Free Theorems for Nested Types

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1 INTRODUCTION

- Bob has forall types. But we have data types. So we each add somethign different to the simply typed lambda calculus. We'll treat simply typed lambda calculus with data types first, and may add poly types later. This will require additional hypotheses on the semantic categories.
- We're not (obviously) using the exponential between functor categories anywhere.
- Couldn't do this before LICS paper? Or could Bob have done it? What's new?
- Introduce notation $\boldsymbol{R}$. Introduce notation $[\boldsymbol{\alpha} := \boldsymbol{R}]$ for $[\alpha_1 := R_1, \ldots, \alpha_k := R_k]$ when the cardinalities of $\boldsymbol{\alpha}$ and $\boldsymbol{R}$ are equal.

1.1 Preliminaries

We write Set for the category of sets and functions.

Definition 1. *The category* Rel *is defined as follows.*

- *An object of* Rel *is a triple* (A, B, R) *where* R *is a relation between the objects* A *and* B *in* Set. *We identify* (A, B, R) *with* R*, and write* $R : \mathsf{Rel}(A, B)$ *when convenient.*
- *A morphism between objects* $R : \mathsf{Rel}(A, B)$ *and* $R' : \mathsf{Rel}(A', B')$ *of* Rel *is a pair* $(f : A \to A', g : B \to B')$ *of morphisms in* Set *such that* $(f\,a, g\,b) \in R'$ *whenever* $(a, b) \in R$.

If $R : \mathsf{Rel}(A, B)$ we write $\pi_1 R$ and $\pi_2 R$ for the *domain* A of R and the *codomain* B of R, respectively. If A : Set, then we write $\mathsf{Eq}_A = (A, A, \{(x, x) \mid x \in A\})$ for the *equality relation* on A.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, we write $[\mathcal{C}, \mathcal{D}]$ for the set of ω-cocontinuous functors from $\mathcal{C}$ to $\mathcal{D}$.

2 THE CALCULUS

2.1 Types

For each $k \geq 0$, we assume a countable set $\mathbb{T}^k$ of *type constructor variables of arity* k, disjoint for distinct k. We use lower case Greek letters for type constructor variables, and write ϕ^k to indicate that $\phi \in \mathbb{T}^k$. When convenient we may write α, β, etc., rather than α^0, β^0, etc., for elements of $\mathbb{T}^0$. The set of all type constructor variables is $\mathbb{T} = \bigcup_{k \geq 0} \mathbb{T}^k$. We further assume an infinite set $\mathbb{V}$ of *type variables* disjoint from $\mathbb{T}$. We write $\overline{\zeta}$ for either a set $\{\zeta_1, \ldots, \zeta_n\}$ of type variables or a set of type constructor variables when the cardinality n of the set is unimportant. If $\mathcal{P}$ is a set of type constructor variables then we write $\mathcal{P}, \overline{\phi}$ for $\mathcal{P} \cup \overline{\phi}$ when $\mathcal{P} \cap \overline{\phi} = \emptyset$. We omit the boldface for a singleton set, thus writing ϕ, rather than $\overline{\phi}$, for $\{\phi\}$.

Definition 2. *Let* V *be a finite subset of* $\mathbb{V}$*, and let* $\mathcal{P}$ *and* $\overline{\alpha}$ *be finite subsets of* $\mathbb{T}$*. The sets* $\mathcal{T}(V)$ *of* type expressions *over* V *and* $\mathcal{F}^{\mathcal{P}}(V)$ *of* type constructor expressions *over* V *are given by:*

$$\mathcal{T}(V) \;::=\; V \mid \mathcal{T}(V) \to \mathcal{T}(V) \mid \forall v.\, \mathcal{T}(V, v) \mid \mathsf{Nat}^{\overline{\alpha}}(\mathcal{F}^{\overline{\alpha}}(V), \mathcal{F}^{\overline{\alpha}}(V))$$

and

$$\mathcal{F}^{\mathcal{P}}(V) \;::=\; \mathcal{T}(V) \mid \mathbb{0} \mid \mathbb{1} \mid \mathcal{P}\overline{\mathcal{F}^{\mathcal{P}}(V)} \mid \mathcal{F}^{\mathcal{P}}(V) + \mathcal{F}^{\mathcal{P}}(V) \mid \mathcal{F}^{\mathcal{P}}(V) \times \mathcal{F}^{\mathcal{P}}(V)$$
$$\mid \left(\mu\phi^k.\lambda\overline{\alpha}.\mathcal{F}^{\mathcal{P},\alpha_1,\ldots,\alpha_k,\phi}(V)\right)\overline{\mathcal{F}^{\mathcal{P}}(V)}$$

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The above notation entails that an application $\tau\tau_1 \dots \tau_k$ is allowed only when τ is a type constructor variable of arity k, or τ is a subexpression of the form $\mu\phi^k.\lambda\alpha_1 \dots \alpha_k.\tau$. Moreover, if τ has arity k then τ must be applied to exactly k arguments. Accordingly, an overbar indicates a sequence of subexpressions whose length matches the arity of the functorial expression applied to it. The fact that functorial expressions are always in *η-long normal form* avoids having to consider β-conversion at the level of type constructors, and the fact that the standard type formers are all defined pointwise avoids having to relate functorial expressions at different kinds.

If $\tau \in \mathcal{F}^{\mathcal{P}}(V)$, if $\mathcal{P}$ contains only type constructor variables of arity 0, and if $k = 0$ for every occurrence of ϕ^k bound by μ in τ, then we say that τ is *first-order*. Otherwise we say that τ is *second-order*. The intuition here is that variables in V can be substituted by any types, but those in $\mathcal{P}$ can only be substituted by type constructors, even if of arity 0. In this case, they'd be substituted by type constructors of arity 0 — i.e., type constants — such as Nat or Bool.

Definition 3. *Let* Γ *be a* type context, *i.e., a finite set of type variables, and let* Φ *be a* type constructor context, *i.e., a finite set of type constructor variables. The formation rules for the set* $\mathcal{T} \subseteq \bigcup_{V \subseteq \mathbb{V}} \mathcal{T}(V)$ *of* well-formed type expressions *are*

$$\frac{}{\Gamma, v; \emptyset \vdash v : \mathcal{T}} \qquad \frac{\Gamma; \emptyset \vdash \sigma : \mathcal{T} \qquad \Gamma; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \vdash \sigma \to \tau : \mathcal{T}}$$

$$\frac{\Gamma, v; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \vdash \forall v.\, \tau : \mathcal{T}} \qquad \frac{\Gamma; \overline{\alpha} \vdash \sigma : \mathcal{F} \qquad \Gamma; \overline{\alpha} \vdash \tau : \mathcal{F}}{\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}} \sigma\, \tau : \mathcal{T}}$$

The formation rules for the set $\mathcal{F} \subseteq \bigcup_{V \subseteq \mathbb{V}, \mathcal{P} \subseteq \mathbb{T}} \mathcal{F}^{\mathcal{P}}(V)$ *of* well-formed type constructor expressions *are*

$$\frac{\Gamma; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \vdash \tau : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi, v \vdash v : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{0} : \mathcal{F}} \qquad \frac{}{\Gamma; \Phi \vdash \mathbb{1} : \mathcal{F}}$$

$$\frac{\Gamma; \Phi \vdash \phi^k : \mathcal{F} \qquad \overline{\Gamma; \Phi \vdash \tau : \mathcal{F}}}{\Gamma; \Phi \vdash \phi^k \overline{\tau}}$$

$$\frac{\Gamma; \Phi, \overline{\alpha} : \mathcal{F} \qquad \overline{\Gamma; \Phi \vdash \tau : \mathcal{F}}}{\Gamma; \Phi \vdash (\mu\phi^k.\lambda\overline{\alpha}.\, \tau)\overline{\tau}}$$

$$\frac{\Gamma; \Phi \vdash \sigma : \mathcal{F} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \vdash \sigma + \tau : \mathcal{F}} \qquad \frac{\Gamma; \Phi \vdash \sigma : \mathcal{F} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \vdash \sigma \times \tau : \mathcal{F}}$$

Our formation rules allow type constructor expressions like $\mathsf{List}\,\gamma = (\mu\beta.\lambda\alpha.\mathbb{1} + \alpha \times \beta)\gamma$ either to be natural in γ or not, according to whether it is well-formed in the context $\emptyset; \gamma$ or $\gamma; \emptyset$. If the former, then we can derive $\vdash \mathsf{Nat}^{\gamma}\mathbb{1}(\mathsf{List}\,\gamma) : \mathcal{T}$. If the latter, then we cannot. Our formation rules also allow the derivation of, e.g., $\delta; \emptyset \vdash \mathsf{Nat}^{\gamma}(\mathsf{List}\,\gamma)\,(\mathsf{Tree}\,\gamma\delta)$, which represents a natural transformation between lists and trees that is natural in γ but not in δ.

Substitution for first-order type constructor expressions is the usual capture-avoiding textual substitution. We write $\tau[\alpha := \sigma]$ for the result of substituting σ for α in τ, and $\tau[\alpha_1 := \tau_1, \dots, \alpha_k := \tau_k]$ for $\tau[\alpha_1 := \tau_1][\alpha_2 := \tau_2, \dots, \alpha_k := \tau_k]$. Substitution for second-order type constructor expressions is given in the next definition.

Definition 4. *If* $\Gamma; \Phi, \phi^k \vdash h[\phi] : \mathcal{F}$ *and* $\Gamma; \Phi, \overline{\alpha} \vdash F : \mathcal{F}$ *with* $\overline{\alpha} = \{\alpha_1, \dots, \alpha_k\}$ *and* $k \geq 1$, *then* $\Gamma; \Phi \vdash h[\phi := F] : \mathcal{F}$, *where the operation* $(\cdot)[\phi := F]$ *of* second-order type constructor substitution *is defined by:*

$$
\begin{array}{lcl}
\tau[\phi := F] & = & \tau \text{ if } \tau \in \mathcal{T} \\
\mathbb{1}[\phi := F] & = & \mathbb{1} \\
\mathbb{0}[\phi := F] & = & \mathbb{0} \\
(\psi^n \overline{\tau})[\phi := F] & = & \begin{cases} \psi^n\, \overline{\tau[\phi := F]} & \text{if } \psi \neq \phi \\ F[\overline{\alpha := \tau[\phi := F]}] & \text{if } \psi = \phi \end{cases} \\
(\sigma + \tau)[\phi := F] & = & \sigma[\phi := F] + \tau[\phi := F] \\
(\sigma \times \tau)[\phi := F] & = & \sigma[\phi := F] \times \tau[\phi := F] \\
((\mu\psi^n.\lambda\overline{\beta}.\,G)\overline{\tau}[\phi := F] & = & (\mu\psi^n.\lambda\overline{\beta}.\,G[\phi := F])\,\overline{\tau[\phi := F]}
\end{array}
$$

Note that, since an arity 0 type constructor is first-order, substitution into it is just the usual textual replacement, i.e., the usual notion of substitution, as expected.

2.2 Terms

We assume an infinite set $\mathcal{V}$ of term variables disjoint from $\mathbb{T}$ and $\mathbb{V}$.

Definition 5. *Let* Γ *be a type context and* Φ *be a type constructor context. A* term context for Γ and Φ *is a finite set of bindings of the form* $x : \tau$*, where* $x \in \mathcal{V}$ *and* $\Gamma; \Phi \vdash \tau : \mathcal{F}$*.*

We adopt the same conventions for denoting disjoint unions in term contexts as in type contexts and type constructor contexts.

Definition 6. *Let* Δ *be a term context for* Γ *and* Φ*. The formation rules for the set of* well-formed terms over Δ *are*

$$
\frac{\Gamma; \emptyset \vdash \tau : \mathcal{T}}{\Gamma; \emptyset \mid \Delta, x : \tau \vdash x : \tau}
\qquad
\frac{\Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \mid \Delta, x : \tau \vdash x : \tau}
$$

$$
\frac{\Gamma; \emptyset \mid \Delta, x : \sigma \vdash t : \tau}{\Gamma; \emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau}
\qquad
\frac{\Gamma; \emptyset \mid \Delta \vdash s : \sigma \to \tau \qquad \Gamma; \emptyset \mid \Delta \vdash t : \sigma}{\Gamma; \emptyset \mid \Delta \vdash st : \tau}
$$

$$
\frac{\Gamma, \alpha; \Phi \vdash \tau : \mathcal{T} \qquad \Gamma, \alpha; \Phi \mid \Delta \vdash t : \tau}{\Gamma; \Phi \mid \Delta \vdash \Lambda\alpha.t : \forall\alpha.\tau}
\qquad
\frac{\Gamma, \alpha; \Phi \vdash \sigma : \mathcal{T} \qquad \Gamma; \Phi \vdash \tau : \mathcal{T} \qquad \Gamma; \Phi \mid \Delta \vdash t : \forall\alpha.\sigma}{\Gamma; \Phi \mid \Delta \vdash t\tau : \sigma[\alpha := \tau]}
$$

No intro $\mathbb{0}$

$$
\frac{\Gamma; \Phi \mid \Delta \vdash t : \mathbb{0} \qquad \Gamma; \Phi \vdash \tau : \mathcal{F}}{\Gamma; \Phi \mid \Delta \vdash \bot_\tau t : \tau}
$$

$$
\frac{}{\Gamma; \Phi \mid \Delta \vdash \top : \mathbb{1}}
$$

No elim $\mathbb{1}$

$$
\frac{\Gamma; \Phi \mid \Delta \vdash s : \sigma}{\Gamma; \Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : \tau}{\Gamma; \Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau}
$$

$$
\frac{\Gamma; \Phi \vdash \tau, \sigma : \mathcal{F} \qquad \Gamma; \Phi \mid \Delta \vdash t : \sigma + \tau \qquad \Gamma; \Phi \mid \Delta, x : \sigma \vdash l : \gamma \qquad \Gamma; \Phi \mid \Delta, y : \tau \vdash r : \gamma}{\Gamma; \Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{x \mapsto l;\ y \mapsto r\} : \gamma}
$$

$$
\frac{\Gamma; \Phi \mid \Delta \vdash s : \sigma \qquad \Gamma; \Phi \mid \Delta \vdash t : \tau}{\Gamma; \Phi \mid \Delta \vdash (s, t) : \sigma \times \tau}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : \sigma \times \tau}{\Gamma; \Phi \mid \Delta \vdash \pi_1 t : \sigma}
\qquad
\frac{\Gamma; \Phi \mid \Delta \vdash t : \sigma \times \tau}{\Gamma; \Phi \mid \Delta \vdash \pi_2 t : \tau}
$$

$$\frac{\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G : \mathcal{T} \qquad \Gamma;\overline{\alpha} \mid \Delta, x : F \vdash t : G}{\Gamma;\emptyset \mid \Delta \vdash L_{\overline{\alpha}}x.t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G}$$

$$\frac{\Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,F\,G \qquad \overline{\Gamma;\Phi \vdash \tau : \mathcal{F}} \qquad \Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}]}{\Gamma;\Phi \mid \Delta \vdash t_{\overline{\tau}}s : G[\overline{\alpha := \tau}]}$$

$$\frac{\Gamma;\Phi \mid \Delta \vdash t : H[\phi := \mu\phi.\lambda\overline{\alpha}.H][\overline{\alpha := A}] \qquad \overline{\Gamma;\Phi \vdash A}}{\Gamma;\Phi \mid \Delta \vdash \mathsf{in}_H\,t : (\mu\phi.\lambda\overline{\alpha}.H)\overline{A}}$$

$$\frac{\Gamma;\overline{\alpha} \vdash F : \mathcal{F} \qquad \Gamma;\phi,\overline{\beta} \vdash H : \mathcal{F} \qquad \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\,H[\phi := F][\overline{\beta := \alpha}]\,F}{\Gamma;\emptyset \mid \Delta \vdash \mathsf{fold}_H\,t : \mathsf{Nat}^{\overline{\alpha}}\,((\mu\phi.\lambda\overline{\beta}.H)\overline{\alpha})\,F}$$

3 INTERPRETING TYPES

Parametricity requires that set interpretations of types are defined concurrently with their relational interpretations. In this section we give the set interpretations for types; in the next section we give their relational interpretations. While the set interpretations are relatively straightforward, their relation interpretations are less so, mainly because of the cocontinuity conditions we must impose to ensure that they are well-behaved. We take some effort to develop these in Section 3.2, which separates Definitions 8 and 19 in space but otherwise has no impact on the fact that they are given by mutually induction.

