<u>ON THE PROBABILITY OF TWO RANDOM INTEGERS BEING COPRIME</u>

DAKSH DHEER, SANSKRITI JHA
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ABSTRACT. This paper attempts to simplify and provide a solution for the popular *Basel Problem*, posed in 1650, which was solved by Leonhard Euler in 1734. In other words, it attempts to answer the question:
What is the probability of two random integers being coprime?

CONDITION FOR COPRIMALITY

When we say two numbers are coprime, we mean to say that they share no common factors. Alternatively, two numbers are coprime if their greatest common divisor (gcd) is 1. Now, before calculating the probability of two random integers being coprime, let us first consider the probability of them being divisible by a single prime, say p. We know from Euclid's division algortihm that any number M can be represented in the form, $M = pk + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, 2, 3, ..., p-1\}$. However, M will be divisible by p if and only if $r = 0$. Thus, in the set of remainders, only one out p choices will allow M to be divisive by p. Hence, the probability that M is divisible by p is exactly $\frac{1}{p}$. Hence, the probability of M and another integer, say L, being simultaneously divisible p is $(\frac{1}{p})(\frac{1}{p}) = \frac{1}{p}^2$.

Consequently, probability of M and L, simultaneously **not** being divisible by p is precisely $1 - \frac{1}{p^2}$.

Now, there is no basis to assume that divisibility by one prime number is anyhow related to the divisibility by any other prime number. We, therefore, assume that events of divisibility by prime numbers (or their complements) are independent. Thus, the probability of two numbers being simultaneously being coprime is:

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)... \tag{1}$$

RELATION BETWEEN AN INFINITE SUM AND AN INFINITE PRODUCT

Consider the following infinite sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$$

Let c be the value that the sum converges to (we shall show the convergence of the sum in a later part of the paper). Thus, we have:

$$c = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \tag{2}$$

Now, take any prime p, multiply equation (2.) by $\frac{1}{p^2}$, and subtract the result obtained from equation (1.). Let us consider the case for $p = 2$. Thus, we get:

$$\left(1 - \frac{1}{2^2}\right) c = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots.$$

We can see that all terms containing a factor of 2 in the denominator get cancelled out. If we repeat the process with $p = 3$, we get the following:

$$\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{2^2}\right) c = \frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \dots.$$

Comparing the above equation with the original value of c, we see that all terms containing a factor of 2 or 3 in the denominator all vanish. Thus, if we repeat this process for all primes p, we get the following result:

$$\prod_p c\left(1 - \frac{1}{p^2}\right) = c\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right) \dots. = \frac{1}{1^2}$$

Hence, we arrive at the following:

$$\frac{1}{c} = \prod_p \left(1 - \frac{1}{p^2}\right) = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1} \tag{3}$$

We shall now evaluate the precise value of c.

Obtaining the Maclaurin series for sine

The actual value of c was initially calculated by famous Swiss mathematician Euler in 1734.

Let us now retrace Euler's steps for calculating the following expression.

$$\begin{aligned} c &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \dots \\ &= 1 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{5} + \dots \end{aligned}$$

This is sum is strictly less than

$$\begin{aligned} S_n &= 1 + \frac{1}{1}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{5} + \dots \\ &= 1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 2 - \frac{1}{n+1} \end{aligned}$$

Now, to evaluate the infinite sum, we may take the limit of the above equation as n tends to infinity to obtain the following fact:

$$c = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots < 2$$

So, we get an intermediate result that Euler's sum is finite and less than 2. This proves our convergence claim made above.

Euler started with the famous Maclaurin series of the sine function:

$$sinx = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{4}$$

Let us attempt to rederive this expression. For this, we shall first write the sine function as an infinite polynomial:

$$sinx = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \tag{5}$$

Putting $x = 0$ above, we obtain $a_0 = 0$.

For obtaining a_1, we differentiate both sides of the equation with respect to x to get:

$$cosx = a_1 + 2a_2x + 3a_3x^2 + \dots$$

We once again put $x = 0$ to obtain $a_1 = 1$

Proceeding similarly, by differentiating equation 5, exactly $(i - 1)$ times and putting $x = 0$ in the result obtained, we can obtain all a_i.

We see that in general,

$$a_i = \begin{cases} 0, & i = 2k, k \in \mathbb{N} \\ -\frac{1}{i!}, & i = 4k-1, k \in \mathbb{N} \\ \\ \frac{1}{i!}, & i = 4k+1, k \in \mathbb{N} \end{cases}$$

Using what we derived above, we obtain the Maclaurin series for sine (equation 4).

An infinite polynomial for sine

The step that Euler performed then was to construct another polynomial of the sine function by using the zeros of the function itself. Using the factor theorem, we know that a polynomial whose zeros are $0, -\pi, \pi$ is $p(x) = (\pi - x)(x)(\pi + x)$
p(x) is, however, just one of the infinite polynomials having zeroes $0, -\pi$ and π. The rest can be obtained by multiplying p(x) by a constant multiple.
We can see that $\frac{1}{\pi^2}$ is the multiple which give us the closest fit to the sine function graph since the slope of the polynomial and the sin function graph is equal at 0 which is 1.

The simplified equation of the polynomial whose zeros are 0, -π, π and which is the closest fit to the sine function graph is

$$p_1(x) = x\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right) = x\left(1 - \frac{x^2}{\pi^2}\right)$$

Proceeding similarly, the polynomial for the middle 5 zeroes, namely $-2\pi, -\pi, 0, \pi, 2\pi$ is

$$p_2(x) = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)$$

Continuing onward, we obtain the 'sine polynomial' as follows:

$$sinx = p(x) = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\left(1 - \frac{x^2}{(4\pi)^2}\right)\dots$$

Expanding the right side of the equation, we get:

$$sinx = x - \left(\frac{1}{(\pi)^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \dots\right)x^3 + \left(\frac{1}{(\pi)^2(2\pi)^2} + \frac{1}{(\pi)^2(3\pi)^2} + \dots\right)x^5 + \dots$$

Finally, we equate the coefficient of x^3 in the above expression with the coefficient of x^3 in the Maclaurin series expansion:

$$-\frac{1}{3!} = -\left(\frac{1}{(\pi)^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \frac{1}{(4\pi)^2}\dots\right)$$

Multiplying both sides of the above equation by $-\pi^2$, we obtain:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \tag{6}$$

Using equations 3 and 6, we arrive at our final result, i.e., the probability of two numbers being simultaneously being coprime is $\frac{1}{c}$:

$$\frac{1}{c} = \prod_p \left(1 - \frac{1}{p^2}\right) = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1} = \frac{6}{\pi^2} \tag{7}$$

This is the solution Euler proposed to the Basel problem, and it is a great example of the elegance and beauty of mathematics.

References

[1] EULER'S INTRODUCTIO IN ANALYSIN INFINITORUM
[2] On the probability that k positive integers are relatively prime, *J.E Nymann*
[3] Remarks on a beautiful relation between direct as well as reciprocal power series