ON THE GENERAL THEORY OF VALUATIONS AND CLASS FIELD THEORY (DRAFT VERSION)

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by

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to

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DECLARATION

I, **Daksh Dheer (Roll No: 23050)**, hereby declare that, this report entitled **"On the General Theory of Valuations and Class Field Theory"** submitted to Indian Institute of Science Education and Research Thiruvananthapuram towards the partial requirement of **Master of Science** in **Mathematics**, is an original work carried out by me under the supervision of **Prof Viji Z. Thomas** and has not formed the basis for the award of any degree or diploma, in this or any other institution or university. I have sincerely tried to uphold academic ethics and honesty. Whenever a piece of external information or statement or result is used then, that has been duly acknowledged and cited.

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CERTIFICATE

This is to certify that the work contained in this project report entitled **"On the General Theory of Valuations and Class Field Theory"** submitted by **Daksh Dheer** (**Roll No: IPHD23050**) to the Indian Institute of Science Education and Research, Thiruvananthapuram towards the partial requirement of **Master of Science (Research)** in **Mathematics** has been carried out by him under my supervision and that it has not been submitted elsewhere for the award of any degree.

Thiruvananthapuram - 695551 Prof. Viji Z. Thomas

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ABSTRACT

Name of the student: **Daksh Dheer** Roll No: **23050**
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In this thesis, we independently develop the theory of valuations in more generality than is traditionally required for algebraic number theory. We begin by defining absolute values (which can be seen as valuations of rank one) before introducing valuations proper and discussing their properties such as the equivalence between valuations and valuation rings. After this, we construct some valuations and move on to discuss their topology – this notion closely relates to dependence of valuations and we make this precise. In the same section, we also show the correspondence among overrings, primes and a class of convex groups, and conclude by proving a general approximation theorem. The remainder of the chapter deals with extensions of valuations and results concering them.

The second half of the thesis will deal with developing abstract class field theory, followed by local class field theory, including the general reciprocity law and generalized cyclotomic theory.

Contents

Contents **vi**

1 Valuation Theory **1**
1.1 Absolute Values 1
1.2 General Valuations 17
1.3 Constructing valuations 25
1.4 Dependence and Topology of Valuations 30
1.5 Extensions of Valuations 39

2 Class Field Theory **56**
2.1 Preliminaries 56

Bibliography **57**

Chapter 1

Valuation Theory

In accordance with the views of Alfréd Rényi, we denote the completion of a proof with ☕.

To begin with, we prove theorems in the theory of valuations in more generality than required in standard algebraic number theory. We base this mainly on the first three chapters, and parts of the fifth chapter of [EP05].

1.1 Absolute Values

Definition 1.1.1. Let K be a field. An ***absolute value*** on K is a map

$$|\cdot| : K \longrightarrow \mathbb{R}$$

satisfying, for all $x, y \in K$:

1. $|x| > 0$ for all $x \neq 0$, and $|0| = 0$,
2. $|xy| = |x||y|$,
3. $|x + y| \leq |x| + |y|$.

Note that from the above axioms, we have $|1|^2 = |1^2| = |1|$, giving $|1| = 1$. Similarly, $|-1|^2 = |(-1)(-1)| = |1| = 1$ implies $|-1| = 1$, whence it follows that $|-x| = |x|$ for all $x \in K$.

Since $|\cdot|$ is a homomorphism on $K^\times$, it follows that $|x^{-1}| = |x|^{-1}$ for $x \neq 0$.

Proposition 1.1.1. *The set $S = \{|n \cdot 1| : n \in \mathbb{Z}\}$ is bounded if and only if $|\cdot|$ satisfies the ultrametric triangle inequality: $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.*

Proof. If $|\cdot|$ satisfies the ultrametric triangle inequality, then for any element $|m \cdot 1| \in S$, we have $|m \cdot 1| = |1 + 1 + \cdots + 1| \leq \max\{|1|, \ldots, |1|\} = 1$.
Conversely, if S is bounded by some constant M, then we have, for any $n \in \mathbb{Z}$

$$|x+y|^n = |(x+y)^n| = \left|\sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu y^{n-\nu}\right| \leq \sum_{\nu=0}^{n} \left|\binom{n}{\nu}\right| |x|^\nu |y|^{n-\nu} \leq M(n+1)\cdot\max\{|x|, |y|\}^n$$

since $|x|^\nu |y|^{n-\nu} \leq \max\{|x|, |y|\}^n$ and $\left|\binom{n}{\nu}\right| \leq M$. Therefore, on taking n-th roots, we get:

$$|x + y| \leq M^{1/n}(n + 1)^{1/n} \cdot \max\{|x|, |y|\}$$

and thus, as $n \to \infty$, we conclude $|x + y| \leq \max\{|x|, |y|\}$.

☕

Definition 1.1.2. An absolute value satisfying the ultrametric triangle inequality

$$|x + y| \leq \max\{|x|, |y|\} \text{ for all } x, y \in K$$

is said to be ***non-Archimedean***. If not, it is said to be ***Archimedean***.

Remark 1. The usual absolute value on $\mathbb{R}$, denoted by $|\cdot|_0$ is Archimedean.

We next consider some important non-Archimedean absolute values.

Example 1.1. For every rational prime p, define the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ by setting $|0|_p \coloneqq 0$ and $\left|p^\nu \frac{m}{n}\right|_p \coloneqq \frac{1}{e^\nu}$, where $m, n \in \mathbb{Z} \setminus \{0\}$ are not divisible by p.

Here, $S = \{|n \cdot 1| : n \in \mathbb{Z}\} = \{e^{-\nu} : \nu \in \mathbb{N}\}$.

Example 1.2. Let k be a field and $q \in k[x]$ be an irreducible polynomial. We define $|\cdot|_q$ on the rational function field $k(x)$ as $|0|_q \coloneqq 0$ and $\left|q^\nu \frac{f}{g}\right|_q \coloneqq \frac{1}{e^\nu}$, where $f, g \in k[x] \setminus \{0\}$ are not divisible by q. This is known as the q-adic absolute value on $k[x]$.

An absolute value $|\cdot|$ on K defines a metric as $d(x, y) = |x - y|$, for all $x, y \in K$ and thus induces a topology on K.

Definition 1.1.3. Two absolute values are ***dependent*** if they induce the same topology on K. If not, they are ***independent***.

Proposition 1.1.2. *Let $|\cdot|_1$ and $|\cdot|_2$ be two nontrivial absolute values on K. The following are equivalent:*

1. *$|\cdot|_1$ and $|\cdot|_2$ are dependent.*

2. $|x|_1 < 1$ *implies* $|x|_2 < 1$.

3. There exists $s > 0$ *such that* $|x|_1 = (|x|_2)^s$ *for all* $x \in K$.

Proof. (1) $\implies$ (2): Suppose $|\cdot|_1$ and $|\cdot|_2$ are dependent, i.e., they induce the same topology on K. Let $x \in K$ be such that $|x|_1 < 1$, i.e., $x \in B_1(0,1)$ — the open unit ball around 0 in the topology induced by $|\cdot|_1$. By hypothesis, there exists $\varepsilon > 0$ such that $B_1(0,\varepsilon) \subset B_2(0,1)$. Since $|x|_1 < 1$, we may choose some positive integer $m \geq 1$ such that $|x^m|_1 = (|x|_1)^m < \varepsilon$, whence $x^m \in B_1(0,\varepsilon) \subset B_2(0,1)$. Thus, $|x^m|_2 = (|x|_2)^m < 1$ and so $|x|_2 < 1$, since m is a positive integer.

(2) $\implies$ (3): Suppose $|x|_1 < 1$ implies $|x|_2 < 1$. Let $y \in K$ be a fixed element satisfying $|y|_1 > 1$. Let $x \in K, x \neq 0$. Then, $\exists \alpha \in \mathbb{R}$ such that $|x|_1 = |y|_1^\alpha$. Let m_i/n_i be a rational sequence converging to α from above; assume that $n_i > 0$. Now, $|x|_1 = |y|_1^\alpha < |y|_1^{m_i/n_i}$, and so $\left|\frac{x^{n_i}}{y^{m_i}}\right|_1 < 1$. By hypothesis, we get $\left|\frac{x^{n_i}}{y^{m_i}}\right|_2 < 1$, whence $|x|_2 \leq |y|_2^{m_i/n_i}$, and therefore $|x|_2 \leq |y|_2^\alpha$. We now use a rational sequence m_i/n_i converging to α from below and proceed similarly to obtain $|x|_2 \geq |y|_2^\alpha$. Thus, we have $|x|_2 = |y|_2^\alpha$. Moreover, we started with $|x|_1 = |y|_1^\alpha$.

Using these two equations, we get, for all nonzero $x \in K$,

$$\alpha = \frac{\log|x|_1}{\log|y|_1} = \frac{\log|x|_2}{\log|y|_2} \implies \frac{\log|x|_1}{\log|x|_2} = \frac{\log|y|_1}{\log|y|_2} =: s,$$

hence, $|x|_1 = |x|_2^s$ and $|y|_1 > 1 \implies |y|_2 > 1$, i.e., $s > 0$.

(3) $\implies$ (1): Suppose $\exists s > 0$ such that $|x|_1 = (|x|_2)^s$ for all $x \in K$.

It suffices to show that every open ball in the topology induced by $|x|_1$ is contained in an open ball in the topology induced by $|x|_2$, and vice versa. Consider $B_1(\alpha, r) = \{x \in K : |x - \alpha|_1 < r\}$. For any $x \in B_1(\alpha, r)$, we have, by hypothesis, $(|x-\alpha|_2)^s < r$, i.e., $|x-\alpha|_2 < r^{1/s}$, i.e., $x \in B_2(\alpha, r^{1/s})$.

Hence, $B_1(\alpha, r) \subset B_2(\alpha, r^{1/s})$. Analogously, it follows that $B_2(\alpha, r) \subset B_1(\alpha, r^s)$ and thus the topologies are the same. ☕

Remark 1.1. We have $|x| = |x - y + y| \leq |x - y| + |y|$. Thus, $|x| - |y| \leq |x - y|$. Exchanging x and y, and using $|y - x| = |x - y|$, it follows that $||x| - |y||_0 \leq |x - y|$, where $|\cdot|_0$ is again the usual absolute value on the real numbers.

Hence, $|\cdot|$ **is a uniformly continuous map from** K**, with the topology given by** $|\cdot|$**, to** $\mathbb{R}$ **with the usual topology defined by** $|\cdot|_0$.

The following theorem is an analogue of the Chinese Remainder Theorem in case of absolute values.

Theorem 1.1.1 (Artin-Whaples Approximation)**.** *Let $|\cdot|_1, |\cdot|_2, \ldots, |\cdot|_n$ be pairwise-independent nontrivial absolute values on a field K. Let $x_1, x_2, \ldots, x_n \in K$ and $\varepsilon > 0$ be given. Then, there exists $x \in K$ such that $|x - x_i|_i < \varepsilon$ for all i.*

Proof. We will prove this in three steps.

Step 1. We shall prove that for every $1 \leq i \leq n$ there exists $a_i \in K$ such that $|a_i|_i > 1$ and $|a_i|_j < 1$ for all $j \neq i$. We show it for $i = 1$ and write $a = a_1$. The argument for every other i follows by replacing 1 with i.

We induct on n: for $n = 2$, the hypothesis states that $|\cdot|_1$and$|\cdot|_2$ are independent. Hence, we use the earlier proposition to conclude that there exist $b, c \in K$ such that $|b|_1 < 1$ and $|b|_2 \geq 1$, $|c|_1 \geq 1$ and $|c|_2 < 1$. Now, take $a = b^{-1}c$, then $|a|_1 = |b^{-1}c|_1 = |b^{-1}|_1|c|_1 > 1$ since $|b|_1 < 1$ and $|c|_1 \geq 1$.

Similarly, $|a|_2 = |b^{-1}c|_2 = |b^{-1}|_2|c|_2 < 1$ since $|b|_2 \geq 1$ and $|c|_2 < 1$. Hence, the claim is true for $n = 2$.

Induction hypothesis: $\exists y \in K$ such that $|y|_1 > 1$ and $|y|_j < 1$ for all $j = 2, \ldots, n-1$.

Applying the $n = 2$ case to $|\cdot|_1$ and $|\cdot|_n$, we get $|z|_1 > 1$ and $|z|_n < 1$ for some $z \in K$. Therefore, if $|y|_n \leq 1$, then $|zy^\nu|_1 > 1$ and $|zy^\nu|_n < 1$ for every integer $\nu \geq 1$. Now, we choose an integer $\nu \geq 1$ such that $|zy^\nu|_j < 1$; this exists because $|y|_j < 1$ for all $j = 2, \ldots, n-1$. For example, we may take any $\nu < \min\limits_{2 \leq j \leq n-1} \left(\dfrac{-\log |z_j|}{\log |y_j|} \right)$.

Then, set $a = zy^\nu$. We have $|a|_1 = |zy^\nu|_1 > 1$ and $|a|_j = |zy^\nu|_j < 1$ for all $j \neq 1$. Hence we are done, under the assumption that $|y|_n \leq 1$.

Suppose now that $|y|_n > 1$. Consider the sequence $w_\nu = \dfrac{y^\nu}{1 + y^\nu}, \nu \in \mathbb{N}$..

We then have

$$\lim_{\nu \to \infty} |w_\nu|_j = \lim_{\nu \to \infty} \frac{|y^\nu|_j}{|1 + y^\nu|_j} = 0 \text{ for } j = 2, \ldots, n-1,$$

because $|y|_j < 1 \implies |y|_j^\nu < 1$ and $|1 + y^\nu|_j > |1|_j + |y^\nu|_j > 1$, i.e., the denominator is bounded below by 1 meanwhile the numerator tends to zero.

Similarly,

$$\lim_{\nu \to \infty} |w_\nu - 1|_j = \lim_{\nu \to \infty} \left| \frac{y^\nu}{1 + y^\nu} - 1 \right|_j = \lim_{\nu \to \infty} \left| \frac{1}{1 + y^\nu} \right|_j = 0 \text{ for } j = 1 \text{ and } n,$$

since $|y|_n > 1$ and $|y|_1 > 1$.

Now,

$$\lim_{\nu \to \infty} |w_\nu - 1|_j < \lim_{\nu \to \infty} |w_\nu|_j = 0 \text{ for } j = 2, \ldots, n-1,$$

and

$$\lim_{\nu\to\infty}|w_\nu - 1|_j = 0 \text{ i.e., } \lim_{\nu\to\infty}|w_\nu|_j = 1, n$$

As a result,

$$\lim_{\nu\to\infty}|zw_\nu|_j = |z|_j \lim_{\nu\to\infty}|w_\nu|_j = 0 \text{ for } j = 2, \ldots, n-1,$$

and

$$\lim_{\nu\to\infty}|zw_\nu|_j = |z|_j \lim_{\nu\to\infty}|w_\nu|_j = |z|_j \text{ for } j = 1, n.$$

Hence, for sufficiently large ν, $a = zw_\nu$ is a valid choice since $|a|_1 = |zw_\nu|_1 = |z|_1|w_\nu|_1 > 1$ (as $\lim_{\nu\to\infty}|zw_\nu|_1 = |z|_1 > 1$) and $|a|_j = |zw_\nu|_j < 1$ (as $\lim_{\nu\to\infty}|zw_\nu|_n = |z|_n < 1$) for all $j \neq 1$.

Step 2. Now we show that for any real $\varepsilon > 0$ and every $1 \leq i \leq n$, there exists $c_i \in K$ with $|c_i - 1|_i < \varepsilon$ and $|c_i|_j < \varepsilon$ for all $j \neq i$. We again only consider the case $i = 1$ in detail.

Let $a \in K$ be as in Step 1. Then the sequence $\left|\frac{a^\nu}{1+a^\nu}\right|_j$ converges to 1 for $j = 1$, and converges to 0 if $j > 1$.

This is because

$$\left|\frac{a^\nu}{1+a^\nu}\right|_1 = \frac{|a^\nu|_1}{|1+a^\nu|_1} \longrightarrow 1$$

since $|a^\nu|_1 > 1 \implies \lim_{\nu\to\infty}|a^\nu|_1 = \lim_{\nu\to\infty}|1+a^\nu|_1 = \infty$ and

$$\left|\frac{a^\nu}{1+a^\nu}\right|_j = \frac{|a^\nu|_j}{|1+a^\nu|_j} \longrightarrow 0$$

since $|a^\nu|_j < 1 \implies \lim_{\nu\to\infty}|a^\nu|_j = 0$ and $\lim_{\nu\to\infty}|1+a^\nu|_j \geq 1$

Thus, for sufficiently large ν, $c_1 = \frac{a^\nu}{1+a^\nu}$ has the required property.

Step 3. Proceeding inductively with step 2, there exist elements $c_1, \ldots, c_n \in K$ such that c_i is close to 1 at $|\cdot|_i$, and for every $j \neq i$, c_i is close to 0 at $|\cdot|_j$. The element

$$x = c_1x_1 + \cdots + c_nx_n$$

is then arbitrarily close to x_i at $|\cdot|_i$ for every $i = 1, \ldots, n$, since

$$|x|_i = |c_1x_1 + \cdots + c_nx_n|_i \leq |c_1|_i|x_1|_i + \ldots |c_n|_i|x_n|_i \longrightarrow |c_i|_i|x_i|_i \longrightarrow |x_i|_i$$

This proves the theorem.

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Definition 1.1.4. A sequence $(x_n)_{n\in\mathbb{N}} \subset K$ is ***Cauchy*** if for every $\varepsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$.

K is called ***complete*** if every Cauchy sequence converges in K.

Note that K is always complete with respect to the trivial absolute value.

Lemma 1.1.1. *Let X be a metric space and $Y \subseteq X$ be a dense subset. Assume every Cauchy sequence in Y converges to a point in X. Then, X is complete.*

Proof. Let $(a_n)_n$ be a Cauchy sequence in X. Since Y is dense, $\forall \varepsilon > 0$, $B(a_n, \varepsilon) \cap Y \neq \emptyset$. Pick $y_n \in B(a_n, \varepsilon) \cap Y$.

Then, $|y_n - a_n| < \varepsilon$ and so $|y_n - y_m| \leq |y_n - a_n| + |a_n - y_m| < 2\varepsilon$, i.e., $(y_n)_n$ is also Cauchy. Since $(y_n)_n \subseteq Y$, we have $y_n \to x \in X$.

Hence, given $\varepsilon > 0$, $|y_n - x| < \varepsilon,\ \forall n \geq M$.

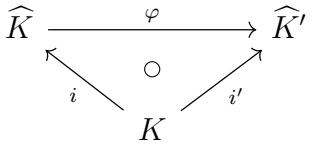
Consider

$$|a_n - x| \leq |a_n - y_n| + |y_n - x| < 2\varepsilon,\ \forall n \text{ large enough.}$$

Hence, $(a_n)_n \to x$, i.e., X is complete. ☕

Theorem 1.1.2. *There exists a field $\widehat{K}$, complete with respect to an absolute value $|\cdot|$, and an embedding $i : K \to \widehat{K}$ such that $|i(x)| = |x|$ for all $x \in K$.*

The image $i(K)$ is dense in $\widehat{K}$. If $(\widehat{K}', i')$ is another such pair, then there exists a unique continuous isomorphism $\varphi : \widehat{K} \to \widehat{K}'$, preserving the absolute value, such that the following diagram commutes:



Proof. **Existence of $\widehat{K}$**: Let $\mathcal{C}$ be the set of Cauchy sequences of elements in K. Clearly, $\mathcal{C}$ is a commutative ring with unity.

Consider the set of null sequences $\mathcal{N} = \{(x_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} x_n = 0\} \subset \mathcal{C}$.

$\mathcal{N}$ is an ideal of $\mathcal{C}$: take any $(a_n)_n \in \mathcal{N}$. Then, $\exists n_0 \in \mathbb{N}$ such that

$$|a_m - a_{n_0+1}| < 1,\ \forall m > n_0.$$

Thus, $|a_m| = |a_m - a_{n_0+1} + a_{n_0+1}| < |a_m - a_{n_0+1}| + |a_{n_0+1}| < 1 + |a_{n_0+1}|,\ \forall m > n_0$, i.e., $(a_n)_n$ is bounded above.

Now, let $(b_n)_n \in \mathcal{N}$ and $\varepsilon > 0$. Then, $\exists n_1' \in \mathbb{N}$ such that

$$|b_n| < \frac{\varepsilon}{1 + |a_{n_0+1}|},\ \forall n > n_1'.$$

Thus, if $n > \max\{n_0, n_1'\},$ then $|a_n b_n| < \varepsilon,$ i.e., $(a_n b_n)_n \in \mathcal{N}$.

We have seen that every sequence from $\mathcal{N}$ admits an upper bound.

Claim: Every sequence from $\mathcal{C} \setminus \mathcal{N}$ also admits a lower bound.

Suppose not: let $(a_n)_n \in \mathcal{C} \setminus \mathcal{N}$ be such that for all $\eta > 0$ and every $n_0 \in \mathbb{N}$, $|a_m| < \eta$ for some $m > n_1'$.

Now, given $\varepsilon > 0$, pick $n_1' \in \mathbb{N}$ such that

$$|a_p - a_q| < \frac{\varepsilon}{2} \ \forall p, q > n_1' \text{ (this is possible since } (a_n)_n \text{ is Cauchy).}$$

Let $m > n_1'$ be such that $|a_m| < \frac{\varepsilon}{2}$ (we are simply choosing $\eta = \frac{\varepsilon}{2}$).

Then, for all $p > n_1'$,

$$|a_p| \leq |a_p - a_m| + |a_m| < \varepsilon,$$

which contradicts the fact that $(a_n)_n \notin \mathcal{N}$. This proves the claim.

Claim: $\mathcal{N}$ is a maximal ideal.

Fix an arbitrary sequence $(a_n)_n \in \mathcal{C} \setminus \mathcal{N}$. Then, there exist some $m_0 \in \mathbb{N}$ and $\eta > 0$ such that $|a_n| > \eta$ for all $n > m_0$.

Define a sequence $(c_n)_n$ by $c_n = 1$ for $1 \leq n \leq M$, and $c_n = a_n^{-1}$ for $n > M$, where M will be chosen later.

Let $\varepsilon > 0$ be given. Since $(a_n)_n$ is Cauchy, there exists $n_0 \in \mathbb{N}$ such that $|a_p - a_q| < \varepsilon\eta^2$ for all $p, q > n_0$.

Set $M = \max\{n_0, m_0\}$. Then, for all $p, q > M$, we have

$$|c_p - c_q| = |a_p^{-1} - a_q^{-1}| = \left|\frac{a_q - a_p}{a_p a_q}\right| = |a_p - a_q| \cdot |a_p^{-1}|^{-1} \cdot |a_q|^{-1} \leq \varepsilon\eta^2 \cdot \frac{1}{\eta} \cdot \frac{1}{\eta} = \varepsilon.$$

This shows that $(c_n)_n$ is Cauchy, hence $(c_n)_n \in \mathcal{C}$. Consider now $(a_n c_n)_n$. We have $|a_n c_n| = 1$ for all $n > m_0$, i.e., $(a_n c_n)_n \to 1$, i.e., $(a_n c_n)_n - 1 \in \mathcal{N}$.

Thus, if we take $\mathcal{N}' = \mathcal{N} + \langle (c_n)_n \rangle$, then $\mathcal{N}' = \mathcal{C}$.

Since $(c_n)_n$ was arbitrary, this means $\mathcal{N}$ is maximal. Hence, $\widehat{K} \coloneqq \mathcal{C}/\mathcal{N}$ is a field. This concludes the proof of existence of $\widehat{K}$.

Existence and density of the embedding: We have an embedding $K \hookrightarrow \widehat{K}$ defined by $x \mapsto (x, x, x, \dots)$.

From the previously seen uniform continuity of $|\cdot|$, we have $||x| - |y||_0 \leq |x - y|$, where $|\cdot|_0$ denotes the usual absolute value.

Thus, for all $(a_n)_n \in \mathcal{N}$, we have $\lim_{n\to\infty} |a_n| = 0$.

Hence, for $\xi = (a_n)_n + \mathcal{N}$, the value $\lim_{n\to\infty} |a_n|$ does not depend on the representative $(a_n)_n$ of ξ. We define

$$\widehat{\xi} \coloneqq \lim_{n\to\infty} |a_n|.$$

By the properties of limits and $|\cdot|$, it follows that $\widehat{K}$ is an absolute value on $\widehat{K}$ that induces $|\cdot|$ on K.

Claim: $i(K)$ is dense in $\widehat{K}$ with respect to $|\cdot|$.

Recall that a subspace Y of a metric space X is *dense* if

$$\forall \varepsilon > 0, x \in X, \exists y \in Y \text{ such that } d(x,y) < \varepsilon, \text{i.e., } B(x,\varepsilon) \cap Y \neq \emptyset.$$

Fix $(x_n)_n \in \mathcal{C}$. Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \ \forall n, m \geq N.$$

In particular, $|x_n - x_N| < \varepsilon \ \forall n \geq N$.