3.1 Interpreting Types as Sets

Definition 7. *A* set environment *maps each type variable to a set, and each type constructor variable of arity k to an element of* $[\mathsf{Set}^k, \mathsf{Set}]$. *A morphism $f : \rho \to \rho'$ from a set environment ρ to a set environment ρ' with $\rho|_{\mathbb{V}} = \rho'|_{\mathbb{V}}$ maps each type variable v to $id_{\rho v}$, and each type constructor variable ϕ of arity k to a natural transformation from the k-ary functor $\rho\phi$ on* Set *to the k-ary functor $\rho'\phi$ on* Set.

When convenient we identify a functor $F : [\mathsf{Set}^0, \mathsf{Set}]$ with the set that is its codomain. With this convention, a set environment maps a type constructor variable of arity 0 to an ω-cocontinuous functor from Set^0 to Set — i.e., to a set — just as it does a type variable. If $\overline{\alpha} = \{\alpha_1, ..., \alpha_k\}$ and $\overline{A} = \{A_1, ..., A_k\}$, then we write $\rho[\overline{\alpha := A}]$ for the set environment ρ' such that $\rho'\alpha_i = A_i$ for $i = 1, ..., k$ and $\rho'\alpha = \rho\alpha$ if $\alpha \notin \{\alpha_1, ..., \alpha_k\}$. We write SetEnv for the collection of all set environments.

If ρ is a set environment we write Eq_ρ for the relation environment such that $\mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v}$ for every type variable or type constructor variable v; see Definition 17 below for the complete definition of a relation environment. The relational interpretations referred to in the condition on the natural transformations in the clause of Definition 8 for types of the form $\mathsf{Nat}^{\overline{\alpha}}\,F\,G$ are given in full in Definition 19. Intuitively, this condition can be thought of as ensuring that set interpretations of such terms are sufficiently uniform.

Definition 8. *Let ρ be a set environment. The* set interpretation $[\![\cdot]\!]^{\mathsf{Set}} : \mathcal{F} \to [\mathsf{SetEnv}, \mathsf{Set}]$ *is defined by*

$$
\begin{aligned}
[\![\Gamma;\emptyset \vdash v]\!]^{\mathsf{Set}}\rho &= \rho v \text{ if } v \in \mathbb{V}\\
[\![\Gamma;\emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\emptyset \vdash \sigma]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho\\
&\textit{need to interpret forall types if we include them}\\
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho &= \{\eta : [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}] \Rightarrow [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := -}]\\
&\qquad \mid \forall \overline{A}, \overline{B} : \mathsf{Set}. \forall \overline{R : \mathsf{Rel}(A,B)}.\\
&\qquad\qquad (\eta_{\overline{A}}, \eta_{\overline{B}}) : [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}] \to [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\overline{\alpha := R}]\}\\
[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}}\rho &= 0\\
[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho &= 1\\
[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho &= (\rho\phi)\,\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}\\
[\![\Gamma;\Phi \vdash \sigma + \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho + [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma;\Phi \vdash \sigma \times \tau]\!]^{\mathsf{Set}}\rho &= [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}}\rho \times [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho\\
[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho &= (\mu T^{\mathsf{Set}}_\rho)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}\\
&\textit{where } T^{\mathsf{Set}}_\rho F = \lambda\overline{A}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}\rho[\phi := F][\overline{\alpha := A}]\\
&\textit{and } T^{\mathsf{Set}}_\rho \eta = \lambda\overline{A}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_\rho[\phi := \eta][\overline{\alpha := id_A}]
\end{aligned}
$$

If ρ is a set environment and $\vdash \tau : \mathcal{F}$ then we may write $[\![\vdash \tau]\!]^{\mathsf{Set}}$ instead of $[\![\emptyset;\emptyset \vdash \tau]\!]^{\mathsf{Set}}\rho$ since the environment is immaterial. Definition 8 ensures that

$$[\![\Gamma;\Phi \vdash F\overline{\tau}]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi,\overline{\alpha} \vdash F\overline{\alpha}]\!]^{\mathsf{Set}}(\rho[\overline{\alpha := [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}}])$$

Moreover, the third fourth clause does indeed define a set. Indeed, local finite presentability of Set and ω-cocontinuity of $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho$ ensure that $\{\eta : [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho \Rightarrow [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho\}$ (which contains $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho$) is a subset of

$$\left\{([\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}])^{([\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}])}\;\middle|\;\overline{S} = (S_1,\ldots,S_{|\overline{\alpha}|}),\text{ and } S_i \text{ is a finite set for } i = 1,\ldots,|\overline{\alpha}|\right\}$$

There are countably many choices for tuples $\overline{S}$, and each of these gives rise to a morphism from $[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}]$ to $[\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := S}]$. But there are only Set-many choices of morphisms between these (or any) two objects because Set is locally small.

In order to make sense of the last clause in Definition 8, we need to know that T^{Set}_ρ is an ω-cocontinuous endofunctor on $[\mathsf{Set}^k, \mathsf{Set}]$, so that it admits a fixed point. Since T^{Set}_ρ is defined in terms of $[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}$, this means that set interpretations of types must be functors. This in turn means that the actions of set interpretations of types on objects and on morphisms in SetEnv are intertwined. In fact, we know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash E : \mathcal{F}$, $[\![\Gamma;\overline{\alpha} \vdash E]\!]^{\mathsf{Set}}$ is actually functorial in $\overline{\alpha}$ and ω-cocontinuous. What remains is to define the actions of each of these functors on morphisms between environments.

Definition 9. *Let $f : \rho \to \rho'$ for set environments ρ and ρ' such that $\rho|_{\mathbb{V}} = \rho'|_{\mathbb{V}}$. The action $[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f$ of $[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}$ on the morphism f is given as follows:*

- *If $\Gamma, v;\emptyset \vdash v$ then $[\![\Gamma, v;\emptyset \vdash v]\!]^{\mathsf{Set}} f = id_{\rho v}$.*
- *If $\Gamma;\emptyset \vdash \sigma \to \tau$ then $[\![\Gamma;\emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Set}} f = id_{[\![\Gamma;\emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Set}}\rho}$.*
- *If $\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G$, then we define $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}} f = id_{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}\rho}$.*

- *If* $\Gamma;\Phi \vdash \mathbb{0}$ *then* $[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Set}} f = id_0$.
- *If* $\Gamma;\Phi \vdash \mathbb{1}$ *then* $[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}} f = id_1$.
- *If* $\Gamma;\Phi \vdash \phi\overline{\tau}$, *then we have that* $[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} f : [\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho' = (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \to (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}$ *is defined by* $[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} f = (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f} = (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f} \circ (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}$. *This equality holds because* $\rho\phi$ *and* $\rho'\phi$ *are functors and* $f\phi : \rho\phi \to \rho'\phi$ *is a natural transformation, so that the following naturality square commutes:*

$$\begin{tikzcd}
(\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\rho\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}"'] & (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \arrow[d, "(\rho'\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}"'] \\
(\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'} \arrow[r, "(f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}"] & (\rho'\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd} \tag{1}$$

- *If* $\Gamma;\Phi \vdash \sigma+\tau$ *then* $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Set}} f$ *is defined by* $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Set}} f(\mathsf{inl}\, x) = \mathsf{inl}\,([\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}} f x)$ *and* $[\![\Gamma;\Phi \vdash \sigma+\tau]\!]^{\mathsf{Set}} f(\mathsf{inr}\, y) = \mathsf{inr}\,([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f y)$.
- *If* $\Gamma;\Phi \vdash \sigma\times\tau$ *then* $[\![\Gamma;\Phi \vdash \sigma\times\tau]\!]^{\mathsf{Set}} f = [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f$.
- *If* $\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$ *then letting* $\sigma^{\mathsf{Set}}_f : T^{\mathsf{Set}}_\rho \to T^{\mathsf{Set}}_{\rho'}$ *be the map*

$$F \mapsto \lambda\overline{A}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} f[\phi := id_F][\overline{\alpha := id_A}]$$

we define

$$\begin{array}{rl}
& [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}} f \\
: & [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho \to [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho' \\
= & (\mu T^{\mathsf{Set}}_\rho)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \to (\mu T^{\mathsf{Set}}_{\rho'})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}
\end{array}$$

by

$$\begin{array}{rl}
& (\mu\sigma^{\mathsf{Set}}_f)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'} \circ (\mu T^{\mathsf{Set}}_\rho)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f} \\
= & (\mu T^{\mathsf{Set}}_{\rho'})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}} f} \circ (\mu\sigma^{\mathsf{Set}}_f)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho}
\end{array}$$

Again, this equality holds because $\mu T^{\mathsf{Set}}_\rho$ *and* $\mu T^{\mathsf{Set}}_{\rho'}$ *are functors and* $f\phi : \rho\phi \to \rho'\phi$ *is a natural transformation, so that the following naturality square commutes:*

$$\begin{tikzcd}
(\mu T^{\mathsf{Set}}_\rho)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \arrow[r, "(\mu\sigma^{\mathsf{Set}}_f)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}"] \arrow[d, "(\mu T^{\mathsf{Set}}_\rho)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f})"'] & (\mu T^{\mathsf{Set}}_{\rho'})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \arrow[d, "(\mu T^{\mathsf{Set}}_{\rho'})\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}f}"'] \\
(\mu T^{\mathsf{Set}}_\rho)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'} \arrow[r, "(\mu\sigma^{\mathsf{Set}}_f)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}"] & (\mu T^{\mathsf{Set}}_{\rho'})\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}\rho'}
\end{tikzcd} \tag{2}$$

3.2 Interpreting Types as Relations

Definition 10. *A* k-ary relation transformer F *is a triple* (F^0, F^1, F^*), *where* $F^0, F^1 : [\mathsf{Set}^k, \mathsf{Set}]$ *are functors,* $F^* : [\mathsf{Rel}^k, \mathsf{Rel}]$ *is a functor, if* $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$ *then* $F^*\overline{R} : \mathsf{Rel}(F^0\overline{A}, F^1\overline{B})$, *and if* $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$ *then* $F^*\overline{(\alpha,\beta)} = (F^0\overline{\alpha}, F^1\overline{\beta})$.

Expanding the last clause of Definition 10 is equivalent to: if $\overline{(a,b) \in R}$ implies $\overline{(\alpha\, a, \beta\, b) \in S}$ then $(c,d) \in F^*\overline{R}$ implies $(F^0\overline{\alpha}\, c, F^1\overline{\beta}\, d) \in F^*\overline{S}$.

When convenient we identify a 0-ary relation transformer (A, B, R) with the relation (A, B, R). It will also be convenient to extend the identification of a relation with its third component as in

Definition 1 to identify a relation transformer $F = (F^0, F^1, F^*)$ with its third component F^*. In this case we may continue to write F^0 and F^1 for $\pi_1 F$ and $\pi_2 F$. We extend these conventions to relation environments, introduced in Definition 17 below, as well.

Definition 11. *The category RT_k of k-ary relation transformers is given by the following data:*

- *An object of RT_k is a relation transformer.*
- *A morphism $\delta : (G^0, G^1, G^*) \to (H^0, H^1, H^*)$ in RT_k is a pair of natural transformations (δ^0, δ^1) where $\delta^0 : G^0 \to H^0$, $\delta^1 : G^1 \to H^1$ such that, for all $\overline{R : \mathsf{Rel}(A, B)}$, if $(x, y) \in G^*\overline{R}$ then $(\delta^0_{\overline{A}} x, \delta^1_{\overline{B}} y) \in H^*\overline{R}$. This is basically a fibred natural transformation, but for heterogeneous relations.*
- *Identity morphisms and composition are inherited from the category of functors on* Set.

Definition 12. *An endofunctor H on RT_k is a triple $H = (H^0, H^1, H^*)$, where*

- *H^0 and H^1 are functors from $[\mathsf{Set}^k, \mathsf{Set}]$ to $[\mathsf{Set}^k, \mathsf{Set}]$*
- *H^* is a functor from RT_k to $[\mathsf{Rel}^k, \mathsf{Rel}]$*
- *for all $\overline{R : \mathsf{Rel}(A, B)}$, $\pi_1((H^*(\delta^0, \delta^1))_{\overline{R}}) = (H^0\delta^0)_{\overline{A}}$ and $\pi_2((H^*(\delta^0, \delta^1))_{\overline{R}}) = (H^1\delta^1)_{\overline{B}}$*
- *The action of H on objects is given by $H\,(F^0, F^1, F^*) = (H^0F^0,\ H^1F^1,\ H^*(F^0, F^1, F^*))$*
- *The action of H on morphisms is given by $H\,(\delta^0, \delta^1) = (H^0\delta^0, H^1\delta^1)$ for $(\delta^0, \delta^1) : (F^0, F^1, F^*) \to (G^0, G^1, G^*)$*

Since the results of applying H to k-ary relation transformers and morphisms between them must again be k-ary relation transformers and morphisms between them, respectively, Definition 12 implicitly requires that the following three conditions hold:

(1) if $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then

$$H^*(F^0, F^1, F^*)\overline{R} : \mathsf{Rel}(H^0F^0\overline{A}, H^1F^1\overline{B})$$

In other words, $\pi_1(H^*(F^0, F^1, F^*)\overline{R}) = H^0F^0\overline{A}$ and $\pi_2(H^*(F^0, F^1, F^*)\overline{R}) = H^1F^1\overline{B}$.

(2) if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then

$$H^*(F^0, F^1, F^*)\overline{(\alpha, \beta)} = (H^0F^0\overline{\alpha}, H^1F^1\overline{\beta})$$

In other words, $\pi_1(H^*(F^0, F^1, F^*)\overline{(\alpha, \beta)}) = H^0F^0\overline{\alpha}$ and $\pi_2(H^*(F^0, F^1, F^*)\overline{(\alpha, \beta)}) = H^1F^1\overline{\beta}$.

(3) if $(\delta^0, \delta^1) : (F^0, F^1, F^*) \to (G^0, G^1, G^*)$ and $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then

$$\text{if } (x, y) \in H^*(F^0, F^1, F^*)\overline{R} \text{ then } ((H^0\delta^0)_{\overline{A}} x, (H^1\delta^1)_{\overline{B}} y) \in H^*(G^0, G^1, G^*)\overline{R}$$

Note, however, that this condition is automatically satisfied because it is implied by the third bullet point of Definition 12.

