mathsf{Eq}_\rho} K_0\,\overline{\mathsf{Eq}_A} = (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0})^*\,\overline{\mathsf{Eq}_A} \tag{1}
$$

and

$$
\begin{aligned}
&[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash J]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_A]}\\
&\qquad = [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash J]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}
\end{aligned} \tag{2}
$$

We prove (1) by induction on n. The case $n = 0$ is trivial, because $T^0_{H,\mathsf{Eq}_\rho} K_0 = K_0$ and $(T^{\mathsf{Set}}_{H,\rho})^0 K_0 = K_0$; the inductive step is proved by the following sequence of equalities:

$$
\begin{aligned}
T^{n+1}_{H,\mathsf{Eq}_\rho} K_0\,\overline{\mathsf{Eq}_A} &= T^{\mathsf{Rel}}_{H,\mathsf{Eq}_\rho}(T^n_{H,\mathsf{Eq}_\rho} K_0)\overline{\mathsf{Eq}_A}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_A]}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}\\
&= \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^{n+1} K_0 \overline{A}}\\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^{n+1} K_0})^*\,\overline{\mathsf{Eq}_A}
\end{aligned}
$$

Here, the third equality is by (2) for $J = H$, the fifth by the induction hypothesis of the IEL on H, and the last is by Lemma 21.

We prove (2) by structural induction on J. The only interesting cases, though, are when $J = \mathsf{Nat}^{\overline{\beta}} G\,K$, when $J = \phi\overline{G}$, and when $J = (\mu\psi.\lambda\overline{\beta}.G)\overline{K}$.

– The case $J = \mathsf{Nat}^{\overline{\beta}} G\, K$ is proved by observing that $\Phi = \overline{\gamma}$ and

$$
\begin{aligned}
&[\![\Gamma; \overline{\gamma}, \phi, \overline{\alpha} \vdash \mathsf{Nat}^{\overline{\beta}} G\, K]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_A]} \\
&= \{\eta : [\![\Gamma; \overline{\gamma}, \phi, \overline{\alpha}, \overline{\beta} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_A]}\,\overline{[\beta := \_]} \Rightarrow \\
&\qquad [\![\Gamma; \overline{\gamma}, \phi, \overline{\alpha}, \overline{\beta} \vdash K]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_A]}\,\overline{[\beta := \_]}\} \\
&= \{\eta : [\![\Gamma; \overline{\beta} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho\overline{[\beta := \_]} \Rightarrow [\![\Gamma; \overline{\gamma}, \phi, \overline{\beta} \vdash K]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho} K_0]\overline{[\beta := \_]}\} \\
&= \{\eta : [\![\Gamma; \overline{\beta} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\beta := \_]} \Rightarrow \\
&\qquad [\![\Gamma; \overline{\gamma}, \phi, \overline{\beta} \vdash K]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\beta := \_]}\} \\
&= [\![\Gamma; \overline{\gamma}, \phi, \overline{\alpha} \vdash \mathsf{Nat}^{\overline{\beta}} G\, K]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}
\end{aligned}
$$

Here, the second equality holds because ϕ must either not appear in J or have arity 0 (since only functorial variables of arity 0 can appear free in bodies of μ types), in which case $\overline{\alpha}$ must be empty, in order for $\Gamma; \overline{\gamma}, \phi, \overline{\alpha} \vdash \mathsf{Nat}^{\overline{\beta}} G\, K$ to be well-typed. The third equality is by (2) for K when ϕ has arity 0.

– The case $J = \phi\overline{G}$ is proved by the sequence of equalities:

$$
\begin{aligned}
&[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash \phi\overline{G}]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_A]} \\
&= T^n_{H,\mathsf{Eq}_\rho} K_0\, \overline{[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho} K_0]\overline{[\alpha := \mathsf{Eq}_A]}} \\
&= T^n_{H,\mathsf{Eq}_\rho} K_0\, \overline{[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}} \\
&= T^n_{H,\mathsf{Eq}_\rho} K_0\, \overline{[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_{\rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}} \\
&= T^n_{H,\mathsf{Eq}_\rho} K_0\, \overline{\mathsf{Eq}_{[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash G]\!]^{\mathsf{Set}} \rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}} \\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0})^*\, \overline{\mathsf{Eq}_{[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash G]\!]^{\mathsf{Set}} \rho[\phi := (T^{\mathsf{Set}}_{H,\rho})^n K_0]\overline{[\alpha := A]}}} \\
&= (\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0})^*\, \overline{[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}} \\
&= [\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash \phi\overline{G}]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^n K_0}]\overline{[\alpha := \mathsf{Eq}_A]}
\end{aligned}
$$

Here, the second equality is by the induction hypothesis for (2) on the Gs, the fourth is by the induction hypothesis for the IEL on the Gs, and the fifth is by the induction hypothesis on n for (1).

– The case $J = (\mu\psi.\,\lambda\overline{\beta}.\,G)\overline{K}$ is proved by the sequence of equalities:

$$
\begin{aligned}
&[\![\Gamma;\Phi,\overline{\gamma},\phi,\overline{\alpha} \vdash (\mu\psi.\,\lambda\overline{\beta}.\,G)\overline{K}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha := \mathsf{Eq}_A}]\\
&= (\mu T_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]})\,\overline{[\![\Gamma;\Phi,\overline{\gamma},\phi,\overline{\alpha} \vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha := \mathsf{Eq}_A}]}\\
&= \varinjlim_{m\in\mathbb{N}} T^m_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}K_0\,(\overline{[\![\Gamma;\Phi,\overline{\gamma},\phi,\overline{\alpha} \vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha := \mathsf{Eq}_A}]})\\
&= \varinjlim_{m\in\mathbb{N}} T^m_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}K_0\,(\overline{[\![\Gamma;\Phi,\overline{\gamma},\phi,\overline{\alpha} \vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha := \mathsf{Eq}_A}]})\\
&= \varinjlim_{m\in\mathbb{N}} T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}K_0\,(\overline{[\![\Gamma;\Phi,\overline{\gamma},\phi,\overline{\alpha} \vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha := \mathsf{Eq}_A}]})\\
&= (\mu T_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]})\,\overline{[\![\Gamma;\Phi,\overline{\gamma},\phi,\overline{\alpha} \vdash K]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha := \mathsf{Eq}_A}]}\\
&= [\![\Gamma;\Phi,\overline{\gamma},\phi,\overline{\alpha} \vdash (\mu\psi.\,\lambda\overline{\beta}.\,G)\overline{K}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha := \mathsf{Eq}_A}]
\end{aligned}
$$

