# ON SOLVING A CUBIC EQUATION

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Abstract. In the 19th century, Évariste Galois proved that all algebraic equations of degree higher than 4 cannot always be solved by radicals, i.e., there does not exist a formula that relates the roots of the equations to their coefficients using the operations of addition, multiplication, subtraction, division, exponentiation and taking nth roots. However, equations of degree 1, 2, 3 and 4 are solvable by radicals. This article attempts to introduce one such formula; specifically, one that can solve a cubic equation.

Keywords: *Cubic Formula, Cubic Discriminant, Depressed cubic, Cardano's formula*

## Introduction

In algebra, an equation of the form $ax^3+bx^2+cx+d=0$ such that $a,b,c,d \in \mathbb{R}$ is called a cubic equation in one variable. The solutions of the equation are called its *roots.*[1] By the Fundamental Theorem of Algebra and the fact that complex roots occur in conjugate pairs, there always exists a real root of every cubic equation. However, unlike the quadratic formula (which enables one to quickly find roots of a quadratic equation), a **cubic formula** is rarely discussed in high-school and most undergraduate courses. We attempt to derive such a formula in this article. [2]
The cubic formula was first published in 1545, by **Girolamo Cardano** in his book, *Ars Magna*. Cardano attributed the formula to **Scipione del Ferro**. However, another mathematician, **Niccolò Tartaglia**, had also independently discovered a formula for solving cubics.

## 1. Depressed Form of a Cubic

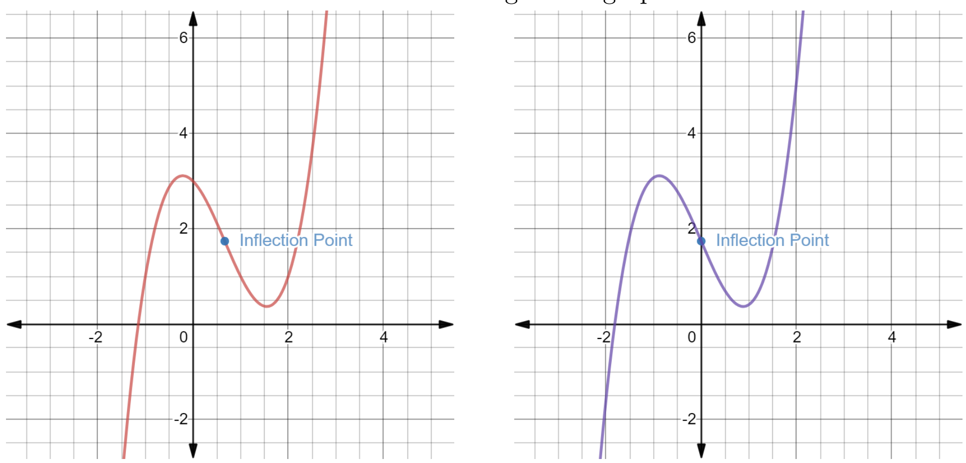
All general cubics can be reduced into a *depressed cubic* of the form

$$y^3+py+q=0$$

which is much easier to solve, and hence, shall be the focus of this article. Reduction of a general cubic into this form can be done by a simple change of variable $x = y - \frac{b}{3a}$. This substitution is motivated by the fact that the inflection point of the general cubic has $-\frac{b}{3a}$ as its abscissa [see Figure 1].(The abscissa of the inflection point can be evaluated by equating the second derivative to zero.) Graphically, this amounts to shifting the coordinate axes such that the inflection point now lies on the y-axis [see Figure 1].

To revert to the original equation, one can simply use the relation $y = x + \frac{b}{3a}$. Thus, there exists a bijection between a general cubic and its depressed form. Hence,

FIGURE 1. Shifting of the graphs




solving the depressed cubic amounts to solving every cubic.

## 2. Number of solutions

We know by the Fundamental Theorem of Algebra, that every cubic has at least one real root. In this section, we shall determine a *cubic discriminant*, which will allow us to decide how many real solutions exist.

Since, complex roots occur in conjugate pairs, a cubic equation has exactly one real root or exactly three real roots, not all distinct. Let us consider the first case, i.e., there exists one real root. This means that the graph of the cubic intersects the x-axis only once [see Figure 2].

As a result, the length of the perpendicular drawn from the origin to the inflection point on the y-axis is always **greater** than the distance between the inflection point and of the local maxima. Let us consider the **positive difference** between these distances. The length of the perpendicular from origin to the inflection point is simply the y-intercept of the curve, $q$.

