Free Theorems for Nested Types

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1 MAP

Definition 1. *Two terms $\Gamma;\Phi\,|\,\Delta \vdash t : \tau$ and $\Gamma;\Phi\,|\,\Delta \vdash t' : \tau$ are semantically equivalent if they have the same set interpretation functor and relational interpretation functor, i.e., for every set environment ρ, we have that*

$$[\![\Gamma;\Phi\,|\,\Delta \vdash t : \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\Phi\,|\,\Delta \vdash t' : \tau]\!]^{\mathsf{Set}}\rho$$

and, for every relation environment ρ, we have that

$$[\![\Gamma;\Phi\,|\,\Delta \vdash t : \tau]\!]^{\mathsf{Rel}}\rho = [\![\Gamma;\Phi\,|\,\Delta \vdash t' : \tau]\!]^{\mathsf{Rel}}\rho$$

and, moreover, the interpretations coincide on morphisms of environments as well.

When proving two terms semantically equivalent, we shall generally only check that their interpretation functors coincide on objects, i.e., on set or relation environments.

We define a composition operation between terms of type Nat and an identity term for functorial types, as convenient shorthands.

Definition 2. *Let $\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\,G$ and $\Gamma;\emptyset\,|\,\Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\,H$ be terms. Then the composition $s \circ t$ of t and s is the term $\Gamma;\emptyset\,|\,\Delta \vdash L_{\overline{\alpha}} x.s_{\overline{\alpha}}(t_{\overline{\alpha}} x) : \mathsf{Nat}^{\overline{\alpha}} F\,H$.*

Lemma 3. *Let $\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\,G$ and $\Gamma;\emptyset\,|\,\Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\,H$ be terms. Then for any set environment ρ, the semantic interpretation of the composition is*

$$[\![\Gamma;\emptyset\,|\,\Delta \vdash s \circ t : \mathsf{Nat}^{\overline{\alpha}} F\,H]\!]^{\mathsf{Set}}\rho = [\![\Gamma;\emptyset\,|\,\Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\,H]\!]^{\mathsf{Set}}\rho \circ [\![\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\,G]\!]^{\mathsf{Set}}\rho$$

Proof. For any set environment ρ and $d : [\![\Gamma;\emptyset \vdash \Delta]\!]^{\mathsf{Set}}\rho$, we have that

$$\begin{aligned}
& [\![\Gamma;\emptyset\,|\,\Delta \vdash s \circ t : \mathsf{Nat}^{\overline{\alpha}} F\,H]\!]^{\mathsf{Set}}\rho d \\
= \;& \lambda \overline{A}.\lambda x.([\![\Gamma;\emptyset\,|\,\Delta \vdash s \circ t : \mathsf{Nat}^{\overline{\alpha}} F\,H]\!]^{\mathsf{Set}}\rho d)_{\overline{A}} x \\
= \;& \lambda \overline{A}.\lambda x.([\![\Gamma;\emptyset\,|\,\Delta \vdash L_{\overline{\alpha}} x.s_{\overline{\alpha}}(t_{\overline{\alpha}} x) : \mathsf{Nat}^{\overline{\alpha}} F\,H]\!]^{\mathsf{Set}}\rho d)_{\overline{A}} x \\
= \;& \lambda \overline{A}.\lambda x.[\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash s_{\overline{\alpha}}(t_{\overline{\alpha}} x) : H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]dx \\
= \;& \lambda \overline{A}.\lambda x.([\![\Gamma;\emptyset\,|\,\Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\,H]\!]^{\mathsf{Set}}\rho d)_{\overline{A}}([\![\Gamma;\overline{\alpha}\,|\,\Delta, x : F \vdash t_{\overline{\alpha}} x : H]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]dx) \\
= \;& \lambda \overline{A}.\lambda x.([\![\Gamma;\emptyset\,|\,\Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\,H]\!]^{\mathsf{Set}}\rho d)_{\overline{A}}(([\![\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\,G]\!]^{\mathsf{Set}}\rho d)_{\overline{A}} x) \\
= \;& [\![\Gamma;\emptyset\,|\,\Delta \vdash s : \mathsf{Nat}^{\overline{\alpha}} G\,H]\!]^{\mathsf{Set}}\rho d \circ [\![\Gamma;\emptyset\,|\,\Delta \vdash t : \mathsf{Nat}^{\overline{\alpha}} F\,G]\!]^{\mathsf{Set}}\rho d \qquad \square
\end{aligned}$$

Definition 4. *Let $\Gamma;\overline{\alpha} \vdash F$ be a type. Then the identity Id_F of F is the term $\Gamma;\emptyset\,|\,\emptyset \vdash L_{\overline{\alpha}} x.x : \mathsf{Nat}^{\overline{\alpha}} F\,F$.*

Lemma 5. *Let $\Gamma;\overline{\alpha} \vdash F$ be a type. Then for any set environment ρ, the semantic interpretation of the identity is*

$$[\![\Gamma;\emptyset\,|\,\emptyset \vdash Id_F : \mathsf{Nat}^{\overline{\alpha}} F\,F]\!]^{\mathsf{Set}}\rho * = Id_{\lambda \overline{A}.[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}]}$$

2020. 2475-1421/2020/1-ART1 $15.00
https://doi.org/

Proof. For any set environment ρ, we have that

$$
\begin{array}{rl}
& [\![\Gamma;\emptyset \,|\, \emptyset \vdash Id_F : \mathsf{Nat}^{\overline{\alpha}} F\, F]\!]^{\mathsf{Set}} \rho * \\
= & [\![\Gamma;\emptyset \,|\, \emptyset \vdash L_{\overline{\alpha}} x.x : \mathsf{Nat}^{\overline{\alpha}} F\, F]\!]^{\mathsf{Set}} \rho * \\
= & \lambda \overline{A}.\lambda x.[\![\Gamma;\overline{\alpha} \,|\, x : F \vdash x : F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}] x \\
= & \lambda \overline{A}.\lambda x.x \\
= & \lambda \overline{A}.Id_{[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}]} \\
= & Id_{\lambda \overline{A}.[\![\Gamma;\overline{\alpha} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}]} \quad \square
\end{array}
$$

The following result shows that terms of type Nat behave like actual natural transformations with respect to their source and target functorial types.