Here, the third equality is by the induction hypothesis for (2) on the Ks, and the fourth equality holds because we can prove that, for all $m \in \mathbb{N}$,

$$
T^m_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}K_0 = T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}K_0 \tag{3}
$$

Indeed, the base case of (3) is trivial because

$$
T^0_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}K_0 = K_0 = T^0_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}K_0
$$

and the inductive case is proved by:

$$
\begin{aligned}
&T^{m+1}_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}K_0\\
&= T_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}(T^m_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}K_0)\\
&= T_{G,\mathsf{Eq}_\rho[\phi:=T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha:=\mathsf{Eq}_A}]}(T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}K_0)\\
&= \lambda\overline{R}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha},\psi,\overline{\beta} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := T^n_{H,\mathsf{Eq}_\rho}K_0][\overline{\alpha := \mathsf{Eq}_A}][\psi := T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}K_0][\overline{\beta := R}]\\
&= \lambda\overline{R}.[\![\Gamma;\overline{\gamma},\phi,\overline{\alpha},\psi,\overline{\beta} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha := \mathsf{Eq}_A}][\psi := T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}K_0][\overline{\beta := R}]\\
&= T_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}(T^m_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}K_0)\\
&= T^{m+1}_{G,\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_{H,\rho})^nK_0}][\overline{\alpha:=\mathsf{Eq}_A}]}K_0
\end{aligned}
$$

Here, the second equality holds by the induction hypothesis for (3) on m. The fourth equality holds because ϕ either does not apear in G, or must have arity 0, in which case $\overline{\alpha}$ must be empty, if ϕ appears in G, and uses (2) for G when ϕ has arity 0.

- $[\![\Gamma;\Phi \vdash F + G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho + [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho} + \mathsf{Eq}_{[\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho + [\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F+G]\!]^{\mathsf{Set}}\rho}$
- $[\![\Gamma;\Phi \vdash F \times G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho \times [\![\Gamma;\Phi \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho} \times \mathsf{Eq}_{[\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi\vdash G]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash F\times G]\!]^{\mathsf{Set}}\rho}$

□

Theorem (Abstraction Theorem). Every well-formed term $\Gamma;\Phi \mid \Delta \vdash t : F$ induces a natural transformation from $[\![\Gamma;\Phi \vdash \Delta]\!]$ to $[\![\Gamma;\Phi \vdash F]\!]$, i.e., a triple of natural transformations

$$([\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}},\ [\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}},\ [\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Rel}})$$

where

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}} \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}} \to [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}$$

has as its component at $\rho : \mathsf{SetEnv}$ a morphism

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}}\rho \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Set}}\rho$$

in Set,

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Rel}} \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}} \to [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}$$

has as its component at $\rho : \mathsf{RelEnv}$ a morphism

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Rel}}\rho \quad : \quad [\![\Gamma;\Phi \vdash \Delta]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma;\Phi \vdash F]\!]^{\mathsf{Rel}}\rho$$

in Rel, and, for all $\rho : \mathsf{RelEnv}$,

$$[\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Rel}}\rho = ([\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}}(\pi_1\rho),\ [\![\Gamma;\Phi \mid \Delta \vdash t : F]\!]^{\mathsf{Set}}(\pi_2\rho)) \tag{4}$$

Proof. By induction on t. The only interesting cases are the cases for abstraction, application, map, in, and fold so we omit the others.

- $\underline{\Gamma;\overline{\gamma} \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G}$ To see that $[\![\Gamma;\overline{\gamma} \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ is a natural transformation from $[\![\Gamma;\overline{\gamma} \vdash \Delta]\!]^{\mathsf{Set}}$ to $[\![\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}$ we need show that, for every $\rho : \mathsf{SetEnv}$, $[\![\Gamma;\overline{\gamma} \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$ is a morphism in Set from $[\![\Gamma;\overline{\gamma} \vdash \Delta]\!]^{\mathsf{Set}}\rho$ to $[\![\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$, and that such family of morphisms is natural. First, we need to show that, for each $\overline{A : \mathsf{Set}}$ and each $d : [\![\Gamma;\overline{\gamma} \vdash \Delta]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]$, we have $([\![\Gamma;\overline{\gamma} \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho\, d)_{\overline{A}}$ $: [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}] \to [\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]$, but this follows easily from the induction hypothesis. That these maps comprise a natural transformation $\eta : [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}] \to [\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$ is clear since $\eta_{\overline{A}} = \mathsf{curry}\,([\![\Gamma;\overline{\gamma},\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}])\, d$ is the component at $\overline{A}$ of the partial specialization to d of the natural transformation $[\![\Gamma;\overline{\gamma},\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := \_}]$. To see that the components of η also satisfy the additional condition needed for η to be in $[\![\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho$, let $\overline{R : \mathsf{Rel}(A, B)}$ and suppose

$$\begin{array}{rcl} (u, v) & \in & [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \\ & = & ([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}], [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := B}], ([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}])^*) \end{array}$$

