ON THE CALCULUS OF SEQUENCES

DAKSH DHEER
(SECOND YEAR, HANSRAJ COLLEGE, 2022)

Abstract. In this paper, we introduce the field of discrete calculus, which is concerned with finding closed forms for expressions for sums of the form $\sum_a^b f(x)$. It might be useful to think of discrete calculus as the calculus of integers, as opposed to the real numbers.
We define the discrete counterparts of the derivative (the difference) and the integral (the sum), and finally introduce the Gregory-Newton formula, the analogue of the Taylor series in differential calculus, which is used to solve various problems, primarily finding the general term of a given sequence. We shall illustrate these applications in the paper.

1. The Discrete Derivative

We define the **discrete derivative**, the counterpart of the regular continuous derivative, of a given function as follows:

$$\Delta f(x) := f(x+1) - f(x)$$

We also define concretely the notion of a **falling power** as follows:
The expression $x^{\underline{m}}$ denotes the m^{th} falling power, where m is a non-negative integer, of x and is defined as:

$$x^{\underline{m}} := x(x-1)(x-2)\cdot\dots\cdot(x-(m-1))$$

As an example, let us calculate

$$\Delta x^{\underline{3}} = (x+1)^{\underline{3}} - x^{\underline{3}} = (x+1)(x)(x-1) - x(x-1)(x-2) = 3x(x-1)$$

.
We define the negative falling powers, in the following way, for maintaining consistency with Theorem 1:

$$x^{\underline{-m}} := \frac{1}{(x+1)(x+2)\cdot\dots\cdot(x+m)}$$

Theorem 1. $\Delta x^{\underline{m}} = m \cdot x^{\underline{m-1}}$

We first prove the case when m is non-negative.

Proof:
$$\begin{aligned}\Delta x^{\underline{m}} &= (x+1)^{\underline{m}} - x^{\underline{m}}\\ &= (x+1)x(x-1)\cdots(x-m+2) - x(x-1)\cdots(x-m+2)(x-m+1)\\ &= (x+1-x+m-1)x(x-1)\cdots(x-m+2)\\ &= m\cdot x^{\underline{m-1}}\end{aligned}$$

For the case when m is negative, we have:

$$\begin{aligned}
\Delta x^{\underline{-m}} &= (x+1)^{\underline{-m}} - x^{\underline{-m}} \\
&= \frac{1}{(x+2)(x+3)\cdot\cdots\cdot(x+m+1)} - \frac{1}{(x+1)(x+2)\cdot\cdots\cdot(x+m)} \\
&= \frac{(x+1)-(x+m+1)}{(x+1)(x+2)\cdot\cdots\cdot(x+m+1)} \\
&= \frac{m}{(x+1)(x+2)\cdot\cdots\cdot(x+m+1)} \\
&= m \cdot x^{\underline{-m-1}}
\end{aligned}$$

□

The higher order discrete derivatives $\Delta^2 f(x)$, $\Delta^3 f(x)$, $\cdots \Delta^n f(x)$, and so on, may be defined similarly using the relation:

$$\Delta^{k+1} f(x) = \Delta(\Delta^k f(x))$$

.

Properties of the Discrete Derivative

$$\Delta[f(x)+g(x)] = \Delta f(x) + \Delta g(x)$$

$$\Delta[\beta \cdot f(x)] = \beta \cdot \Delta f(x), \quad \beta = \text{ constant}$$

$$\begin{aligned}
\Delta[f(x)\cdot g(x)] &= f(x)\cdot\Delta g(x) + g(x+1)\cdot\Delta f(x) \\
&= g(x)\cdot\Delta f(x) + f(x+1)\cdot\Delta g(x) \\
&= f(x)\cdot\Delta g(x) + g(x)\cdot\Delta f(x) + \Delta f(x)\cdot\Delta g(x)
\end{aligned}$$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\Delta f(x) - f(x)\cdot\Delta g(x)}{g(x)\cdot g(x+1)}$$

We also may find out the discrete counterpart of the exponential by solving the difference equation $\Delta c^n = c^n$ as follows:

$$\begin{aligned}
\Delta c^n = c^n &\implies c^{n+1} - c^n = c^n \\
&\implies c^n(c-1) = c^n \\
&\implies (c-1) = 1 \\
&\implies c = 2
\end{aligned}$$

Hence 2^n plays the role of the exponential function in the calculus of sequences.