Now, the other distance can be calculated by taking the difference between the ordinate of the inflection point and the ordinate of the local maxima. We can get the local maxima by equating the first derivative to zero and applying the first derivative test; the local maxima occurs at the point $x = \sqrt{\frac{-p}{3}}$, and thus the ordinate is $y = \frac{2p}{3}\sqrt{\frac{-p}{3}} + q$. Hence, the distance we require is simply $\frac{2p}{3}\sqrt{\frac{-p}{3}}$. Thus the required difference is given by $q - \frac{2p}{3}\sqrt{\frac{-p}{3}}$, and we need to solve the inequality

$$q - \frac{2p}{3}\sqrt{\frac{-p}{3}} > 0$$

which simplifies to

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$$

This is the condition for a cubic equation to have one real root.

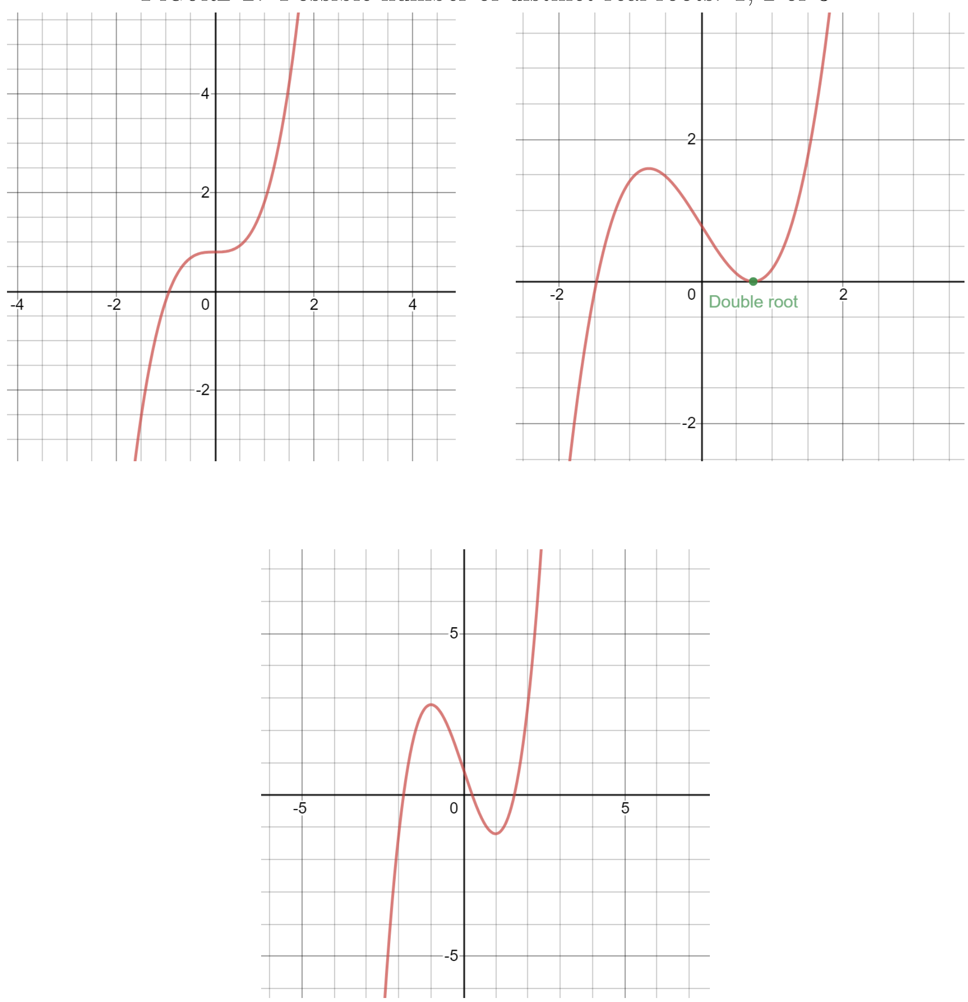
Note that here $p$ represents the slop of the line tangent to the inflection point, which is negative, and thus the quantity $\sqrt{\frac{-p}{3}}$ is real.

Now let us consider the other case, i.e, there exist three real roots (not necessarily distinct). This means that the graph of the cubic intersects the x-axis thrice [see Figure 2].

In case of a repeated root, it shall technically intersect the x-axis twice, but for the sake of brevity, we will consider that equivalent to three intersections.

In this case the length of the perpendicular drawn from the origin to the inflection point on the y-axis is always **smaller** than the distance between the inflection point and of the local maxima. Hence, we can conclude that the condition for existence of three real roots is given by

FIGURE 2. Possible number of distinct real roots: 1, 2 or 3



$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 \ \leq \ 0$$

Hence, this quantity acts as a *discriminant* for cubic equations and so we shall henceforth refer to it as $\Delta$ for ease of writing.