  Then the induction hypothesis and $(d, d) \in [\![\Gamma;\overline{\gamma} \vdash \Delta]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho = [\![\Gamma;\overline{\gamma} \vdash \Delta]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]$ ensure that

$$\begin{array}{rl} & (\eta_{\overline{A}}u, \eta_{\overline{B}}v) \\ = & (\mathsf{curry}\,([\![\Gamma;\overline{\gamma},\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}])\, d\, u, \mathsf{curry}\,([\![\Gamma;\overline{\gamma},\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := B}])\, d\, v) \\ = & \mathsf{curry}\,([\![\Gamma;\overline{\gamma},\overline{\alpha} \mid \Delta, x : F \vdash t : G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}])\,(d, d)\,(u, v) \\ : & [\![\Gamma;\overline{\gamma},\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \end{array}$$

Moreover, to see that $\llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho$ is natural in ρ, let $f : \rho \to \rho'$ and consider the following computation

$$
\begin{aligned}
& \llbracket\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}} f \circ \llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho \\
= {} & \lambda d.\, \llbracket\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}} f(\llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\, d) \\
= {} & \lambda d.\, (\lambda\overline{A}.\, \llbracket\Gamma;\overline{\gamma},\overline{\alpha} \vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id_A}]) \circ \llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\, d \\
= {} & \lambda d\,\overline{A}.\, \llbracket\Gamma;\overline{\gamma},\overline{\alpha} \vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id_A}] \circ (\llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\, d)_{\overline{A}} \\
= {} & \lambda d\,\overline{A}.\, \llbracket\Gamma;\overline{\gamma},\overline{\alpha} \vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id_A}] \circ \llbracket\Gamma;\overline{\gamma},\overline{\alpha}\,|\,\Delta, x : F \vdash t : G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := A}]\, d \\
= {} & \lambda d\,\overline{A}\,x.\, \llbracket\Gamma;\overline{\gamma},\overline{\alpha} \vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id_A}](\llbracket\Gamma;\overline{\gamma},\overline{\alpha}\,|\,\Delta, x : F \vdash t : G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := A}]\, d\, x) \\
= {} & \lambda d\,\overline{A}\,x.\, \llbracket\Gamma;\overline{\gamma},\overline{\alpha}\,|\,\Delta, x : F \vdash t : G\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha := A}](\llbracket\Gamma;\overline{\gamma},\overline{\alpha} \vdash \Delta, x : F\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id_A}]\, d\, x) \\
= {} & \lambda d\,\overline{A}\,x.\, \llbracket\Gamma;\overline{\gamma},\overline{\alpha}\,|\,\Delta, x : F \vdash t : G\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha := A}](\llbracket\Gamma;\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}} f\, d)(\llbracket\Gamma;\overline{\alpha} \vdash F\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id_A}]x) \\
= {} & \lambda d\,\overline{A}\,x.\, \llbracket\Gamma;\overline{\gamma},\overline{\alpha}\,|\,\Delta, x : F \vdash t : G\rrbracket^{\mathsf{Set}}\rho'[\overline{\alpha := A}](\llbracket\Gamma;\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}} f\, d)\, x \\
= {} & \lambda d\,\overline{A}.\, (\llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho'(\llbracket\Gamma;\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}} f\, d))_{\overline{A}} \\
= {} & \llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho' \circ \llbracket\Gamma;\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}} f
\end{aligned}
$$

where the sixth equality is by the naturality of the interpretation of $\Gamma;\overline{\gamma},\overline{\alpha}\,|\,\Delta, x : F \vdash t : G$, which is given by the induction hypothesis, the seventh equality is by currying, and the eighth equality uses the functoriality of $\llbracket\Gamma;\overline{\alpha} \vdash F\rrbracket^{\mathsf{Set}}$ and the fact that the only functorial variables in F are in $\overline{\alpha}$.

- $\underline{\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash t_{\overline{K}}s : G[\overline{\alpha := K}]}$ To see that $\llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash t_{\overline{K}}s : G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}$ is a natural transformation from $\llbracket\Gamma;\Phi,\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}}$ to $\llbracket\Gamma;\Phi,\overline{\gamma} \vdash G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}$ we must show that, for every ρ : SetEnv, $\llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash t_{\overline{K}}s : G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho$ is a morphism from $\llbracket\Gamma;\Phi,\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}}\rho$ to $\llbracket\Gamma;\Phi,\overline{\gamma} \vdash G[\overline{\alpha := K}]\rrbracket^{\mathsf{Se}}$ and that this family of morphisms is natural in ρ. Let $d : \llbracket\Gamma;\Phi,\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}}\rho$. Then

$$
\begin{aligned}
& \llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash t_{\overline{K}}s : G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\,\rho\, d \\
= {} & (\mathsf{eval} \circ \langle(\llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\ \_)_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}}, \llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash s : F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho\rangle)\, d \\
= {} & \mathsf{eval}((\llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\ \_)_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}}\, d, \llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash s : F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho\, d) \\
= {} & \mathsf{eval}((\llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\ d)_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}}, \llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash s : F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho\, d)
\end{aligned}
$$

The induction hypothesis ensures that $(\llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho\ d)_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}}$ has type $\llbracket\Gamma;\overline{\alpha} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma} \vdash K\rrbracket^{\mathsf{Set}}\rho}] \to \llbracket\Gamma;\overline{\gamma},\overline{\alpha} \vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma} \vdash K\rrbracket^{\mathsf{Set}}\rho}]$. Since, in addition, $\llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash s : F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho\, d : \llbracket\Gamma;\Phi,\overline{\gamma} \vdash F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho = \llbracket\Gamma;\Phi,\overline{\gamma},\overline{\alpha} \vdash F\rrbracket^{\mathsf{Set}}\rho[\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}$ $\llbracket\Gamma;\overline{\alpha} \vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma} \vdash K\rrbracket^{\mathsf{Set}}\rho}]$ by Equation (6), we have that $\llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash t_{\overline{K}}s : G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\,\rho\, d$: $\llbracket\Gamma;\Phi,\overline{\gamma},\overline{\alpha} \vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma} \vdash K\rrbracket^{\mathsf{Set}}\rho}] = \llbracket\Gamma;\Phi,\overline{\gamma} \vdash G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho$, as desired.