Definition 13. *If H and K are endofunctors on RT_k, then a* natural transformation $\sigma : H \to K$ *is a pair $\sigma = (\sigma^0, \sigma^1)$, where $\sigma^0 : H^0 \to K^0$ and $\sigma^1 : H^1 \to K^1$ are natural transformations between endofunctors on $[\mathsf{Set}^k, \mathsf{Set}]$ and the component of σ at the k-ary relation transformer F is given by $\sigma_F = (\sigma^0_{F^0}, \sigma^1_{F^1})$.*

Definition 13 entails that $\sigma^i_{F^i}$ must be natural in $F^i : [\mathsf{Set}^k, \mathsf{Set}]$, and, for every F, both $(\sigma^0_{F^0})_{\overline{A}}$ and $(\sigma^1_{F^1})_{\overline{A}}$ must be natural in $\overline{A}$. Moreover, since the results of applying σ to k-ary relation transformers must be morphisms of k-ary relation transformers, Definition 13 implicitly requires that $(\sigma_F)_{\overline{R}} = ((\sigma^0_{F^0})_{\overline{A}}, (\sigma^1_{F^1})_{\overline{B}})$ is a morphism in Rel for any k-tuple of relations $\overline{R : \mathsf{Rel}(A, B)}$, i.e., if $(x, y) \in H^*F\overline{R}$, then $((\sigma^0_{F^0})_{\overline{A}} x, (\sigma^1_{F^1})_{\overline{B}} y) \in K^*F\overline{R}$.

Next, we see that we can compute colimits in RT_k.

Lemma 14. $\varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d) = (\varinjlim_{d\in\mathcal{D}} F^0_d, \varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$

Proof. We first observe that $(\varinjlim_{d\in\mathcal{D}} F^0_d, \varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$ is in RT_k. If $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then $\varinjlim_{d\in\mathcal{D}} F^*_d\overline{R} : \mathsf{Rel}(\varinjlim_{d\in\mathcal{D}} F^0_d\overline{A}, \varinjlim_{d\in\mathcal{D}} F^1_d\overline{B})$ because of how colimits are computed in Rel. Moreover, if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), \ldots, (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then

$$\begin{aligned} & (\varinjlim_{d\in\mathcal{D}} F^*_d)\overline{(\alpha,\beta)} \\ = \; & \varinjlim_{d\in\mathcal{D}} F^*_d\overline{(\alpha,\beta)} \\ = \; & \varinjlim_{d\in\mathcal{D}} (F^0_d\overline{\alpha},\ F^1_d\overline{\beta}) \\ = \; & (\varinjlim_{d\in\mathcal{D}} F^0_d\overline{\alpha},\ \varinjlim_{d\in\mathcal{D}} F^1_d\overline{\beta}) \end{aligned}$$

so $(\varinjlim_{d\in\mathcal{D}} F^0_d, \varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$ actually is in RT_k.

Now to see that $\varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d) = (\varinjlim_{d\in\mathcal{D}} F^0_d, \varinjlim_{d\in\mathcal{D}} F^1_d, \varinjlim_{d\in\mathcal{D}} F^*_d)$, let $\gamma^0_d : F^0_d \to \varinjlim_{d\in\mathcal{D}} F^0_d$ and $\gamma^1_d : F^1_d \to \varinjlim_{d\in\mathcal{D}} F^1_d$ be the injections for the colimits $\varinjlim_{d\in\mathcal{D}} F^0_d$ and $\varinjlim_{d\in\mathcal{D}} F^1_d$, respectively. Then $(\gamma^0_d, \gamma^1_d) : (F^0_d, F^1_d, F^*_d) \to \varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d)$ is a morphism in RT_k because, for all $\overline{R : \mathsf{Rel}(A,B)}$, $((\gamma^0_d)_{\overline{A}}, (\gamma^1_d)_{\overline{B}}) : F^*_d\overline{R} \to \varinjlim_{d\in\mathcal{D}} F^*_d\overline{R}$ is a morphism in Rel. So $\{(\gamma^0_d, \gamma^1_d)\}_{d\in\mathcal{D}}$ are the mediating morphisms of a cocone in RT_k with vertex $\varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d)$. To see that this cocone is a colimiting cocone, let $C = (C^0, C^1, C^*)$ be the vertex of a cocone for $\{(F^0_d, F^1_d, F^*_d)\}_{d\in\mathcal{D}}$ with injections $(\delta^0_d, \delta^1_d) : (F^0_d, F^1_d, F^*_d) \to C$. If $\eta^0 : \varinjlim_{d\in\mathcal{D}} F^0_d \to C^0$ and $\eta^1 : \varinjlim_{d\in\mathcal{D}} F^1_d \to C^1$ are the mediating morphisms in $[\mathsf{Set}^k, \mathsf{Set}]$, then η^0 and η^1 are unique such that $\delta^0_d = \eta^0 \circ \gamma^0_d$ and $\delta^1_d = \eta^1 \circ \gamma^1_d$. We therefore have that $(\eta^0, \eta^1) : \varinjlim_{d\in\mathcal{D}}(F^0_d, F^1_d, F^*_d) \to C$ is the mediating morphism in RT_k. Indeed, for all $\overline{R : \mathsf{Rel}(A,B)}$ and $(x, y) \in \varinjlim_{d\in\mathcal{D}} F^*_d\overline{R}$, there exist d and $(x', y') \in F^*_d\overline{R}$ such that $(\gamma^0_d)_{\overline{A}}x' = x$ and $(\gamma^1_d)_{\overline{B}}y' = y$. But then $(\eta^0_{\overline{A}}x, \eta^1_{\overline{B}}y) = (\eta^0_{\overline{A}}((\gamma^0_d)_{\overline{A}}x'), \eta^1_{\overline{B}}((\gamma^1_d)_{\overline{B}}y')) = ((\delta^0_d)_{\overline{A}}x', (\delta^1_d)_{\overline{B}}y')$, and this pair is in $C^*\overline{R}$ because (δ^0_d, δ^1_d) is a morphism from (F^0_d, F^1_d, F^*_d) to C in RT_k. □

Definition 15. *A functor* $T = (T^0, T^1, T^*)$ *on* RT_k *is* ω-cocontinuous *if* T^0 *and* T^1 *are* ω-*cocontinuous endofunctors on* $[\mathsf{Set}^k, \mathsf{Set}]$ *and* T^* *is an* ω-*cocontinuous functor from* RT_k *to* $[\mathsf{Rel}^k, \mathsf{Rel}]$.

For any k and $R : \mathsf{Rel}(A, B)$, let K^{Rel}_R be the constantly R-valued functor from Rel^k to Rel, and for any k and set A, let K^{Set}_A be the constantly A-valued functor from Set^k to Set. Moreover, let 0 denote either the empty set or the empty relation on the empty set, depending on the context. Observing that, for every k, K^{Set}_0 is initial in the category of functors from Set^k to Set, and similarly for K^{Rel}_0, we have that, for each k, $K_0 = (K^{\mathsf{Set}}_0, K^{\mathsf{Set}}_0, K^{\mathsf{Rel}}_0)$ is initial in the category of k-ary relation transformers. Thus, if $T = (T^0, T^1, T^*) : RT_k \to RT_k$ is an endofunctor on RT_k then we can define μT to be the relation transformer

$$\mu T = \varinjlim_n T^n K_0$$

Then Lemma 14 shows μT is indeed a relation transformer, and that it is given explicitly by

$$\varinjlim_n T^n K_0 = (\mu T^0, \mu T^1, \varinjlim_n (T^n K_0)^*) \tag{3}$$

Lemma 16. *For any* ω-*cocontinuous functor on* RT_k, $\mu T \cong T(\mu T)$.

Proof. We have $T(\mu T) = T(\varinjlim_n (T^n K_0)) \cong \varinjlim_n T(T^n K_0) = \mu T$. □

In fact, the isomorphism in Lemma 16 is given by the morphisms $(in_0, in_1) : T(\mu T) \to \mu T$ and $(in_0^{-1}, in_1^{-1}) : \mu T \to T(\mu T)$ in RT_k. It is worth noting that the latter is always a morphism in RT_k, but the former isn't necessarily a morphism in RT_k unless T is ω-cocontinuous.

Say realizing that not being able to define third components directly, but rather only through the other two components, is an important conceptual contribution. Not all functors on Rel are third components of relation transformers. It's overly restrictive to require that the third component of a functor on RT_k be a functor on all of $[\mathsf{Rel}^k, \mathsf{Rel}]$. For example, we can define $T_\rho F$ when F is a relation transformer, but it is not clear how we could define $T_\rho F$ when $F : [\mathsf{Rel}^k, \mathsf{Rel}]$.

Definition 17. *A* relation environment *maps each type variable to a relation, and each type constructor variable of arity k to a ω-cocontinuous k-ary relation transformer. A morphism $f : \rho \to \rho'$ from a relation environment ρ to a relation environment ρ' such that $\rho|_{\mathbb{V}} = \rho'|_{\mathbb{V}}$ maps each type variable v to $id_{\rho v}$ and each type constructor variable ϕ of arity k to a natural transformation from the k-ary relation transformer $\rho\phi$ to the k-ary relation transformer $\rho'\phi$.*

When convenient we identify a 0-ary relation transformer with the relation (transformer) that is its codomain. With this convention, a relation environment maps a type constructor variable of arity 0 to a 0-ary relation transformer — i.e., to a relation — just as it does a type variable. We write $\rho[\alpha_1 := R_1, ..., \alpha_k := R_k]$ for the relation environment ρ' such that $\rho'\alpha_i = R_i$ for $i = 1, ..., k$ and $\rho'\alpha = \rho\alpha$ if $\alpha \notin \{\alpha_1, ..., \alpha_k\}$. We write RelEnv for the collection of all relation environments. If ρ is a relation environment, we write $\pi_1\rho$ for the set environment mapping each type variable β to $\pi_1(\rho\beta)$ and each type constructor variable ϕ to the functor $(\rho\phi)^0$. The set environment $\pi_2\rho$ is defined analogously.

We define, for each k, the notion of an ω-cocontinuous functor from RelEnv to RT_k:

Definition 18. *An ω-cocontinuous functor $H : [\mathsf{RelEnv}, RT_k]$ is a triple $H = (H^0, H^1, H^*)$, where*

- *H^0 and H^1 are objects in $[\mathsf{SetEnv}, [\mathsf{Set}^k, \mathsf{Set}]]$*
- *H^* is a an object in $[\mathsf{RelEnv}, [\mathsf{Rel}^k, \mathsf{Rel}]]$*
- *for all $\overline{R : \mathsf{Rel}(A, B)}$ and morphisms f in RelEnv, $\pi_1(H^* f\, \overline{R}) = H^0(\pi_1 f)\, \overline{A}$ and $\pi_2(H^* f\, \overline{R}) = H^1(\pi_2 f)\, \overline{B}$*
- *The action of H on ρ in RelEnv is given by $H\rho = (H^0(\pi_1\rho), H^1(\pi_2\rho), H^*\rho)$*
- *The action of H on morphisms $f : \rho \to \rho'$ in RelEnv is given by $Hf = (H^0(\pi_1 f), H^1(\pi_2 f))$*

Spelling out the last two bullet points above gives the following analogues of Conditions (1), (2), and (3) immediately following Definition 12:

(1) if $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then

$$H^*\rho\, \overline{R} : \mathsf{Rel}(H^0(\pi_1\rho)\, \overline{A}, H^1(\pi_2\rho)\, \overline{B})$$

In other words, $\pi_1(H^*\rho\overline{R}) = H^0(\pi_1\rho)\, \overline{A}$ and $\pi_2(H^*\rho\overline{R}) = H^1(\pi_2\rho)\, \overline{B}$.

(2) if $(\alpha_1, \beta_1) \in \mathsf{Hom}_{\mathsf{Rel}}(R_1, S_1), ..., (\alpha_k, \beta_k) \in \mathsf{Hom}_{\mathsf{Rel}}(R_k, S_k)$, then

$$H^*\rho\, \overline{(\alpha, \beta)} = (H^0(\pi_1\rho)\, \overline{\alpha}, H^1(\pi_2\rho)\, \overline{\beta})$$

In other words, $\pi_1(H^*\rho\, \overline{(\alpha, \beta)}) = H^0(\pi_1\rho)\, \overline{\alpha}$ and $\pi_2(H^*\rho\, \overline{(\alpha, \beta)}) = H^1(\pi_2\rho)\, \overline{\beta}$.

(3) if $f : \rho \to \rho'$ and $R_1 : \mathsf{Rel}(A_1, B_1), ..., R_k : \mathsf{Rel}(A_k, B_k)$, then

$$\text{if } (x, y) \in H^*\rho\, \overline{R} \text{ then } (H^0(\pi_1 f)\, \overline{A}\, x, H^1(\pi_2 f)\, \overline{B}\, y) \in H^*\rho'\, \overline{R}$$

Note, however, that this condition is automatically satisfied because it is implied by the third bullet point of Definition 18.

Considering RelEnv as a product $\Pi_{\phi^k \in \mathbb{V} \cup \mathbb{T}} RT_k$, we extend Lemma 14 to compute colimits in RelEnv "componentwise", and similarly extend Definition 15 to give a "componentwise" notion of ω-cocontinuity of functors from RelEnv to RT_k.

We recall from the start of this section that Definition 19 is given mutually inductively with Definition 8. We have:

Definition 19. *Let ρ be a relation environment. The* relation interpretation $[\![\cdot]\!]^{\mathsf{Rel}} : \mathcal{F} \to [\mathsf{RelEnv}, \mathsf{Rel}]$ *is defined by*

$$[\![\Gamma;\emptyset \vdash v]\!]^{\mathsf{Rel}}\rho = \rho v \text{ if } v \in \mathbb{V}$$

$$[\![\Gamma;\emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Rel}}\rho = [\![\Gamma;\emptyset \vdash \sigma]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma;\emptyset \vdash \tau]\!]^{\mathsf{Rel}}\rho$$

need to interpret forall types if we include them

$$\begin{aligned}
[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Rel}}\rho &= \{\eta : [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := -}] \Rightarrow [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := -}]\}\\
&= \{(t,t') \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid\\
&\qquad \forall R_1 : \mathsf{Rel}(A_1,B_1) \ldots R_k : \mathsf{Rel}(A_k,B_k).\\
&\qquad (t_{\overline{A}}, t'_{\overline{B}}) \in ([\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}])^{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]}\}\\
&= \{(t,t') \in [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}(\pi_1\rho) \times [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\,F\,G]\!]^{\mathsf{Set}}(\pi_2\rho) \mid\\
&\qquad \forall R_1 : \mathsf{Rel}(A_1,B_1) \ldots R_k : \mathsf{Rel}(A_k,B_k).\\
&\qquad \forall (a,b) \in [\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}].\\
&\qquad (t_{\overline{A}}a, t'_{\overline{B}}b) \in [\![\Gamma;\overline{\alpha} \vdash G]\!]^{\mathsf{Rel}}\rho[\overline{\alpha := R}]\}
\end{aligned}$$

$$[\![\Gamma;\Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}}\rho = 0$$

$$[\![\Gamma;\Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}}\rho = 1$$

$$[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho = (\rho\phi)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}$$

$$[\![\Gamma;\Phi \vdash \sigma + \tau]\!]^{\mathsf{Rel}}\rho = [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho + [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho$$

$$[\![\Gamma;\Phi \vdash \sigma \times \tau]\!]^{\mathsf{Rel}}\rho = [\![\Gamma;\Phi \vdash \sigma]\!]^{\mathsf{Rel}}\rho \times [\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho$$

$$\begin{aligned}
[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\rho &= (\mu T_\rho)\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}\\
\textit{where } T_\rho &= (T^{\mathsf{Set}}_{\pi_1\rho}, T^{\mathsf{Set}}_{\pi_2\rho}, T^{\mathsf{Rel}}_\rho)\\
\textit{and } T^{\mathsf{Rel}}_\rho F &= \lambda\overline{R}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F][\overline{\alpha := R}]\\
\textit{and } T^{\mathsf{Rel}}_\rho \delta &= \lambda\overline{R}.[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := \delta][\overline{\alpha := id_{\overline{R}}}]
\end{aligned}$$

If ρ is a relational environment and $\vdash \tau : \mathcal{F}$, then we write $[\![\vdash \tau]\!]^{\mathsf{Rel}}$ instead of $[\![\emptyset;\emptyset \vdash \tau]\!]^{\mathsf{Rel}}\rho$ as for set interpretations.

