Homework 2 - Berkeley STAT 157

10/18

Handout 1/29/2019, due 2/5/2019 by 4pm in Git by committing to your repository.

In [3]:
```
from mxnet import nd, autograd, gluon
import numpy as np
import matplotlib.pyplot as plt
```

1. Multinomial Sampling

Implement a sampler from a discrete distribution from scratch, mimicking the function `mxnet.ndarray.random.multinomial`. Its arguments should be a vector of probabilities p. You can assume that the probabilities are normalized, i.e. tha they sum up to 1. Make the call signature as follows:

```
samples = sampler(probs, shape)

probs   : An ndarray vector of size n of nonnegative numbers summing up to 1
shape   : A list of dimensions for the output
samples : Samples from probs with shape matching shape
```

Hints:

1. Use `mxnet.ndarray.random.uniform` to get a sample from $U[0, 1]$.
2. You can simplify things for `probs` by computing the cumulative sum over `probs`.

In [93]:
```
def sampler(probs, shape):
    cumsum = 0
    arr = [0]
    for i in range(len(probs)):
        cumsum = cumsum + probs[i]
        arr = np.append(arr, cumsum)
    sample = nd.random.uniform()
    def search(probs, samples):
        low = 0
        high = len(probs)-1
        mid = (low + high) // 2
        while not (probs[mid] <= sample and probs[mid + 1] > sample):
            if probs[mid] < sample:
                low = mid
            elif probs[mid] > sample:
                high = mid
            elif probs[mid] == sample:
                return mid
            mid = (low + high) // 2
        return mid
    ret = nd.zeros(shape)
    for i in range(shape[0]):
        for j in range(shape[1]):
            sample = nd.random.uniform()
            ret[i, j] = search(arr, sample)
    return ret
# a simple test
sampler(nd.array([0.2, 0.3, 0.5]), (2,3))
```

3/4

Out[93]:
```
[[1. 2. 2.]
 [1. 1. 1.]]
<NDArray 2x3 @cpu(0)>
```

2. Central Limit Theorem

Let's explore the Central Limit Theorem when applied to text processing.

- Download https://www.gutenberg.org/ebooks/84 (https://www.gutenberg.org/files/84/84-0.txt) from Project Gutenberg
- Remove punctuation, uppercase / lowercase, and split the text up into individual tokens (words).
- For the words `a`, `and`, `the`, `i`, `is` compute their respective counts as the book progresses, i.e.

$$n_{\text{the}}[i] = \sum_{j=1}^{i} \{w_j = \text{the}\}$$

- Plot the proportions $n_{\text{word}}[i]/i$ over the document in one plot.
- Find an envelope of the shape $O(1/\sqrt{i})$ for each of these five words.
- Why can we **not** apply the Central Limit Theorem directly?
- How would we have to change the text for it to apply?
- Why does it still work quite well?

1/4 how&why?

In [5]:
```
filename = gluon.utils.download('https://www.gutenberg.org/files/84/84-0.txt')

book = []
def clean(word):
    cleaned_word = ""
    for char in word:
        if char.isalpha():
            cleaned_word = cleaned_word + str(char)
    return cleaned_word.lower()

with open(filename) as f:
    for line in f:
        for word in line.split():
            cleaned_word = clean(word)
            if len(cleaned_word) > 0:
                book = np.append(book, cleaned_word)

print(book[0:100])
## Add your codes here
```