Therefore,

$$\lim_{n\to\infty} |x_n - x_m| \leq \varepsilon \implies \widehat{|(x_n)_n - (x_N)_n + \mathcal{N}|} = \widehat{|(x_n)_n - i(x_N) + \mathcal{N}|} \leq \varepsilon.$$

Hence, given any $(x_n)_n + \mathcal{N} \in \widehat{K}$, we have a corresponding element $i(x_N) \in i(K)$ such that $\widehat{|(x_n)_n + \mathcal{N} - i(x_N)|} \leq \varepsilon$.

Therefore, $i(K)$ is dense in $\widehat{K}$ with respect to $\widehat{|\cdot|}$.

We now prove the completeness of $\widehat{K}$.

We have shown that $i(K)$ is dense in $\widehat{K}$. Hence, if we show that every Cauchy sequence in $i(K)$ converges to a point in $\widehat{K}$, then we are done (by lemma 1.1.1).

Consider a Cauchy sequence $(\mathcal{Z}_n)_n \subseteq i(K)$, i.e., $\mathcal{Z}_n = (z_n, z_n, \dots, z_n) + \mathcal{N}$. We have $\widehat{|\mathcal{Z}_n - \mathcal{Z}_m|} = |z_n - z_m|$, by definition.

Since $(\mathcal{Z}_n)_n$ is Cauchy, $\widehat{|\mathcal{Z}_n - \mathcal{Z}_m|} = |z_n - z_m| < \varepsilon$ for $n, m \geq M$.

Hence, $(z_n)_n = (z_1, z_2, z_3, \dots)$ is a Cauchy sequence in K.

Consider $(z_n)_n + \mathcal{N} \coloneqq \mathcal{Z}^*$. We will show $(\mathcal{Z}_n) \to \mathcal{Z}^*$.

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|z_k - z_n| < \varepsilon/2 \ \forall k, n \geq N$.

Fixing $k \geq N$, we get $|z_k - z_n| < \varepsilon/2 \ \forall n \geq N \implies \lim_{n\to\infty} |z_k - z_n| \leq \varepsilon/2$.

Now,

$$\begin{aligned}
\widehat{|\mathcal{Z}_k - \mathcal{Z}^*|} &= \widehat{|(z_k, z_k, \dots) - (z_1, z_2, \dots) + \mathcal{N}|} \\
&= \widehat{|(z_k - z_1, z_k - z_2, \dots) + \mathcal{N}|} \\
&= \lim_{n\to\infty} |z_k - z_n| \leq \varepsilon/2 < \varepsilon
\end{aligned}$$

Hence, $(\mathcal{Z}_n) \to \mathcal{Z}^*$.

Uniqueness of $(\widehat{K}, i)$: let $(\widehat{K}', i')$ be another pair with the same properties.

For every $\xi = (a_n)_n + \mathcal{N} \in \widehat{K}$, we have a sequence $(i'(a_n))_n$ in $\widehat{K}'$. Since $(a_n)_n$ is Cauchy in K, $(i'(a_n))_n$ is Cauchy in $\widehat{K}'$. Let its limit be $\xi' \in \widehat{K}'$.

Define $\varphi : \widehat{K} \to \widehat{K}' : \xi \mapsto \xi'$. φ is well-defined, by uniqueness of limits.

Given $\eta' \in \widehat{K}'$, there exists a sequence $(i'(b_n))_n \subset i'(K)$ which converges to η'. Consider $(b_n)_n \subset K$ and let $\lim_{n\to\infty} i(b_n) = \eta \in \widehat{K}$. Then, $\varphi(\eta) = \eta'$ by definition.

Hence, φ is surjective.

Now, let $\xi_1, \xi_2 \in \widehat{K}$ with associated sequences $(i'(a'_n))_n$ and $(i'(b'_n))_n$. Then, $\xi_1 + \xi_2$ is associated to $(i'(a'_n) + i'(b'_n))_n = (i'(a'_n + b'_n))_n$, since i' is a homomorphism.

Similarly, $\xi_1\xi_2$ is associated to $(i'(a'_n))_n(i'(b'_n))_n = (i'(a'_n b'_n))_n$.

Thus,

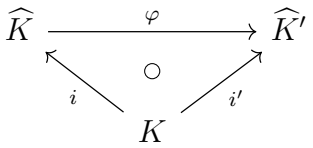
$$\varphi(\xi_1 + \xi_2) = \lim(i'(a'_n + b'_n)) = \lim i'(a'_n) + \lim i'(b'_n) = \varphi(\xi_1) + \varphi(\xi_2).$$

Similarly,

$$\varphi(\xi_1\xi_2) = \varphi(\xi_1)\varphi(\xi_2).$$

Thus, φ is a homomorphism.

Since it is a nontrivial field homomorphism, φ must be injective. Therefore, φ is an isomorphism, and by its very definition, φ makes the diagram below commute.



☕

Remark 1.2. The pair $(\widehat{K}, \widehat{|\cdot|})$ is called a ***completion*** of $(K, |\cdot|)$.

Proposition 1.1.3. *Every Archimedean absolute value on $\mathbb{Q}$ is dependent on the usual one.*

Proof. Let $|\cdot|$ be an Archimedean absolute value on $\mathbb{Q}$. Let $|\cdot|_0$ denote the usual absolute value on $\mathbb{Q}$.

Let $m, n \geq 2$ be arbitrary integers and let $t \geq 1$.

Expand m^t in powers of n: $m^t = \sum_{i=0}^{s} c_i n^i$, where $0 \leq c_i < n$, $c_s \neq 0$.

Now, $|c_i| = |1 + 1 + \cdots + 1| \leq c_i|1| = c_i \leq n$ ($\because |1|^2 = |1^2| = |1| \implies |1| = 1$).

Hence, for each $0 \leq i \leq s$,

$$|m|^t \leq \sum_{i=0}^{s} |c_i||n|^i \leq n \sum_{i=0}^{s} |n|^i \leq n(s+1)\max\{1, |n|^s\},$$

depending on whether $|n| \geq 1$ or $|n| < 1$.

Since $n^s \leq m^t$, we have $s\ln(n) \leq t\ln(m) \implies s \leq t\dfrac{\ln(m)}{\ln(n)}$.

Hence,

$$|m|^t \leq n\left(\frac{t\ln(m)}{\ln(n)}+1\right)\max\{1,|n|^{\frac{t\ln(m)}{\ln(n)}}\} \implies |m| \leq n^{1/t}\left(\frac{t\ln(m)}{\ln(n)}+1\right)^{1/t}\max\left\{1,|n|^{\frac{\ln(m)}{\ln(n)}}\right\}.$$

As $t \to \infty$, we get $|m| \leq \max\left\{1,|n|^{\frac{\ln(m)}{\ln(n)}}\right\}$. Note that we can take t-th root on both sides in the step above since all the quantities above are positive and real.

Suppose $|n| < 1$ for some n. Then by the above inequality, $|m| \leq 1$ for all $m \geq 2$. This contradicts Archimedeanness.

Hence, $\forall n \geq 2$ we must have $|n| \geq 1$. Thus, $|m| \leq |n|^{\frac{\ln(m)}{\ln(n)}}$.

Interchanging m, n above gives $|n| \leq |m|^{\frac{\ln(n)}{\ln(m)}}$, hence we get $|n| = |m|^{\frac{\ln(n)}{\ln(m)}}$.

Hence, if $m > n \geq 2$, then $\frac{\ln(m)}{\ln(n)} > 1$, so $|m| > |n|$.

By definition, $|-m| = |m|$. Therefore, $|mb| > |nb| \implies |m| > |n|$.

Consequently, if $m/n \in \mathbb{Q}$ satisfies $|m/n|_0 < 1$, then $|m/n| < 1$.

Thus, we have shown $|x|_0 < 1 \implies |x| < 1\ \forall x \in \mathbb{Q}$, i.e., $|\cdot|$ and $|\cdot|_0$ are dependent.

☕

Up till now, we have restricted ourselves to Archimedean absolute values. Let us now consider nontrivial, non-Archimedean absolute values on a field K. For future convenience, we use the additive presentation of the absolute value: we define

$$v : K \to \mathbb{R} \cup \{\infty\} \text{ with } v(x) \coloneqq -\ln|x|.$$

Then, the axioms read as follows, for all $x, y \in K$:

1. $v(x) \in \mathbb{R}$ for all $x \neq 0$, and $v(0) = \infty$
2. $v(xy) = v(x) + v(y)$
3. $v(x+y) \geq \min\{v(x), v(y)\}$

Remark 1.3. Note that we only use the additive structure of $\mathbb{R}$ and its ordering, hence we can (and will) generalize this definition later by requiring the target of v to just be an ordered abelian group unioned with a formal symbol ∞.

Now assuming that v is a non-Archimedean absolute value on K, the set

$$\mathcal{O}_v \coloneqq \{x \in K \mid v(x) \geq 0\}$$

is a subring of K. This is easy to check: for any $x, y \in \mathcal{O}_v$, we have

$$v(x \pm y) \geq \min\{v(x), v(\pm y)\} \geq 0 \text{ and } v(xy) = v(x) + v(y) \geq 0.$$

Hence $x \pm y, xy \in \mathcal{O}_v$. (Recall that $|-y| = |y|$.) From $v(x^{-1}) = -v(x)$ (recall $|x^{-1}| = |x|^{-1}$), we find that x is a unit in $\mathcal{O}_v$ if and only if $v(x) = 0$, and that for every $x \in K$, either x or x^{-1} or both lie in $\mathcal{O}_v$.

Definition 1.1.5. An integral domain $\mathcal{O}$ of K satisfying

$$x \in \mathcal{O} \text{ or } x^{-1} \in \mathcal{O} \text{ for all } x \in K^\times$$

is called a ***valuation ring*** of K. Thus $\mathcal{O}_v$ is a valuation ring of K.

Moreover, $\mathcal{M}_v \coloneqq \{x \in K \mid v(x) > 0\} \lhd \mathcal{O}_v$, because for any $x \in \mathcal{O}_v, t \in \mathcal{M}_v$, we have $v(xt) = v(x) + v(t) \geq v(t) > 0$, i.e., the product $xt \in \mathcal{M}_v$.

Now, notice that x is a unit in $\mathcal{O}_v$ if and only if $x, x^{-1} \in \mathcal{O}_v \iff v(x) \geq 0$ and $v(x^{-1}) \geq 0 \iff v(x) \geq 0$ and $-v(x) \geq 0 \iff v(x) = 0$ (since $v(x^{-1}) = -v(x)$).

Moreover, $\mathcal{M}_v$ is the unique maximal ideal because if $\mathfrak{m} \supsetneq \mathcal{M}_v$ was another, then it must contain a unit, and so $\mathfrak{m} = \mathcal{O}_v$.

In summary, $\mathcal{M}_v$ is the unique maximal ideal consisting exactly of the non-units of $\mathcal{O}_v$. Therefore, $\mathcal{O}_v$ is a local ring.

Definition 1.1.6. $\overline{\mathcal{K}}_v \coloneqq \mathcal{O}_v/\mathcal{M}_v$ is called the ***residue class field*** of v. The residue class of $a \in \mathcal{O}_v$ is denoted by $\overline{a}$. Note that v is trivial if and only if $\mathcal{O}_v = K$, and hence also $\overline{\mathcal{K}}_v = K$. The group $v(K^\times)$ is called the ***value group*** of v.

Example 1.3. Consider $K = \mathbb{Q}$ and $v = v_p$, the p-adic valuation. Recall that $v_p(p^\nu \frac{m}{n}) = \nu$. We then have:

$$\mathcal{O}_{v_p} = \{x \in \mathbb{Q} \mid v_p(x) \geq 0\} = \left\{ \frac{a}{b} \in \mathbb{Q} \,\middle|\, a, b \in \mathbb{Z},\ p \nmid b \right\} \text{ and}$$

$$\mathcal{M}_{v_p} = \{x \in \mathbb{Q} \mid v_p(x) > 0\} = \left\{ \frac{ap}{b} \in \mathbb{Q} \,\middle|\, a, b \in \mathbb{Z},\ p \nmid b,\ p \mid a \right\}.$$

Note that $\mathcal{O}_v = \mathbb{Z}_{(p)}$, the localisation of $\mathbb{Z}$ at $p\mathbb{Z}$ and $\mathcal{M}_v = p\mathcal{O}_v = p\mathbb{Z}_{(p)}$. Hence,

$$\overline{\mathcal{K}}_v = \mathcal{O}_v/\mathcal{M}_v = \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} \equiv \mathbb{F}_p.$$

Example 1.4. Take $k[x]$, where k is a field, and the absolute value is given by $|p^\nu f/g|_p = 1/e^\nu$, where $\nu \in \mathbb{Z}$, and $f, g \in k[x] \setminus \{0\}$ are not divisible by p.

Now, $v\left(p^\nu f/g\right) = -\ln|p^\nu f/g|_p = -\ln\left(1/e^\nu\right) = \nu$.

Similar to above, we get $\mathcal{O}_v = k[x]_{(p)}$, the localisation of $k[x]$ at the prime ideal $(p) = pk[x]$. The maximal ideal is $pk[x]_{(p)}$, and $\overline{\mathcal{K}}_v = k[x]_{(p)}/pk[x]_{(p)} = k[x]/pk[x]$.

Proposition 1.1.4. *For any $x, y \in K$ with $v(x) < v(y)$, we have $v(x+y) = v(x)$.*

Proof. Suppose not, i.e., there exist some $x, y \in K$ with $v(x) < v(y)$ such that $v(x+y) \neq v(x)$. Now, since $v(x+y) \geq \min\{v(x), v(y)\}$, we must have $v(x+y) > v(x)$.

Then, since $v(y) > v(x)$ and $v(x+y) > v(x)$, we get

$$v(x) = v((x+y)+(-y)) \geq \min\{v(x+y), v(-y)\} = \min\{v(x+y), v(y)\} > v(x)$$

This is a contradiction. ☕

Remark 1.4. $b \in \mathcal{O}_v$, $\overline{b} = 0$ in $\overline{\mathcal{K}}_v$ if and only if $b \in \mathcal{M}_v$, i.e., $v(b) > 0$, and $\overline{b} \neq 0$ in $\overline{\mathcal{K}}_v$ if and only if $b \in \mathcal{O}_v$ but $b \notin \mathcal{M}_v$, i.e., $v(b) = 0$.

Lemma 1.1.2. *Given a polynomial $f(x) \in R[x]$, where R is a commutative ring with unity, there exists a polynomial $g(x, y) \in R[x, y]$ such that*

$$f(x+y) = f(x) + yf'(x) + y^2 g(x, y).$$

Proof. We have $f(x) = \sum_{i=0}^{n} a_i x^i$, thus $f(x+y) = \sum_{i=0}^{n} a_i (x+y)^i$. Hence,

$$\begin{aligned} f(x+y) &= a_0 + \sum_{i=1}^{n} a_i (x^i + i x^{i-1} y) + \sum_{i=1}^{n} g_i(x, y) y^2 \text{ for } g_i \in R[x, y], \\ &= \sum_{i=0}^{n} a_i x^i + \sum_{i=1}^{n} i a_i x^{i-1} y + g(x, y) y^2, \text{ where } \sum_{i=1}^{n} g_i(x, y) = g(x, y) \\ &= f(x) + yf'(x) + y^2 g(x, y). \end{aligned}$$

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Theorem 1.1.3 (Hensel's Lemma)**.** *Let K be a field complete with respect to a non-Archimedean absolute value v. Let $f \in \mathcal{O}_v[x]$ be a polynomial and $a_0 \in \mathcal{O}_v$ such that $v(f(a_0)) > 2v(f'(a_0))$.*

Then, there exists $a \in \mathcal{O}_v$ with $f(a) = 0$ and $v(a - a_0) > v(f'(a_0))$.

Proof. Let $b_0 = f'(a_0)$ and choose $\varepsilon > 0$ such that $v(f(a_0)) \geq v(b_0^2) + \varepsilon$.

This is possible because $v(f(a_0)) > 2v(f'(a_0)) = 2v(b_0) = v(b_0^2)$

Then, $f(a_0) = b_0^2 c_0$, where $c_0 = \frac{f(a_0)}{b_0^2} \in K$ and $v(c_0) \geq \varepsilon$.

Set $a_1 \coloneqq a_0 - b_0 c_0$. By the lemma above, there exists $d_0 \in K$ such that

$$f(a_1) = f(a_0 - b_0 c_0) = f(a_0) - b_0 c_0 f'(a_1) + (b_0 c_0)^2 d_0 = b_0^2 c_0 d_0.$$

Since $c_0, b_0, c_0 \in \mathcal{O}_v$, $d_0 \in \mathcal{O}_v$. Therefore,

$$v(f(a_1)) = v(b_0^2 c_0^2 d_0) = v(b_0^2) + 2v(c_0 d_0) \geq v(b_0^2) + 2\varepsilon \tag{1}$$

Applying the same procedure to f', we get

$$f'(a_1) = f'(a_0 - b_0 c_0) = f'(a_0) - b_0 c_0 b = b_0 - b_0 c_0 b = b_0(1 - c_0 b) =: b_1 \text{ for some } b \in \mathcal{O}_v.$$

Now, since $v(1 - c_0 b) = v(1) = 0$, as $v(1) < v(c_0 b)$, we have:

$$v(b_1) = v(b_0) + v(1 - c_0 b) = v(b_0).$$

Hence, (1) implies

$$v(f(a_1)) \geq v(b_1^2) + 2\varepsilon, \text{ i.e., } f(a_1) = b_1^2 c_1 \text{ where } v(c_1) > 2\varepsilon.$$

We now repeat the above argument after replacing b_0 with b_1, a_1 with $a_2 := a_1 - b_1 c_1$, and ε with 2ε, to obtain a b_2 with $f'(a_2) = b_2$ such that $v(b_2) = v(b_0)$ and $f(a_2) = b_2^2 c_2$ for some c_2 with $v(c_2) \geq 2^2\varepsilon$. This goes as follows:

We have $f(a_1) = b_1^2 c_1$, where $v(c_1) > 2\varepsilon$, so by the lemma above, there exists $d_1 \in K$ with

$$f(a_2) = f(a_1 - b_1 c_1) = f(a_1) - b_1 c_1 f'(a_2) + (b_1 c_1)^2 d_1 = b_1^2 c_1 d_1.$$

Since $c_1, b_1, c_1 \in \mathcal{O}_v$, we get $d_1 \in \mathcal{O}_v$. Therefore,

$$v(f(a_2)) = v(b_1^2 c_1^2 d_1) = v(b_1^2) + 2v(c_1 d_1) \geq v(b_1^2) + 4\varepsilon \tag{2}$$

Applying the same procedure to f', we get

$$f'(a_2) = f'(a_1 - b_1 c_1) = f'(a_1) - b_1 c_1 b = b_1 - b_1 c_1 b = b_1(1 - c_1 b) =: b_2 \text{ for some } b \in \mathcal{O}_v.$$

Now, $v(1 - c_1 b) = v(1) = 0, \text{as } v(1) < v(c_1 b)$, hence we get

$$v(b_2) = v(b_1) + v(1 - c_1 b) = v(b_1).$$

Therefore, (2) implies

$$v(f(a_2)) \geq v(b_2^2) + 4\varepsilon, \text{ i.e., } f(a_2) = b_2^2 c_2 \text{ where } v(c_2) > 4\varepsilon.$$

Iteratively continuing, we get a sequence $a_{n+1} = a_n - b_n c_n$, where

$$f'(a_{n+1}) = b_{n+1}, v(b_{n+1}) = v(b_0), \ f(a_{n+1}) = b_{n+1}^2 c_{n+1}, \text{ and } v(c_{n+1}) \geq 2^{n+1}\varepsilon.$$

Claim: $(a_n)_n$ is Cauchy: For $m < n$, we have:

$$\begin{aligned} v(a_n - a_m) &= v\left(\sum_{i=m}^{n-1}(a_{i+1} - a_i)\right) \\ &\geq \min_{m\leq i<n} v(a_{i+1} - a_i) \\ &= \min_{m\leq i<n} v(b_i c_i) \geq v(b_0) + 2^m\varepsilon. \end{aligned}$$

Hence, $\lim_{m\to\infty} v(a_n - a_m) = \infty$.

Thus, $\lim_{m\to\infty} |a_n - a_m| = 0$, since $v(x) = -\ln|x| = \ln(1/|x|)$.

Let $a = \lim_{n\to\infty} a_n$. Then, since the polynomial f is continuous, $f(a) = \lim_{n\to\infty} f(a_n)$.

Claim: $(b_n)_n \to f'(a)$: Since $a_n \to a$, we have $f'(a_n) \to f'(a)$, i.e., $b_n \to f'(a)$.

Now, $v(f(a_n)) = v(b_n^2) + v(c_n) \geq 2^n\varepsilon$ for all n.

Hence,

$$\lim_{n\to\infty} v(f(a_n)) = \infty, \text{ i.e., } \lim_{n\to\infty} f(a_n) = 0, \text{ i.e., } f(a) = 0.$$

Furthermore,

$$v(f'(a_n)) = v(b_n) = v(b_0) \implies v(f'(a)) = v(b_0).$$

From $v(a_n - a_m) \geq v(\sum_{i=m}^{n-1}(a_{i+1} - a_i))$, we get $(a_n - a_0) \geq v(b_0) + \varepsilon$.

Hence, for n sufficiently large

$$v(a - a_0) = v((a - a_n) + (a_n - a_0)) \geq v(b_0) + \varepsilon > v(b_0) = v(f'(a_0)).$$

Therefore, our choice $a = \lim_{n\to\infty} a_n$ satisfies the assertions of the theorem. ☕

Corollary 1.1.3.1. *Let K be a field complete with respect to a non-Archimedean absolute value v. Let $f \in \mathcal{O}_v[X]$ be a polynomial having a simple zero $\bar{a}_0$ in $\overline{\mathcal{K}_v}$, i.e., $\bar{f}(\bar{a}_0) = 0$ and $\bar{f}'(\bar{a}_0) \neq 0$. Then, f has a zero $a \in \mathcal{O}_v$ such that $\bar{a} = \bar{a}_0$ in $\overline{\mathcal{K}_v}$.*

Proof. Since $\bar{f}(\bar{a}_0) = 0$, we have $f(a_0) \in \mathcal{M}_v$, i.e., $v(f(a_0)) > 0$ and $f'(a_0) \notin \mathcal{M}_v$, i.e., $v(f'(a_0)) = 0$.

Hence, by the theorem, $\exists a \in \mathcal{O}_v$ such that $f(a) = 0$ and $v(a - a_0) > v(f(a_0)) = 0$, i.e., $a - a_0 \in \mathcal{M}_v$, i.e., $\bar{a} = \bar{a}_0$. ☕

Let $(\widehat{K}, \hat{v})$ denote the completion of (K, v).

Theorem 1.1.4. *Denote by $\mathcal{O}_{\hat{v}}$, $\overline{\mathcal{K}_{\hat{v}}}$ and $\mathcal{O}_v$, $\overline{\mathcal{K}_v}$ the valuation ring and residue class field of $\hat{v}$ and v, respectively. Then, $\overline{\mathcal{K}_v}$, $\overline{\mathcal{K}_{\hat{v}}}$ and $v(K^\times)$, $\hat{v}(\widehat{K}^\times)$ are canonically isomorphic.*

Proof. Consider $\mathcal{O}_{\hat{v}} = \{x \in \widehat{K} : \hat{v}(x) \geq 0\}$.

Thus,

$$\mathcal{O}_{\hat{v}} \cap K = \{x \in K : \hat{v}(x) \geq 0\} = \{x \in K : v(x) \geq 0\} = \mathcal{O}_v$$

since $\hat{v}|_K = v$.

Similarly, $\mathcal{M}_{\hat{v}} \cap K = \mathcal{M}_v$.

Consider the map $\mathcal{O}_v \to \mathcal{O}_{\hat{v}}/\mathcal{M}_{\hat{v}} : a \mapsto \bar{a}$.

It is well-defined since $a - b = 0$ in $\mathcal{O}_v \implies a - b \in \mathcal{M}_v \subseteq \mathcal{M}_{\hat{v}}$, thus $\bar{a} = \bar{b}$. It is clear that the map is a ring morphism as well.

We now show surjectivity: $\forall \bar{x} \in \mathcal{O}_{\hat{v}}/\mathcal{M}_{\hat{v}},\ x + \mathcal{M}_{\hat{v}}$ is an open neighbourhood of x, since it consists of all z such that $\hat{v}(x - z) > 0$, i.e., $\widehat{|x - z|} < 1$.

We know that K is dense in $\widehat{K}$, thus $(x + \mathcal{M}_{\hat{v}}) \cap K \neq \varnothing$.

Consider $y \in (x + \mathcal{M}_{\hat{v}}) \cap K$, then $y \mapsto \bar{x}$.

Hence, given any $\bar{x} \in \mathcal{O}_{\hat{v}}/\mathcal{M}_{\hat{v}}$, $\exists y \in K$ such that $y \mapsto \bar{x}$. Thus, by the first isomorphism theorem, we obtain $\mathcal{O}_v/\mathcal{M}_v \xrightarrow{\sim} \mathcal{O}_{\hat{v}}/\mathcal{M}_{\hat{v}}$; that is $\overline{\mathcal{K}_v} \cong \overline{\mathcal{K}_{\hat{v}}}$.