For the last clause in Definition 19 to be well-defined, we need to know that T_ρ is an ω-cocontinuous endofunctor on RT so that, by Definition 16, it admits a fixed point. Since T_ρ is defined in terms of $[\![\Gamma;\Phi,\phi^k,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}$, this means that relational interpretations of types must be ω-cocontinuous functors from RelEnv to RT_0. This in turn means that the actions of relational interpretations of types on objects and on morphisms in RelEnv are intertwined. In fact, we already know from [Johann and Polonsky 2019] that, for every $\Gamma;\overline{\alpha} \vdash \tau : \mathcal{F}$, $[\![\Gamma;\overline{\alpha} \vdash \tau]\!]^{\mathsf{Rel}}$ is actually functorial in $\overline{\alpha}$ and ω-cocontinuous. We first define the actions of each of these functors on morphisms between environments, and then argue that the functors given by Definitions 19 and 20 are well-defined and have the required properties.

Definition 20. *Let $f : \rho \to \rho'$ for relation environments ρ and ρ' such that $\rho|_{\mathbb{V}} = \rho'|_{\mathbb{V}}$. The action $[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f$ of $[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}$ on the morphism f is given as follows:*

- *If $\Gamma, v; \emptyset \vdash v$ then $[\![\Gamma, v; \emptyset \vdash v]\!]^{\mathsf{Rel}} f = id_{\rho v}$.*
- *If $\Gamma; \emptyset \vdash \sigma \to \tau$ then $[\![\Gamma; \emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Rel}} f = id_{[\![\Gamma;\emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Rel}}\rho}$.*
- *If $\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G$, then we define $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}} f = id_{[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}}\rho}$.*
- *If $\Gamma; \Phi \vdash \mathbb{0}$ then $[\![\Gamma; \Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}} f = id_0$.*
- *If $\Gamma; \Phi \vdash \mathbb{1}$ then $[\![\Gamma; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}} f = id_1$.*
- *If $\Gamma; \Phi \vdash \phi\overline{\tau}$, then we have that $[\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}} f : [\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho \to [\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho' = (\rho\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho} \to (\rho'\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho'}$ is defined by $[\![\Gamma; \Phi \vdash \phi\tau A]\!]^{\mathsf{Rel}} f = (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho'}} \circ (\rho\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} f} = (\rho'\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} f} \circ (f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}}$.*
- *If $\Gamma; \Phi \vdash \sigma + \tau$ then $[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Rel}} f$ is defined by $[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Rel}} f (\mathsf{inl}\, x) = \mathsf{inl}\, ([\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Rel}} f x)$ and $[\![\Gamma; \Phi \vdash \sigma + \tau]\!]^{\mathsf{Rel}} f (\mathsf{inr}\, y) = \mathsf{inr}\, ([\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} f y)$.*
- *If $\Gamma; \Phi \vdash \sigma \times \tau$ then $[\![\Gamma; \Phi \vdash \sigma \times \tau]\!]^{\mathsf{Rel}} f = [\![\Gamma; \Phi \vdash \sigma]\!]^{\mathsf{Rel}} f \times [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} f$.*
- *If $\Gamma; \Phi \vdash (\mu\phi^k.\lambda\overline{\alpha}.H)\overline{\tau}$ then letting $\sigma_f : T_\rho \to T_{\rho'}$ be the map*

$$F \mapsto \lambda\overline{R}.[\![\Gamma; \Phi, \phi, \overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} f[\phi := id_F]\overline{[\alpha := id_R]}$$

we define

$$\begin{array}{cl} & [\![\Gamma; \Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}} f \\ = & (\mu\sigma_f)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho'} \circ (\mu T_\rho)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} f}) \\ = & (\mu T_{\rho'})\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} f} \circ (\mu\sigma_f)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho} \end{array}$$

To see that the functors given by Definitions 19 and 20 are well-defined we must show that $T_\rho F$ is a relation transformer for any relation transformer F, and that $\sigma_f F : T_\rho F \to T_{\rho'} F$ is a morphism of relation transformers for every relation transformer F and every morphism $f : \rho \to \rho'$ in RelEnv.

Lemma 21. *The interpretations in Definitions 19 and 20 are well-defined and, for every $\Gamma; \Phi \vdash \tau$,*

$$[\![\Gamma; \Phi \vdash \tau]\!] = ([\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}, [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}, [\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}})$$

is an ω-cocontinuous functor from RelEnv to RT_0, i.e., is an element of $[\mathsf{RelEnv}, RT_0]$.

Proof. By induction on the structure of τ. The only interesting cases are when $\tau = \phi\overline{\tau}$ and when $\tau = (\mu\phi^k.\lambda\overline{\alpha}.H)\overline{\tau}$. We consider each in turn.

- When $\tau = \Gamma; \Phi \vdash \phi\overline{\tau}$, we have

$$\begin{array}{cl} & \pi_i([\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}}\rho) \\ = & \pi_i((\rho\phi)\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}) \\ = & (\pi_i(\rho\phi))(\overline{\pi_i([\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho)}) \\ = & ((\pi_i\rho)\phi)(\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}(\pi_i\rho)}) \\ = & [\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}(\pi_i\rho) \end{array}$$

and, for $f : \rho \to \rho'$ in RelEnv,

$$\begin{array}{cl} & \pi_i([\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Rel}} f) \\ = & \pi_i((f\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho'}}) \circ \pi_i((\rho\phi)(\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} f})) \\ = & (\pi_i(f\phi))_{\overline{\pi_i([\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho')}} \circ (\pi_i(\rho\phi))(\overline{\pi_i([\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} f)}) \\ = & ((\pi_i f)\phi)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}(\pi_i\rho')}} \circ ((\pi_i\rho)\phi)(\overline{[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Set}}(\pi_i f)}) \\ = & [\![\Gamma; \Phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}(\pi_i f) \end{array}$$

The third equalities of each of the above derivations are by the induction hypothesis. That $[\![\Gamma;\Phi \vdash \phi\overline{\tau}]\!]$ is ω-cocontinuous is an immediate consequence of the facts that Set and Rel are locally finitely presentable, together with Corollary 12 of [Johann and Polonsky 2019].

- When $\tau = (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}$ we first show that $[\![(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]$ is well-defined.
  - $\underline{T_\rho : [RT_k, RT_k]}$: We must show that, for any relation transformer $F = (F^0, F^1, F^*)$, the triple $\overline{T_\rho F = (T^{\mathsf{Set}}_{\pi_1\rho}F^0, T^{\mathsf{Set}}_{\pi_2\rho}F^1, T^{\mathsf{Rel}}_\rho F)}$ is also a relation transformer. Let $\overline{R : \mathsf{Rel}(A,B)}$. Then for $i = 1,2$, we have

$$\begin{aligned}
\pi_i(T^{\mathsf{Rel}}_\rho\, F\,\overline{R}) &= \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}}\rho[\phi := F]\overline{[\alpha := R]})\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i(\rho[\phi := F]\overline{[\alpha := R]}))\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i\rho)[\phi := \pi_i F]\overline{[\alpha := \pi_i R]})\\
&= T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i F)(\overline{\pi_i R})
\end{aligned}$$

  and

$$\begin{aligned}
\pi_i(T^{\mathsf{Rel}}_\rho\, F\,\overline{\gamma}) &= \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := id_F]\overline{[\alpha := \gamma]})\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}}(\pi_i(id_\rho[\phi := id_F]\overline{[\alpha := \gamma]}))\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_{\pi_i\rho}[\phi := id_{\pi_i F}]\overline{[\alpha := \pi_i\gamma]}\\
&= T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i F)(\overline{\pi_i\gamma})
\end{aligned}$$

  Here, the second equality in each of the above chains of equalities is by the induction hypothesis.
  We also have that, for every morphism $\delta = (\delta^0, \delta^1) : F \to G$ in RT_k and all $\overline{R : \mathsf{Rel}(A,B)}$,

$$\begin{array}{cl}
& \pi_i((T^{\mathsf{Rel}}_\rho\delta)_{\overline{R}})\\
= & \pi_i([\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} id_\rho[\phi := \delta]\overline{[\alpha := id_R]})\\
= & [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Set}} id_{\pi_i\rho}[\phi := \pi_i\delta]\overline{[\alpha := id_{\pi_i R}]}\\
= & (T^{\mathsf{Set}}_{\pi_i\rho}(\pi_i\delta))_{\overline{\pi_i R}}
\end{array}$$

  Here, the second equality is by the induction hypothesis. That T_ρ is ω-cocontinuous follows immediately from the induction hypothesis on $[\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]$ and the fact that colimts are computed componentwise in RT.
  - $\underline{\sigma_f = (\sigma^{\mathsf{Set}}_{\pi_1 f}, \sigma^{\mathsf{Set}}_{\pi_2 f}) \text{ is a natural transformation from } T_\rho \text{ to } T_{\rho'}}$: We must show that $(\sigma_f)_F = \overline{((\sigma^{\mathsf{Set}}_{\pi_1 f})_{F^0}, (\sigma^{\mathsf{Set}}_{\pi_2 f})_{F^1})}$ is a morphism in RT_k for all relation transformers $F = (F^0, F^1, F^*)$, i.e., that $((\sigma_f)_F)_{\overline{R}} = (((\sigma^{\mathsf{Set}}_{\pi_1 f})_{F^0})_{\overline{A}}, ((\sigma^{\mathsf{Set}}_{\pi_2 f})_{F_1})_{\overline{B}})$ is a morphism in Rel for all relations $\overline{R : \mathsf{Rel}(A,B)}$. Indeed, we have that

$$((\sigma_f)_F)_{\overline{R}} = [\![\Gamma;\Phi,\phi,\overline{\alpha} \vdash H]\!]^{\mathsf{Rel}} f[\phi := id_F]\overline{[\alpha := id_R]}$$

  is a morphism in RT_0 (and thus in Rel) by the induction hypothesis.

The relation transformer μT_ρ is therefore a fixed point of T_ρ by Lemma 16, and $\mu\sigma_f$ is a morphism in RT_k from μT_ρ to $\mu T_{\rho'}$. (μ is shown to be a functor in [Johann and Polonsky 2019].) So $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}$, and thus $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]$, is well-defined.

To see that $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!] : [\mathsf{RelEnv}, RT_0]$, we must verify three conditions:

– Condition (1) after Definition 18 is satisfied since

$$\begin{aligned}
\pi_i([\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\rho) &= \pi_i((\mu T_\rho)(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho}))\\
&= \pi_i(\mu T_\rho)(\overline{\pi_i([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}}\rho)})\\
&= \mu T^{\mathsf{Set}}_{\pi_i\rho}(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}(\pi_i\rho)})\\
&= [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}(\pi_i\rho)
\end{aligned}$$

The third equality is by Equation 3 and the induction hypothesis.

– Condition (2) after Definition 18 is satisfied since it is subsumed by the previous condition because $k = 0$.

– The third bullet point of Definition 18 is satisfied because

$$\begin{aligned}
&\pi_i([\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}} f)\\
&= \pi_i((\mu T_{\rho'})(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f}) \circ (\mu\sigma_f)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}})\\
&= \pi_i((\mu T_{\rho'})(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f})) \circ \pi_i((\mu\sigma_f)_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho}})\\
&= \pi_i(\mu T_{\rho'})(\overline{\pi_i([\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Rel}} f)}) \circ \pi_i(\mu\sigma_f)_{\overline{\pi_i([\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\rho)}}\\
&= (\mu T^{\mathsf{Set}}_{\pi_i\rho'})(\overline{[\![\Gamma;\Phi \vdash \tau]\!]^{\mathsf{Set}}(\pi_i f)}) \circ (\mu\sigma^{\mathsf{Set}}_{\pi_i f})_{\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}(\pi_i\rho)}}\\
&= [\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}(\pi_i f).
\end{aligned}$$

The fourth equality is by 3 and the induction hypothesis.

As before, that $[\![\Gamma;\Phi \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]$ is ω-concontinuous follows from the facts that Set and Rel are locally finitely presentable, and that colimits in RelEnv are computed componentwise, together with Corollary 12 of [Johann and Polonsky 2019].

□

3.3 The Identity Extension Lemma

Definition 22. *If F is a functor from Set^k to Set, define Eq^*_F as follows. Given $R_1 : \mathsf{Rel}(A_1, B_1)$, ..., $R_k : \mathsf{Rel}(A_k, B_k)$, let $\overline{i_{R_i} : R_i \hookrightarrow A_i \times B_i}$ be the inclusion of R_i as a subset of $A_i \times B_i$. By the universal property of the product, there exists a unique $h_{\overline{A\times B}}$ making the diagram*

$$\begin{tikzcd}
F\overline{A} & F(\overline{A\times B}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] \arrow[d, dashed, "h_{\overline{A\times B}}"] & F\overline{B}\\
& F\overline{A}\times F\overline{B} \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

commute. Let $h_{\overline{R}} : F\overline{R} \to F\overline{A} \times F\overline{B}$ be $h_{\overline{A\times B}} \circ F\overline{i_R}$. Then, define $\mathsf{Eq}^*_F\overline{R}$ as the subobject through which $h_{\overline{R}}$ is factorized by the mono-epi factorization system in Set*, as shown in the following diagram:*

$$\begin{tikzcd}
F\overline{R} \arrow[rr, "h_{\overline{R}}"] \arrow[dr, two heads, "q_{\mathsf{Eq}^*_F\overline{R}}"'] & & F\overline{A}\times F\overline{B}\\
& \mathsf{Eq}^*_F\overline{R} \arrow[ur, hook, "i_{\mathsf{Eq}^*_F\overline{R}}"'] &
\end{tikzcd}$$

Note that $\mathsf{Eq}^*_F\overline{R} : \mathsf{Rel}(F\overline{A}, F\overline{B})$ by construction. Moreover, if $\overline{(\alpha,\beta) : R \to S}$ are morphisms in Rel*, then $\mathsf{Eq}^*_F\overline{(\alpha,\beta)}$ is defined to be $(F\overline{\alpha}, F\overline{\beta})$.*

If F is a functor from Set^k to Set], let $\mathsf{Eq}_F = (F, F, \mathsf{Eq}^*_F)$. Note that if $A : \mathsf{Set}$ then Eq_A is precisely as defined in Section 1.1. This is consistent with the fact that a set can be seen as a 0-ary functor on sets and a relation can be seen as a 0-ary functor on relations.