Lemma 6 (Naturality). *The terms*

$$\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) \circ (L_{\overline{\gamma}} z.x_{\overline{\sigma},\overline{\gamma}} z) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]$$

and

$$\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash (L_{\overline{\gamma}} z.x_{\overline{\tau},\overline{\gamma}} z) \circ ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]$$

are semantically equivalent.

Proof. Let $\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G]\!]^{\mathsf{Set}} \rho$ and $\overline{f : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau]\!]^{\mathsf{Set}} \rho}$. The semantic interpretation of the first term is

$$
\begin{array}{rl}
& [\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y}) \circ (L_{\overline{\gamma}} z.x_{\overline{\sigma},\overline{\gamma}} z) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho \eta \overline{f} \\
= & [\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash L_{\overline{\gamma}} z.((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y})_{\overline{\gamma}} (x_{\overline{\sigma},\overline{\gamma}} z) : \mathsf{Nat}^{\overline{\gamma}} F[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho \eta \overline{f} \\
= & \lambda \overline{C}.\lambda z.[\![\Gamma;\overline{\gamma} \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau}, z \colon F[\overline{\alpha := \sigma}] \vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y})_{\overline{\gamma}} (x_{\overline{\sigma},\overline{\gamma}} z) : G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}] \eta \overline{f} \\
= & \lambda \overline{C}.\lambda z.([\![\Gamma;\emptyset \,|\, \overline{y : \mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau} \vdash (\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_\emptyset \overline{y} : \mathsf{Nat}^{\overline{\gamma}} G[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}]]\!]^{\mathsf{Set}} \rho \overline{f})_{\overline{C}} \\
& ([\![\Gamma;\overline{\gamma} \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G, z \colon F[\overline{\alpha := \sigma}] \vdash x_{\overline{\sigma},\overline{\gamma}} z : G[\overline{\alpha := \sigma}]]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}] \eta z) \\
= & \lambda \overline{C}.\lambda z.([\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{map}_G^{\overline{\sigma},\overline{\tau}} : \mathsf{Nat}^\emptyset (\overline{\mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau})\, (\mathsf{Nat}^{\overline{\gamma}} G[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}])]\!]^{\mathsf{Set}} \rho * \overline{f})_{\overline{C}} \\
& ([\![\Gamma;\emptyset \,|\, x : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G \vdash x]\!]^{\mathsf{Set}} \rho \eta)_{\overline{[\![\Gamma;\overline{\gamma} \vdash \sigma]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}]}, \overline{[\![\Gamma;\overline{\gamma} \vdash \gamma]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}]}} [\![\Gamma;\overline{\gamma} \,|\, z : F[\overline{\alpha := \sigma}] \vdash z]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}] z) \\
= & \lambda \overline{C}.\lambda z.([\![\Gamma;\emptyset \,|\, \emptyset \vdash \mathsf{map}_G^{\overline{\sigma},\overline{\tau}} : \mathsf{Nat}^\emptyset (\overline{\mathsf{Nat}^{\overline{\gamma}} \sigma\, \tau})\, (\mathsf{Nat}^{\overline{\gamma}} G[\overline{\alpha := \sigma}]\, G[\overline{\alpha := \tau}])]\!]^{\mathsf{Set}} \rho * \overline{f})_{\overline{C}} \\
& (\eta_{\overline{[\![\Gamma;\overline{\gamma} \vdash \sigma]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}]}, \overline{C}} z) \\
= & \lambda \overline{C}.\lambda z.[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma := C}]} [\overline{\alpha := f_{\overline{C}}}] (\eta_{\overline{[\![\Gamma;\overline{\gamma} \vdash \sigma]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}]}, \overline{C}} z) \\
= & \lambda \overline{C}.[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma := C}]} [\overline{\alpha := f_{\overline{C}}}] \circ \eta_{\overline{[\![\Gamma;\overline{\gamma} \vdash \sigma]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}]}, \overline{C}}
\end{array}
$$

By naturality of $\eta : \lambda \overline{A}.\lambda \overline{C}.[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}][\overline{\gamma := C}] \to \eta : \lambda \overline{A}.\lambda \overline{C}.[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} \rho[\overline{\alpha := A}][\overline{\gamma := C}]$ we have that

$$[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash G]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma := C}]} [\overline{\alpha := f_{\overline{C}}}] \circ \eta_{\overline{[\![\Gamma;\overline{\gamma} \vdash \sigma]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}]}, \overline{C}} = \eta_{\overline{[\![\Gamma;\overline{\gamma} \vdash \tau]\!]^{\mathsf{Set}} \rho[\overline{\gamma := C}]}, \overline{C}} \circ [\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma := C}]} [\overline{\alpha := f_{\overline{C}}}]$$

