Free Theorems for Nested Types

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1 DEMOTION THEOREMS

Theorem 1. *If* $\Gamma;\Phi,\phi^k \vdash \tau : \mathcal{F}$*, then one can derive* $\Gamma,\psi^k;\Phi \vdash \tau[\phi^k :== \psi^k]$*, where* $\tau[\phi :== \psi]$ *is the textual replacement of* ϕ *in* τ*. In other words, all occurences of* $\phi\overline{\sigma}$ *in* τ *become* $\psi\overline{\sigma}$*.*

Proof. By induction on the structure of τ.

- There is nothing to prove for types in $\mathcal{T}$ because their functorial contexts must be empty.
- Case $\Gamma;\Phi,\alpha \vdash \alpha : \mathcal{F}$. We must derive $\Gamma,\beta;\Phi \vdash \beta : \mathcal{F}$.

$$\dfrac{\dfrac{\Gamma,\beta;\emptyset \vdash \beta : \mathcal{T}}{\Gamma,\beta;\emptyset \vdash \beta : \mathcal{F}}}{\Gamma,\beta;\Phi \vdash \beta : \mathcal{F}}$$

- Case $\Gamma;\Phi,\phi \vdash \mathbb{1} : \mathcal{F}$, $\Gamma;\Phi,\phi \vdash \mathbb{0} : \mathcal{F}$. We can immediately form the required judgments.

- Case $\Gamma;\Phi,\phi \vdash \phi\overline{\tau} : \mathcal{F}$. We must derive $\Gamma,\psi;\Phi \vdash (\phi\overline{\tau})[\phi :== \psi] : \mathcal{F}$. The induction hypothesis gives that $\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi] : \mathcal{F}$ for each τ.

$$\dfrac{\dfrac{\psi \in \{\Gamma,\psi\} \cup \Phi : \mathcal{F} \qquad \overline{\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi] : \mathcal{F}}}{\Gamma,\psi;\Phi \vdash \psi\overline{\tau[\phi :== \psi]} : \mathcal{F}}}{\Gamma,\psi;\Phi \vdash (\phi\overline{\tau})[\phi :== \psi] : \mathcal{F}}$$

  The case for $\Gamma;\Phi,\phi' \vdash \phi\overline{\tau} : \mathcal{F}$, i.e., the case in which the variable being "demoted" only appears in the arguments, works by the same induction.

- Case $\Gamma;\Phi,\phi \vdash (\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau} : \mathcal{F}$. We must derive $\Gamma,\psi;\Phi \vdash ((\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau})[\phi :== \psi] : \mathcal{F}$. The induction hypothesis gives that $\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi] : \mathcal{F}$ for each τ and $\Gamma,\psi;\Phi,\overline{\alpha},\phi' \vdash H[\phi :== \psi] : \mathcal{F}$.

$$\dfrac{\dfrac{\Gamma,\psi;\Phi,\overline{\alpha},\phi' \vdash H[\phi :== \psi] : \mathcal{F} \qquad \overline{\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi] : \mathcal{F}}}{\Gamma,\psi;\Phi \vdash (\mu\phi'.\lambda\overline{\alpha}.H[\phi :== \psi])\overline{\tau[\phi :== \psi]} : \mathcal{F}}}{\Gamma,\psi;\Phi \vdash ((\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau})[\phi :== \psi] : \mathcal{F}}$$

- Case $\Gamma;\Phi,\phi \vdash \sigma \times \tau : \mathcal{F}$. We must derive $\Gamma,\psi;\Phi \vdash (\sigma \times \tau)[\phi :== \psi] : \mathcal{F}$. The induction hypothesis gives that $\Gamma,\psi;\Phi \vdash \sigma[\phi :== \psi] : \mathcal{F}$ and $\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi] : \mathcal{F}$.

$$\dfrac{\dfrac{\Gamma,\psi;\Phi \vdash \sigma[\phi :== \psi] : \mathcal{F} \qquad \Gamma,\psi;\Phi \vdash \tau[\phi :== \psi] : \mathcal{F}}{\Gamma,\psi;\Phi \vdash \sigma[\phi :== \psi] \times \tau[\phi :== \psi] : \mathcal{F}}}{\Gamma,\psi;\Phi \vdash (\sigma \times \tau)[\phi :== \psi] : \mathcal{F}}$$

  The case for $\sigma + \tau$ is analogous.

□

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Note that the next two theorems are proven by simultaneous induction. We are actually only interested in using Theorem 2, but in order to prove the μ case for this theorem, we need Theorem 3 to show that two functors have equal actions on morphisms.