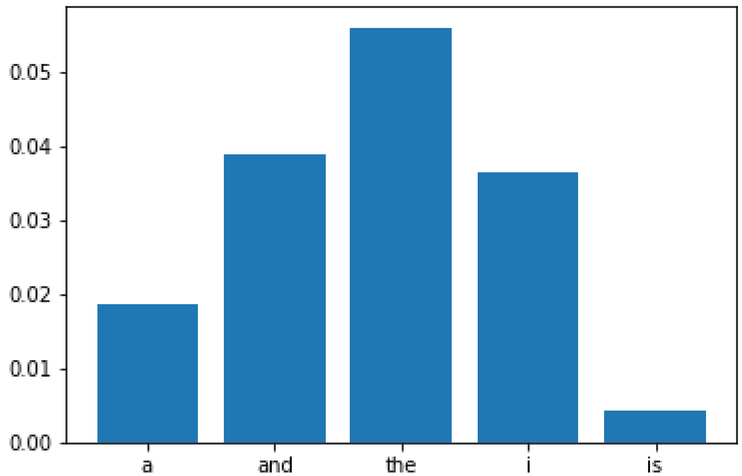
```
['project' 'gutenbergs' 'frankenstein' 'by' 'mary' 'wollstonecraft'
 'godwin' 'shelley' 'this' 'ebook' 'is' 'for' 'the' 'use' 'of' 'anyone'
 'anywhere' 'at' 'no' 'cost' 'and' 'with' 'almost' 'no' 'restrictions'
 'whatsoever' 'you' 'may' 'copy' 'it' 'give' 'it' 'away' 'or' 'reuse' 'it'
 'under' 'the' 'terms' 'of' 'the' 'project' 'gutenberg' 'license'
 'included' 'with' 'this' 'ebook' 'or' 'online' 'at' 'wwwgutenbergnet'
 'title' 'frankenstein' 'or' 'the' 'modern' 'prometheus' 'author' 'mary'
 'wollstonecraft' 'godwin' 'shelley' 'release' 'date' 'june' 'ebook'
 'last' 'updated' 'january' 'language' 'english' 'character' 'set'
 'encoding' 'utf' 'start' 'of' 'this' 'project' 'gutenberg' 'ebook'
 'frankenstein' 'produced' 'by' 'judith' 'boss' 'christy' 'phillips'
 'lynn' 'hanninen' 'and' 'david' 'meltzer' 'html' 'version' 'by' 'al'
 'haines' 'further']
```

In [17]:
```
counts = dict({"a": 0, "and": 0, "the": 0, "i": 0, "is": 0})
for w in book:
    if w in counts:
        counts[w] = counts[w] + 1
for w in counts:
    counts[w] = counts[w] / len(book)
names = list(counts.keys())
values = list(counts.values())

plt.bar(names, values)
```

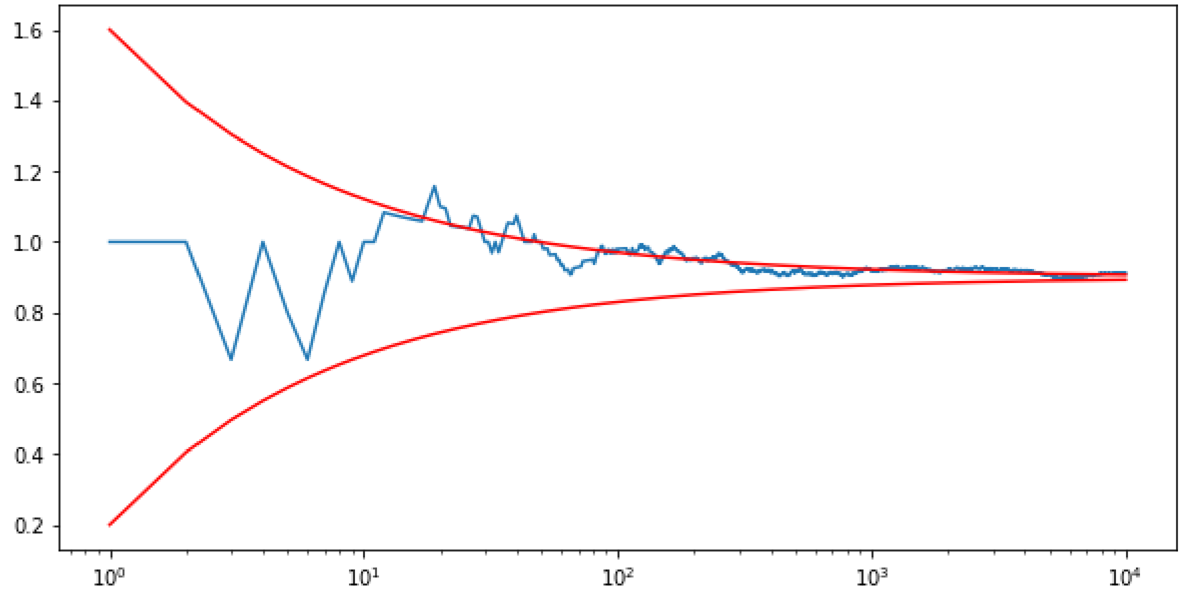
2/4 one plot line graph?

