ON THE CALCULUS OF VARIATIONS

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Abstract. In this paper, we introduce the field of variational calculus, which is concerned with finding the extrema of functionals: mappings from a set of functions to the real numbers . In other words, it answers the question: given a set of initial conditions, which *function* is the most optimal for the given problem? In this paper, we first derive the Euler-Lagrange equation. Then, we use it to prove a well-known fact about planes, and finally move on to solving Johann Bernoulli's Brachistochrone problem.

Keywords: *Brachistochrone, Euler-Lagrange Equation, Functional optimization, Geodesics, Variational Calculus*

Introduction

The calculus of variations is concerned with the discovery of extrema and can thus be considered a subfield of optimization. However, the issues and solutions in this area are significantly different from those involving the extrema of functions of many variables, owing to the domain's nature on the quantity to be optimised.
A **functional** is a relationship between a collection of functions and the real numbers. The calculus of variations is concerned with the discovery of extrema for functionals rather than functions. Thus, the candidates for an extremum are functions rather than vectors in $\mathbb{R}^n$, lending the subject a different character. The functionals are typically defined by definite integrals; the sets of functions are frequently defined by boundary constraints and smoothness requirements introduced during the problem/model development.

1. The Euler-Lagrange Equation

The central problem of the calculus of variations is to find extrema of functionals. The essential notion is as follows: replace the unknown function f in the functional $\mathcal{F}\{f\}$ by $f+\eta$, where $\eta(x)$ is a "small" function. The variation of $\mathcal{F}$ under this change is

$$\delta\mathcal{F} = \mathcal{F}(\{f+\eta\}) - \mathcal{F}(\{f\});$$

If f is the function that defines the functional extremal, the variation in η must disappear to first order. Consider a functional that is entirely dependent on $f(x)$, $f'(x)$, and explicitly on x:

$$\mathcal{F}(\{f\}) = \int_a^b \Phi\left(f(x), f'(x), x\right) dx$$

Clearly,

$$\begin{aligned}\delta\mathcal{F}(\{f\}) &= \int_a^b \left[\Phi\left(f(x)+\eta(x), f'(x)+\eta'(x), x\right) - \Phi\left(f(x), f'(x), x\right)\right] dx \\ &= \int_a^b \left[\frac{\partial\Phi}{\partial f}\eta + \frac{\partial\Phi}{\partial f'}\eta' + \dots\right] dx\end{aligned}$$

After that, we zero off the linear terms. We are now confronted with the challenge of assuming we know the "little" deviation η, but in reality we have no idea what its derivative is. The goal is to eliminate the derivative by re-expressing the the term containing $(\eta)'$. For this, we employ integration by parts:

$$\begin{aligned}\int_a^b \frac{\partial\Phi}{\partial f'}\eta'(x)dx &= \int_a^b \frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\eta\right)dx - \int_a^b \eta\frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\right)dx \\ &= \left(\frac{\partial\Phi}{\partial f'}\eta\right)\Bigg|_a^b - \int_a^b \eta\frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\right)dx\end{aligned}$$

We now see that in the majority of circumstances, we wish to specify the values $f(a)$ and $f(b)$. For instance, in the brachistochrone question that follows, the end points (x, y) are fixed. As a result, the function $\eta(x)$ must disappear at the interval's endpoints, and therefore the perfect derivative term vanishes:

$$\left(\frac{\partial\Phi}{\partial f'}\eta\right)\Bigg|_a^b = 0$$

and we can write, to first order in η,

$$\delta\mathcal{F} = \int_a^b \left[\frac{\partial\Phi}{\partial f} - \frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\right)\right]\eta(x)dx = 0$$

Now the variation $\eta(x)$ can be any function that vanishes at $x = a$ and $x = b$. Hence, it is quite arbitrary. This means that it must be the case such that

$$\frac{\partial\Phi}{\partial f} - \frac{d}{dx}\left(\frac{\partial\Phi}{\partial f'}\right) = 0$$

This is called the **Euler-Lagrange equation**, and is vital to the calculus of variations, as we shall see in the successive sections.

Geodesic on a Plane

As a first application, let us consider a well-known problem whose solution is widely known, to get a sense for how to use the Euler-Lagrange equation.
Let S be a surface, and let p_0, p_1 be two distinct points on S. The geodesic problem concerns finding the curve(s) on S with endpoints p_0, p_1 for which the arclength is minimum. A curve having this property is called a **geodesic**.
Using calculus of variations, we shall show the well-known fact that the solution to the geodesic problem, on a plane, is a straight line, i.e., given two points on a plane, the shortest route connecting them is a straight line.
Let (x_0, y_0) and (x_1, y_1) be two arbitrary points. The arclength of a curve described by $y(x), x \in [x_0, x_1]$ is given by

$$J(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2}dx.$$

The geodesic problem in the plane entails determining the function y such that the arclength is minimum. If y is an extremal for J then the Euler-Lagrange equation must be satisfied; hence,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) - 0 = 0$$

i.e.,

$$\frac{y'}{\sqrt{1+y'^2}} = \text{ constant}$$

The last equation is equivalent to the condition that $y' = c_1$, where c_1 is a constant. Consequently, an extremal for J must be of the form

$$y(x) = c_1 x + c_2,$$

where c_2 is another constant of integration. Thus, the only extremal y is given by $y(x) = c_1 x + c_2$, which describes the line segment from (x_0, y_0) to (x_1, y_1) in the plane (as expected). This proves the claim.