To see that the family of maps comprising $\llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash t_{\overline{K}}s : G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}$ is natural in ρ we need to show that, if $f : \rho \to \rho'$ in SetEnv, then the following diagram commutes, where $g = \llbracket\Gamma;\overline{\gamma}\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}$ and $h = \llbracket\Gamma;\Phi,\overline{\gamma}\,|\,\Delta \vdash s : F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}$:

$$
\begin{tikzcd}
\llbracket\Gamma;\Phi,\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}}\rho \arrow[r, "\llbracket\Gamma;\Phi,\overline{\gamma}\vdash\Delta\rrbracket^{\mathsf{Set}} f"] \arrow[d, "\langle g\rho, h\rho\rangle"'] & \llbracket\Gamma;\Phi,\overline{\gamma} \vdash \Delta\rrbracket^{\mathsf{Set}}\rho' \arrow[d, "\langle g\rho', h\rho'\rangle"] \\
\llbracket\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho \times \llbracket\Gamma;\Phi,\overline{\gamma} \vdash F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho \arrow[r, "\llbracket\Gamma;\overline{\gamma}\vdash\mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}} f \times \llbracket\Gamma;\Phi,\overline{\gamma}\vdash F[\overline{\alpha:=K}]\rrbracket^{\mathsf{Set}} f"] \arrow[d, "\mathsf{eval}\circ((-)_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket\rho}}\times id)"'] & \llbracket\Gamma;\overline{\gamma} \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho' \times \llbracket\Gamma;\Phi,\overline{\gamma} \vdash F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho' \arrow[d, "\mathsf{eval}\circ((-)_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket\rho'}}\times id)"] \\
\llbracket\Gamma;\Phi,\overline{\gamma} \vdash G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho \arrow[r, "\llbracket\Gamma;\Phi,\overline{\gamma}\vdash G[\overline{\alpha:=K}]\rrbracket^{\mathsf{Set}} f"'] & \llbracket\Gamma;\Phi,\overline{\gamma} \vdash G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho'
\end{tikzcd}
$$

The top diagram commutes because g and h are natural in ρ by the induction hypothesis. To see that the bottom diagram commutes, we need to show that $\llbracket\Gamma;\Phi,\overline{\gamma}\vdash G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}} f(\eta_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket\rho}}x) = (\llbracket\Gamma;\overline{\gamma}\vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}} f\,\eta)_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket\rho'}}(\llbracket\Gamma;\Phi,\overline{\gamma}\vdash F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}} f\,x)$ holds for all $\eta\in\llbracket\Gamma;\overline{\gamma}\vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho$ and $x\in\llbracket\Gamma;\Phi,\overline{\gamma}\vdash F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho$, i.e., by remembering the following facts,

$$
\begin{aligned}
\llbracket\Gamma;\Phi,\overline{\gamma}\vdash F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho &= \llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}]\\
\llbracket\Gamma;\Phi,\overline{\gamma}\vdash F[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}} f &= \llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}} f}]\\
\llbracket\Gamma;\Phi,\overline{\gamma}\vdash G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}}\rho &= \llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}]\\
\llbracket\Gamma;\Phi,\overline{\gamma}\vdash G[\overline{\alpha := K}]\rrbracket^{\mathsf{Set}} f &= \llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}} f}]
\end{aligned}
$$

we need to show that

$$
\begin{aligned}
&\llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}} f}]\circ\eta_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}}\\
=\ &\llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id_{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho'}}]\circ\eta_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho'}}\circ\llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}} f}]
\end{aligned}
$$

for all $\eta\in\llbracket\Gamma;\overline{\gamma}\vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G\rrbracket^{\mathsf{Set}}\rho$. But this follows from the naturality of η, which ensures the commutativity of

$$
\begin{tikzcd}
\llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}] \arrow[r, "\eta_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}}"] \arrow[d, "{\llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha:=\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}} f}]}"'] & \llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho}] \arrow[d, "{\llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha:=\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}} f}]}"] \\
\llbracket\Gamma;\overline{\alpha}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho'}] \arrow[r, "\eta_{\overline{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho'}}"] & \llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho'}]
\end{tikzcd}
$$

and the observation that $\llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}} f}]$ is equal to

$$
\llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}} f[\overline{\alpha := id_{\llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}}\rho'}}]\circ\llbracket\Gamma;\overline{\gamma},\overline{\alpha}\vdash G\rrbracket^{\mathsf{Set}} id_\rho[\overline{\alpha := \llbracket\Gamma;\Phi,\overline{\gamma}\vdash K\rrbracket^{\mathsf{Set}} f}]
$$