Proceeding analogously for the second term, we have that

$$
\begin{array}{rl}
& \lambda\overline{C}.\eta_{\overline{\llbracket\Gamma;\overline{\gamma}\vdash\tau\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{C}} \circ \llbracket\Gamma;\overline{\alpha},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\overline{\alpha := f_{\overline{C}}}] \\
= & \lambda\overline{C}.\lambda k.\eta_{\overline{\llbracket\Gamma;\overline{\gamma}\vdash\tau\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{C}}(\llbracket\Gamma;\overline{\alpha},\overline{\gamma}\vdash F\rrbracket^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\overline{\alpha := f_{\overline{C}}}]k) \\
= & \lambda\overline{C}.\lambda k.\eta_{\overline{\llbracket\Gamma;\overline{\gamma}\vdash\tau\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{C}} \\
& ((\llbracket\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}_F^{\overline{\sigma},\overline{\tau}} : \mathsf{Nat}^{\emptyset}(\overline{\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau})\,(\mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,F[\overline{\alpha:=\tau}])\rrbracket^{\mathsf{Set}}\rho * \overline{f})_{\overline{C}}k) \\
= & \lambda\overline{C}.\lambda k.(\llbracket\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G\vdash x\rrbracket^{\mathsf{Set}}\rho\eta)_{\overline{\llbracket\Gamma;\overline{\gamma}\vdash\tau\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{\llbracket\Gamma;\overline{\gamma}\vdash\gamma\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]}} \\
& ((\llbracket\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}_F^{\overline{\sigma},\overline{\tau}} : \mathsf{Nat}^{\emptyset}(\overline{\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau})\,(\mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,F[\overline{\alpha:=\tau}])\rrbracket^{\mathsf{Set}}\rho * \overline{f})_{\overline{C}}k) \\
= & \lambda\overline{C}.\lambda k.(\llbracket\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G\vdash x\rrbracket^{\mathsf{Set}}\rho\eta)_{\overline{\llbracket\Gamma;\overline{\gamma}\vdash\tau\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{\llbracket\Gamma;\overline{\gamma}\vdash\gamma\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]}} \\
& ((\llbracket\Gamma;\emptyset\,|\,\overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash (\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y} : \mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho\overline{f})_{\overline{C}}k) \\
= & \lambda\overline{C}.\lambda k.(\llbracket\Gamma;\overline{\gamma}\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G, z:F[\overline{\alpha:=\tau}]\vdash x_{\overline{\tau},\overline{\gamma}}z\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]\eta) \\
& (\llbracket\Gamma;\overline{\gamma}\,|\,\overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}, k:F[\overline{\alpha:=\sigma}]\vdash ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y})_{\overline{\gamma}}k : F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]\overline{f}k) \\
= & \lambda\overline{C}.\lambda k.(\llbracket\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G\vdash L_{\overline{\gamma}}z.x_{\overline{\tau},\overline{\gamma}}z\rrbracket^{\mathsf{Set}}\rho\eta)_{\overline{C}} \\
& (\llbracket\Gamma;\overline{\gamma}\,|\,\overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}, k:F[\overline{\alpha:=\sigma}]\vdash ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y})_{\overline{\gamma}}k : F[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma:=C}]\overline{f}k) \\
= & \lambda\overline{C}.\lambda k.\llbracket\Gamma;\overline{\gamma}\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G, \overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}, k:F[\overline{\alpha:=\sigma}]\vdash (L_{\overline{\gamma}}z.x_{\overline{\tau},\overline{\gamma}}z)_{\overline{\gamma}}(((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y})_{\overline{\gamma}}k) : G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho[\overline{\gamma} \\
= & \llbracket\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G, \overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash L_{\overline{\gamma}}k.(L_{\overline{\gamma}}z.x_{\overline{\tau},\overline{\gamma}}z)_{\overline{\gamma}}(((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y})_{\overline{\gamma}}k) : \mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho\eta\overline{f} \\
= & \llbracket\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G, \overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash (L_{\overline{\gamma}}z.x_{\overline{\tau},\overline{\gamma}}z)\circ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y}) : \mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho\eta\overline{f}
\end{array}
$$

So, we conclude that

$$
\begin{array}{l}
\llbracket\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G, \overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash ((\mathsf{map}_G^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y})_{\overline{\gamma}}\circ(L_{\overline{\gamma}}z.x_{\overline{\sigma},\overline{\gamma}}z) : \mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho \\
= \llbracket\Gamma;\emptyset\,|\,x:\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}}F\,G, \overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash (L_{\overline{\gamma}}z.x_{\overline{\tau},\overline{\gamma}}z)\circ((\mathsf{map}_F^{\overline{\sigma},\overline{\tau}})_{\emptyset}\overline{y}) : \mathsf{Nat}^{\overline{\gamma}}F[\overline{\alpha:=\sigma}]\,G[\overline{\alpha:=\tau}]\rrbracket^{\mathsf{Set}}\rho
\end{array}
$$

The case for the relational interpretation is analogous. □

We have a special case of Lemma 6 following from the naturality of $\mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := (\lambda\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}$:

Corollary 7. *Let $\Gamma;\phi,\overline{\alpha},\overline{\gamma}\vdash H$ and $\overline{\Gamma;\gamma\vdash\sigma}$ and $\overline{\Gamma;\gamma\vdash\tau}$ be types. Then the terms*

$$
\begin{array}{l}
\Gamma;\emptyset\,|\,\overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash ((\mathsf{map}^{\overline{\sigma},\overline{\tau}}_{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_{\emptyset}\overline{y})\circ(L_{\overline{\gamma}}z.(\mathsf{in}_H)_{\overline{\sigma},\overline{\gamma}}z) \\
\qquad : \mathsf{Nat}^{\overline{\gamma}}H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\sigma}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}
\end{array}
$$

and

$$
\begin{array}{l}
\Gamma;\emptyset\,|\,\overline{y:\mathsf{Nat}^{\overline{\gamma}}\sigma\,\tau}\vdash (L_{\overline{\gamma}}z.(\mathsf{in}_H)_{\overline{\tau},\overline{\gamma}}z)\circ((\mathsf{map}^{\overline{\sigma},\overline{\tau}}_{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_{\emptyset}\overline{y}) \\
\qquad : \mathsf{Nat}^{\overline{\gamma}}H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\sigma}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\tau}
\end{array}
$$

are semantically equivalent.

Proof. The two terms are derived from those in Lemma 6 by instantiating the term variable x with $\mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}$. □

The next lemma states that the map of the composition of two functors is (equivalent to) the composition of their maps.

Lemma 8. *Let*

$$\Gamma;\overline{\psi},\overline{\gamma}\vdash H \qquad \overline{\Gamma;\overline{\alpha},\overline{\gamma},\overline{\phi}\vdash K} \qquad \overline{\Gamma;\overline{\beta},\overline{\gamma}\vdash F} \qquad \overline{\Gamma;\overline{\beta},\overline{\gamma}\vdash G}$$

be types. Then the terms

$$\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}^{\overline{F},\overline{G}}_{H[\overline{\psi:=K}]} : \mathsf{Nat}^{\emptyset}(\overline{\mathsf{Nat}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}FG})(\mathsf{Nat}^{\overline{\gamma}}H[\overline{\psi:=K}][\overline{\phi:=F}]\;H[\overline{\psi:=K}][\overline{\phi:=G}])$$

and

$$\Gamma;\emptyset\,|\,\emptyset\vdash \mathsf{map}_H^{\overline{K[\overline{\phi:=F}]},\overline{K[\overline{\phi:=G}]}}\circ\overline{\mathsf{map}_K^{\overline{F},\overline{G}}} : \mathsf{Nat}^{\emptyset}(\overline{\mathsf{Nat}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}FG})(\mathsf{Nat}^{\overline{\gamma}}H[\overline{\psi:=K[\overline{\phi:=F}]}]\;H[\overline{\psi:=K[\overline{\phi:=G}]}])$$

are semantically equivalent. (Notice that F and G's context is extended with the $\overline{\alpha}$ variables by weakening).