Theorem 2. *If* $\Gamma; \Phi, \phi \vdash \tau : \mathcal{F}$, $\rho : \mathsf{SetEnv}$, *and* $\rho\phi = \rho\psi$, *then*

$$[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho$$

Analogously, if $\rho : \mathsf{RelEnv}$, *and* $\rho\phi = \rho\psi$, *then*

$$[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Rel}}\rho = [\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Rel}}\rho$$

Proof. We prove the case for Set by induction on the structure of τ. The case for Rel proceeds analogously.

- There is nothing to prove for types in $\mathcal{T}$ because their functorial contexts must be empty.
- Case $\Gamma; \Phi, \alpha \vdash \alpha : \mathcal{F}$. Given that $\rho\alpha = \rho\beta$,

$$\begin{aligned} & [\![\Gamma; \Phi, \alpha \vdash \alpha]\!]^{\mathsf{Set}}\rho \\ =\ & \rho\alpha \\ =\ & \rho\beta \\ =\ & [\![\Gamma, \beta; \Phi \vdash \beta]\!]^{\mathsf{Set}}\rho \end{aligned}$$

- Case $\Gamma; \Phi, \phi \vdash \mathbb{1} : \mathcal{F}$, $\Gamma; \Phi, \phi \vdash \mathbb{0} : \mathcal{F}$.

$$\begin{aligned} & [\![\Gamma; \Phi, \phi \vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho \\ =\ & 1 \\ =\ & [\![\Gamma, \psi; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}}\rho \end{aligned}$$

  Similarly for $\mathbb{0}$.
- Case $\Gamma; \Phi, \phi \vdash \phi\overline{\tau} : \mathcal{F}$. The induction hypothesis gives that

$$[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho$$

  for each τ.

$$\begin{aligned} & [\![\Gamma; \Phi, \phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}}\rho \\ =\ & (\rho\phi)\overline{[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}}\rho} \\ =\ & (\rho\phi)\overline{[\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho} \\ =\ & (\rho\psi)\overline{[\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho} \\ =\ & [\![\Gamma, \psi; \Phi \vdash \psi\overline{\tau[\phi :== \psi]}]\!]^{\mathsf{Set}}\rho \\ =\ & [\![\Gamma, \psi; \Phi \vdash (\phi\overline{\tau})[\phi :== \psi]]\!]^{\mathsf{Set}}\rho \end{aligned}$$

  The first and fifth equalities above are by Definition ??. The fourth equality is by equality of the functors $\rho\phi$ and $\rho\psi$.

- Case $\Gamma; \Phi, \phi \vdash (\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau} : \mathcal{F}$. The induction hypothesis gives that

$$[\![\Gamma; \Phi, \phi', \overline{\alpha}, \phi \vdash H]\!]^{\mathsf{Set}}\rho = [\![\Gamma, \psi; \Phi, \phi', \overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}}\rho$$

  and

$$[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}}\rho = [\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}}\rho$$

  for each τ.

$$\begin{aligned}
&[\![\Gamma;\Phi,\phi\vdash(\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}}\rho\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]))\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho}\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]))\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho}\\
&=(\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}])\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho}\\
&=[\![\Gamma,\psi;\Phi\vdash(\mu\phi'.\lambda\overline{\alpha}.H[\phi:==\psi])\overline{\tau[\phi:==\psi]}]\!]^{\mathsf{Set}}\rho\\
&=[\![\Gamma,\psi;\Phi\vdash((\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau})[\phi:==\psi]]\!]^{\mathsf{Set}}\rho
\end{aligned}$$

The first and fifth equalities are by Definition ??. The second equality follows from the following equality:

$$\begin{aligned}
&\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi\vdash H]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]\\
&=\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha}\vdash H[\phi:==\psi]]\!]^{\mathsf{Set}}\rho[\phi':=F][\overline{\alpha:=A}]
\end{aligned}$$

These two functors have the same action on objects by the induction hypothesis on H, and the fact that the extended environment $\rho[\phi':=F][\overline{\alpha:=A}]$ satisfies the required hypothesis. These two functors have the same action on morphisms by the induction hypothesis on H from Theorem 3. Thus they are the same functor with the same fixed point.

