

$\forall; \emptyset \vdash v$ case

$$\begin{aligned}
 & \lambda \bar{\eta} \bar{B}. [\Gamma; \emptyset \vdash \text{map}_{\bar{v}}^{\bar{F}G} : \text{Nat}^{\bar{G}} (\text{Nat}^{\bar{F}\bar{r}} \bar{F} G) (\text{Nat}^{\bar{r}} v v)]_{\rho} * \bar{\eta} \bar{B} \\
 &= \lambda \bar{\eta} \bar{B}. [\Gamma; \bar{e}, \bar{r} \vdash v]_{\text{id}_{\rho[\bar{r}=\bar{B}]}} [\bar{e} := \lambda \bar{A}. \eta_{\bar{A}\bar{B}}] \\
 &= \lambda \bar{\eta} \bar{B}. \text{id}_{\rho v} \\
 &= \lambda \bar{\eta} \bar{B}. \text{id}_{[\Gamma; \bar{e}, \bar{r} \vdash v]_{\rho[\bar{r}=\bar{B}]}} \\
 &= \lambda \bar{\eta} \bar{B}. (\text{id}_{[\Gamma; \bar{e}, \bar{r} \vdash v]_{\rho[\bar{r}=-]}}) \bar{B} \\
 &= \lambda \bar{\eta} \bar{B}. ((\text{curry } \pi_1) *) \bar{B} \\
 &= \lambda \bar{\eta} \bar{B}. ((\text{curry } ([\Gamma; \bar{r} \vdash x:v \vdash x:v]_{\rho[\bar{r}=-]})) *) \bar{B} \\
 &= \lambda \bar{\eta} \bar{B}. [\Gamma; \emptyset \vdash L_{\bar{r}} x. x : \text{Nat}^{\bar{r}} v v]_{\rho} * \bar{\eta} \bar{B}
 \end{aligned}$$

$$\text{MAP}^{\emptyset}_{\text{Nat}^{\overline{B}} \text{ HK}}$$

$$\llbracket \Gamma; \emptyset \vdash \text{MAP}^{\emptyset}_{\text{Nat}^{\overline{B}} \text{ HK}} : \text{Nat}^{\emptyset}(\overline{\text{Nat}^{\emptyset} \text{ FG}}) (\text{Nat}^{\emptyset}(\text{Nat}^{\overline{B}} \text{ HK}) (\text{Nat}^{\overline{B}} \text{ HK})) \rrbracket_{\rho} \begin{array}{c} \Delta \\ \hline \overline{n} \\ \hline \overline{b} \end{array} \begin{array}{c} * \\ * \\ * \end{array}$$

$$= \llbracket \Gamma; \emptyset \vdash \text{Nat}^{\overline{B}} \text{ HK} \rrbracket_{\rho} \text{id}_{\rho}$$

$$= \text{id}_{\llbracket \Gamma; \emptyset \vdash \text{Nat}^{\overline{B}} \text{ HK} \rrbracket_{\rho}}$$

$$= (\lambda _ . \text{id}_{\llbracket \Gamma; \emptyset \vdash \text{Nat}^{\overline{B}} \text{ HK} \rrbracket_{\rho}}) *$$

$$= (\text{curry } \pi_1) *$$

$$= (\text{curry } (\llbracket \Gamma; \emptyset \mid \eta : \text{Nat}^{\overline{B}} \text{ HK} \vdash \eta : \text{Nat}^{\overline{B}} \text{ HK} \rrbracket_{\rho})) *$$

$$= (\llbracket \Gamma; \emptyset \mid \emptyset \vdash L_{\emptyset} \eta . \eta : \text{Nat}^{\emptyset}(\text{Nat}^{\overline{B}} \text{ HK}) (\text{Nat}^{\overline{B}} \text{ HK}) \rrbracket_{\rho} *) *$$

$$\text{map}_H \text{id}_F = \text{id}_{HF} \text{ case}$$

$$\begin{aligned} & \lambda \bar{B}. (\overline{[\Gamma; \emptyset \vdash (\text{MAP}_H^{\bar{F} \bar{F}}) (L_{\bar{\alpha} \bar{\gamma}} x.x) : \text{Nat}^{\bar{\gamma}} H[\bar{e} := \bar{F}] H[\bar{e} := \bar{F}]] \rho *}) \bar{B} \\ &= \lambda \bar{B}. (\overline{[\Gamma; \emptyset \vdash \text{MAP}_H^{\bar{F} \bar{G}} : \text{Nat}^{\bar{\sigma}} (\text{Nat}^{\bar{\sigma} \bar{\gamma}} F G) (\text{Nat}^{\bar{\gamma}} H[\bar{e} := \bar{F}] H[\bar{e} := \bar{G}])] \rho *}) \bar{B} \\ & \quad (\overline{[\Gamma; \emptyset \vdash L_{\bar{\alpha} \bar{\gamma}} x.x : \text{Nat}^{\bar{\sigma} \bar{\gamma}} F F] \rho *}) \bar{B} \\ &= \lambda \bar{B}. ((\lambda \bar{n}. \bar{B}'. [\Gamma; \bar{e}, \bar{\gamma} \vdash H] \text{id}_{\rho[\bar{\gamma} = \bar{B}']} [\bar{e} := \lambda \bar{A}. \eta_{\bar{A} \bar{B}}]) \\ & \quad (\text{curry } ([\Gamma; \bar{e}, \bar{\gamma} \vdash x : F \vdash x : F] \rho[\bar{\alpha} = -] [\bar{\gamma} = -]) *)) \bar{B} \end{aligned}$$