Proof. Throughout this proof we shall use the fact that the variables $\overline{\alpha}$ can be added to the environment of F and G by weakening, even though they do not appear in those types. As a consequence, which is true only because F and G contain no $\overline{\alpha}$'s, $H[\overline{\psi:=_{\overline{\alpha}}K[\overline{\phi:=_{\overline{\beta}}F}]}] = H[\overline{\psi:=_{\overline{\alpha}}K}][\overline{\phi:=_{\overline{\beta}}F}]$ and $H[\overline{\psi:=_{\overline{\alpha}}K[\overline{\phi:=_{\overline{\beta}}G}]}] = H[\overline{\psi:=_{\overline{\alpha}}K}][\overline{\phi:=_{\overline{\beta}}G}]$. Also, observe that a natural transformation $\eta : [\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}FG]\!]^{\mathsf{Set}}\rho$ corresponds, by weakening, to a natural transformation $\eta : [\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}FG]\!]^{\mathsf{Set}}\rho$ which is trivially natural in the α's. Then for every natural transformation $\eta : [\![\Gamma;\emptyset\vdash\mathsf{Nat}^{\overline{\beta},\overline{\gamma}}FG]\!]^{\mathsf{Set}}\rho$, $\overline{C:\mathsf{Set}}$, and $*$ the unique element of the singleton, we have that

$$\begin{aligned}
&([\![\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{map}_H^{\overline{K[\overline{\phi:=F}]},\overline{K[\overline{\phi:=G}]}}\circ\overline{\mathsf{map}_K^{\overline{F},\overline{G}}}]\!]^{\mathsf{Set}}\rho * \overline{\eta})_{\overline{C}}\\
=\;&([\![\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{map}_H^{\overline{K[\overline{\phi:=F}]},\overline{K[\overline{\phi:=G}]}}]\!]^{\mathsf{Set}}\rho * \overline{([\![\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{map}_K^{\overline{F},\overline{G}}]\!]^{\mathsf{Set}}\rho * \overline{\eta})})_{\overline{C}}\\
=\;&[\![\Gamma;\overline{\psi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}Id_{\rho[\overline{\gamma:=C}]}[\overline{\psi:=\lambda\overline{A}.([\![\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{map}_K^{\overline{F},\overline{G}}]\!]^{\mathsf{Set}}\rho * \overline{\eta})_{\overline{A},\overline{C}}}]\\
=\;&[\![\Gamma;\overline{\psi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}Id_{\rho[\overline{\gamma:=C}]}[\overline{\psi:=\lambda\overline{A}.[\![\Gamma;\overline{\alpha},\overline{\gamma},\overline{\phi}\vdash K]\!]^{\mathsf{Set}}Id_{\rho[\overline{\alpha:=A}][\overline{\gamma:=C}]}[\overline{\phi:=\lambda\overline{B}.\eta_{\overline{A},\overline{B},\overline{C}}}]}]\\
=\;&[\![\Gamma;\overline{\psi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}Id_{\rho[\overline{\gamma:=C}]}[\overline{\psi:=\lambda\overline{A}.[\![\Gamma;\overline{\alpha},\overline{\gamma},\overline{\phi}\vdash K]\!]^{\mathsf{Set}}Id_{\rho[\overline{\alpha:=A}][\overline{\gamma:=C}]}[\overline{\phi:=\lambda\overline{B}.\eta_{\overline{B},\overline{C}}}]}]\\
=\;&[\![\Gamma;\overline{\psi},\overline{\gamma}\vdash H]\!]^{\mathsf{Set}}Id_{\rho[\overline{\gamma:=C}]}[\overline{\phi:=\lambda\overline{B}.\eta_{\overline{B},\overline{C}}}][\overline{\psi:=\lambda\overline{A}.[\![\Gamma;\overline{\alpha},\overline{\gamma},\overline{\phi}\vdash K]\!]^{\mathsf{Set}}Id_{\rho[\overline{\alpha:=A}][\overline{\gamma:=C}]}[\overline{\phi:=\lambda\overline{B}.\eta_{\overline{B},\overline{C}}}]}]\\
=\;&[\![\Gamma;\overline{\gamma},\overline{\phi}\vdash H[\overline{\psi:=K}]]\!]^{\mathsf{Set}}Id_{\rho[\overline{\gamma:=C}]}[\overline{\phi:=\lambda\overline{B}.\eta_{\overline{B},\overline{C}}}]\\
=\;&([\![\Gamma;\emptyset\,|\,\emptyset\vdash\mathsf{map}^{\overline{F},\overline{G}}_{H[\overline{\psi:=K}]}]\!]^{\mathsf{Set}}\rho * \overline{\eta})_{\overline{C}}
\end{aligned}$$

The case for the relational interpretation is analogous. $\square$

Type application $\phi\overline{\tau}$ yields a map acting functorially on both the type constructor variable ϕ and its arguments $\overline{\tau}$. Such action consists in mapping along the type constructor variable first and the arguments later, or, equivalently, the arguments first and the type constructor variable later. That the two ways of describing the action are equivalent is due to naturality.