Consider $v(K^\times) \to \hat{v}(\widehat{K}^\times)$. This is a group monomorphism as $K^\times \subset \widehat{K}^\times$.

To see its surjectivity, take $x \in \widehat{K}^\times$, then $\exists z \in K$ such that $\widehat{|z - x|} < |x|$, i.e.,

$$\hat{v}(z - x) > \hat{v}(x) = v(x) \implies \hat{v}(z) = \hat{v}(x), \text{ implying } \hat{v}(z) = v(x).$$

Therefore, as before, we get $v(K^\times) \xrightarrow{\sim} \hat{v}(\widehat{K}^\times)$. ☕

Definition 1.1.7. An absolute value v is called ***discrete*** *(of rank 1)* if $v(K^\times) = (\mathbb{Z}, +)$.

Any $\pi \in K$ with $v(\pi) = 1$ is called a ***uniformizer*** or a ***local parameter*** for v.

Now, if $v(x) = r$, then $v(x\pi^{-r}) = v(x) - rv(\pi) = v(x) - v(x) = 0$, i.e., $x\pi^{-r} = u$, where u is a unit of $\mathcal{O}_v$.

Hence, every $x \in K^\times$ can be written as $x = \pi^r u$.

In particular, any $x \in \mathcal{M}_v$ can be written as $x = u\pi^r$, and thus $\mathcal{M}_v = \langle \pi \rangle$.

Given any other ideal $\mathfrak{a} \lhd \mathcal{O}_v$, $\mathfrak{a} \subseteq \mathcal{M}_v$ (every ideal is contained in a maximal ideal and $\mathcal{M}_v$ is the unique maximal ideal of $\mathcal{O}_v$).

Thus, $\mathfrak{a}$ is principal. Therefore, $\mathcal{O}_v$ is a principal ideal domain, hence factorial (i.e., a unique factorisation domain).

Proposition 1.1.5. *Let v be a discrete absolute value on K with uniformizer π. Then, every $x \in K^\times$ can be written uniquely as a convergent series*

$$x = r_\nu \pi^\nu + r_{\nu+1}\pi^{\nu+1} + r_{\nu+2}\pi^{\nu+2} + \cdots = \lim_{n \to \infty} \sum_{i=\nu}^{n} r_i \pi^i,$$

where $\nu = v(x)$, $r_\nu \neq 0$, and the coefficients r_i are taken from a set $R \subseteq \mathcal{O}_v$ of representatives of the residue classes in the field K_v (i.e., the canonical map $\mathcal{O}_v \to K_v$ induces a bijection of R onto K_v).

Proof. As before, $u = x\pi^{-\nu}$ is a unit in $\mathcal{O}_v$. Choose $r_\nu \in R$ such that $\overline{r_\nu} = \overline{u}$, i.e., $r_\nu - u \in \mathcal{M}_v$, i.e., $v(r_\nu - u) > 0$, and $v(r_\nu - x\pi^{-\nu}) > 0$.

Thus,

$$v(x - r_\nu\pi^\nu) > 0 \implies v(x - r_\nu\pi^\nu + \pi^\nu) > 0$$

which implies

$$v(x - r_\nu\pi^\nu) > v(\pi^\nu) = \nu v(\pi) = \nu.$$

Let $x_1 = x - r_\nu\pi^\nu$ and $\mu = v(x_1) > \nu$.

By the same argument, we can choose $r_\mu \in R$ such that $\overline{r_j} = \overline{x_1}$, and get

$$v(x - (r_\nu\pi^\nu + r_\mu\pi^\mu)) = v(x_1 - r_\mu\pi^\mu) > -v(\pi^{-\mu}) = \mu.$$

Repeating the same and adding zero coefficients (i.e., $\overline{r_\alpha} = \overline{0}$) as necessary, we obtain a series:

$$r_\nu\pi^\nu + r_{\nu+1}\pi^{\nu+1} + \cdots = \sum_{i=0}^{\infty} r_\nu\pi^\nu.$$

Now, $v(x - r_\nu\pi^\nu) > \nu$, and $v(x - (r_\nu\pi^\nu + r_\mu\pi^\mu)) > \mu > \nu$.

Hence, we have an increasing nonconstant sequence

$$y_n \coloneqq v\left(x - \sum_{i=\nu}^{\nu+n} r_i\pi^{r_i}\right) \text{ i.e., } y_1 \leq y_2 \leq \cdots \to \infty.$$

Therefore,

$$v\left(x - \sum_{i=\nu}^{\infty} r_i\pi^{r_i}\right) = \infty, \text{ i.e., } x - \sum_{i=0}^{\infty} r_i\pi^{r_i} = 0.$$

For uniqueness, suppose $x = \sum_{i=0}^{\infty} r_i'\pi^i$, then we have

$$0 = x - x = \sum_{i=\nu}^{\infty}(r_i - r_i')\pi^i$$

with $r_m \neq r_m' \in R$ i.e., $\overline{r_m - r_m'} \neq 0$ for some $m \in \mathbb{N}$. Thus, $v(0) = m$, which is a contradiction. ☕

Remark 1.5. It follows from above that any p-adic number $z \in \mathbb{Q}_p^\times$ has a unique representation of the form

$$z = \sum_{i=m}^{\infty} \alpha_i p^i, \text{ where } m = v_p(z), \alpha_i \in \{0, 1, \ldots, p-1\}.$$

If $z \in \mathbb{Z}_p$, then $v(z) > 0$ and $z = \sum_{i=0}^{\infty} \alpha_i \pi^i = \lim_{n\to\infty} \sum_{i=0}^{n} \alpha_i \pi^i$, i.e., given any $z \in \mathbb{Z}_p$, we have a sequence $z_n \coloneqq \sum_{i=0}^{n} \alpha_i p^i \in \mathbb{Z}_p$ such that $z_n \to z$. **Hence, $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.**

1.2 General Valuations

Definition 1.2.1. We define an ***ordered abelian group*** as an abelian group $(\Gamma, +, 0)$, with a binary relation $\leq$ on Γ such that, for any δ, γ and $\lambda \in \Gamma$, we have:

$$\begin{aligned}
&(1) \quad \gamma \leq \gamma \\
&(2) \quad \gamma \leq \delta,\ \delta \leq \gamma \implies \gamma = \delta \\
&(3) \quad \gamma \leq \delta,\ \delta \leq \lambda \implies \gamma \leq \lambda \\
&(4) \quad \gamma \leq \delta \text{ or } \delta \leq \gamma \\
&(5) \quad \gamma \leq \delta \implies \gamma + \lambda \leq \delta + \lambda.
\end{aligned}$$

Definition 1.2.2. A subgroup $\Delta \subseteq \Gamma$ is ***convex*** if for any $\gamma \in \Gamma$ satisfying $0 \leq \gamma \leq \delta$ with $\delta \in \Delta$, we have $\gamma \in \Delta$.

The collection of all proper convex subgroups of Γ is linearly ordered by inclusion: Given convex subgroups $A, B \subseteq \Gamma$, if $A \subset B$ or $B \subset A$, we are done. Suppose not; pick $a \in A \setminus B$ and $b \in B \setminus A$ and assume without loss of generality that $0 \leq a, b$ (otherwise we work with $-a$ or $-b$). Now, either $a \leq b$ or $a \geq b$. If $a \leq b$, then by convexity $a \in B$, a contradiction. If $a \geq b$, then by convexity $b \in A$, again a contradiction.

The order type of this collection is called the ***rank*** of Γ. Hence, if there are exactly n proper convex subgroups of Γ, then Γ is of rank n. In particular, if $\{0\}$ is the only proper convex subgroup of Γ, then Γ is of rank 1.

Definition 1.2.3. An ordering $\leq$ of an ordered abelian group Γ is ***Archimedean*** if $\forall \gamma, \varepsilon \in \Gamma$ with $\varepsilon > 0, \exists n \in \mathbb{N}$ such that $\gamma \leq n\varepsilon$.

Lemma 1.2.1. *An Archimedean ordered abelian group Γ has rank 1.*

Proof. We need only show that Γ admits no nontrivial proper convex subgroups. Proceeding by contradiction, suppose $\Delta \subseteq \Gamma$ is such a subgroup and let $\delta \in \Delta$. Given any $\gamma \in \Gamma, \exists n \in \mathbb{N}$ such that $\gamma \leq n\delta$, since Γ is Archimedean. Since $\delta \in \Delta$, we have $n\delta = \delta + \delta + \cdots + \delta \in \Delta$, by virtue of Δ being a subgroup. Then, by convexity of Δ, we have $\gamma \in \Delta$ i.e., $\Gamma \subseteq \Delta \implies \Gamma = \Delta$.

This is the required contradiction, as Δ was assumed to be a proper subgroup. ☕

Lemma 1.2.2. *Every nontrivial subgroup Δ of $(\mathbb{R}, +, 0)$ is Archimedean with respect to the canonical ordering $\leq$ induced from $\mathbb{R}$.*

Proof. Let $\gamma, \varepsilon \in \Delta$, $\varepsilon > 0$. Then, $\gamma, \varepsilon \in \mathbb{R}$ and $\leq$ is Archimedean over $\mathbb{R}$, since $\gamma \leq n\varepsilon$ for $n = \lfloor \gamma/\varepsilon \rfloor + 1$.

Now, since Δ is a subgroup and $\varepsilon \in \Delta$, we get $n\varepsilon \in \Delta$. Hence, $\gamma \leq n\varepsilon$ in Δ and so Δ is Archimedean.

Therefore, Δ has rank 1, except for $\Delta = \{0\}$.

The converse is also true: ☕

Lemma 1.2.3. *An ordered abelian group Γ of rank 1 is order-isomorphic to a nontrivial subgroup of $(\mathbb{R}, +, 0)$ with the canonical ordering $\leq$ induced from $\mathbb{R}$.*

Proof. We first show that a rank-one ordered group Γ is Archimedean.

Given $\varepsilon \in \Gamma$, $\varepsilon > 0$, consider

$$\Delta \coloneqq \{\gamma \in \Gamma : -n\varepsilon \leq \gamma \leq n\varepsilon \text{ for some } n \in \mathbb{N}\}.$$

Clearly, $0 \in \Delta$ and $-\delta \in \Delta$ whenever $\delta \in \Delta$.

Suppose $\delta_1, \delta_2 \in \Delta$, then $\pm\delta_1 \leq n_1$ and $\pm\delta_2 \leq n_2$, implying $\pm(\delta_1 + \delta_2) \leq (n_1 + n_2)\varepsilon$, so $\delta_1 + \delta_2 \in \Delta$. Hence, Δ is a subgroup.

Convexity follows because given any $\delta \in \Delta$ and $0 \leq \tau \leq \delta$, we have $\tau \leq \delta \leq n\varepsilon \implies \tau \in \Delta$.

Now, $\varepsilon \in \Delta$ and $\varepsilon \neq 0$, but Γ has rank one, i.e., $\Gamma = \Delta$. Therefore, Γ is Archimedean since Δ is (by definition).

Fix any positive $\varepsilon \in \Gamma$. For each $\alpha \in \Gamma$, let

$$L(\alpha) = \{m/n \in \mathbb{Q} \mid n > 0, m \leq n\alpha\} \text{ and } U(\alpha) = \{m/n \in \mathbb{Q} \mid n > 0, m \geq n\alpha\}.$$

Recall: A ***Dedekind cut*** is a subset X of $\mathbb{Q}$ such that:

a) $\emptyset \neq X \subseteq \mathbb{Q}$, b) $q \in X, r < q \implies r \in X$, c) X has no largest member.

The real number associated to a Dedekind cut X is given by $x \in \mathbb{R}$ such that

$$X = \{\gamma \in \mathbb{Q} \mid \gamma \leq x\} = \{m/n \in \mathbb{Q} \mid m \leq nx\}.$$

In the sequel, we fix any positive $\varepsilon \in \Gamma$. For each $\alpha \in \Gamma$, let

$$L(\alpha) = \{m/n \in \mathbb{N} \mid n > 0, m\varepsilon \leq n\alpha\} \text{ and } U(\alpha) = \{m/n \in \mathbb{N} \mid n > 0, m\varepsilon \geq n\alpha\}.$$

Claim: For each $\alpha \in \Gamma$, $L(\alpha)$ and $U(\alpha)$ define a Dedekind cut.

Since Γ is ordered, either $m\varepsilon \leq n\alpha$ or $m\varepsilon \geq n\alpha$, i.e., every $m/n \in \mathbb{Q}$ lies in $L(\alpha)$ or $U(\alpha)$; i.e., $L(\alpha) \cup U(\alpha) = \mathbb{Q}$.

Clearly, $L(\alpha) \neq \mathbb{Q}$, otherwise $U(\alpha) = \mathbb{Q}$ and $m\varepsilon \leq \alpha$ for all $m \in \mathbb{Z}$, which contradicts the Archimedean property. Similarly, $U(\alpha) \neq \emptyset$.

Now, let $m/n \in L(\alpha)$ and $m'/n' \in U(\alpha)$, then $m\varepsilon \leq n\alpha$ and $m'\varepsilon \geq n'\alpha$, hence

$$mn'\varepsilon \leq n'n\varepsilon = nn'\alpha \leq nm'\varepsilon,$$

which gives $mn' \leq m'n \implies m/n \leq m'/n'$, i.e., for all $\beta \in L(\alpha)$ and $\beta' \in U(\alpha)$, we have $\beta \leq \beta'$.

Consider $q \in L(\alpha)$, i.e., $q\varepsilon \leq \alpha$, hence for every $r < q$, we have $r\varepsilon \leq q\varepsilon \leq \alpha \implies r \in L(\alpha)$. Thus, $L(\alpha)$ is a Dedekind cut. Let its associated real number be $r(\alpha)$.

Consider the map $\Gamma \to (\mathbb{R}, +, 0) : \alpha \mapsto r(\alpha)$. Clearly, $\alpha \leq \beta \implies L(\alpha) \subseteq L(\beta) \implies r(\alpha) \leq r(\beta)$.

Claim: This map is a group monomorphism.

For $\alpha, \beta \in \Gamma$, let $m/n \in L(\alpha)$ and $m'/n' \in L(\beta)$. Thus, $m/n \leq \alpha/\varepsilon$ and $m'/n' \leq \beta/\varepsilon$.

Now, we have $m'/n' \leq \beta/\varepsilon$ if and only if $m'n/n' \leq \beta/\varepsilon$ and similarly, $m/n \leq \alpha/\varepsilon$ if and only if $mn'/nn' \leq \alpha/\varepsilon$. Thus, after replacing m by mn' and m' by $m'n$, we get the same denominators. Hence, we may assume $n = n'$ without loss of generality.

Thus, $m\varepsilon \leq n\alpha$ and $m'\varepsilon \leq n\beta$, which implies $(m + m')\varepsilon \leq n(\alpha + \beta)$, i.e., $(m + m')/n \in L(\alpha + \beta)$. Hence, $r(\alpha + \beta) \geq r(\alpha) + r(\beta)$.

Similarly, $U(\alpha) + U(\beta) \subseteq U(\alpha + \beta)$ implies $r(\alpha + \beta) \leq r(\alpha) + r(\beta)$, i.e., $r(U(\alpha + \beta)) = r(U(\alpha)) + r(U(\beta))$. Thus, r is a group homomorphism.

Now, take $\alpha \in \ker(r)$, i.e., $0 = r(\alpha) = \sup L(\alpha) = \inf U(\alpha)$. Hence, $-1/n \in L(\alpha)$ and $1/n \in U(\alpha)$ for all $n > 0$. Thus, $-\varepsilon \leq n\alpha \leq \varepsilon$ for all $n > 0$, which implies $\alpha = 0$, since Γ is Archimedean. Hence r has trivial kernel, and this proves the claim.

Since we have an order-preserving group monomorphism $\Gamma \hookrightarrow \mathbb{R}$, Γ is order-isomorphic to a nontrivial subgroup, namely its image, of $(\mathbb{R}, +, 0)$ with the canonical ordering $\leq$ induced from $\mathbb{R}$.

☕

Hence, combining the previous two lemmas, we obtain:

Proposition 1.2.1. *An ordered abelian group* Γ *has rank 1 if and only if it is order-isomorphic to a nontrivial subgroup of* $(\mathbb{R}, +, 0)$ *with the canonical ordering* $\leq$ *induced from* $\mathbb{R}$.

☕

Let Γ be an ordered abelian group and $\Delta \subseteq \Gamma$ be convex. Then, $\Delta \lhd \Gamma$ (since Γ is abelian) so Γ/Δ is well-defined as a group.

Γ/Δ can be made an into ordered group by declaring

$$\gamma + \Delta \leq \gamma' + \Delta \text{ if and only if } \gamma < \gamma' \text{ or } \gamma \equiv \gamma' \mod \Delta$$

To check the independence of representatives, take $s \equiv \gamma \mod \Delta$ and $\gamma + \Delta < \gamma' + \Delta$. We will show $s + \Delta < s' + \Delta$.

Let $s - \gamma \in \Delta$ and $s' - \gamma' \in \Delta$. Now, $\gamma < \gamma'$ implies $-\gamma < -\gamma'$, so:

$$s - \gamma < s' - \gamma' \in \Delta \tag{1}$$

Now, $s - \gamma < s' - \gamma'$ implies $-(s - \gamma) < -(s' - \gamma') \in \Delta$, which gives:

$$s' - \gamma' < s - \gamma \in \Delta \tag{2}$$

From (1) and (2), we get $(s' - s) + (s' - \gamma') \in \Delta$, implying $s' - s \in \Delta$. Moreover, $s' \equiv \gamma' \mod \Delta$ and $\gamma < \gamma'$, so $s + \Delta < s' + \Delta$.

Given two ordered abelian groups Γ and Δ, we can order the direct product lexicographically as $(r, s) \leq (r', s')$ if and only if $r \leq r'$ or $r = r'$ and $s \leq s'$ for $r \in \Gamma$ and $s \in \Delta$.

Clearly, $\{0\} \times \Delta$ is a convex subgroup, which is order isomorphic to Δ.

Definition 1.2.4. Let Γ be an ordered abelian group and ∞ be a symbol satisfying $\infty = \infty + r = r + \infty = \infty + \infty \neq r \in \Gamma$.

We define a ***valuation*** v on a field K as a surjective map

$$v : K \twoheadrightarrow \Gamma \cup \{\infty\} \text{ such that } \forall x, y \in K :$$

i) $v(0) = \infty \iff x = 0$,

ii) $v(xy) = v(x) + v(y)$,

iii) $v(x + y) \geq \min\{v(x), v(y)\}$.

- If $\Gamma = \{0\}$, we call v the *trivial valuation.*

- If Γ has rank 1, we call v a *rank-1 valuation.*
- In general, the rank of v is defined as the rank of the value group $v(K^\times)$.

As earlier, we get $v(1) = 0$, $v(x^{-1}) = -v(x)$, $v(-x) = v(x)$, and $v(x) < v(y) \implies v(x+y) = v(x)$.

The set $\mathcal{O}_v \coloneqq \{x \in K \mid v(x) \geq 0\}$ is a valuation ring of K.

The group of units is given by $\mathcal{O}_v^\times = \{x \in K \mid v(x) = 0\}$, and the set of all nonunits $M_v = \{x \in K \mid v(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}_v$.

We define the ***residue class field*** of v as $\overline{\mathcal{K}_v} = \mathcal{O}_v/M_v$.

If K is a subfield of L and w is a valuation on L, we say w ***extends*** v if $w|_K = v$.

Example 1.5. We define the degree valuation v_∞ on $K[x]$, K a field, as $v_\infty(0) = \infty$ and $v_\infty(f/g) = \deg(g) - \deg(f)$. (Note that we set $v_\infty(K) = 0$.)

Clearly, v_∞ is a valuation. Now, f/g is a unit if and only if $v_\infty(f/g) = 0$, i.e., $\deg(f) = \deg(g)$. Hence, if $f = \sum_{i=0}^n c_i x^i$ with $c_i \in k$ and $c_n \neq 0$, then $f(x)/x^n$ is a unit. Since $K \subseteq \mathcal{O}_{v_\infty}$ and $K \cap \mathcal{M}_{v_\infty} = \{0\}$, we can identify K with its image in the residue field.

Moreover, $\overline{f(x)/x^n} = c_n$ because

$$\frac{f(x)}{x^n} - c_n = \frac{f(x) - c_n x^n}{x^n} = \frac{\sum_{i=0}^{n-1} c_i x^i}{x^n}$$
$$\implies v\left(\frac{f(x)}{x^n} - c_n\right) = n - (n-1) = 1 > 0$$

Thus,

$$\frac{f(x)}{x^n} - c_n \in \mathcal{M}_v \implies \overline{\frac{f(x)}{x^n}} = \overline{c_n} = c_n.$$

$$\therefore \mathcal{O}_{v_\infty} = \{f/g : \deg(g) \geq \deg(f)\} \text{ and } \mathcal{M}_{v_\infty} = \{f/g : \deg(g) > \deg(f)\}.$$

Consider $\overline{0} \neq \overline{f/g} \in \overline{\mathcal{K}_{v_\infty}} = \mathcal{O}_{v_\infty}/\mathcal{M}_{v_\infty}$, i.e., $v_\infty(f/g) = 0$, i.e., $\deg(g) = \deg(f)$.

Thus, if $f = \sum_{i=0}^n a_i x^i$ and $g = \sum_{i=0}^n b_i x^i$, then $\overline{f/g} = a_0/b_0$, by the same argument as above, since $f/g = \dfrac{f/x^n}{g/x^n}$.

Hence, we have

$$\overline{\mathcal{K}_{v_\infty}} = \{a/b : a \in K, b \in K^\times\} = K.$$

Proposition 1.2.2. *Let $\mathcal{O} \subseteq K$ be a valuation ring of K. There exists a valuation v on K such that $\mathcal{O} = \mathcal{O}_v$.*

Proof. We have $\mathcal{O}^\times \leq K^\times$. Consider $\Gamma = K^\times/\mathcal{O}^\times$ with operation $x\mathcal{O}^\times + y\mathcal{O}^\times \coloneqq xy\mathcal{O}^\times$.

Define a binary relation $\leq$ on Γ by declaring $\overline{x} = x\mathcal{O}^\times \leq y\mathcal{O}^\times = \overline{y}$ if and only if $y/x \in \mathcal{O}$. This makes Γ into an abelian ordered group: for all $\overline{\gamma}, \overline{\delta}, \overline{\lambda} \in \Gamma$, we have:

1. $\overline{\gamma} \leq \overline{\gamma}$ because $\gamma/\gamma = 1 \in \mathcal{O}^\times$.
2. $\overline{\gamma} \leq \overline{\delta}$ and $\overline{\delta} \leq \overline{\gamma}$ implies $\gamma/\delta \in \mathcal{O}$ and $\delta/\gamma \in \mathcal{O} \implies \gamma$ and δ are units, i.e., $\gamma, \delta \in \mathcal{O}^\times \implies \gamma\delta \in \mathcal{O}^\times \implies \overline{\gamma} = \overline{\delta}$.
3. $\overline{\gamma} \leq \overline{\delta},\ \overline{\delta} \leq \overline{\lambda} \implies \gamma/\delta \in \mathcal{O}$ and $\delta/\lambda \in \mathcal{O} \implies (\gamma/\delta)(\delta/\lambda) = \gamma/\lambda \in \mathcal{O} \implies \overline{\gamma} \leq \overline{\lambda}$.
4. $\overline{\gamma} \leq \overline{\delta}$ or $\overline{\delta} \leq \overline{\gamma}$ because either $\gamma/\delta \in \mathcal{O}$ or $\delta/\gamma \in \mathcal{O}$ (since $\mathcal{O}$ is a valuation ring).
5. $\overline{\gamma} \leq \overline{\delta} \implies \gamma/\delta \in \mathcal{O}$. Now, $\overline{\gamma} + \overline{\lambda} = \overline{\gamma\lambda}$ and $\gamma\lambda/\delta\lambda = \gamma/\delta \in \mathcal{O}$, i.e., $\overline{\gamma\lambda} \leq \overline{\delta\lambda}$, i.e., $\overline{\gamma} + \overline{\lambda} \leq \overline{\delta} + \overline{\lambda}$.

Thus, all axioms of an abelian ordered group hold.
We now define a valuation $v(x) \coloneqq x\mathcal{O}^\times \in \Gamma$, for $x \in K^\times$ and $v(0) \coloneqq \infty$.
Then $v(xy) = v(x) + v(y)$, by definition.
If $v(x) \leq v(y)$, then $y/x \in \mathcal{O} \implies (x+y)/x = 1 + y/x \in \mathcal{O}$.
Thus,

$$v(x+y) \geq v(x) = \min\{v(x), v(y)\}.$$

Hence, v is a well-defined valuation.
Furthermore,

$$\mathcal{O}_v = \{x \in K : v(x) \geq 0\} = \{x \in K : x/1 \in \mathcal{O}\} = \mathcal{O}.$$

☕

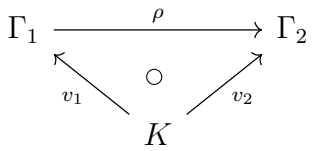
Now, $\mathcal{M} \coloneqq \mathcal{O} \setminus \mathcal{O}^\times$ is the unique maximal ideal of $\mathcal{O}$. This is because $\mathcal{O} = \mathcal{O}_v$ and we have already seen in this case that $\mathcal{M}$ (previously called $\mathcal{M}_v$) is the unique maximal ideal of $\mathcal{O} \setminus \mathcal{O}^\times$.