Lemma 23. *If $F : [\mathsf{Set}^k, \mathsf{Set}]$ then Eq_F is in RT_k.*

Proof. Consider $\overline{(\alpha, \beta) : R \to S}$, where $\overline{R : \mathsf{Rel}(A, B)}$ and $\overline{S : \mathsf{Rel}(C, D)}$. We want to show that there exists a morphism $\epsilon : \mathsf{Eq}^*_F \overline{R} \to \mathsf{Eq}^*_F \overline{S}$ such that

$$\begin{tikzcd}
\mathsf{Eq}^*_F\overline{R} \arrow[r, hook, "i_{\mathsf{Eq}^*_F\overline{R}}"] \arrow[d, "\epsilon"'] & F\overline{A} \times F\overline{B} \arrow[d, "F\overline{\alpha}\times F\overline{\beta}"] \\
\mathsf{Eq}^*_F\overline{S} \arrow[r, hook, "i_{\mathsf{Eq}^*_F\overline{S}}"'] & F\overline{C} \times F\overline{D}
\end{tikzcd}$$

commutes. By hypothesis, there exists $\overline{\gamma : R \to S}$ such that the diagram

$$\begin{tikzcd}
R \arrow[r, hook, "i_R"] \arrow[d, "\gamma"'] & A \times B \arrow[d, "\alpha\times\beta"] \\
S \arrow[r, hook, "i_S"] & C \times D
\end{tikzcd}$$

commutes. Thus, we get the following commutative diagram:

$$\begin{tikzcd}
F\overline{R} \arrow[r, hook, "F\overline{i_R}"] \arrow[d, "F\overline{\gamma}"'] \arrow[rr, bend left, "h_{\overline{R}}"] & F(\overline{A \times B}) \arrow[r, "h_{\overline{A\times B}}"] \arrow[d, "F(\overline{\alpha\times\beta})"] & F\overline{A} \times F\overline{B} \arrow[d, "F\overline{\alpha}\times F\overline{\beta}"] \\
F\overline{S} \arrow[r, hook, "F\overline{i_S}"'] \arrow[rr, bend right, "h_{\overline{S}}"'] & F(\overline{C \times D}) \arrow[r, "h_{\overline{C\times D}}"'] & F\overline{C} \times F\overline{D}
\end{tikzcd}$$

Then, by the left-lifting-property of $q_{\mathsf{Eq}^*_F\overline{R}}$ with respect to $i_{\mathsf{Eq}^*_F\overline{S}}$ given by the epi-mono factorization system, there exists ϵ such that the diagram

$$\begin{tikzcd}
F\overline{R} \arrow[r, two heads, "q_{\mathsf{Eq}^*_F\overline{R}}"] \arrow[d, "F\overline{\gamma}"'] & \mathsf{Eq}^*_F\overline{R} \arrow[r, hook, "i_{\mathsf{Eq}^*_F\overline{R}}"] \arrow[d, dashed, "\epsilon"] & F\overline{A} \times F\overline{B} \arrow[d, "F\overline{\alpha}\times F\overline{\beta}"] \\
F\overline{S} \arrow[r, two heads, "q_{\mathsf{Eq}^*_F\overline{S}}"'] & \mathsf{Eq}^*_F\overline{S} \arrow[r, hook, "i_{\mathsf{Eq}^*_F\overline{S}}"'] & F\overline{C} \times F\overline{D}
\end{tikzcd}$$

commutes. □

Lemma 24. *If $F : [\mathsf{Set}^k, \mathsf{Set}]$ and $A_1, ..., A_k : \mathsf{Set}$, then $\mathsf{Eq}^*_F \overline{\mathsf{Eq}^*_A} = \mathsf{Eq}^*_{F\overline{A}}$.*

Proof. The relation $\overline{\mathsf{Eq}^*_A}$ corresponds to the subobject $\overline{\Delta_A : A \to A \times A}$. Since $h_{\overline{\mathsf{Eq}_A}}$ is the unique morphism making the diagram

$$\begin{tikzcd}
F\overline{A} & F(\overline{A\times A}) \arrow[l, "F\overline{\pi_1}"'] \arrow[r, "F\overline{\pi_2}"] & F\overline{A} \\
& F\overline{A} \arrow[u, "F\overline{\Delta_A}"'] \arrow[ul, equal] \arrow[ur, equal] \arrow[d, dashed, "h_{\overline{\mathsf{Eq}^*_A}}"] & \\
& F\overline{A}\times F\overline{A} \arrow[uul, "\pi_1"] \arrow[uur, "\pi_2"'] &
\end{tikzcd}$$

commute, we have $h_{\overline{\mathsf{Eq}^*_A}} = \Delta_{F\overline{A}}$. Moreover, since $\Delta_{F\overline{A}}$ is a monomorphism, we have that $i_{\mathsf{Eq}^*_F \overline{\mathsf{Eq}^*_A}} = \Delta_{F\overline{A}}$. To conclude we observe that the relation corresponding to the subobject $\Delta_{F\overline{A}}$ is $(\mathsf{Eq}_{F\overline{A}})^*$. $\square$

We now show that the Identity Extension Lemma holds for the interpretations given in Definitions 8 and 19. If ρ is a set environment, define Eq_ρ to be the relation environment such that $\mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v}$ for all $v \in \mathbb{V} \cup \mathbb{T}$. The Identity Extension Lemma can then be stated and proved as follows:

Theorem 25. *If ρ is a set environment, and $\Gamma; \Phi \vdash \tau : \mathcal{F}$, then $[\![\Gamma; \Phi \vdash \tau]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}$.*

Proof. By induction on the structure of τ.

- $[\![\Gamma; \emptyset \vdash v]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_\rho v = \mathsf{Eq}_{\rho v} = \mathsf{Eq}_{[\![\Gamma;\emptyset\vdash v]\!]^{\mathsf{Set}}\rho}$ where $v \in \Gamma$.
- $[\![\Gamma; \emptyset \vdash \sigma \to \tau]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = [\![\Gamma; \emptyset \vdash \sigma]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \to [\![\Gamma; \emptyset \vdash \tau]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_{[\![\Gamma;\emptyset\vdash\sigma]\!]^{\mathsf{Set}}\rho} \to \mathsf{Eq}_{[\![\Gamma;\emptyset\vdash\tau]\!]^{\mathsf{Set}}\rho} = \mathsf{Eq}_{[\![\Gamma;\emptyset\vdash\sigma\to\tau]\!]^{\mathsf{Set}}\rho}$, where the second equality is by the induction hypothesis.
- $\tau = \forall v.\tau_1$
- By definition, $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho$ is the relation on $[\![\Gamma; \emptyset \vdash \mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}} \rho$ relating t and t' if, for all $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ in Rel. To prove that this is equal to $\mathsf{Eq}_{[\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha}}\, F\, G]\!]^{\mathsf{Set}}\rho}$ we need to show that $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$ if and only if $t = t'$ and $(t_{\overline{A}}, t_{\overline{B}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$. The only interesting part of this equivalence is to show that if $(t_{\overline{A}}, t'_{\overline{B}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := R]}$ in Rel for all $R_1 : \mathsf{Rel}(A_1, B_1), \ldots, R_k : \mathsf{Rel}(A_k, B_k)$, then $t = t'$. By hypothesis, $(t_{\overline{A}}, t'_{\overline{A}})$ is a morphism from $[\![\Gamma; \overline{\alpha} \vdash F]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := \mathsf{Eq}_A]}$ to $[\![\Gamma; \overline{\alpha} \vdash G]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho \overline{[\alpha := \mathsf{Eq}_A]}$ in Rel for all $A_1 \ldots A_k$: Set. By the induction hypothesis, it is therefore a morphism from $\mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$ to $\mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash G]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$ in Rel. This means that, for every $x : \mathsf{Eq}_{[\![\Gamma;\overline{\alpha}\vdash F]\!]^{\mathsf{Set}}\rho\overline{[\alpha:=A]}}$, $t_{\overline{A}} x = t'_{\overline{A}} x$. Then, by extensionality, $t = t'$.
- $[\![\Gamma; \Phi \vdash \mathbb{0}]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = 0_{\mathsf{Rel}} = \mathsf{Eq}_{0_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\mathbb{0}]\!]^{\mathsf{Set}}\rho}$
- $[\![\Gamma; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Rel}} \mathsf{Eq}_\rho = 1_{\mathsf{Rel}} = \mathsf{Eq}_{1_{\mathsf{Set}}} = \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\mathbb{1}]\!]^{\mathsf{Set}}\rho}$

- The application case is proved by the sequence of equalities

$$\begin{aligned}
[\![\Gamma;\Phi\vdash\phi\overline{\tau}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mathsf{Eq}_\rho\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= (\mathsf{Eq}_\rho\phi)^*\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{(\rho\phi)\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi\vdash\phi\overline{\tau}]\!]^{\mathsf{Set}}\rho}
\end{aligned}$$

  where the second equality is given by the induction hypothesis, and the third by Lemma 24.
- The fix-point case is proven by the sequence of equalities

$$\begin{aligned}
[\![\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho &= (\mu T_{\mathsf{Eq}_\rho})\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{\mathsf{Eq}_\rho}K_0\overline{[\![\Gamma;\Phi\vdash\tau_1]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho}\\
&= \varinjlim_{n\in\mathbb{N}} T^n_{\mathsf{Eq}_\rho}K_0\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau_1]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0}\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau_1]\!]^{\mathsf{Set}}\rho}}\\
&= \varinjlim_{n\in\mathbb{N}} \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{\varinjlim_{n\in\mathbb{N}}(T^{\mathsf{Set}}_\rho)^nK_0\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi\vdash(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho}
\end{aligned}$$

  where the third equality is by induction hypothesis, the sixth is by Lemma 24 and the fifth equality is because, for every $n\in\mathbb{N}$, the following two statements can be proved by simultaneous induction:

$$T^n_{\mathsf{Eq}_\rho}K_0^{\mathsf{Rel}}\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} = \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0^{\mathsf{Set}}}\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} \tag{4}$$

  and

$$\begin{aligned}
&[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi:=T^n_{\mathsf{Eq}_\rho}K_0^{\mathsf{Rel}}]\overline{[\alpha:=\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&\qquad= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0^{\mathsf{Set}}}]\overline{[\alpha:=\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}
\end{aligned}\tag{5}$$

  We prove (4). The case $n=0$ is trivial, because $T^0_{\mathsf{Eq}_\rho}K_0^{\mathsf{Rel}}=K_0^{\mathsf{Rel}}$ and $(T^{\mathsf{Set}}_\rho)^0K_0^{\mathsf{Set}}=K_0^{\mathsf{Set}}$; the inductive step is proved by the following sequence of equalities:

$$\begin{aligned}
T^{n+1}_{\mathsf{Eq}_\rho}K_0^{\mathsf{Rel}}\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}} &= T^{\mathsf{Rel}}_{\mathsf{Eq}_\rho}(T^n_{\mathsf{Eq}_\rho}K_0^{\mathsf{Rel}})\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi:=T^n_{\mathsf{Eq}_\rho}K_0^{\mathsf{Rel}}]\overline{[\alpha:=\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_\rho[\phi:=\mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^nK_0^{\mathsf{Set}}}]\overline{[\alpha:=\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}]}\\
&= [\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Rel}}\mathsf{Eq}_{\rho[\phi:=(T^{\mathsf{Set}}_\rho)^nK_0^{\mathsf{Set}}]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}\\
&= \mathsf{Eq}_{[\![\Gamma;\Phi,\phi,\overline{\alpha}\vdash H]\!]^{\mathsf{Set}}\rho[\phi:=(T^{\mathsf{Set}}_\rho)^nK_0^{\mathsf{Set}}]\overline{[\alpha:=[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho]}}\\
&= \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^{n+1}K_0^{\mathsf{Set}}\overline{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}\\
&= \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^{n+1}K_0^{\mathsf{Set}}}\overline{\mathsf{Eq}_{[\![\Gamma;\Phi\vdash\tau]\!]^{\mathsf{Set}}\rho}}
\end{aligned}$$

  Here, the third equality is by (5), the fifth by the induction hypothesis on H, and the last by Lemma 24. We prove the induction step of (5) by structural induction on H: the only interesting case, though, is when ϕ is applied, i.e., for $H = \phi\overline{\sigma}$, which is proved by the sequence of equalities

$$
\begin{aligned}
& \llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \phi\overline{\sigma} \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho} K_0^{\mathsf{Rel}}]\overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho}]} \\
&= T^n_{\mathsf{Eq}_\rho} K_0^{\mathsf{Rel}}\, \overline{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := T^n_{\mathsf{Eq}_\rho} K_0^{\mathsf{Rel}}]\overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho}]}} \\
&= T^n_{\mathsf{Eq}_\rho} K_0^{\mathsf{Rel}}\, \overline{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0^{\mathsf{Set}}}]\overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho}]}} \\
&= T^n_{\mathsf{Eq}_\rho} K_0^{\mathsf{Rel}}\, \overline{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_{\rho[\phi := (T^{\mathsf{Set}}_\rho)^n K_0^{\mathsf{Set}}]\overline{[\alpha := \llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho]}}} \\
&= T^n_{\mathsf{Eq}_\rho} K_0^{\mathsf{Rel}}\, \overline{\mathsf{Eq}_{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Set}} \rho[\phi := (T^{\mathsf{Set}}_\rho)^n K_0^{\mathsf{Set}}]\overline{[\alpha := \llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho]}}} \\
&= \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0^{\mathsf{Set}}}\, \overline{\mathsf{Eq}_{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Set}} \rho[\phi := (T^{\mathsf{Set}}_\rho)^n K_0^{\mathsf{Set}}]\overline{[\alpha := \llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho]}}} \\
&= \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0^{\mathsf{Set}}}\, \overline{\llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0^{\mathsf{Set}}}]\overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho}]}} \\
&= \llbracket \Gamma; \Phi, \phi, \overline{\alpha} \vdash \phi\overline{\sigma} \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho[\phi := \mathsf{Eq}_{(T^{\mathsf{Set}}_\rho)^n K_0^{\mathsf{Set}}}]\overline{[\alpha := \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho}]}
\end{aligned}
$$

Here, the second equality is by the induction hypothesis for (5) on the σs, the fourth is by the induction hypothesis on the σs, and the fifth is by the induction hypothesis on n for (4).

- $\llbracket \Gamma; \Phi \vdash \sigma + \tau \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho = \llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho + \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\sigma \rrbracket^{\mathsf{Set}}\rho} + \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho} = \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\sigma \rrbracket^{\mathsf{Set}}\rho + \llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho} = \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\sigma+\tau \rrbracket^{\mathsf{Set}}\rho}$
- $\llbracket \Gamma; \Phi \vdash \sigma \times \tau \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho = \llbracket \Gamma; \Phi \vdash \sigma \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho \times \llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}} \mathsf{Eq}_\rho = \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\sigma \rrbracket^{\mathsf{Set}}\rho} \times \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho} = \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\sigma \rrbracket^{\mathsf{Set}}\rho \times \llbracket \Gamma;\Phi\vdash\tau \rrbracket^{\mathsf{Set}}\rho} = \mathsf{Eq}_{\llbracket \Gamma;\Phi\vdash\sigma\times\tau \rrbracket^{\mathsf{Set}}\rho}$

□

4 INTERPRETING TERMS

If $\Delta = x_1 : \tau_1, ..., x_n : \tau_n$ is a term context for Γ and Φ, then the interpretations $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}}$ and $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}$ are defined by

$$
\begin{aligned}
\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}} &= \llbracket \Gamma; \Phi \vdash \tau_1 \rrbracket^{\mathsf{Set}} \times \ldots \times \llbracket \Gamma; \Phi \vdash \tau_n \rrbracket^{\mathsf{Set}} \\
\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}} &= \llbracket \Gamma; \Phi \vdash \tau_1 \rrbracket^{\mathsf{Rel}} \times \ldots \times \llbracket \Gamma; \Phi \vdash \tau_n \rrbracket^{\mathsf{Rel}}
\end{aligned}
$$

Every well-formed term $\Gamma; \Phi \mid \Delta \vdash t : \tau$ then has a set interpretation $\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}$ as a natural transformation from $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Set}}$ to $\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Set}}$, and a relational interpretation $\llbracket \Gamma; \Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}$ as a natural transformation from $\llbracket \Gamma; \Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}$ to $\llbracket \Gamma; \Phi \vdash \tau \rrbracket^{\mathsf{Rel}}$. These are given in the next two definitions.