Lemma 9 (Map of type application). *Consider the following types*

$$\overline{\Gamma;\phi,\overline{\psi},\overline{\gamma}\vdash\tau} \qquad \Gamma;\overline{\beta},\overline{\gamma}\vdash H \qquad \Gamma;\overline{\beta},\overline{\gamma}\vdash K \qquad \overline{\Gamma;\overline{\alpha},\overline{\gamma}\vdash F} \qquad \overline{\Gamma;\overline{\alpha},\overline{\gamma}\vdash G}$$

Let $\overline{I} = \overline{F}, H$ and $\overline{J} = \overline{G}, K$ be lists of types. Then the terms

$$\Gamma; \emptyset \mid \emptyset \vdash L_\emptyset(x, \overline{y}).L_{\overline{\gamma}} z.x_{\overline{\tau[\overline{\psi:=G}][\phi:=K]},\overline{\gamma}}\Big(((\mathsf{map}_H^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}})_\emptyset \overline{((\mathsf{map}_\tau^{\overline{I},\overline{J}})_\emptyset(x,\overline{y}))})_{\overline{\gamma}} z\Big)$$
$$: \mathsf{Nat}^\emptyset(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H\, K \times \overline{\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G})\,(\mathsf{Nat}^{\overline{\gamma}} H[\overline{\beta := \tau}][\overline{\psi := F}][\phi := H]\ K[\overline{\beta := \tau}][\overline{\psi := G}][\phi := K]) \tag{1}$$

and

$$\Gamma; \emptyset \mid \emptyset \vdash L_\emptyset(x, \overline{y}).L_{\overline{\gamma}} z.((\mathsf{map}_K^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}})_\emptyset \overline{((\mathsf{map}_\tau^{\overline{I},\overline{J}})_\emptyset(x,\overline{y}))})_{\overline{\gamma}}\Big(x_{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\gamma}} z\Big)$$
$$: \mathsf{Nat}^\emptyset(\mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H\, K \times \overline{\mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G})\,(\mathsf{Nat}^{\overline{\gamma}} H[\overline{\beta := \tau}][\overline{\psi := F}][\phi := H]\ K[\overline{\beta := \tau}][\overline{\psi := G}][\phi := K]) \tag{2}$$

are semantically equivalent to $\mathsf{map}^{\overline{I},\overline{J}}_{\phi\overline{\tau}}$.

Proof. To begin with, observe that the terms 1 and 2 are semantically equivalent. This follows from the fact that the terms

$$\Gamma; \emptyset \mid x, \overline{y} \vdash (L_{\overline{\gamma}} w.x_{\overline{\tau[\overline{\psi:=G}][\phi:=K]},\overline{\gamma}} w) \circ \Big((\mathsf{map}_H^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}})_\emptyset \overline{((\mathsf{map}_\tau^{\overline{I},\overline{J}})_\emptyset(x,\overline{y}))}\Big)$$
$$: \mathsf{Nat}^{\overline{\gamma}} H[\overline{\beta := \tau}][\overline{\psi := F}][\phi := H]\, K[\overline{\beta := \tau}][\overline{\psi := G}][\phi := K]$$

and

$$\Gamma; \emptyset \mid x, \overline{y} \vdash \Big((\mathsf{map}_K^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}})_\emptyset \overline{((\mathsf{map}_\tau^{\overline{I},\overline{J}})_\emptyset(x,\overline{y}))}\Big) \circ \Big(L_{\overline{\gamma}} z.x_{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\gamma}} z\Big)$$
$$: \mathsf{Nat}^{\overline{\gamma}} H[\overline{\beta := \tau}][\overline{\psi := F}][\phi := H]\, K[\overline{\beta := \tau}][\overline{\psi := G}][\phi := K]$$

are semantically equivalent by Lemma 6. Thus, it will suffice to prove that any of them, say term 1, is semantically equivalent to $\mathsf{map}^{\overline{I},\overline{J}}_{\phi\overline{\tau}}$.

If ρ is a set environment and $*$ is the unique element of the singleton, we have that the interpretation of the term 1 is given by

$$\begin{aligned}
& [\![\Gamma; \emptyset \mid \emptyset \vdash L_\emptyset(x, \overline{y}).L_{\overline{\gamma}} z.x_{\overline{\tau[\overline{\psi:=G}][\phi:=K]},\overline{\gamma}}(((\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y}))_{\overline{\gamma}} z)]\!]^{\mathsf{Set}}\rho *\\
= \;& \lambda\eta.\lambda\overline{\epsilon}.[\![\Gamma; \emptyset \mid x, \overline{y} \vdash L_{\overline{\gamma}} z.x_{\overline{\tau[\overline{\psi:=G}][\phi:=K]},\overline{\gamma}}(((\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y}))_{\overline{\gamma}} z)]\!]^{\mathsf{Set}}\rho\eta\overline{\epsilon}\\
= \;& \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.\lambda z.[\![\Gamma; \overline{\gamma} \mid x, \overline{y}, z \vdash x_{\overline{\tau[\overline{\psi:=G}][\phi:=K]},\overline{\gamma}}(((\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y}))_{\overline{\gamma}} z)]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}]\eta\overline{\epsilon}z\\
= \;& \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.\lambda z.\eta_{\overline{[\![\Gamma;\overline{\gamma}\vdash\tau[\overline{\psi:=G}][\phi:=K]]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{C}}([\![\Gamma; \overline{\gamma} \mid x, \overline{y}, z \vdash ((\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y}))_{\overline{\gamma}} z]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}]\eta\overline{\epsilon}z)\\
= \;& \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.\lambda z.\eta_{\overline{[\![\Gamma;\overline{\gamma}\vdash\tau[\overline{\psi:=G}][\phi:=K]]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{C}}(([\![\Gamma; \emptyset \mid x, \overline{y} \vdash (\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y})]\!]^{\mathsf{Set}}\rho\eta\overline{\epsilon})_{\overline{C}} z)\\
= \;& \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.\eta_{\overline{[\![\Gamma;\overline{\gamma}\vdash\tau[\overline{\psi:=G}][\phi:=K]]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{C}} \circ ([\![\Gamma; \emptyset \mid x, \overline{y} \vdash (\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y})]\!]^{\mathsf{Set}}\rho\eta\overline{\epsilon})_{\overline{C}}
\end{aligned}$$

where the term variables are typed as $x : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H\, K$, $\overline{y : \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G}$, and $z : H[\overline{\beta := \tau}][\overline{\psi := F}][\phi := H]$.