We define the ***rank*** of $\mathcal{O}$ as $\operatorname{rank}(\Gamma)$.

Example 1.6. Let $\mathcal{O} = K$. Then, $\mathcal{O}^\times = K^\times$ and $\Gamma = \{0\}$, i.e., $v(x) = 0\ \forall x \neq 0$, and thus v is the trivial valuation. In particular, the only valuation ring of a finite field is the trivial one.

Definition 1.2.5. Two valuations $v_i : K \to \Gamma_i \cup \{\infty\}$ $(i = 1, 2)$ are ***equivalent*** if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

Proposition 1.2.3. *Two valuations $v_i : K \to \Gamma_i \cup \{\infty\}$ $(i = 1, 2)$ are equivalent if and only if there exists an order-preserving isomorphism $\rho : \Gamma_1 \to \Gamma_2$ such that $\rho \circ v_1 = v_2$.*



Proof. $\Longrightarrow$: Note that $v_i : K^\times \to \Gamma_i$ is a surjective group morphism with kernel $\mathcal{O}_{v_i}^\times$; thus it induces an isomorphism $\tau_i : K^\times/\mathcal{O}_{v_i}^\times \xrightarrow{\sim} \Gamma_i$ satisfying $\tau_i(x\mathcal{O}_{v_i}^\times) = v_i(x)$.

By hypothesis, $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$, hence $K^\times/\mathcal{O}_{v_1}^\times \cong K^\times/\mathcal{O}_{v_2}^\times$ by the map $\rho = \tau_2 \circ \tau_1^{-1}$, i.e., $\Gamma_1 \cong \Gamma_2$.

$\Longleftarrow$: If there exists an order-preserving isomorphism $\rho : \Gamma_1 \xrightarrow{\sim} \Gamma_2$, then we have:

$$\mathcal{O}_{v_1} = \{x \in K \mid v_1(x) \geq 0\} = \{\rho(x) \in K : (\rho \circ v_1)(x) \geq 0\} = \{x \in K : v_2(x) \geq 0\} = \mathcal{O}_{v_2},$$

since ρ is an isomorphism. ☕

Hence, valuation rings of K correspond one-to-one to valuations of K up to an order-isomorphism of the value group.

Theorem 1.2.1. *(a) Every nontrivial valuation on $\mathbb{Q}$ is a p-adic valuation for some rational prime p.*

(b) Every nontrivial valuation on $k[x]$, trivial on k, is either the degree valuation v_∞ or a p-adic valuation for some irreducible polynomial $p \in k[x]$.

Proof. Let K be either $\mathbb{Q}$ or $k[x]$ and let v be some nontrivial valuation on K. Then, $\mathcal{O}_v \neq K$. In (b), v is trivial on k, thus $k \subseteq \mathcal{O}_v$. We will write $\mathcal{O}$ instead of $\mathcal{O}_v$ in the sequel.

(a) For any valuation v, $v(1) = v(1) + v(1) \implies v(1) = 0$, i.e., $1 \in \mathcal{O}$, so $\mathbb{Z} \subseteq \mathcal{O}$.

Since $\mathcal{O} \neq \mathbb{Q}$, there exists a prime $p \in \mathcal{M}$. Let $q \neq p$ be another prime, then by Bézout's Lemma, there exist $a, b \in \mathbb{Z}$ such that $ap + bq = 1$. Now, $0 = v(ap + bq) \geq \min\{v(ap), v(bq)\}$. Thus, $v(ap) \geq v(bq)$ since $v(a) + v(b) > 0$ ($\because p \notin \mathcal{M}$ and $a \in \mathbb{Z} \subset \mathcal{O}$).

Therefore, $0 \geq v(bq) \implies 0 = v(bq) = v(b) + v(q)$. Since $b \in \mathbb{Z} \subseteq \mathcal{O}$, $v(b) \geq 0$, i.e., $v(q) = -v(b) \leq 0 \implies q \notin \mathcal{M}$.

Hence, all primes other than p are units in $\mathcal{O}$. Now, given a/b with $\gcd(a,b) = 1$, we will show $a/b \in \mathcal{O}$ if and only if $p \nmid b$.

Observation: $v(a/b) = v(a) - v(b) \geq 0$ if and only if $v(a) \geq v(b)$.

Suppose $b = \prod_i q_i^{t_i}$. If $p \nmid b$, then all q_i are units in $\mathcal{O}$. Thus, $v(q_i) = 0$ and so $v(b) = \sum_i t_i v(q_i) = 0$. Thus,

$$v(a/b) = v(a) - v(b) = v(a) \geq 0, \text{since } a \in \mathbb{Z}.$$

Conversely, if $a/b \in \mathcal{O}$, then $v(a) \geq v(b)$. Now, if $p \mid b$, then

$$v(b) = \sum_{q_j \neq p} t_j v(q_j) + tv(p) = tv(p) > 0.$$

Thus, $v(a) \geq v(b) \implies v(a) > 0$.

Since $\gcd(a,b) = 1$, we have $p \nmid a$, but then, by the same argument, $v(a) = \sum t'_j v(q'_j) = 0$, which is a contradiction. Hence, $p \nmid b$.

Therefore, $\mathcal{O} = \mathbb{Z}_{(p)}$ and so v is the p-adic valuation v_p.

(b) If $x \in \mathcal{O}$, then $k[x] \subseteq \mathcal{O}$ and we proceed as in (a), replacing $\mathbb{Z}$ by $k[x]$ and get $v = v_p$ for some irreducible polynomial p.

If $x \notin \mathcal{O}$, then $x^{-1} \in \mathcal{O}$ (property of a valuation ring). Consider $v(x^{-1})$. We have $x \notin \mathcal{O} \implies v(x) < 0 \implies -v(x) > 0 \implies v(x^{-1}) > 0$, i.e., $x^{-1} \in \mathcal{M}$.

Now, for $0 \leq n < m$, $mv(x) = v(x^m) < v(x^n) = nv(x)$ [since $v(x) < 0$].

Since $v(a) = 0 \ \forall a \in k^\times$, we have $v(a_n x^n + \cdots + a_0) = v(a_n x^n)$, because $v\left(\sum_{i=0}^{n-1} a_i x_i\right) \geq \min\{v(a_i x_i)\} = \min\{v(x_i)\}_{i=1}^{n-1}$ and $v(x) < v(y) \implies v(x+y) = v(x)$, (for our case, x corresponds to $a_n x^n$ and y corresponds to $\sum_i^{n-1} a_i x^i$.)

Thus, $v(a_n x^n + \cdots + a_0) = v(a_n x^n) = v(a_n) + v(x^n) = v(x^n) = nv(x)$.

Hence, for any $a_n x^n + \cdots + a_0 \in k[x]$, we get $v(a_n x^n + \cdots + a_0) = nv(x)$, i.e., $v(k[x]^\times) = \mathbb{Z}v(x)$.

Thus, $v(x) \mapsto -1$ gives an isomorphism $v(k[x]^\times) \xrightarrow{\sim} \mathbb{Z}$.

1.3 Constructing valuations

Theorem 1.3.1. *Suppose K is a field, Γ is an ordered subgroup of an ordered group Γ', and $v : K \to \Gamma \cup \{\infty\}$ is a valuation and $\gamma \in \Gamma'$. For $f = \sum_{i=0}^{\infty} a_i x^i \in K[x]$, define*

$$w(f) \coloneqq \begin{cases} \infty, & \text{if } f = 0 \\ \min\limits_{0 \le i \le n} \{v(a_i) + i\gamma\}, & \text{otherwise} \end{cases}$$

For $f, g \in K[x] \setminus \{0\}$, let $w(f/g) = w(f) - w(g)$.

The above equations define a valuation $w : K[x] \to \Gamma' \cup \{\infty\}$ on $K[x]$ that extends v.

Proof. For $f, g \in K[x] \setminus \{0\}$, let $n = \max\{\deg(f), \deg(g)\}$ and $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{n} b_j x^j$.

Then, $f + g = \sum_{i=0}^{n} (a_i + b_j) x^i$ and we get

$$\begin{aligned} w(f+g) &= v(a_i + b_i) + i\gamma \ge \min\{v(a_i), v(b_i)\} + i\gamma \\ &= \min\{v(a_i) + i\gamma,\ v(b_i) + i\gamma\} = \min\{w(f), w(g)\}. \end{aligned} \tag{1}$$

Now,

$$fg = \left(\sum_{i=0}^{n} a_i x^i \right) \left(\sum_{j=0}^{m} b_j x^j \right) = \sum_{k=0}^{n+m} c_k x^k \text{ where } c_k = \sum_{i+j=k} a_i b_j.$$

For every $i + j = k$, we have

$$v(a_i b_j) + k\gamma = v(a_i) + v(b_j) + (i+j)\gamma = v(a_i) + i\gamma + v(b_j) + j\gamma \ge \omega(f) + \omega(g).$$

Hence,

$$v(c_k) = v \left(\sum_{i+j=k} a_i b_j \right) \ge \min_{i+j=k} \{v(a_i b_j)\}.$$

Hence,

$$v(c_k) + k\gamma \ge \min_{i+j=k} \{v(a_i b_j)\} + k\gamma = \min_{i+j=k} \{v(a_i b_j) + k\gamma\}.$$

Therefore,

$$w(f) + w(g) \le \min_{i \le k} \{v(a_i b_i) + k\gamma\} = w(fg).$$

To show the other inequality, let

$$i_0 = \min\{i : v(a_i) + i\gamma = w(f)\}, j_0 = \min\{v(b_j) + j\gamma = w(g)\} \text{ and } k_0 = i_0 + j_0.$$

Consider

$$c_{k_0} = \sum_{i+j=k_0} a_i b_i = \sum_{i \le i_0} a_i b_i + a_{i_0} b_{i_0} + \sum_{i \ge i_0} a_i b_i.$$

Note that whenever $i < i_0$, $v(a_i) + i\gamma \geq w(f)$, by definition of i_0..

Hence, in the teal sum, we have

$$\begin{aligned} v(a_i b_j) + k_0\gamma &= v(a_i) + i\gamma + v(b_j) + j\gamma \\ &\geq v(a_i) + i\gamma + w(g) \\ &> w(f) + w(g) \end{aligned}$$

Similarly, in the violet sum,

$$\begin{aligned} v(a_i b_j) + k_0\gamma &= v(a_i) + i\gamma + v(b_j) + j\gamma \\ &\geq w(f) + v(a_j) + j\gamma \\ &> w(f) + w(g) \end{aligned}$$

Hence,

$$v\left(\sum_{i \le i_0} a_i b_i\right) > w(f) + w(g) - k_0 \text{ and } v\left(\sum_{i \ge i_0} a_i b_i\right) > w(f) + w(g) - k_0$$

and so

$$\min\left\{v\left(\sum_{i \le i_0} a_i b_i\right), v\left(\sum_{i \ge i_0} a_i b_i\right)\right\} > w(f) + w(g) - k_0 = v(a_{i_0} b_{i_0}).$$

Recall that $v(x) < v(y) \implies v(x+y) = v(x)$ whence we deduce:

$$v(c_{k_0}) = v\left(\sum_{i \le i_0} a_i b_i + a_{i_0} b_{i_0} + \sum_{i \ge i_0} a_i b_i\right) = v(a_{i_0} b_{i_0}).$$

Therefore,

$$v(c_{k_0}) + k_0\gamma = v(a_{i_0} b_{i_0}) + k_0\gamma = w(f) + w(g)$$

Thus,

$$w(fg) \geq w(f) + w(g) \implies w(fg) = w(f) + w(g) \qquad (**)$$

Note that we have proved $(*)$ and $(**)$ for any $f, g \in K[x] \setminus \{0\}$, the polynomial ring but not for any $f, g \in K(x) \setminus \{0\}$, the rational function field.

The map $w : K[x] \to \Gamma' \cup \{\infty\}$ is well-defined:

If $f_1/g_1 = f_1/g_2$, then $f_1g_2 = f_2g_1$. Thus, $w(f_1g_2) = w(f_2g_1) \implies w(f_1) + w(g_2) = w(f_2) + w(g_1) \implies w(f_1) - w(g_1) = w(f_2) - w(g_2) \implies w(f_1/g_1) = w(f_2/g_2)$.

To extend $(*)$ and $(**)$ to $K(x) \setminus \{0\}$, take $h_1, h_2 \in K(x) \setminus \{0\}$. Let g be a common denominator of h_1 and h_2: $h_1 = f_1/g$ and $h_2 = f_2/g$ where $g, f_i \in K[x] \setminus \{0\}$.

Then,

$$\begin{aligned} w(h_1 + h_2) = w\left(\frac{f_1 + f_2}{g}\right) &\coloneqq w(f_1 + f_2) - w(g) \\ &\geq \min\{w(f_1), w(f_2)\} - w(g) \\ &= \min\{w(f_1) - w(g), w(f_2) - w(g)\} \\ &= \min\{w(f_1/g), w(f_2/g)\} \\ &= \min\{w(h_1), w(h_2)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} w(h_1h_2) = w\left(\frac{f_1f_2}{g^2}\right) &\coloneqq w(f_1) + w(f_2) - 2w(g) \\ &= w(f_1) + w(f_2) - 2w(g) \\ &= w(f_1/g) + w(f_2/g) \\ &= w(h_1) + w(h_2). \end{aligned}$$

Corollary 1.3.1.1. *Let $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ be a valuation. There is exactly one extension w of v to $K(x)$ such that $w(x) = 0$ and $\overline{x}$ is transcendental over $\overline{\mathcal{K}}$, the residue class field of v. We have $\overline{K(x)} = \overline{\mathcal{K}}(\overline{x})$ and $w(K(x)^\times) = \Gamma$.*

*This valuation w is called the **Gauss extension** of v.*

Proof. For uniqueness, let $f = \sum_{i=0}^n a_i x^i \in K[x] \setminus \{0\}$. Pick $k \leq n$ such that $v(a_k) = \min\limits_{0 \leq i \leq n} v(a_i)$.

Write $f = a_k \sum_{i=0}^n b_i x^i = a_k g$ where $b_i = a_i/a_k$ and $v(b_i) \geq 0$.

Thus,

$$w(g) = w\left(\sum_{i=0}^n b_i x^i\right) = \min\{w(b_i) + iw(x)\} = \min\{v(b_i)\} \geq 0.$$

Now, $\overline{g} = \sum \overline{b_i}\overline{x}^i \neq 0$ since $b_k = a_k/a_k = 1$ and $\overline{x}$ is transcendental over $\overline{\mathcal{K}}$. Hence

$$g \notin \mathcal{M}_w, \text{ i.e., } g \in \mathcal{O}^\times{}_w \implies w(g) = 0 \implies w(f) = w(a_k) + w(g) = v(a_k).$$

Thus, $w(f) = \min_{0 \le i \le n} v(a_i)$.

For existence, let $f \in K[x]$ and define $w(f)$ as earlier by taking $\Gamma' = \Gamma$ and $\gamma = 0$.

Then, $w(x) = \min\{v(1) + 0\} = 0$.

Claim: $\overline{x}$ is transcendental over $\overline{\mathcal{K}}$.

Suppose, for a contradiction, that $\overline{x}$ is algebraic over $\overline{\mathcal{K}}$.

Then, $\exists a_i \in \mathcal{O}_v$ such that $\sum \overline{a_i}\, \overline{x}^i = \sum \overline{a_i x^i} = 0$.

Thus, $\sum a_i x^i \in \mathcal{M}_w$, i.e., $w\left(\sum a_i x^i\right) = \min\{v(a_i)\} > 0$, i.e., $v(a_i) > 0\ \forall i \implies a_i = 0$, so the only polynomial in $\overline{\mathcal{K}}$ having $\overline{x}$ as a root is the zero polynomial. This proves the claim.

Moreover, $w(K(x)^\times) = \Gamma$ because given any $f \in K(x)^\times$,

$$w(f) = \min\{v(a_i)\} \in \Gamma, \text{ i.e., } w(K(x)^\times) \subseteq \Gamma$$

Conversely, for any $t \in \Gamma$, $\exists a \in K^\times$ such that $v(a) = t$ (by surjectivity of v).
Hence, $\Gamma \subseteq w(K(x)^\times) \implies w(K(x)^\times) = \Gamma$.

Claim: $\overline{K(x)} = \overline{\mathcal{K}}(\overline{x})$.

Let $h \in \mathcal{O}_w^\times$, say $h = f_1/f_2$ with $f_i \in K[x] \setminus \{0\}$.
As before, write $f_i = c_i g_i$, $c_i \in K^\times$ and $g_i \in \mathcal{O}_v^\times$.
Thus,

$$h = c_1 g_1 / c_2 g_2 = c \cdot (g_1/g_2) \text{ where } c = c_1/c_2 \in K^\times.$$

Moreover, $h, g_1, g_2 \in \mathcal{O}_w^\times \implies c = hg_2/g_1 \in \mathcal{O}_w^\times$, whence we conclude $\overline{h} = \overline{hg_2/g_1} \in \overline{\mathcal{K}}(\overline{x})$.

This proves the inclusion $\overline{K(x)} \subset \overline{\mathcal{K}}(\overline{x})$

Conversely, let $\overline{h}(\overline{x}) \in \overline{\mathcal{K}}(\overline{x})$ i.e., $\overline{h} \in \mathcal{O}_w/\mathcal{M}_w \implies h \in \mathcal{O}_w$. Since h is nonzero in $\overline{\mathcal{K}}$, we have $h \notin \mathcal{M}_w \implies h \in \mathcal{O}_w^\times \implies \overline{h} \in \overline{K(x)}$, which yields $\overline{K(x)} \supset \overline{\mathcal{K}}(\overline{x})$.

☕

Corollary 1.3.1.2. *Let $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$, Γ be an ordered subgroup of an ordered group Γ', and $\gamma \in \Gamma'$ has the property that if $n \in \mathbb{Z}$ satisfies $n\gamma \in \Gamma$, then $n = 0$.*

Then, there is exactly one valuation w on $K(x)$ extending v, with $w(x) = \gamma$, and we have $\overline{K(x)} = \overline{\mathcal{K}}$ and $w(K(x)^\times) = \Gamma \oplus \mathbb{Z}\gamma$.

Proof. The existence of w follows from the previous theorem.

For uniqueness, consider an $f \in K[x]$, say $f = \sum_{i=0}^n a_i x^i,\ a_i \in K$.

Now, for each $i \leq n, w(a_i x^i) = v(a_i) + iw(x) = v(a_i) + i\gamma$.

For $i \neq j$ and $a_i \neq 0 \neq a_j$, we claim $w(a_i x^i) \neq w(a_j x^j)$:

Suppose this is not the case. Then,

$$w(a_i x^i) = w(a_j x^j) \implies v(a_i) + i\gamma = v(a_j) + j\gamma \implies (i-j)\gamma = v(a_j) - v(a_i).$$

Since $i - j \in \mathbb{Z}$ and $(i-j)\gamma \in \Gamma$ we have $i - j = 0 \implies i = j$, a contradiction.

Thus,

$$w(f) = \min_{0 \leq i \leq n} \{w(a_i x^i)\} = \min_{0 \leq i \leq n} \{v(a_i) + i\gamma\},$$

which is uniquely determined (by the claim). Hence, w is uniquely determined on $K[x]$ and hence on $K(x)$.

Moreover, $w(K(x)^\times) = \Gamma \oplus \mathbb{Z}\gamma$ because given any $f \in K(x)^\times$, $w(f) = \min\{v(a_i) + i\gamma\} \in \Gamma \oplus \mathbb{Z}\gamma$, i.e., $w(K(x)^\times) \subseteq \Gamma \oplus \mathbb{Z}\gamma$.

Conversely, for any $t + i\gamma$, there exists $a \in K^\times$ such that $v(a) = t$, and hence $v(ax^i) = t + i\gamma$, so $\Gamma \oplus \mathbb{Z}\gamma \subseteq w(K(x)^\times) \implies w(K(x)^\times) = \Gamma \oplus \mathbb{Z}\gamma$.

Claim: $\overline{K(x)} = \overline{\mathcal{K}}$.

We first show that every $f \in K[x] \setminus \{0\}$ is of the form $f = ax^n(1+u)$ where $a \in K^\times$, $u \in K(x)$, $n \in \mathbb{Z}$, and $w(u) > 0$.

For this, write $f = \sum_{i=0}^n a_i x^i$, with $a_i \in K$. Now, $\exists!$ i_0 such that

$$w(f) = \min_{0 \leq i \leq n} \{w(a_i x^i)\} = w(a_{i_0} x^{i_0}) = v(a_{i_0}) + i_0\gamma.$$

Thus,

$$f = a_{i_0} x^{i_0} \left(\sum_{i \neq i_0}^{n} \frac{a_i x^i}{a_{i_0} x^{i_0}} + 1 \right) =: a_{i_0} x^{i_0} u.$$

Now,

$$w\left(\frac{a_i x^i}{a_{i_0} x^{i_0}}\right) = w(a_i x^i) - w(a_{i_0} x^{i_0}) > 0 \text{ for } i \neq i_0,$$

hence

$$w(u) = w\left(\sum_{i \neq i_0}^{n} \frac{a_i x^i}{a_{i_0} x^{i_0}} + 1 \right) = w\left(\sum_{i \neq i_0}^{n} \frac{a_i x^i}{a_{i_0} x^{i_0}} \right) \geq \min_{i \neq i_0} \left\{ w\left(\frac{a_i x^i}{a_{i_0} x^{i_0}} \right) \right\} > 0,$$

whence, $w(u) > 0$.

Second, we consider any $h \in K(x) \setminus \{0\}$, and write $h = f/g$, with $f, g \in K[x] \setminus \{0\}$.

Write $f = ax^m(1+u)$ and $g = bx^n(1+u')$, with $a, b \in K^\times$, $m, n \in \mathbb{N}$, and $w(u), w(u') > 0$. Then

$$h = \frac{f}{g} = \frac{a}{b}(x^{m-n}) \left(\frac{1+u}{1+u'} \right) = cx^r \left(1 + \frac{u - u'}{1+u'} \right),$$

where $c = a/b \in K^\times$ and $r = m - n \in \mathbb{Z}$. Since $w(u') > 0$, $w(1+u') = 0$; therefore $w(u - u'/1 + u') = w(u - u') - w(1 + u') = w(u - u') > 0$. By the same argument as before, there exists $u'' \in K(x)$ with $w(u'') > 0$ such that $h = cx^r(1 + u'')$.

We now show that $K(x) = \overline{\mathcal{K}}(\overline{x})$. Let $h \in \mathcal{O}_w^\times$ and write $h = cx^r(1 + u'')$ as described above. We then have

$$0 = w(h) = w(cx^r(1 + u'')) = v(c) + r\gamma,$$

whence $r\gamma = -v(c) \in \Gamma$. By assumption, $r = 0$. Consequently, $v(c) = 0$. Therefore $\overline{h} = \overline{c}(1 + \overline{u''}) = \overline{c} \in \overline{\mathcal{K}}$ ($\because \overline{u''} = 0$ since $w(u'') > 0$).

1.4 Dependence and Topology of Valuations

Definition 1.4.1. Two valuation rings $\mathcal{O}_1, \mathcal{O}_2 \subseteq K$ are ***dependent*** if $\mathcal{O}_1\mathcal{O}_2 \neq K$, where $\mathcal{O}_1\mathcal{O}_2$ denotes the smallest subring of K containing both $\mathcal{O}_1, \mathcal{O}_2$.

We also define the ***dependence class of*** $\mathcal{O}$ as

$$[\mathcal{O}] \coloneqq \{\mathcal{O}' \subseteq K : \mathcal{O}' \text{is a nontrivial valuation ring dependent on } \mathcal{O}\Big\}.$$

Note that every overring of a valuation ring is also a valuation ring: if $\mathcal{O} \subseteq \mathcal{O}'$ and $\mathcal{O}$ is a valuation ring, then $\forall x \in K$, $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$, i.e., $x \in \mathcal{O}'$ or $x^{-1} \in \mathcal{O}'$.

Such an overring $\mathcal{O}'$ of $\mathcal{O}$ is called a ***coarsening*** of $\mathcal{O}$.

Two dependent valuation rings $\mathcal{O}_1, \mathcal{O}_2 \subseteq K$ always have a *lowest common coarsening*, namely $\mathcal{O}_1\mathcal{O}_2$.

The set of overrings $\mathcal{O}'$ of $\mathcal{O}$ in K is linearly ordered by inclusion: Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are overrings of $\mathcal{O}$ in K. If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, we are done. Suppose not. Let $a \in \mathcal{O}_1 \setminus \mathcal{O}_2$ and $b \in \mathcal{O}_2$.

Now, $a/b \notin \mathcal{O}$, otherwise $a/b \in \mathcal{O} \subseteq \mathcal{O}_2 \implies a = b(a/b) \in \mathcal{O}_2$, but $a \notin \mathcal{O}_2$. Since $\mathcal{O}$ is a valuation ring, $(a/b)^{-1} = b/a \in \mathcal{O} \subseteq \mathcal{O}_1$. Hence, $b = a(b/a) \in \mathcal{O}_1$, i.e., $\mathcal{O}_2 \subseteq \mathcal{O}_1$.