Definition 26. *If ρ is a set environment and $\Gamma;\Phi \mid \Delta \vdash t : \tau$ then $\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}\rho$ is defined as follows:*

$$
\begin{array}{lcl}
\llbracket \Gamma;\emptyset \mid \Delta, x : \tau \vdash x : \tau \rrbracket^{\mathsf{Set}}\rho & = & \pi_{|\Delta|+1} \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau \rrbracket^{\mathsf{Set}}\rho & = & \mathsf{curry}(\llbracket \Gamma;\emptyset \mid \Delta, x : \sigma \vdash t : \tau \rrbracket^{\mathsf{Set}}\rho) \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash st : \tau \rrbracket^{\mathsf{Set}}\rho & = & \mathsf{eval} \circ \langle \llbracket \Gamma;\emptyset \mid \Delta \vdash s : \sigma \to \tau \rrbracket^{\mathsf{Set}}\rho, \llbracket \Gamma;\emptyset \mid \Delta \vdash t : \sigma \rrbracket^{\mathsf{Set}}\rho \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash L_{\boldsymbol{\alpha}} x.t : \mathsf{Nat}^{\boldsymbol{\alpha}}\, F\, G \rrbracket^{\mathsf{Set}}\rho & = & \mathsf{curry}(\llbracket \Gamma;\boldsymbol{\alpha} \mid \Delta, x : F \vdash t : G \rrbracket^{\mathsf{Set}}\rho[\overline{\alpha := \_}]) \\
\llbracket \Gamma;\Phi \mid \Delta \vdash t_{\tau} s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\rho & = & \mathsf{eval} \circ \langle (\llbracket \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket\rho\ \_)_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho}}, \\
 & & \qquad \llbracket \Gamma;\Phi \mid \Delta \vdash s : F[\overline{\alpha := \tau}] \rrbracket\rho \rangle
\end{array}
$$

Add rules for $\forall$ if we include it

$$
\begin{array}{lcl}
\llbracket \Gamma;\Phi \mid \Delta, x : \tau \vdash x : \tau \rrbracket^{\mathsf{Set}}\rho & = & \pi_{|\Delta|+1} \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \bot_{\tau} t : \tau \rrbracket\rho & = & !^{0}_{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \mathbb{0} \rrbracket^{\mathsf{Set}}\rho \textit{ where} \\
 & & \quad !^{0}_{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho} \textit{ is the unique morphism from } 0 \\
 & & \quad \textit{to } \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \top : \mathbb{1} \rrbracket^{\mathsf{Set}}\rho & = & !_{1}^{\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}\rho} \textit{ where } !_{1}^{\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}\rho} \\
 & & \quad \textit{is the unique morphism from } \llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}\rho \textit{ to } 1 \\
\llbracket \Gamma;\Phi \mid \Delta \vdash (s,t) : \sigma \times \tau \rrbracket^{\mathsf{Set}}\rho & = & \llbracket \Gamma;\Phi \mid \Delta \vdash s : \sigma \rrbracket^{\mathsf{Set}}\rho \times \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \pi_1 t : \sigma \rrbracket^{\mathsf{Set}}\rho & = & \pi_1 \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau \rrbracket^{\mathsf{Set}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \pi_2 t : \sigma \rrbracket^{\mathsf{Set}}\rho & = & \pi_2 \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau \rrbracket^{\mathsf{Set}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{x \mapsto l;\, y \mapsto r\} : \gamma \rrbracket^{\mathsf{Set}}\rho & = & \mathsf{eval} \circ \langle \mathsf{curry}\, [\llbracket \Gamma;\Phi \mid \Delta, x : \sigma \vdash l : \gamma \rrbracket^{\mathsf{Set}}\rho, \\
 & & \qquad\qquad \llbracket \Gamma;\Phi \mid \Delta, y : \tau \vdash r : \gamma \rrbracket^{\mathsf{Set}}\rho], \\
 & & \qquad \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma + \tau \rrbracket^{\mathsf{Set}}\rho \rangle \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau \rrbracket^{\mathsf{Set}}\rho & = & \mathsf{inl} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash s : \sigma \rrbracket^{\mathsf{Set}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau \rrbracket^{\mathsf{Set}}\rho & = & \mathsf{inr} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{in}\, t : (\mu\phi^k.\lambda\boldsymbol{\alpha}.H)\boldsymbol{A} \rrbracket^{\mathsf{Set}}\rho & = & \mathit{in} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : H[\phi^k := \mu\phi^k.\lambda\boldsymbol{\alpha}.H][\boldsymbol{\alpha} := \boldsymbol{A}] \rrbracket^{\mathsf{Set}}\rho \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash \mathsf{fold}_{H,F}\, t : \mathsf{Nat}^{\boldsymbol{\alpha}}\, ((\mu\phi.\lambda\boldsymbol{\beta}.H)\boldsymbol{\alpha})\, F \rrbracket^{\mathsf{Set}}\rho & = & \mathit{fold} \circ \llbracket \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\boldsymbol{\alpha}}\, (H[\phi := F][\boldsymbol{\beta} := \boldsymbol{\alpha}])\, F \rrbracket^{\mathsf{Set}}\rho
\end{array}
$$

Definition 27. *If ρ is a relation environment and $\Gamma;\Phi \mid \Delta \vdash t : \tau$ then $\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho$ is defined as follows:*

$$
\begin{array}{lcl}
\llbracket \Gamma;\emptyset \mid \Delta, x : \tau \vdash x : \tau \rrbracket^{\mathsf{Rel}}\rho & = & \pi_{|\Delta|+1} \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash \lambda x.t : \sigma \to \tau \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{curry}(\llbracket \Gamma;\emptyset \mid \Delta, x : \sigma \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho) \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash st : \tau \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{eval} \circ \langle \llbracket \Gamma;\emptyset \mid \Delta \vdash s : \sigma \to \tau \rrbracket^{\mathsf{Rel}}\rho, \llbracket \Gamma;\emptyset \mid \Delta \vdash t : \sigma \rrbracket^{\mathsf{Rel}}\rho \rangle \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash L_{\boldsymbol{\alpha}} x.t : \mathsf{Nat}^{\boldsymbol{\alpha}}\, F\, G \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{curry}(\llbracket \Gamma;\boldsymbol{\alpha} \mid \Delta, x : F \vdash t : G \rrbracket^{\mathsf{Rel}}\rho[\overline{\alpha := \_}]) \\
\llbracket \Gamma;\Phi \mid \Delta \vdash t_\tau s : G[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{eval} \circ \langle (\llbracket \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}}\, F\, G \rrbracket\rho\; \_)_{\overline{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}\rho}}, \\
 & & \qquad \llbracket \Gamma;\Phi \mid \Delta \vdash s : F\overline{[\alpha := \tau]} \rrbracket\rho \rangle
\end{array}
$$

Add rules for $\forall$ if we include it

$$
\begin{array}{lcl}
\llbracket \Gamma;\Phi \mid \Delta, x : \tau \vdash x : \tau \rrbracket^{\mathsf{Rel}}\rho & = & \pi_{|\Delta|+1} \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \bot_\tau t : \tau \rrbracket\rho & = & !^0_{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}\rho} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \mathbb{0} \rrbracket^{\mathsf{Rel}}\rho \text{ where} \\
 & & \quad !^0_{\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}\rho} \text{ is the unique morphism from } 0 \\
 & & \quad \text{to } \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \top : \mathbb{1} \rrbracket^{\mathsf{Rel}}\rho & = & !_1^{\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}\rho} \text{ where } !_1^{\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}\rho} \\
 & & \quad \text{is the unique morphism from } \llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}\rho \text{ to } 1 \\
\llbracket \Gamma;\Phi \mid \Delta \vdash (s,t) : \sigma \times \tau \rrbracket^{\mathsf{Rel}}\rho & = & \llbracket \Gamma;\Phi \mid \Delta \vdash s : \sigma \rrbracket^{\mathsf{Rel}}\rho \times \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \pi_1 t : \sigma \rrbracket^{\mathsf{Rel}}\rho & = & \pi_1 \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \pi_2 t : \sigma \rrbracket^{\mathsf{Rel}}\rho & = & \pi_2 \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma \times \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{x \mapsto l;\, y \mapsto r\} : \gamma \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{eval} \circ \langle \mathsf{curry}\, [\llbracket \Gamma;\Phi \mid \Delta, x : \sigma \vdash l : \gamma \rrbracket^{\mathsf{Rel}}\rho, \\
 & & \qquad \llbracket \Gamma;\Phi \mid \Delta, y : \tau \vdash r : \gamma \rrbracket^{\mathsf{Rel}}\rho], \\
 & & \qquad \llbracket \Gamma;\Phi \mid \Delta \vdash t : \sigma + \tau \rrbracket^{\mathsf{Rel}}\rho \rangle \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{inl}\, s : \sigma + \tau \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{inl} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash s : \sigma \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{inr}\, t : \sigma + \tau \rrbracket^{\mathsf{Rel}}\rho & = & \mathsf{inr} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\Phi \mid \Delta \vdash \mathsf{in}\, t : (\mu\phi^k.\lambda\boldsymbol{\alpha}.H)\boldsymbol{A} \rrbracket^{\mathsf{Rel}}\rho & = & \mathit{in} \circ \llbracket \Gamma;\Phi \mid \Delta \vdash t : H[\phi^k := \mu\phi^k.\lambda\boldsymbol{\alpha}.H][\boldsymbol{\alpha} := \boldsymbol{A}] \rrbracket^{\mathsf{Rel}}\rho \\
\llbracket \Gamma;\emptyset \mid \Delta \vdash \mathsf{fold}_{H,F}\, t : \mathsf{Nat}^{\boldsymbol{\alpha}}\, ((\mu\phi.\lambda\boldsymbol{\beta}.H)\boldsymbol{\alpha})\, F \rrbracket^{\mathsf{Rel}}\rho & = & \mathit{fold} \circ \llbracket \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\boldsymbol{\alpha}}\, (H[\phi := F][\boldsymbol{\beta} := \boldsymbol{\alpha}])\, F \rrbracket^{\mathsf{Rel}}\rho
\end{array}
$$

If t is closed, i.e., if $\emptyset;\emptyset \mid \emptyset \vdash t : \tau$, then we write $\llbracket t : \tau \rrbracket^{\mathsf{Set}}$ instead of $\llbracket \emptyset;\emptyset \mid \emptyset \vdash t : \tau \rrbracket^{\mathsf{Set}}$, and similarly for $\llbracket \emptyset;\emptyset \mid \emptyset \vdash t : \tau \rrbracket^{\mathsf{Rel}}$.

The set and relation interpretations of every well-formed term are well-defined, and are actually natural transformations.

Lemma 28. *For every well-formed term $\Gamma;\Phi \mid \Delta \vdash t : \tau$, its set interpretation $\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}$ is well-defined and gives a natural transformation from $\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}$ to $\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}$. Similarly, its relational interpretation $\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}$ is well-defined and gives a natural transformation from $\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Rel}}$ to $\llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Rel}}$.*

We will need to know that type interpretations respect type substitution. That's what all these little lemmas will establish. We will also ultimately want to know that term interpretations respect type susbstitution, and that term interpretations respect term substitution.

Proof. The type application case will need the following lemma:

$$\llbracket \Gamma;\Phi \vdash F[\overline{\alpha := \tau}] \rrbracket^{\mathsf{Set}}\rho = \llbracket \Gamma;\Phi, \boldsymbol{\alpha} \vdash F \rrbracket^{\mathsf{Set}}\rho\overline{[\alpha := \llbracket \Gamma;\Phi \vdash \tau \rrbracket^{\mathsf{Set}}\rho]}$$

and probably a similar lemma for the relation interpretations.

The in case will need the following lemma:

$$\llbracket \Gamma;\Phi \vdash H[\phi := \mu\phi.\lambda\alpha.H][\alpha := A] \rrbracket^{\mathsf{Set}}\rho = \llbracket \Gamma;\Phi,\phi,\alpha \vdash H \rrbracket^{\mathsf{Set}}\rho[\phi := \mu T^{\mathsf{Set}}_\rho][\alpha := \llbracket \Gamma;\Phi \vdash A \rrbracket^{\mathsf{Set}}\rho]$$

The fold case will need to use the conditions on the natural transformations obtained from the hypothesis to verify those obtained from the conclusion. (Perhaps other cases too.) □

4.1 The Abstraction Theorem

Since the Abstraction Theorem is a special case of soundness of the interpretation, it follows from Lemma 28. Indeed, we first observe that, by Lemma 21, $(\llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}, \llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Set}}, \llbracket \Gamma;\Phi \vdash \Delta \rrbracket^{\mathsf{Rel}})$ is a functor from RelEnv to RT_0, which we denote by $\llbracket \Gamma;\Phi \vdash \Delta \rrbracket$. We then have:

Theorem 29. *Every well-formed term* $\Gamma;\Phi \mid \Delta \vdash t : \tau$ *induces a natural transformation from* $\llbracket \Gamma;\Phi \vdash \Delta \rrbracket$ *to* $\llbracket \Gamma;\Phi \vdash \tau \rrbracket$*, i.e., a triple of natural transformations*

$$(\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}, \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}, \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}})$$

such that, for all ρ : RelEnv*,*

$$\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Rel}}\rho = (\llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}(\pi_1\rho), \llbracket \Gamma;\Phi \mid \Delta \vdash t : \tau \rrbracket^{\mathsf{Set}}(\pi_2\rho))$$

Proof. A straightforward proof by induction on the judgement $\Gamma;\Phi \mid \Delta \vdash t : \tau$, using Definitions 26 and 27, together with the facts that the cartesian structure of Rel is derived from that of Set and that initial algebras in Rel are computed in terms of initial algebras in Set. □

We now show that the interpretation given in Sections ??, 3.2, and 4 define a logical relation. Indeed, the Abstraction Theorem is the special case of Lemma 28 for closed terms.

Theorem 30. *If* $\vdash \tau : \mathcal{F}$ *and* $\vdash t : \tau$*, then* $(\llbracket t : \tau \rrbracket^{\mathsf{Set}}, \llbracket t : \tau \rrbracket^{\mathsf{Set}}) \in \llbracket \tau \rrbracket^{\mathsf{Rel}}$.