Observe that the terms

$$(\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y})$$

and

$$(\mathsf{map}_H^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}})_\emptyset \overline{((\mathsf{map}_\tau^{\overline{I},\overline{J}})_\emptyset(x,\overline{y}))}$$

are semantically equivalent because of Lemma 8. Then for all $\eta : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H\, K]\!]^{\mathsf{Set}}\rho$ and $\epsilon : [\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\alpha},\overline{\gamma}} F\, G]\!]^{\mathsf{Set}}\rho$, we have that

$$
\begin{array}{cl}
& ([\![\Gamma;\emptyset \mid x,\overline{y} \vdash (\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y})]\!]^{\mathsf{Set}}\rho\eta\overline{\epsilon})_{\overline{C}} \\
= & ([\![\Gamma;\emptyset \mid x,\overline{y} \vdash (\mathsf{map}^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}}_{H})_\emptyset(\overline{(\mathsf{map}^{\overline{I},\overline{J}}_{\tau})_\emptyset(x,\overline{y})})]\!]^{\mathsf{Set}}\rho\eta\overline{\epsilon})_{\overline{C}} \\
= & \big([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{\tau[\overline{\psi:=F}][\phi:=H]},\overline{\tau[\overline{\psi:=G}][\phi:=K]}}_{H}]\!]^{\mathsf{Set}}\rho * \overline{([\![\Gamma;\emptyset \mid x,\overline{y} \vdash (\mathsf{map}^{\overline{I},\overline{J}}_{\tau})_\emptyset(x,\overline{y})]\!]^{\mathsf{Set}}\rho\eta\overline{\epsilon})}\big)_{\overline{C}} \\
= & [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\overline{\beta := ([\![\Gamma;\emptyset \mid x,\overline{y} \vdash (\mathsf{map}^{\overline{I},\overline{J}}_{\tau})_\emptyset(x,\overline{y})]\!]^{\mathsf{Set}}\rho\eta\overline{\epsilon})_{\overline{C}}}] \\
= & [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\overline{\beta := [\![\Gamma;\phi,\overline{\psi},\overline{\gamma} \vdash \tau]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\overline{\psi := \lambda\overline{A}.\epsilon_{\overline{A},\overline{C}}}][\phi := \lambda\overline{B}.\eta_{\overline{B},\overline{C}}]}]
\end{array}
$$

Observe that, for each τ,

$$
[\![\Gamma;\overline{\gamma} \vdash \tau[\overline{\psi := G}][\phi := K]]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}]
$$

is equal to

$$
[\![\Gamma;\phi,\overline{\psi},\overline{\gamma} \vdash \tau]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\psi := [\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\alpha := \_}]}][\phi := [\![\Gamma;\overline{\beta},\overline{\gamma} \vdash K]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\beta := \_}]]
$$

because of Lemma **??** (references lemma in draft document). Moreover, observe that $\lambda\overline{B}.\eta_{\overline{B},\overline{C}}$ is a natural transformation

$$
\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}] \Rightarrow \lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash K]\!]^{\mathsf{Set}}\rho[\overline{\beta := B}][\overline{\gamma := C}]
$$

and $\lambda\overline{A}.\epsilon_{\overline{A},\overline{C}}$ is a natural transformation

$$
\lambda\overline{A}.[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash F]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}][\overline{\gamma := C}] \Rightarrow \lambda\overline{A}.[\![\Gamma;\overline{\alpha},\overline{\gamma} \vdash G]\!]^{\mathsf{Set}}\rho[\overline{\alpha := A}][\overline{\gamma := C}]
$$

for any $\overline{C : \mathsf{Set}}$. Then we have that

$$
\begin{array}{cl}
& [\![\Gamma;\emptyset \mid \emptyset \vdash L_\emptyset(x,\overline{y}).L_{\overline{\gamma}} z.x_{\overline{\tau[\overline{\psi:=G}][\phi:=K]},\overline{\gamma}}\big(\overline{((\mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]})_\emptyset(x,\overline{y}))_{\overline{\gamma}} z}\big)]\!]^{\mathsf{Set}}\rho * \\
= & \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.\eta_{\overline{[\![\Gamma;\overline{\gamma}\vdash\tau[\overline{\psi:=G}][\phi:=K]]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}]},\overline{C}} \circ \big([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{I},\overline{J}}_{H[\overline{\beta:=\tau}]}]\!]^{\mathsf{Set}}\rho * \eta\overline{\epsilon}\big)_{\overline{C}} \\
= & \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.\eta_{\overline{[\![\Gamma;\phi,\overline{\psi},\overline{\gamma}\vdash\tau]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\psi:=[\![\Gamma;\overline{\alpha},\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\alpha:=\_}]}][\phi:=[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\beta:=\_}]]},\overline{C}} \\
& \circ[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\overline{\beta := [\![\Gamma;\phi,\overline{\psi},\overline{\gamma} \vdash \tau]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\overline{\psi := \lambda\overline{A}.\epsilon_{\overline{A},\overline{C}}}][\phi := \lambda\overline{B}.\eta_{\overline{B},\overline{C}}]}] \\
= & \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.\eta_{\overline{[\![\Gamma;\phi,\overline{\psi},\overline{\gamma}\vdash\tau]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\psi:=[\![\Gamma;\overline{\alpha},\overline{\gamma}\vdash G]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\alpha:=\_}]}][\phi:=[\![\Gamma;\overline{\beta},\overline{\gamma}\vdash K]\!]^{\mathsf{Set}}\rho[\overline{\gamma:=C}][\overline{\beta:=\_}]]},\overline{C}} \\
& \circ(\lambda\overline{B}.[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}}\rho[\overline{\gamma := C}][\overline{\beta := B}])(\overline{[\![\Gamma;\phi,\overline{\psi},\overline{\gamma} \vdash \tau]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\overline{\psi := \lambda\overline{A}.\epsilon_{\overline{A},\overline{C}}}][\phi := \lambda\overline{B}.\eta_{\overline{B},\overline{C}}]}) \\
= & \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.[\![\Gamma;\phi,\overline{\psi},\overline{\gamma} \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} Id_{\rho[\overline{\gamma:=C}]}[\phi := \lambda\overline{B}.\eta_{\overline{B},\overline{C}}][\overline{\psi := \lambda\overline{A}.\epsilon_{\overline{A},\overline{C}}}] \\
= & \lambda\eta.\lambda\overline{\epsilon}.\lambda\overline{C}.\big([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{I},\overline{J}}_{\phi\overline{\tau}}]\!]^{\mathsf{Set}}\rho * \eta\overline{\epsilon}\big)_{\overline{C}} \\
= & [\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{\overline{I},\overline{J}}_{\phi\overline{\tau}}]\!]^{\mathsf{Set}}\rho *
\end{array}
$$

where the third equality is given by the definition of functorial action of the semantic interpretation for type application, in Definition **??** (the definition of the action of set interpretations of types on morphisms in SetEnv).