Theorem 1.4.1. *Overrings $\mathcal{O}'$ of $\mathcal{O}$ correspond one-to-one with prime ideals of $\mathcal{O}$.*

Proof. Let $\mathcal{O}$ be a fixed nontrivial valuation ring of K and $\mathcal{O}' \subseteq K$ be an overring.

Then, $\mathcal{O} \subseteq \mathcal{O}' \implies \mathcal{M}' \subseteq \mathcal{M}$ because if $x \in \mathcal{M}'$, then $x^{-1} \notin \mathcal{O}'$, hence $x^{-1} \notin \mathcal{O}$, and so $x \in \mathcal{M}$. Note that $\mathcal{M}$ consists exactly of non-units of $\mathcal{O}$.

Now, $\mathcal{M}'$ is prime in $\mathcal{O}'$, i.e., $\mathcal{M}'\mathcal{O} \subseteq \mathcal{M}'$. Since $\mathcal{O} \subseteq \mathcal{O}'$, we have $\mathcal{M}'\mathcal{O} \subseteq \mathcal{M}'$, i.e., $\mathcal{M}'$ is also prime in $\mathcal{O}$.

Here, given an overring $\mathcal{O}' \supseteq \mathcal{O}$, we have a prime ideal $M' \triangleleft \mathcal{O}$.

Conversely, if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}$, then $\mathcal{O}_\mathfrak{p} = \{a/b : a \in \mathcal{O}, b \notin \mathfrak{p}\}$ is an overring of $\mathcal{O}$ with maximal ideal $\mathfrak{p}\mathcal{O}_\mathfrak{p}$.

Claim: The correspondence is one-to-one, i.e., $\mathcal{O}_{M'} = \mathcal{O}'$ and $\mathfrak{p}\mathcal{O}_\mathfrak{p} = \mathfrak{p}$.

We need to show that every $x \in \mathcal{O}'$ is of the form a/b, $a \in \mathcal{O}$ and $b \notin \mathcal{M}'$. If $x \in \mathcal{O}$, then $x = x/1$ and we are done. Suppose not. Then, $x^{-1} \in \mathcal{O}$ so $x^{-1} \in \mathcal{M} - \mathcal{M}'$ (since $\mathcal{M}'$ consists of non-units of $\mathcal{O}'$). Thus, $x = 1/x^{-1}$ and we are done.

Hence, $\mathcal{O}' \subseteq \mathcal{O}_{\mathcal{M}'}$.

Now, let $a/b \in \mathcal{O}_{\mathcal{M}'}$, i.e., $a \in \mathcal{O}$ and $b \notin \mathcal{M}'$, so b is a unit in $\mathcal{O}'$,i.e., $b^{-1} \in \mathcal{O}'$. Hence, $a/b = ab^{-1} \in \mathcal{O}' \implies \mathcal{O}_{\mathcal{M}'} \subseteq \mathcal{O}$. Therefore, $\mathcal{O}_{\mathcal{M}'} = \mathcal{O}'$.

Now, given $a/b \in \mathfrak{p}\mathcal{O}_\mathfrak{p}$, we have $a \in \mathfrak{p}$ and $b \notin \mathfrak{p}$. Since $\mathfrak{p} \subseteq \mathcal{M}$, $b \notin \mathcal{M}$, i.e., b is a unit in $\mathcal{O}$. Hence, $b^{-1} \in \mathcal{O}$, so $a/b = ab^{-1} \in \mathfrak{p}$ ($\mathfrak{p}$ is prime and $a \in \mathfrak{p}$). Thus, $\mathfrak{p}\mathcal{O}_\mathfrak{p} \subseteq \mathfrak{p}$.

Given $a \in \mathfrak{p}$, $a = a/1 \in \mathfrak{p}\mathcal{O}_\mathfrak{p}$, so $\mathfrak{p} \subseteq \mathfrak{p}\mathcal{O}_\mathfrak{p}$. Hence, $\mathfrak{p}\mathcal{O}_\mathfrak{p} = \mathfrak{p}$.

Therefore, the correspondence is one-to-one. ☕

Remark 1.6. If $\mathcal{O}' = K$, then $\mathcal{M}'$ is trivial and we have $K = \mathcal{O}_{(0)} = \{a/b : a \in \mathcal{O}, b \neq 0\}$, which is the quotient field of $\mathcal{O}$. Therefore, K is the quotient field for every valuation ring of K.

Lemma 1.4.1. *Let $\mathcal{O}$ be a nontrivial valuation ring of K corresponding to the valuation $v : K \longrightarrow \Gamma \cup \{\infty\}$. Then, there is a one-to-one correspondence of convex subgroups $\Delta \subseteq \Gamma$ with prime ideals $\mathfrak{p}$ of $\mathcal{O}$, and hence with overrings $\mathcal{O}_\mathfrak{p} \supseteq \mathcal{O}$.*

This correspondence is given by

$$\Delta \mapsto \mathfrak{p}_\Delta \coloneqq \{\, x \in \mathcal{O} : v(x) \notin \Delta \,\},$$
$$\mathfrak{p} \mapsto \Delta_\mathfrak{p} \coloneqq \{\gamma \in \Gamma : \gamma = v(x) \text{ or } v(x^{-1}) \text{ for some } x \in \mathcal{O} \setminus \mathfrak{p}\}.$$

In particular, if $\mathcal{O}$ has finite rank, then this rank coincides with the Krull dimension of $\mathcal{O}$.

Proof. Consider $\Delta \mapsto \mathfrak{p}_\Delta$. Now, $\mathfrak{p}_\Delta \triangleleft \mathcal{O}$ since $\forall t \in \mathcal{O}$ and $x \in \mathfrak{p}_\Delta$, we have

$$v(xt) = v(x) + v(t) \geq v(x) \notin \Delta \implies v(xt) \notin \Delta \text{ (by convexity of}\Delta).$$

To show $\mathfrak{p}_\Delta$ is prime, consider $xy \in \mathfrak{p}_\Delta$, i.e. $v(xy) = v(x) + v(y) \notin \Delta$.

If $x \in \mathfrak{p}_\Delta$ or $y \in \mathfrak{p}_\Delta$, we are done. Suppose not. Then $v(x) \in \Delta$ and $v(y) \in \Delta$. Thus, $v(xy) = v(x) + v(y) \in \Delta$(by convexity), a contradiction.

Hence, $\mathfrak{p}_\Delta$ is a prime ideal.

To show that $\Delta_\mathfrak{p}$ is convex, we take $0 \leq \gamma \leq \delta$ with $\delta \in \Delta_\mathfrak{p}$.

Since v is surjective, we have $\gamma = v(y)$ for some $y \in K$ but since $0 \leq \gamma$, it must be the case that $y \in \mathcal{O}$. This already gives that $\gamma \in \Delta_\mathfrak{p}$. Hence, convexity of $\Delta_\mathfrak{p}$ follows. Moreover, by definition, we have $\Delta_\mathfrak{p} = -\Delta_\mathfrak{p}$, hence we need only prove closure under addition to ensure that $\Delta_\mathfrak{p}$ is a subgroup.

Let $\gamma, \delta \in \Delta_\mathfrak{p}$. We may assume without loss of generality that $0 \leq \gamma \leq \delta$, otherwise we work with $-\gamma$ or $-\delta$ as required. Then, we have $\gamma = v(x)$ and $\delta = v(y)$ for some $x, y \in \mathcal{O} \setminus \mathfrak{p}$. Consider now $\gamma + \delta = v(x) + v(y) = v(xy)$. Since $\mathfrak{p}$ is prime in $\mathcal{O}$, we have $x, y \in \mathcal{O} \setminus \mathfrak{p} \implies xy \in \mathcal{O} \setminus \mathfrak{p}$. Hence, $\gamma + \delta = v(xy) \in \Delta_\mathfrak{p}$, and thus $\Delta_\mathfrak{p}$ is a convex subgroup of Γ.

We now show that the two mappings are mutually inverse, i.e., $\Delta_{\mathfrak{p}_\Delta} = \Delta$ and $\mathfrak{p}_{\Delta_\mathfrak{p}} = \mathfrak{p}$.

- $\underline{\mathfrak{p}_{\Delta_\mathfrak{p}} \subseteq \mathfrak{p}}$: Let $x \in \mathfrak{p}_{\Delta_\mathfrak{p}}$ and suppose $x \notin \mathfrak{p}$. Then, $v(x) \in \Delta_\mathfrak{p}$, which contradicts $x \in \mathfrak{p}_{\Delta_\mathfrak{p}}$.

- $\underline{\mathfrak{p} \subseteq \mathfrak{p}_{\Delta_\mathfrak{p}}}$: Let $x \in \mathfrak{p}$ and suppose $x \notin \mathfrak{p}_{\Delta_\mathfrak{p}}$. Then, $v(x) \in \Delta_\mathfrak{p}$ and so $v(x) = v(y)$ or $v(y^{-1})$ for some $y \in \mathcal{O} \setminus \mathfrak{p}$. Since $x \in \mathfrak{p} \subset \mathcal{O}, v(x) \geq 0 \implies v(x) = v(y)$.

  Then, $v(xy^{-1}) = 0 \implies xy^{-1} \notin \mathcal{M} \implies x^{-1}y \in \mathcal{O}$.

  Since $x \in \mathfrak{p}$ and $x^{-1}y \in \mathcal{O},\ x(x^{-1}y) = y \in \mathfrak{p}$, a contradiction.

- $\underline{\Delta \subseteq \Delta_{\mathfrak{p}_\Delta}}$: Let $\lambda \in \Delta$. Since v is surjective, we have $\lambda = v(x)$ for some $x \in K$. Now, since Δ is a subgroup, $-\lambda \in \Delta$, thus both $v(x)$ and $v(x^{-1}) \in \Delta$. Moreover, since $\mathcal{O} \subset K$ is a valuation ring, we have x or $x^{-1} \in \mathcal{O}$. Suppose first that $x \in \mathcal{O}$ i.e., $\lambda = v(x) \geq 0$. Further, $\lambda \in \Delta \implies x \notin \mathfrak{p}_\Delta$. Hence, we get $x \in \mathcal{O} \setminus \mathfrak{p}_\Delta \implies \lambda = v(x) \in \Delta_{\mathfrak{p}_\Delta}$. Now, suppose $x^{-1} \in \mathcal{O}$ i.e., $\lambda = v(x) \leq 0$. Since $v(x^{-1}) = -\lambda \in \Delta \implies x^{-1} \notin \mathfrak{p}_\Delta$. Hence, we get $x^{-1} \in \mathcal{O} \setminus \mathfrak{p}_\Delta \implies -\lambda = v(x^{-1}) \in \Delta_{\mathfrak{p}_\Delta} \implies \lambda \in \Delta_{\mathfrak{p}_\Delta}$.

- $\underline{\Delta_{\mathfrak{p}_\Delta} \subseteq \Delta}$: Let $\lambda \in \Delta_{\mathfrak{p}_\Delta}$, i.e., $\lambda = v(t)$ or $v(t^{-1}),\ t \in \mathcal{O} \setminus \mathfrak{p}_\Delta$. Now $t \notin \mathfrak{p}_\Delta$ gives $v(t) \in \Delta$. Since Δ is a subgroup, $-v(t) = v(t^{-1}) \in \Delta$. Thus, $\lambda \in \Delta$.

This concludes the proof. ☕

Corollary 1.4.1.1. *Let $\mathcal{O} \subseteq K$ be a nontrivial valuation ring. Then* $\operatorname{rank}(\mathcal{O}) = 1$ *if and only if $\mathcal{O}$ is a maximal subring of K.*

Proof. $\operatorname{rank}(\mathcal{O}) = 1$ if and only if $\mathcal{O}$ has one convex subgroup; this is equivalent to having one prime ideal, which in turn implies that $\mathcal{O}$ has a single overring. Since K is an overring, it must be the only one; i.e., $\mathcal{O} \subseteq S \subseteq K \implies S = K$ or $S = \mathcal{O}$. Thus, $\mathcal{O}$ is a maximal subring of K. ☕

Now, let $\mathcal{O}$ be a nontrivial valuation ring of K corresponding to the valuation $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$. Assume $\mathfrak{p} \lhd \mathcal{O}$ is prime with corresponding convex subgroup $\Delta \subseteq \Gamma$.

The canonical valuation $v_\mathfrak{p}$ corresponding to $\mathcal{O}_\mathfrak{p}$ induces an order-preserving group homomorphism

$$\begin{aligned} \phi : K^\times/\mathcal{O}^\times &\to K^\times/\mathcal{O}_\mathfrak{p}^\times, \\ x\mathcal{O}^\times &\mapsto x\mathcal{O}_\mathfrak{p}^\times. \end{aligned}$$

It is well-defined because $\mathcal{O} \subseteq \mathcal{O}_\mathfrak{p} \implies \mathcal{O}^\times \subseteq \mathcal{O}_\mathfrak{p}^\times$.

Now, $\ker\phi = \{x\mathcal{O}^\times : x\mathcal{O}_\mathfrak{p}^\times = \mathcal{O}_\mathfrak{p}^\times\} = \{x\mathcal{O}^\times : x \in \mathcal{O}_\mathfrak{p}^\times\} = \mathcal{O}_\mathfrak{p}^\times/\mathcal{O}^\times$.

We want to show that $v_\mathfrak{p}$ is obtained from v by dividing Γ by Δ, i.e.

$$w : K \twoheadrightarrow \Gamma \cup \{\infty\} \twoheadrightarrow \Gamma/\Delta_\mathfrak{p} \cup \{\infty\} : \ x \mapsto v(x) + \Delta_\mathfrak{p}$$

is the same as

$$v_\mathfrak{p} : K \twoheadrightarrow \Gamma_\mathfrak{p} \cup \{\infty\}, \quad \text{where } \Gamma_\mathfrak{p} = K^\times/\mathcal{O}_\mathfrak{p}^\times.$$

We first consider $\mathcal{O}_w$:

$\mathcal{O}_w = \{x \in K : w(x) \geq 0\} = \{x \in K : v(x) + \Delta \geq \Delta\} = \{x \in K : v(x) > 0 \text{ or } v(x) \in \Delta\}$.

Let $x \in \mathcal{O}_w$. Then, $v(x) > 0$ or $v(x) \in \Delta$. If $v(x) > 0$, then $x \in \mathcal{M}_v \subseteq \mathcal{O}_v \subseteq \mathcal{O}_{v_\mathfrak{p}}$.

Suppose not, i.e., $v(x) \leq 0$. Then since $x \in \mathcal{O}_w$, we must have $v(x) \in \Delta$, i.e., there exists $t \in \mathcal{O} \setminus \mathfrak{p}$ such that $v(x) = v(t^{-1})$ ($\because v(x) \leq 0$). Hence, $v(xt) \geq 0 \implies xt \notin \mathcal{M}_v \implies xt$ and $x^{-1}t^{-1} \in \mathcal{O}_v$.

Thus, $x = xt/t \in \mathcal{O}_\mathfrak{p}$, hence $\mathcal{O}_w \subseteq \mathcal{O}_\mathfrak{p}$.

Conversely, given $x = a/b \in \mathcal{O}_\mathfrak{p}$, i.e., $a \in \mathcal{O}$ and $b \in \mathcal{O} \setminus \mathfrak{p}$, we have

$$v(x) = v(a) - v(b) = v(a) + v(b^{-1}) = v(a) + \delta \text{ where } \delta \in \Delta$$

thus $v(x) + \Delta = v(a) + \Delta \geq \Delta$, since $v(a) \geq 0$ as $a \in \mathcal{O}$.

Therefore, $x \in \mathcal{O}_w$, i.e., $\mathcal{O}_\mathfrak{p} \subseteq \mathcal{O}_w$, whence $\mathcal{O}_\mathfrak{p} = \mathcal{O}_w$, i.e., $v_\mathfrak{p} = w$.

Therefore, $\Gamma/\Delta_\mathfrak{p} = (K^\times/\mathcal{O}^\times)/\Delta_\mathfrak{p} \cong \Gamma_\mathfrak{p} = K^\times/\mathcal{O}_\mathfrak{p}^\times$

The residue homomorphism $\varphi_\mathfrak{p} : \mathcal{O}_\mathfrak{p} \to \mathcal{O}_\mathfrak{p}/\mathfrak{p} = \overline{\mathcal{K}}_\mathfrak{p}$ maps the valuation ring $\mathcal{O}$ to a valuation ring $\overline{\mathcal{O}} = \mathcal{O}/\mathfrak{p}$ in $\overline{\mathcal{K}}_\mathfrak{p}$: let $\overline{x} \in \overline{\mathcal{K}}_\mathfrak{p}$, i.e., $\overline{x} = x + \mathfrak{p}$, $x \in \mathcal{O}_\mathfrak{p}$. Now, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. If $x \in \mathcal{O}$, then $\overline{x} \in \overline{\mathcal{O}}$. Otherwise, if $x^{-1} \in \mathcal{O}$, then $\overline{x^{-1}} = \overline{x}^{-1} \in \overline{\mathcal{O}}$ and we are done.

Defining $\overline{v} : \overline{\mathcal{K}}_\mathfrak{p} \twoheadrightarrow \Delta \cup \{\infty\} : \overline{v}(\overline{x}) := v(x)$ yields a valuation on $\overline{\mathcal{K}}_\mathfrak{p}$ corresponding to $\overline{\mathcal{O}}$, with residue class field $\overline{\mathcal{O}}/\overline{\mathcal{M}} = \overline{\mathcal{K}}$ since $\overline{\mathcal{O}}/\overline{\mathcal{M}} = (\mathcal{O}/\mathfrak{p})/(\mathcal{M}/\mathfrak{p}) = \mathcal{O}/\mathcal{M}$.

Hence, passing from $\mathcal{O}$ to a coarsening $\mathcal{O}_\mathfrak{p}$ yields two valuations, namely

$$v_\mathfrak{p} : K \twoheadrightarrow \Gamma/\Delta_\mathfrak{p} \cup \{\infty\} \text{ with valuation ring } \mathcal{O}_\mathfrak{p}, \text{ and}$$

$$\overline{v}_\mathfrak{p} : \overline{\mathcal{K}}_\mathfrak{p} \twoheadrightarrow \Delta_\mathfrak{p} \cup \{\infty\} \text{ on the residue class field of } v_\mathfrak{p}.$$

The last process can be reversed, i.e., given a valuation $v' : K \twoheadrightarrow \Gamma' \cup \{\infty\}$ and another valuation on the residue class field of v', say $\overline{v} : \overline{\mathcal{K}}_{v'} \twoheadrightarrow \Delta' \cup \{\infty\}$, we can define a 'composition' of v' with $\overline{v}$:

Let $\mathcal{O} = \varphi^{-1}(\mathcal{O}_{\overline{v}})$ where $\mathcal{O}_{v'} \xrightarrow{\varphi} \overline{\mathcal{K}}_{v'}$ is the canonical residue homomorphism.

$\mathcal{O}$ is a valuation ring of K and $\mathcal{O}_{v'}$ is a coarsening of $\mathcal{O}$:

$\mathcal{O} = \varphi^{-1}(\mathcal{O}_{\overline{v}}) = \{x \in \mathcal{O}_{v'} : \varphi(x) = x\mathcal{M}_{v'} \in \mathcal{O}_{\overline{v}}\} = \{x \in \mathcal{O}_{v'} : \overline{v}(x\mathcal{M}_{v'}) \geq 0\}$.

Let $x \in K$. If $x \in \mathcal{O}$, then we are done. Otherwise, $x \notin \mathcal{O}$, i.e., $\overline{v}(x\mathcal{M}_{v'}) < 0$ i.e., $x\mathcal{M}_{v'}$ is a unit in $\mathcal{O}_{\overline{v}}$, i.e., $x^{-1}\mathcal{M}_{v'} \in \mathcal{O}_{\overline{v}}$. Hence, $\overline{v}(x^{-1}\mathcal{M}_{v'}) \geq 0 \implies x^{-1} \in \mathcal{O}$.

Thus, $\mathcal{O}$ is a valuation ring of K and itm follows that $\mathcal{O} \subseteq \mathcal{O}_{v'}$.

Let $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ be the canonical valuation corresponding to $\mathcal{O}$. Since $\mathcal{O}_{v'}$ is a coarsening, there exists a prime $\mathfrak{p} \lhd \mathcal{O}$ such that $\mathcal{O}_{v'} = \mathcal{O}_\mathfrak{p}$. Hence, $v' = v_\mathfrak{p}$, $\mathcal{M}_{v'} = \mathfrak{p}\mathcal{O}_\mathfrak{p}$ and $\overline{\mathcal{K}}_{v'} = \mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p} = \overline{\mathcal{K}}_\mathfrak{p}$. As before, $v_\mathfrak{p} : K \twoheadrightarrow \Gamma/\Delta_\mathfrak{p} \cup \{\infty\}$, and $\Gamma' \cong \Gamma/\Delta_\mathfrak{p}$.

Again, as before, we have $\overline{v}_\mathfrak{p} : \overline{K}_\mathfrak{p} \to \Delta_\mathfrak{p} \cup \{\infty\}$ with valuation ring $\mathcal{O}_{\overline{v}_\mathfrak{p}}$. Note that $\overline{v}_\mathfrak{p}(xM_{v'}) := v(x)$, and so $\overline{v}_\mathfrak{p}(xM_{v'}) \geq 0 \implies v(x) \geq 0$, i.e., $\varphi(\mathcal{O}_{\overline{v}_\mathfrak{p}}) \subseteq \mathcal{O}$.

If $v(x) \geq 0$, then $\overline{v}_\mathfrak{p}(xM_{v'}) \geq 0$, so $\mathcal{O} \subseteq \varphi^{-1}(\mathcal{O}_{\overline{v}_\mathfrak{p}})$, i.e., $\varphi^{-1}(\mathcal{O}_{\overline{v}_\mathfrak{p}}) = \mathcal{O} = \varphi(\mathcal{O}_{v'})$. Hence, $\mathcal{O}_{\overline{v}_\mathfrak{p}} = \mathcal{O}_{v'}$.

Note that we have used the following fact above:
If $f : A \to B$ is surjective and $f^{-1}(B_1) = f^{-1}(B_2)$, then $B_1 = B_2$.
Hence, $\overline{v}_\mathfrak{p} = \overline{v}$; so Δ' is isomorphic to a convex subgroup $\Delta_\mathfrak{p} \subset \Gamma$, and $\Gamma' \cong \Gamma/\Delta'$, where $\Gamma = K^\times/\mathcal{O}^\times$.

We call this v the ***composition*** of v' with $\overline{v}$.

We now consider the topology induced by a valuation on a field K. We begin by showing that two valuations are dependent if and only if they induce the same topology.

Given a valuation $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$, for any $\gamma \in \Gamma$ and each $a \in K$, we define $\mathcal{U}_\gamma(a) := \{x \in K : v(x-a) > \gamma\} = \{x \in K : |x-a| < e^{-\gamma}\}$.

These sets form a basis of open neighborhoods of a, i.e.,

(i) $a \in \mathcal{U}_\gamma(a)$

(ii) $\forall\, \mathcal{U}_\gamma, \mathcal{U}_\alpha, \exists\, \mathcal{U}_\delta$ such that $\mathcal{U}_\delta \subset \mathcal{U}_\gamma \cap \mathcal{U}_\alpha$

In fact, we have the following properties:

(i) $a \in \mathcal{U}_\gamma(a)$

(ii) $\mathcal{U}_\gamma(a) \cap \mathcal{U}_\alpha(a) = \mathcal{U}_\delta(a)$ where $\delta = \max\{\gamma, \alpha\}$

(iii) For any $b \in \mathcal{U}_\gamma(a)$ with $b \neq a$, $v(b-a) = \gamma' > \gamma$ implies $\mathcal{U}_{\gamma'}(b) \subset \mathcal{U}_\gamma(a)$

The first two follow directly from the definition. The third holds since $v(x-b) > \gamma' = v(b-a)$ implies $v(x-a) = v((x-b)+(b-a)) = \min\{v(x-b), v(b-a)\} = v(b-a) = \gamma' > \gamma$ i.e., $\forall x \in \mathcal{U}_{\gamma'}(b)$, we get $x \in \mathcal{U}_\gamma(a)$.

We consider the topology defined by this basis of neighborhoods, i.e., a set $\mathcal{U}$ is open if $\forall x \in \mathcal{U}, \exists\, \mathcal{U}_\gamma(x)$ such that $x \in \mathcal{U}_\gamma(x) \subset \mathcal{U}$.

The topology constructed above is Hausdorff: let $a \neq b$. Let $v(b-a) = \gamma'$. Consider $\mathcal{U}_{\gamma'}(a) = \{x \in K : v(x-a) > \gamma'\}$. Clearly, $b \notin \mathcal{U}_{\gamma'}(a)$. Pick some $\gamma > \gamma'$ and consider $\mathcal{U}_\gamma(b)$.

Claim: $\mathcal{U}_{\gamma'}(a) \cap \mathcal{U}_\gamma(b) = \emptyset$.

