# ENGSCI 211 - Mathematical Modelling 2

## Assignment 1 - Ordinary Differential Equations

Semester One 2023

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## Introduction

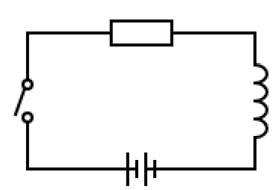
This assignment focuses on the mathematical and numerical solution of ordinary differential equations (ODEs) that can be used to model the flow of current through simple electronic circuits. It is not necessary for you to have an understanding of electronic circuits in order to complete this assignment. Rather, the focus is on solution methods for second-order linear constant coefficient non-homogeneous ODEs and interpretation of the mathematical and numerical solution behaviour.

This assignment will require you to use all aspects of the content taught in the ODE modules, including the method of complementary function and particular integral, numerical solution methods, and the Laplace transform method. You will also be required to write some code in MATLAB, which should be tested with and submitted to MATLAB Grader.

## Task 1: RL Circuit

### Background

An RL circuit is one that contains a resistor and an inductor connected in series. When the switch is closed, an input voltage $V_{\text{in}}$ is applied across the circuit.



Kirchoff's law states that the sum of voltages around the circuit must be zero, meaning that the input voltage must equal the sum of voltage drops across the resistor, $V_R$, and the inductor, $V_L$:

$$V_R + V_L = V_{\text{in}} \tag{1}$$

The voltage drop across a resistor is given by Ohm's law:

$$V_R = Ri \tag{2}$$

where $R$ is the resistance (measured in Ohms, $\Omega$) and $i$ is the current (measured in Amperes, $A$). The voltage drop across an inductor is given by:

$$V_L = L\frac{di}{dt} \tag{3}$$

where $L$ is the inductance (measured in Henries, $H$) and $t$ is time (measured in seconds, $s$). Substituting equations (2) and (3) into (1) gives the ODE:

$$Ri + L\frac{di}{dt} = V_{\text{in}} \tag{4}$$

which can be solved for the current as a function of time. We will assume in this task that the circuit is being powered by direct current (DC), rather than alternating current (AC), such that $V_{\text{in}}$ can be treated as a constant in time.

## Tasks To Complete

Complete the following tasks about the RL circuit:

1.1 Find the complementary function, $i_c$, and particular integral, $i_p$, for the ODE in equation (4). This should use a similar approach to a second-order linear constant coefficient non-homogeneous ODE. Use $i_c$ and $i_p$ to write down the general solution, $i$, leaving $R$, $L$ and $V_{\text{in}}$ as unknown constants.

1.2 Write a MATLAB function, `didt_rl`, that calculates $\frac{di}{dt}$. You will need to consider how to appropriately re-arrange equation (4) to obtain an expression for $\frac{di}{dt}$. The function must have the following input/output (I/O) variables:

- Input 1 (float): $t$. While not used in the function, we provide it by convention.
- Input 2 (float): $i$
- Input 3 (float): $R$
- Input 4 (float): $L$
- Input 5 (float): $V_{\text{in}}$

- Output 1 (float): $\frac{di}{dt}$

Ensure that your function has suitable header comments, detailing the function purpose and the I/O variables.

1.3 Consider the specific example of an RL circuit with of $R = 4$ and an inductance of $L = 1$. At $t = 0$, when the circuit has no initial current, the switch is closed and a constant input voltage of $V_{\text{in}} = 18$ is applied to the circuit.

Write a MATLAB script to solve this RL circuit for $0 \leq t \leq 5$, using the Improved Euler method with a time step of $\Delta t = 0.1$. This script should make use of your previously written MATLAB function, `didt_rl`.

1.4 By considering your mathematical and numerical solutions from Tasks 1.1 and 1.3, comment on the behaviour of the RL circuit current as a function of time. By considering Ohm's law for the input voltage, $V_{\text{in}} = Ri$, does the long-term behaviour of these solutions make sense?

### What to Upload

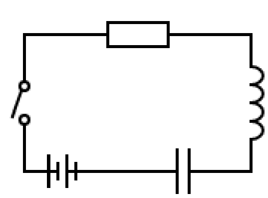
1. Upload your answers for Tasks 1.1 and 1.4 to the Task 1 assignment page on Canvas.
2. Upload your MATLAB function, from Task 1.2, to MATLAB Grader.
3. Upload your MATLAB script, from Task 1.3, to MATLAB Grader.

Your mark for the coding tasks will be determined from how many of the test cases are passed.

## Task 2: RLC Circuit

### Background

An RLC circuit is one that contains a resistor, an inductor and a capacitor connected in series. When the switch is closed an input voltage is applied across the circuit.