The Gregory-Newton Formula

Theorem 2. *Given a function $f(x)$, which is (n+1)-times discretely differentiable, we have*

$$f(x) = f(a) + \frac{\Delta f(a)}{\Delta x}\frac{(x-a)^{\underline{1}}}{1!} + \frac{\Delta^2 f(a)}{\Delta x^2}\frac{(x-a)^{\underline{2}}}{2!} + \cdots + \frac{\Delta^n f(a)}{\Delta x^n}\frac{(x-a)^{\underline{n}}}{n!} + R_n$$

The remainder R_n is given by

$$R_n = \frac{f^{\underline{n+1}}(\eta)(x-a)^{\underline{n+1}}}{(n+1)!}$$

where η lies between a and x.

We shall prove the above theorem for the simpler case where $f(x)$ is a polynomial and $a = 0$:
Proof: If $f(x)$ is a polynomial of degree n we can write it as a factorial polynomial, i.e.

$$f(x) = A_0 + A_1 x^{\underline{1}} + A_2 x^{\underline{2}} + A_3 x^{\underline{3}} + \cdots + A_n x^{\underline{n}}$$

Then

$$\Delta f(x) = A_1 + 2A_2 x^{\underline{1}} + 3A_3 x^{\underline{2}} + \cdots + nA_n x^{\underline{n-1}}$$

$$\Delta^2 f(x) = 2!A_2 + 3\cdot 2A_3 x^{\underline{1}} + \cdots + n(n-1)A_n x^{\underline{n-1}}$$
$$\Delta^n f(x) = n!A_n$$

Putting $x = 0$ in the above equations we find

$$A_0 = f(0), \quad A_1 = \Delta f|_{x=0}, \quad A_2 = \frac{1}{2!}\Delta^2 f\bigg|_{x=0}, \quad \ldots, \quad A_n = \frac{1}{n!}\Delta^n f\bigg|_{x=0}$$

which we may rewrite as

$$A_0 = f(0), \quad A_1 = \Delta f(0), \quad A_2 = \frac{1}{2!}\Delta^2 f(0), \quad \ldots, \quad A_n = \frac{1}{n!}\Delta^n f(0)$$

Using these, we obtain:

$$f(x) = f(0) + \Delta f(0)x^{\underline{1}} + \frac{1}{2!}\Delta^2 f(0)x^{\underline{2}} + \cdots + \frac{1}{n!}\Delta^n f(0)x^{\underline{n}} \qquad \square$$

The more general formula may be obtained by replacing x by $x - a$, and then putting $x = a$.

Let us write the above formula more compactly by writing $u_0 = f(0)$ in the following way:

$$u_k = u_0 + \Delta u_0 x^{\underline{1}} + \frac{1}{2!}\Delta^2 u_0 x^{\underline{2}} + \cdots + \frac{1}{n!}\Delta^n u_0 x^{\underline{n}}$$

Finding terms of Sequences

Let us consider the following problem:

Problem. The first 5 terms of a sequence are given by 2, 7, 16, 35, 70. Find the general term of the sequence.
We may represent the sequence by $u_0, u_1, u_2, \ldots$ where the general term is u_n. The given terms can be represented in the below table.

n	0	1	2	3	4
u_n	2	7	16	35	70

To calculate the discrete derivatives, let us construct the *difference table* for the given sequence:

u_n	Δu_n	$\Delta^2 u_n$	$\Delta^3 u_n$	$\Delta^4 u_n$
2				
	5			
7		4		
	9		6	
16		10		0
	19		6	
35		16		
	35			
70				

Since $u_0 = 2, \Delta u_0 = 5$ $\Delta^2 u_0 = 4, \Delta^3 u_0 = 6, \Delta^4 u_0 = 0$, then by the Gregory-Newton formula, we have

$$\begin{aligned} u_n &= u_0 + \frac{\Delta u_0 n^{\underline{1}}}{1!} + \frac{\Delta^2 u_0 n^{\underline{2}}}{2!} + \frac{\Delta^3 u_0 n^{\underline{3}}}{3!} + \frac{\Delta^4 u_0 n^{\underline{4}}}{4!} \\ &= 2 + 5n + \frac{4n(n-1)}{2} + \frac{6n(n-1)(n-2)}{6} \\ &= n^3 - n^2 + 5n + 2 \end{aligned}$$

Hence, the general term of the sequence is given by $u_n = n^3 - n^2 + 5n + 2$

We assume here that there *is* some law governing the formation of the terms in the sequence, and that we can determine this law using only the limited information supplied by the first 5 terms, namely that the fourth and higher order differences are zero. Any law of formation is theoretically possible. Thus, we need to keep in mind these two inherent assumptions we made about the sequence.