Suppose not, i.e., $\exists x \in K$ such that $v(x-a) > \gamma'$ and $v(x-b) > \gamma$.

Then, $\gamma' = v(b-a) = v((x-b)-(x-a)) = \min\{v(x-b), v(x-a)\}$. Since $v(x-a) > \gamma'$, we have $v(x-b) = \gamma'$, but $v(x-b) > \gamma$ and $\gamma' < \gamma$ — a contradiction.

$\Gamma = \{0\}$ if and only if $\mathcal{U}_\gamma(a) = \{a\}$ for all $a \in K$ and all $\gamma \in \Gamma$.

($\implies$): if $\Gamma = \{0\}$, then $v(x) = 0$ for all x, i.e., $x \in \mathcal{O}^\times$ for all x, i.e., $K = \mathcal{O}^\times$.

Let $x \in \mathcal{U}_\gamma(a) = \{x \in K : v(x-a) > \gamma\}$ where $x \neq a$. Let $\gamma = t\mathcal{O}^\times$. Since $K = \mathcal{O}^\times, t \in \mathcal{O}^\times$, i.e., $\gamma = 0$, which is a contradiction. Hence, $\mathcal{U}_\gamma(a) = \{a\}$.

($\impliedby$): Let $\mathcal{U}_\gamma(a) = \{a\}$ for all $a \in K$ and all $\gamma \in \Gamma$.

Thus, any $x \in K$ satisfying $v(x-a) > \gamma$ must be a itself. Suppose $v(x) \neq 0$ for some x. Let $v(x) = \gamma$. Then, $\mathcal{U}_\gamma(0) = \{y \in K \mid v(y) > 0\} = \{0\}$.

Since $v(x) \neq 0$, either $v(x) > 0$ or $v(x) < 0$. If $v(x) > 0$, then $x \in \mathcal{U}_\gamma(0)$, i.e., $x = 0$. Otherwise, $v(x) < 0$, i.e., $0 < -v(x)$, i.e., $0 < v(x^{-1})$, i.e., $x^{-1} \in \mathcal{U}_\gamma(0)$, i.e., $x^{-1} = 0$, i.e., $x = \infty$.

Hence, $v(x) \neq 0$ implies $x = 0$ or ∞. Thus, $\Gamma = \{0\}$

Therefore, v is trivial if and only if the induced topology is discrete.

Remark 1.7.

1. The following sets are open:

$$S_1 = \{x \mid v(x-a) \geq \gamma\},\ S_2 = \{x \mid v(x-a) \leq \gamma\},\ S_3 = \{x \mid v(x-a) = \gamma\}$$

2. The field operations are continuous with respect to this topology.

Theorem 1.4.2. *Two nontrivial valuation rings $\mathcal{O}_1, \mathcal{O}_2$ of K are dependent if and only if they induce the same topology on K.*

Proof. Since two dependent valuation rings have a common nontrivial coarsening $\mathcal{O}_1\mathcal{O}_2$, to show they induce the same topology, it suffices to consider only the case $\mathcal{O}_1 \subseteq \mathcal{O}_2$.

Let $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ be a valuation on $\mathcal{O}_1$. Then by Lemma 1.2.4, there exists a convex subgroup $\Delta \subseteq \Gamma$ connected with $\mathcal{O}_2$ such that $v_2 : K^\times \twoheadrightarrow \Gamma \twoheadrightarrow \Gamma/\Delta = \Gamma_2$ is a valuation on $\mathcal{O}_2$. Since $\mathcal{O}_1 \neq K$, $\Gamma_2 \neq \{0\}$.

Write

$$\mathcal{U}_\gamma(0) = \{a \in K \mid v(a) > \gamma\}$$

and

$$\mathcal{U}_{\gamma+\Delta}(0) = \{a \in K \mid v_2(a) > \gamma + \Delta\} = \{a \in K \mid v(a) > d\ \forall d \equiv \gamma \bmod \Delta\}$$

We have $\mathcal{U}_{\gamma+\Delta}(0) \subseteq \mathcal{U}_\gamma(0)$ since $v(a) > \gamma \implies a \in \mathcal{U}_{\gamma+\Delta}(0)$.

On the other hand, $v(a) > 2\gamma$ implies $v_2(a) \geq 2\gamma + \Delta$. Hence for $\gamma > 0$, if $v_2(a) \leq \gamma + \Delta$, then $\gamma + \Delta \geq 2\gamma + \Delta$, i.e., $\gamma \in \Delta$. Contrapositively, this implies, $0 < \gamma \notin \Delta \implies v(a) \leq 2\gamma$, i.e., $\mathcal{U}_\gamma(0) \subseteq \mathcal{U}_{\gamma+\Delta}(0)$.

Therefore, the topologies are equivalent.

Conversely, if $\mathcal{O}_1, \mathcal{O}_2$ induce the same topology on K, then $\mathcal{M}_2$ is an open neighborhood of zero in the topology induced by $\mathcal{O}_1$. Thus, $\mathcal{M}_2$ contains an open set around 0, i.e., $\exists a \in K^\times$ such that

$$\{x \in K \mid v(x) > \gamma\} = a\mathcal{M}_1 \subseteq \mathcal{M}_2, \text{ where } \gamma = v(a^{-1}).$$

Claim: $\mathcal{O}_1 \setminus \mathcal{M}_2$ is multiplicatively closed in $\mathcal{O}_1$:
Suppose $y_1, y_2 \in \mathcal{O}_1 \setminus \mathcal{M}_2$ and $y_1y_2 \notin \mathcal{O}_1 \setminus \mathcal{M}_2 \implies y_1y_2 \in \mathcal{M}_2$.
Thus, $v_2(y_1y_2) > 0 \implies v_2(y_1) > v_2(y_2^{-1})$.
Since $y_1, y_2 \notin \mathcal{M}_2$, $v_2(y_1) \leq 0$ and $v_2(y_2) \leq 0$.
Thus, $v_2(y_1^{-1}) \geq 0$ but $v_2(y_1) > v_2(y_2^{-1}) \geq 0$, i.e., $v_2(y_1) > 0$.
This is a contradiction. Hence, $\mathcal{O}_1 \setminus \mathcal{M}_2$ is multiplicatively closed.

Thus, we may form the ring $\mathcal{O}_3 := \left\{\frac{x}{y} : x \in \mathcal{O}_1, y \in \mathcal{O}_1 \setminus \mathcal{M}_2\right\}$.

Clearly, $\mathcal{O}_1 \subset \mathcal{O}_3$, hence $\mathcal{O}_3$ is a valuation ring.

Now, for $x \in \mathcal{O}_2 \setminus \{0\}$, $x^{-1} \notin \mathcal{M}_2$ (since x^{-1} is invertible in $\mathcal{O}_2$).

Thus, $x = 1/x^{-1} \in \mathcal{O}_3$, if $x^{-1} \in \mathcal{O}_1$. If $x^{-1} \notin \mathcal{O}_1$, then $x \in \mathcal{O}_1$, and so $x \in \mathcal{O}_3$ ($\because \mathcal{O}_1 \subseteq \mathcal{O}_3$). Hence, in any case, $x \in \mathcal{O}_3$.

Thus, $\mathcal{O}_3$ contains $\mathcal{O}_2$ as well. Therefore, $\mathcal{O}_3 \supseteq \mathcal{O}_1\mathcal{O}_2$.

We need only show $\mathcal{O}_3 \neq K$ to conclude that $\mathcal{O}_1, \mathcal{O}_2$ are dependent.

Take $z \in \mathcal{M}_1 \setminus \{0\}$. Then, $1/az \notin \mathcal{O}_3$ because if $1/az \in \mathcal{O}_3$, then $1/az = x/y$, for some $x \in \mathcal{O}_1$, $y \in \mathcal{O}_1 \setminus \mathcal{M}_1$, implying $y = a(zx) \in a\mathcal{M}_1 \subseteq \mathcal{M}_2$ — contradiction.

Thus, $1/ax \in K \setminus \mathcal{O}_3$, and hence $\mathcal{O}_3 \neq K$.

☕

Theorem 1.4.3 (Approximation Theorem)**.** *Suppose $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ are pairwise independent valuation rings of K. For every $1 \leq i \leq n$, let $v_i : K \twoheadrightarrow \Gamma_i \cup \{\infty\}$ be a valuation on $\mathcal{O}_i$.*

Then, for any $a_1, a_2, \ldots, a_n \in K$ and $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, \ldots, \gamma_n \in \Gamma_n$, there exists $x \in K$ such that $v_i(x - a_i) > \gamma_i$ for all $i \in \{1, 2, \ldots, n\}$.

Proof. For every i, pick positive $\delta_i \in \Gamma_i$ satisfying $\delta_i \geq \gamma_i$ and $-\delta_i \leq v_i(a_1), v_i(a_2), \ldots, v_i(a_n)$. Some new restrictions will be imposed on each δ_i during the proof.

Consider the open sets

$$M_i = \{x \in K : 2\delta_i < v_i(x)\} \text{ and}$$

$$A_i = \{x \in K : -2\delta_i \leq v_i(x)\}.$$

Note that $\gamma \leq v(x)$ and $\gamma \leq v(y) \implies \gamma \leq \min\{v(x), v(y)\} < v(x \pm y)$, hence M_i and A_i are closed under addition and subtraction.

Claim: We may choose δ_i such that $M_1 \cap \bigcap_{j=2}^{n}(K \setminus A_j) \neq \emptyset$.

We induct on n. For $n = 2$, suppose $M_1 \cap (K \setminus A_2) = \emptyset$, then $M_1 \subseteq A_2$. Now, choosing $c_i \in M_i$ $(i = 1, 2)$, we have $c_iA_i \subseteq \mathcal{M}_i$ and $c_i\mathcal{M}_i \subseteq M_i$.

Thus, $M_1 \subseteq A_2 \implies c_2c_1\mathcal{M}_1 \subseteq c_2M_2 \subseteq c_2A_2 \subseteq \mathcal{M}_2$.

Taking $a = c_2c_1$ gives $a\mathcal{M}_1 \subseteq \mathcal{M}_2$. Proceeding as in Theorem 1.4.2, we get $\mathcal{O}_1$ and $\mathcal{O}_2$ are dependent — contradiction.

For $n > 2$, by induction hypothesis, $\exists r \in M_1 \cap (K \setminus A_j)$. We choose $\delta_3, \delta_4, \ldots, \delta_n$ large enough for $r \in A_j$ $\forall j = 3, 4, \ldots, n$, i.e.

$$\delta_i \geq \gamma_i,\ \delta_i \leq v_i(a_1), v_i(a_2), \ldots, v_i(a_n), \text{ and } -2\delta_i \leq v_i(x)\ \forall j = 3, 4, \ldots, n.$$

Thus, r has the following properties:

$$2\delta_i < v_1(r),\ -2\delta_2 \leq v_2(r) \text{ and } -2\delta_i \leq v_i(r)\ \forall j = 3, 4, \ldots, n.$$

By induction hypothesis, $\exists s \in M_1 \cap \bigcap\limits_{3\leq j\leq n} (K \setminus A_j)$.

If $s \notin A_2$, then $s \in M_1 \cap \bigcap\limits_{j=2}^{n} (K \setminus A_j)$ and the claim is proved.

Suppose not. Then, $s \in A_2$ and $r \notin A_2 \implies s + r \notin A_2$.

Further, $s \notin A_j$ and $r \in A_j\ \forall j = 3, 4, \ldots, n \implies s + r \notin A_j\ \forall j = 3, 4, \ldots, n$.

Therefore, $s + r \in M_1 \cap \bigcap\limits_{j=2}^{n} (K \setminus A_j)$ and the claim is proved.

Analogously, by replacing M_1 with M_i and $\bigcap\limits_{j=2}^{n} (K \setminus A_j)$ with $\bigcap\limits_{j\neq i} (K \setminus A_j)$ and repeating the argument, we get $M_i \cap \bigcap\limits_{j\neq i} (K \setminus A_j) \neq \emptyset$.

Now, for any $x \in K \setminus A_j$, we have $v_j(1 + x) = v_j(x)$ since $v_j(1) = 0 > -2\delta_j > v_j(x)$.

Thus, $-2\delta_j > v_j(1 + x) \implies 2\delta_j < v_j(1/(1 + x)) \implies 1/(1 + x) \in M_j$

Further, if $x \in M_i$, then $v_i\left(\dfrac{x}{1+x}\right) = v_i(x) - v_i(1 + x) = v_i(x)$, since $v_i(1) = 0 < v_i(x) \implies v_i(1 + x) = v_i(1) = 0$.

Thus, $\frac{x}{1+x} \in M_i$. Hence, $\dfrac{1}{1+x} = 1 - \dfrac{x}{1+x} \in (1 + M_i)$.

Therefore,

$$\frac{1}{1+x} \in (1 + M_i) \cap \bigcap_{j\neq i} M_j \text{ for any } x \in M_i \cap \bigcap_{j\neq i} (K \setminus A_j)$$

Hence, $(1 + M_i) \cap \bigcap\limits_{j\neq i} M_j \neq \emptyset$.

We now choose $d_i \in (1 + M_i) \cap \bigcap\limits_{j\neq i} M_j$ and set $x \coloneqq a_1 d_1 + \cdots + a_n d_n$.

Since $d_i - 1 \in M_i$ and $d_j \in M_j\ \forall j \neq i$, $v_i(d_i - 1) > 2\delta_i$ and $v_i(d_j) > 2\delta_i$. Therefore,

$$\begin{aligned} v_i(x - a_i) &= v_i(a_1 d_1 + \cdots + a_i(d_i - 1) + \cdots + a_n d_n) \\ &> \min_{1\leq j\leq n} \{v_i(a_j) + 2\delta_i\} \geq -\delta_i + 2\delta_i = \delta_i \geq \gamma_i. \end{aligned}$$

1.5 Extensions of Valuations

We now show that for every extension L/K, a valuation on K may be extended to L.

Theorem 1.5.1 (Chevalley)**.** *For a field K, let $R \subseteq K$ be a subring and let $\mathfrak{p} \lhd R$ be prime. Then, there exists a valuation ring $\mathcal{O} \subseteq K$ such that $R \subseteq \mathcal{O}$ and $\mathcal{M} \cap R = \mathfrak{p}$.*

Proof. Consider

$$\Sigma \coloneqq \{(A, I) : \mathfrak{p}R_{\mathfrak{p}} \subseteq A \subseteq K, \text{ and } \mathfrak{p}R_{\mathfrak{p}} \subseteq I \subset A, \text{ where } A \text{ is a ring and } I \lhd A\}.$$

Since $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}) \in \Sigma$, it is nonempty. We partially order Σ as

$$(A_1, I_1) \leq (A_2, I_2) \iff A_1 \subseteq A_2 \text{ and } I_1 \subseteq I_2.$$

Claim: For any chain $\{(A_i, I_i)\}_{i \in J}$ of such pairs, then $\left(\bigcup_i A_i, \bigcup_i I_i\right)$ gives an upper bound.

For any $i \neq j$, assume without loss of generality that $A_i \subseteq A_j$ which implies $A_i \cup A_j = A_j$. Hence $\bigcup_i A_i = A_{i'}$ for some $i' \in J$, so $\bigcup_i A_i$ is a ring.

To see that $I = \bigcup_j I_j$ is an ideal of A, let $x, y \in I$. Say $x \in I_n$ and $y \in I_m$, and suppose $I_m \subseteq I_n$. Then $x, y \in I_n$ and so $x + y,\ xy \in I_n \subseteq I$. Hence closure under addition and multiplication follows. Clearly, $-x \in I_n \subseteq I$, hence I contains additive inverses. It also contains 0 and 1.

Now, to show $IR \subseteq I$, let $r \in R$ and $i \in I$. Then $i \in I_n$ for some n. Thus $ri \in I_n \subseteq I \implies IR \subseteq I$.

We have $R \subseteq R_{\mathfrak{p}} \subseteq \mathcal{O}$ and $\mathfrak{p}R_{\mathfrak{p}}$ is the unique maximal ideal of $R_{\mathfrak{p}}$. Hence $\mathcal{M} \cap R_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}}$.

Conversely, since $\mathfrak{p}R_{\mathfrak{p}} \subseteq \mathcal{M}$, we get $\mathcal{M} \cap R_{\mathfrak{p}} \supseteq \mathfrak{p}R_{\mathfrak{p}}$, thus $\mathfrak{p}R_{\mathfrak{p}} = \mathcal{M} \cap R_{\mathfrak{p}}$.

Claim: $\mathfrak{p} = \mathcal{M} \cap R$: Let $x \in \mathfrak{p}$, then $x \in \mathfrak{p}R_{\mathfrak{p}} \implies x \in \mathcal{M} \cap R_{\mathfrak{p}} \implies x \in \mathcal{M} \implies x \in \mathcal{M} \cap R$. Hence, $\mathfrak{p} \subseteq \mathcal{M} \cap R$.

Let $x \in \mathcal{M} \cap R$, $x \in R \subseteq R_{\mathfrak{p}}$, and $x \in \mathcal{M} \implies x \in \mathfrak{p}R_{\mathfrak{p}}$. But $\mathfrak{p}R_{\mathfrak{p}} \cap R = \mathfrak{p} \implies x \in \mathfrak{p}$. Hence, $\mathcal{M} \cap R \subseteq \mathfrak{p}$.

Thus, we need only show that $\mathcal{O}$ is a valuation ring. Now, $(\mathcal{O}, \mathcal{M})$ is a local ring because if there is another maximal ideal $\mathcal{M}'$ of $\mathcal{O}$, then $\mathcal{M}' \subseteq \mathcal{M}$ or $\mathcal{M} \subseteq \mathcal{M}'$. By maximality in Σ, $\mathcal{M}' \subseteq \mathcal{M}$, but since $\mathcal{M}'$ is a maximal ideal, $\mathcal{M}' = \mathcal{M}$.

Suppose $\mathcal{O}$ is not a valuation ring, then $\exists\, x \in K^\times$ with $x, x^{-1} \notin \mathcal{O}$. Thus, $\mathcal{O} \subsetneq \mathcal{O}[x]$ and $\mathcal{O} \subsetneq \mathcal{O}[x^{-1}]$.

The maximality of $(\mathcal{O}, \mathcal{M})$ implies $\mathcal{M}\mathcal{O}[x] = \mathcal{O}[x]$ (otherwise $(\mathcal{O}, \mathcal{M}) \leq (\mathcal{O}[x], \mathcal{M}\mathcal{O}[x])$). Similarly, $\mathcal{M}\mathcal{O}[x^{-1}] = \mathcal{O}[x^{-1}]$, and hence there exist $a_0, \ldots, a_n, b_0, \ldots, b_m \in \mathcal{M}$

such that

$$1 = \sum_{i=0}^{n} a_i x^i = \sum_{j=0}^{m} b_j x^{-j},$$

with n, m minimal. Without loss of generality, assume $m \leq n$.

Now, $b_0 \in \mathcal{M} \implies \sum_{j=1}^{m} b_j x^{-j} = 1 - b_0 \in \mathcal{O}^\times$ (since R is a local ring $\implies$ for all $r \in R$, either r or $1 - r$ is invertible, so $1 - b_0 \in \mathcal{O}^\times$).

Put $c_i = b_i(1 - b_0)^{-1}$ to get

$$1 = \Big(\sum_{j=1}^{m} b_j x^{-j}\Big)(1 - b_0)^{-1} = \sum_{j=1}^{m} c_j x^{-j}.$$

Hence, $x^n = \sum_{j=1}^{m} c_j x^{n-j}$. Using $1 = \sum_{i=0}^{n} a_i x^i$, we get

$$1 = \sum_{i=0}^{n} a_i x^i + \sum_{j=1}^{m} a_n c_j x^{n-j}.$$

But $m \leq n$, so $n - j \geq 0$, for all $j \leq m$. Also $n - j < n$, hence we have an expression for 1 with exponent of x being at most $n - 1$, which contradicts the minimality of n.

Therefore, $\mathcal{O}$ is a valuation ring. ☕

Definition 1.5.1. Let K_2/K_1 be a field extension and $\mathcal{O}_i \subseteq K_i$ be valuation rings. We say $\mathcal{O}_2$ is a ***prolongation*** or ***extension*** of $\mathcal{O}_1$ (or $\mathcal{O}_2$ lies over $\mathcal{O}_1$) if $\mathcal{O}_2 \cap K_1 = \mathcal{O}_1$, denoted as $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$.

In this case, we have

$$\mathcal{M}_2 \cap K_1 = \mathcal{M}_1,\ \mathcal{O}_2 \cap K_1 = \mathcal{O}_1, \text{ and } \mathcal{O}_2^\times \cap K_1 = \mathcal{O}_1^\times \implies \mathcal{O}_1 = \mathcal{O}_2 \cap K_1.$$

Given a field extension K_2/K_1 and a valuation ring $\mathcal{O}_1 \subseteq K_1$, we see that $\mathcal{O}_1 := \mathcal{O}_2 \cap K_1$ is a valuation ring of K_1 such that $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$.

Theorem 1.5.2. *Let K_2/K_1 be a field extension and $\mathcal{O}_1 \subseteq K_1$ be a valuation ring. Then, there is a prolongation $\mathcal{O}_2 \subseteq K_2$ of $\mathcal{O}_1$.*

Proof. We have $\mathcal{O}_1$ is a subring of K_1 and $\mathcal{M}_1 \lhd \mathcal{O}_1$ is maximal, hence prime. Thus, by Chevalley's Theorem, there exists a valuation ring $\mathcal{O}_2 \subseteq K_2$ with $\mathcal{O}_2 \cap K_1 = \mathcal{O}_1$ and $\mathcal{M}_2 \cap K_1 = \mathcal{M}_1$.

Claim. The maximal ideal of $\mathcal{O}_2 \cap K_1$ is $\mathcal{M}_2 \cap K_1 = \mathcal{M}_1$.

Let $x \in \mathcal{M}_2 \cap K_1$ and $a \in \mathcal{O}_2 \cap K_1$ be arbitrary. We will show that $ax \in \mathcal{M}_2 \cap K_1 = \mathcal{M}_1$.

Now, $a \in K_1 \implies a \in \mathcal{O}_1$ or $a^{-1} \in \mathcal{O}_1$. If $a \in \mathcal{O}_1$, then $ax \in \mathcal{M}_1$, and we are done.

Otherwise, $a^{-1} \in \mathcal{O}_1$. We already have $ax \in \mathcal{M}_2$, since $a \in \mathcal{O}_2$, $x \in \mathcal{M}_2$, and $\mathcal{M}_2 \subset \mathcal{O}_2$. Thus, we need only show that $ax \in \mathcal{O}_1$. Suppose not, i.e., $ax \notin \mathcal{O}_1$.

Then,

$$(ax)^{-1} = a^{-1}x^{-1} \in \mathcal{O}_1 \text{ (since } \mathcal{O}_1 \subseteq K_1 \text{ is a valuation ring).}$$

Now,

$$x \in \mathcal{M}_1 \subset \mathcal{O}_1 \implies (a^{-1}x^{-1})x = a^{-1} \in \mathcal{M}_1 = \mathcal{M}_2 \cap K_1 \implies a^{-1} \in \mathcal{M}_2.$$

So a^{-1} is not invertible in $\mathcal{O}_2$, i.e., $(a^{-1})^{-1} = a \notin \mathcal{O}_2$ — contradiction.. Hence the claim is true.

Since local rings with the same maximal ideal must coincide, we have $\mathcal{O}_2 \cap K_1 = \mathcal{O}_1$, and we are done.

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Theorem 1.5.3. *1. Every valuation ring $\mathcal{O}$ of a field K is integrally closed in K.*

2. Let $D \subseteq K$ be a subring and $\mathbb{V} = \{\mathcal{O} : D \subseteq \mathcal{O}$ and $\mathcal{M} \cap D$ is a maximal ideal of $D\}$, where $\mathcal{O}$ is a valuation ring. The integral closure R of D in K equals

$$R_1 := \bigcap_{\mathcal{O} \in \mathbb{V}} \mathcal{O}.$$

Proof. 1. Let $x \in K$ with $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x_n = 0$ for some $a_0, \ldots, a_{n-1} \in \mathcal{O}$.

If $x \in \mathcal{O}$, we are done. Suppose not. Then $x \notin \mathcal{O}$, i.e., $x^{-1} \in \mathcal{M}$, so

$$-x^n = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$$

implies

$$-1 = a_0 x^{-n} + \cdots + a_{n-1}x^{-1} \in \mathcal{M},$$

but $\mathcal{M}$ is a proper ideal — contradiction.

2. Let $x \in K$ be integral over D, i.e., $x^n + \sum_{i=0}^{n-1} a_i x^i = 0$ for some $a_i \in D$.

Now, $D \subseteq \mathcal{O} \implies a_i \in \mathcal{O} \implies D \subseteq \mathcal{O}$ for all $\mathcal{O} \in \mathbb{V}$, so $D \subseteq R_1$. Hence, each $a_i \in R_1$, and so x is integral over R_1. By (1), each $\mathcal{O} \in \mathbb{V}$ is integrally closed in K.

Thus, $x \in \mathcal{O}$ for all $\mathcal{O} \in \mathbb{V}$, so $x \in R_1$. Thus, $R \subseteq R_1$.