## 3. Completing the cube

As the counterpart to completing the square technique in the derivation of the quadratic formula, let us attempt the same in case of a cubic equation.

Consider the identity

$$(v+u)^3 = v^3 + u^3 + 3vu(v+u)$$

Rearranging, we obtain

$$(v+u)^3 - 3vu(v+u) - (v^3 + u^3) = 0$$

Comparing this with our depressed cubic, we can deduce the following relations:

$$p = -3vu$$

$$q = -(v^3 + u^3)$$

$$y = v + u$$

Hence, if we obtain u and v, our original equation is a perfect cube and our equation is effectively solved.

Solving the above equations, for u and v, and subsequently adding them, we get

$$\boldsymbol{y = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\frac{q^2}{4} + \frac{p^3}{27}}}}$$

More compactly, the same formula can be written as

$$\boldsymbol{y = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\Delta}}}$$

The above is called *Cardano's Formula.*[3]

Note that we are assuming the existence of a solution by the Fundamental Theorem of Algebra and moreover, the system of equations can be algebraically reduced to a quadratic equation in $v^3$. Hence, it can be solved using the quadratic formula.

## 4. A Complex Detour and an example

One may notice the fact that the cases of one real root or three non-distinct real roots, i.e., when $\Delta > 0$ or $\Delta = 0$, respectively, correspond to cube roots of real numbers and can thus be calculated.
However, in the case of existence of three distinct real roots, $\Delta < 0$ and hence we are required to take the sum of two complex numbers. This might seem contradictory and impossible - obtaining three real roots by adding up two complex numbers - but it is possible since the two complex numbers being added are conjugates of one another.

Furthermore, a cube root of a complex numbers will result in three different complex numbers and hence we shall have three pairs of $u$ and $v$. Adding corresponding values of $u$ and $v$ in a pair will give us the three required roots.

Now, let us solve an actual cubic equation using this method.

Consider the following equation:

$$x^3 - 26x^2 + 193x - 420 = 0$$

Here, $a = 1$, $b = -26$, $c = 193$, and $d = -420$. Thus, replacing $x$ by $y - \frac{b}{3a}$, i.e., $y + \frac{26}{3}$ and simplifying, we get:

$$y^3 - \frac{97}{3}y - \frac{1330}{27} = 0$$

Now,

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \left(\frac{\frac{-1330}{27}}{2}\right)^2 + \left(\frac{\frac{-97}{3}}{3}\right)^3$$

$$\Delta = \frac{442225}{729} - \frac{912673}{729} = -\frac{470448}{729} < 0$$

Thus, there exist 3 real distinct roots.

Now, we know that

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\Delta}}$$

Therefore, putting values we get:

$$y = \sqrt[3]{-\frac{\frac{-1330}{27}}{2} + \sqrt[2]{-\frac{470448}{729}}} + \sqrt[3]{-\frac{\frac{-1330}{27}}{2} - \sqrt[2]{-\frac{470448}{729}}}$$

$$y = \sqrt[3]{\frac{1330}{54} + \sqrt{\frac{44^2}{3}}\, i} + \sqrt[3]{\frac{1330}{54} - \sqrt{\frac{44^2}{3}}\, i}$$

Calculating these cube roots (one method of doing this can be first converting them to their polar forms and then carrying out the operations; other methods exist too, but the final answers remain the same), we get three different sums of conjugate pairs of complex numbers. Adding them up, we get the following values for $y$:

$$y = -\frac{5}{3},\quad -\frac{14}{3} \text{ and } \frac{19}{3}$$

Now, using the relation, $x = y - \frac{b}{3a} = y + \frac{26}{3}$, we obtain the solutions to our original equation as:

$$x = 7,\ 4 \text{ and } 15$$

## 5. The General cubic formula

We have seen before that every general cubic can be reduced to a depressed cubic and we derived a formula for solving such a reduced equation [4]. However, since the mapping from general cubics to depressed cubics is bijective, we can also derive a direct formula for a general cubic as follows:

$$\boldsymbol{x = -\frac{b}{3a} + \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt[2]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}}$$
$$\boldsymbol{+ \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt[2]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}}$$

## Conclusions

We come to the conclusion that every general cubic equation can be reduced to a depressed cubic and solved using the formula derived above.
In case of three real distinct roots, the cubic formula outputs answers as sum of complex conjugates and thus, it is beyond the curfew of high school mathematics and is rarely discussed at that level, even though it is algebraically calculable.

## References

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