The Brachistochrone

The history of calculus of variations began with a problem set by Johann Bernoulli (1696) as a challenge to the mathematical community, specifically to his brother Jacob. Johann's problem was to **identify the form of a wire along which a bead initially at rest glides under gravity from one end to the other in the shortest amount of time.** The wire's ends are given, and the bead's motion is considered to be frictionless.

Let us use Cartesian coordinates to model the problem with positive $z-$axis taken to be in the direction of gravitational force. Let (a, z_0) be the initial position and (b, z_1) denote the final position of the bead such that $a < b$ and $z_0 < z_1$. We must determine, among the curves that have (a, z_0) and (b, z_1) as endpoints, the curve on which the bead slides down in the smallest amount of time. We know that the velocity of an object, starting at zero initial velocity, is given by

$$v(x) = \sqrt{2gz(x)}$$

for all x. Now we denote by $s(x)$ the length of the path from 0 to $(x, z(x))$. Then

$$s(x) = \int_0^x \sqrt{1+z'(\hat{x})^2}\,d\hat{x},$$

implying that

$$\frac{ds}{dx} = \sqrt{1+z'(x)^2}$$

Moreover, the length L of the whole path is given by

$$L = \int_a^b \sqrt{1+z'(x)^2}\,dx$$

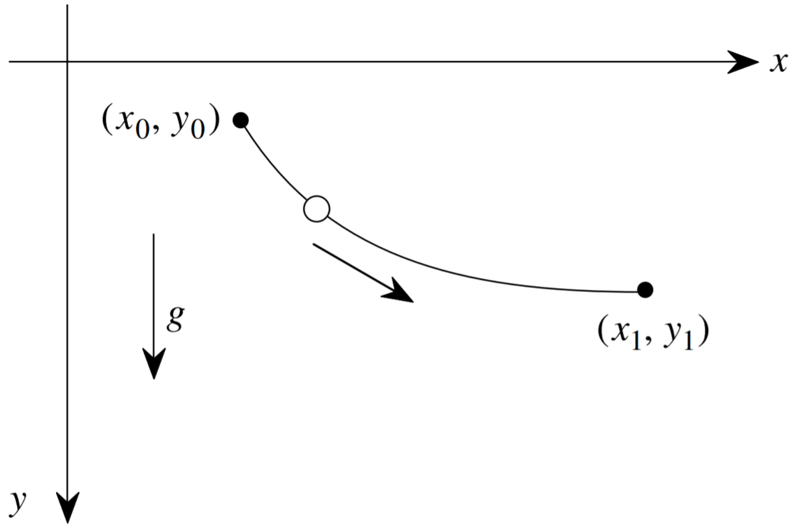

FIGURE 1. The brachistochrone problem

Now we switch from the space variable x to the time variable t. By definition of velocity, we have

$$v(t) = \frac{ds}{dt}$$

or

$$\frac{dt}{ds} = \frac{1}{v(s)}$$

Therefore, if we denote by T the total travel time, we obtain (after some changes of variables)

$$T = \int_0^T dt = \int_0^L \frac{1}{v(s)} ds = \int_a^b \frac{1}{\sqrt{2gz(x)}} \sqrt{1 + z'(x)^2} dx.$$

Thus we can formulate the brachistochrone problem as the minimization o functional

$$F(z) := \int_a^b \frac{\sqrt{1 + z'(x)^2}}{\sqrt{2gz(x)}} dx$$

subject to the constraints $z(a) = z_a$ and $z(b) = z_b$.
The function Φ is

$$\Phi(z, z') = \sqrt{\frac{1 + (z'(x))^2}{z(x)}}$$

(the factor $2g$ is immaterial) and the Euler-Lagrange equation becomes

$$\frac{\partial \Phi}{\partial z} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial z'}\right)$$

$$\equiv -\frac{1}{2}\sqrt{\frac{1 + (z'(x))^2}{z^3(x)}} - \frac{d}{dx}\left(\frac{z'(x)}{\sqrt{z(x)\left[1 + (z'(x))^2\right]}}\right) = 0$$

To simplify the differential equation, we take z for the variable of integration. Then letting $\dot{x} = dx/dz$ we have

$$\delta \int_{z_a}^{z_b} dz \sqrt{\frac{1+\dot{x}^2}{z}} = 0$$

or (since there is no explicit dependence on x)

$$\frac{d}{dz}\left(\frac{\dot{x}}{1+\dot{x}^2}\sqrt{\frac{1+\dot{x}^2}{z}}\right) = 0.$$

This can be integrated immediately, and squared, to give the first-order equation

$$\frac{1}{z} \cdot \frac{\dot{x}^2}{1+\dot{x}^2} = A^2$$

which leads to the separable form

$$dx = \pm dz \sqrt{\frac{A^2 z}{1 - A^2 z}}$$

We want the $-$ sign because on physical grounds the altitude $z(x)$ must decrease monotonically with x. The solution is the equation of a cycloid,

$$A^2 x + B = \sqrt{A^2 z\,(1 - A^2 z)} - \sin^{-1}\sqrt{A^2 z}$$

where the constants A^2 and B must be adjusted to match the initial and final conditions. Thus, we have solved the Brachistochrone problem!

Conclusion

In this paper, we showed the use of variational calculus to derive the Euler-Lagrange equation which has widespread utility over all areas of science. We also conclusively proved facts about geodesics and presented a solution to the brachistochrone problem. Thus, the calculus of variations is an important and elegant field of mathematics with a lot of applications.

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