Finally, the proof for the relation interpretation is analogous to the above proof for the set interpretation. ◻

Lemma 10. *The terms*

$$
\Gamma;\emptyset \mid x : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := F][\overline{\alpha := \beta}]\, F \vdash ((\mathsf{fold}_{H,F})_\emptyset x) \circ \mathsf{in}_H : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, F
$$

and

$$\begin{aligned}\Gamma;\emptyset \mid x : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := F][\overline{\alpha := \beta}]\, F \vdash x \circ \big((\mathsf{map}^{(\mu\phi.\lambda\overline{\alpha}H)\overline{\beta},F}_{H[\overline{\alpha:=\beta}]})_\emptyset((\mathsf{fold}_{H,F})_\emptyset x)\big)&\\ : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}]\, F&\end{aligned}$$

are semantically equivalent.

Proof. Let ρ be a set environment, $\overline{B}$ and $\overline{C}$ be sets, and η be a natural transformation in $[\![\Gamma;\emptyset \vdash \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := F][\overline{\alpha := \beta}]\, F]\!]^{\mathsf{Set}}\rho$. Then we have that

$$\begin{aligned}
& ([\![\Gamma;\emptyset \mid x \vdash ((\mathsf{fold}_{H,F})_\emptyset x)\circ \mathsf{in}_H]\!]^{\mathsf{Set}}\rho\eta)_{\overline{B},\overline{C}}\\
= \;& ([\![\Gamma;\emptyset \mid x \vdash (\mathsf{fold}_{H,F})_\emptyset x]\!]^{\mathsf{Set}}\rho\eta)_{\overline{B},\overline{C}} \circ ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H]\!]^{\mathsf{Set}}\rho *)_{\overline{B},\overline{C}}\\
= \;& ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}_{H,F}]\!]^{\mathsf{Set}}\rho * \eta)_{\overline{B},\overline{C}} \circ (in_{T_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}\\
= \;& (fold_{T_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}))_{\overline{B}} \circ (in_{T_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}\\
= \;& ((fold_{T_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})) \circ in_{T_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}\\
= \;& \big((\lambda\overline{A}.\eta_{\overline{A},\overline{C}}) \circ (T_{\rho[\overline{\gamma:=C}]}(fold_{T_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})))\big)_{\overline{B}}\\
= \;& (\lambda\overline{A}.\eta_{\overline{A},\overline{C}})_{\overline{B}} \circ \big(T_{\rho[\overline{\gamma:=C}]}(fold_{T_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}))\big)_{\overline{B}}\\
= \;& \eta_{\overline{B},\overline{C}} \circ [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\alpha:=B}][\overline{\gamma:=C}]}[\phi := fold_{T_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}})]\\
= \;& \eta_{\overline{B},\overline{C}} \circ [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\alpha:=B}][\overline{\gamma:=C}]}[\phi := \lambda\overline{A'}.(fold_{T_{\rho[\overline{\gamma:=C}]}}(\lambda\overline{A}.\eta_{\overline{A},\overline{C}}))_{\overline{A'}}]\\
= \;& \eta_{\overline{B},\overline{C}} \circ [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\alpha:=B}][\overline{\gamma:=C}]}[\phi := \lambda\overline{A'}.([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H]\!]^{\mathsf{Set}}\rho * \eta)_{\overline{A'},\overline{C}}]\\
= \;& \eta_{\overline{B},\overline{C}} \circ [\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\alpha:=B}][\overline{\gamma:=C}]}[\phi := \lambda\overline{A'}.([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H]\!]^{\mathsf{Set}}\rho * \eta)_{\overline{A'},\overline{B},\overline{C}}]\\
= \;& \eta_{\overline{B},\overline{C}} \circ \big([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta},F}_H]\!]^{\mathsf{Set}}\rho * ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}^F_H]\!]^{\mathsf{Set}}\rho * \eta)\big)_{\overline{B},\overline{C}}\\
= \;& \eta_{\overline{B},\overline{C}} \circ \big([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}^{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta},F}_H]\!]^{\mathsf{Set}}\rho * ([\![\Gamma;\emptyset \mid x \vdash (\mathsf{fold}^F_H)_\emptyset x]\!]^{\mathsf{Set}}\rho\eta)\big)_{\overline{B},\overline{C}}\\
= \;& ([\![\Gamma;\emptyset \mid x \vdash x]\!]^{\mathsf{Set}}\rho\eta)_{\overline{B},\overline{C}} \circ ([\![\Gamma;\emptyset \mid x \vdash (\mathsf{map}^{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta},F}_H)_\emptyset((\mathsf{fold}^F_H)_\emptyset x)]\!]^{\mathsf{Set}}\rho\eta)_{\overline{B},\overline{C}}\\
= \;& ([\![\Gamma;\emptyset \mid x \vdash x \circ \big((\mathsf{map}^{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta},F}_H)_\emptyset((\mathsf{fold}^F_H)_\emptyset x)\big)]\!]^{\mathsf{Set}}\rho\eta)_{\overline{B},\overline{C}}
\end{aligned}$$

where the tenth equality is given by weakening. □

It is a general property of the semantic *in* to be invertible, its inverse being given in terms of *fold*. We shall prove that this fact holds for the semantic interpretation of the syntactic in and fold.

The next lemma states the syntactic analogue of the fact that, for an endofunctor H, the composition $in_H \circ fold_H(H\, in_H)$ is the identity on the fixed point μH.