- $\underline{\Gamma;\overline{\alpha}\mid\Delta\vdash \mathsf{map}^{\overline{F},\overline{G}}_H\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]}$ To see that

$$\llbracket\Gamma;\overline{\alpha}\mid\Delta\vdash \mathsf{map}^{\overline{F},\overline{G}}_H\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]\rrbracket^{\mathsf{Set}}$$

is a natural transformation from

$$\llbracket\Gamma;\overline{\alpha}\vdash\Delta\rrbracket^{\mathsf{Set}}$$

to

$$\llbracket\Gamma;\overline{\alpha}\vdash \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]\rrbracket^{\mathsf{Set}}$$

we need to show that

$$
\begin{aligned}
&\llbracket\Gamma;\overline{\alpha}\mid\Delta\vdash \mathsf{map}^{\overline{F},\overline{G}}_H\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]\rrbracket^{\mathsf{Set}}\,\rho\,d\\
:\ &\llbracket\Gamma;\overline{\alpha}\vdash \mathsf{Nat}^{\overline{\gamma}}\,H[\overline{\phi :=_{\overline{\beta}} F}]\,H[\overline{\phi :=_{\overline{\beta}} G}]\rrbracket^{\mathsf{Set}}\rho
\end{aligned}
$$

for all $\rho : \mathsf{SetEnv}$ and $d : \llbracket\Gamma;\overline{\alpha}\vdash\Delta\rrbracket^{\mathsf{Set}}\rho$, and that this family of morphisms is natural in ρ. For this, $\underline{\text{we first note that } \llbracket\Gamma;\overline{\phi},\overline{\gamma}\vdash H\rrbracket^{\mathsf{Set}}}$ is a functor from SetEnv to Set and, for any $\overline{B}$, $id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.(\llbracket\Gamma;\overline{\alpha}\mid\Delta\vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,F\,G\rrbracket^{\mathsf{Set}}\rho d)_{\overline{A}\,\overline{B}}}]$ is a morphism in SetEnv from

$$\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}]$$

to

$$\rho[\overline{\gamma := B}][\overline{\phi := \lambda\overline{A}.\llbracket\Gamma;\overline{\alpha},\overline{\gamma},\overline{\beta}\vdash G\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma := B}][\overline{\beta := A}]}]$$

so that

$$
\begin{aligned}
&(\llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash \mathsf{map}_H^{\overline{F},\overline{G}}\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}}\, \rho\, d)_{\overline{B}} \\
&\qquad = \llbracket \Gamma; \overline{\phi}, \overline{\gamma} \vdash H \rrbracket^{\mathsf{Set}} id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.(\llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G \rrbracket^{\mathsf{Set}} \rho d)_{\overline{A}\,\overline{B}}}]
\end{aligned}
$$

which is indeed a morphism from

$$\llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := F}] \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := B}]$$

to

$$\llbracket \Gamma; \overline{\alpha}, \overline{\gamma} \vdash H[\overline{\phi := G}] \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := B}]$$

This family of morphisms is natural in $\overline{B}$: if $\overline{f : B \to B'}$ then, writing η for

$$\llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash \mathsf{map}_H^{\overline{F},\overline{G}}\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}}\, \rho\, d$$

the naturality of $\llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G \rrbracket^{\mathsf{Set}} \rho\, d$, together with the fact that composition of environments is computed componentwise, ensure that the following naturality diagram for η commutes:

$$
\begin{array}{ccc}
\llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := F}] \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := B}] & \xrightarrow{\eta_{\overline{B}}} & \llbracket \Gamma; \overline{\alpha}, \overline{\gamma} \vdash H[\overline{\phi := G}] \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := B}] \\
{\scriptstyle \llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := F}] \rrbracket^{\mathsf{Set}} id_{\rho}[\overline{\gamma := f}]}\big\downarrow & & \big\downarrow{\scriptstyle \llbracket \Gamma; \overline{\alpha}, \overline{\gamma} \vdash H[\overline{\phi := G}] \rrbracket^{\mathsf{Set}} id_{\rho}[\overline{\gamma := f}]} \\
\llbracket \Gamma; \overline{\gamma} \vdash H[\overline{\phi := F}] \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := B'}] & \xrightarrow{\eta_{\overline{B'}}} & \llbracket \Gamma; \overline{\alpha}, \overline{\gamma} \vdash H[\overline{\phi := G}] \rrbracket^{\mathsf{Set}} \rho[\overline{\gamma := B'}]
\end{array}
$$

That, for all $\rho : \mathsf{SetEnv}$ and $d : \llbracket \Gamma; \overline{\alpha} \vdash \Delta \rrbracket^{\mathsf{Set}} \rho$, η satisfies the additional condition needed for it to be in $\llbracket \Gamma; \overline{\alpha} \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}} \rho$ follows from the fact that

$$\llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G \rrbracket^{\mathsf{Set}} \rho d$$

satisfies the extra condition needed for it to be in its corresponding $\llbracket \Gamma; \overline{\alpha} \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G \rrbracket^{\mathsf{Set}} \rho$. For the naturality of $\llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash \mathsf{map}_H^{\overline{F},\overline{G}}\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}}$, consider $f : \rho \to \rho'$. We need to prove that

$$
\begin{aligned}
&\llbracket \Gamma; \overline{\alpha} \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}} f \circ \llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash \mathsf{map}_H^{\overline{F},\overline{G}}\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}}\, \rho \\
&\qquad = \llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash \mathsf{map}_H^{\overline{F},\overline{G}}\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}}\, \rho' \circ \llbracket \Gamma; \overline{\alpha} \vdash \Delta \rrbracket^{\mathsf{Set}}\, f
\end{aligned}
$$

i.e., that

$$
\begin{aligned}
&\llbracket \Gamma; \overline{\alpha} \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}} f (\llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash \mathsf{map}_H^{\overline{F},\overline{G}}\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}}\, \rho\, d) \\
&\qquad = \llbracket \Gamma; \overline{\alpha} \mid \Delta \vdash \mathsf{map}_H^{\overline{F},\overline{G}}\,\overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}] \rrbracket^{\mathsf{Set}}\, \rho' (\llbracket \Gamma; \overline{\alpha} \vdash \Delta \rrbracket^{\mathsf{Set}}\, f\, d)
\end{aligned}
$$

for any $d : [\![\Gamma;\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\rho$. That is shown by the following calculations:

$$
\begin{aligned}
& [\![\Gamma;\overline{\alpha} \vdash \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}} f([\![\Gamma;\overline{\alpha} \mid \Delta \vdash \mathsf{map}^{\overline{F},\overline{G}}_H\, \overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\, \rho\, d) \\
= \; & \lambda\overline{B}.\, [\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}} f[\overline{\gamma := id_B}] \\
& \quad \circ([\![\Gamma;\overline{\alpha} \mid \Delta \vdash \mathsf{map}^{\overline{F},\overline{G}}_H\, \overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\, \rho\, d)_{\overline{B}} \\
= \; & \lambda\overline{B}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} f[\overline{\gamma := id_B}][\overline{\phi := \lambda\overline{A}.\, [\![\Gamma;\overline{\alpha},\overline{\beta},\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} f[\overline{\beta := id_A}][\overline{\gamma := id_B}]}] \\
& \quad \circ [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.([\![\Gamma;\overline{\alpha} \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}} \rho\, d)_{\overline{A}\,\overline{B}}}] \\
= \; & \lambda\overline{B}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} f[\overline{\gamma := id_B}] \\
& \quad [\overline{\phi := \lambda\overline{A}.\, [\![\Gamma;\overline{\alpha},\overline{\beta},\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} f[\overline{\beta := id_A}][\overline{\gamma := id_B}] \circ ([\![\Gamma;\overline{\alpha} \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}} \rho\, d)_{\overline{A}\,\overline{B}}}] \\
= \; & \lambda\overline{B}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho'[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.([\![\Gamma;\overline{\alpha} \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}} f([\![\Gamma;\overline{\alpha} \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}}\, \rho\, d))_{\overline{A}\,\overline{B}}}] \\
= \; & \lambda\overline{B}.\, [\![\Gamma;\overline{\phi},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} id_{\rho'[\overline{\gamma := B}]}[\overline{\phi := \lambda\overline{A}.([\![\Gamma;\overline{\alpha} \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G]\!]^{\mathsf{Set}} \rho'([\![\Gamma;\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\, f\, d))_{\overline{A}\,\overline{B}}}] \\
= \; & [\![\Gamma;\overline{\alpha} \mid \Delta \vdash \mathsf{map}^{\overline{F},\overline{G}}_H\, \overline{t} : \mathsf{Nat}^{\overline{\gamma}}\, H[\overline{\phi :=_{\overline{\beta}} F}]\, H[\overline{\phi :=_{\overline{\beta}} G}]]\!]^{\mathsf{Set}}\, \rho'([\![\Gamma;\overline{\alpha} \vdash \Delta]\!]^{\mathsf{Set}}\, f\, d)
\end{aligned}
$$

where the third equality is given by composition of morphisms of environments and the fifth equality is given by the naturality of $\Gamma;\overline{\alpha} \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, F\, G$, which we have by the induction hypothesis.

- $\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}$ To see that if $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$ then $[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\, \rho\, d$ is in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\, \rho$, we first note that, for all $\overline{B}$ and $\overline{C}$, $([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\, \rho\, d)_{\overline{B}\,\overline{C}} \;=\; (in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}})_{\overline{B}}$ maps $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] = T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\,\overline{B}$ to $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] = (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\,\overline{B}$. Secondly, we observe that $[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\; (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\, \rho\, d = \lambda\overline{B}\,\overline{C}.\,(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}})_{\overline{B}}$ is natural in $\overline{B}$ and $\overline{C}$, since naturality of in with respect to its functor argument and naturality of $in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}$ ensure that the following diagram commutes for all $\overline{f : B \to B'}$ and $\overline{g : C \to C'}$:

$$
\begin{tikzcd}
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\,\overline{B} \arrow[r, "(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}})_{\overline{B}}"] \arrow[d, "T^{\mathsf{Set}}_{H,id_\rho[\overline{\gamma := g}]}\,(\mu T^{\mathsf{Set}}_{H,id_\rho[\overline{\gamma := g}]})\,\overline{B}"] & (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]})\,\overline{B} \arrow[d, "(\mu T^{\mathsf{Set}}_{H,id_\rho[\overline{\gamma := g}]})\,\overline{B}"] \\
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]})\,\overline{B} \arrow[r, "(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}})_{\overline{B}}"] \arrow[d, "T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]})\,\overline{f}"] & (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]})\,\overline{B} \arrow[d, "\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}\,\overline{f}"] \\
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]})\,\overline{B'} \arrow[r, "(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}})_{\overline{B'}}"] & (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]})\,\overline{B'}
\end{tikzcd}
$$

That, for all ρ : SetEnv and $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\,d$$

satisfies the additional property needed for it to be in

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho$$

let $\overline{R : \mathsf{Rel}(B,B')}$ and $\overline{S : \mathsf{Rel}(C,C')}$ follows from the fact that

$$\begin{array}{rl} & \big(\,([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\,d)_{\overline{B},\overline{C}},\\ & \quad([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\;(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\,\rho\,d)_{\overline{B'},\overline{C'}}\,\big)\\ = & \big(\,(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}})_{\overline{B}},\,(in_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}})_{\overline{B'}}\,\big)\end{array}$$

has type

$$\begin{array}{rl} & \big(\,T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\,\overline{B} \to (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\,\overline{B},\\ & \quad T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}\,(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{B'} \to (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\,\overline{B'}\,\big)\\ = & [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}] \to\\ & \quad [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\beta := R}][\overline{\gamma := S}]\end{array}$$