We will need to go back and add typing rules for well-formed terms involving $\mathsf{map}^{\mathcal{F}}$ and $\mathsf{map}^{\mathcal{T}}$ in Def 5, set and relational interpretations of these maps (just the actual functorial actions), and cases for map to all of our proofs thus far having to do with terms.

Next we will want to sanity-check our model by showing that term interps respect conversion rules. These are

- $\lambda x.t = \lambda y.t[x := y]$
- $L_{\boldsymbol{\alpha}} x.t = L_{\boldsymbol{\beta}} y.(t[\boldsymbol{\alpha} := \boldsymbol{\beta}][x := y])$
- $(\lambda x.t)s = t[x := s]$
- $(L_{\boldsymbol{\alpha}} x.t)_{\boldsymbol{\tau}} s = t[\boldsymbol{\alpha} := \boldsymbol{\tau}][x := s]$
- $\pi_i(t_1, t_2) = t_i$
- $\mathsf{case}\ \mathsf{inl}\ t\ \mathsf{of}\ \{x_1 \mapsto t_1;\ x_2 \mapsto t_2\} = t_i[x_i := t]$
- and other conversion rules as on page 18 of MFPS paper
- perhaps add weakening rules explicitly here?

- All of the above are shorthands for saying that the interps of the LHSs are the same as the interps of the RHSs. For this conversion rule: $\mathsf{fold}\ k\ (\mathsf{in}\ t) = k\ (\mathsf{map}\ (\mathsf{fold}\ k)\ t)$, we can't express it in syntax. So what we really want to say here is that some semantic equivalent of this syntacic rule holds. And similarly for the next rules.
- Maybe we want to show that $(\llbracket \mu\alpha.F[\alpha] \rrbracket, \llbracket \mathsf{in} \rrbracket)$ is an initial $\llbracket F \rrbracket$-algebra in the model? See Birkedal and Mogelberg Section 5.4. As part of this we would have the next bullet point, plus some other intermediate results as in 5.17, 5.18, and 5.19 there. We would also need representations of map functions. Perhaps we can define them syntactically as in Plotkin and Abadi section 2.1? (But isn't this precisely what we tried?)
- $\mathsf{fold}_H\ \mathsf{in}_H x = x$ (Intuitively, this is the syntactic counterpart to initiality of in.)
- $\mathsf{map}^{\mathcal{F}}_H\ \overline{(L_{\boldsymbol{\alpha}} x.x)} = L_{\bigcup \boldsymbol{\alpha}} x.x$ for all H

- $\mathsf{map}^{\mathcal{F}}_{H}\,(\overline{L_{\boldsymbol{\alpha}} x.\eta_{\boldsymbol{\alpha}}(\mu_{\boldsymbol{\alpha}} x)}) = L_{\cup\,\boldsymbol{\alpha}} x.(\mathsf{map}^{\mathcal{F}}_{H}\,\overline{\eta})_{\cup\,\boldsymbol{\alpha}}((\mathsf{map}^{\mathcal{F}}_{H}\,\overline{\mu})_{\cup\,\boldsymbol{\alpha}}\, x)$
- $\lambda x.\mathsf{map}^{\mathcal{F}}_{G}\,\overline{f}\,(\eta_{\overline{\sigma}} x) = \lambda x.\eta_{\overline{\tau}}\,(\mathsf{map}^{\mathcal{F}}_{F}\,\overline{f}\,x)$ (note that $..\vdash f : \mathsf{Nat}^{\emptyset}\,F\,G$)
- $\mathsf{map}^{\mathcal{F}}_{H}(\overline{\mathsf{map}^{\mathcal{F}}_{K_i}\,t_i}) = \mathsf{map}^{\mathcal{F}}_{H[\overline{\psi := K}]}\,\overline{t}$
- $\mathsf{map}^{\mathcal{F}}_{\phi}\,\eta = \eta$

Note that there are no computation rules for types because types are always fully applied in our syntax.

Show $[\![\Gamma;\emptyset \vdash \sigma \to \tau]\!] = [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\emptyset}\,\sigma\,\tau]\!]$. Oh, this doesn't appear to hold. Unfolding the definitions, the latter appears to impose a commutativity condition ($[\![\Gamma \vdash \tau]\!]^{\mathsf{Rel}}(\mathsf{Eq}\,\rho) \circ \eta = \eta \circ [\![\Gamma \vdash \sigma]\!]^{\mathsf{Rel}}(\mathsf{Eq}\,\rho)$) that the former does not require.

Other sanity checks?

Note that our calculus does not support Church encodings of data types like pair or sum or list types because all of the "forall"s in our calculus must be at the top level. Nevertheless, our calculus does admit actual sum and product and list types because they are coded by μ-terms in our calculus. We just don't have an equivalence of these types and their Church encodings in our calculus, that's all.

5 FREE THEOREMS FOR NESTED TYPES

We can use the results of Section 4.1 to prove interesting results about nested types. To this end, let α_i have arity n_i for $i = 1, ..., k$, and suppose further that $\emptyset;\boldsymbol{\alpha} \vdash E : \mathcal{F}$, that $F = \lambda \boldsymbol{A}.[\![\emptyset;\boldsymbol{\alpha} \vdash E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$, and that $F^* = \lambda \boldsymbol{R}.[\![\emptyset;\boldsymbol{\alpha} \vdash E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$.

The next proposition is the only place where we use the syntactic structure of E.

Propagate contexts?

Proposition 31. *If $(\beta_i, \gamma_i) \in \mathsf{Hom}_{\mathsf{Rel}^{n_i}}(R_i, R'_i)$ for $i = 1, ..., k$, then $(F\boldsymbol{\beta}, F\boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}}(F^*\boldsymbol{R}, F^*\boldsymbol{R'})$.*

Proof. By induction on the structure of E.

- If $\emptyset;\boldsymbol{\alpha} \vdash E : \mathcal{T}$, then the functor F is constant in $\boldsymbol{\alpha}$. Since F therefore maps every morphism in Set to id, we need only show that $(id, id) \in \mathsf{Hom}_{\mathsf{Rel}}(F^*\boldsymbol{R}, F^*\boldsymbol{R'})$ for all $\boldsymbol{R}$ and $\boldsymbol{R'}$. But since the functor F^* is also constant in $\boldsymbol{\alpha}$, this holds trivially.
- $E = \mathbb{0}$. Similar to previous case.
- $E = \mathbb{1}$. Similar to previous case.
- $E = E_1 * E_2$. If $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A}, \boldsymbol{B})$, $\boldsymbol{R'} : \mathsf{Rel}(\boldsymbol{A'}, \boldsymbol{B'})$, $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}^{\boldsymbol{n}}}(\boldsymbol{R}, \boldsymbol{R'})$, and $(x, y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, then $x \in [\![\vdash E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$ and $y \in [\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$, so $x = (x_1, x_2)$ where $x_i \in [\![\emptyset;\boldsymbol{\alpha} \vdash E_i]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$ and $y = (y_1, y_2)$ where $y_i \in [\![E_i]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$. Therefore $(x_1, y_1) \in [\![\emptyset;\boldsymbol{\alpha} \vdash E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$ and $(x_2, y_2) \in [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$. Using the induction hypothesis twice we get that $([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_1, [\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_1) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$ and $([\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_2, [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_2) \in [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $(([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_1, [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_2), ([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_1, [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_2)) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \times [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $(([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta} \times [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\beta})(x_1, x_2), ([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma} \times [\![E_2]\!]^{\mathsf{Set}}\boldsymbol{\gamma})(y_1, y_2)) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \times [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, i.e., $([\![E]\!]^{\mathsf{Set}}\boldsymbol{\beta}x, [\![E]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$.
- $E = E_1 + E_2$. If $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A}, \boldsymbol{B})$, $\boldsymbol{R'} : \mathsf{Rel}(\boldsymbol{A'}, \boldsymbol{B'})$, $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}^{k}}(\boldsymbol{R}, \boldsymbol{R'})$, and $(x, y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, then $x \in [\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}] = [\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}] + [\![E_2]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$ and $y \in [\![E]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}] = [\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}] + [\![E_2]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$. Since $(x, y) \in [\![E]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, we must have either $x = \mathsf{inl}\,x_1$ for $x_1 \in [\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$, $y = \mathsf{inl}\,y_1$ for $y_1 \in [\![E_1]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$, and $(x_1, y_1) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, or $x = \mathsf{inr}\,x_2$ for $x_2 \in [\![E_2]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$, $y = \mathsf{inr}\,y_2$ for $y_2 \in [\![E_2]\!]^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$, and $(x_2, y_2) \in [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$. We prove the result for the first case; the second is analogous. By the induction hypothesis, $([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_1, [\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_1) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$, so $(\mathsf{inl}\,([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\beta}x_1), \mathsf{inl}\,([\![E_1]\!]^{\mathsf{Set}}\boldsymbol{\gamma}y_1)) \in [\![E_1]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] + [\![E_2]\!]^{\mathsf{Rel}}[\boldsymbol{\alpha} :=$

$\boldsymbol{R}'$], i.e., $(\llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\beta}(\mathsf{inl}\,x_1), \llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma}(\mathsf{inl}\,y_1)) \in \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}']$, i.e., $(\llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\beta}x, \llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma}y) \in \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}']$.

- $E = \phi^m E_1...E_m$. Suppose $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A},\boldsymbol{B})$, $\boldsymbol{R}' : \mathsf{Rel}(\boldsymbol{A}',\boldsymbol{B}')$, $(\boldsymbol{\beta},\boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\boldsymbol{R},\boldsymbol{R}')$, $R_\phi = (R^0_\phi, R^1_\phi, R^*_\phi)$, and $R'_\phi = ({R'_\phi}^0, {R'_\phi}^1, {R'_\phi}^*)$. If
$$(x,y) \in \llbracket \phi^m E_1...E_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}] = R^*_\phi(\llbracket E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])...(\llbracket E_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])$$
(since $\phi \in \boldsymbol{\alpha}$), then
$$x \in R^0_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])...(\llbracket E_m\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])$$
and
$$y \in R^1_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])...(\llbracket E_m\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])$$
Since $(\boldsymbol{\beta},\boldsymbol{\gamma}) \in \mathsf{Hom}(\boldsymbol{R},\boldsymbol{R}')$, the induction hypothesis gives that, for each $i = 1,\dots,m$, $(w,z) \in \llbracket E_i\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$ implies $(\llbracket E_i\rrbracket^{\mathsf{Set}}\boldsymbol{\beta}w, \llbracket E_i\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma}z) \in \llbracket E_i\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}']$, i.e., $(\llbracket E_i\rrbracket^{\mathsf{Set}}\boldsymbol{\beta}, \llbracket E_i\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}}(\llbracket E_i\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}], \llbracket E_i\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'])$. The remark after Definition 10 thus gives that $(R^0_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\beta})...(\llbracket E_m\rrbracket^{\mathsf{Set}}\boldsymbol{\beta}), R^1_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma})...(\llbracket E_m\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma})) \in \mathsf{Hom}_{\mathsf{Rel}}(R^*_\phi(\llbracket E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])...(\llbracket E_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]), R^*_\phi(\llbracket E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'])...(\llbracket E_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}']))$. Then since $(x,y) \in R^*_\phi(\llbracket E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])...(\llbracket E_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])$, we have that
$$\begin{aligned} &(R^0_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\beta})...(\llbracket E_m\rrbracket^{\mathsf{Set}}\boldsymbol{\beta})x, R^1_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma})...(\llbracket E_m\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma})y) \\ \in\ & R^*_\phi(\llbracket E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'])...(\llbracket E_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}']) \end{aligned} \tag{6}$$
By hypothesis, $(\beta_\phi, \gamma_\phi) : R^*_\phi \to {R'_\phi}^*$. Since β_ϕ and γ_ϕ are natural transformations, this gives that for all $\boldsymbol{S} : \mathsf{Rel}(\boldsymbol{C},\boldsymbol{D})$, $((\beta_\phi)_{\boldsymbol{C}}, (\gamma_\phi)_{\boldsymbol{D}}) \in \mathsf{Hom}_{\mathsf{Rel}}({R_\phi}^*\boldsymbol{S}, {R'_\phi}^*\boldsymbol{S})$. Letting $\boldsymbol{S} = (\llbracket E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}']),\dots,(\llbracket E_m\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R}'])$, $\boldsymbol{C} = (\llbracket E_1\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}']),\dots,(\llbracket E_m\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}'])$, and $\boldsymbol{D} = (\llbracket E_1\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}']),\dots,(\llbracket E_m\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}'])$, and noting that
$$(R^0_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\beta})...(\llbracket E_m\rrbracket^{\mathsf{Set}}\boldsymbol{\beta})x, R^1_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma})...(\llbracket E_m\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma})y) \in R^*_\phi\boldsymbol{S}$$
by Equation 8, our hypothesis gives that
$$\begin{aligned} &((\beta_\phi)_{\boldsymbol{C}}(R^0_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\beta})...(\llbracket E_m\rrbracket^{\mathsf{Set}}\boldsymbol{\beta})x), (\gamma_\phi)_{\boldsymbol{D}}(R^1_\phi(\llbracket E_1\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma})...(\llbracket E_m\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma})y)) \\ \in\ & {R'}^*_\phi\boldsymbol{S} = {R'}^*_\phi(\llbracket E_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'])...(\llbracket E_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}']) = \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}'] \end{aligned} \tag{7}$$
Using the definition of the action of $\llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\beta}$ on morphisms (see Diagram 1) twice — once with instantiations $\rho = \boldsymbol{A}$, $\rho' = \boldsymbol{A}'$, $f = \boldsymbol{\beta}$ and $\phi\rho = R^0_\phi$, and once with instantiations $\rho = \boldsymbol{B}$, $\rho' = \boldsymbol{B}'$, $f = \boldsymbol{\gamma}$ and $\phi\rho = R^1_\phi$ — Equation 7 is exactly $(\llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\beta}x, \llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma}y) \in \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}']$.
- $E = (\mu\phi^m.\lambda\delta_1...\delta_m.h)T_1...T_m$. Suppose $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A},\boldsymbol{B})$, $\boldsymbol{R}' : \mathsf{Rel}(\boldsymbol{A}',\boldsymbol{B}')$, $(\boldsymbol{\beta},\boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\boldsymbol{R},\boldsymbol{R}')$, and $(x,y) \in F^*\boldsymbol{R} = \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$. If $(x,y) \in \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$, then $x \in \llbracket E\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]$ and $y \in \llbracket E\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]$. Consider the relation transformers (L^0, L^1, L^*) and (G^0, G^1, G^*), where
$$\begin{aligned} L^0 &= \mu(H \mapsto \lambda\boldsymbol{X}.\llbracket h\rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}]) \\ L^1 &= \mu(H \mapsto \lambda\boldsymbol{X}.\llbracket h\rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}]) \\ L^* &= \mu(W \mapsto \lambda\boldsymbol{S}.\llbracket h\rrbracket^{\mathsf{Rel}}[\phi := W][\boldsymbol{\delta} := \boldsymbol{S}][\boldsymbol{\alpha} := \boldsymbol{R}]) \\ G^0 &= \mu(H \mapsto \lambda\boldsymbol{X}.\llbracket h\rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}']) \\ G^1 &= \mu(H \mapsto \lambda\boldsymbol{X}.\llbracket h\rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}']) \\ G^* &= \mu(W \mapsto \lambda\boldsymbol{S}.\llbracket h\rrbracket^{\mathsf{Rel}}[\phi := W][\boldsymbol{\delta} := \boldsymbol{S}][\boldsymbol{\alpha} := \boldsymbol{R}']) \end{aligned}$$
Then $(x,y) \in L^*(\llbracket T_1\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])...(\llbracket T_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])$, i.e., $x \in L^0(\llbracket T_1\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])...(\llbracket T_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{A}])$ and $y \in L^1(\llbracket T_1\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}])...(\llbracket T_m\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{B}])$. Lemma **??** ensures that each $i = 1,\dots m$,