Kirchoff's law states that the sum of voltages must satisfy:

$$V_R + V_L + V_C = V_{\text{in}} \tag{5}$$

where $V_R$ and $V_L$ are the same as in equations (2) and (3) for an RL circuit, and the voltage drop across the capacitor is given by:

$$V_C = \frac{Q}{C} \tag{6}$$

where $Q$ is the charge across the capacitor and $C$ is the capacitance (measured in Farads, $F$, which is different from a Faraday). It is most common to use an alternating current (AC) with an RLC circuit, in which the input voltage can be described by:

$$V_{\text{in}} = V_0 \sin(\omega t) \tag{7}$$

where $V_0$ and $\omega$ are related to the maximum amplitude and the frequency, respectively, of the oscillating input voltage. Substituting in equations (2), (3), (6) and (7) into (5) gives the following first-order ODE:

$$L\frac{di}{dt} + Ri + \frac{Q}{C} = V_0 \sin(\omega t) \tag{8}$$

However, this ODE is not yet ready to solve as the charge $Q$ is not a constant. Instead, the charge across a capacitor varies with time and is related to the current by:

$$i = \frac{dQ}{dt} \tag{9}$$

Therefore, by taking the first-order derivative with respect to time of each term in (8), it is possible to derive a second-order ODE for an RLC circuit:

$$\frac{d}{dt}\left(L\frac{di}{dt} + Ri + \frac{Q}{C}\right) = \frac{d}{dt}\left(V_0 \sin(\omega t)\right) \tag{10}$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = V_0\, \omega \cos(\omega t) \tag{11}$$

As this is a second-order linear constant coefficient non-homogeneous ODE, the current in the circuit can exhibit similar behaviour to the car suspension system covered in lectures. This includes different types of damping and resonance.

## Tasks to Complete

Complete the following tasks about an RLC circuit:

2.1 Write the second-order ODE in equation (11) as a system of first-order ODEs, $\frac{d\vec{i}}{dt}$. You do not need to include initial conditions, as none are specified, and you should leave $R$, $L$, $C$,

$V_0$ and $\omega$ as unknowns.

2.2 Write a MATLAB function, `didt_rlc`, that calculates $\frac{d\vec{i}}{dt}$. You should make use of your system of first-order ODEs from Task 2.1. The function must have the following I/O:

- Input 1 (double): $t$
- Input 2 (1D array): $\vec{i}$
- Input 3 (double): $R$
- Input 4 (double): $L$
- Input 5 (double): $C$
- Input 6 (double): $V_0$
- Input 7 (double): $\omega$
- Output 1 (1D array, column vector): $\frac{d\vec{i}}{dt}$

Ensure that this function has suitable header comments.

2.3 Consider the specific example of an RLC circuit that has an inductance of $L = 1$, a resistance of $R = 2$, a capacitance of $C = \frac{1}{5}$, an input voltage with $V_0 = 17$ and $\omega = 2$, and the initial conditions $i(0) = 2$, $\frac{di}{dt}(0) = 2$.

Write a MATLAB script to solve this RLC circuit for $0 \leq t \leq 10$, using the built-in `ode45` function with a time step of $\Delta t = 0.1$. This script should make use of your previously written MATLAB function, `didt_rlc`. If you are unfamiliar with `ode45` (it was used in ENGGEN 131, which most students have taken), you can find more information on it here. This page includes all the relevant information and examples needed to complete this task.

2.4 Find the general and total solution for the specific RLC circuit given in Task 2.3. Comment on the expected behaviour of the current as a function of time, including whether or not there is resonance in the system. Include an accompanying plot produced by MATLAB (the plotting code does **not** need to be submitted), ensuring that it includes a title and axes labels.

## What to Upload

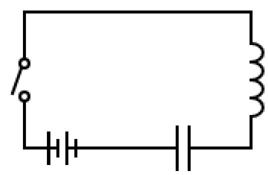
- Upload your answers for Tasks 2.1 and 2.4 to the Task 2 assignment page on Canvas. Include your plot (no need for the code) produced by MATLAB for Task 2.4.
- Upload your MATLAB function, from Task 2.2, to MATLAB Grader.
- Upload your MATLAB script, from Task 2.3, to MATLAB Grader.

Your mark for the coding tasks will be determined from how many of the test cases are passed.

# Task 3: LC Circuit

## Background

An LC circuit is similar to an RLC circuit, except that the resistor has been removed.



It is therefore governed by the same ODE as an RLC circuit, but with $R = 0$:

$$L\frac{d^2i}{dt^2} + \frac{1}{C}i = \frac{dV_{\text{in}}}{dt} \tag{12}$$

When driven by an AC current, $V_{\text{in}} = V_0 \sin(\omega t)$, this ODE can be written as:

$$L\frac{d^2i}{dt^2} + \frac{1}{C}i = V_0\,\omega\cos(\omega t) \tag{13}$$

## Tasks to Complete

3.1 Use the Laplace transform method to solve the ODE in equation (13) with an inductance of $L = 1$, a capacitance of $C = \frac{1}{4}$, an input voltage with $V_0 = 17$ and $\omega = 2$, and the initial conditions $i(0) = 2$, $\frac{di}{dt}(0) = 2$.

Comment on whether there is any resonance in the system, and on the type of damping that the system would exhibit with a zero input voltage.

## What to Upload

1. Upload your answer for Task 3.1 to the Task 3 assignment page on Canvas.