- $\underline{\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)}$ Since Φ is empty, to see that $[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)]\!]^{\mathsf{Set}}$ is a natural transformation $[\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}$ to

$$[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F]\!]^{\mathsf{Set}}$$

we need only show that, for all ρ : SetEnv, the unique $d : [\![\Gamma;\emptyset \vdash \emptyset]\!]^{\mathsf{Set}}\rho$, and all $\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F]\!]^{\mathsf{Set}}\rho$,

$$[\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F]\!]^{\mathsf{Set}}\,\rho\,d\,\eta$$

has type $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F]\!]^{\mathsf{Set}}\,\rho$ i.e., for any $\overline{B}$ and $\overline{C}$,

$$([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F]\!]^{\mathsf{Set}}\,\rho\,d\,\eta)_{\overline{B}\,\overline{C}}$$

is a morphism from $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] = (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\overline{B}$ to $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$. To see this, note that η is a natural transformation from

$$\begin{array}{rl} & \lambda\overline{B}\,\overline{C}.\,[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]\\ = & \lambda\overline{B}\,\overline{C}.\,T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}\,(\lambda\overline{A}.\,[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C}])\,\overline{B}\end{array}$$

to

$$\begin{array}{rl} & \lambda\overline{B}\,\overline{C}.\,(\lambda\overline{A}.\,[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C}])\overline{B}\\ = & \lambda\overline{B}\,\overline{C}.\,[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]\end{array}$$

and thus for each $\overline{B}$ and $\overline{C}$,

$$([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F]\!]^{\mathsf{Set}}\,\rho\,d\,\eta)_{\overline{B}\,\overline{C}}$$

is a morphism from $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] = (\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\overline{B}$ to $[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$.

To see that this family of morphisms is natural in $\overline{B}$ and $\overline{C}$, we observe that the following diagram commutes for all $\overline{f : B \to B'}$ and $\overline{g : C \to C'}$:

$$
\begin{tikzcd}
(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]})\overline{B} \arrow[r, "(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C}}))_{\overline{B}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{H,id_{\rho}[\overline{\gamma:=g}]})\overline{B}"'] & {[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\beta:=B}]} \arrow[d, "{[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho}[\overline{\gamma:=g}][\overline{\beta:=id_B}]}"] \\
(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\overline{B} \arrow[r, "(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}}))_{\overline{B}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\overline{f}"'] & {[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C'}][\overline{\beta:=B}]} \arrow[d, "{[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho}[\overline{\gamma:=id_{C'}}][\overline{\beta:=f}]}"] \\
(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]})\overline{B'} \arrow[r, "(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}}))_{\overline{B'}}"] & {[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C'}][\overline{\beta:=B'}]}
\end{tikzcd}
$$

Indeed, naturality of $fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C'}})$ ensures that the bottom diagram commutes. To see that the top one commutes we first observe that, given a natural transformation $\Theta : H \to K : [\mathsf{Set}^k, \mathsf{Set}] \to [\mathsf{Set}^k, \mathsf{Set}]$, the fixpoint natural transformation $\mu\Theta : \mu H \to \mu K : \mathsf{Set}^k \to \mathsf{Set}$ is defined to be $fold_H(\Theta\,(\mu K) \circ in_K)$, i.e., the unique morphism making the following diagram commute:

$$
\begin{tikzcd}
H(\mu H) \arrow[r, "H(\mu\Theta)"] \arrow[dd, "in_H"'] & H(\mu K) \arrow[d, "\Theta(\mu K)"] \\
 & K(\mu K) \arrow[d, "in_K"] \\
\mu H \arrow[r, "\mu\Theta"'] & \mu K
\end{tikzcd}
$$

Taking $\Theta = T^{\mathsf{Set}}_{H,f} : T^{\mathsf{Set}}_{H,\rho} \to T^{\mathsf{Set}}_{H,\rho'}$ thus gives that, for any $f : \rho \to \rho'$ in SetEnv,

$$
in_{T^{\mathsf{Set}}_{H,\rho'}} \circ T^{\mathsf{Set}}_{H,f}(\mu T^{\mathsf{Set}}_{H,\rho'}) \circ T^{\mathsf{Set}}_{H,\rho}(\mu T^{\mathsf{Set}}_{H,f}) \;=\; \mu T^{\mathsf{Set}}_{H,f} \circ in_{T^{\mathsf{Set}}_{H,\rho}} \tag{5}
$$

Next, note that the action of the functor $\lambda\overline{B}.\lambda\overline{C}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]$ on the morphisms $\overline{f : B \to B'}, \overline{g : C \to C'}$ is given by

$$
\begin{array}{ll}
 & [\![\Gamma;\overline{\beta},\overline{\gamma}\vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}} id_{\rho}[\overline{\beta := f}][\overline{\gamma := g}] \\
= & [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}} id_{\rho}[\overline{\alpha := f}][\overline{\gamma := g}][\phi := \lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=A}]}[\overline{\gamma := g}]] \\
= & [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\gamma:=C'}][\phi:=\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C'}]]}[\overline{\alpha := f}] \\
 & \quad \circ [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\alpha:=B}][\phi:=\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta:=A}][\overline{\gamma:=C'}]]}[\overline{\gamma := g}] \\
 & \quad \circ [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}} id_{\rho[\overline{\alpha:=B}][\overline{\gamma:=C}]}[\phi := \lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=A}]}[\overline{\gamma := g}]] \\
= & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\overline{f} \\
 & \quad \circ \big(T^{\mathsf{Set}}_{H,id_{\rho}[\overline{\gamma:=g}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}}\rho[\overline{\beta := A}][\overline{\gamma := C'}])\big)_{\overline{B}} \\
 & \quad \circ \big(T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\lambda\overline{A}.[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta:=A}]}[\overline{\gamma := g}])\big)_{\overline{B}}
\end{array}
$$

So if η is a natural transformation such that $\eta_{\overline{B},\overline{C}}$ has type

$$[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash H[\phi := F][\overline{\alpha := \beta}]]\!]^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}] \to [\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C}]$$

then, by naturality,

$$\begin{aligned}
& [\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho}[\overline{\beta := f}][\overline{\gamma := g}] \circ \eta_{\overline{B},\overline{C}} \\
= \; & \eta_{\overline{B'},\overline{C'}} \circ T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}(\lambda \overline{A}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\beta := A}][\overline{\gamma := C'}])\overline{f} \\
& \circ (T^{\mathsf{Set}}_{H, id_{\rho}[\overline{\gamma := g}]}(\lambda \overline{A}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\beta := A}][\overline{\gamma := C'}]))_{\overline{B}} \\
& \circ (T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}(\lambda \overline{A}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}]))_{\overline{B}}
\end{aligned}$$

