Practical Parametricity for GADTs

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Abstract goes here

1 TYPING RULES

Definition 1. *The formation rules for the set* $\mathcal{F} \subseteq \bigcup_{V \subseteq \mathbb{T}, P \subseteq \mathbb{F}} \mathcal{F}^P(V)$ *of* well-formed types *are*

$$\frac{}{\Gamma;\Phi \vdash \mathbb{0}} \qquad \frac{}{\Gamma;\Phi \vdash \mathbb{1}}$$

$$\frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G}{\Gamma;\Phi \vdash F + G} \qquad \frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G}{\Gamma;\Phi \vdash F \times G}$$

$$\frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G}{\Gamma;\emptyset \vdash \mathsf{Nat}^{\Phi} F\, G}$$

$$\frac{\phi^k \in \Gamma \cup \Phi \qquad \overline{\Gamma;\Phi \vdash F}}{\Gamma;\Phi \vdash \phi^k \overline{F}}$$

$$\frac{\Gamma;\overline{\gamma^0},\overline{\alpha^0},\phi^k \vdash F \qquad \overline{\Gamma;\Phi,\overline{\gamma^0} \vdash G}}{\Gamma;\Phi,\overline{\gamma^0} \vdash (\mu\phi^k.\lambda\overline{\alpha^0}.\,F)\,\overline{G}}$$

$$\frac{\Gamma;\Phi,\overline{\alpha}^0 \vdash F \qquad \overline{\Gamma;\overline{\alpha}^0 \vdash K} \qquad \overline{\Gamma;\Phi \vdash A}}{\Gamma;\Phi \vdash (\mathsf{Lan}^{\overline{\alpha}^0}_{\overline{K}} F)\,\overline{A}}$$

Definition 2. *Let* Δ *be a term context for* Γ *and* Φ*. The formation rules for the set of* well-formed terms over Δ *are*

$$\frac{\Gamma;\Phi \vdash F}{\Gamma;\Phi \,|\, \Delta, x : F \vdash x : F} \qquad \frac{\Gamma;\Phi \,|\, \Delta \vdash t : \mathbb{0} \qquad \Gamma;\Phi \vdash F}{\Gamma;\Phi \,|\, \Delta \vdash \bot_F t : F} \qquad \frac{}{\Gamma;\Phi \,|\, \Delta \vdash \top : \mathbb{1}}$$

$$\frac{\Gamma;\Phi \,|\, \Delta \vdash s : F}{\Gamma;\Phi \,|\, \Delta \vdash \mathsf{inl}\, s : F + G} \qquad \frac{\Gamma;\Phi \,|\, \Delta \vdash t : G}{\Gamma;\Phi \,|\, \Delta \vdash \mathsf{inr}\, t : F + G}$$

$$\frac{\Gamma;\Phi \vdash F, G \qquad \Gamma;\Phi \,|\, \Delta \vdash t : F + G \qquad \Gamma;\Phi \,|\, \Delta, x : F \vdash l : K \qquad \Gamma;\Phi \,|\, \Delta, y : G \vdash r : K}{\Gamma;\Phi \mid \Delta \vdash \mathsf{case}\, t\, \mathsf{of}\, \{x \mapsto l;\ y \mapsto r\} : K}$$

$$\frac{\Gamma;\Phi \,|\, \Delta \vdash s : F \qquad \Gamma;\Phi \,|\, \Delta \vdash t : G}{\Gamma;\Phi \,|\, \Delta \vdash (s, t) : F \times G} \qquad \frac{\Gamma;\Phi \,|\, \Delta \vdash t : F \times G}{\Gamma;\Phi \,|\, \Delta \vdash \pi_1 t : F} \qquad \frac{\Gamma;\Phi \,|\, \Delta \vdash t : F \times G}{\Gamma;\Phi \,|\, \Delta \vdash \pi_2 t : G}$$

2020. 2475-1421/2020/1-ART1 $15.00
https://doi.org/

$$\frac{\Gamma;\Phi \vdash F \qquad \Gamma;\Phi \vdash G \qquad \Gamma;\Phi \mid \Delta, x : F \vdash t : G}{\Gamma;\emptyset \mid \Delta \vdash L_{\Phi} x.t : \mathsf{Nat}^{\Phi}\, F\, G}$$

$$\frac{\overline{\Gamma;\Phi,\overline{\beta} \vdash K} \qquad \Gamma;\emptyset \mid \Delta \vdash t : \mathsf{Nat}^{\overline{\psi}}\, F\, G \qquad \Gamma;\Phi \mid \Delta \vdash s : F[\overline{\psi :=_{\overline{\beta}} K}]}{\Gamma;\Phi \mid \Delta \vdash t_{\overline{K}} s : G[\overline{\psi :=_{\overline{\beta}} K}]}$$

$$\frac{\Gamma;\Phi,\overline{\phi} \vdash H \qquad \overline{\Gamma;\Phi,\overline{\beta} \vdash F} \qquad \overline{\Gamma;\Phi,\overline{\beta} \vdash G}}{\Gamma;\emptyset \mid \emptyset \vdash \mathsf{map}_H^{\overline{F},\overline{G}} : \mathsf{Nat}^{\emptyset}\, (\overline{\mathsf{Nat}^{\Phi,\overline{\beta}}\, F\, G})\, (\mathsf{Nat}^{\Phi}\, H[\overline{\phi :=_{\overline{\beta}} F}]\; H[\overline{\phi :=_{\overline{\beta}} G}])}$$

$$\frac{\Gamma;\Phi,\phi,\overline{\alpha} \vdash H}{\Gamma;\emptyset \mid \emptyset \vdash \mathsf{in}_H : \mathsf{Nat}^{\Phi,\overline{\beta}} H[\phi :=_{\overline{\beta}} (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}][\overline{\alpha := \beta}]\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}}$$