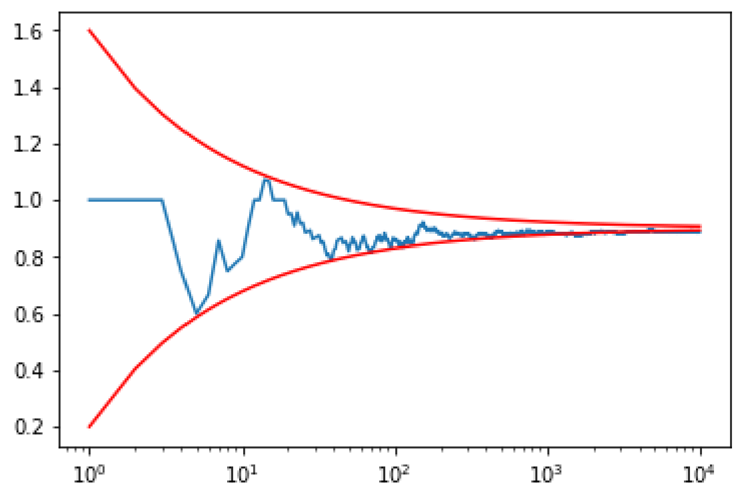
Out[17]: <BarContainer object of 5 artists>

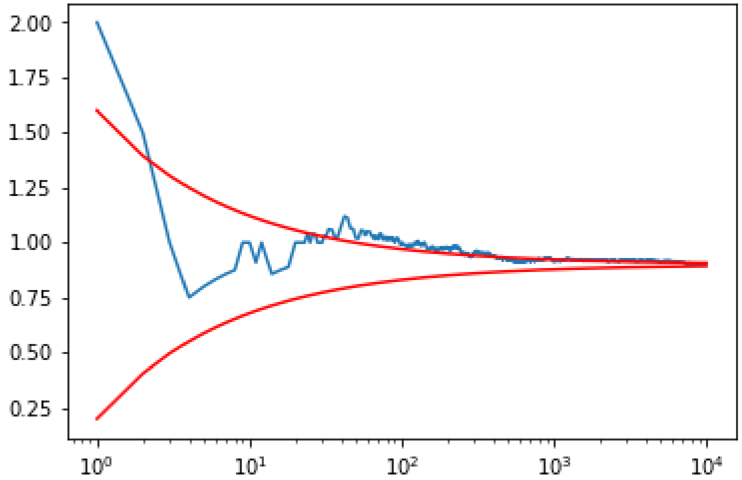


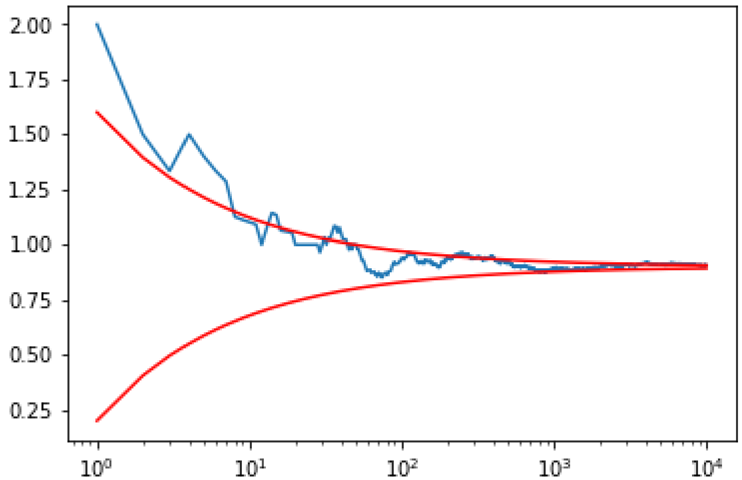
In [41]:
```
tmp = np.random.uniform(size=(10000,10))
x = 1.0 * (tmp > 0.3) + 1.0 * (tmp > 0.8)
mean = 1 * 0.5 + 2 * 0.2
variance = 1 * 0.5 + 4 * 0.2 - mean**2
print('mean {}, variance {}'.format(mean, variance))
# cumulative sum and normalization
y = np.arange(1,10001).reshape(10000,1)
z = np.cumsum(x,axis=0) / y
plt.figure(figsize=(10,5))
for i in range(10):
    plt.semilogx(y,z[:,i])
    plt.semilogx(y,(variance**0.5) * np.power(y,-0.5) + mean,'r')
    plt.semilogx(y,-(variance**0.5) * np.power(y,-0.5) + mean,'r')
    plt.show()
```