~~$$\lambda \bar{B}. (\lambda \bar{n}. \bar{B}'. [\Gamma; \bar{e}, \bar{\gamma} \vdash H] \text{id}_{\rho[\bar{\gamma} = \bar{B}']} [\bar{e} := \lambda \bar{A}. \eta_{\bar{A} \bar{B}}])$$~~

$$\begin{aligned} &= \lambda \bar{B}. ((\lambda \bar{n}. \bar{B}'. [\Gamma; \bar{e}, \bar{\gamma} \vdash H] \text{id}_{\rho[\bar{\gamma} = \bar{B}']} [\bar{e} := \lambda \bar{A}. \eta_{\bar{A} \bar{B}}]) \\ & \quad (\text{id}_{[\Gamma; \bar{e}, \bar{\gamma} \vdash F] \rho[\bar{\alpha} = -] [\bar{\gamma} = -]})) \bar{B} \end{aligned}$$

$$= \lambda \bar{B}. (\lambda \bar{B}'. [\Gamma; \bar{e}, \bar{\gamma} \vdash H] \text{id}_{\rho[\bar{\gamma} = \bar{B}']} [\bar{e} := \lambda \bar{A}. \text{id}_{[\Gamma; \bar{e}, \bar{\gamma} \vdash F] \rho[\bar{\alpha} = \bar{A}] [\bar{\gamma} = \bar{B}']}]) \bar{B}$$

$$= \lambda \bar{B}. [\Gamma; \bar{e}, \bar{\gamma} \vdash H] \text{id}_{\rho[\bar{\gamma} = \bar{B}']} [\bar{e} := \lambda \bar{A}. \text{id}_{[\Gamma; \bar{e}, \bar{\gamma} \vdash F] \rho[\bar{\alpha} = \bar{A}] [\bar{\gamma} = \bar{B}]}]$$

$$= \lambda \bar{B}. \text{id}_{[\Gamma; \bar{e}, \bar{\gamma} \vdash H] \rho[\bar{\gamma} = \bar{B}]} [\bar{e} := \lambda \bar{A}. [\Gamma; \bar{e}, \bar{\gamma} \vdash F] \rho[\bar{\alpha} = \bar{A}] [\bar{\gamma} = \bar{B}]]$$

$$= \lambda \bar{B}. \text{id}_{[\Gamma; \bar{\gamma} \vdash H[\bar{e} := \bar{F}]] \rho[\bar{\gamma} = \bar{B}]}$$

$$= \lambda \bar{B}. (\text{curry } \pi, *) \bar{B}$$

$$= \lambda \bar{B}. (\text{curry } ([\Gamma; \bar{\gamma} \vdash x : H[\bar{e} := \bar{F}] \vdash x : H[\bar{e} := \bar{F}]] \rho[\bar{\gamma} = -]) *) \bar{B}$$

$$= \lambda \bar{B}. [\Gamma; \emptyset \vdash L_{\bar{\gamma}} x.x : \text{Nat}^{\bar{\gamma}} H[\bar{e} := \bar{F}] H[\bar{e} := \bar{F}]] \rho * \bar{B}$$

$$\frac{\text{map}_H f \circ g = \text{map}_H f \circ \text{map}_H g \text{ case}}{\text{Let } MT = \text{Nat}^{\delta} (\text{Nat}^{\delta \bar{\delta}} F G) (\text{Nat}^{\delta} H[\bar{e} \equiv F] H[\bar{e} \equiv G])} \left| \begin{array}{l} f: \text{Nat}^{\delta \bar{\delta}} G K \\ g: \text{Nat}^{\delta \bar{\delta}} F G \end{array} \right.$$

$$\lambda \bar{b}. [\Gamma; \emptyset | \emptyset \vdash L_{\bar{y}} x. ((\text{MAP}_H^{GK})_{\emptyset} f)_{\bar{y}} (((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{y}} x) : \text{Nat}^{\delta} H[\bar{e} \equiv F] H[\bar{e} \equiv K]]_{\rho \bar{y} \equiv \bar{b}} \bar{b}$$

