Free Theorems for Nested Types

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1 SHORT CUT FUSION FOR ARBITRARY ADTS

THEOREM 1. *Let* $\overline{\vdash \tau : \mathcal{F}}$, *let* $\vdash \tau' : \mathcal{F}$, *let* $\overline{\alpha}; \beta \vdash F : \mathcal{F}$, *and let* $\beta; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\, \beta)\, \beta$. *If we regard*

$$\begin{array}{rcl} H & = & [\![\emptyset; \beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \\ G & = & [\![\beta; \emptyset \mid \emptyset \vdash g : \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset}\, F[\overline{\alpha := \tau}]\, \beta)\, \beta]\!]^{\mathsf{Set}} \end{array}$$

as functors in β, *then for every* $B \in H[\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}}$ *we have*

$$fold_H\, B\, (G\, \mu H\, in_H) = G\, [\![\vdash \tau']\!]^{\mathsf{Set}}\, B$$

PROOF. We first note that the type of g is well-formed, since $\emptyset; \beta \vdash F[\overline{\alpha := \tau}] : \mathcal{F}$ so our promotion theorem gives that $\beta; \emptyset \vdash F[\overline{\alpha := \tau}] : \mathcal{F}$, and $\emptyset; \beta \vdash \beta : \mathcal{F}$ so that our promotion theorem gives $\beta; \emptyset \vdash \beta : \mathcal{F}$. From these facts we deduce that $\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}\, F[\overline{\alpha := \tau}]\, \beta : \mathcal{T}$, and thus that $\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\, \beta)\, \beta : \mathcal{T}$.

Theorem ?? gives that, for any relation environment ρ and any $(a, b) \in [\![\beta; \emptyset \vdash \emptyset]\!]^{\mathsf{Rel}} \rho = 1$, eliding the only possible instantiations of a and b gives that

$$(G\,(\pi_1 \rho), G\,(\pi_2 \rho)) \in [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\, \beta)\, \beta]\!]^{\mathsf{Rel}} \rho$$

Since

$$\begin{array}{rl} & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset}(\mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\, \beta)\, \beta]\!]^{\mathsf{Rel}} \rho \\ = & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\, \beta]\!]^{\mathsf{Rel}} \rho \to \rho\beta \end{array}$$

we have that if $(A, B) \in [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\, \beta]\!]^{\mathsf{Rel}} \rho$ then

$$(G\,(\pi_1 \rho)\, A, G\,(\pi_2 \rho)\, B) \in \rho\beta$$

Now note that

$$[\![\vdash \mathsf{fold}^{\tau'}_{F[\overline{\alpha := \tau}]} : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\emptyset}\, F[\overline{\alpha := \tau}][\beta := \tau']\, \tau')\, (\mathsf{Nat}^{\emptyset}\, (\mu\beta.F[\overline{\alpha := \tau}]\, \tau')]\!]^{\mathsf{Set}} = fold_H$$

and consider the instantiation

$$\begin{array}{rcl} A & = & in_H : H(\mu H) \to \mu H \\ B & : & H[\![\vdash \tau']\!]^{\mathsf{Set}} \to [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & = & \langle fold_H\, B \rangle \end{array}$$

(Note that all the types here are well-formed.) This gives

$$\begin{array}{rcl} \pi_1 \rho\beta & = & [\![\vdash \mu\beta.F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \;=\; \mu H \\ \pi_2 \rho\beta & = & [\![\vdash \tau']\!]^{\mathsf{Set}} \\ \rho\beta & : & \mathsf{Rel}(\pi_1 \rho\beta, \pi_2 \rho\beta) \\ A & : & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\, \beta]\!]^{\mathsf{Set}}(\pi_1 \rho) \\ B & : & [\![\beta; \emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\, \beta]\!]^{\mathsf{Set}}(\pi_2 \rho) \end{array}$$

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since

$$
\begin{array}{rcl}
A = in_H & : & H(\mu H) \longrightarrow \mu H \\
& = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\mu[\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}) \longrightarrow \mu[\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \\
& = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_1\rho) \longrightarrow [\![\emptyset;\beta \vdash \beta]\!]^{\mathsf{Set}}(\pi_1\rho) \\
& = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(\pi_1\rho) \longrightarrow [\![\beta;\emptyset \vdash \beta]\!]^{\mathsf{Set}}(\pi_1\rho) \qquad \text{Daniel's trick; now a theorem} \\
& = & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Set}}(\pi_1\rho)
\end{array}
$$

where "Daniel's trick" is the observation that a functor can be seen as non-functorial when we only care about its action on objects. This is now a theorem. We also have

$$
\begin{array}{rcl}
(A,B) = (in_H, B) & \in & [\![\beta;\emptyset \vdash \mathsf{Nat}^{\emptyset} F[\overline{\alpha := \tau}]\,\beta]\!]^{\mathsf{Rel}}\rho \\
& = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\rho[\beta := \langle fold_H\, B\rangle] \longrightarrow \langle fold_H\, B\rangle \\
& = & [\![\beta;\emptyset \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\langle fold_H\, B\rangle \longrightarrow \langle fold_H\, B\rangle \\
& = & [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Rel}}\langle fold_H\, B\rangle \longrightarrow \langle fold_H\, B\rangle \qquad \text{Daniel's trick; now a theorem} \\
& = & \langle [\![\emptyset;\beta \vdash F[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}}(fold_H\, B)\rangle \longrightarrow \langle fold_H\, B\rangle \qquad \text{by the graph lemma} \\
& = & \langle map_H\,(fold_H\, B)\rangle \longrightarrow \langle fold_H\, B\rangle
\end{array}
$$

since if $(x,y) \in \langle map_H\,(fold_H\, B)\rangle$, i.e., if $map_H\,(fold_H\, B)\, x = y$, then $fold_H\, B\,(in_H\, x) = B\, y = B\,(map_H\,(fold_H\, B)\, x)$ by the definition of $fold_H$ as a (indeed, the unique) morphism from in_H to B. Thus,

$$(G\,(\pi_1\rho)\, A, G\,(\pi_2\rho)\, B) \in \langle fold_H\, B\rangle$$

i.e.,

$$fold_H\, B\,(G\,(\pi_1\rho)\, in_H) = G\,(\pi_2\rho)\, B$$

Since β is the only free variable in G, this simplifies to

$$fold_H\, B\,(G\,\mu H\, in_H) = G\,[\![\vdash \tau']\!]^{\mathsf{Set}}\, B$$

□