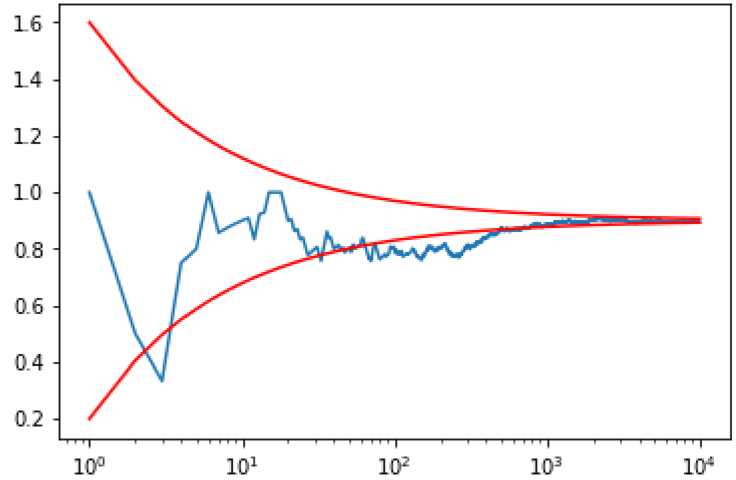
```
mean 0.9, variance 0.49
```

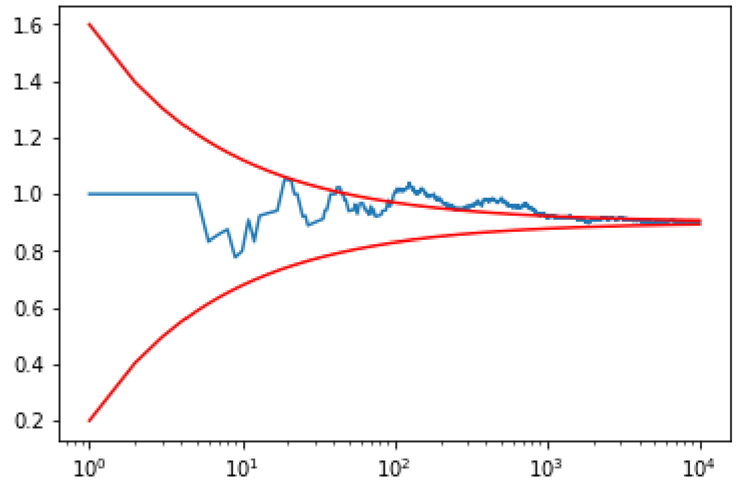


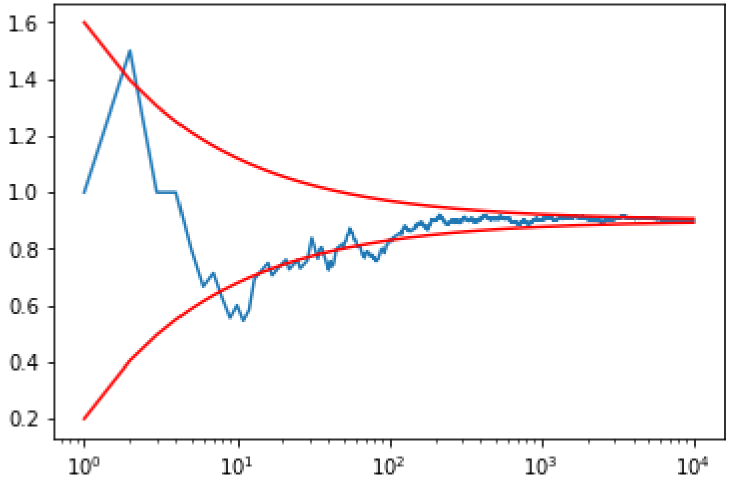


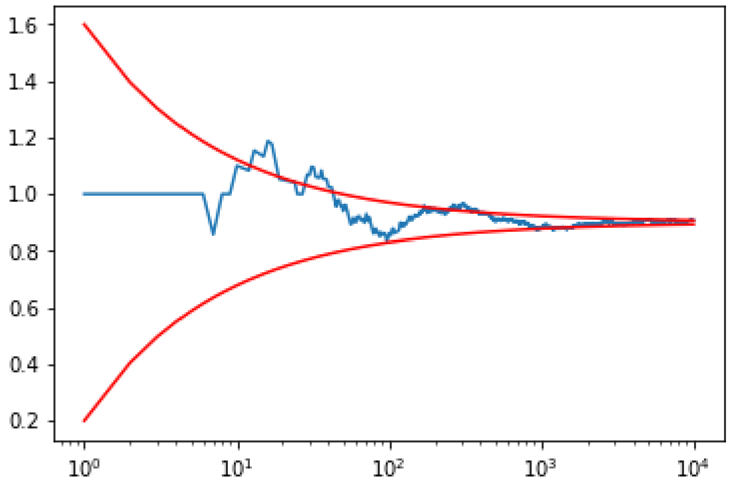

2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
10^0
10^1
10^2
10^3
10^4

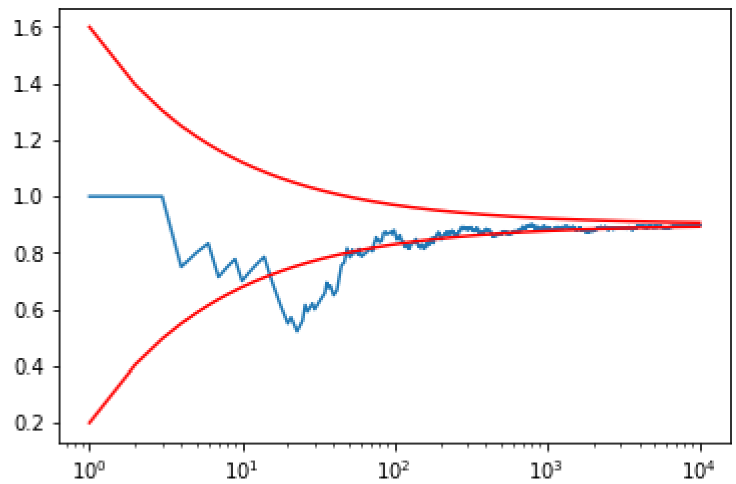

2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
10^0
10^1
10^2
10^3
10^4

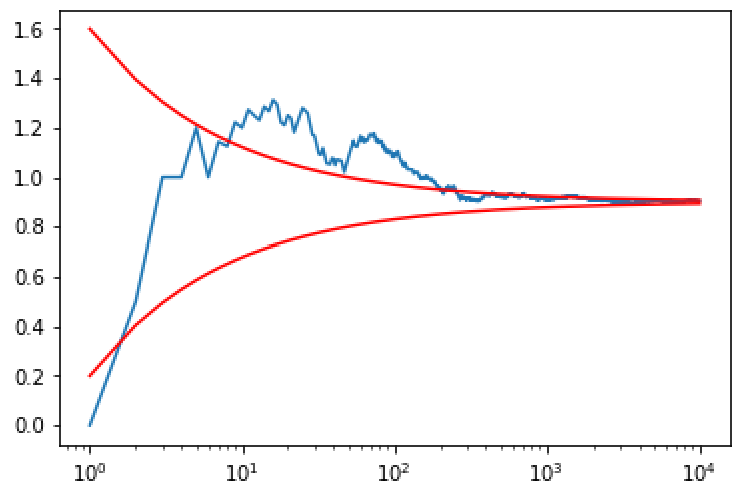

1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
10^0
10^1
10^2
10^3
10^4


1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
10^0
10^1
10^2
10^3
10^4


1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
10^0
10^1
10^2
10^3
10^4


1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
10^0
10^1
10^2
10^3
10^4





We can't use CLT because the test is sequenced and there are patterns in word usage (word distribution is not iid). We could randomize the text for it to follow the CLT. The CLT still works pretty well because we have a large sample size.