Lemma 11. *The terms*

$$\begin{aligned}\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H \circ (\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \mathsf{in}_H)&\\ : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}&\end{aligned}$$

and

$$\Gamma;\emptyset \mid \emptyset \vdash Id_{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}} : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\,(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}$$

are semantically equivalent.

Proof. Let ρ be a set environment, $\overline{B}$ and $\overline{C}$ be sets, and $*$ be the unique element of the singleton. Then we have that

$$
\begin{aligned}
& ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H \circ (\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \mathsf{in}_H)]\!]^{\mathsf{Set}}\rho *)_{\overline{B},\overline{C}}\\
= \; & ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H]\!]^{\mathsf{Set}}\rho *)_{\overline{B},\overline{C}}\\
& \circ([\![\Gamma;\emptyset \mid \emptyset \vdash (\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \mathsf{in}_H)]\!]^{\mathsf{Set}}\rho *)_{\overline{B},\overline{C}}\\
= \; & (in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}\\
& \circ\big([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]}]\!]^{\mathsf{Set}}\rho *\\
& ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}]\!]^{\mathsf{Set}}\rho * ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H]\!]^{\mathsf{Set}}\rho *))\big)_{\overline{B},\overline{C}}\\
= \; & (in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}\\
& \circ\big(fold_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}\, \lambda\overline{A}.([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}]\!]^{\mathsf{Set}}\rho * ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H]\!]^{\mathsf{Set}}\rho *))_{\overline{A},\overline{C}}\big)_{\overline{B}}
\end{aligned}
$$

Notice that

$$
\begin{aligned}
& \lambda\overline{A}.([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}]\!]^{\mathsf{Set}}\rho * ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H]\!]^{\mathsf{Set}}\rho *))_{\overline{A},\overline{C}}\\
= \; & \lambda\overline{A}.[\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\alpha:=A}][\overline{\gamma:=C}]}[\phi := \lambda\overline{B'}.([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H]\!]^{\mathsf{Set}}\rho *)_{\overline{A},\overline{B'},\overline{C}}]\\
= \; & \lambda\overline{A}.[\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\alpha:=A}][\overline{\gamma:=C}]}[\phi := \lambda\overline{B'}.([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H]\!]^{\mathsf{Set}}\rho *)_{\overline{B'},\overline{C}}]\\
= \; & \lambda\overline{A}.[\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\alpha:=A}][\overline{\gamma:=C}]}[\phi := \lambda\overline{B'}.(in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}})_{\overline{B'}}]\\
= \; & \lambda\overline{A}.[\![\Gamma;\phi,\overline{\alpha},\overline{\gamma} \vdash H]\!]^{\mathsf{Set}} Id_{\rho[\overline{\alpha:=A}][\overline{\gamma:=C}]}[\phi := in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}]\\
= \; & \lambda\overline{A}.T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}\, in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}\, \overline{A}\\
= \; & T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}\, in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}
\end{aligned}
$$

Then we use the above calculation to compute

$$
\begin{aligned}
& (in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}\\
& \circ\big(fold_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}\, (\lambda\overline{A}.([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}]\!]^{\mathsf{Set}}\rho * ([\![\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H]\!]^{\mathsf{Set}}\rho *))_{\overline{A},\overline{C}})\big)_{\overline{B}}\\
= \; & (in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}})_{\overline{B}} \circ \big(fold_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}\, (T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}\, in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}})\big)_{\overline{B}}\\
= \; & \big(in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}} \circ fold_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}}\, (T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}\, in_{T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}})\big)_{\overline{B}}\\
= \; & (Id_{\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]}})_{\overline{B}}\\
= \; & Id_{(\mu T^{\mathsf{Set}}_{\rho[\overline{\gamma:=C}]})\overline{B}}\\
= \; & Id_{[\![\Gamma;\overline{\beta},\overline{\gamma} \vdash (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\!]^{\mathsf{Set}}\rho[\overline{\beta:=B}][\overline{\gamma:=C}]}\\
= \; & ([\![\Gamma;\emptyset \mid \emptyset \vdash Id_{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}]\!]^{\mathsf{Set}}\rho *)_{\overline{B},\overline{C}}
\end{aligned}
$$

where the first equality follows from the previous calculation, the third equality is a semantic property of *fold* and *in*, and the last equality is by Lemma 5.

The proof for the relation interpretation is analogous. □

The next lemma states the syntactic analogue of the fact that, for an endofunctor H, the composition $fold_H(H in_H) \circ in_H$ is the identity on the fixed point $H(\mu H)$.

Lemma 12. *The terms*

$$
\begin{aligned}
\Gamma;\emptyset \mid \emptyset \vdash & (\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \mathsf{in}_H) \circ \mathsf{in}_H\\
& : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}} H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]
\end{aligned}
$$

and

$$\Gamma;\emptyset \mid \emptyset \vdash Id_{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]} : \mathsf{Nat}^{\overline{\beta},\overline{\gamma}}\, H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]\; H[\phi := (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]$$

are semantically equivalent.

Proof. The term

$$\Gamma;\emptyset \mid \emptyset \vdash (\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \mathsf{in}_H) \circ \mathsf{in}_H$$

because of Lemma 10, is semantically equivalent to

$$\begin{aligned}\Gamma;\emptyset \mid \emptyset \vdash ((&\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \mathsf{in}_H)\\ &\circ (\mathsf{map}_H^{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta},H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}]})_\emptyset\\ &((\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \mathsf{in}_H))\end{aligned}$$

which, because of functoriality (map notes, pages 1–2), is semantically equivalent to

$$\begin{aligned}\Gamma;\emptyset \mid \emptyset \vdash (&\mathsf{map}_H^{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta},(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset\\ &(\mathsf{in}_H \circ (\mathsf{fold}_{H,H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]})_\emptyset((\mathsf{map}_H^{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha:=\beta}],(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset \mathsf{in}_H))\end{aligned}$$

which, because of Lemma 11, is semantically equivalent to

$$\Gamma;\emptyset \mid \emptyset \vdash (\mathsf{map}_H^{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta},(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})_\emptyset(Id_{(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}})$$

which, because of functoriality (map notes, page 3), is semantically equivalent to

$$\Gamma;\emptyset \mid \emptyset \vdash Id_{H[\phi:=(\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}]}$$ □