- Case $\Gamma;\Phi,\phi\vdash\sigma\times\tau:\mathcal{F}$. The induction hypothesis gives that

$$[\![\Gamma;\Phi,\phi\vdash\sigma]\!]^{\mathsf{Set}}\rho=[\![\Gamma,\psi;\Phi\vdash\sigma[\phi:==\psi]]\!]^{\mathsf{Set}}\rho$$

and

$$[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho=[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho$$

$$\begin{aligned}
&[\![\Gamma;\Phi,\phi\vdash\sigma\times\tau]\!]^{\mathsf{Set}}\rho\\
&=[\![\Gamma;\Phi,\phi\vdash\sigma]\!]^{\mathsf{Set}}\rho\times[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho\\
&=[\![\Gamma,\psi;\Phi\vdash\sigma[\phi:==\psi]]\!]^{\mathsf{Set}}\rho\times[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho\\
&=[\![\Gamma,\psi;\Phi\vdash\sigma[\phi:==\psi]\times\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho\\
&=[\![\Gamma,\psi;\Phi\vdash(\sigma\times\tau)[\phi:==\psi]]\!]^{\mathsf{Set}}\rho
\end{aligned}$$

The case for $\sigma+\tau$ is analogous.

□

THEOREM 3. *If $\Gamma;\Phi,\phi\vdash\tau:\mathcal{F}$, and if $f:\rho\to\rho'$, is a morphism of set environments such that $\rho\phi=\rho\psi=\rho'\phi=\rho'\psi$, and $f\phi=f\psi=id_{\rho\phi}$, then*

$$[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}f=[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}f$$

Analogously, if $f:\rho\to\rho'$, is a morphism of relation environments such that $\rho\phi=\rho\psi=\rho'\phi=\rho'\psi$, and $f\phi=f\psi=id_{\rho\phi}$, then

$$[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Rel}}f=[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Rel}}f$$

PROOF. We prove the case for Set by induction on the structure of τ. The case for Rel proceeds analogously.

- There is nothing to prove for types in $\mathcal{T}$ because their functorial contexts must be empty.

- Case $\Gamma; \Phi, \alpha \vdash \alpha : \mathcal{F}$. Given that $\rho\alpha = \rho\beta$,

$$\begin{aligned}
& [\![\Gamma; \Phi, \alpha \vdash \alpha]\!]^{\mathsf{Set}} f \\
=\ & id_{\rho\alpha} \\
=\ & id_{\rho\beta} \\
=\ & [\![\Gamma, \beta; \Phi \vdash \beta]\!]^{\mathsf{Set}} f
\end{aligned}$$

- Case $\Gamma; \Phi, \phi \vdash \mathbb{1} : \mathcal{F}$, $\Gamma; \Phi, \phi \vdash \mathbb{0} : \mathcal{F}$.

$$\begin{aligned}
& [\![\Gamma; \Phi, \phi \vdash \mathbb{1}]\!]^{\mathsf{Set}} f \\
=\ & id_1 \\
=\ & [\![\Gamma, \psi; \Phi \vdash \mathbb{1}]\!]^{\mathsf{Set}} f
\end{aligned}$$

  Similarly for $\mathbb{0}$.

- Case $\Gamma; \Phi, \phi \vdash \phi\overline{\tau} : \mathcal{F}$. The induction hypothesis gives that

$$[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}} f = [\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} f$$

  for each τ.

$$\begin{aligned}
& [\![\Gamma; \Phi, \phi \vdash \phi\overline{\tau}]\!]^{\mathsf{Set}} f \\
=\ & (f\phi)_{\overline{[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}} \rho'}} \circ (\rho\phi)\overline{[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}} f} \\
=\ & (id_{\rho\phi})_{\overline{[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}} \rho'}} \circ (\rho\phi)\overline{[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}} f} \\
=\ & (\rho\phi)\overline{[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}} f} \\
=\ & (\rho\psi)\overline{[\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} f} \\
=\ & (id_{\rho\psi})_{\overline{[\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} \rho'}} \circ (\rho\psi)\overline{[\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} f} \\
=\ & (f\psi)_{\overline{[\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} \rho'}} \circ (\rho\psi)\overline{[\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} f} \\
=\ & [\![\Gamma, \psi; \Phi \vdash \psi\overline{\tau[\phi :== \psi]}]\!]^{\mathsf{Set}} f \\
=\ & [\![\Gamma, \psi; \Phi \vdash (\phi\overline{\tau})[\phi :== \psi]]\!]^{\mathsf{Set}} f
\end{aligned}$$

- Case $\Gamma; \Phi, \phi \vdash (\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau} : \mathcal{F}$. The induction hypothesis gives that

$$[\![\Gamma; \Phi, \phi', \overline{\alpha}, \phi \vdash H]\!]^{\mathsf{Set}} f = [\![\Gamma, \psi; \Phi, \phi', \overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}} f$$

  and

$$[\![\Gamma; \Phi, \phi \vdash \tau]\!]^{\mathsf{Set}} f = [\![\Gamma, \psi; \Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} f$$

for each τ.

