Free Theorems for Nested Types

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1 FREE THEOREM FOR TYPE OF POLYMORPHIC IDENTITY

Suppose $\vdash g : \mathsf{Nat}^\alpha\, \alpha\, \alpha$, let $G^{\mathsf{Set}} = [\![\vdash g : \mathsf{Nat}^\alpha\, \alpha\, \alpha]\!]^{\mathsf{Set}}$, and let $G^{\mathsf{Rel}} = [\![\vdash g : \mathsf{Nat}^\alpha\, \alpha\, \alpha]\!]^{\mathsf{Rel}}$. By Theorem **??**, $(G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) = G^{\mathsf{Rel}}\rho$. Thus, for all $\rho \in \mathsf{RelEnv}$ and any $(a, b) \in [\![\vdash \emptyset]\!]^{\mathsf{Rel}}\rho = 1$, eliding the only possible instantiations of a and b gives that

$$\begin{array}{rcl} (G^{\mathsf{Set}}, G^{\mathsf{Set}}) \;=\; (G^{\mathsf{Set}}(\pi_1\rho), G^{\mathsf{Set}}(\pi_2\rho)) & \in & [\![\vdash \mathsf{Nat}^\alpha\, \alpha\, \alpha]\!]^{\mathsf{Rel}}\rho \\ & = & \{\eta : id \Rightarrow id\} \\ & = & \{(\eta_1 : id \Rightarrow id, \eta_2 : id \Rightarrow id)\} \end{array}$$

That is, G^{Set} is a natural transformation from the identity functor on Set to itself.

Now let S be any set. If $S = \emptyset$, then there is exactly one morphism $id_S : S \to S$, so $G^{\mathsf{Set}}_S : S \to S$ must be id_S. If $S \neq \emptyset$, then if a is any element of S and $K_a : S \to S$ is the constantly a-valued morphism on S, then instantiating the naturality square implied by the above equality gives that $G^{\mathsf{Set}}_S \circ K_a = K_a \circ G^{\mathsf{Set}}_S$, i.e., $G^{\mathsf{Set}}_S\, a = a$, i.e., $G^{\mathsf{Set}}_S = id_S$. Putting these two cases together we have that for every $S : \mathsf{Set}$, $G^{\mathsf{Set}}_S = id_S$, i.e., G^{Set} is the identity natural transformation for the identity functor on Set. So every closed term g of closed type $\mathsf{Nat}^\alpha\, \alpha\, \alpha$ always denotes the identity natural transformation for the identity functor on Set, i.e., every closed term g of type $\mathsf{Nat}^\alpha\, \alpha\, \alpha$ denotes the polymorphic identity function.

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