Conversely, let $x \notin R$. We will show $x \notin R_1$.

First, observe $x \notin R[x^{-1}]$, otherwise $x = \sum_{i=0}^{m} b_i x^{-i}$, $b_i \in R$, then

$$x^{m+1} = \sum_{i=0}^{m} b_i x^{m-i} = b_0 x^m + b_1 x^{m-1} + \cdots + b_m, \quad b_i \in R.$$

i.e., x is integral over R, and so integral over D — contradiction. Hence, $x \notin R[x^{-1}]$.

Since x^{-1} is not invertible in $R[x^{-1}]$, we have $x^{-1} \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $R[x^{-1}]$.

By Chevalley's Theorem, there exists a valuation ring $\mathcal{O} \subseteq K$ such that $R[x^{-1}] \subseteq \mathcal{O}$ and $\mathcal{M} \cap R[x^{-1}] = \mathfrak{m}$. Since $x \in \mathfrak{m} = \mathcal{M} \cap R[x^{-1}]$, we get $x \in \mathcal{M} \implies x \notin \mathcal{O}$.

Finally, we have $\mathcal{O} \in \mathbb{V}$, i.e., $\mathcal{M} \cap D \lhd D$ is maximal. This follows easily as contraction from an integral extension preserves maximality. Since $x \notin \mathcal{O}$, it follows that $x \notin R_1$, and we are done.

☕

Corollary 1.5.3.1. *Let L/K be any field extension and $\mathcal{O}$ be a valuation ring of K. Let R be the integral closure of $\mathcal{O}$ in L. Then, letting $\mathcal{O}'$ range over the set of prolongations of $\mathcal{O}$ to L, we have*

$$R = \bigcap_{\mathcal{O}' \supseteq \mathcal{O}} \mathcal{O}'.$$

Proof. For each prolongation $\mathcal{O}'$, let $\mathcal{M}'$ be its maximal ideal. Then, by definition, $\mathcal{M} \cap \mathcal{O} = \mathcal{M}$. Conversely, if $\mathcal{O}' \supseteq \mathcal{O}$ has maximal ideal $\mathcal{M}'$ satisfying $\mathcal{M} \cap \mathcal{O} = \mathcal{M}$, then $\mathcal{O}' \cap K = \mathcal{O}$. Hence, the set $\mathbb{V}$ is exactly the set of prolongations of $\mathcal{O}$ to L. The result thus follows from the theorem.

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Let $(K_1, \mathcal{O}_1) \subset (K_2, \mathcal{O}_2)$ be an arbitrary extension of valued fields. For each $\mathcal{O}_i$, let $v_i : K \twoheadrightarrow \Gamma_i \cup \{\infty\}$ be a valuation and recall that its restriction $v_i : K_i^\times \twoheadrightarrow \Gamma_i$

is a group homomorphism with kernel $\mathcal{O}_i^\times$ and so $K_i^\times/\mathcal{O}_i^\times \cong \Gamma_i$. Further, the composite map

$$K_1^\times \hookrightarrow K_2^\times \twoheadrightarrow K_2^\times/\mathcal{O}_2^\times \cong \Gamma_2$$

has kernel $K_1^\times \cap \mathcal{O}_2^\times = \mathcal{O}_1^\times$.

Thus, $\Gamma_1 \cong K_1^\times/\mathcal{O}_1^\times \hookrightarrow K_2^\times/\mathcal{O}_2^\times \cong \Gamma_2$ is a well-defined group homomorphism.

Hence, we may regard Γ_1 as an ordered subgroup of Γ_2. Define the ***ramification index*** of this extension as

$$e := e(\mathcal{O}_2/\mathcal{O}_1) := [\Gamma_2 : \Gamma_1].$$

As before, considering $\mathcal{O}_1 \hookrightarrow \mathcal{O}_2 \twoheadrightarrow \mathcal{O}_2/\mathcal{M}_2 = \mathcal{K}_2$ has kernel $\mathcal{O}_1/\mathcal{M}_2 = \mathcal{M}_1$, we get a well-defined map $\mathcal{K}_1 \hookrightarrow \mathcal{K}_2$. Define the ***residue degree*** of this extension as

$$f := f(\mathcal{O}_2/\mathcal{O}_1) = [\mathcal{K}_2 : \mathcal{K}_1].$$

Now, if $e(\mathcal{O}_2/\mathcal{O}_1) = f(\mathcal{O}_2/\mathcal{O}_1) = 1$, then the extension $\mathcal{O}_2/\mathcal{O}_1$ is called ***immediate***.

Example 1.7. A completion $(\widehat{K}, \widehat{\mathcal{O}})$ of a rank-one valued field $(K, \mathcal{O})$ is an immediate extension, as seen previously.

Remark 1.8. e and f are multiplicative: if $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2) \subseteq (K_3, \mathcal{O}_3)$, then

$$e(\mathcal{O}_3/\mathcal{O}_1) = e(\mathcal{O}_3/\mathcal{O}_2)\, e(\mathcal{O}_2/\mathcal{O}_1)$$

and

$$f(\mathcal{O}_3/\mathcal{O}_1) = f(\mathcal{O}_3/\mathcal{O}_2)\, f(\mathcal{O}_2/\mathcal{O}_1),$$

because degrees of extensions are multiplicative.

Lemma 1.5.1. *Suppose $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$, and for $i = 1, 2$, let v_i be the valuation corresponding to $\mathcal{O}_i$. Choose $\omega_1, \ldots, \omega_f \in \mathcal{O}_2$ and $\pi_1, \ldots, \pi_e \in K_2^\times$ so that:*

1. the residues $\overline{\omega}_1, \ldots, \overline{\omega}_f \in \overline{\mathcal{K}}_2$ are linearly independent over $\overline{\mathcal{K}}_1$;

2. the values $v_2(\pi_1), \ldots, v_2(\pi_e)$ are representatives of the distinct cosets of Γ_2/Γ_1.

Then for all $a_{ij} \in K_1$,

$$v_2\left(\sum_{i=1}^{f}\sum_{j=1}^{e} a_{ij}\omega_i\pi_j\right) = \min\{\, v_2(a_{ij}\omega_i\pi_j) \mid 1 \le i \le f,\, 1 \le j \le e \,\}.$$

In particular, the products

$$\{\omega_i\pi_j \mid i = 1, \ldots, f,\, j = 1, \ldots, e\}$$

are linearly independent over K_1.

Proof. Let $a_{ij} \in K_1$ not all zero, and pick $1 \leq I \leq f$ and $1 \leq J \leq e$ such that

$$v_2(a_{IJ}\omega_I\pi_J) = \min_{i,j}\{\, v_2(a_{ij}\omega_i\pi_j)\,\}.$$

Claim. $v_2(a_{IJ}\pi_J) < v_2(a_{ij}\pi_j)$ for all $j \neq J$.
If not, then there exist $j \neq J$ with $v_2(a_{ij}\pi_j) = v_2(a_{IJ}\pi_J)$; i.e.

$$v_2(a_{IJ}) + v_2(\pi_J) = v_2(a_{IJ}) + v_2(\pi_j).$$

Now, $v_2(\pi_j) - v_2(\pi_J) = v_2(a_{IJ}) - v_2(a_{ij}) \in \Gamma_1$ implying $v_2(\pi_j) \equiv v_2(\pi_J) \mod \Gamma_1$, which contradicts the second hypothesis. This proves the claim.

Now, let $z = \sum_{i=1}^{f}\sum_{j=1}^{e} a_{ij}\omega_i\pi_j$. We know by the strong triangle inequality that $v_2(z) \geq \min_{i,j} v_2(a_{ij}\omega_i\pi_j)$. If equality holds, we are done.

Suppose not, i.e. $v_2(z) > \min_{i,j} v_2(a_{ij}\omega_i\pi_j)$.

Note that

$$v_2(a_{ij}\omega_i\pi_j) = v_2(a_{ij}\pi_j) + v_2(\omega_i) \geq v_2(a_{ij}\pi_j) \quad (\text{since } \omega_i \in \mathcal{O}_2).$$

Hence $v_2(z) \geq v_2(a_{ij}\pi_j) \geq v_2(a_{IJ}\pi_J)$. Therefore $z(a_{IJ}\pi_J)^{-1} \in \mathcal{M}_2$.

Moreover, by the earlier claim, $a_{ij}\pi_j\omega_i(a_{IJ}\pi_J)^{-1} \in \mathcal{M}_2$ for all $j \neq J$.

Thus

$$z(a_{IJ}\pi_J)^{-1} - \sum_{i=1}^{f}\sum_{j\neq J} a_{ij}\pi_j\omega_i(a_{IJ}\pi_J)^{-1} \in \mathcal{M}_2,$$

and this expression equals $\sum_{i=1}^{f} a_{iJ}(a_{IJ})^{-1}\omega_i$, because

$$z - \sum_{j\neq J}\sum_{i=1}^{f} a_{ij}\pi_j\omega_i = \sum_{i=1}^{f} a_{iJ}\omega_i\pi_J.$$

Hence, $\sum_{i=1}^{f} a_{iJ}(a_{IJ})^{-1}\omega_i \in \mathcal{M}_2$, thus its image in $\mathcal{K}_2$ is zero, i.e., $\sum_{i=1}^{f} \overline{a_{iJ}(a_{IJ})^{-1}}\overline{\omega_i} = \overline{0}$, which contradicts hypothesis 1.

☕

Corollary 1.5.3.2. *Suppose $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$ and $n = [K_2 : K_1] < \infty$. Then, $e, f < \infty$ and $ef < n$.*

Proof. This follows because $\{\omega_i\pi_j \mid i = 1, \ldots, f,\ j = 1, \ldots, e\}$ is a linearly independent set of cardinality ef. ☕

Theorem 1.5.4. *Let $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$ with K_2 algebraic over K_1. Then*

1. *For every $\gamma \in \Gamma_2$, there exists $n \in \mathbb{N}$ such that $n\gamma \in \Gamma_1$, i.e. Γ_2/Γ_1 is a torsion group.*

2. *$\overline{\mathcal{K}_2}$ is an algebraic extension of $\overline{\mathcal{K}_1}$.*

Proof. 1. Let $\gamma \in \Gamma_2$ be arbitrary. Since v_2 is surjective, choose $x \in K_2^\times$ with $v_2(x) = \gamma$. Let $L = K_1(x) \subseteq K_2$. Because x is algebraic over K_1, the extension L/K_1 is finite. Let $\Gamma = v(L^\times) \subseteq \Gamma_2$. Clearly, we have $(L, \mathcal{O}) \subset (K_2, \mathcal{O}_2)$.

Claim: $(K_1, \mathcal{O}_1) \subset (L, \mathcal{O})$.

Consider $\mathcal{O} \cap K_1 = \mathcal{O}_2 \cap K_1(x) \cap K_1 = (\mathcal{O}_2 \cap K_1) \cap K_1(x) = \mathcal{O}_1 \cap K_1(x) = \mathcal{O}_1$, because $\mathcal{O}_2 \cap K_1 = \mathcal{O}_1$ and $\mathcal{O}_1 \subseteq K_1 \subseteq K_1(x)$.

Therefore, we have $(K_1, \mathcal{O}_1) \subseteq (L, \mathcal{O}) \subseteq (K_1, \mathcal{O}_2)$.

Now, $L = K_1(x)$ and x is algebraic over K_1, hence $[L : K_1] < \infty$. Thus, from the previous corollary, we have $[\Gamma_L : \Gamma_1] < \infty$. Clearly, $\Gamma_1 = v(K^\times) \subset v(L^\times) = \Gamma$ and both are abelian groups. Thus $\Gamma_1 \lhd \Gamma$. Therefore Γ/Γ_1 is a group and it is finite. Therefore, $\exists n \in \mathbb{N}$ such that for all $\gamma \in \Gamma$ we have $n\gamma \in \Gamma_1$, showing that Γ_2/Γ_1 is torsion.

2. Take any $x \in \mathcal{O}_2^\times$ and define L as before, then proceeding similarly as before, we obtain $\overline{L}/\overline{\mathcal{K}_1}$ is finite (by the previous corollary) and hence $\overline{x}$ is algebraic over $\overline{\mathcal{K}_1}$.

Lemma 1.5.2. *Suppose $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ are valuation rings of a field K with maximal ideals $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$. Let $R = \bigcap_{i=1}^n \mathcal{O}_i$ and $\mathfrak{p}_i = R \cap \mathcal{M}_i$. Then $\forall\, i,\ R_{\mathfrak{p}_i} = \mathcal{O}_i$.*

Proof. We first show that $R_{\mathfrak{p}_i} \subseteq \mathcal{O}_i$.

Let $a/b \in R_{\mathfrak{p}_i}$ with $a, b \in R$ and $b \notin \mathfrak{p}_i$. Then $b \notin \mathcal{M}_i$, so $b^{-1} \in \mathcal{O}_i^\times$. Since $a \in R \subseteq \mathcal{O}_i$, we have $ab^{-1} = a/b \in \mathcal{O}_i$.

Conversely, let $a \in \mathcal{O}_i$ and set $I_a = \{j \mid a \in \mathcal{O}_j\}$. Write $\alpha_j = a + \mathcal{M}_j \in \overline{\mathcal{K}_j}$ for each $j \in I_a$.

Choose a prime $p \in \mathbb{N}$ such that for all $j \in I_a$ we have $p > \mathrm{char}(\overline{\mathcal{K}_j})$ and α_j is not a primitive pth root of unity. Set

$$b = 1 + a + \cdots + a^{p-1}.$$

Observe that

$$\alpha_j = \overline{a} = 1 \text{ implies } \overline{b} = 1 + 1 + \cdots + 1 = p \neq 0 \text{ in } \overline{\mathcal{K}_j}$$

$$\alpha_j = \overline{a} \neq 1 \text{ implies } \overline{b} = \frac{1-\alpha_j^p}{1-\alpha_j} \neq 0 \text{ in } \overline{\mathcal{K}_j}.$$

Thus, $b \notin \mathcal{M}_j$ in both cases, hence $b \in \mathcal{O}_j^\times \ \forall j \in I_a$.

For $j \in \{1, 2, \ldots, n\} \backslash I_a$, $a \notin \mathcal{O}_j \implies a^{-1} \in \mathcal{O}_j$ and $a^{-1} \in \mathcal{M}_j$, since a^{-1} is a nonunit of $\mathcal{O}_j$. Thus,

$$1 + a^{-1} + \cdots + a^{-(p-1)} \in \mathcal{O}_j^\times,$$

implying

$$b^{-1} = a^{-(p-1)}\big(1 + a^{-1} + \cdots + a^{-(p-1)}\big)^{-1} \in \mathcal{O}_j,$$

and so

$$ab^{-1} = a^{-(p-2)}\big(1 + a^{-1} + \cdots + a^{-(p-1)}\big)^{-1} \in \mathcal{O}_j.$$

Thus for all $j = 1, \ldots, n$ we have b^{-1}, $ab^{-1} \in \mathcal{O}_j$. Therefore b^{-1}, $ab^{-1} \in R$, and $b^{-1} \notin \mathcal{M}_i \cap R = \mathfrak{p}_i$, since $b \in \mathcal{O}_i^\times$.

Hence $a = \dfrac{ab^{-1}}{b^{-1}} \in R_{\mathfrak{p}_i}$. ☕

Theorem 1.5.5. *Suppose $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ are valuation rings of a field K with maximal ideals $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$. Let $R = \bigcap_{i=1}^n \mathcal{O}_i$ and $\mathfrak{p}_i = R \cap \mathcal{M}_i$. Further, suppose that $\mathcal{O}_i \nsubseteq \mathcal{O}_j$ for all $i \neq j$. Then*

1. *for all $i \neq j$, $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$;*
2. *$\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ is the set of all maximal ideals of R;*
3. *for each n-tuple $(a_1, \ldots, a_n) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$, there exists an $a \in R$ with $a - a_i \in \mathcal{M}_i$.*

Proof. 1. Suppose not, i.e., $\exists i \neq j$ with $\mathfrak{p}_i \subseteq \mathfrak{p}_j$. By the previous lemma, $\mathcal{O}_j = R_{\mathfrak{p}_j} \subseteq R_{\mathfrak{p}_i} = \mathcal{O}_i$, but this contradicts the hypothesis.

2. Claim: every ideal $\mathfrak{a} \neq R$ is contained in some $\mathfrak{p}_i$, $i = 1, \ldots, n$.

   Suppose not, i.e., there exists an ideal $\mathfrak{a} \neq R$ such that for each $i = 1, \ldots, n$, there exists $a_i \in \mathfrak{a} \setminus \mathfrak{p}_i$. For each $i \neq j$, use (1) to pick $b_{ij} \in \mathfrak{p}_i \setminus \mathfrak{p}_j$. Then

   $$c_j := \prod_{i \neq j} b_{ij} \in \mathfrak{p}_i \setminus \mathfrak{p}_j, \text{ for every } j = 1, \ldots, n.$$

   Consequently, $a_j c_j \in \mathfrak{p}_i$ for all $i \neq j$ and $a_j c_j \notin \mathfrak{p}_j$. This is because $\mathfrak{p}_j$ is prime, since it is the contraction of the prime ideal $\mathcal{M}_j$ to R.

   Consider

   $$d := \sum_{j=1}^n a_j c_j \notin \mathfrak{p}_i \text{ for all } i = 1, \ldots, n.$$

Now, $d \in \mathfrak{a} \subset R$ but $d \notin \mathfrak{p}_i = R \cap \mathcal{M}_i \implies d \notin \mathcal{M}_i$, implying $d^{-1} \in \mathcal{O}_i$, for every i such that $1 \leq i \leq n$.

Hence $d^{-1} \in R$, yielding $1 = dd^{-1} \in \mathfrak{a}$, a contradiction, since $\mathfrak{a} \neq R$. This proves the claim.

Now, note that the above claim holds for any ideal. In particular, it holds for maximal ideals, whence we conclude that every maximal ideal is equal to some $\mathfrak{p}_i$, $i = 1, \ldots, n$. Thus, the set of maximal ideals in a subset of $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. But by (1.), all the $\mathfrak{p}_i$ are distinct, hence the set of maximal ideals must be exactly equal to $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

3. For $i \neq j$, $\mathfrak{p}_i + \mathfrak{p}_j = R$, since the only maximal ideals are $\mathfrak{p}_i$ and $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$. Therefore, by the Chinese Remainder Theorem, the canonical map

$$R \longrightarrow R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_n$$

is surjective. We know, for each i, $R_{\mathfrak{p}_i}/\mathfrak{p}_i R_{\mathfrak{p}_i} \cong R/\mathfrak{p}_i$, and $R_{\mathfrak{p}_i} = \mathcal{O}_i$, it follows that

$$R \longrightarrow \mathcal{O}_1/\mathcal{M}_1 \times \cdots \times \mathcal{O}_n/\mathcal{M}_n$$

is surjective. Hence, given any $(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) \in \prod_i \mathcal{O}_i/\mathcal{M}_i,$, there exists $a \in R$ such that $a \equiv a_i \mod \mathcal{M}_i$ for all $i \implies a - a_i \in \mathcal{M}_i$.

Lemma 1.5.3. *Suppose K_2/K_1 is an algebraic extension of fields and $\mathcal{O} \subset K_1$ is a valuation ring. Let $\mathcal{O}', \mathcal{O}'' \subset K_2$ be valuation rings lying over $\mathcal{O}$.*
If $\mathcal{O}' \subset \mathcal{O}''$, then $\mathcal{O}' = \mathcal{O}''$.

Proof. We have $\mathcal{O}' \cap K_1 = \mathcal{O}'' \cap K_1 = \mathcal{O}$ and $\mathcal{O}' \subset \mathcal{O}''$, which implies $\mathcal{M}'' \subset \mathcal{M}'$. Hence, we have $\mathcal{O}'/\mathcal{M}'' = \overline{\mathcal{O}'} \subset \mathcal{O}''/\mathcal{M}'' = \overline{\mathcal{K}''}$. Consider the natural order-preserving homomorphism $\mathcal{O}'' \twoheadrightarrow \overline{\mathcal{K}''}$.

The image of $\mathcal{O}'$ under this map is $\mathcal{O}'/\mathcal{M}'' = \overline{\mathcal{O}'}$.

Claim: $\overline{\mathcal{O}'}$ is a valuation ring of $\overline{\mathcal{K}''}$: let $\overline{x} \in \overline{\mathcal{K}''}$ be arbitrary; look at its preimage $x \in \mathcal{O}''$. If $x \in \mathcal{O}'$, then $\overline{x} \in \overline{\mathcal{O}'}$ and we are done. If not, then we must have $x^{-1} \in \mathcal{O}'$, as $\mathcal{O}'$ is a valuation ring. Then, $\overline{x^{-1}} = \overline{x}^{-1} \in \overline{\mathcal{O}'}$. Thus, given any $\overline{x} \in \overline{\mathcal{K}''}$, we must have either $\overline{x} \in \overline{\mathcal{O}'}$ or $\overline{x}^{-1} \in \overline{\mathcal{O}'}$ — this proves the claim.

We will now show that $\overline{\mathcal{O}'} = \overline{\mathcal{K}''}$. First, to see that $\overline{\mathcal{O}'}$ is a field, take any nonzero $x \in \overline{\mathcal{O}'}$. We have seen previously that the extension $\overline{\mathcal{K}''}/\overline{\mathcal{K}}$ of residue class fields is algebraic, hence there exist $a_0, a_1, \ldots, a_r \in \overline{\mathcal{K}}$ such that $a_0 + a_1 x + \cdots + a_r x^r = 0$ in $\overline{\mathcal{O}'}$. Assume without loss of generality that $a_0 \neq 0$. Note that since $a_0 \in \overline{\mathcal{K}} = \mathcal{O}'/\mathcal{M}'$, which is a field, hence its inverse a_0^{-1} is also in $\mathcal{O}'/\mathcal{M}' \subset \mathcal{O}'/\mathcal{M}'' = \overline{\mathcal{O}}$.

Thus, the element $y = -a_0^{-1}(a_1 + \cdots a_r x^{r-1}) \in \overline{\mathcal{O}'}$. It is clear that $xy = 1$, hence the claim follows.

Thus, $\overline{\mathcal{O}'}$ is a field and hence must equal its field of fractions $\overline{\mathcal{K}''}$. We then have $\mathcal{M}''$ is a maximal ideal of $\mathcal{O}'$, but $\mathcal{O}'$ is a local ring so $\mathcal{M}' = \mathcal{M}''$ whence it follows that $\mathcal{O}' = \mathcal{O}''$. ☕

Given an algebraic extension K_2 of K_1 and a valuation ring $\mathcal{O}_1$ of K_1, there may exist infinitely many valuation rings of K_2 lying over $\mathcal{O}_1$. In certain cases, there is a natural bound.

Let $K_2 \cap K_1^s = \{x \in K_2 \mid x \text{ is separable over } K_1\}$. The field $K_2 \cap K_1^s$ is a separable extension of K_1.

Definition 1.5.2. $[K_2 \cap K_1^s : K_1]$ is called the ***degree of separability*** of K_2 over K_1. Moreover, $[K_2 : K_2 \cap K_1^s]$ is called the ***degree of inseparability*** of K_2 over K_1.

Every $x \in K_2 \setminus (K_2 \cap K_1^s)$ is purely inseparable over $K_2 \cap K_1^s$.

We use the following notations:

$$[K_2 : K_1]_s = [K_2 \cap K_1^s : K_1] \qquad \text{and} \qquad [K_2 : K_1]_i = [K_2 : K_2 \cap K_1^s].$$

Theorem 1.5.6. *Let K_2 be algebraic over K_1, and $[K_2 : K_1]_s < \infty$. Let $\mathcal{O}$ be a valuation ring of K_1. Then the number n of all prolongations of $\mathcal{O}$ to K_2 is finite, and $n \leq [K_2 : K_1]_s$.*

Proof. Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be distinct prolongations of $\mathcal{O}$ to K_2, with maximal ideals $\mathcal{M}_1, \ldots, \mathcal{M}_m$, respectively. Since they are distinct, they must be pairwise incomparable, by the previous lemma.

Consider now the elements $e_j := (0, \ldots 1, \ldots, 0) \in \prod_{i=1}^m \mathcal{O}_i$. For every j, using Theorem 1.5.5 (3), we conclude the existence of a c_j such that $c_j - 1 \in \mathcal{M}_j$ and $c_i \in \mathcal{M}_j$ for $i \neq j$. Hence, we obtain $c_1, \ldots, c_m$ such that for all $i, j \in \{1, \ldots, m\}$,

$$c_j - 1 \in \mathcal{M}_j \text{ and } c_i \in \mathcal{M}_j \text{ for } i \neq j.$$

We consider two cases:

(i) char $K_1 = p > 0$: we pick $k \in \mathbb{N}$ large enough to ensure the separability of $c_1^{p^k}, \ldots, c_m^{p^k}$ over K_1.

(ii) char $K_1 = 0$: we proceed as above and simply take $k = 0$.