As a special case when $\overline{f = id_B}$ we have

$$\begin{aligned}
& \lambda \overline{B}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := B}]}[\overline{\gamma := g}] \circ \lambda \overline{B}.\eta_{\overline{B},\overline{C}} \\
= \; & \lambda \overline{B}.\eta_{\overline{B},\overline{C'}} \circ T^{\mathsf{Set}}_{H, id_{\rho}[\overline{\gamma := g}]}(\lambda \overline{A}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\beta := A}][\overline{\gamma := C'}]) \\
& \qquad \circ T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}(\lambda \overline{A}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} id_{\rho[\overline{\beta := A}]}[\overline{\gamma := g}])
\end{aligned} \tag{6}$$

Finally, to see that the top diagram in the diagram on page 11 commutes we first note that functoriality of $T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}$, naturality of $T^{\mathsf{Set}}_{H, id_{\rho}[\overline{\gamma := g}]}$, the universal property of $fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}(\lambda \overline{A}.\eta_{\overline{A},\overline{C'}})$ and Equation 5 ensure that the following diagram commutes:

$$\begin{tikzcd}
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}) \arrow[rr, "T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}(fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}(\lambda \overline{A}.\eta_{\overline{A},\overline{C'}}) \circ \mu T^{\mathsf{Set}}_{H, id_{\rho}[\overline{\gamma := g}]})"] \arrow[dd, "\mathsf{in}_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}}"'] & & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]}(\lambda \overline{B}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C'}]) \arrow[d, "T^{\mathsf{Set}}_{H, id_{\rho}[\overline{\gamma := g}]}(\lambda \overline{B}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C'}])"] \\
& & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}(\lambda \overline{B}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C'}]) \arrow[d, "\lambda \overline{A}.\eta_{\overline{A},\overline{C'}}"] \\
\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C}]} \arrow[r, "\mu T^{\mathsf{Set}}_{H, id_{\rho}[\overline{\gamma := g}]}"'] & \mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]} \arrow[r, bend right, "fold_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma := C'}]}}(\lambda \overline{A}.\eta_{\overline{A},\overline{C'}})"'] & \lambda \overline{B}.[\![\Gamma; \overline{\beta}, \overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\beta := B}][\overline{\gamma := C'}]
\end{tikzcd} \tag{7}$$

Next, we note that functoriality of $T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}$, Equation 6, and the universal property of $\mathit{fold}_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})$ ensure that the following diagram commutes:

$$
\begin{tikzcd}
T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}) \arrow[rr, "T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\mathit{id}_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}]\circ\mathit{fold}_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}))"] \arrow[dd, "\mathsf{in}_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}"'] & & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "T^{\mathsf{Set}}_{H,\mathit{id}_{\rho[\overline{\gamma:=g}]}}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}])"] \\
& & T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}(\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]) \arrow[d, "\lambda\overline{A}.\eta_{\overline{A},\overline{C'}}"] \\
\mu T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]} \arrow[r, bend right, "\mathit{fold}_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})"'] & \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}] \arrow[r, bend right, "\lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\mathit{id}_{\rho[\overline{\beta:=B}]}[\overline{\gamma:=g}]"'] & \lambda\overline{B}.\llbracket\Gamma;\overline{\beta},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C'}]
\end{tikzcd}
\tag{8}
$$

Combining the equations entailed by 7 and 8, we get that the top diagram in the diagram on page 11 commutes, as desired. To see that, for all ρ : SetEnv, $d \in \llbracket\Gamma;\emptyset\vdash\emptyset\rrbracket^{\mathsf{Set}}\rho$, and $\eta : \llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi:=_{\overline{\beta}}F][\overline{\alpha:=\beta}]\,F\rrbracket^{\mathsf{Set}}\rho$,

$$\llbracket\Gamma;\emptyset\mid\emptyset\vdash\mathsf{fold}^F_H : \mathsf{Nat}^{\emptyset}\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,H[\phi:=_{\overline{\beta}}F][\overline{\alpha:=\beta}]\,F)\,(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F\rrbracket^{\mathsf{Set}}\,\rho\,d\,\eta$$

satisfies the additional condition needed for it to be in $\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,F)\rrbracket^{\mathsf{Set}}\rho$, let $\overline{R:\mathsf{Rel}(B,B')}$ and $\overline{S:\mathsf{Rel}(C,C')}$. Since η satisfies the additional condition needed for it to be in $\llbracket\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\,(H[\phi:=F][\overline{\alpha:=\beta}])\,F\rrbracket^{\mathsf{Set}}\rho$,

$$(\,(\mathit{fold}_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\,\eta_{\overline{A}\,\overline{C}}))_{\overline{B}},\,(\mathit{fold}_{T^{\mathsf{Set}}_{H,\rho[\overline{\gamma:=C'}]}}(\lambda\overline{A}.\eta_{\overline{A}\,\overline{C'}}))_{\overline{B'}}\,)$$

has type

$$
\begin{aligned}
& (\mu T_{H,\mathsf{Eq}_\rho[\overline{\gamma:=S}]})\,\overline{R}\to\llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma:=S}][\overline{\beta:=R}]\\
= \;& (\mu T_{H,\mathsf{Eq}_\rho[\overline{\gamma:=S}]})\,\overline{\llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash\beta\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma:=S}][\overline{\beta:=R}]}\to\llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma:=S}][\overline{\beta:=R}]\\
= \;& \llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma:=S}][\overline{\beta:=R}]\to\llbracket\Gamma;\overline{\gamma},\overline{\beta}\vdash F\rrbracket^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\gamma:=S}][\overline{\beta:=R}]
\end{aligned}
$$

□