$$= \lambda \bar{b}. (\text{Curry} ([\Gamma; \bar{y} | x: H[\bar{e} \equiv F] \vdash ((\text{MAP}_H^{GK})_{\emptyset} f)_{\bar{y}} (((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{y}} x) : H[\bar{e} \equiv K]]_{\rho \bar{y} \equiv \bar{b}}] *) \bar{b}$$

$$= \lambda \bar{b}. (\text{Curry} ([\Gamma; \bar{y} | x: H[\bar{e} \equiv F] \vdash ((\text{MAP}_H^{GK})_{\emptyset} f)_{\bar{y}} (((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{y}} x) : H[\bar{e} \equiv K]]_{\rho \bar{y} \equiv \bar{b}}] *)$$

$$= \lambda \bar{b} d. [\Gamma; \bar{y} | x: H[\bar{e} \equiv F] \vdash ((\text{MAP}_H^{GK})_{\emptyset} f)_{\bar{y}} (((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{y}} x) : H[\bar{e} \equiv K]]_{\rho \bar{y} \equiv \bar{b}} d$$

$$= \lambda \bar{b} d. ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash (\text{MAP}_H^{GK})_{\emptyset} f : \text{Nat}^{\delta} H[\bar{e} \equiv G] H[\bar{e} \equiv K]]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{b}} \\ ([\Gamma; \bar{y} | x: H[\bar{e} \equiv F] \vdash ((\text{MAP}_H^{FG})_{\emptyset} g)_{\bar{y}} x : H[\bar{e} \equiv G]]_{\rho \bar{y} \equiv \bar{b}} d)$$

$$= \lambda \bar{b} d. ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash \text{MAP}_H : MT]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{b}} ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash f : \text{Nat}^{\delta \bar{\delta}} G K]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{b}} \\ ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash ((\text{MAP}_H^{FG})_{\emptyset} g) : \text{Nat}^{\delta} H[\bar{e} \equiv F] H[\bar{e} \equiv G]]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{b}} ([\Gamma; \bar{y} | x: H[\bar{e} \equiv F] \vdash x]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{b}}$$

$$= \lambda \bar{b} d. ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash \text{MAP}_H : MT]_{\rho \bar{y} \equiv \bar{b}} d)_{\emptyset} ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash f : \text{Nat}^{\delta \bar{\delta}} G K]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{b}} \\ ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash \text{MAP}_H : MT]_{\rho \bar{y} \equiv \bar{b}} d)_{\emptyset} ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash g : \text{Nat}^{\delta \bar{\delta}} F G]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{b}} (d)$$

$$= \lambda \bar{b} d. (\lambda \bar{c}. [\Gamma; \bar{e}, \bar{y} \vdash H] \text{id}_{\rho \bar{y} \equiv \bar{b}} \text{id}_{\bar{y} \equiv \bar{c}} [e := \lambda \bar{A}. ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash f : \text{Nat}^{\delta \bar{\delta}} G K]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{A} \bar{c}}])_{\bar{b}} \\ ((\lambda \bar{c}. [\Gamma; \bar{e}, \bar{y} \vdash H] \text{id}_{\rho \bar{y} \equiv \bar{b}} \text{id}_{\bar{y} \equiv \bar{c}} [e := \lambda \bar{A}. ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash g : \text{Nat}^{\delta \bar{\delta}} F G]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{A} \bar{c}}])_{\bar{b}} d)$$

$$= \lambda \bar{b} d. ([\Gamma; \bar{e}, \bar{y} \vdash H] \text{id}_{\rho \bar{y} \equiv \bar{b}} [e := \lambda \bar{A}. ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash f : \text{Nat}^{\delta \bar{\delta}} G K]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{A} \bar{b}}])_{\bar{b}} \\ ([\Gamma; \bar{e}, \bar{y} \vdash H] \text{id}_{\rho \bar{y} \equiv \bar{b}} [e := \lambda \bar{A}. ([\Gamma; \emptyset | x: H[\bar{e} \equiv F] \vdash g : \text{Nat}^{\delta \bar{\delta}} F G]_{\rho \bar{y} \equiv \bar{b}} d)_{\bar{A} \bar{b}}])_{\bar{b}} d)$$

$$= \lambda \bar{b}. [\Gamma; \bar{e}, \bar{y} \vdash H] \text{id}_{\rho \bar{y} \equiv \bar{b}} [e := \lambda \bar{A}. (\lambda \bar{c}. \bar{c} \bar{d} d ([\Gamma; \emptyset | x: F \vdash f : \text{Nat}^{\delta \bar{\delta}} G K]_{\rho \bar{A} \equiv \bar{c}} \bar{y} \equiv \bar{b}) d)_{\bar{A} \bar{b}} \\ ([\Gamma; \emptyset | x: F \vdash g : \text{Nat}^{\delta \bar{\delta}} F G]_{\rho \bar{A} \equiv \bar{c}} \bar{y} \equiv \bar{b}) d)_{\bar{A} \bar{b}} d)_{\bar{A} \bar{b}}]$$

$$\text{map}_H f \circ g = \text{map}_H f \circ \text{map}_H g \quad \text{continued}$$