$$
\begin{aligned}
&[\![\Gamma;\Phi,\phi \vdash (\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau}]\!]^{\mathsf{Set}} f\\
&= (\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}} f[\phi' := id_F][\overline{\alpha := id_A}]))_{\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}}\rho'}}\\
&\quad \circ (\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}} \rho[\phi' := F][\overline{\alpha := A}]))\overline{[\![\Gamma;\Phi,\phi\vdash\tau]\!]^{\mathsf{Set}} f}\\
&= (\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}} f[\phi' := id_F][\overline{\alpha := id_A}]))_{\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho'}}\\
&\quad \circ (\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}} \rho[\phi' := F][\overline{\alpha := A}]))\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi :== \psi]]\!]^{\mathsf{Set}} f}\\
&= (\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}} f[\phi' := id_F][\overline{\alpha := id_A}]))_{\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho'}}\\
&\quad \circ (\mu(\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}} \rho[\phi' := F][\overline{\alpha := A}]))\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi :== \psi]]\!]^{\mathsf{Set}} f}\\
&= (\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}} f[\phi' := id_F][\overline{\alpha := id_A}]))_{\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi:==\psi]]\!]^{\mathsf{Set}}\rho'}}\\
&\quad \circ (\mu(\lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}} \rho[\phi' := F][\overline{\alpha := A}]))\overline{[\![\Gamma,\psi;\Phi\vdash\tau[\phi :== \psi]]\!]^{\mathsf{Set}} f}\\
&= [\![\Gamma,\psi;\Phi \vdash (\mu\phi'.\lambda\overline{\alpha}.H[\phi :== \psi])\overline{\tau[\phi :== \psi]}]\!]^{\mathsf{Set}} f\\
&= [\![\Gamma,\psi;\Phi \vdash ((\mu\phi'.\lambda\overline{\alpha}.H)\overline{\tau})[\phi :== \psi]]\!]^{\mathsf{Set}} f
\end{aligned}
$$

The first and fifth equalities are by Definition **??**. The third equality is by the equality of the arguments to the first μ operator:

$$
\begin{aligned}
&\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}} f[\phi' := id_F][\overline{\alpha := id_A}]\\
&\quad = \lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}} f[\phi' := id_F][\overline{\alpha := id_A}]
\end{aligned}
$$

By the induction hypothesis on H and the fact that the morphism $f[\phi' := id_F][\overline{\alpha := id_A}] : \rho[\phi' := F][\overline{\alpha := A}] \to \rho'[\phi' := F][\overline{\alpha := A}]$ still satisfies the required hypotheses. The fourth equality is by the equality of the arguments to the second μ operator:

$$
\begin{aligned}
&\lambda F.\lambda\overline{A}.[\![\Gamma;\Phi,\phi',\overline{\alpha},\phi \vdash H]\!]^{\mathsf{Set}} \rho[\phi' := F][\overline{\alpha := A}]\\
&\quad = \lambda F.\lambda\overline{A}.[\![\Gamma,\psi;\Phi,\phi',\overline{\alpha} \vdash H[\phi :== \psi]]\!]^{\mathsf{Set}} \rho[\phi' := F][\overline{\alpha := A}]
\end{aligned}
$$

These two functors have the same action on objects by the induction hypothesis on H from Theorem 2, and they have the same action on morphisms by the induction hypothesis on H from this theorem. Thus they are the same functor with the same fixed point.

- Case $\Gamma;\Phi,\phi \vdash \sigma \times \tau : \mathcal{F}$. The induction hypothesis gives that

$$[\![\Gamma;\Phi,\phi \vdash \sigma]\!]^{\mathsf{Set}} f = [\![\Gamma,\psi;\Phi \vdash \sigma[\phi :== \psi]]\!]^{\mathsf{Set}} f$$

and

$$[\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}} f = [\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} f$$

$$
\begin{aligned}
&[\![\Gamma;\Phi,\phi \vdash \sigma \times \tau]\!]^{\mathsf{Set}} f\\
&= [\![\Gamma;\Phi,\phi \vdash \sigma]\!]^{\mathsf{Set}} f \times [\![\Gamma;\Phi,\phi \vdash \tau]\!]^{\mathsf{Set}} f\\
&= [\![\Gamma,\psi;\Phi \vdash \sigma[\phi :== \psi]]\!]^{\mathsf{Set}} f \times [\![\Gamma,\psi;\Phi \vdash \tau[\phi :== \psi]]\!]^{\mathsf{Set}} f\\
&= [\![\Gamma,\psi;\Phi \vdash \sigma[\phi :== \psi] \times \tau[\phi :== \psi]]\!]^{\mathsf{Set}} f\\
&= [\![\Gamma,\psi;\Phi \vdash (\sigma \times \tau)[\phi :== \psi]]\!]^{\mathsf{Set}} f
\end{aligned}
$$

The case for $\sigma + \tau$ is analogous.

□