2. The Discrete Integral

The discrete counterpart of the anti-derivative, or the indefinite integral, is the **indefinite sum**. Let $g(x)$ be a given function and let $f(x)$ be a function satisfying

$\Delta f(x) = g(x)$, then the class of functions satisfying the property

$$\sum g(x)\delta x = f(x) + C$$

where C is an arbitrary constant, is called the *indefinite sum* of $g(x)$.

We denote the discrete anti-derivative operator by Δ^{-1}, which is analogous to the continuous anti-derivative (or integral) operator $\int$, for notational convenience. Thus, we have

$$\Delta^{-1}g(x) = \sum g(x)\delta x = f(x) + C$$

It follows from Theorem 1 and the definition of the discrete anti-derivative that, for $m \neq 1$

$$\Delta^{-1}x^{\underline{m}} = \frac{x^{\underline{m+1}}}{m+1}$$

Properties of the Discrete Anti-Derivative

$$\Delta^{-1}[f(x) + g(x)] = \Delta^{-1}f(x) + \Delta^{-1}g(x)$$

$$\Delta^{-1}[\beta \cdot f(x)] = \beta \cdot \Delta^{-1}f(x), \quad \beta = \text{ constant}$$

$$\Delta^{-1}[f(x) \cdot \Delta g(x)] = f(x) \cdot g(x) - \Delta^{-1}[g(x+1) \cdot \Delta f(x)]$$

Let us now work towards defining the **discrete definite integral**. Let $g(x)$ be a given function, and let $f(x)$ be a function satisfying $\Delta f(x) = g(x)$, then

$$\Delta^{-1}g(x) \mid_a^b := f(b) - f(a)$$

is defined as the *definite sum* of $g(x)$.

Now, we have the **Fundamental Theorem of Discrete Calculus**:

Theorem 3. $\sum_a^{b-1} g(x) = \Delta^{-1}g(x) \mid_a^b$

Proof: We have $\Delta f(x) = f(x+1) - f(x) = g(x)$, thus

$$\begin{aligned}
\sum_a^{b-1} g(x) &= \sum_a^{b-1} f(x+1) - f(x) \\
&= f(a+1) - f(a) + f(a+2) - f(a+1) + ... + f(b) - f(b-1) \\
&= f(b) - f(a) = \Delta^{-1}g(x) \mid_a^b
\end{aligned}$$

□

Finding sums of Series

While the discrete derivative helped us with sequences, using the Gregory-Newton formula, it seems fitting that the discrete integral shall help us calculate sums of series. Let us see a few examples to illustrate this:

Problem. Compute the sum $\sum_{x=1}^{n}\sum_{y=1}^{n}(x+y)^2$

We shall convert the exponent to a falling power by observing that $z^{\underline{2}} = z(z-1) = z^2 - z = z^2 - z^{\underline{1}}$, i.e., $z^2 = z^{\underline{2}} + z^{\underline{1}}$. Replacing z by $x+y$, we obtain:

$$\begin{aligned}
\sum_{x=1}^{n}\sum_{y=1}^{n}(x+y)^2 &= \sum_{1}^{n+1}\sum_{1}^{n+1}(x+y)^2\delta y\delta x = \sum_{1}^{n+1}\sum_{1}^{n+1}(x+y)^{\underline{2}} + (x+y)^{\underline{1}}\ \delta y\delta x \\
&= \sum_{1}^{n+1} \left.\frac{(x+y)^{\underline{3}}}{3}\right|_1^{n+1} + \left.\frac{(x+y)^{\underline{2}}}{2}\right|_1^{n+1}\ \delta x \\
&= \sum_{1}^{n+1} \frac{(x+n+1)^{\underline{3}}}{3} + \frac{(x+n+1)^{\underline{2}}}{2} - \frac{(x-1)^{\underline{3}}}{3} - \frac{(x-1)^{\underline{2}}}{2}\ \delta x \\
&= \left.\frac{(x+n+1)^{\underline{4}}}{12} + \frac{(x+n+1)^{\underline{3}}}{6} - \frac{(x-1)^{\underline{4}}}{12} - \frac{(x-1)^{\underline{3}}}{3}\right|_1^{n+1} \\
&= \frac{(2n+2)^{\underline{4}}}{12} + \frac{(2n+2)^{\underline{3}}}{6} - \frac{(n+2)^{\underline{4}}}{6} - \frac{(n+2)^{\underline{3}}}{3}
\end{aligned}$$

Problem. Sum the series $\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + \cdots$ to n terms.

We conclude that the nth term of the sequence associated with the series is $a_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$. Now we observe that

$$(x-2)^{\underline{-3}} = \frac{1}{x(x+2)(x+4)}$$

Thus, we may choose $h = 2$ to obtain the required sum by using the Fundamental Theorem as follows:

$$\begin{aligned}
\sum_{1}^{2n-1}(x-2)^{\underline{-3}} &= \Delta^{-1}(x-2)^{\underline{-3}}\Big|_1^{2n+1} \\
&= \left.\frac{(x-2)^{\underline{-2}}}{(-2)(2)}\right|_1^{2n+1} = \left[\frac{(2n-1)^{\underline{-2}}}{-4}\right] - \left[\frac{(-1)^{\underline{-2}}}{-4}\right] \\
&= \left[-\frac{1}{4}\cdot\frac{1}{(2n+1)(2n+3)}\right] - \left[-\frac{1}{4}\cdot\frac{1}{(1)(3)}\right] \\
&= \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}
\end{aligned}$$

If we wish to sum the series to infinity, we may simply let $n \longrightarrow \infty$ and obtain the infinite sum $\sum_{1}^{\infty}\frac{1}{(2n-1)(2n+1)(2n+3)} = \frac{1}{12}$

Stirling Numbers of the Second Kind

Definition: Stirling Numbers of the Second Kind count the number of ways of partitioning n distinct objects into k non-empty sets. For example, there are seven ways to partition a set having four elements into two subsets: $\{a,b,c\}\cup\{d\}$,

$\{a,b,d\}\cup\{c\}$, $\{d,b,c\}\cup\{a\}$, $\{a,d,c\}\cup\{b\}$, $\{a,b\}\cup\{d,c\}$, $\{c,b\}\cup\{d,a\}$, and $\{a,b\}\cup\{d,c\}$. Thus, $\left\{ \begin{array}{c} 4 \\ 2 \end{array} \right\} = 7$.

Theorem 4. *Recursive identity for Stirling Numbers II*

$$\left\{ \begin{array}{c} m \\ k \end{array} \right\} = \left\{ \begin{array}{c} m-1 \\ k-1 \end{array} \right\} + k \left\{ \begin{array}{c} m-1 \\ k \end{array} \right\}$$

Note: We define $\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} := 1$

The most important theorem for calculating Stirling Numbers II exactly is as follows:

Theorem 5.

$$\left\{ \begin{array}{c} m \\ k \end{array} \right\} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \left(\begin{array}{c} k \\ i \end{array} \right)(k-i)^m = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \frac{k!}{i!(k-i)!}(k-i)^m$$

Theorem 6. *Conversion of exponents to falling powers*

$$x^m = \sum_{k=0}^{m} \left\{ \begin{array}{c} m \\ k \end{array} \right\} x^{\underline{k}}$$

where $\left\{ \begin{array}{c} m \\ k \end{array} \right\}$ *is a Stirling number of the second kind.*

References

[1] M Spiegel, *Schaum's Outline of Calculus of Finite Differences and Difference Equations*, United States: Tata Mcgraw-Hill Publishing Company Limited, 1971. ISBN 9780070602182.

[2] Article. Finite Calculus: A Tutorial for Solving Nasty Sums

[3] R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete mathematics*, AddisonWesley, Reading, MA, 1989.