Claim: These m elements are linearly independent over K_1

Suppose not. Then there exist a_i, not all zero, such that $\sum_{i=1}^{m} a_i c_i^{p^k} = 0$. Pick $j \leq n$ such that $v(a_j) = \min_{i \leq m} v(a_i)$. Note that $a_j \neq 0$, otherwise $v(a_j) = \infty \implies v(a_i) = \infty \implies a_i = 0$ for all i. Thus, $c_j^{p^k} = -\sum_{i \neq j} a_i c_i^{p^k} \in \mathcal{M}_j$.

Hence, $v_j(c_j^{p^k}) = p^k v_j(c_j) > 0 \implies v_j(c_j) > 0 \implies c_j \in \mathcal{M}_j$. In particular, we have both $c_j - 1 \in \mathcal{M}_j$ and $c_j \in \mathcal{M}_j \implies 1 \in \mathcal{M}_j$, which is a contradiction.

This proves the claim and hence $m \leq [K_2 : K_1]_s$.

Note that we may repeat the above argument with $m+1$ distinct valuations and thus similarly show that $m + 1 \leq [K_2 : K_1]_s$. Proceeding inductively, we can conclude that there can only be finitely many extensions of $\mathcal{O}$ to K_2 since $[K_2 : K_1]_s < \infty$. Moreover, this number must be bounded above by $[K_2 : K_1]_s$ as seen above. This concludes the proof. ☕

Corollary 1.5.6.1. *Suppose K_2 is a purely inseparable extension of K_1. Then every valuation ring $\mathcal{O}$ of K_1 has exactly one prolongation to K_2.*

Lemma 1.5.4. *Suppose K is a field with a non-trivial valuation v. For every polynomial*

$$g(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n \in K[x]$$

and every γ in the value group Γ of v, there exists a separable polynomial

$$h(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} + x^n \in K[x]$$

such that $v(a_i - b_i) > \gamma$ for every i with $0 \leq i < n$.

Proof. Let $y_0, y_1, \ldots, y_{n-1}$ be indeterminates over K. Construct

$$f_y(x) = f_{y_0, y_1, \ldots, y_{n-1}}(x) = \sum_{i=0}^{n-1} (a_i + y_i) x^i + x^n \in K(y_0, y_1, \ldots, y_{n-1})[x].$$

Consider the resultant $\mathrm{Res}(f_y, f_y') = \mathrm{disc}(f)$. By the algebraic independence of $y_0, y_1, \ldots, y_{n-1}$ over K, it follows that $\mathrm{Res}(f_y, f_y')$ is a nontrivial polynomial R in $K[y_0, y_1, \ldots, y_{n-1}]$.

Since v is nontrivial, $\exists x \in K$ such that $v(x) = \alpha \neq 0$. Then, $v(x^n) = n\alpha \neq 0 \ \forall n \in \mathbb{N}$. Hence, $v(K) = \Gamma$ is infinite. Therefore, K cannot be a finite field. Hence $\forall \delta \in \Gamma, \{x \in K \mid v(x) > \delta\}$ is an infinite set, otherwise it would be a closed set.

Claim: For a nonconstant polynomial $g(x_1, \ldots, x_n)$ in $K[x_1, \ldots, x_n]$ and an infinite subset $M \subseteq K$, $\exists (m_1, \ldots, m_n) \in M^n$ such that $g(m_1, \ldots, m_n) \neq 0$.

We prove this by induction on n. For $n = 1$, we have a polynomial $g(x) \in K[x]$. Suppose $g(x) = 0\ \forall x \in M$, since M is infinite, g has infinite roots, i.e. $g \equiv 0$, a contradiction. Thus, there exists $y \in M$ such that $g(y) \neq 0$.

We now assume the claim for $n = k - 1$ and prove it for $n = k$. We have $g(x_1, \ldots, x_n)$. Fix some $y_n \in M$ and look at $g(x_1, x_2, \ldots, y_n) \in K[x_1, x_2, \ldots, x_{n-1}]$. By induction hypothesis, we have $(y_1, \ldots, y_{n-1}) \in M^{n-1}$ such that $g(y_1, y_2, \ldots, y_n) \neq 0$. Thus, we have a $(y_1, y_2, \ldots, y_n) \in M^n$ such that $g(y_1, \ldots, y_n) \neq 0$.

Applying the claim with $M = \{x \in K \mid v(x) > \delta\}$ and $g = R$, we get $c_0, c_1, \ldots, c_n \in M$ such that $R(c_0, c_1, \ldots, c_n) \neq 0$. Now, $c_i \in M \implies v(c_i) > \delta$.

Set $h(x) := x^n + \sum_{i=0}^{n-1}(a_i + c_i)x^i = (a_0 + c_0) + (a_1 + c_1)x + \ldots + x^n$.

It follows that $\mathrm{Res}(h, h') = \mathrm{disc}(h) \neq 0.$, i.e., h and h' have no common roots, i.e., $h(x)$ has no repeated roots, i.e., h is a separable polynomial.

We shall use this lemma to prove the following theorem:

Theorem 1.5.7. *Suppose K is a separably closed field and $\mathcal{O}$ is a proper valuation ring of K. Let $\widetilde{K}$ be an algebraic closure of K and let $\widetilde{\mathcal{O}}$ be the unique extension of $\mathcal{O}$ to $\widetilde{K}$. Then $\widetilde{\mathcal{O}}/\mathcal{O}$ is an immediate extension. In particular, the residue class field $\overline{\mathcal{K}}$ of $\mathcal{O}$ is algebraically closed, and the value group Γ of $\mathcal{O}$ is divisible, i.e., for every $\gamma \in \Gamma$ and any $n \in \mathbb{N} \setminus \{0\}$, there exists $\delta \in \Gamma$ such that $n\delta = \gamma$.*

Proof. Let $\widetilde{\Gamma}$ and $\overline{\widetilde{\mathcal{K}}}$ denote the value group and residue class field of $\widetilde{\mathcal{O}}$.

Note that the multiplicative group $\widetilde{K}^\times$ of $\widetilde{K}$ is divisible. To see this, let $a \in \widetilde{K}^\times$ and $n \in \mathbb{N}$. Consider $g(x) = x^n - a$. Since $\widetilde{K}$ is an algebraic closure of K, $g(x)$ has a root in $\widetilde{K}$, say b. Then, $b^n = a$, whence we conclude that $\widetilde{K}^\times$ is divisible.

Now, $\widetilde{\Gamma} = \widetilde{K}^\times/\widetilde{\mathcal{O}}^\times$ is a quotient of a divisible group, hence it is also divisible. Similarly, for $a_0, a_1, \ldots, a_n \in \widetilde{\mathcal{O}},\ n > 0,\ a_n \in \widetilde{\mathcal{O}}^\times$, the polynomial $\overline{f}(x) = a_0 + a_1x + \ldots + a_nx^n \in \overline{\widetilde{\mathcal{K}}}[x]$ has a root in $\widetilde{K}$ since $f(x) = a_0 + a_1x + \ldots + a_nx^n$ has a root in $\widetilde{\mathcal{O}}$ because $\widetilde{\mathcal{O}}$ is integrally closed in $\widetilde{K}$.

Claim: $\overline{K} = \overline{\widetilde{\mathcal{K}}}$.

Take $x \in \widetilde{\mathcal{O}}^\times$ and let g be the minimal polynomial of $\overline{x}$ over $\overline{\mathcal{K}}$. Consider $g \in \mathcal{O}[x]$, say $g = a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + x^n$. By the previous lemma, pick a separable polynomial $h(x) = x^n + \sum_{i=0}^{n-1} b_ix^i$ such that $v(a_i - b_i) > 0\ \forall 1 \leq i < n$, then $g(x) - h(x) \in \mathcal{O}[x]$.

Hence, $h(x) = g(x) - (g(x) - h(x)) \in \mathcal{O}[x]$. Further, note that $v(a_i - b_i) > 0 \implies g - h \in \mathcal{M}[x]$.

Now, h is separable, hence it has a root z in K. We know that $\mathcal{O}$ is integrally closed in K, hence $z \in \mathcal{O}$.

Now, $\overline{g}(\overline{z}) = \overline{g(z)} = \overline{g(z) - h(z)} = \overline{\sum_{i=0}^{n-1}(a_i - b_i)z^i} = 0$.

As $\overline{g}$ is the minimal polynomial of $\overline{x}$ over $\overline{\mathcal{K}}$, it follows that $\overline{g}$ has degree one and hence $\overline{x} \in \overline{\mathcal{K}}$. This gives $\widetilde{\overline{\mathcal{K}}} \subseteq \overline{\mathcal{K}}$.

Conversely, $\mathcal{O} \subseteq \widetilde{\mathcal{O}}$ and $\mathcal{M} \subseteq \widetilde{\mathcal{M}} \implies \mathcal{O}/\mathcal{M} \subseteq \widetilde{\mathcal{O}}/\widetilde{\mathcal{M}} \implies \overline{\mathcal{K}} \subseteq \widetilde{\overline{\mathcal{K}}}$.

Hence, $\overline{\mathcal{K}} = \widetilde{\overline{\mathcal{K}}}$.

Now, we will show $\Gamma = \widetilde{\Gamma}$. Let $\delta \in \widetilde{\Gamma}$, then by Theorem 1.5.4, $\exists n > 1$ and $a \in K$ such that $n\delta = v(a) \in \Gamma$ where v is a valuation corresponding to $\mathcal{O}$. Take $\delta > 0$ without loss of generality and so $v(a) = n\delta > 0 \implies a \in \mathcal{O}$.

Let $g(x) = x^n - a$. By the previous lemma, there exists a separable polynomial $h(x) = x^n + \sum_{i=0}^{n-1} b_i x^i$ with $v(a_i - b_i) > n\delta \ \forall i = 1, 2, \ldots, n$.

Note that the a_i here are all zero except $a_0 = -a$.

As before, h has a root $z \in \mathcal{O}$ and we have

$$v(g(z)) = v(g(z) - h(z)) \geq \min_{0 \leq i < n} \{v(a_i - b_i) + iv(z)\} > n\delta + \min_{0 \leq i < n} \{iv(z)\} = n\delta = v(a),$$

because $v(z) \geq 0 \implies z \in \mathcal{O}$.

Hence, $v(g(z)) > v(a) \implies v(z^n - a) > v(a) \implies v(z^n) = v(z^n - a + a) = v(a) = n\delta \implies v(z) = \delta \in \Gamma$.

Hence, $\widetilde{\Gamma} \subseteq \Gamma$. As before, $\Gamma \subseteq \widetilde{\Gamma}$ follows trivially.

Therefore $\widetilde{\mathcal{O}}/\mathcal{O}$ is an immediate extension and we are done.

We now show that the integral closure of a valuation ring in an algebraic extension of its field of fractions has a localisation property in the following sense:

Theorem 1.5.8. *Let L be an algebraic extension of a field K, and let $\mathcal{O}$ be a valuation ring of K. Denote by R the integral closure of $\mathcal{O}$ in L and let $\mathcal{O}'$ be a prolongation of $\mathcal{O}$ to L. If $\mathcal{M}'$ is the maximal ideal of $\mathcal{O}'$ and $\mathfrak{m} = \mathcal{M}' \cap R$, then $R_{\mathfrak{m}} = \mathcal{O}'$.*

Proof. The containment $R_{\mathfrak{m}} \subset \mathcal{O}'$ is easy: take any $a/b \in R_{\mathfrak{m}}$ i.e., $a \in R, b \in R \setminus \mathfrak{m} \subset \mathcal{M}$. Since $b \notin \mathcal{M}$, it follows that $b^{-1} \in \mathcal{O}'$. Now, $a \in R$ and $\mathcal{O}'$ is integrally closed in L, whence we get $a \in \mathcal{O}' \implies ab^{-1} = a/b \in \mathcal{O}'$.

For the converse, let $x \in \mathcal{O}'$. Set $K_2 := K(x)$, $R_2 := R \cap K_2$, $\mathfrak{m}_2 := \mathfrak{m} \cap K_2$, $\mathcal{O}_2' := \mathcal{O}' \cap K_2$, and $\mathcal{M}_2 := \mathcal{M} \cap K_2$.

Since R is the integral closure of $\mathcal{O}$ in L and $K_2 \subset L$, it follows that R_2 is the integral closure of $\mathcal{O}$ in K_2. Hence, via Corollary 1.5.3.1, we obtain $R_2 = \bigcap \mathcal{O}''$ where $\mathcal{O}''$ ranges over all prolongations of $\mathcal{O}$ to K_2. Note that since x is algebraic over K, we have $[K_2 : K] < \infty \implies [K_2 : K]_s < \infty$. We have seen before that

in this case $n < [K_2 : K]_s$, i.e., n is finite. Hence, applying Lemma 1.5.2. with $\mathfrak{p}_2 = R_2 \cap \mathcal{M}_2 = (R \cap K_2) \bigcap (\mathcal{M} \cap K_2) = (R \cap \mathcal{M}) \bigcap K_2 = \mathfrak{m} \cap K_2 = \mathfrak{m}_2$, we get $\mathcal{O}_2 = (R_2)_{\mathfrak{m}_2}$. Therefore, there exist $a, b \in R_2$ with $b \notin \mathfrak{m}_2$ such that $x = ab^{-1}$. Clearly, $ab^{-1} \in R_{\mathfrak{m}} \implies x \in R_{\mathfrak{m}}$, yielding $\mathcal{O}' \subset R_{\mathfrak{m}}$.

☕

We now focus on normal extensions and prove some results on prolongations of a fixed valuation ring.

Theorem 1.5.9. *Suppose L/K is a finite normal extension of fields, with $G = Aut(L/K)$. Suppose $\mathcal{O}$ is a valuation ring of K, and $\mathcal{O}'$ and $\mathcal{O}''$ are valuation rings in L extending $\mathcal{O}$. Then $\mathcal{O}'$ and $\mathcal{O}''$ are conjugate over K, i.e., there exists $\sigma \in G$ with $\sigma\mathcal{O}' = \mathcal{O}''$.*

Proof. We have $K \subset L \cap K^s \subset L$. Set $L^s := L \cap K^s$. By Corollary 1.5.6.1., every prolongation of $\mathcal{O}$ to L^s can be uniquely extended to L. We also know from Galois theory that there is a canonical isomorphism $\mathrm{Aut}(L/K) \xrightarrow{\sim} \mathrm{Aut}(L^s/K)$. Hence, it suffices to consider the case where L/K is separable.

Let

$$H' = \{\sigma \in G : \sigma\mathcal{O}' = \mathcal{O}'\} \leq G$$

$$H'' = \{\sigma \in G : \sigma\mathcal{O}'' = \mathcal{O}''\} \leq G$$

Claim: For every $\sigma \in H'$, we have $\sigma\mathcal{M}' = \mathcal{M}'$.

This follows because $\sigma\mathcal{O}' = \mathcal{O}'$ and if $\sigma(x) \in \mathcal{O}' \setminus \mathcal{M}'$ for some $x \in \mathcal{M}'$ then $\sigma(x)$ is a unit in $\mathcal{O}'$, i.e., $\sigma(x)^{-1} = \sigma(x^{-1}) \in \mathcal{O}' = \sigma\mathcal{O}' \implies x^{-1} \in \mathcal{O}'$ which contradicts the fact that $x \in \mathcal{M}'$.

Now, $\sigma\mathcal{O}'$ is a valuation ring of L since $\sigma\mathcal{O}' = \mathcal{O}'$. By the above argument, it also follows that its maximal ideal is $\sigma\mathcal{M}'$.

Similarly, for every $\tau \in H''$, we have $\tau\mathcal{O}''$ is a valuation ring of L with maximal ideal $\tau\mathcal{M}''$.

We next write G as the following disjoint unions:

$$G = \coprod_{i=1}^{n} H'\sigma_i^{-1} = \coprod_{j=1}^{m} H''\tau_j^{-1}$$

It then suffices to simply show that there exist i, j such that $\sigma_i\mathcal{O}' \subseteq \tau_j\mathcal{O}''$ or $\sigma_i\mathcal{O}' \supseteq \tau_j\mathcal{O}''$, since this implies $\sigma_i\mathcal{O}' = \tau_j\mathcal{O}''$, by Lemma 1.5.3. We then have $\tau_j^{-1}\sigma_i\mathcal{O}' = \mathcal{O}''$, thus finishing the proof.

Thus, we need only show the existence of such i, j. Proceeding by contradiction, we assume no such i, j exist. Since $\{\sigma_i^{-1}\}_{i=1}^n$ is a transversal for G/H', we must

have $\sigma_k\mathcal{O}' \nsubseteq \sigma_t\mathcal{O}'$ for all $k \neq t$ otherwise we would obtain $\sigma_k\mathcal{O}' = \sigma_t\mathcal{O}'$ using Lemma 1.5.3.

Similarly $\tau_k\mathcal{O}'' \nsubseteq \tau_t\mathcal{O}''$ for all $k \neq t$. Set

$$R = \left(\bigcap_{i=1}^{n} \sigma_i\mathcal{O}'\right) \bigcap \left(\bigcap_{j=1}^{m} \tau_j\mathcal{O}''\right)$$

Using Theorem 1.5.5.(3) with the tuple $(1, \ldots, 1, 0, \ldots, 0) \in \prod_{i=1}^{n} \sigma_i\mathcal{O}' \times \prod_{j=1}^{m} \tau_j\mathcal{O}''$, we obtain $a \in R$ such that

$$a - 1 \in \sigma_i(\mathcal{M}') \text{ for } i = 1, \ldots, n \text{ and}$$

$$a \in \tau_j(\mathcal{M}') \text{ for } j = 1, \ldots, m.$$

Hence, for $\sigma \in G_1$, write $\sigma = \rho\sigma_i^{-1} \in H'\sigma_i \implies \rho \in H'$.

This yields $\sigma(a-1) = \rho\sigma_i^{-1}(a-1) \in \rho\sigma_i^{-1}(\sigma_i(\mathcal{M}')) = \rho(\mathcal{M}') = \mathcal{M}'$.

Similarly, $\sigma(a) \in \mathcal{M}''$ for all $\sigma \in G$.

Now, $N_{L/K}(a) = \prod_{\sigma \in G_1} \sigma(a) \in (\mathcal{M}'' + 1) \cap K = \mathcal{M} + 1$, where the last equality follows by observing that $1 + m' \in K \implies m' \in K \cap \mathcal{M}' = \mathcal{M}$ for any $m' \in \mathcal{M}'$.

Furthermore, we also have $N_{L/K}(a) = \prod_{\sigma \in G_1} \sigma(a) \in \mathcal{M}'' \cap K = \mathcal{M}$.

This is a contradiction, and we are done.

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Theorem 1.5.10 (Conjugation Theorem)**.** *Suppose L/K is an arbitrary normal extension of fields, $\mathcal{O}$ is a valuation ring of K, and $\mathcal{O}'$ and $\mathcal{O}''$ are valuation rings in L extending $\mathcal{O}$. Then there exists $\sigma \in Aut(L/K)$ with $\sigma(\mathcal{O}') = \mathcal{O}''$.*

Proof. Consider Σ, the set of ordered pairs (K_1, σ_1) where K_1 is a normal extension of L/K, i.e. $L/K_1/K$ with K_1/K normal, $\mathcal{O}_1' := \mathcal{O}' \cap K_1$, $\mathcal{O}_1'' := \mathcal{O}'' \cap K_1$ and $\sigma_i \in \mathrm{Aut}(K_1/K)$ such that $\sigma_1(\mathcal{O}_1') = \mathcal{O}_1''$.

Clearly, $\Sigma \neq \emptyset$ since $(K, \mathrm{id}) \in \Sigma$. We may partially order Σ as

$$(K_1, \sigma_1) \leq (K_2, \sigma_2) \iff K_1 \subseteq K_2 \text{ and } \sigma_2|_{K_1} = \sigma_1.$$

Take a chain $(K_i, \sigma_i)_{i \in \mathbb{N}}$ in Σ.

Consider (K, σ) where $K := \bigcup K_i$ and $\sigma|_{K_i} := \sigma_i$.

Since K_m is a composition of countably many normal extensions contained in L, K_m is also a normal extension of L/K. Thus, $(K, \sigma) \in \Sigma$ and we have an upper bound, whence, by Zorn's Lemma, Σ has a maximal element, say (K_m, σ_m).

We have $K \subseteq K_m \subseteq L$ and $\sigma_m(\mathcal{O}_m') = \mathcal{O}_m''$, where $\mathcal{O}_m' = \mathcal{O}' \cap K_m$ and $\mathcal{O}_m'' = \mathcal{O}'' \cap K_m$.

Claim: $K_m = L$.

Suppose not. Pick $\alpha \in L \setminus K_m$. Let f be the minimal polynomial of α over K_m and $N = SF_L(f) \cdot K_m = SF_{K_m}(f) \subseteq L$, where $SF_L(f)$ denotes the splitting field of f over L.

We extend σ_m to an algebraic closure $\widetilde{K}/K$. Then, since N and L are normal, $\sigma_m(L) = L$ and $\sigma_m(N) = N$.

Let $\mathcal{O}^* := \mathcal{O}' \cap N$ and $\mathcal{O}^{**} := \sigma_m^{-1}(\mathcal{O}'' \cap N)$. Consider

$$\mathcal{O}^* \cap K_m = \mathcal{O}' \cap N \cap K_m = \mathcal{O}' \cap (SF_{K_m}(f) \cap K_m) = \mathcal{O}' \cap K_m = \mathcal{O}'_m$$

Similarly,

$$\begin{aligned}
\mathcal{O}^{**} \cap K_m &= \sigma_m^{-1}(\mathcal{O}'' \cap N) \cap K_m \\
&= \sigma_m^{-1}(\mathcal{O}'' \cap N) \cap \sigma_m^{-1}(\sigma_m(K_m)) \\
&= \sigma_m^{-1}(\mathcal{O}'' \cap N \cap \sigma_m(K_m)) \\
&= \sigma_m^{-1}(\mathcal{O}'' \cap (SF_{K_m}(f) \cap K_m)) \\
&= \sigma_m^{-1}(\mathcal{O}'' \cap K_m) = \sigma_m^{-1}(\mathcal{O}''_m) = \mathcal{O}'_m
\end{aligned}$$

We now apply the previous theorem with $L \mapsto N$, $K \mapsto K_m$, $\mathcal{O}' \mapsto \mathcal{O}^*$ and $\mathcal{O}'' \mapsto \mathcal{O}^{**}$. Note that N is a splitting field, hence normal. We then get an automoprhism $\sigma \in \mathrm{Aut}(N/K_m)$ with $\sigma(\mathcal{O}^*) = \mathcal{O}^{**}$.

Then, $(\sigma_m \circ \sigma)(\mathcal{O}' \cap N) = \sigma_m(\sigma(\mathcal{O}^*)) = \sigma_m(\mathcal{O}^{**}) = \mathcal{O}'' \cap N$.

Thus, $(N, \sigma_m \circ \sigma) > (K_m, \sigma_m)$ and $(N, \sigma_m \circ \sigma) \in \Sigma$.

This contradiction proves the claim.

Finally, we conclude this section by collecting some useful properties of normal extensions:

Proposition 1.5.1. *Let N be a normal extension of a field K, $\mathcal{O}$ a valuation ring of K, and $\mathcal{O}'$ a valuation ring of N lying over $\mathcal{O}$. Let $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ and $v' : N \twoheadrightarrow \Gamma' \cup \{\infty\}$ be valuations corresponding to $\mathcal{O}$ and $\mathcal{O}'$, respectively, and assume that $v'|_K = v$. Denote the residue class field of $\mathcal{O}'$ by $\overline{N}$.*

1. *For $\sigma \in Aut(N/K)$, the map $v' \circ \sigma$ is the unique valuation of N corresponding to $\sigma^{-1}(\mathcal{O}')$ and Γ'. In particular, if $\sigma(\mathcal{O}') = \mathcal{O}'$, then $v' \circ \sigma = v'$.*

2. *$\overline{N}/\overline{\mathcal{K}}$ is a normal extension.*

3. *The map $\sigma^{-1}(\mathcal{O}') \to \overline{N} : x \mapsto \sigma(x)$ is a ring homomorphism which induces a $\overline{\mathcal{K}}$-isomorphism from $\sigma^{-1}(\mathcal{O}')/\sigma^{-1}(M') \to \overline{N}$ satisfying $\overline{\sigma}(u + \sigma^{-1}(\mathcal{M}')) = \overline{\sigma(u)}$ for every $u \in \sigma^{-1}(\mathcal{O}')$. In particular, if $\sigma(\mathcal{O}') = \mathcal{O}'$, then $\overline{\sigma} \in Aut(\overline{N}/K)$.*

4. $e(\sigma^{-1}(\mathcal{O}')/\mathcal{O}) = e(\mathcal{O}'/\mathcal{O})$ *and* $f(\sigma^{-1}(\mathcal{O}')/\mathcal{O}) = f(\mathcal{O}'/\mathcal{O})$, *for every* $\sigma \in Aut(N/K)$.

Proof. ☕

Chapter 2

Class Field Theory

In this chapter, we shall develop abstract class field theory, followed by local class field theory. We closely follow the treatment given in [Neu99], particularly, we aim to cover the majority chapters IV–V and portions from chapter VI.

2.1 Preliminaries

We quickly introduce notions from Infinite Galois Theory and recall some results, particularly, projective limits and the absolute Galois group of a field,

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