3. Denominator-layout notation

We used the numerator-layout notation for matrix calculus in class, now let's examine the denominator-layout notation.

Given $x, y \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, we have

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}, \quad \frac{\partial \mathbf{y}}{\partial x} = \left[\frac{\partial y_1}{\partial x}, \frac{\partial y_2}{\partial x}, \ldots, \frac{\partial y_m}{\partial x} \right]$$

and

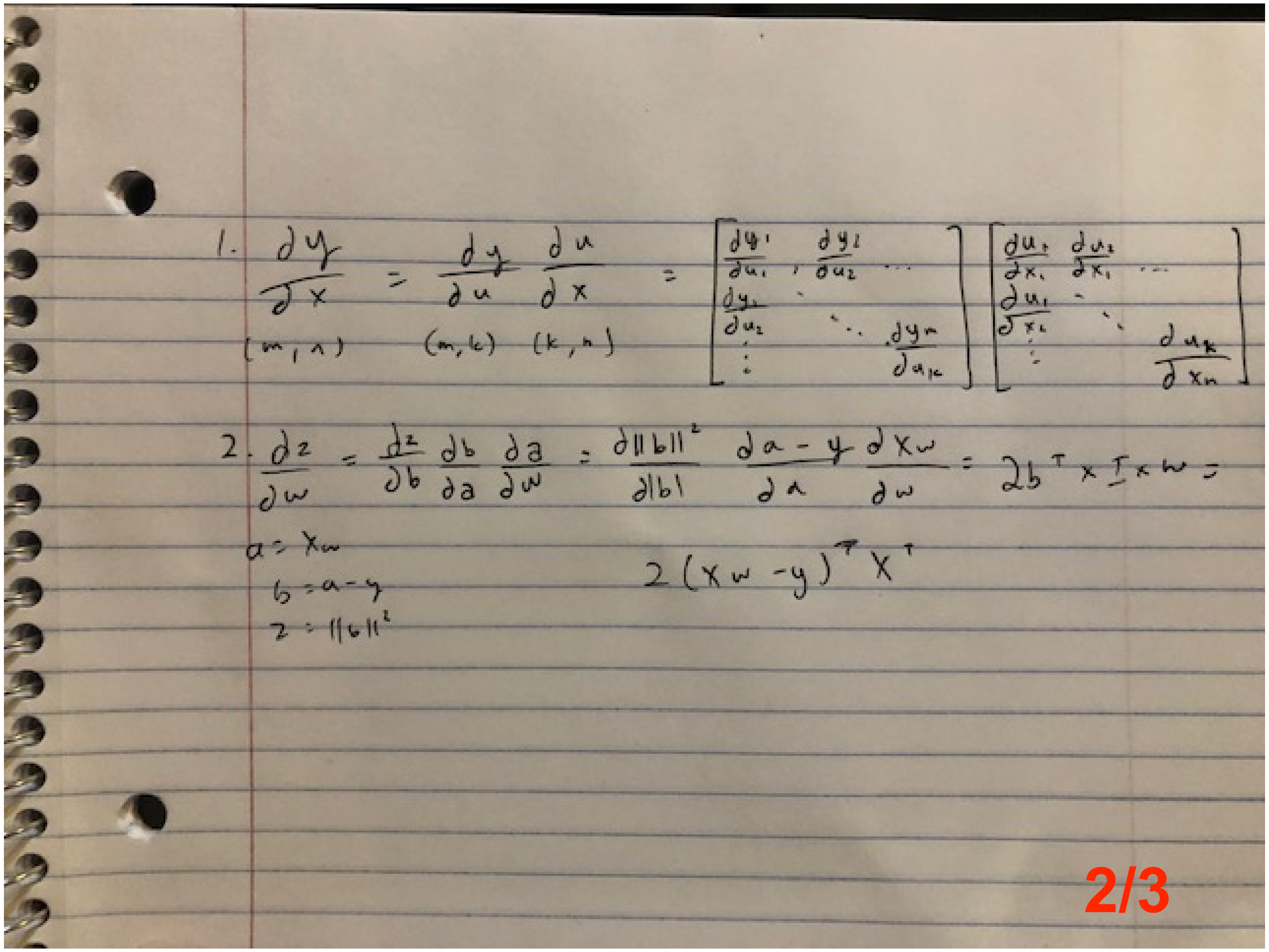
$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} \\ \frac{\partial \mathbf{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{y}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1}, \ldots, \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_m}{\partial x_2} \\ \vdots \\ \frac{\partial y_1}{\partial x_n}, \frac{\partial y_2}{\partial x_n}, \ldots, \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Questions:

1. Assume $\mathbf{y} = f(\mathbf{u})$ and $\mathbf{u} = g(\mathbf{x})$, write down the chain rule for $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$
2. Given $\mathbf{X} \in \mathbb{R}^{m \times n}$, $\mathbf{w} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, assume $z = \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$, compute $\frac{\partial z}{\partial \mathbf{w}}$.

In [21]:
```
from IPython.display import Image
Image(filename='hw2/IMG_0046.jpg')
```

Out[21]:



1. $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial u_1}, & \frac{\partial y_1}{\partial u_2} & \cdots \\ \frac{\partial y_2}{\partial u_2} & & \ddots & \\ \vdots & & & \frac{\partial y_m}{\partial u_k} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots \\ \frac{\partial u_1}{\partial x_2} & & \ddots & \\ \vdots & & & \frac{\partial u_k}{\partial x_n} \end{bmatrix}$

$(m,n) \quad (m,k) \quad (k,n)$

2. $\frac{\partial z}{\partial w} = \frac{\partial z}{\partial b}\frac{\partial b}{\partial a}\frac{\partial a}{\partial w} = \frac{\partial \|b\|^2}{\partial |b|}\frac{\partial a-y}{\partial a}\frac{\partial Xw}{\partial w} = 2b^T \times I \times w =$

$a = Xw$

$b = a - y$

$z = \|b\|^2$

$2(Xw-y)^T X^T$

2/3

4. Numerical Precision

Given scalars `x` and `y`, implement the following `log_exp` function such that it returns a numerically stable version of

$$-\log\left(\frac{e^x}{e^x + e^y}\right)$$

In [86]:
```
def log_exp(x, y):
    return -nd.log(nd.exp(x)) + nd.log(nd.exp(x) + nd.exp(y))
```

Test your codes with normal inputs:

In [87]:
```
x, y = nd.array([2]), nd.array([3])
z = log_exp(x, y)
z
```

Out[87]:
```
[1.3132617]
<NDArray 1 @cpu(0)>
```

Now implement a function to compute $\partial z/\partial x$ and $\partial z/\partial y$ with `autograd`

In [90]:
```
def grad(forward_func, x, y):
    x.attach_grad()
    y.attach_grad()
    with autograd.record():
        z = forward_func(x, y)
    z.backward()
    print('x.grad =', x.grad)
    print('y.grad =', y.grad)
```

Test your codes, it should print the results nicely.

In [91]:
```
grad(log_exp, x, y)
```

```
x.grad =
[-0.7310586]
<NDArray 1 @cpu(0)>
y.grad =
[0.7310586]
<NDArray 1 @cpu(0)>
```

But now let's try some "hard" inputs

In [92]:
```
x, y = nd.array([50]), nd.array([100])
grad(log_exp, x, y)
```

```
x.grad =
[-1.]
<NDArray 1 @cpu(0)>
y.grad =
[nan]
<NDArray 1 @cpu(0)>
```

Does your code return correct results? If not, try to understand the reason. (Hint, evaluate `exp(100)`). Now develop a new function `stable_log_exp` that is identical to `log_exp` in math, but returns a more numerical stable result.

In [9]:
```
def stable_log_exp(x, y):
    ## Add your codes here
    pass

grad(stable_log_exp, x, y)
```

```
x.grad = None
y.grad = None
```

2/3