$(\llbracket T_i \rrbracket^{\mathsf{Set}}, \llbracket T_i \rrbracket^{\mathsf{Set}}, \llbracket T_i \rrbracket^{\mathsf{Rel}})$ is a relation transformer, so the induction hypothesis gives that $(w, z) \in \llbracket T_i \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$ implies $(\llbracket T_i \rrbracket^{\mathsf{Set}} \boldsymbol{\beta} w, \llbracket T_i \rrbracket^{\mathsf{Set}} \boldsymbol{\gamma} z) \in \llbracket T_i \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$ for all $i = 1, \ldots, m$, i.e., $(\llbracket T_i \rrbracket^{\mathsf{Set}} \boldsymbol{\beta}, \llbracket T_i \rrbracket^{\mathsf{Set}} \boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}}(\llbracket T_i \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}], \llbracket T_i \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}])$. The remark after Definition 10 thus gives that

$$\begin{array}{ll} & (L^0(\llbracket T_1 \rrbracket^{\mathsf{Set}} \boldsymbol{\beta}) \ldots (\llbracket T_m \rrbracket^{\mathsf{Set}} \boldsymbol{\beta}), L^1(\llbracket T_1 \rrbracket^{\mathsf{Set}} \boldsymbol{\gamma}) \ldots (\llbracket T_m \rrbracket^{\mathsf{Set}} \boldsymbol{\gamma})) \\ \in & \mathsf{Hom}_{\mathsf{Rel}}(L^*(\llbracket T_1 \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]) \ldots (\llbracket T_m \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]), \\ & \qquad L^*(\llbracket T_1 \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \ldots (\llbracket T_m \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}])) \end{array}$$

Then since $(x, y) \in L^*(\llbracket T_1 \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]) \ldots (\llbracket T_m \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}])$, we have that

$$\begin{array}{ll} & (L^0(\llbracket T_1 \rrbracket^{\mathsf{Set}} \boldsymbol{\beta}) \ldots (\llbracket T_m \rrbracket^{\mathsf{Set}} \boldsymbol{\beta}) x, L^1(\llbracket T_1 \rrbracket^{\mathsf{Set}} \boldsymbol{\gamma}) \ldots (\llbracket T_m \rrbracket^{\mathsf{Set}} \boldsymbol{\gamma}) y) \\ \in & L^*(\llbracket T_1 \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \ldots (\llbracket T_m \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \end{array} \tag{8}$$

Now, note that for every functor H and sequence of sets $\boldsymbol{X}$,

$$\begin{array}{rcl} \eta^0_{H,\boldsymbol{X}} & = & \llbracket h \rrbracket^{\mathsf{Set}}[\phi := id][\boldsymbol{\delta} := \boldsymbol{id}][\boldsymbol{\alpha} := \boldsymbol{\beta}] \\ & : & \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}] \to \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}] \end{array}$$

is a morphism in Set^k, so

$$\begin{array}{rcl} \eta^0 & = & (H \mapsto \lambda \boldsymbol{X}.\, \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{\beta}]) \\ & & : (H \mapsto \lambda \boldsymbol{X}.\, \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}]) \\ & & \quad \to (H \mapsto \lambda \boldsymbol{X}.\, \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}]) \end{array}$$

is a morphism (i.e., a higher-order natural transformation) between higher-order functors between functors on $\mathsf{Set}^m \to \mathsf{Set}$: indeed, for every natural transformation $f : H \to H'$ we have that

$$\begin{tikzcd} \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}] \arrow[r, "\eta^0_{H,\boldsymbol{X}}"] \arrow[d, "{\llbracket h \rrbracket^{\mathsf{Set}}[\phi := f][\boldsymbol{\delta} := \boldsymbol{id_X}][\boldsymbol{\alpha} := \boldsymbol{id_A}]}"'] & \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}] \arrow[d, "{\llbracket h \rrbracket^{\mathsf{Set}}[\phi := f][\boldsymbol{\delta} := \boldsymbol{id_X}][\boldsymbol{\alpha} := \boldsymbol{id_{A'}}]}"'] \\ \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H'][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A}] \arrow[r, "\eta^0_{H',\boldsymbol{X}}"] & \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H'][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{A'}] \end{tikzcd} \tag{9}$$

commutes because the vertical arrows are the $\boldsymbol{A}$ and $\boldsymbol{A'}$ components of the natural transformation $\llbracket h \rrbracket^{\mathsf{Set}}[\phi := f][\boldsymbol{\delta} := \boldsymbol{id_X}][\boldsymbol{\alpha} := \boldsymbol{id\_}]$ induced by f between the functors $\llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \_]$ and $\llbracket h \rrbracket^{\mathsf{Set}}[\phi := H'][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \_]$. Similarly, if

$$\begin{array}{rcl} \eta^1_{H,\boldsymbol{X}} & = & \llbracket h \rrbracket^{\mathsf{Set}}[\phi := id][\boldsymbol{\delta} := \boldsymbol{id}][\boldsymbol{\alpha} := \boldsymbol{\gamma}] \\ & : & \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}] \to \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B'}] \end{array}$$

and

$$\begin{array}{rcl} \eta^1 & = & (H \mapsto \lambda \boldsymbol{X}.\, \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{\gamma}]) \\ & & : (H \mapsto \lambda \boldsymbol{X}.\, \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B}]) \\ & & \quad \to (H \mapsto \lambda \boldsymbol{X}.\, \llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{B'}]) \end{array}$$

then η^1 is a morphism between higher-order functors between functors on $\mathsf{Set}^m \to \mathsf{Set}$. Since μ is functorial, it has an action on morphisms, so $\mu\eta^0 : L^0 \to G^0$ and $\mu\eta^1 : L^1 \to G^1$ are well-defined. Moreover, since $(\boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathsf{Hom}_{\mathsf{Rel}}(\boldsymbol{R}, \boldsymbol{R'})$, the following diagram commutes:

$$\begin{tikzcd}[column sep=huge]
L^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}]) \arrow[r, "(\mu\eta^0)(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])"] \arrow[d, leftrightarrow, "L^*(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R'}])"] & G^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}]) \arrow[d, leftrightarrow, "G^*(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R'}])"'] \\
L^1(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}]) \arrow[r, "(\mu\eta^1)(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])"] & G^1(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}])
\end{tikzcd} \tag{10}$$

Together with Equation 8, Equation 10 gives

$$\begin{array}{l}
(\,(\mu\eta^0)(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])(L^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])x), \\
\quad (\mu\eta^1)(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])(L^1(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])y)\,) \\
\qquad \in\; G^*(\llbracket T_1 \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{R'}]) \\
\qquad =\; \llbracket (\mu\phi.\lambda\boldsymbol{\delta}.h)\boldsymbol{T} \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \\
\qquad =\; \llbracket E \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]
\end{array} \tag{11}$$

We also have that if ψ is a fresh type constructor variable, then

$$\llbracket \psi T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}][\psi := L^0] \;=\; L^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])$$

and

$$\llbracket \psi T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}][\psi := G^0] \;=\; G^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])$$

so that

$$\begin{array}{rl}
 & \llbracket \psi T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}][\psi := \mu\eta^0] \\
= & (\mu\eta^0)(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}]) \circ L^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]) \\
: & L^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A}]) \to G^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{A'}])
\end{array} \tag{12}$$

Similarly,

$$\begin{array}{rl}
 & \llbracket \psi T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}][\psi := \mu\eta^1] \\
= & (\mu\eta^1)(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}]) \circ L^1(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]) \\
: & L^1(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B}]) \to G^1(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{B'}])
\end{array} \tag{13}$$

Rewriting Equation 11 using Equations 12 and 13 gives

$$(\llbracket \psi T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}][\psi := \mu\eta^0]x, \llbracket \psi T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}][\psi := \mu\eta^1]y) \in \llbracket E \rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \tag{14}$$

Now we have that

$$\begin{array}{rl}
 & \llbracket \psi T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}][\psi := \mu\eta^0] \\
= & \mu\eta^0(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]) \\
= & \mu(H \mapsto \lambda\boldsymbol{X}.\,\llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{\beta}])(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]) \\
= & \llbracket (\mu\phi.\lambda\boldsymbol{\delta}.h)T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]
\end{array}$$

and

$$\begin{array}{rl}
 & \llbracket \psi T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}][\psi := \mu\eta^1] \\
= & \mu\eta^1(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]) \\
= & \mu(H \mapsto \lambda\boldsymbol{X}.\,\llbracket h \rrbracket^{\mathsf{Set}}[\phi := H][\boldsymbol{\delta} := \boldsymbol{X}][\boldsymbol{\alpha} := \boldsymbol{\gamma}])(\llbracket T_1 \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}])\ldots(\llbracket T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]) \\
= & \llbracket (\mu\phi.\lambda\boldsymbol{\delta}.h)T_1 \ldots T_m \rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]
\end{array}$$

so (14) becomes

$$(\llbracket(\mu\phi.\lambda\boldsymbol{\delta}.h)T_1 \ldots T_m\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\beta}]x, \llbracket(\mu\phi.\lambda\boldsymbol{\delta}.h)T_1 \ldots T_m\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{\gamma}]y) \in \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}] \tag{15}$$

i.e., $(\llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\beta}x, \llbracket E\rrbracket^{\mathsf{Set}}\boldsymbol{\gamma}y) \in \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R'}]$.

□

With the following standard definition, we can prove that our interpretations give rise to a Graph Lemma.

Definition 32. *If $f : A \to B$ then the relation $\langle f\rangle : \mathsf{Rel}(A, B)$ is defined by $(x, y) \in \langle f\rangle$ iff $fx = y$.*

Note that $\langle id_B\rangle = \mathsf{Eq}_B$.

Theorem 33. *If $f_i : A_i \to B_i$ for $i = 1, \ldots, k$ then $F^*\langle f\rangle_1 \ldots \langle f\rangle_k = \langle F f_1 \ldots f_k\rangle$.*

Proof. First observe that

$$((f_1, \ldots, f_k), (id_{B_1}, \ldots, id_{B_k})) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\langle\boldsymbol{f}\rangle, \mathbf{Eq}_{\boldsymbol{B_i}})$$

and

$$((id_{A_1}, \ldots, id_{A_k}), (f_1, \ldots, f_k)) \in \mathsf{Hom}_{\mathsf{Rel}^k}(\mathbf{Eq}_{\boldsymbol{A_i}}, \langle\boldsymbol{f}\rangle)$$

Applying Proposition 31 to each of these observations gives that

$$(F\boldsymbol{f}, F\boldsymbol{id}_{B_i}) \in \mathsf{Hom}_{\mathsf{Rel}}(F^*\langle\boldsymbol{f}\rangle, F^*\mathbf{Eq}_{\boldsymbol{B_i}}) \tag{16}$$

and

$$(F\boldsymbol{id}_{A_i}, F\boldsymbol{f}) \in \mathsf{Hom}_{\mathsf{Rel}}(F^*\mathbf{Eq}_{\boldsymbol{A_i}}, F^*\langle\boldsymbol{f}\rangle) \tag{17}$$

Expanding Equation 16 gives that if $(x, y) \in F^*\langle\boldsymbol{f}\rangle$ then $(F\boldsymbol{f}x, F\boldsymbol{id}_{B_i}y) \in F^*\mathbf{Eq}_{B_i} = \llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \mathbf{Eq}_{B_i}] = \mathsf{Eq}_{\llbracket E\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha}:=\boldsymbol{B_i}]} = \mathsf{Eq}_{F\boldsymbol{B}}$, where the penultimate equality holds by Theorem 25. That is, if $(x, y) \in F^*\langle\boldsymbol{f}\rangle$ then $(F\boldsymbol{f}x, y) \in \mathsf{Eq}_{F\boldsymbol{B}}$, i.e., if $(x, y) \in F^*\langle\boldsymbol{f}\rangle$ then $F\boldsymbol{f}x = y$, i.e., if $(x, y) \in F^*\langle\boldsymbol{f}\rangle$ then $(x, y) \in \langle F\boldsymbol{f}\rangle$. Thus $F^*\langle\boldsymbol{f}\rangle \subseteq \langle F\boldsymbol{f}\rangle$.

Similar analysis of Equation 17 gives that $\langle F\boldsymbol{f}\rangle \subseteq F^*\langle\boldsymbol{f}\rangle$. □

Inlining the definitions of F and F^* in the statement of Theorem 33 gives

$$\llbracket E\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \langle\boldsymbol{f}\rangle] = \langle\llbracket E\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]\rangle \tag{18}$$

We can use Equation 18 to prove that the set interpretation of a closed term of (closed) type $\mathsf{Nat}^{\boldsymbol{\alpha}}\, F\, G$ is a natural transformation.

Theorem 34. *If $\vdash t : \mathsf{Nat}^{\boldsymbol{\alpha}}\, F\, G$ and $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$, then $\llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{B}} \circ \llbracket F\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] = \llbracket G\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] \circ \llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{A}}$.*

Proof. Theorem 30 ensures that $(\llbracket t\rrbracket^{\mathsf{Set}}, \llbracket t\rrbracket^{\mathsf{Set}}) \in \llbracket\mathsf{Nat}^{\boldsymbol{\alpha}}\, F\, G\rrbracket^{\mathsf{Rel}}$, i.e., that for all $\boldsymbol{R} : \mathsf{Rel}(\boldsymbol{A}, \boldsymbol{B})$, $x : F\boldsymbol{A}$, and $x' : F\boldsymbol{B}$, if $(x, x') \in \llbracket F\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$ then $(\llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{A}}x, \llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{B}}x') \in \llbracket G\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \boldsymbol{R}]$. If $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$, then taking $\boldsymbol{R} = \langle\boldsymbol{f}\rangle$ and instantiating gives that if $(x, x') \in \llbracket F\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \langle\boldsymbol{f}\rangle]$ then $(\llbracket t\rrbracket^{\mathsf{Set}}_{\mathbf{A}}x, \llbracket t\rrbracket^{\mathsf{Set}}_{\mathbf{B}}x') \in \llbracket G\rrbracket^{\mathsf{Rel}}[\boldsymbol{\alpha} := \langle\boldsymbol{f}\rangle]$. By Equation 18 this is the same as the requirement that if $(x, x') \in \langle\llbracket F\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]\rangle$ then $(\llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{A}}x, \llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{B}}x') \in \langle\llbracket G\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]\rangle$ i.e., that if $x' = \llbracket F\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]x$ then $\llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{B}}x' = \llbracket G\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}](\llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{A}})$, i.e., that $\llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{B}}(\llbracket F\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}]x) = \llbracket G\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}](\llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{A}}x)$ for all $x : F\boldsymbol{A}$, i.e., that $\llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{B}} \circ \llbracket F\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] = \llbracket G\rrbracket^{\mathsf{Set}}[\boldsymbol{\alpha} := \boldsymbol{f}] \circ \llbracket t\rrbracket^{\mathsf{Set}}_{\boldsymbol{A}}$. □

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