$$\frac{\Gamma;\phi,\Phi,\overline{\alpha} \vdash H \qquad \Gamma;\Phi,\overline{\beta} \vdash F}{\Gamma;\emptyset \mid \emptyset \vdash \mathsf{fold}_H^{F} : \mathsf{Nat}^{\emptyset}\, (\mathsf{Nat}^{\Phi,\overline{\beta}}\, H[\phi :=_{\overline{\beta}} F][\overline{\alpha := \beta}]\, F)\, (\mathsf{Nat}^{\Phi,\overline{\beta}}\, (\mu\phi.\lambda\overline{\alpha}.H)\overline{\beta}\, F)}$$

$$\frac{\Gamma;\Phi,\overline{\alpha} \vdash F \qquad \overline{\Gamma;\overline{\alpha} \vdash K} \qquad \overline{\Gamma;\Phi \vdash A} \qquad \Gamma;\Phi \mid \Delta \vdash t : F[\overline{\alpha := A}]}{\Gamma;\Phi \mid \Delta \vdash \int_{K,F} t : (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}}\, F)\, \overline{K[\overline{\alpha := A}]}}$$

$$\frac{\Gamma;\emptyset \mid \Delta \vdash \eta : \mathsf{Nat}^{\Phi,\overline{\alpha}}\, F\, G[\overline{\beta := K}] \qquad \overline{\Gamma;\Phi \vdash A} \qquad \Gamma;\Phi \mid \Delta \vdash t : (\mathsf{Lan}^{\overline{\alpha}}_{\overline{K}} F)\, \overline{A}}{\Gamma;\Phi \mid \Delta \vdash \partial_F^{G,\overline{K}} \eta\, t : G[\overline{\beta := A}]}$$

Things to check:

- Are these Nat-types ωCPOs? An argument similar to the one in the POPL submission should hold.
- Does IEL hold if we allow functoriality in variables of arity > 0?
- Does AT hold if we allow functoriality in variables of arity > 0?

Sum and product intro and elim rules should be annotated with constituent types for consistency.

We should have a computation rule along the lines of: If $\eta : \mathsf{Nat}^{\overline{\alpha}} F\, G[\overline{\beta := K}]$ then

$$\begin{array}{rl} & (\partial_F^{G,\overline{K}} \eta)_{\overline{K[\overline{\alpha := A}]}} \circ (\int_{K,F})_{\overline{A}} \;\longrightarrow\; \eta_{\overline{A}} \\ : & F[\overline{\alpha := A}] \to G[\overline{\beta := \overline{K[\overline{\alpha := A}]}}] \\ = & F[\overline{\alpha := A}] \to G[\overline{\beta := K}][\overline{\alpha := A}] \end{array}$$

Some questions/issues:

- Can we write zipBush and appendBush with ∂ and $\int$? We could already represent the uncurried type of appendBush (although not its curried type), but couldn't recurse over both input bushes because folds take natural transformations as inputs.
- More generally, how do we compute with ∂ and $\int$? Can we use the colimit formulation of Lans (see Lemma 6.3.7 of [Riehl 2016]) to get a handle on this?

- What is the connection between exponentials and natural transformations? (Should we assume only small objects are exponentiable?) Do we want the former or the latter for computational purposes? (I suspect the latter.)
  [From nlab: In a functor category D^C , a natural transformation $\alpha : F \to G$ is exponentiable if (though probably not "only if") it is cartesian and each component $\alpha_c : Fc \to Gc$ is exponentiable in D. Given $H \to F$ we define $(\Pi_\alpha H)c = \Pi_{\alpha_c}(Hc)$; then for $u : c \to c'$ to obtain a map $\Pi_{\alpha_c}(Hc) \to \Pi_{\alpha_{c'}}(Hc')$ we need a map $\alpha^*_{c'}(\Pi_{\alpha_c}(Hc)) \to Hc'$. But since α is cartesian, $\alpha^*_{c'}(\Pi_{\alpha_c}(Hc)) \cong \alpha^*_c(\Pi_{\alpha_c}(Hc))$, so we have the counit $\alpha^*_c(\Pi_{\alpha_c}(Hc)) \to Hc$ that we can compose with Hu.]
- After we understand what we can do with Lans and folds on GADTs we might want to try to extend calculus with term-level fixpoints. This would give a categorical analogue for GADTs of [Pitts 1998, 2000] for ADTs. Would it also more accurately reflect how GADTs are used in practice, or are functions over GADTs usually folds? Investigate applications in the literature and/or in implementations.
- What kind of category can handle Lan types? Can ωCPO? This category is locally ω_1-presentable (but not locally finitely presentable); see 1.18(2) of [Adámek and Rosický 1994]). The ω_1-presentable objects are the ωCPOs cardinality less than ω_1, i.e., the countable ωCPOs.
- ωCPO is a natural choice for modeling general recursion. We know $(Lan^\gamma_C \mathbb{1})D$ is $C \to D$ for any closed type C. (Also for select classes of open types?) So can model $\mathsf{Nat} \to \gamma$. But the functor $NX = \mathsf{Nat} \to X$ isn't ω-cocontinuous. It also doesn't preserve ω_1-presentable objects, i.e., countable ωCPOs since $\mathsf{Nat} \to \mathsf{Nat}$ is not countable. So we cannot have a functor like N as the subscript to Lan and expect the resulting Lan to be ω_1-cocontinuous.
- What functors can be subscripts to Lan and produce ω_1-cocontinuous functors? We can use functors that preserve presentable objects by theorem in [Johann and Polonsky 2019], and possibly others as well. These include polynomial functors, ADTs and nested types seen as functors, certain (which?) GADTs seen as functors? How big can GADTs get?

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