(first line same as last line on previous page)

$$\text{MAP}_H^{FK}: \text{Nat}^{\sigma}(\text{Nat}^{\sigma\delta} F K) (\text{Nat}^{\sigma} H[\overline{e:=F}] H[\overline{e:=K}])$$

$$= \lambda \overline{B}. [\Gamma; \overline{e}, \overline{\delta} \vdash H] \text{id}_{\rho[\overline{\delta:=\overline{\delta}}]} [\overline{e:=}]$$

$$\lambda \overline{A}. (\lambda \overline{C} \overline{D} d. ([\Gamma; \overline{\sigma} | x: F \vdash f: \text{Nat}^{\sigma\delta} G K] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}] d)_{\overline{C} \overline{D}} ([\Gamma; \overline{\sigma} | x: F \vdash g: \text{Nat}^{\sigma\delta} F G] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}] d)_{\overline{C} \overline{D}} d)_{\overline{A} \overline{B}}]$$

$$= (\lambda \eta \overline{B}. [\Gamma; \overline{e}, \overline{\delta} \vdash H] \text{id}_{\rho[\overline{\delta:=\overline{\delta}}]} [\overline{e:=\lambda \overline{A}. \eta_{\overline{A} \overline{B}}]}) (\lambda \overline{C} \overline{D} d. ([\Gamma; \overline{\sigma} | x: F \vdash f: \text{Nat}^{\sigma\delta} G K] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}] d)_{\overline{C} \overline{D}} ([\Gamma; \overline{\sigma} | x: F \vdash g: \text{Nat}^{\sigma\delta} F G] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}] d)_{\overline{C} \overline{D}} d))$$

$$= ([\Gamma; \overline{\sigma} | \overline{\sigma} \vdash \text{MAP}_H^{FK}] \rho *) (\lambda \overline{C} \overline{D} d. ([\Gamma; \overline{\sigma} | x: F \vdash f: \text{Nat}^{\sigma\delta} G K] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}] d)_{\overline{C} \overline{D}} ([\Gamma; \overline{\sigma}, \overline{\delta} | x: F \vdash g_{\overline{\sigma} \overline{\delta}} x] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}] d)_{\overline{C} \overline{D}} d) \quad \uparrow : G$$

$$= ([\Gamma; \overline{\sigma} | \overline{\sigma} \vdash \text{MAP}_H^{FK}] \rho *) (\lambda \overline{C} \overline{D} d. ([\Gamma; \overline{\sigma}, \overline{\delta} | x: F \vdash f_{\overline{\sigma} \overline{\delta}} (g_{\overline{\sigma} \overline{\delta}} x): K] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}] d)_{\overline{C} \overline{D}} d)$$

$$= ([\Gamma; \overline{\sigma} | \overline{\sigma} \vdash \text{MAP}_H^{FK}] \rho *) (\lambda \overline{C} \overline{D} (\text{curry} ([\Gamma; \overline{\sigma}, \overline{\delta} | x: F \vdash f_{\overline{\sigma} \overline{\delta}} (g_{\overline{\sigma} \overline{\delta}} x): K] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}]) *))$$

$$= ([\Gamma; \overline{\sigma} | \overline{\sigma} \vdash \text{MAP}_H^{FK}] \rho *) (\text{curry} ([\Gamma; \overline{\sigma}, \overline{\delta} | x: F \vdash f_{\overline{\sigma} \overline{\delta}} (g_{\overline{\sigma} \overline{\delta}} x): K] \rho[\overline{A:=\overline{C}}] [\overline{\delta:=\overline{D}}]) *))$$

$$= ([\Gamma; \overline{\sigma} | \overline{\sigma} \vdash \text{MAP}_H^{FK}] \rho *) ([\Gamma; \overline{\sigma} | \overline{\sigma} \vdash L_{\overline{\sigma} \overline{\delta}} x. f_{\overline{\sigma} \overline{\delta}} (g_{\overline{\sigma} \overline{\delta}} x): \text{Nat}^{\sigma\delta} F K] \rho *)$$

$$= [\Gamma; \overline{\sigma} | \overline{\sigma} \vdash (\text{MAP}_H^{FK}) (L_{\overline{\sigma} \overline{\delta}} x. f_{\overline{\sigma} \overline{\delta}} (g_{\overline{\sigma} \overline{\delta}} x)): \text{Nat}^{\sigma\delta} H[\overline{e:=F}] H[\overline{e:=K}]] \rho *$$